FORMS OF HOMOGENEOUS SPHERICAL VARIETIES

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Abstract

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Let G be a connected reductive algebraic group, spherical G-varieties are generalizations of symmetric G-spaces bearing nice properties on their compactifications. Over an algebraically closed field of characteristic 0, spherical varieties are classified by the Luna-Vust theory (spherical embeddings) together with combinatorial objects called the Luna data (homogeneous spherical varieties). A homogeneous spherical G-variety X can be determined, up to isomorphisms, by its corresponding Luna datum $\Lambda_{(G,X)}$.

In the first part of this work, Galois cohomology is used to study the spherical varieties over a general field k of characteristic 0, called k-forms of spherical varieties. We start from a homogeneous spherical G-variety X defined over k, with quasi-split G, then it is proven that there is a one-to-one correspondence between the set of k-forms (G', X') with a group G' which is quasi-split over k, up to k-isomorphisms, and the (continuous) cocycle classes in the first Galois cohomology of the automorphism group of the Luna datum, $\mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G,X)}))$.

As an application, in the second part, the Luna data satisfying the transitivity of the automorphism group action on the set of spherical roots are classified. With the transitivity condition, the k-forms corresponding to the sets of the first Galois cohomology of the automorphism group of these Luna data contains all the spherical varieties over k which is of k-rank 1, according to the main theorem in the first part.

献给我的父母

To my parents

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Chapter 1

Introduction

1.1 Spherical Varieties

Let Ω be an algebraically closed field, and let G be a connected algebraic group over Ω.

In a work of De Concini and Procesi [DCP83], a special equivariant compactification of symmetric homogeneous spaces $G/\mathcal{N}_G(G^\theta)$ is studied. Here G is semisimple and G^θ is the subgroup of fixed elements in G by an involution $\theta : G \longrightarrow G$.

This compactification has several nice properties, which are later used to define a certain kind of G-varieties:

Definition 1.1.1. Let G be semisimple, a G-variety X is called **wonderful** if it is smooth, complete, with an open G-orbit whose complement in X is $\bigcup_{i=1}^{r} D_i$, the union of r smooth prime G-divisors with normal crossings and non-empty intersections.

In the same year, Luna and Vust in [LV83] developed a general theory of equivariant compactifications of homogeneous G-varieties. They also found certain good properties in the situations where the homogeneous variety contains an open Borel orbit, which explains the behavior of symmetric cases. Then such a condition is called "sphericity":

Definition 1.1.2. Let k be a field, and let G be a geometrically connected (a variety Y over k is geometrically connected if its base change $Y_{\bar{k}}$ is connected) reductive algebraic group over k. A geometrically connected G-variety X is called **spherical** if $X_{\bar{k}} = X \times_{\mathrm{Spec} k} \mathrm{Spec}\bar{k}$, the base change to the algebraic closure $\bar{k}$ of k, is a spherical $G_{\bar{k}}$-variety, that is, $X_{\bar{k}}$ is normal and there is a Zariski open (dense) Borel orbit in $X_{\bar{k}}$ for a Borel subgroup of $G_{\bar{k}}$.

Moreover, there are several other equivalent definitions of a spherical G-variety, by conditions such as finiteness of the set of Borel orbits in X, multiplicity-freeness of $k[X]$ as a representation of G when X is affine , etc.

And later the following theorem is shown by Luna revealing the relation between these two classes of G-varieties:

Theorem 1.1.3 ([Lun96]). *A wonderful G-variety is spherical.*

Further relations between them are shown in the classification theory of spherical varieties.

1.2 Classification of Spherical Varieties

That X contains an open Borel orbit implies that the G-orbit containing the open Borel orbit is also open. Thus the task can be divided into two parts: the homogeneous spherical varieties $H\backslash G$ (with right G-action) for some "spherical subgroup" H, and the embeddings of $H\backslash G$ into X.

1.2.1 Luna-Vust Theory

In [LV83], the work of Luna and Vust also contains the classification of spherical embeddings over algebraically closed field Ω (of arbitrary characteristic). This is known as the Luna-Vust theory.

As the original paper also contains a lot of results not related to spherical embeddings, most of the following results about Luna-Vust theory are from Knop's survey [Kno91].

Definition 1.2.1. Let G be a connected reductive group defined over Ω, and let B be a Borel subgroup of G. Let X be a normal G-variety together with a G-equivariant open embedding $H\backslash G \hookrightarrow X$.

- For a G-variety Z, define $\mathcal{D}(Z) := \{B\text{-stable prime divisors of } Z\}$,
- $\mathcal{V}(Z) := \{G\text{-stable valuations on } \Omega(Z)^{\times}\}$, and $\mathcal{V}$ stands for $\mathcal{V}(H\backslash G)$,
- Let Y be a G-orbit in X, $\mathcal{D}_Y(X) := \{D \in \mathcal{D}(X) : Y \subseteq D\}$,
- $\mathcal{F}_Y(X) := \{(D \cap (H\backslash G)) \in \mathcal{D}(H\backslash G) : D \in \mathcal{D}_Y(X) \text{ is not } G\text{-stable}\}$,
- Let $\mathcal{X}(B)$ be the weight lattice of B, and

  $$\Xi := \{\chi_f \in \mathcal{X}(B) : \chi_f \text{ is associated to } B\text{-semiinvariant function } f \in \Omega(H\backslash G)^{(B)}\},$$

- By [LV83, 7.4 Proposition], by evaluating $v \in \mathcal{V}$ on Ξ, there is an injection $\hat{\rho} : \mathcal{V} \longrightarrow \mathcal{Q}$.
- There is a map $\overline{\rho} : \mathcal{D}(H\backslash G) \longrightarrow \mathcal{Q}$ given in the following way. For every $D \in \mathcal{D}(H\backslash G)$, there is a valuation v_D on $\Omega(H\backslash G)^{\times}$. Being evaluated on Ξ, the valuation v_D produces $\overline{\rho}(D)$.
- $\mathcal{B}_Y(X) := \{\hat{\rho}(v_D) : \text{for some } G\text{-stable } D \in \mathcal{D}_Y(X) \text{ with the induced valuation } v_D \in \mathcal{V}\}$,
- $\mathcal{C}_Y(X) \subseteq \mathcal{Q}$ a cone generated by $\overline{\rho}(\mathcal{F}_Y(X))$ and $\mathcal{B}_Y(X)$.

A spherical embedding X is called simple if there is only one closed G-orbit in X. Let Y be this orbit, then denote $\mathcal{F}_Y(X)$, $\mathcal{B}_Y(X)$, and $\mathcal{C}_Y(X)$ by $\mathcal{F}(X)$, $\mathcal{B}(X)$, and $\mathcal{C}(X)$, respectively.

Given a cone $\mathcal{C} \subseteq \mathcal{Q}$, let $C^\vee = \{\alpha \in \mathcal{Q}^\vee : \alpha(v) \geq 0$ for all $v \in \mathcal{C}\}$ be the dual cone, then a face of $\mathcal{C}$ is a cone $\mathcal{C}' \subseteq \mathcal{Q}$ of the form $\mathcal{C}' = \mathcal{C} \cap \{v \in \mathcal{Q} : \alpha(v) = 0$ for some $\alpha \in \mathcal{C}^\vee\}$, and the relative interior $\mathcal{C}^\circ$ of $\mathcal{C}$ is $\mathcal{C}$ with all proper faces removed.

Definition 1.2.2. A **colored cone** is a pair $(\mathcal{C}, \mathcal{F})$ where $\mathcal{C} \subseteq \mathcal{Q}$ and $\mathcal{F} \subseteq \mathcal{D}(\mathcal{H}\backslash\mathcal{G})$, satisfying: $\mathcal{C}$ is generated by $\overline{\rho}(\mathcal{F})$ and finitely many elements in $\hat{\rho}(\mathcal{V})$, and $\mathcal{C}^\circ \cap \hat{\rho}(\mathcal{V}) \neq \emptyset$, where $\mathcal{C}^\circ$ is the relative interior of $\mathcal{C}$. A colored cone is called **strictly convex** if $\mathcal{C}$ is strictly convex and $0 \notin \overline{\rho}(\mathcal{F})$.

Theorem 1.2.3 ([LV83, 8.10, Proposition]). *The map $X \mapsto (\mathcal{C}(X), \mathcal{F}(X))$ is a bijection between isomorphism classes of simple embeddings and strictly convex colored cones.*

In general, the spherical embeddings are classified by colored fans.

Definition 1.2.4. Given a colored cone $(\mathcal{C}, \mathcal{F})$, a face of $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}_0, \mathcal{F}_0)$, where $\mathcal{C}_0$ is a face of $\mathcal{C}$, $\mathcal{C}_0^\circ \cap \hat{\rho}(\mathcal{V}) \neq \emptyset$ and $\mathcal{F}_0 = \mathcal{F} \cap \overline{\rho}^{-1}(\mathcal{C}_0)$.

Definition 1.2.5. A **colored fan** is a nonempty finite set $\mathfrak{F}$ of colored cones, satisfying the conditions: Every face of $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$ belongs to $\mathfrak{F}$, and for every $v \in \hat{\rho}(\mathcal{V})$ there is at most one $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$ with $v \in \mathcal{C}^\circ$.

For a spherical embedding X, let $\mathfrak{F}(X) := \{(\mathcal{C}_Y(X), \mathcal{F}_Y(X)) : Y \subseteq X$ is a G-orbit$\}$.

Theorem 1.2.6 ([Kno91, 3.3, Theorem]). *The map $X \mapsto \mathfrak{F}(X)$ induces a bijection between isomorphism classes of embeddings and strictly convex colored fans.*

For a general field k of characteristic 0, as it is perfect, $k^{\text{sep}} = \bar{k}$, this makes it possible to apply the theory of Galois actions to Luna-Vust theory over the algebraic closure.

Let $\Gamma = \mathrm{Gal}(\bar{k}/k)$ be the absolute Galois group. For a spherical embedding $H\backslash G \hookrightarrow X$, defined over k, up to an k isomorphism, this corresponds to the equivariant embedding $(H\backslash G)_{\bar{k}} \hookrightarrow X_{\bar{k}}$ with a Γ-action, that is, for any $\gamma \in \Gamma$, the following diagram commutes.

$$\begin{tikzcd}
 & G_{\bar{k}} \times X_{\bar{k}} \arrow[rr] \arrow[dd] & & X_{\bar{k}} \arrow[dd] \\
G_{\bar{k}} \times (H\backslash G)_{\bar{k}} \arrow[ur, hook] \arrow[rr] \arrow[dd] & & (H\backslash G)_{\bar{k}} \arrow[ur, hook] \arrow[dd] & \\
 & G_{\bar{k}} \times X_{\bar{k}} \arrow[rr] & & X_{\bar{k}} \\
G_{\bar{k}} \times (H\backslash G)_{\bar{k}} \arrow[ur, hook] \arrow[rr] & & (H\backslash G)_{\bar{k}} \arrow[ur, hook] &
\end{tikzcd} \tag{1.1}$$

The vertical arrows are isomorphisms induced by $\gamma \in \Gamma$.

A Γ-action on an embedding $X_{\bar{k}}$ induces an action on the colored fan $\mathfrak{F}(X_{\bar{k}})$. Under $\gamma \in \Gamma$, a $G_{\bar{k}}$-orbit is mapped to a $G_{\bar{k}}$-orbit, leading to a match on colored cones in the colored fans, and for each colored cone, γ induces a bijection on the generators of $\mathcal{C}_Y$ to $\mathcal{C}_{\gamma Y}$, and this eventually induces an automorphism of $\mathfrak{F}(X_{\bar{k}})$. Conversely, by [Kno91, 4.1, Theorem], an automorphism of $\mathfrak{F}(X_{\bar{k}})$ induced by $\gamma \in \Gamma$ can be extended to an automorphism of the spherical embedding $X_{\bar{k}}$. Thus a Γ-action on the colored fan $\mathfrak{F}(X_{\bar{k}})$ induces a Γ-action on the spherical embedding $X_{\bar{k}}$. Conversely, we say that a spherical embedding $\overline{X}$ over $\bar{k}$ admits a k-form X if X is a G-variety, with a spherical embedding $(H\backslash G)_{\bar{k}} \hookrightarrow X_{\bar{k}}$ and $X_{\bar{k}}$ is $G_{\bar{k}}$-isomorphic to $\overline{X}$, and we have,

Theorem 1.2.7 ([Hur11, Proposition 2.20, 2.21, Theorem 2.23, 2.26])**.** *A spherical embedding $\overline{X}$ defined over $\bar{k}$ admits a k-form X if and only if*

1. *the colored fan $\mathfrak{F}(\overline{X})$ is Γ-stable, i.e., $(\sigma(\mathcal{C}), \sigma(\mathcal{F})) \in \mathfrak{F}(\overline{X})$ for all $\sigma \in \Gamma$ and*

$(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}(\overline{X})$,

2. *for every* $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}(\overline{X})$, *the colored fan consisting of* $(\sigma(\mathcal{C}), \sigma(\mathcal{F}))_{\sigma\in\Gamma}$ *and their faces is quasi-projective. A colored fan* $\mathfrak{F}$ *is called quasi-projective if there exists a collection* $(l_{(\mathcal{C},\mathcal{F})})_{(\mathcal{C},\mathcal{F})\in\mathfrak{F}}$ *of linear forms on* $\mathcal{Q}$ *satisfying*

   - $\forall(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$, $\forall(\mathcal{C}', \mathcal{F}') \in \mathfrak{F}$, $l_{(\mathcal{C},\mathcal{F})} = l_{(\mathcal{C}',\mathcal{F}')}$ *over* $\mathfrak{C} \cap \mathfrak{C}'$.
   - $\forall(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$, $\forall x \in \mathcal{C}^\circ \cap \hat{\rho}(\mathcal{V})$, $\forall(\mathcal{C}', \mathcal{F}') \in \mathfrak{F} - (\mathcal{C}, \mathcal{F})$, $l_{(\mathcal{C},\mathcal{F})}(x) > l_{(\mathcal{C}',\mathcal{F}')}(x)$.

1.2.2 Homogeneous Spherical Varieties

There is a complete classification of homogeneous spherical varieties over algebraically closed fields of characteristic 0. Let Ω be such a field, the classification is based on Akhiezer's work [Akh83] in classifying of rank 1 wonderful varieties over Ω.

In [Lun01], Luna conjectured that there is a bijection between the set of Ω-isomorphism classes of homogeneous spherical G-varieties and the combinatorial data called Luna data (originally called augmented spherical system, see Definition 2.2.2), known as Luna Conjecture. The conjecture is proven under the contributions of Bravi, Cupid-Foutou, Losev, Luna and Pezzini in [Lun01, BP05, Bra07, Los09, BCF10, BP14, BP16], which can be concluded as in Theorem 2.3.3.

This is where this work begins. We go further for a general field k of characteristic 0, and study the k-forms of a spherical pair (G, X).

Similar questions are investigated by Akhiezer and Cupit-Foutou in their works [ACF14], [Akh15], [CF15], and quite recently the work of Borovoi and Gagliardi [BG17].

1.3 Outline

In the rest of the discussions, the fields are always of characteristic 0, but for simplicity in notations, in some chapters we assume the base field is algebraically closed.

The classification mentioned in Section 1.2.2 is briefly reviewed in Chapter 2, where Luna datum is defined and the theorem of classification is stated.

Before applying Galois cohomology for possible Galois actions on spherical pairs, the groups of automorphisms are studied in Chapter 3. There the base field Ω is algebraically closed. Several relative automorphism groups are defined (in Section 3.1) and the relations between these groups are investigated (Proposition 3.2.3). With the condition of spherical closedness (Definition 3.3.6), the sequence above becomes split.

The results above are applied in Chapter 4. In this chapter the spherical pairs are defined over k, a field of characteristic 0 and not necessarily algebraically closed. Let Γ be the absolute Galois group, non-abelian Galois cohomology theory is applied to the automorphism groups in order to look for k-forms. The first result is that when $\mathrm{H}^1(k, \cdot)$ is applied to $\mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}})$, the k-forms of the spherical pair (G, X) over k can be mapped bijectively to $\mathrm{H}^1(k, \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}}))$, up to k-isomorphisms. Furthermore, in the case where $(G_{\bar{k}}, X_{\bar{k}})$ is spherically closed, there is a connection between k-forms (G, X) where G is quasi-split with the first Galois cohomology of the automorphism group of the combinatorial data classifying homogeneous spherical varieties. Then we can eventually show in Theorem 4.2.4 that this correspondence is a bijection.

Theorem 4.2.4. *Let G be a connected reductive group defined over k, and G is quasi-split. Let X be a spherically closed homogeneous spherical G-variety, then there is a bijection between the set of k-forms (G', X') with quasi-split G', and $\mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}))$. The k-form (G', X') is unique up to k-isomorphisms.*

Several examples are investigated by showing explicitly the correspondence between k-forms with quasi-split group and the class of 1-cocycles in $\mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}))$.

Finally, in Chapter 5, a special case of $\mathrm{Aut}(\Lambda)$ is studied, for some abstractly defined Λ with a group $\mathrm{Aut}(\Lambda)$ acting on Σ transitively. This condition implies that any prime relative rank 1 quasi-split k-form (G, X) (with k structure such that there is only one Galois orbit on Σ_X) has its Luna datum in the list. After some reduction steps (Section 5.4), such a list of the spherical systems of adjoint type is given (Theorem 5.5.1).

1.4 Notations

Throughout the rest of the chapters, all fields are of characteristic 0. Two base fields are considered: Ω is always algebraically closed, and k is general and has algebraic closure $\bar{k}$. Once assigned, the base field is used consistently for the whole chapter.

G denotes a connected reductive algebraic group over the base field. And X is the corresponding homogeneous (spherical) G-variety. Group actions are supposed to be right actions, thus the quotient is in terms of $H\backslash G$, to distinguish, the difference between two sets U and V is denoted by $U - V$ instead of $U \backslash V$. And the cardinality of a set U is denoted by $\#U$. For any algebraic group S, we use $\mathcal{X}(S)$ or $\mathcal{X}_S^*$ to denote the group of characters of S. The pairing between weights $\mathcal{X}^*$ and coroots $\mathcal{X}_*$ (and also their combinations) of G is denoted by $\langle \alpha^\vee, \sigma \rangle$ where $\alpha^\vee$ the coroot of α satisfying $\langle \alpha^\vee, \alpha \rangle = 2$, σ can be a linear combination of roots.

Chapter 2

Luna Data and Classification of Spherical Varieties

In this part, the groups and varieties are defined over an algebraically closed field Ω of characteristic 0.

2.1 Universal Cartan Group

Definition 2.1.1. Let G be a reductive algebraic group over Ω, the **universal Cartan group** is defined to be a torus A over Ω, such that for any chosen Borel subgroup B, there is an isomorphism $i_B : B/N \longrightarrow A$, where N is the unipotent subgroup of B, satisfying the following property: for any other Borel subgroup B' and the unique morphism $\phi : B/N \longrightarrow B'/N'$ induced by the conjugation by an element $g \in G$ (a different choice of g induces the same isomorphism of the quotient), the following diagram commutes

$$\begin{tikzcd} B/N \arrow[r, "i_B"] \arrow[d, "\phi"'] & A \\ B'/N'. \arrow[ur, "i_{B'}"'] & \end{tikzcd}$$

Recall the root datum defined for the pair (G, T), where T is a maximal torus in G.

Definition 2.1.2. Given a reductive group G and a maximal torus $T \subseteq G$, the **root datum** of the pair (G, T) is the quadruple $\Psi = (\mathcal{X}_T^*, \Phi_T, (\mathcal{X}_*)_T, \Phi_T^\vee)$, where

- $\mathcal{X}_T^* = \mathcal{X}^*(T) = \mathrm{Hom}(T, \mathbb{G}_\mathrm{m})$ is the free abelian group of characters of T,
- $\Phi_T \subseteq \mathcal{X}_T^*$ is the root system consisting of the nontrivial characters which appear as eigencharacters in the adjoint representation of T in the Lie algebra $\mathfrak{g}$,
- $(\mathcal{X}_*)_T = \mathcal{X}_*(T) = \mathrm{Hom}(\mathbb{G}_\mathrm{m}, T)$ is the dual of $\mathcal{X}_T^*$, the free abelian group of one parameter subgroups of T,
- $\Phi_T^\vee \subseteq (\mathcal{X}_*)_T$ is the root system consisting of the unique homomorphisms $\alpha^\vee : \mathbb{G}_\mathrm{m} \longrightarrow T$ corresponding to $\alpha \in \Phi$ in the following way: let $T_\alpha = (\ker \alpha)^\circ$, the identity component of the kernel of α, and let $G_\alpha = (\mathcal{Z}_G(T_\alpha))'$ be the derived subgroup of the centralizer of T_α; G_α is a semi-simple group of rank 1 with maximal torus a subgroup of T (thus isomorphic to SL_2 or PSL_2), and thus $\alpha^\vee$ is defined to be the unique homomorphism $\mathbb{G}_\mathrm{m} \longrightarrow G_\alpha$ such that $T = (\mathrm{Im}\ \alpha^\vee)T_\alpha$ and $\langle \alpha^\vee, \alpha \rangle = 2$.

Furthermore, a choice of Borel subgroup $B \subseteq G$ containing the chosen maximal torus T induces a set of positive roots $\Phi^+ \subseteq \Phi$ together with a set of positive simple roots $S \subseteq \Phi^+$.

And an isomorphism between two root data $\Psi_1 \longrightarrow \Psi_2$ is a quadruple of isomorphisms carrying $(\mathcal{X}_1^*, \Phi_1, (\mathcal{X}_*)_1, \Phi_1^\vee)$ to $(\mathcal{X}_2^*, \Phi_2, (\mathcal{X}_*)_2, \Phi_2^\vee)$ in a compatible way, precisely, an isomorphism $\xi : \mathcal{X}_1^* \longrightarrow \mathcal{X}_2^*$ with $\Phi_2 = \xi(\Phi_1)$, and the dual of its inverse $(\xi^{-1})^\vee : (\mathcal{X}_*)_1 \longrightarrow (\mathcal{X}_*)_2$ with $\Phi_2^\vee = (\xi^{-1})^\vee(\Phi_1^\vee)$.

Then we redefine the root datum on universal Cartan of a reductive group G. Let B be a Borel subgroup of G and $T \subseteq B$ be a maximal torus, there is an unique isomorphism $\eta_{(B,T)} : T \longrightarrow A$ factoring through B. Let $\eta_{(B,T)} = i_B \circ \iota$ where $\iota : T \longrightarrow B$ is the inclusion. Consider that $T \cap N = \{e\}$ where e is the identity element in G.

Definition 2.1.3. Let A be the universal Cartan of G, choose T and B as above. Let $\mathcal{X}_A^* := \mathcal{X}^*(A)$, and $(\mathcal{X}_*)_A := \mathcal{X}_*(A)$. Then there are isomorphisms $\eta^* : \mathcal{X}_T^* \longrightarrow \mathcal{X}_A^*$ and $\eta_* : (\mathcal{X}_*)_T \longrightarrow (\mathcal{X}_*)_A$ induced by the isomorphism $\eta = \eta_{(B,T)}$. Let $\Phi_A := \eta^*(\Phi_T)$ and $\Phi_A^\vee := \eta_*(\Phi_T^\vee)$, then $\Psi_A := (\mathcal{X}_A^*, \Phi_A, (\mathcal{X}_*)_A, \Phi_A^\vee)$ is a root datum, and $\eta^\flat_{(B,T)} : \Psi_T \longrightarrow \Psi_A$ is an isomorphism of root data.

Furthermore, the image of the set of positive roots under η^* can be defined as positive simple roots $S_A := \eta^*(S)$.

To prove it is well defined, it suffices to show:

Proposition 2.1.4. *Let B, B' be two Borel subgroups of G, and $T \subseteq B$, $T' \subseteq B'$ be two maximal tori, respectively. Then the following diagrams commute.*

$$\begin{tikzcd} \Psi_T \arrow[r, "\eta^\flat_{(B,T)}"] \arrow[d, "\phi^\flat"'] & \Psi_A \\ \Psi_{T'} \arrow[ur, "\eta^\flat_{(B',T')}"'] & \end{tikzcd} \qquad \begin{tikzcd} S_T \arrow[r, "\eta^\flat_{(B,T)}"] \arrow[d, "\phi^\flat"'] & S_A \\ S_{T'}. \arrow[ur, "\eta^\flat_{(B',T')}"'] & \end{tikzcd} \tag{2.1}$$

Proof. The root datum Ψ_A contains covariant and contravariant objects. Therefore, the objects are investigated separately.

First consider the following diagram about the lattice of characters $\mathcal{X}^*$. There is an element $g \in G$ such that $\mathrm{Int}(g)(T) = g^{-1}Tg = T'$ and $\mathrm{Int}(g)(B) = B'$ (Borel subgroups are conjugate, so are maximal tori). A different choice of such g induces the same homomorphism $T \longrightarrow T'$. (A different $g' = bg$, $b \in B$, and furthermore, $b \in T$).

$$\begin{tikzcd}
 & T \arrow[dl, "\chi"'] \arrow[dd, "\mathrm{Int}(g)"'] & \\
\mathbb{G}_{\mathrm{m}} & & A \arrow[ul, "\eta^{-1}"'] \arrow[dl, "(\eta')^{-1}"] \\
 & T' \arrow[ul, "\chi'"] &
\end{tikzcd}$$

This graph shows the relation (2.1) for the lattice of characters. As the diagram commutes, the two characters of A defined through T and T' are identical. That is, for $\chi = \chi' \circ \mathrm{Int}(g)$, $\eta^{-1} \circ \chi = (\eta')^{-1} \circ \chi'$. And it does the same for the root system Φ.

A similar argument is valid also for cocharacters and the dual root systems. $\square$

Given an isomorphism of reductive groups $h : G \longrightarrow G'$, with the corresponding isomorphism of their universal Cartan $h_A : A \longrightarrow A'$, there is an induced isomorphism $h^* : \Psi_A \longrightarrow \Psi_{A'}$, note that Ψ_A is defined for G and $\Psi_{A'}$ is defined for G'. And with a Borel subgroup $B \subseteq G$ chosen, let $B' = h(B)$, then there is a bijection $h^* : S \longrightarrow S'$ between the corresponding sets of positive simple roots.

Definition 2.1.5. Let G be a reductive algebraic group, and X be a spherical G-variety. Let $B \subseteq G$ be a Borel subgroup, and $N \subseteq B$ its unipotent radical, $\mathring{X}_B$ be the open B-orbit, and $R_X := \mathring{X}_B/N$. Then the universal Cartan group A of G acts on R_X via the isomorphism to B/N. Moreover, this action factors through the quotient $A_X \simeq B_x \backslash B/N$ of A, called the **universal Cartan group of the spherical variety** X. The rank r of A_X is called the (absolute) rank of the spherical variety X.

In the definition, a different choice of B_x does not affect A_X, since the quotient defined by $B_{x'}$ is conjugate to A_X by an element of A, and A is abelian.

2.2 Luna Datum

First we define spherical datum abstractly over a root datum Ψ (this is called an augmented spherical system in [Lun01]) together with a set of positive simple roots S.

Definition 2.2.1. Let Ψ be a root datum where the root system Φ is reduced (the only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α itself and $-\alpha$), with a choice of the set S of the positive simple roots. The set of (Ψ, S)-spherical roots of adjoint type, denoted by $\Sigma_{\mathrm{ad}}(S)$, is the set of $\sigma \in \mathbb{N}S$ such that:

- either $\sigma = \alpha + \beta$ where $\alpha, \beta \in S$ are orthogonal (σ is said to be of type aa),
- or σ and its support set $\mathrm{supp}(\sigma) = \{\alpha \in S : \sigma = \sum_{\alpha \in S} n_\alpha \alpha, n_\alpha \neq 0\}$ is in the following table:

type of support	σ	type of σ
A_1	α	a
A_1	2α	$2a$
A_n, $n \geq 2$	$\sum_{i=1}^{n} \alpha_i$	$a(n)$
B_n, $n \geq 2$	$\sum_{i=1}^{n} \alpha_i$	$b(n)$
B_n, $n \geq 2$	$2\sum_{i=1}^{n} \alpha_i$	$2b(n)$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	b
C_n, $n \geq 3$	$\alpha_1 + \left(2\sum_{i=2}^{n-1} \alpha_i\right) + \alpha_n$	$c(n)$
D_n, $n \geq 3$	$\left(2\sum_{i=1}^{n-2} \alpha_i\right) + \alpha_{n-1} + \alpha_n$	$d(n)$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	f
G_2	$2\alpha_1 + \alpha_2$	g
G_2	$4\alpha_1 + 2\alpha_2$	$2g$
G_2	$\alpha_1 + \alpha_2$	g'

We give alias to some spherical roots which may have more than 1 origins: $d(2) = aa$, $b(1) = a(1)$, $2b = 2a$, and $c(2) = b(2)$.

Let $\Sigma(S)$ be the set of $\sigma \in \frac{1}{2}\mathbb{N}S$, such that:

- either $\sigma \in \Sigma_{\mathrm{ad}}(S)$,
- or $2\sigma \in \Sigma_{\mathrm{ad}}(S)$, $\sigma \in \mathcal{X}^*$, and 2σ is of type aa, b or $d(n)$ where $n \geq 3$. The non-adjoint type spherical roots are called to be of type $\frac{1}{2}aa = \frac{1}{2}d(2)$, $\frac{1}{2}b$ or $\frac{1}{2}d(n)$, respectively.

Definition 2.2.2. Given a root datum $\Psi = (\mathcal{X}^*, \Phi, \mathcal{X}_*, \Phi^\vee)$ of reductive algebraic group G with a set of positive roots S, a **spherical datum** associated to Ψ is a quintuple $(S^p, \Sigma, \mathscr{A}, \Xi, \rho)$ such that $S^p \subseteq S$, Σ is a linearly independent set of B-weights which is a subset of $\Sigma(S)$, Ξ is a free abelian subgroup of $\mathcal{X}^*$ containing Σ, $\mathscr{A}$ is a finite set, and $\rho : \mathscr{A} \longrightarrow \Xi^\vee$ is a map, satisfying the following axioms:

(A1) $\forall\ D \in \mathscr{A}$, $\rho(D)(\alpha) \leq 1$ for every $\alpha \in \Sigma$, equality holds if and only if $\alpha \in S \cap \Sigma$.

(A2) $\forall\ \alpha \in S \cap \Sigma$, $\mathscr{A}(\alpha) := \{D \in \mathscr{A} | \rho(D)(\alpha) = 1\} = \{D_\alpha^+, D_\alpha^-\}$, and $\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee$.

(A3) $\mathscr{A} = \cup_{\alpha \in S \cap \Sigma} \mathscr{A}(\alpha)$.

(Σ1) If $2\alpha \in \Sigma \cap 2S$, then $\frac{1}{2}\langle \alpha^\vee, \beta \rangle$ is a non-positive integer, $\forall \beta \in \Sigma \backslash \{2\alpha\}$. Furthermore, $\alpha \notin \Xi$ and $\frac{1}{2}\langle \alpha^\vee, \beta \rangle$ is an integer for all $\beta \in \Xi$.

(Σ2) If $\alpha, \beta \in S$ are orthogonal and $\alpha + \beta$ belongs to Σ or 2Σ, then $\langle \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle$, $\forall \gamma \in \Xi$.

(S1) For all $\alpha \in \Sigma$, there is a wonderful G variety X of rank 1 with $S_X^p = S^p$, and $\Sigma_X = \{\alpha\}$.

(S2) $\forall \alpha \in S^p$ and $\beta \in \Xi$, $\langle \alpha^\vee, \beta \rangle = 0$.

A spherical datum is denoted by $\mathscr{L}$. The rank of $\mathscr{L}$ is the rank of Ξ as a $\mathbb{Z}$-module. The triplet $(\Psi, S, \mathscr{L})$ is called a Luna datum, denoted by Λ.

Remark.

1. This definition uses Definition 2.2.1. The list there is a full list is obtained from the classification of the spherical varieties over Ω of rank 1. The classification is given in [Akh83], and the list (including the non-adjoint part) can be found in [Was96, Table 1].
2. According to [BP16, 1.4.1, Definition], a map $\overline{\rho} : \Delta \longrightarrow \Xi^\vee$ can be defined compatible with the map ρ. Let $D \in \Delta(\alpha)$ for some $\alpha \in S$, and let $\sigma \in \Xi$,

$$\overline{\rho}(D)(\sigma) = \begin{cases} \rho(D)(\sigma) & \text{if } D \in \mathscr{A}, \\ \frac{1}{2}\langle \alpha^\vee, \sigma \rangle & \text{if } \alpha \in S \cap \frac{1}{2}\Sigma, \\ \langle \alpha^\vee, \sigma \rangle & \text{otherwise.} \end{cases}$$

   This map is the same one as that mentioned in the last chapter.

Definition 2.2.3. Given Luna data $\Lambda = (\Psi, S, \mathscr{L})$ and $\Lambda' = (\Psi', S', \mathscr{L}')$, Λ and Λ' are **isomorphic** if

- there is an isomorphism $i_R : \Psi \longrightarrow \Psi'$ consisting of $i^* : \mathcal{X}^* \longrightarrow (\mathcal{X}^*)'$ and $i_* : \mathcal{X}_* \longrightarrow \mathcal{X}_*{}'$, satisfying that $i^*(\Xi) = \Xi'$, $i^*(\Sigma) = \Sigma'$ and $i^*(S^p) = (S^p)'$,
- there is a bijection $i_A : \mathscr{A} \longrightarrow \mathscr{A}'$, such that $\rho' \circ i_A = i_* \circ \rho$, i.e., the following diagram commutes.

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\rho"] \arrow[d, "i_A"'] & \Xi^\vee \arrow[d, "i_*|_\Xi"] \\ \mathscr{A}' \arrow[r, "\rho'"] & (\Xi')^\vee \end{tikzcd}$$

And the morphism $i = (i_R, i_A)$ is called an isomorphism from Λ to Λ'.

The following definition gives the construction of a spherical datum from a given spherical G-variety X, or say, from a **spherical pair** (G, X).

Definition 2.2.4. Given a reductive group G and a spherical G-variety X, let B be a Borel subgroup, $N \subseteq B$ its unipotent radical, and $\mathring{X}_B$ be the open B-orbit. Let A_X be the universal Cartan of X, and let S be the set of positive simple roots of G with respect to B, define

- $\Xi_X := \mathcal{X}^*(A_X)$,
- Δ_X the set of B-stable prime divisors in X which are not G-stable, such divisor D is called a **color**. Furthermore, for a positive simple root $\alpha \in S$, define $\Delta_X(\alpha) = \{D \in \Delta_X : DP_\alpha \neq D, P_\alpha \text{ is the parabolic subgroup determined by } B \text{ and } \alpha\}$, and for $S' \subseteq S$, $\Delta_X(S') = \bigcup_{\alpha \in S'} \Delta_X(\alpha)$,
- let P_X be the maximal parabolic subgroup of G preserving $\mathring{X}_B$, and let $S^p_X \subseteq S$ be the positive simple roots of the Levi subgroup of P_X. By [BP14], this set is actually the set of $\alpha \in S$, such that $\Delta_X(\alpha) = \emptyset$,
- $\mathscr{A}_X = \Delta_X(S \cap \Sigma_X)$,
- $\rho^\circ_X : \Delta_X \longrightarrow (\Xi_X)^*$ consists of $D \mapsto v_D|_{\Xi_X}$, where v_D is the valuation of $\Omega(X)^*$ induced by the color $D \in \Delta_X$, and let $\rho_X = \rho^\circ_X|_{\mathscr{A}_X}$,
- let $\mathcal{V}(X)$ be the cone of G-stable valuations of $\Omega(X)^*$, and $\mathcal{V}(X)_\Xi$ be the cone of the valuations restricted on Ξ^*_X, and let $\mathcal{V}(X)^\vee = \{x \in \Xi_X \otimes \mathbb{Q} : \langle x, v\rangle \leq 0, \forall v \in \mathcal{V}(X)\}$ be the negative dual cone of $\mathcal{V}(X)_\Xi$. The minimal set of simple elements in Ξ_X which spans $\mathcal{V}^-$ is called the set of **spherical roots** is denoted Σ_X.

The quintuple $\mathscr{L}_X := (S^p_X, \Sigma_X, \mathscr{A}_X, \Xi_X, \rho_X)$ is called the spherical datum of the spherical G-variety X with respect to the choice of Borel subgroup B. It is indeed a spherical datum according to [Lun01].

Remark. By definition, $\Delta_X(\alpha)$ contains at most 2 elements, and $S \cap \Sigma_X = \{\alpha \in S : \#\Delta_X(\alpha) = 2\}$, $S^p_X = \{\alpha \in S : \Delta_X(\alpha) = \emptyset\}$.

Given a spherical G-variety X, and two Borel subgroups B and B', there is a canonical isomorphism $i_{B',B} : \mathscr{L}_{(X,B)} \longrightarrow \mathscr{L}_{(X,B')}$ between the spherical data defined for B and B',respectively. Hence, up to an isomorphism, the spherical datum (thus further the Luna datum) is defined independently of the choice of a Borel subgroup.

Definition 2.2.5. Let (G, X) and (G', X') be two spherical pairs, an **isomorphism** $m : (G, X) \longrightarrow (G', X')$ is an isomorphism of group actions, i.e., it is a pair (m_G, m_X), where $m_G : G \longrightarrow G'$ is an isomorphism of groups, and $m_X : X \longrightarrow X'$ is an isomorphism of varieties, which makes the following diagram commute:

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "(m_G, m_X)"'] & X \arrow[d, "m_X"] \\ G' \times X' \arrow[r] & X' \end{tikzcd}$$

Given an isomorphism of spherical pairs $m = (m_G, m_X) : (G, X) \longrightarrow (G', X')$, there is an induced isomorphism $m_* : \Lambda_{(G,X)} \longrightarrow \Lambda_{(G',X')}$ of their Luna data. The isomorphism of corresponding root data and that of the sets of positive simple roots are determined by the isomorphism of groups, according to the theory of algebraic groups. To construct the isomorphism of spherical data, first take a Borel subgroup $B \subseteq G$, with image $B' = m_G(B)$ a Borel subgroup of G'. Consider that the universal Cartan A_X is isomorphic to the double quotient $B_x \backslash B/N$ for some $x \in \mathring{X}_B$, and for the universal Cartan $A_{X'} = B_{x'} \backslash B'/N'$, choose $N' = m_G(N)$, $x' = m_X(x)$, and $B_{x'} = \mathrm{Stab}_{B'}(m_X(x))$. It turns out that $B_{x'} = m_G(B_x)$ because of the compatibility of m_G and m_X, where for $b \in B_x$, $m_G(b).m_X(x) = m_X(b.x) = m_X(x)$. Hence the isomorphism $A_X \longrightarrow A_{X'}$ is induced by $m_G|_B$.

Note that the existence of the induced isomorphism does not mean that spherical datum is functorial, as it works only for isomorphisms.

2.3 Classification over Ω

The following theorem gives the classification of homogeneous spherical G-varieties based on the spherical data.

Theorem 2.3.1 ([Los09, Lun01, BP16]). *Let $\mathscr{L}$ be a spherical datum over (Ψ, S) where Ψ and S are the root datum and a set of positive simple roots of a reductive algebraic group G. Then there is a homogeneous spherical G-variety X together with a isomorphism $\lambda_X : \mathscr{L}_X \longrightarrow \mathscr{L}$, and for any other such spherical G-variety X' and the corresponding $\lambda_{X'} : \mathscr{L}_{X'} \longrightarrow \mathscr{L}$, there is a G-equivariant isomorphism $\varphi : X \longrightarrow X'$, such that the following diagram commutes:*

$$\begin{tikzcd} \mathscr{L}_X \arrow[r, "\lambda_X"] \arrow[d, "\varphi_*"'] & \mathscr{L} \\ \mathscr{L}_{X'} \arrow[ur, "\lambda_{X'}"'] & \end{tikzcd} \tag{2.2}$$

where φ_* is the induced isomorphism of spherical data by φ.

Proof. This is a collection of the main results of [Los09], [Lun01], and [BP16].

The diagram 2.3 can be obtained from [Los09, 1, theorem], which shows the existence of a G-equivariant morphism between X_1 and X_2 from an isomorphism between the corresponding Luna data. For the existence of a spherical variety corresponding to a spherical datum, [Lun01] provides a reduction to the same result for wonderful varieties, and proves the type A cases. [BP16] follows the same strategy and completes the proof by figuring out all unknown cases after considering the previous works (see [BP16, 2.6, Section] for more details) according to the classification of primitive spherical systems given by [Bra13]. $\square$

The isomorphism φ mentioned in Theorem 2.3.1 is not unique for X and X'.

Proposition 2.3.2. *Let $\mathrm{Aut}^G(X)$ be the group of G-equivariant automorphisms of X, and $\mathfrak{A}^\sharp_X := \{\iota \in \mathrm{Aut}^G(X) : \iota$ preserves all B-stable divisors of $X\}$. For each $\iota \in \mathfrak{A}^\sharp_X$, the induced automorphism ι_* of $\mathscr{L}_X$ according to the previous theorem is the identity morphism $\mathrm{id}_{\mathscr{L}_X}$. Therefore, the isomorphsm of spherical varieties in the previous theorem, given an isomorphism of the corresponding spherical systems, is unique up to $\mathfrak{A}^\sharp_X$.*

Proof. $\mathfrak{A}^\sharp_X$ acts trivially on G, thus it acts trivially on Ψ_G and S_G. By the definition of $\mathscr{L}_X$ (Definition 2.2.4), the elements in S^p_X, Σ_X and Ξ_X are fixed by $\mathrm{Aut}^G(X)$ (because they can be considered as linear combinations of positive simple roots, which are fixed by $\mathrm{Aut}^G(X)$). Thus $\mathfrak{A}^\sharp_X$ acts trivially on $\mathscr{A}_X$, which induces that the $\mathfrak{A}^\sharp_X$-action on ρ_X is also trivial. □

Remark. An element $\iota \in \mathfrak{A}^\sharp_X$ pre-composed with a choice of φ determines all the choices of G-isomorphisms $X \longrightarrow X'$ making diagram 2.2 commute.

Theorem 2.3.3. *Let $\Lambda = (\Psi, S, \mathscr{L})$ be a Luna datum. Then there exists a homogeneous spherical pair (G, X) together with an isomorphism $\mu : \Lambda_{(G,X)} \longrightarrow \Lambda$. Moreover, for any other such pair (G', X') with μ', there is an isomorphism of spherical pairs $m = (m_G, m_X) : (G, X) \longrightarrow (G', X')$ such that the induced isomorphism $m_* : \Lambda_{(G,X)} \longrightarrow \Lambda_{(G',X')}$ makes the following diagram commute:*

$$\begin{tikzcd} \Lambda_{(G,X)} \arrow[r, "\mu"] \arrow[d, "m_*"'] & \Lambda \\ \Lambda_{(G',X')} \arrow[ur, "\mu'"'] & \end{tikzcd} \tag{2.3}$$

Proof. This is a generalization of Theorem 2.3.1.

According to the theory of reductive groups, the group G can be constructed when the root datum is given, and then Theorem 2.3.1 can be applied to find the spherical pair (G, X).

Furthermore, an isomorphism $m_G : G \to G'$ can be given for any other spherical pair (G', X') whose Luna datum is isomorphic to Λ, which makes X' a spherical G-variety (G acts on X through the isomorphism to G'), thus Theorem 2.3.1 implies an isomorphism $(id_G, m_X) : (G, X) \longrightarrow (G, X')$ satisfying diagram 2.3 for spherical pairs (G, X) and (G, X'). Finally, composed with (m_G, id_X), the isomorphism $m = (m_G, m_X)$ is obtained. And the following diagrams commute.

$$
\begin{tikzcd}
(G,X) \arrow[d, "{(id_G, m_X)}"'] \arrow[dr, dashed, "m"] & \\
(G,X') \arrow[r, "{(m_G, id_X)}"'] & (G',X')
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda_{(G,X)} \arrow[r, "\mu"] \arrow[d, "{(id_G, m_X)_*}"'] \arrow[dr, dashed, "m_*"] & \Lambda \\
\Lambda_{(G,X')} \arrow[ur, "\mu^\circ"] \arrow[r, "{(m_G, id_X)_*}"'] & \Lambda_{(G',X')} \arrow[u, "\mu'"']
\end{tikzcd}
\tag{2.4}
$$

□

Remark. There are various choices of $m : (G, X) \longrightarrow (G', X')$ making the diagram 2.3 commute. These isomorphisms can be identified with automorphisms of (G, X) which induce The group of such isomorphisms is denoted by $\mathfrak{S}_X$, and more details will be discussed in next chapter.

Chapter 3

Automorphisms of Spherical Varieties

Let k be a field of characteristic 0, G be a connected reductive algebraic group defined over k, and X be a homogeneous spherical G-variety over k. Denote the Galois group $\mathrm{Gal}(\bar{k}/k)$ by Γ.

Starting from a spherical G-variety X defined over k, which is, a pair of k-forms (G, X) where G is quasi-split(G has a Borel subgroup B defined over k), and $X(k) \neq \emptyset$, the target is to find out any other possible k-forms X' of X which is spherical under the same k-form of the group G.

By [Ser97], the k-forms of a k-variety X are classified by the first Galois cohomology of its automorphisms $\mathrm{H}^1(\Gamma, \mathrm{Aut}(X_{\bar{k}}))$.

Before going into the details of first Galois cohomology, it is necessary to investigate the automorphisms of spherical $G_{\bar{k}}$-varieties, and from now on, unless specified, the base change $G_{\bar{k}}$ is denoted by G, and similarly $X_{\bar{k}}$ by X.

3.1 Definitions

Let G be a reductive algebraic group over $\bar{k}$, X be a homogeneous spherical G-variety, as stated above.

Definition 3.1.1. An **automorphism** of a spherical pair (G, X) is an isomorphism of spherical pair (G, X) to itself.

Recall that the isomorphisms between spherical pairs are defined in Definition 2.2.5.

Definition 3.1.2. An automorphism $\sigma \in \mathrm{Aut}(G, X)$ is called **inner** if it is of the form $(g_0, x) \mapsto (g^{-1}g_0g, xg)$ for some $g \in G$. The group of inner automorphisms of (G, X) are denoted by $\mathrm{Inn}(G, X)$.

By the definitions above, there is an injection: $\mathrm{Inn}(G, X) \longrightarrow \mathrm{Aut}(G, X)$, thus

Definition 3.1.3. The cokernel of the injection $\mathrm{Inn}(G, X) \longrightarrow \mathrm{Aut}(G, X)$ is called the group of **outer automorphisms**, denoted by $\mathrm{Out}(G, X)$.

Hence the following sequence is exact:

$$1 \longrightarrow \mathrm{Inn}(G, X) \longrightarrow \mathrm{Aut}(G, X) \longrightarrow \mathrm{Out}(G, X) \longrightarrow 1$$

Consider the automorphism group of G, an automorphism $\sigma \in \mathrm{Aut}(G, X)$ induces $\sigma_G \in \mathrm{Aut}(G)$. Moreover, if σ is an inner automorphism, then the corresponding σ_G takes any group element to its g-conjugation for some group element $g \in G$, hence $\sigma_G \in \mathrm{Inn}(G)$. From diagram chasing, there is a map $\mathrm{Out}(G, X) \longrightarrow \mathrm{Out}(G)$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Inn}(G, X) \arrow[r] \arrow[d, "p_i"'] & \mathrm{Aut}(G, X) \arrow[r] \arrow[d, "p_a"'] & \mathrm{Out}(G, X) \arrow[r] \arrow[d, "p_o"'] & 1 \\
1 \arrow[r] & \mathrm{Inn}(G) \arrow[r] & \mathrm{Aut}(G) \arrow[r] & \mathrm{Out}(G) \arrow[r] & 1
\end{tikzcd}$$

Consider that $\ker p_i = \mathcal{Z}(G)/\mathrm{Stab}_{\mathcal{Z}(G)}(x)$ for an $x \in X$ (a different choice x' makes it $\mathcal{Z}(G)/\mathrm{Stab}_{\mathcal{Z}(G)}(x')$, however $\mathrm{Stab}_{\mathcal{Z}(G)}(x') = g^{-1}(\mathrm{Stab}_{\mathcal{Z}(G)}(x))g$ for some $g \in G$, which is just $\mathrm{Stab}_{\mathcal{Z}(G)}(x)$). And $\ker p_a = \mathrm{Aut}^G(X)$, the group of automorphisms of (G, X) whose restriction on G is identity.

In particular, when $X = H\backslash G$ is a homogeneous spherical G-variety, with H the stabilizer of a point, $\mathrm{Aut}^G(X) \simeq \mathcal{N}_G(H)/H$. For any point $x \in X$, represented in terms of Hg with $g \in G$, $nH \in \mathrm{Aut}^G(X)$ acts by $nH.Hg = nHg = H(ng)$. Furthermore, the stabilizer of x under this action is trivial.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \ker p_i \arrow[r] \arrow[d] & \mathrm{Aut}^G(X) \arrow[d] & & \\
1 \arrow[r] & \mathrm{Inn}(G,X) \arrow[r] \arrow[d, "p_i"] & \mathrm{Aut}(G,X) \arrow[r] \arrow[d, "p_a"] & \mathrm{Out}(G,X) \arrow[r] \arrow[d, "p_o"] & 1 \\
1 \arrow[r] & \mathrm{Inn}(G) \arrow[r] & \mathrm{Aut}(G) \arrow[r] & \mathrm{Out}(G) \arrow[r] & 1
\end{tikzcd} \tag{3.1}$$

The diagram above is exact.

3.2 Automorphisms of Luna Data

According to Definition 2.2.2 and Definition 2.2.3, an automorphism of a Luna datum $\Lambda = (\Psi, S, \mathscr{L})$ is determined by $\sigma = (\sigma_{\mathcal{X}^*}, \sigma_{\mathscr{A}})$, an isomorphism of the pair $(\mathcal{X}^*, \mathscr{A})$ such that $\sigma_{\mathcal{X}^*}$ preserves Φ, S, S^p, Σ, Ξ as subsets, and $\sigma_{\mathscr{A}}$ satisfies that $\sigma_{\mathscr{A}}(\mathscr{A}(\alpha)) = \mathscr{A}(\sigma_{\mathcal{X}^*}(\alpha))$, and $\sigma_{\mathcal{X}^*} \circ \rho \circ \sigma_{\mathscr{A}}^{-1} = \rho$. And all the axioms in the definition of Λ remain valid in the image of the automorphism.

For each automorphism $\sigma \in \mathrm{Aut}(G, X)$, there is a corresponding automorphism of the universal Cartan group $\sigma_C : A \longrightarrow A$ which implies an automorphism on the Luna data of X. Hence there is a morphism $\alpha : \mathrm{Aut}(G, X) \longrightarrow \mathrm{Aut}(\Lambda_{(G,X)})$. In fact, for any $\sigma \in \mathrm{Aut}(G, X)$, there is an inner automorphism $\tau \in \mathrm{Inn}(G, X)$ such that $\tau \circ \sigma$ preserves the chosen Borel subgroup B and maximal torus T. Then the automorphism of $\Lambda_{(G,X)}$ induced by σ is defined by the automorphism induced by $\tau \circ \sigma$. If a different inner automorphism τ' is chosen which also preserves the chosen B and T, then $\tau^{-1} \circ \tau' = \mathrm{Int}(b)$ for some $b \in B$, which acts trivially on Ψ_G, S_G and fixes all the colors (as they are B-stable). Hence, the morphism $\alpha : \mathrm{Aut}(G, X) \longrightarrow \mathrm{Aut}(\Lambda_{(G,X)})$ is well defined explicitly in this way.

The morphism α is an epimorphism (surjective) according to Theorem 2.3.3. Let η be an automorphism of $\Lambda_{(G,X)}$, where $\Lambda_{(G,X)}$ is the Luna datum of (G, X), the theorem shows that there is an isomorphism $(G, X) \longrightarrow (G, X)$ inducing η, hence,

Proposition 3.2.1. $\alpha : \mathrm{Aut}(G, X) \longrightarrow \mathrm{Aut}(\Lambda)$ *is surjective.*

Proposition 3.2.2. *The kernel of* α *is* $\ker(\alpha) = \mathfrak{S}_X = \mathfrak{A}_X^\sharp \times^{\mathcal{Z}(G)} \mathrm{Inn}(G, X)$ *(recall that* $\mathfrak{S}_X$ *is first mentioned in the remark after Theorem 2.3.3), where* $\mathcal{Z}(G) \subseteq G \to \mathrm{Inn}(G, X)$ *and* $\mathcal{Z}(G)$ *maps into* $\mathfrak{A}_X^\sharp$ *through a quotient. Thus the following sequence is exact,*

$$1 \longrightarrow \mathfrak{A}_X^\sharp \times^{\mathcal{Z}(G)} \mathrm{Inn}(G, X) \longrightarrow \mathrm{Aut}(G, X) \longrightarrow \mathrm{Aut}(\Lambda_{(G,X)}) \longrightarrow 1 \tag{3.2}$$

Proof. $\mathfrak{A}_X^\sharp$ is defined in Proposition 2.3.2. $\mathfrak{A}_X^\sharp$ acts trivially on both G and the set $\mathscr{A}_X$, hence $\mathfrak{A}_X^\sharp \subseteq \ker \alpha$. An inner automorphism $\iota \in \mathrm{Inn}(G, X)$ acts on Ψ_G through an inner automorphism of G, thus the action is trivial by the theory of reductive groups. Moreover, by previous discussions, ι acts on $\mathscr{L}_X$ the ι by $\iota \circ \tau$ with $\tau \in \mathrm{Inn}(G, X)$

to fix a Borel subgroup, thus acts trivially on all colors. Therefore, $\mathrm{Inn}(G, X)$ acts trivially on $\Lambda_{(G,X)}$.

Then for any automorphism $\sigma \in \ker(\alpha)$, there is a $\sigma \in \mathrm{Inn}(G, X)$ such that $\sigma_s = \sigma_i \circ \sigma$ acts trivially on G and $\sigma_s \in \ker(\alpha)$ as well. Hence $\sigma_s \in \mathrm{Aut}^G(X)$. Also, σ_s acts trivially on $\mathscr{A}_X$, so $\sigma_s \in \mathfrak{A}^\sharp_X$. That is, any element in $\ker(\alpha)$ is the composition of an element in $\mathfrak{A}^\sharp_X$ with an inner automorphism of the pair (G, X), and vice versa.

Finally, the intersection of $\mathfrak{A}^\sharp_X$ and $\mathrm{Inn}(G, X)$ is isomorphic to $\mathrm{Stab}_{\mathcal{Z}(G)}(x)$. □

There is a morphism $\mathfrak{S}_X \longrightarrow \mathrm{Inn}(G)$ through $\mathrm{Inn}(G, X)$. And by definition of automorphisms of Luna data, there is a morphism $\mathrm{Aut}(G, X) \longrightarrow \mathrm{Aut}(\Psi_G)$. Thus there is diagram similar to diagram 3.1,

Proposition 3.2.3. *Let $\mathrm{Aut}^{\rho_X}(\mathscr{A}_X)$ be the subgroup of $\mathrm{Aut}(\Lambda_{(G,X)})$ consisting of the automorphisms whose actions on Ψ_G, S_G, S^p_X, Σ_X, and Ξ_X are trivial (thus the only nontrivial parts are the action on $\mathscr{A}_X$, permitting elements whose image under ρ coincides. The following diagram commutes and all the rows and columns are exact.*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathfrak{A}^\sharp_X \arrow[r] \arrow[d] & \mathrm{Aut}^G(X) \arrow[r] \arrow[d] & \mathrm{Aut}^{\rho_X}(\mathscr{A}_X) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathfrak{S}_X \arrow[r] \arrow[d] & \mathrm{Aut}(G, X) \arrow[r] \arrow[d] & \mathrm{Aut}(\Lambda_{(G,X)}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{Inn}(G) \arrow[r] & \mathrm{Aut}(G) \arrow[r, shift right] & \mathrm{Aut}(\Psi_G) \arrow[r] \arrow[l, dashed, shift right, "s"'] & 1
\end{tikzcd} \tag{3.3}$$

The bottom short exact sequence splits, with the splitting morphism s.

Proof. From diagram 3.1 and the discussions above, the second and the third rows are exact and the there is a morphism between the short exact sequences. Moreover, the splitting morphism of the third row is from the theory of reductive groups. Then it is sufficient to show that $\mathfrak{A}_X^\sharp$ and $\mathrm{Aut}^{\rho_X}(\mathscr{A}_X)$ are the corresponding kernels. The rest exactness and commutativity comes from the Snake lemma.

$\mathfrak{S}_X \longrightarrow \mathrm{Inn}(G)$ factors through $\mathrm{Inn}(G,X)$, thus the kernel is $\mathcal{Z}_{\mathcal{Z}(G)}(x) \times^{\mathcal{Z}(G)} \mathfrak{A}_X^\sharp = \mathfrak{A}_X^\sharp$.

For $\mathrm{Aut}(\Lambda_{(G,X)}) \longrightarrow \mathrm{Aut}(\Psi_G)$, as the kernel acts trivially on Ψ_G, it also acts trivially on S_X^p, Σ_X, Ξ_X and ρ_X. Hence the kernel consists automorphisms of $\mathscr{A}_X$ while fixing ρ_X. (It is nontrivial since there can be colors with the same valuation.) $\square$

Furthermore, consider the case that a Borel subgroup B and a pinning Pin (also known as *épinglage*) is given. A pinning is a system of isomorphisms $\{u_\alpha : \alpha \in S_G\}$ where $u_\alpha : \mathbb{G}_{\mathrm{a}} \longrightarrow U_\alpha$ is an isomorphism, for each positive simple root α, from the additive group $\mathbb{G}_{\mathrm{a}}$ to the unipotent subgroup U_α corresponding to α. An automorphism of Luna datum $\Lambda_{(G,X)}$ induces an automorphism of Ψ_G. And an automorphism of Ψ_G induces a unique automorphism of the corresponding G, with Borel subgroup B and pinning Pin fixed. Thus any B-orbit remains to be a B-orbit. Hence the unique open B-orbit $\mathring{X}_B$ is fixed. Let $\mathrm{Aut}(G, B, X, \mathring{X}_B, \mathrm{Pin})$ denote the set of automorphisms of (G, X) fixing a Borel subgroup B (thus the open B-orbit $\mathring{X}_B$ is automatically fixed), and a pinning Pin, then there is a homomorphism

$$\alpha' : \mathrm{Aut}(G, B, X, \mathring{X}_B, \mathrm{Pin}) \longrightarrow \mathrm{Aut}(\Lambda_{(G,X)}).$$

Proposition 3.2.4. *With given B, $\mathring{X}_B$ and* Pin*, the homomorphism α' is surjective and with a kernel* $\ker \alpha' = \mathfrak{A}_X^\sharp$*, i.e., the following sequence is exact.*

$$0 \longrightarrow \mathfrak{A}_X^\sharp \longrightarrow \mathrm{Aut}(G, B, X, \mathring{X}_B, \mathrm{Pin}) \longrightarrow \mathrm{Aut}(\Lambda_{(G,X)}) \longrightarrow 0 \tag{3.4}$$

Proof. First consider the exact sequence 3.2, the homomorphism $\alpha : \mathrm{Aut}(G, X) \longrightarrow \mathrm{Aut}(\Lambda_{(G,X)})$ is surjective, then each fiber of α admits an action of $\mathrm{Inn}(G, X)$. Hence there is an automorphism of (G, X) fixing the chosen B and Pin, thus it is an element of $\mathrm{Aut}(G, B, X, \mathring{X}_B, \mathrm{Pin})$. $\ker(\alpha') = \ker(\alpha) \cap \mathrm{Aut}(G, B, X, \mathring{X}_B, \mathrm{Pin})$, hence $\ker(\alpha') = \mathfrak{A}^\sharp$ as $\xi \in \mathrm{Inn}(G, X)$ does not preserve B or Pin unless ξ is the image of $\mathcal{Z}(G)$. □

Corollary 3.2.5. *If* $\mathfrak{A}^\sharp_X$ *is trivial, then* $\mathrm{Aut}(G, B, X, \mathring{X}_B, \mathrm{Pin})$ *is isomorphic to* $\mathrm{Aut}(\Lambda_{(G,X)})$. *That is, there is a canonical lifting of any* $\xi \in \mathrm{Aut}(\Lambda_{(G,X)})$ *to* $\mathrm{Aut}(G, B, X, \mathring{X}_B, \mathrm{Pin})$.

3.3 Spherical Closedness

Definition 3.3.1. A **spherical system** is a spherical datum $\mathscr{L} = (S^p, \Sigma, \mathscr{A}, \Xi, \rho)$ where $\Xi = \langle \Sigma \rangle_{\mathbb{Z}}$ is generated by the set of spherical roots Σ as a $\mathbb{Z}$-module. A spherical system is determined by $(S^p, \Sigma, \mathscr{A}, \rho)$ and denoted by $\mathscr{S}$.

Remark. Since $\Xi = \langle \Sigma \rangle_{\mathbb{Z}}$, the map ρ is determined by the Cartan pairing. Spherical systems are used to classify wonderful varieties (the homogeneous spherical varieties having a wonderful compactification).

An automorphism of $\mathscr{S}$ as a Luna datum is considered to be an automorphism of the spherical system $\mathscr{S}$.

According to Theorem 1.2 in [Kno96], let X be a homogeneous spherical G-variety, there is a canonical inclusion $\mathrm{Hom}(\Xi_X/\langle \Sigma_X \rangle_{\mathbb{Z}}, \bar{k}^*) \hookrightarrow \mathrm{Aut}^G(X)$, in the sense that for each $t \in \mathrm{Hom}(\Xi_X/\langle \Sigma_X \rangle_{\mathbb{Z}}, \bar{k}^*)$, the corresponding G-automorphism φ_t of X acts on each element $f_\chi \in \Omega(X)^{(B)}$ with eigencharacter $\chi \in \Xi_X$ by $\varphi_t(f_\chi) = t(\chi) f_\chi$. Let $\mathfrak{T}$ denote $\mathrm{Hom}(\Xi_X/\langle \Sigma_X \rangle_{\mathbb{Z}}, \bar{k}^*)$ as a subgroup of $\mathrm{Aut}^G(X)$.

Proposition 3.3.2. $\mathfrak{T} \subseteq \mathfrak{A}^\sharp_X$.

Proof. According to [PB87, 5.2, Corollaire], $\mathrm{Aut}^G(X)$ is diagonalizable. Moreover, [Kno96, 5.5, Theorem] shows that $\mathrm{Aut}^G(X)$ is a subgroup of $\mathrm{Hom}(\Xi_X, k^\times)$, thus $\mathcal{X}(\mathrm{Aut}^G(X))$ can be considered as a quotient of Ξ_X.

As $\mathfrak{T}$ and $\mathfrak{A}^\sharp_X$ are subgroups of $\mathrm{Aut}^G(X)$, there are homomorphisms $\pi_{\mathfrak{T}} : \Xi_X \twoheadrightarrow \mathcal{X}(\mathfrak{T})$ and $\pi_{\mathfrak{A}^\sharp} : \Xi_X \twoheadrightarrow \mathcal{X}(\mathfrak{A}^\sharp_X)$. By definition of $\mathfrak{T}$, $\ker \pi_{\mathfrak{T}} = \langle \Sigma_X \rangle_{\mathbb{Z}}$. And by [Kno96, 7.5, Theorem], there is a root system $\Delta^\sharp_X \subseteq \Xi_X$, such that $\mathfrak{A}^\sharp_X = \bigcap_{\alpha \in \Delta^\sharp_X} \ker_{A_X} \alpha$. Thus $\ker \pi_{\mathfrak{A}^\sharp} = \langle \Delta^\sharp_X \rangle_{\mathbb{Z}}$ as characters of universal Cartan A_X whose restriction on $\mathfrak{A}^\sharp_X$ is trivial.

As $\Delta^\sharp_X$ is a root system on the lattice $\langle \Sigma_X \rangle_{\mathbb{Z}}$, $\langle \Delta^\sharp_X \rangle_{\mathbb{Z}} \subseteq \langle \Sigma_X \rangle_{\mathbb{Z}}$. Therefore, $\mathfrak{T} \subseteq \mathfrak{A}^\sharp_X$. □

Proposition 3.3.3. *Let $Y := \mathfrak{A}' \backslash X$, where $\mathfrak{A}' \subseteq \mathfrak{A}^\sharp_X$, then $\mathfrak{A}^\sharp_Y = \mathfrak{A}' \backslash \mathfrak{A}^\sharp_X$.*

Proof. Let $H = \mathrm{Stab}_G(x)$ be the stabilizer of $x \in X$, by [Kno96, 7.4, Corollary], $\mathfrak{A}^\sharp_X = H \backslash H^\sharp$ for some $H^\sharp \subseteq \mathcal{N}_G(H)$ which acts trivially on $\mathcal{X}(H)$. And let $\mathfrak{A}' = H \backslash H'$ with $H' \subseteq H^\sharp$, then it suffices to show that $(H')^\sharp = H^\sharp$.

Let $H_0 = [H, H]$, denote $D := H_0 \backslash H$, $D' := H_0 \backslash H'$ and $D^\sharp := H_0 \backslash H^\sharp$. First we show $D^\sharp$ is diagonalizable, and so is $D' \subseteq D^\sharp$.

Let D^0 be the connected component of D containing identity, and $H_1 \subseteq H$ be its preimage. $[H : H_1]$ is finite, so H_1 is spherical. $H^\sharp \subseteq \mathcal{N}_G(H_1)$, then $D^0 \backslash D^\sharp = H_1 \backslash H^\sharp$ is diagonalizable.

$D^\sharp$ is linear, thus can be embedded into some GL_N. Let $L = \mathcal{Z}(D^0)$, it is a Levi subgroup as D^0 is a torus. L is a Levi subgroup, and in particular a connected reductive subgroup, containing $D^\sharp$, and $D^0 \subseteq \mathcal{Z}(L)$. As $D^0 \backslash D^\sharp$ is diagonalizable, the image of $D^\sharp$ in $L/\mathcal{Z}(L)$ is diagonalizable (contained in a maximal torus). Therefore, $D^\sharp$ is contained in a maximal torus of L. Thus $D^\sharp$ is diagonalizable, and so is D'.

Then, let $R' := \mathcal{X}(H') = \mathcal{X}(D')$ and $R := \mathcal{X}(H) = \mathcal{X}(D)$, R is a quotient of R'. By definition, $(H')^\sharp$ acts trivially on R', and also acts trivially on R. Hence $(H')^\sharp$ preserves $H = \{h \in H' : \chi(h) = 1, \forall \chi \in \ker(R' \to R)\}$. Therefore, by [Kno96, 7.4, Corollary], as $(H')^\sharp$ acts trivially on $\mathcal{X}(H)$, $(H')^\sharp = H^\sharp$, thus $\mathfrak{A}_Y^\sharp = H'\backslash H^\sharp = \mathfrak{A}'\backslash\mathfrak{A}_X^\sharp$. □

The following corollary shows a special case.

Corollary 3.3.4. *For* $Z := X/\mathfrak{A}_X^\sharp$*,* $\mathfrak{A}_Z^\sharp = \{1\}$*.*

Corollary 3.3.5. *Let* Z *be* $X/\mathfrak{A}_X^\sharp$*, same as that in the previous corollary, then* $\Xi_Z = \langle \Sigma_Z \rangle_{\mathbb{Z}}$*.*

Consider $\mathfrak{T}_Z \subseteq \mathfrak{A}_Z^\sharp$, hence $\mathfrak{T}_Z = \Xi_Z/\langle \Sigma_Z \rangle_{\mathbb{Z}} = \{1\}$.

Definition 3.3.6. A spherical G-variety X is called **spherically closed** if $\mathfrak{A}_X^\sharp$ is trivial. The spherical G-variety $Z = X/\mathfrak{A}_X^\sharp$ is called the **spherical closure** of X.

Proposition 3.3.7. $\Lambda_{(G,Z)}$ *is a spherical system, and* $\mathrm{Aut}(G, B, Z, \mathring{Z}_B, \mathrm{Pin})$ *is isomorphic to* $\mathrm{Aut}(\Lambda_{(G,Z)})$*, that is, for any automorphism of* $\Lambda_{(G,Z)}$*, there is a canonical automorphism of the spherical pair* (G, Z) *stabilizing* B*,* $\mathring{Z}_B$ *and* Pin*.*

The first statement is directly from Corollary 3.3.5, then consider Proposition 3.3.3 and 3.2.4, as $\mathfrak{A}_Z^\sharp$ is trivial, $\mathrm{Aut}(G, B, Z, \mathring{Z}_B, \mathrm{Pin}) \longrightarrow \mathrm{Aut}(\Lambda_{(G,Z)})$ is an isomorphism, and $\Lambda_{(G,Z)}$ is in fact a spherical system.

Chapter 4

k-Forms of Spherical Varieties

Let k be a field of characteristic 0, and $\bar{k}$ be its algebraic closure (also its separable closure as k is a perfect field), with Galois group Γ.

In this chapter, the k-forms of spherical varieties will be discussed. With the condition that G is a quasi-split reductive group over k, the k-forms of spherically closed homogeneous spherical G-varieties can be described by combinatorial data.

4.1 Galois Cohomology

This section is devoted to recalling the definitions and basic properties of Galois cohomology:

Definition 4.1.1. Given a group $\mathcal{H}$ (not necessarily abelian) with a left group action by Γ, the zeroth group cohomology of Γ with coefficients in $\mathcal{H}$ is the subgroup of fixed elements $\{h \in \mathcal{H} | \gamma(h) = h, \forall \gamma \in \Gamma\}$, denoted by $\mathrm{H}^0(\Gamma, \mathcal{H}) := \mathcal{H}^\Gamma$. And the **first group cohomology** of Γ with coefficients in $\mathcal{H}$ is the set of equvialence classes of the cocycles $C^1 = \{f : \Gamma \longrightarrow \mathcal{H} \mid f(\gamma_1\gamma_2) = f(\gamma_1)[\gamma_1.(f(\gamma_2))]\ \}$ under the equivalence relation that $f \sim g$ if there exists $c \in \mathcal{H}$ such that $f(\gamma) = c^{-1}g(\gamma)(\gamma.c)$, denoted by $\mathrm{H}^1(\Gamma, \mathcal{H}) := C^1/\sim$. If Γ and $\mathcal{H}$ are topological groups, and Γ acts continuously, then the first continuous group cohomology $\mathrm{H}^1_c(\Gamma, \mathcal{H})$ are equivalent classes of continuous cocycles in $C^1_c = \{f \in C_1 : f \text{ is continuous}\}$.

Definition 4.1.2. Let Γ be the Galois group $\mathrm{Gal}(\bar{k}/k)$, and $\mathcal{H}$ be a group with a continuous Γ-action on it. Then define the i-th **Galois cohomology** $\mathrm{H}^i(k, \mathcal{H})$ ($i = 0$ or 1) to be the continuous group cohomology $\mathrm{H}^i_c(\Gamma, \mathcal{H})$, respectively, with the given Γ-action.

Lemma 4.1.3. *Given a variety X defined over k, and $X_{\bar{k}} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ its base change to $\bar{k}$. Then the Galois group Γ acts on $X_{\bar{k}}$, hence acts on $\mathrm{Aut}(X_{\bar{k}})$.*

For $\gamma \in \Gamma$, the action of γ on $X_{\bar{k}}$ is given by the universal property of fiber products, shown in the following diagram:

$$
\begin{tikzcd}
 & X_{\bar{k}} \arrow[r, dashed, "\gamma"] \arrow[dl] \arrow[dr] & X_{\bar{k}} \arrow[dll] \arrow[dr] & \\
X \arrow[dr] & & \mathrm{Spec}(\bar{k}) \arrow[r, "\gamma"] \arrow[dl] & \mathrm{Spec}(\bar{k}) \arrow[dll] \\
 & \mathrm{Spec}(k) & &
\end{tikzcd}
$$

And this action induces a Γ-action on $\mathrm{Aut}_{\bar{k}}(X_{\bar{k}})$ by conjugation, i.e., $(\gamma.f)(x) = \gamma(f(\gamma^{-1}x))$ where $f \in \mathrm{Aut}_{\bar{k}}(X_{\bar{k}})$ and $\gamma \in \Gamma$. And without special emphases, $\mathrm{Aut}(X_{\bar{k}})$ will be used to represent $\mathrm{Aut}_{\bar{k}}(X_{\bar{k}})$, the $\bar{k}$-morphisms of $X_{\bar{k}}$.

Definition 4.1.4. Given an algebraic variety X defined over k, a variety Y over k is called a k-form of X if $X_{\bar{k}}$ is isomorphic to $Y_{\bar{k}}$ as $\bar{k}$-varieties.

It is known that the k-forms of a variety X over k can be classified by the first Galois cohomology of $\mathrm{Aut}(X_{\bar{k}})$, as stated in the following theorem.

Theorem 4.1.5. *Given a variety X defined over k, there is a bijection between* $\mathrm{H}^1(k, \mathrm{Aut}(X_{\bar{k}}))$ *and the set of k-forms of X up to k-isomorphisms, and X is mapped to the canonical point in* $\mathrm{H}^1(k, \mathrm{Aut}(X_{\bar{k}}))$. *Here X and X' are equivalent if there is a k-isomorphism $m : X \longrightarrow X'$ and the $\bar{k}$-isomorphism is the lifting of m via the universal property of the base change.*

Proof. For any k-form X' of X, the isomorphism $X_{\bar{k}} \longrightarrow X'_{\bar{k}}$ implies a Galois action on $X_{\bar{k}}$ and on $\mathrm{Aut}(X_{\bar{k}})$. Let $\mu' : \Gamma \longrightarrow \mathrm{Aut}(X_{\bar{k}})$ be the induced action from X', and μ the original action of Γ on $X_{\bar{k}}$.(Thus μ and μ' are group homomorphisms.)

Let $f(\gamma) = \mu'(\gamma)\mu(\gamma^{-1})$, then

$$\begin{aligned} f(\gamma_1\gamma_2) &= \mu'(\gamma_1\gamma_2)\mu((\gamma_1\gamma_2)^{-1}) \\ &= \mu'(\gamma_1)\mu'(\gamma_2)\mu(\gamma_2^{-1})\mu(\gamma_1^{-1}) \\ &= \mu'(\gamma_1)\mu(\gamma_1^{-1})\mu(\gamma_1)\mu'(\gamma_2)\mu(\gamma_2^{-1})\mu(\gamma_1^{-1}) \\ &= f(\gamma_1)\mu(\gamma_1)f(\gamma_2)\mu(\gamma_1^{-1}) \\ &= f(\gamma_1)[\gamma_1.f(\gamma_2)]. \end{aligned}$$

Hence f is a cocycle according to Definition 4.1.1.

For two k-forms X_1 and X_2 of X, if they are k-isomorphic to each other, that is, there is a morphism $m : X_1 \longrightarrow X_2$ over k, and the induced isomorphism $\bar{m} : (X_1)_{\bar{k}} \longrightarrow (X_2)_{\bar{k}}$ brings the Γ action on $(X_1)_{\bar{k}}$ to that on $(X_2)_{\bar{k}}$. That is, let $\mu_i : \Gamma \longrightarrow \mathrm{Aut}((X_i)_{\bar{k}})$ for $i = 1, 2$, then $\mu_1(\gamma) = \bar{m}^{-1}\mu_2(\gamma)\bar{m}$. Let $\tilde{\mu}_i = m_i^{-1}\mu_i m_i$ be the induced Galois action on $X_{\bar{k}}$, where $m_i : X_{\bar{k}} \longrightarrow (X_i)_{\bar{k}}$ is the $\bar{k}$-isomorphism for each

$(X_i)_{\bar{k}}$, and let $f_i(\gamma) = \tilde{\mu}_i(\gamma)\mu(\gamma^{-1})$. Then

$$\begin{aligned}
f_2(\gamma) &= \tilde{\mu}_i(\gamma)\mu(\gamma^{-1}) \\
&= m_2^{-1}\mu_2(\gamma)m_2\mu(\gamma^{-1}) \\
&= m_2^{-1}\bar{m}^{-1}\mu_1(\gamma)\bar{m}m_2\mu(\gamma^{-1}) \\
&= m_2^{-1}\bar{m}^{-1}m_1\tilde{\mu}_1(\gamma)m_1^{-1}\bar{m}m_2\mu(\gamma^{-1}) \\
&= [(m_1^{-1}\bar{m}m_2)^{-1}\tilde{\mu}_1(\gamma)\mu(\gamma^{-1})][\mu(\gamma)(m_1^{-1}\bar{m}m_2)\mu(\gamma^{-1})] \\
&= (m_1^{-1}\bar{m}m_2)^{-1}f_1(\gamma)[\gamma.((m_1^{-1}\bar{m}m_2))].
\end{aligned}$$

Therefore, $f_1 \sim f_2$ as cocycles, thus $[f_1] = [f_2]$ in $\mathrm{H}^1(k, \mathrm{Aut}(X_{\bar{k}}))$.

On the other hand, let $f \in C^1$, and μ the Γ-action on $X_{\bar{k}}$ as above, then $f\mu : \Gamma \longrightarrow \mathrm{Aut}(X_{\bar{k}})$ is a homomorphism by the previous calculation, thus it is a Γ action on $X_{\bar{k}}$. Hence the variety fixed by Γ X' is a k-form of X.

For two maps f_1 and f_2 equivalent by $c \in \mathrm{Aut}(X_{\bar{k}})$, $f_2\mu = c^{-1}f_2(\gamma.c)\mu = c^{-1}(f_2\mu)c$ provides two γ actions which are conjugate to each other, hence the k-forms they provide are k-isomorphic to each other. □

4.2 Forms of Spherical Varieties

Definition 4.2.1. A k-form of the spherical pair (G, X) is a k-form of the action morphism $G \times X \longrightarrow X$, i.e., a pair (G', X') where the reductive algebraic group G' over k acts on the variety X' over k, such that there exist $\bar{k}$-isomorphisms $t_G : G_{\bar{k}} \longrightarrow$

$G'_{\bar{k}}$ and $t_X : X_{\bar{k}} \longrightarrow X'_{\bar{k}}$ making the following diagram commutes:

$$\begin{tikzcd}
G_{\bar{k}} \times X_{\bar{k}} \arrow[r] \arrow[d, "t_G \times t_X"'] & X_{\bar{k}} \arrow[d, "t_X"] \\
G'_{\bar{k}} \times X'_{\bar{k}} \arrow[r] & X'_{\bar{k}}
\end{tikzcd}$$

The conditions above can be considered as the condition of an $\bar{k}$-isomorphism between the actions.

Theorem 4.2.2. *Given a spherical pair (G, X) defined over k, then the k-forms of a spherical pair (G, X) are classified by $\mathrm{H}^1(k, \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}}))$ up to k-isomorphisms.*

This is a direct conclusion of Theorem 4.1.5.

Further, the forms corresponding to the Galois actions on the Luna data will be investigated.

We start from the following diagram induced from 3.3.

Lemma 4.2.3. *Given a spherical pair (G, X) defined over k, by applying the cohomology long exact sequence to the diagram 3.3, the following diagram can be obtained.*

$$\begin{tikzcd}
\mathrm{Aut}^{\rho_{X_{\bar{k}}}}(\mathscr{A}_{X_{\bar{k}}})^{\Gamma} \arrow[r] \arrow[d] & \mathrm{H}^1(k, \mathfrak{A}^{\sharp}_{X_{\bar{k}}}) \arrow[r] \arrow[d] & \mathrm{H}^1(k, \mathrm{Aut}^{G_{\bar{k}}}(X_{\bar{k}})) \arrow[r] \arrow[d] & \mathrm{H}^1(k, \mathrm{Aut}^{\rho_{X_{\bar{k}}}}(\mathscr{A}_{X_{\bar{k}}})) \arrow[d] & \\
\mathrm{Aut}(\Lambda_{X_{\bar{k}}})^{\Gamma} \arrow[r] \arrow[d] & \mathrm{H}^1(k, \mathfrak{S}_{X_{\bar{k}}}) \arrow[r] \arrow[d] & \mathrm{H}^1(k, \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}})) \arrow[r] \arrow[d, "\alpha"'] & \mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})})) \arrow[d] & \\
\mathrm{Aut}(\Psi_{G_{\bar{k}}})^{\Gamma} \arrow[r] & \mathrm{H}^1(k, \mathrm{Inn}(G_{\bar{k}})) \arrow[r] & \mathrm{H}^1(k, \mathrm{Aut}(G_{\bar{k}})) \arrow[r, shift left] & \mathrm{H}^1(k, \mathrm{Aut}(\Psi_{G_{\bar{k}}})) \arrow[l, dashed, shift left] \arrow[r] & 1
\end{tikzcd} \tag{4.1}$$

This diagram depends on the choice of the k-form (G, X).

Remark. Since non-abelian first group cohomogolies are considered are considered as pointed sets, the exactness of a sequence

$$A \longrightarrow B \longrightarrow C$$

at B is defined in the sense that the fiber containing the base point $b \in B$ is the image of A. Thus the previous diagram depends on the choice of original spherical pair (G, X) defined over k.

This diagram can be further expanded to left, but only this part will be used.

Proof. For each row in Diagram 4.1, consider that the "long exact sequence" in the non-abelian Galois cohomology case which just involves $\mathrm{H}^0(k, *)$ and $\mathrm{H}^1(k, *)$, thus the row sequences are exact. Similarly, each column sequence is exact. However, as the original columns in Diagram 4.1 are not short exact, the exactness for each column in the above diagram only holds at the middle point.

Furthermore, in the last row there is a split morphism of the epimorphism (surjective map) $\mathrm{H}^1(k, \mathrm{Aut}(G_{\bar{k}})) \longrightarrow \mathrm{H}^1(k, \mathrm{Aut}(\Psi_{G_{\bar{k}}}))$, this is from the theory of reductive algebraic groups over k. Details can be found in [Ser97] and [Spr79].

The diagram commutes since the base diagram 3.1 commutes. $\square$

In the rest of the discussion, only the k-form of a particular kind of homogeneous spherical varieties will be discussed.

Recall that a connected reductive group G over k is called split if it has a maximal torus which is split over k.

G is called a *quasi-split* reductive group over k if there is a Borel subgroup $B \subseteq G$ defined over k.

Theorem 4.2.4. *Let G be a connected reductive group defined over k, and G is quasi-split. Let X be a spherically closed homogeneous spherical G-variety, then there is a bi-*

jection between the set of k-forms (G', X') with quasi-split G', and $\mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}))$. *The k-form (G', X') is unique up to k-isomorphisms.*

By Proposition 3.3.2, $\mathfrak{T}$ is trivial, so $\Sigma_{X_{\bar{k}}}$ spans $\Xi_{X_{\bar{k}}}$, thus the spherical datum of $X_{\bar{k}}$ is in fact a spherical system.

Proof. By Definition 3.3.6, $\mathfrak{A}^{\sharp}_{X_{\bar{k}}}$ is trivial, thus the first column of the diagram 3.3

$$1 \longrightarrow \mathfrak{A}^{\sharp}_{X_{\bar{k}}} \longrightarrow \mathfrak{S}_{X_{\bar{k}}} \longrightarrow \mathrm{Inn}(G_{\bar{k}}) \longrightarrow 1$$

implies that $\mathfrak{S}_{X_{\bar{k}}}$ is isomorphic to $\mathrm{Inn}(G_{\bar{k}})$. The homomorphism $\mathrm{Inn}(G_{\bar{k}}) \longrightarrow \mathfrak{S}_{X_{\bar{k}}}$ maps $\mathrm{Int}(g_0)$, the inner automorphism of $G_{\bar{k}}$ by $g_0 \in G_{\bar{k}}$, to the inner automorphism of $(G_{\bar{k}}, X_{\bar{k}})$ given by $(g, x) \mapsto (g_0^{-1} g g_0, x g_0)$.

Then the sequence 3.2 becomes

$$1 \longrightarrow \mathrm{Inn}(G_{\bar{k}}) \longrightarrow \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}}) \xrightarrow{\pi} \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}) \longrightarrow 1. \tag{4.2}$$

Thus there is an exact sequence of cohomologies(as pointed sets):

$$\begin{aligned} 0 \longrightarrow (\mathrm{Inn}(G_{\bar{k}}))^{\Gamma} &\longrightarrow (\mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}}))^{\Gamma} \longrightarrow (\mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}))^{\Gamma} \\ \xrightarrow{\delta} \mathrm{H}^1(k, \mathrm{Inn}(G_{\bar{k}})) &\longrightarrow \mathrm{H}^1(k, \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}})) \xrightarrow{\mathrm{H}^1(\pi)} \mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})})). \end{aligned} \tag{4.3}$$

By Proposition 3.2.4, $m : \mathrm{Aut}(G_{\bar{k}}, B_{\bar{k}}, X_{\bar{k}}, (\mathring{X}_{\bar{k}})_{B_{\bar{k}}}, \mathrm{Pin}) \longrightarrow \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})})$ is an isomorphism. As $i : \mathrm{Aut}(G_{\bar{k}}, B_{\bar{k}}, X_{\bar{k}}, (\mathring{X}_{\bar{k}})_{B_{\bar{k}}}, \mathrm{Pin}) \hookrightarrow \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}})$ is an inclusion, there is a homomorphism $u = i \circ m^{-1} : \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}) \longrightarrow \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}})$, which makes the sequence 4.2 right split.

For each class of cocycles $[s] \in \mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}))$, $\mathrm{H}^1(u)([s])$ corresponds to the k-isomorphism class of a k-form $(G^{[s]}, X^{[s]})$ of (G, X). The induced Γ action on $(G^{[s]}_{\bar{k}}, X^{[s]}_{\bar{k}})$ preserves $B^{[s]}_{\bar{k}}$, $(\mathring{X}^{[s]}_{\bar{k}})_{B^{[s]}_{\bar{k}}}$, and Pin. Thus $G^{[s]}$ admits a Borel subgroup $B^{[s]}$ defined over k, which makes $G^{[s]}$ quasi-split.

To show the uniqueness, let (G_1, X_1) and (G_2, X_2) be two k-forms of (G, X) with G_1 and G_2 quasi-split, and lives on the same fiber $(\mathrm{H}^1(\pi))^{-1}([s])$. Without loss of generality, let (G_1, X_1) be the k-form given by the class $\mathrm{H}^1(u)([s])$.

In the following discussion in this proof, denote (G_1, X_1) by (G, X) (now G is just quasi-split, not the same form in the statement of the theorem, which is split) and (G_2, X_2) by (G', X'). Then from the exactness of the sequence

$$\mathrm{H}^1(k, \mathrm{Inn}(G_{\bar{k}})) \xrightarrow{\mathrm{H}^1(\iota)} \mathrm{H}^1(k, \mathrm{Aut}(G_{\bar{k}}, X_{\bar{k}})) \xrightarrow{\mathrm{H}^1(\pi)} \mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})})),$$

(G', X') lives on the fiber of $\mathrm{H}^1(\pi)$ over the base point of $\mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G_{\bar{k}}, X_{\bar{k}})}))$, thus in the image of $\mathrm{H}^1(\iota)$. Let $[\xi] \in \mathrm{H}^1(k, \mathrm{Inn}(G_{\bar{k}}))$ twisting G to G', via the $\bar{k}$-isomorphism $m : (G'_{\bar{k}}, X'_{\bar{k}}) \longrightarrow (G_{\bar{k}}, X_{\bar{k}})$.

As G and G' are both quasi-split, let B and B' be their Borel subgroups over k. $m(B'_{\bar{k}})$ is a Borel subgroup in $G_{\bar{k}}$, thus conjugate to $B_{\bar{k}}$. So there is an inner automorphism of $G_{\bar{k}}$ (conjugating by some element $g \in G_{\bar{k}}$) post-composed to m by which induces a $\bar{k}$-isomorphism mapping $B'_{\bar{k}}$ to $B_{\bar{k}}$. However, twisting by such an inner automorphism gives a cocycle which is equivalent as before. Hence without loss of generality, $m(B'_{\bar{k}}) = B_{\bar{k}}$.

So $[\xi]$ is valued in $\mathcal{N}_{G_{\bar{k}}/\mathcal{Z}(G_{\bar{k}})}(B_{\bar{k}}/\mathcal{Z}(G_{\bar{k}})) = B_{\bar{k}}/\mathcal{Z}(G_{\bar{k}})$. Thus to show the uniqueness, it suffices to show that $\mathrm{H}^1(k, B_{\bar{k}}/\mathcal{Z}(G_{\bar{k}})) = 1$.

Let $U = \mathscr{R}_{\mathrm{u}}(B/\mathcal{Z}(G)) \simeq \mathscr{R}_{\mathrm{u}}(B)$ be the unipotent radical of B (also of $B/\mathcal{Z}(G)$) over k, and let $T = (B/\mathcal{Z}(G))/U$ which is a k-torus. U is k-split, and admits a sequence of normal subgroups U_i such that$U_0 = U$, $U_n = \{e\}$ for some n, and

$U_i/U_{i+1} \simeq \mathbb{G}_{\mathrm{a}}$. Thus $\mathrm{H}^1(k, U_{\bar{k}})$ is trivial ([Ser97, Chapter III, Proposition 6]), and it is sufficient to show $\mathrm{H}^1(k, T)$ is trivial.

The quotient torus T is $\mathrm{Res}_{R/k}\mathbb{G}_{\mathrm{m}}$, the restriction of scalar (a.k.a. Weil restriction) of $(\mathbb{G}_{\mathrm{m}})_{/R}$ to k for a finite étale k-algebra R. Then by Hilbert's theorem 90, $\mathrm{H}^1(k, \mathrm{Res}_{R/k}\mathbb{G}_{\mathrm{m}}) = \{1\}$, which induces the uniqueness of the choice of k-form $(G^{[s]}, X^{[s]})$ for $[s] \in \mathrm{H}^1(\mathrm{Aut}(k, \Lambda_{(G_{\bar{k}}, X_{\bar{k}})}))$ where $G^{[s]}$ is quasi-split. $\square$

$\mathfrak{A}^{\sharp}_{X_{\bar{k}}}$ admits a Γ-action inherited from that on $\mathrm{Aut}^{G_{\bar{k}}}(X_{\bar{k}})$, which makes it possible to define $\mathfrak{A}^{\sharp}_X$ over k. Moreover, the Γ-action on $\mathfrak{A}^{\sharp}_{X_{\bar{k}}}$ is compatible with that on $X_{\bar{k}}$, thus $\mathfrak{A}^{\sharp}_X$ acts on X.

Corollary 4.2.5. *Given a homogeneous spherical pair (G, X), the spherical closure $Z := X/\mathfrak{A}^{\sharp}_X$ of X can be defined over k, thus the k-forms (G', Z') of (G, Z) is given by Theorem 4.2.4.*

4.3 Examples

Let k be a field of characterisitc 0. In this section, three examples will be calculated. Given a split k-form (G, X), the other k-forms (G', X') with quasi-split G', up to k-isomorphisms, are assigned to each cocycle class in $\mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{G_{\bar{k}}, X_{\bar{k}}}))$, with Γ acts on $\mathrm{Aut}(\Lambda_{G_{\bar{k}}, X_{\bar{k}}})$ trivially (as G is split), and write this action as a left action. Some of the corresponding Luna data will be mentioned in the next chapter.

The Galois cohomology can be calculated by the following result on group cohomology.

Proposition 4.3.1. *Let Γ be a group, and A be a group with a trivial Γ-action, then*

$$\mathrm{H}^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A)/A\text{-}conj.$$

Particularly, if A is abelian, $\mathrm{H}^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A)$.

Proof. Let f be a 1-cocycle, then $f(\gamma_1 \circ \gamma_2) = f(\gamma_1) \cdot {}^{\gamma_1}(f(\gamma_2)) = f(\gamma_1) \cdot f(\gamma_2)$ as Γ acts on A trivially. Thus the set of 1-cocycles is $\mathrm{Hom}(\Gamma, A)$.

By definition, two cocycles f and g are equivalent if there is an element $a \in A$ such that $f(\gamma) = a^{-1} \cdot g(\gamma) \cdot ({}^{\gamma}a) = a^{-1} \cdot g(\gamma) \cdot a$. $\square$

4.3.1 The First Example

Let $G = \mathrm{SL}_2$, and $X = \mathbb{P}^1 \times \mathbb{P}^1 - (\mathbb{P}^1)^{\mathrm{diag}}$, G acts on X diagonally (on the two $\mathbb{P}^1$ components separately). The generic stabilizer is T, the split maximal torus.

The Luna datum corresponds to Case **a-A-1.**. The automorphism group is $\mathrm{Aut}(\Lambda) = \mathbb{Z}/2\mathbb{Z}$.

Proposition 4.3.2. *The first Galois cohomology* $\mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z})$ *with trivial Galois action on* $\mathbb{Z}/2\mathbb{Z}$ *is the pointed set*

$$\{k\} \cup \{E : \textit{quadratic extensions of } k\},$$

with the base point k.

Proof. By Proposition 4.3.1, a cocycle class is a homomorphism $f : \Gamma \longrightarrow \mathbb{Z}/2\mathbb{Z} = \{1, \xi\}$.

If the image of f is $\{1\}$, then $\ker(f) = \Gamma$, the fixed field of $\bar{k}$ by Γ is k. It is the base point since a trivial homomorphism induces the same action as the original one on $\mathbb{Z}/2\mathbb{Z}$.

Otherwise, f is surjective. So $\ker(f) \subseteq \Gamma$ is a subgroup of index 2, and any such subgroup has a fiexed field E of $\bar{k}$ which is a quadratic extension of k. $\square$

The base point $k \in \mathrm{H}^1(k, \mathrm{Aut}(\Lambda))$ corresponds to the split form of (G, X) where we started with.

For a quadratic extension E of k, denote the corresponding spherical pair over k by (G', X'), the induced Galois group acts through the automorphism group $\mathbb{Z}/2\mathbb{Z}$, which means that the pair (G', X') is split over E. Let the Galois group $\mathrm{Gal}(E/k) = \{1, \sigma\}$.

Since $\mathrm{Aut}(\Psi_{\mathrm{SL}_2})$ is trivial, there is only one quasi-split k-form, which is SL_2 itself, up to k-isomorphisms.

Since (G', X') is isomorphic to (G, X) over E, $X'_E \simeq X_E = \mathbb{P}^1_E \times \mathbb{P}^1_E - (\mathbb{P}^1_E)^{\mathrm{diag}}$, with $\mathrm{Gal}(E/k)$-action given by $\sigma.(x, y) = ((^\sigma x), (^\sigma y)).J = ((^\sigma y), (^\sigma x))$ where x, $y \in \mathbb{P}^1_E$, and $x \neq y$. The variety is $\mathrm{Res}_{E/k}(\mathbb{P}^1_E) - \mathbb{P}^1_k$.

The generic stabilizer H' is a maximal torus T'. Let $([x : 1], [^\sigma x : 1])$ be a point in X', with $x \in E$. Furthermore, we require $(^\sigma x) = -x$. The k-points of G' in the form of matrices are

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc = 1$, a, b, c, $d \in k$. Then $([x : 1], [-x : 1]).g = ([x : 1], [-x : 1])$ implies that

$$([ax + c : bx + d], [-ax + c : -bx + d]) = ([x : 1], [-x : 1]),$$

thus $\dfrac{ax + c}{bx + d} = x$, the other one $\dfrac{-ax + c}{-bx + d} = -x$ is directly its image under σ, so it is enough to conclude $bx^2 - (a - d)x - c = 0$. Since $x \notin k$, $b \neq 0$. So $a - d = b\mathrm{Tr}_{E/k}(x) = 0$ and $c = -b\mathcal{N}_{E/k}(x)$. With the condition $ad - bc = 1$, we have $a^2 + b^2\mathcal{N}_{E/k}(x) = 1$. Therefore, for point $([x : 1], [-x : 1]) \in X'$, its stabilizer H' is a (non-split) torus of the form

$$\left\{ \begin{pmatrix} a & b \\ -b\mathcal{N}_{E/k}(x) & a \end{pmatrix} : a^2 + b^2\mathcal{N}_{E/k}(x) = 1 \right\}.$$

4.3.2 Group as a Spherical Variety

Let G_0 be a split geometrically connected reductive algebraic group defined over k, let $G = G_0 \times G_0$, and $X = G_0/\mathcal{Z}(G_0)$ as a variety. The action of G on X, given a point $[x_0] \in X$ where $x_0 \in G_0$, satisfies that $[x_0].(g_1, g_2) = [g_1^{-1} x g_2]$, then $H = (G_0)^{\text{diag}} \times \mathcal{Z}(G_0)$.

In this example, only those group G_0 with trivial $\text{Aut}(\Psi_{G_0})$, thus $\text{Aut}(\Lambda) = \mathbb{Z}/2\mathbb{Z}$. So the first Galois cohomology is given by Proposition 4.3.2. Actually, those G_0 with extra symmetries will fail to have this group as $\text{Aut}(\Lambda)$. There are still such k-forms constructed below, but those will fail to be all of the possible k-forms in this case.

For the base point k in $\text{H}^1(k, \mathbb{Z}/2\mathbb{Z})$, the k-form corresponding to it is still the split form, (G, X) itself.

For the other points, let E be a quadratic extension of k, which is a non-base point in $\text{H}^1(k, \mathbb{Z}/2\mathbb{Z})$. Let $\text{Gal}(E/k) = \{1, \sigma\}$, then the (G', X') is split over E, and $\text{Gal}(E/k)$ acts on G'_E in the way that $\sigma.(g_1, g_2) = ({}^{\sigma}g_2, {}^{\sigma}g_1)$, (again, the Galois action on (G_E, X_E) defining (G, X) is denoted by $({}^{\sigma}x)$, that defining (G', X') is denoted by $\sigma.x$,) thus the k-form $G' = \text{Res}_{E/k}(G_{0,E})$. This is the only quasi-split k-form which is split over E. Because G_0 does not provide any quasi-split k-form other than itself ($\text{Aut}(\Psi_{G_0})$ is trivial).

Thus the action of $\text{Gal}(E/k)$ on X'_E which is compatible with that on G_E is, $\sigma.x = {}^{\sigma}x^{-1}$. This defines X' over k, whose k-points are $x \in P(G_0)$ satisfying $x = ({}^{\sigma}x^{-1})$.

For the generic stabilizer H', choose $[x]$ to be identity class $\mathcal{Z}(G_0)$ in X', an element $(g, {}^{\sigma}g) \in H'$ satisfies that $[x].(g, {}^{\sigma}g) = [g^{-1} x\,({}^{\sigma}g)]$, hence $g^{-1} \cdot ({}^{\sigma}g) \in \mathcal{Z}(G_{0,E})$, thus $H' = G_{0,k} \times \text{Res}_{E/k}(\mathcal{Z}(G_{0,E}))$.

Particularly, if G_0 is taken to be SL_2, then its corresponding Luna datum is shown in case **aa-A-1.**.

4.3.3 A Non-Abelian Automorphism Group

Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, and $H = (\mathrm{SL}_2)^{\mathrm{diag}}$. Then X is isomorphic to $\mathrm{SL}_2 \times \mathrm{SL}_2$ with G action given in the form $(x_1, x_2).(g_1, g_2, g_3) = (g_3^{-1} x_1 g_1, g_3^{-1} x_2 g_2)$.

This example corresponds to the case **a-A-3.**, the automorphism group is $\mathrm{Aut}(\Lambda) = S_3$.

Proposition 4.3.3. *The first Galois cohomology* $\mathrm{H}^1(k, S_3)$ *with trivial Galois action is the pointed set*

$$\{k\} \cup \{E : \textit{extensions of } k \textit{ that is quadratic, cubic, or with Galois group } S_3\}.$$

The base point is k.

Proof. Let Γ denote the Galois group, a homomorphism $f : \Gamma \longrightarrow S_3$ induces an isomorphism between $\Gamma/\ker(f) \longrightarrow \mathrm{Im}(f)$.

If the image of f is trivial, then $\ker(f) = \Gamma$, the fixed field of $\ker(f)$ is k, which is the base point in $\mathrm{H}^1(k, S_3)$.

The automorphism group of S_3 is isomorphic to the group of inner automorphisms of S_3.

For each homomorphism $f : \Gamma \longrightarrow S_3$, a homomorphism g is equivalent to f as a cocycle when $f = a^{-1}ga$ for some $a \in S_3$, acoording to the definition. So $\ker(f) = \ker(g)$. Conversely, for any f and g homomorphisms from Γ to S_3, with $\ker(f) = \ker(g)$, then $\mathrm{Im}(f)$ is isomorphic to $\mathrm{Im}(g)$, thus there is an automorphism of S_3 (thus it is an inner automorphism) mapping $\mathrm{Im}(f)$ to $\mathrm{Im}(g)$. Therefore, f and g are equivalent as cocycles in $\mathrm{H}^1(k, S_3)$.

Consider that the image $\mathrm{Im}(f)$ can only be of order 2, 3, and 6. Thus the corresponding kernels determine the field extensions of type quadratic, cubic and with Galois group S_3, respectively. $\square$

The form corresponding to k: In this case, $(G', X') = (G, X)$. This is the split form, given at the beginning of this example.

The form corresponding to a quadratic extension E: By the previous proposition, we may choose the cocycle f such that the image of f in S_3 is $\Gamma_2 = \{(1), (12)\}$, with $\mathrm{Gal}(E/k) = \{1, \sigma\}$, and $f(\sigma) = (12)$. The form (G', X') is split over E. Thus Γ_2 acts on G'_E by $\sigma.(g_1, g_2, g_3) = ({}^{\sigma}g_2\, , {}^{\sigma}g_1\, , {}^{\sigma}g_3)$. Then $G' = \mathrm{Res}_{E/k}(\mathrm{SL}_{2,E}) \times \mathrm{SL}_{2,k}$.

Let G'_E acts on X'_E by $(x_1, x_2).(g_1, g_2, g_3) = (g_3^{-1}x_1g_1, g_3^{-1}x_2g_2)$, then the Γ_2 action on X'_E is $\sigma.(x_1, x_2) = ({}^{\sigma}x_2\, , {}^{\sigma}x_1)$. So $X' = \mathrm{Res}_{E/k}(\mathrm{SL}_{2,E})$.

The generic stabilizer, therefore, is $H' = \mathrm{SL}_{2,k}$, embedded as $(1, g) \in G' = \mathrm{Res}_{E/k}(\mathrm{SL}_{2,E}) \times \mathrm{SL}_{2,k}$.

The form corresponding to a cubic extension E: Choose a cocycle f whose image is $\Gamma_3 = \{(1), (123), (132)\} \subseteq S_3$, with $\mathrm{Gal}(E/k) = \{1, \sigma, \sigma^2\}$, and $f(\sigma) = (123)$. (Actually, $f(\sigma) = (132)$ only gives another equivalent cocycle. So, without loss of generality, $f(\sigma) = (123)$ can be chosen.) The form (G', X') is split over E and quasi-split over k.

The group $G' = \mathrm{Res}_{E/k}(\mathrm{SL}_{2,E})$ determined by the Γ_3 action on G'_E, where in terms of a generator σ $\sigma.(g_1, g_2, g_3) = ({}^{\sigma}g_2\, , {}^{\sigma}x_3\, , {}^{\sigma}x_1)$.

X' is given by the following discussion. Since $[x_1, x_2, x_3] = (x_3^{-1}x_1, x_3^{-1}x_2)$, and $\sigma.[x_1, x_2, x_3] = [{}^{\sigma}x_2\, , {}^{\sigma}x_3\, , {}^{\sigma}x_1]$, so $\sigma.(x_1, x_2) = (({}^{\sigma}x_1^{-1}\, {}^{\sigma}x_2), (({}^{\sigma}x_1^{-1}))$, thus Galois stable condition is $x_2 = {}^{\sigma}x_1^{-1}$, and $x_1 = ({}^{\sigma}x_1^{-1})({}^{\sigma^2}x_1^{-1})$, which means $X' = \{x \in \mathrm{SL}_{2,E} : x({}^{\sigma}x_1)({}^{\sigma^2}x_1) = I\}$, where I is the identity element in $\mathrm{SL}_{2,E}$.

Choose $x = I$, let $(x, {}^{\sigma}x^{-1}) = (I, I)$, then the stabilizer H' is given by the condition that, for $(g, {}^{\sigma^2}g\, , {}^{\sigma}g)$, the following condition holds, $(({}^{\sigma}g^{-1}g), ({}^{\sigma}g^{-1}) \cdot ({}^{\sigma^2}g)) = (I, I)$, thus $g = {}^{\sigma}g$, which means $H' = \mathrm{SL}_{2,k}$.

The form corresponding to an extension E with Galois group S_3: Let f be a cocycle corresponding to E, with $\Gamma_6 = \mathrm{Gal}(E/k)$ generated by $\{a, b\}$, and without loss of generality, we may let $f(a) = (12)$, $f(b) = (13)$.

The action of Γ on G'_E can be given by $a.(g_1, g_2, g_3) = ((^a g_2), (^a g_1), (^a g_3))$ and $b.(g_1, g_2, g_3) = ((^b g_3), (^b g_2), (^b g_1))$. Thus we have the condition $g_2 = {^a g_1}$, $g_3 = {^b g_1}$, and $g_3 = {^a g_3}$, $g_2 = {^a g_2}$.

Let $g_3 \in \mathrm{SL}_2(E)$, such that $(^a g_3) = g_3$, let $g_1 = (^b g_3)$, and $g_2 = (^a g_1) = (^{ab} g_3)$, then $(^b g_2) = (^{bab} g_3) = (^{baba} g_3) = (^{ab} g_3) = g_2$. So denote the fixed field of $\{1, a\}$ by E_a, $G'(k) \simeq \mathrm{SL}_2(E_a)$, and $G' = \mathrm{Res}_{E/k}(\mathrm{SL}_{2,E_a})$. And choosing any other subfield of E of index 2 will just produce a k-isomorphic group.

For the variety X', the Galois group Γ_6 acts on X'_E by $a.(x_1, x_2) = ((^a x_2), (^a x_1))$, and $b.(x_1, x_2) = ((^b x_1^{-1}), (^b x_1^{-1} \cdot^b x_2))$. The second identity can be obtained by choosing a representative in $X'_E = H'_E \backslash G'_E$, let $[y_1 : y_2 : y_3]$ represent $(x_1, x_2) = (y_3^{-1} y_1, y_3^{-1} y_2)$, then $b.[y_1 : y_2 : y_3] = [(^b y_3) : (^b y_2) : (^b y_1)] = ((^b y_1^{-1} \cdot^b y_3), (^b y_1^{-1} \cdot^b y_2)) = ((^b x_1^{-1}), (^b x_1^{-1} \cdot^b x_2))$. These two conditions imply the following conditions characterizing X': $(x_1, x_2) \in X'_E \simeq \mathrm{SL}_{2,E} \times \mathrm{SL}_{2,E}$ such that $x_2 = (^a x_1)$, and $x_1 \cdot (^b x_1) = I$, $x_1 \cdot (^{ba} x_1) \cdot (^{baba} x_1) = I$. (Recall that $\{1, ba, baba\}$ is the index-2 subgroup of Γ_6.)

At last, let H' be the stabilizer of $(I, I) \in X'$ in G'. Then let $((^b g), (^{ab} g), g) \in H'$, then it satisfies that $(g^{-1}(^b g), g^{-1}(^b g)) = (I, I)$, thus $g = (^b g) = (^a g)$. So $H' = \mathrm{SL}_{2,k}$, embedded in $G' = \mathrm{Res}_{E_a/k}(\mathrm{SL}_2)$ as a subgroup.

Chapter 5

Spherical Systems with the Transitivity Condition

This chapter is devoted to a full classification of spherical systems whose group of automorphisms acts transitively on the set of spherical roots.

5.1 Motivation

From Theorem 4.2.4, for a given spherically closed spherical pair (G, X) over k, with G quasi-split, the forms (G', X') of (G, X) with quasi-split group G' can be obtained from studying $\mathrm{H}^1(k, \mathrm{Aut}(\Lambda_{(G,X)}))$.

In this application, only a baby model is considered.

Definition 5.1.1. Let k be a field of characteristic 0 with absolute Galois group Γ. Given a spherical pair (G, X) over k, the k-rank of (G, X) is the rank of $(\Xi_X)^\Gamma$ as a $\mathbb{Z}$-module.

Proposition 5.1.2. *Let (G, X) be a spherically closed spherical pair defined over k, then the k-rank of (G, X) is the number of Γ-orbits in Σ_X.*

Consider that for spherically closed (G, X), $\Xi_X = \langle \Sigma_X \rangle_{\mathbb{Z}}$, thus the sum of the spherical roots in an Γ-orbit lies in $(\Xi_X)^\Gamma$. Conversely, any Γ-invariant element in Ξ_X corresponds to finitely many Γ-orbits in Σ_X written in terms of a linear combination of spherical roots.

The Galois group Γ acts on Λ through $\mathrm{Aut}(\Lambda)$, then all the spherically closed spherical pairs of k-rank 1 have Luna data with the following property:

- The automorphism group $\mathrm{Aut}(\Lambda)$ acts transitively on the set Σ of spherical roots.

This property is called the **transitivity property**.

5.2 More on Spherical Systems

As defined in Definition 3.3.1, a spherical system is a Luna datum with $\Xi = \langle \Sigma \rangle_{\mathbb{Z}}$, thus in this chapter, a spherical system $\mathscr{S}$ is said to consist of Ψ and $(S^p, \Sigma, \mathscr{A}, \rho)$.

5.2.1 Properties

Let Ψ be a root datum, and S be the set of positive simple roots.

Definition 5.2.1. A spherical system $\mathscr{S}$ associated to (Ψ, S) is called of **adjoint type** if for every spherical root $\sigma \in \Sigma$, written in the form $\sigma = \sum_{\alpha \in S} n_\alpha \alpha$, the coefficients n_α are all integral. Equivalently, this is to say $\Sigma \subseteq \Sigma_{\mathrm{ad}}(S)$ (see Definition 2.2.1).

Definition 5.2.2. Let $\mathscr{S}$ be a spherical system associated to (Ψ, S), and let $\sigma \in \Sigma$ be a spherical root. σ can be written as a linear combination of positive simple roots, $\sigma = \sum_{\alpha \in S} n_\alpha \alpha$, then the **support** of σ is the set of α with $n_\alpha \neq 0$, denoted by $\mathrm{supp}(\alpha)$. Let $\Sigma' \subseteq \Sigma$ be a set of spherical roots, then the support of Σ' is $\mathrm{supp}(\Sigma') = \bigcup_{\sigma \in \Sigma'} \mathrm{supp}(\sigma)$. A spherical Φ-system $\mathscr{S}$ is called **cuspidal** if the $\mathrm{supp}(\Sigma) = S$.

Definition 5.2.3. Let $\mathscr{S} = (S^p, \Sigma, \mathscr{A}, \rho)$ be a spherical system associated to (Ψ, S), let $S' \subseteq S$ be a set of positive simple roots, and Ψ' be a sub-root datum of Ψ, and Ψ' contains S' as the set of positive simple roots, the **localization** of $\mathscr{S}$ to S' is a spherical system $\mathscr{S}' = ((S^p)', \Sigma', \mathscr{A}', \rho')$ where $(S^p)' = S^p \cap S'$, $\Sigma' = \{\sigma \in \Sigma : \mathrm{supp}(\sigma) \subseteq S'\}$, $\mathscr{A}' = \bigcup_{\alpha \in \Sigma' \cap S'} \Delta(\alpha)$, and ρ' is the restriction of ρ on $\mathscr{A}'$.

Definition 5.2.4. Let $\mathscr{S} = (S^p, \Sigma, \mathscr{A}, \rho)$ be a spherical system associated to (Ψ, S), let (Ψ', S') be a (based) root datum containing (Ψ, S), then the **induction** of $\mathscr{S}$ to (Ψ', S') is the spherical system $\mathscr{S}' = ((S^p)', \Sigma', \mathscr{A}', \rho')$ associated to (Ψ', S'), where $\mathscr{A}' = \mathscr{A}$, $\rho' = \rho$, and $(S^p)' = S^p$, $\Sigma' = \Sigma$ are just the same set as before but considered as subsets of S and $\mathbb{N}S$, respectively.

Definition 5.2.5. Let $\mathscr{S}$ be a spherical Ψ-system, if either the Dynkin diagram of Ψ is connected, or for each pair of distinct connected components of the Dynkin diagram with positive simple roots S_1 and S_2, there is a color $D \in \Delta(\alpha_1) \cap \Delta(\alpha_2)$, where $\alpha_1 \in S_1$ and $\alpha_2 \in S_2$.

Definition 5.2.6. A spherical system $\mathscr{S}$ is called **prime** if it is cuspidal, connected, and of adjoint type.

5.2.2 Luna Diagrams

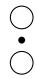
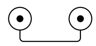
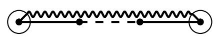
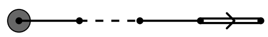
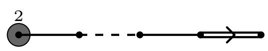
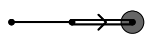
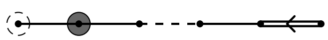
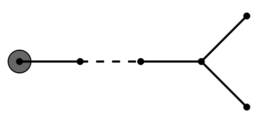
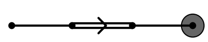
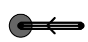
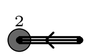
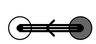
A Luna diagram is a visualization of a spherical system (thus, there is a one-to-one correpsondence between Luna diagrams and spherical systems), which is the Dynkin diagram of the corresponding root datum together with some decorations. The decorations contains the information of colors, thus uniquely determine the set of spherical roots and the Cartan pairing.

The following table (Table 5.1) provides a list of spherical roots of adjoint type in Luna diagrams together with their supports (hence on only part of the underlying Dynkin diagram).

In Luna diagrams, circles and shaded circles are attached to each prime simple root (black vertices in the base Dynkin diagram), representing the colors lying in $\Delta(\alpha)$ for each root α. Colors are in 3 "genres" based on its PGL_2-model. For each color $D \in \Delta(\alpha)$ as a prime divisor of X, let $P_\alpha \supseteq B$ be the parabolic subgroup corresponding to α, and N_α the radical of P_α, then $\mathring{X}_B P_\alpha / N_\alpha$ is a homogeneous spherical $\tilde{G} = \mathrm{PGL}_2$-variety, thus one of the following four cases: $T\backslash\tilde{G}$ with maximal torus T, $\mathcal{N}(T)\backslash\tilde{G}$, $U\backslash\tilde{G}$ with a unipotent group U and $\tilde{G}\backslash\tilde{G}$. In the 4 cases, the positive simple roots and the corresponding colors are called of genre T, N, U, and G, respectively (also called genre a, $2a$, b and p). In Luna diagrams, colors of genre U are drawn surrounding the root, centered at the vertex representing the root. Colors of genre N are drawn below the root. And colors of genre T are drawn above and below the root, where the above one representing D^+_α satisfying $\rho(D^+_\alpha)(\sigma) \in \{-1, 0, 1\}$ for every spherical root σ. And an angle sign ($<$ or $>$) is attached to D^+_α for each spherical root σ not orthogonal to α if $\rho(D^+_\alpha)(\sigma) = -1$. Some colors may belong to $\Delta(\alpha)$ for more than one root α, thus there is one circle drawn for each root, and all these circles representing the same color are connected by solid lines. Other decorations such as numbers above a color or wavy lines connecting roots are used to denote different types of spherical roots.

The set S^p is the set of all positive simple roots (vertices in the graph) which has no color (shaded or not shaded circles) attached above, below or surrounding it. And the set $\mathscr{A}$ is the set of all colors attached to the positive simple roots to which 2 colors are attached. Those circles connected by lines are considered as one color. So in the following detailed discussions, S^p, $\mathscr{A}$ and ρ will be mentioned only if necessary.

Table 5.1: Spherical Roots of Adjoint Type in Luna Diagrams

Type	Luna Diagram	Spherical Root
a		α
$2a$		2α
aa		$\alpha + \alpha'$
$a(n)$		$\alpha_1 + \alpha_2 + \cdots + \alpha_n$, $n \geq 2$
$b(n)$		$\alpha_1 + \alpha_2 + \cdots + \alpha_n$, $n \geq 2$
$2b(n)$		$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$, $n \geq 2$
b		$\alpha_1 + 2\alpha_2 + 3\alpha_3$
$c(n)$[1]		$\alpha_1 + (\sum_{i=2}^{n-1} 2\alpha_i) + \alpha_n$, $n \geq 3$
$d(n)$		$(\sum_{i=1}^{n-2} 2\alpha_i) + \alpha_{n-1} + \alpha_n$, $n \geq 3$
f		$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
g		$2\alpha_1 + \alpha_2$
$2g$		$4\alpha_1 + 2\alpha_2$
g'		$\alpha_1 + \alpha_2$

[1] The dashed circle around α_1 means that there can be a genre U color attached to α_1, making it the same spherical root but of a different spherical system.

5.3 Properties of Transitivity

The transitive action of $\mathrm{Aut}(\mathscr{S})$ suggests some properties, which helps in the classification.

Lemma 5.3.1. *Let $\mathscr{S}$ be a spherical Φ-system with $\mathrm{Aut}(\mathscr{S})$ acting on Σ transitively, then all the spherical roots $\sigma \in \Sigma$ belong to the same type in Table 5.1.*

Under an automorphism of the spherical system, a spherical root is mapped to a spherical root of the same type.

Consequently, the classification can be divided into several different cases based on the types of spherical roots.

Lemma 5.3.2. *Let $\mathscr{S}$ be a spherical Φ-system, there is a group homomorphism*

$$\eta : \mathrm{Aut}(\mathscr{S}) \longrightarrow \mathrm{Aut}(\Phi)$$

with $\ker(\eta) \simeq \prod_{\alpha \in \Sigma \cap S} \mathrm{Aut}(\Delta(\alpha))$*, where an automorphism of $\Delta(\alpha)$ respects the Cartan pairing.*

Proof. By the definition of automorphisms of a spherical system, the group $\mathrm{Aut}(\mathscr{S})$ acts on Φ, and is compatible with Σ and Δ, that is, $\forall \xi \in \mathrm{Aut}(\mathscr{S})$, and any spherical root a spherical root $\sigma = \sum_{i=1}^{n} c_i\alpha_i$, ξ acts linearly, i.e., $\xi(\sigma) = \sum_{i=1}^{n} c_i\xi(\alpha_i)$ and $\xi(D) \in \Delta(\xi(\alpha))$ for any positive simple root α and any color $D \in \Delta(\alpha)$. Hence $\ker(\eta)$ acts on S^p and Σ trivially, and acts on $\mathscr{A}$ through $\prod_{\alpha \in \Sigma \cap S} \mathrm{Aut}(\Delta(\alpha))$. Consider that for each $\xi_A \in \prod_{\alpha \in \Sigma \cap S} \mathrm{Aut}(\Delta(\alpha))$, there is a $\xi \in \mathrm{Aut}(\mathscr{S})$ where ξ acts on Φ trivially and on $\mathscr{A}$ by ξ_A, then $\ker(\eta) \simeq \prod_{\alpha \in \Sigma \cap S} \mathrm{Aut}(\Delta(\alpha))$. $\square$

Proposition 5.3.3. *The kernel of η acts on Σ trivially.*

The kernel $\ker(\eta)$ fixes each positive simple root, so it also fixes each spherical root.

Corollary 5.3.4. *Let $\mathscr{S}$ be a spherical Φ-system. If $\forall \alpha \in S$, $\#\Delta(\alpha) < 2$, then the morphism η is injective.*

In this case, $\ker(\eta)$ is trivial.

5.4 Reductions

5.4.1 Adjoint Type

From the classification of spherical varieties of rank 1 (see [Akh83]), outside the set of the spherical roots of adjoint type, the spherical roots can only be half of the spherical roots of type aa, b, and $d(n)$, $n \geq 3$ from Table 5.1. Concretely, they are of the form $\frac{1}{2}\alpha + \frac{1}{2}\alpha'$, where α is orthogonal to α' of the same length, $\frac{1}{2}\alpha_1 + \alpha_2 + \frac{3}{2}\alpha_3$, supported in a root datum of type B, and $(\sum_{i=1}^{n-2} \alpha_i) + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n$, for $n \geq 3$, supported in a root system of type D.

Thus based on Lemma 5.3.1, for a spherical Φ-system $\mathscr{S}$ of non-adjoint type with $\mathrm{Aut}(\mathscr{S})$ acting transitively on Σ, Σ contains only one type of spherical roots. Define $\mathscr{S}' = (S^p, \Sigma', \mathscr{A})$ by assigning $\Sigma' = \{2\sigma : \sigma \in \Sigma\}$. $\mathscr{S}'$ is a spherical Φ-system of adjoint type, and $\mathrm{Aut}(\mathscr{S}') = \mathrm{Aut}(\mathscr{S})$ acts transitively on Σ'.

This construction also helps to find general cases from the cases of adjoint type.

5.4.2 Cuspidality

For an arbitrary spherical Φ-spherical system $\mathscr{S}$, let $S' = \mathrm{supp}(\Sigma)$, and let $\mathscr{S}'$ be the localization of $\mathscr{S}$ to S'. From the construction, they have the same set of spherical roots, $\Sigma' = \Sigma$. If $\mathrm{Aut}(\mathscr{S})$ acts transitively on Σ, then $\mathrm{Aut}(\mathscr{S}')$ acts transitively on Σ'. This is because an automorphism of $\mathscr{S}$ preserves $\mathrm{supp}(\Sigma)$ and induces an automorphism of $\mathscr{S}'$, that is, $\mathrm{Aut}(\mathscr{S})$ is a subgroup of $\mathrm{Aut}(\mathscr{S}')$.

Conversely, for a cuspidal spherical Φ-system $\mathscr{S}$ with $\mathrm{Aut}(\mathscr{S})$ acting transitively on Σ, any induction $\mathscr{S}'$ of $\mathscr{S}$ to a root system Φ' remains to have $\mathrm{Aut}(\mathscr{S}')$ acting transitively on $\Sigma' = \Sigma$ if $\mathrm{Aut}^{\Sigma}(\Phi') = \{m \in \mathrm{Aut}(\Phi') : m \text{ preserves } \Sigma\}$ acts transitively on Σ.

5.4.3 Connectedness

Let the spherical systems mentioned in this section be cuspidal.

Let $\mathscr{S}$ be a spherical system with $\mathrm{Aut}(\mathscr{S})$ acting transitively on Σ. Let $\mathscr{S}_i$ be a connected component of $\mathscr{S}$, then $\mathscr{S}_i$ with $\mathrm{Aut}(\mathscr{S}_i) = \mathrm{Stab}_{\mathrm{Aut}(\mathscr{S})}(\mathscr{S}_i)$-action is a connected spherical system. By transitivity, any two such components $\mathscr{S}_i$ and $\mathscr{S}_j$ are conjugate by a group element $\gamma \in \Gamma$. And Γ_i acts transitively on Σ_i, the set of spherical roots in $\mathscr{S}_i$.

Thus any spherical system $\mathscr{S}$ with $\mathrm{Aut}(\mathscr{S})$ acting transitively on Σ consists of finitely many copies of a connected spherical system having the same property. And $\mathrm{Aut}(\mathscr{S}) = \mathrm{Aut}(\mathscr{S}_i) \wr S_n$, the wreath product of $\mathrm{Aut}(\mathscr{S}_i)$ by S_n, where $\mathscr{S}_i$ is a connected component of $\mathscr{S}$, and n denotes the number of connected components in $\mathscr{S}$.

5.5 Prime Cases

Thanks to the previous reductions, to finish the classification, it is sufficient to provide the full list of prime spherical systems with automorphism group acting transitively on set of spherical roots.

Theorem 5.5.1. *A prime spherical system $\mathscr{S}$ of adjoint type with the transitivity condition is either a spherical system of rank 1, or one of the spherical systems listed in Table 5.2. The table also includes the geometric realizations over an algebraically* *closed field* Ω *with* $\mathrm{char}(\Omega) = 0$.

The geometric realizations are from [Akh83, Was96, Lun01, BP05, Bra07], the generic stabilizers are $H = \mathcal{Z}(G)H^\flat$, *with* $H^\flat$ *listed in the tables.*

Table 5.2: Spherical Systems with Nontrivial Transitive Action on Spherical Roots by Automorphisms

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
a-A-2.	$(\mathsf{A}_1)^n$ $n \geq 2$	$S^p = \emptyset$, $\Sigma = \{\alpha_1, \dots, \alpha_n\}$, $\mathscr{A} = \{D^+, D_1^-, \dots, D_n^-\}$.	S_n	$(\mathrm{SL}_2)^n$	$\prod_{i=1}^{n} \begin{pmatrix} a & x_i \\ & a^{-1} \end{pmatrix}$ *with* $\sum_{i=1}^{n} x_i = 0$.
a-A-3.	$(\mathsf{A}_1)^3$	$S^p = \emptyset$, $\Sigma = \{\alpha_1, \alpha_2, \alpha_3\}$, $\mathscr{A} = \{D_{12}, D_{13}, D_{23}\}$	S_3	$(\mathrm{SL}_2)^3$	$(\mathrm{SL}_2)^{\mathrm{diag}}$
a-A-4.		*This case is given by the correspondence introduced in Proposition 5.5.12. The remark after it reveals the geometric realization.*			
2a-A-2.	A_2	$S^p = \emptyset$, $\Sigma = \{2\alpha_1, 2\alpha_2\}$, $\mathscr{A} = \emptyset$.	$\mathbb{Z}/2\mathbb{Z}$	SL_3	SO_3
aa-A-2.	$(\mathsf{A}_2)^2$	$S^p = \emptyset$, $\Sigma = \{\alpha_1 + \alpha_1', \alpha_2 + \alpha_2'\}$, $\mathscr{A} = \emptyset$.	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathrm{SL}_3)^2$	$(\mathrm{SL}_3)^{\mathrm{diag}}$

Table 5.2: Continued

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
a(n)-A-2.	A_{2n}	$S^p = \{\alpha_2, \dots \alpha_{n-1},$ $\alpha_{n+2}, \dots \alpha_{2n-1}\},$ $\Sigma = \{\sum_{i=1}^{n} \alpha_i, \sum_{i=n+1}^{2n} \alpha_i\},$ $\mathscr{A} = \emptyset.$	$\mathbb{Z}/2\mathbb{Z}$	SL_{2n+1}	$\begin{pmatrix} c_1A_1 & 0 & 0 \\ 0 & * & 0 \\ * & 0 & c_2A_2 \end{pmatrix}$ $A_i \in \mathrm{SL}_n,\ c_i \in \mathbb{G}_\mathrm{m}$
a(n)-A-3.	A_3	$S^p = \emptyset,$ $\Sigma = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3\},$ $\mathscr{A} = \emptyset.$	$\mathbb{Z}/2\mathbb{Z}$	SL_4	$\begin{pmatrix} c & 0 & 0 \\ M_1 & A & 0 \\ * & M_2 & c^{-1} \end{pmatrix}$ $A \in \mathrm{SL}_2,\ c_i \in \mathbb{G}_\mathrm{m}$ $M_1 + M_2^t = 0$
a(n)-D-1.	D_n	$S^p = \{\alpha_2, \dots, \alpha_{n-2}\},$ $\Sigma = \{\sum_{i=1}^{n-1} \alpha_i, \alpha_n + \sum_{i=1}^{n-2} \alpha_i\},$ $\mathscr{A} = \emptyset.$	$\mathbb{Z}/2\mathbb{Z}$	Spin_{2n}	$\begin{pmatrix} cA & 0 & 0 & 0 \\ M & 1 & 0 & 0 \\ -M & 0 & 1 & 0 \\ * & * & * & * \end{pmatrix}$ *for* $A \in \mathrm{SL}_{n-1},\ c_i \in \mathbb{G}_\mathrm{m}$

Table 5.2: Continued

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
a(n)-D-2.	D_4	$S^p = \emptyset$, $\Sigma = \{\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_2+\alpha_4\}$, $\mathscr{A} = \emptyset$.	S_3	Spin_8	$\begin{pmatrix} c^2 & 0 & 0 & 0 & 0 & 0 \\ M_1 & cA & 0 & 0 & 0 & 0 \\ * & M_2 & 1 & 0 & 0 & 0 \\ * & M_3 & 0 & * & 0 & 0 \\ * & * & * & * & * & 0 \\ 0 & * & * & * & * & * \end{pmatrix}$ *for* $A \in \mathrm{Sp}_2$, $c \in \mathbb{G}_\mathrm{m}$, $M_1^t + M_2 + M_3 = 0$.
a(n)-D-3.	D_4	$S^p = \{\alpha_2\}$, $\Sigma = \{\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_4$, $\alpha_3+\alpha_2+\alpha_4\}$, $\mathscr{A} = \emptyset$.	S_3	Spin_8	G_2

Table 5.2: Continued

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
d(n)-E.	E_6	$S^p = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $\Sigma = \{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ $+\alpha_5,\ 2\alpha_6 + 2\alpha_5$ $+2\alpha_3 + \alpha_2 + \alpha_4\}$, $\mathscr{A} = \emptyset$.	$\mathbb{Z}/2\mathbb{Z}$	E_6	F_4
d(3)-A-2.	A_5	$S^p = \{\alpha_1, \alpha_3, \alpha_5\}$, $\Sigma = \{\alpha_1 + 2\alpha_2 + \alpha_3$, $\alpha_3 + 2\alpha_4 + \alpha_5\}$, $\mathscr{A} = \emptyset$.	$\mathbb{Z}/2\mathbb{Z}$	SL_6	Sp_6

Remark. The following table (Table 5.3) provides the automorphism groups of the spherical systems of rank 1.

Table 5.3: Spherical Systems of Rank 1

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
Rank 1 *with Nontrivial Automorphism Group*					
a-A-1.	A_1	$S^p = \emptyset$, $\Sigma = \{\alpha\}$, $\mathscr{A} = \{D^+, D^-\}$.	$\mathbb{Z}/2\mathbb{Z}$	SL_2	T *(Maximal Torus)*
aa-A-1.	$(\mathsf{A}_1)^2$	$S^p = \emptyset$, $\Sigma = \{\alpha_1 + \alpha_2\}$, $\mathscr{A} = \emptyset$.	$\mathbb{Z}/2\mathbb{Z}$	$(\mathrm{SL}_2)^2$	SL_2
a(n)-A-1.	A_n	$S^p = \{\alpha_2, \dots \alpha_{n-1}\}$, $\Sigma = \{\sum_{i=1}^{n} \alpha_i\}$, $\mathscr{A} = \emptyset$.	$\mathbb{Z}/2\mathbb{Z}$	SL_{n+1}	GL_n
d(n)-D.	D_n	$S^p = \{\alpha_2, \dots, \alpha_n\}$, $\Sigma = \{(\sum_{i=1}^{n-2} 2\alpha_i) + \alpha_{n-1} + \alpha_n\}$, $\mathscr{A} = \emptyset$.	$\mathbb{Z}/2\mathbb{Z}$	Spin_{2n}	Spin_{2n-1}

Table 5.3: Continued

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
d(3)-A-1.	A_3	$S^p = \{\alpha_1, \alpha_3\}$, $\Sigma = \{\alpha_1 + 2\alpha_2 + \alpha_3\}$, $\mathscr{A} = \emptyset$.	$\mathbb{Z}/2\mathbb{Z}$	SL_4	Sp_4
Rank 1 *with Trivial Automorphism Group*					
2a-A-1.	A_1	$S^p = \emptyset$, $\Sigma = \{2\alpha_1\}$, $\mathscr{A} = \emptyset$.	1	SL_2	$\mathcal{N}(\mathrm{T})$
b-B.	B_n	$S^p = \{\alpha_1, \alpha_2\}$, $\Sigma = \{\alpha_1 + 2\alpha_2 + 3\alpha_3\}$, $\mathscr{A} = \emptyset$.	1	Spin_7	G_2
b(n)-B.	B_n	$S^p = \{\alpha_2, \ldots, \alpha_n\}$, $\Sigma = \{\sum_{i=1}^n \alpha_i\}$, $\mathscr{A} = \emptyset$.	1	Spin_{2n+1}	Spin_{2n}

Table 5.3: Continued

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
2b(n)-B.	B_n	$S^p = \{\alpha_2, \dots, \alpha_n\}$, $\Sigma = \{\sum_{i=1}^{n} \alpha_i\}$, $\mathscr{A} = \emptyset$.	1	Spin_{2n+1}	$\mathcal{N}(\mathrm{Spin}_{2n})$
c(n)-C-1.	C_n	$S^p = \{\alpha_1, \alpha_3, \dots, \alpha_n\}$, $\Sigma = \{\alpha_1 + (\sum_{i=2}^{n-1} 2\alpha_i) + \alpha_n\}$, $\mathscr{A} = \emptyset$.	1	Sp_{2n}	$\mathrm{SL}_2 \times \mathrm{Sp}_{2n-2}$
c(n)-C-2.	C_n	$S^p = \{\alpha_3, \dots, \alpha_n\}$, $\Sigma = \{\alpha_1 + (\sum_{i=2}^{n-1} 2\alpha_i) + \alpha_n\}$, $\mathscr{A} = \emptyset$.	1	Sp_{2n}	$B \times \mathrm{Sp}_{2n-2}$, *where* B *is the Borel subgroup of* SL_2.
f-F.	F_4	$S^p = \{\alpha_1, \alpha_2, \alpha_3\}$, $\Sigma = \{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\}$, $\mathscr{A} = \emptyset$.	1	F_4	Spin_9
g-G.	G_2	$S^p = \{\alpha_2\}$, $\Sigma = \{2\alpha_1 + \alpha_2\}$, $\mathscr{A} = \emptyset$.	1	G_2	SL_3

Table 5.3: Continued

Index	Φ	$\mathscr{S}$	$\mathrm{Aut}(\mathscr{S})$	G	$H^\flat$
2g-G.	G_2	$S^p = \{\alpha_2\}$, $\Sigma = \{4\alpha_1 + 2\alpha_2\}$, $\mathscr{A} = \emptyset$.	1	G_2	$\mathscr{N}(\mathrm{SL}_3)$
g'-G.	G_2	$S^p = \emptyset$, $\Sigma = \{\alpha_1 + \alpha_2\}$, $\mathscr{A} = \emptyset$.	1	G_2	$L = \mathbb{G}_{\mathrm{m}} \times \mathrm{SL}_2$ $\mathfrak{u} = K \oplus K^2$

By 5.3.1, there is only one type of spherical roots in such spherical varieties, thus a discussion on each possible type of spherical roots proves this theorem.

Lemma 5.5.2. *Let $\mathscr{S}$ be a prime spherical Φ-system with* $\mathrm{Aut}(\mathscr{S})$ *acting transitively on Σ, if σ is not of type a or aa, for any $\sigma \in \Sigma$, then the Φ has a connected Dynkin diagram.*

Proof. Suppose Φ has a Dynkin diagram with more than 1 connected components, then there is a color $D \in \Delta$, such that $D \in \Delta(\alpha_1) \cap \Delta(\alpha_2)$ for $\alpha_i \in S$ belonging to two different connected components in the Dynkin diagram. Recall that by the definition of spherical systems, this happens only in the case that either $\alpha_i \in \Sigma$, or $\alpha_1 \perp \alpha_2$ with $\alpha_1 + \alpha_2 \in \Sigma$.

Therefore, unless all spherical roots are of type a or aa, the underlying Dynkin diagram is connected. □

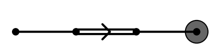
The cases of type a and aa will be discussed at the end.

5.5.1 Type f, g and 2g

These three spherical roots can only live on their corresponding Dynkin diagrams (F_4 or G_2). And by Lemma 5.5.2, the Dynkin diagram is connected. Then by Lemma 5.3.4, the automorphism group of each spherical system is trivial.

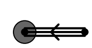
Hence the spherical systems $\mathscr{S}$ with $\mathrm{Aut}(\mathscr{S})$ acting transitively on Σ, that have spherical roots of types f, g or $2g$, are:

f-F. $\Sigma = \{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4\}$.



(5.1)

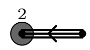
g-G. $\Sigma = \{2\alpha_1 + \alpha_2\}$.



(5.2)

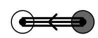
Remark. This spherical system is not spherically closed based on [BP14]. But it has a spherical closure shown below.

2g-G. $\Sigma = \{4\alpha_1 + 2\alpha_2\}$.



(5.3)

g'-G. $\Sigma = \{\alpha_1 + \alpha_2\}$.



(5.4)

And $\mathrm{Aut}(\mathscr{S}) = \{1\}$ for all the three cases above.

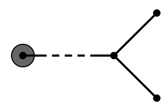
5.5.2 Type d(n), $n > 3$

The spherical root of type $\mathbf{d(n)}$ for $n > 3$ has a positive simple root α_{n-2} such that there are 3 positive simple roots (different from α_{n-2} itself) which are not orthogonal to it. Hence the underlying root system Φ is D_n with $n > 3$ or E_6, E_7, E_8.

Among these root systems, $\mathrm{Aut}(\mathsf{D}_4) \simeq S_3$, $\mathrm{Aut}(\mathsf{D}_n) \simeq \mathrm{Aut}(\mathsf{E}_6) \simeq \mathbb{Z}/2\mathbb{Z}$ for $n > 4$, and $\mathrm{Aut}(\Phi) = \{1\}$ for the rest of them.

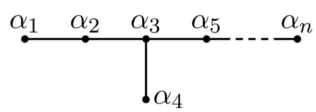
Over root system D_n ($n \geq 4$): There is only 1 spherical root of type $\mathbf{d(n)}$. Otherwise if there are two spherical roots σ_1 of type $\mathbf{d(n_1)}$ and σ_2 of type $\mathbf{d(n_2)}$, with $n_1 \geq n_2$, then both $\mathrm{supp}(\sigma_1)$ and $\mathrm{supp}(\sigma_2)$ contain $\{\alpha_{n-2}, \alpha_{n-3}, \alpha_{n-1}, \alpha_n\}$, where α_{n-3}, α_{n-1}, and α_n are the three positive simple roots non-orthogonal to α_{n-2}. Since $\mathscr{S}$ is cuspidal, $\mathrm{supp}(\sigma_1) = S$. Then $\mathrm{supp}(\sigma_1) \supseteq \mathrm{supp}(\sigma_2)$. Consider that one of the positive simple root $\alpha \in \mathrm{supp}(\sigma_2)$ has a nonempty $\Delta(\alpha)$, so $\alpha \notin S^p$. Then $S^p = S - \{\alpha_1, \alpha_k\}$ for some $1 < k \leq n$. This violates the axiom (S1), since all rank 1 wonderful varieties with spherical root σ of type $\mathbf{d(n)}$ has a S^p containing a corresponding α_k (it may have a different name but in the same position relatively to σ). Therefore, D_m admits no more than 1 spherical roots of type $\mathbf{d(n)}$.

d(n)-D. $\Phi = \mathsf{D}_n$, $n \geq 4$, with $\Sigma = \{2\sum_{i=1}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n\}$. $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$, where ξ is the automorphism of $\mathscr{S}$ induced by the automorphism of D_n which switches α_{n-1} and α_n, and fixes other positive simple roots. So ξ acts trivially on Σ. From now on, the ξ will be used to denote both the automorphism of Φ and the induced automorphism of $\mathscr{S}$.



(5.5)

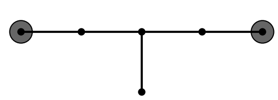
For the cases Φ is of type E: Future discussions will follow the labelling of E_n shown below:



(5.6)

Consider that the support of one spherical root of type $d(n)$ fails to cover $S(E_n)$ for $n = 6, 7, 8$, so there are at least two spherical roots of the same type $d(n)$. Again, for one of the spherical roots, to make the root α in its support that $\Delta(\alpha) \neq \emptyset$ out of the support of all other spherical roots, there can only be 2 spherical roots, with their colors lying at α_1 and α_n, the two positive simple roots lying at the end of two "long" legs of the Dynkin diagram. But it is only in E_6 that these two spherical roots are of the same type.

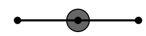
d(n)-E. $\Phi = \mathsf{E}_6$, $\Sigma = \{2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, 2\alpha_6 + 2\alpha_5 + 2\alpha_3 + \alpha_2 + \alpha_4\}$. The automorphism group $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$, where ξ exchanges α_1 with α_6, α_2 with α_5, and leaves the others fixed. So ξ swaps the two spherical roots, and $\mathrm{Aut}(\mathscr{S})$ acts on Σ transitively.



(5.7)

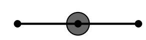
5.5.3 Type d(3)

Recall that type **d(3)** spherical roots have support of type $\mathsf{D}_3 \simeq \mathsf{A}_3$, with the corresponding Luna diagram:



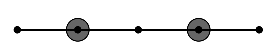
Over root systems of type A_n: here $n \geq 3$, and $\#\mathrm{Aut}(\mathsf{A}_n) = 2$, so there are at most 2 spherical roots of type **d(3)**.

d(3)-A-1. First, with only one spherical root of type **d(3)**, the only cuspidal spherical system is over root system A_3, with $\Sigma = \{\alpha_1 + 2\alpha_2 + \alpha_3\}$. Let ξ be the automorphism of A_3 swapping α_1 with α_3 and leaves α_2 fixed. It fixes Σ and is the only nontrivial automorphism of A_3. So $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$, acting on Σ trivially.



(5.8)

d(3)-A-2. Then the case with $\#\Sigma = 2$, the root system is $\Phi = \mathsf{A}_5$, and $\Sigma = \{\alpha_1 + 2\alpha_2 + \alpha_3, \alpha_3 + 2\alpha_4 + \alpha_5\}$. (It cannot be A_6 since in that case $\alpha_3 \in S^p$ fails to be orthogonal to $\alpha_4 + 2\alpha_5 + \alpha_6$, violates (S2).) $\Sigma = \{\alpha_{n+1} + 2\alpha_{n+2} + \alpha_{n+3}\}$. The nontrivial automorphism ξ of A_5 carrying α_i to α_{6-i} switches the two spherical roots, so $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$.



To build cuspidal spherical systems, all the positive simple roots should be of the same length, so the underlying root system cannot be B_n, C_n, F_4, or G_2.

For D_4, two or more spherical roots of type $d(3)$ canot live on D_4. Suppose there are two such roots, without loss of generality, let them be $\alpha_1+2\alpha_2+\alpha_3$ and $\alpha_1+2\alpha_2+\alpha_4$, then $\langle \alpha_4^\vee, \alpha_1 + 2\alpha_2 + \alpha_3 \rangle = -2 \neq 0$, which violates the Axiom (S2).

For D_n with $n > 4$ (D_3 is considered of type A), to obtain cuspidality, there are at least 2 spherical roots, and one of which is $\alpha_{n-2}+2\alpha_{n-1}+\alpha_n$. But the nontrivial automorphism of D_n fixes all the spherical roots, as it switches α_{n-1} with α_n and fixes all the others. So there are no spherical varieties of type D with spherical roots of type D_3, whose automorphism group acts transitively on the set of spherical roots.

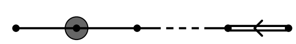
Finally, for root system E_n, based on the labelling given in Figure 5.6, in order to make the spherical system cuspidal, two of the spherical roots should be $\sigma_1 = \alpha_1+2\alpha_2+\alpha_3$, and σ_2 be either $\alpha_4+2\alpha_3+\alpha_2$ or $\alpha_4+2\alpha_3+\alpha_5$. However, in both of these two cases, there is a color associated to α_3, thus α_3 is not in S^p, which causes a contradiction to axiom (S1). So there are no type E cases.

5.5.4 Type c(n), b, b(n) and 2b(n)

This family is similar to the cases of type **f**, **g** and **2g**. The required lengths of positive simple roots make the underlying root systems just be their supports. And as $\mathrm{Aut}(\mathsf{B}_n) = \mathrm{Aut}(\mathsf{C}_n) = \{1\}$, the automorphism groups $\mathrm{Aut}(\mathscr{S}) = \{1\}$ for all of them.

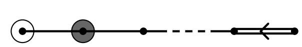
Here is the list:

c(n)-C-1. Here $n \geq 3$, and $\Sigma = \{\alpha_1 + (\sum_{i=2}^{n-1} 2\alpha_i) + \alpha_n\}$.



(5.9)

c(n)-C-2. Here $n \geq 3$, and $\Sigma = \{\alpha_1 + (\sum_{i=2}^{n-1} 2\alpha_i) + \alpha_n\}$.

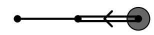


(5.10)

These two spherical systems are different since the latter one has $S^p_{c(n)\text{-}1} = \{\alpha_3, \alpha_4, \ldots, \alpha_n\}$

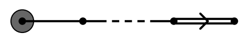
while the former one has $S^p_{c(n)\text{-}2} = \{\alpha_1, \alpha_3, \alpha_4, \ldots, \alpha_n\}$.

b-B. $\Sigma = \{\alpha_1 + 2\alpha_2 + 3\alpha_3\}$.



(5.11)

b(n)-B. Here $n \geq 2$, and $\Sigma = \{\sum_{i=1}^{n} \alpha_i\}$.



(5.12)

However, this is not spherically closed based on [BP14], with $\mathfrak{A}^\sharp \simeq \mathbb{Z}/2\mathbb{Z}$. And its spherical closure is the following one.

2b(n)-B. Here $n \geq 2$, and $\Sigma = \{\sum_{i=1}^{n} 2\alpha_i\}$.

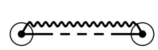
(5.13)

5.5.5 Type a(n)

The support of a spherical root of type **a(n)** is a set of positive simple roots with the same length, so it cannot survive on cuspidal spherical systems with B_n, C_n, F_4, or G_2 as underlying root system. So spherical systems over root systems A_n, D_n, and E_n will be studied.

Over root system A_n: Here $n \geq 2$, for A_1 does not admit any spherical root of type $a(n)$. Considering $\#\mathrm{Aut}(\mathsf{A}_n) = 2$, there are at most 2 spherical roots of type $a(n)$.

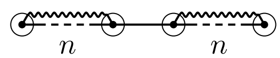
a(n)-A-1. $n \geq 2$. $\Sigma = \{\sum_{i=1}^{n} \alpha_i\}$. This is the only case with one spherical root of type $a(n)$. And $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$ with ξ induced by the nontrivial automorphism of A_n. $\mathrm{Aut}(\mathscr{S})$ acts trivially on Σ.



(5.14)

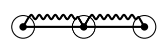
There are two different situations with 2 spherical roots.

a(n)-A-**2.** Here $n \geq 2$. $\Sigma = \{\sum_{i=1}^{n} \alpha_i, \sum_{i=n+1}^{2n} \alpha_i\}$. This is the cuspidal spherical system with 2 spherical roots of type $a(n)$ with maximal rank. $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$, where ξ acts on Φ as the one above, so it permutes the two spherical roots.



(5.15)

a(n)-A-**3.** Over $\Phi = \mathsf{A}_3$, there is one situation that the support of two spherical roots intersects. $\Sigma = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$. Similarly, $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$ where ξ acts on Φ by exchanging α_1 with α_3 and fixing α_2. So the action of $\mathrm{Aut}(\mathscr{S})$ on Σ by exchanging the two spherical roots.



(5.16)

There are no further cases over root systems of type A, according to the following lemma about the intersecting support of spherical roots of type **a(n)**.

Lemma 5.5.3. *Let σ_1, σ_2 be distinct spherical roots of type **a(n)**,* $\mathrm{supp}(\sigma_1) \cap \mathrm{supp}(\sigma_2)$ *is nonempty if and only if $n \geq 2$, and $\sigma_1 = \alpha_{i-n+2} + \alpha_{i-n+3} + \cdots + \alpha_i + \alpha_k$, $\sigma_2 = \alpha_{i-n+2} + \alpha_{i-n+3} + \cdots + \alpha_i + \alpha_l$ where α_k and α_l are different positive simple roots where $\langle \alpha_i^\vee, \alpha_k \rangle = \langle \alpha_i^\vee, \alpha_l \rangle = -1$.*

Proof. Let $I = \mathrm{supp}(\sigma_1) \cap \mathrm{supp}(\sigma_2)$, where σ_1, σ_2 are of type **a(n)** with the same $n \geq 2$ (for $n = 1$, nontrivial intersection of the support implies that the spherical roots are identical). According to Axiom $(S1)$, let $\alpha \in I$, if $\Delta(\alpha) = 1$, α is located in one of the two "ends" of $\mathrm{supp}(\sigma_i)$ for $i = 1, 2$; otherwise if $\Delta(\alpha) = 1$, α is not at the end in both supports.

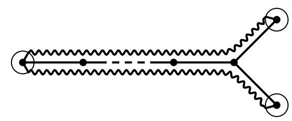
If the intersection I is of size 1, we show the only case is **1(n)**-A-**3.**. First, we show the only positive simple root $\alpha \in I$ satisfies $\Delta(\alpha) = 1$. Otherwise, α has no color corresponding to it, thus the 4 roots in $\mathrm{supp}(\sigma_1)$ and $\mathrm{supp}(\sigma_2)$ are distinct (the intersection I does not contain any of them). Thus α has 4 distinct positive simple roots not orthogonal to it except itself, which does not happen to root systems. And if $n > 2$, there is a nonempty S^p and then there is a root $\beta \in \mathrm{supp}(\sigma_1) \cap S^p$ which fails to be orthogonal to I, so β is not orthogonal to σ_2, hence violates axiom (S2). Therefore, only two spherical roots of type **a(2)** can have a intersection of size 1 on their supports. This is **a(n)-A-3.**.

If $\#I > 1$, the σ_1 and σ_2 share only one color. (If the other color is shared, the two spherical roots are identical. And if no color is shared, there should be 4 "ends", roots with only one positive simple root non-orthogonal to it except itself, in the Dynkin diagram.) And the intersection contains S^p. First, $I \cap S^p \neq \emptyset$ as I contains only one positive simple root with color and has at least one more element. Second, if S^p is not a subset of I, then there is a positive simple root $\alpha \in S^p \cap (\mathrm{supp}(\sigma_1) \backslash I)$ and not orthogonal to every element in I. Then $\alpha \notin \mathrm{supp}(\sigma_2)$, and $\langle \alpha^\vee, \sigma_2 \rangle < 0$, which violates axiom (S2). □

Therefore $\Phi = \mathsf{D}_{n+1}$. The only situation of type A is when $n = 2$, $\mathsf{D}_3 = \mathsf{A}_3$, as there are at most 2 spherical roots over root systems of type A.

Over root system D_n: Here $n \geq 4$. The $n < 4$ situations are considered as of type A. And one single spherical root of type $a(n)$ is not enough to cover $S(\mathsf{D}_n)$ since there are 3 "ends", but there are only 2 in support of σ of type $a(n)$. The first case is a spherical system with 2 spherical roots.

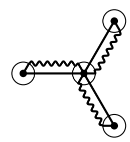
a(n)-D-1. $\Phi = \mathsf{D}_{n+1}$, and $\Sigma = \{\sum_{i=1}^{n-1} \alpha_i, \alpha_n + \sum_{i=1}^{n-2} \alpha_i\}$. Based on Lemma 5.5.3, this is the only case with two spherical roots. Here $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$, where ξ is induced by the automorphism of D_{n+1} swapping α_{n-1} with α_n and leaving other positive simple roots fixed. ξ acts on Σ by exchanging the two spherical roots.



$$(5.17)$$

If there are more than 2 spherical roots, the base root system is D_4, as $\#\mathrm{Aut}(\mathsf{D}_m) = 2$ for $m > 4$. The spherical roots can only be of type $a(2)$ or $a(3)$.

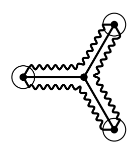
a(n)-D-2. $\Sigma = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\}$. And $\mathrm{Aut}(\mathscr{S}) \simeq \mathrm{Aut}(\mathsf{D}_4) \simeq \mathrm{S}_3$, it acts on Σ by permutation.



$$(5.18)$$

With spherical roots of type **a(2)**, this is the only cuspidal case, as there are only 3 edges in the Dynkin diagram, each can carry one spherical root of type **a(2)**.

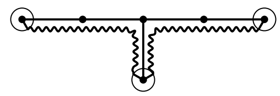
a(n)-D-3. $\Sigma = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4\}$. And $\mathrm{Aut}(\mathscr{S}) \simeq \mathrm{Aut}(\mathsf{D}_4) \simeq \mathrm{S}_3$, it acts on Σ by permutation.



$$(5.19)$$

With $\#\Sigma \geq 3$, this is the only one with spherical roots of type **a(3)**. And D_3 does not admit more than 3 spherical roots of type **a(3)**.

Over root system E_n: According to the same reason that E_n also has 3 ends on it, so there should be at least 2 spherical roots in the system. Considering $\#\mathrm{Aut}(\mathsf{E}_6) = 2$ and $\#\mathrm{Aut}(\mathsf{E}_n) = 1$ for others, Φ cannot be E_7 or E_8. Let $\Phi = \mathsf{E}_6$, to make the spherical system cuspital and to satisfy the transitivity condition, one may obtain the following diagram



NOT a spherical system

with "spherical roots" $\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\sigma_2 = \alpha_4 + \alpha_3 + \alpha_5 + \alpha_6$ (following the labelling in Graph 5.6). However, a simple check that $\langle \alpha_2^\vee, \sigma_2 \rangle = -1$ shows that the axiom (S2) fails to hold. Therefore, there are no such spherical systems over E_6.

5.5.6 Type 2a

Φ still has a connected Dynkin diagram. As $\#\mathrm{supp}(\sigma) = 1$ for σ of type **2a**, the cuspidality implies that $\Sigma = 2S = \{2\alpha : \alpha \in S\}$.

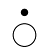
Lemma 5.5.4. *Let $\mathscr{S}$ be a connected cuspidal spherical Φ-system with all its spherical roots of type **a** (or **2a**) and a transitive $\mathrm{Aut}(\mathscr{S})$-action on Σ, then Φ is $(\mathsf{A}_1)^n$ or $(\mathsf{A}_2)^n$ for some $n \geq 1$.*

Proof. As $\mathrm{Aut}(\mathscr{S})$ acts transitively on $\Sigma = S$, (or $\Sigma = 2S$) all positive simple roots are of the same length. Hence Φ can be only of types A, D, or E. In each of those root systems, there is a positive simple root α "at the end", i.e., there is no more than one other positive simple root being non-orthogonal to it. Hence all the positive simple roots are at the end, so the connected root systems can only be A_1 or A_2. For the non-connected situations, together with the transitivity, Φ can only be $(\mathsf{A}_1)^n$ or $(\mathsf{A}_2)^n$ for some $n \geq 1$. $\square$

As each color belongs to $\Delta(\alpha)$ for only one α, the connected spherical system implies that the Dynkin diagram is connected. (See Lemma 5.5.2.) Hence $\Phi = \mathsf{A}_1$ or

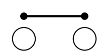
A_2.

2a-A-1. $\Phi = \mathsf{A}_1$, $\Sigma = \{2\alpha\}$. And $\mathrm{Aut}(\mathscr{S}) = \{1\}$.



(5.20)

2a-A-2. $\Phi = \mathsf{A}_2$, $\Sigma = \{2\alpha_1, 2\alpha_2\}$. And $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\} \simeq \mathrm{Aut}(S)$. ξ acts on Σ by exchanging the two spherical roots.



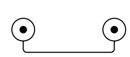
(5.21)

5.5.7 Type aa

Let $\sigma \in \Sigma$ be a spherical root of type aa, then $\sigma = \alpha + \alpha'$ where α and α' are of the same length. The condition that $\mathrm{Aut}(\mathscr{S})$ acts transitively on Σ implies that all the elements in S are of the same length, and each of them is an "end" of the root system (each has only one non-orthogonal root but itself). Therefore, Φ is $(\mathsf{A}_1)^n$ or $(\mathsf{A}_2)^n$.

If $\Phi = (\mathsf{A}_1)^n$, then $n = 2$, otherwise the spherical system is not connected.

aa-A-1. $\Phi = (\mathsf{A}_1)^2$. $\Sigma = \{\alpha + \alpha'\}$. $\mathrm{Aut}(\mathscr{S}) = \mathbb{Z}/2\mathbb{Z}$, the nontrivial element acts on S by exchanging α and α', then fixes the spherical root $\alpha + \alpha'$.

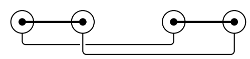


(5.22)

Moreover, let $\sigma_1 = \alpha_1 + \alpha_1'$ and $\sigma_2 = \alpha_2 + \alpha$, based on the axiom ($\Sigma 2$), $\langle \alpha_1^\vee, \sigma_2 \rangle = \langle (\alpha_1')^\vee, \sigma_2 \rangle = -1$, $\alpha = \alpha_2'$. Hence the only possible root system is $(\mathsf{A}_2)^2$, shown below.

If $\Phi = (\mathsf{A}_3)^n$, we show $n = 2$. Otherwise, if $n = 1$, the only two positive simple roots are not orthogonal, A_3 alone does not admit a type aa spherical root. If $n \geq 3$, choose spherical root $\alpha_1 + \alpha_1'$, let α_2 belongs to $\mathrm{supp}(\sigma_2)$, by Axiom ($\Sigma 2$), $\langle \alpha_1^\vee, \sigma_2 \rangle = \langle (\alpha_1)'^\vee, \sigma_2 \rangle = -1$, thus $\sigma_2 = \alpha_2 + \alpha_2'$, but in this case the spherical system fails to be connected for $n \geq 3$.

aa-A-2. $\Phi = (\mathsf{A}_2)^2$. $\Sigma = \{\alpha_1 + \alpha'_1, \alpha_2 + \alpha'_2\}$. And $\mathrm{Aut}(\mathscr{S}) = (\mathbb{Z}/2\mathbb{Z})^2$. two generators acts on S by exchanging α_i with α'_i for each i, and exchanging α_1 with α'_1, α_2 with α'_2, respectively.



(5.23)

5.5.8 Type a

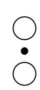
In this case, all the spherical roots are of the form $\alpha \in S$, i.e., $S = \Sigma$.

By Lemma 5.5.4, the underlying root system $\Phi = (\mathsf{A}_1)^n$ or $(\mathsf{A}_2)^n$ for some $n \geq 1$.

First, for $\Phi = (\mathsf{A}_1)^n$,

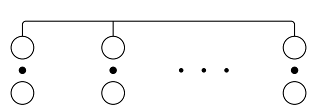
Proposition 5.5.5. *A connected cuspidal spherical* $(\mathsf{A}_1)^n$*-system* $\mathscr{S}$ *with transitive* $\mathrm{Aut}(\mathscr{S})$ *action on* Σ *belongs to one of the following classes:* **a-A-1.**, **a-A-2.**, *or* **a-A-3.**

a-A-1. $\Phi = \mathsf{A}_1$. $\Sigma = \{\alpha_1\}$, $\mathscr{A} = \{D_1^+, D_1^-\}$. With $\rho(D_1^+)(\alpha_1) = \rho(D_1^-)(\alpha_1) = 1$. $\mathrm{Aut}(\mathscr{S}) \simeq \mathbb{Z}/2\mathbb{Z}$. $\mathrm{Aut}(\mathscr{S}) = \{1, \xi\}$, where ξ swaps D_1^+ and D_1^-.



(5.24)

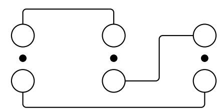
a-A-2. $\Phi = (\mathsf{A}_1)^n$, $n \geq 2$. $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$, $\mathscr{A} = \{D^+, D_1^-, D_2^-, \ldots, D_n^-\}$, where $\Delta(\alpha_i) = \{D^+, D_i^-\}$ for all $\alpha_i \in \Sigma$. Then $\rho(D^+)(\alpha_i) = 1$ for all $1 \leq i \leq n$, and $\rho(D_i^-)(\alpha_i) = 1$, $\rho(D_i^-)(\alpha_j) = -1$ for all $1 \leq i \leq n$ and $i \neq j$. $\mathrm{Aut}(\mathscr{S}) \simeq S_n$.



(5.25)

a-A-3. $\Phi = (\mathsf{A}_1)^3$. $\Sigma = \{\alpha_1, \alpha_2, \alpha_3\}$, $\mathscr{A} = \{D_{12}, D_{13}, D_{23}\}$, where $\Delta(\alpha_1) =$

$\{D_{12}, D_{13}\}$, $\Delta(\alpha_2) = \{D_{12}, D_{23}\}$, $\Delta(\alpha_3) = \{D_{13}, D_{23}\}$. $\mathrm{Aut}(\mathscr{S}) \simeq \mathbb{Z}/2\mathbb{Z}$.



(5.26)

Proof of Proposition 5.5.5. As in the cases where $\Phi = (\mathsf{A}_1)^n$, all the spherical roots are equivalent under the transitive action of $\mathrm{Aut}(\mathscr{S})$, without loss of generality, spherical root α_1 and $\Delta(\alpha_1)$ are chosen to be the starting point of the discussion (in fact, any other positive simple root can be chosen).

First, if no colors in $\Delta(\alpha_1)$ belong to $\Delta(\sigma)$ for any other spherical root σ, then the spherical system fails to be connected. Hence no spherical roots other than α_1 exist, so by cuspidality, $\Phi = \mathsf{A}_1$. This is the case **a-A-1.**

If there is only one color $D_1^+ \in \Delta(\alpha_1)$ belonging to some other $\Delta(\sigma)$, then it also belongs to $\Delta(\sigma')$ for all the other spherical roots σ', to make the spherical system connected. Denote this color by D^+, the Axiom (A1) implies that $\rho(D^+)(\sigma) = 1$ for any spherical root σ, and the valuations induced by other colors are determined by Axiom (A2). This is the case shown in **a-A-2.**

If both colors D_1^+ and D_1^- belong to some other set of colors corresponding to other spherical roots, it will not happen that $\Delta(\alpha_1) = \Delta(\alpha_2)$, as $D_1^+ = D_2^+$ implies that $\rho(D_1^+) = (1, 1, \dots) \in \Xi^\vee$, hence $\rho(D_1^-) = (1, -1, \dots)$ and $\rho(D_2^-) = (-1, 1, \dots)$, which means $D_1^- \neq D_2^-$.

So let $D_1^+ = D_2^+$ and $D_1^- = D_3^-$, then $D_2^- = D_3^+$ by the following discussion. By the assumption, $\rho(D_2^+)(\alpha_1) = 1$, by Axiom (A2), $\rho(D_2^-)(\alpha_1) = -1$. Similarly, $\rho(D_1^+)(\alpha_2) = 1$, then $\rho(D_1^-)(\alpha_2) = -1$. Recall that $D_1^- = D_3^-$, $\rho(D_3^+)(\alpha_2) = 1$. The valuation determined by D_2^+ can be calculated similarly. Let $\Xi_3 \subseteq \Xi$ be the sublattice generated by α_1, α_2 and α_3, then the restriction of $\rho(D)$ (D is any of the colors mentioned above) on $\Xi_3^\vee$ are:

- $\rho(D_1^+) = (1, 1, -1)$,
- $\rho(D_1^-) = (1, -1, 1)$,
- $\rho(D_2^-) = (-1, 1, 1)$,
- $\rho(D_3^+) = (-1, 1, 1)$.

By Axiom (A1), $D_3^+ \in \Delta(\alpha_2)$ and $D_2^- \in \Delta(\alpha_3)$, then it can only be $D_2^- = D_3^+$. It is easy to check that the triplet $(S^p, \Sigma, \mathscr{A})$ where $S^p = \emptyset$, $\Sigma = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathscr{A} = \{D_1^+ = D_2^+, D_1^- = D_3^-, D_2^- = D_3^+\}$ is a spherical system, which is the case **a-A-3.** listed above.

However, with this structure, no more spherical roots can be attached. Suppose there is one more spherical root α_4 of type **a** in the spherical system. By connectedness, there is an identification between a color in $\Delta(\alpha_4)$ and one of the three colors mentioned above. Without loss of generality, assume $D_4^+ = D_1^+ = D_2^+$, then $\rho(D_4^+) = (1, 1, -1, 1)$ restricted on $\Xi_4^\vee$ where $\Xi_4 \subseteq \Xi$ is the sublattice generated by α_1, α_2, α_3 and α_4, and $\rho(D_4^-) = (-1, -1, 1, 1)$ by Axiom (A2). This suggests that the color D_4^- is identified to a color in $\Delta(\alpha_3)$, but D_4^- is a color neither in $\Delta(\alpha_1)$ nor $\Delta(\alpha_2)$. But $D_3^+ \in \Delta(\alpha_2)$ and $D_3^- \in \Delta(\alpha_1)$, there is no choice for the identification of D_4^-. Hence **a-A-3.** is the only possible spherical system in this class. □

Then only the cases that $\Phi = (\mathsf{A}_2)^n$ are left to be investigated. Let the positive simple roots be $\alpha_{i,j}$ where $i \in \{1, 2, \ldots, n\}$ denotes the index of the A_2 component and $j \in \{1, 2\}$ denotes the index of the root in the A_2 component.

Let $\Phi = (\mathsf{A}_2)^n$, denote by $\mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}$ the set of isomorphism classes of connected cuspidal spherical Φ-systems $\mathscr{S}$ with spherical roots of type **a** and $\mathrm{Aut}(\mathscr{S})$ acting transitively on Σ.

To investigate the set $\mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}$, the following concept will be used.

Definition 5.5.6. Given a graph $\mathcal{G} = (V, E)$, where V is the set of vertices, E is the set of edges. Let $\mathcal{G}_0 = (V_0, E_0)$ be the graph of isolated edges such that $E_0 \simeq E$, where

$V_0 = \{v_e^i : e \in E_0, i = 0,1\}$ and $e \in E_0$ connects v_e^0 and v_e^1. A **formation** of $\mathcal{G}$ is a morphism of graphs $f_\mathcal{G} : \mathcal{G}_0 \longrightarrow \mathcal{G}$ which is a bijection on $E_0 \longrightarrow E$. An isomorphism between formations of $\mathcal{G}$ is an isomorphism between the morphisms $f_\mathcal{G} : \mathcal{G}_0 \to \mathcal{G}$ and $f'_\mathcal{G} : \mathcal{G}'_0 \to \mathcal{G}$, i.e., an isomorphism $\xi_0 : \mathcal{G}_0 \to \mathcal{G}'_0$ together with an automorphism ξ of $\mathcal{G}$ such that the following diagram commutes,

$$\begin{tikzcd}
\mathcal{G}_0 \arrow[r, "\xi_0"] \arrow[d, "f_\mathcal{G}"'] & \mathcal{G}'_0 \arrow[d, "f'_\mathcal{G}"] \\
\mathcal{G} \arrow[r, "\xi"] & \mathcal{G}
\end{tikzcd}$$

An automorphism of $f_\mathcal{G}$ is an isomorphism from $f_\mathcal{G}$ to itself. The set of automorphisms of a formation is denoted by $\mathrm{Aut}(f_\mathcal{G})$.

Lemma 5.5.7. *For a connected graph $\mathcal{G}$, any two formations $f_\mathcal{G}$ and $f'_\mathcal{G}$ are isomorphic to each other (as morphisms of graphs). Therefore, the two groups $\mathrm{Aut}(f_\mathcal{G})$ and $\mathrm{Aut}(f'_\mathcal{G})$ are isomorphic.*

Proof. Let $\mathcal{G}_0 = (V_0, E_0)$ and $\mathcal{G}'_0 = (V'_0, E'_0)$ denote the graphs corresponding to the formations $f_\mathcal{G}$ and $f'_\mathcal{G}$, respectively. An isomorphism between $f_\mathcal{G}$ and $f'_\mathcal{G}$ can be constructed in the following way.

Denote the isomorphism to be constructed (μ, id), where $\mu : \mathcal{G}_0 \longrightarrow \mathcal{G}'_0$ be an isomorphism of graphs (to be constructed), and $id : \mathcal{G} \longrightarrow \mathcal{G}$ be the identity morphism of $\mathcal{G}$. The definition of formations induces a bijection between E_0 and E'_0 through E, the set of edges of $\mathcal{G}$. For those $e_0 \in \mathcal{G}_0$ such that $f_\mathcal{G}(e_0)$ is not a loop, let $e'_0 \in E'_0$ be the corresponding edge by the bijection $E_0 \longrightarrow E'_0$, then $v_e^0 = f_\mathcal{G}(v_{e_0}^0)$ is different from the vertex $v_e^1 = f_\mathcal{G}(v_{e_0}^1)$. Also, $v_e^0 = f'_\mathcal{G}(v_{e'_0}^0)$, and $v_e^1 = f'_\mathcal{G}(v_{e'_0}^1)$. Let $\mu(v_{e_0}^0) = v_{e'_0}^0$, and $\mu(v_{e_0}^1) = v_{e'_0}^1$, then the images of all vertices connected by such e_0's are given. Otherwise, for those e_0 such that $e = f_\mathcal{G}(e_0)$ is a loop, choose any bijection between $\{v^0_{e_0}, v^1_{e_0}\}$ and $\{v^0_{e'_0}, v^1_{e'_0}\}$ as μ restricted on $\{v^0_{e_0}, v^1_{e_0}\}$. Thus μ is an isomorphism between $\mathcal{G}_0$ and $\mathcal{G}'_0$.

The induced map $(\mu, id) : f_\mathcal{G} \longrightarrow f'_\mathcal{G}$ is an isomorphism of graph formations because the construction of μ guarantees that all every edge and vertex in $\mathcal{G}_0$ and their μ-images in $\mathcal{G}'_0$ match after being passed to $\mathcal{G}$ by the corresponding formations.

Furthermore, the conjugation by (μ, id) on automorphisms of $f_\mathcal{G}$ is an isomorphism between $\mathrm{Aut}(f_\mathcal{G})$ and $\mathrm{Aut}(f'_\mathcal{G})$. □

Lemma 5.5.8. *There is a forgetful map* $\mathrm{Aut}(f_\mathcal{G}) \longrightarrow \mathrm{Aut}(\mathcal{G})$ *by choosing the underlying automorphism of* $\mathcal{G}$ *in an automorphism of* $f_\mathcal{G}$*. This map admits a splitting* $\mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Aut}(f_\mathcal{G})$*.*

Proof. The forgetful map comes from "forgetting" the formation structure in an automorphism of $f_\mathcal{G}$, that is, for an automorphism (ξ_0, ξ) of $f_\mathcal{G}$, its image in $\mathrm{Aut}(\mathcal{G})$ is chosen to be ξ.

For the splitting morphism, it is sufficient to construct $(\xi_0, \xi) \in \mathrm{Aut}(f_\mathcal{G})$ from an automorphism $m \in \mathrm{Aut}(\mathcal{G})$ with $\xi = m$.

Given a formation $f_\mathcal{G}$, for each $m \in \mathrm{Aut}(\mathcal{G})$, the image $\mu = (\mu|_{\mathcal{G}_0}, \mu|_\mathcal{G}) = (\xi_0, \xi) \in \mathrm{Aut}(f_\mathcal{G})$ of m under the splitting morphism should satisfy the following conditions:

1. $\xi = m$,
2. $\xi_0|_E = (f_\mathcal{G}|_{E_0})^{-1} \circ m|_E \circ (f_\mathcal{G}|_E)$,
3. $\xi_0(\{v^0_e, v^1_e\}) = \{v^0_{\xi_0(e)}, v^1_{\xi_0(e)}\}$ for each $e \in E_0$,
4. $\xi_0(f_\mathcal{G}^{-1}(v)) = f_\mathcal{G}^{-1}(m(v))$ for each $v \in V$.

There is always such an automorphism of $f_\mathcal{G}$ satisfying these conditions. For each $v_0 \in V_0$ and the edge $e_0 \in E_0$ connecting v_0, the cardinality of the set $\{v^0_{e_0}, v^1_{e_0}\} \cap f_\mathcal{G}^{-1}(f_\mathcal{G}(v_0))$ is either 2 (if $f_\mathcal{G}(e_0)$ is a loop in $\mathcal{G}$) or 1 (otherwise), and its image under ξ_0, given by the conditions 3 and 4, is of the same cardinality by the definition of $\mathrm{Aut}(\mathcal{G})$. If the cardinality is 1, there is only one choice of the image of v_0; and if it is 2, the two choices can give two different elements in $\mathrm{Aut}(f_{\mathcal{G}})$. For each $f_{\mathcal{G}}(e_0)$ which is a loop, the choice needed to make is to choose an element from the set of bijections $\mathrm{Isom}(\{v^0_{e_0}, v^1_{e_0}\}, \{v^0_{\xi_0(e_0)}, v^1_{\xi_0(e_0)}\})$. After fixing all the choices above, the morphism ξ_0 is uniquely determined together with conditions 1 to 4 listed above. Thus (ξ_0, ξ) is the image of m under the splitting morphism. $\square$

Remark. The splitting lemma does not apply in the category of groups, so the existence of the right splitting does not induce that $\mathrm{Aut}(\mathcal{G})$ is a direct summand of $\mathrm{Aut}(f_{\mathcal{G}})$.

Definition 5.5.9. Let $\mathfrak{G}$ be the set of connected graphs $\mathcal{G} = (V, E)$, such that for a formation $f_{\mathcal{G}} : \mathcal{G}_0 \longrightarrow \mathcal{G}$ where $\mathcal{G}_0 = (V_0, E_0)$, $\mathrm{Aut}(f_{\mathcal{G}})$ acts transitively on the set V_0.

Remark. Note that the condition above holds for every formation of $\mathcal{G}$ if it holds for one.

The following proposition shows a different condition on $\mathrm{Aut}(\mathcal{G})$ to verify whether a graph is in the set $\mathfrak{G}$.

Proposition 5.5.10. *Let $\mathcal{G} = (V, E)$ be a graph, $\mathcal{G} \in \mathfrak{G}$ if and only if $\mathrm{Aut}(\mathcal{G})$ acts transitively on E, and there exists an edge $e \in E$ connecting to vertices v_1, v_2, such that $\mu(v_1) = v_2$ and $\mu(v_2) = v_1$ for some $\mu \in \mathrm{Aut}(\mathcal{G})$.*

Proof. To show necessity, for a graph $\mathcal{G} \in \mathfrak{G}$, with a formation $f_{\mathcal{G}}$ satisfying that $\mathrm{Aut}(f_{\mathcal{G}})$ acts transitively on V_0. By definition, an automorphism of $f_{\mathcal{G}}$ induces an automorphism of $\mathcal{G}$. For an edge $e \in E$, consider the preimage of e under $f_{\mathcal{G}}$, and denote that edge also by $e \in E_0$. Let v^0_e, v^1_e be two vertices in V_0 that e connects, then the image of $\{v^0_e, v^1_e\}$ under an automorphism of $f_{\mathcal{G}}$ is a set $\{v^0_{e'}, v^1_{e'}\}$ for another edge $e' \in E_0$ (to preserve the structure of the formation). Thus the transitivity of $\mathrm{Aut}(f_{\mathcal{G}})$ on V_0 induces the transitivity of that on E_0, hence $\mathrm{Aut}(\mathcal{G})$ acts transitively on E. The condition that $\mathrm{Aut}(f_{\mathcal{G}})$ acts on V_0 transitively shows that $\mathrm{Aut}(\mathcal{G})$ acts on V transitively. Thus the second condition is satisfied.

Then, to show sufficiency, let $\mathcal{G}$ be a graph satisfying the condition mentioned in the proposition. If $\#V = 1$, $\mathcal{G}$ is an n-rose, where $n = \#E$, the automorphism group $\mathrm{Aut}(f_{\mathcal{G}})$ consisting of permutations of E, and for each edge $e \in E_0$, which connects $\{v_e^0, v_e^1\}$, there is an automorphism of $f_{\mathcal{G}}$ swapping the two vertices. Thus $\mathrm{Aut}(f_{\mathcal{G}}) = (\mathbb{Z}/2\mathbb{Z}) \wr (\mathrm{Aut}(\mathcal{G})) = (\mathbb{Z}/2\mathbb{Z}) \wr (S_n)$ where S_n is the symmetric group on E, and it acts on V_0 transitively.

When $\#V > 1$, there is no loop (edges connecting only one vertex) in $\mathcal{G}$, otherwise every edge is a loop, and according to connectedness, the graph is just a rose. The first condition induces that $\mathrm{Aut}(f_{\mathcal{G}})$ acts transitively on E_0 transitively. The second condition implies that for each $e \in E_0$ connecting v_e^0 and v_e^1, there is an automorphism of $f_{\mathcal{G}}$ swapping the two vertices. Thus the transitivity on E_0 implies the transitivity of the action of $\mathrm{Aut}(f_{\mathcal{G}})$ on V_0. □

Corollary 5.5.11. *For a graph $\mathcal{G} = (V, E) \in \mathfrak{G}$, the automorphism of a formation $f_{\mathcal{G}}$ of $\mathcal{G}$ is*

$$\mathrm{Aut}(f_{\mathcal{G}}) = \begin{cases} (\mathbb{Z}/2\mathbb{Z}) \wr (S_n) & \textit{if } \#V = 1, \textit{ and } n = \#E, \\ \mathrm{Aut}(\mathcal{G}) & \textit{otherwise.} \end{cases}$$

Proof. If there is a loop in $\mathcal{G} \in \mathfrak{G}$, then every edge is a loop by Proposition 5.5.10 , thus $\#V = 1$. Also if $\#V = 1$, every edge is a loop. So in this case, for every edge $e \in E$, there is a $\mathbb{Z}/2\mathbb{Z}$ symmetry in $\mathrm{Aut}(f_{\mathcal{G}})$ for every edge. Thus the subgroup of $\mathrm{Aut}(f_{\mathcal{G}})$ which acts on $\mathcal{G}$ trivially is $(\mathbb{Z}/2\mathbb{Z})^n$. And $\mathrm{Aut}(\mathcal{G}) = S_n$ acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by $\xi(a_e) = a_{\xi^{-1}(e)}$ where $\xi \in \mathrm{Aut}(\mathcal{G})$, and a_e is nontrivial only on e-th component of $(\mathbb{Z}/2\mathbb{Z})^n$. Thus $\mathrm{Aut}(f_{\mathcal{G}}) = (\mathbb{Z}/2\mathbb{Z}) \wr (S_n)$.

Otherwise, the subgroup of $\mathrm{Aut}(f_{\mathcal{G}})$ which acts on $\mathcal{G}$ trivially is the trivial group, thus $\mathrm{Aut}(f_{\mathcal{G}}) = \mathrm{Aut}(\mathcal{G})$. $\square$

a-A-4.

Proposition 5.5.12. *There is a bijection* $i : \mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2} \longrightarrow \mathfrak{G}$, *that with* $\mathcal{G} = i(\mathscr{S})$, $\mathrm{Aut}(\mathscr{S}) \simeq \mathrm{Aut}(f_{\mathcal{G}})$ *for a formation* $f_{\mathcal{G}}$ *of* $\mathcal{G}$, *where*

i: Given $\mathscr{S}$, *such that* $[\mathscr{S}] \in \mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}$, $i(\mathscr{S}) = (V, E)$ *where* $V = \{D^+ : \rho(D^+)(\sigma) \geq 0, \forall \sigma \in \Sigma\}$, *and* $E = \{l_{\alpha_1,\alpha_2} : l_{\alpha_1,\alpha_2}$ *connects vertices* D_1 *and* $D_2 \in V$, *where* $D_i \in \Delta(\alpha_i), \langle \alpha_1^\vee, \alpha_2 \rangle < 0\}$.

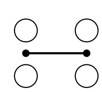
i^{-1}*: Given* $\mathcal{G} = (V, E) \in \mathfrak{G}$, *for any formation of* $\mathcal{G}$, $f_{\mathcal{G}} : \mathcal{G}_0 \longrightarrow \mathcal{G}$, *let* $n = \#E$, *and* $\Phi = (\mathsf{A}_2)^n$ *with positive simple roots identified to* V_0 *of* $\mathcal{G}_0$. *Then* $\mathscr{S} = (S^p, \Sigma, \mathscr{A})$, *where* $S^p = \emptyset$, $\Sigma = V_0 = S$, *and* $\mathscr{A} = \{D_v^+ : v \in V\} \cup \{D_\sigma^- : \sigma \in \Sigma\}$. *For each* $\alpha \in S$, $\Delta(\alpha) = \{D^+_{f_{\mathcal{G}}(\alpha)}, D^-_\alpha\}$, *and the valuations are:*

$$\rho(D_v^+)(\sigma) = \begin{cases} 1 & \text{if } v \in \Delta(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho(D_\alpha^-)(\sigma) = \begin{cases} 1 & \text{if } \sigma = \alpha, \\ 0 & \text{if } \langle \alpha^\vee, \sigma \rangle = 0 \text{ and } \Delta(\alpha) \cap \Delta(\sigma) = \emptyset, \\ -2 & \text{if } \langle \alpha^\vee, \sigma \rangle = -1 \text{ and } \Delta(\alpha) \cap \Delta(\sigma) \neq \emptyset, \\ -1 & \text{otherwise.} \end{cases}$$

Here are several examples:

Example 5.5.13. $\Phi = \mathsf{A}_2$. $S^p = \emptyset$, $\Sigma = \{\alpha_1, \alpha_2\}$, and $\mathscr{A} = \{D_1^+, D_1^-, D_2^+, D_2^-\}$, where $\Delta(\alpha_i) = \{D_i^+, D_i^-\}$.

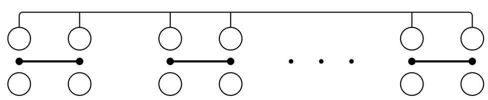


(5.27)

It corresponds to the formation of the following graph $\mathcal{G}$

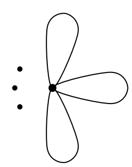
And the automorphism group is $\mathrm{Aut}(\mathscr{S}) \simeq \mathbb{Z}/2\mathbb{Z}$.

Example 5.5.14. $\Phi = (\mathsf{A}_2)^n$. $S^p = \emptyset$, $\Sigma = \{\alpha_{i,j} : i = 1, 2, \ldots, n, \text{ and } j = 1, 2\}$, $\mathscr{A} = \{D^+\} \cup \{D^-_{i,j} : i = 1, 2, \ldots, n, \text{ and } j = 1, 2\}$, where $\Delta(\alpha_{i,j}) = \{D^+, D^-_{i,j}\}$. The map ρ is given by the images of $\mathscr{A}$ under it, expressed in pairs of numbers, given by being paired with α_1 and α_2, respectively: $\rho(D^+_1) = (1, 0)$, $\rho(D^+_2) = (0, 1)$, $\rho(D^-_1) = (1, -1)$, and $\rho(D^-_2) = (-1, 1)$.



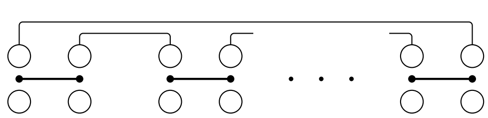
(5.28)

It corresponds to a formation of $\mathcal{G}$, where $\mathcal{G}$ is an n-rose:



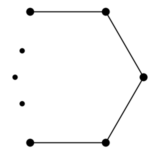
The automorphism group is $\mathrm{Aut}(\mathscr{S}) \simeq (\mathbb{Z}/2\mathbb{Z}) \wr S_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, with S_n permutes on A_2-components and each $\mathbb{Z}/2\mathbb{Z}$ acts on the corresponding component.

Example 5.5.15. $\Phi = (\mathsf{A}_2)^n$. $S^p = \emptyset$, $\Sigma = \{\alpha_{i,j} : i = 1, 2, \ldots, n, \text{ and } j = 1, 2\}$, $\mathscr{A} = \{D^+_{12}, D^+_{23}, \ldots, D^+_{n1}\} \cup \{D^-_{i,j} : i = 1, 2, \ldots, n, \text{ and } j = 1, 2\}$, where $\Delta(\alpha_{i,1}) = \{D^+_{i-1,i}, D^-_{i,1}\}$ and $\Delta(\alpha_{i,2}) = \{D^+_{i,i+1}, D^-_{i,1}\}$.



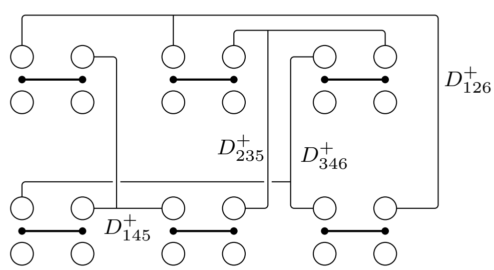
(5.29)

It corresponds to a formation of the polygon with n-edges:



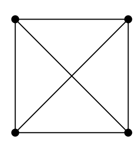
The group of automorphisms is $\mathrm{Aut}(\mathscr{S}) = D_n$, the dihedral group acting on a n-gon.

Example 5.5.16. Let $\Phi = (\mathsf{A}_2)^6$. The spherical system consists of $S^p = \emptyset$, $\Sigma = \{\alpha_{i,j} : i = 1, 2, \ldots, 6, \text{ and } j = 1, 2\}$, $\mathscr{A} = \{D^+_{126}, D^+_{145}, D^+_{235}, D^+_{346}\} \cup \{D^-_{ij} : i = 1, 2, \ldots, 6, \text{ and } j = 1, 2\}$.



(5.30)

It corresponds to the formation of the complete graph with 4 vertices:



The group of automorphisms is $\mathrm{Aut}(\mathscr{S}) = S_4$, permuting the 4 positive-decorated colors.

Remark (Geometric Realizations). When the spherical system is not as in Example 5.5.13, the spherical varieties corresponding to the spherical $(\mathsf{A}_2)^n$-system $\mathscr{S}$ is determined by $G = (\mathrm{SL}_3)^n$ and generic stabilizer $H = \mathcal{Z}(G) \cdot (A \cdot U)$ where

$$U = \prod_{i=1}^{n} \begin{pmatrix} 1 & & \\ x_{i,1} & 1 & \\ * & x_{i,2} & 1 \end{pmatrix}$$

satisfies the following conditions: for each $D^+ \in \mathscr{A}$ such that $\rho(D^+)(\alpha) \geq 0$, $\forall \alpha \in \Sigma$ (i.e., D^+ is in V),

$$\sum_{D^+ \in \Delta(\alpha_{i,j})} x_{i,j} = 0.$$

And

$$A = \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix}$$

acting diagonally on each component.

And for the spherical system in Example 5.5.13, $G = \mathrm{SL}_3$ and

$$H = \begin{pmatrix} * & & \\ 0 & * & \\ * & 0 & * \end{pmatrix}.$$

Proof of Proposition 5.5.12. Similar to the discussion for $\Phi = (\mathsf{A}_1)^n$ cases, the spherical Φ-systems with $\Phi = (\mathsf{A}_2)^n$ are discussed case by case according to the number ν of colors in $\Delta(\sigma)$ for spherical root σ which also belongs to $\Delta(\sigma')$ for some other spherical root, which can only be 0, 1 or 2.

Case 1: If $\nu = 0$, then the positive simple roots in the spherical system is connected only when the underlying Dynkin diagram is connected, hence $\Phi = \mathsf{A}_2$, and no colors in the spherical system belong to more than one $\Delta(\alpha)$. On the graph formation side, this spherical system corresponds to the graph $\mathcal{G}$ with $V = \{v_1, v_2\}$, $E = \{e\}$ where e connects v_1 and v_2. With the condition $\nu = 0$, which means there are no vertices shared by a pair of distinct edges, there is at most 1 edge, hence $\mathcal{G}$ is the only possible graph in this case. And $\mathcal{G}_0 = \mathcal{G}$, $\mathrm{Aut}(\mathscr{S}) \simeq \mathrm{Aut}(f_{\mathcal{G}}) \simeq \{1, \sigma\}$ with σ exchanging α_1 with α_2. This is shown in Example 5.5.13. Also, it is easy to see that the images of colors under ρ are exactly the ones given in the proposition. And it is the only one in its isomorphism class, if the "positive" labels of colors are given to the ones with greater ρ-values in each $\Delta(\alpha)$. Consider the fact that this case is the only $\mathcal{G} \in \mathfrak{G}$ that there is no edges in $\mathcal{G}$ which is adjacent to any edge through a vertex. As $\mathfrak{G}$ requires any graph in it to be connected, then such a graph can only have one edge and exactly two vertices connected to it. And for the rest of the discussion, it may be assumed that all edges in $\mathcal{G}$ are adjacent to some other edge (can be itself) through a vertex.

Case 2: $\nu = 1$. This case contains the rest of the valid spherical systems and graphs. So we may assume that the spherical varieties mentioned below are not of the form in Example 5.5.13, and graphs does not contain the corresponding one.

It is necessary to check that the maps i and i^{-1} (it will be shown that it is the inverse of i) given in the proposition are maps between $\mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}$ and $\mathfrak{G}$, then the facts that $i \circ i^{-1} = id_{\mathcal{G}}$ and $i^{-1} \circ i = id_{\mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}}$ can show i^{-1} is the "true" inverse. At last, the automorphism groups will be discussed.

As $\nu = 1$, every positive color D^+_α also belongs to $\Delta(\alpha')$ for some other $\alpha' \in S$, and the corresponding D^-_α belongs to $\Delta(\alpha)$ only. Then for any spherical system $\mathscr{S}$ with $[\mathscr{S}] \in \mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}$, let $\mathcal{G}$ be its image under i. The connectedness of $\mathscr{S}$ means that the edges in $\mathcal{G}$ (components of the underlying Dynkin diagram) are connected by the colors (common vertices between edges). Thus $\mathcal{G}$ is connected. Moreover, let $\mathcal{G}_0$ be the underlying Dynkin diagram of $\mathscr{S}$, the map of graphs $f_{\mathscr{S}} : \mathcal{G}_0 \longrightarrow \mathcal{G}$ induced by the Luna diagram is a graph formation of $\mathcal{G}$, and the transitivity condition shows that $\mathcal{G} \in \mathfrak{G}$.

Conversely, a similar discussion shows that for each $\mathcal{G} \in \mathfrak{G}$, if $\mathscr{S} = i^{-1}(\mathcal{G})$ is a spherical system, then the isomorphism class $[\mathscr{S}] \in \mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}$. To show $\mathscr{S}$ is a spherical system, it suffices to check the Axioms (A1), (A2) and (A3), since (S1) is implied by the type **a** condition, and other axioms are not applicable. (A1) is implied directly from the assignment of the Cartan pairing. (A3) is from the construction of $\mathscr{A}$ and $\Delta(\alpha)$ for each α. For (A2), let $v \in \Delta(\alpha)$,

$$(\rho(D_v^+) + \rho(D_\alpha^-))(\sigma) = \begin{cases} 1+1=2 & \text{if } \sigma = \alpha, \\ 0+0=0 & \text{if } \langle \alpha^\vee, \sigma \rangle = 0 \text{ and } \Delta(\alpha) \cap \Delta(\sigma) = \emptyset, \\ 1+(-2)=-1 & \text{if } \langle \alpha^\vee, \sigma \rangle = -1 \text{ and } \Delta(\alpha) \cap \Delta(\sigma) \neq \emptyset, \\ 1+(-1)=0 & \text{if } \langle \alpha^\vee, \sigma \rangle = 0 \text{ and } \Delta(\alpha) \cap \Delta(\sigma) \neq \emptyset, \\ 0+(-1)=-1 & \text{if } \langle \alpha^\vee, \sigma \rangle = -1 \text{ and } \Delta(\alpha) \cap \Delta(\sigma) = \emptyset. \end{cases}$$

Hence, $\rho(D_v^+) + \rho(D_\alpha^-) = \alpha^\vee$, i.e., (A2) holds.

Then for the compositions, it is easy to see that $i \circ i^{-1}(\mathcal{G})$ produces $\mathcal{G}$. Conversely, if a formation is chosen to be the $f_{\mathscr{S}}$ defined above, $i^{-1} \circ i$ is an identity on spherical systems in $\mathfrak{S}^{\mathbf{a}}_{\mathsf{A}_2}$. Otherwise, if another formation f of $i(\mathscr{S})$ is chosen, the isomorphism between f and $f_{\mathscr{S}}$ (Lemma 5.5.7) implies an isomorphism between the spherical systems produced by i^{-1}, however, they still live in the same isomorphism class.

For the automorphisms, as $\nu = 1$, each vertex attached to an edge in $\mathcal{G}_0$ belongs to only one image under the formation map. In this procedure $V_0 = \Sigma$, the action of $\mathrm{Aut}(\mathscr{S})$ on Σ is considered as the same action on V_0. By the previous construction, $\mathrm{Aut}(\mathscr{S})$ acts on the formation, hence $\mathrm{Aut}(\mathscr{S}) \subseteq \mathrm{Aut}(f_{\mathcal{G}})$. On the other direction, $\mathrm{Aut}(f_{\mathcal{G}})$ acts on Σ and S^p , then it defines the action on $\mathscr{A}$ by $\xi_f(D_v^+) = D^+_{\xi_f(v)}$ and $\xi_f(D_\alpha^-) = D^-_{\xi_0(\alpha)}$, for $\xi_f = (\xi_0, \xi) \in \mathrm{Aut}(f_{\mathcal{G}})$. So $\mathrm{Aut}(\mathscr{S}) \supseteq \mathrm{Aut}(f_{\mathcal{G}})$, hence they are isomorphic.

Case 3: $\nu = 2$. There are no spherical systems satisfying $\nu = 2$. Suppose there exists one. Let $\Delta(\alpha_{1,1}) = \{D_{1,1}^+, D_{1,1}^-\}$, $D_{1,1}^+ \in \Delta(\sigma_1)$, and $D_{1,1}^- \in \Delta(\sigma_2)$ where $\sigma_i \neq \alpha_{1,1}$ (there can be more than one possible such σ_is, just choose one of them in the discussion). And let the colors D_1^- and D_2^+ satisfy the conditions that $\Delta(\sigma_1) = \{D_{1,1}^+, D_1^-\}$, and $\Delta(\sigma_2) = \{D_{1,1}^-, D_2^+\}$. In the following discussion, σ_1 and σ_2 can be exchanged, so without loss of generality, σ_1 is chosen instead of "one of the σ_i's". There are 4 situations for σ_1 and σ_2 to be considered:

1. $\sigma_1 = \sigma_2$. This violates the axiom (A2) as $\rho(D_{1,1}^+)(\sigma_1) = \rho(D_{1,1}^-)(\sigma_1) = 1$.
2. $\sigma_1 = \alpha_{1,2}$, i.e., $\langle \alpha_{1,1}^\vee, \sigma_1 \rangle = -1$. $\rho(D_{1,1}^+)(\alpha_{1,2}) = 1$, hence $\rho(D_{1,1}^-)(\alpha_{1,2}) = -2$. To make $\rho(D_{1,1}^-) + \rho(D_2^+) = \sigma_2^\vee$, $\rho(D_2^+)(\alpha_{1,2}) = 2$ which violates the axiom (A1).
3. $\langle \sigma_1^\vee, \sigma_2 \rangle = -1$. In this case, let $\sigma_1 = \alpha_{2,1}$, and $\sigma_2 = \alpha_{2,2}$. $(\rho(D_{1,1}^+)+\rho(D_{1,1}^-))(\alpha_{1,2}) = -1$, hence $\rho(D_{1,1}^\pm)(\alpha_{1,2}) = 0$ or -1. Choose $\rho(D_{1,1}^+)(\alpha_{1,2}) = 0$ (the other case is equivalent to a swap of $D_{1,1}^\pm$), then $\rho(D_{1,1}^-)(\alpha_{1,2}) = -1$, and $\rho(D_2^+)(\alpha_{1,2}) = 1$. Part of the valuations are given in the following table:

	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{2,1}$	$\alpha_{2,2}$
$\rho(D_{1,1}^+)$	1	0	1	-1
$\rho(D_{1,1}^-)$	1	-1	-1	1
$\rho(D_1^-)$	-1	0	1	0
$\rho(D_2^+)$	-1	1	0	1
$\rho(D_{1,2}^-)$	0	1	0	-1

   However, by the assumption of case 3, $D_1^- \in \Delta(\sigma_3)$ for some σ_3 other than the 4 spherical roots mentioned in the table. Considering $\rho(D_1^-)(\sigma_3) = 1$, then $\rho(D_{1,1}^+)(\sigma_3) = -1$, and $\rho(D_{1,1}^-)(\sigma_3) = 1$, hence $D_3^+ = D_{1,1}^-$. Furthermore, $\rho(D_2^+)(\sigma_3) = -1$ and $\rho(D_{1,2}^-)(\sigma_3) = 1$. But $D_{1,2}^-$ can not be identified with $D_{1,1}^-$

or D_1^-. This violates the axiom (A1).

4. $\langle \sigma_1^\vee, \alpha_{1,1}\rangle = \langle \alpha_{1,1}^\vee, \sigma_2\rangle = \langle \sigma_1^\vee, \sigma_2\rangle = 0$. From the discussion of **a-A-3.**, the color D_1^- is identical to D_2^+, and the valuations are:

	$\alpha_{1,1}$	σ_1	σ_2
$\rho(D_{1,1}^+)$	1	1	-1
$\rho(D_{1,1}^-)$	1	-1	1
$\rho(D_1^-) = \rho(D_2^+)$	-1	1	1

Without loss of generality, let $\rho(D_{1,1}^+)(\alpha_{1,2}) = 0$, then $\rho(D_{1,1}^-)(\alpha_{1,2}) = -1$, and $\rho(D_2^+)(\alpha_{1,2}) = 1$. Hence $D_2^+ \in \Delta(\alpha_{1,2})$. Let $\Delta(\alpha_{1,2}) = \{D_2^+, D'\}$, then $\rho(D')(\alpha_{1,1}) = 1$. This goes back to the situation 2 above.

Therefore, case 3 does not provide any possible spherical systems. $\square$

Thus Theorem 5.5.1 is proven.

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