RELIABILITY MODELING, TESTING AND OPTIMIZATION OF SYSTEMS WITH MIXTURES OF ONE-SHOT UNITS

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ABSTRACT OF THE DISSERTATION

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One-shot units are usually produced and stored (or used as standby) in batches until retrieved. These units can be defined as a system which may experience degradations or sudden failures during its storage period. To assess the reliability performance of the units, reliability tests are repeatedly (in non-identical pattern) and randomly conducted across the lifetime of the units; where corresponding actions are taken afterwards. The continuous arrival of batches and conduct of tests induce the system contains a mixture of nonhomogeneous units, which is defined as a general "*k*-out-of-*n*: F system" with *k* and *n* nonhomogeneous and time-dependent. In this dissertation, we propose models to investigate the reliability metrics of the system under a variety of scenarios. Extensive simulation studies are performed to validate the models.

Failure or degradation caused by thermal fatigue is a pervasive phenomenon during the one-shot units' storage period. The Birnbaum-Saunders (BS) distribution is specifically developed for describing mechanical fatigue failures, but limited in describing a variety of

hazard functions. Hence it is reasonable to investigate whether the generalized form of BS (GBS) distribution can be extended for modeling the plastic deformation induced by thermal cyclic stresses and providing reliability metrics of units subject to thermal fatigue. In this dissertation, we investigate system reliability metrics when subjecting to thermal fatigue failure by adopting the GBS accelerated model.

The one-shot units might experience competing failure modes during its storage period. Specifically, repeated thermal cyclic tests (TCTs) are randomly conducted; at the end of an arbitrary TCT, the unit's failure is observed either when any of its failure modes occurs suddenly or when any of its degradation modes reach its "failure threshold". Under such circumstances, unit's failure data cannot be described by a single failure time distribution; instead, a competing failure model which considers multiple failure modes is adopted to assess unit's reliability metrics. The units' potential failure modes as well as the reliability metrics of the system under competing failure modes are investigated in this dissertation.

Due to the characteristics of one-shot units and recent advances in technology and materials, one-shot units are usually highly reliable and it is impractical to obtain one-shot units' failure (degradation) data under operating conditions. Accelerated life testing (ALT) is an efficient approach to obtain failure/degradation observations in a much shorter time period and utilize the test data to predict reliability metrics under normal operating conditions. We develop physics-statistics-based models and obtain optimal sequential accelerated non-destructive test (NDT) plans under different scenarios. The efficiency of the NDT plans is validated by comparing the system reliability metrics obtained under accelerated and normal conditions.

NDT assesses unit's functionality without permanent damage in order to demonstrate the unit's reliability. However, one cannot make decisions regarding system reliability by only depending on NDT results because NDT does not fully perform the unit's functionality. In contrast, destructive testing (DT) fully tests the unit's functionality but destroys the units. This intensifies the need to investigate hybrid reliability tests that include both NDT and DT. In this dissertation, an optimal sequential hybrid reliability testing plan is designed and the results of the tests are utilized to improve the accuracy of the system reliability metrics estimation. We validate that by conducting hybrid reliability test, the unit's lifetime parameters approach their true values. As the reliability estimation converge, we decrease the number of units tested in DTs and eventually perform NDT only.

There exists many situations that a specific number of one-units are used consecutively when put into operational use. Therefore, it becomes interesting and challenging to determine the characteristics and sequence of the one-shot units to be launched such that the operational use of the launched units is optimized. Defining the launched one-shot units as a system, we investigate the reliability metrics of the system to optimize the system's operational use at arbitrary time by formulating an optimization problem which is applicable to a variety of objectives. We also provide the bounds of the system's successful operational probability estimation.

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CHAPTER 1

INTRODUCTION

1.1 Research Background

The development of new technologies and global competition emphasize the need for accurate estimation or prediction of a unit's quality. Reliability is one of the most important quality characteristics of interest. Reliability is defined as the probability that a unit operates for a given period of time (design lifetime) under the designed operating conditions. Although modeling of systems reliability has been extensively investigated, the reliability modeling of one-shot units is limited which motivates the research of this dissertation.

1.1.1 Reliability of One-shot Units and Systems Composed of One-shot Units

1.1.1.1 Reliability of One-shot Units

There exist some types of units that can only perform its function once as its use is normally accompanied by irreversible reactions, e.g. chemical reaction or physical destruction. After the use, the units are destroyed or require extensive repair. Units with such a property are referred to as one-shot units. Missiles, airbags and most of the military weapons are typical examples of one-shot units. An important characteristic of one-shot units is its long-term storage period or standby status before delivered to users or deployed. To illustrate, military weapons are conventionally stored for long periods, often in excess of decades until there is a need. The auto airbag is another example of one-shot units as an airbag stays in standby status during a vehicle operating life and inflates rapidly (in millisecond) when an accident or collision happens in order to provide the auto occupant(s) with protection. One-shot unit's storage life (or time in standby status) approximately equals to its operational life.

Generally, the reliability of one-shot units is defined as the probability that they successfully perform required function when needed. Initially, reliability of one-shot units is simply assigned a time-independent probability. However, reliability of one-shot units needs to consider the effect of aging since the units experience a variety of potential failure modes which are caused by different failure mechanisms and are dependent on the length of the storage/standby periods. These failure modes may inhibit one-shot unit's ability to perform its function. For example, the integrated circuit (IC) board of the missile's electronic guidance system might no longer function properly as the temperature fluctuation incudes the crack of the solder joints to an unacceptable level (thermal fatigue), or as its resistance degrades and eventually reaches a critical point, or due to a failure without indicator. Meanwhile, the IC board might fail suddenly with no indicator of failure. Determining the status of one-shot units during the long storage/standby period under different types of failure modes, evaluating the reliability of stored one-shot units and ensuring the proper function of one-shot units when needed become an important issue.

1.1.1.2 Systems Composed of One-shot Units

Batches of one-shot units are produced and stored to form a system. Current research on one-shot units' reliability is sparse by addressing single (non-repeated) test and assuming that all units are homogenous. In real life, the successive arrivals of batches and the repeated reliability tests result in a mixture of nonhomogeneous units with different characteristics at any point of time. Instead of the exact lifetime data, the testing results only show the number and the characteristics of the testing units. Such data can be addressed as generalized binary data. As reliability tests (non-destructive or other types) are repeated during the life horizon of the units, appropriate decisions are made regarding the failed units during the test. Specifically, there could be the following possible decisions regarding the failed units:

1. Failed units are repaired and placed back into the system with a higher failure rate;
2. Failed units are discarded after certain consecutive number of failures;
3. Failed units are discarded after certain number of failures.

Consequently, the characteristics of the units in the system are continuously changing. Specifically, the population size changes as the new batches arrive and failed units might be discarded. In the system, some units are newly arrived whereas some units are aged; some are repaired multiple times and might have high failure rates and some have never failed.

The extensive experience with units' reliability testing indicates that all tests require some kind of combinatorial testing such as "*k*-out-of-*n*" system. In this dissertation, the system with a mixture of different units can be defined as a generalized "$k(t)$-out-of-$n(t)$: F system". Specifically, in this dissertation, the "$k(t)$-out-of-$n(t)$: F system" is a generalization of traditional "*k*-out-of-*n*: F system" as both $k(t)$ and $n(t)$ increase with time. Characteristic of such a system is described in details and the reliability metrics evaluation under different scenarios becomes challenging. Computationally efficient approximations that evaluate the system reliability metrics over an extended time horizon as well as large batch sizes have significant practical values and should be investigated. Additionally, not much attention has been paid to investigate the effect of population's non-homogeneity on reliability prediction. This is addressed in details in this dissertation.

For the system described above, there could be many testing approaches and testing scenarios to conduct the reliability tests during the storage period (under either normal or accelerated conditions) such that the unit's/system's reliability metrics are assessed. The reliability tests can be performed by testing the entire population; however, with the continuous arrival of units, the cost of testing a large population becomes high and testing samples which represent the population mixture proves to be a viable alternative. Different types of reliability tests under different scenarios, as well as their advantages, disadvantages, and applications, are discussed later in this dissertation.

In real life, times of arrivals of batches, times to perform the reliability tests, and the batch sizes are not arbitrary. Instead, they could be described by specific probabilistic distributions. To illustrate, the arrival of batches and the conduct of the reliability tests may follow nonhomogeneous Poisson distributions and the batch size may follow a uniform or normal distribution. Under such circumstances, the potential condition and reliability of the system can be obtained by applying a stochastic approach (SA). The details are investigated in the dissertation.

1.1.2 Reliability Testing

Performing reliability testing is an effective approach to demonstrate or predict the reliability of the units or systems, which is a function of time. Specifically, to determine the reliability of one-shot units, one can perform either DT or NDT under either normal or accelerated conditions.

DT involves the actual use of the units and determines whether the units perform the expected function or not. Its use is limited since performing such a test “destroys” the unit or requires high repair cost, the units are discarded after the test. Accordingly, it is not appropriate to conduct DT using large samples. Consequently, performing the test on a limited number of one-shot units inevitably results in sampling errors.

Another type of reliability test, NDT, assesses the unit's reliability without causing permanent damage. The NDT is used to ensure that the testing units continue to perform their functions after the test. The procedures during NDT incorporate inspecting, testing, and evaluating individual units or assemblies. For example, the electronic guidance subsystem of a missile may be subjected to a functional test or electric field test such as thermal cyclic test (TCT) to determine its reliability. In a TCT, the testing systems are heated at a temperature and kept for a certain dwell time, then cooled to a temperature to complete a testing cycle. Obviously, there exists no limit on the number of testing units except test time and cost. However, it is difficult to make decisions regarding system reliability by completely relying on the results of NDT since it does not fully perform all functions of the units.

Most of the research in the literature utilizes and analyzes the failure data obtained from DTs, which destroys (consumes) the one-shot units. In this dissertation, we consider repeated NDTs (under either normal or accelerated conditions) during the storage period of the one-shot units which do not affect the units' functionality. We also utilize the units' lifetime models to predict the units' reliability behavior, by assuming that test data only show the number and combination of the failed units at the end of each test when repeated NDTs are performed.

Besides performing NDT, it is also challenging but interesting to hybridize NDT and DT to demonstrate and predict the reliability of the system, which has not been addressed and investigated in previous research. To conclude, the reliability metrics of the system composed of one-shot units can be assessed by considering a variety of combinations of reliability tests. Specifically, we investigate the following testing scenarios in this dissertation:

1. Conducting a sequence of normal NDTs on the entire population/selected samples;
2. Conducting a sequence of accelerated NDTs on the entire population/selected samples;
3. Conducting a sequence of normal hybrid reliability tests on the selected samples.

Specifically, when scenario 3 (conducting hybrid reliability tests that utilize NDTs and DTs) is considered, the following two specific approaches are taken into consideration:

a. Each NDT and DT are performed on two different samples;

b. Each NDT and DT are performed using the same sample;

Units failed in the NDTs are either repaired and tested in DTs, or repaired and placed back into the system, or discarded.

The above tests are repeatedly performed during the entire storage period of one-shot units in order to determine the reliability metrics of the system regularly. Once the type of the reliability test is determined (NDT, or hybrid reliability test), designing the optimal sequential reliability testing plan based on sampling is necessary and challenging. To illustrate, it is reasonable to minimize the number of units destroyed during the tests, i.e., the number of units assigned to the DTs, while obtaining accurate reliability metrics estimation.

When designing the optimal sequential NDTs plans under accelerated conditions, the difference between the reliability estimation obtained under accelerated and normal conditions needs to be minimized, or/and the test durations need to be reduced. Investigation of the above problems is unique and has significant contributions.

It is difficult to continuously monitor one-shot units' status. In such a case, reliability tests are performed at discrete time intervals and only the number of occurrences (say failures) during an interval is known, i.e., no information between observation time points is available. Panel count data which include the number of event occurrences between observation time points are collected. In this dissertation, the collected panel count data demonstrate the testing units' nonhomogeneous characteristics, which are general. Reliability estimation is then obtained by considering the general panel count data.

1.1.3 Accelerated Life Testing (ALT) Models and Optimal Testing Plans

1.1.3.1 ALT Models

There are many situations that neither exact nor binary failure/degradation data of highly-reliable units under normal operating conditions are attainable during its expected life, e.g., the electronic units (transistors, resistors, integrated circuits, and capacitors, etc.) show low failure rates during their lifetime and only a small number of failures occur if the reliability tests are conducted at normal conditions. An effective approach to obtain failure data within a reasonable period of time is to consider accelerated life testing, which induces failures quickly by subjecting the units to severer stresses conditions than normal operating conditions. The accelerated data are then used to estimate the unit's reliability metrics under normal stresses. In this dissertation, we intend to investigate the use of the ALT in estimating the reliability of one-shot units.

The accuracy of reliability metrics estimation is highly dependent on: 1) a proper ALT model that models the unit's life characteristics, reflects the effect of applied stresses on the units' reliability metrics, and accurately relates the failure data under the severer stresses and normal stresses, and 2) an optimal ALT plan that specifically determines the details of implementing the ALT.

Elsayed (2012) classifies ALT models mainly into three categories: statistics-based models (including parametric and non-parametric models), physics-statistics-based models, and physics-experimental-based models.

Statistic-based models are generally used when the exact relationship between the applied stresses and failure times of the units cannot be determined based on the physics or chemistry principles of the units. Either parametric models (failure time distributions) or non-parametric models (linear, proportional-hazards (PH), and proportional-odds (PO), etc.) can be adopted. However, if only a small number of units are tested for a short time, evaluation process will be biased since insufficient failure data are collected. Besides, only analyzing the unit's lifetime based on statistic-based models is insufficient since statistics-based models fail to consider the unit's failure mechanism.

The physics-based models are effective in estimating unit's reliability metrics and addressing subtle performance problems (e.g., when only very few units are tested) once the unit's specific failure mechanisms are known; especially when the changes in the unit's physics are closely related to the unit's lifetime model parameters. However, the uncertainty of modeling (caused by the variation of the unit's material property, operating conditions, and the variation during the unit's manufacturing process) is not considered in most of the physics-based models. This is addressed in the next models.

Physics-statistics-based models consider both the statistics-based models and physics-based models by describing the physics of unit's failure mechanisms and considering the uncertainty during the modeling process. Arrhenius model, Eyring model, and inverse power rule model, are typical examples of physics-statistics-based models. We intend to investigate physics-statistics-based model to estimate the reliability of one-shot units.

In some cases, the unit exhibits degradation indicator before failure. In such cases, we monitor the degradation indicator with time (under normal or accelerated conditions) and utilize the degradation data to estimate the unit's/system's expected time to failure (time to reach a threshold of the degradation level) and other reliability metrics. We refer to the test where degradation data are collected at accelerated conditions as accelerated degradation testing (ADT). Elsayed (2012) summarizes typical ADT models, similar to the ALT models: the physics-based models such as resistor degradation mode, laser degradation model and hot-carrier degradation model; the stochastic degradation models such Brownian motion models, Inverse Gaussian models and Gamma models; and the physics-statistics-based models which include the characteristics of the previous two models. In this dissertation, we investigate the system reliability metrics under the scenario that the units are subject to different types of failure modes.

1.1.3.2 Optimal Testing Plans

Once the ALT (ADT) models are determined, an optimal ALT (ADT) plan is designed. The reliability metrics obtained via extrapolation in accelerated stresses levels and reduced test duration are inevitably less accurate than those obtained under normal reliability tests. The motivation of the optimal ALT (ADT) plan design is to obtain reliability metrics estimates at normal operating conditions as accurate as possible. Specifically, an ALT (ADT) plan needs to be designed to optimize a specific objective (usually minimizing the error of certain reliability metrics estimates at normal operating conditions) while satisfying given constraints.

Usually, some or all of the following decision variables need to be determined: the applied stresses (temperature, electric field, radiation, etc.), the method of stress application (constant stress, step stress, cyclic stress, ramp-step stress, and triangular-cyclic stress, etc.), the applied stresses levels, the number of testing units to be allocated to each stress level, the test duration, and the time to perform the test.

In this dissertation, different types of reliability tests are successively performed and corresponding optimal sequential testing plans based on sampling need to be designed (when the reliability tests are performed under accelerated stresses). Different from the traditional ALT plan that considers the reliability metrics of individual units, we propose sequential ALT plans by taking the system reliability metrics into account, which is significantly different from existing ALT plans and is realistic. Moreover, the optimal ALT plans are designed sequentially, i.e., the design of current testing plan is dependent on the previous testing plans, which incorporates and generalizes the traditional design of single ALT plan.

1.1.4 One-shot Units Optimal Operational Use

One-shot units such as missiles, airbags, and most of the military weapons are deployed after long terms of storage (or standby). It is important to ensure the stored units operate its function properly when needed. In this dissertation, we investigate the optimization of one-shot units' operational use at arbitrary time.

There exists many situations that the one-units are used consecutively when put into operational use. To illustrate, a certain number of one-shot units are selected from the stored population and launched in sequence. The population has a mixture of one-shot units with nonhomogeneous characteristics due to the units' different arrival times and the conduct of the sequential reliability tests during its storage period. Therefore, it becomes interesting and challenging to determine the characteristics and sequence of the one-shot units to be launched such that the operational use of the launched units is optimized. Defining the launched one-shot units as a system, the reliability metrics of the system (e.g., the probability that the system achieves the successful operation, the expected number of successfully launched units) need to be investigated. In this dissertation, we optimize the system's operational use at arbitrary time by formulating an optimization problem which is applicable to a variety of objectives. We also provide the bounds of the system's successful operational probability estimation and develop a simulation study to validate the proposed approach.

1.2 Dissertation Organization

The dissertation is organized as follows: In chapter 2, we provide a detailed review of related literature. We summarize the literature, analyze its contributions and limitations, and highlight the motivation as well as the uniqueness of the problems investigated in the dissertation.

In chapter 3, we propose the problem, describe the system in details, and develop effective models to estimate the system reliability metrics under different scenarios by either testing the population or testing the sample. The effect of aging on the system reliability is also taken into consideration. Defining the system as a generalized "$k(t)$-out-of-$n(t)$: F" system, we develop analytical expressions of the system reliability metrics. We also propose several computationally effective alternatives to investigate the system reliability metrics when the batch size is large or when the reliability tests are performed extensively over the time horizon. Extensive simulation studies are studied in chapter 3 to validate the proposed models under different scenarios.

In chapter 4, the problem investigated in chapter 3 is solved by using a stochastic approach. We assume that specific probabilistic distributions are used to describe the batches' arrival times, batches' sizes, and the times to conduct the tests, then the system reliability metrics are investigated. The potential characteristics of the system (the number of units in the system, the ages of the units, the times to perform the reliability tests, and the times to repair failed units, etc.) become complicated. The expected system reliability metrics are then compared with those investigated in chapter 3. Reliability metrics based on sampling is also discussed.

In chapter 5, we address the system reliability metrics under thermal fatigue. Modeling units' thermal fatigue life due to cyclic temperature fluctuation based on Coffin-Mason (CM) principle has been extensively investigated. However, sparse research assesses the thermal fatigue life by providing the reliability metrics of components/systems under thermal fatigue. We investigate a GBS distribution and its performance in predicting fatigue failure caused by thermal cyclic stresses. We then apply the GBS distribution to model the reliability metrics of a system with mixtures of nonhomogeneous one-shot units subject to thermal fatigue. An extensive simulation model is developed to validate the system reliability metrics accuracy. Numerical examples are presented to illustrate the use of the models.

In chapter 6, the reliability metrics of the system under competing failure modes are investigated. The unit's failure is observed either when any of its failure modes occurs suddenly (failure modes without indicators of failure) or when any of its degradation modes (which exhibit indicators that eventually lead to failure) reach its "failure threshold". Meanwhile, the unit is repaired either when it fails between two reliability tests or when one of its degradation modes reaches the predetermined "repair threshold", where the "repair threshold" is lower than the "failure threshold". We study the units' potential failure modes, its reliability at arbitrary time and the reliability metrics of the system under a generalized competing failure modes.

In chapter 7, we address the design of optimal sequential accelerated NDTs plans. We first study the individual unit's reliability behavior under accelerated conditions by developing a statistics-physics-based lifetime distribution, which directly relates the applied stresses to the system reliability metrics. We then propose the optimal design of sequential accelerated NDTs, taking system's/sample's reliability metrics under different testing scenarios into account. Specifically, we determine the optimal times to perform the tests, the applied stresses levels, and the test durations. The uncertainty during the sampling procedure as well as the units' characteristics are considered. We show that a well-designed sequential accelerated NDT is an effective approach to reduce the test durations while providing accurate reliability prediction with negligible consequences on the residual lives of the units and other system reliability metrics. Moreover, we numerically illustrate that the sample size has no effect on the accelerated NDTs plan design, in the long run.

In chapter 8, we consider the optimization of sequential hybrid reliability tests (including DT and NDT) based on sampling with the objective of minimizing the number of units used in the DT while obtaining accurate estimates of the reliability metrics. Specifically, we determine the optimal times to perform the tests and the number of units assigned to the two types of testing in each hybrid test. Consequently, after conducting a number of hybrid tests, we decrease the sample size of the DT as the accuracy of reliability metrics estimation improves and eventually we conduct NDT only. The optimization problem is studied under several testing scenarios. The proposed methods are validated by numerical illustrations and extensive simulation studies.

In chapter 9, we optimize the operational use of the one-shot units at arbitrary time during its storage. Some units (with nonhomogeneous characteristics) are selected from storage and launched or operationally used at arbitrary time as needed. Referring to the launched units as a system, we optimize the system's operational use by determining the selected units' characteristics (number of units selected from each batch) and its launching sequence. The system reliability metrics such as the expected number of successfully launched units, the average and variance of the probability that the system achieves a successful operation, and the expected time of the system's k^{th} failure are considered in the optimum selection procedure. Considering the units' inhomogeneity, we provide the confidence bounds of the probability that the system achieves a successful operation based on the units' optimal selection and launching sequence.

In chapter 10, we give conclusive remarks of our research.

CHAPTER 2

LITERATURE REVIEW

In this chapter, we present a detailed overview of research related to the problem being investigated in this dissertation. We summarize the limitations of current research and highlight the motivation and the uniqueness of the research in the dissertation. We first review the research on the storage life and reliability metrics of one-shot units. There is extensive work on thermal fatigue failure, however, limited research is done on the reliability metrics of systems with mixtures of one-shot units when subjecting to thermal fatigue. We review current study on thermal fatigue and summarize existing approaches and models on thermal fatigue. We then review the research on the reliability evaluation and optimization of different types of *k*-out-of-*n* systems. Most of the existing research is based on the assumptions that units in the system are identical, which is limited. There is few research on the reliability metrics of systems composed of one-shot units. Evaluation of system's reliability through the conduct of NDT is limited. There is no investigation on hybridizing NDT and DT when conducting reliability tests. We then present a thorough review of literature about the current ALT models and design of optimal ALT plans. Limited research is carried on the design, analysis, and application of sequential ALT plans when evaluating one-shot unit's storage reliability during its lifetime. We also review units' lifetime models under different failure modes related to the proposed research but not as general as the lifetime models we investigate in the dissertation.

In section 2.1, we review all related literature in details. In section 2.2, we summarize the limitations of the literature and highlight the importance and uniqueness of the research in this dissertation.

2.1 Literature Review

Approaches to estimate the reliability of one-shot electro-explosive units are introduced and current practices which can result in suspect data are discussed. Besides, remedial action which leads to accurate and dependable methods of reliability estimation is suggested (Peckham, 1965). Several priors in the Bayesian approach (with three different prior settings) for the predictions of reliability metrics of electro-explosive units at operating conditions under the exponential lifetime distribution are compared (Fan *et al*., 2009). The deterioration of stored one-shot units is examined and the reliability levels of individual units which together with the inspection regime give a particular reliability level in the delivered units are established (Newby, 2008). Apparently, no research is carried on the reliability metrics on systems that composed of nonhomogeneous one-shot units.

Approaches and difficulties of estimating the total life (which approximately equals to the storage life) of one-shot unit are investigated as follows: prognostics and health monitoring (PHM) systems are entirely applied in design and storage stages of tactical missile. Engineering approaches to the life degradation factor analysis and life prediction process are investigated (Li *et al*., 2014). The analysis results are used by U.S. Army personnel and contractors in evaluating current missile programs, the results are also used in the design of future missile systems. The storage-life modeling method of electric steering gear based on competing failure modes is studied, including the function and structure principle, the main performance parameters and failure thresholds of electric steering gear. The sensitivity of the various components of the electric steering gear is also analyzed in order to determine the degradation paths of the components (Deng *et al*., 2014). A case study of condition-based remaining storage life prediction for gyros in the inertial navigation system is presented (Wang *et al*., 2014) on the basis of the condition monitoring data by considering the slow degradation that occurs when the system is in storage. The previously mentioned literature only analyzes the storage lifetime of individual one-shot units rather than the reliability metrics of systems that compose one-shot units. Moreover, the effect of time on unit's storage life and characteristics is not addressed during the analysis.

Failure or degradation caused by thermal fatigue is a pervasive phenomenon. Temperature fluctuation in thermal fatigue can quickly lead to strains which are much higher than the elastic limit thus cause plastic strains and deformation in each cycle. Therefore, most mechanical stress fatigue models that deal with high-cycle failure (e.g., the S-N curve, Miner's law) are not readily applicable for modeling thermal fatigue failure. Instead, Coffin-Manson (CM) model and its modified versions (e.g., Norris-Landzberg model, Engelmaier's model and Darveaux's model) are widely adopted when assessing the unit's cycle-to-thermal-failure especially for low-cycle fatigue, where the CM model considers the loads in terms of plastic strain rather than stress.

Darveaux's model (McPherson, 2010) considers the strain rate effect under various ramp rates for the prediction of thermal fatigue lifetime of the solder interconnections. The derivation and application of the CM model to different types of materials (ductile, plastic, and brittle) are discussed and illustrated in McPherson (McPherson, 2010). The parameters of the CM model for solder fatigue life prediction are evaluated and different reference temperatures are simulated to investigate the effect of temperature on solder fatigue life prediction (Che and Pang, 2013)

The parameters of Engelmaier's model are recalibrated (Salmela *et al*., 2005) to contrast the fatigue data. A rapid life-prediction simulation approach for solder joints based on Engelmaier's model is developed (Qi *et al*., 2009) for combined temperature cycle and vibration conditions. The life expectancy of solder interconnect under simple temperature cycles using the Engelmaier model is predicted (Chai *et al*., 2014). The characteristic lifetime data are contrasted with a recalibrated Engelmaier's model (Putaala *et al*., 2012). Norris-Landzberg's model is recalibrated to provide an accurate estimation of units' lifetime under mechanical fatigue (Pan *et al*., 2005). It is also indicated that Norris–Landzberg's model fits the experimental data accurately and the error is less than 6% in the lead-free assemblies (Vasudevan and Fan, 2008). In this paper, the CM model is used to validate the accelerated statistical model that we propose.

Thermal cyclic test (TCT) is performed to obtain the units' fatigue life data which are then used for units' fatigue life modeling. When performing TCT, the testing units' material properties and the TCT plan (e.g., dwell time, heating and cooling rate and temperature amplitude) have a direct impact on the units' fatigue life. It shows that the damage per thermal cycle increases with the temperature amplitude (Qi *et al*., 2006). It is also validated that a faster ramp rate generates more fatigue failures per cycle and thus decreases the testing units' fatigue life (Zhai and Blish, 2003), (Qi *et al*., 2006), (Darveaux, 2002), (Chaparala *et al*., 2005) and (Ghaffarian, 2000). However, the effect of the ramp rate and the dwell time on testing units' life are negligible due to the stress relaxation at maximum cycle temperature (Schubert *et al*., 2002).

Accelerated thermal cyclic test (ATCT) is applied where the testing samples are subjected to a larger temperature fluctuation than the actual situation in order to induce sufficient failures. A comprehensive set of ATCTs are performed on vehicles (Ma *et al*., 2011) to analyze the impact of solder alloy characteristics on ATCT acceleration factors. By simulating ATCTs under different conditions, it is determined that the ATCT acceleration factor is independent of sample composition (Bosco *et al*., 2016). An ATCT is introduced (Yang *et al*., 2008) to approximate the operating conditions, showing that the low thermal conductivity and high specific heat of lead-free solder cause a short thermal fatigue life. Experimental and modeling results of surviving material combinations that occur at various temperatures are described (Shapiro *et al*., 2010). Different plated through holes (PTH) cycle-to-failure (CTF)-temperature models are evaluated for its effectiveness in determining acceleration factors for PTH fatigue life prediction under different conditions (Xie *et al*., 2008).

The Birnbaum-Saunders (BS) distribution is specifically developed for describing mechanical fatigue failures. It is noted that the BS distribution can be applied even when the assumption of the BS is relaxed (Desmond, 1985), i.e., the crack increment in a certain cycle not only depends on the applied stresses but is also affected by the accumulated crack size. It is stated that the BS distribution is more flexible than the Inverse Gaussian (IG) distribution (Bhattacharyya and Fries, 1982). Owen proposes a generalized BS (GBS) distribution by introducing a second shape parameter (Owen, 2006).

Reliability tests are performed during the storage period of one-shot units. Instead of continuous monitoring of the units, reliability tests are carried out at discrete times; correspondingly, the test data (panel count data) only show the status (fail, survive, or other states) of testing units. The following literature covers recent studies on the analysis of panel count data: the statistical analysis of interval-censored failure time data with applications are explored, three different data sets (Breast Cancer, Hemophilia, and AIDS data) are used to numerically illustrate both parametric and nonparametric methods of analysis (Sun and Fang, 2003). A family of mixed Poisson likelihood regression method for longitudinal interval count data is applied (Thall, 1988) to estimate and test the unit's failure rate over a time of a particular event. Meanwhile, a related empirical Bayesian estimation of random-effect parameters is proposed. A method for fitting the proportional hazards (PH) regression model when utilizing interval-censored observations is developed (Finkelstein, 1986). The Poisson assumption of panel count data when the observation time or process is related to the underlying recurrent event process is relaxed and a robust method for regression analysis of panel count data is applied (Zhao *et al.*, 2013).

Meanwhile, the asymptotic properties of the resulting estimates are discussed and numerical illustrations are proposed to demonstrate the use of the approach in practical situations. Nonparametric estimation procedures for the marginal mean function of a counting process are developed based on periodic observations (Hu *et al*., 2009). Obviously, most of the current research on panel count data emphasizes on the optimization of parameter estimation. Little attention has been paid to the application of panel count data to unit's lifetime models and reliability prediction.

It is assumed that the system is good if and only if less (no less) than *k* out of the *n* units fail (survive). There is an extensive work on the reliability of traditional *k*-out-of-*n*: F (G) systems: The reliability of a *k*-out-of-*n* system based on expandable reliability block diagram is calculated (Chang and Zhao, 2013), which is very efficient and convenient for engineering application. A fast and robust reliability evaluation algorithm based on conditional probabilities is developed (Amari *et al*., 2009), which is computationally efficient and general for calculating the exact reliability of large multi-state-*k*-out-of-*n* systems. An effective approach is proposed (Tian *et al*., 2008) for obtaining the "reliability bounds" of complex *k*-out-of-*n* systems with a large number of components and possible states by focusing on the probability of the system in certain states, which provides the range of the system reliability in a much shorter computation time.

Consecutive-*k*-out-of-*n*: F (G) system is a special kind of *k*-out-of-*n*: F (G) system, where the system works if and only if less (no less) than *k* consecutive units fail (survive). The reliability evaluation and optimization of such a system have been investigated as follows: Recent advances in methods of reliability evaluation, importance and optimal stochastic orderings of the units in consecutive-*k*-out-of-*n* systems are reviewed comprehensively (Eryilmaz, 2010). Optimal system reliability design of consecutive-*k*-out-of-*n* systems is investigated, both invariant and variant optimal design under different circumstances are proposed (Zuo, 1989). Zuo's work is generalized and extended to multi-dimensional consecutive-*k*-out-of-*n* systems (Kuo and Zuo, 2003); other types of *k*-out-of-*n* and consecutive-*k*-out-of-*n* systems are meanwhile addressed, e.g., the *s*-stage-*k*-out-of-*n* systems, linear and circular *m*-consecutive-*k*-out-of-*n* systems and the *k*-within-consecutive-*m*-out-of-*n* systems. The optimal component arrangement for a multi-state consecutive-*k*-out-of-*n*: F system with the objective of maximizing the expectation of the system state is considered (Akiba *et al*., 2011) when the selected units are arranged to the positions in the system. Graphical Evaluation and Review Technique (GERT) is applied (Agarwal *et al*., 2007) to deal with the reliability of *m*-consecutive-*k*-out-of-*n*: F system with (*k*-1)-step Markov dependence and *m*-consecutive-*k*-out-of-*n*: F system with Block-*k* dependence. The time efficiency of GERT is validated by illustrative numerical examples. The Stein-Chen method is employed (Godbole, 1993) to obtain the Poisson approximation for a linear *m*-consecutive-*k*-out-of-*n* system, considering the units in the system are stationary Markov dependent. The reliability and residual lifetime of linear and circular consecutive-*k*-out-of-*n* systems with nonhomogeneous components lifetimes are investigated (Salehi *et al*., 2011), i.e., the lifetimes of components are independent but nonhomogeneous. However, the units in the system are non-repairable and investigation of system's current status relies on system's previous status.

Performing reliability tests is an effective process to assess and predict the reliability of various types of units. Reliability tests might either destroy the testing unit's serviceability (DT) or not (NDT). The optimization of the NDT program is studied (Hunt and Wester, 2013) for a missile inventory system. Specifically, the number of testing units in each NDT is optimized with the constraint of frequency of NDTs. However, the reliability of individual missile is considered to be independent of the time and age, which is unrealistic. The test planning methods for designing accelerated destructive degradation tests is studied from different aspects (Shi *et al*., 2009), including non-Bayesian and Bayesian methods. The effect of sample size and test duration on the optimal plan is shown by conducting sensitivity analysis. The reliability of NDT techniques for the inspection of pipeline welds employed in the petroleum industry is evaluated (Carvalho *et al*., 2008). Tests on specimen made from pipelines containing defects are carried out and artificial neural networks (ANN) in the detection and automatic classification of the defects are used. To conclude, the use of NDT for reliability estimation of one-shot units is limited and the hybrid reliability testing that combines NDT and DT has not been addressed so far.

There are many situations when the failure data of the test units are lacking especially when the reliability test is performed under normal operating conditions. In such cases, ALT is applied by subjecting the units to severer-than-normal stresses, such that failures are induced in a much shorter time and accelerated data are then utilized with appropriate ALT model to estimate the units' lifetime under normal conditions.

Statistics-based models, physics-statistics-based models, and physics-experimental-based models are discussed (Elsayed, 2012). An overall review of ALT models that are widely used in engineering practice is provided (Escobar and Meeker, 2006), including both statistics-based models and physics-based models.

Several frequently used types of testing stresses and corresponding failure mechanisms that the stresses might induce are introduced (Elsayed, 2012). The design of ALT plans under different conditions and constrains are summarized; the concept and detailed application of the equivalent ALT plans are explained (Elsayed, 2012). The methods and principles of designing optimal ALT plans with two experimental factors without interactions are investigated (Escobar and Meeker, 1995). Escobar and Meeker's work is extended by designing the optimal ALT plan with two types of stresses assuming possible interactions between them (Park and Yum, 1996). It is widely accepted that if the ALT is performed under multiple stresses levels, the number of testing units should be allocated inversely proportional to the applied stresses levels.

An ALT plan needs to be designed to accurately estimate the unit's (system's) reliability metrics while reducing the testing time. Extensive work can be found in designing ALT plan on regular units (which is different from one-shot units) by assuming different types of ALT models and different objective functions (see (Liao and Elsayed, 2010), (Zhu and Elsayed, 2013), (Zhao and Elsayed, 2005), (Elsayed and Jiao, 2002), (Zhang, 2007), (Nelson and Meeker, 1978) , etc.). The application of ALT on one-shot units is limited: an optimal ALT plan is designed by assuming the testing unit' lifetime follows a Weibull distribution (Balakrishnan and Ling, 2013). However, the work assumes only a single accelerated DT is conducted and the units are homogeneous. Moreover, the testing plan is designed by considering the individual unit's reliability, which is limited. There is no research investigating the design of optimal accelerated NDT plan based on the system's reliability metrics with mixture of nonhomogeneous one-shot units.

The current research on ALT plan design only addresses the single optimal testing plan without considering the possibility that a sequential ALTs are performed. Besides, the testing scenario that failed units are repaired, placed back into the system, and are subjected to the next ALT has not been addressed yet.

When evaluating the reliability of the system or units (under either normal or accelerated stresses), the unit's lifetime model (physics-based model, statistics-based model, or physics-statistics-based model) is of great significance. A probabilistic physics of failure approach for estimating tube rupture frequency in steam generator of water reactor is presented (Chatterjee and Modarres, 2012). A methodology for the implementation of physics-based models of unit lifetimes is described (Hall and Strutt, 2003), treating the uncertain parameters as random variables which can be described by certain statistical distribution and sampled using Monte Carlo methods. The application of physics-statistics-based model to ALT, where the model parameter (usually scale parameter) is dependent on the applied stresses has not been extensively investigated. Current work on physics models or physics-statistics-based models does not pay attention to the lifetime of one-shot units, which is limited.

2.2 Limitations of Literature

A thorough review of the literature shows that the research on storage life and storage reliability of one-shot units is limited as current work only focuses on identical one-shot units. Moreover, the investigation of one-shot unit's reliability metrics is limited to the individual units. The effect of successive reliability tests on the unit's reliability has not been investigated. In additional, most of the current work only assigns a constant value as the reliability of the one-shot units, where the effect of time is not considered in the reliability modeling of the one-shot units. There is also no investigation on the reliability modeling of mixtures of nonhomogeneous one-shot units when subjecting to a variety types of failure modes such as thermal fatigue, degradation, and/or competing failure modes.

Although ALT models and optimization of ALT plan design have been widely and extensively investigated, they are limited to testing the units as a onetime ALT and all testing units are identical and non-repairable. Designing optimal sequential ALT plans for the system which considers the one-shot units' nonhomogeneous characteristics has not been addressed. The consequence of applying accelerated NDT needs to be discussed.

Significant work has been done on the reliability modeling of different types of systems. However, the characteristics of the units in the system are assumed to be identical, stationary, and known beforehand. This dissertation investigates the generalized $k(t)$-out-of-$n(t)$ systems composed of units whose characteristics vary and are nonhomogeneous, which is a generalization of previous work.

Although extensive research has been done on the reliability optimization of a variety of $k(t)$-out-of-$n(t)$ and consecutive-$k(t)$-out-of-$n(t)$ systems, literature review indicates that there is no research on nonhomogeneous one-shot units. Moreover, current research is based on the assumptions that the characteristics of the units in the system are known and the units are non-repairable, which fails to provide a generalized result.

The hybridization of NDT and DT when performing reliability tests has not been studied. Moreover, the effect of sequential hybrid reliability tests on the estimates of system reliability metrics has not be considered. We investigate hybrid reliability tests by performing NDT and DT under different testing scenarios. By investigating the design of optimal sequential hybrid tests, we increase the accuracy of system reliability estimation and the unit's lifetime parameter.

CHAPTER 3

RELIABILTY MODELING AND PREDICTION OF SYSYTEMS WITH MIXTURES OF UNITS

One-shot units such as missiles, airbags, and most military weapons, are usually produced in batches and then stored (or used as standby) for a long time before deployment. These units can be defined as a system which may experience degradation or failure during its storage period due to corrosion, thermal fatigue, or repeated shock loads. It is important to ensure these units or systems perform its function when needed regardless of its storage (standby) duration. Conducting NDTs during the storage period is an effective way to assess their functionality. Moreover, these tests are repeated during the entire life horizon of the unit (system). Corresponding actions (either repair or discard failed units) can be taken. The testing is randomly performed over time, and the population becomes a mixture because of the different ages of the units and the conduct of previous multiple test. Specifically, the population might contain previously repaired units, recently arrived units, or units that arrived at different time periods into the storage area).

Once all units in the storage are defined as a system, the reliability of the individual units which can be either identical or nonhomogeneous with different characteristics (age, number of repairs, failure rate, etc.) have a direct effect on the system's reliability metrics. A system with a mixture of different units can be defined as a general "$k(t)$-out-of-$n(t)$: F system". Traditional k-out-of-n: F systems assume all units in the system are identical and the system is good if and only if less than k out of the n units fail. However, when units arrive at different times and NDTs are performed repeatedly, both $k(t)$ and $n(t)$ become mixtures of nonhomogeneous units and increase with time. Details of such a system are described in this chapter.

The remainder of the chapter is organized as follows: section 3.1 presents the problem and proposes models for different scenarios. Predicting the distribution of failed units when considering the effect of aging on the system is presented. We then develop general expressions of the system reliability metrics in section 3.2. In section 3.3, predicting distribution of failed units as well as the expected number of failures by sampling are investigated. Three approaches are provided in section 3.4 to estimate system reliability over an extended time horizon as well as large batch sizes, their limitations and applicable conditions are discussed. A general simulation model is developed in section 3.5 to validate the proposed models and approaches. Several examples are provided in section 3.6 to numerically illustrate the use of the models.

3.1 Reliability Evaluation of Systems with Mixtures of Units

One-shot units are produced in batches and kept in storage until needed. Specifically, the i^{th} batch of units with size n_i arrives into the storage at time w_i immediately after production. NDTs are conducted at arbitrary time during the entire life horizon of the units by either testing all units in the storage area or by testing selected samples. The m^{th} test is performed at time t_m, which is independent of the unit's arrival time w_i. The test is assumed to be instantaneous (duration of the test is ignored). Failed units are either repaired and placed back into the system, or removed after a predetermined number of repairs (consecutive or not). The failure rates are increasing functions of the number of repairs, and the effect of aging on the failure rate is also considered. All units in the storage are defined as a system which is composed of nonhomogeneous units that arrived at different times, subjected to different series of tests, and repaired at different times. In this chapter, the system fails at time t, if $k(t)$ or more units fail, which is referred to as "$k(t)$-out-of-$n(t)$: F" system. This criterion is commonly used in many reliability testing such as reliability acceptance test and reliability demonstration test. Note that in this chapter, $k(t)$ is not fixed and increases as $n(t)$ increases with the continuous arrival of batches.

We investigate the distribution of failed units at the end of an arbitrary test when all units in the system are tested. The following are necessary notations:

w_i : time when the i^{th} batch arrives in storage;

t_m : time when the m^{th} test is performed, $m = 1, 2, 3...$;

K^m : total number of failed units observed at t_m ;

n_i : size of the i^{th} batch;

z_i: number of tests performed before arrival of the i^{th} batch;

$F^x(\cdot) / f^x(\cdot)$: *cdf*/*pdf* of unit lifetime after the x^{th} repair;

$K_{i,\underset{\sim}{j}}^{m}$ / $R_{i,\underset{\sim}{j}}^{m}$:number of failed/survived units at t_m , the $K_{i,\underset{\sim}{j}}^{m}$ / $R_{i,\underset{\sim}{j}}^{m}$ units are from the i^{th} batch and observed to fail at test sequence $\underset{\sim}{j}$; note that $\underset{\sim}{j}$ does not exist when $j = 0$;

$j(\alpha)$: the α^{th} NDT in test sequence $\underset{\sim}{j}$;

$\#\{\underset{\sim}{j}\}$: the number of tests in sequence $\underset{\sim}{j}$;

s_y : time when the y^{th} failure of the unit is observed; and

$\hat{s}_y$: last test time before s_y , if $s_y = t_y$, $\hat{s}_y = t_{y-1}$;

N_i^m : a certain batch state, which is a combination of units with different characteristics in the i^{th} batch;

N^m : a certain system state, which is a combination of units with different characteristics (certain states for all batches).

Figure 3.1 illustrates the relationship between $\hat{s}_y$ and s_y , where s_y could be the time of any of the first (m-1) tests. Note that $\hat{s}_y$ is uniquely determined by s_y , while s_{y-1} could be any test time of the first (y-1) tests.

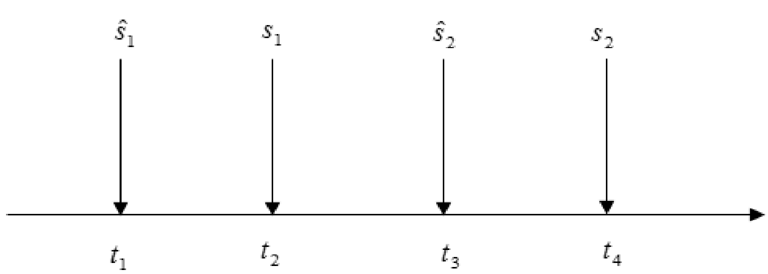

Figure 3.1 Relationship between $\hat{s}_y$ and s_y

There could be numerous possible system states when the tests are performed depending on the size and arrival time of the batches in the system. Generally, when the m^{th} test is performed, number of possible system states equals to the number of feasible solutions of the following equation sets. Meanwhile, number of solutions of the i^{th} row of the following equation sets is the number of possible states of the i^{th} batch:

$$
\begin{array}{ll}
\sum_{\forall \underset{\sim}{j}} \left(K_{1,\underset{\sim}{j}}^{m} + R_{1,\underset{\sim}{j}}^{m} \right) = n_1 & \text{Row } 1 \\
\vdots & \vdots \\
\sum_{\forall \underset{\sim}{j}} \left(K_{i,\underset{\sim}{j}}^{m} + R_{i,\underset{\sim}{j}}^{m} \right) = n_i & \text{Row } i
\end{array}
$$

We numerically show how the system states and batch states reflect by $K_{i,\underset{\sim}{j}}^{m}$ and $R_{i,\underset{\sim}{j}}^{m}$ $\forall i, \underset{\sim}{j}$ at time t_2 (the end of the 2$^{\text{nd}}$ test), assuming there are two batches ($i = 1, 2$) in the system by t_2, each batch has 5 units ($n_1 = n_2 = 5$). Specifically, the first batch arrives before the first NDT and the second batch arrives between the first and second NDT (see Figure 3.2). The system reliability metrics under different scenarios are analyzed in details from section 3.1.1, 3.1.2, and 3.1.3, respectively.

3.1.1 Distribution of Failed Units under "No-Removal" Scenario

In this section, we study the mixtures of units under the "no-removal" scenario, i.e., a failed unit is repaired and placed back into the population which results in a higher failure rate of the system. Figure 3.2 shows the failed and repaired units at the end of the first two NDTs under "no-removal" scenario, where we simplify the scenario by assuming only one batch arrives between two tests without loss of generality. It is assumed that t_1, t_2, n_1, n_2 , w_1 and w_2 are arbitrary.

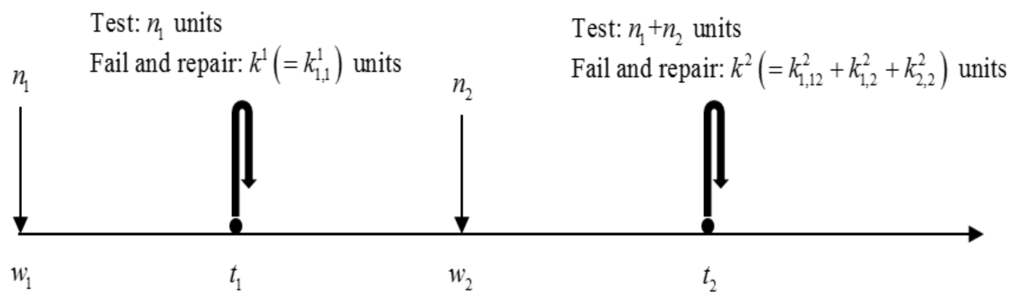

Figure 3.2 Test procedure under "no-removal" scenario

To obtain the probability of having exactly k^m (k^m ranges from 0 to $\sum_{\forall\{i;w_i \le t_m\}} n_i$) failures among all the units at test time ($P\left(K^m = k^m\right)$), we analyze the system state at time t_m , all system states that yield k^m failures (system state that satisfies $\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} k^m_{i,\underset{\sim}{j}} = k^m$) are considered and its probabilities are calculated and summed; consequently, the sum of the probabilities is $P\left(K^m = k^m\right)$ (term A of Eq. (3.1)). Since all batches are independent, the probability of certain system state (term B) equals to the product (term C) of the probabilities of all possible states of all batches. The probability of a batch's certain state (term D) is obtained by using a multinomial distribution (term D), equaling to number of trials/ways to generate the specific batch state (term E), times the probability of one "trail" to generate the specific batch state (certain combination of units with different characteristics (term F)):

$$
\begin{aligned}
P\left(K^{m}=k^{m}\right) &= P\left(\left(\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} K^{m}_{i,\underset{\sim}{j}}\right)=k^{m}\right)\\
&= \sum_{a}\sum_{b}\left[P\left(K^{m}_{i,\underset{\sim}{j}}=k^{m}_{i,\underset{\sim}{j}}, R^{m}_{i,\underset{\sim}{j}}=r^{m}_{i,\underset{\sim}{j}};\forall i,\underset{\sim}{j}\right)\right]\\
&= \overbrace{\sum_{a}\sum_{b}\underbrace{\left\{\prod_{\forall i}^{\underset{\sim}{C}}\overbrace{\left[\underbrace{\frac{(n_i)!}{\prod_{\forall \underset{\sim}{j}}\left(k^{m}_{i,\underset{\sim}{j}}\right)!\left(r^{m}_{i,\underset{\sim}{j}}\right)!}}_{E}\cdot\underbrace{P\left(K^{m}_{i,\underset{\sim}{j}}=k^{m}_{i,\underset{\sim}{j}}, R^{m}_{i,\underset{\sim}{j}}=r^{m}_{i,\underset{\sim}{j}};\forall \underset{\sim}{j}\right)}_{F}\right]}^{D}\right\}}_{B}}^{A}
\end{aligned}
\tag{3.1}
$$

where

$$
a=\left\{\left(\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} k^{m}_{i,\underset{\sim}{j}}\right)=k^{m}\right\}
$$

$$
b=\left\{\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} r^{m}_{i,\underset{\sim}{j}}=\left(\sum_{\forall i} n_i\right)-k^{m}\right\}
$$

Term F is calculated in Eq. (2) as:

$$
\begin{aligned}
P\left(K^{m}_{i,\underset{\sim}{j}}=k^{m}_{i,\underset{\sim}{j}}, R^{m}_{i,\underset{\sim}{j}}=r^{m}_{i,\underset{\sim}{j}};\forall \underset{\sim}{j}\right) &= \left(F\left(t_m-w_i\right)-F\left(t_{m-1}-w_i\right)\right)^{k^{m}_{i,1}}\left(1-F\left(t_m-w_i\right)\right)^{r^{m}_{i,0}}\\
&\cdot\prod_{\forall \underset{\sim}{j}}\left(\begin{array}{l}\left[\begin{array}{l}\left(F\left(t_{j(1)}-w_i\right)-F\left(t_{j(1)-1}-w_i\right)\right)\cdot L\cdot\\ \left(F^{j}\left(t_m-j(j)\right)-F^{j}\left(t_{m-1}-j(j)\right)\right)\end{array}\right]^{k^{m}_{i,\underset{\sim}{j}}}\\ \cdot\left[\begin{array}{l}\left(F\left(t_{j(1)}-w_i\right)-F\left(t_{j(1)-1}-w_i\right)\right)\cdot L\cdot\\ \left(1-F^{j}\left(t_m-j(j)\right)\right)\end{array}\right]^{r^{m}_{i,\underset{\sim}{j}}}\end{array}\right)
\end{aligned}
\tag{3.2}
$$

where $L = \prod_{a=1}^{j\text{-}1}\left(F^{a}\left(t_{j(\alpha+1)} - t_{j(\alpha)}\right) - F^{a}\left(t_{j(\alpha+1)-1} - t_{j(\alpha)}\right)\right)$.

Note that s_1 could be any test selected from $\left(z_i+1,\ldots m-1\right)$, and s_a could be any test selected from $\left(s_{a-1}+1,\ldots,m-1\right)$. Moreover, constraint (3.3) must be satisfied:

$$\sum_{\forall \underset{\sim}{j}}\left(k_{i,\underset{\sim}{j}}^{m} + r_{i,\underset{\sim}{j}}^{m}\right) = n_i \quad \forall i \tag{3.3}$$

The expected number of failures at t_m is calculated accordingly:

$$E\left(K^{m}\right) = \sum_{k^{m}=0}^{\sum_{\forall i} n_i} k^{m} \cdot P\left(K^{m} = k^{m}\right) \tag{3.4}$$

3.1.2 Distribution of Failed Units under “*p*-Consecutive-Failure-Removal” Scenario

Now consider the same system discussed in the previous scenario with the exception that when a unit experiences the p^{th} consecutive failure, it is removed from the system without repair. The following notations are necessary to obtain the distribution of failures:

$\underset{\sim}{j}'$: test sequence indicating the time when the failures are observed where $\#\left\{\underset{\sim}{j}\right\} < p$;

$\underset{\sim}{j}''$: test sequence indicating the time when the failures are observed, where $\#\left\{\underset{\sim}{j}\right\} \geq p$ and there are less than p consecutive tests in $\underset{\sim}{j}''$;

$\underset{\sim}{j}'''$: test sequence indicating the time when the j failures are observed, where $\#\left\{\underset{\sim}{j}\right\} \geq p$ and there are p or more consecutive tests in $\underset{\sim}{j}'''$; and

$\underset{\sim}{j}$: union of $\underset{\sim}{j}'$, $\underset{\sim}{j}''$ and $\underset{\sim}{j}'''$.

Compared to Eq. (3.1), $P\left(K^m = k^m\right)$ changes since system states that include units with p or more consecutive failures (combinations of $k^m_{i,\underset{\sim}{j}}$ and $r^m_{i,\underset{\sim}{j}}$ units including $\underset{\sim}{j}'''$) are not counted. The term B remains unchanged as explained in section 3.1.1.

$$P\left(K^m = k^m\right) = P\left(\left(\sum_{\forall i}\sum_{\forall \underset{\sim}{j}'} K^m_{i,\underset{\sim}{j}}\right) + I \cdot \left(\sum_{\forall i}\sum_{\forall \underset{\sim}{j}''} K^m_{i,\underset{\sim}{j}}\right) = k^m\right)$$
$$= \overbrace{\sum_a\sum_b}^{A} \left\{\prod_{\forall i}\left[\frac{\left[(n_i)!\right] \cdot \overbrace{P\left(K^m_{i,\underset{\sim}{j}} = k^m_{i,\underset{\sim}{j}}, R^m_{i,\underset{\sim}{j}} = r^m_{i,\underset{\sim}{j}}; \forall j, \underset{\sim}{l}_j\right)}^{B}}{\prod_{\forall \underset{\sim}{j}}\left(k^m_{i,\underset{\sim}{j}}\right)!\left(r^m_{i,\underset{\sim}{j}}\right)!}\right]\right\} \tag{3.5}$$

where $\begin{cases} I = 0 & \text{if } z_i \geq m - p + 1 \\ I = 1 & \text{if } z_i < m - p + 1 \end{cases}$, and terms a and b are interpreted in section 3.1.1.

In additional, constraint (3.6) must be satisfied:

$$\left[\sum_{\forall \underset{\sim}{j}'} \left(k^m_{i,\underset{\sim}{j}'} + r^m_{i,\underset{\sim}{j}'}\right)\right] + I \cdot \left[\sum_{\forall \underset{\sim}{j}''} \left(k^m_{i,\underset{\sim}{j}''} + r^m_{i,\underset{\sim}{j}''}\right) + \sum_{\forall \underset{\sim}{j}'''} \left(k^m_{i,\underset{\sim}{j}'''} + r^m_{i,\underset{\sim}{j}'''}\right)\right] = n_i \ \forall i \tag{3.6}$$

3.1.3 Distribution of Failed Units under "*p*-Failure-Removal" Scenario

If a unit is removed from the system when it experiences the p^{th} failure (either consecutive or not), system states including units with p or more failures ($k^m_{i,\underset{\sim}{j}}$ and $r^m_{i,\underset{\sim}{j}}$ units including either $\underset{\sim}{j}''$ or $\underset{\sim}{j}'''$) are not counted, the failure distribution of the system is obtained via Eq. (3.7):

$$\begin{aligned} P\left(K^m = k^m\right) &= P\left(\left(\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} K^m_{i,\underset{\sim}{j}}\right) = k^m\right) \\ &= \sum_a \sum_b \left\{ \prod_{\forall i} \left[\frac{\left[(n_i)!\right] P\left(K^m_{i,\underset{\sim}{j}} = k^m_{i,\underset{\sim}{j}}, R^m_{i,\underset{\sim}{j}} = r^m_{i,\underset{\sim}{j}}; \forall \underset{\sim}{j}\right)}{\prod_{\forall \underset{\sim}{j}} \left(k^m_{i,\underset{\sim}{j}}\right)! \left(r^m_{i,\underset{\sim}{j}}\right)!} \cdot \right] \right\} \end{aligned} \tag{3.7}$$

where terms a and b are interpreted in section 3.1.1.

Comparing the system failure distributions under the above three scenarios, we expect most failures when there is no removal (scenario 1). An intuitive explanation is that keeping repaired units which have higher failure rates increases the expected number of failures. In section 3.6, we provide a numerical comparison of system reliability metrics under the three scenarios.

3.1.4 Distribution of Failed Units Considering Effect of Aging

Currently, units stored for long periods of time experience aging, which has a direct effect on its failure rates. Assuming an one-shot unit has a Weibull storage lifetime in Eq.(3.8), its failure rate increases due to change in the Weibull shape parameter. In other words, the shape parameter is an increasing function of the unit's age. Under such circumstance, term B in Eq. (3.1) is obtained by Eq. (3.9) as:

$$F\left(t;\theta,\gamma\left(t\right)\right)=1-e^{\left(-\left(\frac{t}{\theta}\right)^{\gamma(t)}\right)} \tag{3.8}$$

$$\begin{aligned}
&P\left(K_{i,\underset{\sim}{j}}^{m}=k_{i,\underset{\sim}{j}}^{m},R_{i,\underset{\sim}{j}}^{m}=r_{i,\underset{\sim}{j}}^{m};\forall\underset{\sim}{j}\right)\\
&=\left(\int_{0}^{t_m-w_i} f\left(\delta;\theta,\gamma\left(\delta\right)\right)d\delta-\int_{0}^{t_{m-1}-w_i} f\left(\delta;\theta,\gamma\left(\delta\right)\right)d\delta\right)^{k_{i,\underset{\sim}{1}}^{m}}\left(1-\int_{0}^{t_m-w_i} f\left(\delta;\theta,\gamma\left(\delta\right)\right)d\delta\right)^{r_{i,0}^{m}}\\
&\cdot\prod_{\forall\underset{\sim}{j}}\left(\begin{array}{l}
\left[\begin{array}{l}
\left(\int_{w_i}^{s_1} f\left(\delta-w_i;\theta,\gamma\left(\delta-w_i\right)\right)d\delta-\int_{w_i}^{\hat{s}_1} f\left(\delta-w_i;\theta,\gamma\left(\delta-w_i\right)\right)d\delta\right)\\
\cdot L\\
\cdot\left(\int_{s_j}^{t_m} f^{j}\left(\delta-s_j;\theta,\gamma\left(\delta-w_i\right)\right)d\delta-\int_{s_j}^{t_{m-1}} f^{j}\left(\delta-s_j;\theta,\gamma\left(\delta-w_i\right)\right)d\delta\right)
\end{array}\right]^{k_{i,\underset{\sim}{j}}^{m}}\\
\left[\begin{array}{l}
\left(\int_{w_i}^{s_1} f\left(\delta-w_i;\theta,\gamma\left(\delta-w_i\right)\right)d\delta-\int_{w_i}^{\hat{s}_1} f^{j}\left(\delta-w_i;\theta,\gamma\left(\delta-w_i\right)\right)d\delta\right)\\
\cdot L\cdot\left(1-\int_{s_j}^{t_m} f^{j}\left(\delta-s_j;\theta,\gamma\left(\delta-w_i\right)\right)d\delta\right)
\end{array}\right]^{r_{i,\underset{\sim}{j}}^{m}}
\end{array}\right)
\end{aligned} \tag{3.9}$$

where

$$L=\prod_{\forall a}\left(\left(\int_{s_a}^{s_{a+1}} f^{a}\left(\delta-s_a;\theta,\gamma\left(\delta-w_i\right)\right)d\delta-\int_{s_a}^{\hat{s}_{a+1}} f^{a}\left(\delta-s_a;\theta,\gamma\left(\delta-w_i\right)\right)d\delta\right)\right)$$

The above model assumes that a failed unit is repaired to as-good-as new condition. However, the effect of aging and repairs on the failure rate is cumulative. Accordingly, $P\left(K^m=k^m\right)$ under different scenarios can be obtained by applying the models in section 3.1.

3.2 System Reliability

Wide applications of fault-tolerant systems, e.g., *k*-out-of-*n* systems, are found in industrial and military systems where it is not necessarily that all units are required to be operational. In this chapter, the system is operational when less than $q\%$ of units of the system fail, which can be defined as a generalized "*k*(*t*)-out-of-*n*(*t*)" system. Clearly, both *k*(*t*) and *n*(*t*) are functions of time, i.e., they experience step increases when new batches arrive. Specifically, we have: $k\left(t\right)=\left\lfloor q\%\cdot\left(\sum_{\forall\{i,w_i\le t\}} n_i\right)\right\rfloor$ and $n\left(t\right)=\sum_{\forall\{i,w_i\le t\}} n_i$. Eq. (3.10) calculates the system reliability at arbitrary time:

$$R_m\left(t\right)=\sum_{k^m=0}^{\left\lfloor q\%\cdot\left(\sum_{\forall i,w_i\le t} n_i\right)\right\rfloor} P\left(K^m=k^m\right) \tag{3.10}$$

where $R_m(t)$ is the reliability of the system at any time between (t_{m-1}, t_m) and $m = 1, 2, \ldots$. Specifically, system reliability is a piecewise function as shown in Eq. (3.11). Expression of system reliability changes according to the increasing number of tests since the potential system states increase significantly with the performance of tests. The reliability is expressed as

$$R(t) = \begin{cases} R_1(t) & 0 \le t \le t_1 \\ R_2(t) & t_1 < t \le t_2 \\ \vdots & \\ R_m(t) & t_{m\text{-}1} < t \le t_m \\ \vdots & \end{cases} \tag{3.11}$$

Intuitively, the system reliability shows a small step increment when new units are introduced, or when the system is tested and the failed units are repaired.

Reliability prediction models are proposed in this section to deal with systems experiencing complex states, taking into account the age distribution of the units, repaired/unrepaired units, removal of failing units, and the conduct of multiple tests. These models can be applied to different cases, e.g., when the repair of the units results in as-good-as new units (with unchanged failure rates), or when the repair only restores the units to as-good-as old. Moreover, in reality, previous test results are known, i.e., $k_{i,\underset{\sim}{j}}^{m-1}$ and $r_{i,\underset{\sim}{j}}^{m-1}\ \forall i, \underset{\sim}{j}$ are known when predicting system reliability metrics at t_m; under such circumstance, there is no need to investigate potential system states in the previous tests, consequently, $P\left(\left(K_{i,\underset{\sim}{j}}^{m} = k_{i,\underset{\sim}{j}}^{m}, R_{i,\underset{\sim}{j}}^{m} = r_{i,\underset{\sim}{j}}^{m}; \forall i, \underset{\sim}{j}\right) \middle| \left(K_{i,\underset{\sim}{j}}^{m-1} = k_{i,\underset{\sim}{j}}^{m-1}, R_{i,\underset{\sim}{j}}^{m-1} = r_{i,\underset{\sim}{j}}^{m-1}; \forall i, \underset{\sim}{j}\right)\right)$ can be similarly obtained.

Theoretically, system reliability metrics can be obtained using the above models under any scenarios. However, for large t_m and/or, the extensive system states complicate the analytical expression. Later, we present approaches for reliability estimation under such circumstances.

3.3 System Reliability by Sampling

When the size of the total population in the system is large, it is impractical to test all units in the system. A possible approach to estimate the reliability of the system is to perform sampling. What fundamentally matters for the statistic from a random sample is that it represents the population mixture. Selecting a sample size proportional to the batch size and investigating effective relationship between the population and the samples become a feasible approach to estimate the reliability of the system.

Additional notations:

$S_{K^m_{i,\underset{\sim}{j}}} / S_{R^m_{i,\underset{\sim}{j}}}$: number of failed/survived units that are selected from $K^m_{i,\underset{\sim}{j}}$ and $R^m_{i,\underset{\sim}{j}}$ at t_m ;

$P\left(N^m\right)$: probability of being in system state N^m , i.e., the term B in Eq. (3.1) ($K^m_{i,\underset{\sim}{j}} = k^m_{i,\underset{\sim}{j}}, R^m_{i,\underset{\sim}{j}} = r^m_{i,\underset{\sim}{j}}; \forall i, \underset{\sim}{j}$);

s_i : number of units selected from the i^{th} batch; and

K_s^m : number of selected units that fail at t_m .

Assuming all failed units are repaired and placed back into the population, the failure distribution of the sample is obtained by Eq. (3.12):

$$P\left(K_s^m = k_s^m\right) = \sum_{\forall N^m} P\left(\left(K_s^m = k_s^m\right) \middle| N^m\right) \cdot P\left(N^m\right) \tag{3.12}$$

where

$$\begin{aligned} P\left(K_s^m = k_s^m \middle| N^m\right) &= \sum_{\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} s_{k_{i,\underset{\sim}{j}}^m} = k_s^m} P\left(S_{K_{i,\underset{\sim}{j}}^m} = s_{k_{i,\underset{\sim}{j}}^m}, S_{R_{i,\underset{\sim}{j}}^m} = s_{r_{i,\underset{\sim}{j}}^m} \middle| N^m; \forall i, \underset{\sim}{j} \right) \\ &= \sum_{\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} s_{k_{i,\underset{\sim}{j}}^m} = k_s^m} \frac{\prod_{\forall i}\prod_{\forall \underset{\sim}{j}} \binom{k_{i,\underset{\sim}{j}}^m}{s_{k_{i,\underset{\sim}{j}}^m}} \binom{r_{i,\underset{\sim}{j}}^m}{s_{r_{i,\underset{\sim}{j}}^m}}}{\prod_{\forall i} \binom{n_i}{s_i}} \end{aligned} \tag{3.13}$$

Eq. (3.13) calculates the expected distribution of failures of a sample. Since repeated tests are performed and repeated samples are selected, in the long run, the expectation of the sample's reliability metrics represents the population's metrics without loss of generality, as on average units with different characteristics are proportionally selected in the samples. This is validated in section 3.5.

3.4 Alternative System Reliability Evaluation Approaches

A computationally efficient approach is required when m or n_i are large. We provide three time-saving approaches, summarize the limitations and applicable conditions of the three approaches in this section.

3.4.1 Approach 1 (A1)

This approach only considers the effects of some of the repairs or/and aggregates the arrival times of different batches. Since the batch arrival time and the test time are independent, we can use the exact arrival time of each batch while considering some of the repairs, or aggregate the arrival times of different batches, but take all repairs into consideration. Moreover, this approach obtains system reliability metrics conditional on the previous system metrics, e.g., system's reliability performances obtained in the m^{th} test are utilized to predict the system's reliability in the following test. Intuitively, it provides accurate estimation especially for early tests but underestimates the system reliability as m increases because some repairs are overlooked. To conclude, it applies to the situation when $\frac{t}{\theta} < 1$ or when the time increment between two adjacent tests is small. A reasonable explanation is when $\frac{t}{\theta} < 1$, the probability of failure is small and the effect of the repair is insignificant. We numerically illustrate an application of A1 in section 3.6.

3.4.2 Approach 2 (A2)

One of the most frequently investigated reliability metrics is the expected number of failures during a specified time period. In this section, we apply the renewal process (RP) since the observed time scale is discrete. We first investigate the scenario when the repair of the units has no effect on units' failure rates. The following additional notations are necessary:

M_i^m : the number of expected failures of one unit from the i^{th} batch by t_m; and

$M\left(t_{\rho,m}\right)$: the number of expected failures of one unit in the interval $\left[t_\rho, t_m\right]$.

The number of expected failures until time t_m for a unit from the i^{th} batch is derived as:

$$M_i^m = \left[1 + M\left(t_{z_i+1,\, m}\right)\right] \int_0^{t_{z_i+1}-w_i} f(t)\,dt + I \cdot \left\{ \sum_{\alpha=1}^{m-z_i-1} \left[1 + M\left(t_{z+a+1,\, m}\right)\right] \int_{t_{z_i+a}-w_i}^{t_{z_i+a+1}-w_i} f(t)\,dt \right\} \tag{3.14}$$

where $I = \begin{cases} 1 & \text{if } z_i < m-1 \\ 0 & \text{if } z_i = m-1 \end{cases}$, and

$$M\left(t_{\rho,m}\right) = \sum_{\alpha=1}^{m-\rho} \left[1 + M\left(t_{\rho+a,m}\right)\right] \int_{t_{\rho+a-1}-t_\rho}^{t_{\rho+a}-t_\rho} f(t)\,dt$$

$$M\left(t_{m,\,m}\right) = 0 \;\; \forall i$$

The integral $\int_{t_{z_i+a}-w_i}^{t_{z_i+a+1}-w_i} f(t)\,dt$ is the probability that the first failure of the unit occurs in the interval $\left(t_{z_i+a}, t_{z_i+a+1}\right)$.

RP can be applied to the general case when the failure rate of the unit is an increasing function of the number of repairs experienced by the unit. We define the following notations:

$f^{\varphi}(t)$: *pdf* of unit lifetime after φ failures; and

$M^{\varphi}\left(t_{\rho,m}\right)$: the number of expected failures of a unit during interval $\left[t_{\rho}, t_m\right]$ with φ failures.

The number of expected failures until t_m for one unit in the i^{th} batch is obtained recursively by:

$$M_i^m = \left[1 + M^1\left(t_{z_i+1,\,m}\right)\right] \int_0^{t_{z_i+1}-w_i} f^0(t)\,dt + I \cdot \left\{ \sum_{\alpha=1}^{m-z_i-1} \left[1 + M^1\left(t_{z_i+a+1,m}\right)\right] \int_{t_{z+a}-w_i}^{t_{z+a+1}-w_i} f^0(t)\,dt \right\} \quad (3.15)$$

where $I = \begin{cases} 1 & \text{if } z_i < m-1 \\ 0 & \text{if } z_i = m-1 \end{cases}$, and

$$M^{\varphi}\left(t_{\rho,m}\right) = \sum_{\alpha=1}^{m-\rho} \left[1 + M^{\varphi+1}\left(t_{\rho+a,m}\right)\right] \int_{t_{\rho+a-1}-t_{\rho}}^{t_{\rho+a}-t_{\rho}} f^{\varphi}(t)\,dt \quad \forall \varphi$$

$$M^{\varphi}\left(t_{m,\,m}\right) = 0 \quad \forall \varphi$$

For the system, we have:

$$E_{system}\left(K^m\right) = \sum_{\forall i} n_i \cdot M_i^m \tag{3.16}$$

$$E_{system}\left(K^{m\text{-}1}\right) = \sum_{\forall i} n_i \cdot M_i^{m-1} \tag{3.17}$$

The total number of expected failures at t_m equals to $E_{system}\left(K^m\right) - E_{system}\left(K^{m-1}\right)$. Applying Eqs. (3.16) and (3.17), the distribution of failures in the m^{th} test is obtained by first calculating the average failure probability p_m of a single unit: $p_m = \dfrac{E_{system}\left(K^m\right) - E_{system}\left(K^{m\text{-}1}\right)}{\sum_{\forall i} n_i}$.

Accordingly, $P\left(K^m = k^m\right)$ is calculated as:

$$P\left(K^m = k^m\right) = \begin{pmatrix} \sum_{\forall i} n_i \\ k^m \end{pmatrix} \left(p_m\right)^{k^m} \left(1 - p_m\right)^{\left(\sum_{\forall i} n_i\right) - k^m} \tag{3.18}$$

3.4.3 Approach 3 (A3)

The complexity of the models discussed in section 3.1 is caused by the investigation of the system's detailed conditions. Specifically, listing all feasible solutions to $\sum_{\forall \underset{\sim}{j}} \left(K_{i,\underset{\sim}{j}}^m + R_{i,\underset{\sim}{j}}^m\right) = n_i$ and calculating corresponding probabilities become complex as n_i and m increase. We derive a simplified model which yields accurate reliability prediction by defining K_i^m and R_i^m as the number of failed and survived units at t_m from the i^{th} batch.

We only consider the effect of aging of the units on the failure distribution and $P\left(K_i^m = k_i^m, R_i^m = r_i^m\right)$ and $P(K^m = k^m)$ are respectively determined as:

$$P(K^m = k^m) = \sum_{\left(\sum_{\forall i} k_i^m\right) = k^m} \; \sum_{\left(\sum_{\forall i} r_i^m\right) = \left(\sum_{\forall i} n_i\right) - k^m} \prod_{\forall i} \left[\frac{\left[(n_i)!\right] P\left(K_i^m = k_i^m, R_i^m = r_i^m\right)}{\left(k_i^m\right)!\left(r_i^m\right)!} \right] \tag{3.19}$$

where

$$P\left(K_i^m = k_i^m, R_i^m = r_i^m\right) = \begin{bmatrix} \left(F\left(t_m - w_i\right) - F\left(t_{m-1} - w_i\right)\right) \\ + \sum_{j=2}^{m-z_i} \sum_{\forall a} \begin{bmatrix} \left(F\left(s_1 - w_i\right) - F\left(\hat{s}_1 - w_i\right)\right) \\ \prod_{a=1}^{j-1} \left(F^a\left(s_{a+1} - s_a\right) - F^a\left(\hat{s}_{a+1} - s_a\right)\right) \\ \left(F^j\left(t_m - s_j\right) - F^j\left(t_{m-1} - s_j\right)\right) \end{bmatrix} \end{bmatrix}^{k_i^m} \cdot$$

$$\begin{bmatrix} \left(F\left(t_m - w_i\right) - F\left(t_{m-1} - w_i\right)\right) \\ + \sum_{j=2}^{m-z_i} \sum_{\forall a} \begin{bmatrix} \left(F\left(s_1 - w_i\right) - F\left(\hat{s}_1 - w_i\right)\right) \\ \prod_{a=1}^{j-1} \left(F^a\left(s_{a+1} - s_a\right) - F^a\left(\hat{s}_{a+1} - s_a\right)\right) \\ \left(1 - F^j\left(t_m - s_j\right)\right) \end{bmatrix} \end{bmatrix}^{r_i^m}$$

In this section, we propose three computationally efficient approaches to estimate the system reliability. A1 is limited as it only predicts the system reliability metrics accurately at its early stages. A2 calculates the system failure distribution within short time under arbitrary condition and provides results close to the exact solutions. A3 gives exact system reliability metrics while masking detailed system conditions. A2 and A3 are compared with a simulation model in section 3.5 to validate its feasibility.

3.5 Simulation Model

In this section, we develop a simulation model to validate the exact model, A2, and A3. We introduce the objective and the procedure of the simulation study in section 3.5.1. In section 3.5.2, we numerically obtain the reliability metrics using the simulation model and compare the results with those of three proposed approaches. We then predict the system reliability metrics with large size batches and extensive testing over time, based on A2.

3.5.1 Simulation Model

The objective of this simulation model is to validate that:

a. The exact model and A3 obtain accurate and identical system reliability metrics prediction;
b. A2 provides accurate system reliability metrics prediction for large t_m and n_i.

To validate the exact model, A2, and A3 comprehensively, we apply *r* (*r* is usually large) replications of simulation, each replication with one set of randomly generated test parameters (batch size n_i, batch arrival time w_i, number of batches between two adjacent tests α_y, test time t_y, where $y = 1, 2, ..., m$). In each replication, we obtain the reliability metrics ($E\left(K^m\right)$ and $P\left(K^m = k^m\right)$) using the three approaches and the simulation model. Afterwards, we obtain and compare the mean and standard deviation of the r replications.

The test parameters are randomly generated with:

$\alpha_y \sim unidrnd\left(\alpha_L, \alpha_U\right), y = 1, 2, ..., m$;

$t_y = t_{y-1} + rand(t_L, t_U), y = 1, 2, ..., m$;

$n_i \sim unidrnd\left(n_L, n_U\right) \ \ \forall i$; and

$w_i \sim rand\left(t_{y-1}, t_y\right), \quad \forall \ w_i \in \left(t_{y-1}, t_y\right)$

The failure time of the units has the following Weibull parameters:

$$F^x\left(t;\theta,\gamma\right) = 1 - e^{-\left(\frac{t}{100}\right)^{\gamma_x}} \tag{3.20}$$

where x represents the effect of number of previous repairs on the failure rate as discussed in section 3.1 and γ_x is the shape parameter which is dependent on x.

To reflect the scenario, we randomly generate α_y (the number of batches arriving between the y^{th} and the $(y\text{-}1)^{th}$ test for any y). The set of generated $\alpha_y \ \forall y$ is fixed once generated and used for all r replications of parameter (t_y, n_i, w_i) generations. Note that $t_y \forall y$ are first generated, then the batch arrival times and sizes ($w_i, n_i \ \forall i$) are generated accordingly.

We apply Eq. (3.4), Eqs. (3.16)-(3.17), and Eq.(3.19) to obtain $E\left(K^m\right)$ using the exact model, A2, and A3, respectively. We specifically express the number of failures at end of the m^{th} test (K^m) as the sum of units that fail once or more before t_m and the units that never failed until t_m. We then obtain the number of failures at the end of one iteration which is repeated for 10^3 times to simulate the system conditions under a certain set of realistic test parameters.

The following notations are used during one iteration in the simulation procedure:

j': the $j'^{\text{ th}}$ step in the iteration to obtain $K^m_{i,\underset{\sim}{j}}$;

$j(j)$/ $j(j-1)$/ $j(j-j')$/ $j(j-j'+1)$: j^{th} (last)/$(j-1)^{\text{th}}$/ $(j-j')^{\text{th}}$/$(j-j'+1)^{\text{th}}$ element (time point) in test sequence $\underset{\sim}{j}$;

$K^{j(j)}_{i,\left(\underset{\sim}{j}-1\right)}$: the number of failed units at the end of the $j(j)^{\text{th}}$ test; the $K^{j(j)}_{i,\left(\underset{\sim}{j}-1\right)}$ units are in the i^{th} batch and fail in sequence $\left(\underset{\sim}{j}-1\right)$; where $\left(\underset{\sim}{j}-1\right)$ is composed of the first $\left(\#\left\{\underset{\sim}{j}\right\}-1\right)$ tests in $\underset{\sim}{j}$.

- **Estimation of** $K^m_{i,\underset{\sim}{1}}$ $\forall i$:

1. Generate $\sum_{y=1}^{m}\alpha_y$ groups of random failure times that follow Weibull distribution with $x=0$, the i^{th} group has n_i random failure times.

2. For the i^{th} group, count the number of failure times that occurred between $\left(t_{m-1}-w_i, t_m-w_i\right)$, record as $K_{i,\underset{\sim}{1}}^{m}$ $\forall i$.

3. Do 2 $\forall i$, record $K_{i,\underset{\sim}{1}}^{m}$ $\forall i$.

- **Estimation of** $K_{i,\underset{\sim}{j}}^{m} \forall i, \underset{\sim}{j}\ \forall \#\left\{\underset{\sim}{j}\right\} > 1$:

1. For any $i, \underset{\sim}{j}$, generate $K_{i,\left(\underset{\sim}{j}-1\right)}^{j(j)}$ random failure times that follow Weibull distribution with $x = j-1$.

2. Count the number of failure times between $\left(t_{m-1}-t_{j(j)}, t_m-t_{j(j)}\right)$, record as $K_{i,\underset{\sim}{j}}^{m}$.

3. Do 2 $\forall\ i, \underset{\sim}{j}$, record $K_{i,\underset{\sim}{j}}^{m}$ $\forall\ i, \underset{\sim}{j}$.

The following are procedures of the estimation of $K_{i,\underset{\sim}{j}-1}^{j(j)}$:

1. Generate $K_{i,\underset{\sim}{j}-2}^{j(j-1)}$ random failure times that follow Weibull distribution.

2. Count the number of failure times that occurred between $\left(t_{j(\hat{j})}-t_{j(j-1)}, t_{j(j)}-t_{j(j-1)}\right)$, record as $K_{i,\underset{\sim}{j}-1}^{j(j)} \forall i, \underset{\sim}{j}$.

3. Do 2 $\forall\ i, \underset{\sim}{j}$, record $K_{i,\underset{\sim}{j}-1}^{j(j)} \forall i, \underset{\sim}{j}$.

Accordingly, $K_{i,\underset{\sim}{j}}^{m} \forall i, \underset{\sim}{j}$ can be traced back to the n_i random failure times generated when estimating $K_{i,\underset{\sim}{1}}^{m}$. The number of failures in this iteration is obtained accordingly.

The above procedure is iterated 10^3 times, yielding 10^3 K^m in one replication of the simulation run. Counting the occurring frequency of possible K^m during the 10^3 iteration, we obtain $P\left(K^m = k^m\right)$ at the end of the m^{th} test. The distribution is a tabulation of frequencies of each possible k^m, ranging from 0 to $\sum_{\forall i} n_i$. Specifically, $P\left(K^m = k^m\right) = \dfrac{\text{frequency of } k^m}{10^3}$.

The number of failures at the end of the m^{th} test in one replication, with one set of generated test parameters is obtained by taking the mean of number of failures at the end of the 10^3 iterations.

To compare the three proposed approaches, we calculate $E\left(K^m\right)$ and $P\left(K^m = k^m\right)$ in each replication and obtain the mean of the r replications $E\left(K^m\right)$ using the exact model, A2, and A3. Figure 3.3 outlines the procedure of the comparison.

r replications

a fixed set of α_y + r sets of generated $\begin{cases} w_i,\ n_i\ \forall i \\ t_y,\ y=1, \ldots, m \end{cases}$

1 replication (1 set of $\begin{cases} w_i, n_i \forall i \\ t_y, y=1,\ldots m \end{cases}$ + the fixed set of α_y)

$E\left(K^m\right)$:
- Exact model (Eq.(3.4))
- A2 (Eqs.(3.15),(3.16))
- A3 (based on Eq. (3.18))
- Simulation model =mean of 10^3 iterations' numbers of failures

$P\left(K^m = k^m\right)$
- Exact model (Eq.(3.1))
- A2 (Eq.(3.17))
- A3 (Eq. (3.18))
- Simulation model = mean of 10^3 iterations' failure frequencies

$\Downarrow$

Means and standard deviations of expected number of failures in r replications using exact model, A2, A3, and simulation model

Figure 3.3 Procedure of the comparison among exact model, A2, A3, and simulation model

This simulation model is general and can be applied to other scenarios. When applying "*p*-failure-removal" and "*p*-consecutive-failure-removal" scenarios, all procedures remain the same except at the end of each iteration.

3.5.2 Simulation Results and Validation

We apply the simulation procedure and compare $E\left(K^m\right)$ numerically, assuming the following parameters:

$r = 200$

$m = 10$

$\alpha_y \sim rand(1,3), y = 1,2,...m$

$t_y = t_{y-1} + rand(9,11), y = 1,2,...m$

$n_i \sim rand(24,26) \quad \forall i$

$w_i \sim rand(t_{y-1}, t_y), \quad \text{for } w_i \in (t_{y-1}, t_y)$

$\gamma_x = 4 - 0.1x$

The average and standard deviations of $E(K^m)$ at the end of first 10 tests using the three approaches and simulation model, are shown in Tables 3.1 and 3.2.

Table 3.1 Means of expected number of failures at the end of first 10 tests using the exact model, A2, A3, and the simulation model

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
Exact model	0.003	0.038	0.188	0.632	1.639	3.593	6.858	11.821	18.480	26.753
A2	0.003	0.038	0.191	0.635	1.642	3.598	6.864	11.829	18.490	26.767
A3	0.003	0.038	0.188	0.632	1.639	3.593	6.858	11.821	18.480	26.753
Simulation	0.002	0.038	0.190	0.629	1.638	3.595	6.833	11.798	18.454	26.693

Table 3.2 Standard deviations of expected number of failures at the end of first 10 tests using the exact model, A2, A3, and the simulation model

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
Exact model	0.001	0.008	0.034	0.104	0.260	0.444	0.838	1.291	1.895	2.554
A2	0.001	0.008	0.035	0.104	0.261	0.445	0.838	1.292	1.895	2.555
A3	0.001	0.008	0.034	0.104	0.260	0.444	0.838	1.291	1.895	2.554
Simulation	0.002	0.011	0.039	0.106	0.259	0.450	0.831	1.313	1.899	2.553

The means and standard deviations of $E\left(K^m\right)$ obtained by the three approaches and simulation model are approximately equal, which validates the exact model, A2, and A3. It is also observed that more iterations (10^4 or 10^5 times) result in closer simulation results to the real system situation, and the two results (simulation model and exact model (A3)) are closer to each other.

In Figure 3.4, we predict the expected probability of failure of one unit at the end of first 50 tests ($m = 50$) using A2 with:

$w_1 = 0,\ w_i = 10i - 5$ for $i = 2, ...40$;

$n_i = 100\ \forall i$; $\alpha_y = 1$; and $t_y = 10y\ \forall y$

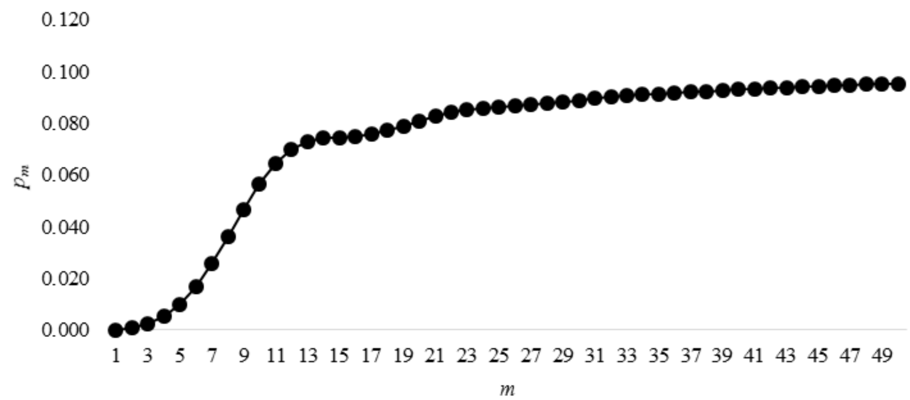

Figure 3.4 Expected probability of failure for one unit in the first 50 tests based on A2

Similar comparisons are extended to the "*p*-consecutive-failure removal" and "*p*-failure-removal" scenarios, which indeed validate the exact models derived in section 3.1.

3.6 Numerical Illustrations

3.6.1 Distribution of Failures under Different Scenarios

Four batches of units arrive in the storage area at times w_1=0, $w_i = 10i - 5, i = 2, 3, 4$ with population sizes $n_1 = 10$, $n_2 = 9$, $n_3 = 8$ and $n_4 = 7$, respectively. Tests are conducted on all units in storage at times $t_y = 10y, y = 1,2,3,4$. Units are i.i.d. with the following lifetime distribution in Eq. (3.21):

$$F(t;\theta) = 1 - e^{-\left(\frac{t}{\theta}\right)^4};\ F^{x=1}(t;\theta) = 1 - e^{-\left(\frac{t}{\theta}\right)^3};$$
$$F^{x=2}(t;\theta) = 1 - e^{-\left(\frac{t}{\theta}\right)^2};\ F^{x\geq 3}(t;\theta) = 1 - e^{-\left(\frac{t}{\theta}\right)} \tag{3.21}$$

where $\theta = 50$, and $F^x(t;\theta)$ represents the cumulative distribution (*cdf*) of the unit after the x^{th} repair.

The distributions of the failed units at time t_4 under "no-removal", "2-failure-removal", and "2-consecutive-failure-removal" scenarios are shown in Figure 3.5. An important observation is that the peak of the failure distribution curve shifts to the left when removal happens. This is due to the fact that keeping a large proportion of aged units in the system (no-removal scenario) increases the probability of having a larger number of failures.

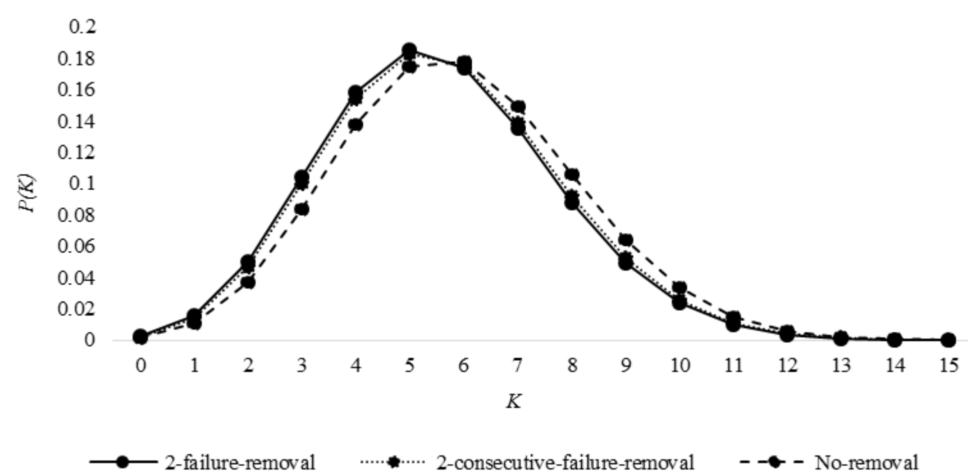

Figure 3.5 Distribution of failed units under different scenarios

3.6.2 System Reliability Estimation for both Testing using the Entire Population and by Sampling

Four batches of units arriving in storage at times w_1 =0, $w_i = 10i - 5, i = 2,3,4$, the batch sizes are n_1 =6, n_2 =12, n_3 =18, and n_4 =24, respectively. The lifetime of the units follows Weibull distribution as shown in Eq. (3.21). We select the units proportionally, i.e., pre-determined proportion (1/2, 1/4 and 1/6) of units are chosen randomly from each batch (recorded as $s1, s2, s3, s4$) to constitute the sample. Assuming that tests are performed at $t_y = 10y, y = 1,2,3,4$, we calculate $E\left(K_s^4\right)$ and $P\left(K_s^4 = k_s^4\right)$ at t_4 in Figure 3.6, assuming all failed units are repaired and returned to storage. It is clear that reduction of the sample size "compresses" the curve such that the peak of the curve moves to the left. The expected number of failures of the total population and samples with different sizes are obtained below.

$$E_{s_1=12,s_2=24,s_3=36,s_4=48\text{ (all population)}}\left[K_s{}^4\right] = 2.08$$

$$E_{1/2\text{ populaion}}\left[K_s{}^4\right] = 1.04$$

$$E_{1/3\text{ populaion}}\left[K_s{}^4\right] = 0.70$$

$$E_{1/6\text{ populaion}}\left[K_s{}^4\right] = 0.35$$

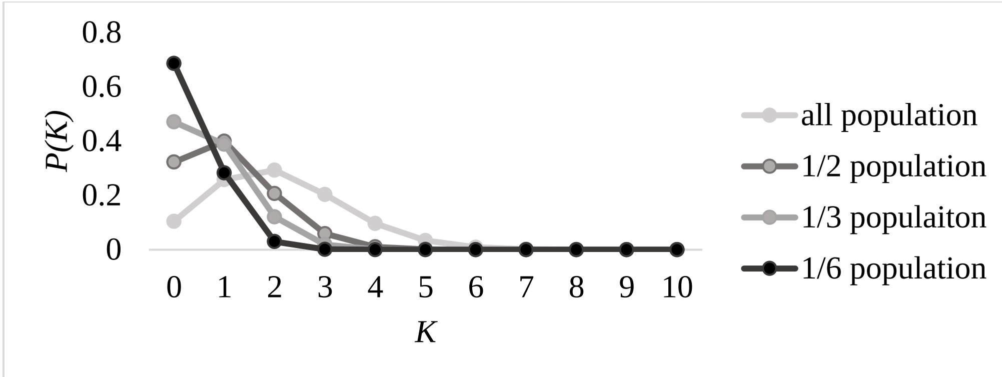

Figure 3.6 Distribution of failed units with different sample sizes and total population under “no-removal” scenario

Obviously, the expected number of failures is proportional to the sample size. Extending to the “*p*-consecutive-failure-removal” and “*p*-failure-removal” scenario, we validate that indeed the samples accurately represent the population.

3.6.3 An Illustration of A1

Applying A1, we predict the failure distribution of the system at time t_4 with $w_1 < t_1 < w_2 < t_2 < w_3 < t_3 < w_4 < t_4$. We only consider the 2nd and 4th tests and repairs. Accordingly, we calculate the aggregated batches’ arrival times:

$$w_{12} = \frac{w_1 \cdot n_1 + w_2 \cdot n_2}{n_1 + n_2}; w_{34} = \frac{w_3 \cdot n_3 + w_4 \cdot n_4}{n_3 + n_4}$$

Figures 3.7 and 3.8 are procedures of the original problem and A1.

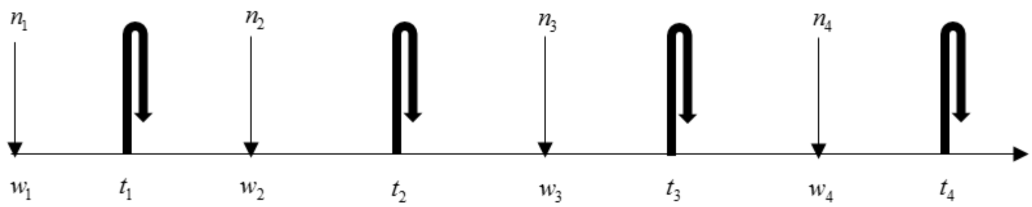

Figure 3.7 Procedure of the original problem by t_4

Figure 3.8 Procedure of A1 by t_4

Since A1 relies on the previous system reliability metrics, we predict the distribution of failures at time t_2 first:

$$P\left(K^2 = k^2\right) = \binom{n_1 + n_2}{k^2} \left(\left[F\left(t_2 - w_{12}\right) \right]^{k^2} \left[R\left(t_2 - w_{12}\right) \right]^{n_1 + n_2 - k^2} \right)$$

Failure distribution at t_4 is obtained conditionally on system status at time t_2 :

$$P\left(K^4 = k^4 \middle| K^2 = k^2\right) = \left[\sum_{k^2=0}^{n_1+n_2} P\left(K^4 = k^4 \middle| K^2 = k^2\right) \cdot \, P\left(K^2 = k^2\right) \right]$$

where

$$P\left(K^4 = k^4 \middle| K^2 = k^2\right) = \sum_{k^{4,1}+k^{4,2}+k^{4,3}=k^4} \left\{ \begin{array}{l} \binom{n_1+n_2-k^2}{k^{4,1}} \left[\dfrac{F\left(t_4-w_{12}\right)-F\left(t_2-w_{12}\right)}{R\left(t_2-w_{12}\right)} \right]^{k^{4,1}} \\ \left[1-\dfrac{F\left(t_4-w_{12}\right)-F\left(t_2-w_{12}\right)}{R\left(t_2-w_{12}\right)} \right]^{n_1+n_2-k^2-k^{4,1}} \\ \binom{k^2}{k^{4,2}} \left[F^1\left(t_4-t_2\right) \right]^{k^{4,2}} \left[1-F^1\left(t_4-t_2\right) \right]^{k^2-k^{4,2}} \\ \binom{n_3+n_4}{k^{4,3}} \left[F\left(t_4-w_{34}\right) \right]^{k^{4,3}} \left[1-F\left(t_4-w_{34}\right) \right]^{n_3+n_4-k^{4,3}} \end{array} \right\}$$

3.7 Conclusions

In this chapter, we investigate effective approaches to estimate the reliability metrics of a system with nonhomogeneous one-shot units under different scenarios, by performing consecutive reliability tests and taking into account the units' different characteristics during their storage period. Moreover, we show that the reliability metrics of such systems can be obtained by sampling from the total population. The approaches obtain accurate system reliability metrics but require extensive computation time when the batch size is large or when many tests are performed. In such situations, we propose three computationally efficient approaches that provide effective and accurate system reliability estimations while either masking some of the system conditions or yielding negligible estimation bias. An extensive simulation model validates that the proposed exact model estimate system reliability accurately; meanwhile the proposed approaches also yield accurate and immediate system reliability predictions.

CHAPTER 4

RELIABILTY MODELING OF SYSYTEMS WITH MIXTURE OF UNITS: A STOCHASTIC APPROACH

A general "$k(t)$-out-of-$n(t)$: F" system with mixtures of nonhomogeneous one-shot units is proposed and its reliability metrics are investigated in chapter 3. The one-shot units are produced in batches at arbitrary time and kept in storage until needed. During the entire life horizon of the units, reliability tests are conducted at arbitrary time. This original approach is named as "OA".

Under some circumstances, we can describe the arrival times of the batches, the sizes of the batches, as well as the times to perform the reliability tests using specific probabilistic distributions. The number of units in the system, the ages of the units, and the times to perform the tests become stochastic at arbitrary time. In this chapter, we propose a stochastic approach (SA) to investigate the system reliability metrics.

In section 4.1, we introduce the system and propose models to estimate the system reliability metrics. The distribution of failed units is studied. We assume that the batch arrival times, batch sizes, and test times respectively follow specific probabilistic distributions. The entire population (regardless of the size) is tested. In section 4.2, we predict the distribution of failed units as well as the expected number of failures by testing a sample from the population. Several examples are provided in section 4.3 to numerically compare the system reliability metrics obtained using both the OA and the SA.

4.1 Failure Distribution of Systems with Mixtures of Units Using the SA

One-shot units are produced in batches and kept in storage until needed. The batches of units arrive into the storage immediately after production according to a compound non-homogeneous Poisson distribution (CNPD). Specifically, the number of units in each batch follows a uniform distribution and the batches themselves arrive according to non-homogeneous Poisson distribution (NPD). The time to perform the NDT follows an NPD and the testing is conducted during the entire life horizon of the units by either testing all units in the storage area or by testing selected samples. Specifically, we have:

- The number of units in the i^{th} batch follows a uniform distribution $\forall i$, i.e., $N_i \sim U(a,b)$, where *a* and *b* are the lower and upper limits of the distribution.
- The arrival time of the batch follows a non-homogeneous Poisson distribution, i.e., $W_i \sim NPD(\lambda_B(\cdot))$ where $\lambda_B(\cdot)$ is the parameter of the distribution.
- The time to perform the NDT follows a non-homogeneous Poisson distribution, i.e., $T_y \sim NPD(\lambda_T(\cdot))$ where $\lambda_T(\cdot)$ is the parameter of the distribution.

The NDTs are assumed to be instantaneous (duration of the test is ignored). The batch arrival time and the test time are independent (i.e., W_i and T_y are two independent random variables). Failed units are either repaired and placed back into the system, or removed after a predetermined number of repairs (consecutive or not). All units in the storage are defined as a system which is composed of nonhomogeneous one-shot units. We consider that the system fails if $q\%$ or more units of the entire population fail. The system state at arbitrary time t is defined as:

$\Psi(t)$: the possible system state at time t, where $\Psi(t)$ is defined by the number of batches (of units) in the system, the batch sizes, the batch arrival times, the number of NDTs conducted, and the test times. Specifically, $\Psi(t)=\varphi(t)$ where $\varphi(t)$ is a specific system state.

We investigate the distribution of failed units at arbitrary time t when all units in the system are tested. In sections 4.1.1 and 4.1.2, we investigate the system reliability metrics under two testing scenarios, respectively. In the first scenario we assume that the failed units are discarded. In the second scenario we assume that the failed units are repaired and placed back into the system with higher failure rates.

4.1.1 Reliability Metrics of the Systems Using SA without Repairs

In this section, we investigate the system's failure distribution, assuming no repair is performed. The following are necessary notations:

$B(t)$: the total number of batches of units in the system until time t, specifically, $B(t)=b(t)$ where $b(t)=0,1,...$;

$K(t)$: the total number of failed units at time t, specifically, $K(t)=k(t)$ where $k(t)=0,1,...$;

$K_i(t)$: the number of failed units in the i^{th} batch at time t, specifically, $K_i(t)=k_i(t)$ where $k_i(t)=0,1,...,b$ and $\left(\sum_{\forall i} k_i(t)\right)=k(t)$

$L(t)$: the number of units remaining in the system at time t (when units are removed after a predetermined number of repairs), specifically, $L(t)=l(t)$ where $l(t)=0,1,...$;

$f_{W_i,N_i;\ i=1,2,...,b(t)}(\cdot)$: joint conditional distribution of the batch arrival times and batch sizes.

Without the repair, the system state at time t is determined by the number of batches, the arrival times of the batches, and the batch sizes. The probability of having $k(t)$ failures at time t is obtained by considering the probability of all system states that might yield $k(t)$ failures at time t (i.e., all system states with a total number of units in the system at time t greater than $k(t)$). The probability of having $k(t)$ failures under such a system state is:

$$P\left[K(t)=k(t)\right]=\sum_{\forall\left(\varphi(t):\ \sum_{\forall i} n_i \geq k(t)\right)}\left\{P\left[\Psi(t)=\varphi(t)\right]\cdot \overbrace{P\begin{bmatrix}K(t)=k(t)\\ \middle|\Psi(t)=\varphi(t)\end{bmatrix}}^{A}\right\} \tag{4.1}$$

where

$$P\left[K(t)=k(t)\middle|\Psi(t)=\varphi(t)\right]=P\left(\left(\sum_{\forall i}K_i(t)\right)=k(t)\middle|\Psi(t)=\varphi(t)\right)$$

$$=\sum_{\left(\sum_{\forall i}k_i(t)\right)=k(t)}\left[P\left(K_i(t)=k_i(t)\right);\forall i\middle|\Psi(t)=\varphi(t)\right] \tag{4.2}$$

$$=\sum_{\left(\sum_{\forall i}k_i(t)\right)=k(t)}\left[\prod_{\forall i}\left[\binom{n_i}{k_i(t)}\left[F(t-w_i)\right]^{k_i(t)}\right]\right]$$

Eq. (4.2) is the probability of having $k(t)$ failures in the system under a specific system state. We now address the probability of having a specific system state. Letting $\Lambda_B(t)=\int_0^t \lambda_B(\tau)\,d\tau$, we obtain the probability of realizing an arbitrary system state ($P\left[\Psi(t)=\varphi(t)\right]$) as:

$$P\left[\Psi(t)=\varphi(t)\right]$$

$$=\sum_{b(t)=0}^{\infty}\sum_{n_1=a}^{b}\cdots\sum_{n_{b(t)}=a}^{b}\left\{\int_0^t\int_{w_1}^t\cdots\int_{w_{b(t)-1}}^t\left[\frac{\overbrace{P\left(B(t)=b(t)\right)}^{A}\cdot}{\underbrace{f_{W_i,N_i;\,i=1,2,\ldots,b(t)}\left(\begin{matrix}w_i,n_i;i=1,2,\ldots,b(t)\\ \middle|B(t)=b(t)\end{matrix}\right)}_{B}}\right]dw_i\ \ \forall i\right\} \tag{4.3}$$

where the terms A and B are expressed as:

$$P\left(B(t)=b(t)\right)=\frac{\exp\left(-\Lambda_B(t)\right)\left[\Lambda_B(t)\right]^{b(t)}}{\left[b(t)\right]!} \tag{4.4}$$

and

$$f_{W_i,N_i;\, i=1,2,\ldots,b(t)}\left(w_i, n_i; i=1,2,\ldots,b(t) \middle| B(t)=b(t)\right)$$

$$= \left\{ \prod_{i=1}^{b(t)} \left[\lambda_B\left(w_i\right) e^{-\left[\Lambda_B\left(w_{i+1}\right)-\Lambda_B\left(w_i\right)\right]} \right] \right\} e^{-\Lambda_B(t)-\Lambda_B\left(w_{b(t)}\right)} \cdot \left[\frac{1}{\left(b-a\right)^{b(t)}} \right] \tag{4.5}$$

$$= e^{-\Lambda_B(t)} \left[\prod_{i=1}^{b(t)} \lambda_B\left(w_i\right) \right] \left[\frac{1}{\left(b-a\right)^{b(t)}} \right]$$

Eq. (4.4) estimates the probability of having $b(t)$ batches in the system until time t. Eq. (4.5) is the conditional joint *pdf* of the batches' arrival times and sizes, given $b(t)$ batches arrive into the system until time t. Due to the complexity of the integration, we obtain the numerical solution of Eq. (4.5) by approximating the continuous variables by its discrete forms in Eq. (4.6), where we partition the time interval $\left(0,t\right]$ into small time intervals with length Δw. To illustrate, the numerical solution of Eq. (4.6) is infinitely close to but does not equal to 0. However, it is known that numerical realization of Eq. (4.5) equals to 0 if W_i is strictly considered as a continuous variable.

$$f_{W_i,N_i;\ i=1,2,\ldots,b(t)}\left(w_i,n_i;i=1,2,\ldots,b(t)\middle|B(t)=b(t)\right)$$

$$=P\left(N_i=n_i,W_i=w_i;i=1,2,\ldots,b(t)\middle|B(t)=b(t)\right)$$

$$=\frac{1}{(b-a)^{b(t)}}\cdot\left[P\left(W_{B(t)}=w_{b(t)}\middle|W_{B(t)-1}=w_{b(t)-1},\ldots,W_1=w_1\right)\cdot P\left(W_{B(t)-1}=w_{b(t)-1},\ldots,W_1=w_1\right)\right]$$

$$=\frac{1}{(b-a)^{b(t)}}\cdot\prod_{i=2}^{b(t)}P\left(W_i=w_i\middle|W_{i-1}=w_{i-1}\right)\cdot P\left(W_1=w_1\right)$$

$$\approx\frac{1}{(b-a)^{b(t)}}\left[\begin{array}{l}P\left(w_{b(t)}-\frac{\Delta w}{2}\le W_{b(t)}\le w_{b(t)}+\frac{\Delta w}{2}\middle|w_{b(t)-1}-\frac{\Delta w}{2}\le W_{b(t)-1}\le w_{b(t)-1}+\frac{\Delta w}{2}\right)\\ \ldots P\left(\begin{array}{l}w_2-\frac{\Delta w}{2}\le W_2\le w_2+\frac{\Delta w}{2}\\ \left|w_1-\frac{\Delta w}{2}\le W_1\le w_1+\frac{\Delta w}{2}\right.\end{array}\right)P\left(w_1-\frac{\Delta w}{2}\le W_1\le w_1+\frac{\Delta w}{2}\right)\end{array}\right]$$

$$\to 0$$

(4.6)

The probability of having $p\%$ of the units fail, the expected number of failures, and the system reliability at time t, can be obtained in the similar way in Eqs. (4.7)-(4.9), respectively:

$$P\left[K(t)=p\%\left(\sum_{\forall i}N_i\right)\right]=\sum_{\forall\varphi(t)}\left\{P\left[\Psi(t)=\varphi(t)\right]\cdot\overbrace{P\left[\begin{array}{l}K(t)=p\%\left(\sum_{\forall i}N_i\right)\\ \left|\Psi(t)=\varphi(t)\right.\end{array}\right]}^{A}\right\} \tag{4.7}$$

$$E_{system}\left[K(t)\right]=\sum_{\forall\varphi(t)}\left\{P\left[\Psi(t)=\varphi(t)\right]\cdot\overbrace{\left[E_{system}\left[K(t)\right]\middle|\Psi(t)=\varphi(t)\right]}^{A}\right\} \tag{4.8}$$

$$\begin{aligned}R_{system}(t)&=\sum_{\forall\varphi(t)}\left\{P\left[\Psi(t)=\varphi(t)\right]\cdot\overbrace{\left[R_{system}(t)\middle|\Psi(t)=\varphi(t)\right]}^{A}\right\}\\&=\sum_{\forall\varphi(t)}\left\{P\left[\Psi(t)=\varphi(t)\right]\cdot\overbrace{\left[\sum_{k(t)=0}^{q\%\left(\sum_{\forall i}n_i\right)}P\left[K(t)=k(t)\right]\middle|\Psi(t)=\varphi(t)\right]}^{A}\right\}\end{aligned} \tag{4.9}$$

The terms A in Eqs. (4.7)-(4.9) are respectively the probability of having $p\%$ of the entire population fail, the expected number of failures, and the reliability of the system at time t, conditional on a specific system state. These three reliability metrics can be obtained by referring to Eq. (4.2). Specifically, $P\left[K(t)=p\%\left(\sum_{\forall i}N_i\right)\middle|\Psi(t)=\varphi(t)\right]$ is directly obtained via Eq. (4.4) by replacing $k(t)$ with $p\%\left(\sum_{\forall i}N_i\right)$, and the remaining two reliability metrics are obtained as:

$$\left[E_{system}\left[K(t)\right]\middle|\Psi(t)=\varphi(t)\right]=\sum_{k(t)=0}^{\left(\sum_{\forall i}n_i\right)\left|\left(\Psi(t)=\varphi(t)\right)\right.}P\left[K(t)=k(t)\middle|\Psi(t)=\varphi(t)\right]\cdot k(t) \tag{4.10}$$

and

$$\left[R_{system}(t)\middle|\Psi(t)=\varphi(t)\right]=\sum_{k(t)=0}^{\left(q\%\cdot\sum_{\forall i}n_i\right)\left|\left(\Psi(t)=\varphi(t)\right)\right.} P\left[K(t)=k(t)\middle|\Psi(t)=\varphi(t)\right] \tag{4.11}$$

4.1.2 Failure Distribution and Reliability of Systems Using SA with Repairs

Consider the scenario when the conduct of tests is taken into consideration. The following additional notations are necessary:

$T(t)$: the number of NDTs conducted until time t; specifically, $T(t)=t(t)$ where $t(t)=0,1,\ldots$;

B_y: the number of batches arrive between the (y-1)th and yth NDT; specifically, $B_y=b_y$ where $b_y=0,1,\ldots$;

μ_B: the expected number of batches in the system until time t;

μ_T: the expected number of NDTs conducted until time t.

Specifically, the system state at time t is determined by the number of NDTs conducted, the test times, the number of batches that arrive between two arbitrary NDTs, all batches' sizes, and all batches' arrival times. We obtain the probability of realizing a specific system state $\varphi(t)$ by applying the law of total probability as follows:

$$
\begin{aligned}
&P\left[\Psi(t)=\varphi(t)\right] \\
&= \overbrace{\sum_{t(t)=0}^{\infty}}^{A} \left[\left[\overbrace{\sum_{b_1=0}^{\infty} \cdots \sum_{b_{t(t)}=0}^{\infty} \sum_{b_{t(t)+1}=0}^{\infty}}^{B} \overbrace{\int_0^t \int_{t_1}^t \cdots \int_{t_{t(t)}}^t}^{C} \overbrace{\sum_{n_i=a;\forall i}^{b}}^{D} \overbrace{\int_0^t \int_{w_1}^{t_1} \cdots \int_{w_{b_1-1}}^{t} \cdots \int_{t_{t(t)-1}}^{t_{n(t(t)-1)}} \cdots\cdots \int_{w_{b_{t(t)}-1}}^{t_{n(t(t))}}}^{E} \cdots \right]\right] \\
&\cdot \left\{ \left\{ \begin{array}{l}
\overbrace{P\left(T(t)=t(t)\right)}^{F} \cdot \\
\overbrace{P\left(B_1=b_1,\ldots B_{t(t)+1}=b_{t(t)+1}\right)}^{F} \\
\overbrace{f_{T_y}\left(t_y; y=1,2,\ldots,t(t) \middle| T(t)=t(t)\right)}^{G} \cdot \\
\overbrace{f_{W_i,N_i,i=1,\ldots,b_1+\ldots b_{t(t)+1}} \left(\begin{array}{l} \left(w_i, n_i; i=1,\ldots,b_1+\ldots b_{t(t)+1}\right) \\ \left| B_1=b_1,\ldots B_{t(t)+1}=b_{t(t)+1} \right. \end{array} \right)}^{H} \cdot
\end{array} \right\} dt_y dw_i \forall i, y \right\}
\end{aligned}
\tag{4.12}
$$

where

$$
\Lambda_T(t)=\int_0^t \lambda_T(\tau)d\tau \tag{4.13}
$$

$$
P\left(T(t)=t(t)\right)=\frac{\exp\left(-\Lambda_T(t)\right)\left[\Lambda_T(t)\right]^{t(t)}}{\left[t(t)\right]!} \tag{4.14}
$$

$$
\begin{aligned}
f_{T_y}\left(t_y; y=1,2,\ldots,t(t) \middle| T(t)=t(t)\right) &= \left\{ \prod_{y=1}^{t(t)} \left[\lambda_T(t_y) e^{-\left[\Lambda_T(t_{y+1})-\Lambda_T(t_y)\right]} \right] \right\} e^{-\Lambda_T(t)-\Lambda_T\left(t_{t(t)}\right)} \\
&= e^{-\Lambda_T(t)} \left[\prod_{i=1}^{t(t)} \lambda_T(t_y) \right]
\end{aligned}
\tag{4.15}
$$

$$P\left(B_1 = b_1, \ldots B_{t(t)+1} = b_{t(t)+1}\right)$$

$$= \left\{ \prod_{y=1}^{t(t)} \frac{1}{\left(b_y\right)!\left(b_{t(t)+1}\right)!} \exp\left[-\left[\Lambda_B\left(t_y\right) - \Lambda_B\left(t_{y-1}\right)\right]\right]\left[\left[\Lambda_B\left(t_y\right) - \Lambda_B\left(t_{y-1}\right)\right]\right]^{b_y} \right\} \tag{4.16}$$

$$\cdot \exp\left[-\left[\Lambda_B(t) - \Lambda_B\left(t_{t(t)}\right)\right]\right]\left[\left[\Lambda_B(t) - \Lambda_B\left(t_{t(t)}\right)\right]\right]^{b_{t(t)+1}}$$

and

$$f_{W_i, N_i, i=1,\ldots,b_1+\ldots b_{t(t)+1}}\left(\left(w_i, n_i; i = 1, \ldots, b_1 + \ldots b_{t(t)+1}\right) \middle| B_1 = b_1, \ldots B_{t(t)+1} = b_{t(t)+1}\right)$$

$$= \left\{ \prod_{y=1}^{t(t)} \prod_{i=1}^{b_y} \begin{bmatrix} \lambda_B\left(w_i\right) e^{-\left[\Lambda_B\left(w_{i+1}\right) - \Lambda_B\left(w_i\right)\right]} \\ e^{-\left[\Lambda_B\left(w_{b_{y(1)}}\right) - \Lambda_B\left(t_{y-1}\right)\right]} \\ e^{-\left[\Lambda_B\left(t_y\right) - \Lambda_B\left(w_{b_{y(b_y)}}\right)\right]} \end{bmatrix} \right\} e^{-\Lambda_B(t) - \Lambda_B\left(w_{b_{t(t)+1}}\right)} \cdot \left[\frac{1}{(b-a)^{\sum_{y=1}^{t(t)+1} b_y}} \right]$$

$$= e^{-\Lambda_B(t)} \left[\prod_{i=1}^{\sum_{y=1}^{t(t)+1} b_y} \lambda_B\left(w_i\right) \right] \left[\frac{1}{(b-a)^{\sum_{y=1}^{t(t)+1} b_y}} \right] \tag{4.17}$$

In Eq. (4.12), the term A is the sum of the probability of potential numbers of NDTs conducted until time t. Similarly, the term B considers the probability of potential numbers of batches that arrive between two arbitrary NDTs, given $t(t)$ tests are performed until time t. The term C considers the times that the $t(t)$ is performed. Terms D and E include all possible batches’ arrival times and sizes.

In Eq. (4.13), we obtain the expected number of NDTs conducted during an arbitrary time interval. We then investigate the probability that $t(t)$ NDTs are conducted until time t in Eq. (4.14). Eq. (4.15) provides the conditional joint *pdf* of the times when the tests are conducted, given $T(t) = t(t)$. In Eq. (4.16), we obtain the probability of having a specific number of batches arrive between two arbitrary NDTs. Similarly, we provide the conditional joint *pdf* of the batches' arrival times and sizes in Eq. (4.17).

The system reliability metrics are obtained by applying Eq. (4.1) and Eqs. (4.7)-(4.9). Under the scenario discussed in this section, the term A in Eq. (4.1) is obtained by referring to the calculation of $P\left(K^m = k^m\right)$ in chapter 3. Accordingly, the probability of having *p*% of the entire population fail, the expected number of failures, and the reliability of the system at time *t* can be obtained.

The above models are applicable to other scenarios, e.g., the units are removed after *p* failures (consecutive or not), or units are repaired to as-good-as old ones. Under such scenarios, the probability that defines the system state given in Eq. (4.9) still applies and the system reliability metrics under a specific system state are obtained by referring to the models in chapter 3 (section 3.1.2 and 3.1.3, respectively). Assuming the units are discarded after *p* failures, we investigate the expected number of units in the system at arbitrary time t :

$$E\left[L(t)\right] = \sum_{\forall \varphi(t)} \left\{ P\left[\Psi(t) = \varphi(t)\right] \cdot E\left[L(t) \middle| \Psi(t) = \varphi(t)\right] \right\} \tag{4.18}$$

where

$$E\left[L(t)\middle|\Psi(t)=\varphi(t)\right]=\sum_{\forall i}n_i\left\{\sum_{j=0}^{\min\{t(t)-z_i,p\}}\sum_{\forall a}\begin{bmatrix}\left(F\left(s_1-w_i\right)-F\left(\hat{s}_1-w_i\right)\right)\\ \prod_{a=1}^{j-1}\left(F^a\left(s_{a+1}-s_a\right)-F^a\left(\hat{s}_{a+1}-s_a\right)\right)\\ \left(F^j\left(t_{t(t)}-s_j\right)-F^j\left(t_{t(t)-1}-s_j\right)\right)\end{bmatrix}\right\} \quad (4.19)$$

In Eq. (4.19), s_1 could be any test selected from $\left(z_i+1,...,t(t)-1\right)$ and s_a could be any NDT selected from $\left(s_{a-1}+1,...,t(t)-1\right)$.

It is expected that the system reliability metrics obtained by using the OA should approximately equal to the expected system reliability obtained by using the SA as shown numerically in section 4.3.

4.2 Reliability Metrics of Systems with Mixture of Units Using the SA by Sampling

It is impractical to test all units in the system when the size of the total population in the system is large. Selecting a sample that represents the population mixture and investigating the relationship between the population and the sample is a feasible approach to obtain the reliability metrics of the population. We randomly select q_s% of the units and conduct the NDT test at an arbitrary time t. Letting:

$S(t)$: the sample state at time *t*, where a sample state is determined by the number of units, the ages of the units, the number of tests and repairs performed on the units. Specifically, $S(t)=s(t)$ where $s(t)$ is a specific sample state;

$K_s(t)$: the number of failed units in the sample, specifically, $K_s(t)=k_s(t)$.

We obtain the expected failure distribution of the sample by applying the law of total probability in Eq. (4.20):

$$P\left[K_s(t)=k_s(t)\right]=\sum_{\forall s(t)} P\left[S(t)=s(t)\right]\cdot P\begin{bmatrix}\left(K_s(t)=k_s(t)\right)\\ \Big|\left[S(t)=s(t)\right]\end{bmatrix}$$

$$=\sum_{\forall s(t)}\sum_{\forall k(t)} P\left[K(t)=k(t)\right]\cdot P\begin{bmatrix}S(t)=s(t)\\ \big|K(t)=k(t)\end{bmatrix}\cdot P\begin{bmatrix}\left(K_s(t)=k_s(t)\right)\\ \Big|\left[S(t)=s(t)\right]\end{bmatrix}$$

$$=\sum_{\forall s(t)}\sum_{\forall k(t)}\sum_{\forall \varphi(t)}\left\{\begin{array}{l}\overbrace{P\left[\Psi(t)=\varphi(t)\right]}^{A}\cdot\\ \overbrace{P\left[K(t)=k(t)\big|\Psi(t)=\varphi(t)\right]}^{B}\cdot\\ \overbrace{P\left[S(t)=s(t)\big|K(t)=k(t)\right]}^{C}\cdot\\ \overbrace{P\left[\left(K_s(t)=k_s(t)\right)\Big|\left[S(t)=s(t)\right]\right]}^{D}\end{array}\right\}$$

$$
= \sum_{\forall s(t)} \sum_{\forall k(t)} \sum_{\forall \varphi(t)} \left\{
\begin{array}{l}
\overbrace{P\left[\Psi(t)=\varphi(t)\right]}^{A}
\overbrace{P\left[\begin{array}{l}
K_{i,j,\underset{\sim}{l}_j}(t)=k_{i,j,\underset{\sim}{l}_j}(t), \\
R_{i,j,\underset{\sim}{l}_j}(t)=r_{i,j,\underset{\sim}{l}_j}(t); \\
\middle| \Psi(t)=\varphi(t); \forall i, j, \underset{\sim}{l}_j
\end{array}\right]}^{B} \\
\overbrace{P\left[\left(\begin{array}{l}
S_{K_{i,j,l_j}}(t)=s_{k_{i,j,l_j}}(t), \\
S_{R_{i,j,l_j}}(t)=s_{r_{i,j,l_j}}(t)
\end{array}\right); \middle| \begin{array}{l}
\left(\begin{array}{l}
K_{i,j,\underset{\sim}{l}_j}(t)=k_{i,j,\underset{\sim}{l}_j}(t), \\
R_{i,j,\underset{\sim}{l}_j}(t)=r_{i,j,\underset{\sim}{l}_j}(t);
\end{array}\right) \\
\forall i, j, \underset{\sim}{l}_j
\end{array}\right]}^{C} \\
\overbrace{P\left[\left(K_s(t)=k_s(t)\right) \middle\| \left[S(t)=s(t)\right]\right]}^{D}
\end{array}
\right\}
$$

(4.20)

The terms A and B have been described in details in section 4.1. With the characteristics of all the units in the system known, the probability of realizing a specific sample state (Term C in Eq. (4.20)) can be obtained by applying the hypergeometric distribution in Eq. (4.21):

$$
P\left[\left(S(t)=s(t)\right) \middle\| \left[K(t)=k(t)\right]\right] = \frac{\prod_{\forall i} \prod_{\forall j} \prod_{\forall \underset{\sim}{l}_j} \binom{k_{i,j,\underset{\sim}{l}_j}(t)}{s_{k_{i,j,l_j}}(t)} \binom{r_{i,j,\underset{\sim}{l}_j}(t)}{s_{r_{i,j,l_j}}(t)}}{\binom{\sum_{\forall i} n_i}{q_s\% \left(\sum_{\forall i} n_i\right)}} \tag{4.21}
$$

Term D is an indicator function with binary values (0 or 1), specifically:

$$
P\left[\left(K_s(t)=k_s(t)\right) \middle\| \left[S(t)=s(t)\right]\right] = \begin{cases} 1 & \text{if } \sum_{\forall i} \sum_{\forall j} \sum_{\forall \underset{\sim}{l}_j} s_{k_{i,j,l_j}}(t) = k_s(t) \\ 0 & \text{otherwise} \end{cases} \tag{4.22}
$$

The expected number of failures in the sample is:

$$E\left[K_s(t)\right] = \sum_{\forall \varphi(t)} \left\{ E\left[K_s(t)\right] \middle| \Psi(t) = \varphi(t) \right\} \cdot P\left[\Psi(t) = \varphi(t)\right]$$

$$= \sum_{\forall \varphi(t)} \left\{ \sum_{k_s(t)=0}^{\sum_{\forall i} n_i | (\Psi(t)=\varphi(t))} \left\{ P\left[K_s(t) = k_s(t)\right] \middle| \Psi(t) = \varphi(t) \right\} \cdot k_s(t) \right\} \cdot P\left[\Psi(t) = \varphi(t)\right]$$

(4.23)

Since repeated tests are extensively performed, the sample represents the population's mixture in the long run without loss of generality. On the average, the units with different characteristics are proportionally selected. This is also numerically validated in section 4.3.

4.3 Numerical Comparisons

In this section, we numerically compare the system reliability metrics (reliability and expected number of failures) using the SA and OA. The reliability metrics obtained by the two approaches are compared by first testing the entire population and then testing the sample. We validate that: 1) the sample represents the population when applying the SA, and 2) the system reliability obtained by the SA can be accurately predicted by those obtained by the OA, and vice versa.

4.3.1 System Reliability Metrics Estimation by Testing the Total Population

We assume that arrival times of the batches, the conduct of the NDT tests, the batch size, and the unit's failure time distribution have the following distributions:

$W_i \sim \text{NPP}\left(\lambda_B(t)\right),$ where $\lambda_B(t) = 1.2 \times 10^{-3} t;$

$T_y \sim \text{NPP}\ \left(\lambda_T(t)\right);$ where $\lambda_T(t) = 1 \times 10^{-3} t$

$N_i \sim$ Discrete Distribution, specifically, $P(N_i = 10) = P(N_i = 20) = P(N_i = 30) = \frac{1}{3}$

Unit's failure time distribution: $F^j(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{r(j)}}$; $\theta = 70$

We analyze the system reliability metrics at time $t = 100$. Theoretically, there could be an arbitrary number of tests performed and number of arrivals of batches until $t = 100$ depending on their probability distributions. Table 4.1 shows some of the probabilities:

Table 4.1 Probabilities of system's numbers of arrived batches and performed tests

$b(100)$	$P\left[b(100)\right]$	$t(100)$	$P\left[t(100)\right]$
0	0.0498	0	0.1353
1	0.1494	1	0.2702
2	0.2240	2	0.2702
3	0.2240	3	0.1804

4	0.1680	4	0.0902
5	0.1008	5	0.0361
6	0.0504	6	0.0120
7	0.0216		

We also calculate the means of the above distributions, which are the parameters of the OA:

$\mu_B(100) = \Lambda_B(100) = 3; \mu_{W_1} = 50,\ \mu_{W_2} = 71,\ \mu_{W_3} = 87;$

$\mu_T(100) = \Lambda_T(100) = 2; \mu_{T_1} = 58;\ \mu_{T_2} = 82;$

$\mu_{N_1} = \mu_{N_2} = \mu_{N_3} = 20.$

Applying the OA, two NDTs are performed at time $t_1 = 58$ and $t_2 = 82$. Three batches arrive at time $w_1 = 50,\ w_2 = 71,$ and $w_3 = 87$ respectively. We now calculate the system reliability at an arbitrary time before $t = 100$ using the two approaches. When calculating the system reliability metrics using the SA, we use the discrete approximation proposed in Eq. (4.6). Note that since the life of the system using the OA starts at $w_1 = 50$, the comparison of the system reliabilities starts from $t = 50$ to 100.

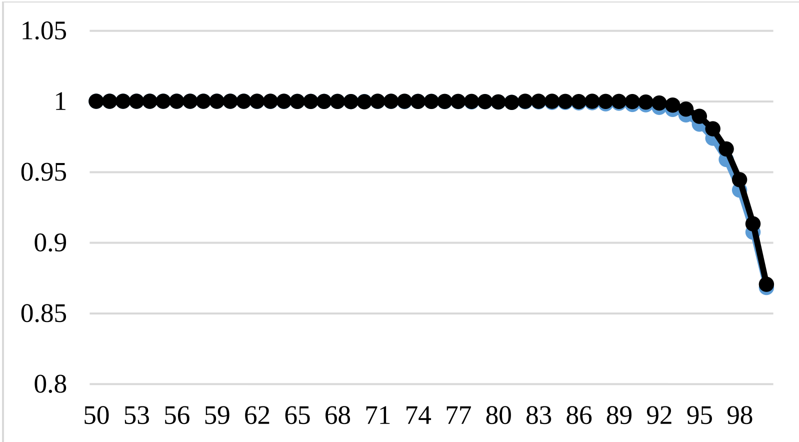

Figure 4.1 System reliability using the OA and SA over time

As shown in Figure 4.1, the two approaches (the black curve shows the system reliability obtained by the OA and the blue curve shows the system reliability obtained by the SA) yield approximately equal system reliability functions, which validates the assumption that the system reliability obtained by one approach can be predicted by using the other.

4.3.2 System Reliability Metrics Estimation by Sampling

In this section, we estimate the reliability metrics of the system using two approaches by either testing the samples or testing the population. We validate that sample represents the population mixture when applying both approaches in terms of the estimated expected number of failures.

Assuming the arrivals of batches and the conduct of NDTs follow the same distributions in section 4.3.1. We estimate and plot the expected numbers of failures when testing all units, choosing $q_s\% = 50\%$, and $q_s\% = 20\%$ in each test as shown in Figure 4.2.

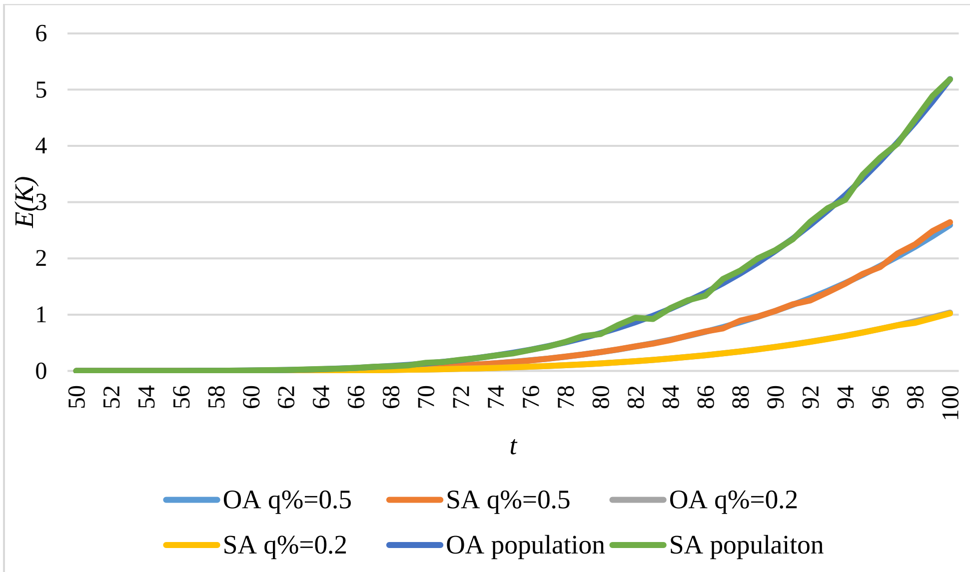

Figure 4.2. Expected numbers of failures with different sample sizes and total population using the two approaches

We observe that the expected numbers of failures obtained by the two approaches are approximately equal.

4.4 Conclusions

In this chapter, we investigate the system reliability metrics using a stochastic approach. We analyze the system's failure distribution, the expected number of failures, and the system reliability. The investigation is based on either testing the entire population or testing a selected sample. We also propose an approximation to obtain the system reliability metrics numerically. The numerical comparison between the reliability metrics obtained using the OA and SA show that the system reliability metrics obtained by the SA can be approximately predicted by those obtained by the OA, by either testing all the units or testing a sample, and vice versa.

CHAPTER 5

RELIABILITY MODELING OF MIXTURES OF ONE-SHOT UNITS UNDER THERMAL CYCLIC STRESSES

Failure or degradation caused by thermal fatigue is a pervasive phenomenon. Repeated fluctuations of temperature (and in some cases humidity) expand and contract the volume of a solid body. The expansion and contraction produce thermal stresses (and strains) which lead to the material deformation. Unlike the cyclic mechanical stresses that normally occur at stress levels which are much lower than the material's yield/proof stresses, temperature fluctuation in thermal fatigue can quickly lead to strains which are much higher than the elastic limit thus cause plastic strains and deformation in each cycle. The accumulation of plastic strains initiates and propagates cracks, which eventually cause thermal fatigue failure with a host of potential failure modes within a small number of temperature fluctuation cycles.

Most mechanical stress fatigue models that deal with high-cycle failure (e.g., the S-N curve, Miner's law) are not readily applicable for modeling thermal fatigue failure. Instead, Coffin-Manson (CM) model and its modified versions are widely adopted when assessing the unit's cycle-to-thermal-failure especially for low-cycle fatigue, where the CM model considers the loads in terms of plastic strain rather than stress.

As stated earlier, the CM model and its extensions have been used in reliability evaluation of units subject to thermal fatigue. However, they neglect the uncertainty during the unit's failure process and thus fail to provide the units' lifetime distribution and other reliability metrics under thermal fatigue. Moreover, they investigate the individual unit's fatigue life performance but do not provide other reliability metrics of a system which is a collection of units arranged in a specific configuration. Therefore, development of statistics-based model is useful in providing overall system reliability metrics.

The Birnbaum-Saunders (BS) distribution is specifically developed for describing mechanical fatigue. Hence it is reasonable to investigate whether the BS distribution can be extended for modeling the plastic deformation induced by thermal stresses and providing reliability metrics of units subject to thermal fatigue. In this chapter, we validate the performance of a generalized BS (GBS) distribution in modeling thermal fatigue. We then investigate the reliability metrics of the system when subjected to thermal fatigue failure.

The remainder of the chapter is organized as follows: Section 5.1 presents the development of the GBS distribution. The characteristics, properties and the hazard function of the GBS distribution are investigated. In section 5.2, we examine the accuracy and robustness of the GBS distribution in predicting unit's thermal fatigue failure under arbitrary conditions. Section 5.3 introduces the conduct of accelerated thermal cyclic tests (ATCTs) and a system with mixtures of nonhomogeneous one-shot units subjected to a sequence of ATCTs. Effective models are developed to explore system reliability metrics (e.g., distributions of failed units, reliability and expected number of failed units) at arbitrary time by sampling. A simulation model is developed to validate the proposed models. Section 5.4 provides numerical examples to illustrate the use of the models.

5.1 Generalized Birnbaum-Saunders (GBS) Distribution

The Birnbaum-Saunders (BS) distribution is proposed to model fatigue lifetime of units when a dominant crack (caused by repeated cycles of mechanical stress) surpasses or reaches a predetermined crack length threshold. Each cycle of stress leads to an incremental increase in the accumulated crack length. After the n^{th} cycle, the accumulated crack length is assumed to be normally distributed with $n\mu$ and variance $n\sigma^2$. It does not exceed a critical length ω with probability:

$$\Pr\left(X \le \omega\right) = \Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) \tag{5.1}$$

If the number of cycles is denoted in terms of unit's fatigue life *T*, the cumulative density function (CDF) of *T* is:

$$P\left(T \le t\right) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{0.5} - \left(\frac{\beta}{t}\right)^{0.5}\right]\right\} \tag{5.2}$$

where α is the shape parameter and β is the scale parameter respectively:

$$\alpha = \frac{\sigma}{\sqrt{\omega\mu}},\ \ \beta = \frac{\omega}{\mu};\ \ \alpha > 0,\ \ \beta > 0\,.$$

Adding the second shape parameter λ (Owen, 2004), the GBS distribution has the following CDF and hazard function as presented in Eqs. (5.3) and (5.4):

$$F\left(t;\alpha,\beta,\lambda\right)=\Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{\lambda}-\left(\frac{\beta}{t}\right)^{\lambda}\right]\right\} \tag{5.3}$$

$$h(t;\alpha,\beta,\lambda)=\frac{\lambda\left[\left(\frac{t}{\beta}\right)^{\lambda}+\left(\frac{t}{\beta}\right)^{-\lambda}\right]\exp\left\{-\frac{1}{2a^{2}}\left[\left(\frac{t}{\beta}\right)^{\lambda}+\left(\frac{t}{\beta}\right)^{-\lambda}-2\right]\right\}}{\sqrt{2\pi}\alpha t\left\{\Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{\beta}\right)^{-\lambda}-\left(\frac{t}{\beta}\right)^{\lambda}\right]\right\}\right\}} \tag{5.4}$$

where α and λ are GBS shape parameters and β is GBS scale parameter. The BS distribution is a special case of GBS with $\lambda=0.5$.

Utilizing Newton's generalized binomial theorem, we obtain the r^{th} moment of the GBS distribution in Eq. (5.5):

$$E\left(T^{r}\right)=\beta^{r}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\begin{pmatrix}\frac{r}{\lambda}\\ k\end{pmatrix}\begin{pmatrix}\frac{r-k\lambda}{2\lambda}\\ s\end{pmatrix}\cdot\left(I_{1}+I_{2}\right) \tag{5.5}$$

where

$$I_{1}=2^{\frac{-2s-k-1}{2}}\left[\Gamma\left(\frac{2s+k+1}{2}\right)-\Gamma\left(\frac{2s+k+1}{2},\frac{2}{\alpha^{2}}\right)\right]\cdot\left[\left(-1\right)^{2s+k}+1\right];$$

$$I_{2}=\frac{a^{\frac{r-2s}{\lambda}}}{2^{\frac{r-2s\lambda+\lambda}{2\lambda}}}\left[\left(-1\right)^{\frac{r-2s\lambda}{\lambda}}+1\right]\Gamma\left(\frac{r-2s\lambda+1}{2\lambda},\frac{2}{\alpha^{2}}\right)$$

Note that when $z < -\frac{2}{a}$, the GBS moments do not exist.

The BS hazard function is unimodally upside-down, which limits BS distribution in modeling flexible hazard functions. We show that GBS covers three types of failure conditions. Specifically, the following are necessary and sufficient conditions which enable GBS distribution to model diverse hazard functions:

1. The hazard function can be either increasing or multimodal when $\lambda > 0.5$;
2. The hazard function can be either upside-down or multimodal when $\lambda < 0.5$;
3. The hazard function is always upside-down when $\lambda = 0.5$.

5.2 Reliability Prediction Models for Thermal Fatigue Data

In this section, the GBS performance in predicting unit's thermal fatigue life based on accelerated fatigue failure data is presented and compared with the commonly used CM model. We show that GBS distribution models thermal fatigue data as accurate as CM model and provides more reliability metrics accurately.

5.2.1 The Coffin-Manson (CM) Model

Temperature cycling causes thermal expansion and contraction which induce thermo-mechanical stresses. It is considered that each thermal cycle causes plastic deformation which damages the units. Low-cycle (stress conditions that require a few hundred or thousand cycles to produce failure) fatigue data for either ductile or brittle material are effectively modeled using CM model as:

$$N_f = A(\Delta T)^b \tag{5.6}$$

where

N_f is the number of cycle-to-failure;

ΔT is the applied temperature amplitude in Kelvin;

b is an empirically dependent exponent; and

A is the material-dependent parameter.

The CM model deterministically calculates the number of cycle-to-failure. However, it is worth noting that the empirically determined exponent b usually follows certain probabilistic distribution due to the variation of the testing environment. Without loss of generality, we assume that $b \sim N(\mu_b, \sigma_b)$, where $\mu_b = -1 \sim -3$ for ductile material, $-4 \sim -6$ for hard material (or plastics) and $-7 \sim -9$ for brittle material. Accordingly, unit's cycle-to-failure becomes non-deterministic.

5.2.2 GBS Performance in Predicting Thermal Faituge Life

There are many situations when the units' failure data are not attainable under operating conditions. Under such circumstances, ALT is conducted to induce failures in a much shorter time and the accelrated failure data are then used to predict the unit's reliability performance under operating (or arbitrary) conditions. We investigate the GBS performance in predicting units' lifetime under normal conditions based on the accelerated thermal fatigue data.

We assume the scale parameter β (characteristic life) of the GBS distribution is determined by the applied temperature amplitude and other material/environmental-dependent parameters. Specifically, the GBS accelerated model is derived as:

$$F(t) = \Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{A(\Delta T)^{b}}\right)^{\lambda} - \left(\frac{A(\Delta T)^{b}}{t}\right)^{\lambda}\right]\right\} \tag{5.7}$$

The GBS accelerated model's parameters can be estimated based on the accelerated fatigue failure data generated by the CM model. The estimated parameters are then used to predict the fatigue life under normal conditions. Specifically, GBS accelerated model and the CM model respectively predict units' expected life under thermal conditions ΔT_N as:

$$E_{GBS}\left(N_{f\_N};\Delta T_N;\hat{\alpha},\hat{A},\hat{b},\hat{\lambda}\right) = \int_{t=0}^{\infty} t\cdot f_{GBS}\left(t;\Delta T_N;\hat{\alpha},\hat{A},\hat{b},\hat{\lambda}\right)dt \tag{5.8}$$

$$\bar{N}_{f\_N} = \bar{N}_{f\_A}\left(\frac{\Delta T_N}{\Delta T_A}\right)^b = A\left(\Delta T_A\right)^b\left(\frac{\Delta T_N}{\Delta T_A}\right)^b \tag{5.9}$$

where

$\bar{N}_{f\_A}, \bar{N}_{f\_N}$ is the average number of cycles to failure under accelerated (normal) conditions;

$\Delta T_A, \Delta T_N$ is the temperature fluctuation under accelerated (normal) conditions; and

$E_{GBS}\left(N_{f\_N}; \Delta T_N; \hat{\alpha}, \hat{A}, \hat{b}, \hat{\lambda}\right)$ is the expected life at normal operating conditions obtained using the GBS model.

The estimated expected life under normal conditions using the two models is compared in section 5.4. Moreover, the GBS models provide additional reliability metrics. To illustrate, units' reliability at time *t* under thermal stresses ΔT_N is predicted by GBS model in Eq. (5.10), which is not attainable by the CM model:

$$\hat{R}\left(t; \Delta T_N; \hat{\alpha}, \hat{A}, \hat{b}, \hat{\lambda}\right) = 1 - \Phi\left\{\frac{1}{\hat{\alpha}}\left[\left(\frac{t}{\hat{A}\left(\Delta T_N\right)^{\hat{b}}}\right)^{\hat{\lambda}} - \left(\frac{\hat{A}\left(\Delta T_N\right)^{\hat{b}}}{t}\right)^{\hat{\lambda}}\right]\right\} \tag{5.10}$$

Similarly, the units' harazrd function, *pdf*, the p^{th} percentile life, time-to-k^{th}-failure and other reliability metrics can be predicted by the GBS accelerated model.

In this section, we verify that the GBS distribution, though developed to model mechanical fatigue, provides accurate prediction of thermal fatigue failure under arbitrary conditions. Besides, the GBS distribution enables the estimation of other reliability metrics that are not attainable by the CM model as stated earlier. It is worth noting that the GBS distribution not only applies to thermal cyclic stresses, but could also be generally used in modeling the fatigue data caused by mechanical stresses, acoustic stresses and/or other types of cyclic stresses.

5.3 Reliability Metrics of Systems with Mixtures of One-Shot Units

As discussed in section 5.2, the GBS distribution can be utilized to model and predict thermal fatigue by providing the reliability metrics of units subject to thermal stresses. In this section, sequence of thermal cyclic tests is performed as non-destructive reliability tests to assess the reliability metrics of a system with mixtures of nonhomogeneous one-shot units. The CM model does not provide reliability metrics of a system when it is subjected to thermal fatigue. Instead, the GBS distribution is versatile and provides systems' reliability metrics of interest.

5.3.1 ATCT and Equivalent Test Duration

In a normal TCT, testing units are subjected to a temperature (T_N^U) and kept for a certain dwell time. It then cooled to a temperature (T_N^L) to complete a testing cycle. However, there are many situations when “highly” reliable units’ failure data are lacking when the tests are performed under normal operating conditions. Under such circumstances, ATCTs are conducted with higher temperature amplitudes to induce more failures for accurate reliability prediction in a short duration with negligible consequences on the testing units’ reliability metrics. Specifically, with the designed accelerated testing plan (considering the testing units’ reliability metrics, material properties and testing conditions), units are subjected to a higher temperature T_A^U ($T_N^U < T_A^U$) and cooled to a lower temperature T_A^L ($T_N^L > T_A^L$). To illustrate, during the ATCT, IC solder joints of a missile’s electronic guidance system are subjected to cycles of temperature fluctuations whose amplitude is higher than that when the solder joints undergo normal power-up and power-down cycles; therefore, more failures occur.

Compared with TCT, ATCT produces higher thermal-mechanical stresses and causes more damage. Specifically, τ cycles of temperature fluctuation produces $D_N(\tau)$ damage under normal conditions and $D_A(\tau)$ damage under accelerated conditions, where $D_N(\tau) < D_A(\tau)$. Equivalently, under normal conditions, the testing unit needs τ^E cycles of temperature fluctuation to produce $D_A(\tau)$ damage, where τ^E is defined as the “equivalent test duration” or “equivalent number of cycles”. Obviously, $\tau^E > \tau$. Equivalent test duration relates testing units’ reliability metrics under normal conditions to those under accelerated conditions.

5.3.2 System Reliability Metrics under ATCTs

One-shot units (such as missiles) are produced in batches and kept in storage until needed. Specifically, the i^{th} batch of units with size n_i arrives into the storage at time w_i immediately after production. ATCTs are repeatedly (not in identical pattern) conducted at arbitrary time during the entire life horizon of the units. The m^{th} test is performed at time t_m for τ_m cycles of temperature fluctuation. Failed units are repaired and placed back into the system. The failure rate of a unit is an increasing function of the number of repairs. All units in the storage are defined as a system which is composed of nonhomogeneous units that arrived at different times, subjected to different ATCTs and repaired (when applicable) at different times. The system fails if a certain percentage or more units fail, which is referred to as a generalized "$k(t)$-out-of-$n(t)$: F" system.

The reliability metrics of a system composed of one-shot units are investigated in chapter 3, assuming the entire population is tested in each test under normal conditions. However, accelerated TCTs are performed repeatedly by testing samples as stated earlier. In the following, we analyze the system reliability metrics at arbitrary time under such a scenario. In each ATCT, p% of the population are tested under accelerated applied stress (temperature amplitude) at arbitrary levels (lower than a given threshold).

To obtain the system reliability metrics, the system state needs to be investigated first. The system state at time t is determined by the number of nonhomogeneous units with different characteristics (e.g., arrival time, the ATCT sequences under which the units are tested, and the ATCT sequences under which the units' failures are observed). To obtain the system state and the distribution of failed units, we define the following:

$F_x(\cdot), f_x(\cdot)$: *cdf*/*pdf* of unit's lifetime after the x^{th} repair;

$\lambda(x)$: the GBS shape parameter after the x^{th} repair;

w_i: time when i^{th} batch arrives in storage;

n_i: size of the i^{th} batch;

τ_β: duration of the β^{th} ATCT, $\beta = 1, 2, ..., m, ...$

t_β: time when the β^{th} ATCT is conducted;

$\Delta T_N, \Delta T_A$: the temperature amplitude under normal/accelerated conditions;

z_i: the number of ATCT s conducted before the arrival of the i^{th} batch;

$\underset{\sim}{\pi}$: the ATCT sequence under which the units are tested;

$\#\{\underset{\sim}{\pi}\}$: the number of tests in sequence $\underset{\sim}{\pi}$; $\#(\pi) = \pi$;

$\pi(a)$: the a^{th} test in sequence $\underset{\sim}{\pi}$;

$\underset{\sim}{j}$: the ATCT sequence under which the units' failure(s) is observed;

$\#\left(\underset{\sim}{j}\right)$: the number of tests in sequence $\underset{\sim}{j}$; $\#\left(\underset{\sim}{j}\right) = j$;

$j(\alpha)$: the a^{th} test in sequence $\underset{\sim}{j}$;

$N^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}$: the number of units with characteristics $i, \underset{\sim}{\pi}, \underset{\sim}{j}$ at time t_m. Specifically, the units are from the i^{th} batch, tested in sequence $\underset{\sim}{\pi}$ and fails in sequence $\underset{\sim}{j}$ (until the m^{th} test); $\underset{\sim}{j}$ is composed of arbitrary $\#\left(\underset{\sim}{j}\right)$ tests/elements in $\underset{\sim}{\pi}$; $N^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} = n^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}$;

$t_{j(\alpha)}$: time when the unit's a^{th} failure is observed;

$K^m(t)$: the number of failed units at arbitrary time $t \in \left[t_m + \tau_m, t_{m+1}\right)$, $K^m(t) = k^m(t)$;

$K^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t), R^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)$: the number of failed/survived units (with characteristics $i, \underset{\sim}{\pi}, \underset{\sim}{j}$) at time t; $K^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)$ and $R^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)$;

$\tau^E_\beta\left(i, \underset{\sim}{j}|\underset{\sim}{\pi}\right)$: equivalent test duration of the β^{th} ATCT for units with characteristics $i, \underset{\sim}{\pi}, \underset{\sim}{j}$;

For an arbitrary m, $\pi(a)$ could be any element in $\left(z_i + 1, \ldots, m\right)$ and $j(a)$ could be any of the elements in sequence $\underset{\sim}{\pi}$. Note that $j(a) - 1$ is uniquely determined by $j(a)$ while $j(a-1)$ could be any element in $\left\{\pi(1), \ldots, \pi(s-1) : \pi(s) = j(a)\right\}$.

There are numerous potential system states at arbitrary time t, depending on the batch sizes and number of previously performed ATCTs. Generally, the number of potential system states at time t equals to the number of feasible solutions to the following equations.

Meanwhile, the number of solutions of the i^{th} row of the following equations is the number of possible states of the i^{th} batch.

$$\sum_{\#\{\underset{\sim}{\pi}\}=0}^{m-z_1} \sum_{\forall \underset{\sim}{j} \in \underset{\sim}{\pi}} \left(K^m_{1,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) + R^m_{1,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) \right) = n_1 \qquad \text{Row 1}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\sum_{\#\{\underset{\sim}{\pi}\}=0}^{m-z_i} \sum_{\forall \underset{\sim}{j} \in \underset{\sim}{\pi}} \left(K^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) + R^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) \right) = n_i \qquad \text{Row } i$$

We numerically show how the system states and batch states are reflected by $K^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)$ and $R^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\ \forall i, \underset{\sim}{j}, \underset{\sim}{\pi}$ at time $t \in (t_1 + \tau_1, t_2)$, assuming two batches in the system at time t with $n_1 = n_2 = 5$. Specifically, $w_1 \le t_1 + \tau_1 \le w_2 < t$. The number of potential system states at time t is obtained by solving all feasible solutions to:

$$K^1_{1,1|1} + K^1_{1,0|1} + K^1_{1,0|0} + R^1_{1,1|1} + R^1_{1,0|1} + R^1_{1,0|0} = 5 \tag{5.11}$$

$$K^1_{2,0|0} + R^1_{2,0|0} = 5 \tag{5.12}$$

We always have: $K^1_{1,1|1}(t) + K^1_{1,0|1}(t) + K^1_{1,0|0}(t) + K^1_{2,0|0}(t) = K^1(t)$. In the following, we list a possible system state, which yields one failure at time t ($K^1(t) = 1$). Batch states are marked in the system state.

$$
\left.
\begin{array}{c}
(n_1 = 5) \\
+ \\
(n_2 = 5)
\end{array}
\right\}
\left\{
\begin{array}{l}
\left.
\begin{array}{l}
K^1_{1,1|1}(t) = 0 \text{ (\# of failed units in batch 1 at } t\text{, tested in } 1^{\text{st}} \text{ test,} \\
\qquad\qquad \text{observed failure at } t_1 + \tau_1\text{)} \\
K^1_{1,0|1}(t) = 1 \text{ (\# of failed units in batch 1 at } t\text{, tested in } 1^{\text{st}} \text{ test,} \\
\qquad\qquad \text{no failure observed)} \\
K^1_{1,0|0}(t) = 0 \text{ (\# of failed units in batch 1 at } t\text{,} \\
\qquad\qquad \text{no test thus no failure observed)} \\
R^1_{1,1|1}(t) = 0 \text{ (\# of survived units in batch 1 at } t\text{, tested in } 1^{\text{st}} \text{ test,} \\
\qquad\qquad \text{observed failure at } t_1 + \tau_1\text{)} \\
R^1_{1,0|1}(t) = 0 \text{ (\# of survived units in batch 1 at } t\text{, tested in } 1^{\text{st}} \text{ test,} \\
\qquad\qquad \text{no failure observed)} \\
R^1_{1,0|0}(t) = 0 \text{ (\# of survived units in batch 1 at } t\text{,} \\
\qquad\qquad \text{no test thus no failure observed)}
\end{array}
\right\rangle \text{batch 1 state} \\
\left.
\begin{array}{l}
K^1_{2,0|0}(t) = 0 \text{ (\# of failed units in batch 2 at } t\text{,} \\
\qquad\qquad \text{no test thus no failure observed)} \\
R^1_{2,0|0}(t) = 0 \text{ (\# of survived units in batch 2 at } t\text{,} \\
\qquad\qquad \text{no test thus no failure observed)}
\end{array}
\right\rangle \text{batch 2 state}
\end{array}
\right.
$$

The distribution of the failed units (the probability of having $k^m(t)$ failures at time $t \in [t_m + \tau_m, t_{m+1})$) is obtained by considering all potential system states whose sum of failed units with nonhomogeneous characteristics equals to $k^m(t)$:

$$
\begin{aligned}
P\left(K^m(t) = k^m(t)\right) &= P\left(\sum_{\forall i, \underset{\sim}{\pi}, \underset{\sim}{j}} K^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = k^m(t) \right) \\
&= \overbrace{\sum_a \sum_b}^{A} P\left(K^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t), R^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t);\ \forall i, \underset{\sim}{\pi}, \underset{\sim}{j} \right) \\
&= \sum_a \sum_b \prod_{\forall i} \overbrace{ \overbrace{ \left(\frac{(n_i)!}{\left[k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right]! \left[r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right]! \forall \underset{\sim}{\pi}, \underset{\sim}{j}} \right) \cdot \left[P \left(\begin{array}{l} K^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t), \\ R^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) \end{array} \right); \atop \forall \underset{\sim}{\pi}, \underset{\sim}{j} \right] }^{C} }^{B}
\end{aligned}
$$

$$= \sum_a \sum_b \prod_{\forall i} \left(\frac{(n_i)!}{\left[k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right]! \left[r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right]! \forall \underset{\sim}{\pi}, \underset{\sim}{j}} \right) \cdot \left[\overbrace{P\left(k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right)}^{D} \right]^{k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)} \cdot \left[\overbrace{P\left(r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right)}^{E} \right]^{r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)}$$

(5.13)

where

$$\sum_{\#\{\underset{\sim}{\pi}\}=0}^{m-z_i} \sum_{\forall \underset{\sim}{j} \in \underset{\sim}{\pi}} \left(k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) + r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) \right) = n_i \quad \forall i$$

$$a = \left(\sum_{\forall i, \underset{\sim}{\pi}, \underset{\sim}{j}} k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) \right) = k^m(t)$$

$$b = \left(\sum_{\forall i, \underset{\sim}{\pi}, \underset{\sim}{j}} r^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) \right) = \left(\sum_{\forall \{i; w_i \le t\}} n_i \right) - k^m(t)$$

To obtain the probability that the system has $k^m(t)$ ($k^m = 0,1, ..., \sum_{\forall \{i; w_i \le t\}} n_i$) failures at time t ($P\left(K^m(t) = k^m(t)\right)$), we consider all system states that yield $k^m(t)$ failures (system states that satisfy $\sum_{\forall i} \sum_{\forall \underset{\sim}{\pi}} \sum_{\forall \underset{\sim}{j}} k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = k^m(t)$) and sum these probabilities as $P\left(K^m(t) = k^m(t)\right)$ (term A in Eq. (5.13)). The probability of a specific system state (term B) is obtained by taking the product of the probabilities of all batch states. The probability of a batch's state (term C) is obtained by a multinomial distribution. Terms D and E are respectively the probability that the unit with characteristic $i, \underset{\sim}{\pi}, \underset{\sim}{j}$ fails and survives at time

t . Term D is presented in Eq. (5.14). Term E is obtained by replacing term D4 in Eq. (5.14)

with $R_{j(j)}\left(t-t_{j(j)}-\sum_{\beta=j(j);\,\beta\in\underset{\sim}{\pi},}^{\pi(\pi)-1}\tau_\beta+\sum_{\beta=j(j)+1}^{\pi(\pi)}\tau_\beta^E\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)\right)$.

$$
\begin{aligned}
P\left(k^m_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right) &= \overbrace{(p\%)^{\#\{\underset{\sim}{\pi}\}}(1-p\%)^{m-z_i-\#\{\underset{\sim}{\pi}\}}}^{D1}\cdot \\
&\overbrace{\begin{gathered}
\overbrace{\left[\begin{array}{l}
F\left(t_{j(1)}-w_i-\sum\limits_{\beta=\pi(1);\,\beta\in\underset{\sim}{\pi},}^{j(1)-1}\tau_\beta+\sum\limits_{\beta=\pi(1);\,\beta\in\underset{\sim}{\pi},}^{j(1)}\tau_\beta^E\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)\right)\\
-F\left(t_{\substack{\pi(s-1)\\ :\,\pi(s)=j(1)}}-w_i-\sum\limits_{\beta=\pi(1);\,\beta\in\underset{\sim}{\pi},}^{\pi(s-1)-1}\tau_\beta+\sum\limits_{\beta=\pi(1);\,\beta\in\underset{\sim}{\pi},}^{\pi(s-1)}\tau_\beta^E\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)\right)
\end{array}\right]}^{D3}\\
\cdot\prod_{\alpha=2}^{j(j)}\left[\begin{array}{l}
F\left(t_{j(\alpha)}-t_{j(\alpha-1)}-\sum\limits_{\beta=j(\alpha-1);\,\beta\in\underset{\sim}{\pi},}^{j(\alpha)-1}\tau_\beta+\sum\limits_{\beta=j(\alpha-1)+1;\,\beta\in\underset{\sim}{\pi},}^{j(\alpha)}\tau_\beta^E\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)\right)\\
-F\left(t_{\substack{\pi(s-1)\\ :\,\pi(s)=j(\alpha)}}-t_{j(\alpha-1)}-\sum\limits_{\beta=j(\alpha-1);\,\beta\in\underset{\sim}{\pi},}^{\pi(s-1)-1}\tau_\beta+\sum\limits_{\beta=j(\alpha-1)+1;\,\beta\in\underset{\sim}{\pi},}^{\pi(s-1)}\tau_\beta^E\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)\right)
\end{array}\right]
\end{gathered}}^{D2}\\
&\cdot\overbrace{\left[\begin{array}{l}
F\left(t-t_{j(j)}-\sum\limits_{\beta=j(j);\,\beta\in\underset{\sim}{\pi},}^{\pi(\pi)-1}\tau_\beta+\sum\limits_{\beta=j(j)+1}^{\pi(\pi)}\tau_\beta^E\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)\right)\\
-F\left(t_{\pi(\pi)}-t_{j(j)}-\sum\limits_{\beta=j(j);\,\beta\in\underset{\sim}{\pi},}^{\pi(\pi)-1}\tau_\beta+\sum\limits_{\beta=j(j)+1}^{\pi(\pi)}\tau_\beta^E\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)\right)
\end{array}\right]}^{D4}
\end{aligned}
$$

(5.14)

where $F_x(t)=\Phi\left\{\frac{1}{\alpha}\left[\left(\frac{t}{A(\Delta T)^b}\right)^{\lambda(x)}-\left(\frac{A(\Delta T)^b}{t}\right)^{\lambda(x)}\right]\right\}$

Term D1 in Eq.(5.14) is the probability that the unit in the i^{th} batch is tested in sequence $\underset{\sim}{\pi}$, where we assume without loss of generality that an arbitrary unit is selected and tested with equal probability p% in each ATCT. Term D2 is the probability that the unit with characteristics $i, \underset{\sim}{\pi}, \underset{\sim}{j}$ fails at time t. In the following, we explain term D3:

$$D3 = \overbrace{F\left(t_{j(1)} - w_i - \overbrace{\sum_{\beta=\pi(1);\, \beta\in\underset{\sim}{\pi},}^{j(1)-1} \tau_\beta}^{3} + \overbrace{\sum_{\beta=\pi(1);\, \beta\in\underset{\sim}{\pi},}^{j(1)} \tau_\beta^E \left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right)}^{4} \right)}^{1}$$
$$- \overbrace{F\left(\overbrace{t_{\substack{\pi(s-1) \\ :\, \pi(s)=j(1)}}}^{5} - w_i - \overbrace{\sum_{\beta=\pi(1);\, \beta\in\underset{\sim}{\pi},}^{\pi(s-1)-1} \tau_\beta}^{6} + \overbrace{\sum_{\beta=\pi(1);\, \beta\in\underset{\sim}{\pi},}^{\pi(s-1)} \tau_\beta^E \left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right)}^{7} \right)}^{2}$$

Term D3 calculates the probability that the unit's first failure is observed at time $t_{j(1)} + \tau_{j(1)}$.

Specifically:

Term 1 is the probability the unit fails before $t_{j(1)} + \tau_{j(1)}$;

Term 2 is the probability that the unit fails before $t_{\substack{\pi(s-1) \\ :\, \pi(s)=j(1)}} + \tau_{\substack{\pi(s-1) \\ :\, \pi(s)=j(1)}}$, where $t_{\substack{\pi(s-1) \\ :\, \pi(s)=j(1)}} + \tau_{\substack{\pi(s-1) \\ :\, \pi(s)=j(1)}}$ is the time when the last ATCT (that tests the unit before the unit's first failure is observed) ends;

Term 3 is the sum of the durations of all ATCTs that test the unit before the unit's first failure;

Term 4 is the sum of these ATCTs' equivalent durations under accelerated conditions;

Term 5 is the time when the last ATCT (that tests the unit before the unit's first failure is observed) starts;

Terms 6 and 7 are the durations and equivalent durations of all ATCTs that test the unit with characteristics $i, \underset{\sim}{\pi}, \underset{\sim}{j}$ before time $t_{\pi(s-1):\,\pi(s)=j(1)}$, respectively.

Note that the equivalent test duration is always dependent on a specific unit's characteristics, i.e., $\tau_{\beta}^{E}\left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right)$ is dependent on $i, \underset{\sim}{\pi}, \underset{\sim}{j}$ with fixed β. The iterative procedures and an illustration of calculating equivalent duration of the β^{th} ATCT ($\tau_{\beta}^{E}\left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right)$) for a unit with characteristics $i, \underset{\sim}{\pi}, \underset{\sim}{j}$ can be obtained by the following steps:

Step 1: Find y and σ such that $j(y) < \beta \le j(y+1)$ and $\pi(\sigma) < \beta \le \pi(\sigma+1)$; meanwhile, we always have: $j(y) = \pi(s) \le \pi(\sigma)$. If $y \ge 1$, continue with step 2; if $y = 0$, i.e., $\beta \le j(1)$, move to step 3.

Step 2: Calculate $\tau_{\beta}^{E}\left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right)$ via Eq. (5.15)

$$\begin{aligned} & F_{GBS}\left(t_{\beta} - \left(t_{\pi(s)} + \tau_{\pi(s)}\right) - \sum_{\alpha=\pi(s+1)}^{\pi(\sigma)} \tau_{\alpha} + \sum_{\alpha=\pi(s+1)}^{\pi(\sigma)} \tau_{\alpha}^{E}\left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right) + \tau_{\beta}^{E}\left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right); \Delta T_{N}\right) \\ & = F_{GBS}\left(t_{\beta-\pi(s)}^{E}\left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right) + \tau_{\beta}; \Delta T_{A}\right) \end{aligned} \tag{5.15}$$

where $t_{\beta-\pi(s)}^{E}\left(i, \underset{\sim}{j} \middle| \underset{\sim}{\pi}\right)$ is obtained in Eq. (5.16):

$$F_{GBS}\left(t^{E}_{\beta-\pi(s)}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right);\Delta T_A\right)=F_{GBS}\left(t_\beta-\left(t_{\pi(s)}+\tau_{\pi(s)}\right)-\sum_{\alpha=\pi(s+1)}^{\pi(\beta-1)}\tau_\alpha+\sum_{\alpha=\pi(s+1)}^{\pi(\beta-1)}\tau^{E}_{\alpha}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right);\Delta T_N\right)\quad(5.16)$$

where $\tau^{E}_{\alpha}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right),\alpha=\pi(s+1),...,\pi(\beta-1)$ in Eq. (5.15) is obtained iteratively according to the above procedure.

Step 3: Calculate $\tau^{E}_{\beta}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)$ via Eq. (5.17):

$$F_{GBS}\left(t_\beta-w_i-\sum_{\alpha=\pi(1)}^{\pi(\sigma)}\tau_\alpha+\sum_{\alpha=\pi(1)}^{\pi(\sigma)}\tau^{E}_{\alpha}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)+\tau^{E}_{\beta}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right);\Delta T_N\right)=F_{GBS}\left(t^{E}_{\beta-w_i}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)+\tau_\beta;\Delta T_A\right)\quad(5.17)$$

where $t^{E}_{\beta-w_i}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right)$ is obtained in Eq. (5.18):

$$F_{GBS}\left(t^{E}_{\beta-w_i}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right);\Delta T_A\right)=F_{GBS}\left(t_\beta-w_i-\sum_{\alpha=\pi(1)}^{\pi(\sigma)}\tau_\alpha+\sum_{\alpha=\pi(1)}^{\pi(\sigma)}\tau^{E}_{\alpha}\left(i,\underset{\sim}{j}\middle|\underset{\sim}{\pi}\right);\Delta T_N\right)\quad(5.18)$$

The expected number of failures at arbitrary time $t\in\left[t_m+\tau_m,t_{m+1}\right)$ is obtained as:

$$E\left(K^m(t)\right)=\sum_{k^m(t)=0}^{\sum_{\forall i}n_i}k^m(t)\cdot P\left(K^m(t)=k^m(t)\right)\quad(5.19)$$

Note that the testing results are known under some circumstances, i.e., $n^{m}_{i,j|\pi}\ \forall i,\underset{\sim}{\pi},\underset{\sim}{j}$ are known by time $t_m+\tau_m$. The model proposed in Eq. (5.13) can be simplified and applied:

$$P\left(\left(K^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = k^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t), R^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t) = r^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t); \forall i, \underset{\sim}{j}, \underset{\sim}{\pi}\right) \middle| \left(N^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} = n^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \forall i, \underset{\sim}{j}, \underset{\sim}{\pi}\right)\right)$$

$$= \sum_{a}\sum_{b}\prod_{\forall i,\underset{\sim}{\pi},\underset{\sim}{j}} \binom{n^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}}{k^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)} \cdot \left[\overbrace{P\left(k^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right) \middle| N^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} = n^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}}^{A}\right]^{k^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)} \cdot \left[\overbrace{P\left(r^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)\right) \middle| N^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} = n^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}}^{B}\right]^{r^{m}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(t)}$$

(5.20)

where terms a and b are the same as presented in Eq. (5.14), and terms A and B can be obtained conditionally by referring to Eq. (5.14).

The potential number of system states increases with the conduct of the ATCTs. Therefore, the analytical expression of the system reliability is a piecewise function as the conduct of the ATCTs. We also note that the system reliability shows a step increment when new batches are introduced or when the ATCT is conducted and failed units are repaired. An illustration is given in section 5.5.

5.4 Simulation Model

We develop a simulation model to validate the proposed model in section 5.3. We apply *r* replications and compare the mean of system's expected number of failures of the *r* replications (using the simulation model and the proposed model). Each replication with one set of randomly generated testing parameters (batch size n_i, batch arrival time w_i, test time t_β, test duration τ_β, where $\beta = 1, 2, ..., m$) runs for 10^4 iterations. In each iteration, the number of failed units is recorded as $K^m(t)$. The reliability metrics ($E\left(K^m(t)\right)$ and $P\left(K^m(t)=k^m(t)\right)$) in one replication are respectively obtained by Eqs. (5.21) and (5.22):

$$E\left(K^m(t)\right)=\frac{\sum_{10^4} K^m(t)}{10^4} \tag{5.21}$$

$$P\left(K^m(t)=k^m(t)\right)=\frac{\text{frequency of } K^m(t)=k^m(t)}{10^4} \tag{5.22}$$

We illustrate the procedures of estimating $K^1(t)$ and $K^2(t)$ in one iteration as follows, where $K^m(t)\,\forall m$ can be estimated iteratively. We assume that $w_1=0<t_1+\tau_1<w_2<t_2$.

- ***Estimation of*** $K^1(t)$:

1. Generate n_1 random failure times follow GBS distribution with given parameters and temperature amplitude ΔT_N ; randomly select $n^1_{1,0|0}=p\%\cdot n_1$ failure times; within the $n^1_{1,0|0}$ failure times, record the number of failure times between $\left(0,t_1+\tau_1^E(1,0|0)\right]$ as $n^1_{1,1|1}$.
2. Generate $n^1_{1,1|1}$ failure times, record the number of failure times between $(0,t-t_1-\tau_1]$ as $K^1_{1,1|1}(t)$.
3. From the $n^1_{1,0|0}-n^1_{1,1|1}=n^1_{1,0|1}$ failure times, record the number of failure times between $\left(t_1+\tau_1^E(1,0|0),t-\tau_1+\tau_1^E(1,0|0)\right]$ as $K^1_{1,0|1}(t)$.
4. Generate $n_1-n^1_{1,0|0}$ failure times, record the number of failure times between $(0,t]$ as

$K^{1}_{1,0|0}(t)$.

5. $K^{1}(t)=K^{1}_{1,0|0}(t)+K^{1}_{1,0|1}(t)+K^{1}_{1,1|1}(t)$.

- ***Estimation of*** $K^{2}(t)$

1. Generate $n^{1}_{1,1|1}$ failure times from $n_1-s^{1}_{1,0|0}$;

1.1. Randomly select $p\%$;

1.1.1 Record the number of failure times between $\left(0,t_2-t_1-\tau_1+\tau_2^E\left(1,1|1\right)\right]$ as $n^{2}_{1,12|12}$; generate $n^{2}_{1,12|12}$ random failures times and record the number of random failures between $\left(0,t-t_2-\tau_2\right)$ as $K^{2}_{1,12|12}$;

1.1.2 Record the number of failure times between $\left(t_2-t_1-\tau_1+\tau_2^E\left(1,1|1\right),t-t_1-\tau_1-\tau_2+\tau_2^E\left(1,1|1\right)\right]$ as $K^{2}_{1,1|12}$.

1.2 From the remaining $1-p\%$ failure times, record the number of failures between $\left(0,t-t_1-\tau_1\right)$ as $K^{2}_{1,1|1}$.

2. From the $n^{1}_{1,0|1}$ failure times:

2.1 Randomly select $p\%\cdot n^{1}_{1,0|1}$;

2.1.1 Record the number of failure times between $\left(t_1+\tau_1^E\left(1,0|0\right),t_2-\tau_1+\tau_1^E\left(1,0|0\right)+\tau_2^E\left(1,0|1\right)\right]$ as $n^{1}_{1,1|12}$; generate $n^{1}_{1,1|12}$ random failures and record the number of failures between $\left(0,t-t_2-\tau_2\right]$ as $K^{1}_{1,2|12}$;

2.1.2 From $p\%\cdot n^1_{1,0|1} - n^1_{1,1|12}$ failure times, record the number of failure times between

$$\left(t_2-\tau_1+\tau_1^E\left(1,0|0\right)+\tau_2^E\left(1,0|1\right),t-\tau_1-\tau_2+\tau_1^E\left(1,0|0\right)+\tau_2^E\left(1,0|1\right)\right] \text{ as } K^1_{1,0|12};$$

2.2 From the remaining $\left(1-p\%\right)\cdot n^1_{1,0|1}$ failure times, record the number of failure times between $\left(t_1+\tau_1^E\left(1,0|0\right),t-\tau_1+\tau_1^E\left(1,0|0\right)\right)$ as $K^1_{1,0|1}$.

3. From the $n^1_{1,0|0}$ failure times:

3.1 Randomly select $p\%\cdot n^1_{1,0|0}$;

3.1.1 Record the number of failure times between $\left(0,t_2+\tau_2^E\left(1,0|0\right)\right]$ as $n^1_{1,2|2}$; generate $n^1_{1,2|2}$ failure times and record the number of failure times between $\left(0,t-t_2-\tau_2\right]$ as $K^2_{1,2|2}$;

3.1.2 Record the number of failure times between $\left(t_2+\tau_2^E\left(1,0|0\right),t-\tau_2+\tau_2^E\left(1,0|0\right)\right]$ as $K^2_{1,0|2}$;

3.2 From the remaining failure times, record the number of failures between $\left(0,t\right]$ as $K^2_{1,0|0}$.

4. $K^2\left(t\right)=K^2_{1,12|12}\left(t\right)+K^2_{1,1|12}\left(t\right)+K^2_{1,2|12}\left(t\right)+K^2_{1,1|1}\left(t\right)+K^2_{1,0|1}\left(t\right)+K^2_{1,2|2}\left(t\right)+K^2_{1,0|2}\left(t\right)+K^2_{1,0|0}\left(t\right)$

General procedures of estimating $K^m\left(t\right)\forall m$ in one simulation iteration can be obtained iteratively.

In this section, we conduct sequential ATCTs on batches of one-shot units with nonhomogeneous characteristics. We propose accurate and effective models to obtain reliability metrics of the mixtures of units, taking into account the effect of thermal fatigue and sampling uncertainty. The proposed models can be applied to a variety of testing scenarios as discussed in chapter 3.

5.5 Numerical Illustrations and Validation of the Proposed Models

5.5.1 GBS Performance in Predicting Fatigue Life

Assuming that $A = 1\times 10^7, \mu_b = -2, \sigma_b = 0.01$ and $\Delta T_A = 194$, we generate a group of accelerated fatigue data based on the CM model which are then used for the estimation of GBS parameters. The parameters are then used to predict the expected fatigue life under normal conditions. Table 5.1 shows the predicted fatigue life by the CM model and the GBS accelerated model under different normal conditions. The comparisons validate the performance of GBS distribution in predicting fatigue life under normal conditions.

Table 5.1 Expected fatigue life predicted by the CM model and the GBS model

ΔT_N	CM model	GBS model
64	2441.4063	2442.9241
104	924.5562	927.7419
144	482.2531	481.4050

184	295.3686	295.8766

5.5.2 System Reliability

In section 5.3, the system reliability metrics at arbitrary time $t \in [t_m + \tau_m, t_{m+1})$ $\forall m$ are investigated. In the following, the system reliability until $t > t_4 + \tau_4$ is plotted for different $p\%$ values with the following parameters:

$q\% = 20\%$; $n_i = 20,\ i = 1,2,3,4,5$;

$w_1 = 0, w_2 = 150, w_3 = 350;\ t_1 = 100, t_2 = 250, t_3 = 450, t = 460; \tau_1 = \tau_2 = \tau_3 = 20;$

$A = 6\times10^6, b = -2, \alpha = 1, \lambda(x) = 0.5 - 0.001x$; and $\Delta T_N = 104, \Delta T_A = 194$.

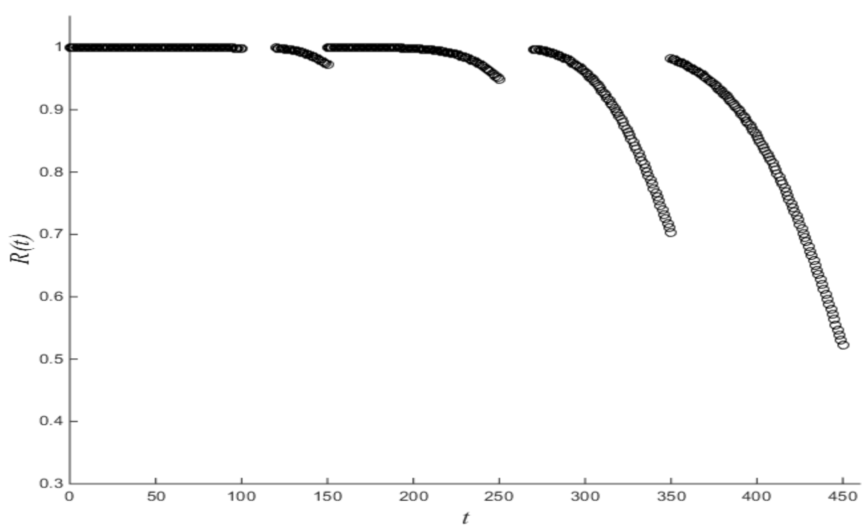

Figure 5.1 System reliability until time *t* with $p\% = 50\%$

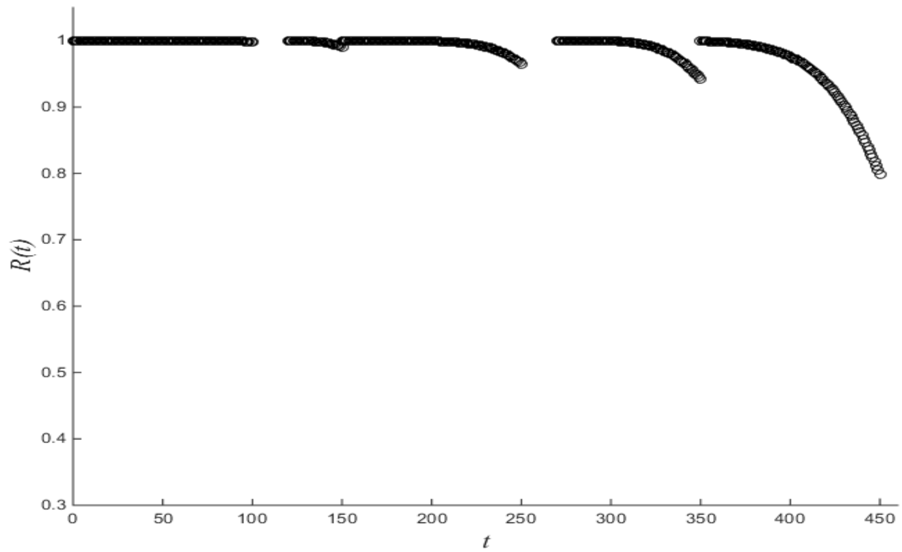

Figure 5.2 System reliability until time t with $p\% = 90\%$

Figures 5.1 and 5.2 show the system reliabilities for $p\% = 50\%$ and $p\% = 90\%$, respectively. They also show that selecting and testing (and repairing failed units) a large number of units improves the system reliability metrics. Meanwhile, the system reliability shows a step increment when new batches arrive or when the tests are conducted.

5.5.3 Expected Number of Failures by the Proposed Model and the Simulation Model

Applying the simulation model and the proposed model, we calculate $E\left(K^m\left(t\right)\right)$ at arbitrary time $t \in \left(t_m + \tau_m, t_{m+1}\right)$ for $m = 1, 2, 3$. The testing parameters are generated with simple distributions as described in section 5.4. We validate the proposed model for arbitrary values of n_i by considering:

a) $n_i = 150 \ \forall i$;

b) $n_1 = 50,\ n_2 = 100,\ n_1 = 150$; i.e., the system has a large number of newly arrived units;

c) $n_1 = 150, n_2 = 100,\ n_3 = 50$; i.e., the system has a large number of aged units.

The average $E\left(K^m\left(t\right)\right)$ for $m = 1, 2, 3$ using the proposed model and simulation model with batch sizes in a), b) and c) are shown in Tables 5.2-5.4.

Table 5.2 Expected number of failures at arbitrary time between the first three tests based on proposed model and simulation model with batch sizes in *a*)

	t_1	t_2	t_3
Proposed model	0.0243	0.0978	0.1703
Simulation model	0.0243	0.0981	0.1709

Table 5.3 Expected number of failures at arbitrary time between the first three tests based on proposed model and simulation model with batch sizes in *b*)

	t_1	t_2	t_3
Proposed model	0.0501	0.1836	0.2773
Simulation model	0.0502	0.1839	0.2779

Table 5.4 Expected number of failures at arbitrary time between the first three tests based on proposed model and simulation model with batch sizes in *c*)

	t_1	t_2	t_3
Proposed model	0.0718	0.2692	0.3978
Simulation model	0.0719	0.2694	0.3974

5.6 Conclusions

In this chapter, we propose accurate and effective approaches that model the reliability metrics of a system subjecting to thermal cyclic fatigue by utilizing the GBS distribution. The system has a mixture of nonhomogeneous one-shot units that arrive at different times and are subjected to sequential ATCTs. We demonstrate that the GBS distribution, though developed for mechanical fatigue failure, is suitable for modelling thermal fatigue data. Moreover, compared with the commonly used CM model, we show that the GBS distribution provides additional reliability metrics and assesses the reliability at the system level that could not be accomplished by the CM model. The proposed models' flexibility and robustness in estimating/predicting the system reliability metrics are validated by a simulation model.

CHAPTER 6

RELIABILITY MODELING OF MIXTURES OF ONE-SHOT UNITS UNDER COMPETING FAILURE MODES

There exists many situations that a unit fails in different failure modes which are caused by different failure mechanisms. For example, the integrated circuit (IC) board might no longer function properly due to either the fluctuations of temperature that initiates and propagates cracks in the solder joints to an unacceptable level, or due to degradation of resistors with time that eventually reaches a critical point and the IC seizes to function. Under such circumstances, the failure data cannot be described by a single failure time distribution; instead, a competing failure model which considers multiple failure modes is adopted to assess unit's reliability metrics. Under competing failure modes, a unit fails when any of the failure mechanisms reaches its failure state.

The one-shot unit might experience competing failure modes during its storage period. Repeated thermal cyclic tests (TCTs) are randomly conducted. At the end of an arbitrary TCT, the unit's failure is observed either when any of its failure modes occurs suddenly (failure modes without indicators of failure) or when any of its degradation modes (which exhibit indicators that eventually lead to failure) reach its "failure threshold". Meanwhile, the unit is repaired either when it fails between two tests or when one of its degradation modes reaches the predetermined "repair threshold", where the "repair threshold" is lower than the "failure threshold".

In this chapter, the reliability metrics of the system under competing failure modes are investigated. In section 6.1, we study the units' potential failure modes and its reliability at arbitrary time. In section 6.2, we investigate the reliability metrics (such as failure distribution, reliability function, time-to-the k^{th} failure) of the system under competing failure modes, where batches of one-shot units arrive and sequential TCTs are conducted. In section 6.3, we investigate the system reliability metrics under a generalized scenario where the units may exhibit M failure modes (that occur suddenly) and N degradation modes. In section 6.4, we develop a simulation model to validate the models proposed in sections 6.2 and 6.3. In section 6.5 we illustrate the use of the proposed models with a numerical example. In section 6.6, we conclude our work.

6.1 Unit's Reliability under Competing Failure Modes

The one-shot units might fail suddenly (without indications that failure is about to occur) at arbitrary time during its storage period, which can be characterized by certain lifetime distributions (e.g., Weibull distribution); besides, the repeated fluctuations of temperature produce thermal stresses and initiate and propagate cracks of the units (e.g., solid solder joints in an IC board) in each cycle. The crack is observable (as an indicator of failure) and eventually causes thermal fatigue failure when it reaches the failure threshold. Moreover, the resistance of the units degrades with time under arbitrary temperature, which eventually reaches an unacceptable level and causes the failure of the unit. Specifically, the one-shot units in the system might fail due to any of the following three reasons:

a) Failure occurs suddenly without an indicator; the failure time follows a Weibull distribution and its CDF is given by Eq. (6.1);

$$F_W(t) = 1 - \exp\left(-\left(\frac{t}{\theta}\right)^{\gamma}\right) \tag{6.1}$$

b) Crack is initiated and propagated due to the thermal cyclic stresses; the failure time due to crack propagation is modeled by a GBS distribution as presented in Eq. (6.2);

$$P\left(\Delta C(t) \geq \Delta C_F\right) = F_{GBS}\left(t; \Delta C_F\right) = \Phi\left\{\frac{\sqrt{\Delta C_F}}{\alpha}\left[\left(\frac{t}{\beta\sqrt{\Delta C_F}}\right)^{\lambda} - \left(\frac{t}{\beta\sqrt{\Delta C_F}}\right)^{-\lambda}\right]\right\} \tag{6.2}$$

where

α, β and λ are the GBS parameters which depend only on unit's material property;

$\Delta C(t)$ is the crack length at time t; and

ΔC_F is the failure threshold of the crack.

c) The resistor changes (degrades) under arbitrary temperature as described in Eq. (6.3):

$$\Delta R(t) = R_0 \cdot \left(\frac{t}{\phi\, e^{\frac{E}{kT}}}\right)^{m} \tag{6.3}$$

where

$\Delta R(t)$ is the change in resistance (degradation level) at time t;

R_0 is the initial value of resistance;

ϕ and m are constants;

T is the temperature (in Kelvin);

k is the Boltzmann constant and equals to 8.62×10^{-5}; and

E is the activation energy which depends on the unit's material property.

It is worth noting that in the resistance change model, the empirically determined exponent ϕ usually follows certain probability distribution either due to the variation of the testing environment or due to the material property inconsistency. Without loss of generality, we assume that ϕ follows a normal distribution, i.e., $\phi \sim N\left(\mu_\phi, \sigma_\phi\right)$. In Eq. (6.4), we calculate the probability that the resistance degrades to its failure threshold (ΔR_F) before time t:

$$P\left(\Delta R(t) > \Delta R_F\right) = P\left(\phi < \sqrt[m]{\frac{\Delta R_F \cdot \exp\left(\frac{Em}{kT}\right)}{R_0 t^m}}\right) = \Phi\left(\frac{\exp\left(\frac{E}{kT}\right)\sqrt[m]{\frac{\Delta R_F}{R_0 t^m}} - \mu_\phi}{\sigma_\phi}\right) \tag{6.4}$$

As stated earlier, the degradation process provides indicators of failure which can be utilized to prevent potential failures. Specifically, units are repaired to as good as new either when it fails abruptly or when either of the above two degradation modes (crack growth or resistance degradation) reaches its repair threshold. It is without loss of generality to assume the repair threshold is lower than the failure threshold. Respectively, the unit in the system needs repair at time t due to the crack growth or the resistance degradation with the probabilities calculated in Eqs. (6.5) and (6.6) respectively:

$$F_{GBS}\left(t;\Delta C_R\right)=\Phi\left\{\frac{\sqrt{\Delta C_R}}{\alpha}\left[\left(\frac{t}{\beta\sqrt{\Delta C_R}}\right)^{\lambda}-\left(\frac{t}{\beta\sqrt{\Delta C_R}}\right)^{-\lambda}\right]\right\} \tag{6.5}$$

$$P\left(\Delta R(t)>\Delta R_R\right)=P\left(\phi<\sqrt[m]{\frac{\Delta R_R\cdot\exp\left(\frac{Em}{kT}\right)}{R_0 t^m}}\right)=\Phi\left(\frac{\exp\left(\frac{E}{kT}\right)\sqrt[m]{\frac{\Delta R_R}{R_0 t^m}}-\mu_\phi}{\sigma_\phi}\right) \tag{6.6}$$

At arbitrary time t, the units can (only) be in any of the following three states:

State 1: the unit is good and needs no repair; i.e., the unit does not fail suddenly and neither of the degradation levels reaches its repair threshold;

State 2: the unit is good but needs repair; i.e., the unit does not fail suddenly and at least one of the degradation modes' levels reaches its repair threshold; however, neither of the degradation levels reaches its failure threshold;

State 3: the unit fails and needs repair; i.e., the unit either fails suddenly or at least one of the degradation levels reaches its failure threshold.

At arbitrary time *t*, letting $P_1(t)$, $P_2(t)$ and $P_3(t)$ respectively denote the probability that the unit is in state 1, 2 and 3, then:

$$P_1(t) = \left[1 - F_W(t)\right] \cdot \left[P\left(\Delta C(t) < \Delta C_R\right)\right] \cdot \left[P\left(\Delta R(t) < \Delta R_R\right)\right]$$
$$= \left[1 - F_W(t)\right] \cdot \left[1 - F_{GBS}(t; \Delta C_R)\right] \cdot \left[1 - \Phi\left(\frac{t \exp\left(\frac{E}{kT}\right)\sqrt[m]{\frac{\Delta R_R}{R_0}} - \mu_\phi}{\sigma_\phi}\right)\right] \tag{6.7}$$

$$P_2(t) = \left[1 - F_W(t)\right] \cdot \begin{bmatrix} \left(P\left(\Delta C(t) < \Delta C_F\right) - P\left(\Delta C(t) < \Delta C_R\right)\right) \cdot P\left(\Delta R(t) < \Delta R_F\right) \\ + \\ P\left(\Delta C(t) < \Delta C_R\right) \cdot \left(P\left(\Delta R(t) < \Delta R_F\right) - P\left(\Delta R(t) < \Delta R_R\right)\right) \end{bmatrix}$$
$$= \left[1 - F_W(t)\right] \cdot \begin{bmatrix} \left(F_{GBS}(t; \Delta C_F) - F_{GBS}(t; \Delta C_R)\right) \cdot \left[1 - \Phi\left(\frac{\exp\left(\frac{E}{kT}\right)\sqrt[m]{\frac{\Delta R_F}{R_0 t^m}} - \mu_\phi}{\sigma_\phi}\right)\right] \\ + \\ \left(1 - F_{GBS}(t; \Delta C_R)\right) \cdot \begin{pmatrix} \Phi\left(\frac{\exp\left(\frac{E}{kT}\right)\sqrt[m]{\frac{\Delta R_R}{R_0 t^m}} - \mu_\phi}{\sigma_\phi}\right) \\ -\Phi\left(\frac{\exp\left(\frac{E}{kT}\right)\sqrt[m]{\frac{\Delta R_F}{R_0 t^m}} - \mu_\phi}{\sigma_\phi}\right) \end{pmatrix} \end{bmatrix}$$

(6.8)

$$P_3(t) = F_W(t) + \left[1 - F_W(t)\right] \cdot \left(P\left(\Delta C(t) \geq \Delta C_F\right) + P\left(\Delta C(t) < \Delta C_F\right) P\left(\Delta R(t) > \Delta R_F\right) \right)$$

$$= F_W(t) + \left[1 - F_W(t)\right] \cdot \begin{pmatrix} F_{GBS}\left(t; \Delta C_F\right) \\ + \\ \left(1 - F_{GBS}\left(t; \Delta C_F\right)\right) \cdot \Phi\left(\dfrac{t \exp\left(\dfrac{E}{kT}\right) \sqrt[m]{\dfrac{\Delta R_F}{R_0}} - \mu_\phi}{\sigma_\phi} \right) \end{pmatrix}$$

(6.9)

6.2 System Reliability Metrics under Competing Failure Modes

As discussed in the previous chapters, one-shot units (such as missiles) are produced in batches and kept in storage until needed. Specifically, the i^{th} batch of units with size n_i arrives into the storage at time w_i immediately after production. TCTs are conducted at arbitrary times during the entire life horizon of the units. The m^{th} TCT is performed at time t_m by testing the entire population. The units in storage are subject to three failure modes as discussed in section 6.1. Units are repaired and placed back into the system either when: 1) they fail suddenly between two tests; or 2) when crack length reaches the repair threshold ΔC_R; or 3) when resistance changes to the repair threshold ΔR_R. The system fails if a certain percentage or more units fail, which is referred to as a generalized "*k*(*t*)-out-of-*n*(*t*): F" system.

In this section, we investigate the system reliability metrics at arbitrary time t under the competing failure modes. The following notations are defined:

z_i : the number of TCTs conducted before the arrival of the i^{th} batch;

$\underset{\sim}{j}$: the TCT sequence under which the units' repair(s) is conducted;

$\#\left(\underset{\sim}{j}\right)$: the number of tests in sequence $\underset{\sim}{j}$; $\#\left(\underset{\sim}{j}\right) = j$;

$j(\alpha)$: the a^{th} TCT in sequence $\underset{\sim}{j}$;

$\Delta C\left(j(\alpha)\right)$: the crack length at the end of the a^{th} TCT;

$\Delta R\left(j(\alpha)\right)$: the resistance change at the end of the a^{th} TCT;

K^m : the number of failed units at the end of the m^{th} TCT;

$K^m_{i,\underset{\sim}{j}}, R^m_{i,\underset{\sim}{j}}$: the number of failed (survived) units with characteristics $i, \underset{\sim}{j}$ at time t_m; $\underset{\sim}{j}$ is composed of arbitrary $\#\left(\underset{\sim}{j}\right)$ tests in the discrete set $\{z_i+1,...,m\}$; specifically, $K^m_{i,\underset{\sim}{j}} = k^m_{i,\underset{\sim}{j}}$ and $R^m_{i,\underset{\sim}{j}} = r^m_{i,\underset{\sim}{j}}$;

$t_{j(\alpha)}$: time when the unit's a^{th} repair is conducted.

6.2.1 System Failure Distribution and Reliability under Competing Failure Modes

The distribution of the failed units at the end of the m^{th} TCT is obtained in Eq. (6.10) by considering all potential possibilities that yield k^m failures:

$$\begin{aligned} P\left(K^m = k^m\right) &= P\left(\sum_{\forall i,\underset{\sim}{j}} K^m_{i,\underset{\sim}{j}} = k^m\right) \\ &= \sum_a \sum_b P\left(K^m_{i,\underset{\sim}{j}} = k^m_{i,\underset{\sim}{j}}, R^m_{i,\underset{\sim}{j}} = r^m_{i,\underset{\sim}{j}};\ \forall i, \underset{\sim}{j}\right) \\ &= \sum_a \sum_b \prod_{\forall i} \left(\frac{(n_i)!}{\left(k^m_{i,\underset{\sim}{j}}!\right)\left(r^m_{i,\underset{\sim}{j}}!\right)\forall \underset{\sim}{j}}\right) \cdot \left[P\left(K^m_{i,\underset{\sim}{j}} = k^m_{i,\underset{\sim}{j}}, R^m_{i,\underset{\sim}{j}} = K^m_{i,\underset{\sim}{j}}\right); \forall \underset{\sim}{j}\right] \\ &= \sum_a \sum_b \prod_{\forall i} \left(\frac{(n_i)!}{\left(k^m_{i,\underset{\sim}{j}}!\right)\left(k^m_{i,\underset{\sim}{j}}!\right)\forall \underset{\sim}{j}}\right) \cdot \left[P\left(k^m_{i,\underset{\sim}{j}}\right)\right]^{k^m_{i,\underset{\sim}{j}}} \cdot \left[P\left(r^m_{i,\underset{\sim}{j}}\right)\right]^{r^m_{i,\underset{\sim}{j}}} \end{aligned} \tag{6.10}$$

where

$$a = \left(\sum_{\forall i,\underset{\sim}{j}} k^m_{i,\underset{\sim}{j}}\right) = k^m;\ b = \left(\sum_{\forall i,\underset{\sim}{j}} r^m_{i,\underset{\sim}{j}}\right) = \left(\sum_{\forall\{i;w_i \le t\}} n_i\right) - k^m$$

We obtain the expected number of failures at the end of the m^{th} TCT as shown in Eq. (6.11)

$$E\left(K^m\right) = \sum_{k^m=0}^{\sum_{\forall i} n_i} k^m \cdot P\left(K^m = k^m\right) \tag{6.11}$$

Defining the entire population as a generalized "$k(t)$-out-of-$n(t)$*:* F" system, where $k(t) = \left\lfloor q\% \cdot \left(\sum_{\forall\{i,w_i \le t\}} n_i\right)\right\rfloor$ and $n(t) = \sum_{\forall\{i,w_i \le t\}} n_i$. The reliability of the system at time $t \in \left[t_m,\ t_{m+1}\right)$ ($m = 1, 2, \ldots$) is obtained using Eq. (6.12):

$$R_m(t) = \sum_{k^m=0}^{q\% \cdot \sum_{\forall i} n_i} P\left(K^m(t) = k^m(t)\right) \tag{6.12}$$

The analytical expression of the system reliability is a piecewise function of the time of the conduct of the TCTs. We also note that the system reliability shows a step increment when new batches are introduced or when the TCT is conducted and units are repaired. An illustration is given in section 6.5.

6.2.2 System's Time-to-k^{th}-failure

One-shot units are deployed after long terms of storage. A sample with certain number of one-shot units is selected from the stored population and deployed for use. In the following, we obtain the *pdf* of the time-to-k^{th}-failure ($f_k(t)$) of an arbitrarily selected sample (with nonhomogeneous units) under competing failure modes. We first define the following:

$s_{i,\underset{\sim}{j}}^m$: the number of selected one-shot units with characteristics $i, \underset{\sim}{j}$ at the end of the m^{th} TCT;

$s_{\underset{\sim}{i'},\underset{\sim}{j'}}^m$: the number of selected one-shot units with characteristics $i', \underset{\sim}{j}'$ at the end of the m^{th} TCT, specifically, the k^{th} failed one-shot unit has the characteristics $i', \underset{\sim}{j}'$;

$k_{i,\underset{\sim}{j}}^m$: the number of selected one-shot units (with characteristics $i, \underset{\sim}{j}$ at the end of the m^{th} TCT) that fail between time t_m and t;

$k_{\underset{\sim}{i}',\underset{\sim}{j}'}^{m}$: the number of selected one-shot units (with characteristics $i', \underset{\sim}{j}'$ at the end of the m^{th} TCT) that fail between time t_m and t;

$r_{i,\underset{\sim}{j}}^{m}$: the number of selected one-shot units (with characteristics $i, \underset{\sim}{j}$ at the end of the m^{th} TCT) that survive until time t;

$r_{i',\underset{\sim}{j}'}^{m}$: the number of selected one-shot units (with characteristics $i', \underset{\sim}{j}'$ at the end of the m^{th} TCT) that survive until time t;

The reliability of the unit with characteristics $i, \underset{\sim}{j}$ at arbitrary time $t \in [t_m,\ t_{m+1})$ is obtained by Eq. (6.13):

$$R_{i,\underset{\sim}{j}}(t) = \frac{\left[1 - F_W\left(t - t_{j(j)}\right)\right] \cdot P\left(\Delta C\left(t - t_{j(j)}\right) < \Delta C_F\right) \cdot P\left(\Delta R\left(t - t_{j(j)}\right) < \Delta R_F\right)}{\left[1 - F_W\left(t_m - t_{j(j)}\right)\right] \cdot P\left(\Delta C\left(t_m - t_{j(j)}\right) < \Delta C_F\right) \cdot P\left(\Delta R\left(t_m - t_{j(j)}\right) < \Delta R_F\right)} \qquad (6.13)$$

Accordingly, the expected time-to-k^{th}-failure of the sample is expressed as $E(T_k) = t_m + \int_0^\infty f_k(t)\,dt$, where in Eq. (6.14),

$$f_k(t) = \sum_{i',\underset{\sim}{j}'} \sum_{\forall i,\underset{\sim}{j}\left(\sum_{\forall i}\sum_{\forall \underset{\sim}{j}} k_{i,\underset{\sim}{j}}^{m}\right) = k-1} \frac{(k-1)!}{\prod_{\forall i}\prod_{\forall \underset{\sim}{j}}\left(\left(k_{i,\underset{\sim}{j}}^{m}\right)!\right)} \cdot \left\{ \begin{array}{l} \left[s_{i',\underset{\sim}{j}'}^{m} \begin{pmatrix} s_{i',\underset{\sim}{j}'}^{m} - 1 \\ k_{i',\underset{\sim}{j}'}^{m} - 1 \end{pmatrix} \cdot \left(1 - R_{i,\underset{\sim}{j}}(t)\right)^{k_{i',\underset{\sim}{j}'}^{m}} \left(R_{i,\underset{\sim}{j}}(t)\right)^{r_{i',\underset{\sim}{j}'}^{m}} \right] \cdot \dfrac{d\left[1 - R_{i,\underset{\sim}{j}}(t)\right]}{dt} \\ \cdot \left[\prod_{\forall i \text{ except } i'} \prod_{\forall \underset{\sim}{j} \text{ except } \underset{\sim}{j}'} \left(\begin{pmatrix} s_{i,\underset{\sim}{j}}^{m} \\ k_{i,\underset{\sim}{j}}^{m} \end{pmatrix} \left(1 - R_{i,\underset{\sim}{j}}(t)\right)^{k_{i,\underset{\sim}{j}}^{m}} \left(R_{i,\underset{\sim}{j}}(t)\right)^{r_{i,\underset{\sim}{j}}^{m}} \right) \right] \end{array} \right\}$$

(6.14)

When the units exhibit competing failure modes, then the sample's time-to-k^{th}-failure is expected to occur earlier than that when the system exhibits a single failure mode.

6.3 Generalization of System Reliability Metrics Modeling

In this section, we generalize the scenario discussed in section 6.2 and investigate the system reliability metrics when the units exhibit M failure modes without indictors and N degradation modes. At the end of the m^{th} TCT, the unit's failure might be observed either when any of the M failures occurs suddenly between the $(m\text{-}1)^{th}$ and the m^{th} TCTs or when any of the N degradation modes reaches its failure threshold. Similarly, the unit is repaired either when any of the M failures occurs or when any of the N degradation modes reaches its repair threshold. Under such circumstances, the system's failure distribution can be obtained by referring to Eq. (6.10); however, the probability that the unit has specific characteristics $i, \underset{\sim}{j}$ and fails (survives) at the end of the m^{th} TCT needs to be reinvestigated.

We define the following and calculate the $P\left(k_{i,\underset{\sim}{j}}^{m}\right)$ and $P\left(r_{i,\underset{\sim}{j}}^{m}\right)$ by referring to Eq. (6.14).

$R_p\left(t\right)$: the unit's reliability due to the p^{th} failure mode (without an indicator) at time t, $p = 1, 2, ..., M$;

$\Delta R_q\left(t\right)$: the level of the q^{th} degradation mode at time t, $q = 1, 2, ..., N$;

ΔR_q^F : the failure threshold of the q^{th} degradation mode;

ΔR_q^R : the repair threshold of the q^{th} degradation mode;

$\underset{\sim}{q}$: a set of degradation modes (selected from the N degradation modes) which reach its repair thresholds at the end of the TCTs in sequence $\underset{\sim}{j}$.

Appendix A shows the calculation of system reliability metrics.

6.4 Simulation Model

In this section, we develop a simulation model to validate the proposed models in sections 6.2 and 6.3. For a comprehensive validation, we apply r replications and compare the system reliability metrics (e.g., reliability, expected number of failures) obtained by the proposed models and the simulation model in each replication. Each replication has one set of randomly generated testing parameters (batch size, batch arrival time, test time and temperatures). In one replication, the simulating procedure is repeated for 10^3 iterations. The steps of simulating the number of failed units in the 1st batch (assuming $w_1 = 0$) at the end of the 1st (k_1^1) and 2nd (k_1^2, where $k_1^2 = k_{1,0}^2 + k_{1,1}^2$) TCTs in one iteration are illustrated in the flow charts in Figures 6.1 and 6.2. We define the following notations:

$\underset{\sim}{v}_{i,\underset{\sim}{j}\_W}^m$: vector composed of random failure times at the m^{th} TCT (with characteristics $i, \underset{\sim}{j}$) generated based on the Weibull distribution;

$\underset{\sim}{v}_{i,\underset{\sim}{j}\_W}^m(\xi)$: the ξ^{th} element in vector $\underset{\sim}{v}_{i,\underset{\sim}{j}\_W}^m$;

$\underset{\sim}{v}^{m}_{i,\underset{\sim}{j}\_GBS}$: vector composed of random failure times (with characteristics $i, \underset{\sim}{j}$) generated based on the GBS distribution;

$\underset{\sim}{v}^{m}_{i,\underset{\sim}{j}\_GBS}(\xi)$: the ξ^{th} element in vector $\underset{\sim}{v}^{m}_{i,\underset{\sim}{j}\_GBS}$;

$\underset{\sim}{v}^{m}_{i,\underset{\sim}{j}\_Resistor}$: vector composed of random failure times (with characteristics $i, \underset{\sim}{j}$) generated based on the resistance change model;

$\underset{\sim}{v}^{m}_{i,\underset{\sim}{j}\_Resistor}(\xi)$: the ξ^{th} element in vector $\underset{\sim}{v}^{m}_{i,\underset{\sim}{j}\_Resistor}$.

Simulating random failures at time t_1:

Generate the vector $\underset{\sim}{v}^{1}_{1,0} = \begin{pmatrix} \underset{\sim}{v}^{1}_{1,0\_W} \\ \underset{\sim}{v}^{1}_{1,0\_GBS} \\ \underset{\sim}{v}^{1}_{1,0\_Resistor} \end{pmatrix} = \left(\underset{\sim}{v}^{1}_{1,0}(1),\ldots,\underset{\sim}{v}^{1}_{1,0}(\xi),\ldots,\underset{\sim}{v}^{1}_{1,0}(n_1)\right)$, where

$\underset{\sim}{v}^{1}_{1,0\_W} = \left(\underset{\sim}{v}^{1}_{1,0\_W}(1),\ldots,\underset{\sim}{v}^{1}_{1,0\_W}(n_1)\right)$; the n_1 elements in $\underset{\sim}{v}^{1}_{1,0\_W}$ are random failure times generated based on the Weibull distribution;

$\underset{\sim}{v}^{1}_{1,0\_GBS} = \left(\underset{\sim}{v}^{1}_{1,0\_GBS}(1),\ldots,\underset{\sim}{v}^{1}_{1,0\_GBS}(n_1)\right)$; the n_1 elements in $\underset{\sim}{v}^{1}_{1,0\_GBS}$ are random failure times generated based on the GBS distribution;

$\underset{\sim}{v}^{1}_{1,0\_Resistor} = \left(\underset{\sim}{v}^{1}_{1,0\_Resistor}(1),\ldots,\underset{\sim}{v}^{1}_{1,0\_Resistor}(n_1)\right)$; the n_1 elements in $\underset{\sim}{v}^{1}_{1,0\_Resistor}$ are random failure times generated based on the resistor change model.

↓

In the vector $\underset{\sim}{v}^{1}_{1,0}$, count the number of columns $\underset{\sim}{v}^{1}_{1,0} = \begin{pmatrix} \underset{\sim}{v}^{1}_{1,0\_W} \\ \underset{\sim}{v}^{1}_{1,0\_GBS} \\ \underset{\sim}{v}^{1}_{1,0\_Resistor} \end{pmatrix}$ satisfying at least one of the following and record the number as k_1^1:

1) $\underset{\sim}{v}^{1}_{1,0\_W}(\xi) > t_1$;

2) $\underset{\sim}{v}^{1}_{1,0\_GBS}(\xi) < \dfrac{\Delta CF}{t_1}$;

3) $\underset{\sim}{v}^{1}_{1,0\_Resistor}(\xi) < \left(\dfrac{D_u}{D_u + D_l}\exp\left(\dfrac{E}{kT_u}\right)\sqrt[m]{\dfrac{\Delta R_R}{R_0 t_1^m}} + \dfrac{D_l}{D_u + D_l}\exp\left(\dfrac{E}{kT_l}\right)\sqrt[m]{\dfrac{\Delta R_R}{R_0 t_1^m}}\right)$

Figure 6.1 Steps of simulating the number of failures in the 1st batch after the 1st TCT

Simulating random failures at time t_2 :

In the vector $y_{1,0}^{1}$, count the number of columns $y_{1,0}^{1}(\xi)=\begin{pmatrix} y_{1,0\_W}^{1}(\xi) \\ y_{1,0\_GBS}^{1}(\xi) \\ y_{1,0\_Resistor}^{1}(\xi) \end{pmatrix}$ satisfying at least one of the following and record as $k_{1,0}^{2}$:

1) $t_1 < y_{1,0\_W}^{1}(\xi) \le t_2$;

2) $\dfrac{\Delta CF}{t_2} < y_{1,0\_GBS}^{1}(\xi) \le \dfrac{\Delta CF}{t_1}$;

3)
$$\left(\frac{D_u}{D_u+D_l}\exp\left(\frac{E}{kT_u}\right)\sqrt[m]{\frac{\Delta R_F}{R_0 t_2^m}}+\frac{D_l}{D_u+D_l}\exp\left(\frac{E}{kT_l}\right)\sqrt[m]{\frac{\Delta R_F}{R_0 t_2^m}}\right)$$
$$< y_{1,0\_Resistor}^{1}(\xi) <$$
$$\left(\frac{D_u}{D_u+D_l}\exp\left(\frac{E}{kT_u}\right)\sqrt[m]{\frac{\Delta R_F}{R_0 t_1^m}}+\frac{D_l}{D_u+D_l}\exp\left(\frac{E}{kT_l}\right)\sqrt[m]{\frac{\Delta R_F}{R_0 t_1^m}}\right)$$

In the vector $y_{1,0}^{1}$, count the number of columns $y_{1,0}^{1}(\xi)=\begin{pmatrix} y_{1,0\_W}^{1}(\xi) \\ y_{1,0\_GBS}^{1}(\xi) \\ y_{1,0\_Resistor}^{1}(\xi) \end{pmatrix}$ satisfying at least one of the following and record as rk^{1}:

1) $y_{1,0\_W}^{1}(\xi) \le t_1$;

2) $y_{1,0\_GBS}^{1}(\xi) \ge \dfrac{\Delta C_R}{t_1}$;

3)
$$y_{1,0\_Resistor}^{1}(\xi) < \left(\frac{D_u}{D_u+D_l}\exp\left(\frac{E}{kT_u}\right)\sqrt[m]{\frac{\Delta R_R}{R_0 t_1^m}}+\frac{D_l}{D_u+D_l}\exp\left(\frac{E}{kT_l}\right)\sqrt[m]{\frac{\Delta R_R}{R_0 t_1^m}}\right)$$

Generate the vector $\underset{\sim}{v}_{1,1}^{2}=\begin{pmatrix}\underset{\sim}{v}_{1,1\_W}^{2}\\ \underset{\sim}{v}_{1,1\_GBS}^{2}\\ \underset{\sim}{v}_{1,1\_Resistor}^{2}\end{pmatrix}=\left(\underset{\sim}{v}_{1,1}^{2}(1),\ldots,\underset{\sim}{v}_{1,1}^{2}(\xi),\ldots,\underset{\sim}{v}_{1,1}^{2}(rk^{1})\right)$, where

$\underset{\sim}{v}_{1,1\_W}^{2}=\left(\underset{\sim}{v}_{1,1\_W}^{2}(1),\ldots,\underset{\sim}{v}_{1,1\_W}^{2}(rk^{1})\right)$; the rk^{1} elements in $\underset{\sim}{v}_{1,1\_W}^{2}$ are random failure times generated based on the Weibull distribution

$\underset{\sim}{v}_{1,1\_GBS}^{2}=\left(\underset{\sim}{v}_{1,1\_GBS}^{2}(1),\ldots,\underset{\sim}{v}_{1,1\_GBS}^{2}(rk^{1})\right)$; the rk^{1} elements in $\underset{\sim}{v}_{1,1\_GBS}^{2}$ are random failure times generated based on the GBS distribution

$\underset{\sim}{v}_{1,1\_Resistor}^{2}=\left(\underset{\sim}{v}_{1,1\_Resistor}^{2}(1),\ldots,\underset{\sim}{v}_{1,1\_Resistor}^{2}(rk^{1})\right)$; the rk^{1} elements in $\underset{\sim}{v}_{1,1\_Resistor}^{2}$ are random failure times generated based on the resistor change model

In the vector $\underset{\sim}{v}_{1,1}^{2}$, count the number of columns $\underset{\sim}{v}_{1,1}^{2}(\xi)=\begin{pmatrix}\underset{\sim}{v}_{1,1\_W}^{2}(\xi)\\ \underset{\sim}{v}_{1,1\_GBS}^{2}(\xi)\\ \underset{\sim}{v}_{1,1\_Resistor}^{2}(\xi)\end{pmatrix}$ satisfying at least one of the following and record as $k_{1,1}^{2}$:

1) $\underset{\sim}{v}_{1,1\_W}^{2}(\xi)\le(t_2-t_1)$;

2) $\underset{\sim}{v}_{1,1\_GBS}^{2}(\xi)\ge\dfrac{\Delta C_F}{t_2-t_1}$;

3)

$$\underset{\sim}{v}_{1,1\_Resistor}^{2}(\xi)<\left(\frac{D_u}{D_u+D_l}\exp\left(\frac{E}{kT_u}\right)\sqrt[m]{\frac{\Delta R_F}{R_0(t_2-t_1)^m}}+\frac{D_l}{D_u+D_l}\exp\left(\frac{E}{kT_l}\right)\sqrt[m]{\frac{\Delta R_F}{R_0(t_2-t_1)^m}}\right)$$

Figure 6.2 Steps of simulating the number of failures in the 1st batch at the end of the 2nd TCT

6.5 A Numerical Illustration

In this section, we numerically obtain the system reliability at arbitrary time *t* until the end of the 3rd TCT, under competing failure modes.

Batches of one-shot units are produced and stored at times $w_1 = 0,\ w_2 = 200$ and $w_3 = 400$ with batch sizes $n_1 = n_2 = n_3 = 20$, respectively. TCTs are conducted at times $t_1 = 100,\ t_2 = 300$ and $t_3 = 500$. In each temperature fluctuation cycle, the entire population is subjected to temperature ($T_U = 350K$) for $D_U = 10s$, and then cooled to a temperature ($T_L = 250K$) for $D_L = 20s$, where $D_L,\ D_U$ are the dwell time in seconds at the lower and upper temperatures, respectively. Units in the system might fail due to any of the following reasons:

1. The sudden failure time is modeled by a Weibull distribution with CDF $F_W = 1 - \exp\left(-\left(\frac{t}{\theta}\right)^{\gamma}\right)$ where $\theta = 450$ and $\gamma = 2$;
2. The probability of failure due to crack length $\Delta C(t)$ reaching a predetermined failure threshold ΔC_F due to the thermal stresses is modeled as:

$$P\left(\Delta C(t) \geq \Delta C_F\right) = \Phi\left\{\frac{\sqrt{\Delta C_F}}{\alpha}\left[\left(\frac{t}{A(T_U - T_L)^B\,\Delta C_F}\right)^{\lambda} - \left(\frac{t}{A(T_U - T_L)^B\,\Delta C_F}\right)^{-\lambda}\right]\right\},$$

where

$\Delta C_F = 40\%$, i.e., the crack length reaches 40% of the unit's entire length;

$\alpha = 2,\ A = 8\times 10^{13},\ B = \text{-}5,\ \lambda = 0.5,\ T_U = 350K,\ T_L = 250K$; and

3. The change of the resistance reaches a predetermined failure threshold ΔR_F with the probability:

$$P\left(\Delta R(t) \geq \Delta R_F\right) = \Phi\left(\frac{\sqrt[m]{\dfrac{\Delta R_F}{R_0 t^m}}\left[\dfrac{D_U}{D_U + D_L}\exp\left(\dfrac{E}{kT_U}\right) + \dfrac{D_L}{D_U + D_L}\exp\left(\dfrac{E}{kT_L}\right)\right] - \mu_\phi}{\sigma_\phi}\right)$$

where

$$\Delta R_F = 100\Omega,\ R_0 = 200\Omega,\ \ E = 0.3\text{eV}, k = 8.62 \times 10^{-5}\,\text{eV/K};$$

$$m = 0.5,\ \mu_\phi = 2, \sigma_\phi = 0.02$$

Repairs are conducted: 1) when the unit is observed to fail suddenly between two tests, or 2) when the repair threshold for the crack length $\Delta C_R = 20\%$ is reached, or 3) when the repair threshold for the resistance change $\Delta R_R = 40\Omega$ is reached. Defining the system as a "*k*(*t*)-out-of-*n*(*t*)*:* F" system with $k(t) = 25\% \cdot n(t)$, we plot the system reliability at arbitrary time *t* until the end of the 3rd TCT in Figure 6.3. Besides the competing failure modes, we also plot the system reliability under each failure mode independently to identify the dominant failure/degradation mode that causes system's failure.

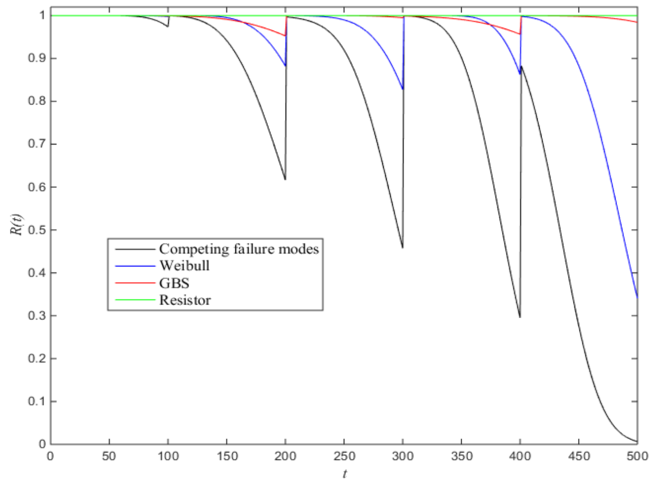

Figure 6.3 System reliability until the 3rd TCT under different failure modes

It is observed that the "sudden" failure, characterized by the Weibull distribution, is the dominant failure mode. However, when θ changes from 450 to 800 and A changes from 8×10^{13} to 3×10^{13}, the crack caused by the thermal fatigue becomes the dominant failure mode, as plotted in Figure 6.4.

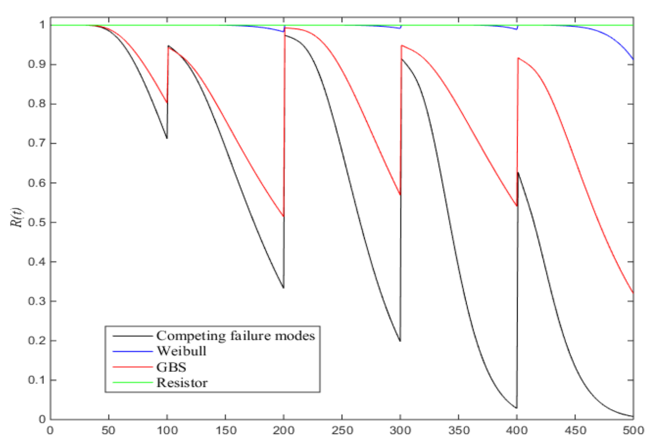

Figure 6.4 System reliability until the 3rd TCT under different failure modes

6.6 Conclusions

In this chapter, we propose efficient and effective approaches that model the reliability metrics of systems subject to competing failure modes. The system has a mixture of nonhomogeneous one-shot units that arrive at different times and subjected to sequential TCTs. We develop and utilize statistics-based models and physics-statistics-based models to characterize the failure/degradation modes. We state that more frequent NDT is required to minimize the probability of system failure and validate that competing failure modes result in worse system reliability than single failure mode. Moreover, the proposed models' accuracy in estimating the system reliability metrics is validated by a simulation model.

CHAPTER 7

OPTIMAL SEQUENTIAL ALT PLANS

In many situations, it is difficult or impossible to obtain failure data under normal conditions for highly reliable units. In such cases, accelerated life testing is applied by subjecting the units to severer-than-normal stresses such that failures are induced in a much shorter time. The accelerated data are utilized to estimate the units' lifetime and other reliability metrics under normal conditions by using ALT models. The accuracy of reliability estimation depends on the ALT models that relate the failure data under accelerated conditions to that under normal conditions and the design of the ALT plans.

Statistics-based models, physics-statistics-based models, and physics-experimental-based models are typical types of ALT models which relate the failure data at accelerated conditions to the normal conditions and predict the units' reliability under normal stresses. Statistics-based models utilize historical failure data but neglect the physics of the units' failure mechanism, moreover, they do not apply to the situations when insufficient data are available. Physics-based models are effective when the change in the unit's physics (either due to the unit's material properties, or due to the unit's working environments) is precisely related to the reliability model parameters, however, it neglects the uncertainty during the units' failure processes. Physics-statistics-based models combine both the statistics-based models and the physics-based models by predominately considering unit's failures mechanism as well as taking the uncertainty during the modeling process into account.

When selecting ALT models based on the units' physics and statistical properties, an optimal ALT plan is required to improve the accuracy and efficiency of the reliability estimation. Specifically, an ALT plan optimizes a specific objective (usually specific reliability metrics) meanwhile meets given constraints. An appropriate optimization criterion for the ALT plan is important as it reflects the purpose of the ALT plan, ensures the accelerated stress levels are within engineering range, as well as ensures that the reliability estimation is accurate.

Usually, some or all of the types and levels of the applied stresses, the number of testing units, the time to perform the test, and test duration need to be determined. These factors are mainly decided by the functions that the units will be performing under normal conditions. In this chapter, we design a sequence of accelerated NDTs plans, considering the effect of population's non-homogeneity. Only a sample is randomly selected and tested in each test. The results show that a well-designed sequential accelerated NDT is an effective approach for accurate reliability prediction with negligible effect on the residual lives of the units and other system reliability metrics.

The accelerated NDTs plans are designed under the following two testing scenarios:

Testing scenario 1 (TS1): The NDTs (duration of the test is considered) are sequentially performed under accelerated conditions at arbitrary time by testing the selected samples, failed units are discarded;

Testing scenario 2 (TS2): The NDTs (duration of the test is considered) are repeatedly performed under accelerated conditions at arbitrary time by testing the selected samples, failed units are repaired and placed back in the population with higher failure rates.

The remainder of this chapter is organized as follows: Section 7.1 investigates the lifetime distribution of an individual unit in the system by developing a physics-statistics-based model, which directly relates the applied stresses to system reliability metrics. In section 7.2, we design a sequence of accelerated NDTs plans by considering the population's non-homogeneity under TS1 and TS2. A numerical example is provided in section 7.3 to demonstrate the design of the optimal sequential ALT plans. In section 7.4, we summarize the conclusions of this chapter.

7.1 Power-Law Humidity Model

In chapter 3, we apply the general notations $F^{x}\left(\cdot\right)$ and $f^{x}\left(\cdot\right)$ to relate the system reliability metrics to individual unit's lifetime when calculating a specific system state's probability ($P\left(K_{i,\underset{\sim}{j}}^{m}=k_{i,\underset{\sim}{j}}^{m},R_{i,\underset{\sim}{j}}^{m}=r_{i,\underset{\sim}{j}}^{m};\forall i,\underset{\sim}{j}\right)$). The specific form of unit's lifetime model, which reflects unit's characteristics and applied stresses, is required. In this section, we develop a physics-statistics-based model to analyze the individual unit's lifetime, which relates the unit's lifetime under arbitrary environment to its working environment.

We consider the one-shot units that are affected by both temperature and humidity follow a physics-based power-law model. The model relates the relative humidity (*RH*) and temperature (T) to the unit's lifetime as shown in Eq. (7.1):

$$TF = A_0 \left(RH\right)^{-n} \exp\left(\frac{Q}{K_B T}\right) \tag{7.1}$$

Where *TF* is the time to failure; A_0 is the process (material) dependent parameter and serves to produce a distribution of the unit's time-to-failure; *RH* and *T* are the relative humidity and temperature, respectively; n is the power-law exponent; K_B is the Boltzmann constant and equals to 8.62×10^{-5} and Q is the activation energy constant which depends on the material.

The power-law humidity model determines the time to failure once the environmental stresses and the unit's specific property-dependent parameters are known. The parameters of the model can follow specific probabilistic distributions due to the units' production processes variability, the variation of the unit's material properties, and the environments under which the units are tested. Without loss of generality, we assume that A_0 follows lognormal distribution with known mean and variance: $A_0 \sim \log N\left(u_{A_0}, \sigma_{A_0}^2\right)$. Similarly, the power-law exponent n is normally distributed with $n \sim N\left(u_n, \sigma_n{}^2\right)$, which reflects the uncertainty effect of *RH* on the unit's reliability. Taking the logarithm of both sides of Eq.(7.1), we obtain:

$$\ln TF \sim N\left(u_{A_0} - \ln RH \cdot u_n + \frac{Q}{K_B T}, \sigma_{A_0}{}^2 - \ln^2 RH \cdot \sigma^2{}_n\right), \text{ i.e. } TF \sim \log N\left(\mu, \sigma^2\right)$$

where

$$\mu = u_{A_0} - \ln RH \cdot u_n + \frac{Q}{K_B T} \text{ and } \sigma = \sqrt{\sigma_{A_0}{}^2 - \ln^2 RH \cdot \sigma^2{}_n}$$

Specifically, the *cdf* and reliability of the units in the system are respectively:

$$F\left(t\right) = \frac{1}{2} + \frac{1}{2} erf\left[\frac{\ln t - \mu}{\sqrt{2}\sigma}\right] \tag{7.2}$$

and

$$R\left(t\right) = \frac{1}{2} - \frac{1}{2} erf\left[\frac{\ln t - \mu}{\sqrt{2}\sigma}\right] \tag{7.3}$$

Eqs. (7.2) and (7.3) are used in section 7.2 to design the optimal sequential accelerated NDTs plans.

7.2 Optimal Sequential Accelerated NDTs Plans

There exists some circumstances that the reliability metrics of highly reliable units are not attainable within a short testing period under working (normal) conditions. As an alternative, conducting reliability tests under accelerated conditions yields acceptable reliability metrics estimates in a shorter test duration. In this section, we design the optimal sequential testing plans for a sequence of accelerated NDTs (including *m* NDTs, where *m* could be an arbitrary value) with the objective of minimizing the difference between units' reliability metrics estimates under normal and accelerated conditions while controlling the testing duration within an acceptable level. In each accelerated NDT plan, we determine the test duration as well as the applied stress levels; taking into account the effect of the units' nonhomogeneous and time-dependent characteristics and the previously conducted accelerated NDTs. We design the optimal sequential testing plans based on the sample's/system's reliability metrics under TS2 and TS3. At the end of this section, we numerically illustrate how the first three accelerated NDTs are designed sequentially.

7.2.1 Mean Residual Life (MRL), Scenarios and Notations

When a unit has the lognormal lifetime distribution as given in Eq.(8), its MRL at time t can be derived as the function of: time t, number of its prior repairs x, applied stresses $\underset{\sim}{Z}$ (RH and T), and the material-dependent parameters (A_0 and n):

$$MRL(t)=\frac{\left[e^{\frac{2\mu+\sigma^2}{2}}\left\{1-\Phi\left(\frac{\ln t-\left(\mu+\sigma^2\right)}{\sigma}\right)\right\}\right]}{1-\Phi\left(\frac{\ln t-\mu}{\sigma}\right)}-t \qquad (7.4)$$

where

$$\mu = \mu_{A_0}(x) - (\ln RH)\cdot \mu_n(x) + \frac{Q}{K_B T} \text{ and } \sigma = \sqrt{\sigma_{A_0}^2 - (\ln RH)^2 \cdot \sigma_n^2}$$

We formulate the optimization problem with the following statements:

a. In each accelerated NDT, q_s ($0 < q_s \le 1$) of the entire population are selected and tested; the "stratified sampling" approach (samples are selected proportionally according to the units' characteristics) is adopted such that the sample represents the population's characteristics;
b. Samples are placed back into the population after the accelerated NDTs;
c. The applied stresses (temperature and relative humidity) and the test duration need to be determined in each accelerated NDT;
d. The unit has a lifetime that follows the log-normal distribution and the MRL(t) as given in Eq. (7.4).

We use the subscripts A and N to refer to reliability metrics associated with the accelerated and normal NDTs, respectively. The y^{th} accelerated NDT plan is affected by the previously designed (y-1) NDT plans. The following notations are applicable for the problems formulated under TS2 and TS3:

t_y : time when the y^{th} accelerated NDT starts, $y = 1, 2, ..., m$;

τ^y : duration of the y^{th} accelerated NDT;

τ_U^y, τ_L^y : upper and lower bounds of the y^{th} accelerated NDT duration;

$\underset{\sim}{Z}^y$: the applied stresses (relative humidity RH^y and temperature T^y) levels in the y^{th} accelerated NDT;

$\underset{\sim}{Z}_N^y$: the applied stresses levels under normal operating conditions;

$\underset{\sim}{Z}_U^y$: the upper bounds of the applied stresses levels in the y^{th} accelerated NDT;

N^y : the system state at t_y, which is a combination of units with different characteristics in the system, specifically, $N^y = n^y$ where n^y is a specific system state;

S^y : the sample state at t_y, which is a combination of units with different characteristics in the sample, specifically, $S^y = s^y$ where s^y is a specific sample state;

$\left[P_A\left(K^y = k^y\right); \underset{\sim}{Z}^y, \tau^y\right]$, $\left[TMRL_A\left(s^y\right); \underset{\sim}{Z}^y, \tau^y\right]$,$\left[R_A^y; \underset{\sim}{Z}^y, \tau^y\right]$, $\left[E_A^y; \underset{\sim}{Z}^y, \tau^y\right]$: the probability of having k^y failed units, expected total mean residual life (TMRL) of the units in the sample (with sample state s^y), the system reliability, and the expected number of failed units at the end of the y^{th} accelerated NDT, with applied stresses $\underset{\sim}{Z}^y\left(RH^y, T^y\right)$ and test duration τ^y, respectively;

$\left[P_N\left(K^y = k^y\right); \underset{\sim}{Z}_N^y, \tau^y\right]$, $\left[TMRL_N\left(s^y\right); \underset{\sim}{Z}_N^y, \tau^y\right]$,$\left[R_N^y; \underset{\sim}{Z}_N^y, \tau^y\right]$, $\left[E_N^y; \underset{\sim}{Z}_N^y, \tau^y\right]$: the probability of having k^y failed units, expected total mean residual life (TMRL) of the units in the sample (with sample state s^y), the system reliability, and the expected number of failed units at the end of the y^{th} normal NDT, with normal stresses $\underset{\sim}{Z}_N^y\left(RH_N^y, T_N^y\right)$ and test duration τ^y, respectively;

q_1^y : the ratio of the sample's expected TMRL after the y^{th} accelerated and normal NDT;

q_2^y : the ratio of the system reliability after the y^{th} accelerated and normal NDT;

q_3^y : the ratio of the expected number of failed units after the y^{th} accelerated and normal NDT;

7.2.2 Optimization of Sequential Accelerated NDTs under TS1

We first design the optimal sequential ALTs under TS1, i.e., at the end of each accelerated NDT, the sample is placed back into the system without repairs. Under such circumstance, the units in the system have different reliability characteristics due to their different arrival times and accelerated NDT sequences under which they are tested. The following additional notations are necessary for the optimization problem under TS1:

$N_{i,\underset{\sim}{\pi}}^y$: at time t_y, the number of units from the i^{th} batch which are subjected to accelerated NDT sequence $\underset{\sim}{\pi}$. Specifically, under TS1, $\sum_{\forall \underset{\sim}{\pi}} n_{i,\underset{\sim}{\pi}}^y = n_i \ \forall i$ always holds, where $n_{i,\underset{\sim}{\pi}}^y$ ranges from 0 to n_i;

$S_{i,\underset{\sim}{\pi}}^y$: the number of testing units selected from $N_{i,\underset{\sim}{\pi}}^y$ at t_y, specifically, $S_{i,\underset{\sim}{\pi}}^y = s_{i,\underset{\sim}{\pi}}^y, \forall i, \underset{\sim}{\pi}$. Note that $s_{i,\underset{\sim}{\pi}}^y \le n_{i,\underset{\sim}{\pi}}^y \forall i, \underset{\sim}{\pi}$ and $\sum_{\forall i,\underset{\sim}{\pi}} S_{i,\underset{\sim}{\pi}}^y = q_s\% \cdot \left(\sum_{\forall i,\underset{\sim}{\pi}} N_{i,\underset{\sim}{\pi}}^y \right)$ always holds;

$\#\{\underset{\sim}{\pi}\}$: the number of elements/NDTs in NDT sequence $\underset{\sim}{\pi}$, i.e., the number of accelerated NDTs the $N_{i,\underset{\sim}{\pi}}^y$ units subject to;

$K_{i,\underset{\sim}{\pi}}^{y,A}\left(K_{i,\underset{\sim}{\pi}}^{y,N}\right)$: the number of failed units out of $N_{i,\underset{\sim}{\pi}}^{y}$ at the end of the y^{th} accelerated (normal) NDT;

$R_{i,\underset{\sim}{\pi}}^{y,A}\left(R_{i,\underset{\sim}{\pi}}^{y,N}\right)$: the number of survived units out of $N_{i,\underset{\sim}{\pi}}^{y}$ at the end of the y^{th} accelerated (normal) NDT;

$MRL_A\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)\left(MRL_N\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}_N^{y},\tau^{y}\right)\right)$: mean residual life of any of the $S_{i,\underset{\sim}{\pi}}^{y}$ units at the end of the y^{th} accelerated (normal) NDT;

$t_A\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)\left(t_N\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}_N^{y},\tau^{y}\right)\right)$: "true" age of any of the $S_{i,\underset{\sim}{\pi}}^{y}$ units at the end of the y^{th} accelerated (normal) NDT;

$R_A\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)\left(R_N\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}_N^{y},\tau^{y}\right)\right)$: reliability of any of the $S_{i,\underset{\sim}{\pi}}^{y}$ units at its true age $t_A\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)\left(t_N\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}_N^{y},\tau^{y}\right)\right)$;

τ^{π}: sum of test durations of all NDTs in sequence $\underset{\sim}{\pi}$;

$\underset{\sim}{Z}^{\pi}$ and τ^{π}: accelerated stresses under all NDTs in NDT sequence $\underset{\sim}{\pi}$;

$\tau^{y}\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)$: "equivalent" test duration of the y^{th} accelerated NDT for any of the $S_{i,\underset{\sim}{\pi}}^{y}$ units;

$\tau^{\pi}\left(S_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{\pi},\tau^{\pi}\right)$: sum of "equivalent" test durations of all accelerated NDTs in sequence $\underset{\sim}{\pi}$ (in which the $S_{i,\underset{\sim}{\pi}}^{y}$ units are tested) for any of the $S_{i,\underset{\sim}{\pi}}^{y}$ units;

$f^{x}(\cdot)$: *pdf* of the units' lifetime distribution (lognormal) after its x^{th} repair in Eq. (7.1);

The optimal testing plans for the first m accelerated NDTs are obtained successively by first solving for $\underset{\sim}{Z}^1$, t_1, and τ^1 in the first accelerated NDT; then optimal decision is found for the second accelerated NDT. Generally, the y^{th} accelerated NDT plan is affected by previous (y-1) NDTs plans and the system state at time t_y. We investigate the procedure of optimizing the y^{th} accelerated NDT, where $y = 1, \ldots, m$. Specifically, the applied stresses (temperature and relative humidity) in the y^{th} test, the time to conduct the y^{th} test, and the y^{th} test duration are determined by optimizing the following problem:

Min

$$\sum_{k^y=0}^{\sum_{\forall i} n_i} \left\| \left[P_A\left(K^y = k^y\right); \underset{\sim}{Z}^y, \tau^y \right] - \left[P_N\left(K^y = k^y\right); \underset{\sim}{Z}_N^y, \tau^y \right] \right\| \tag{7.5}$$

Subject to

$$\underset{\sim}{Z}_N^y < \underset{\sim}{Z}^y \leq \underset{\sim}{Z}_U^y \tag{7.6}$$

$$\tau^y \leq \tau_U^y \text{ and } t_L^y \leq t_y \leq t_U^y \quad \forall y = i \tag{7.7}$$

$$\sum_{\forall s^y} P\left(S^y = s^y\right) \cdot \left(TMRL_A\left(s^y; \underset{\sim}{Z}^y, \tau^y\right)\right) \geq q_1^y \cdot \sum_{\forall s^y} P\left(S^y = s^y\right) \cdot \left(TMRL_N\left(s^y; \underset{\sim}{Z}_N^y, \tau^y\right)\right) \tag{7.8}$$

$$\left[E_A^y; \underset{\sim}{Z}^y, \tau^y \right] \geq q_2^y \cdot \left[E_N^y; \underset{\sim}{Z}_N^y, \tau^y \right] \tag{7.9}$$

$$\left[R_A^y; \underset{\sim}{Z}^y, \tau^y \right] \geq q_3^y \cdot \left[R_N^y; \underset{\sim}{Z}_N^y, \tau^y \right] \tag{7.10}$$

$$k \left| \left\{ \text{Max } P_A\left(K^y = k^y\right); \underset{\sim}{Z}^y, \tau^y \right\} = k \right| \left\{ \text{Max } P_N\left(K^y = k^y\right); \underset{\sim}{Z}_N^y, \tau^y \right\} \tag{7.11}$$

To minimize the effect of acceleration on the system reliability metrics, we minimize the sum of the difference between the failure distributions at the end of the y^{th} accelerated NDT and normal NDT (Eq. (7.5)). Constraint (7.6) ensures that applied stresses' levels are higher-than-normal and lower than specified thresholds such that failure modes remain unchanged. The test duration of the accelerated NDT should not exceed its upper bound and the test should be performed within certain time intervals (Eq. (7.7)). We require the total expected mean residual life (TMRL) of the units in the sample at the end of the y^{th} accelerated NDT equals to or longer than q_1^y of total expected mean residual life after the y^{th} normal NDT (constraint (7.8)). According to constraint (7.9), the system's expected number of failures after the y^{th} accelerated NDT should not be less than q_2^y of that after the y^{th} normal NDT. Similarly, the difference between system reliability after the y^{th} accelerated NDT and the y^{th} normal NDT is less than q_3^y according to constraint (7.10). In constraint (7.11), the peaks (maxima) of the number of failures distributions after the y^{th} accelerated and normal NDT are the same. This optimization problem can be solved by using nonlinear optimization programming.

Assuming $S^y = s^y$, i.e., $S_{i,\underset{\sim}{\pi}}^y = s_{i,\underset{\sim}{\pi}}^y \forall i, \underset{\sim}{\pi}$, the TMRLs of the units in the sample equals to:

$$TMRL_A\left(s^y; \underset{\sim}{Z}^y, \tau^y\right) = \sum_{\forall i,\underset{\sim}{\pi}} s_{i,\underset{\sim}{\pi}}^y \cdot \underbrace{MRL_A\left(s_{i,\underset{\sim}{\pi}}^y; \underset{\sim}{Z}^y, \tau^y\right)}_{A} \tag{7.12}$$

Term A in Eq. (7.12) (the MRL of any of the $s_{i,\underset{\sim}{\pi}}^y$ units at the end of the y^{th} accelerated NDT) is obtained as:

$$MRL_A\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)=\frac{1}{R_A\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)}\int_{t_A\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)}^{\infty}\tau f\left(\tau\right)d\tau-\underbrace{t_A\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)}_{A} \tag{7.13}$$

Term A in Eq. (7.13) is obtained as shown in Eq. (7.14) by reducing the sum of all accelerated NDTs' durations in sequence $\underset{\sim}{\pi}$ (term A in Eq. (7.14)), and adding the sum of their corresponding "equivalent" test durations (term B and C in Eq. (7.14)):

$$t_A\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)=t_y-w_i-\underset{A}{\tau^{\pi}}+\underbrace{\tau^{\pi}\left(s_{i,\underset{\sim}{\pi}}^{\pi};\underset{\sim}{Z}^{\pi},\tau^{\pi}\right)}_{B}+\underbrace{\tau^{y}\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)}_{C} \tag{7.14}$$

Term B in Eq. (7.14) can be obtained by letting the reliability of any of the $s_{i,\underset{\sim}{\pi}}^{y}$ units at t_y (if all NDTs in sequence $\underset{\sim}{\pi}$ are accelerated) equals to the reliability of any of the $s_{i,\underset{\sim}{\pi}}^{y}$ units at its true age (if all NDTs in sequence $\underset{\sim}{\pi}$ are under normal conditions):

$$R\left(t_{y-1}-w_i;\underset{\sim}{Z}^{\pi},\tau^{\pi}\forall\text{ NDT in }\underset{\sim}{\pi}\right)=R\left(t_{y-1}-w_i-\tau^{\pi}+\tau^{\pi}\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{\pi},\tau^{\pi}\right);\underset{\sim}{Z}_N^{\pi},\tau_N^{\pi},\tau^{\pi}\forall\text{ NDT in }\underset{\sim}{\pi}\right)$$
(7.15)

Similarly, the equivalent duration of the y^{th} NDT for any of the $s_{i,\underset{\sim}{\pi}}^{y}$ units is obtained as:

$$R\left(t_y-w_i+\tau^{y};\underset{\sim}{Z}^{\pi},\tau^{\pi}\forall\underset{\sim}{\pi},\underset{\sim}{Z}^{y},\tau^{y}\right)=R\left(\begin{array}{l}t_y-w_i-\tau^{\pi}+\tau^{\pi}\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{\pi},\tau^{\pi}\right)+\tau^{y}\left(s_{i,\underset{\sim}{\pi}}^{y};\underset{\sim}{Z}^{y},\tau^{y}\right)\\;\underset{\sim}{Z}_N^{\pi},\tau_N^{\pi},\tau^{\pi}\forall\ \underset{\sim}{\pi},\underset{\sim}{Z}_N^{y},\tau^{y}\end{array}\right) \tag{7.16}$$

Then $R_A\left(s_{i,\underset{\sim}{\pi}}^{y}; \underset{\sim}{Z}^{y}, \tau^{y}\right)$ is obtained. The MRL of any of the $s_{i,\underset{\sim}{\pi}}^{y}$ units at the end of the y^{th} accelerated NDT is accordingly obtained.

Terms B and C in Eq. (7.14) reflect the effect of applied stresses on the units' reliability metrics as described in Eq. (7.2). The expected TMRL of the sample at the end of the y^{th} normal NDT can be calculated by following the same steps in Eqs.(7.12)-(7.14), only replacing term C in Eq. (7.14) with τ^{y}.

Eqs. (7.12)-(7.14) obtain the sample's TMRL under a specific sample state s^{y}. Obviously, there exists many potential sample states at t_y and the probability of having a specific sample state ($P\left(S^{y} = s^{y}\right)$) needs to be investigated. Under TS1, $P\left(S^{y} = s^{y}\right)$ is obtained by applying the law of total probability. Specifically, for arbitrary s^{y} we consider all possible system states ($N^{y} = n^{y}$, i.e., $N_{i,\underset{\sim}{\pi}}^{y} = n_{i,\underset{\sim}{\pi}}^{y} \forall i, \underset{\sim}{\pi}$) that yield s^{y}. Term A in Eq. (7.17)) uses a hypergeometric distribution to obtain all possible s^{y} under system state n^{y}. All possible system states that yield s^{y} are considered and the conditional probabilities (term B) are summed as:

$$P\left(S^{y}=s^{y}\right)=\sum_{\forall n^{y}}\left[\underbrace{P\left(S^{y}=s^{y}\left|N^{y}=n^{y}\right.\right)}_{A}\cdot\underbrace{P\left(N^{y}=n^{y}\right)}_{B}\right]$$

$$=\sum_{\forall\left(n_{i,\underset{\sim}{\pi}}^{y}\forall i,\underset{\sim}{\pi}\right)}\left[\underbrace{P\left(\left(S_{i,\underset{\sim}{\pi}}^{y}=s_{i,\underset{\sim}{\pi}}^{y}\forall i,\underset{\sim}{\pi}\right)\middle|\left(N_{i,\underset{\sim}{\pi}}^{y}=n_{i,\underset{\sim}{\pi}}^{y}\forall i,\underset{\sim}{\pi}\right)\right)}_{A}\cdot\underbrace{P\left(N_{i,\underset{\sim}{\pi}}^{y}=n_{i,\underset{\sim}{\pi}}^{y}\forall i,\underset{\sim}{\pi}\right)}_{B}\right]$$

$$=\sum_{\forall\left(n_{i,\underset{\sim}{\pi}}^{y}\forall i,\underset{\sim}{\pi}\right)}\left[\underbrace{\frac{\prod_{\forall i}\prod_{\forall\underset{\sim}{\pi}}\binom{n_{i,\underset{\sim}{\pi}}^{y}}{s_{i,\underset{\sim}{\pi}}^{y}}}{\binom{\sum_{\forall i}n_{i}}{q_{s}\%\cdot\sum_{\forall i}n_{i}}}}_{A}\cdot\underbrace{P\left(N_{i,\underset{\sim}{\pi}}^{y}=n_{i,\pi}^{y}\forall i,\underset{\sim}{\pi}\right)}_{B}\right] \tag{7.17}$$

A specific system state is realized with the probability shown in Eq. (7.18), where the numerator is the probability of having a specified system state and the denominator is the sum of all potential system states' probabilities. All potential system states are obtained by solving Eqs. (7.19)-(7.21).

$$P\left(N^{y}=n^{y}\right)=P\left(N_{i,\underset{\sim}{\pi}}^{y}=n_{i,\underset{\sim}{\pi}}^{y}\forall i,\underset{\sim}{\pi}\right)$$

$$=\frac{\prod_{\forall i}\left[\frac{\left(n_{i}\right)!}{\prod_{\forall\underset{\sim}{\pi}}\left(n_{i,\underset{\sim}{\pi}}^{y}\right)!}\prod_{\forall\underset{\sim}{\pi}}\left[\left(q_{s}\right)^{\#\{\underset{\sim}{\pi}\}}\left(1-q_{s}\right)^{y-z_{i}-\#\{\underset{\sim}{\pi}\}}\right]^{n_{i,\underset{\sim}{\pi}}^{y}}\right]}{\sum_{\forall\left(N_{i,\underset{\sim}{\pi}}^{y}=n_{i,\underset{\sim}{\pi}}^{y}\forall i,\underset{\sim}{\pi}\right)\text{ satisfying Eqs.(7.19-7.21)}}\prod_{\forall i}\left[\frac{\left(n_{i}\right)!}{\prod_{\forall\underset{\sim}{\pi}}\left(n_{i,\underset{\sim}{\pi}}^{y}\right)!}\prod_{\forall\underset{\sim}{\pi}}\left[\begin{matrix}\left(q_{s}\right)^{\#\{\underset{\sim}{\pi}\}}\\\left(1-q_{s}\right)^{y-z_{i}-\#\{\underset{\sim}{\pi}\}}\end{matrix}\right]^{n_{i,\underset{\sim}{\pi}}^{y}}\right]} \tag{7.18}$$

Assuming that in each accelerated NDT, any unit is equally selected and tested with probability q_s, the following equations always hold:

$$\sum_{\forall \underset{\sim}{\pi}} N_{i,\underset{\sim}{\pi}}^{y} = n_i \quad \forall i \tag{7.19}$$

$$\sum_{\forall i} \sum_{\forall \underset{\sim}{\pi} \text{ including } z} N_{i,\underset{\sim}{\pi}}^{y} = \left(\sum_{\forall \{i; w_i \le t_z\}} n_i \right) \cdot q_s \quad z = z_i, z_{i+1}, \ldots, y \text{ ; } z \text{ represents the } z^{\text{th}} \text{ test} \tag{7.20}$$

$$N_{i,\underset{\sim}{\pi}}^{y} \ge 0 \quad \forall i, \underset{\sim}{\pi} \tag{7.21}$$

In Eq. (7.19), the sum of number of units with different characteristics in the i^{th} batch equals to the i^{th} batch's size. Eq. (7.20) guarantees that in each accelerated NDT， q_s units of the total population are selected and tested. Moreover, $N_{i,\underset{\sim}{\pi}}^{y}$ should be non-negative (constraint (7.21)). To illustrate, assuming there are two batches in the system when the third NDT starts (t_3), where the 1st batch arrives before the first NDT and the 2nd batch arrives between the 1st and 2nd NDT, we obtain the number of all possible system states at t_3 by solving the following set of equations:

$$N_{1,12}^{3} + N_{1,1}^{3} + N_{1,2}^{3} + N_{1,0}^{3} = n_1$$

$$N_{2,2}^{3} + N_{2,0}^{3} = n_2$$

$$N_{1,12}^{3} + N_{1,1}^{3} = q_s \cdot n_1$$

$$N_{1,12}^{3} + N_{1,2}^{3} + N_{2,2}^{3} = q_s \cdot \left(n_1 + n_2 \right)$$

$$N_{1,12}^{3},\ N_{1,1}^{3},\ N_{1,2}^{3},\ N_{1,0}^{3},\ N_{2,2}^{3},\ N_{2,0}^{3} \ge 0$$

Distribution of failed units at the end of the y^{th} accelerated NDT ($P_A\left(K^y = k^y\right); \underset{\sim}{Z}^y, \tau^y$) can be obtained using Eq. (7.22) by enumerating all possible system states which satisfy $\left(\sum_{\forall i,\underset{\sim}{\pi}} K_{i,\underset{\sim}{\pi}}^{y,A}\right) = k^y$ and summing their probabilities.

$$\left(P_A\left(K^y = k^y\right); \underset{\sim}{Z}^y, \tau^y\right)$$

$$= \left(P_A\left(\sum_{\forall i,\underset{\sim}{\pi}} K_{i,\underset{\sim}{\pi}}^{y,A} = k^y\right); \underset{\sim}{Z}^y, \tau^y\right)$$

$$= \sum_{\left(\sum_{\forall i,\underset{\sim}{\pi}} k_{i,\underset{\sim}{\pi}}^{y,A}\right) = k^y} \; \sum_{\left(\sum_{\forall i,\underset{\sim}{\pi}} r_{i,\underset{\sim}{\pi}}^{y,A}\right) = \left(\sum_{\forall i} n_i\right) - k^y} \left\{ \prod_{\forall i} \frac{(n_i)!}{\prod_{\forall \underset{\sim}{\pi}}\left[\left(n_{i,\underset{\sim}{\pi}}^y\right)!\right]} \cdot \left[\prod_{\forall \underset{\sim}{\pi}} \begin{pmatrix} n_{i,\underset{\sim}{\pi}}^y \\ k_{i,\underset{\sim}{\pi}}^{y,A} \end{pmatrix} \left[F\left(t_A\left(s_{i,\underset{\sim}{\pi}}^y; \underset{\sim}{Z}^y, \tau^y\right)\right)\right]^{k_{i,\underset{\sim}{\pi}}^{y,A}} \left[R\left(t_A\left(s_{i,\underset{\sim}{\pi}}^y; \underset{\sim}{Z}^y, \tau^y\right)\right)\right]^{r_{i,\underset{\sim}{\pi}}^{y,A}} \right] \right\} \tag{7.22}$$

Similarly, the distribution of failed units at the end of the y^{th} normal NDT is obtained in Eq. (7.23):

$$\left(P_N\left(K^y = k^y\right); \underset{\sim}{Z}_N^y, \tau^y\right)$$

$$= \left(P_N\left(\sum_{\forall i,\underset{\sim}{\pi}} K_{i,\underset{\sim}{\pi}}^{y,N} = k^y\right); \underset{\sim}{Z}_N^y, \tau^y\right)$$

$$= \sum_{\left(\sum_{\forall i,\underset{\sim}{\pi}} k_{i,\underset{\sim}{\pi}}^{y,N}\right)=k^y} \sum_{\left(\sum_{\forall i,\underset{\sim}{\pi}} r_{i,\underset{\sim}{\pi}}^{y,N}\right)=\left(\sum_{\forall i} n_i\right)-k^y} \left\{\prod_{\forall i} \frac{(n_i)!}{\prod_{\forall \underset{\sim}{\pi}}\left[\left(n_{i,\underset{\sim}{\pi}}^y\right)!\right]} \cdot \left[\prod_{\forall \underset{\sim}{\pi}} \begin{matrix} \begin{pmatrix} n_{i,\underset{\sim}{\pi}}^y \\ k_{i,\underset{\sim}{\pi}}^{y,N} \end{pmatrix} \\ \left[F\left(t_N\left(s_{i,\underset{\sim}{\pi}}^y; \underset{\sim}{Z}_N^y, \tau^y\right)\right)\right]^{k_{i,\underset{\sim}{\pi}}^{y,N}} \\ \left[R\left(t_N\left(s_{i,\underset{\sim}{\pi}}^y; \underset{\sim}{Z}_N^y, \tau^y\right)\right)\right]^{r_{i,\underset{\sim}{\pi}}^{y,N}} \end{matrix}\right]\right\} \tag{7.23}$$

Accordingly, the expected number of failed units and the system reliability at the end of the y^{th} accelerated (normal) NDT can be obtained. The optimization problem is flexible and can be generalized to consider different scenarios.

7.2.3 Optimization of Sequential Accelerated NDTs Plans under TS2

We now design a sequence of optimal accelerated NDTs plans under TS2, where failed units are repaired and placed back in storage. The following notations are necessary:

$K_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\left(R_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\right)$: number of failed (survived) units in the system at the end of the $(y\text{-}1)^{\text{th}}$ accelerated test, the $K_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\left(R_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\right)$ units are from the i^{th} batch and observed to fail j times (which are observed at test sequence $\underset{\sim}{j}$) until t_{y-1}, $\underset{\sim}{j}$ is composed of arbitrary j NDTs/elements that are selected from the NDT sequence $\underset{\sim}{\pi}$ and $\underset{\sim}{\pi}$ is the NDT sequence in which the units are tested.

$SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$: the number of units selected from the $K^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(R^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$ units that are tested at t_y; under TS2, a certain S^y is determined by the values of $SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}$ and $SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(R^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$ $\forall i, \underset{\sim}{j}, \underset{\sim}{\pi}$, specifically, $SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} = sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}$ and $SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} = sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}$ $\forall i, \underset{\sim}{j}, \underset{\sim}{\pi}$

$MRL_A\left(SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\left(MRL_A\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\right)$: MRL of any of the $SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$ units at the end of the y^{th} accelerated NDT;

$MRL_N\left(SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\left(MRL_N\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\right)$: MRL of any of the $SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$ units at the end of the y^{th} normal NDT;

$t_A\left(SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\left(t_A\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\right)$: true age of any of the $SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$ units at the end of the y^{th} accelerated NDT;

$t_N\left(SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\left(t_N\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\right)$: true age of any of the $SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$ units at the end of the y^{th} normal NDT;

$R_A\left(SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\left(R_A\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\right)$: reliability of any of the $SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\left(SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\right)$ units at its true age $t_A\left(SK^{y-1}_{i,j,\underset{\sim}{l}_j|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\left(t_A\left(SR^{y-1}_{i,j,\underset{\sim}{l}_j|\underset{\sim}{\pi}}; \underset{\sim}{Z}^y, \tau^y\right)\right)$;

$R_N\left(SK_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)\left(R_N\left(SR_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)\right)$: reliability of any of the $SK_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\left(SR_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\right)$units at its true age $t_A\left(SK_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)\left(t_A\left(SR_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)\right)$;

$\tau_A^y\left(SK_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)\left(\tau_A^y\left(SR_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)\right)$: equivalent test duration of the y^{th} accelerated NDT for any of the $SK_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\left(SR_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\right)$units;

$\tau^{j(j)}$: duration of the $j(j)^{\text{th}}$ NDT.

Under TS2, units in the system have different arrival times, testing sequences $\underset{\sim}{\pi}$, and failing sequences $\underset{\sim}{j}$. With the same objective and constraints as shown in Eqs. (7.5)-(7.11), the sample's TMRL with sample state $S^y = s^y$ (i.e., $SK_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1} = sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}$, $SR_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1} = sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}$; $\forall i,\underset{\sim}{j},\underset{\sim}{\pi}$) is calculated as:

$$TMRL_A(s^y;\underset{\sim}{Z}^y,\tau^y)=\sum_{\forall i,\underset{\sim}{j},\underset{\sim}{\pi}}\left\{sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\cdot MRL_A\left(SK_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)+sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\cdot MRL_A\left(SR_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)\right\} \quad (7.24)$$

When the y^{th} accelerated NDT starts, unit that fails in the (y-1)$^{\text{th}}$ accelerated NDT has a lifetime $t_y-\left(t_{y-1}+\tau^{y-1}\right)$ as it is repaired at the end of the (y-1)$^{\text{th}}$ test. Unit that survives in the (y-1)$^{\text{th}}$ NDT has a lifetime $t_y-\left(t_{j(j)}+\tau^{j(j)}\right)$ since its latest failure is observed at the end of the $j(j)^{\text{th}}$ NDT. MRLs of units with different characteristics can be obtained. Specifically, MRL of any of the $sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}}$ units at the end of the y^{th} accelerated NDT is obtained in Eq. (7.25):

$$MRL_A\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)=\frac{1}{R_A\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)}\int_{t_A\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)}^{\infty}\tau f^j(\tau)\,d\tau-\underbrace{t_A\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)}_{A} \quad (7.25)$$

Similarly, MRL of any of the $sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}}$ units at the end of the y^{th} accelerated NDT is obtained by Eq. (7.26):

$$MRL_A\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)=\frac{1}{R_A\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)}\int_{t_A\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)}^{\infty}\tau f^j(\tau)\,d\tau-\underbrace{t_A\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y,\tau^y\right)}_{A} \quad (7.26)$$

Replacing the subscript *A* with *N*, we have the MRL of any of the $sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}}$ ($sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}}$) units at the end of the y^{th} normal NDT:

$$MRL_N\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)=\frac{1}{R_N\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)}\int_{t_N\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)}^{\infty}\tau f^j(\tau)\,d\tau-\underbrace{t_N\left(sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)}_{A} \quad (7.27)$$

and

$$MRL_N\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)=\frac{1}{R_N\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)}\int_{t_N\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)}^{\infty}\tau f^j(\tau)\,d\tau-\underbrace{t_N\left(sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\tau}};\underset{\sim}{Z}^y_N,\tau^y\right)}_{A} \quad (7.28)$$

Terms A in Eqs. (7.25)-(7.28) (true age of any of the $sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}$ ($sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}$) at the end of the y^{th} accelerated (normal) NDT) are respectively obtained as:

$$t_A\left(sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)=t_y+\underbrace{\tau_A^y\left(sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\right)}_{A}-\left(t_{y-1}+\tau^{y-1}\right) \tag{7.29}$$

$$t_A\left(sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)=t_y-\underbrace{\tau^{\underset{\sim}{\pi}(j(j))}+\tau^{\underset{\sim}{\pi}(j(j))}\left(sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}_N^{\underset{\sim}{\pi}(j(j))},\tau^{\underset{\sim}{\pi}(j(j))}\right)}_{A}+\underbrace{\tau^y\left(sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}_N^{\underset{\sim}{\pi}(j(j))},\tau^{\underset{\sim}{\pi}(j(j))}\right)}_{B} \tag{7.30}$$

$$t_N\left(sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}^y,\tau^y\right)=t_y+\tau^y-\left(t_{y-1}+\tau^{y-1}\right) \tag{7.31}$$

$$t_N\left(sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}_N^y,\tau^y\right)=t_y+\tau^y-\tau^{\underset{\sim}{\pi}(j(j))}+\tau^{\underset{\sim}{\pi}(j(j))}\left(sr_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1};\underset{\sim}{Z}_N^{\underset{\sim}{\pi}(j(j))},\tau^{\underset{\sim}{\pi}(j(j))}\right) \tag{7.32}$$

Specifically, $\tau_A^y\left(sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\right)$ in Eq. (7.29) is calculated by letting:

$$R\left(t_y+\tau^y-t_{y-1}-\tau^{y-1};\underset{\sim}{Z}^y,\tau^y\right)=R\left(t_y+\tau_A^y\left(sk_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\right)-t_{y-1}-\tau^{y-1};\underset{\sim}{Z}_N^y,\tau^y\right) \tag{7.33}$$

Terms A and B in Eq. (7.30) can be obtained similarly.

Under TS2, the probability of yielding a specific sample state s^y at time t_y is calculated by considering all possible system states N^{y-1} at time t_{y-1} as shown in Eq. (7.34):

$$
\begin{aligned}
P\left(S^{y}=s^{y}\right) &= \sum_{\forall n^{y-1}} P\left(S^{y}=s^{y} \middle| N^{y-1}=n^{y-1}\right)\cdot \underbrace{P\left(N^{y-1}=n^{y-1}\right)}_{A} \\
&= \sum_{\forall\left(k^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \text{ and } r^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\ \forall i,\underset{\sim}{j},\underset{\sim}{\pi}\right)} P\left(\begin{array}{l}\left(SK^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \text{ and } SR^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\ \forall i,\underset{\sim}{j},\underset{\sim}{\pi}\right) \\ \middle|\left(K^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=k^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \text{ and } R^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=r^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\ \forall i,\underset{\sim}{j},\underset{\sim}{\pi}\right)\end{array}\right) \\
&\qquad \cdot \underbrace{P\left(\begin{array}{l}K^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=k^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \text{ and } R^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=r^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \\ \forall i,\underset{\sim}{j},\underset{\sim}{\pi}\end{array}\right)}_{A} \\
&= \sum_{\forall\left(k^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \text{ and } r^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \forall i,\underset{\sim}{j},\underset{\sim}{\pi}\right)} \frac{\prod_{\forall i}\prod_{\forall \underset{\sim}{j}}\prod_{\forall \underset{\sim}{\pi}} \binom{k^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}}{sk^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}}\binom{r^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}}{sr^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}}}{\binom{\sum_{\forall i} n_i}{q_s\cdot\sum_{\forall i} n_i}} \cdot \\
&\qquad P\left(K^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=k^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}} \text{ and } R^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}=r^{y-1}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}\ \forall i,\underset{\sim}{j},\underset{\sim}{\pi}\right)
\end{aligned}
\tag{7.34}
$$

Term A in Eq. (7.34) (units' states (fail or survive) when the (y-1)$^{\text{th}}$ NDT ends) is obtained by Eq. (7.35). It is affected by the characteristics of the units that are tested in previous (y-1) NDTs (term B in Eq. (7.35)), which has been investigated by Eq. (7.18).

Details of term A in Eq. (7.34) (the units' states (fail or survive) when the (y-1)$^{\text{th}}$ NDT ends, conditional on term B) can be obtained by referring to the models in chapter 3, where the failure probabilities of units with different characteristics are determined by their "true" ages.

$$P\left(N^{y-1}=n^{y-1}\right)=\sum_{\forall\left(N_{i,\underset{\sim}{\pi}}^{y-1}=n_{i,\underset{\sim}{\pi}}^{y-1}\ \forall i,\underset{\sim}{\pi}\right)}\underbrace{P\left(\begin{array}{l}\left(K_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}=k_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\text{ and }R_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}=r_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y-1}\ \forall i,\underset{\sim}{j},\underset{\sim}{\pi}\right)\\ \left|N_{i,\underset{\sim}{\pi}}^{y-1}=n_{i,\underset{\sim}{\pi}}^{y-1}\ \forall i,\underset{\sim}{\pi}\right.\end{array}\right)}_{A} \cdot P\left(N_{i,\underset{\sim}{\pi}}^{y-1}=n_{i,\underset{\sim}{\pi}}^{y-1}\ \forall i,\underset{\sim}{\pi}\right) \tag{7.35}$$

The distribution of failed units at the end of the y^{th} accelerated NDT $\left(P_A\left(K^y=k^y\right);\underset{\sim}{Z}^y,\tau^y\right)$ is obtained based on Eq. (7.34) by replacing $y-1$ with y, then all $P\left(N^y=n^y\right)$ satisfying $\left(\sum_{\forall i,\underset{\sim}{j},\underset{\sim}{\pi}}K_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}^{y}\right)=k^y$ are summed. The expected number of failed units $\left(E_A;\underset{\sim}{Z}^y,\tau^y\right)$ and the system reliability $\left(R_A;\underset{\sim}{Z}^y,\tau^y\right)$ are obtained accordingly.

Different scenarios can be developed based on TS2. For example, failed units are either repaired to as-good-as new or to as-good-as old. Meanwhile, failed units are either repaired and placed back in storage, or discarded after a specified number of failures.

7.3 A Numerical Illustration

In this section, we numerically design the optimal testing plans for the first three accelerated NDTs under TS2 and TS3, respectively. We describe how the sequential accelerated NDTs are conducted as well as the procedure of designing the sequential optimal testing plans as follows:

- Three batches of one-shot units arrive into storage at times $w_1 = 0,\ w_2 = 300,\ w_3 = 600$, respectively; the three batch sizes are $n_1 = 100, n_2 = 100, n_3 = 100$, respectively.
- The unit's lifetime T follows lognormal distribution, specifically, $T \sim \log N\left(\mu, \sigma^2\right)$; where

  $\mu = \mu_{A_0} - \ln RH \cdot \mu_n + \dfrac{Q}{K_B T}$; $\mu_{A_0} = -10 - 0.001x$; $\mu_N = 2 + 0.0001x$; $Q = 0.5\text{eV}$; and $K_B = 8.26 \times 10^{-5}$

  $\sigma = \sqrt{\left(\sigma_{A_0}\right)^2 - \left(\ln RH\right)^2 \cdot \left(\sigma_n\right)^2}$ $\sigma_{A_0} = 2$ and $\sigma_n = 0.01$;
- The 1st, 2nd, and the 3rd accelerated NDT are conducted at times $t_1 = 150$, $t_2 = 450$ and $t_3 = 750$; the test durations are respectively τ_1, τ_2 and τ_3 (need to be determined). In each NDT, 20% of the population is selected and tested ($q_s^1 = q_s^2 = q_s^3 = 20\%$); i.e., 20, 40, and 60 units are tested under the first three NDTs, respectively. The upper bound of each test duration is $\tau_U = 100$; meanwhile, the following requirement needs to be satisfied when designing τ_1, τ_2 and τ_3 : $t_1 + \tau_1 < w_2,\ w_2 < t_2 + \tau_2 < w_3,\ w_3 < t_3 + \tau_3 < t_U$; $t_U = 850$.
- The temperature and the relative humidity in each of the three tests need to be determined ($RH^y, T^y; y = 1,2,3$); the lower and upper bounds of the relative humidity and temperature are: $RH_U^y = 0.7,\ RH_N^y = 0.3,\ T_U^y = 373K,\ T_N^y = 293K$;
- The system is operational if no more than 20% of the entire population fail;

- The ratio of the testing sample's MRLs after the 1st, 2nd, and 3rd accelerated and normal NDT needs to be respectively greater than $0.9, 0.9,$ and 0.9 (and less than $1, 1$ and 1);
- The ratio of the system reliability after the 1st, 2nd, and 3rd accelerated and normal NDT needs to be respectively greater than $0.9, 0.9,$ and 0.9 (and less than $1, 1,$ and 1);
- The ratio of the expected number of failed units after the 1st, 2nd, and 3rd accelerated and normal NDT needs to be respectively greater than $1.15, 1.15$ and 1.15.

Under each testing scenario, we first determine the optimal testing plan for the 1st NDT; accordingly, the equivalent test duration of the 1st accelerated NDT for units with different characteristics, the system/sample reliability metrics are calculated and considered in the design of the optimal testing plan for the 2nd accelerated NDT; followed by the 3rd accelerated NDT.

We obtain the optimal testing plans for the first three accelerated NDTs under TS2 and TS3 in Tables 7.1 and 7.2. In Table 7.3, we quantify the differences between the accelerated and normal NDTs by calculating the objective values of the optimization problem (differences between the distributions of failed units after the first three accelerated and normal NDTs) under TS2 and TS3.

We observe that TS2 and TS3 yield the same optimal testing plan for the 1st accelerated NDT, as no repair is conducted when designing the 1st NDT; besides, the distributions of failed units after accelerated and normal NDTs are approximately the same as the differences between the distributions of failed units after the accelerated and normal NDTs for $y = 1, 2, 3$ are negligible as shown in Table 7.3.

As required, the ratios of the system reliability after the accelerated and normal tests as well as the ratios of the sample's TMRL after the accelerated and normal tests approximately equal to 1 ($q_1^y = 0.9,\ q_2^y = 0.9;\ y = 1, 2, 3$); which implies that the optimal accelerated NDTs have negligible effect on the system reliability metrics within a short test duration. Furthermore, as the sample represents the population characteristics, q_s^y does not affect the y^{th} optimal testing plan as the sampling error does not exist.

Table 7.1 Optimal testing plans for the first three accelerated NDTs under TS2

	1^{st} NDT	2^{nd} NDT	3^{rd} NDT
RH^y	0.41	0.34	0.34
T^y	294	300	300
τ^y	13	34	57

Table 7.2 Optimal testing plans for the first three accelerated NDTs under TS3

	1^{st} NDT	2^{nd} NDT	3^{rd} NDT
RH^y	0.41	0.44	0.39
T^y	294	294	297
τ^y	13	23	35

Table 7.3 Objective function values after the first three accelerated and normal NDTs under TS2 and TS3

	1st NDT	2nd NDT	3rd NDT
TS2	1.53e-5	6.98e-4	2.68e-3
TS3	1.53e-5	2.71e-5	1.62e-3

7.4 Conclusions

In this chapter, we study the individual unit's reliability behavior by developing a statistics-physics-based model, which directly relates the applied stresses to the system reliability metrics. We then propose the optimal design of a sequence of accelerated NDTs on the system level, taking system's/sample's reliability metrics under different testing scenarios into account. The uncertainty during the sampling procedure as well as the units' different time-dependent characteristics are considered. We show that the accelerated NDTs have insignificant effect on the system reliability metrics while reducing the test durations.

CHAPTER 8

OPTIMAL DESIGN OF HYBRID SEQUENTIAL TESTING PLANS

NDT is conducted to determine the functionality of the units without permanent damage. It is also used to estimate the units' reliability. However, it is difficult to make decisions regarding system's reliability by completely using the results of the NDT since the total functionality of the units are not fully tested. In contrast, destructive test (DT) is conducted to observe the failure times of the units. This results in a more accurate estimation of the units' reliability. However, after the DT, testing units are destroyed. This amplifies the need to investigate the hybrid reliability testing which utilizes the advantages of both the NDT and DT. In this chapter, we assume the units selected from the system are subjected to a sequence of hybrid reliability testing. It is of great interest to optimally design the hybrid sequential reliability testing which results in more accurate reliability estimation and minimizes the number of units subjected to DT based on sampling. After conducting a number of hybrid testing, we decrease the sample size of the DT as the accuracy of reliability metrics estimation improves. Eventually, we conduct NDT only. We investigate the problem under four testing scenarios. The proposed methods are validated through numerical illustrations and extensive simulation studies.

This chapter is organized as follows: in sections 8.1 and 8.2, we design the optimal sequential hybrid reliability testing plans. We formulate the optimization problem, update the unit's lifetime parameter based on the testing results and improve the accuracy of system reliability prediction. We assume that NDT and DT are performed on two separate samples in each hybrid test in section 8.1. In section 8.2, we assume that each hybrid test is performed on the same sample (three testing scenarios are assumed and investigated in details). In section 8.3, we develop extensive simulation studies to validate the optimization problems and methods. In section 8.4, we provide a numerical example to illustrate the proposed methods.

8.1 Optimal Sequential Hybrid Reliability Testing Plans based on Two Separated Samples

One-shot units are produced in batches and kept in storage until placed into actual use. Specifically, the i^{th} batch of units with size n_i arrives into the storage at time w_i immediately after production. A sequence of hybrid reliability tests are conducted at arbitrary time during the entire life horizon of the units by testing selected samples. Specifically, in each hybrid reliability test NDT and DT are performed simultaneously on two selected samples. The y^{th} hybrid reliability test is performed at time t_y. The reliability tests are assumed to be instantaneous (duration of the test is ignored). Units subjected to the DTs are removed from the system after the tests. Units that fail in the NDT are repaired and placed back into the system. The units in the storage are subjected to different series of hybrid reliability tests and thus have different characteristics. The system is referred to a generalized "*k*-out-of-*n*: F" system.

We assume the lifetime of the units in the system follows exponential distribution. The following are necessary notations:

$F^{x}(\cdot)\left(f^{x}(\cdot)\right)$: *cdf* (*pdf*) of the unit lifetime after the x^{th} repair;

t_y : time when the y^{th} hybrid test is performed, $y = 1, 2, \ldots, m$;

n_i^y : number of units in the i^{th} batch when the y^{th} test starts;

N_y : total number of units in the system at time t_y;

$\underset{\sim}{\pi}$: sequence of NDTs that the testing units are subjected to;

$\#\{\underset{\sim}{\pi}\}$: number of NDTs in sequence $\underset{\sim}{\pi}$;

$M_i^y\left(\underset{\sim}{\pi}; \lambda_{y-1}\right)$: number of expected failures of a unit (in the i^{th} batch) until time t_y based on λ_{y-1}, the unit is subjected to NDT sequence $\underset{\sim}{\pi}$;

$\left(p_y; \lambda_{y-1}\right)$: failure probability of the unit in the system at time t_y based on λ_{y-1};

ΔR : threshold of the difference between the reliability estimate of the system and the reliability estimation of the sample;

$s_N^y\left(s_D^y\right)$: number of testing units assigned to the y^{th} NDT (DT);

$K_N^y\left(K_D^y\right)$: number of units failed in the y^{th} NDT (DT), respectively; $K_N^y = k_N^y$ $\left(K_D^y = k_D^y\right)$; $k_N^y = 0, 1, \ldots, s_N^y$ $(k_D^y = 0, 1, \ldots, s_D^y)$;

$q_N\%$: percentage of units assigned to the y^{th} NDT, $q_N\%$ is constant $\forall y$; specifically, $s_N^y = q_N\% \cdot N_y$;

$q\%$: the system fails if $q\%$ or more units fail;

λ_y : unit's updated lifetime parameter after the y^{th} hybrid test; the initial value of the parameter is recorded as λ;

λ_{true} : the true value of unit's lifetime parameter;

$\Delta t_L \left(\Delta t_U\right)$: lower and upper bounds of the time interval between two reliability tests;

$\bar{k}_N^y \left(\bar{k}_D^y\right)$: the sample assigned to the y^{th} NDT (DT) is considered to fail if $\bar{k}_N^y \left(\bar{k}_D^y\right)$ or more units fail.

In section 8.1, two samples are selected in each hybrid test. One sample is for the NDT and the other is for the DT. The following two factors significantly affect the testing results and the accuracy of system reliability estimation:

1. Time when the y^{th} hybrid test is performed (t_y);

2. The number of units in the samples of the y^{th} NDT and DT, respectively (s_N^y and s_D^y).

The design of the y^{th} test is affected by the previous (y-1) tests and the system state at time t_y. Specifically, the optimal solutions for the first *m* tests are obtained by first obtaining

t_1, s_N^1 and s_D^1 in the 1st test. Based on t_1, s_N^1, s_D^1 and the first test's results, the units' lifetime parameter is updated from λ to λ_1. An optimal testing design is then found for the second hybrid test and the units' lifetime parameter is updated from λ_1 to λ_2. A sequence of optimal hybrid testing can be determined by applying the above procedures. In the following, we describe the steps when optimizing the y^{th} hybrid test, where $y = 1, ..., m$.

Step 1: *Determine the optimal time to start the y^{th} test and the number of testing units assigned to the y^{th} NDT and DT. Note that the optimization of the test plan is affected by λ_{y-1} , which is updated based on the results of previous* (y-1) *tests;*

Specifically, the time to start the y^{th} hybrid reliability test and the number of units assigned to the y^{th} test (t_y, s_N^y, and s_D^y) are determined by solving the following optimization problem:

$$\textbf{\textit{Min}}\ s_D^y \tag{8.1}$$

Subject to

$$\Delta t_L \le t_y - t_{y-1} \le \Delta t_U \tag{8.2}$$

$$s_N^y = q_N \% \cdot N_y \tag{8.3}$$

$$\left| R^{y}_{sample}\left(\bar{k}^{y}_{D}\text{-out-of-}s^{y}_{D};t_{y},\lambda_{y-1}\right) - R^{y}_{system}\left(\left(q\%\cdot N_{y}\right)\text{-out-of-}N_{y};t_{y},\lambda_{y-1}\right)\right| \leq \Delta R \tag{8.4}$$

where

$$N_{y} = \left(\sum_{\forall\{i;w_i \leq t_y\}} n_i\right) - \sum_{\omega=1}^{y-1} s^{\omega}_{D} \tag{8.5}$$

The objective (Eq. (8.1)) is to minimize the number of units assigned to the y^{th} DT. There exist an upper and a lower bound for the time interval between two hybrid tests (Eq. (8.2)). Note that for $y=1$, Eq. (8.2) is written as: $t_1 \leq \Delta t_U$.

In the y^{th} hybrid test, we randomly select $q_N\%$ of the units in the system to perform NDT (Eq. (8.3). It is required that the difference between the reliability estimation of the system and the sample should not be greater than a predetermined threshold such that the sample represents the population's characteristics in the y^{th} DT (Eq. (8.4)). The total number of units in the system equals to the total number of arrived units in the system minus the total number of units assigned to the previous $(y-1)$ DTs (Eq. (8.5)). The reliability of the system is defined in Eq. (8.6):

$$R^{y}_{system}\left(\left(q\%\cdot N_{y}\right)\text{-out-of-}N_{y};t_{y},\lambda_{y-1}\right) = \sum_{k^{y}=0}^{q\%\cdot N_{y}} P\left(K^{y} = k^{y}\right) \tag{8.6}$$

Similarly, the reliability of the sample subjected to the y^{th} DT is obtained using Eq. (8.7) as:

$$R_{sample}^{y}\left(\bar{k}_D^{\,y}\text{-out-of-}s_D^y;t_y,\lambda_{y-1}\right)=\sum_{k_D^y=0}^{\bar{k}_D^y}P\left(K_D^y=k_D^y\right) \tag{8.7}$$

The failure distributions corresponding to Eqs. (8.6) and (8.7) are obtained by applying the models proposed in section 3.2. Moreover, an alternative approach to approximately obtain the failure distributions is using the binomial distribution. Specifically:

$$P\left(K^y=k^y\right)=\binom{N_y}{k^y}\left(p_y;\lambda_{y-1}\right)^{k^y}\left(1-\left(p_y;\lambda_{y-1}\right)\right)^{N_y-k^y} \tag{8.8}$$

$$P\left(K_D^y=k_D^y\right)=\binom{s_D^y}{k_D^y}\left(p_y;\lambda_{y-1}\right)^{k_D^y}\left(1-\left(p_y;\lambda_{y-1}\right)\right)^{s_D^y-k_D^y} \tag{8.9}$$

The system's failure probability $\left(p_y;\lambda_{y-1}\right)$ is obtained by applying the renewal process. Specifically:

$$\left(p_y;\lambda_{y-1}\right)=\frac{1}{N_y}\left\{\sum_{\forall\{i;w_i\le t_y\}}\left\{\left(n_i^y\right)\left[\left(q_N\,\%\right)^{\#\{\pi\}}\left(1-q_N\,\%\right)^{y-z_i-\#\{\pi\}}\right]M_i^y\left(\underset{\sim}{\pi};\lambda_{y-1}\right)\right\}\right\} \tag{8.10}$$

The number of units in the i^{th} batch when the y^{th} test starts is obtained by solving Eq. (8.11) recursively. We specifically have $n_i^1=n_i$.

$$n_i^y = n_i^{y-1} - \frac{s_D^{y-1} \cdot n_i^{y-1}}{N_{y-1}} \tag{8.11}$$

The number of expected number of failures that one unit (from the i^{th} batch) experiences until time t_y, which is tested in NDT sequence $\underset{\sim}{\pi}$, is found by applying the RP in Eq. (8.12):

$$\begin{aligned} M_i^y\left(\underset{\sim}{\pi};\lambda_{y-1}\right) &= \left[1+M\left(t_{z_i+1,\,y}\right)\right]\int_0^{t_{z_i+1}-w_i}\left[f\left(t\right);\lambda_{y-1}\right]dt \\ &\quad + I\cdot\left\{\sum_{\alpha=1}^{\{\underset{\sim}{\pi}:m-z_i-1\}}\left[\left[1+M\left(t_{z_i+a+1,\,y}\right)\right]\int_{t_{z_i+a}-w_i}^{t_{z_i+a+1}-w_i}\left[f\left(t\right);\lambda_{y-1}\right]dt\right]\right\} \end{aligned} \tag{8.12}$$

where

$$I = \begin{cases} 1 & \text{if } z_i < m-1 \\ 0 & \text{if } z_i = m-1 \end{cases}$$

$$M\left(t_{\rho,m}\right) = \sum_{\alpha=1}^{m-\rho}\left[1+M\left(t_{\rho+a,m}\right)\right]\int_{t_{\rho+a-1}-t_\rho}^{t_{\rho+a}-t_\rho} f\left(t\right)dt$$

and $M\left(t_{m,\,m}\right)=0\ \forall i$

Step 2: *Perform the y^{th} hybrid test and obtain the number of failed units in the y^{th} NDT and DT. Update the parameter of the unit's lifetime from λ_{y-1} to λ_y based on the results of the test;*

Intuitively, we expect $\frac{k_N^y}{s_N^y} \le \frac{k_D^y}{s_D^y}$ since DT fully performs the testing units' functionalities. All potential failures are detected in the DT. The number of failed units in the y^{th} NDT is obtained as:

$$k_N^y = E\left(K_N^y;\lambda_{y-1}\right) = \sum_{\hat{k}_N^y=0}^{s_N^y} P\left(K_N^y = \hat{k}_N^y;\lambda_{y-1}\right)\cdot \hat{k}_N^y \tag{8.13}$$

where

$$P\left(K_N^y = \hat{k}_N^y;\lambda_{y-1}\right) = \binom{s_N^y}{\hat{k}_N^y}\left(p_y;\lambda_{y-1}\right)^{\hat{k}_N^y}\left(1-\left(p_y;\lambda_{y-1}\right)\right)^{s_N^y-\hat{k}_N^y}$$

Utilizing the results of the y^{th} NDT and DT, we update the unit's lifetime parameter from λ_{y-1} to λ_y using Eqs. (8.14)-(8.17):

$$\frac{k_N^y + k_D^y}{s_N^y + s_D^y} = \frac{E\left(K_N^y;\lambda_y\right)}{s_N^y} = \frac{\sum_{\hat{k}_N^y=0}^{s_N^y} P\left(K_N^y = \hat{k}_N^y;\lambda_y\right)\cdot \hat{k}_N^y}{s_N^y} \tag{8.14}$$

where

$$P\left(K_N^y = \hat{k}_N^y;\lambda_y\right) = \binom{s_N^y}{\hat{k}_N^y}\left(p_y;\lambda_y\right)^{\hat{k}_N^y}\left(1-\left(p_y;\lambda_y\right)\right)^{s_N^y-\hat{k}_N^y} \tag{8.15}$$

$$\left(p_y;\lambda_y\right) = \frac{1}{N_y}\left\{\sum_{\forall\{i;w_i\le t_y\}}\left\{n_i^y\left[\left(q_N\,\%\right)^{\#\{\underset{\sim}{\pi}\}}\left(1-q_N\,\%\right)^{y-z_i-\#\{\underset{\sim}{\pi}\}}\right]M_i^y\left(\underset{\sim}{\pi};\lambda_y\right)\right\}\right\} \tag{8.16}$$

$$M_i^y\left(\underset{\sim}{\pi};\lambda_y\right)$$
$$=\left[1+M\left(t_{z_i+1,\,y}\right)\right]\int_0^{t_{z_i+1}\text{-}w_i}\left[f\left(t\right);\lambda_y\right]dt+I\cdot\left\{\sum_{\alpha=1}^{\{\underset{\sim}{\pi}:m\text{-}z_i\text{-}1\}}\left[\left[1+M\left(t_{z_i+a+1,\,y}\right)\right]\int_{t_{z_i+a}\text{-}w_i}^{t_{z_i+a+1}\text{-}w_i}\left[f\left(t\right);\lambda_y\right]dt\right]\right\} \tag{8.17}$$

Assuming the units have an exponentially distributed lifetime distribution, the term $\left(f\left(t\right);\lambda_y\right)$ in Eq. (8.17) is $\left(f\left(t\right);\lambda_y\right)=\lambda_y e^{-\lambda_y t}$.

Step 3*: Determine the optimal time to start the* $(y+1)^{\text{th}}$ *test and the number of testing units assigned to the* $(y+1)^{\text{th}}$ *test. The optimal testing plan is based on* λ_y *.*

Based on the unit's updated lifetime parameter λ_y , the optimal t_{y+1}, s_N^{y+1} and s_D^{y+1} are determined by solving the same optimization problem formulated in step 1. The units' lifetime parameter is then updated from λ_y to λ_{y+1} based on the $(y+1)^{\text{th}}$ test's results.

The accuracy of the system reliability estimation improves by applying the above procedure. We gradually decrease the sample size of the DT and expect there exists a y' such that $0\le\frac{k_D^{y'}}{s_D^{y'}}-\frac{k_N^{y'}}{s_N^{y'}}\le\Delta$. We then conduct NDT only as the difference between $\lambda_{y'}$ and its true value λ_{true} becomes negligible.

The optimization problem formulated above is based on the assumption that units that fail in the NDT are repaired and placed back into the system. Under some circumstances, failed units are discarded after the NDT, therefore the total number of units in the system at time t_y is continuously updated after the NDTs as follows:

$$N_y = \left(\sum_{\forall \{i; w_i \le t_y\}} n_i \right) - \sum_{\omega=1}^{y-1} s_D^{\omega} - \sum_{\omega=1}^{y-1} k_N^{\omega} \qquad (8.18)$$

where the last term in Eq. (8.18) is the total number of units that failed in the previous (*y*-1) NDTs.

8.2 Optimal Sequential Hybrid Testing Plans based on One Sample

In this section, we design the sequential testing plans by assuming that only one sample of the units is tested in each hybrid test. Specifically, the sample is first subjected to the NDT and then is subjected to DT. Specifically, the following testing scenarios in the y^{th} hybrid test are investigated:

Testing scenario a (TSa): The sample (with sample size s_y) is first subjected to the y^{th} NDT. Units that fail in the y^{th} NDT (k_N^y) are repaired and placed back into the system, i.e., only units that survive in the y^{th} NDT ($s_y - k_N^y$) are subjected to the y^{th} DT;

Testing scenario b (TSb): The sample (with sample size s_y) is first subjected to the y^{th} NDT. Units that fail in the y^{th} NDT (k_N^y) are repaired and placed back into the sample with a higher failure rate, i.e., units that are subjected to the y^{th} DT are a mixture of the units that fail in the y^{th} NDT (k_N^y) and the units that survive in the y^{th} NDT ($s_y - k_N^y$);

Testing scenario c (TSc): The sample (with sample size s_y) is first subjected to the y^{th} NDT. Units that fail in the y^{th} NDT (k_N^y) are discarded, i.e., only units that survive in the DT ($s_y - k_N^y$) are subjected to the y^{th} DT;

We now design the optimal sequential hybrid testing plans for the above three scenarios. The following two decision variables need to be determined:

1. The time to perform the y^{th} hybrid test (t_y);

2. The number of units assigned to the y^{th} test (s_y).

8.2.1 Optimal Sequential Hybrid Testing Plans for the TSa

We describe the three steps when optimizing the y^{th} hybrid test:

Step 1: *Determine the optimal time to start the y^{th} test and the number of testing units based on* λ_{y-1}*;*

Specifically, the time to start the y^{th} test and the number of testing units (t_y and s_y) are determined by solving the following optimization problem, which applies to arbitrary y between 0 and m.

Min s_y (8.19)

Subject to

$$\Delta t_L \le t_y - t_{y-1} \le \Delta t_U \tag{8.20}$$

$$\left| R_{sample}^{y}\left(\bar{k}_N^{\,y}\text{-out-of-}s_y; t_y, \lambda_{y-1}\right) - R_{system}^{y}\left(\left(q\% \cdot N_y\right)\text{-out-of-}N_y; t_y, \lambda_{y-1}\right)\right| \le \Delta R \tag{8.21}$$

where

$$N_y = \left(\sum_{\forall\{i; w_i \le t_y\}} n_i\right) - \sum_{\omega=1}^{y-1}\left(s_\omega - k_N^{\omega}\right) \tag{8.22}$$

The objective (Eq. (8.19)) is to minimize the number of units assigned to the y^{th} test, which minimizes the number of units assigned to the DTs. There exist an upper and a lower bound for the time interval between the two hybrid tests (Eq. (8.20)).

The difference between the reliability of the system and the sample (when performing NDT) should not be greater than a predetermined threshold (Eq. (8.21)). Under TSa, the total number of units in the system at time t_y equals to the total number of units arrived into the system minus the units that survived in the previous (y-1) NDTs (those units are tested in the previous (y-1) DTs), which is reflected by Eq. (8.22). The reliabilities of the system and sample are given in Eqs. (8.23) and (8.24).

$$R_{system}^{y}\left(\left(q\%\cdot N_y\right)\text{-out-of-}N_y;t_y,\lambda_{y-1}\right)=\sum_{\hat{k}_y=0}^{q\%\cdot N_y}P\left(K_y=\hat{k}_y\right) \tag{8.23}$$

$$R_{sample}^{y}\left(\bar{k}_N^{y}\text{-out-of-}s_y;t_y,\lambda_{y-1}\right)=\sum_{\hat{k}_N^{y}=0}^{\bar{k}_N^{y}}P\left(K_N^{y}=\hat{k}_N^{y}\right) \tag{8.24}$$

The failure distributions in Eqs. (8.23) and (8.24) are obtained approximately by using a binomial distribution. Specifically:

$$P\left(K^{y}=k^{y}\right)=\binom{N_y}{k^{y}}\left(p_y;\lambda_{y-1}\right)^{k^{y}}\left(1-\left(p_y;\lambda_{y-1}\right)\right)^{N_y-k^{y}} \tag{8.25}$$

$$P\left(K_N^{y}=\hat{k}_N^{y}\right)=\binom{s_y}{\hat{k}_N^{y}}\left(p_y;\lambda_{y-1}\right)^{\hat{k}_N^{y}}\left(1-\left(p_y;\lambda_{y-1}\right)\right)^{s_y-\hat{k}_N^{y}} \tag{8.26}$$

The system's failure probability $\left(p_y;\lambda_{y-1}\right)$ is obtained as:

$$\left(p_y;\lambda_{y-1}\right)=\frac{1}{N_y}\left\{\sum_{\forall\{i;w_i\le t_y\}}\left\{n_i\sum_{\forall\underset{\sim}{\pi}}\left[\begin{array}{l}\overbrace{\prod_{\forall\omega\in\underset{\sim}{\pi}}\left(\frac{s_\omega}{N_\omega}\right)\prod_{\forall\omega\in\{y-z_i-\underset{\sim}{\pi}\}}\left(1-\frac{s_\omega}{N_\omega}\right)}^{A}\\ \left[\begin{array}{l}\overbrace{\left(F\left(t_{\underset{\sim}{\pi}(\pi'-1)}-w_i\right);\lambda_{\underset{\sim}{\pi}(\pi'-2)}\right)}^{B}\\ \prod_{\forall\omega\in\underset{\sim}{\pi}}\left(F\left(t_\omega-\hat{t}_\omega\right);\lambda_{\omega-1}\right)\end{array}\right]\\ \left[\overbrace{F\left(t_y-t_{\underset{\sim}{\pi}_\pi}\right);\lambda_{y-1}}^{C}\right]\end{array}\right]\right\}\right\} \tag{8.27}$$

where in Eq. (8.27), $\underset{\sim}{\pi}_a$ represents the a^{th} element in sequence $\underset{\sim}{\pi}_a$.

The terms A, B, and C are:

A: the probability that the units in the i^{th} batch are tested in NDT sequence $\underset{\sim}{\pi}_a$;

B: the probability that the units are observed to fail in a specific sequence π', given they are tested in NDT sequence $\underset{\sim}{\pi}_a$;

C: the probability that the units with characteristics described in B fail in the y^{th} NDT.

The number of units in the i^{th} batch when the y^{th} test starts can be obtained by solving Eq. (8.28) recursively:

$$n_i^y = n_i^{y-1} - \frac{s_{y-1} \cdot n_i^{y-1}}{N_{y-1}} \tag{8.28}$$

Step 2*:*

Step 2.1*: Subject the* s_y *units to the* y^{th} *NDT, obtain the number of failed unit and record it as* k_N^y*. Repair failed units and place them back into the system;*

Step 2.2*: Subject the* $\left(s_y - k_N^y\right)$ *units (that survive in the* y^{th} *NDT) to the* y^{th} *DT and obtain the number of units fail in the* y^{th} *DT, record it as* k_D^y*;*

Step 2.3*: Update the unit's lifetime parameter from* λ_{y-1} *to* λ_y *based on the results of the* y^{th} *test.*

It is expected that $k_D^y > 0$ since not all potential failures can be detected in NDT. The number of failed units in the y^{th} NDT is obtained via Eq. (8.29):

$$k_N^y = E\left(K_N^y; \lambda_{y-1}\right) = \sum_{\hat{k}_N^y = 0}^{s_y} P\left(K_N^y = \hat{k}_N^y; \lambda_{y-1}\right) \cdot \hat{k}_N^y \tag{8.29}$$

where $P\left(K_N^y = \hat{k}_N^y; \lambda_{y-1}\right)$ is presented in Eq. (8.26).

Under all three scenarios in section 8.2, the number of units fail in the y^{th} DT (k_D^y) is dependent on the number of units fail in the y^{th} NDT (k_N^y). Based on the result of the y^{th} DT, we update the unit's lifetime parameter from λ_{y-1} to λ_y:

$$k_D^y = \sum_{\hat{k}_D^y=0}^{s_y-k_N^y} P\left(K_D^y = \hat{k}_D^y\right)\cdot \hat{k}_D^y \tag{8.30}$$

where

$$P\left(K_D^y = \hat{k}_D^y\right) = \binom{s_y - k_N^y}{s_y - k_N^y - \hat{k}_D^y}\left[\frac{\overbrace{1-p\left(t_y;\lambda_y\right)}^{A}}{\underbrace{1-p\left(t_y;\lambda_{y-1}\right)}_{B}}\right]^{s_y-k_N^y-\hat{k}_D^y}\left[\frac{p\left(t_y;\lambda_y\right)-p\left(t_y;\lambda_{y-1}\right)}{1-p\left(t_y;\lambda_{y-1}\right)}\right]^{\hat{k}_D^y} \tag{8.31}$$

In Eq. (8.31), term A is the probability that the unit survives in the y^{th} DT and the term B is the probability that the unit survives in the y^{th} NDT. The ratio of terms A and B conditionally provides the unit's probability of surviving in the y^{th} DT. We then combine Eqs. (8.30) and (8.31) to update the units' lifetime parameter.

Step 3: *Determine the optimal time to start the* $(y+1)^{\text{th}}$ *test and the number of testing units based on* λ_y.

The optimal solution of t_{y+1} and s_{y+1} is determined by solving the same optimization problem formulated in step 1. We then update the units' lifetime parameter from λ_y to λ_{y+1} by following the procedures in step 2.

Intuitively, after conducting a number of hybrid tests, we expect there exists a y' such that $\frac{k_D^{y'}}{s_{y'} - k_N^{y'}} \leq \Delta$. Specifically, the proportion of units that fail in the y'^{th} DT approximately equals to 0 because the y'^{th} NDT reveals the entire population's characteristics and demonstrates the unit's reliability performance. In other words, units that survive in the y'^{th} NDT are expected to survive in the y'^{th} DT since the difference between $\lambda_{y'}$ and its true value λ_{true} is negligible. There is no need to perform DT in the subsequent tests as NDT provides accurate estimation of the system reliability.

8.2.2 Optimal Sequential Hybrid Testing Plans for the TSb

In the TSb (units that fail in the NDT are repaired and subjected to the DT), the following additional notations are necessary:

k_{D-FN}^{y} : the number of units that fail in the y^{th} NDT and the y^{th} DT;

k_{D-RN}^{y} : the number of units that survive in the y^{th} NDT and fail in the y^{th} DT.

The three steps of the y^{th} hybrid test for the TSb are:

Step 1: *This step is the same as that proposed in section* 6.2.1, *except the following two system characteristics are different:*

1) The total number of units in the system at time t_y equals to the total number of units arrived into the system minus the total number of units that tested in the previous (y-1) reliability tests (Eq. (8.32)):

$$N_y = \left(\left(\sum_{\forall\{i; w_i \le t_y\}} n_i \right) - \sum_{\omega=1}^{y-1} s_\omega \right) \tag{8.32}$$

Obviously, N_y affects the testing plans as the system reliability is affected by the total number of units in the system.

2) The system's failure probability $\left(p_y; \lambda_{y-1}\right)$ is now obtained as:

$$\left(p_y; \lambda_{y-1}\right) = \frac{1}{N_y} \left\{ \sum_{\forall\{i; w_i \le t_y\}} \left[n_i^y F\left(t_y - w_i\right) \right] \right\} \tag{8.33}$$

where

$$n_i^y = n_i^{y-1} - \frac{s_{y-1} \cdot n_i^{y-1}}{N_{y-1}} \tag{8.34}$$

Step 2: *Steps* 2.1 *and* 2.3 *for the TSb are the same as those proposed for the TSa. Step* 2.2 *is different:*

Step 2.2*: Subject the* s_y *units to the* y^{th} *DT* (*among these* s_y *units:* k_N^y *units fail in the* y^{th} *NDT are repaired with higher failure rates,* $\left(s_y - k_N^y\right)$ *units survive in the* y^{th} *NDT*) *and obtain the number of units fail in the* y^{th} *DT, record as* k_D^y *;*

The number of failed units in the y^{th} NDT (k_N^y) is obtained by referring to Eq. (8.29). The number of units fail in the y^{th} DT is dependent on k_N^y. Based on Eqs. (8.35)-(8.38), we update the unit's lifetime parameter from λ_{y-1} to λ_y:

$$k_D^y = \sum_{\hat{k}_D^y=0}^{s_y} P\left(K_D^y = \hat{k}_D^y\right) \cdot \hat{k}_D^y \tag{8.35}$$

where

$$P\left(K_D^y = \hat{k}_D^y\right) = P\left(K_{D-FN}^y + K_{D-RN}^y = \hat{k}_D^y\right) = \sum_{k_{D-FN}^y + k_{D-RN}^y = k_D^y} \begin{bmatrix} P\left(K_{D-FN}^y = k_{D-FN}^y\right) \\ P\left(K_{D-RN}^y = k_{D-RN}^y\right) \end{bmatrix} \tag{8.36}$$

$$P\left(K_{D-FN}^y = k_{D-FN}^y\right) = \binom{k_N^y}{k_{D-FN}^y} \left[p\left(t_y;\lambda_y\right)\right]^{k_{D-FN}^y} \left[1 - p\left(t_y;\lambda_y\right)\right]^{k_N^y - k_{D-FN}^y} \tag{8.37}$$

and

$$P\left(K_{D-RN}^{y}=k_{D-RN}^{y}\right)$$
$$=\binom{s_y-k_N^y}{s_y-k_N^y-k_{D-RN}^y}\left[\frac{1-p\left(t_y;\lambda_y\right)}{1-p\left(t_y;\lambda_{y-1}\right)}\right]^{s_y-k_N^y-k_{D-RN}^y}\left[\frac{p\left(t_y;\lambda_y\right)-p\left(t_y;\lambda_{y-1}\right)}{1-p\left(t_y;\lambda_{y-1}\right)}\right]^{k_{D-RN}^y} \tag{8.38}$$

Eqs. (8.37) and (8.38) obtain the distributions of failed units in the y^{th} DT with different characteristics.

Step 3: *Determine the optimal time to start the* $(y+1)^{\text{th}}$ *test and the number of testing units based on* λ_y.

Based on the unit's lifetime parameter λ_y, the optimal t_{y+1} and s_{y+1} are determined by solving the same optimization problem formulated in step 1. We then update the units' lifetime parameter from λ_y to λ_{y+1} based on the test results.

In the TSb, there exists a y' so that $\frac{k_{D-RN}^{y'}}{s_y-k_N^y}\le\Delta$ after a number of hybrid tests. The y'^{th} NDT provides accurate estimation of the system's reliability metrics. Consequently, there is no need to perform DT in subsequent reliability tests.

8.2.3 Optimal Sequential Hybrid Testing Plans for the TSc

The three steps of the problem formulation for the TSc (units that fail in the NDT are discarded) are the same as those proposed in section 8.1.1. Only the total number of units in the system at time t_y is different as shown in Eq. (8.39):

$$N_y = \left(\left(\sum_{\forall \{i; w_i \le t_y\}} n_i \right) - \sum_{\omega=1}^{y-1} s_\omega \right) \tag{8.39}$$

8.3 Simulation Model

In this section, we develop a simulation model to validate the methods investigated in section 8.1 and 8.2. We introduce the objectives and the procedures of the simulation model. We then generate the number of failed units in the sequential reliability tests and update the units' lifetime parameter by applying the simulation model.

We perform the simulation model and observe if the unit's lifetime parameter approaches its true value during the sequential hybrid tests. The following three testing scenarios are addressed:

i) Each hybrid test is performed on two separate samples, units that fail in the NDT are repaired and placed back into the system;

ii) Each hybrid test is performed on one sample, units that fail in the NDT are repaired and placed back into the system;

iii) Each hybrid test is performed on one sample, units that fail in the NDT are repaired and placed back into the sample (TSb).

To validate the proposed methods, we replicate the simulation model 1000 times. In each replication, we obtain the number of failed units (in the simulation model, they are measured as random failure times) in the sequential hybrid tests. We then calculate the average number of failures in the 1000 replications. The simulated number of failures is used to update the units' lifetime parameter. We illustrate the first two hybrid reliability tests by the simulation model.

The following notations are necessary:

$s^{y}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(N)\left(s^{y}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(D)\right)$: the number of units selected from the i^{th} batch and fail in NDT (DT) sequence $\underset{\sim}{j}$ in each replication at time t_y. These units are tested in NDT (DT) sequence $\underset{\sim}{\pi}$;

$k^{y}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(N)\left(k^{y}_{i,\underset{\sim}{j}|\underset{\sim}{\pi}}(D)\right)$: the number of failed units selected from the i^{th} batch and fail in NDT (DT) sequence $\underset{\sim}{j}$ in each replication at time t_y. These units are tested in NDT (DT) sequence $\underset{\sim}{\pi}$;

$\bar{s}^{y}_{i,j|\underline{x}}(N)\left(\bar{s}^{y}_{i,j|\underline{x}}(D)\right)$: the average number of 1000 $s^{y}_{i,j|\underline{x}}(N)\left(s^{y}_{i,j|\underline{x}}(D)\right)$;

$\bar{k}^{y}_{i,j|\underline{x}}(N)\left(\bar{k}^{y}_{i,j|\underline{x}}(D)\right)$: the average number of 1000 $k^{y}_{i,j|\underline{x}}(N)\left(k^{y}_{i,j|\underline{x}}(D)\right)$:.

Under testing scenario i), the number of failures in the first two hybrid tests (in one replication) is simulated in the following:

- Estimation of k_N^1 and k_D^1:

1. Generate n_1 random failure times that follow exponential distribution with parameter λ; randomly select s_N^1 failure times from n_1; within the s_N^1 failure times, count the number of failure times that occur between $(0,t_1]$; record it as k_N^1.
2. Generate n_1 random failure times $\sim \exp(\lambda_{true})$; randomly select s_D^1 failure times from n_1; within the s_D^1 failure times, count the number of failure times that occur between $(0,t_1]$; record it as k_D^1.
3. Update λ to λ_1.

- Estimation of k_N^2 and k_D^2

5. Generate $n_1 - s_N^1 - s_D^1$ random failure times $\sim \exp(\lambda_1)$; randomly select $s^2_{1,0|0}(N)$ failure times from $n_1 - s_N^1 - s_D^1$; count the number of failure times that occur between $(0,t_2]$; record it as $k^2_{1,2|2}(N)$;

6. Generate k_N^1 random failure times $\sim \exp(\lambda_1)$; randomly select $s_{1,1|1}^2(N)$ failure times from k_N^1; count the number of failure times that occur between $(0, t_2 - t_1]$; record it as $k_{1,12|12}^2(N)$;

7. Generate $s_N^1 - k_N^1$ random failure times $\sim \exp(\lambda_1)$; randomly select $s_{1,0|1}^2(N)$ failure times from $s_N^1 - k_N^1$; count the number of failure times that occur between $(t_1, t_2]$; record it as $k_{1,2|12}^2(N)$;

8. Generate n_2 random failure times $\sim \exp(\lambda_1)$; randomly select $s_{2,0|0}^2(N)$ failure times from n_2; count the number of failure times that occur between $(0, t_2 - w_2]$; record as $k_{2,1|2}^2(N)$;

The following constraint must be satisfied in steps 1-4:

$$s_{1,0|0}^2(N) + s_{1,1|1}^2(N) + s_{1,0|1}^2(N) + s_{2,0|0}^2(N) = s_N^2 .$$

Meanwhile, we have: $k_N^2 = k_{1,2|2}^2(N) + k_{1,2|12}^2(N) + k_{1,12|12}^2(N) + k_{2,1|2}^2(N)$

9. Generate $n_1 - s_N^1 - s_D^1$ random failure times $\sim \exp(\lambda_{true})$; randomly select $s_{1,0|0}^2(D)$ failure times from $n_1 - s_N^1 - s_D^1$; count the number of failure times that occur between $(0, t_2]$; record it as $k_{1,2|2}^2(D)$;

10. Generate k_N^1 random failure times $\sim \exp(\lambda_{true})$; randomly select $s_{1,1|1}^2(D)$ failure times from k_N^1; count the number of failure times that occur between $(0, t_2 - t_1]$; record it as $k_{1,12|12}^2(D)$;

11. Generate $s_N^1 - k_N^1$ random failure times $\sim \exp(\lambda_{true})$; randomly select $s_{1,0|1}^2(D)$ failure times from $s_N^1 - k_N^1$; count the number of failure times that occur between $(t_1, t_2]$; record it as $k_{1,2|12}^2(D)$;

12. Generate n_2 random failure times $\sim \exp(\lambda_{true})$; randomly select $s_{2,0|0}^2(D)$ failure times from n_2; count the number of failure times that occur between $(0, t_2 - w_2]$; record it as $k_{2,1|2}^2(D)$;

The following constraint must be satisfied in steps 5-8:

$s_{1,0|0}^2(D) + s_{1,1|1}^2(D) + s_{1,0|1}^2(D) + s_{2,0|0}^2(D) = s_D^2$.

Meanwhile, we have: $k_N^2 = k_{1,2|2}^2(D) + k_{1,2|12}^2(D) + k_{1,12|12}^2(D) + k_{2,1|2}^2(D)$

13. Update λ_1 to λ_2.

In the above procedures, we simulate and obtain k_N^1, k_D^1, k_N^2 and k_D^2 in one replication. The replication is repeated for 1000 times. The means of the above four terms are calculated and compared with those obtained from the proposed methods.

For the testing scenario ii (TSb), k_N^1, k_D^1, k_N^2 and k_D^2 in one replication are generated as follows:

- Estimation of k_N^1 and k_D^1:

1. Generate n_1 random failure times $\sim \exp(\lambda)$; randomly select s_1 failure times from n_1; count the number of failure times that occur between $(0, t_1]$; record it as k_N^1;
2. Generate $s_1 - k_N^1$ random failure times $\sim Uniform\left(0, \dfrac{t_1}{1-\left[\dfrac{R(t_1;\lambda_{true})}{R(t_1;\lambda)}\right]}\right)$; count the number of failure times that occur between $(0, t_1]$; record it as k_D^1;
3. Update λ to λ_1.

- Estimation of k_N^2 and k_D^2

1. Generate $n_1 - s_1$ random failure times $\sim \exp(\lambda_1)$; randomly select $s_{1,0|0}^2(N)$ failure times from $n_1 - s_1$; count the number of failure times that occur between $(0, t_2]$; record it as $k_{1,2|2}^2(N)$;
2. Generate k_N^1 random failure times $\sim \exp(\lambda_1)$; randomly select $s_{1,1|1}^2(N)$ failure times from k_N^1; count the number of failure times that occur between $(0, t_2 - t_1]$; record it as $k_{1,12|12}^2(N)$;

3. Generate n_2 random failure times $\sim \exp(\lambda_1)$; randomly select $s^2_{2,0|0}(N)$ failure times from n_2; count the number of failure times that occur between $(0, t_2 - w_2]$; record it as $k^2_{2,1|2}(N)$;

The following constraint must be satisfied in steps 1-3:

$s^2_{1,0|0}(N) + s^2_{1,1|1}(N) + s^2_{2,0|0}(N) = s^2_N$.

Meanwhile, we have: $k^2_N = k^2_{1,2|2}(N) + k^2_{1,12|12}(N) + k^2_{2,1|2}(N)$.

4. Generate $s^2_{1,0|0} - k^2_{1,2|2}(N)$ random failure times $\sim Uniform\left(0, \dfrac{t_2}{1-\left[\dfrac{R(t_2;\lambda_{true})}{R(t_2;\lambda_1)}\right]}\right)$; count the number of failure times that occur between $(0, t_2]$; record it as $k^2_{1,2|2}(D)$;

5. Generate $s^2_{1,1|1} - k^2_{1,12|12}(N)$ random failure times $\sim Uniform\left(0, \dfrac{t_2 - t_1}{1-\left[\dfrac{R(t_2 - t_1;\lambda_{true})}{R(t_2 - t_1;\lambda_1)}\right]}\right)$; count the number of failure times that occur between $(0, t_2 - t_1]$; record it as $k^2_{1,12|12}(D)$;

6. Generate $s^2_{2,0|0} - k^2_{1,2|2}(N)$ random failure times $\sim Uniform\left(0, \dfrac{t_2 - w_2}{1-\left[\dfrac{R(t_2 - t_1;\lambda_{true})}{R(t_2 - t_1;\lambda_1)}\right]}\right)$; count the number of failure times that occur between $(0, t_2 - w_2]$; record it as $k^2_{2,2|2}(D)$;

The following constraint must be satisfied in steps 5-6:

$s_{1,0|0}^2(D)+s_{1,1|1}^2(D)+s_{1,0|1}^2(D)+s_{2,0|0}^2(D)=s_D^2$.

Meanwhile, we have: $k_D^2=k_{1,2|2}^2(D)+k_{1,12|12}^2(D)+k_{2,2|2}^2(D)$.

7. Update λ_1 to λ_2.

The simulation procedures for the TSc are similar to the procedures for the TSb. However, units fail in the NDT are also repaired and subjected to the DT under TSc. Accordingly, when considering the total number of failures in DT under TSc, additional random failure times need to be generated and taken into consideration. Specifically:

- Estimation of k_N^1 and k_D^1:

1. k_N^1 is generated by following the same steps under TSb.
2. $k_D^1=k_{D\_FN}^1+k_{D\_RN}^1$,where $k_{D\_RN}^1$ is generated by following the same procedures as generating k_D^1 under TSb and $k_{D\_FN}^1$ is obtained by:

Generating k_N^1 random failure times $\sim\exp(\lambda_{true})$; counting the number of failure times that occur between $(0,t_1]$; recording as $k_{D\_FN}^1$.

- Estimation of k_N^2 and k_D^2:

1. Generate $n_1 - s_1$ random failure times $\sim \exp(\lambda_1)$; randomly select $s^2_{1,0|0}(N)$ failure times from $n_1 - s_1$; count the number of failure times that occur between $(0, t_2]$; record it as $k^2_{1,1,2|0}(N)$;

2. Generate n_2 random failure times $\sim \exp(\lambda_1)$; randomly select $s^2_{2,0|0}(N)$ failure times from n_2; count the number of failure times that occur between $(0, t_2 - w_2]$; record it as $k^2_{2,1,2|0}(N)$;

The following constraint must be satisfied in steps 1-2:

$s^2_{1,0|0}(N) + s^2_{2,0|0}(N) = s^2_N$.

Meanwhile, we have: $k^2_N = k^2_{1,1,2|0}(N) + k^2_{2,1,2|0}(N)$.

3. Generate $k^2_{1,1,2|0}(N)$ random failure times $\sim \exp(\lambda_{true})$; count the number of failure times that occur between $(0, t_2]$; record it as $k^2_{1,1,2|0}(D)$;

4. Generate $k^2_{2,1,2|0}(N)$ random failure times $\sim \exp(\lambda_{true})$; count the number of failure times that occur between $(0, t_2 - w_2]$; record it as $k^2_{2,1,2|0}(D)$;

We have $k^2_D = k^2_{1,1,2|0}(D) + k^2_{2,1,2|0}(D) + k^2_{D\_RN}$, where $k^2_{D\_RN}$ is generated by following the same procedures when generating k^2_D under TSb.

8.4 A Numerical Illustration

In this section, we illustrate the optimal sequential hybrid reliability testing design for different testing scenarios and numerically compare the optimal testing plans obtained from proposed methods with those obtained from the simulation model.

We show the design of the first four hybrid reliability tests with the following parameters**:**

$n_i = 200;\ w_i = 200i;\ q_N\% = 10\%;\ \Delta R = 10^{-6};$

$\lambda = 2\times 10^{-3};\ \lambda_{true} = 2.5\times 10^{-3};\ \Delta t_L = 50,\ \Delta t_U = 200$

In Tables 8.1-8.6, we show the optimal testing plans, the unit's updated lifetime parameter λ and other characteristics. We apply the proposed methods and the simulation model for the three testing scenarios, respectively. Specifically, Tables 8.1-8.6 show the optimal testing plans and unit's updated lifetime for the testing scenarios investigated in section 8.1, TSa and TSb in section 8.2, respectively:

Table 8.1 First four optimal reliability tests based on the method proposed in section 8.1 for the testing scenario investigated in section 8.1

Reliability Test	1	2	3	4
t_y	182	391	596	799

λ_y	2.35×10^{-3}	2.41×10^{-3}	2.44×10^{-3}	2.46×10^{-3}
s_D^y	38	37	40	44
Proportion of s_D^y in the population	0.19	0.10	0.06	0.04
s_N^y	20	36	53	69

Table 8.2 First four optimal reliability tests based on the simulation model for the testing scenario investigated in section 8.1

Reliability Test	1	2	3	4
t_y	182	398	598	792
λ_y	2.35×10^{-3}	2.42×10^{-3}	2.45×10^{-3}	2.46×10^{-3}
s_D^y	38	38	41	46
Proportion of s_D^y in the population	0.19	0.10	0.06	0.04
s_N^y	20	36	53	69

Table 8.3 First four optimal reliability tests based on the method proposed in section 8.2 for the TSa

Reliability Test	1	2	3	4
t_y	182	334	565	734
Updated λ_y	2.35×10^{-3}	2.43×10^{-3}	2.46×10^{-3}	2.47×10^{-3}
s_D^y	38	28	30	19
Proportion of s_D^y in the population	0.19	0.08	0.06	0.03
s_N^y	20	36	53	70

Table 8.4 First four optimal reliability tests based on the simulation model for the TSa

Reliability Tests	1	2	3	4
t_y	182	339	569	739
Updated λ_y	2.35×10^{-3}	2.45×10^{-3}	2.47×10^{-3}	2.48×10^{-3}
s_D^y	38	30	31	20
Proportion of s_D^y in the population	0.19	0.08	0.06	0.03
s_N^y	20	36	53	70

Table 8.5 First four optimal reliability tests based on the method proposed in section 8.2 for the TSb

Reliability Test	1	2	3	4
t_y	182	334	565	734
Updated λ_y	2.35×10^{-3}	2.43×10^{-3}	2.46×10^{-3}	2.47×10^{-3}
s_D^y	38	28	30	19
Proportion of s_D^y in the population	0.19	0.08	0.06	0.03
s_N^y	20	36	53	70

Table 8.6 First four optimal reliability tests based on the simulation model for the TSb

Reliability Tests	1	2	3	4
t_y	182	342	552	729
Updated λ_y	2.35×10^{-3}	2.44×10^{-3}	2.45×10^{-3}	2.46×10^{-3}
s_D^y	38	29	29	18
Proportion of s_D^y in the population	0.19	0.08	0.06	0.03
s_N^y	20	36	53	70

From the tables, we have the following observations:

1) $\lambda_y - \lambda_{y-1}$ decreases as y increases, which shows that the unit's parameter converges;

2) λ_4 approximately equals to λ_{true}, which shows the efficiency of the methods;

3) The proportion of units assigned to the DTs shows a decreasing trend, which also demonstrates the efficiency of the methods.

4) The proposed methods and the simulation model yield approximately the same updated lifetime parameter, which validates the accuracy of the methods.

8.5 Conclusions

In this chapter, we design optimal hybrid sequential reliability testing plans during the one-shot units' storage life by performing NDT and DT in each reliability test. We determine the optimal times to conduct the reliability tests as well as the numbers of units assigned to the hybrid tests under different testing scenarios, such that the number of available units after the tests is maximized. Based on the results of the sequential hybrid tests, the accuracy of the system reliability estimation is improved such that DT is no longer needed after a number of hybrid tests. We also develop a simulation model to validate the accuracy and efficiency of the proposed methods. We use a numerical example to illustrate the proposed methods.

CHAPTER 9

OPTIMAL SELECTION AND OPERATIONAL SEQUENCING OF NONHOMOGENEOUS ONE-SHOT UNITS

One-shot units such as missiles, airbags, and most of the military weapons are deployed after long terms of storage (or standby). It is important to ensure the stored units operate its function properly when needed. In this chapter, we investigate the optimization of one-shot units' operational use at arbitrary time.

There exists many situations that the one-units are used consecutively when put into operational use. To illustrate, a certain number of one-shot units (say, s) are selected from the stored population and launched in sequence. Successful operation is achieved when the: 1) first k one-shot units are successfully launched and 2) no less than consecutive r out of the remaining (s-k) one-shot units are successfully launched. The population has a mixture of one-shot units with nonhomogeneous characteristics due to the units' different arrival times and the conduct of the sequential reliability tests during its storage period. Therefore, it becomes interesting and challenging to determine the characteristics and sequence of the one-shot units to be launched such that the operational use of the s launched units is optimized. Defining the s launched one-shot units as a system, the reliability metrics of the system (e.g., the probability that the system achieves the successful operation, the expected number of successfully launched units) need to be investigated. In this chapter, we optimize the system's operational use at arbitrary time by formulating an optimization problem which is applicable to a variety of objectives. We also provide the bounds of the system's successful operational probability estimation and develop a simulation study to validate the proposed approach.

The chapter is organized as follows: in section 9.1, we state the problem and scenarios in details and calculate the reliability of the one-shot units when they are put into operational use. In section 9.2, we investigate the expected time-to-k^{th}-failure of the system (with nonhomogeneous units) and formulate an optimization problem to maximize the operational use of the system. Specifically, we determine the characteristics and launching sequence of the selected units by considering the system's reliability metrics. In section 9.3, we analyze the detailed characteristics of the one-shot units and investigate the bounds of the system's probability of successful operation based on the optimal selection and launching sequence obtained in section 9.2. In section 9.4, we develop a simulation model to validate the optimality of the solution(s) obtained in section 9.2. In section 9.5, we provide several numerical examples to illustrate the application of the optimization problem. We then conclude our work in section 9.6.

9.1. One-shot Units' Operational Use Reliability

One-shot units are produced and stored in batches at different times. To guarantee the one-shot units' operational use when needed, instantaneous NDTs are repeatedly conducted during its entire storage life horizon. The following notations are needed when obtaining the expected number of one-shot units with specific characteristics:

t_m : the time when the m^{th} NDT is conducted;

n_i : the number of units in the i^{th} batch;

w_i : the arrival time of the i^{th} batch;

$n_{i,\underset{\sim}{j}}^{m}$: the number of one-shot units with characteristics $i, \underset{\sim}{j}$ (i.e., the one-shot unit is in the i^{th} batch and fails in NDT sequence $\underset{\sim}{j}$) at time t_m ;

$\#\{\underset{\sim}{j}\}$: the number of tests in sequence $\underset{\sim}{j}$;

$j(\alpha)$: the α^{th} test in NDT sequence $\underset{\sim}{j}$; $j(j)$ is the last test in $\underset{\sim}{j}$;

$R_{i,\underset{\sim}{j}}(t)$: reliability of the one-shot unit (with characteristics $i, \underset{\sim}{j}$ at time t_m) at time t;

$F_x(\cdot)$: CDF of the one-shot unit's lifetime after its x^{th} repair; $F(\cdot)$ is the CDF of units that never repaired.

The expected numbers of one-shot units (with specific characteristics) at time t_m are obtained in Eqs. (9.1) and (9.2), where $j(j) = m$ means the unit fails at time t (its j^{th} failure is observed in the m^{th} NDT) and $j(j) < m$ means the unit has survived at time t.

$$E\left(n_{i,\underset{\sim}{j}}^{m}; j(j) = m\right) = n_i \cdot \begin{bmatrix} \left(F\left(t_{j(1)} - w_i\right) - F\left(t_{j(1)-1} - w_i\right)\right) \cdot \\ \prod_{a=1}^{\#\{\underset{\sim}{j}\}-1} \left(F_a\left(t_{j(\alpha)} - t_{j(\alpha-1)}\right) - F_a\left(t_{j(\alpha)-1} - t_{j(\alpha-1)}\right)\right) \cdot \\ \left(F_{\#\{\underset{\sim}{j}\}}\left(t_m - t_{j(j)}\right) - F_{\#\{\underset{\sim}{j}\}}\left(t_{m-1} - t_{j(j)}\right)\right) \end{bmatrix} \tag{9.1}$$

$$E\left(n_{i,\underset{\sim}{j}}^{m}; j\left(j\right) < m\right) = n_i \cdot \begin{bmatrix} \left(F\left(t_{j(1)} - w_i\right) - F\left(t_{j(1)-1} - w_i\right)\right) \cdot \\ \prod_{a=1}^{\#\left\{\underset{\sim}{j}\right\}-1} \left(F_a\left(t_{j(\alpha)} - t_{j(\alpha-1)}\right) - F_a\left(t_{j(\alpha)-1} - t_{j(\alpha-1)}\right)\right) \cdot \\ R_{\#\left\{\underset{\sim}{j}\right\}}\left(t_m - t_{j(j)}\right) \end{bmatrix} \tag{9.2}$$

The reliability of the one-shot unit (with characteristics $i, \underset{\sim}{j}$) at time t ($t > t_m$) is accordingly calculated:

$$R_{i,\underset{\sim}{j}}\left(t\right) = \frac{R_{\#\left\{\underset{\sim}{j}\right\}}\left(t - t_{j(j)}\right)}{R_{\#\left\{\underset{\sim}{j}\right\}}\left(t_m - t_{j(j)}\right)} \tag{9.3}$$

Under most circumstances, the one-shot units within the same batch have different characteristics. Specifically, they have different repair conditions (the number of repairs $\#\left\{\underset{\sim}{j}\right\}$ and the when the repairs are conducted ($\underset{\sim}{j}$)). Based on Eqs. (9.1)-(9.3), we obtain the mean and variance of the reliability ($\bar{R}_i\left(t\right)$ and $Var\left(R_i\left(t\right)\right)$) of the one-shot units in the i^{th} batch:

$$\bar{R}_i\left(t\right) = \frac{\sum_{\forall \underset{\sim}{j}} \left[\frac{R_{\#\left\{\underset{\sim}{j}\right\}}\left(t - t_{j(j)}\right)}{R_{\#\left\{\underset{\sim}{j}\right\}}\left(t_m - t_{j(j)}\right)} \cdot E\left(n_{i,\underset{\sim}{j}}^{m}\right)\right]}{n_i} \tag{9.4}$$

$$Var\left(R_i\left(t\right)\right) = \frac{\sum_{\forall \underset{\sim}{j}} \left[\left[\frac{R_{\#\left\{\underset{\sim}{j}\right\}}\left(t - t_{j(j)}\right)}{R_{\#\left\{\underset{\sim}{j}\right\}}\left(t_m - t_{j(j)}\right)} - \bar{R}_i\left(t\right)\right]^2 \cdot E\left(n_{i,\underset{\sim}{j}}^{m}\right)\right]}{n_i} \tag{9.5}$$

9.2 Optimal Selection and Operational Sequencing of One-shot Units

The stored one-shot units (with nonhomogeneous characteristics) are put into operational use whenever needed (say, at time t). Under most circumstances, a specific number of one-shot units (say, s, where $s \le \sum_{\forall i} n_i$) are selected and put into operational use. The successful operation is achieved once 1) the first k one-shot units are successfully launched and 2) consecutive r out of the rest (s-k) one-shot units are successfully launched. It is without loss of generality to define the selected one-shot units as a "first-hit-k-consecutive-r-out-of-(s-k)" system, where the time of selecting and launching the one-shot units is negligible. In this section, we investigate the time-to-k^{th}-failure of the system with nonhomogeneous one-shot units and optimize the system's operational use.

9.2.1 Optimal Selection and Operational Sequencing

In this section, we formulate an optimization problem to obtain the optimal selection and launching sequence of the one-shot units. We first define the following:

$\underset{\sim}{v}$: the sequence to launch the selected one-shot units, $\underset{\sim}{v}$ is presented in terms of selected one-shot unit's batch number;

$v(\alpha)$: the batch number of the α^{th} launched one-shot unit;

$\bar{R}_{v(\alpha)}(t)$: expected reliability of the α^{th} launched one-shot unit at time t;

$Var\left(R_{v(\alpha)}(t)\right)$: variance of the reliability of the α^{th} launched one-shot unit at time t;

β: one-shot units in the β^{th} batch have the β^{th} lowest average reliability;

$\bar{R}_{system}\left(t; v(\alpha)\forall\alpha\right)$: expected probability that the system operates successfully at time t;

$Var\left(R_{system}\left(t; v(\alpha)\forall\alpha\right)\right)$: variance of the probability that the system operates successfully at time t ;

s_β : the number of one-shot units selected from the β^{th} batch; $\sum_{\forall\beta} s_\beta = s$.

To optimize the operational use of the system (e.g., maximize the expected probability that the system achieves the successful operation), the following two variables need to be determined:

1) The characteristics of the selected one-shot units; i.e., the number of one-shot units selected from each batch; and
2) The sequence to launch the selected one-shot units.

The above two variables determine the characteristics of the α^{th} launched one-shot units $\forall\alpha$ ($\bar{R}_{v(\alpha)}(t)$ and $Var\left(R_{v(\alpha)}(t)\right)$). Note that the one-shot units' launching sequence is dependent on its characteristics. The expected probability that the system achieves the successful operation at arbitrary time $t \geq t_m$ ($m = 1, 2, \ldots$) can be maximized by solving the following problem:

Max

$$\bar{R}_{system}\left(t; v(\alpha)\forall\alpha\right) \tag{9.6}$$

s.t.

$$\frac{n_\beta - s_\beta}{n_{\beta+1} - s_{\beta+1}} \le \frac{n_\beta \cdot \bar{R}_\beta(t)}{n_{\beta+1} \cdot \bar{R}_{\beta+1}(t)}; \quad \forall \beta \tag{9.7}$$

$$E_{system}\left(T_{s-k-r+1}; v(\alpha)\forall\alpha\right) \ge t \tag{9.8}$$

$$E_{system}\left(\text{number of sucessfully launched one-shot units; } t; v(\alpha)\forall\alpha\right) \ge k + r \tag{9.9}$$

$$Var\left(R_{system}\left(t; v(\alpha)\forall\alpha\right)\right) \le Var_{\text{threshold}} \tag{9.10}$$

$$\sum_{\forall\beta} s_\beta = s \tag{9.11}$$

$$\#\left\{R_{v(\alpha)}(t) : \bar{R}_{v(\alpha)}(t) = \bar{R}_\beta(t)\right\} = s_\beta \tag{9.12}$$

The objective is to maximize the expected probability that the system successfully operates at arbitrary time t (Eq. (9.6)). Intuitively, the objective can be achieved by selecting one-shot units in a decreasing order in terms of its average reliability at time t ($\bar{R}_\beta(t)$). However, the reliability metrics of unselected one-shot units also need to be considered for future operational use. Therefore, we consider that a minimum number of less-reliable one-shot units need to be selected (Eq. (9.7)). In Eq. (9.8), the system's expected-time-to-(s-k+r+1)$^{\text{th}}$-failure must be later than time t. Similarly, the expected number of successfully launched one-shot units at time t should be greater than (k+r) (Eq. (9.9)). As the characteristics of the one-shot units are nonhomogeneous, the randomness of selected one-shot units' reliability metrics needs to be considered (Eq. (9.10)). Eqs. (9.11) and (9.12) indicate that the launched one-shot units are selected from different batches.

Note that the above formulated optimization problem is general as the objective can be exchanged with some of the constraints. To illustrate, the optimal operational use can be realized by minimizing the variance of probability of the system's success; meanwhile, the expected probability that the system achieves a successful operation becomes a constraint which should be less than a given threshold:

Min

$$Var\left(R_{system}\left(t;v\left(\alpha\right)\forall\alpha\right)\right) \tag{9.13}$$

s.t.

$$\bar{R}_{system}\left(t;v\left(\alpha\right)\forall\alpha\right)\geq\bar{R}_{threshold}\text{; and} \tag{9.14}$$

Constraints as shown in Eqs. (9.7)-(9.8), (9.10)-(9.11).

We can also maximize the expected number of successfully launched one-shot units; meanwhile the expected probability that system achieves a successful operation should be less than a given threshold:

Max

$$E_{\text{system}}\left(\text{number of sucessfully launched one-shot units; } t;v\left(\alpha\right)\forall\alpha\right) \tag{9.15}$$

s.t.

$$\bar{R}_{system}\left(t;v\left(\alpha\right)\forall\alpha\right)\geq\bar{R}_{threshold}\text{; and} \tag{9.16}$$

Constraints as shown in Eqs. (9.7)-(9.9), (9.11).

The probability that a "first-hit-*k*-consecutive-*r*-out-of-(*s*-*k*)" system successfully operates is obtained by taking the product of the probability that the first *k* launched one-shot units are successful (term A in Eq. (9.17)) and the probability that at least *r* out of the rest (*s*-*k*) one-shot units are consecutively launched (term B). Specifically:

$$R_{system}\left(t;v(\alpha)\forall\alpha\right)=\overbrace{\prod_{\alpha=1}^{k}R_{v(\alpha)}(t)}^{A}\cdot\overbrace{\left\{\sum_{\substack{\forall\underset{\sim}{\hat{r}}:1)\#\{\underset{\sim}{\hat{r}}\}=r\\ 2)\ r\ \text{consecutive elements in }\underset{\sim}{\hat{r}}}}^{s-k}\left[\prod_{\forall v(\alpha)\in\underset{\sim}{\hat{r}}}R_{v(\alpha)}(t)\prod_{\forall v(\alpha)\notin\underset{\sim}{\hat{r}}}\left(1-R_{v(\alpha)}(t)\right)\right]\right\}}^{B}$$

(9.17)

where $\underset{\sim}{\hat{r}}$ is the sequence of successfully launched one-shot units out of the (*s*-*k*) one-shot units and $\#\{\underset{\sim}{\hat{r}}\}$ is the number of elements (one-shot units) in $\{\underset{\sim}{\hat{r}}\}$. In term B, we consider all possible combinations of launched one-shot units' status (successful or not), where there need to be $\#\{\underset{\sim}{\hat{r}}\}$ ($\#\{\underset{\sim}{\hat{r}}\}=r,r+1,\dots,s-k$) successfully launched one-shot units with at least *r* one-shot units are consecutive.

As stated earlier, the probability that the system successfully operates is not deterministic. We respectively obtain the expected probability that the system successfully operates and the expected number of successfully launched one-shot units in Eq. (9.18) and (9.19):

$$R_{system}\left(t;v(\alpha)\forall\alpha\right)=\overbrace{\prod_{\alpha=1}^{k}\bar{R}_{v(\alpha)}(t)}^{A}\cdot\overbrace{\left\{\sum_{\substack{\forall\underset{\sim}{\hat{r}}:1)\#\{\underset{\sim}{\hat{r}}\}\geq r\\ 2)\ r\ \text{consecutive elements in }\underset{\sim}{\hat{r}}}}\left[\prod_{\forall v(\alpha)\in\underset{\sim}{\hat{r}}}\bar{R}_{v(\alpha)}(t)\prod_{\forall v(\alpha)\notin\underset{\sim}{\hat{r}}}\left(1-\bar{R}_{v(\alpha)}(t)\right)\right]\right\}}^{B}$$

(9.18)

$$E_{\text{system}}\left(\text{number of successfully launched one-shot units; } t;\nu\left(\alpha\right)\forall\alpha\right)$$

$$= \overbrace{\sum_{\alpha=1}^{k}\bar{R}_{\nu(\alpha)}\left(t\right)}^{A} + \overbrace{\left\{\sum_{\substack{\forall \hat{\underset{\sim}{r}}:1)\#\{\hat{\underset{\sim}{r}}\}\geq r\\ 2)\ r \text{ consecutive elements in } \hat{\underset{\sim}{r}}}} \#\left\{\hat{\underset{\sim}{r}}\right\}\cdot\left[\prod_{\forall\nu(\alpha)\in\hat{\underset{\sim}{r}}}\bar{R}_{\nu(\alpha)}\left(t\right)\prod_{\forall\nu(\alpha)\notin\hat{\underset{\sim}{r}}}\left(1-\bar{R}_{\nu(\alpha)}\left(t\right)\right)\right]\right\}}^{B} \tag{9.19}$$

In Eq. (9.15), the expected number of successfully launched one-shot units in the first k one-shot units (term A) and in the remaining (s-k) one-shot units (term B) are respectively calculated and summed. Similarly, referring to $Var\left(\prod_{\forall i}X_i\right) = \prod_{\forall i}\left(Var\left(X_i\right)+\left(E\left(X_i\right)\right)^2\right)-\prod_{\forall i}\left(E\left(X_i\right)\right)^2$ and $Var\left(\sum_{\forall i}X_i\right)=\sum_{\forall i}Var\left(X_i\right)$ if X_i are independent of each other, we obtain the variance of the probability that the system operates successfully in Eq. (9.20):

$$\begin{aligned} &Var\left(R_{\text{system}}\left(t;R_{\nu_\alpha}\left(t\right),\alpha=1,2,...,s\right)\right)\\ &=\left[Var\left(A\right)+\left(E\left(A\right)\right)^2\right]\left[Var\left(B\right)+\left(E\left(B\right)\right)^2\right]-\left(E\left(A\right)\right)^2\left(E\left(B\right)\right)^2 \end{aligned} \tag{9.20}$$

where

$$Var\left(A\right)=\prod_{\alpha=1}^{k}\left(Var\left(R_{\nu(\alpha)}\left(t\right)\right)+\left(\bar{R}_{\nu(\alpha)}\left(t\right)\right)^2\right)-\prod_{\alpha=1}^{k}\left(\bar{R}_{\nu(\alpha)}\left(t\right)\right)^2$$

$$E\left(A\right)=\prod_{\alpha=1}^{k}\bar{R}_{\nu(\alpha)}\left(t\right)$$

$$Var\left(B\right)=\sum_{\substack{\forall \hat{\underset{\sim}{r}}:1)\#\{\hat{\underset{\sim}{r}}\}\geq r\\ 2)\ r \text{ consecutive elements in } \hat{\underset{\sim}{r}}}}\left[\prod_{\alpha=k+1}^{s}\left(Var\left(R_{\nu(\alpha)}\left(t\right)\right)+\left(\bar{R}_{\nu(\alpha)}\left(t\right)\right)^2\right)-\prod_{\alpha=k+1}^{s}\left(\bar{R}_{\nu(\alpha)}\left(t\right)\right)^2\right]$$

$$E\left(B\right)=\sum_{\substack{\forall \hat{\underset{\sim}{r}}:1)\#\{\hat{\underset{\sim}{r}}\}\geq r\\ 2)\ r \text{ consecutive elements in } \hat{\underset{\sim}{r}}}}\left[\prod_{\forall\nu(\alpha)\in\hat{\underset{\sim}{r}}}\bar{R}_{\nu(\alpha)}\left(t\right)\prod_{\forall\nu(\alpha)\notin\hat{\underset{\sim}{r}}}\left(1-\bar{R}_{\nu(\alpha)}\left(t\right)\right)\right]$$

The optimal solution(s) can be obtained by either searching for all feasible solutions or using a nonlinear optimal algorithm. It is noted that some constraints are associated with the characteristics of the selected units while some constraints are associated with units' characteristics and launching sequence.

9.2.2 Time-to-k^{th}-failure of a Population with Nonhomogeneous One-shot Units

Current research on time-to-k^{th}-failure (T_k) of a population assumes that the population has homogeneous units. In this section, we investigate T_k of a population with nonhomogeneous one-shot units. Specifically, we investigate the system's expected time-to-k^{th}-failure since the last nondestructive test time t_m.

9.2.2.1 Probability Density Function (*pdf*) of System's Time-to-k^{th}-Failure

The characteristics of the one-shot units have an important impact on system's time-to-k^{th}-failure. We obtain the probability that T_k happens at time t ($t > t_m$) with one-shot units' characteristics known (i.e., the number of selected one-shot units with different characteristics are known). We first define the following:

$s_{i,\underset{\sim}{j}}^{m}$:the number of selected one-shot units with characteristics $i, \underset{\sim}{j}$ at time t_m ;

$$s_{i,\underset{\sim}{j}}^{m} \le E\left(n_{i,\underset{\sim}{j}}^{m}\right) \ \forall i, \underset{\sim}{j}$$

$s^{m}_{\underset{\sim}{i}',\underset{\sim}{j}'}$: the number of selected one-shot units with characteristics $i', \underset{\sim}{j}'$ at time t_m, specifically, the k^{th} failed one-shot units has the characteristics $i', \underset{\sim}{j}'$;

$k^{m}_{i,\underset{\sim}{j}}$: the number of selected one-shot units (with characteristics $i, \underset{\sim}{j}$ at time t_m) that fail between time t_m and t;

$k^{m}_{\underset{\sim}{i}',\underset{\sim}{j}'}$: the number of selected one-shot units (with characteristics $i', \underset{\sim}{j}'$ at time t_m) that fail between time t_m and t;

$r^{m}_{i,\underset{\sim}{j}}$: the number of selected one-shot units (with characteristics $i, \underset{\sim}{j}$ at time t_m) that survive until time t;

$r^{m}_{i',\underset{\sim}{j}'}$: the number of selected one-shot units (with characteristics $i', \underset{\sim}{j}'$ at time t_m) that survive until time t;

We obtain the *pdf* of T_k ($f_{T_k}(t)$) in Eq. (9.21):

$$f_{T_k}(t)=\overbrace{\sum_{i',\underset{\sim}{j}'}}^{A}\overbrace{\sum_{\forall i,\underset{\sim}{j}}}^{B}\;\underset{\left(\sum_{\forall i}\sum_{\forall \underset{\sim}{j}}k_{i,\underset{\sim}{j}}^{m}\right)=k-1}{}\frac{(k-1)!}{\prod_{\forall i}\prod_{\forall \underset{\sim}{j}}\left(\left(k_{i,\underset{\sim}{j}}^{m}\right)!\right)\left(\left(r_{i,\underset{\sim}{j}}^{m}\right)!\right)}$$

$$\cdot\left\{\overbrace{\left[s_{i',\underset{\sim}{j}'}^{m}\begin{pmatrix}s_{i',\underset{\sim}{j}'}^{m}-1\\ k_{i',\underset{\sim}{j}'}^{m}-1\end{pmatrix}\cdot\left(1-\frac{R_j\left(t-t_{j(j)}\right)}{R_j\left(t_m-t_{j(j)}\right)}\right)^{k_{i',\underset{\sim}{j}'}^{m}}\left(\frac{R_j\left(t-t_{j(j)}\right)}{R_j\left(t_m-t_{j(j)}\right)}\right)^{r_{i',\underset{\sim}{j}'}^{m}}\frac{\left[f_j\left(t-t_{j(j)}\right)R_j\left(t_m-t_{j(j)}\right)-f_j\left(t_m-t_{j(j)}\right)R_j\left(t-t_{j(j)}\right)\right]}{\left[R_j\left(t_m-t_{j(j)}\right)\right]^2}\right]}^{C}\right.$$

$$\left.\cdot\overbrace{\left[\prod_{\forall i \text{ except } i'}\prod_{\forall \underset{\sim}{j} \text{ except } \underset{\sim}{j}'}\left(\begin{pmatrix}s_{i,\underset{\sim}{j}}^{m}\\ k_{i,\underset{\sim}{j}}^{m}\end{pmatrix}\left(1-\frac{R_j\left(t-t_{j(j)}\right)}{R_j\left(t_m-t_{j(j)}\right)}\right)^{k_{i,\underset{\sim}{j}}^{m}}\left(\frac{R_j\left(t-t_{j(j)}\right)}{R_j\left(t_m-t_{j(j)}\right)}\right)^{r_{i,\underset{\sim}{j}}^{m}}\right)\right]}^{D}\right\} \qquad (9.21)$$

where $r_{i,\underset{\sim}{j}}^{m}=s_{i,\underset{\sim}{j}}^{m}-k_{i,\underset{\sim}{j}}^{m}$ and $r_{i',\underset{\sim}{j}'}^{m}=s_{i',\underset{\sim}{j}'}^{m}-k_{i',\underset{\sim}{j}'}^{m}-1$.

The term A in Eq.(9.21) considers and sums of all possibilities that the k^{th} failed one-shot unit has specific characteristics $i',\underset{\sim}{j}'$, where $i',\underset{\sim}{j}'$ could be any of the $i,\underset{\sim}{j}$. The term B considers the possible characteristics of previously failed (*k*-1) one-shot units, where the (*k*-1) failures occur between time t_m and t. The term C is the probability that out of the $s_{i',\underset{\sim}{j}'}^{m}$ one-shot units: $k_{i',\underset{\sim}{j}'}^{m}$ one-shot units fail between time t_m and t, one one-shot unit fails at time t and $r_{i',\underset{\sim}{j}'}^{m}$ one-shot units survive until time t. The term D calculates the probability that: out of the $s_{i,\underset{\sim}{j}}^{m}$ one-shot units (with all potential characteristics except $i',\underset{\sim}{j}'$), $k_{i,\underset{\sim}{j}}^{m}$ one-shot units fail between time t_m and t, and $r_{i,\underset{\sim}{j}}^{m}$ one-shot units survive until time t.

9.2.2.2 Expected Time-to-k^{th}-Failure of the System

Based on Eq. (9.21), the expected time-to-k^{th}-failure of the system ($E(T_k)$) can be obtained as $E(T_k)=t_m+\int_0^{\infty} f_k(t)\,dt$. Note that $E(T_k)$ can only be numerically obtained under most circumstances, except that when the unit's lifetime is exponentially distributed (whose failure rate is constant). Defining:

λ_x : the one-shot unit's failure rate after its x^{th} repair;

s_{m+y} : the number of one-shot units selected from the y^{th} batch that arrives after the m^{th} NDT;

w_{m+y} : the arrival time of the y^{th} batch that arrives after the m^{th} NDT.

We iteratively obtain $E(T_k)$ of the system:

***Step* 1:**

If $\displaystyle\sum_{\alpha=0}^{k-1} \frac{1}{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot \left(s_{i,\underset{\sim}{j}}^{m}\right) - \alpha \cdot \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot \left(s_{i,\underset{\sim}{j}}^{m}\right)}{\sum_{\forall i,\underset{\sim}{j}} \left(s_{i,\underset{\sim}{j}}^{m}\right)}} \le w_{m+1}$; where $\alpha = 0,\ldots,k-1$

Stop, $$E(T_k) = \sum_{\alpha=0}^{k-1} \frac{1}{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^{m} - \alpha \cdot \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^{m}}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^{m}}} .$$

Otherwise go to step 2;

***Step* 2:**

Find k_1^* such that:

$$w_{m+1} \in \left[\sum_{\alpha=0}^{k_1^*-1} \frac{1}{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m - \alpha \cdot \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m}}, \sum_{\alpha=0}^{k_1^*} \frac{1}{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m - \alpha \cdot \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m}} \right) \text{ and}$$

$$E\left(T_{k_1^*+1}\right) = \min \left\{ \sum_{\alpha=0}^{k_1^*} \frac{1}{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m - \alpha \cdot \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m}}, \frac{1}{\lambda_0 \cdot s_{m+1}} + w_{m+1} \right\}.$$

If

$$E\left(T_{k_1^*+1}\right) + \sum_{\alpha=0}^{k-k_1^*-1} \frac{1}{\left[\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1} - k_1^* \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m} - \alpha \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1}}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m + s_{m+1}} \right]} \le w_{m+2},$$

Stop,

$$E\left(T_k\right) = E\left(T_{k_1^*+1}\right) + \sum_{\alpha=0}^{k-k_1^*-1} \frac{1}{\left[\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1} - k_1^* \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m} - \alpha \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1}}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m + s_{m+1}} \right]};$$

Otherwise, go to step 3;

***Step* 3**

Find k_2^* such that:

$$w_{m+2} \in \left[\begin{array}{l} E\left(T_{k_1^*+1}\right) + \sum_{\alpha=0}^{k_2^*-k_1^*-1} \frac{1}{\left[\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1} - k_1^* \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m} - \alpha \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1}}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m + s_{m+1}} \right]}, \\ E\left(T_{k_1^*+1}\right) + \sum_{\alpha=0}^{k_2^*-k_1^*} \frac{1}{\left[\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1} - k_1^* \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m} - \alpha \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1}}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m + s_{m+1}} \right]} \end{array} \right)$$

and

$$E\left(T_{k^{*2}+1}\right) = \min \left\{ \begin{array}{l} E\left(T_{k_1^*+1}\right) + \sum_{\alpha=0}^{k_2^*-k_1^*} \frac{1}{\left[\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1} - k_1^* \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m} - \alpha \frac{\sum_{\forall i,\underset{\sim}{j}} \lambda_{\#\{\underset{\sim}{j}\}} \cdot s_{i,\underset{\sim}{j}}^m + \lambda_0 \cdot s_{m+1}}{\sum_{\forall i,\underset{\sim}{j}} s_{i,\underset{\sim}{j}}^m + s_{m+1}} \right]}, \\ \frac{1}{\lambda_0 \cdot s_{m+2}} + w_{m+2} \end{array} \right\}$$

If

$$E\left(T_{k^{*2}+1}\right)+\sum_{\alpha=0}^{k_2^*-1}\frac{1}{\left[\begin{array}{l}\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m+\lambda_0\cdot\left(s_{m+1}+s_{m+2}\right)-k_1^*\frac{\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}}s_{i,\underset{\sim}{j}}^m}\\ -\left(k_2^*-k_1^*\right)\frac{\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m+\lambda_0\cdot s_{m+1}}{\sum_{\forall i,\underset{\sim}{j}}s_{i,\underset{\sim}{j}}^m+s_{m+1}}-\alpha\frac{\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m+\lambda_0\cdot\left(s_{m+1}+s_{m+2}\right)}{\sum_{\forall i,\underset{\sim}{j}}s_{i,\underset{\sim}{j}}^m+s_{m+1}+s_{m+2}}\end{array}\right]}\le w_{m+2},$$

Stop,

$$E\left(T_k\right)=E\left(T_{k_2^*+1}\right)+\sum_{\alpha=0}^{k-k_2^*-1}\frac{1}{\left[\begin{array}{l}\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m+\lambda_0\cdot\left(s_{m+1}+s_{m+2}\right)-k_1^*\frac{\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m}{\sum_{\forall i,\underset{\sim}{j}}s_{i,\underset{\sim}{j}}^m}\\ -\left(k_2^*-k_1^*\right)\frac{\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m+\lambda_0\cdot s_{m+1}}{\sum_{\forall i,\underset{\sim}{j}}s_{i,\underset{\sim}{j}}^m+s_{m+1}}-\alpha\frac{\sum_{\forall i,\underset{\sim}{j}}\lambda_{\#\{\underset{\sim}{j}\}}\cdot s_{i,\underset{\sim}{j}}^m+\lambda_0\cdot\left(s_{m+1}+s_{m+2}\right)}{\sum_{\forall i,\underset{\sim}{j}}s_{i,\underset{\sim}{j}}^m+s_{m+1}+s_{m+2}}\end{array}\right]}$$

Otherwise, continue with step 4.

…

The above steps iterates until $E\left(T_k\right)$ is found. Note that the number of steps until stop is dependent on the batches' arrival times and sizes.

9.3 Bounds of System's Successful Probability

As the one-shot units within a batch are nonhomogeneous, the system's successful operational probability based on the optimal selection and launching sequence (which only shows the launched one-shot units' batch numbers) varies within a certain range depending on the specific unit selected from the batch. In this section, we investigate the lower and upper bounds of system's successful operational probability by considering the one-shot units' repair conditions (the number of repairs and when the repairs are conducted).

9.3.1 Bounds of System's Successful Probability of Operation under Constant Failure Rate

If the one-shot units have an exponentially distributed lifetime, the reliability of one-shot units with characteristics $i, \underset{\sim}{j}$ at time t is presented by Eq. (9.22). It can be observed the reliability of the one-shot units (with characteristics $i, \underset{\sim}{j}$) at time t is independent of i and $\underset{\sim}{j}$.

$$R_{i,\underset{\sim}{j}}(t) = \frac{R_{\#\{j\}}\left(t - t_{j(j)}\right)}{R_{\#\{j\}}\left(t_m - t_{j(j)}\right)} = \frac{\exp\left(-\lambda_{\#\{j\}}\left(t - t_{j(j)}\right)\right)}{\exp\left(-\lambda_{\#\{j\}}\left(t - t_{j(j)}\right)\right)} = \exp\left(-\lambda_{\#\{j\}}\left(t - t_m\right)\right) \tag{9.22}$$

Generally, one-shot unit's reliability at time t decreases when its number of repairs increases as presented in Eq. (9.23):

$$R_{i,\underset{\sim}{j}:\#\{\underset{\sim}{j}\}=0}(t) > R_{i,\underset{\sim}{j}:\#\{\underset{\sim}{j}\}=1}(t) > R_{i,\underset{\sim}{j}:\#\{\underset{\sim}{j}\}=2}(t) > \ldots > R_{i,\underset{\sim}{j}:\#\{\underset{\sim}{j}\}=m-z_i}(t); \forall i \tag{9.23}$$

Based on the optimal selection and launching sequence obtained in section 9.2, the system's successful operational probability is guaranteed to be within the range in Eq. (9.24):

$$R_{system}\left(t;v\left(\alpha\right)\forall\alpha\right)\in\begin{bmatrix}R_{system}\left(t;v\left(\alpha;\#\left\{\underset{\sim}{j}\right\}=m-z_{v(\alpha)}\right),\forall\alpha\right)\\R_{system}\left(t;v\left(\alpha;\#\left\{\underset{\sim}{j}\right\}=0\right),\forall\alpha\right)\end{bmatrix}\tag{9.24}$$

where $v\left(\alpha;\#\left\{\underset{\sim}{j}\right\}=x\right)$ represents the α^{th} launched one-shot unit that has been repaired x repairs.

The lower bound of the system's successful operational probability is achieved if selecting one-shot units with the largest number of repairs in the designated batches and the upper bound is achieved by selecting one-shot units with no prior repairs.

9.3.2 Bounds of System's Successful Probability under Time-dependent Failure Rate

If the one-shot units' lifetime follows Weibull distribution with its shape parameter $\gamma_{\#\{j\}}$ dependent on the one-shot units' number of repairs $\#\left\{j\right\}$, the reliability of the one-shot units at time t is dependent on: the time when its last failure is observed and its number of repairs. Specifically, the unit's reliability at time t is an increasing function of $j\left(j\right)$ and a decreasing function of the unit's number of prior repair (see Eq. (9.25)):

$$\begin{aligned}&\overbrace{\underbrace{R_{i,\underset{\sim}{j}:j(j)=j(1)=m}\left(t\right)}_{\text{fails (and repaired) once}}\geq\underbrace{R_{i,\underset{\sim}{j}:j(j)=j(2)=m}\left(t\right)}_{\text{fails (and repaired) twice}}\geq\ldots\geq\underbrace{R_{i,\underset{\sim}{j}:j(j)=j(m-z_i)=m}\left(t\right)}_{\text{fails (and repaired) } m-z_i \text{ times}}}^{\text{last failure observed at time } t_m}\\&\geq\overbrace{R_{i,\underset{\sim}{j}:j(j)=j(1)=m-1}\left(t\right)\geq R_{i,\underset{\sim}{j}:j(j)=j(2)=m-1}\left(t\right)\geq\ldots\geq R_{i,\underset{\sim}{j}:j(j)=j(m-z_i-1)=m-1}\left(t\right)}^{\text{last failure observed at time } t_{m-1}}\geq\ldots\\&\geq\overbrace{R_{i,\underset{\sim}{j}:j(1)=j(1)=1}\left(t\right)}^{\text{one failure observed at time } t_{z_i+1}}\geq\overbrace{R_{i,\underset{\sim}{j}:\#\left\{\underset{\sim}{j}\right\}=0}\left(t\right)}^{\text{no failure observed}}\end{aligned}\tag{9.25}$$

We randomly generate the batches' arrival times, the times to conduct the sequential NDTs, and the Weibull shape parameters $\gamma_{\#\{j\}}$. The generation runs for 10^3 iterations. Specifically,

$$w_1 = 0; \; t_1 = randbetween(0,50); \; w_2 = t_1 + randbetween(0,50);$$

$$t_2 = w_2 + randbetween(0,50); w_3 = t_2 + randbetween(0,50);$$

$$w_4 = w_3 + randbetween(0,50); t_3 = w_4 + randbetween(0,50); t = t_3 + randbetween(0,50);$$

$$\gamma_0 = randbetween(2,4); \gamma_{\#\{j\}} = r_{\#\{j\}-1} - 0.1 \times randbetween(1,3); \#\{j\} = 1,2,3;$$

We calculate the means of $R_{i,\underset{\sim}{j}}(t)$ of the 1000 iterations $\forall i, \underset{\sim}{j}$. We numerically validate that the one-shot unit's reliability is lower when more repairs are conducted and/or when the one-shot unit's last repair happens earlier. The reliabilities of the one-shot units in the four batches with different characteristics and the average reliabilities of the one-shot units in the four batches at time t are shown in Table 9.1.

Table 9.1 Reliabilities of the one-shot units (with different characteristics) at time t

Reliabilities of one-shot units in the 1st batch		Reliabilities of one-shot units in the 2nd batch		Reliabilities of one-shot units in the 3rd batch		Reliabilities of one-shot units in the 4th batch	
$R_{1,0}(t)$	0.9805	$R_{2,0}(t)$	0.9896	$R_{3,0}(t)$	0.9960	$R_{4,0}(t)$	0.9982
$\bar{R}_1(t)$	0.9813	$\bar{R}_2(t)$	0.9897	$\bar{R}_3(t)$	0.9960	$\bar{R}_4(t)$	0.9982

$R_{1,1}(t)$	0.9827	$R_{2,2}(t)$	0.9912	$R_{3,3}(t)$	0.9993	$R_{4,3}(t)$	0.9993
$R_{1,12}(t)$	0.9900	$R_{2,23}(t)$	0.9991				
$R_{1,2}(t)$	0.9912	$R_{2,3}(t)$	0.9993				
$R_{1,123}(t)$	0.0980						
$R_{1,13}(t)$ $R_{1,23}(t)$	0.9991						
$R_{1,3}(t)$	0.9993						

Therefore, Eq. (9.25) is numerically proven. Based on the optimal launching sequence obtained in section 9.2, the system's successful operational probability is guaranteed to be within the range presented in Eq. (9.26):

$$R_{system}\left(t;v(\alpha)\forall\alpha\right)\in\begin{bmatrix} R_{system}\left(t;v\left(\alpha;\#\left\{\underset{\sim}{j}\right\}=0\right),\forall\alpha\right); \\ R_{system}\left(t;v\left(\alpha;\underset{\sim}{j}:j(j)=j(1)=m=0\right),\forall\alpha\right)\end{bmatrix} \tag{9.26}$$

where $v\left(\alpha;\underset{\sim}{j}:j(j)=j(1)=m=0\right)$ represents the α^{th} launched one-shot unit which is selected from the designated batch with its last (also the 1$^{\text{st}}$) repair conducted at time t_m. The lower bound of the system's successful operational probability is achieved by selecting one-shot units with no prior repairs in the designated batches (i.e., one-shot units with the "longest" age) and the upper bound is achieved by selecting one-shot units with one failure (and repair) at time t_m (i.e., one-shot units with the shortest age ($t - t_m$)).

9.4 Simulation Model

In this section, we develop a simulation model to validate the optimality of the solution(s) obtained in section 9.2. The steps of the simulation model are introduced as follows:

Step 1: Find all sequences satisfying the constraint of Eq. (9.7).

Step 2: Find all sequences satisfying Eq. (9.8):

Step 2.1: For a specific sequence $\underset{\sim}{v}$ with $s_i \forall i$, randomly determine $s^m_{i,\underset{\sim}{j}} \forall \underset{\sim}{j}$ such that

$$\sum_{\forall \underset{\sim}{j}} s^m_{i,\underset{\sim}{j}} = s_i \text{ and } s^m_{i,\underset{\sim}{j}} \le E\left(n^m_{i,\underset{\sim}{j}}\right).$$

Step 2.2: Generate $s^m_{i,\underset{\sim}{j}}$ failure times $\forall i, \underset{\sim}{j}$ where the $s^m_{i,\underset{\sim}{j}}$ failure times follow the distribution with CDF $F_{\#\{\underset{\sim}{j}\}}(\cdot)$; note that the $s^m_{i,\underset{\sim}{j}} \forall i, \underset{\sim}{j} : j(\underset{\sim}{j}) < m$ failure times need to be greater than $t_m - t_{j(\underset{\sim}{j})}$;

Step 2.3: Add t_m to the $s^m_{i,\underset{\sim}{j}}$ failure times $\forall i, \underset{\sim}{j} : j(\underset{\sim}{j}) = m$; add $t_{j(\underset{\sim}{j})}$ to the $s^m_{i,\underset{\sim}{j},\underset{\sim}{l}_j}$ failure times $\forall i, \underset{\sim}{j} : j(\underset{\sim}{j}) < m$; sort all failure times in an increasing order and record the α^{th} failure time as T_α;

Step 2.4: Repeat steps 2.2-2.3 for 10^5 times and calculate the average of $T_\alpha \forall\alpha$ as

$E_{system}\left(T_\alpha\right)$; let $I_1 = 1$ if $E_{system}\left(T_{(s-k-r+1)}\right) > t$; otherwise $I_1 = 0$.

Step 2.5: Eliminate all sequences whose $I_1 = 0$.

Step 3: Simulate $\bar{R}_{system}\left(t; v(\alpha)\forall\alpha\right)$:

Step 3.1： For $v(\alpha)$ in a specific sequence $\underset{\sim}{v}$, generate $100 \times E\left(n^m_{v(\alpha),\underset{\sim}{j}}\right)$ failure times $\forall\underset{\sim}{j}$,

where $100 \times E\left(n^m_{v(\alpha),\underset{\sim}{j}}\right) \times R_{v(\alpha),\underset{\sim}{j}}(t)$ failure times with index 1 and

$100 \times E\left(n^m_{v(\alpha),\underset{\sim}{j}}\right) \times F_{v(\alpha),\underset{\sim}{j}}(t)$ failure times with index 0;

Step 3.2: randomly select one number out of $100 \times E\left(n^m_{v(\alpha),\underset{\sim}{j}}\right)$, record its index as $I_{v(\alpha)}$ ($I_{v(\alpha)}$ =1 or 0);

Step 3.3: Repeat steps 3.1-3.2 $\forall\alpha$;

Step 3.4: Record $I_{system} = 1$ if 1) $I_{v(\alpha)} = 1$ for $\alpha = 1,\ldots,k$ and 2) at least consecutive r

$I_{v(\alpha)}$=1 for α=$k+1,\ldots,s$; otherwise $I_{system} = 0$.

Step 3.5: Repeat steps 3.1-3.4 10^6 times, we have: $\bar{R}_{system}\left(t; v(\alpha)\forall\alpha\right) = \dfrac{\sum_{\forall 10^6} I_{system}}{10^6}$ and

$$E_{\text{system}}\left(\text{number of successfully launched one-shot units; } t; v(\alpha)\forall\alpha\right) = \frac{\sum_{\forall 10^6}\left(\sum_{\alpha=1}^{k} I_{v(\alpha)} + I \cdot \sum_{\alpha=k+1}^{s} I_{v(\alpha)}\right)}{10^6}$$

where $I = 1$ if 2) in step 3.4 is satisfied and $I = 0$ otherwise;

Step 3.6: the variance of the probability that the system operates successfully is obtained accordingly.

Step 4: Among the qualified sequences, select the sequence $\underset{\sim}{v}_{optimal}$ with the largest $\bar{R}_{system}\left(t;v\left(\alpha\right)\forall\alpha\right)$.

We examine if the $\underset{\sim}{v}_{optimal}$ obtained by the simulation model is the same as that obtained in section 9.2. The simulation model generally applies when the objective is exchanged with some of the constraints as discussed in section 9.2.

9.5 Numerical Illustrations

In this section, we numerically illustrate the optimal selecting and launching sequencing of one-shot units from the population. We also show the bounds of system's successful operational probability based on the optimal sequence(s). The following testing parameters and one-shot units' lifetime distributions are given as:

$N = 6, k = 2, r = 2$; $\bar{R}_{threshold} = 0.5$; $Var_{threshold} = 0.05$;

Testing parameters a): The one-shot units' lifetime follows Exponential distribution:

$F_j\left(t\right) = 1 - \exp\left(-\lambda_j t\right)$, where $\lambda_0 = 0.0001; \lambda_1 = 0.0003; \lambda_2 = 0.0005; \lambda_3 = 0.001;$

$w_1 = 0, w_2 = 1200, w_3 = 2500, w_4 = 3000; t_1 = 1000; t_2 = 2000; t_3 = 3500;$

$n_1 = 3, n_2 = 4, n_3 = 5, n_4 = 6$

Testing parameters b): The one-shot units' lifetime follows Weibull distribution:

$F_j(t) = 1 - \exp\left(-\left(\frac{t}{\theta}\right)^{\gamma_j}\right)$, where $\gamma_0 = 2.5; \gamma_1 = 2.8; \gamma_2 = 3; \gamma_3 = 3.2;$

$w_1 = 0, w_2 = 200, w_3 = 275, w_4 = 350; t_1 = 100; t_2 = 250; t_3 = 400.$

$n_1 = 4, n_2 = 5, n_3 = 5, n_4 = 6;$

The following three objectives are considered:

Objective 1 (O1): Maximize the expected probability of system's successful operation;

Objective 2 (O2): Maximize the expected number of successfully launched one-shot units;

Objective 3 (O3): Minimize the variance of the system's successful probability.

The expected number of one-shot units with different characteristics and its reliability at time t are calculated. We use the one-shot units in the 1st batch to illustrate the model, where $R_x(\cdot)$ is the reliability of the units with x repairs.

$$E\left(n_{1,123}^3\right) = n_1 \cdot F\left(t_1 - w_1\right) \cdot F_1\left(t_2 - t_1\right) \cdot F_2\left(t_3 - t_2\right); \qquad R_{1,123}(t) = R_4\left(t - t_3\right);$$

$$E\left(n_{1,12}^3\right) = n_1 \cdot F\left(t_1 - w_1\right) \cdot F_1\left(t_2 - t_1\right) \cdot R_2\left(t_3 - t_2\right); \qquad R_{1,12}(t) = \frac{R_3\left(t - t_2\right)}{R_3\left(t_3 - t_2\right)};$$

$$E\left(n_{1,13}^{3}\right)=n_1\cdot F\left(t_1-w_1\right)\cdot\left[F_1\left(t_3-t_1\right)-F_1\left(t_2-t_1\right)\right]; \qquad R_{1,13}\left(t\right)=R_2\left(t-t_3\right);$$

$$E\left(n_{1,23}^{3}\right)=n_1\cdot\left[F\left(t_2-w_1\right)-F\left(t_1-w_1\right)\right]\cdot F_1\left(t_3-t_2\right); \qquad R_{1,23}\left(t\right)=R_3\left(t-t_3\right);$$

$$E\left(n_{1,1}^{3}\right)=n_1\cdot F\left(t_1-w_1\right)\cdot R_1\left(t_3-t_1\right); \qquad R_{1,1}\left(t\right)=\frac{R_2\left(t-t_1\right)}{R_2\left(t_3-t_1\right)};$$

$$E\left(n_{1,2}^{3}\right)=n_1\cdot\left[F\left(t_2-w_1\right)-F\left(t_1-w_1\right)\right]\cdot R_1\left(t_3-t_2\right); \qquad R_{1,2}\left(t\right)=\frac{R_3\left(t-t_2\right)}{R_3\left(t_3-t_2\right)};$$

$$E\left(n_{1,3}^{3}\right)=n_1\cdot\left[F\left(t_3-w_1\right)-F\left(t_2-w_1\right)\right]; \qquad R_{1,3}\left(t\right)=R_1\left(t-t_3\right);$$

$$E\left(n_{1,0}^{3}\right)=n_1\cdot R\left(t_3-w_1\right); \qquad R_{1,0}\left(t\right)=\frac{R_1\left(t-w_1\right)}{R_1\left(t_3-w_1\right)}.$$

Accordingly, we have:

$$\bar{R}_1\left(t\right)=\frac{\sum_{\forall 1,\underset{\sim}{j}} E\left(n_{1,\underset{\sim}{j}}^{3}\right)\cdot R_{1,\underset{\sim}{j}}\left(t\right)}{n_1} \text{ and } Var\left[R_1\left(t\right)\right]=\frac{\left[R_{1,\underset{\sim}{j}}\left(t\right)-\bar{R}_1\left(t\right)\right]^2\cdot E\left(n_{1,\underset{\sim}{j}}^{3}\right)}{n_1}$$

We then solve for the optimal solutions for different objectives with different testing parameters. Generally, it is observed the batches' arrival times and sizes, units' failure rate, the times to conduct the NDTs, and the time to select and launch the units affect the optimal selection and launching sequence(s). Several illustrations are shown in Tables 9.2-9.4. The optimal sequences are presented in terms of the selected units' batch numbers. The sequence (3, 4, 1, 2, 2, 1) means that the 1st launched unit is selected from the 3rd batch, the 2nd launched unit is selected from the 4th batch, and so on.

Table 9.2 One-shot units' optimal selection and launching sequences with testing parameters *a*) at time $t = 4000$

Objective	Optimal Sequence(s)	Objective Function Value
O1	(3, 4, 1, 2, 2, 1) or (4, 3, 1, 2, 2, 1)	0.8784
O2	(1, 1, 2, 4, 3, 2) or (1, 1, 2, 3, 4, 2)	5.5616
O3	(1, 1, 4, 2, 2, 3)	0.0112

Table 9.3 One-shot units' optimal selection and launching sequences with testing parameters *a*) at time $t = 6500$

Objective	Optimal Sequence	Objective Function Value
O1	(3, 4, 1, 2, 2, 1) or (4, 3, 1, 2, 2, 1)	0.6391
O2	(1, 1, 2, 3, 4, 2) or (1, 1, 2, 4, 3, 2)	4.7369
O3	(3, 4, 2, 1, 1, 2) or (4, 3, 2, 1, 1, 2)	0.0281

Table 9.4 One-shot units' optimal selection and launching sequences with testing parameters *b*) at time $t = 500$

Objective	Optimal Sequence	Objective Function Value

O1	(2, 3, 1, 1, 2, 1) or (3, 2, 1, 1, 2, 1)	0.6941
O2	(1, 1, 1, 2, 3, 2) or (1, 1, 2, 3, 2, 1)	4.8623
O3	(2, 3, 1, 1, 1, 2) or (3, 2, 1, 1, 1, 2)	0.0357

With testing parameter a) at time $t = 4000$, the system achieves its highest successful probability at an average value of 0.8747, with sequence (3, 4, 1, 2, 2, 1) or (4, 3, 1, 2, 2, 1). Specifically, the average reliabilities of the one-shot units in the four batches at time $t = 4000$ are respectively:

$\bar{R}_1(t = 4000) = 0.9156$; $\bar{R}_2(t = 4000) = 0.9303$;

$\bar{R}_3(t = 4000) = 0.9426$; $\bar{R}_4(t = 4000) = 0.9468$

We observe that: $R_{i,0}(t = 4000) = 0.9512 > \bar{R}_i(t = 4000)\ \forall i$. Therefore, the upper bound of the system's probability of successful operation is 0.8986 by selecting the one-shot units that have not experienced repairs in the designated batch. Similarly, the lower bound of the system's probability of successful operation is 0.6041 when selecting the one-shot units with the highest numbers of repairs in the designated batches (Eq. (9.27)):

$R_{1,123}(t = 4000) = 0.6065$; $R_{2,23}(t = 4000) = 0.7788$;

$$R_{3,3}(t = 4000) = 0.8607;\ R_{4,3}(t = 4000) = 0.8607. \tag{9.27}$$

With testing parameters *b*) at time $t = 500$, the system achieves the highest successful probability at an average value of 0.6941 with launching sequence (2, 3, 1, 1, 2, 1) or (3, 2, 1, 1, 2, 1). In Eq. (9.28), we calculate the reliabilities of the one-shot units (that fail once at time t_3) in the four batches at time $t = 500$:

$$R_{1,3}\left(t=500\right)=R_{2,3}\left(t=500\right)=R_{3,3}\left(t=500\right)=R_{4,3}\left(t=500\right)=0.9890; \tag{9.28}$$

We also obtain the reliabilities of one-shot units that have not experienced repairs in the four batches in Eq. (9.29):

$$R_{1,0}\left(t=500\right)=0.6521; R_{2,0}\left(t=500\right)=0.8372;$$

$$R_{3,0}\left(t=500\right)=0.9007; R_{4,0}\left(t=500\right)=0.9549; \tag{9.29}$$

Selecting and launching the one-shot units in sequence (2, 3, 1, 1, 2, 1) or (3, 2, 1, 1, 2, 1), the system's probability of successful operation is guaranteed to be within the range of $\left[0.6071, 0.9778\right]$, where the upper bound is achieved if the one-shot units with one failure at time t_3 (Eq. (9.28)) are selected from the designated batches and the lower bound is achieved if the one-shot units that have not experienced repairs are selected (Eq. (9.29)).

9.6 Conclusions

In this chapter, we investigate the optimal operational use of the one-shot units at arbitrary time. We determine the characteristics of the one-shot units selected from the stored population and the launching sequence to achieve the one-shot units' optimal successful operation. The reliability metrics of the launched one-shot units and a variety of objectives are considered. The bounds of the launched one-shot units' probability of successful operation based on the optimal launching sequence are provided. We also develop a simulation model to validate the optimality of the solution(s) obtained by the formulated optimization problem.

CHAPTER 10

CONCLUSIONS

In this dissertation, we investigate several interesting and challenging problems regarding the reliability metrics of systems with mixtures of nonhomogeneous one-shot units. Batches of one-shot units arrive into storage at arbitrary time and a sequence of reliability tests are conducted randomly to assess the reliability metrics of the units. Defining the stored one-shot units as a system, we investigate the reliability metrics of the system under a variety of failure modes such as failures with no indicator, thermal fatigue failures, and competing failure modes. We also investigate the optimal design of sequential reliability testing plans by considering different types of reliability tests. The one-shot units are retrieved from storage and launched at arbitrary time during its storage, we also investigate the optimal operational use of the one-shot units in this dissertation. In the following, we briefly provide conclusions of this dissertation.

In chapter 3, we propose effective approaches to assess the reliability metrics of a generalized "*k*-out-of-*n*: F" system with nonhomogeneous one-shot units under different scenarios, taking into account the units' characteristics during its storage period. We show that aging has an important effect on the system reliability metrics. We also show that the reliability metrics of such systems can be obtained by either testing the entire population or testing selected samples. We propose that the system reliability metrics can be estimated accurately and effectively even when the batch size is large or when the reliability tests are performed extensively over the time horizon, by developing three computationally effective alternatives which either mask some of the system conditions or yield negligible estimation bias. An extensive simulation model validates that the proposed approaches estimate system reliability metrics accurately.

In chapter 4, we propose stochastic approaches to assess the system reliability metrics when the batches' arrival time and sizes, and the time to conduct the reliability tests can be described by specific probabilistic distributions. We show that the system reliability metrics obtained by the stochastic approaches can be approximately predicted by those obtained by the approaches proposed in chapter 3, by either testing the entire population or testing a sample, and vice versa.

In chapter 5, we propose accurate and effective approaches that model the system reliability metrics subjecting to thermal fatigue by utilizing the GBS distribution, where the system has a mixture of nonhomogeneous one-shot units that are subjected to sequential ATCTs. We demonstrate that the GBS distribution, though developed for mechanical fatigue failure, is suitable for modelling thermal fatigue data. Moreover, compared with the commonly used CM model, we show that the GBS distribution provides additional reliability metrics and assesses the reliability at the system level which could not be accomplished by the CM model. The proposed models' flexibility and robustness in estimating/predicting the system reliability metrics are validated by a simulation model.

In chapter 6, we propose efficient and effective approaches that model the reliability metrics of systems subject to competing failure modes, where the system has a mixture of nonhomogeneous one-shot units subject to sequential TCTs. We develop and utilize statistics-based models and physics-statistics-based models to characterize the failure/degradation modes. We validate that competing failure modes result in worse system reliability metrics than single failure mode. Moreover, the proposed models' accuracy in estimating the system reliability metrics is validated by a simulation model.

In chapter 7, we study the unit's reliability metrics by developing a statistics-physics-based model and show the temperature and humidity have significant effect on the system reliability metrics. We then propose the optimal design of sequential accelerated NDTs on the system level, taking system's/sample's reliability metrics under different testing scenarios into account. We numerically prove that: 1) a randomly selected sample represents the population's characteristics in the long run when designing the optimal testing plans; and 2) the sample size has negligible consequence on the design of optimal testing plans. We also show that the accelerated NDTs have insignificant effect on the system reliability metrics while reducing the test durations.

In chapter 8, we design optimal sequential hybrid reliability testing plans during the one-shot units' storage life under different scenarios, by performing both NDT and DT in each reliability test. We therefore show that the number of available units after the tests can be maximized by designing the optimal testing plan. We also prove that the accuracy of the system reliability metrics estimation is improved based on the results of the sequential hybrid tests, such that DT is no longer needed after a number of hybrid tests.

In chapter 9, we achieve the optimal operational use of the one-shot units at arbitrary time. The characteristics of the one-shot units when launched are obtained. We optimize the launched one-shot units' optimal successful operation by determining its characteristics and launching sequence. We show that the probability that the launched one-shot units achieves a successful operation is guaranteed to be within a certain range based on the proposed optimal launching sequence, by providing the bounds of the estimates of the probability.

APPENDIX

The probability that the unit has specific characteristics are obtained in Eqs. (A1) and (A2):

$$P\left(k_{i,j}^{m}\right)=\left\{\begin{array}{l} R_W\left(j(1)\right)\cdot\left\{\begin{array}{l}\left[\int_{\Delta C_{z_i+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(1)}=\Delta C_R}^{\infty} P\begin{pmatrix}\Delta C\left(j(1)\right)=\Delta C_{j(1)},\\ \ldots,\\ \Delta C\left(z_i+1\right)=\Delta C_{z_i+1}\end{pmatrix} d\Delta C_{z_i+1}\ldots d\Delta C_{j(1)}\right]\\ \left[\int_{\Delta R_{z_i+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(1)}=\Delta R_{j(1)-1}}^{\infty} P\begin{pmatrix}\Delta R\left(j(1)\right)=\Delta R_{j(1)},\\ \ldots,\\ \Delta R\left(z_i+1\right)=\Delta R_{z_i+1}\end{pmatrix} d\Delta R_{z_i+1}\ldots d\Delta R_{j(1)}\right]\\ +\\ \left[\int_{\Delta C_{z_i+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(1)}=\Delta C_{j(1)-1}}^{\Delta C_R} P\begin{pmatrix}\Delta C\left(j(1)\right)=\Delta C_{j(1)},\\ \ldots,\\ \Delta C\left(z_i+1\right)=\Delta C_{z_i+1}\end{pmatrix} d\Delta C_{z_i+1}\ldots d\Delta C_{j(1)}\right]\\ \left[\int_{\Delta R_{z_i+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(1)}=\Delta R_R}^{\infty} P\begin{pmatrix}\Delta R\left(j(1)\right)=\Delta R_{j(1)},\\ \ldots,\\ \Delta R\left(z_i+1\right)=\Delta R_{z_i+1}\end{pmatrix} d\Delta R_{z_i+1}..d\Delta R_{j(1)}\right]\end{array}\right\}\\ +\left[F_W\left(t_{j(1)}\right)-F_W\left(t_{j(1)-1}\right)\right]P\left(\Delta C\left(j(1)-1\right)<\Delta C_R\right)P\left(\Delta R\left(j(1)-1\right)<\Delta R_R\right)\end{array}\right\}$$

$$
\cdot\prod_{\alpha=2}^{j}\left\{\begin{aligned}
&R_W\left(t_{j(\alpha)}-t_{j(\alpha-1)}\right)\left\{\begin{aligned}
&\left[\int_{\Delta C_{j(\alpha-1)+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(\alpha)}=\Delta C_R}^{\infty}P\begin{pmatrix}\Delta C\left(j(\alpha)\right)\\=\Delta C_{j(\alpha)},\\\ldots,\\\Delta C\left(j(\alpha-1)+1\right)\\=\Delta C_{j(\alpha-1)+1}\end{pmatrix}d\Delta C_{j(\alpha-1)+1}\ldots d\Delta C_{j(\alpha)}\right]\\
&\left[\int_{\Delta R_{j(\alpha-1)+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(\alpha)}=\Delta R_{j(\alpha)-1}}^{\infty}P\begin{pmatrix}\Delta R\left(j(\alpha)\right)\\=\Delta R_{j(\alpha)},\\\ldots,\\\Delta R\left(j(\alpha-1)+1\right)\\=\Delta R_{j(\alpha-1)+1}\end{pmatrix}d\Delta R_{j(\alpha-1)+1}\ldots d\Delta R_{j(\alpha)}\right]\\
&+\\
&\left[\int_{\Delta C_{j(\alpha-1)+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(\alpha)}=\Delta C_{j(\alpha)-1}}^{\Delta C_R}P\begin{pmatrix}\Delta C\left(j(\alpha)\right)\\=\Delta C_{j(\alpha)},\\\ldots,\\\Delta C\left(j(\alpha-1)+1\right)\\=\Delta C_{j(\alpha-1)+1}\end{pmatrix}d\Delta C_{j(\alpha-1)+1}\ldots d\Delta C_{j(\alpha)}\right]\\
&\left[\int_{\Delta R_{j(\alpha-1)+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(\alpha)}=\Delta R_R}^{\infty}P\begin{pmatrix}\Delta R\left(j(\alpha)\right)\\=\Delta R_{j(\alpha)},\\\ldots,\\\Delta R\left(j(\alpha-1)+1\right)\\=\Delta R_{j(\alpha-1)+1}\end{pmatrix}d\Delta R_{j(\alpha-1)+1}\ldots d\Delta R_{j(\alpha)}\right]
\end{aligned}\right\}\\
&+\left[F_W\left(t_{j(\alpha)}-t_{j(\alpha-1)}\right)-F_W\left(t_{j(\alpha)-1}-t_{j(\alpha-1)}\right)\right]\\
&\left[P\left(\Delta C\left(\left(j(\alpha)-1\right)-j(\alpha-1)\right)<\Delta C_R\right)P\left(\Delta R\left(\left(j(\alpha)-1\right)-j(\alpha-1)\right)<\Delta R_R\right)\right]
\end{aligned}\right\}
$$

$$
\cdot\left\{\begin{aligned}
&R_W\left(t_m-t_{j(j)}\right)\left\{\begin{aligned}
&\left[\int_{\Delta C_{j(j)+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_m=\Delta C_F}^{\infty}P\begin{pmatrix}\Delta C(m)=\Delta C_m,\\\ldots,\\\Delta C\left(j(j)+1\right)=\Delta C_{j(j)+1}\end{pmatrix}d\Delta C_{j(j)+1}\ldots d\Delta C_m\right]\\
&\left[\int_{\Delta R_{j(j)+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_m=\Delta R_{m-1}}^{\infty}P\begin{pmatrix}\Delta R(m)=\Delta R_m,\\\ldots,\\\Delta R\left(j(j)+1\right)=\Delta R_{j(j)+1}\end{pmatrix}d\Delta R_{j(j)+1}\ldots d\Delta R_m\right]\\
&+\\
&\left[\int_{\Delta C_{j(j)+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_m=\Delta C_{m-1}}^{\Delta C_F}P\begin{pmatrix}\Delta C(m)=\Delta C_m,\\\ldots,\\\Delta C\left(j(j)+1\right)=\Delta C_{j(j)+1}\end{pmatrix}d\Delta C_{j(j)+1}\ldots d\Delta C_m\right]\\
&\left[\int_{\Delta R_{j(j)+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_m=\Delta R_F}^{\infty}P\begin{pmatrix}\Delta R(m)=\Delta R_m,\\\ldots,\\\Delta R\left(j(j)+1\right)=\Delta R_{j(j)+1}\end{pmatrix}d\Delta R_{j(j)+1}\ldots d\Delta R_m\right]
\end{aligned}\right\}\\
&+\left[F_W\left(t_m-t_{j(j)}\right)-F_W\left(t_{m-1}-t_{j(j)}\right)\right]P\left(\Delta C\left(t_{m-1}-t_{j(j)}\right)<\Delta C_R\right)P\left(\Delta R\left(t_{m-1}-t_{j(j)}\right)<\Delta R_R\right)
\end{aligned}\right\}
$$

(A1)

$$
P\left(r^{m}_{i,\underset{\sim}{j}}\right)=\left\{\begin{array}{l}
R_W\left(j(1)\right)\cdot\left\{\begin{array}{l}
\left[\int_{\Delta C_{z_i+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(1)}=\Delta C_R}^{\infty}P\begin{pmatrix}\Delta C\left(j(1)\right)=\Delta C_{j(1)},\ldots,\\ \Delta C\left(z_i+1\right)=\Delta C_{z_i+1}\end{pmatrix}d\Delta C_{z_i+1}\ldots d\Delta C_{j(1)}\right]\\
\left[\int_{\Delta R_{z_i+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(1)}=\Delta R_{j(1)-1}}^{\infty}P\begin{pmatrix}\Delta R\left(j(1)\right)=\Delta R_{j(1)},\ldots,\\ \Delta R\left(z_i+1\right)=\Delta R_{z_i+1}\end{pmatrix}d\Delta R_{z_i+1}\ldots d\Delta R_{j(1)}\right]\\
+\\
\left[\int_{\Delta C_{z_i+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(1)}=\Delta C_{j(1)-1}}^{\Delta C_R}P\begin{pmatrix}\Delta C\left(j(1)\right)=\Delta C_{j(1)},\ldots,\\ \Delta C\left(z_i+1\right)=\Delta C_{z_i+1}\end{pmatrix}d\Delta C_{z_i+1}\ldots d\Delta C_{j(1)}\right]\\
\left[\int_{\Delta R_{z_i+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(1)}=\Delta R_R}^{\infty}P\begin{pmatrix}\Delta R\left(j(1)\right)=\Delta R_{j(1)},\ldots,\\ \Delta R\left(z_i+1\right)=\Delta R_{z_i+1}\end{pmatrix}d\Delta R_{z_i+1}..d\Delta R_{j(1)}\right]
\end{array}\right\}\\
+\left[F_W\left(t_{j(1)}\right)-F_W\left(t_{j(1)-1}\right)\right]P\left(\Delta C\left(j(1)-1\right)<\Delta C_R\right)P\left(\Delta R\left(j(1)-1\right)<\Delta R_R\right)
\end{array}\right\}
$$

$$
\cdot\prod_{\alpha=2}^{j}\left\{\begin{array}{l}
R_W\left(t_{j(\alpha)}-t_{j(\alpha-1)}\right)\left\{\begin{array}{l}
\left[\int_{\Delta C_{j(\alpha-1)+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(\alpha)}=\Delta C_R}^{\infty}P\begin{pmatrix}\Delta C\left(j(\alpha)\right)\\ =\Delta C_{j(\alpha)},\\ \ldots,\\ \Delta C\left(j(\alpha-1)+1\right)\\ =\Delta C_{j(\alpha-1)+1}\end{pmatrix}d\Delta C_{j(\alpha-1)+1}\ldots d\Delta C_{j(\alpha)}\right]\\
\left[\int_{\Delta R_{j(\alpha-1)+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(\alpha)}=\Delta R_{j(\alpha)-1}}^{\infty}P\begin{pmatrix}\Delta R\left(j(\alpha)\right)\\ =\Delta R_{j(\alpha)},\\ \ldots,\\ \Delta R\left(j(\alpha-1)+1\right)\\ =\Delta R_{j(\alpha-1)+1}\end{pmatrix}d\Delta R_{j(\alpha-1)+1}\ldots d\Delta R_{j(\alpha)}\right]\\
+\\
\left[\int_{\Delta C_{j(\alpha-1)+1}=0}^{\Delta C_R}\cdots\int_{\Delta C_{j(\alpha)}=\Delta C_{j(\alpha)-1}}^{\Delta C_R}P\begin{pmatrix}\Delta C\left(j(\alpha)\right)\\ =\Delta C_{j(\alpha)},\\ \ldots,\\ \Delta C\left(j(\alpha-1)+1\right)\\ =\Delta C_{j(\alpha-1)+1}\end{pmatrix}d\Delta C_{j(\alpha-1)+1}\ldots d\Delta C_{j(\alpha)}\right]\\
\left[\int_{\Delta R_{j(\alpha-1)+1}=0}^{\Delta R_R}\cdots\int_{\Delta R_{j(\alpha)}=\Delta R_R}^{\infty}P\begin{pmatrix}\Delta R\left(j(\alpha)\right)\\ =\Delta R_{j(\alpha)},\\ \ldots,\\ \Delta R\left(j(\alpha-1)+1\right)\\ =\Delta R_{j(\alpha-1)+1}\end{pmatrix}d\Delta R_{j(\alpha-1)+1}\ldots d\Delta R_{j(\alpha)}\right]
\end{array}\right\}\\
+\left[F_W\left(t_{j(\alpha)}-t_{j(\alpha-1)}\right)-F_W\left(t_{j(\alpha)-1}-t_{j(\alpha-1)}\right)\right]\\
\left[P\left(\Delta C\left(\left(j(\alpha)-1\right)-j(\alpha-1)\right)<\Delta C_R\right)P\left(\Delta R\left(\left(j(\alpha)-1\right)-j(\alpha-1)\right)<\Delta R_R\right)\right]
\end{array}\right\}
$$

$$
\cdot\left\{ R_W\left(t_m - t_{j(j)}\right)\left\{ \begin{array}{l} \left[\int_{\Delta C_{j(j)+1}=0}^{\Delta C_R}\int_{\Delta C_m=\Delta C_{m-1}}^{\Delta C_F} P\begin{pmatrix} \Delta C(m)=\Delta C_m, \\ ..., \\ \Delta C\left(j(j)+1\right)=\Delta C_{j(j)+1} \end{pmatrix} d\Delta C_{j(j)+1}...d\Delta C_m \right] - \\ \left[\int_{\Delta R_{j(j)+1}=0}^{\Delta R_R}...\int_{\Delta R_m=\Delta R_{m-1}}^{\Delta R_F} P\begin{pmatrix} \Delta R(m)=\Delta R_m, \\ ..., \\ \Delta R\left(j(j)+1\right)=\Delta R_{j(j)+1} \end{pmatrix} d\Delta R_{j(j)+1}...d\Delta R_m \right] \end{array} \right\} \right\}
$$

(A2)

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