VARIOUS MINIMIZATION PROBLEMS INVOLVING THE TOTAL VARIATION IN ONE DIMENSION

BY THOMAS SZNIGIR

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Dr. Haim Brezis

and approved by

New Brunswick, New Jersey

October, 2017

ABSTRACT OF THE DISSERTATION

Various Minimization Problems Involving the Total Variation in One Dimension

by Thomas Sznigir

Dissertation Director: Dr. Haim Brezis

We consider certain minimization problems in one dimension. The first one is the ROF filter, which was originally introduced in the context of image processing. For the one-dimensional case, we show that the problem can be reformulated as a variational inequality, and use this to extend existing regularity results. In addition, we look at the jump set of solutions and investigate its behavior as certain parameters are changed. The second functional to be considered arises in the context of regularized interpolation. The second problem is in the context of regularized interpolation, and the functional to be minimized uses the total variation as a penalty term. This problem is shown to be ill-posed with multiple solutions, and the set of solutions is described. Next, we introduce further regularization methods that lead to unique solutions, and use these regularized solutions to determine special solutions of the original problem. Finally, we consider the functional in the space L^2. To investigate it, the lower semicontinuous envelope is constructed. We then characterize the minimizers of the LSC envelope.

Table of Contents

Abstract ii

1. ROF Filter in One Dimension 1

1.1. Introduction and Basic Results 1

1.1.1. Background 1

1.1.2. The Problem 2

1.1.3. Special Case: Step Functions 7

1.1.4. Stability Under Convergence of f 12

1.2. The Dual Problem and Regularity 17

1.2.1. Dual Problem 17

1.2.2. Regularity 23

1.3. Dependence on the Fidelity Parameter 24

1.3.1. The Jump Set 30

2. Regularized Interpolation 37

2.1. The Problem 37

2.1.1. Introduction 37

2.1.2. Minimizing in $W^{1,1}$ 38

2.1.3. Change of Parameters 45

2.1.4. On the Term $|u(x_i) - f_i|^q$ 45

2.1.5. Adding an Extra Term 54

2.2. Regularization 56

2.2.1. Introduction 56

2.2.2. The term $\int_0^1 |u'|^p$ 57

2.2.3. Adding the term $\epsilon \int_0^1 |u'|^2$ 62

2.2.4. Adding the term $\epsilon \int_0^1 ||u''||^2$ 66

2.3. The Lower Semicontinuous Envelope 80

2.3.1. Introduction 80

2.3.2. On LSC envelopes 81

2.3.3. Constructing the LSC Envelope 84

2.3.4. The Minimizers of G 95

2.3.5. Minimizing $G[u] + \int_0^1 u^2$ 101

Appendix A. Functions of Bounded Variation 114

A.1. Properties of BV Functions 119

Appendix B. Variational Inequalities 124

Appendix C. Approximation of L^2 Functions by Step Functions 128

References 131

Bibliograpy 131

Chapter 1

ROF Filter in One Dimension

1.1 Introduction and Basic Results

1.1.1 Background

Given a function f and a parameter $A > 0$, the problem of minimizing

$$J[u] = \int_\Omega |Du| + A \int_\Omega (f-u)^2 \tag{1.1}$$

was posed by Rudin, Osher, and Fatemi in [21] in the context of image processing, where it was formulated as a constrained minimization problem. The two-dimensional case was the focus originally, but the methods used obtained existence and regularity results in other dimensions as well.

Briefly, the method used was to show that the level sets $\{u > t\}$ of the minimizer satisfied the following minimization problem:

$$\min_{E \subset \Omega} \left\{ Per(E, \Omega) + \int_E (t - f(x))\, dx \right\}$$

Next, one could use properties of the level sets to deduce results about regularity. We will not go into the details here. See [14] for an overview.

Here, we will concentrate on the one dimensional case. In higher dimensions, the behavior of the singular part of the derivative of a BV function can be very complicated. In 1-D, the picture is much simpler. In fact, we can always find nice representatives of BV functions which will let us consider pointwise behavior. This enables us to obtain some stronger results in one dimension.

The plan is as follows. For the remainder of Section 1.1, we will show that unique minimizers of the ROF filter exist, provide a basic regularity result, and give a solution in the special case where f is a step function, with the viewpoint of later using step functions to approximate more general functions. In Section 1.2, we will show that the problem is equivalent to a certain variational inequality, and use this to deduce some further regularity results. In Section 1.3, we will consider the effect of varying the parameter A, in particular showing that the sizes of the jumps of the minimizer are nondecreasing in A.

For convenience, the relevant background information on functions of bounded variation can be found in Appendix A.

1.1.2 The Problem

Here, we will define the ROF filter in one dimension. We prove existence of a unique solution and provide a basic regularity result. Consider the following problem:

Problem 1.1.1. Let $f \in L^2$ and $A > 0$. Minimize the functional

$$J[u] = \int_0^1 |Du| + A \int_0^1 (f-u)^2 \tag{1.2}$$

over the space $BV([0,1])$.

Proposition 1.1.2. *There exists a unique solution to Problem 1.1.1.*

Proof: The functional J in (1.2) is nonnegative. Let $\{u_n\}$ be a minimizing sequence. Then quantities $J[u_n]$ are bounded. Hence, there is some $M > 0$ such that

$$\int_0^1 (f-u_n)^2 < M$$
$$\int_0^1 |Du_n| < M$$

for $n = 1, 2, ...$ This implies that $\{u_n\}$ is bounded in BV. Hence, there is a subsequence $\{u_{n_k}\}$ converging strongly in L^2 to some $u_0 \in BV$. In Corollary A.10 it is shown that the total variation is lower semicontinuous in L^2, and so J is also lower semicontinuous in L^2. Hence,

$$J[u_0] \leq \liminf_{n\to\infty} J[u_n]$$

Since J is strictly convex we conclude that u_0 is the unique minimizer of J in BV.

□

Since the minimizer is of bounded variation, it may contain jumps. In higher dimensions it has been shown that if $f \in BV$ then the jump set of the minimizer u is contained in the jump set of f, except possibly for a set whose $(n-1)$-dimensional Hausdorff measure is zero (see [12]). In one dimension, we have a stronger result:

Proposition 1.1.3. *Let $f \in L^2$ and u be the corresponding solution of Problem 1.1.1. If $x_0 \in (0,1)$ such that $f(x_0-0)$ and $f(x_0+0)$ exist and are finite, then*

$$|u(x_0+0) - u(x_0-0)| \leq |f(x_0+0) - f(x_0-0)| \tag{1.3}$$

Moreover,

$$\mathrm{sgn}(u(x_0+0) - u(x_0-0)) \cdot \mathrm{sgn}(f(x_0+0) - f(x_0-0)) \geq 0$$

Proof: Without loss of generality we may suppose $f(x_0-0) \leq f(x_0+0)$.

If $u(x_0-0) = u(x_0+0)$, then u does not have a jump at x_0 and we're done. Suppose that $u(x_0-0) \neq u(x_0+0)$.

Step 1: We will show $u(x_0-0) < u(x_0+0)$. We do this by contradiction.

Suppose $u(x_0-0) > u(x_0+0)$. Since $f(x_0-0) \leq f(x_0+0)$, this means that either $u(x_0-0) > f(x_0-0)$ or $u(x_0+0) < f(x_0+0)$. The cases are similar, so we will consider the case $u(x_0-0) > f(x_0-0)$.

Let $\lambda = \max\{f(x_0-0), u(x_0+0)\}$ and $h = u(x_0-0) - \lambda$. By our assumptions, $h > 0$. We will construct a function $\tilde{u}$ such that $J[\tilde{u}] < J[u]$.

Choose $\eta > 0$ such that

$$\begin{aligned} u(x) - u(x_0 + 0) &> \frac{h}{3}, \\ |u(x) - f(x)| &> \frac{2h}{3}, \\ |f(x) - f(x_0 - 0)| &< \frac{h}{3} \end{aligned} \tag{1.4}$$

whenever $x \in (x_0 - \eta, x_0)$. Note that

$$u(x_0 + 0) \leq \max\{f(x_0 - 0), u(x_0 + 0)\} < u(x_0 - 0)$$

Let

$$\tilde{u}(x) = \begin{cases} \lambda & \text{if } x_0 - \eta \leq x \leq x_0 \\ u(x) & \text{otherwise} \end{cases} \tag{1.5}$$

The conditions (1.4) imply that on $(x_0 - \eta, x_0)$,

$$\lambda - f(x) \geq -\frac{h}{3},$$

$$f(x) \leq \lambda + \frac{h}{3} < u(x) - \frac{h}{3}$$

Hence,

$$\int_0^1 (\tilde{u} - f)^2 < \int_0^1 (u - f)^2$$

Since $u(x_0 + 0) \leq \lambda \leq u(x_0 - 0)$ on a closed interval containing x_0, we may apply Proposition A.16, and so it holds that

$$\int_0^1 |D\tilde{u}| \leq \int_0^1 |Du|$$

It follows that $J[\tilde{u}] < J[u]$ and so u could not have been the minimizer. The case where $u(x_0 + 0) < f(x_0 + 0)$ is similar. Thus, we may conclude that $u(x_0 - 0) < u(x_0 + 0)$.

Step 2: We will show that if u is a solution to Problem 1.1.1, then $u(x_0 - 0) < u(x_0 + 0)$ implies $f(x_0 - 0) \leq \tilde{u}(x_0 - 0) < \tilde{u}(x_0 + 0) \leq f(x_0 + 0)$.

We distinguish three (not necessarily exclusive) possible arrangements of the left- and right-hand limits of u with those of f, namely:

(i) $f(x_0 - 0) \leq u(x_0 - 0) < u(x_0 + 0) \leq f(x_0 + 0)$

(ii) $u(x_0 - 0) < f(x_0 - 0)$

(iii) $u(x_0 + 0) > f(x_0 + 0)$

Since $u(x_0 - 0) < u(x_0 + 0)$ and $f(x_0 - 0) \leq f(x_0 + 0)$, at least one of (i)-(iii) must be true. If (i) holds, then there is nothing to prove. We will show that neither (ii) nor (iii) can hold under the hypothesis $u(x_0 - 0) < u(x_0 + 0)$. Since (ii) and (iii) are very similar, we will prove our result for (ii).

Suppose then that $u(x_0 - 0) < f(x_0 - 0)$. We will show that this leads to a contradiction. The method of proof will depend on whether $u(x_0 + 0) < f(x_0 - 0)$ or $u(x_0 + 0) \geq f(x_0 - 0)$. We therefore distinguish two cases:

Case 1: $u(x_0 + 0) < f(x_0 - 0)$

Since $u(x_0 - 0) < u(x_0 + 0) < f(x_0 - 0)$, there exists $\eta > 0$ such that

$$f(x) > u(x_0 + 0) > u(x) \tag{1.6}$$

whenever $x \in [x_0 - \eta, x_0]$. Let

$$\tilde{u}(x) = \begin{cases} u(x_0 + 0) & \text{if } x_0 - \eta \leq x \leq x_0 \\ u(x) & \text{otherwise} \end{cases}$$

From (1.6) it follows that $|\tilde{u}(x) - f(x)| < |u(x) - f(x)|$ whenever $x \in [x_0 - \eta, x_0]$. Hence,

$$\int_0^1 (\tilde{u} - f)^2 < \int_0^1 (u - f)^2$$

By Proposition A.16,

$$\int_0^1 |D\tilde{u}| \leq \int_0^1 |Du|$$

This implies $J[\tilde{u}] < J[u]$, which is a contradiction.

Case 2: $u(x_0 + 0) \geq f(x_0 + 0)$.

Let $\alpha = f(x_0 - 0) - u(x_0 - 0)$. Then the assumption (ii) implies $\alpha > 0$. Choose $\eta > 0$ such that

$$f(x) > f(x_0 - 0) - \frac{\alpha}{2} > u(x) \tag{1.7}$$

whenever $x \in [x_0 - \eta, x_0]$. Let

$$\tilde{u}(x) = \begin{cases} f(x_0 - 0) - \dfrac{\alpha}{2} & \text{if } x_0 - \eta \leq x \leq x_0 \\ u(x) & \text{otherwise} \end{cases}$$

By (1.7), $|\tilde{u}(x) - f(x)| < |u(x) - f(x)|$ whenever $x \in [x_0 - \eta, x_0]$. Hence,

$$\int_0^1 (\tilde{u} - f)^2 < \int_0^1 (u - f)^2$$

Moreover, since $u(x_0 - 0) \leq f(x_0 - 0) - \frac{\alpha}{2} \leq u(x_0 + 0)$, we may apply Proposition A.16 and deduce that

$$\int_0^1 |D\tilde{u}| \leq \int_0^1 |Du|$$

This implies $J[\tilde{u}] < J[u]$, which is a contradiction.

We have obtained a contradiction both cases, so statement (ii) is not consistent with the assumption $u(x_0 - 0) < u(x_0 + 0)$. We may similarly prove that statement (iii) is not consistent with this either. Hence, statement (i) must hold, i.e.

$$f(x_0 - 0) \leq u(x_0 - 0) < u(x_0 + 0) \leq f(x_0 + 0)$$

Note that the statement $f(x_0 - 0) < f(x_0 + 0)$ is a consequence of our assumptions. This implies (1.3).

We have also shown that if $f(x_0 - 0) < f(x_0 + 0)$, then either u is continuous or

$$f(x_0 - 0) \leq u(x_0 - 0) < u(x_0 + 0) \leq f(x_0 + 0)$$

This implies that either $\text{sgn}(u(x_0+0)-u(x_0-0)) = \text{sgn}(f(x_0+0)-f(x_0-0))$ or else $\text{sgn}(u(x_0+0)-u(x_0-0)) = 0$. Hence,

$$\text{sgn}(u(x_0+0)-u(x_0-0)) \cdot \text{sgn}(f(x_0+0)-f(x_0-0)) \geq 0$$

□

Remark: It was shown that if the minimizer satisfies $u(x_0-0) < u(x_0+0)$, then $f(x_0-0) \leq u(x_0-0) < u(x_0+0) \leq f(x_0+0)$. In particular, if one of the one-sided limits of u does not fall into the interval between $f(x_0-0)$ and $f(x_0+0)$, then u must be continuous at x_0.

1.1.3 Special Case: Step Functions

Here, we analyze solutions to Problem 1.1.1 in the case where f is a step function. In this case, the minimizer has a simple form. We provide a method for computing this minimizer.

Proposition 1.1.4. *Let* $0 = x_0 < x_1 < .... < x_n < x_{n+1} = 1$ *and* $E_i = [x_i, x_{i+1})$. *Denote by* χ_{E_i} *the indicator function of* E_i. *If there exist constants* $c_0, c_1, ..., c_n$ *such that*

$$f(x) = \sum_{i=0}^{n} c_i \chi_{E_i}(x) \tag{1.8}$$

then there exist numbers $\lambda_0, \lambda_1, ..., \lambda_n$ *such that the function*

$$u(x) = \sum_{i=0}^{n} \lambda_i \chi_{E_i}(x) \tag{1.9}$$

is the solution of Problem 1.1.1.

Proof: Fix $v \in BV$ and $0 \leq i \leq n$. Choose a left-continuous representative of v and let

$$\rho_i = \inf_{E_i} |v(x) - c_i| \tag{1.10}$$

There exists a sequence $\{x_k\}$ of points in E_i such that

$$\lim_{k\to\infty} |v(x) - c_i| = \rho_i$$

This sequence has a limit point y_0 in the closure of E_i. Since $v \in BV$, left- and right-hand limits exist at every point. This implies that either $|v(y_0 + 0) - c_i| = \rho_i$ or $|v(y_0 - 0) - c_i| = \rho_i$.

If $|v(y_0+0)-c_i| = \rho_i$, let $\lambda_i = v(y_0+0)$. Otherwise, let $\lambda_i = v(y_0-0)$. Then $\lambda_i = c_i \pm \rho_i$. Now consider the function

$$\tilde{v}_i(x) = \begin{cases} \lambda_i & \text{if } x \in E_i \\ v(x) & \text{otherwise} \end{cases}$$

By construction,

$$\begin{aligned} \int_0^1 (\tilde{v}_i - f)^2 &= \int_0^{x_i} (\tilde{v}_i - f)^2 + \int_{E_i} (\tilde{v}_i - f)^2 + \int_{x_{i+1}}^1 (\tilde{v}_i - f)^2 \\ &= \int_0^{x_i} (v - f)^2 + \int_{E_i} (\lambda_i - f)^2 + \int_{x_{i+1}}^1 (v - f)^2 \\ &\le \int_0^{x_i} (v - f)^2 + \int_{E_i} (v - f)^2 + \int_{x_{i+1}}^1 (v - f)^2 \\ &= \int_0^1 (v - f)^2 \end{aligned} \tag{1.11}$$

with the inequality being strict if v is nonconstant on E_i.

Since y_0 is in the closure of E_i and λ_i equals one of the one-sided limits of v at y_0, we may apply Proposition A.16. Thus,

$$\int_0^1 |D\tilde{v}_i| \le \int_0^1 |Dv|$$

Hence,

$$J[\tilde{v}_i] \le J[v]$$

with strict inequality if v is nonconstant on E_i.

Thus, we cannot do better than a constant on E_i. This holds for $i = 1, 2, ..., n$. Hence, (1.2) is minimized by a piecewise constant function if f is piecewise constant, and this is the unique minimizer.

□

This effectively reduces the problem to a finite-dimensional one:

Problem 1.1.5. Let $0 = x_0 < x_1 < .... < x_n < x_{n+1} = 1$ and let $\Delta x_i = x_{i+1} - x_i$. Given numbers $c_0, c_1, ..., c_n$, minimize the function

$$\mathfrak{J}(\lambda_0, \lambda_1, ..., \lambda_n) = \sum_{i=0}^{n} |\lambda_{i+1} - \lambda_i| + A \sum_{i=0}^{n} (\lambda_i - c_i)^2 \Delta x_i \tag{1.12}$$

Next, let us compute an explicit solution. First, we have a very basic result:

Lemma 1.1.6. *Let* $0 = x_0 < x_1 < .... < x_n < x_{n+1} = 1$ *and let numbers* $c_0, c_1, ..., c_n$ *be given. Let* $\lambda_0, \lambda_1, ..., \lambda_n$ *be the corresponding solution of Problem 1.1.5. If* $c_i \leq c_{i+1}$*, then*

$$\lambda_i \leq \lambda_{i+1}$$

Similarly, if If $c_i \geq c_{i+1}$*, then*

$$\lambda_i \geq \lambda_{i+1}$$

Proof: The numbers $c_0, c_1, ..., c_n$ define a function f be as in (1.8). Then, by Proposition 1.1.4, the corresponding solution to Problem 1.1.1 is a function u as in (1.9), with coefficients $\lambda_0, \lambda_1, ..., \lambda_n$ which are the solutions to Problem 1.1.5.

Hence, we may apply Proposition 1.1.3. Since $f \in BV$, this implies that

$$\text{sgn}(u(x+0) - u(x-0)) \cdot \text{sgn}(f(x+0) - f(x-0)) \geq 0$$

for any $x \in (0, 1)$. In particular, we may fix i and let $x = x_i$. In this case, $u(x_i - 0) = \lambda_i$, $u(x_i + 0) = \lambda_{i+1}$, $f(x_i - 0) = c_i$, and $f(x_i + 0) = c_{i+1}$. Hence,

$$\operatorname{sgn}(\lambda_{i+1} - \lambda_i) \cdot \operatorname{sgn}(c_{i+1} - c_i) \geq 0$$

Thus, if $c_i \leq c_{i+1}$, then $\operatorname{sgn}(\lambda_{i+1} - \lambda_i) \geq 0$, so $\lambda_i \leq \lambda_{i+1}$. Likewise, if $c_i \geq c_{i+1}$, then $\lambda_i \geq \lambda_{i+1}$.

□

Next, we try to compute the values of the λ_i. Differentiation of $\mathfrak{J}$ gives:

$$\frac{\partial \mathfrak{J}}{\partial \lambda_i} = \operatorname{sgn}(\lambda_i - \lambda_{i+1}) + \operatorname{sgn}(\lambda_i - \lambda_{i-1}) + 2A(\lambda_i - c_i)\Delta x_i$$

with the obvious modification if $i = 0$ or $i = n$. If the values of $\operatorname{sgn}(\lambda_i - \lambda_{i+1})$ and $\operatorname{sgn}(\lambda_i - \lambda_{i-1})$ are known, then we can solve for λ_i. If $\lambda_i \neq \lambda_{i-1}$ and $\lambda_i \neq \lambda_{i+1}$, then

$$\lambda_i = c_i - \frac{\operatorname{sgn}(\lambda_i - \lambda_{i+1}) + \operatorname{sgn}(\lambda_i - \lambda_{i-1})}{2A\Delta x_i} \tag{1.13}$$

It is also possible that $\lambda_i = \lambda_{i-1}$ or $\lambda_i = \lambda_{i+1}$. Suppose there is a number τ_{jk} with the property that $\lambda_i = \tau_{jk}$ whenever $j \leq i \leq k$, and $\lambda_{j-1} \neq \tau_{jk}$ and $\lambda_{k+1} \neq \tau_{jk}$. We want to minimize

$$\sum_{i=0}^{j-1} |\lambda_{i+1} - \lambda_i| + A \sum_{i=0}^{j-1} (\lambda_i - c_i)^2 \Delta x_i + \sum_{i=k+1}^{n} |\lambda_{i+1} - \lambda_i| + A \sum_{i=k+1}^{n} (\lambda_i - c_i)^2 \Delta x_i$$
$$+ |\lambda_{j-1} - \tau_{jk}| + |\tau_{jk} - \lambda_{k+1}| + A \sum_{i=j}^{k} (\tau_{jk} - c_i)^2 \Delta x_i \tag{1.14}$$

We can differentiate with respect to τ_{jk} and find that it vanishes when

$$\tau_{jk} = \frac{\sum_{i=j}^{k} c_i \Delta x_i}{x_{k+1} - x_j} - \frac{\operatorname{sgn}(\tau_{jk} - \lambda_{k+1}) + \operatorname{sgn}(\tau_{jk} - \lambda_{j-1})}{2A(x_{k+1} - x_j)} \tag{1.15}$$

Remark: Observe that if f is the step function taking the value c_i on E_i, then the first term in the expression for τ_{jk} is the average of f from x_j to x_{k+1}.

Let us recall the combinatorial notion of a composition. A composition of positive integer N is a finite sequence $a_1, a_2, ..., a_k$ of positive integers such that $\sum_{i=1}^{k} a_i = N$. A solution of Problem 1.1.5 determines a composition $n+1$. If we let m be the greatest index such that $\lambda_0 = \lambda_1 = ... = \lambda_{m-1}$, then $a_1 = m$. In the same way we can determine $a_2, a_3, ..., a_k$. Let

$$s_j = \sum_{i=1}^{j} a_j$$

Let the sets $A_1, A_2, ..., A_k$ be defined by

$$A_j = \bigcup_{i=s_{j-1}}^{s_j} E_i \tag{1.16}$$

We will say that these sets form a partition of $[0, 1]$ into contiguous blocks, and refer to A_j as a block.

We summarize the above in the following:

Proposition 1.1.7. *Let* $0 = x_0 < x_1 < .... < x_n < x_{n+1} = 1$ *and let numbers* $c_0, c_1, ..., c_n$ *be given. Let* $\lambda_0, \lambda_1, ..., \lambda_n$ *be the corresponding solution of Problem 1.1.5. Let* $a_1, a_2, ..., a_k$ *be the induced composition of* $n+1$ *and* $A_1, A_2, ..., A_k$ *the corresponding partition of* $[0, 1]$ *into contiguous blocks. If* $E_i \subset A_j$*, then*

$$\lambda_i = \tau_{(s_{j-1} s_j)}$$

where $\tau_{(s_{j-1} s_j)}$ *is given as in (1.15).*

In fact, we may apply Lemma 1.1.6 and write

$$\tau_{jk} = \frac{\sum_{i=j}^{k} c_i \Delta x_i}{x_{k+1} - x_j} - \frac{\operatorname{sgn}(f_k - f_{k+1}) + \operatorname{sgn}(f_j - f_{j-1})}{2A(x_{k+1} - x_j)} \tag{1.17}$$

This gives a way to compute the solution to Problem 1.1.1 if we know the appropriate composition of $n + 1$ in advance. We have not provided a way to determine this from the initial conditions. However, it is possible to brute force the solution by trying every possible composition and using Lemma 1.1.6 to determine the values of $\operatorname{sgn}(\lambda_{i+1} - \lambda_i)$. There are 2^n compositions of $n + 1$, so this method takes exponential time.

1.1.4 Stability Under Convergence of f

The space of step functions is dense in L^2 (see Proposition C.2). We would like to approximate the function f in Problem 1.1.1 with step functions, with the aim of using the simple form of the solutions to deduce properties of the general case. This question will be taken up in Section 3. To facilitate our discussion there, we will present some results on the behavior of J and the minimizers when the function f is varied. In particular, we will show that if there is a sequence f_n converging to f in L^2, then the corresponding sequence of minimizers converges strictly in BV.

To this end, we introduce some notation. Given $f \in L^2$, let

$$J^f[u] = \int_0^1 |Du| + A\int_0^1 (u-f)^2$$

and let u^f denote the corresponding minimizer. If we have a sequence $\{f_n\}$, we may use J^n and u^n in lieu of J^{f_n} and u^{f_n}.

We have chosen to use superscripts here to denote dependence on f. In Section 3, we will also investigate dependence on A, and for that application we will use subscripts.

Lemma 1.1.8. *If $\{f_n\} \subset L^2$ and $f_n \to f_0$ in L^2, then $J^n[u^n] \to J^0[u^0]$.*

Proof: For any $u \in BV$,

$$\begin{aligned} |J^0[u] - J^n[u]| &= A\left|\int_0^1 \left[(f_0-u)^2 - (f_n-u)^2\right]\right| \\ &= A\left|\int_0^1 \left[(f_0^2 - f_n^2) + 2u(f_n - f_0)\right]\right| \\ &= A\left|\int_0^1 (f_0 + f_n + 2u)(f_n - f_0)\right| \\ &\leq A||f_0 + f_n + 2u||_2 ||f_n - f_0||_2 \end{aligned} \tag{1.18}$$

Since $f_0, u \in L^2$, and $\{f_n\}$ is a bounded subset of L^2, there exists M such that $||f_0 + f_n + 2u||_2 < M$ for $n = 1, 2, ...$ Hence, (1.18) implies

$$\lim_{n\to\infty} |J^0[u] - J^n[u]| = 0$$

Let $\epsilon > 0$. Then, for sufficiently large n,

$$|J^0[u^0] - J^n[u^0]| < \epsilon$$

Since $J^n[u^n] \leq J^n[u^0]$ for all n, this implies $J^n[u^n] \leq J^0[u^0] + \epsilon$. Hence,

$$\limsup_{n\to\infty} J^n[u^n] \leq J^0[u^0] \tag{1.19}$$

Denote the average value of f by $\bar{f}$. The minimizer u^f satisfies

$$\begin{aligned} \int_0^1 (f - u^f)^2 &\leq J^f[u^f] \\ &\leq J^f[\bar{f}] \\ &= \int_0^1 (f - \bar{f})^2 \end{aligned}$$

It follows that

$$\begin{aligned} ||u^f||_{L^2} &\leq ||f||_{L^2} + ||u^f - f||_{L^2} \\ &\leq ||f||_{L^2} + ||f - \bar{f}||_{L^2} \\ &\leq 2||f||_{L^2} \end{aligned} \tag{1.20}$$

By (1.18), this implies

$$\begin{aligned} |J^0[u^n] - J^n[u^n]| &\leq A||f_0 + f_n + 2u||_2\,||f_n - f_0||_2 \\ &\leq (||f_0||_2 + 5||f_n||_2)||f_n - f_0||_2 \\ &\leq (6||f_0||_2 + 5\epsilon)\epsilon \end{aligned}$$

for sufficiently large n. Thus, for some constant C and ϵ small, $J^0[u^n] \leq J^n[u^n] + C\epsilon$ for sufficiently large n. Since u^0 is a minimizer of J^0, for any n, $J^0[u^0] \leq J^0[u^n]$. Hence,

$$J^0[u^0] \leq \liminf_{n\to\infty} J^n[u^n]$$

It follows from (1.19) that

$$\lim_{n\to\infty} J^n[u^n] = J^0[u^0]$$

□

Proposition 1.1.9. *Under the same hypothesis as in Lemma 1.1.8, $u^n \to u^0$ in L^2.*

Proof: First, we find that

$$|J^0[u^n] - J^0[u^0]| \leq |J^0[u^n] - J^n[u^n]| + |J^n[u^n] - J^0[u^0]|$$

Both quantities on the right hand side to go 0 as $n \to \infty$, so we conclude that

$$\lim_{n\to\infty} |J^0[u^n] - J^0[u^0]| = 0 \tag{1.21}$$

Next, we compute

$$\begin{aligned} J^0[u^0] &\leq J^0\left[\frac{u^n + u^0}{2}\right] \\ &= \int_0^1 \left|D\left(\frac{u^0 + u^n}{2}\right)\right| + A\int_0^1 \left(f_0 - \frac{u^n + u^0}{2}\right)^2 \\ &= \frac{1}{2}\int_0^1 |D(u^n + u^0)| + \frac{A}{2}\int_0^1 ((f_0 - u^0)^2 + (f_0 - u^n)^2) - A\int_0^1 \left(\frac{u^n}{2} - \frac{u^0}{2}\right)^2 \\ &\leq \frac{1}{2}(J^0[u^n] + J^0[u^0]) - A\int \left(\frac{u^n}{2} - \frac{u^0}{2}\right)^2 \end{aligned} \tag{1.22}$$

Hence

$$J^0[u^0] \leq J^0[u^n] - 2A\int \left(\frac{u^n}{2} - \frac{u^0}{2}\right)^2$$

Taking the limit on both sides, it follows from (1.21) and the fact that $A > 0$ that

$$\lim_{n\to\infty} \int \left(\frac{u^n}{2} - \frac{u^0}{2}\right)^2 \leq 0$$

Since this quanity is nonnegative, the limit must be 0.

□

Proposition 1.1.10. *The sequence $\{u^n\}$ converges strictly to u^0 in BV.*

Proof: Lemma 1.1.8 and Proposition 1.1.9 imply that

$$\lim_{n\to\infty}\int_0^1 |Du^n| = \int_0^1 |Du^0|$$

Moreover,

$$\int_0^1 |u^n - u| \leq \left(\int_0^1 (u^n - u)^2\right)^{\frac{1}{2}}$$

whence it follows that $u^n \to u$ in L^1. Therefore, $\{u^n\}$ converges strictly to u^0 in BV.

□

Next, let us recall the notion of Γ-convergence (see [1], Definition 6.12):

Definition 1.1.11. Let (X, d) be a metric space and let $F, F_1, F_2, ... : X \to [0, \infty]$ be functions. We say that the sequence $\{F_n\}$ **Γ-converges** to F if the following two conditions are satisfied:

(i) for any sequence $\{x_n\}$ in X converging to x, the following holds:

$$\liminf_{n\to\infty} F_n(x_n) \geq F(x)$$

(ii) for any $x \in X$ there exists a sequence $\{x_n\}$ converging to x such that

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x)$$

Given a sequence of functions $\{f_n\}$ converging in L^2 to some function f, we would like to determine whether the functionals J^{f_n} Γ-converge to the functional J^f. The definition requires us to define the functionals J^{f_n} on a metric space. The norm on BV is not convenient for this purpose. Instead, we can use strict convergence to define a metric on the space $BV([0, 1])$, namely

$$\rho(u, v) = \int_0^1 |u - v| + \left|\int_0^1 |Du| - \int_0^1 |Dv|\right| \tag{1.23}$$

In this context, we have the following result:

Proposition 1.1.12. *Let* $\{f_n\}$ *be a sequence of functions in* L^2 *converging to some function* f*. Then the functionals* J^{f_n}*, as defined on the metric space* $(BV([0,1]),\rho)$*,* Γ*-converge to* J^f*.*

Proof: Let $\{f_n\}$ be a sequence of functions in L^2 converging to some function f, and consider the corresponding functionals $J^f, J^{f_1}, J^{f_2}, \ldots$. We will check that conditions (i) and (ii) of Definition 1.1.11 are satisfied.

Begin with condition (i). Let $\{v_n\}$ be a sequence strictly converging in BV to a limit function v. This implies that there exists some $M > 0$ such that

$$\int_0^1 |Dv_n| \leq M$$

for $n = 1, 2, \ldots$. Moreover, these functions are all bounded. Since the sequence $\{v_n\}$ converges in L^1 to v, for sufficiently large n it must be the case that

$$||v||_{L^\infty} \geq ||v_n||_{L^\infty} - 2M$$

Hence, the quantities $||v_n||_{L^\infty}$ are uniformly bounded. From the inequality

$$\int_0^1 |v_n - v|^2 \leq ||v - v_n||_{L^\infty} \int_0^1 |v_n - v|$$

it follows that $v_n \to v$ in L^2. Since $f_n \to f$ in L^2 as well, it must be the case that

$$\lim_{n\to\infty} \int_0^1 (f_n - v_n)^2 = \int_0^1 (f - v)^2$$

Since the total variation is lower semicontinuous in L^2, it is true that

$$\liminf_{n\to\infty} \int_0^1 |Dv_n| \geq \int_0^1 |Dv|$$

Hence,

$$\liminf_{n\to\infty} J^{f_n}[v_n] = J^f[v]$$

This concludes the proof of condition (i).

Next, we consider condition (ii). Let $v \in BV$ be given. Since $f_n \to f$ in L^2,

$$\lim_{n\to\infty} \int_0^1 (f_n - v)^2 = \int_0^1 (f - v)^2$$

Hence,

$$\lim_{n\to\infty} J^{f_n}[v] = J^f[v]$$

Condition (ii) is then satisfied by taking the constant sequence defined by $v_k \equiv v$ for $k = 1, 2, ...$

Therefore, the sequence $\{J^{f_n}\}$ Γ-converges to J^f.

□

1.2 The Dual Problem and Regularity

We will reformulate our problem in terms of a variational inequality. Specifically, we relate Problem 1.1.1 to the following:

Problem 1.2.1. Given $f \in L^2$, find $v \in H_0^1([0,1])$ that minimizes

$$L[v] = \frac{1}{2}\int_0^1 (v')^2 + 2A\int_0^1 fv' \tag{1.24}$$

under the constraint $|v| \leq 1$.

See Appendix B for analysis of Problem 1.2.1.

1.2.1 Dual Problem

The following result connects Problem 1.1.1 and Problem 1.2.1:

Theorem 1.2.2. *Let u be the solution of Problem 1.1.1 and v the solution of Problem 1.2.1. Then*

$$v' = 2A(u - f) \tag{1.25}$$

Before presenting the proof, we will give some motivation.

Motivation

The connection between Problem 1.1.1 and Problem 1.2.1 was suggested by H. Brezis in [7]. Here we will present a formal calculation, without concern over whether each step can be rigorously justified. The actual proof relies on a somewhat different argument and will be presented afterward.

We begin by approximating the absolute value function $x \mapsto |x|$. We take a C^2 approximation by interpolating a polynomial on a small interval $[-\epsilon, \epsilon]$. This is given by

$$j_\epsilon(x) = \begin{cases} -x & \text{if } x < -\epsilon \\ -\dfrac{x^4}{8\epsilon^3} + \dfrac{3x^2}{4\epsilon} + \dfrac{3\epsilon}{8} & \text{if } -\epsilon \le x \le \epsilon \\ x & \text{if x} > \epsilon \end{cases}$$

Consider the modified functional

$$J_\epsilon[u] = \int_0^1 j_\epsilon(u') + 2A \int_0^1 (u-f)^2$$

This has the Euler-Lagrange equation

$$-(j_\epsilon'(u'))' + 2A(u-f) = 0 \tag{1.26}$$

with the natural boundary condition $u'(0) = u'(1) = 0$. Define $\beta_\epsilon = j_\epsilon'$, and introduce the new unknown $v = \beta_\epsilon(u')$. Note that the range of β_ϵ is $[-1,1]$, so we always have $|v| \le 1$. Moreover, since u' vanishes at 0 and 1 and $\beta_\epsilon(0) = 0$, we must also have $v(0) = v(1) = 0$. Substituting, this satisfies

$$-v' + 2A(u-f) = 0$$

Differentiate this expression to obtain

$$-v'' + 2Au' - 2Af' = 0$$

Observe that $\beta_\epsilon(u')$ has range $[-1,1]$, and is in fact one to one on $\beta_\epsilon^{-1}((-1,1))$. So we can define $\gamma_\epsilon = \beta_\epsilon^{-1}$ on $(-1,1)$ and extend by continuity to $[-1,1]$. Then $\gamma_\epsilon(v) = u'$, so

$$-v'' + 2A\gamma_\epsilon(v) - 2Af' = 0$$

Now we let $\epsilon \to 0$. Then $\gamma_\epsilon \to 0$ on $(-1,1)$, so

$$-v'' - 2Af' = 0$$

This is the Euler-Lagrange equation corresponding to the functional

$$L[v] = \frac{1}{2}\int_0^1 (v')^2 - 2A\int_0^1 f'v$$

We need $v \in H^1$ for this to be defined. We have observed above that $v(0) = v(1) = 0$, so we want to minimize over the space H_0^1, and we also have the constraint $|v| \leq 1$. In other words, we recover Problem 1.2.1.

Next, we provide the actual proof.

Main Result

Let $f \in L^2$ be given and let v be the corresponding solution to Problem 1.2.1. Then f has a distributional derivative $f' \in H^{-1}$. By Theorem B.3, there exists a Radon measure μ concentrated on the set where $|v| = 1$ such that

$$-v'' = 2Af' + \mu \tag{1.27}$$

in the sense of distributions. Moreover, $\mu \llcorner \{x : v(x) = 1\}$ is nonpositive and $\mu \llcorner \{x : v(x) = -1\}$ is nonnegative. There exists $\hat{u} \in BV$ such that

$$D\hat{u} = -2A\mu \tag{1.28}$$

and

$$v' = 2A(\hat{u} - f) \tag{1.29}$$

We will show that this $\hat{u}$ is the minimizer of (1.2). Recall first the notion of a subdifferential:

Definition 1.2.3. Let I be a functional on $L^2([0,1])$. The **subdifferential** of I at u, denoted $\partial I(u)$, is the set of $h \in L^2$ such that

$$I[w] - I[u] \geq \int_0^1 h(w-u) \tag{1.30}$$

for all $w \in L^2$.

Extend (1.2) to L^2 as follows:

$$J[u] = \begin{cases} \int_0^1 |Du| + A\int_0^1 (f-u)^2 & \text{if } u \in BV \\ +\infty & \text{otherwise} \end{cases} \tag{1.31}$$

Thus extended, J is convex and lower semicontinuous over L^2, so we may show that $\hat{u}$ is a minimizer by demonstrating that $0 \in \partial J(\hat{u})$.

Let $V(u) = V(u, [0,1])$, the total variation of u over $[0,1]$. The full subdifferential of the total variation has been computed in [3]. We shall only need the following special case:

Lemma 1.2.4. *Let v and $\hat{u}$ be as above. Then $-v' \in \partial V(\hat{u})$.*

Proof: We need to show that

$$\int_0^1 |Dw| - \int_0^1 |D\hat{u}| \geq -\int_0^1 v(w-\hat{u})$$

for all $w \in L^2$. If $w \notin BV$, then the left hand side is infinite and so the inequality holds. Suppose then that $w \in BV$.

Let $D\hat{u}^+$ and $D\hat{u}^-$ be the positive and negative components of $D\hat{u}$, respectively. As noted above, $D\hat{u}^+$ is concentrated on the set $\{x : v(x) = 1\}$ and $D\hat{u}^-$ is concentrated on the set $\{x : v(x) = -1\}$. Hence,

$$\int_0^1 vD\hat{u} = \int_0^1 |D\hat{u}|$$

Integrating by parts, it follows that

$$\int_0^1 |D\hat{u}| = -\int_0^1 \hat{u}v'$$

Since $v \in H_0^1$ and $|v| \leq 1$, there exists a sequence $\{\phi_n\} \subset C_0^\infty([0,1])$ such that $||\phi_n||_{L^\infty} \leq 1$ for every n and $\phi_n \to v$ in the H^1 norm. Thus, for any $w \in BV$,

$$\lim_{n\to\infty} \int_0^1 w\phi' = \int_0^1 wv'$$

Hence,

$$-\int_0^1 wv' \leq \sup \left\{ \int_0^1 \psi' w \right\}$$

where the supremum is taken over all $\psi \in C_0^\infty$ such that $|\psi(x)| \leq 1$ for all $x \in [0,1]$. This implies

$$\int_0^1 |Dw| \geq -\int_0^1 wv'$$

Therefore,

$$\int_0^1 |Dw| - \int_0^1 |D\hat{u}| \geq \int_0^1 -v'(w - \hat{u})$$

□

Proof of Theorem 1.2.2: Let

$$I[u] = A\int_0^1 (u-f)^2 \tag{1.32}$$

This has a Gâteaux derivative

$$I'(u;h) = 2A\int_0^1 h(u-f)$$

Since $J = V + I$, this means (see [16], Chapter I, Proposition 5.3)

$$\partial J(\hat{u}) = \partial V(\hat{u}) + \{2A(\hat{u} - f)\} \tag{1.33}$$

By Lemma 1.2.4, $-v' \in \partial V(\hat{u})$. Hence, $0 \in \partial J(\hat{u})$. Therefore, $\hat{u}$ is a minimizer of J, which is unique by Proposition 1.1.2.

□

Corollary 1.2.5. *Let* $\Lambda = \{x : |v(x)| = 1\}$. *Then* $u = f$ *on* Λ *and* u *is constant on any connected component of* $[0,1] \setminus \Lambda$.

Corollary 1.2.6. *There exists* $\eta > 0$ *such that* u *is constant on* $[0, \eta]$ *and* $[1-\eta, 1]$.

Proof: Since $v \in H_0^1$, $v(0) = v(1) = 0$, and by continuity it follows that there exists η such that $|v| < 1$ on $[0, \eta]$ and $[1-\eta, 1]$. The result follows from Corollary 1.2.5.

□

Corollary 1.2.7. *There exist finitely many intervals* $E_1, E_2, ..., E_n$ *such that* $\cup E_i = [0,1]$ *and* $u|_{E_i}$ *is monotone for* $i = 1, 2, ..., n$.

Proof: If $\Lambda = \emptyset$, then u is constant and we're done. Suppose, then, that Λ is nonempty. Let $x_1 = \min\{x : |v(x)| = 1\}$. Without loss of generality we may assume $v(x_1) = 1$. Let E_1 be the maximal subinterval of $[0,1]$ containing x_1 and satisfying $v(x) > -1$ for all $x \in E_1$. Then $Du \llcorner E_1$ is a nonnegative measure, so u is nondecreasing on E_1. If $E_1 = [0,1]$, we're done.

If not, let $x_2 = \sup\{x : x \in E_1\}$. Then $v(x_2) = -1$. Let E_2 be the maximal subinterval of $[0,1]$ containing x_2 and satisfying $v(x) < 1$ for all $x \in E_2$. Then $Du \llcorner E_2$ is a nonpositive measure, so u is nonincreasing on E_2. Note that while E_1 and E_2 are not disjoint, $E_1 \cap E_2$ is a connected subset of $[0,1] \setminus \Lambda$, so u is constant there by Corollary 2.5.

By repeating this process, we get a sequence of intervals $E_1, E_2, ...$ such that u is nondecreasing on sets of the form E_{2k+1} and nonincreasing on sets of the form E_{2k}. Moreover, for every i, there exists $x \in E_i$ such that $v(x) = (-1)^{i+1}$. Since $v \in H_0^1$, it must be the case that $v \in BV$, which then implies that this process terminates after finitely many steps.

□

1.2.2 Regularity

Theorem 1.2.2 enables us to apply results from the theory of variational inequalities directly to Problem 1.1.1. For the remainder of this subsection, we shall use u to denote the solution to Problem 1.1.1.

The following regularity result was shown in [13]. It was originally proven for open, convex domains of finite perimeter in dimension $N \leq 7$.

Theorem 1.2.8. *If $f \in C^{0,\beta}$ locally in $(0,1)$ for some $\beta \in (0,1]$, then $u \in C^{0,\beta}$ locally in $(0,1)$.*

In one dimension, we have the following:

Proposition 1.2.9. *If $f \in W^{1,p}$ for some $1 \leq p \leq \infty$, then $u \in W^{1,p}$.*

Proof: We have $f' \in L^p$. In our case, the results of [10] imply $v'' \in L^p$. From (1.27) it then follows that $u' \in L^p$. Hence, $u \in W^{1,p}$.

□

If we impose some extra regularity on f, we can push this a little bit further.

Proposition 1.2.10. *If $f \in C^{0,1}$ and $f' \in BV$, then $u \in C^{0,1}$ and $u' \in BV$.*

Proof: By a result in [9], since $f' \in BV$, the solution v of Problem 1.2.1 has the property that $v'' \in BV$. From (1.27) it then follows that $u' \in BV$.

□

If f is nonconstant, then u need not have continuous derivatives, as shown in the following example:

Example: Let $f(x) = x$. We compute the solution of Problem 1.2.1. On $[0,1] \setminus \Lambda$, v must satisfy

$$-v'' = 2A$$

This means that v is a quadratic polynomial on any subinterval of $[0,1] \setminus \Lambda$. We have shown that $v \in C^1((0,1))$. This can only be satisfied if Λ has at most one connected component.

If $A < 4$, then $v(x) = Ax(1-x)$ satisfies the Euler-Lagrange equation and lies in the constraint set, so that is the solution and we have $u \equiv 0$. Suppose $A > 4$. We want to find a point ξ_1 and constants B_1, C_1 such that

$$v(x) = -Ax^2 + B_1 x + C_1$$

on $[0, \xi_1]$, $v(0) = 0$, $v(\xi_1) = 1$, and $v'(\xi_1) = 0$. This is satisfied by $\xi_1 = A^{-\frac{1}{2}}$, $C_1 = 0$, and $B_1 = 2\sqrt{A}$. We have a similar problem of finding an interval $[\xi_2, 1]$ over which v is a quadratic polynomial. In this case, it is found that $\xi_2 = 1 - \xi_1$.

From this, we can compute the solution to Problem 1.2.1. It is found to be

$$u(x) = \begin{cases} \dfrac{1}{\sqrt{A}} & \text{if } 0 \le x \le \xi_1 \\ x & \text{if } \xi_1 \le x \le \xi_2 \\ 1 - \dfrac{1}{\sqrt{A}} & \text{if } \xi_2 \le x \le 1 \end{cases}$$

This is not differentiable at ξ_1 or ξ_2, and so $u \notin C^1$.

1.3 Dependence on the Fidelity Parameter

We investigate the dependence of Problem 1.1.1 on the parameter A. When we wish to emphasize the dependence on A, we will use the notation

$$J_A[u] = \int_0^1 |Du| + A \int_0^1 (u-f)^2 \tag{1.34}$$

and the corresponding minimizer will be denoted u_A. We will be making use of sequences of the form $\{A_n\}$, and to keep notation simple we may write u_n and J_n in lieu of u_{A_n} and J_{A_n}, respectively.

Let us consider the effect of changes in A on the solution of Problem 1.1.1.

Lemma 1.3.1. *If $A_1 \le A_2$, then $J_{A_1}[u_{A_1}] \le J_{A_2}[u_{A_2}]$.*

Proof: We have

$$\begin{aligned} J_{A_1}[u_{A_1}] &\le J_{A_1}[u_{A_2}] \\ &= \int_0^1 |Du_{A_2}| + A_1 \int_0^1 (u_{A_2} - f)^2 \\ &\le \int_0^1 |Du_{A_2}| + A_1 \int_0^1 (u_{A_2} - f)^2 + (A_2 - A_1) \int_0^1 (u_{A_2} - f)^2 \\ &= J_{A_2}[u_{A_2}] \end{aligned}$$

□

Proposition 1.3.2. *Let f be given. Then the mapping $A \mapsto J_A[u_A]$ is continuous for $A > 0$.*

Proof: If f is constant, then we're done. So suppose f is nonconstant. Fix A_0. Let $\epsilon > 0$, let $\bar{f}$ be the mean value of f, and let

$$b = \int_0^1 (f - \bar{f})^2$$

Note that for any $A > 0$,

$$J_A[\bar{f}] = A \int_0^1 (f - \bar{f})^2$$

and so the minimizer u_A must satisfy

$$\int_0^1 (f - u_A)^2 \le \int_0^1 (f - \bar{f})^2 \tag{1.35}$$

Choose A such that $A_0 < A < A_0 + \frac{\epsilon}{b}$. Then:

$$0 \le J_A[u_A] - J_{A_0}[u_0]$$

by Lemma 1.3.1. Since u_A minimizes J_A,

$$\begin{aligned} J_A[u_A] - J_{A_0}[u_{A_0}] &\le J_A[u_{A_0}] - J_{A_0}[u_{A_0}] \\ &= (A - A_0) \int_0^1 (f - u_{A_0})^2 \\ &\le (A - A_0) b \\ &< \epsilon \end{aligned}$$

Hence, the mapping is continuous from the right.

Now choose A such that $A_0 > A > A_0 - \frac{\epsilon}{b}$. Then

$$\begin{aligned} 0 &\leq J_{A_0}[u_{A_0}] - J_A[u_A] \\ &\leq J_{A_0}[u_A] - J_A[u_A] \\ &\leq (A_0 - A)\int_0^1 (f - u_A)^2 \\ &\leq (A_0 - A)b \\ &< \epsilon \end{aligned}$$

This implies continuity from the left. Therefore, $A \mapsto J_A[u_A]$ is continuous.

□

Proposition 1.3.3. *If the sequence $\{A_n\}$ converges to A_0, then*

$$\lim_{n\to\infty} \int_0^1 (u_{A_n} - u_{A_0})^2 = 0$$

Proof: By Lemma 1.3.1, there exists $M > 0$ such that

$$J_{A_n}[u_{A_n}] \leq M$$

for $n = 1, 2, ...$ Recalling the expression for J_{A_n}, (1.1.1), this means that sequence of minimizers $\{u_{A_n}\}$ is bounded in BV, so there is a subsequence converging strongly in L^2 to some $u_\infty \in L^2 \cap BV$. Relabel the subsequence as $\{u_{A_n}\}$. Lower semicontinuity of J_{A_0} in L^2 implies

$$J_{A_0}[u_\infty] \leq \liminf_{n\to\infty} J_{A_0}[u_{A_n}]$$

We may write

$$\begin{aligned} J_{A_0}[u_{A_n}] &= \int_0^1 |Du_{A_n}| + A_n \int_0^1 (f - u_{A_n})^2 \\ &= \int_0^1 |Du_{A_n}| + A_0 \int_0^1 (f - u_{A_n})^2 + (A_n - A_0)\int_0^1 (f - u_{A_n})^2 \qquad (1.36) \\ &= J_{A_n}[u_{A_n}] + (A_n - A_0)\int_0^1 (f - u_{A_n})^2 \end{aligned}$$

Since $A_n \to A$ and $\{u_{A_n}\}$ is bounded in L^2, taking the limit inferior on both sides implies

$$\liminf_{n\to\infty} J_{A_0}[u_{A_n}] = \liminf_{n\to\infty} J_{A_n}[u_{A_n}]$$

By Proposition 1.3.2, $\liminf_{n\to\infty} J_{A_n}[u_{A_n}] = J_{A_0}[u_{A_0}]$. Hence,

$$J_{A_0}[u_\infty] \leq J_{A_0}[u_{A_0}]$$

Since u_0 is the unique minimizer of J_{A_0}, this implies $u_\infty = u_0$. Hence, for any sequence $\{A_n\}$ converging to A_0, the sequence $\{u_{A_n}\}$ has a subsequence converging in L^2 to u_0.

$\square$

Corollary 1.3.4. *Let $\{A_n\}$ be a sequence of positive numbers converging to some $A_0 > 0$. Then*

$$\lim_{n\to\infty} \int_0^1 (f - u_{A_n})^2 = \int_0^1 (f - u_{A_0})^2$$

Corollary 1.3.5. *Under the hypotheses of Corollary 1.3.4,*

$$\lim_{n\to\infty} \int_0^1 |Du_{A_n}| = \int_0^1 |Du_{A_0}|$$

Proof: Follows from Corollary 1.3.4 and Proposition 1.3.3.

Corollary 1.3.6. *Both $\displaystyle\int_0^1 (f - u_A)^2$ and $\displaystyle\int_0^1 |Du_A|$ are continuous functions of A for $A > 0$.*

Remark: The above continuity can be used to extend $J_A[u_A]$ to the case $A = 0$, where the minimizer is a constant. In fact, the minimizer of $\int_0^1 (f - c)^2$ over all constants c is the average of f over $[0, 1]$.

Proposition 1.3.7. *$\displaystyle\int_0^1 (f - u_A)^2$ is a nonincreasing in A.*

Proof: Suppose, on the contrary, that there exist A_1 and A_2 such that $A_1 < A_2$ and

$$\int_0^1 (f - u_1)^2 < \int_0^1 (f - u_2)^2$$

Extending (1.2) by continuity to the case $A = 0$, where u_A is the constant $\bar{f}$, it follows from (1.35) that

$$\begin{aligned} \int_0^1 (f - u_A)^2 &\geq \int_0^1 (f - u_1)^2 \\ \int_0^1 (f - u_A)^2 &\geq \int_0^1 (f - u_2)^2 \end{aligned} \tag{1.37}$$

Hence, by continuity, there exists $0 < A_0 < A_2$ such that

$$\int_0^1 (f - u_0)^2 = \int_0^1 (f - u_2)^2$$

Since u_0 minimizes J_{A_0} and u_2 minimizes J_{A_2}, we must also have

$$\int_0^1 |Du_0| = \int_0^1 |Du_2|$$

This implies that u_0 minimizes J_{A_2} and u_2 minimizes J_{A_0}. Since minimizers are unique, we must have $u_0 = u_2$. Let:

$$\alpha_i = \int_0^1 (f - u_i)^2, \qquad \beta_i = \int_0^1 |Du_i|, \qquad i = 0, 1$$

Then

$$\alpha_0 + A_0\beta_0 < \alpha_1 + A_0\beta_1$$

and

$$\alpha_0 + A_2\beta_0 < \alpha_1 + A_2\beta_1$$

Since $\alpha_i + A\beta_i$ is affine in A and $A_0 < A_1 < A_2$, it must also be true that

$$\alpha_0 + A_1\beta_0 < \alpha_1 + A_1\beta_1$$

Hence, $J_{A_1}[u_0] < J_{A_1}[u_1]$, contradicting our assumption that u_1 is a minimizer.

□

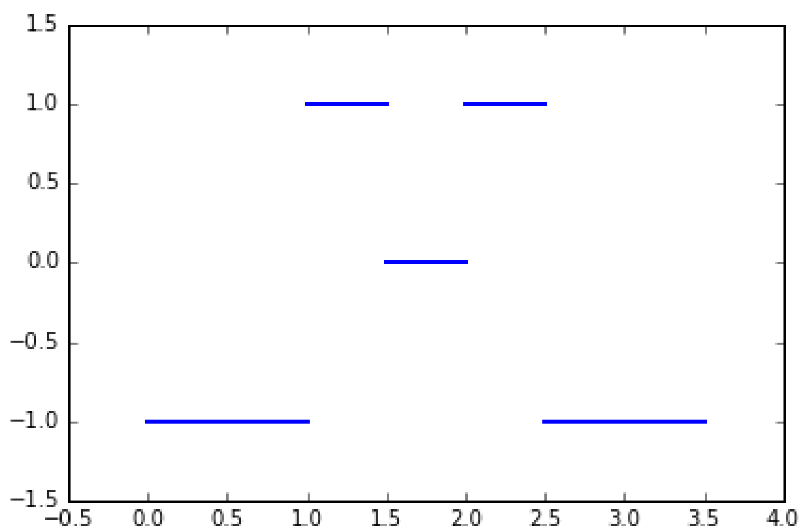
Corollary 1.3.8. *The total variation $V(u_A)$ is a nondecreasing function of A.*

As the following example shows, Proposition 1.3.7 need not hold locally.

Example: Let

$$f(x) = \begin{cases} -1 & \text{if } 0 \leq x < 1 \text{ or } \frac{5}{2} \leq x \leq \frac{7}{2} \\ 0 & \text{if } \frac{3}{2} \leq x < 2 \\ 1 & \text{if } 1 \leq x < \frac{3}{2} \text{ or } 2 \leq x < \frac{5}{2} \end{cases}$$

For reference, here is a plot:



Since f is a step function, we may compute an exact solution, as in Section 1.1.3. For this example, we need only worry about λ_3.

When $A = 2$,

$$\lambda_3 = 0$$

However, when $A = 4$, we find that

$$\lambda_3 = \frac{1}{2}$$

Thus, letting u_2 denote the minimizer corresponding to $A = 2$ and u_4 denote the minimizer corresponding to $A = 4$,

$$\int_{\frac{3}{2}}^{2} (u_2 - f)^2 < \int_{\frac{3}{2}}^{2} (u_4 - f)^2$$

1.3.1 The Jump Set

In contrast to the counterexample at the end of the previous section, we will show that the sizes of the jumps of the minimizers are nondecreasing in A. First we will prove it for step functions, and then we will use that to prove the general case.

Lemma 1.3.9. *Let* $0 = x_0 < x_1 < .... < x_n < x_{n+1} = 1$ *and* $E_i = [x_i, x_{i+1})$. *Let* $c_0, c_1, ..., c_n$ *be real numbers and let*

$$f(x) = \sum_{i=0}^{n} c_i \chi_{E_i}(x)$$

Let u_A *be the corresponding solution of Problem 1.1.1. Then*

$$u_A(x) = \sum_{i=0}^{n} \lambda_i(A) \chi_{E_i}(x)$$

and the mapping $A \mapsto \lambda_i(A)$ *is continuous for* $i = 0, 1, 2, ..., n$.

Proof: This follows from Proposition 1.1.7.

For convenience, when the value of A is fixed, we may write λ_i in lieu of $\lambda_i(A)$.

Proposition 1.3.10. *Let* $0 = x_0 < x_1 < .... < x_n < x_{n+1} = 1$ *and* $E_i = [x_i, x_{i+1})$. *Let* $c_0, c_1, ..., c_n$ *be real numbers and let*

$$f(x) = \sum_{i=0}^{n} c_i \chi_{E_i}(x)$$

Write the minimizer u_A *in the form*

$$u_A(x) = \sum_{i=0}^{n} \lambda_i(A) \chi_{E_i}(x)$$

If $A_1 < A_2$*, then*

$$|\lambda_i(A_1) - \lambda_{i+1}(A_1)| \leq |\lambda_i(A_2) - \lambda_{i+1}(A_2)|$$

Proof: If $\lambda_i(A_1) = \lambda_{i+1}(A_1)$, there is nothing to prove. So suppose without loss of generality that $\lambda_i(A_1) > \lambda_{i+1}(A_1)$.

We show that this implies

$$c_i \geq \lambda_i > \lambda_{i+1} \geq c_{i+1}$$

Suppose not. If $\lambda_i > c_i$, let

$$\tilde{\lambda}_i = \max\{c_i, \lambda_{i+1}\}$$

Then

$$(\tilde{\lambda}_i - c_i)^2 < (\lambda_i - c_i)^2$$

and

$$|\lambda_{i-1} - \tilde{\lambda}_i| + |\tilde{\lambda}_i - \lambda_{i+1}| < |\lambda_{i-1} - \lambda_i| + |\lambda_i - \lambda_{i+1}|$$

which contradicts our hypothesis that u was a minimizer. Hence, $c_i \geq \lambda_i$. We may similarly show that $\lambda_{i+1} \geq c_{i+1}$

Next, we show that $\lambda_{i+1}(A)$ is a nonincreasing function of A. We consider the various cases:

Case 1: $\lambda_{i+1} \neq \lambda_{i+2}$.

This means that λ_{i+1} is not part of any larger contiguous block. By continuity this will be true for all A in some interval $(A_0 - \delta, A_0 + \delta)$. If $i + 1 = n$, then

$$\lambda_n = c_n + \frac{1}{2A\Delta x_n)}$$

The plus sign comes from the hypothesis $\lambda_i > \lambda_{i+1}$. This is a nonincreasing function of A. If $i+1 \neq n$, then either

$$\lambda_{i+1} = c_{i+1} + \frac{1}{A\Delta x_{i+1}}$$

or

$$\lambda_{i+1} = c_{i+1}$$

Either way, $\lambda_{i+1}(A) \geq \lambda_{i+1}(A+\eta)$ for $0 \leq \eta \leq \delta$.

Case 2: There exists $\delta > 0$ such that λ_{i+1} is part of a block that remains both maximal and contiguous for $A \in [A_0, A_0 + \delta)$.

There exists k such that $\lambda_{i+1} = \lambda_{i+2} = ... = \lambda_k$. Let $\tau_{(i+1)k}$ denote their common value. As shown in Proposition 2.7,

$$\tau_{(i+1)k} = \frac{\sum_{j=i+1}^{k} c_j \Delta x_j}{x_{k+1} - x_{i+1}} - \frac{\text{sgn}(\tau_{(i+1)k} - \lambda_{k+1}) + \text{sgn}(\tau_{(i+1)k} - \lambda_i)}{2A(x_{k+1} - x_{i+1})}$$

By hypothesis, $\text{sgn}(\tau_{(i+1)k} - \lambda_i) = -1$. Hence,

$$c_i \geq \lambda_i > \tau_{(i+1)k} \geq \frac{\sum_{j=i+1}^{k} c_j \Delta x_j}{x_{k+1} - x_{i+1}}$$

If the last two terms are equal, then $\lambda_{k+1} < \tau_{(i+1)k}$, and by continuity there is some η such that

$$\lambda_i(A) > \tau_{(i+1)k}(A) > \lambda_{k+1}(A)$$

for all $A \in [A_0, A_0 + \eta)$. Hence, equality continues to hold and λ_{i+1} is nonincreasing in some neighborhood of A_0.

On the other hand, if

$$\tau_{(i+1)k} > \frac{\sum_{j=i+1}^{k} c_j \Delta x_j}{x_{k+1} - x_{i+1}}$$

then $\mathrm{sgn}(\tau_{(i+1)k} - \lambda_{k+1})$ is constant on some half-open interval containing A_0 and our expression for $\tau_{(i+1)k}$ shows it is nonincreasing there.

Case 3: The maximal contiguous block containing λ_{i+1} splits at A_0. More precisely, suppose $\lambda_{i+1} = \lambda_{i+2} = ... = \lambda_k$ and $\lambda_k \neq \lambda_{k+1}$. Then there is a decreasing sequence $A_n \to A_0$ and a sequence $\{a_n\}$ where $i+1 < a_j \leq k$ for $j = 1, 2, ...$ such that $\lambda_{a_n}(A_n)\lambda_{i+1}(A_n)$.

Lemma 1.3.9 and the formulas from Proposition 1.1.7 imply that blocks cannot recombine. Every A determines a corresponding solution u_A of Problem 1.1.1, which in turn determines a partition of $[0,1]$ into contiguous blocks as in Proposition 1.1.7. There are only finitely many such partitions, so there are only finitely many values of A for which blocks split.

Hence, there exists $\delta > 0$ such that no splits occur on $(A_0 - \delta, A_0)$ or $(A_0, A_0 + \delta)$. Then Cases 1 and 2 apply, so λ_{i+1} is nonincreasing on $(A_0 - \delta, A_0)$ or $(A_0, A_0 + \delta)$. The result follows by continuity.

Similarly, we may show λ_i is nondecreasing. Therefore, $|\lambda_i - \lambda_{i+1}|$ is nondecreasing.

□

We will prove the general case by an approximation argument. As we will be varying both the function f and the fidelity constant A, we will introduce some notation to simplify our expressions. Our functional will be denoted

$$J_A^f[u] = \int_0^1 |Du| + A \int_0^1 (u - f)^2 \tag{1.38}$$

and the corresponding minimizer will be denoted u_A^f. As before, if we have indices as in f_m or A_n, we will use J_n^m and u_n^m to simplify things.

Theorem 1.3.11. *If $f \in BV([0,1])$ and $A_1 < A_2$, then*

$$|u_{A_1}(x+0) - u_{A_1}(x-0)| \leq |u_{A_2}(x+0) - u_{A_2}(x-0)| \tag{1.39}$$

for all $x \in (0,1)$.

Proof: Fix $x_0 \in (0,1)$ and let $0 < A_1 < A_2$. Let $u_1 = u_{A_1}$ and $u_2 = u_{A_2}$. If f is continuous at x_0, then so are u_1 and u_2 and we're done. Suppose then that f has a jump at x_0.

Define a sequence $\{f_k\}$ of step functions such that

$$\lim_{k\to\infty} \int_0^1 (f - f_k)^2 = 0$$

The f_k in turn determine sequences $\{u_1^k\}$ and $\{u_2^k\}$ corresponding to solutions of Problem 1.1.1 for A_1 and A_2. These are step functions, so Du_i^k is a purely atomic measure for $i = 1, 2$ and $k = 1, 2, \ldots$. Proposition 3.10 implies that $|Du_1^k|(\{x\}) \leq |Du_2^k|(\{x\})$ for any $x \in (0,1)$.

Hence, for any nonnegative $\psi \in C_0^\infty$,

$$\int_0^1 \psi \, d(|Du_1^k|) \leq \int_0^1 \psi \, d(|Du_2^k|)| \tag{1.40}$$

By Proposition 1.1.10, the sequences $\{u_1^k\}$ and $\{u_2^k\}$ converge strictly in BV, so they also converge weak* in BV. Hence,

$$\lim_{k\to\infty} \int_0^1 \psi \, d(|Du_i^k|) = \int_0^1 \psi \, d(|Du_i|)$$

for $i = 1, 2$. By (1.40), this implies

$$\int_0^1 \psi \, d(|Du_1|) \leq \int_0^1 \psi \, d(|Du_2|)| \tag{1.41}$$

for all $\psi \in C_0^\infty([0,1])$.

Now suppose, on the contrary, that $|u_1(x_0+0) - u_1(x_0-0)| > |u_2(x_0+0) - u_2(x_0-0)|$. Let

$$\epsilon = |u_1(x_0+0) - u_1(x_0-0)| - |u_2(x_0+0) - u_2(x_0-0)|$$

We can decompose u_1 into the sum of a continuous function g_1 and a pure jump function j_1. We may enumerate the jumps of j_1. Excluding a possible jump at x_0, let $y_1, y_2, \ldots$ denote the other jump points. There exists $\delta_1 > 0$ such that

$$\sum_{y_i \in (x_0-\delta_1, x_0+\delta_1)} |u(y_i+0) - u(y_i-0)| < \frac{\epsilon}{6}$$

and, since Dg_1 is a diffuse measure, we may further require that

$$\int_{x_0-\delta_1}^{x_0+\delta_1} |Dg_1| < \frac{\epsilon}{6}$$

Thus,

$$\left| |u_1(x_0+0) - u_1(x_0-0)| \int_{x_0-\delta_1}^{x_0+\delta_1} |Du_1| \right| < \frac{\epsilon}{3} \tag{1.42}$$

We may similarly decompose u_2 into $g_2 + j_2$ and likewise choose δ_2 such that

$$\left| |u_2(x_0+0) - u_2(x_0-0)| - \int_{x_0-\delta_2}^{x_0+\delta_2} |Du_2| \right| < \frac{\epsilon}{3} \tag{1.43}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Let ϕ be a continuous function supported on the interval $(x_0 - \delta, x_0 + \delta)$ such that $\phi(x_0) = 1$ and $0 \leq \phi(x) \leq 1$ for all $x \in [0, 1]$. Then

$$\begin{aligned} \int_0^1 \phi \, d(|Du_1|) &= \int_{x_0-\delta}^{x_0+\delta} \phi \, d(|Du_1|) \\ &= |u_1(x_0+0) - u_1(x_0-0)| + \int_{(x_0-\delta, 0+\delta)\setminus\{x_0\}} \phi \, d(|Du_1|) \end{aligned} \tag{1.44}$$

Since $0 \leq \phi \leq 1$, it follows from (1.42) that

$$0 \leq \int_{(x_0-\delta, x_0+\delta)\setminus\{x_0\}} \phi \, d(|Du_1|) < \frac{\epsilon}{3}$$

Likewise,

$$\int_0^1 \phi \, d(|Du_2|) = |u_2(x_0+0) - u_2(x_0-0)| + \int_{(x_0-\delta, x_0+\delta)\setminus\{x_0\}} \phi \, d(|Du_2|) \tag{1.45}$$

and

$$0 \leq \int_{(x_0-\delta, x_0+\delta)\setminus\{x_0\}} \phi \, d(|Du_2|) < \frac{\epsilon}{3}$$

Hence,

$$\int_0^1 \phi\, d(|Du_1|) - \int_0^1 \phi\, d(|Du_2|)$$
$$\geq |u_1(x_0+0) - u_1(x_0-0)| - |u_2(x_0+0) - u_2(x_0-0)| - \frac{2\epsilon}{3} \quad (1.46)$$

This is strictly positive, which is a contradiction. Therefore,

$$|u_1(x_0+0) - u_1(x_0-0)| \leq |u_2(x_0+0) - u_2(x_0-0)| \quad (1.47)$$

The choice of x_0 was arbitrary, so this holds for all $x \in (0,1)$.

□

Chapter 2

Regularized Interpolation

2.1 The Problem

2.1.1 Introduction

The classical interpolation problem is as follows: given a finite sequence of abscissas $x_1 < x_2 < ... < x_n$ and a set of numbers $f_1, f_2, ..., f_n$, find a function u such that

$$u(x_i) = f_i \qquad \forall i = 1, 2..., n \tag{2.1}$$

We relax this requirement. Instead of mandating equality, we trade goodness of fit off against a penalty term that imposes regularity. Specifically, we consider the following functional:

$$F[u] = A \sum_{i=1}^{n} |f_i - u(x_i)|^2 + \int_0^1 |Du| \tag{2.2}$$

This was posed by H. Berestycki in [5]. To investigate its minimizers, we must choose an appropriate function space. A minimizer would have to be of bounded variation. In addition, it would have to be continuous at $x_1, x_2, ..., x_n$ in order to make sense of the first term in (2.2). We begin by investigating minimizers in the space $W^{1,1}$. Consider the following problem:

Problem 2.1.1. Let $0 < x_1 < x_2 < ... < x_n < 1$ be given, and to every x_i associate a number f_i. Find a function $u \in W^{1,1}$ that minimizes (2.2).

It will be shown that this has many solutions in $W^{1,1}$. In the remainder of Section 2.1, we will prove that these minimizers have certain characteristics in common, namely that they are monotone on the intervals (x_i, x_{i+1}) for $i = 1, 2, ..., (n-1)$ and moreover share a common value at the x_i. Afterward, we consider some variants of the problem and see how things change.

In Section 2.2, we will introduce regularization terms to force a unique solution. Our first method will replace the total variation term in (2.2) by the L^p norm of the derivative, for $p > 1$, which will be shown to have a unique solution in $W^{1,p}$. Our other approach will consider the modifications

$$\begin{aligned} F[u] + \epsilon \int_0^1 |u'|^2 \\ F[u] + \epsilon \int_0^1 |u''|^2 \end{aligned} \tag{2.3}$$

which will force a unique solution to the problem in H^1 and H^2, respectively. Then we will consider the behavior as $\epsilon \to 0$.

In Section 2.3, instead of restricting the space, we expand our scope to L^2. As (2.2) is badly behaved there, we will construct its lower-semicontinuous envelope and investigate that. We will end by partially combining this problem with the ROF filter from Chapter 1 (we take $f = 0$). While in Section 2.1 we show that

$$u \mapsto F[u] + \int_0^1 u^2$$

does not have a minimizer in $W^{1,1}$, in Section 2.3 we show that by replacing F with its lower semicontinuous envelope, the new problem has a minimizer in $L^2 \cap BV$.

Finally, a word on notation. We will associate $x_1, x_2, ..., x_n$ and $f_1, f_2, ..., f_n$ as ordered pairs (x_i, f_i). These will be called *control points.* For simplicity, we will take $A = 1$ in (2.2) except where otherwise noted.

2.1.2 Minimizing in $W^{1,1}$

We consider Problem 2.1.1. The space $W^{1,1}$ is not reflexive. Thus, a bounded sequence need not have a subsequence weakly converging to a limit in $W^{1,1}$. In general, minimizing sequences of (2.2) can at best be expected to have a weak limit in BV.

As a result, we will not use general principles to prove existence. Instead, we will explicitly construct solutions in $W^{1,1}$. To this end, we begin by considering a finite-dimensional problem in terms of the numbers $u(x_i)$ appearing in the first term on the right-hand side in (2.2).

We introduce the auxiliary problem:

Problem 2.1.2. Given control points $(x_1, f_1), ..., (x_n, f_n)$, find $u_1, u_2, ..., u_n$ that minimize the function

$$\mathfrak{F}(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} |u_{i+1} - u_i| \tag{2.4}$$

This is a finite, strictly convex and coercive function on R^n, so a unique minimizer exists. We shall henceforth use the symbols $U_1, U_2, ..., U_n$ to denote the points that solve Problem 2.1.2. When convenient, we may represent them as an n-dimensional vector

$$\mathbf{U} = (U_1, U_2, ..., U_n) \tag{2.5}$$

Proposition 2.1.3. *A function $u \in W^{1,1}$ is a solution to Problem 2.1.1 if and only if the numbers $u(x_1), u(x_2), ..., u(x_n)$ are solutions to Problem 2.1.2, and u is monotone on the intervals (x_i, x_{i+1}) and constant on the intervals $(0, x_1)$ and $(x_n, 1)$.*

Proof: Let $u \in W^{1,1}$. Since u is continuous, the total variation is equivalent to the classical pointwise variation. In particular,

$$\int_0^1 |Du| \geq \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)|$$

Hence,

$$\mathfrak{F}(u(x_1), u(x_2), ..., u(x_n)) \leq F[u]$$

This implies that the minimum of F is bounded from below by the minimum of $\mathfrak{F}$, i.e. that for any u

$$F[u] \geq \mathfrak{F}(U_1, U_2, ..., U_n) \tag{2.6}$$

If u is monotone on the intervals (x_i, x_{i+1}), then

$$\int_0^1 |Du| = \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)|$$

and so

$$F[u] = \mathfrak{F}(u(x_1), u(x_2), ..., u(x_n))$$

Therefore, if $u(x_i) = U_i$ for every i, and the monotonicity condition holds, then equality is attained in (2.6). Hence, F attains its lower bound at u and so u is a minimizer.

Conversely, if u is not monotone on some interval $[x_i, x_{i+1}]$, then

$$\int_0^1 |Du| > \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)|$$

and $F[u] > \mathfrak{F}(u(x_1), u(x_2), ..., u(x_n))$, so u cannot be a minimizer.

Likewise, if $u(x_i) \neq U_i$ for some i, then

$$F[u] \geq \mathfrak{F}(u(x_1), u(x_2), ..., u(x_n)) > \mathfrak{F}(U_1, U_2, ..., U_n)$$

Since it is possible to attain equality, as noted above, this implies that u cannot be a minimizer.

□

Thus, Problem 2.1.1 can be solved by first solving Problem 2.1.2, then interpolating the values with *any* monotone functions. Hence, there are many possible solutions to Problem 2.1.1.

On the Solution of Problem 2.1.2

Here we provide an explicit method to find the exact solution of Problem 2.1.2. We have shown that a unique solution exists.

Let control points $(x_1, f_1), (x_2, f_2), ..., (x_n, f_n)$ be given and let $U_1, U_2, ..., U_n$ denote the solution to Problem 1.2.

There are similarities to the methods used in Section 1.1.3 to find minimizers of (1.1.1) for step functions. We may recall the notions of a composition and a partition into contiguous blocks. The difference here is that instead of intervals, we focus on the discrete set of points $x_1, x_2, ..., x_n$.

A **maximal contiguous block** will be a set of indices for which

$$U_i = U_{i+1} = ... = U_{i+k}$$

and either $i = 1$ or $U_{i-1} \neq U_i$, and either $i + k = n$ or $U_{i+k} \neq U_{i+k+1}$. As a convention, all future contiguous blocks will be assumed maximal unless otherwise noted. If a contiguous block does not contain 1 or n, we shall call it an **interior block**.

A solution of Problem 1.1.2 therefore partitions the set $\{1, 2, ..., n\}$ into contiguous blocks $C_1, C_2, ..., C_m$.

We introduce some more notation. For a maximal contiguous block C_l, the average of the ordinates of the corresponding control points will be denoted by

$$\bar{f}_l = \frac{1}{|C_l|} \sum_{i \in C_l} f_i \tag{2.7}$$

where $|C_l|$ is the cardinality of C_l. For simplicity, we may drop the subscript l if no confusion will result.

We will begin by computing the value of the solution on points corresponding to a given block. To do this, we will have need of the following results:

Lemma 2.1.4. *For any* $i = 1, 2, ..., n - 1$*, if* $f_i = f_{i+1}$*, then* $U_i = U_{i+1}$.

Proof: Suppose i is chosen so that $f_i = f_{i+1}$. Without loss of generality, assume

$$|U_i - f_i| \leq |U_{i+1} - f_{i+1}|$$

Let $\tilde{U}_{i+1} = U_i$. Then

$$|\tilde{U}_{i+1} - f_{i+1}|^2 \leq |U_{i+1} - f_i|^2$$

and

$$\begin{aligned} |\tilde{U}_{i+1} - U_i| + |U_{i+2} - \tilde{U}_{i+1}| &= |U_{i+2} - U_i| \\ &\leq |U_{i+1} - U_i| + |U_{i+2} - U_{i+1}| \end{aligned}$$

Hence,

$$\mathfrak{F}(U_1, U_2, ..., U_i, \tilde{U}_{i+1}, ..., U_n) \leq \mathfrak{F}(U_1, U_2, ..., U_i, U_{i+1}, ..., U_n)$$

By hypothesis, $\mathbf{U}$ minimizes $\mathfrak{F}$. Since the minimizer is unique, it must follow that

$$\tilde{U}_{i+1} = U_{i+1}$$

and therefore

$$U_{i+1} = U_i$$

□

Lemma 2.1.5. *For any* $i = 1, 2, ..., n-1$, *if* $U_i \neq U_{i+1}$ *then*

$$\text{sgn}(U_i - U_{i+1}) = \text{sgn}(f_i - f_{i+1}) \tag{2.8}$$

Proof: Lemma 2.1.4 implies that under these hypotheses, $f_i \neq f_{i+1}$. Without loss of generality we may assume $f_i < f_{i+1}$.

Compare (2.4) with the (1.12) from Part I. We may apply Lemma 1.1.6 in Part I to show that $f_i < f_{i+1}$ implies $U_i \leq U_{i+1}$. Since we have assumed $U_i \neq U_{i+1}$, this implies $U_i < U_{i+1}$, as desired.

□

Proposition 2.1.6. *Let* $C = \{j, j+1, ..., j+k\}$ *be an interior maximal contiguous block. Then* $U_j, U_{j+1}, ...U_{j+k}$ *have the common value*

$$c = \bar{f} + \frac{\operatorname{sgn}(f_{j+k+1} - f_{j+k}) - \operatorname{sgn}(f_j - f_{j-1})}{2(k+1)} \tag{2.9}$$

Proof: By hypothesis, the following hold:

$$U_j = U_{j+1} = ... = U_{j+k}$$

$$U_j \neq U_{j-1}$$

$$U_{j+k} \neq U_{j+k+1}$$

Let $c = U_j$. Then

$$\begin{aligned} \mathfrak{F}(U_1, U_2, ..., U_n) &= \mathfrak{F}(U_1, ..., U_j, U_{j+1}, ..., U_{j+k}, ..., U_n) \\ &= \mathfrak{F}(U_1, ..., U_{j-1}, c, ..., c, U_{j+k+1}, ..., U_n) \\ &= \sum_{i \notin C} |u_i - f_i|^2 + \sum_{i=j}^{j+k} |c - f_i|^2 + \sum_{i=1}^{j-2} |u_{i+1} - u_i| \end{aligned}$$

By hypothesis, this is the minimum. This is differentiable in a neighborhood of c, so it must satisfy

$$\frac{\partial \mathfrak{F}}{\partial c} = 0$$

Now,

$$\frac{\partial \mathfrak{F}}{\partial c} = 2 \sum_{i=j}^{j+k} (c - f_i) + \operatorname{sgn}(c - U_{j-1}) - \operatorname{sgn}(U_{j+k+1} - c)$$

Setting this equal to zero, and rearranging,

$$2(k+1)c = 2 \sum_{i=j}^{j+k} f_i - \operatorname{sgn}(c - U_{j-1}) + \operatorname{sgn}(U_{j+k+1} - c)$$

Hence

$$\begin{aligned} c &= \frac{1}{k}\sum_{i=j}^{j+k} f_i + \frac{\operatorname{sgn}(U_{j+k+1}-c) - \operatorname{sgn}(c-U_{j-1})}{2k} \\ &= \bar{f} + \frac{\operatorname{sgn}(U_{j+k+1}-c) - \operatorname{sgn}(c-U_{j-1})}{2(k+1)} \end{aligned} \tag{2.10}$$

Now, $U_{j+k} = c$ and $U_j = c$, so Lemma 1.5 implies

$$c = \bar{f} + \frac{\operatorname{sgn}(f_{j+k+1} - f_{j+k}) - \operatorname{sgn}(f_j - f_{j-1})}{2(k+1)}$$

□

The following corollaries give the value of c when one of the endpoints of the contiguous block is 1 or n.

Corollary 2.1.7. *If $k < n$ and $C = \{1, 2, ..., k\}$ is a maximal contiguous block, then $U_1, U_2, ...U_k$ have the common value*

$$c = \bar{f} + \frac{\operatorname{sgn}(f_{k+1} - f_k)}{2(k+1)} \tag{2.11}$$

Corollary 2.1.8. *If $j > 1$ and $C = \{j, j+1, ..., n\}$ is a maximal contiguous block, then $U_j, U_{j+1}, ...U_n$ have the common value*

$$c = \bar{f} - \frac{\operatorname{sgn}(f_j - f_{j-1})}{2(k+1)} \tag{2.12}$$

Corollary 2.1.9. *If $C = \{1, 2, ..., n\}$ is a maximal contiguous block, then $U_1, U_2, ..., U_n$ have the common value*

$$c = \bar{f} \tag{2.13}$$

The formulas given here assume that we already know the decomposition of $\{1, 2, ..., n\}$ into contiguous blocks. However, there are only finitely many such partitions, so in principle the correct one can be found with brute force. That is, one may try every possible partition into contiguous blocks and use the above formulas to obtain a corresponding trial solution. One of these must be the minimizer, and this can be determined by direct substitution.

2.1.3 Change of Parameters

We may generalize Problem 2.1.1 by adjusting some parameters. The first will be to consider terms of the form

$$|u(x_i) - f_i|^q$$

in lieu of

$$|u(x_i) - f_i|^2$$

We will also consider what happens when $A \neq 1$ in (2.2).

2.1.4 On the Term $|u(x_i) - f_i|^q$

The first generalization we shall consider is to replace the terms $|u(x_i) - f_i|^2$ with $|u(x_i) - f_i|^q$, giving us the modified functional

$$F_q[u] = \sum_{i=1}^{n} |f_i - u(x_i)|^q + \int_0^1 |Du| \tag{2.14}$$

To ensure convexity, we require $q \geq 1$. The case $q = 1$ is undesirable, as can be seen from the following example:

Example: Let $q = 1$ and suppose we have two control points (x_1, f_1) and (x_2, f_2). Suppose $f_1 < f_2$. We wish to minimize

$$F_1[u] = |f_1 - u(x_1)| + |f_2 - u(x_2)| + \int_0^1 |Du|$$

For any $u \in W^{1,1}$,

$$\int_0^1 |Du| \geq |u(x_2) - u(x_1)|$$

Hence,

$$F_1[u] \geq |f_1 - u(x_1)| + |f_2 - u(x_2)| + |u(x_2) - u(x_1)|$$

for any u. If we choose points such that

$$f_1 \leq u_1 \leq u_2 \leq f_2$$

and interpolate a function u that is monotone in the interval (x_1, x_2) and constant otherwise, then we will have

$$F_1[u] = f_2 - f_1$$

So any choice of u_1 and u_2 satisfying the above inequality leads to a minimizer.

To avoid this situation, we will take $q > 1$.

To find a minimizer, we use the same device as we used for $q = 2$ case. We introduce the auxiliary function

$$\mathfrak{F}_q(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} |u_i - f_i|^q + \sum_{i=1}^{n-1} |u_{i+1} - u_i|$$

and find its minimizers. Since $q > 1$, the problem is strictly convex and so a unique solution exists. Differentiating,

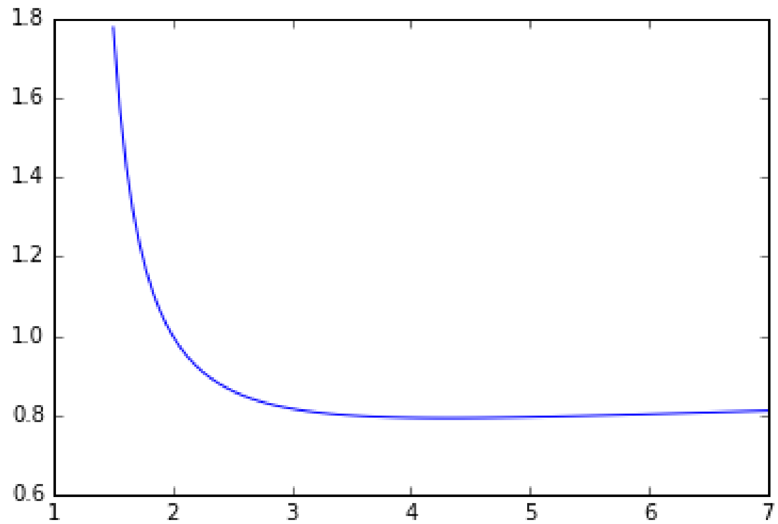
$$\frac{\partial \mathfrak{F}}{\partial u_i} = q|u_i - f_i|^{q-1} \operatorname{sgn}(u_i - f_i) + \operatorname{sgn}(u_i - u_{i-1}) - \operatorname{sgn}(u_{i+1} - u_i)$$

Finding a complete solution is similar to the case $q = 2$, so we will not pursue it. Instead, we will assume that $u_i \neq u_{i-1}$ and $u_{i+1} \neq u_i$ in the solution. Setting the derivative equal to 0 and solving for u_i, we obtain

$$u_i = f_i - \operatorname{sgn}(u_i - f_i) \left(\frac{\operatorname{sgn}(u_i - u_{i-1}) - \operatorname{sgn}(u_{i+1} - u_i)}{q} \right)^{\frac{1}{q-1}}$$

We may elimiate the appearance u_i in the right-hand side by considering the configuration of f_{i-1}, f_i, and f_{i+1} as we did earlier.

As we can see, adjusting q will change the distance between the control points and the minimizer. The following plot shows the distance as a function of q:



As q increases, the function levels off. For smaller values of q, the distance seems to grow, but in practice this just means the minimizer tends toward a horizontal line.

If one wants to control the closeness of fit to the control points, q is not a useful parameter to adjust.

The Fidelity Parameter

Here, we will consider what happens when the parameter A is varied. Let

$$F_A[u] = A \sum_{i=1}^{n} |f_i - u(x_i)|^2 + \int_0^1 |Du| \tag{2.15}$$

As before, we introduce an auxiliary function

$$\mathfrak{F}_A(u_1, u_2, ..., u_n) = A \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} |u_{i+1} - u_i| \tag{2.16}$$

Minimizing this is essentially the same as the $A = 1$ case. We therefore just give the modified result. For any contiguous block of length m,

$$c = \bar{f} + \frac{\operatorname{sgn}(f_{j+m+1} - \bar{f}) - \operatorname{sgn}(\bar{f} - f_{j-1})}{2Am} \tag{2.17}$$

where $\bar{f}$ is as in (2.9). In particular, when $m = 1$, we see that the gap between a control point and the minimizer is either 0 or $\frac{1}{A}$.

As A goes to 0, the quantity A^{-1} grows without bound. Hence, eventually the solution will consist of a single contiguous block. In that situation, we will have

$$u \equiv \frac{1}{n} \sum_{i=1}^{n} f_i$$

which is just the arithmetic mean of the f_i.

As $A \to \infty$, the quantity A^{-1} goes to 0, so in the limit we expect to get a function that interpolates the control points. The following proposition shows that any such interpolant, monotone on the intervals (x_i, x_{i+1}), can be so obtained.

Proposition 2.1.10. *Let control points $(x_1, f_1), ..., (x_n, f_n)$ be given. Let v be continuous at x_i and attain the value $v(x_i) = U_i$ for $i = 1, 2, ..., n$. If v monotone on the intervals (x_i, x_{i+1}) and constant on the intervals $(0, x_1)$ and $(x_n, 1)$, then there exists a sequence $\{u_k\}$ such that*

$$\lim_{k \to \infty} \int_0^1 |u_k - v| = 0$$

and a sequence $A_k \to \infty$ such that u_k is a minimizer of F_{A_k} for every k.

Before commencing the proof, we will have need of the following lemma:

Lemma 2.1.11. *Let g be a bounded, monotone function on $[0, 1]$. Then there exists a sequence $\{v_k\} \subset W^{1,1}$ of monotone functions such that*

$$\lim_{k \to \infty} \int_0^1 |v_k - g| = 0$$

and $v_k(0) = g(0 + 0)$ and $v_k(1) = g(1 - 0)$ for every k.

Proof: There is a representative of g that satisfies $g(0) = g(0 + 0)$ and $g(1) = g(1 - 0)$. Pick one such representative, which will also be called g. Since g is monotone, it can be decomposed as

$$g = g^a + g^j + g^c$$

where g^a is absolutely continuous, g^j is a jump function, and g^c is a Cantor function. Moreover, all three will be monotone as well. We will approximate each component separately.

For the absolutely continuous component g^a, it is already in $W^{1,1}$ so we let $\alpha_k = g^a$.

Next we consider the jump function g^j. Since g is bounded, for every k there can be at most finitely many jumps larger than $\frac{1}{k}$. Let $M = |g^j(1) - g^j(0)|$. This is finite since g is bounded.

Since $g \in BV$, the sum of the sizes of the jumps of g converges absolutely. Define a finite sequence $0 = s_0 < s_1 < ... < s_m = 1$ satisfying the following conditions:

i): g is continuous at s_i for every i

ii): Intervals (s_i, s_{i+1}) containing jumps of size at least $\dfrac{1}{k}$ have total length at most $\dfrac{1}{kM}$

iii): If the interval (s_i, s_{i+1}) does not contain a jump of size at least $\dfrac{1}{k}$, then

$$|g^j(s_{i+1}) - g^j(s_i)| \leq \frac{1}{k}$$

Let β_k be the piecewise-linear interpolant of g^j with nodes at the s_i. Condition (iii) implies that

$$|\beta_k(x) - g^j(x)| < \frac{1}{k}$$

except on a set of total length at most $\frac{1}{kM}$. Hence,

$$\begin{aligned} \int_0^1 |\beta_k - g^j| &\leq \left(1 - \frac{1}{kM}\right)\frac{1}{k} + \frac{1}{kM}M \\ &\leq \frac{2}{k} \end{aligned}$$

The function $\beta_k \in W^{1,1}$ since it is continuous and piecewise linear with finitely many components.

Finally, we approximate the Cantor component g^c. Let $N = |g^c(1) - g^c(0)|$. Let

$$h_k = \frac{1}{Nk^2}$$

and let $s_i = ih_k$ for $i = 0, 1, ..., Nk^2$. Let γ_k be the piecewise-linear interpolant of g^c with nodes at the s_i. This is possible because g^c is continuous. There are at most Nk subintervals where

$$|g^c(s_{i+1}) - g^c(s_i)| \geq \frac{1}{k}$$

and these have total length at most $\frac{1}{k}$. As before, we obtain

$$\begin{aligned} \int_0^1 |g^c - \gamma_k| &\leq \left(1 - \frac{1}{k}\right)\frac{1}{k} + \frac{1}{k} \\ &\leq \frac{2}{k} \end{aligned}$$

Let $v_k = \alpha_k + \beta_k + \gamma_k$. Then

$$\begin{aligned} \int_0^1 |v_k - g| &\leq \int_0^1 |\alpha_k - g^a| + \int_0^1 |\beta_k - g^j| + \int_0^1 |\gamma_k - g^c| \\ &\leq \frac{4}{k} \end{aligned}$$

Hence,

$$\lim_{k\to\infty} \int_0^1 |v_k - g| = 0$$

By construction, we have $v_k(1) = g(1)$ and $v_k(0) = g(0)$, so the result is proven.

□

Clearly, this remains true if the interval $[0, 1]$ is replaced by any other interval $[t_1, t_2]$. We proceed to the proof of the proposition.

Proof of Proposition 2.1.10: Let v be given. For every k, choose A_k such that

$$|f_i - U_i(A_k)| < \frac{1}{k}$$

For every interval (x_i, x_{i+i}), let $\{v_k^i\}$ be defined as in Lemma 2.1.11, i.e. let $v_k^i \in W^{1,1}$ be monotone on (x_i, x_{i+1}) and satisfy $v_k^i(x_i) = v(x_i)$ and $v_k^i(x_{i+1}) = v(x_{i+1})$, and suppose that

$$\lim_{k\to\infty} \int_{x_i}^{x_{i+1}} |v_k^i - v| = 0$$

Suppose without loss of generality that $f_i \le f_{i+1}$. Then the interval $(U_i(A_k), U_{i+1}(A_k))$ is a subset of the interval (f_i, f_{i+1}). Hence, there is a point $p_i(A_k) \in (U_i(A_k), U_{i+1}(A_k))$ and a number $\eta_i(A_k) \ge 0$ such that the mapping

$$\phi_k^i(y) = \eta_i(A_k)(y - p_i(A_k)) \tag{2.18}$$

satisfies

$$\begin{aligned} \phi_k^i(f_i) &= U_i(A_k) \\ \phi_k^i(f_{i+1}) &= U_{i+1}(A_k) \end{aligned} \tag{2.19}$$

Then, since $|f_i - U_i(A_k)| < \dfrac{1}{k}$ and v_k^i is a monotone interpolant of (x_i, f_i) and (x_{i+1}, f_{i+1}), we have

$$|\phi_k^i(v_k^i(x)) - v_k^i(x)| < \frac{1}{k}$$

whenever $x \in (x_i, x_{i+1})$.

Define a sequence $\{u_k\}$ as follows:

$$u_k(x) = \begin{cases} U_1(A_k) & \text{if } 0 \le x \le x_1 \\ \phi_k^i(v_k^i(x)) & \text{if } x \in [x_i, x_{i+1}] \\ U_n(A_k) & \text{if } x_n \le x \le 1 \end{cases}$$

By construction, $u_k \in W^{1,1}$, $u_k(x_i) = U_i(A_k)$ for every i, and u_k is monotone on intervals of the form (x_i, x_{i+1}) and constant on the intervals $(0, x_1)$ and $(x_n, 1)$. Therefore, u_k is a minimizer of F_{A_k} for every k. Moreover,

$$\begin{aligned}\int_0^1 |u_k - v| &\leq (U_1(A_k) - f_1)(x_1) + (U_n(A_k) - f_n)(1 - x_n) \\ &\quad + \sum_{i=1}^{n-1} \left[\int_{x_i}^{x_{i+1}} |\phi_k^i(v_k^i(x)) - v_k^i(x)| + \int_{x_i}^{x_{i+1}} |v_k^i(x) - v(x)| \right] \\ &\leq \frac{x_1}{k} + \frac{1 - x_n}{k} + (n-1)\left(\frac{1}{k} + \frac{4}{k}\right) \\ &\leq \frac{1 + 5(n-1)}{k}\end{aligned}$$

Hence, for any $\epsilon > 0$,

$$\int_0^1 |u_k - v| < \epsilon$$

whenever

$$k > \frac{1 + 5(n-1)}{\epsilon}$$

Therefore,

$$\lim_{k\to\infty} \int_0^1 |u_k - v| = 0$$

□

Proposition 2.1.12. *Let v satisfy the hypotheses of Proposition 2.1.10, and let $\{u_k\}$ satisfy the conclusion of Proposition 2.1.10. Then $\{u_k\}$ converges to v strictly in BV.*

We have already shown convergence in L^1, so we need to show that

$$\lim_{k\to\infty} \int_0^1 |Du_k| = \int_0^1 |Dv|$$

The values of $u_k(x_i)$ are given by (2.17). When the maximal block contining i has length 1, (2.17) implies

$$|u_k(x_i) - f_i| \leq \frac{1}{A_k} \tag{2.20}$$

For sufficiently large values of A, the maximal contiguous blocks all have length 1. Hence, there exists some N such that for $k > N$, (2.20) holds for $i = 1, 2, ..., n$. Suppose then that $k > N$. Since

$$|u_k(x_{i+1}) - u_k(x_i)| \le |u_k(x_{i+1}) - f_{i+1}| + |f_{i+1} - f_i| + |f_i - u_k(x_i)|$$

it follows that

$$\begin{aligned} ||u_k(x_{i+1}) - u_k(x_i)| - |f_{i+1} - f_i|| &\le |u_k(x_{i+1}) - f_{i+1}| + |f_i - u_k(x_i)| \\ &\le \frac{2}{A_k} \end{aligned} \tag{2.21}$$

Since u_k is the solution of Problem 2.1.1 with $A = A_k$, it is monotone on the intervals (x_i, x_{i+1}). This means

$$\int_0^1 |Du_k| = \sum_{i=1}^{n-1} |u_k(x_{i+1}) - u_k(x_i)|$$

From (2.21) it follows that

$$\left| \int_0^1 |Du_k| - \sum_{i=1}^{n-1} |f_{i+1} - f_i| \right| \le \frac{2(n-1)}{A_k} \tag{2.22}$$

Moreover, since v interpolates the points (x_i, f_i) and is monotone in intervals (x_i, x_{i+1}), we have

$$\int_0^1 |Dv| = \sum_{i=1}^{n-1} |f_{i+1} - f_i|$$

Hence,

$$\left| \int_0^1 |Du_k| - \int_0^1 |Dv| \right| \le \frac{2(n-1)}{A_k} \tag{2.23}$$

As $k \to \infty$, $A_k \to \infty$, whence it follows that

$$\lim_{k \to \infty} \int_0^1 |Du_k| = \int_0^1 |Dv|$$

□

2.1.5 Adding an Extra Term

We consider a modification of Problem 2.1.1, something of a combination between this and the ROF filter. Namely, suppose we wanted to minimize

$$H[u] = F[u] + \int_0^1 u^2 \tag{2.24}$$

The functional H is strictly convex, so if a minimizer exists it will be unique. Here we are confronted with the problem that a minimizing sequence in $W^{1,1}$ need not converge in $W^{1,1}$. That a minimizer need not exist in $W^{1,1}$ can be seen from the following counterexample:

Example: Consider the case with two control points

$$(x_1, f_1) = \left(\frac{1}{4}, 0\right)$$
$$(x_2, f_2) = \left(\frac{3}{4}, 5\right)$$

First, we will show that the minimizer is not a constant. To compute the minimum among constant functions, we minimize

$$h(c) = (c-0)^2 + (c-5)^2 + c^2$$

Differentiating,

$$h'(c) = 6c - 10$$

Hence, the minimum occurs at

$$c = \frac{5}{3}$$

and so the best we can do with a constant function $u \equiv c$ is

$$\begin{aligned} h[u] &= \frac{5}{3}^2 + \left(\frac{5}{3} - 5\right)^2 + \frac{5}{3}^2 \\ &= \frac{50}{3} \end{aligned}$$

Now let

$$v(x) = \begin{cases} \frac{1}{3} & \text{if } 0 \le x < \frac{2}{5} \\ \frac{40}{3}\left(x - \frac{2}{5}\right) + \frac{1}{3} & \text{if } \frac{2}{5} \le x < \frac{3}{5} \\ 3 & \text{if } \frac{3}{5} \le x \le 1 \end{cases}$$

Then

$$\begin{aligned} H[v] &= \frac{1}{3}^2 + (3-5)^2 + \frac{8}{3} + \frac{2}{5}\left(\frac{1}{3}^2 + 3^2\right) + \int_{\frac{2}{5}}^{\frac{3}{5}} \left(\frac{40}{3}\left(x - \frac{2}{5}\right) + \frac{1}{3}\right)^2 \\ &= \frac{1}{9} + 4 + \frac{8}{3}\frac{164}{45} + \frac{91}{135} \\ &= \frac{1498}{135} \end{aligned}$$

which is better than we got with the optimal constant. Therefore, no constant function can be the minimizer.

Next, we show that there is no minimizer in $W^{1,1}$. Suppose, on the contrary, that there exists $u \in W^{1,1}$ that minimizes H. As we have shown above, it is nonconstant. It is clear from the choice of control points that any minimizer must be nonnegative. Continuity assures us that on at least one of the intervals $(0, x_1)$, (x_1, x_2), or $(x_2, 1)$, there exist two points a_1 and a_2 such that $u(a_1) < u(a_2)$. Either $a_1 < a_2$ or $a_1 > a_2$.

If $a_1 < a_2$, then by continuity there exists a point γ such that $a_1 < \gamma < a_2$ and $u(\gamma) > u(a_1)$. Let y be the equation of the line connecting the points $(\gamma, u(a_1))$ and $(a_2, u(a_2))$ Define a function $\tilde{u}$ as follows:

$$\tilde{u} = \begin{cases} u(x) & \text{if } x < a_1 \text{ or } x > a_2 \\ u(a_1) & \text{if } a_1 \le x \le \gamma \\ \min\{u(x), y(x)\} & \text{if } \gamma < x \le a_2 \end{cases}$$

This function still in $W^{1,1}$. The interval where $u \neq \tilde{u}$ does not contain any of the control points, so

$$|\tilde{u}(x_i) - f_i|^2 = |u(x_i) - f_i|^2$$

for $i = 1, 2$. Moreover, $\tilde{u}$ is monotone on the interval (a_1, a_2) and equals u outside the interval and on the endpoints, so

$$\int_0^1 |D\tilde{u}| \leq \int_0^1 |Du|$$

On the interval (a_1, a_2), $0 \leq \tilde{u} \leq u$ and $\tilde{u}(\gamma) < u(\gamma)$, so continuity implies

$$\int_0^1 \tilde{u}^2 < \int_0^1 u^2$$

Therefore, $H[\tilde{u}] < H[u]$ and so u cannot be the minimizer. The case where $a_1 > a_2$ is handled similarly. Thus, we conclude that there is no minimizer in $W^{1,1}$.

In the above example, it appears that we may obtain a sequence of functions converging to a step function. However, H is not defined for a function that has jumps at the control points, so the problem of minimizing H has no solution. We will revisit this in Section 3.

2.2 Regularization

2.2.1 Introduction

As seen in Section 1, Problem 2.1.1 admits many solutions in the space $W^{1,1}$. In this chapter, we consider ways to regularize the problem that lead to a unique solution. We will consider the following modifications on (2.2). First, we will replace the total variation by an L^p norm of the derivative, obtaining

$$F_p[u] = \sum_{i=1}^{n} |u(x_i) - f_i|^2 + \int_0^1 |u'|^p$$

The minimizer will be shown to be piecewise-linear, and to converge uniformly to a limit as $p \to 1$.

Next, consider a slightly different approach where the approximations are all in $W^{1,2}$, namely

$$F_\epsilon[u] = \sum_{i=1}^{n} |u(x_i) - f_i|^2 + \int_0^1 |Du| + \epsilon \int_0^1 |u'|^2$$

The behavior will be shown to be similar to the previous regularization method.

The final method will look for approximations in H^2. We will seek to minimize

$$F_\epsilon[u] = \sum_{i=1}^{n} |u(x_i) - f_i|^2 + \int_0^1 |Du| + \epsilon \int_0^1 |u''|^2$$

It will be shown that as $\epsilon \to 0$ the minimizers converge in H^2 to a minimizer of (2.2). In fact, the limit will be the solution of a certain variational inequality.

2.2.2 The term $\int_0^1 |u'|^p$

Modify Problem 2.1.1 to a functional defined on $W^{1,p}$.

Problem 2.2.1. Let $p > 1$. Minimize

$$F_p[u] = \sum_{i=1}^{n} |u(x_i) - f_i|^2 + \int_0^1 |u'|^p \tag{2.25}$$

over the space $W^{1,p}$.

Proposition 2.2.2. *Problem 2.2.1 admits a unique solution in* $W^{1,p}$.

Proof: Since the space $W^{1,p}$ is reflexive, it is sufficient to show that F_p is continuous, strictly convex, and coercive.

The mapping

$$u \mapsto \int_0^1 |u'|^p$$

is continuous and strictly convex.

Next, the Dirac delta, $\delta_{x_i}(u) = u(x_i)$, is a continuous linear functional on $W^{1,p}$, so

$$u \mapsto \sum_{i=1}^{n} |\delta_{x_i}(u) - f_i|^2$$

is continuous and convex. Hence, F_p is continuous and convex.

To prove coercivity, we may write

$$u(x) = u(x_1) + \int_{x_1}^{x} u'(t)\,dt$$

By Holder's inequality,

$$\int_0^1 |u'(t)|\,dt \leq \left(\int_0^1 |u'|^p\right)^{\frac{1}{p}}$$

Hence,

$$\sup_{[0,1]} |u| - u(x_1) \leq \left(\int_0^1 |u'|^p\right)^{\frac{1}{p}}$$

This proves coercivity.

Thus, F_p is continuous, coercive, and convex, whence it follows that a unique minimizer exists.

□

Proposition 2.2.3. *The solution of Problem 2.2.1 is a piecewise-linear function with nodes at* $x_1, x_2, ..., x_n$. *It is constant on the intervals* $(0, x_1)$ *and* $(x_n, 1)$.

Proof: By Proposition 2.2.2, there exists a unique solution to Problem 2.2.1. Call it u. Then it takes values $u_1, u_2, ..., u_n$ at $x_1, x_2, ..., x_n$, respectively. If we restrict our focus to the interval (x_i, x_{i+1}), $1 \leq i \leq n-1$, u must minimize

$$I[u] = \int_{x_i}^{x_{i+1}} |u'|^p$$

with boundary values $u(x_i) = u_i$ and $u(x_{i+1}) = u_{i+1}$.

Under these boundary conditions,

$$|u_{i+1} - u_i| \leq \int_{x_i}^{x_{i+1}} |Du|$$

and, since u is absolutely continuous, the right-hand side is just the integral of $|u'|$. Making this substitution and applying Holder's inequality, we find that

$$|u_{i+1} - u_i| \leq \int_{x_i}^{x_{i+1}} |u'| \leq |x_{i+1} - x_i|^{(1-\frac{1}{p})} \left(\int_{x_i}^{x_{i+1}} |u'|^p \right)^{\frac{1}{p}}$$

Raising each side to the power of p,

$$|u_{i+1} - u_i|^p \leq |x_{i+1} - x_i|^{p-1} \int_{x_i}^{x_{i+1}} |u'|^p$$

Hence,

$$\int_{x_i}^{x_{i+1}} |u'|^p \geq \frac{|u_{i+1} - u_i|^p}{|x_{i+1} - x_i|^{p-1}} \tag{2.26}$$

This holds for any $u \in W^{1,p}$ satisfying the boundary conditions. Equality holds if and only if, on the interval (x_i, x_{i+1}), u is the straight line connecting the points (x_i, u_i) and (x_{i+1}, u_{i+1}). We can repeat this for $i = 1, 2, ..., n-1$.

On the interval $(0, x_1)$, u must minimize

$$\int_0^{x_1} |u'|^p$$

under the boundary constraint $u(x_1) = u_1$. A constant is the best we can do. Similarly for the interval $(x_n, 1)$.

Thus, a piecewise-linear function is a solution to Problem 2.2.1.

□

A continuous piecewise linear function is entirely determined by its value at the nodes. Proposition 2.2.3 therefore provides a way to reduce Problem 2.2.1 to a finite-dimensional problem.

Proposition 2.2.4. *Fix p and let control points $(x_1, f_1), ..., (x_n, f_n)$ be given. Let u_p be the corresponding solution of Problem 2.1. For $i = 1, 2, ..., n$, let $U_i(p) = u_p(x_i)$. Then the numbers $U_1(p), U_2(p), ..., U_n(p)$ are the unique minimizers of the following function:*

$$\mathfrak{F}_p(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} \frac{|u_{i+1} - u_i|^p}{(x_{i+1} - x_i)^{p-1}} \tag{2.27}$$

Proof: By Proposition 2.2.3, u_p is piecewise linear. For $i = 1, 2, ..., n$,

$$u_p|_{[x_i,x_{i+1}]} = \frac{U_{i+1}(p) - U_i(p)}{x_{i+1} - x_i}(x - x_i) + U_i(p)$$

and therefore

$$u_p'|_{[x_i,x_{i+1}]} = \frac{U_{i+1}(p) - U_i(p)}{x_{i+1} - x_i}$$

and u_p' vanishes on $(0, x_1)$ and $(x_n, 1)$. Thus,

$$\begin{aligned}\int_0^1 |u_p'|^p &= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \left|\frac{U_{i+1}(p) - U_i(p)}{x_{i+1} - x_i}\right|^p \\ &= \sum_{i=1}^{n-1} (x_{i+1} - x_i) \left|\frac{U_{i+1}(p) - U_i(p)}{x_{i+1} - x_i}\right|^p \\ &= \sum_{i=1}^{n-1} \frac{|U_{i+1}(p) - U_i(p)|^p}{|x_{i+1} - x_i|^{p-1}}\end{aligned}$$

Hence

$$F_p[u_p] = \sum_{i=1}^{n} |U_i(p) - f_i|^2 + \sum_{i=1}^{n-1} \frac{|U_{i+1}(p) - U_i(p)|^p}{(x_{i+1} - x_i)^{p-1}} \tag{2.28}$$

and so

$$F_p[u_p] = \mathfrak{F}_p(U_1(p), U_2(p), ..., U_n(p))$$

Next we show that the numbers $U_1(p), ..., U_n(p)$ uniquely minimize $\mathfrak{F}_p$. Suppose $v_1, v_2, ..., v_n$ are real numbers that satisfy

$$\mathfrak{F}_p(v_1, v_2, ..., v_n) \leq \mathfrak{F}_p(U_1(p), U_2(p), ..., U_n(p))$$

Let v be a continuous, piecewise linear function with nodes at the x_i such that $v(x_i) = v_i$ and v constant on the intervals $(0, x_1)$ and $(x_n, 1)$. By same argument we used for u_p, we have

$$F_p[v] = \mathfrak{F}_p(v(x_1), v(x_2), ..., v(x_n))$$

Therefore,

$$F_p[v] \leq F_p[u_p]$$

However, we have shown that F_p has a unique minimizer u_p. Hence, $v = u_p$ and so $v_i = U_i(p)$ for $i = 1, 2, ..., n$.

□

Remark: When $p = 2$,

$$\mathfrak{F}_2(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} \frac{|u_{i+1} - u_i|^2}{(x_{i+1} - x_i)}$$

and

$$\frac{\partial \mathfrak{F}_2}{\partial u_i} = 2(u_i - f_i) + \frac{2}{x_i - x_{i-1}}(u_i - u_{i-1}) - \frac{2}{x_{i+1} - x_i}(u_{i+1} - u_i)$$

Hence, setting

$$\nabla \mathfrak{F}_2 = 0$$

leads to a system of linear equations, so the problem is easy to solve explicitly.

So far, we have shown that that the regularized problem has a unique solution. We now wish to investigate what happens as $p \to 1$ and see how the regularized problem relates to Problem 2.1.1.

Proposition 2.2.5. *For $p > 1$, let u_p be the solution to Problem 2.2.1. The limit as $p \to 1$ of u_p is the piecewise-linear function passing through the points $U_1, U_2, ..., U_n$ which minimize the function*

$$\mathfrak{F}(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} |u_{i+1} - u_i|$$

Proof: Introduce the auxiliary function

$$g(u_1, u_2, ..., u_n, p) = \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} \frac{|u_{i+1} - u_i|^p}{(x_{i+1} - x_i)^{p-1}} \tag{2.29}$$

This is continuous in $\mathbb{R}^n \times (0, \infty)$. Let $\{p_j\}$ be a sequence such that $p_1 > p_2 > ...$ and $p_j \to 1$. Let $\{\mathbf{U}_j\}$ denote the corresponding minimizer of $\mathfrak{F}_{p_j}$. Clearly, $||\mathbf{U}_j||$ is uniformly bounded, so the sequence $\{\mathbf{U}_j\}$ has a point of accumulation. Let $\mathbf{U}_\infty$ denote one such point of accumulation, and let $\mathbf{U}_0$ denote the minimizer of $\mathfrak{F}$.

Suppose $\mathbf{U}_\infty \neq \mathbf{U}_0$. Let $D = g(\mathbf{U}_\infty, 1) - g(\mathbf{U}_0, 1)$. By hypothesis, $D > 0$. Let $\eta = ||\mathbf{U}_\infty - \mathbf{U}||$.

By continuity of g, there exists δ such that $0 < \delta < \dfrac{\eta}{2}$ and

$$|g(\mathbf{U}, p) - g(\mathbf{U}_0, 1)| < \frac{D}{2}$$

whenever

$$\max\{||\mathbf{U} - \mathbf{U}_0||, |p - 1|\} < \delta$$

This implies that a sufficiently small neighborhood of $\mathbf{U}_\infty$ can contain at most finitely many of the $\mathbf{U}_j$. Therefore, the only possible point of accumulation for the sequence $\{\mathbf{U}_j\}$ is $\mathbf{U}_0$, which must then be the limit as the sequence is bounded. This holds for any sequence of minimizers corresponding to a sequence $\{p_j\}$ such that $p_j \to 1$. Hence,

$$u_p(x_i) \to U_i$$

as $p \to 1$ for any $i = 1, 2, ..., n$. Since u_p is piecewise linear for every $p > 1$, this implies that u_p converges uniformly to a limit function u, which is the piecewise linear interpolant of $\mathbf{U}_0$.

$\square$

2.2.3 Adding the term $\epsilon \int_0^1 |u'|^2$

Consider now a new regularization method. Instead of adjusting the exponent as in the previous section, we take (2.2) and add an extra term to ensure a unique minimizer in $W^{1,2}$.

Problem 2.2.6. Let $\epsilon > 0$. Minimize

$$F_\epsilon[u] = \sum_{i=1}^{n} |u(x_i) - f_i|^2 + \int_0^1 |Du| + \epsilon \int_0^1 |u'|^2 \tag{2.30}$$

over the space $W^{1,2}$.

Proposition 2.2.7. *Problem 2.2.6 admits a unique solution in* $W^{1,2}$*.*

Proof: It is sufficient to prove that F_ϵ is strictly convex, coercive, and lower semicontinuous. The proof of Proposition 2.1.2 shows that the functional

$$G_\epsilon[u] = \sum_{i=1}^{n} |u(x_i) - f_i|^2 + \epsilon \int_0^1 |u'|^2 \tag{2.31}$$

is already stricly convex, coercive, and continuous in $W^{1,2}$. The total variation is nonnegative, convex, lower semicontinuous in $W^{1,2}$. Hence, the sum of (2.31) and the total variation is strictly convex, coercive, and continuous in $W^{1,2}$.

□

Proposition 2.2.8. *The solution of Problem 2.2.6 is a piecewise-linear function with nodes at* $x_1, x_2, ..., x_n$*. It is constant on the intervals* $(0, x_1)$ *and* $(x_n, 1)$*.*

Proof: Let $\hat{u}_\epsilon$ be the solution of Problem 2.2.6. For $i = 1, 2, ..., n$, let $U_i(\epsilon) = \hat{u}_\epsilon(x_{i+1})$ and let w be the piecewise-linear interpolant of these points. Let

$$K(\epsilon) = \left\{ u \in W^{1,2} : \, u(x_i) = U_i(\epsilon) \quad \text{for} \;\; i = 1, 2, ..., n \right\} \tag{2.32}$$

Given i, the minimizer of

$$u \mapsto \epsilon \int_{x_i}^{x_{i+1}} |u'|^2$$

under the boundary conditions $u(x_j) = U_j(\epsilon)$ for $j = i, i+1$ is a straight line, and so its minimizer over $K(\epsilon)$ is just w. Moreover, w is a minimizer of the total variation functional over the set $K(\epsilon)$.

Given $u \in K$,

$$F_\epsilon[u] = \sum_{i=1}^{n} |U_i(\epsilon) - f_i|^2 + \int_0^1 |Du| + \epsilon \int_0^1 |u'|^2$$

The summation term on the right-hand side is constant on K. Then

$$\inf_{u\in K} F_\epsilon[u] \geq \sum_{i=1}^{n} |U_i(\epsilon) - f_i|^2 + \inf_{u\in K} \int_0^1 |Du| + \epsilon \inf_{u\in K} \int_0^1 |u'|^2$$

As noted above, w minimizes the right-hand side. Hence,

$$F_\epsilon[\hat{u}] \geq \sum_{i=1}^{n} |U_i(\epsilon) - f_i|^2 + \int_0^1 |Dw| + \epsilon \int_0^1 |w'|^2$$

so $F_\epsilon[w] \leq F_\epsilon[\hat{u}]$. Since $\hat{u}$ is the unique minimizer, it follows that $w = \hat{u}$. Hence $\hat{u}$ is piecewise-linear.

□

Proposition 2.2.9. *Fix p and let control points $(x_1, f_1), ..., (x_n, f_n)$ be given. Let u_ϵ be the corresponding solution of Problem 2.2.6. For $i = 1, 2, ..., n$, let $U_i(\epsilon) = u_\epsilon(x_i)$. Then the numbers $U_1(\epsilon), U_2(\epsilon), ..., U_n(\epsilon)$ are the unique minimizers of the following function:*

$$\mathfrak{F}_\epsilon(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} |u_{i+1} - u_i| + \epsilon \sum_{i=1}^{n-1} \frac{|u_{i+1} - u_i|^2}{x_{i+1} - x_i} \tag{2.33}$$

Proof: This uses the same method as the proof of Proposition 2.2.4.

Next, we examine what happens as $\epsilon \to 0$. Let $\mathfrak{F}$ be as in (2.4)

Lemma 2.2.10. *Let u_ϵ be the minimizer of F_ϵ. Let $U_i(\epsilon) = u_\epsilon(x_i)$ for $i = 0, ..., n+1$. Finally, let $\boldsymbol{U}$ denote the corresponding minimizer of (2.4). Then*

$$\lim_{\epsilon\to 0} U_i(\epsilon) = U_i$$

Proof: Given $\epsilon > 0$, we may consider the collection of points $\{u_i(\epsilon)\}$ as components of a vector $\mathbf{U}(\epsilon)$. Being points on the minimizer of the regularized problem, they must be bounded. Consider now a sequence $\{\epsilon_j\}$ tending to 0 as $j \to \infty$. For convenience, we will write $\mathbf{U}_j$ instead of $\mathbf{U}(\epsilon_j)$.

The sequence $\{\mathbf{U}_j\}$ is bounded in $\mathbb{R}^n$, so it has at a point of accumulation as $j \to \infty$. Let $\mathbf{U}_\infty$ denote one such point of accumulation, and u_∞ denote its piecewise linear interpolant.

Suppose that $\mathbf{U}_\infty$ does not minimize $\mathfrak{F}$. Let $\mathbf{U}_0$ be the minimizer of the function $\mathfrak{F}$ and interpolate a piecewise linear function u_0. Let

$$M = \int_0^1 |u_0'|^2$$

Then

$$F_\epsilon[u_0] = \mathfrak{F}(\mathbf{u}_0) + \epsilon M$$

Since $\mathbf{U}_0$ is the minimizer of $\mathfrak{F}$ and $\mathbf{U}_0 \neq \mathbf{U}_\infty$, $\mathfrak{F}[\mathbf{U}_\infty] > \mathfrak{F}[\mathbf{U}_0]$. Thus, whenever

$$\epsilon < \frac{\mathfrak{F}(\mathbf{U}_\infty) - \mathfrak{F}(\mathbf{U}_0)}{M}$$

it will also be the case that

$$F_\epsilon[u_\infty] - F_\epsilon[u_0] \geq \mathfrak{F}(\mathbf{U}_\infty) - \mathfrak{F}(\mathbf{U}_0) - \epsilon M$$

As $\epsilon \to 0$, the right-hand side is bounded away from zero. Hence, $\mathbf{U}_\infty$ cannot be a point of accumulation of the sequence $\{\mathbf{U}_j\}$. For any such sequence, the only possible point of accumulation is therefore $\mathbf{U}_0$, and so this must be the limit.

□

Once we know that the solution passes through the points u_i, we can show that it is piecewise linear.

Proposition 2.2.11. *Let u_ϵ be the minimizer of F_ϵ. Then $\lim_{\epsilon\to 0} u_\epsilon$ exists and is a piecewise linear function passing through the points $U_1, U_2, ..., U_n$ that minimize $\mathfrak{F}$*

Proof: By Lemma 2.2.10, the points $U_i(\epsilon)$ converge to U_i. Since the functions are piecewise linear whose nodes have abscissas x_i, the interpolants also converge.

□

2.2.4 Adding the term $\epsilon \int_0^1 ||u''||^2$

Our third and final regularization method is similar to the previous one. We now add a term that controls the second derivative, and so the regularized problem will have a solution in H^2. Specifically, we investigate the following problem:

Problem 2.2.12. : Let $\epsilon > 0$. Minimize

$$F_\epsilon[u] = \sum_{i=1}^{n} |u(x_i) - f_i|^2 + \int_0^1 |Du| + \epsilon \int_0^1 |u''|^2 \tag{2.34}$$

over the space $H^2([0,1])$.

We will prove existence of a unique solution, then investigate the behavior of the solutions as $\epsilon \to 0$. There are minimizers of (2.2) besides the piecewise-linear ones we encountered earlier in this chapter.

Consider, for example, a piece-wise cubic interpolant w such that $w(x_i) = U_i$ and $w'(x_i) = 0$ for $i = 1, 2, .., n$. This is monotone on the intervals (x_i, x_{i+1}), and so is also a minimizer of (2.2). Moreover, $w \in C^1$ and, in fact, $w'' \in L^\infty$, so $w \in H^2$. Thus, we expect that the solutions of Problem 2.2.12 will be uniformly bounded in H^2 as $\epsilon \to 0$. We therefore expect there to be a weak limit $v \in C^1$. We will show, in fact, that the solutions of Problem 2.2.12 converge strongly to v in the H^2 norm, and we give a characterization of v.

Proposition 2.2.13. *There exists a unique solution to Problem 2.2.12.*

Proof: If $n = 1$, then the minimizer is just $u \equiv f_1$.

Suppose $n > 1$. Since the space H^2 is reflexive, it is sufficient to show that F_ϵ is lower semicontinuous, strictly convex, and coercive in H^2.

Each term in (2.34) is continuous in H^2, so F_ϵ is continuous.

Next, we consider convexity. We know (2.34) is convex. We show it is strictly convex. Let $u_1, u_2 \in H^2([0,1])$. If $u_1'' \equiv u_2''$, then

$$u_1(x) = u_2(x) + Ax + B$$

for some constants A and B. If $u_1(x_i) = u_2(x_i)$ for $i = 1, 2$, then $u_1 \equiv u_2$. If not, then for any $t \in (0, 1)$,

$$\begin{aligned} &|tu_1(x_1) + (1-t)u_2(x_1) - f_1|^2 + |tu_1(x_2) + (1-t)u_2(x_2) - f_2|^2 \\ &< t\left(|u_1(x_1) - f_1|^2 + |u_1(x_2) - f_2|^2\right) + (1-t)\left(|u_2(x_1) - f_1|^2 + |u_2(x_2) - f_2|^2\right) \end{aligned} \tag{2.35}$$

and therefore

$$F_\epsilon[tu_1 + (1-t)u_2] < tF_\epsilon[u_1] + (1-t)F_\epsilon[u_2]$$

If u_1'' and u_2'' are not identically equal, then, for $t \in (0, 1)$,

$$\int_0^1 |tu_1'' + (1-t)u_2''|^2 < t\int_0^1 |u_1''|^2 + (1-t)\int_0^1 |u_2''|^2$$

Thus, (2.34) is strictly convex.

Finally, we must show coercivity. Let $M > 0$ and suppose $F_\epsilon[u] < M$. Then

$$\int_0^1 |u''|^2 < M,$$
$$\int_0^1 |Du| < M,$$

and

$$|u(x_1) - f_1|^2 < M$$

Hence,

$$\begin{aligned} \sup_{[0,1]} |u| &\leq |u(x_1)| + \int_0^1 |Du| \\ &\leq |u(x_1) - f_1| + |f_1| + \int_0^1 |Du| \\ &\leq \sqrt{M} + |f_1| + M \end{aligned} \tag{2.36}$$

This implies

$$\int_0^1 u^2 \leq \left(|f_1| + M + \sqrt{M}\right)^2$$

Next, we have the interpolation inequality

$$\int_0^1 |u'|^2 \leq C\left(\int_0^1 u^2 + \int_0^1 |u''|^2\right) \tag{2.37}$$

We have shown that the right-hand side is bounded by a constant depending only on the control points and the bound on $F_\epsilon[u]$. Consolidating the constants into a number D, we conclude that

$$||u||_{H^2} < D$$

This is true of any u such that $F_\epsilon[u] < M$. Hence, F_ϵ is coercive.

Thus, we have shown F_ϵ is lower semicontinuous, strictly convex, and coercive. Therefore, Problem 2.2.12 has a unique solution.

□

Next, we will investigate the behavior as $\epsilon \to 0$. To this end, it will be convenient to introduce the auxiliary problem:

Problem 2.2.14. Let control points $(x_1, f_1), ..., (x_n, f_n)$ be given. Let $U_1, U_2, ..., U_n$ minimize (2.4). Find a function $v \in H^2$ that minimizes

$$L[v] = \int_0^1 |v''|^2$$

under the constraints

$$v(x_i) = U_i, \qquad i = 1, 2, ..., n$$

and such that v is monotone over intervals of the form $[x_i, x_{i+1}]$ and constant on the intervals $(0, x_1)$ and $(x_n, 1)$.

Lemma 2.2.15. *Problem 2.2.14 has a unique solution.*

Proof: If there is only one control point, then the solution is clearly the constant $u \equiv f_1$.

Suppose then that $n > 1$. Let K be the subset of H^2 satisfying the constraints of Problem 2.2.14. To prove a unique solution of Problem 2.2.14 exists, it is sufficient to show that K is closed and convex, and that the functional L is strictly convex, coercive, and continuous on K. Continuity of L on H^2 is known.

Next, we show that K is convex. Let $u_1, u_2 \in K$. Then, for $i = 1, 2, ..., n$ and $t \in [0, 1]$,

$$\begin{aligned} tu_1(x_i) + (1-t)u_2(x_i) &= tU_i + (1-t)U_i \\ &= U_i \end{aligned}$$

The constraints imply that on any interval (x_i, x_{i+1}), u_1 and u_2 are both either monotone increasing or monotone decreasing. Fix i and suppose without loss of generality that u_1 and u_2 are increasing. Then if $x_1 < z_1 < z_2 < x_{i+1}$, we will have $u_1(z_2) \geq u_1(z_1)$ and $u_2(z_2) \geq u_2(z_1)$. Hence, for $t \in [0, 1]$,

$$tu_1(z_2) + (1-t)u_2(z_2) \geq tu_1(z_1) + (1-t)u_2(z_1)$$

and so monotonicity is preserved. Thus, if u_1 and u_2 satisfy the constraints, then so too does $tu_1 + (1-t)u_2$, and we therefore conclude that K is convex.

In the proof of Proposition 2.2.13, we noted that if $u_1, u_2 \in H^2$, $u_1(x_i) = u_2(x_i)$ for $i = 1, 2$, and $u_1'' \equiv u_2''$, then $u_1 \equiv u_2$. This implies that L is strictly convex on K when $n \geq 2$.

Finally, we consider coercivity. If $u \in K$, then the monotonicity constraints imply that

$$\min\{f_i\} \leq u(x) \leq \max\{f_i\} \quad \forall x \in [0, 1]$$

Hence, the quantity

$$\int_0^1 u^2$$

is uniformly bounded for all $u \in K$. From (2.37) it then follows that if $u \in K$ and $L[u]$ is bounded, then so to is $||u||_{H^2}$. This proves coercivity.

We have shown that K is convex, and that on K the functional L is continuous, stricly convex, and coercive. Therefore, Problem 2.2.14 has a unique solution.

$\square$

The following result shows that in the limit as $\epsilon \to 0$, the solution of Problem 2.2.12 converges to the solution of Problem 2.2.14.

Theorem 2.2.16. *Fix control points $(x_1, f_1), (x_2, f_2), ..., (x_n, f_n)$. Let u_ϵ minimize F_ϵ and let v be the solution to Problem 2.2.14. Then*

$$\lim_{\epsilon \to 0^+} ||u_\epsilon - v||_{H^2} = 0 \tag{2.38}$$

To prove this, we will first need some intermediate results.

Lemma 2.2.17. *For $\epsilon > 0$, let u_ϵ be the solution of Problem 2.2.12. Let $U_i(\epsilon) = u_\epsilon(x_i)$ for $i = 0, ..., n+1$. Finally, let U_i denote the corresponding points of the minimizer of*

$$\mathfrak{F}(u_1, u_2, ..., u_n) = \sum_{i=1}^{n} |u_i - f_i|^2 + \sum_{i=1}^{n-1} |u_{i+1} - u_i|$$

Then

$$\lim_{\epsilon \to 0^+} U_i(\epsilon) = U_i$$

Proof: For a given ϵ, we may consider the collection of points $\{U_i(\epsilon\}$ as components of a vector $\mathbf{U}(\epsilon)$. For sufficiently small values of ϵ, $||\mathbf{U}(\epsilon)||$ is uniformly bounded. To see this, let u_0 be the piecewise cubic interpolant of the points $(x_1, U_1), (x_2, U_2), ..., (x_n, U_n)$ such that

$$u_0'(x_i) = 0$$

for $i = 1, 2, ..., n$ and which is constant on the intervals $(0, x_1)$ and $(x_n, 1)$. Then

$$F_\epsilon[u_0] = \sum_{i=1}^{n} |U_i - f_i|^2 + \sum_{i=1}^{n-1} |U_{i+1} - U_i| + \epsilon \int_0^1 |u_0''|^2$$

Let

$$M = \int_0^1 |u_0''|^2$$

Then, for any $0 < \epsilon \leq 1$,

$$\begin{aligned} F_\epsilon(u_\epsilon) &\leq F_\epsilon(u_0) \\ &\leq \sum_{i=1}^{n} |U_i - f_i|^2 + \sum_{i=1}^{n-1} |U_{i+1} - U_i| + M \end{aligned}$$

The boundedness of $||\mathbf{U}(\epsilon)||$ follows.

Consider now a sequence $\{\epsilon_j\}$ tending to 0 as $j \to \infty$. For convenience, we will write $\mathbf{U}_j$ instead of $\mathbf{U}(\epsilon_j)$. Since the vectors $\mathbf{U}_j$ are a bounded subset of $\mathbb{R}^n$, $\{\mathbf{U}_j\}$ has at least one point of accumulation. Let $\mathbf{U}_\infty$ denote one such point of accumulation. Let $\mathfrak{F}$ be as in Problem 2.2.14. Suppose that $\mathbf{U}_\infty$ does not minimize $\mathfrak{F}$. Let u_∞ be an arbitrary function whose values at the x_i are the corresponding points of $\mathbf{U}_\infty$.

For any ϵ,

$$F_\epsilon[u_0] = \mathfrak{F}(\mathbf{U}_0) + \epsilon M$$

Since $\mathbf{U}_0$ is the unique minimizer of $\mathfrak{F}$, $\mathfrak{F}(\mathbf{U}_\infty) > \mathfrak{F}(\mathbf{U}_0)$. Whenever

$$\epsilon < \frac{\mathfrak{F}(\mathbf{U}_\infty) - \mathfrak{F}(\mathbf{U}_0)}{M}$$

it will also be the case that

$$F_\epsilon[u_\infty] - F_\epsilon[u_0] \geq \mathfrak{F}(\mathbf{U}_\infty) - \mathfrak{F}(\mathbf{U}_0) - \epsilon M$$

As $\epsilon \to 0$, the right-hand side is bounded away from zero, so $\mathbf{U}_\infty$ cannot be a point of accumulation of the sequence $\{\mathbf{U}_j\}$. For any such sequence, the only possible point of accumulation is therefore $\mathbf{U}_0$, and so this must be the limit.

$\square$

Lemma 2.2.18. *Let u_ϵ be the solution to Problem 2.2.12. Let $\boldsymbol{U}$ minimize $\mathfrak{F}$. Then*

$$\lim_{\epsilon\to 0^+}\int_0^1 |Du_\epsilon| = \sum_{i=1}^{n-1}|U_{i+1}-U_i|$$

Proof: Let $\{\epsilon_j\}$ be a nonincreasing sequence of positive numbers converging to zero. For any j,

$$\int_0^1 |Du_{\epsilon_j}| \geq \sum_{i=1}^{n-1}|U_{i+1}(\epsilon_j)-U_i(\epsilon_j)|$$

where, as before, we have $U_i(\epsilon_j) = u_{\epsilon_j}(x_i)$. We may take the limit inferior on both sides and apply Lemma 2.17 to obtain

$$\liminf_{j\to\infty}\int_0^1 |Du_{\epsilon_j}| \geq \sum_{i=1}^{n-1}|U_{i+1}-U_i| \tag{2.39}$$

We want to show equality holds in the limit. We proceed by contradiction.

Suppose, on the contrary, that

$$\limsup_{j\to\infty}\int_0^1 |Du_{\epsilon_j}| > \sum_{i=1}^{n-1}|U_{i+1}-U_i|$$

and let

$$\eta = \limsup_{j\to\infty}\left(\int_0^1 |Du_{\epsilon_j}|\right) - \sum_{i=1}^{n-1}|U_{i+1}-U_i| \tag{2.40}$$

Passing to a subsequence if necessary we may assume that

$$\int_0^1 |Du_{\epsilon_j}| \geq \sum_{i=1}^{n-1}|U_{i+1}(\epsilon_j)-U_i(\epsilon_j)|$$

for all j.

Let u_0 be as in the proof of Lemma 2.2.17 and let

$$M = \int_0^1 |u_0''|^2$$

Then

$$F_\epsilon[u_0] = \mathfrak{F}(u_0(x_1), u_0(x_2), ..., u_0(x_n)) + \epsilon M$$

By Lemma 2.2.17, for any $\delta > 0$, we may find some N such that whenever $j > N$,

$$\left| \sum_{i=1}^{n} |U_i(\epsilon_j) - f_i|^2 - \sum_{i=1}^{n} |U_i - f_i|^2 \right| < \delta$$

Let $\delta = \dfrac{\eta}{3}$ and choose α such that

$$\alpha < \frac{\eta}{3M}$$

For sufficiently large j, $\epsilon_j < \alpha$. When this holds,

$$F_{\epsilon_j}[u_{\epsilon_j}] - F_{\epsilon_j}[u_0] > \frac{\eta}{3}$$

which is a contradiction since u_{ϵ_j} minimizes F_{ϵ_j}. Therefore,

$$\limsup_{j\to\infty} \int_0^1 |Du_{\epsilon_j}| \leq \sum_{i=1}^{n-1} |U_{i+1} - U_i|$$

It then follows from (2.39) that

$$\lim_{j\to\infty} \int_0^1 |Du_{\epsilon_j}| = \sum_{i=1}^{n-1} |u_{i+1} - u_i|$$

□

Lemma 2.2.19. *Let v be the solution of Problem 2.2.14. For every $\epsilon > 0$, let u_ϵ be a solution to Problem 2.2.12. Then*

$$\int_0^1 |u_\epsilon''|^2 \leq \int_0^1 |v''|^2 \tag{2.41}$$

Proof: The constraints of Problem 2.2.14 imply that v is a solution of Problem 2.1.1. Hence,

$$\sum_{i=1}^{n} |v(x_i) - f_i|^2 + \int_0^1 |Dv| \leq \sum_{i=1}^{n} |u_\epsilon(x_i) - f_i|^2 + \int_0^1 |Du_\epsilon|$$

for any $\epsilon > 0$. Since u_ϵ minimizes (2.34) it must be true that

$$\int_0^1 |u_\epsilon''|^2 \leq \int_0^1 |v''|^2$$

□

Lemma 2.2.20. *For every sequence* $\{\epsilon_k\}$ *such that* $\epsilon_k \downarrow 0$*, the corresponding sequence* $\{u_{\epsilon_k}\}$ *of solutions of Problem 2.2.12 has a weakly convergent subsequence.*

Proof: The space H^2 is a Hilbert space, so the result will follow if we prove that the sequence $\{u_{\epsilon_k}\}$ is bounded. Lemmas 2.2.17 and 2.2.18 show that $||u_{\epsilon_k}||_{L^2}$ is bounded as $k \to \infty$. Lemma 2.2.19 shows that

$$\int_0^1 |u_\epsilon''|^2$$

is bounded. This implies that the H^2 norms are also bounded, as desired.

□

Proposition 2.2.21. *Let* u_∞ *be a weak limit of some sequence* $\{u_{\epsilon_k}\}$ *for which* $\epsilon_k \downarrow 0$*. Then*

$$u_\infty(x_i) = U_i, \quad i = 1, 2, ..., n$$

and

$$\int_0^1 |Du_\infty| = \sum_{i=1}^{n-1} |U_{i+1} - U_i|$$

Proof: Let

$$\delta_x[f] = f(x)$$

where $x \in [0, 1]$. This is a continuous linear functional on $H^2([0, 1])$, so by weak convergence and Lemma 2.2.17

$$\begin{aligned} u_\infty(x_i) &= \delta_{x_i}[u_\infty] \\ &= \lim_{k\to\infty} \delta_{x_i}[u_{\epsilon_k}] \\ &= U_i \end{aligned} \tag{2.42}$$

Next, since the total variation is convex and lower semicontinuous on H^2, it is weakly lower semicontinuous. This means that

$$\int_0^1 |Du_\infty| \leq \liminf_{k\to\infty} \int_0^1 |Du_{\epsilon_k}|$$

By Lemma 2.2.18, this means

$$\int_0^1 |Du_\infty| \leq \sum_{i=1}^{n-1} |U_{i+1} - U_i|$$

Since $u_\infty(x_i) = U_i$ for $i = 1, 2, ..., n$, it must also be true that

$$\int_0^1 |Du_\infty| \geq \sum_{i=1}^{n-1} |U_{i+1} - U_i|$$

Therefore, equality holds.

□

Proposition 2.2.22. *For any sequence $\epsilon_k \downarrow 0$, the corresponding sequence of minimizers $\{u_{\epsilon_k}\}$ converges weakly to a function v, which is the solution to Problem 2.2.14.*

Proof: Let u_∞ be a limit point of the sequence $\{u_{\epsilon_k}\}$. The mapping

$$u \mapsto \int_0^1 |u''|^2$$

is continuous in H^2, so

$$\int_0^1 |u''_\infty|^2 \leq \liminf_{k\to\infty} \int_0^1 |u''_{\epsilon_k}|^2$$

By Lemma 2.2.20, this implies

$$\int_0^1 |u''_\infty|^2 \leq \int_0^1 |v''|^2$$

Proposition 2.2.21 says that u_∞ satisfies the constraints on Problem 2.2.14. As v is the unique minimizer under those constraints, the above inequality implies $u_\infty = v$.

Thus, any subsequence of $\{u_{\epsilon_k}\}$ will have a sub-subsequence converging weakly to v. Therefore v must be the weak limit of the full sequence.

□

We now complete the proof of Theorem 2.2.16.

Proof of Theorem 2.2.16: Compact imbedding of H^2 into C^1 and Proposition 2.2.22 imply that the sequence $\{u_{\epsilon_k}\}$ converges in C^1 to v. Hence,

$$\lim_{k\to\infty} \int_0^1 |u_{\epsilon_k} - v|^2 = 0$$

$$\lim_{k\to\infty} \int_0^1 |u'_{\epsilon_k} - v'|^2 = 0$$

By Lemma 2.2.21,

$$\int_0^1 |u''_{\epsilon_k}|^2 \leq \int_0^1 |v''|^2$$

Hence,

$$||v||_{H^2} \geq \limsup_{k\to\infty} ||u_{\epsilon_k}||_{H^2}$$

Since H^2 is uniformly convex and u_{ϵ_k} converges weakly to v, by Proposition 3.32 of [6] this implies that the convergence is strong. These results hold for any sequence $\{u_{\epsilon_k}\}$ for which $\epsilon_k \downarrow 0$. Therefore,

$$\lim_{\epsilon\to 0^+} ||u_\epsilon - v||_{H^2} = 0$$

□

The Solution of Problem 2.2.14

Suppose $\hat{u}$ is a solution to Problem 2.2.14. For $j = 1, 2, ..., n$, let $p_j = \hat{u}'(x_j)$. On any interval (x_i, x_{i+1}), $\hat{u}$ minimizes

$$\int_{x_i}^{x_{i+1}} |u''|^2$$

subject to the boundary conditions

$$u(x_i) = U_i, \qquad u'(x_i) = p_i$$

$$u(x_{i+1}) = U_{i+1}, \qquad u'(x_{i+1}) = p_{i+1}$$

and the constraint that u be monotone on (x_i, x_{i+1}).

We may therefore focus on a fixed interval between two control points. If we can find the minimizer with arbitrary p_i and p_{i+1}, then we can treat the full problem as a question of finding appropriate values of $p_1, p_2, ..., p_n$. To simplify our analysis, we translate and rescale the interval $[x_i, x_{i+1}]$ to be the unit interval. We will treat the case where the function is monotone decreasing and take $U_i = 1$ and $U_{i+1} = 0$. Thus we consider:

Problem 2.2.23. Let numbers $p_0, p_1 \leq 0$ be given. Find a function $\hat{v} \in H^2$ that minimizes

$$L[v] = \int_0^1 |v''|^2 \tag{2.43}$$

under the boundary conditions

$$\begin{aligned} v(0) = 1, &\qquad v'(0) = p_0 \\ v(1) = 0, &\qquad v'(1) = p_1 \end{aligned} \tag{2.44}$$

and the constraint $v' \leq 0$.

Proposition 2.2.24. *There exists a unique solution to Problem 2.2.23.*

Proof: Let K be the subset of H^2 satisfying the boundary conditions (2.2.4 and the constraint. Then K is closed and convex. Since p_0 and p_1 are nonpositive, it is possible to find a nonincreasing smooth function taking on the prescribed boundary conditions. Hence, K is nonempty.

We want to show that L is continuous, strictly convex, and coercive over K in the H^2 topology. Given two functions v_1 and v_2 that satisfy the (2.2.4), if $v_1'' \equiv v_2''$, then

$$v_1'(x) = p_0 + \int_0^x v_1''(t)\,dt$$
$$v_2'(x) = p_0 + \int_0^x v_2''(t)\,dt$$

Hence, $v_1' \equiv v_2'$. Likewise, $v_1 \equiv v_2$. Hence, the functional L is strictly convex over K.

To show coercivity, we observe that, given $v \in K$,

$$v'(x) = p_0 + \int_0^x v''(t)\,dt$$
$$v(x) = 1 + \int_0^x \left(p_0 + \int_0^t v_2''(s)\,ds\right)\,dt$$

Hence, boundedness of $L[v]$ implies boundedness of v, so L is coercive.

Thus, a unique solution exists.

□

As Problem 2.2.23 is an obstacle problem, there will be a coincidence set Λ such that $\hat{v}'$ vanishes on Λ and

$$\frac{\partial^4}{\partial x^4}\hat{v} = 0$$

on $(0, 1) \setminus \Lambda$.

Proposition 2.2.25. *Let $\hat{v}$ be the solution of Problem 2.2.23. If $p_0 = p_1 = 0$, then $\hat{v}$ is a cubic polynomial. Otherwise, Λ is closed and connected and the restriction of $\hat{v}$ to any connected component of $[0, 1] \setminus \Lambda$ is a cubic polynomial.*

Proof: We consider three cases:

Case 1: $p_0 = p_1 = 0$.

Consider the Hermite interpolating polynomial

$$P(x) = 2x^2(x-1) - x^2 + 1$$

This minimizes an unconstrained version of Problem 2.2.23. Its derivative is a quadratic that vanishes at 0 and 1, so it cannot change sign on $(0,1)$. Hence, $P' \leq 0$ and so satisfies the constraints of Problem 2.2.23. Therefore, it must be the minimizer.

Case 2: $p_0 < 0$ and $p_1 = 0$.

Let us consider a modified version of Problem 2.2.23 where the constraint $v' \leq 0$ is replaced by $v \geq 0$ on $[0,1]$. The set of functions satisfying this constraint is convex, so we still get existence of a unique solution. Let $\hat{u}$ be this solution.

Any monotone function satisfying the boundary conditions is certainly nonnegative on $[0,1]$. Thus, if the $\hat{u}$ is monotone, it must also solve Problem 2.2.23.

Let $\xi = \min\{x \in [0,1] : \hat{u}(x) = 0\}$. The boundary conditions (2.2.4) imply $0 < \xi \leq 1$. On the interval $[\xi, 1]$, $\hat{u}$ minimizes L with homogeneous boundary data, so it must vanish identically. On $[0,\xi)$, $\hat{u} > 0$ and so must satisfy

$$\frac{\partial^4}{\partial x^4}\hat{u} = 0$$

Hence, it is a cubic polynomial. On the boundary of Λ in $(0,1)$, we require that the second derivative be continuous. Hence, to find ξ we solve:

$$\hat{u}(\xi) = \hat{u}'(\xi) = \hat{u}''(\xi) = 0$$

whence it follows that

$$\hat{u} = \begin{cases} C(x-\xi)^3 & \text{if } 0 \leq x < \xi \\ 0 & \text{if } \xi \leq x \leq 1 \end{cases} \tag{2.45}$$

This is a monotone function, so it also solves Problem 2.2.23.

Case 3: $p_0 < 0$ and $p_1 < 0$.

If $\hat{v}'$ is strictly positive, then

$$\frac{\partial^4}{\partial x^4}\hat{v} = 0$$

on $[0,1]$ and so $\hat{v}$ is a cubic polynomial. If not, then $\hat{v}'(\xi) = 0$ for some $\xi \in (0,1)$. Let $\eta = \hat{v}(\xi)$. In that case, we can split the problem into two components, minimizing L on $[0,\xi]$ and $[\xi,1]$ with respective boundary conditions

$$\begin{aligned} v(0) &= 1, \quad & v'(0) &= p_0 \\ v(\xi) &= \eta, \quad & v'(\xi) &= 0 \end{aligned}$$

and

$$\begin{aligned} v(\xi) &= \eta, \quad & v'(\xi) &= 0 \\ v(1) &= 0, \quad & v'(1) &= p_1 \end{aligned}$$

These can be separately solved using covered by Case 2. Hence, there exist $\xi_1 \in (0,\xi]$ and $\xi_2 \in [\xi,1)$ such that $\hat{v}'$ vanishes on $[\xi_1,\xi_2]$, and $\hat{v}$ when restricted to $[0,\xi_1]$ or $[\xi_2,1]$ is a cubic polynomial.

□

2.3 The Lower Semicontinuous Envelope

2.3.1 Introduction

Let us return to Problem 2.1.1. We may extend (2.2) to L^2 as follows:

$$F[u] = \begin{cases} \displaystyle\sum_{i=1}^{n} |u(x_i) - f_i|^2 + \int_0^1 |Du| & \text{if } u \in W^{1,1} \\ \infty & \text{otherwise} \end{cases} \tag{2.46}$$

This functional is not lower-semicontinuous in the L^2 topology, as can be seen from the following example:

Example: Let $x_1 = \frac{1}{2}$ and $f_1 = 2$. Let $v \equiv 0$ and define the sequence $\{v_k\}$ by

$$v_k(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2} - \frac{1}{k} \text{ or } x > \frac{1}{2} + \frac{1}{k} \\ k\left(x - \frac{1}{2} + \frac{1}{k}\right) & \text{if } \frac{1}{2} - \frac{1}{k} \le x \le \frac{1}{2} \\ -k\left(x - \frac{1}{2} - \frac{1}{k}\right) & \text{if } \frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{k} \end{cases}$$

Then $v_k \to v$ in the L^2 norm and $F[v_k] = 3$ for every k, but $F[v] = 4$.

To analyze the problem in the L^2 topology, we construct the lower semicontinuous envelope of (2.46).

2.3.2 On LSC envelopes

Definition 2.3.1. The **lower semi-continuous envelope** in L^2 of a functional F on $W^{1,1}$ is the map $G : L^2 \to \mathbb{R} \cup \{-\infty, \infty\}$ defined by

$$G[u] = \inf \left\{ \liminf_{k\to\infty} F[u_k] \right\} \tag{2.47}$$

where the infimum is taken over all sequences $\{u_k\} \subset W^{1,1}$ such that

$$\lim_{k\to\infty} ||u_k - u||_{L^2} = 0$$

The density of $W^{1,1}$ in L^2 ensures that G is well-defined. Before constructing the LSC envelope of F, we will first prove some statements directly from the definition.

Lemma 2.3.2. *The lower semicontinuous envelope of a functional is lower semicontinuous.*

Proof: Let $v \in L^2$ and consider a sequence $\{v_j\} \subset L^2$ such that $v_j \to v$. We want to show

$$G[v] \le \liminf_{j\to\infty} G[v_j]$$

Let $\epsilon > 0$. There exists some a subsequence $\{v_{j_k}\}$ such that

$$\int_0^1 (v_{j_k} - v)^2 < \frac{\epsilon}{2k}$$

For every k, there exists $u_k \in W^{1,1}$ such that

$$\int_0^1 (v_{j_k} - u_k)^2 < \frac{\epsilon}{2k}$$

and

$$G[v_{j_k}] \geq F[u_k] - \epsilon \tag{2.48}$$

Then

$$\begin{aligned} \int_0^1 (v - u_k)^2 &\leq \int_0^1 (v_{j_k} - v)^2 + \int_0^1 (v_{j_k} - u_k)^2 \\ &< \frac{\epsilon}{k} \end{aligned} \tag{2.49}$$

Hence, the sequence $\{u_k\}$ converges to v in L^2. By (2.47) and (2.48),

$$\begin{aligned} G[v] &\leq \liminf_{k\to\infty} F[u_k] \\ &\leq \liminf_{k\to\infty} G[v_{j_k}] + \epsilon \end{aligned} \tag{2.50}$$

This holds for all $\epsilon > 0$, so

$$G[v] \leq \liminf_{k\to\infty} G[v_{j_k}]$$

Hence, G is lower semicontinuous in L^2.

□

Proposition 2.3.3. *Let F be as in (2.46) and let G be its lower semicontinuous envelope. Then*

$$G[u] < \infty$$

if and only if $u \in BV$.

Proof: If $u \in BV$, then there exists a sequence $\{u_k\} \subset W^{1,1}$ that strictly converges to u, i.e. that satisfies

$$\lim_{k\to\infty} \int_0^1 |u - u_k| = 0$$

and

$$\int_0^1 |Du| = \lim_{k\to\infty} \int_0^1 |Du_k|$$

Compact embedding of BV into L^2 (Theorem A.14) then ensures $\{u_k\}$ converges to u in the L^2 topology.

Moreover, since $\{u_k\}$ is bounded in BV and u_k is continuous for every k, there exists $M > 0$ such that

$$\sum_{i=1}^{n} |f_i - u_k(x_i)|^2 < M$$

for every k. Hence, the numbers $F[u_k]$ are bounded, and (2.47) implies that $G[u]$ is finite.

For any $u \in L^2$, if $G[u]$ is finite then there exists a sequence $\{u_k\}$ converging to a u in L^2 and a number K such that

$$\int_0^1 |Du_k| < K$$

for every k. The total variation of a function is lower semicontinuous in the L^2 topology, whence it follows

$$\int_0^1 |Du| < K$$

and therefore $u \in BV$.

□

Proposition 2.3.4. *Let $u \in L^2$. If there exists a sequence $\{u_k\} \subset W^{1,1}$ such that $u_k \to u$ in L^2 and u_k is a minimizer of (2.46) for $k = 1, 2, ...$, then u is a minimizer of (2.47).*

Proof: By hypothesis, $F[u_i] = F[u_j]$ for all $i, j = 1, 2, \ldots.$ Let C denote their common value. Then $G[u] = C$. Hence,

$$\inf_{v \in L^2} G[v] \leq C$$

Now let $w \in L^2$ and $\{w_m\} \subset W^{1,1}$ be some sequence converging to w in L^2. Then $F[w_m] \geq C$ for $m = 1, 2, \ldots.$ Hence,

$$\liminf_{m \to \infty} F[w_m] \geq C$$

This holds for every such sequence. By (2.47), this implies $G[w] \geq C$. The choice of w was arbitrary, so

$$\inf_{v \in L^2} G[v] \geq C$$

Therefore,

$$\inf_{v \in L^2} G[v] = C$$

Since $G[u] = C$, this means u is a minimizer of G.

□

2.3.3 Constructing the LSC Envelope

The construction of the LSC envelope draws on the idea used in the example in Section 2.3.1. There, we had $f_i - v(x_i) > 1$ and chose an approximating sequence $\{v_k\}$ that converged to v in L^2 and satisfied $f_i - v_k(x_i) = 1$. The increase in variation was offset by the decrease in the distance squared to f_i. This is the approach we will use to construct minimizing sequences of (2.47). Proposition 2.3.3 shows that we expect G to be finite for any $u \in BV$, so in particular it may have jumps at the x_i. In general, the value $G[u]$ will depend on the configuration of f_i and the left- and right-hand limits of u at x_i. This will require us to consider the various cases.

A convenient formula for the lower semicontinuous envelope has been suggested by H. Brezis ([8]). To this end, we introduce some notation.

Define a function

$$\Phi(t) = \begin{cases} t^2 & \text{if } t \le 1 \\ 2t - 1 & \text{if } t > 1 \end{cases} \tag{2.51}$$

Given $u \in BV([0,1])$, for any $x \in (0,1)$ let

$$j(u)(x) = [\min\{u(x-0), u(x+0)\}, \max\{u(x-0), u(x+0)\}] \tag{2.52}$$

This takes a function u and associates it with the interval between its left- and right-hand limits at x. These always exist for a function of bounded variation, so $j(u)$ is well-defined.

Let $d : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to [0, \infty)$ be the distance function between a point and a set.

Theorem 2.3.5. *Let F be as in (3.1). Then its lower semicontinuous envelope is given by*

$$G[u] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du| \tag{2.53}$$

Our method of proof has two stages. First, we will show that the proposed formula is a lower bound for G. Then, we show that for any $u \in BV$, we can construct a sequence $\{u_k\}$ converging to u in L^2 such that

$$\lim_{k\to\infty} F[u_k] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

To establish the lower bound, we introduce an auxiliary problem:

Problem 2.3.6. For a fixed control point (x_i, f_i) and a given function $u \in BV$, minimize the function

$$g_i(c) = |c - f_i|^2 + |c - u(x_i - 0)| + |c - u(x_i + 0)| \tag{2.54}$$

This is quadratic in c, so there's a unique minimizer.

Proposition 2.3.7. *The minimum attained in Problem 2.3.6 is given by*

$$\inf_{c\in\mathbb{R}}\{g_i(c)\} = \Phi\left(d(f_i, j(u)(x_i))\right) + |u(x_i-0) - u(x_i+0)|$$

Proof: We consider separately the cases $f_i \in j(u)(x_i)$ and $f_i \notin j(u)(x_i)$.

Case 1: $f_i \in j(u)(x_i)$

For any real number c,

$$|c - u(x_i-0)| + |c - u(x_i+0)| \geq |u(x_i+0) - u(x_i-0)|$$

From (2.54) it follows that

$$g_i(c) \geq |u(x_i+0) - u(x_i-0)| \tag{2.55}$$

for any c. When $c = f_i$,

$$\Phi\left(d(f_i, j(u)(x_i))\right) = 0$$

Hence,

$$g_i(f_i) = |u(x_i+0) - u(x_i-0)|$$

and by (2.55) this must be the minimum.

Case 2: $f_i \notin j(u)(x_i)$

If $c \in j(u)(x_i)$, then

$$|c - u(x_i-0)| + |c - u(x_i+0)| = |u(x_i+0) - u(x_i-0)|$$

and the minimum of the term

$$|c - f_i|^2$$

under the constraint $c \in j(u)(x_i)$ occurs at one of the endpoints, i.e. at $c = u(x_i + 0)$ or $c = u(x_i - 0)$. Then

$$g_i(c) = d(f_i, j(u)(x_i))^2 + |u(x_i + 0) - u(x_i - 0)|$$

On the other hand, if $c \notin j(u)(x_i)$, then we compute

$$g_i'(c) = 2(c - f_i) + \text{sgn}(c - u(x_i - 0)) + \text{sgn}(c - u(x_i + 0))$$

and it moreover holds that

$$\text{sgn}(c - u(x_i - 0)) = \text{sgn}(c - u(x_i + 0))$$

setting the derivative of g_i equal to 0 and solving for c, it follows that

$$c = f_i \mp 1$$

depending on whether $\text{sgn}(c - u(x_i - 0)) = \pm 1$. If this is the minimizer, that implies $d(f_i, j(u)(x_i)) \geq 1$.

We combine these cases. Let c^* denote the minimizer of g_i. Then

$$|c^* - f_i| = \begin{cases} d(f_i, j(u)(x_i)) & \text{if } d(f_i, j(u)(x_i)) < 1 \\ 1 & \text{otherwise} \end{cases}$$

Moreover,

$$|c^* - u(x_i - 0)| + |c^* - u(x_i + 0)| = \begin{cases} |u(x_i + 0) - u(x_i - 0)| & \text{if } c^* \in j(u)(x_i) \\ 2d(c^*, j(u)(x_i)) + |u(x_i + 0) - u(x_i - 0)| & \text{if } c^* \notin j(u)(x_i) \end{cases}$$

Now, if $c^* \notin j(u)(x_i)$, then $|c^* - f_i| = 1$ and

$$d(c^*, j(u)(x_i)) = d(f_i, j(u)(x_i)) - 1$$

Hence

$$g_i(c^*) = \begin{cases} d(f_i, j(u)(x_i))^2 + |u(x_i+0) - u(x_i-0)| & \text{if } d(f_i, j(u)(x_i)) < 1 \\ 1 + 2(d(f_i, j(u)(x_i)) - 1) + |u(x_i+0) - u(x_i-0)| & \text{if } d(f_i, j(u)(x_i)) \geq 1 \end{cases}$$

Comparing this with (2.51), we conclude

$$g_i(c^*) = \Phi\left(d(f_i, j(u)(x_i))\right) + |u(x_i - 0) - u(x_i + 0)|$$

□

Lemma 2.3.8. *For any* $u \in BV$*,*

$$G[u] \geq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

Proof: Let $u \in BV$ be given and let $\{u_k\} \subset W^{1,1}$ be such that $u_k \to u$ in the L^2 norm. Passing to a subsequence if necessary, this implies that $\{u_k\}$ converges pointwise a.e.

Let $\epsilon > 0$. Since $u \in BV$, there exists $\delta > 0$ such that, for $i = 0, 1, ..., n$,

$$|u(x_i + 0) - u(x)| < \frac{\epsilon}{2}$$

whenever $0 < x - x_i < \delta$, and for $i = 1, 2, ..., n+1$

$$|u(x_i - 0) - u(x)| < \frac{\epsilon}{2}$$

whenever $0 < x_i - x < \delta$.

For $i = 0, 1, ..., n$, let a_i be chosen such that

$$x_i < a_i < x_i + \delta,$$

u is continuous at a_i, and

$$\lim_{k\to\infty} u_k(a_i) = u(a_i)$$

Since u has at most countably many discontinuities, we may always find such an a_i.

Similarly, for $i = 1, 2, ..., n+1$, we may choose b_i be chosen such that

$$x_i - \delta < b_i < x_i,$$

u is continuous at b_i, and

$$\lim_{k\to\infty} u_k(b_i) = u(b_i)$$

Hence, for sufficiently large k and $i = 1, 2, ..., n$,

$$|u_k(x_i) - u_k(a_i)| + |u_k(x_i) - u_k(b_i)| \geq |u_k(x_i) - u(x_i - 0)| + |u_k(x_i) - u(x_i + 0)| - \epsilon \quad (2.56)$$

Recalling (2.54), it follows from Proposition 2.3.7 that

$$|u_k(x_i) - f_i|^2 + |u_k(x_i) - u_k(a_i)| + |u_k(x_i) - u_k(b_i)| \\ \geq \Phi\left(d(f_i, j(u)(x_i))\right) + |u(x_i - 0) - u(x_i + 0)| - 2\epsilon \quad (2.57)$$

Since u_k is continuous, we have

$$\int_0^1 |Du_k| \geq \sum_{i=0}^{n} \int_{a_i}^{b_{i+1}} |u_k'| + \sum_{i=1}^{n} |u_k(x_i) - u_k(a_i)| + |u_k(x_i) - u_k(b_i)|$$

Lower semicontinuity of the total variation in L^2 ensures that for sufficiently large k,

$$\sum_{i=0}^{n} \int_{a_i}^{b_{i+1}} |Du_k| \geq \sum_{i=0}^{n} \int_{a_i}^{b_{i+1}} |Du| - \epsilon$$

Hence, for sufficiently large k,

$$F[u_k] \geq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \sum_{i=1}^{n} |u(x_i - 0) - u(x_i + 0)| + \sum_{i=0}^{n} \left(\int_{a_i}^{b_{i+1}} |Du|\right) - (2n-1)\epsilon$$

Now, as $\epsilon \to 0$, $(b_i - a_i) \to 0$, so

$$\lim_{\epsilon\to 0} \left(\sum_{i=0}^{n} \int_{a_i}^{b_{i+1}} |Du| + \sum_{i=1}^{n} |u(x_i - 0) - u(x_i + 0)|\right) = \int_0^1 |Du| \quad (2.58)$$

Hence,

$$F[u_k] \geq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

This is true for sufficiently large k, so in particular

$$\liminf_{k\to\infty} F[u_k] \geq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

This is true for any sequence $\{u_k\}$ satisfying the requirements of Definition 2.3.1. Therefore,

$$G[u] \geq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

$\square$

Having established the lower bound, we show that minimizing sequences can always be constructed. First, we will handle the special case where $u \in W^{1,1}$.

Lemma 2.3.9. *Let* $u \in W^{1,1}$. *Then*

$$G[u] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

Proof: We construct a minimizing sequence for (2.47).

For $i = 1, 2, ..., n$, let C_i be the minimizer of Problem 2.3.6. Let

$$d = \max_{1\leq i\leq n} |x_{i+1} - x_i|$$

If $C_i \neq u(x_i)$, let ψ_i^k be a piecewise linear function with nodes at

$$\left(x_i - \frac{1}{2dk}, u\left(x_i - \frac{1}{2dk}\right)\right), (x_i, C_i), \quad \text{and } \left(x_i + \frac{1}{2dk}, u\left(x_i + \frac{1}{2dk}\right)\right)$$

Otherwise, if $C_i = u(x_i)$ let $\psi_i^k = u$.

Define a sequence $\{u_k\}$ by

$$u_k(x) = \begin{cases} \psi_i^k(x) & \text{if } |x_i - x| < \dfrac{1}{2dk} \\ u(x) & \text{otherwise} \end{cases} \tag{2.59}$$

Then $|u_k(x_i) - f_i| \leq 1$ for $i = 1, 2, ..., n$. By (2.51), this implies

$$F[u_k] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u_k)(x_i))\right) + \int_0^1 |Du_k|$$

By Lemma 2.3.8, this implies

$$G[u_k] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u_k)(x_i))\right) + \int_0^1 |Du_k|$$

If $C_i = u(x_i)$ for every i, then $u_k = u$ so there's nothing to prove.

Suppose that $C_j \neq u(x_j)$ for at least one index j. Let $\{m_1, m_2, ..., m_l\}$ be the collection of indices such that $C_{m_i} \neq u(x_{m_i})$. Let $A_k = \{x |\, u_k(x) \neq u(x)\}$. By (2.59), this is a finite union of intervals, so we may write

$$\begin{aligned} G[u_k] = \sum_{i=1}^{n} |C_i - f_i|^2 + \int_{[0,1]\setminus A_k} |u'| \\ + \sum_{i=1}^{l} \left[\left| C_{m_i} - u\left(x_{m_i} - \frac{1}{2dk}\right) \right| + \left| C_{m_i} - u\left(x_{m_i} + \frac{1}{2dk}\right) \right| \right] \end{aligned} \tag{2.60}$$

Now we compute the limit inferior. Since u is absolutely continuous and the measure of A_k goes to zero as $k \to \infty$,

$$\lim_{k\to\infty} \int_{[0,1]-A_k} |Du| = \int_0^1 |Du|$$

Moreover, continuity of u ensures that

$$\lim_{k\to\infty} \sum_{i=1}^{l} \left[\left| C_{m_i} - u\left(x_{m_i} - \frac{1}{2dk}\right) \right| + \left| C_{m_i} - u\left(x_{m_i} + \frac{1}{2dk}\right) \right| \right] = 2\sum_{i=1}^{l} |C_{m_i} - u(x_{m_i})|$$

Hence,

$$\lim_{k\to\infty} G[u_k] = \sum_{i=1}^{n} |C_i - f_i|^2 + \int_0^1 |Du| + 2\sum_{i=1}^{l} |C_{m_i} - u(x_{m_i})|$$

For $i = m_1, m_2, ..., m_l$,

$$\Phi\left(d(f_i, j(u)(x_i))\right) = |C_i - f_i|^2 + 2|C_i - u(x_i)|$$

Otherwise, $C_i = u(x_i)$ and so

$$\Phi\left(d(f_i, j(u)(x_i))\right) = |C_i - f_i|^2$$

Hence,

$$\lim_{k\to\infty} G[u_k] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du| \tag{2.61}$$

By Lemma 2.3.2, we have

$$\liminf_{k\to\infty} G[u_k] \geq G[u]$$

Applying Lemma 2.3.8 then shows that

$$G[u] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

□

Proof of Theorem 2.3.3: Let $u \in BV$ and suppose there exists $\delta > 0$ such that u is continuous on $(x_i - \delta, x_i + \delta)$ for $i = 1, 2, ..., n$. We will approximate u by absolutely continuous functions.

Let $a_i = x_i - \frac{\delta}{2}$ and let $b_i = x_i + \frac{\delta}{2}$. For $l = 1, 2, ..., n-1$, define functions $v_l^k \in W^{1,1}([b_l, a_{l+1}])$ satisfying

$$\begin{aligned} v_l^k(b_l) &= u(b_l) \\ v_l^k(a_{l+1}) &= u(a_{l+1}) \\ \int_{b_l}^{a_{l+1}} |u - v_l^k|^2 &\leq \frac{1}{(n+1)k} \\ \int_{b_l}^{a_{l+1}} |Dv_l^k| &\leq \int_{b_l}^{a_{l+1}} |Du| + \frac{1}{(n+1)k} \end{aligned} \tag{2.62}$$

It was shown in Propositions 2.1.10 and 2.1.12 that the above conditions can be satisfied for monotone functions on $[b_l, a_{l+1}]$. Since $u \in BV$, it is the difference of two monotone functions, and so we may find functions v_l^k satisfying the above conditions.

Let $v_0^k \in W^{1,1}([0, a_1])$ satisfy

$$\begin{aligned} v_0^k(a_1) &= u(a_1) \\ \int_0^{a_1} |u - v_0^k|^2 &\leq \frac{1}{(n+1)k} \\ \int_0^{a_1} |Dv_0^k| &\leq \int_0^{a_1} |Du| + \frac{1}{(n+1)k} \end{aligned} \tag{2.63}$$

Let $v_n^k \in W^{1,1}([b_n, 1])$ satisfy

$$\begin{aligned} v_l^k(b_n) &= u(b_n) \\ \int_{b_n}^{1} |u - v_n^k|^2 &\leq \frac{1}{(n+1)k} \\ \int_{b_n}^{1} |Dv_n^k| &\leq \int_{b_n}^{1} |Du| + \frac{1}{(n+1)k} \end{aligned} \tag{2.64}$$

Define a sequence $\{u_k\}$ by

$$u_k(x) = \begin{cases} v_0^k(x) & \text{if } x < a_1 \\ v_l^k(x) & \text{if } 1 \leq l < n \text{ and } b_l < x < a_{l+1} \\ v_n^k(x) & \text{if } x > b_n \\ u(x) & \text{otherwise} \end{cases}$$

Then $u_k \to u$ in the L^2 norm, $\Phi\left(d(f_i, j(u_k)(x_i))\right) = \Phi\left(d(f_i, j(u)(x_i))\right)$ for $i = 1, 2, ..., n$, and, since u is continuous at the points a_i and b_i, we also have

$$\int_0^1 |Du_k| \leq \int_0^1 |Du| + \frac{1}{k}$$

for $i = 1, 2, ..., n$.

By Lemma 2.3.9,

$$G[u_k] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u_k)(x_i))\right) + \int_0^1 |Du_k|$$

Hence,

$$G[u_k] \leq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du| + \frac{1}{k}$$

Lower semicontinuity of G in L^2 therefore ensures

$$G[u] \leq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

and by Lemma 2.3.8 equality holds.

Finally, we consider the general case of $u \in BV$. Let $C_1, C_2, ..., C_n$ be the corresponding solutions of Problem 2.3.6. Let $D = \max\{|x_i - x_{i-1}|\}$. Define a sequence $\{u_k\}$ by

$$u_k(x) = \begin{cases} C_i(x) & \text{if } |x - x_i| < \dfrac{1}{2Dk} \\ u(x) & \text{otherwise} \end{cases}$$

The functions u_k are continuous in a neighborhood of x_i for $i = 1, 2, .., n$ and for every $k = 1, 2, ...$ As we have shown, this implies

$$G[u_k] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u_k)(x_i))\right) + \int_0^1 |Du_k|$$

Since $u_k(x_i) = C_i$, we have

$$G[u_k] = \sum_{i=1}^{n} |f_i - C_i|^2 + \int_0^1 |Du_k|$$

Now,

$$\lim_{k\to\infty} \int_0^1 |Du_k| = \int_0^1 |Du| + 2d(C_i, j(u_k)(x_i))$$

Hence,

$$\lim_{k\to\infty} G[u_k] = \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

By construction, $u_k \to u$ in L^2. Lower semicontinuity of G in L^2 implies

$$G[u] \leq \sum_{i=1}^{n} \Phi\left(d(f_i, j(u)(x_i))\right) + \int_0^1 |Du|$$

Lemma 2.3.8 then implies that equality holds.

□

2.3.4 The Minimizers of G

Now we characterize the functions which minimize G. Recall that in Section 2.1, it was shown that for Problem 2.1.1, there were numbers $U_1, U_2, ..., U_n$ such that any solution u satisfied $u(x_i) = U_i$ and that u was moreover monotone in the intervals (x_i, x_{i+1}). We will show that an even larger class of functions minimizes G. Minimizers will still have to be monotone in intervals (x_i, x_{i+1}), but the endpoints will, loosely speaking, only have to lie between U_i and U_{i+1}.

Our first result connects the numbers C_i which minimize Problem 2.3.6 and the numbers U_i corresponding to a minimizer of F.

Proposition 2.3.10. *Let control points $(x_1, f_1), (x_2, f_2), ..., (x_n, f_n)$ be given. Let u be a minimizer of G, and let the numbers $C_1, C_2, ..., C_n$ denote the corresponding minimizers of Problem 2.3.6. If $U_1, U_2, ..., U_n$ are the minimizers of Problem 1.2, then*

$$C_i = U_i$$

for $i = 1, 2, ..., n$.

Proof: Let v be a continuous piecewise linear function interpolating the points (x_i, C_i), which is constant on the intervals $(0, x_1)$ and $(x_n, 1)$. Then v is also a minimizer of G and $G[v] = F[v]$.

Suppose, for the sake of contradiction, that $C_i \neq U_i$ for some i. In this circumstance, v is not a minimizer of F.

Let w be a minimizer of F. Then $F[w] < F[v]$. Hence,

$$G[w] \leq F[w] < F[v] = G[v]$$

so v could not be a minimizer of G, which is a contradiction.

□

Theorem 2.3.11. *Let control points* $(x_1, f_1), (x_2, f_2), ..., (x_n, f_n)$ *be given. Let* $U_1, U_2, ..., U_n$ *be the numbers that minimize Problem 2.1.1, and let*

$$B_i = [\min\{U_i, U_{i+1}\}, \max\{U_i, U_{i+1}\}]$$

A function $u \in BV$ *is a minimizer of* G *if and only if it satisfies the following conditions:*

1. $u(x_i + 0) \in B_i$ *and* $u(x_{i+1} - 0) \in B_i$ *for* $i = 1, 2, ..., n-1$
2. u *is monotone on* (x_i, x_{i+1}) *for* $i = 1, 2, .., .n-1$.
3. $\operatorname{sgn}(U_{i+1} - U_i) \cdot \operatorname{sgn}(u(x_{i+1} - 0) - u(x_i + 0)) \geq 0$.
4. $u \equiv U_1$ *on* $(0, x_1)$.
5. $u \equiv U_n$ *on* $(x_n, 1)$.

Remark: Conditions 1 and 2 imply that instead of having to pass through the points U_i, a minimizer of G need only have its range in the interval between U_i and U_{i+1}. Condition 3 ensures that u is decreasing if $U_{i+1} < U_i$ and increasing if $U_i < U_{i+1}$.

Proof: First we prove that each condition is necessary. Suppose $u \in BV$ is a minimizer. It has a left-continuous representative, which we also denote u.

Begin with Condition 2. To see that Condition 2 holds, first observe that

$$\int_0^1 |Du| \geq \sum_{i=1}^{n-1} |u(x_{i+1} - 0) - u(x_i + 0)| + \sum_{i=1}^{n} |u(x_i - 0) - u(x_i + 0)|$$

and equality holds iff u is monotone on the intervals (x_i, x_{i+1}). Since the numbers $\Phi\left(d(f_i, j(\tilde{u})(x_i))\right))$ depend only on the values of the left- and right-hand limits of u at the x_i, it is clear that only a function monotone on (x_i, x_{i+1}) can minimize G.

Now consider Condition 1. For $i = 1, 2, ..., n$, choose $\lambda_i \in j(u)(x_i)$ such that

$$|f_i - \lambda_i| = d(f_i, j(u)(x_i)) \tag{2.65}$$

Let w be a piecewise-linear function with nodes at the points (x_i, λ_i). Then $G[w] \leq G[u]$. Let v be a minimizer of (2.2). Then $G[v] = F[v]$. Moreover, $G[w] = G[v]$ since both are minimizers. Explicitly,

$$G[v] = \sum_{i=1}^{n} |f_i - U_i|^2 + \sum_{i=1}^{n-1} |U_{i+1} - U_i|$$

$$G[w] = \sum_{i=1}^{n} \Phi\left(|f_i - \lambda_i|\right) + \sum_{i=1}^{n-1} |\lambda_{i+1} - \lambda_i|$$

If $|f_i - \lambda_i| \leq 1$ for $i = 12, ..., n$, then

$$\Phi\left(|f_i - \lambda_i|\right) = |f_i - \lambda_i|^2$$

This would imply that $\lambda_1, \lambda_2, ..., \lambda_n$ are solutions to Problem 2.1.2, so by uniqueness we must perforce have $\lambda_i = U_i$.

Suppose, on the other hand, that for some index k, $|f_k - \lambda_k| > 1$. Then

$$\Phi\left(|f_k - \lambda_k|\right) = 2\left(|f_k - \lambda_k| - 1\right) + 1$$

Then there exists α_k such that $|f_k - \alpha_k| = 1$ and

$$|\lambda_k - \lambda_{k-1}| + |\lambda_{k+1} - \lambda_k| + 2\left(|f_k - \lambda_k| - 1\right) \geq |\alpha_k - \lambda_{k-1}| + |\lambda_{k+1} - \alpha_k| \tag{2.66}$$

We may do this for any such index k. Now let

$$\tilde{\lambda}_i = \begin{cases} \alpha_i & \text{if } |f_i - \lambda_i| > 1 \\ \lambda_i & \text{if } |f_k - \lambda_k| \leq 1 \end{cases}$$

Then

$$
\begin{aligned}
G[w] &\geq \sum_{i=1}^{n} \Phi\left(|f_i - \tilde{\lambda}_i|\right) + \sum_{i=1}^{n-1} |\tilde{\lambda}_{i+1} - \tilde{\lambda}_i| \\
&= \sum_{i=1}^{n} |f_i - \tilde{\lambda}_i|^2 + \sum_{i=1}^{n-1} |\tilde{\lambda}_{i+1} - \tilde{\lambda}_i|
\end{aligned}
\tag{2.67}
$$

Since $G[w] = G[v]$, this implies

$$
\sum_{i=1}^{n} |f_i - U_i|^2 + \sum_{i=1}^{n-1} |U_{i+1} - U_i| \geq \sum_{i=1}^{n} |f_i - \tilde{\lambda}_i|^2 + \sum_{i=1}^{n-1} |\tilde{\lambda}_{i+1} - \tilde{\lambda}_i|
$$

However, the numbers U_i minimize this quantity and the minimizer is unique, so this implies equality holds $\tilde{\lambda}_i = U_i$.

Hence, equality must hold in (2.66). This implies Condition 1.

Next, consider Condition 3. Fix i and assume without loss of generality that $U_i \leq U_{i+1}$. Suppose, on the contrary, that $u(x_i + 0) > u(x_{i+1} - 0)$. Conditions 1 and 2 assure us that there exists some $\gamma \in [U_i, U_{i+1}]$ and some $x_0 \in (x_i, x_{i+1})$ such that $u(x_i) = \gamma$, and that $u(x_i + 0) \geq \gamma \geq u(x_{i+1} - 0)$. Let

$$
\tilde{u}(x) = \begin{cases} \gamma & \text{if } x \in (x_i, x_{i+1}) \\ u(x) & \text{otherwise} \end{cases}
$$

Then

$$
\Phi\left(d(f_i, j(\tilde{u})(x_i))\right) \leq \Phi\left(d(f_i, j(u)(x_i))\right),
$$

$$
\Phi\left(d(f_i, j(\tilde{u})(x_{i+1}))\right) \leq \Phi\left(d(f_i, j(u)(x_{i+1}))\right),
$$

$$
\int_0^1 |D\tilde{u}| < \int_0^1 |Du|
$$

whence it follows that $G[\tilde{u}] < G[u]$, a contradiction. Therefore, Condition 3 holds.

On to Condition 4. Let $\lambda = u(x_1 - 0)$. If u is nonconstant on $(0, x_1)$, then let

$$
\tilde{u}(x) = \begin{cases} \lambda & \text{if } x < x_1 \\ u(x) & \text{otherwise} \end{cases}
$$

Then $d(f_i, j(u)(x_i)) = d(f_i, j(\tilde{u})(x_i))$ for $i = 1, 2, ...n$, but, by Proposition A.16,

$$\int_0^1 |Du| > \int_0^1 |D\tilde{u}|$$

so $G[\tilde{u}] < G[u]$. Now that we know u must be constant, we find the optimal constant. We claim this is C_1. If $|C_1 - f_1| \leq 1$, then $C_1 \in j(u)(x_i)$. If $u(x_1 + 0) = C_1$, then it's clear that taking $\lambda = C_1$ will minimize the total variation. If $u(x_1 + 0) \neq C_1$, then λ must be chosen so that $C_1 \in j(u)(x_i)$, and setting $\lambda = C_1$ will minimize the total variation.

If $|C_1 - f_1| > 1$, then $C_1 \notin j(u)(x_i)$. Hence,

$$\Phi\left(d(f_i, j(u)(x_i))\right) = 2d(C_i, j(u)(x_i)) + 1$$

Meanwhile, the total variation of u, isolating terms depending on λ, is

$$\int_0^1 |Du| = |u(x_1 + 0) - \lambda| + \int_{x_1}^1 |Du|$$

Thus, we seek to minimize the quantity

$$2d(C_i, j(u)(x_i)) + |u(x_1 + 0) - \lambda|$$

which can be rewritten as

$$2 \min\{|u(x_1 + 0) - C_i|, |\lambda - C_i|\} + |u(x_1 + 0) - \lambda|$$

and it is clear that the minimum occurs when $\lambda = C_i$.

Finally, Condition 5 is entirely analgous to Condition 4. Thus, we have shown that Conditions 1-5 are necessary for a minimizer. We proceed to prove that they are sufficient.

Let $u \in BV$ satisfy Conditions 1-5. Let $D = \max\{|x_i - x_{i+1}|\}$. In Lemma 2.1.11, we gave a method for constructing a sequence $\{v_k\} \subset W^{1,1}$ of monotone functions that converge in L^1 to a given bounded, monotone function v on an interval (a, b), such that

$v_k(a) = v(a+0)$ and $v_k(b) = v(b-0)$. Since we are on a bounded interval, these will also converge in L^2.

For $i = 1, 2, ..., n-1$, let α_i^k be chosen such that

$$x_i < \alpha_i^k < x_i + \frac{1}{2Dk}$$

and where u is continuous at α_i^k.

Likewise, let β_i^k be chosen such that

$$x_{i+1} - \frac{1}{2Dk} < \beta_i^k < x_{i+1}$$

and where u is continuous at β_i^k.

On every interval (α_i^k, β_i^k), let $v_i^k \in W^{1,1}$ be monotone and satisfy $v_i^k(\alpha_i^k) = u(\alpha_i^k + 0)$, $v_i^k(\beta_i^k) = u(\beta_i^k - 0)$, and

$$\int_{\alpha_i^k}^{\beta_i^k} |v_i^k - u|^2 < \frac{1}{(n-1)k}$$

For $i = 1, 2, ..., n-2$, let ψ_i^k be the piecewise linear function connecting the points $(\beta_i^k, u(\beta_i^k)$, (x_{i+1}, U_{i+1}), and $(\alpha_{i+1}^k, u(\alpha_{i+1}^k)$. Let ψ_0^k be the line connecting the points (x_1, U_1) and $(\alpha_1^k, u(\alpha_1^k)$ and ψ_{n-1}^k the line connecting the points $(\beta_{n-1}^k, u(\beta_{n-1}^k)$ and (x_n, U_n). Define a sequence $\{u_k\}$ by

$$u_k(x) = \begin{cases} U_1 & \text{if } x < x_1 \\ \psi_0^k & \text{if } x_1 \leq x \leq \alpha_1^k \\ v_i^k & \text{if } \alpha_i^k \leq x \leq \beta_i^k \\ \psi_i^k & \text{if } \beta_i^k \leq x \leq \alpha_{i+1}^k \\ \psi_{n-1}^k & \text{if } \beta_{n-1}^k \leq x \leq x_n \\ U_n & \text{if } x > x_n \end{cases} \tag{2.68}$$

For every k, u_k is continuous, monotone on the intervals (x_i, x_{i+1}), and $u(x_i) = U_i$. It also takes the constant value U_1 on $(0, x_1)$ and the constant U_n on $(x_n, 1)$. Hence, it is a minimizer of the functional F.

Moreover, we have

$$\sum_{i=1}^{n-1} \int_{\alpha_i^k}^{\beta_i^k} |u_k - u|^2 < \frac{1}{k}$$

Since u satisfies Conditions 4 and 5, we have $u = u_k$ on the intervals $(0, x_1)$ and $(x_n, 1)$. Hence,

$$\int_0^1 |u_k - u|^2 < \frac{1}{k} + \sum_{i=1}^{n-1} \int_{\beta_i^k}^{\alpha_{i+1}^k} |u_k - u|^2 \tag{2.69}$$

The functions u_k are uniformly bounded and $\alpha_{i+1}^k - \beta_i^k < \frac{1}{Dk}$, so the term

$$\sum_{i=1}^{n-1} \int_{\beta_i^k}^{\alpha_{i+1}^k} |u_k - u|^2$$

vanishes as $k \to \infty$. Hence, $u_k \to u$ in the L^2 norm. This means u is the L^2 limit of minimizers of F, and by Proposition 2.3.4 this implies u is a minimizer of G.

□

Notice that we proved the sufficiency of Conditions 1-5 by constructing an approximating sequence of minimizers of F. This implies the following corollary:

Corollary 2.3.12. *If $u \in BV$ is a minimizer of G, then there exists a sequence $\{u_k\}$ converging to u in L^2 such that u_k is a minimizer of F for every k.*

2.3.5 Minimizing $G[u] + \int_0^1 u^2$

In Section 2.1.5, we considered the functional (2.24), and showed that it did not have a minimizer in $W^{1,1}$. Here, we will reconsider it in terms of the functional G instead of F. Thus, we have the following problem:

Problem 2.3.13. Let control points $(x_1, f_1), ..., (x_n, f_n)$ be given. Find $u \in L^2$ that minimizes the functional

$$H[u] = G[u] + \int_0^1 u^2 \tag{2.70}$$

Proposition 2.3.14. *There exists a unique solution to Problem 2.3.13.*

Proof: Since L^2 is reflexive, it is sufficient to show that H is lower semicontinuous, strictly convex, and coercive.

The functional

$$u \mapsto \int_0^1 u^2$$

is coercive and strictly convex and continuous. The functional G is convex, nonnegative, lower semicontinuous in the L^2 norm. Therefore, H is coercive, strictly convex, and lower semicontinuous in L^2.

□

Proposition 2.3.15. *If $\{v_k\} \subset W^{1,1}$ is a minimizing sequence of (2.24), and if $\hat{u}$ is the unique minimizer of H, then*

$$\lim_{k\to\infty} \int_0^1 |v_k - \hat{u}|^2 = 0$$

Proof: Since G is the lower semicontinuous envelope of F, there exists a sequence $\{u_k\} \subset W^{1,1}$ such that $\lim_{k\to\infty} \int_0^1 |u_k - \hat{u}|^2 = 0$ and $\lim_{k\to\infty} F[u_k] = G[\hat{u}]$. Hence,

$$\lim_{k\to\infty} F[u_k] + \int_0^1 u_k^2 = G[\hat{u}] + \int_0^1 \hat{u}^2 \tag{2.71}$$

and so $\{u_k\}$ is a minimizing sequence.

Let $\{v_k\} \subset W^{1,1}$ be an arbitrary minimizing sequence. By coercivity, $\{v_k\}$ is bounded in $W^{1,1}$, and compact embedding of $W^{1,1}([0,1])$ into $L^2([0,1])$ ensures that there is a subsequence converging strongly to some $v \in L^2$. Relabeling if necessary, we may denote this subsequence by $\{v_k\}$. Since G is the lower semicontinuous envelope of F, we have

$$G[v] \leq \liminf F[v_k]$$

Since $\{v_k\}$ converges strongly in L^2, $\lim_{k\to\infty} ||v_k||_{L^2} = ||v||_{L^2}$; therefore,

$$G[v] + \int_0^1 v^2 \leq \liminf \left\{ F[v_k] + \int_0^1 v_k^2 \right\}$$

We have taken $\{v_k\}$ to be a minimizing sequence, so

$$\liminf \left\{ F[v_k] + \int_0^1 v_k^2 \right\} = \liminf \left\{ F[u_k] + \int_0^1 u_k^2 \right\}$$

Hence,

$$G[v] + \int_0^1 v^2 \leq G[\hat{u}] + \int_0^1 \hat{u}^2$$

The function $\hat{u}$ is the unique minimizer, whence it follows that $\hat{u} = v$.

□

The minimizers can be characterized as follows:

Proposition 2.3.16. *Let u be the solution to Problem 2.3.13. Then u is constant on (x_i, x_{i+1}) for $i = 1, 2, ..., n-1$ as well as the intervals $(0, x_1)$ and $(x_n, 1)$.*

Proof: Suppose, on the contrary, that u minimizes H and there exists k such that no representative of u is constant on (x_k, x_{k+1}). There exist $\alpha, \beta \in (x_k, x_{k+1})$ such that u is continuous at α and β and

$$|u(\alpha)| < |u(\beta)| \tag{2.72}$$

Without loss of generality, we may suppose $\alpha < \beta$. There exists $y_0 \in [\alpha, \beta]$ and a sequence $\{y_k\} \subset [\alpha, \beta]$ converging to y_0 such that

$$\lim_{k\to\infty} |u(y_k)| = \inf_{y\in[\alpha,\beta]} |u(y)|$$

Let λ denote this value. Then either $\lambda = u(y_0 + 0)$ or $\lambda = u(y_0 - 0)$. Let

$$\tilde{u}(x) = \begin{cases} \lambda & \text{if } \alpha \leq x \leq \beta \\ u(x) & \text{otherwise} \end{cases} \tag{2.73}$$

The only change happens in a closed subinterval of (x_k, x_{k+1}), so in particular we have, for $i = 1, 2, ..., n$,

$$u(x_i + 0) = \tilde{u}(x_i + 0)$$

$$u(x_i - 0) = \tilde{u}(x_i - 0)$$

and therefore

$$\Phi\left(d(f_i, j(u)(x_i))\right) = \Phi\left(d(f_i, j(\tilde{u})(x_i))\right)$$

The function $\tilde{u}$ satisfes the hypotheses of Proposition A.16. Hence,

$$\int_0^1 |D\tilde{u}| \leq \int_0^1 |Du|$$

Moreover, on the interval $[\alpha, \beta]$, $|\tilde{u}| \leq |u|$. Hence,

$$\int_0^1 |\tilde{u}|^2 \leq \int_0^1 |u|^2$$

Using (2.72) and the continuity of u at β, this inequality must be strict. This implies

$$G[\tilde{u}] + \int_0^1 |\tilde{u}|^2 < G[u] + \int_0^1 |u|^2$$

which is a contradiction.

□

We conclude this section by considering explicit solutions to Problem 2.3.13.

Finding an explicit solution

Since the minimizers are piecewise-constant, we may treat the functional as being defined on $\mathbb{R}^{n+1}$. Specifically, a point $(a_0, a_1, ..., a_n)$ shall be identified with the function

$$u(x) = \sum_{i=0}^{n} a_i \chi_{E_i}(x)$$

where $E_i = (x_i, x_{i+1})$ for $i = 1, 2, .., n$ and χ_{E_i} is the corresponding indicator function. As we will be considering functions depending on finitely many parameters, it makes sense to define an analogue of (2.52) in terms of these parameters. To that end, define a function

$$\text{¡}(a, b) = (\min\{a, b\}, \max\{a, b\}) \tag{2.74}$$

Let

$$\Delta x_i = x_{i+1} - x_i$$

Let $\mathbf{a} = (a_0, a_1, ..., a_n)$ and $\mathbf{f} = (f_1, f_2, ..., f_n)$. We can write

$$\mathfrak{H}(\mathbf{f}, \mathbf{a}) = \sum_{i=1}^{n} \Phi\left(d(f_i, \text{¡}(a_{i-1}, a_i))\right) + \sum_{i=0}^{n} |a_{i+1} - a_i| + \sum_{i=0}^{n} a_i^2 \Delta x_i \tag{2.75}$$

where for future convenience we have made explicit the dependence on the f_i. When the f_i are understood to be fixed, we shall use the notation

$$\mathfrak{H}(a_0, a_2, ..., a_n) = \sum_{i=1}^{n} \Phi\left(d(f_i, \text{¡}(a_{i-1}, a_i))\right) + \sum_{i=0}^{n} |a_{i+1} - a_i| + \sum_{i=0}^{n} a_i^2 \Delta x_i$$

Problem 2.3.17. Given control points $(x_1, f_1), ..., (x_n, f_n)$, find numbers $A_1, A_2, ..., A_n$ such that $\mathfrak{H}$ attains a minimum.

Since (2.75) is nonnegative and goes to infinity as $||\mathbf{a}|| \to \infty$, it attains a minimum. In fact, it is strictly convex in $\mathbf{a}$ so the minimum is unique.

Our solution to Problem 2.3.17 will be developed in the following way: we will show that if $f_1, f_2, ..., f_n$ are sufficiently large, then the minimizer takes a specific form depending only on $x_1, x_2, ..., x_n$. Next, we will show that if we have two sets of control points which differ only in the ordinate of the k^{th} point, then the minimizers are equal except possibly on the intervals (x_{k-1}, x_k) and (x_k, x_{k+1}). We may thus transform the minimizer of a special case to a minimizer for arbitrary $f_1, f_2, ..., f_n$.

Proposition 2.3.18. *Let* $D = \max_i \left\{ \frac{1}{\Delta x_i} \right\}$. *If* $f_1, f_2, ..., f_n > D + 1$, *and if*

$$A_i = \frac{1}{\Delta x_i}$$

for $i = 1, 2, ..., n - 1$, *and*

$$A_0 = \frac{1}{2\Delta x_0},$$
$$A_n = \frac{1}{2\Delta x_n},$$

then $u = \sum_{i=0}^{n} A_i \chi_{E_i}$ *is the minimizer of* $\mathfrak{H}$.

Proof: Suppose first that the numbers Δx_i are all distinct. Let $A_0 = \frac{1}{2\Delta x_0}$, $A_n = \frac{1}{2\Delta x_n}$, and $A_i = \frac{1}{\Delta x_i}$ for $i = 1, 2, ..., n - 1$. For any i between 1 and $n - 1$, we have

$$\frac{\partial \mathfrak{H}}{\partial a_i}|_{(A_0, A_1, ..., A_n)} = \frac{\partial \Phi}{\partial a_i}\left(d(f_i, \mathfrak{i}(A_{i-1}, A_i)\right) + \frac{\partial \Phi}{\partial a_i}\left(d(f_{i+1}, \mathfrak{i}(A_i, A_{i+1})\right)$$
$$+ \operatorname{sgn}(A_i - A_{i-1}) - \operatorname{sgn}(A_{i+1} - A_i) + 2$$

If $A_i > A_{i+1}$, then $\frac{\partial \Phi}{\partial a_i}\left(d(f_{i+1}, \mathfrak{i}(A_i, A_{i+1})\right) = -2$ and $\operatorname{sgn}(A_{i+1} - A_i) = -1$. Otherwise, $\frac{\partial \Phi}{\partial a_i}\left(d(f_{i+1}, \mathfrak{i}(A_i, A_{i+1})\right) = 0$ and $\operatorname{sgn}(A_{i+1} - A_i) = 1$. In either case,

$$\frac{\partial \Phi}{\partial a_i}\left(d(f_{i+1}, \mathfrak{i}(A_i, A_{i+1})\right) - \operatorname{sgn}(A_{i+1} - A_i) = -1$$

and similarly

$$\frac{\partial \Phi}{\partial a_i}\left(d(f_i, \mathfrak{i}(A_{i-1}, A_i)\right) + \operatorname{sgn}(A_i - A_{i-1}) = -1$$

Hence,

$$\frac{\partial \mathfrak{H}}{\partial a_i}(A_0, A_1, ..., A_n) = 0$$

If $i = 0$, then

$$\frac{\partial \mathfrak{H}}{\partial a_0}|_{(A_0, A_1, ..., A_n)} = \frac{\partial \Phi}{\partial a_i}\left(d(f_{i+1}, \mathfrak{i}(A_0, A_1)\right) - \operatorname{sgn}(A_1 - 0) + 1$$

which also vanishes by the above argument, and similarly for $i = n$. Thus,

$$\nabla \mathfrak{H}(A_0, A_1, ..., A_n) = 0$$

Since $\mathfrak{H}$ is convex, this must be the minimizer.

We proceed to the general case, where the numbers Δx_i need not be distinct. Given f_i for $i = 1, 2, ..., n$, let

$$h(x_1, x_2, ..., x_n; a_0, a_1, ..., a_n) = \sum_{i=1}^{n} \Phi\left(d(f_i, \mathfrak{i}(a_{i-1}, a_i))\right) + \sum_{i=0}^{n} |a_{i+1} - a_i| + \sum_{i=1}^{n-1} a_i^2(x_{i+1} - x_i) + a_0^2 x_1 + a_n^2(1 - x_n)$$

This is continuous in both the x_i and the a_i. Let $x_1, x_2, ..., x_n$ be given. If Δx_i are not all distinct, suppose there exist numbers $\alpha_0, ..., \alpha_n$ such that

$$h(x_1, x_2, ..., x_n; \alpha_0, \alpha_1, ..., \alpha_n) < h(x_1, x_2, ..., x_n; A_0, A_1, ..., A_n) \tag{2.76}$$

and let η denote the difference between the right hand side and the left hand side. If we take $0 < \epsilon < \frac{\eta}{2}$, then by continuity there exists $\delta > 0$ such that if $\max\{|x_i - \tilde{x}_i|\} < \delta$, and $\max\{|a_i - \tilde{a}_i|\} < \delta$, then

$$|h(x_1, x_2, ..., x_n; a_0, a_1, ..., a_n) - h(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n; \tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_n)| < \epsilon$$

There exist $\hat{x}_1, \hat{x}_2, ..., \hat{x}_n$ such that the numbers $\Delta\hat{x}_i$ are all distinct, $\max\{|x_i - \hat{x}_i|\} < \delta$, and $\max\{|A_i - \hat{A}_i|\} < \delta$, where $\hat{A}_i = (\Delta\hat{x}_i)^{-1}$ for $i = 1, 2, ..., n-1$, $\hat{A}_0 = (2\Delta\hat{x}_0)^{-1}$, and $\hat{A}_n = (2\Delta\hat{x}_n)^{-1}$. Then

$$|h(x_1, x_2, ..., x_n; a_0, a_1, ..., a_n) - h(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n; \hat{A}_0, \hat{A}_1, ..., \hat{A}_n)| < \epsilon$$

$$|h(x_1, x_2, ..., x_n; a_0, \alpha_1, ..., \alpha_n) - h(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n; \alpha_0, \alpha_1, ..., \alpha_n)| < \epsilon$$

This then implies

$$h(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n; \hat{A}_0, \hat{A}_1, ..., \hat{A}_n) > h(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n; \alpha_0, \alpha_1, ..., \alpha_n)$$

However, since the $\Delta\hat{x}_i$ are distinct it must be the case that

$$h(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n; \hat{A}_0, \hat{A}_1, ..., \hat{A}_n) < h(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n; \alpha_0, \alpha_1, ..., \alpha_n)$$

which is a contradiction. Hence, the numbers α_i cannot exist.

□

Lemma 2.3.19. *Let $x_1, x_2, ..., x_n$ be fixed. Define a mapping $T : [0,\infty)^n \to \mathbb{R}^{n+1}$ as follows: given a vector $\boldsymbol{f} = (f_1, f_2, ..., f_n)$, let $T(\boldsymbol{f})$ satisfy*

$$\mathfrak{H}(\boldsymbol{f}, T(\boldsymbol{f})) = \inf_{\boldsymbol{A}} \mathfrak{H}(\boldsymbol{f}, \boldsymbol{A}) \tag{2.77}$$

Then T is continuous.

Proof: Let $\mathbf{f}_0$ be given and consider a sequence $\{\mathbf{f}_k\}$ converging to $\mathbf{f}_0$. Let $\mathbf{A}_i = T(\mathbf{f}_i)$. From Proposition 2.3.18 it follows that the range of T is bounded, and therefore the sequence $\{\mathbf{A}_i\}$ has a limit point $\mathbf{A}_0$. There is a subsequence $\{\mathbf{A}_{k_j}\}$ converging to $\mathbf{A}_0$. Let

$$\epsilon = \mathfrak{H}(\mathbf{f}_0, \mathbf{A}_0)) - \mathfrak{H}(\mathbf{f}_0, T(\mathbf{f}_0))$$

By hypothesis, $\epsilon \geq 0$.

Suppose, on the contrary, that $\epsilon > 0$. Continuity of $\mathfrak{H}$ implies that there exists some $\delta > 0$ such that

$$|\mathfrak{H}(\mathbf{f}_0, T(\mathbf{f}_0)) - \mathfrak{H}(\mathbf{f}, \mathbf{A})| < \frac{\epsilon}{3}$$

whenever $||\mathbf{f}_0 - \mathbf{f}|| + ||T(\mathbf{f}_0) - \mathbf{A}|| < \delta$, and

$$|\mathfrak{H}(\mathbf{f}_0, \mathbf{A}_0)) - \mathfrak{H}(\mathbf{f}, \mathbf{A}))| < \frac{\epsilon}{3}$$

whenever $||\mathbf{f}_0 - \mathbf{f}|| + ||\mathbf{A}_0 - \mathbf{A}|| < \delta$.

Hence, for sufficiently large j

$$\mathfrak{H}(\mathbf{f}_{k_j}, \mathbf{A}_{k_j})) - \mathfrak{H}(\mathbf{f}_{k_j}, T(\mathbf{f}_{k_j})) > \frac{\epsilon}{3}$$

which is a contradiction. Hence, we must have $\mathbf{A}_0 = T(\mathbf{f}_0)$.

Thus, any sequence $\{\mathbf{f}_k\}$ converging to $\mathbf{f}_0$ has a subsequence for which $T(\mathbf{f}_k) \to T(\mathbf{f}_0)$. This implies that if

$$\lim_{k\to\infty} \mathbf{f}_k = \mathbf{f}_0$$

then

$$\lim_{k\to\infty} T(\mathbf{f}_k) = T(\mathbf{f}_0)$$

whence it follows that T is continuous at $\mathbf{f}_0$. The choice of f_0 was arbitrary, so T is continuous on its domain, as desired.

□

The following proposition shows that if a minimizer is found with a given set of control points, then changing the ordinate of a control point only has a local effect on the new minimizer.

Proposition 2.3.20. *Let control points* $(x_1, f_1), ..., (x_n, f_n)$ *be given and let* $u = \sum_{i=0}^{n} A_i \chi_{E_i}$ *be the minimizer of (2.70). Let* $k \in [0, n]$ *be an integer and* $\kappa \geq 0$*. Define a new set of control points* $(x_1, f_1), ..., (x_k, f_k), (x_{k+1}, \kappa), (x_{k+2}, f_{k+2}) ..., (x_n, f_n)$ *Let* $\tilde{u} = \sum_{i=0}^{n} \tilde{A}_i \chi_{E_i}$ *the corresponding minimizer of (2.70). If* $i \neq k$ *and* $i \neq k+1$*, then*

$$\tilde{A}_i = A_i$$

Proof: We proceed by cases:

Case 1: $A_k > A_{k-1}$ and $\tilde{A}_k \geq A_{k-1}$

Suppose first that the second inequality is strict. Let $D = A_k - A_{k-1}$. By continuity, there exists $\delta > 0$ such that if $|f_{k+1} - \kappa| < \delta$, then $|A_k - \tilde{A}_k| < \frac{1}{2}D$ and $|A_{k-1} - \tilde{A}_{k-1}| < \frac{1}{2}D$. Let

$$S_k = \sum_{i=1} k - 1\Phi\left(d(f_i, \mathfrak{i}(A_{i-1}, A_i))\right) + \sum_{i=0}^{k-2} |A_{i+1} - A_i| + \sum_{i=0}^{k-1} A_i^2 \Delta x_i$$
$$\tilde{S}_k = \sum_{i=1} k - 1\Phi\left(d(f_i, \mathfrak{i}(\tilde{A}_{i-1}, \tilde{A}_i))\right) + \sum_{i=0}^{k-2} |\tilde{A}_{i+1} - \tilde{A}_i| + \sum_{i=0}^{k-1} \tilde{A}_i^2 \Delta x_i$$

We claim that $S_k = \tilde{S}_k$. Without loss of generality, suppose $A_k \geq \tilde{A}_k$. Consider three subcases:

<u>Subcase A</u>: $f_k > A_k$

In this case, we have $\Phi\left(d(f_k, \mathfrak{i}(A_{k-1}, A_k))\right) = \Phi\left(d(f_i, \mathfrak{i}(\tilde{A}_{k-1}, A_k))\right)$ and $\Phi\left(d(f_k, \mathfrak{i}(\tilde{A}_{k-1}, \tilde{A}_k))\right) = \Phi\left(d(f_i, \mathfrak{i}(A_{k-1}, \tilde{A}_k))\right)$. If $S_k \neq \tilde{S}_k$, then uniqueness of the minimizer implies

$$S_k + \Phi\left(d(f_k, \mathfrak{i}(A_{k-1}, A_k))\right) + A_k - A_{k-1} < \tilde{S}_k + \Phi\left(d(f_i, \mathfrak{i}(\tilde{A}_{k-1}, A_k))\right) + A_k - \tilde{A}_{k-1}$$
$$\tilde{S}_k + \Phi\left(d(f_k, \mathfrak{i}(\tilde{A}_{k-1}, \tilde{A}_k))\right) + \tilde{A}_k - \tilde{A}_{k-1} < S_k + \Phi\left(d(f_i, \mathfrak{i}(A_{k-1}, \tilde{A}_k))\right) + \tilde{A}_k - A_{k-1}$$

This simplifies to

$$S_k + A_{k-1} < \tilde{S}_k + \tilde{A}_{k-1}$$
$$\tilde{S}_k + \tilde{A}_{k-1} < S_k + A_{k-1}$$

which is a contradiction.

<u>Subcase B</u>: $f_k < \tilde{A}_k$

In this case, we have $\Phi\left(d(f_k, \mathfrak{i}(A_{k-1}, A_k))\right) = \Phi\left(d(f_i, \mathfrak{i}(A_{k-1}, \tilde{A}_k))\right)$ and $\Phi\left(d(f_k, \mathfrak{i}(\tilde{A}_{k-1}, \tilde{A}_k))\right) = \Phi\left(d(f_i, \mathfrak{i}(\tilde{A}_{k-1}, A_k))\right)$. If $S_k \neq \tilde{S}_k$, then uniqueness of the minimizer implies

$$S_k + \Phi\left(d(f_k, \mathfrak{i}(A_{k-1}, A_k))\right) + A_k - A_{k-1} < \tilde{S}_k + \Phi\left(d(f_i, \mathfrak{i}(\tilde{A}_{k-1}, A_k))\right) + A_k - \tilde{A}_{k-1}$$

$$\tilde{S}_k + \Phi\left(d(f_k, \mathfrak{i}(\tilde{A}_{k-1}, \tilde{A}_k))\right) + \tilde{A}_k - \tilde{A}_{k-1} < S_k + \Phi\left(d(f_i, \mathfrak{i}(A_{k-1}, \tilde{A}_k))\right) + \tilde{A}_k - A_{k-1}$$

Subracting the right hand side of the second inequality from the left hand side of the first inequality, and subtracting the left hand side of the second inequality from the right hand side of the first inequality reduces to

$$A_k - \tilde{A}_k < A_k - \tilde{A}_k$$

which is a contradiction.

Subcase C: $\tilde{A}_k \leq f_k \leq A_k$

In this case, $\Phi\left(d(f_k, \mathfrak{i}(A_{k-1}, A_k))\right) = \Phi\left(d(f_i, \mathfrak{i}(\tilde{A}_{k-1}, A_k))\right) = 0$ and $\Phi\left(d(f_k, \mathfrak{i}(\tilde{A}_{k-1}, \tilde{A}_k))\right) = \Phi\left(d(f_i, \mathfrak{i}(A_{k-1}, \tilde{A}_k))\right)$. If $S_k \neq \tilde{S}_k$, then uniqueness of the minimizer implies

$$S_k + \Phi\left(d(f_k, \mathfrak{i}(A_{k-1}, A_k))\right) + A_k - A_{k-1} < \tilde{S}_k + \Phi\left(d(f_i, \mathfrak{i}(\tilde{A}_{k-1}, A_k))\right) + A_k - \tilde{A}_{k-1}$$

$$\tilde{S}_k + \Phi\left(d(f_k, \mathfrak{i}(\tilde{A}_{k-1}, \tilde{A}_k))\right) + \tilde{A}_k - \tilde{A}_{k-1} < S_k + \Phi\left(d(f_i, \mathfrak{i}(A_{k-1}, \tilde{A}_k))\right) + \tilde{A}_k - A_{k-1}$$

This simplifies to

$$S_k + A_{k-1} < \tilde{S}_k + \tilde{A}_{k-1}$$

$$\tilde{S}_k + \tilde{A}_{k-1} < S_k + A_{k-1}$$

which is a contradiction.

In any case, we have $S_k = \tilde{S}_k$. Since minimizers are unique, and we did not modify any of the f_i except f_{k+1}, we must have $\tilde{A}_i = A_i$ for $i = 0, 1, ..., k-1$.

This is valid for any $\kappa \in (f_{k+1} - \delta, f_{k+1} + \delta)$. By Lemma 1, the conclusion will still be true if κ is in the closed interval $[f_{k+1} - \delta, f_{k+1} + \delta]$. Next, we extend this result outside the interval. For if $\kappa = f_{k+1} - \delta$ or $\kappa = f_{k+1} + \delta]$ and $\tilde{A}_k > A_{k-1}$, we may repeat the above argument and find a larger closed interval I where $\tilde{A}_i = A_i$ for $i = 0, 1, ..., k-1$ provided $\kappa \in I$.

Consider now the maximal interval K with the property that $\tilde{A}_k > A_{k-1}$ and $\tilde{A}_i = A_i$ for $i = 0, 1, ..., k-1$ whenever $\kappa \in K$. We claim that the infimum m of K has the property that $\tilde{A}_k = A_{k-1}$ and $\tilde{A}_i = A_i$ for $i = 0, 1, ..., k-1$. Lemma 1 already assures us that $\tilde{A}_k \geq A_{k-1}$ and $\tilde{A}_i = A_i$ for $i = 0, 1, ..., k-1$. Suppose $\tilde{A}_k > A_{k-1}$. Then we may repeat the above argument to find some δ such that the result holds in $K \cup (m-\delta, m+\delta)$, contradicting the maximality of K.

This concludes Case 1.

Case 2: $A_k < A_{k-1}$ and $\tilde{A}_k \leq A_{k-1}$

This is similar to Case 1.

Case 3: $A_k = A_{k-1}$ and $\tilde{A}_k \neq A_{k-1}$

This follows from Cases 1 and 2 by interchanging f_{k+1} and κ.

Case 4: $A_k > A_{k-1}$ and $\tilde{A}_k < A_{k-1}$

We may split this into two sub-problems. By continuity, there exists $\hat{\kappa}$ where the corresponding minimizer satisfies $\hat{A}_k = A_{k-1}$, and by Case 1 we have $\hat{A}_i = A_i$ for $i = 0, 1, ..., k-1$. Then we may apply Case 3 with $\hat{\kappa}$ in place of f_{k+1} and $\hat{A}_k$ and $\hat{A}_{k-1}$ in place of A_k and A_{k-1}, respectively.

Case 5: $A_k < A_{k-1}$ and $\tilde{A}_k > A_{k-1}$

This is similar to Case 4.

Case 6: $A_k = A_{k-1}$ and $\tilde{A}_k = A_{k-1}$

We have $\tilde{A}_k = A_k$ and the rest follows by uniqueness of minimizers.

So far, we have shown that in all cases, $\tilde{A}_i = A_i$ for $i = 0, 1, ..., k-1$. Similar arguments hold for $i = k+2, k+3, ..., n$. Hence, we have $\tilde{A}_i = A_i$ provided $i \neq k$ and $i \neq k+1$, as desired.

$\square$

In light of Proposition 2.3.22, we pose the following problem:

Problem 2.3.21. Given control points $(x_1, f_1), ..., (x_n, f_n)$, and $A_0, A_1, ..., A_{k-2}, A_{k+1}, ..., A_n$, find A_k and A_{k+1} that minimize

$$
\begin{aligned}
h_k(a, b) = \Phi\left(d(f_i, \text{¡}(A_{k-2}, a)\right) + \Phi\left(d(f_i, \text{¡}(a, b)\right) + \Phi\left(d(f_i, b, A_{k+1})\right) \\
+ |a - A_{k-2}| + |b - a| + |A_{k+1} - b| + a^2\Delta x_k + b^2\Delta x_{k+1}
\end{aligned}
$$

We have

$$
\frac{\partial h_k}{\partial a} = \frac{\partial \Phi}{\partial a}\left(d(f_i, \text{¡}(A_{k-1}, a))\right) + \frac{\partial \Phi}{\partial a}\left(d(f_i, \text{¡}(a, b)) + \operatorname{sgn}(a - A_{k-2})\right) - \operatorname{sgn}(b - a) + 2a\Delta x_k
$$

$$
\frac{\partial h_k}{\partial b} = \frac{\partial \Phi}{\partial b}\left(d(f_i, \text{¡}(a, b))\right) + \frac{\partial \Phi}{\partial b}\left(d(f_i, b, A_{k+1})\right) + \operatorname{sgn}(b - a) - \operatorname{sgn}(A_{k+2} - b) + 2b\Delta x_k
$$

In general, the derivative of $\Phi\left(d(f_i, \text{¡}(\alpha, \beta))\right)$ depends on the configuration of f_i, α, and β. If $\alpha \leq f_i \leq \beta$, then the partial derivatives with respect to α and β vanish. If $\alpha < \beta < f_i$, then $\frac{\partial \Phi}{\partial \alpha} = 0$ and

$$
\frac{\partial \Phi}{\partial \beta} = \begin{cases} 2(f - \beta) & \text{if } f - \beta < 1 \\ -2 & \text{otherwise} \end{cases}
$$

We may impose a definite configuration of a and b with respect to the given numbers $A_{k-2}, A_{k+1}, f_{k-1}, f_k$, and f_{k+1}. We assign definite values to the terms involving the signum function as well as Φ. This makes the minimization problem straightforward and gives us a candidate solution. Since a unique minimizer exists, it defines some configuration, and by iterating over all possible configurations we are sure to find it.

Appendix A

Functions of Bounded Variation

Begin with the classical definition:

Definition A.1. The **variation** of a function u over an interval $[a, b]$ is

$$V_a^b(u) = \sup \left\{ \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)| \right\} \tag{A.1}$$

where the supremum is taken over all finite sequences $a \leq x_1 < x_2 < \ldots < x_n \leq b$.

A function is said to be of bounded variation if $V_a^b(u)$ is finite. The space of functions of bounded variation is denoted $BV([a, b])$.

If $a \leq c \leq b$, then $V_a^c + V_c^b = V_a^b$.

We have the following result due to Jordan:

Proposition A.2. If $u \in BV([a, b])$, then u can be written as the difference of monotone increasing functions.

Proof: Let $T(x) = V_a^x(u)$. Then T and $T - u$ are nondecreasing and $u = T - (T - u)$.

□

Corollary A.3. If $u \in BV([a, b])$, then for all $x \in (a, b)$ the left- and right-hand limits $u(x - 0)$ and $u(x + 0)$ exist and are finite.

Proof: The left- and right-hand limits exist for any monotone function. Since the taking of limits is finitely additive, the result follows from Proposition A.2.

□

We will want to compute the variation of functions in some L^p space. Definition A.1 is very sensitive to changes in the function value at individual points, and in fact we may modify any function on a set of measure zero to ensure that the supremum in (A.1) is infinite. On the other hand, if we take some equivalence class $[u] \in L^p$, and there is a representative $u \in BV$, then by Corollary A.3 we may find a left-continuous representative, which we may take to be right-continuous at a. This choice in fact minimizes the pointwise variation over the equivalence class $[u]$. Thus, when we refer to the variation of some $u \in L^p$, we will refer to the variation of this representative.

Let $u \in BV$ and let ψ be continuous and compactly supported in (a, b). Let $\delta > 0$. To any finite sequence $a \leq x_1 < x_2 < ... < x_{n+1} \leq b$ satisfying $\max\{x_{i+1} - x_i\} < \delta$, and to any corresponding finite sequence $t_1, t_2, ..., t_n$ satisfying $t_i \in [x_i, x_{i+1}]$, we associate the quantity

$$\sum_{i=1}^{n} \psi(t_i)(u(x_{i+1}) - u(x_i)) \tag{A.2}$$

Now, the difference between any two such sums is bounded above by

$$2 \sup \{|\psi(t) - \psi(s)| : \, |t - s| < \delta\} \, V_a^b(u)$$

This implies that the sums converge to a limit $\delta \to 0$. The limit is the Riemann-Stieltjes integral

$$L_u[\psi] = \int_a^b \psi \, du$$

This is a linear functional on $C_0([a, b])$. Since

$$|L_u(\psi)| \leq ||\psi||_{L^\infty} V_a^b(u),$$

it follows that L_u is bounded. Recall the Riesz Representation Theorem:

Theorem A.4. Let X be a locally compact, separable metric space. Let $F : C_0(X) \to \mathbb{R}$ be bounded and linear. There exists a unique Radon measure μ on X such that

$$F(g) = \int_X g \, d\mu$$

for all $g \in C_0(X)$.

Thus, there exists a Radon measure Du such that

$$L_u(\psi) = \int_a^b \psi \, d(Du) \tag{A.3}$$

for all $\psi \in C_0$.

Proposition A.5. Let $\phi \in C_0^1([a, b])$ and $u \in BV$. Then

$$\int_a^b \phi \, d(Du) = -\int_a^b u\phi' \, dx$$

Proof: Consider a finite sequence $a = x_1 < x_2 < ... < x_{n+1} = b$. By the Mean Value Theorem, for every i there exists $t_i \in (x_i, x_{i+1})$ such that $\phi(x_{i+1}) - \phi(x_i) = \phi'(t_i)(x_{i+1} - x_i)$. Hence,

$$\begin{aligned}
\sum_{i=1}^{n} u(x_i)\phi'(t_i)(x_{i+1} - x_i) &= \sum_{i=1}^{n} u(x_i)(\phi(x_{i+1}) - \phi(x_i)) \\
&= \sum_{i=1}^{n-1} \phi(x_{i+1})(u(x_i) - u(x_{i+1})) + u(x_n)\phi(b) - u(a)\phi(a) \\
&= \sum_{i=1}^{n} \phi(x_{i+1})(u(x_i) - u(x_{i+1}))
\end{aligned}$$

where the last equality follows from the fact that $\phi(a) = \phi(b) = 0$ Now, $u\phi'$ is continuous outside of a countable set and hence Riemann integrable. If given $\delta > 0$ we require that our finite sequence satisfy $x_{i+1} - x_i < \delta$, then as $\delta \to 0$,

$$\sum_{i=1}^{n} u(x_i)\phi'(t_i)(x_{i+1} - x_i) \to \int_a^b u\phi' \, dx$$

Also, since ϕ is continuous and u is of bounded variation,

$$\sum_{i=1}^{n} \phi(x_{i+1})(u(x_i) - u(x_{i+1})) \to -\int_a^b \phi \, du$$

where the term on the right is a Riemann-Stieltjes integral. The result follows from (A.3).

□

Corollary A.6. Du is the distributional derivative of u.

Given Radon measure μ, we can consider the total variation measure $|\mu|$. For any Borel measurable set E,

$$|\mu|(E) = \sup \left\{ \sum_{i=1}^{\infty} |\mu|(E_i) \right\}$$

where the supremum is taken over all countable partitions of E into measurable sets.

If $\phi \in C_0^1$ and $||\phi||_{L^\infty} \leq 1$, then

$$\int_a^b d(|Du|) \geq \int_a^b \phi \, d(Du)$$

From Proposition 2 it follows that

$$\int_a^b d(|Du|) \geq \int_a^b u \, \phi'$$

Define a functional

$$V(u, [a, b]) = \sup \left\{ \int_a^b u\phi' \, dx : \phi \in C_0^1([a, b]) \text{ and } ||\phi||_{L^\infty} \leq 1 \right\} \tag{A.4}$$

Proposition A.7. Let $u \in BV([a, b])$. Then

$$V(u, [a, b]) = |Du|((a, b))$$

Proof: There exist disjoint Borel-measurable sets $A \subset (a, b)$ and $B \subset (a, b)$ and positive Radon measures Du^+ and Du^- such that Du^+ is concentrated on A, Du^- is concentrated on B, $Du = Du^+ - Du^-$, and $|Du| = Du^+ + Du^-$.

Let $\epsilon > 0$. By inner regularity of Radon measures, there exist compact sets $A_\epsilon \subset A$ and $B_\epsilon \subset B$ such that

$$Du^+(A \setminus A_\epsilon) < \frac{\epsilon}{2}$$

$$Du^-(B \setminus B_\epsilon) < \frac{\epsilon}{2}$$

There exists a function $\phi \in C_0^1$ such that $\phi\,|_{A_\epsilon} = 1$, $\phi\,|_{B_\epsilon} = -1$, and $||\phi||_{L^\infty} \leq 1$. Then

$$\int_a^b \phi d(Du) \geq \int_a^b d|Du| - \epsilon$$

□

We also have a converse:

Proposition A.8. Let $u \in L^1([a,b])$. If $V(u,[a,b]) < \infty$, then there exists a Radon measure Du which is the distributional derivative of u and satisfies $V(u,[a,b]) = |Du|((a,b))$.

Proof: Define a functional

$$J[\phi] = \int_a^b u\phi'\,dx$$

Since $V(u,[a,b]) < \infty$, J is bounded. In fact, it is bounded on a dense subset of unit ball in C_0, so we may extend it to a bounded linear functional on all of C_0. Hence, we may apply the Riesz Representation Theorem to deduce the existence of a Radon measure Du satisfying

$$\int_a^b \phi d(Du) = -\int_a^b u\phi'\,dx$$

This implies that Du is the distributional derivative of u.

The proof of Proposition 3 shows that $V(u,[a,b]) = |Du|((a,b))$.

□

The connection between $V(u,[a,b])$ and $V_a^b(u)$ is provided by the following:

Proposition A.9. Let $u \in L^1([a,b])$. There exists a representative of u, also denoted u, such that

$$V(u,[a,b]) = V_a^b(u) \tag{A.5}$$

Proof: See [1], Theorem 3.27

Corollary A.10. The total variation functional is convex and lower semicontinuous in L^p for $1 \le p \le \infty$.

Proof: By (A.4) and Proposition A.9, the total variation is the pointwise supremum of a family of linear functionals that are continuous on L^p for $1 \le p \le \infty$. Hence, it is convex and lower semicontinuous in L^p.

□

We now have an expression for the total variation that does not depend on the choice of representative of u. Henceforth, we will refer to (A.4) as the total variation, and to Definition A.1 as the pointwise variation. Moreover, we will use the notation

$$V(u,[a,b]) = \int_a^b |Du| \tag{A.6}$$

to denote the total variation of u.

We have the following integration by parts formula

Proposition A.11. If $u \in BV([a,b])$ and $v \in W^{1,1}([a,b])$, then

$$\int_a^b uv'\,dx + \int_0^1 v\,u(Df) = u(b-0)v(b) - u(a+0)v(a)$$

Proof: See [17], 2.9.24.

A.1 Properties of BV Functions

We may endow $BV([a,b])$ with the norm

$$||u||_{BV} = \int_a^b |u| + \int_a^b |Du|$$

It can be shown that BV is a Banach space under this norm. However, it is too strong for our purposes, as it greatly limits the ways in which BV functions can be approximated. For example, there is no smooth approximation to the Heaviside function in the BV norm.

Instead, we will use a weaker form of convergence.

Definition A.12. Let $u \in BV([a,b])$. A sequence $\{u_n\} \subset BV([a,b])$ **weakly* converges** in BV to u if

$$\lim_{n\to\infty} \int_a^b |u_n - u| = 0 \tag{A.7}$$

and, for all $\phi \in C_0([a,b])$,

$$\lim_{n\to\infty} \int_a^b \phi \, d(Du_n) = \int_a^b \phi \, d(Du)$$

If moreover

$$\lim_{n\to\infty} \int_a^b |Du_n| = \int_a^b |Du| \tag{A.8}$$

we say $\{u_n\}$ **strictly converges** to u.

Using this sense of convergence, we have smooth approximations to BV functions.

Proposition A.13. Let $u \in BV([a,b])$. There exists a sequence $\{\phi_n\} \subset C^\infty([a,b])$ that strictly converges to u in BV.

Proof: See [1].

Next, we have the following result on compact embedding of BV into L^p spaces:

Theorem A.14. Let $\{u_k\} \subset BV([a,b])$ and suppose there exists M such that

$$\int_a^b |u_k| + \int_a^b |Du_k| < M$$

for $k = 1, 2, ...$ Then there exists $u \in BV$ and a subsequence u_{k_j} such that

$$\lim_{j\to\infty} \int_a^b |u_{k_j} - u|^p = 0$$

whenever $1 \leq p < \infty$.

Proof: See [1] for a proof in the case $p = 1$. The general case follows from Holder's inequality and the inclusion $BV([a, b]) \subset L^\infty([a, b])$.

□

Another useful property is the decomposition of BV functions. Let $u \in BV$. By the Radon-Nikodym-Lebesgue Theorem, there exists a measurable function u' and a measure Du^s concentrated on a set of Lebesgue measure zero, such that

$$Du = u'dx + Du^s$$

The singular part Du^s can be further decomposed into a purely atomic measure Du^j and a diffuse measure Du^c. We may write $Du^a = u'dx$ as well.

These measures may be interpreted as follows:

Du^a is the absolutely continuous component, and there is an absolutely continuous function u^a such that u' is its almost-everywhere derivative and satisfies

$$Du^a(E) = \int_E u'(x)\,dx$$

for any measurable set E.

Next, Du^j is known as the jump component. It corresponds to a jump function u^j.

Finally Du^c is known as the Cantor component, and corresponds to a Cantor function u^c. This terminology is taken from the classical example of the Cantor-Vitali function.

This leads to a decomposition of u, namely

$$u = u^a + u^j + u^c$$

We conclude this section with a result about modifying BV functions on intervals.

Lemma A.15. Let $u \in BV([a,b])$ and let $c \in (a,b)$. Let $x_0 \in [c,b]$. For any

$$\lambda \in [\min\{u(x_0-0), u(x_0+0)\}, \max\{u(x_0-0), u(x_0+0)\}]$$

define the function

$$u_\lambda(x) = \begin{cases} \lambda & \text{if } c \leq x \leq d \\ u(x) & \text{otherwise} \end{cases}$$

If $x_0 = b$, take $\lambda = u(b-0)$. Then $u_\lambda \in BV([a,b])$ and

$$\int_a^b |Du_\lambda| \leq \int_a^b |Du|$$

Proof: First, note that

$$\int_a^b |Du| = \int_a^c |Du| + \int_c^b |Du| + |u(c+0) - u(c-0)| \tag{A.9}$$

If $x_0 = b$, then

$$u_\lambda(x) = \begin{cases} u(b-0) & \text{if } c \leq x \leq b \\ u(x) & \text{otherwise} \end{cases}$$

and

$$\int_a^b |Du_\lambda| = \int_a^c |Du| + |u(b-0) - u(c-0)|$$

Now,

$$\int_c^b |Du| \geq |u(c+0) - u(b-0)|$$

and by the triangle inequality

$$|u(c+0) - u(c-0)| + |u(c+0) - u(b-0)| \geq |u(b-0) - u(c-0)|$$

Hence,

$$\int_a^b |Du_\lambda| \le \int_a^b |Du|$$

Next, suppose $x_0 < b$. Then

$$\int_c^b |Du| \ge |u(c+0) - u(x_0-0)| + |u(x_0+0) - u(x_0-0)| + |u(x_0+0) - u(b-0)|$$

Our hypotheses imply

$$|u(x_0+0) - u(x_0-0)| = |\lambda - u(x_0-0)| + |u(x_0+0) - \lambda|$$

Applying the triangle inequality, it follows that

$$\int_c^b |Du| + |u(c+0) - u(c-0)| \ge |u(c-0) - \lambda| + |\lambda - u(b-0)|$$

Hence,

$$\int_a^b |Du_\lambda| \le \int_a^b |Du|$$

□

Similarly, we may prove the following:

Proposition A.16. Let $u \in BV([a,b])$. Let $x_0 \in [a,b]$ and let $[c,d]$ be an interval containing x_0. For

$$\lambda \in [\min\{u(x_0-0), u(x_0+0)\}, \max\{u(x_0-0), u(x_0+0)\}]$$

define the function

$$u_\lambda(x) = \begin{cases} \lambda & \text{if } c \le x \le d \\ u(x) & \text{otherwise} \end{cases}$$

Then $u_\lambda \in BV([a,b])$ and

$$\int_a^b |Du_\lambda| \le \int_a^b |Du|$$

Appendix B

Variational Inequalities

Problem B.1. Given $g \in H^{-1}$, find $v \in H_0^1([0,1])$ that minimizes

$$L[v] = \int_0^1 (v')^2 - \langle g, v \rangle \tag{B.1}$$

under the constraint $|v| \leq 1$.

Define the **constraint set**

$$K = \left\{ v \in H_0^1([0,1]) : ||v||_{L^\infty} \leq 1 \right\} \tag{B.2}$$

If a solution v of B.1 were to exist, then, given $\epsilon \in (0,1]$ and $u \in K$, then since K is convex $v + \epsilon(u - v) \in K$ and hence

$$L[v] \leq L[v + \epsilon(u - v)]$$

That is,

$$\int_0^1 (v')^2 - \langle g, v \rangle \leq \int_0^1 \left(v' + \epsilon(u' - v')\right)^2 - \langle g, v + \epsilon(u - v) \rangle$$

Take the difference

$$L[v + \epsilon(u - v)] - L[v] = \frac{\epsilon^2}{2} \int_0^1 (u' - v')^2 + \epsilon \int_0^1 v'(u' - v') - \epsilon \langle g, u - v \rangle$$

This is nonnegative for all $\epsilon \in (0,1]$, whence it follows that

$$\int_0^1 v'(u' - v') \geq \langle g, u - v \rangle \tag{B.3}$$

for all $u \in K$. An expression of this form is known as a **variational inequality**. In general, we have the following result:

Theorem B.2. Let H be a Hilbert space and $K \subset H$ closed and convex. Let $a(u, v)$ be a coercive bilinear form on H and F a continuous linear functional on H. Then there exists a unique $v \in K$ such that

$$a(v, u - v) \geq F(u - v) \tag{B.4}$$

for all $u \in K$.

For a proof, see [20], Chapter II, Theorem 2.1.

In the context of Problem B.1,

$$a(u, v) = \int_0^1 v'u'$$

Coercivity follows from Poincare's inequality. Since K is closed and convex, this implies that there is a unique $v \in K$ that satisfies B.3. Then v is a unique solution of Problem B.1.

Let $\Lambda = \{x : |v(x)| = 1\}$. This is known as the **coincidence set**. Since $v \in H_0^1([0, 1])$, it follows that Λ is a closed subset of $(0, 1)$. Let $x_0 \in (0, 1) \setminus \Lambda$. Let $0 < \delta < d(x_0, \Lambda)$ and let $B_\delta(x_0)$ denote the ball of radius δ centered at x_0.

Let $\phi \in C^\infty([0, 1])$ such that $\mathrm{supp}(\phi) \subset B_\delta(x_0)$. For sufficiently small ϵ, $v \pm \epsilon\phi \in K$. Letting $u = v + \epsilon\phi$ and substituting into B.3, it follows that

$$\int_0^1 v'\phi' \geq \langle g, \phi \rangle$$

If instead we set $u = v - \epsilon\phi$, then it would follow that

$$\int_0^1 v'\phi' \leq \langle g, \phi \rangle$$

Hence,

$$\int_0^1 v'\phi' = \langle g, \phi \rangle$$

This holds for all $\phi \in C^\infty([0,1])$ such that $\text{supp}(\phi) \subset B_\delta(x_0)$. Repeating this for all $x_0 \in (0,1) \setminus \Lambda$ we conclude that

$$-v'' = g \tag{B.5}$$

on $(0,1) \setminus \Lambda$, where the equality is in the sense of distributions.

This is not yet the full picture. We want to see what happens to v'' on Λ, especially on its boundary. We have the following (cf. [20], Chapter II, Theorem 6.9):

Theorem B.3. Let v be the solution to Problem B.1. Then there exists a Radon measure μ concentrated on Λ such that

$$-v'' = \mu + g$$

in the sense of distributions. Moreover, $\mu \llcorner \{x : v(x) = -1\}$ is nonnegative and $\mu \llcorner \{x : v(x) = 1\}$ is nonpositive.

We shall make use of the following:

Theorem B.4. Let X be an open subset of $\mathbb{R}$. If F is a continuous linear functional on $C_0^\infty(X)$ with $F(\phi) \geq 0$ for all nonnegative $\phi \in C_0^\infty(X)$, then there exists a positive measure μ such that

$$F(\phi) = \int_X \phi \, d\mu \tag{B.6}$$

For a proof, see [19], Theorem 2.1.7.

Proof of Theorem B.3: Let $X = \{x : v(x) < \frac{2}{3}$. The mapping

$$\phi \mapsto \int_X v'\phi' - \langle g, \phi \rangle$$

is continuous and linear for all $\phi \in C_0^\infty(X)$. If ϕ is nonnegative on X, then $v + \epsilon\phi \in K$ for sufficiently small ϵ. As above, we use the variational inequality to deduce that

$$\int_X v'\phi' - \langle g, \phi \rangle \geq 0$$

Hence, there exists a nonnegative Radon measure μ^+ such that

$$\int_X v'\phi' - \langle g, \phi \rangle = \int_X \phi \, d\mu^+$$

From B.5 it follows that μ^+ is concentrated on $X \cap \Lambda$.

Similarly, if we let $Y = \{x : v(x) > -\frac{2}{3}$, we deduce the existence of a nonpositive Radon measure $-\mu^-$ concentrated on $X \cap \Lambda$ that satisfies

$$\int_Y v'\phi' - \langle g, \phi \rangle = -\int_Y \phi \, d\mu^-$$

for all $\phi \in C_0^\infty(Y)$.

Letting $\mu = \mu^+ - \mu^-$, we can combine the above and deduce that

$$\int_0^1 v'\phi' - \langle g, \phi \rangle = \int_0^1 \phi \, d\mu$$

for all $\phi \in C_0^\infty([0, 1])$. Hence,

$$-v'' = \mu + g$$

in the sense of distributions.

□

Appendix C

Approximation of L^2 Functions by Step Functions

We will show that step functions are dense in L^2. We show that smooth functions can be approximated by step functions in the L^2 norm, and then use the density of C^∞ in L^2.

Lemma C.1. : Let $\phi \in C^\infty([0,1])$. For all $\epsilon > 0$, there exist disjoint intervals $E_1, E_2, ..., E_k$, whose union is the interval $[0,1]$, and numbers $\lambda_1, \lambda_2, ..., \lambda_k$ such that the step function

$$g_\epsilon(x) = \sum_{i=1}^{k} \lambda_i \chi_{E_i}(x) \tag{C.1}$$

satisfies

$$\int_0^1 |\phi - g_\epsilon|^2 < \epsilon \tag{C.2}$$

Proof: Suppose $\phi \in C^\infty([0,1])$ and let

$$M = \sup_{x \in [0,1]} \{|\phi'(x)|\}$$

Let $\epsilon > 0$. Choose a natural number N such that

$$N^2 > \frac{M^2}{\epsilon}$$

Define a sequence $\{x_0, x_1, ..., x_N\}$ where

$$x_i = \frac{i}{N}$$

Define a function g_ϵ by

$$g_\epsilon(x) = \sum_{i=1}^{N} \phi(x_{i-1})\chi_{[x_{i-1},x_i)}(x) \tag{C.3}$$

Then, on the interval $[x_{i-1}, x_i)$,

$$||g_\epsilon - \phi||_{L^\infty} \leq \frac{M}{N}$$

Hence,

$$\int_{x_{i-1}}^{x_i} (g_\epsilon - \phi)^2 \leq \frac{1}{N}\left(\frac{M}{N}\right)^2 \tag{C.4}$$

We may write

$$\int_0^1 (g_\epsilon - \phi)^2 = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (g_\epsilon - \phi)^2$$

By (C.4), this implies

$$\begin{aligned} \int_0^1 (g_\epsilon - \phi)^2 &\leq \sum_{i=1}^{N} \frac{1}{N}\left(\frac{M}{N}\right)^2 \\ &= \left(\frac{M}{N}\right)^2 \\ &< \epsilon \end{aligned} \tag{C.5}$$

□

Proposition C.2. Let $f \in L^2([0,1])$. For all $\epsilon > 0$, there exists a step function g_ϵ such that

$$\int_0^1 |f - g_\epsilon^2| < \epsilon \tag{C.6}$$

Proof: Suppose $f \in L^2([0,1])$ and let $\epsilon > 0$. The density of $C^\infty([0,1])$ in L^2 ensures that there exists a smooth function ϕ such that

$$\int_0^1 (\phi - f)^2 < \frac{\epsilon}{2}$$

By Lemma C.1, there exists g_ϵ such that

$$\int_0^1 (\phi - g_\epsilon)^2 < \frac{\epsilon}{2}$$

Therefore,

$$\int_0^1 |f - g_\epsilon^2| < \epsilon$$

$\square$

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000

[2] F. Andreu, C. Ballester, V. Caselles, J. Mazon. Minimizing Total Variation Flow. *Differential and Integral Equations* **14**(3): 321-360, 2001

[3] G. Bellettini, V. Caselles, and M. Novaga. Total Variation Flow in $\mathbb{R}^n$. *Journal of Differential Equations* **184**:475-525, 2002.

[4] G. Bellettini, V. Caselles, and M. Novaga. Explicit Solutions of the Eigenvalue Problem -div$\left(\frac{Du}{|Du|}\right) = u$. *SIAM J. Math. Anal.* **36**: 1095-1129, 2005.

[5] H. Berestycki. Personal communication to H. Brezis, Sept 2012.

[6] H. Brezis. *Functional Analysis, Sobolev Spaces, and Partial Differential Equations.* Springer, New York, 2010.

[7] H. Brezis. Personal communication to T. Sznigir, Jan 2016.

[8] H. Brezis. Personal communication to T. Sznigir, Jan 2017.

[9] H. Brezis and D. Kinderlehrer. The smoothness of solutions to nonlinear variational inequalities. *Indiana Univ. J. Math.* **23**:831-844, 1974.

[10] H. Brezis and W. Strauss. Semi-linear second-order elliptic equations in L^1. *J. Math. Soc. Japan* **25**(4): 565-590, 1973.

[11] G. Buttazzo, M. Giaquinta, and S. Hildebrandt. *One-dimensional Variational Problems: an Introduction.* Oxford University Press, Oxford, 1998.

[12] V. Caselles, A. Chambolle, and M. Novaga. The Discontinuity Set of Solutions of the TV Denoising Problem and some Extensions. *Multiscale Model. Simul.* **6**(3): 879-894, 2007.

[13] V. Casseles, A. Chambolle, and M. Novaga. Regularity for Solutions of the Total Variation Denoising Problem. *Rev. Mat. Iberoamericana* **27**: 233-252, 2011.

[14] V. Caselles, A. Chambolle, and M. Novaga. Total Variation in Imaging, in O. Scherzer, ed., *Handbook of Mathematical Methods in Imaging.* Springer, New York, 2015, 1455-1499

[15] A. Chambolle and P.-L. Lions. Image Recovery via Total Variation Minimization and Related Problems. *Numer. Math.* **76**:167-188, 1998.

[16] I. Ekeland, R. Temam. *Convex Analysis and Variational Problems.* North-Holland, Amsterdam, 1976.

[17] H. Federer. *Geometric Measure Theory.* Springer, 1969.

[18] A. Friedman. *Variational Principles and Free-Boundary Problems.* Wiley, New York, 1982.

[19] L. Hormander. *The Analysis of Linear Partial Differential Operators I.* Springer-Verlag, Berlin, 1990.

[20] D. Kinderlehrer, G. Stampacchia. *An Introduction to Variational Inequalities and their Applications.* SIAM, New York, 2000.

[21] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D* **60**:259-268, 1992.