RISK FILTERING AND RISK-AVERSE CONTROL OF PARTIALLY OBSERVABLE MARKOV JUMP PROCESSES

by

RUOFAN YAN

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ABSTRACT OF THE DISSERTATION

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Dissertation Director: Paul Feehan and Andrzej Ruszczyński

In this dissertation, we provide a theory of time-consistent dynamic risk measures for partially observable Markov jump processes in continuous time. By introducing risk filters, which are new two-stage risk measures, we show that the risk measure of a partially observable system can be represented as a risk measure of a fully observable system that is defined by a g-evaluation. The innovation process of the original system is the Brownian Motion driving the new fully observable system. Furthermore, we introduce a risk-averse control problem for the partially observable system and we derive a risk-averse dynamic programming equation.

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Dedication

To my parents...

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Chapter 1

Introduction

The problem of evaluating a partially observable problem in continuous time consists of two components. One component is the estimation of the unobservable part. Roughly speaking, this means that we want to give the best estimate based on our currently available information. A natural way of achieving this is to estimate the unobservable process via the conditional expectation of the observable process, since conditional expectation gives an unbiased estimator with the smallest mean squared error. The other question is how an estimation of our target function is evaluated. The classical way is via taking expectation. In this dissertation, we focus on the Wonham filter, a specific case of a partially observable system, and we will derive our risk framework regarding this problem such that it is a generalization of the classical method for both issues mentioned above.

The estimation of the unobservable process is actually a filtering problem. There are several well-known results for the optimal filtering problems, for example the Kalman–Bucy [45] and Wonham filters [63]. The optimal filters in both cases are characterized by stochastic differential equations (SDE). A more general result was obtained by Zakai [66] and Kushner [35, 36], in which the filter was represented by a stochastic partial differential equation (SPDE). The Kusher equation is a stochastic nonlinear partial differential equation, the investigation of which would be difficult. In contrast, the Zakai equation is a stochastic linear partial differential equation, and it describes a non-normalized filtering density, which corresponds one-to-one to the filtering density. The Zakai equation is pretty important for both theory and applications [6, 1, 11, 29].

With the optimal filtering, it is natural to define the corresponding optimal control problem. In the case of linear systems, Wonham [64] introduced the so-called separation theorem. In the more general settings, some crucial results were obtained by Beneš and Karatzas [60] and

Fleming and Pardoux [61]. However, the results about the optimal control for partially observable problems are mainly concerned with the expected cost functionals; our work addresses the problem of constructing risk measures and risk-averse control of such systems, which appears to be unexplored.

In the financial area, everyone, including traders and investors, are exposed to various kinds of risks. Therefore, finding a way to quantify the riskiness of a position is very helpful for decision making. For this reason, the idea of risk measures was proposed in the literature. Artzner *et al.* [39] first introduced the concept of a coherent risk measure. Föllmer and Schied [20] and independently Frittelli and Rosazza Gianin [26] broadened the class of coherent risk measures, defining convex risk measures. Then, Ruszczyński and Shapiro [53] considered optimization problems involving convex risk measures.

Further developments in this area proceeded to dynamic settings. A dynamic risk measure is a very natural and important concept. For example, in financial markets, risky payoffs are usually spread over different dates, and the risk should be measured at each time based on the updated information. Even if the payoff occurs at one or few times, our perpective on its risk changes in time. In the discrete time setting, Bion-Nadal [10], Detlefsen and Scandolo [15] and Ruszczyński and Shapiro [51] considered conditional risk measures. Coherent dynamic risk measures were explored by Delbaen [14] and Artzner *et al.* [40]. Then, Cheridito *et al.* [41, 13], Riedel [48], Frittelli and Rosazza Gianin [27] studied convex dynamic risk measures. Furthermore, as for the control with a risk averse problem in the discrete time, Ruszczyński *et al.* [50, 19, 12] developed Markov risk measures and risk averse dynamic programming equations as well as computational methods.

In the continuous time setting, Coquet, Hu, Mémin, and Peng [17] discovered that a time-consistent dynamic risk measure for a Brownian filtration can be represented by the solution of a corresponding backward stochastic differential equation (BSDE). Important works along these lines include Peng [42], Barrieu and El Karoui [8, 7] and Delbaen, Peng and Rosazza Gianin [18]. As for control problems, while numerous books [31, 24, 43] discuss traditional stochastic control, where the objective functional is defined as an expectation, the area of risk-averse case is largely unerexplored. Ruszczyński and Yao [3, 2] considered the risk-averse case with a coherent risk measure given by a g-evaluation. Therefore, by the result of [17], they changed the problem into a decoupled forward–backward stochastic differential equation system (FBSDE), and obtained the related risk-averse dynamic programming equation, the corresponding HJB equation, as well as numerical approximation results.

The contribution of this paper is the study of continuous time risk-averse control problems with a partially observable state. In our setting, the unobservable process is a Markov jump process while the observable one is given by a diffusion process. By introducing a special risk structure called the risk filter, we transform the partially observable system into a fully observable system of stochastic differential equations. Then we explore the properties of risk filters to derive an equivalent controlled forward–backward system. Finally, we derive the dynamic programming equation in the case of piecewise-constant controls.

The thesis is organized as follows: In Chapter 2, we introduce the background of risk measures including the static risk measure as well as dynamic risk measures in both discrete time and continuous time. In Chapter 3, we cover the necessary background on optimal filtering. We propose the general equation for optimal nonlinear filter and its applications: the Kalman-Bucy filter and Wonham filter. We then extend to the equation of interpolation and extrapolation of filter problem. In Chapter 4, we start to introduce our framework of risk filtering. We first discuss the case with only terminal cost function and then generalize to the case with running cost. In Chapter 5, we further introduce control into our model and propose the risk averse control problem with risk filtering setting. In Chapter 6, we develop dynamic programming w.r.t. to the control problem and also the optimal control problem. Finally in Chapter 7, we sumarize this thesis.

Chapter 2

Risk Measures

When people are making decisions under uncertainty, they will always have a measure as a criterion. One of the most widely used measures is the expected value of some cost or profit. In probability theory, the expected value, intuitively, is the long-run average value of repetitions of the experiment it represents. So the advantage of using the expected value is that it captures the average behavior of the uncertain outcome and also it is convenient and easy to use. However, in some cases, the simple average is not a desired a result for your decision making. One of the example is the following, the famous ”St. Petersburg paradox”.

Example 2.0.1. *(St. Petersburg Paradox) A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake starts at 2 dollars and is doubled every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot. Thus the player wins 2 dollars if tails appears on the first toss, 4 dollars if heads appears on the first toss and tails on the second, 8 dollars if heads appears on the first two tosses and tails on the third, and so on. Mathematically, the player wins 2k dollars, where k equals number of tosses (k must be a whole number and greater than zero). What would be a fair price to pay the casino for entering the game?*

As for the above problem, if you use the expected value as your criterion for decision making, you will be in big trouble. The expected return is the following:

$$E = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \cdots = \infty$$

which means, we would like to pay an infinite amount for this game. But is it a good decision? The answer is no. The reason is simple, you only have 1.6% probability of winning \$64 or more, which means, with more than 98.4% chance, your win is no more than \$32. Therefore, definitely you are not willing to pay much for this game. Then, we should have some other criterion for our decision making. One possible approach is the famous Expected Utility Theory. A common utility function to use is $\log(x)$ (we choose the base to be 10). It is a function of the gamblers total wealth w, and the concept of diminishing marginal utility of money is built into it. The expected utility hypothesis posits that a utility function exists whose expected net change from accepting the gamble is a good criterion for real people's behavior. For each possible event, the change in utility $\log(\textit{wealth after the event}) - \log(\textit{wealth before the event})$ will be weighted by the probability of that event occurring. The intuition with this formula is easy to understand: For example, a poor person whose initial wealth is \$10 and who wins \$10,000 has a different perspective from a rich person whose initial wealth is \$100,000 and who wins the same amount of money. Actually, the importance of winning this amount for the poor person is much higher than for the rich person. But in the measure of expected value, both gain of \$10,000, while in contrast, in the expected utility theory, the utility for the poor person is

$$\log(10010) - \log(10) = 3.00$$

while the utility for the rich person is

$$\log(110000) - \log(100000) = 1.04.$$

The result is consistent with our intuition.

Now let c be the cost charged to enter the game. The expected incremental utility of the lottery now converges to a finite value:

$$c = \sum_{k=1}^{\infty} \frac{1}{2^k}[\ln(2^k + w) - \ln(w)] < \infty$$

which is a fair price for the player with total wealth w.

Remark 2.0.2. *This paradox takes its name from its resolution by Daniel Bernoulli [9], however, the problem was invented by Daniel's cousin, Nicolas Bernoulli, who first stated it in a letter to Pierre Raymond de Montmort in 1713.*

2.1 The Concept of a Static Risk Measures

Risk Measures were introduced as means to quantify the riskness of financial positions and provide a criterion to determine whether the risk is acceptable or not.

2.1.1 Value at Risk and related risk measures

The expected utility function as mentioned in the above example is a widely used risk measure, especially in economic theory. As for the area of finance, there is another very popular static risk measure, called "Value at Risk"(*VaR*).

Value at risk(*VaR*) (Duffle and Pan [16]) is a measure of the risk of investments. It estimates how much a set of investments might lose, given normal market conditions, in a set time period such as a day. *VaR* is typically used by firms and regulators in the financial industry to gauge the amount of assets needed to cover possible losses. Formally, *VaR* with confidence level p is defined such that the probability of a loss greater than *VaR* is less than or equal to p while the probability of a loss less than *VaR* is less than or equal to $1-p$.

Mathematically, given a confidence level p, if X is the underlying(e.g., the price of a portfolio)), then $VaR_p(X)$ is the negative of the p-quantile, i.e.:

$$VaR_p(X) = \inf\{x \in \mathbb{R} : P(X + x < 0) \leq 1 - p\} = \inf\{x \in \mathbb{R} : 1 - F_X(-x) \geq p\}$$

where $F_X(\cdot)$ is the cdf as usual in probability theory.

Compared to expected value, which only focuses on the average, *VaR* pays attention to the tail of the distribution, which quantifies the probability of unlikely and undesirable outcomes, such as catastrophe or large loss. Since these measures help us to better understand and manage risk, they are called "risk measures". There are other *VaR* related risk measures [37], for example, Expected Shortfall, Conditional Value At Risk (*CVaR*).

2.1.2 Markowitz Portfolio Theory

Markowitz proposed that portfolio risk is equal to the variance of the portfolio returns. His setting is the following:

Assume we have a portfolio consisting of n assets, with return r_i for $i \in \{1, 2, ..., n\}$. Denote by $\mu_i = E(r_i)$, $m = (\mu_1, \mu_2, ...\mu_n)^{\mathbf{T}}$ and the covariance matrix of their return to be Σ. If the weight associated with each asset i is ω_i, and we define $\omega = \{\omega_1, .., \omega_n\}$. Then the return and the variance of the portfolio is given by:

$$E = m^{\mathbf{T}}\omega, \quad Var = \frac{1}{2}\omega^{\mathbf{T}}\Sigma\omega$$

Now since variance is the measure of risk in Markowitz's setting, the mathematical formulation of the resulting optimization problem is the following:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\omega^{\mathbf{T}}\Sigma\omega \\ \text{such that} \quad & m^{\mathbf{T}}\omega \geq \mu_b, \ e^{\mathbf{T}}m = 1 \end{aligned}$$

Based on the Markowitz portfolio risk measurement, Sharpe [54] invented the Sharpe Ratio:

$$Sharpe = \frac{\mu_p - r_f}{\sigma_p}$$

where μ_p is the portfolio return, r_f is the risk free rate of return, and σ_p is the volatility (standard deviation) of the portfolio. Sharpe ratio can be interpreted as the excess return above the risk free rate per unit of risk, where risk is measured by Markowitz Portfolio Theory. The Sharpe ratio provides a portfolio risk measure in terms of determining the quality of the portfolio return at a given level of risk.

2.1.3 Coherent Risk Measure

A significant milestone in risk measurement, which is the concept of a coherent risk measure, was achieved by Artzner et al [38, 39], who are the first to propose the axioms of risk measurement. These axioms have far reaching implications. It is no longer possible to assign an arbitrary function as risk measure unless it satisfies all these axioms. From this perspective, *VaR* is no longer a coherent risk measure while *CVaR* is. Further developments in this area include: *convex risk measure*, which was first introduced by Heath [28] and later by Föllmer and Schied [20] and Frittelli and Rosazza Gianin [26] in more general spaces of random variables; *representation theorems for sublinear and convex risk measures* by Föllmer and Schied [21, 20] and independently, Frittelli [25] and Frittelli and Rosazza Gianin [26]. Additional interesting results about convex risk measures can be found in [22] by Föllmer and Schied. Specifically, by choosing the acceptance set according to the lost functions of the investors, they proposed the convex risk measures "dependent" on the preferences of the agents.

Definition 2.1.1. *(Convex / Coherent Risk Measure [23]) A mapping $\rho : \mathcal{Z} \to \mathbb{R} \cup \{\infty\}$ is called a convex risk measure if $\rho(0)$ is finite and $\rho(\cdot)$ satisfies the following axioms:*

(i) ***Monotonicity****: If $X \leq Y$, then $\rho(X) \leq \rho(Y) \quad \forall X, Y \in \mathcal{Z}$.*

(ii) ***Translation Invariance**: $\rho(X + m) = \rho(X) + m \quad \forall X \in \mathcal{Z},\ m \in \mathbb{R}$.*

(iii) ***Convexity**: $\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y) \quad \forall X, Y \in \mathcal{Z},\ 0 \le \lambda \le 1$.*

A convex risk measure is called a coherent risk measure, if it further satisfies:

(i) ***Positive Homogeneity**: $\rho(\lambda X) = \lambda\rho(X) \quad \forall X \in \mathcal{Z},\ \lambda \ge 0$.*

Under the assumption of positive homogeneity, the Convexity of a monetary risk measure is equivalent to:

(i) ***Subadditivity**: $\rho(X + Y) \le \rho(X) + \rho(Y) \quad \forall X, Y \in \mathcal{Z}$.*

Each axiom listed above has its own explanation. Assume the above $\rho(X)$ measures the future loss of the portfolio X. Then, *Monotonicity* means that high loss is associated with high risk. *Translation Invariance* can be understood as that the investment in the riskless bond bears no loss with probability 1. Hence we must always receive the initial amount invested. The *Convexity* or *Subadditivity* states that if you diversify your portfolio, it will reduce the risk, which reflects the core idea of risk management. Finally, *Positive Homogeneity* ensures that we cannot increase or decrease risk by investing differing amounts in the same stock; in other words the risk arises from the stock itself and is not a function of the quantity purchased.

Under the framework of coherent risk measure, as mentioned at the beginning, *VaR* is no longer a candidate since it does not satisfy the *Subadditivity* property.

Remark 2.1.1. *The following are some popular used coherent risk measures:*

- ***CVaR**:*

$$CVaR = E(X|X > VaR_p(X))$$

- ***Mean-semideviation** ([4]):*

$$\rho(X) = E(X) + \alpha(E(Z - E[Z])^+)^p)^{\frac{1}{p}}, \quad \alpha \in [0,1], p \ge 1$$

- ***Entropic risk measure** ([22])*

$$\rho(X) = \frac{1}{p}\log(E(e^{pX})), \quad p > 0$$

2.2 Dynamic Risk Measures in Discrete Time

The above section is mainly about risk measures within one stage, i.e., the problem of quantifying today's risk of our financial position w.r.t. a future maturity date. Such measures are called "static risk measures."

Another interesting and prospective direction is the theory of multi-stage risk measures, or dynamic risk measures. In this setting, we estimate the risk of our position at different times between today and the maturity. There are two directions within this theory, discrete time models and continuous time models. We start with the discrete time models.

Definition 2.2.1. *(Conditional Risk Measure) Given a probability space* $(\Omega, \mathcal{F}, P)$ *with filtrations* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$. *We also have adapted sequence of random variables* $Z_1, Z_2, ..., Z_T$ *such that* $Z_t \in \mathcal{Z}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, P)$ *and* $\mathcal{Z}_{t,T} = \mathcal{Z}_t \times \cdots \times \mathcal{Z}_T$. *Then a mapping* $\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t$ *is called Conditional Risk Measure if it satisfies the following monotonicity condition:*

$$\rho_{t,T}(Z) \leq \rho_{t,T}(W) \quad \forall Z, W \in \mathcal{Z}_{t,T} \text{ such that } Z \leq W$$

Based on the Conditional Risk Measure, we can define the Dynamic Risk Measure.

Definition 2.2.2. *(Dynamic Risk Measure) Dynamic Risk Measure is a sequence of conditional risk measures*

$$\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t \quad t = 1, 2, ..., T$$

For example:

$$\rho_{1,T}(Z_1, Z_2, Z_3, ..., Z_T) \in \mathcal{Z}_1 = \mathbb{R}$$

$$\rho_{2,T}(Z_2, Z_3, ..., Z_T) \in \mathcal{Z}_2$$

$$\rho_{2,T}(Z_3, ..., Z_T) \in \mathcal{Z}_3$$

$$......$$

The key issue in the dynamic setting is the consistency over time, which has been studied in various references (e.g. [40, 33, 34]), in the past. The definition we proposed is similar to that in [50].

Definition 2.2.3. *(Time-Consistency) A dynamic risk measure* $\{\rho_{t,T}\}_{t=1}^{T}$ *is time-consistent if for all* $\tau < \theta$ *the relation*

$$Z_k = W_k, \quad k = \tau, ..., \theta - 1$$

and

$$\rho_{\theta,T}(Z_\theta, ..., Z_T) \le \rho_{\theta,T}(W_\theta, ..., W_T)$$

imply that

$$\rho_{\tau,T}(Z_\tau, ..., Z_T) \le \rho_{\tau,T}(W_\tau, ..., W_T)$$

The understanding of the concept of time consistency is intuitive: If Z is at least as good as W from future time θ and Z and W are identical between τ and $\theta - 1$, then at earlier time τ, Z should be at least as good as W as well. Then we have the following Risk-Averse Equivalence Theorem regarding the structure of time consistency.

Theorem 2.2.1. *(Risk-Averse Equivalence Theorem [50]) Suppose* $\{\rho_{t,T}\}_{t=1}^{T}$ *satisfies the condition:*

$$\rho_{t,T}(Z_t, Z_{t+1}, ..., Z_T) = Z_t + \rho_{t,T}(0, Z_{t+1}, ..., Z_T)$$
$$\rho(0, 0, ..., 0) = 0$$

Then it is time-consistent if and only if for all $\tau \le \theta$*:*

$$\rho_{\tau,T}(Z_\tau, ..., Z_\theta, ..., Z_T) = \rho_{\tau,\theta}(Z_\tau, ..., Z_{\theta-1}, \rho_{\theta,T}(Z_\theta, ..., Z_T))$$

Based on the above theorem, if we further define one-step conditional risk measures ρ_t : $\mathcal{Z}_{t+1} \to \mathcal{Z}_t$:

$$\rho_t(Z_{t+1}) = \rho_{t,T}(0, Z_{t+1}, 0, ..., 0)$$

Then, we dynamically evaluate the risk $\rho_{t,T}$ at time t by recursively evaluating the one-step conditional risk measures.

Theorem 2.2.2. *(Nested Decomposition Theorem [50]) Suppose* $\{\rho_{t,T}\}_{t=1}^{T}$ *is time-consistent and satisfies the condition:*

$$\rho_{t,T}(Z_t, Z_{t+1}, ..., Z_T) = Z_t + \rho_{t,T}(0, Z_{t+1}, ..., Z_T)$$
$$\rho(0, 0, ..., 0) = 0$$

Then for all t we have the representation:

$$\rho_{t,T}(Z_t, Z_{t+1}, ..., Z_T) = Z_t + \rho_t(Z_{t+1} + \rho_{t+1}(Z_{t+2} + ... + \rho_{T-1}(Z_T)))$$

From the above theorem, we can see that it gives an algorithm to evaluate the risk in a backward induction, which is a dynamic setting. And it involves only one-step conditional risk measures.

Example 2.2.3. *(The mean semideviation model [5, 52])*

$$\rho_t(Z_{t+1}) = E(Z_{t+1}|\mathcal{F}_t) + kE[((Z_{t+1} - E[Z_{t+1}|\mathcal{F}_t])^+)^p|\mathcal{F}_t]^{\frac{1}{p}}$$

where $k \in [0, 1]$ *may be* $\mathcal{F}_t$*- measurable and* $p \geq 1$.

If it involves optimal decision making in solving Multistage Risk-Averse Optimization Problems, the following Interchangeability Principle will be used.

Proposition 2.2.4. *(Interchangeability Principle [50]) Given a probability space* $(\Omega, \mathcal{F}, P)$ *with filtration* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$. *If we denote by* $x_t(\omega)$ $\omega \in \Omega,\ t = 1, 2, ..., T$ *to be the decision variable at time t. And we further assume that* $x_t(\cdot)$ *is not anticipated, i.e.* $x_t(\cdot)$ *is* $\mathcal{F}_t$*-measurable. Also, the cost at each stage is defined by* $Z_t(x_t),\ t = 1, 2, .., T$. *Then we have the following:*

$$\begin{aligned}&\min_{x_1, x_2, ..., x_T} \{Z_1(x_1) + \rho_1(Z_2(x_2) + \rho_2(Z_3(x_3) + ... + \rho_{T-1}(Z_T)) \cdots))\}\\&= \min_{x_1}\{Z_1(x_1) + \rho_1(\min_{x_2} Z_2(x_2) + \rho_2(\min_{x_3} Z_3(x_3) + ... + \rho_{T-1}(\min_{x_T}(Z_T)) \cdots))\}\end{aligned}$$

2.3 Dynamic Risk Measures in Continuous Time

Dynamic risk measures in the continuous time setting starts from Peng et al's [17] contribution of connecting time consistent dynamic risk measure with a corresponding backward stochastic differential equation(BSDE). We will go through the concepts of nonlinear expectation, g-evaluation and BSDE in this section. Also, since law invariance plays a crucial role in the partially observable problem, which is our main material, we shall first start this section with the concept of a law invariant risk measure, namely a risk estimator.

2.3.1 Risk Estimators

Given a probability space $(\Omega, \mathcal{F}, P)$, and a space $\mathcal{Z}$ of measurable real functions on $(\Omega, \mathcal{F})$, a *risk measure* is a real-valued function on $\mathcal{Z}$. In our further considerations we focus on $\mathcal{Z} = \mathcal{L}_2(\Omega, \mathcal{F}, P)$.

Definition 2.3.1. *A risk measure $\rho : \mathcal{L}_2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is law invariant with respect to the probability measure P, if for any random variables $Z \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ and $Z' \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ that have the same distribution under P, we have $\rho(Z) = \rho(Z')$.*

Remark 2.3.1. *A law invariant risk measure $\rho(Z)$ can be written as $\rho(\pi_Z)$, where $\pi_Z = P \circ Z^{-1}$ is the distribution of Z on $\mathbb{R}$. Also, for any measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(Z) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$ if $Z \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$, the risk measure $\rho(f(Z))$ can be written as $\rho(f, \pi_Z)$.*

Definition 2.3.2. *A law invariant risk measure is called a* risk estimator.

The concept of risk estimator will play a crucial role in the main setting and we will discuss it later on.

2.3.2 Dynamic Risk Measures

Consider now a filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ and the corresponding spaces $\mathcal{Z}_t = \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$, $0 \le t \le T$.

Definition 2.3.3. *A* conditional risk measure *is a measurable mapping $\rho_{t,T} : \mathcal{Z}_T \to \mathcal{Z}_t$, where $0 \le t \le T$.*

(i) *It is* ***monotonic****, if $\xi \le \xi'$ a.s. implies that $\rho_{t,T}[\xi] \le \rho_{t,T}[\xi']$ a.s.;*

(ii) *It is* ***normalized****, if $\rho_{t,T}[0] = 0$;*

(iii) *If has the* ***generalized constant preservation property****, if $\rho_{t,T}(\xi) = \xi$ for all $\xi \in \mathcal{Z}_t$;*

(iv) *It has the* ***translation property****, if $\rho_{t,T}[\xi + \eta] = \xi + \rho_{t,T}[\eta]$, for all $\xi \in \mathcal{Z}_t$, $\eta \in \mathcal{Z}_T$;*

(v) *It has the* ***local property****, if $\rho_{t,T}[\, 1_A \xi \,] = 1_A \rho_{t,T}[\, \xi \,]$, for all $\xi \in \mathcal{Z}_T$ and all $A \in \mathcal{F}_t$.*

Observe that a normalized conditional risk measure having the translation property has the generalized constant preservation property as well.

Definition 2.3.4. *A* dynamic risk measure *is a collection of conditional risk measures $\{\rho_{t,T}\}_{0 \le t \le T}$. We say that it is monotonic, normalized, has the generalized constant preservation property, the* *translation property, or the local property, if all its conditional risk measures have the corresponding properties* (i), (ii), (iii)*, or* (v) *of Definition 2.3.3. It is* ***time consistent****, if for all* $0 \le t \le s \le T$ *and all* $\xi \in \mathcal{Z}_T$ *we have* $\rho_{t,T}(\xi) = \rho_{t,T}(\rho_{s,T}(\xi))$*.*

For any $0 \le t \le s \le T$, we can define $\rho_{t,s} : \mathcal{Z}_s \to \mathcal{Z}_t$ such that $\rho_{t,s}(\eta) = \rho_{t,T}(\eta)$ for any $\eta \in \mathcal{Z}_s$. Therefore, a family of conditional risk measures $\{\rho_{t,s}\}_{0 \le t \le s \le T}$ is available.

Remark 2.3.2. *Under the generalized constant preservation property, time consistency can be equivalently expressed as follows: for all* $0 \le t \le s \le T$*, if* $\rho_{s,T}(Y) \le \rho_{s,T}(Y')$ *then* $\rho_{t,T}(Y) \le \rho_{t,T}(Y')$*. Indeed, we immediately see that* $\rho_{s,T}(Y) = \rho_{s,T}(Y')$ *implies* $\rho_{t,T}(Y) = \rho_{t,T}(Y')$ *and thus* $\rho_{t,T}(Y) = \varphi(\rho_{s,T}(Y))$ *for all* $Y \in \mathcal{Z}_T$*, for some operator* $\varphi : \mathcal{Z}_s \to \mathcal{Z}_t$*. Setting* $Y \in \mathcal{Z}_s$*, we get* $\rho_{t,T}(Y) = \varphi(Y)$*, and thus* $\varphi(\cdot) = \rho_{t,s}(\cdot)$*.*

2.3.3 Nonlinear Expectations

We start from basic properties on nonlinear expectations.

Definition 2.3.5. *For* $0 \le T < \infty$*, a* nonlinear expectation *is a functional:* $\gamma_{0,T} : \mathcal{L}_2(\Omega, \mathcal{F}_T, P) \to \mathbb{R}$ *satisfying the following properties:*

(i) ***Strict Monotonicity****: If* $\eta \ge \eta'$ *a.s., then* $\gamma_{0,T}[\eta] \ge \gamma_{0,T}[\eta']$*; If* $\eta \ge \eta'$ *a.s., then* $\gamma_{0,T}[\eta] = \gamma_{0,T}[\eta']$ *if and only if* $\eta = \eta'$ *a.s..*

(ii) ***Constant Preservation****:* $\gamma_{0,T}[c] = c \quad \forall c \in \mathbb{R}$

Lemma 2.3.3. *For any* $0 \le t \le T$*, and* $\eta, \eta' \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$*. If*

$$\gamma_{0,T}[\eta 1_A] = \gamma_{0,T}[\eta' 1_A] \quad \forall A \in \mathcal{F}_t$$

then $\eta = \eta'$ *a.s.*

Proof. We choose $A = \{\eta_1 > \eta_2\} \in \mathcal{F}_t$, since $(\eta_1 - \eta_2)1_A \ge 0$, and $\gamma_{0,T}[\eta_1 1_A] = \gamma_{0,T}[\eta_2 1_A]$, it follows that $\eta_1 1_A = \eta_2 1_A$ *a.s.* Thus $\eta_2 \ge \eta_1$ *a.s.* Similarly, we can also prove that $\eta_1 \ge \eta_2$ *a.s.* Therefore, $\eta_1 = \eta_2$ *a.s.*. □

Based on the above lemma, we can further define an $\mathcal{F}$-consistent nonlinear expectation.

Definition 2.3.6. *For the given filtration $\{\mathcal{F}_t\}, 0 \le t \le T$, a nonlinear expectation $\gamma_{0,T}[\cdot]$ is $\mathcal{F}$-consistent if for every $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ and every $t \in [0, T]$, there exists a random variable $\zeta \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$ such that*

$$\gamma_{0,T}[\eta 1_A] = \gamma_{0,T}[\zeta 1_A] \quad \forall A \in \mathcal{F}_t$$

Remark 2.3.4. *According to Lemma 2.3.3, ζ is uniquely defined; we denote it by $\gamma_{t,T}[\eta]$. This can be understood as the* nonlinear conditional expectation *of η at time t. We can further define $\gamma_{0,t} : \mathcal{L}_2(\Omega, \mathcal{F}_t, P) \to \mathbb{R}$ such that $\gamma_{0,t}[\eta] = \gamma_{0,T}[\eta]$ for all $t \in [0, T]$ and $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$. This can be interpreted as follows: since $\mathcal{F}_t$ has all the information about η, even if $\mathcal{F}_T$ contains more information, our estimation of η remains the same. Furthermore, for any $0 \le t \le s \le T$, we can define $\gamma_{t,s} : \mathcal{L}_2(\Omega, \mathcal{F}_s, P) \to \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$ such that $\gamma_{t,s}(\eta) = \gamma_{t,T}(\eta)$ for any $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_s, P)$. The interpretation for this is similar to $\gamma_{0,t}$. Therefore, a family of $\mathcal{F}$-consistent nonlinear expectations $\{\gamma_{t,s}, 0 \le t \le s \le T\}$ is defined.*

Proposition 2.3.5. *Let $\gamma_{0,T}(\cdot)$ be defined in Definition* 2.3.5*, for each $0 \le t \le T$ and $\xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, there exists a $\gamma_{t,T}(\cdot) \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$ satisfying Definition* 2.3.6*, then $\gamma_{t,T}(\cdot)$ satisfies the following axioms:*

- ***Monotonicity:*** *$\gamma_{t,T}[\xi] \ge \gamma_{t,T}[\xi']$ a.s. if $\xi \ge \xi'$ a.s.*
- ***Constant Preservation:*** *$\gamma_{t,T}[\xi] = \xi$ if $\xi \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$*
- ***Local Property:*** *for each t, $\gamma_{t,T}[1_A \xi] = 1_A \gamma_{t,T}[\xi]$ $\forall A \in \mathcal{F}_t$*
- ***Time Consistency:*** *$\gamma_{s,t}[\gamma_{t,T}[\xi]] = \gamma_{s,T}[\xi]$ for $0 \le s \le t \le T$*

Proof. We first prove Monotonicity. Define $\xi_t = \gamma_{t,T}[\xi]$ and $\xi'_t = \gamma_{t,T}[\xi']$. Assuming that $\xi, \xi' \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ and $\xi' \le \xi$. Let $A \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$. Then by the monotonicity property of $\gamma_{0,T}[\cdot]$, we have:

$$\gamma_{0,T}[\xi'_t 1_A] = \gamma_{0,T}[\xi' 1_A] \le \gamma_{0,T}[\xi_t 1_A] = \gamma_{0,T}[\xi_t 1_A]$$

Now we take $A = \{\xi'_t > \xi_t\}$, then if $P(A) > 0$, by the strict monotonicity property of $\gamma_{0,T}[\cdot]$, we have

$$\gamma_{0,T}[\xi'_t 1_A] > \gamma_{0,T}[\xi_t 1_A]$$

which is a contradiction. Hence, $P(A) = 0$, which implies $\xi'_t \le \xi_t$ *a.s.*

Second, the constant preservation property is trivial and straightforward by recalling the Definition 2.3.6 and Lemma 2.3.3.

Third, we prove the Local Property. For each $B \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$, we have:

$$\begin{aligned}\gamma_{0,T}[\gamma_{t,T}[\xi 1_A]1_B] &= \gamma_{0,T}[\xi 1_A 1_B]\\ &= \gamma_{0,T}[\gamma_{t,T}[\xi]1_{A\cap B}]\\ &= \gamma_{0,T}[[\gamma_{t,T}[\xi]1_A]1_B]\end{aligned}$$

Therefore, by the uniqueness property, the local property holds.

Finally we shall prove the Time Consistency Property. For any $A \in \mathcal{F}_s$, we have $A \in \mathcal{F}_t$. Thus,

$$\begin{aligned}\gamma_{0,T}[\gamma_{s,T}[\gamma_{t,T}[\xi]]1_A] &= \gamma_{0,T}[\gamma_{t,T}[\xi]1_A]\\ &= \gamma_{0,T}[\xi 1_A]\\ &= \gamma_{0,T}[\gamma_{s,T}[\xi]1_A]\end{aligned}$$

Therefore, $\gamma_{s,T}[\xi] = \gamma_{s,T}[\gamma_{t,T}[\xi]] = \gamma_{s,t}[\gamma_{t,T}[\xi]]$. □

Lemma 2.3.6. *For any $\xi, \xi' \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, $\forall A \in \mathcal{F}_t$, and for each $t \in [0, T]$, we have:*

$$\gamma_{t,T}[\xi 1_A + \xi' 1_{A^C}] = \gamma_{t,T}[\xi]1_A + \gamma_{t,T}[\xi']1_{A^C}$$

Proof.

$$\begin{aligned}\gamma_{t,T}[\xi 1_A + \xi' 1_{A^C}] &= \gamma_{t,T}[\xi 1_A + \xi' 1_{A^C}]1_A + \gamma_{t,T}[\xi 1_A + \xi' 1_{A^C}]1_{A^C}\\ &= \gamma_{t,T}[\xi 1_A + \xi' 1_{A^C} 1_A] + \gamma_{t,T}[\xi 1_A + \xi' 1_{A^C} 1_{A^C}]\\ &= \gamma_{t,T}[\xi 1_A] + \gamma_{t,T}[\xi' 1_{A^C}]\\ &= \gamma_{t,T}[\xi]1_A + \gamma_{t,T}[\xi']1_{A^C}\end{aligned}$$

□

2.3.4 Backward Stochastic Differential Equations

Let's first introduce the framework of BSDE by a simple example. For $\forall \xi \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, by the Martingale Representation Theorem, there exists $\eta_t \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$ such that $\xi = E[\xi] +$

$\int_0^T \eta_t dW_t$. For here, the filtration $\{\mathcal{F}_t\}$ is the augmented filtration generated by the Brownian Motion W_t. Denote

$$Y_t := E[\xi] + \int_0^t \eta_s dW_s, \quad Z_t := \eta_t$$

Then (Y_t, Z_t) are $\mathcal{F}_t$-adapted and they satisfy the following:

$$dY_t = Z_t dW_t, \quad Y_T = \xi$$

This is actually a simple BSDE, and the pair (Y, Z) is called the solution of this BSDE.

More generally and formally, let the filtration be generated by a d-dimensional Brownian motion $\{W_t\}_{0\le t\le T}$. On the probability space $(\Omega, \mathcal{F}, P)$ with a Brownian filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, we consider the following one-dimensional backward stochastic differential equation (BSDE):

$$-dY_t = g(t, Y_t, Z_t)\, dt - Z_t\, dW_t, \quad Y_T = \eta. \tag{2.1}$$

Here, $\eta \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ is called the *terminal condition* and the measurable function $g : [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is called the *driver* (or generator); we always assume that $g(t, Y_t, Z_t)$ is $\mathcal{F}_t$-measurable for all $t \in [0, T]$. We define two relevant spaces of processes:

- $\mathcal{S}^{2,m}[t, T]$ is the set of functions $Y : [t, T] \times \Omega \to \mathbb{R}^m$ that are adapted and such that $\mathbb{E}[\sup_{t\le s\le T} |Y_s|^2] < \infty$; for $m = 1$, we write it as $\mathcal{S}^2[t, T]$;
- $\mathcal{H}^{2,m}[t, T]$ is the set of functions $Y : [t, T] \times \Omega \to \mathbb{R}^m$ that are adapted and such that $\mathbb{E}[\int_t^T |Y_s|^2\, ds] < \infty$; for $m = 1$, we write it as $\mathcal{H}^2[t, T]$.

Definition 2.3.7. *If a pair of processes* $(Y, Z) \in \mathcal{S}^2[0, T] \times \mathcal{H}^{2,d}[0, T]$ *satisfies equation* (2.1)*, we say that the pair* (Y, Z) *is the solution of the corresponding BSDE.*

Assumption 2.3.7. *The function g is jointly Lipschitz in* (y, z)*. i.e., a constant* $K > 0$ *exists such that for all* $t \in [0, T]$*, all* $y_1, y_2 \in \mathbb{R}$ *and all* $z_1, z_2 \in \mathbb{R}^d$*, we have*

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \le K(|y_1 - y_2| + |z_1 - z_2|) \quad a.s.,$$

and the process $g(\cdot, 0, 0) \in \mathcal{H}^2[0, T]$.

In order to introduce the existence and uniqueness result, we shall state the following lemma first.

Lemma 2.3.8. *Under Assumption* 2.3.7*, if* (Y,Z) *is a solution of the orignial BSDE, Then*

$$\|(Y,Z)\|^2 \leq CE[|\eta|^2 + \int_0^T |g(t,0,0)^2 dt|]$$

where

$$\|(Y,Z)\|^2 := E\{\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt\}$$

Theorem 2.3.9. *If Assumption* 2.3.7 *is satisfied, the existence and uniqueness of the solution of* (2.1) *can be guaranteed (see J. Zhang [67]).*

Proof. We first prove the uniqueness of the solution. Suppose the pairs $(Y^i, Z^i), i = 1,2$ are the solutions. And it suffices to prove that

$$\|(Y^1 - Y^2, Z^1 - Z^2)\| = 0$$

Denote $\Delta Y_t := Y_t^1 - Y_t^2, \Delta Z_t := Z_t^1 - Z_t^2$, then

$$\begin{aligned}\Delta Y_t &= \int_t^T [g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)]ds - \int_t^T \Delta Z_s dW_s \\ &= \int_t^T [\alpha_s \Delta Y_s + \beta_s \Delta Z_s]ds - \int_t^T \Delta Z_s dW_s\end{aligned}$$

where

$$\alpha_t = \begin{cases} \frac{g(t,Y_t^1,Z_t^1)-g(t,Y_t^2,Z_t^1)}{\Delta Y_t}, & \text{if } \Delta Y_t \neq 0 \\ 0, & \text{if } \Delta Y_t = 0 \end{cases} \qquad \beta_t = \begin{cases} \frac{g(t,Y_t^2,Z_t^1)-g(t,Y_t^2,Z_t^2)}{\Delta Z_t}, & \text{if } \Delta Z_t \neq 0 \\ 0, & \text{if } \Delta Z_t = 0 \end{cases} \tag{2.2}$$

are bounded. Then by Lemma 2.3.8, we get $\|(\Delta Y, \Delta Z)\|^2 \leq 0$, the uniqueness is proved.

Then, let's prove the existence by Picard Iteration. Denote

$$Y_t^0 = \eta + \int_t^T g(s,0,0)ds - \int_t^T Z_s^0 dW_s$$

and for $n = 1, 2, ...,$

$$Y_t^n = \eta + \int_t^T g(s, Y_s^{n-1}, Z_s^{n-1})ds - \int_t^T Z_s^n dW_s$$

By induction, one can easily show that $\|(Y^n, Z^n)\| < \infty, n = 0,1,2,....$ Denote $\Delta Y_t^n := Y_t^n - Y_t^{n-1}, \Delta Z_t := Z_t^n - Z_t^{n-1}$. Then

$$\Delta Y_t^n = \int_t^T [\alpha_s^{n-1} \Delta Y_s^{n-1} + \beta_s^{n-1} \Delta Z_s^{n-1}]ds - \int_t^T \Delta Z_s^n dW_s$$

where α^n, β^n are defined in a similar way as in the (2.2) and are bounded. By applying Itô formula to $e^{\gamma t}|\Delta Y_t^n|^2$ with $\gamma > 0$ a constant and by noting that $\Delta Y_T^n = 0$, we have:

$$
\begin{aligned}
& E[e^{\gamma t}|\Delta Y_t^n|^2 + \gamma \int_t^T e^{\gamma s}|\Delta Y_s^n|^2 ds + \int_t^T e^{\gamma s}|\Delta Z_s^n|^2 ds] \\
&= E[2 \int_t^T [e^{\gamma s}\Delta Y_s^n[\alpha_s^{n-1}\Delta Y_s^{n-1} + \beta_s^{n-1}\Delta Z_s^{n-1}]]ds] \\
&\leq E[C \int_t^T [e^{\gamma s}|\Delta Y_s^n|[|\Delta Y_s^{n-1}| + |\Delta Z_s^{n-1}|]]ds] \\
&\leq E[C_0 \int_t^T e^{\gamma t}|\Delta Y_s^n|^2 ds + \frac{1}{4T}\int_t^T e^{\gamma s}|\Delta Y_s^{n-1}|^2 ds + \frac{1}{4}\int_t^T e^{\gamma s}|\Delta Z_s^{n-1}|^2 ds]
\end{aligned}
$$

where C_0 is a constant and here by choosing $\gamma = C_0$, we have:

$$
\begin{aligned}
& E[e^{\gamma t}|\Delta Y_t^n|^2 + \int_t^T e^{\gamma s}|\Delta Z_s^n|^2 ds] \\
&\leq E[\frac{1}{4T}\int_t^T e^{\gamma s}|\Delta Y_s^{n-1}|^2 ds + \frac{1}{4}\int_t^T e^{\gamma s}|\Delta Z_s^{n-1}|^2 ds] \\
&\leq \frac{1}{4}[\sup_{0\leq s\leq T} e^{\gamma s}E|\Delta Y_s^{n-1}|^2 + \int_0^T e^{\gamma s}E|\Delta Z_s^{n-1}|^2 ds]
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sup_{0\leq t\leq T} e^{\gamma t}E|\Delta Y_t^n|^2 &\leq \frac{1}{4}[\sup_{0\leq t\leq T} e^{\gamma t}E|\Delta Y_t^{n-1}|^2 + \int_0^T e^{\gamma t}E|\Delta Z_t^{n-1}|^2 dt] \\
\int_0^T e^{\gamma t}|\Delta Z_t^n|^2 dt &\leq \frac{1}{4}[\sup_{0\leq t\leq T} e^{\gamma t}E|\Delta Y_t^{n-1}|^2 + \int_0^T e^{\gamma t}E|\Delta Z_t^{n-1}|^2 dt]
\end{aligned}
$$

Define

$$
\|(Y,Z)\|_\gamma^2 := E[\sup_{0\leq t\leq T} e^{\gamma t}|Y_t|^2 + \int_0^T e^{\gamma t}|Z_t|^2 dt]
$$

Then we have the following relation:

$$
\|(\Delta Y^n, \Delta Z^n)\|_\gamma^2 \leq \frac{1}{2}\|(\Delta Y^{n-1}, \Delta Z^{n-1})\|_\gamma^2
$$

Hence,

$$
|(\Delta Y^n, \Delta Z^n)\|_\gamma^2 \leq \frac{1}{2^{n-1}}|(\Delta Y^1, \Delta Z^1)\|_\gamma^2 = \frac{C}{2^n}
$$

Also, noticing that

$$
|(Y^1 + Y^2, Z^1 + Z^2)\|_\gamma \leq |(Y^1, Z^1)\|_\gamma + \|(Y^2, Z^2)\|_\gamma
$$

For any $n < m$,

$$
\|(Y_t^n - Y_t^m, Z_t^n - Z_t^m)\|_\gamma \leq \sum_{j=n+1}^{m} \|(\Delta Y^j, \Delta Z^j)\|_\gamma \leq \sum_{j=n+1}^{m} \frac{C}{2^{\frac{j}{2}}} \leq \frac{C}{2^{\frac{n}{2}}}
$$

Therefore,

$$\|(Y_t^n - Y_t^m, Z_t^n - Z_t^m)\|_\gamma \to 0 \quad \text{as } n, m \to \infty$$

Thus, there exists (Y, Z) such that

$$\sup_{0\le t\le T} e^{\gamma t} E|Y_t^n - Y_t|^2 + \int_0^T e^{\gamma t}|Z_t^n - Z_t|^2 dt \to 0 \quad \text{as } n \to \infty$$

Finally, by letting $n \to \infty$, we conclude that the pair (Y, Z) satisfies the original BSDE.

□

We also have the following stability and comparison result for BSDE.

Theorem 2.3.10. *(Stability) Assume*

$$Y_t = \eta + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

and

$$Y_t^n = \eta_n + \int_t^T g_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_n$$

where $\eta, \eta_n \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, both g and g_n satisfies Assumption 2.3.7 *with a common Lipschitz constant for all n. Assume further that*

$$\lim_{n\to\infty} E[|\eta_n - \eta|^2 + \int_0^T |g_n(t, 0, 0) - g(t, 0, 0)|^2 dt] = 0$$

and that $\forall(t, y, z)$, $g_n(t, y, z) \to g(t, y, z)$, $P - a.s.$ Then,

$$\lim_{n\to\infty} \|(Y^n - Y, Z^n - Z)\| = 0$$

Theorem 2.3.11. *(Comparison) Assume Assumption* 2.3.7 *hold for the drivers g_i, $i = 1, 2$ and that $\eta_1, \eta_2 \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$. Also, let (Y^i, Z^i) bet the solution of the following BSDE:*

$$Y_t^i = \eta_i + \int_t^T g_i(s, Y_s^i, Z_s^i)ds - \int_t^T Z_s^i dW_s \quad i = 1, 2.$$

Assume further that $\eta_1 > \eta_2$ a.s. and that for $\forall(t, y, z), g_1(t, y, z) \ge g_2(t, y, z)$ a.s.. Then, $Y_t^1 > Y_t^2$ $\forall t$, a.s.. In particular, $Y_0^1 \ge Y_0^2$.

Proof. Denote

$$\Delta Y_t := Y_t^1 - Y_t^2; \quad \Delta Z_t := Z_t^1 - Z_t^2; \quad \Delta\xi := \xi_1 - \xi_2$$

Then,

$$\Delta Y_t = \Delta\xi + \int_t^T [g_1(s, Y_s^1, Z_s^1) - g_2(s, Y_s^2, Z_s^2)]ds - \int_t^T \Delta Z_s dW_s$$
$$= \Delta\xi + \int_t^T [\alpha_s \Delta Y_s + \beta_s \Delta Z_s + \hat{g}_s]ds - \int_t^T \Delta Z_s dW_s$$

where

$$\alpha_s := \frac{g_1(s, Y_s^1, Z_s^1) - g_1(s, Y_s^2, Z_s^1)}{\Delta Y_s} \quad \beta_s := \frac{g_1(s, Y_s^2, Z_s^1) - g_1(s, Y_s^2, Z_s^2)}{\Delta Z_s}$$

are bounded and

$$\hat{g}_t := g_1(t, Y_t^2, Z_t^2) - g_2(t, Y_t^2, Z_t^2)$$

By recalling that we have

$$\Delta\xi \geq 0, \quad \text{and} \quad \hat{g}_t \geq 0 \quad a.s.\ \forall t$$

We get $\Delta Y_t \geq 0$ *a.s.* from the basic result for linear BSDE (J. Zhang [67]). □

Remark 2.3.12. *Some comments on the comparison theorem:*

- *Comparison theorem holds true only when dim$(Y) = 1$, but W can be high dimensional.*
- *In general, we do not have $g_1(t, Y_t^1, Z_t^1) \geq g_2(t, Y_t^2, Z_t^2)$.*
- *We cannot claim that $Z_t^1 \geq Z_t^2$ since actually Z can be in high dimensional.*

We also have the following useful result (Peng [42]).

Proposition 2.3.13. *Suppose g satisfies Assumption 2.3.7, for any $t_0 \in [0, T]$ and any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, the process $g(\cdot, y, z)$ is adapted to $\mathcal{F}_t^{t_0} = \sigma(W_\tau, t_0 \leq \tau \leq t)$ on the interval $[t_0, T]$ and $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T^{t_0}, P)$. Then the solution (Y, Z) of the BSDE (2.1) is also $\mathcal{F}_t^{t_0}$-adapted on $[t_0, T]$. In particular, Y_{t_0} and Z_{t_0} are deterministic.*

Proof. Let (Y', Z') be the solution of $\mathcal{F}_t^{t_0}$-adapted solution, on the interval $[t_0, T]$ of the BSDE

$$Y_t' = \eta + \int_t^T g(s, Y_s', Z_s')ds - \int_t^T Z_s' dW_s^0$$

where we denote $W_t^0 \equiv W_t - W_{t_0}$. Observe that $(W_s^0)_{t_0 \leq s \leq T}$ is a Brownian Motion that is adapted to $\mathcal{F}_t^{t_0}$ on the interval $[t_0, T]$. However, on the other hand, the process $(Y_t', Z_t')_{t_0 \leq s \leq T}$ is also $\mathcal{F}_t$-adapted and we have the following:

$$\int_t^T Z_s' dW_s = \int_t^T Z_s' dW_s^0, \quad t \in [t_0, T]$$

Therefore, the solution (Y, Z) of the original BSDE coincides with (Y', Z') on the interval $[t_0, T]$ by the uniqueness of the solution of the BSDE. Hence the pair (Y, Z) is $\mathcal{F}_t^{t_0}$-adapted. In particular, Y_{t_0} and Z_{t_0} are deterministic. □

2.3.5 g-Evaluations

Under Assumption 2.3.7, we can move on to the definition of the g-evaluation.

Definition 2.3.8. *For each $0 \leq t \leq T$ and $\eta \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$, the g-evaluation at time t is the operator $\rho^g_{t,T} : \mathcal{L}^2(\Omega, \mathcal{F}_T, P) \to \mathcal{L}^2(\Omega, \mathcal{F}_t, P)$ defined as follows:*

$$\rho^g_{t,T}[\eta] = Y_t,$$

where $(Y, Z) \in \mathcal{S}^2[0, T] \times \mathcal{H}^{2,d}[0, T]$ is the unique solution of (2.1).

The following theorem clarifies the relation between g-evaluations and dynamic risk measures, as well as $\mathcal{F}$-consistent nonlinear expectations.

Theorem 2.3.14. *If the driver g satisfies Assumption 2.3.7 and $g(\cdot, y, 0) \equiv 0$ for all $y \in \mathbb{R}$, then the the system of g-evaluations $\{\rho^g_{t,T}\}_{0 \leq t \leq T}$ is a monotonic, time-consistent, generalized constant preserving risk measure having the local property (actually a system of $\mathcal{F}$-consistent nonlinear expectation). Moreover, for any $t \in (0, T]$, any $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$, we have $\lim_{s \uparrow t} \rho^g_{s,t}[\eta] = \eta$.*

Proof. First, the Monotonicity Property is direct from the Comparison Theorem above.

As for the Local Property, we multiply 2.1 by 1_A on both hand sides on the interval $[t, T]$, and then we get

$$\begin{aligned} Y_t 1_A &= Y_T 1_A + \int_t^T 1_A g(t, Y_s, Z_s) ds - \int_t^T 1_A Z_s dW_s \\ &= Y_T 1_A + \int_t^T g(t, Y_s 1_A, Z_s 1_A) ds - \int_t^T 1_A Z_s dW_s \end{aligned}$$

where the last equation is true because of the fact that $g(\cdot, y, 0) \equiv 0$. This implies that $(1_A Y_s, 1_A Z_s)$ for $s \in [t, T]$ is actually a solution of the above BSDE. Therefore we have:

$$1_A \rho^g_{t,T}[\eta] = \rho^g_{t,T}[1_A \eta]$$

As for the Time Consistency Property, it follow from the uniqueness of the solution of BSDE that for any $s \leq t \leq T$,:

$$\rho^g_{s,T}[\eta] = \rho^g_{s,t}[Y_t] = \rho^g_{s,t}[\rho^g_{t,T}[\eta]]$$

As for the Generalized Constant Preservation Property, we consider the solution (y, z) of equation (2.1) defined on $[t, T]$ with $\eta \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$:

$$Y_t = \eta + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

Then by definition, we have $Y_t = \rho^g_{t,T}(\eta)$. Also, since $g(\cdot, y, 0) \equiv 0$ for all $y \in \mathbb{R}$, it is easy to check $(Y_t, Z_t) = (\eta, 0)$. Therefore,

$$\rho^g_{t,T}(\eta) = \eta, \quad \forall \eta \in \mathcal{L}_2(\Omega, \mathcal{F}_t, P)$$

Finally, due to the continuity property of Y_t w.r.t. t, we have:

$$\lim_{s \uparrow t} \rho^g_{s,t}[\eta] = \eta$$

□

Proposition 2.3.15. *We assume that the drivers g_1 and g_2 satisfy Assumption* 2.3.7*, then if g_1 dominates g_2 in the following sense:*

$$g_1(t, y, z) - g_1(t, y', z') \leq g_2(t, y - y', z - z') \quad \forall y, y' \in \mathbb{R}, \quad \forall z, z' \in \mathbb{R}^d$$

then, $\rho^{g_1}_{t,T}[\cdot]$ is dominated by $\rho^{g_2}_{t,T}[\cdot]$ in the following sense, for each $\eta, \eta' \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$:

$$\rho^{g_1}_{t,T}[\eta] - \rho^{g_1}_{t,T}[\eta'] \leq \rho^{g_2}_{t,T}[\eta - \eta']$$

In particular, if g is dominated by itself, then $\rho^g_{t,T}[\cdot]$ is also dominated by itself.

Proof. We consider three BSDE such that the first two are with same driver g_1 but different terminal conditions η and η' while the last one is with driver g_2 and terminal condition $\eta - \eta'$.

$$-dY_s = g_1(s, Y_s, Z_s)ds - Z_s dW_s \quad Y_T = \eta$$

$$-dY'_s = g_1(s, Y'_s, Z'_s)ds - Z'_s dW_s \quad Y_T = \eta'$$

$$-d\tilde{Y}_s = g_2(s, \tilde{Y}_s, \tilde{Z}_s)ds - \tilde{Z}_s dW_s \quad \tilde{Y}_T = \eta - \eta'$$

From the first two equations, we have:

$$-d(Y_s - Y'_s) = (g_1(s, Y_s, Z_s) - g_1(s, Y'_s, Z'_s))ds - (Z_s - Z'_s)dW_s \quad Y_T - Y'_T = \eta - \eta'$$

and if we denote by $\bar{Y}_s = Y_s - Y'_s$, $\bar{Z}_s = Z_s - Z'_s$ and $\bar{g}_s = g_1(s, Y_s, Z_s) - g_1(s, Y'_s, Z'_s)$, then we have

$$-d\bar{Y}_s = \bar{g}_s ds - \bar{Z}_s dW_s \quad \bar{Y}_T = \eta - \eta'$$

By the Comparison Theorem and by noticing that

$$\bar{g}_s = g_1(s, Y_s, Z_s) - g_1(s, Y'_s, Z'_s) \leq g_2(s, Y_s - Y'_s, Z_s - Z'_s) = g_2(s, \bar{Y}_s, \bar{Z}_s)$$

we have:

$$\bar{Y}_t = Y_t - Y'_t \leq \tilde{Y}_t$$

Then the conclusion follows by recalling the definition of g-evaluation.

□

On the other hand, we have the following remarkable result due to [17].

Theorem 2.3.16. *Suppose the filtration is defined by a d-dimensional Brownian motion, and a dynamic risk measure $\{\rho_{t,T}\}_{0\leq t\leq T}$ is monotonic, time consistent, has the local property, the normalization property (a system of $\mathcal{F}$-consistent nonlinear expectation), the translation property, and*

$$\rho_{0,T}[\xi + \eta] - \rho_{0,T}[\xi] \leq \rho^{\mu,\nu}_{0,T}[\eta], \quad \forall \xi, \eta \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P),$$

where $\rho^{\mu,\nu}_{0,T}[\cdot]$ is a g-evaluation with $g(t, y, z) = \mu|y| + \nu|z|$ for some $\mu > 0, \nu > 0$. Then a driver g satisfying Assumption 2.3.7 with $g(\cdot, y, 0) \equiv 0$ for all $y \in \mathbb{R}$ exists, such that each $\rho_{t,T}[\cdot]$ is a g-evaluation.

From now on, we shall consider only risk measures defined by g-evaluations. To ensure their desired properties, we may impose additional conditions on the driver g.

Assumption 2.3.17. *The driver g satisfies the following conditions for almost all $t \in [0, T]$:*

(i) *g is deterministic and independent of y, i.e., $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and $g(\cdot, 0) \equiv 0$;*

(ii) *$g(t, \cdot)$ is convex for all $t \in [0, T]$;*

(iii) *$g(t, \cdot)$ is positively homogeneous for all $t \in [0, T]$.*

The desired properties of the g-evaluation are obtained by the following theorem.

Theorem 2.3.18. *Suppose g satisfies Assumption 2.3.7 and Assumption 2.3.17*(i). *Then the system of the corresponding g-evaluations $\{\rho^g_{t,s}\}_{0\leq t\leq s\leq T}$ is normalized, monotonic and has the translation property. If, additionally, condition* (ii) *of Assumption 2.3.17 is satisfied, then $\rho^g_{t,s}$ has the property of convexity: for all $\eta, \eta' \in \mathcal{L}^2(\Omega, \mathcal{F}_s, P)$ and all $\beta \in \mathcal{L}^\infty(\Omega, \mathcal{F}_t, P)$ such that $0 \leq \beta \leq 1$, we have*

$$\rho^g_{t,s}(\beta\eta + (1-\beta)\eta') \leq \beta\rho^g_{t,s}(\eta) + (1-\beta)\rho^g_{t,s}(\eta') \quad a.s.$$

If g also satisfies (iii) *of Assumption* 2.3.17*, then $\rho^g_{t,s}$ also has the property of positive homogeneity: for all $\eta \in \mathcal{L}^2(\Omega, \mathcal{F}_s, P)$ and all $\beta \in \mathcal{L}^\infty(\Omega, \mathcal{F}_t, P)$ such that $\beta \geq 0$, we have*

$$\rho^g_{t,s}(\beta\eta) = \beta\rho^g_{t,s}(\eta) \quad a.s.$$

In the following chapters, we shall drop the superscript g from the dynamic risk measures defined by g-evaluations; the driver will be obvious from the context.

Chapter 3

Optimal Filter

In the process of solving the partially observable problem, the conditional expectation plays a key role, since it gives an unbiased estimation with the smallest mean squared error. Therefore, there are many works focusing on the deduction of the representations of the conditional expectations under different assumptions. Kolmogorov [32] and Wiener [62] first did the fundamental work of constructing the optimal estimator within the linear framework. Further work obtained by themselves as well as others can be found in [44, 49, 65]. When it comes to the field of optimal nonlinear filtering, Stratonovich [58, 59] derived the first general result on the construction of an optimal nonlinear estimator for Markov process based on the theory of conditional Markov processes. Other works in the nonlinear filtering area including: Wonham [63], Kushner [35, 57], Shiryaev [55, 56, 57].

In general, filtering theory is concerned with the following problem. Suppose we have a signal process θ_t, which we cannot observe directly. Instead, what we can observe is another process ξ_t which is correlated with θ_t. We can understand the process ξ_t as "signal plus white noise", which means

$$d\xi_t = a(\theta_t)dt + \sigma_t dW_t$$

where W_t is a Wiener process and ξ_t is related with θ_t by this dynamics. Given a time t, we can only observe $\{\xi_t, 0 \leq s \leq t\}$, therefore, our task is to estimate θ_t based on the information from $\{\xi_{s\leq t}\}$. For any function $h(\cdot)$ w.r.t θ_t, the best estimation in the sense of smallest mean square error, is given by the conditional expectation,

$$\pi_t(h) = E(h(\theta_t)|\mathcal{F}_t^{\xi})$$

where $\mathcal{F}_t^{\xi} = \sigma\{\theta_s,\ 0 \leq s \leq t\}$.

The goal in this type of problem is to find an explicit expression for $\pi_t(h)$ in terms of

$\{\xi_{s\le t}\}$. In particular, we want to seek to express $\pi_t(h)$ as the solution of a stochastic differential equation driven by the observable part ξ_t. The organization of this chapter is the following: we first introduce a theorem about the dynamics of $\pi_t(h)$ in the general sense, then gives two famous types of filtering, i.e. Kalman-Bucy filter and Wonham filter. Finally, we introduce the interpolation and extrapolation problem at the end.

3.1 General Equations of Optimal Nonlineare Filter

In this section, let's give a general formula for the optimal filter problem, this is one of the main theories from the reference [47]. The Kalman-Bucy filter and the Wonham filter's result can be reached by applying this general theorem. We will discuss them later on.

3.1.1 Basic Setting

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space with a nondecreasing family of right continuous sub σ-algebras $\{\mathcal{F}_t\}_{0\le t\le T}$ and let θ_t be an unobservable process measurable w.r.t. the whole filtration $\mathcal{F}_t$ while let ξ_t be an observable process measurable w.r.t $\mathcal{F}_t$ with the following dynamics:

$$\xi_t = \xi_0 + \int_0^t A_s(\omega)ds + \int_0^t B_s(\xi)dW_s$$

where $(W_s, \mathcal{F}_t)$ is a Wiener process and the process $(A_t(\omega), \mathcal{F}_t)$ and $(B_t(\xi), \mathcal{F}_t)$ are assumed to satisfy the following:

$$P(\int_0^T |A_t(\omega)|dt < \infty) = 1$$
$$P(\int_0^T |B_t^2(\xi)|dt < \infty) = 1$$

Then we further assume the following conditions:

$$|B_t(x) - B_t(y)|^2 \le L_1 \int_0^t [x_s - y_s]^2 dK(s) + L_2[x_t - y_t]^2$$
$$B_t^2(x) \le L_1 \int_0^t (1 + x_s^2)dK(s) + L_2(1 + x_t^2)$$

where $K(t)$ is nondereasing right continuous function lies between 0 and 1, and both L_1 and L_2 are nonnegative constants.

So now, suppose we have a $\mathcal{F}_t$-measurable function h_t of (θ, ξ), and $Eh_t^2 < \infty$. Then, a natural way to characterize h_t based on the information of $\xi_s,\ s \leq t$ is by $\pi_t(h) = E(h_t|\mathcal{F}_t^{\xi})$. If we further assume that h_t has the following representation:

$$h_t = h_0 + \int_0^t H_s ds + x_t$$

where $(x_t, \mathcal{F}_t)$ is a martingale and $(H_t, \mathcal{F}_t)$ is a random process such that $\int_0^T |H_s| ds < \infty\ a.s.$

3.1.2 Equation for the Optimal Nonlinear Filter

Under all the assumption above, we can have a characterize for $\xi_s,\ s \leq t$, which is the following main theorem. ([47])

Theorem 3.1.1. *With all the assumption above satisfies, as well as the following:*

$$\sup_{0 \leq t \leq T} Eh_t^2 < \infty$$

$$\int_0^T EH_t^2 dt < \infty$$

$$B_t^2(x) \geq C > 0$$

Then, for each $t \in [0, T]$, we have P a.s.

$$\pi_t(h) = \pi_0(h) + \int_0^t \pi_s(H) ds + \int_0^t \pi_s(D) + \frac{\pi_s(hA) - \pi_s(h)\pi_s(A)}{B_s(\xi)} d\overline{W}_s$$

where

$$\overline{W}_t = \int_0^t \frac{d\xi_s - \pi_s(A) ds}{B_s(\xi)}, \quad D_t = \frac{d\langle x, W\rangle_t}{dt}$$

$\overline{W}_t$ is a Wiener process w.r.t $\mathcal{F}_t^{\xi}$ and D_t is a process measurable w.r.t. $\mathcal{F}_t$.

Remark 3.1.2. *If we consider the problem of estimating the unobservable component θ_t of the two-dimensional diffusion Markov process (θ_t, ξ_t), $0 \leq t \leq T$, based on the observable process ξ_t, then the representation result of $\pi_t(h)$ can be found in Shiryaev [57] and Liptser and Shiryaev [46].*

3.1.3 Kalman-Bucy Filter

We will give two specific applications of the above main theorem, the first one is the Kalman-Bucy Filter, and the second one is the Wonham Filter. As you might have already noticed, the dynamics of $\pi_t(h)$ in the main theorem is not a SDE in general. However, in these two cases, it yields a Stochastic Differential Equation.

The Kalman-Bucy Filter is a continuous time counterpart to the discrete time Kalman Filter. As with the Kalman Filter, the Kalman-Bucy Filter is designed to estimate unmeasured states of a process, usually for the purpose of controlling one or more of them.

In the setting of Kalman-Bucy Filters, the processes θ_t and ξ_t are two one-dimensional Gaussian random processes given by the following linear equations:

$$d\theta_t = a(t)\theta_t dt + b(t)dW_1(t)$$

$$d\xi_t = A(t)\theta_t dt + B(t)dW_2(t)$$

where $(W_1(t), \mathcal{F}_t)$ and $(W_2(t), \mathcal{F}_t)$ are two independent Wiener processes and θ_0 and ξ_0 are $\mathcal{F}_0$-measurable. In Kalman-Bucy setting, we want to have an estimation of the unobservable process θ_t based on the observable process ξ_t, which will provide partial information. We would like to have the estimation at each moment t in the optimal way, which is in the sense that mean square error is minimized. Therefore, this estimation for θ_t coincides with the conditional expectation

$$m_t = E(\theta_t|\mathcal{F}_t^{\xi})$$

the mean square error is denoted by

$$\gamma_t = E(\theta_t - m_t)^2$$

The following theorem gives a closed system of dynamic equations with respect to m_t and γ_t so that we can construct the optimal filter.

Theorem 3.1.3. *Let (θ_t, ξ_t), $0 \leq t \leq T$ be a two-dimensional Gaussian process as mentioned above. Also, the following conditions satisfiy:*

$$\int_0^T |a(t)|dt < \infty, \quad \int_0^T b^2(t)dt < \infty$$

$$\int_0^T |A(t)|dt < \infty, \quad \int_0^T B^2(t)dt < \infty$$

$$\int_0^T A^2(t)dt < \infty, \quad B^2(t) \geq C > 0, \quad 0 \leq t \leq T$$

Then the conditional expectation $m_t = E(\theta_t|\mathcal{F}_t^{\xi})$ *and the mean square error* $\gamma_t = E(\theta_t - m_t)^2$ *satisfy the system of equations:*

$$dm_t = a(t)m_t dt + \frac{\gamma_t A(t)}{B^2(t)}(d\xi_t - A(t)m_t dt)$$
$$\dot{\gamma}_t = 2a(t)\gamma_t - \frac{A^2(t)\gamma_t^2}{B^2(t)} + b^2(t)$$

with $m_0 = E(\theta_0|\xi_0)$, $\gamma_0 = E(\theta_0 - m_0)^2$. *Moreover, the above system has a unique continuous solution.*

3.2 Wonham Filter

From now on, we focus on another kind of filter which is called the Wonham Filter. And all our later risk measure and risk filter arguments are focusing on this specific class of filters. Unlike the Kalman-Bucy filter, the signal process θ_t in this model is not defined as the solution of a stochastic differential equation. Instead, it is a Markov Jump Process, which takes a finite number of values and has piecewise constant sample paths. The observable process ξ_t is given by a diffusion process which involes the signal process in its drift part. The simplification of the unobservable process matters, since if θ_t takes a finite number of values $\alpha_1, \alpha_2, ..., \alpha_n$, then estimation of any $\pi_t(h)$ is equivalent of the knowledge of conditional probability distribution $P(\theta_t = \alpha_i|\mathcal{F}_t^{\xi})$, $i = 1, 2, ..., n$. Therefore, the dynamics of $P(\theta_t = \alpha_i|\mathcal{F}_t^{\xi})$ is the main concern in the Wonham filter.

3.2.1 Equations of the Optimal Nonlinear Filter

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space with a nondecreasing family of right continuous sub σ-algebras $\{\mathcal{F}_t\}_{0\le t\le T}$. Let the process $\{\theta_t\}_{0\le t\le T}$, adapted to $\{\mathcal{F}_t\}_{0\le t\le T}$, be a real right continuous Markov process with values in a finite set $E = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$; it is called a *Markov jump process*. The values of the process $\{\theta_t\}$ are not observed, but at time $t = 0$ we know the prior probabilities $p_i = P[\theta_0 = \alpha_i]$, $i = 1, \ldots, n$.

Let $\{W_t\}_{0\le t\le T}$, be a standard Wiener process, also adapted to $\{\mathcal{F}_t\}_{0\le t\le T}$, independent of θ. The observed process $\{\xi_t\}_{0\le t\le T}$ follows the SDE:

$$d\xi_t = A(\theta_t, t)\, dt + B(t)\, dW_t, \quad 0 \le t \le T, \tag{3.1}$$

with the initial value ξ_0, which is $\mathcal{F}_0$-measurable and independent of θ_0.

Assumption 3.2.1. *We assume the following conditions:*

(i) $\mathbb{E}[\xi_0^2] < \infty$;

(ii) *The functions* $A(\alpha, t)$ *and* $B(t)$ *satisfy for all* $\alpha \in E$ *and* $0 \le t \le T$ *the following conditions:* $|A(\alpha, t)| \le L_1$, *and* $0 < C \le B^2(t) \le L_1$, *where* L_1 *and* C *are constants.*

Now we define the posterior probabilities:

$$\pi_i(t) = P[\theta_t = \alpha_i | \mathcal{F}_t^\xi], \quad i = 1, \ldots, n, \tag{3.2}$$

where $\{\mathcal{F}_t^\xi\}$ is the filtration generated by the observable process $\{\xi_t\}$. The vector $\pi(t) = \{\pi_1(t), \pi_2(t), \ldots, \pi_n(t)\}$ may be regarded as our estimate of θ_t based on the information from the σ-subalgebra $\mathcal{F}_t^\xi$. To characterize the dynamics of the n-dimensional process $\pi(t)$, we introduce a condition on the transition rates of the unobservable process θ_t.

Assumption 3.2.2. *Functions* $\lambda_{\alpha\beta}(t)$, $0 \le t \le T$, $\alpha, \beta \in E$, *and a constant* K *exist, such that (uniformly over* α, β, *and* t*),* $|\lambda_{\alpha\beta}(t)| \le K$ *and*

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[P(\theta_{t+\Delta t} = \beta \,|\, \theta_t = \alpha) - \delta(\alpha, \beta) \right] = \lambda_{\alpha\beta}(t), \tag{3.3}$$

where $\delta(\alpha, \beta)$ *is the Kronecker function (equal 1 if* $\alpha = \beta$*, and 0 otherwise).*

The matrix $\Lambda = (\lambda_{\alpha_i\alpha_j})_{i,j=1}^n$ is the infinitesimal generator of the process θ.

The following theorem characterizes the movement of the n-dimensional process $\pi(t)$.

Theorem 3.2.3 ([47]). *Let Assumptions* 3.2.1 *and* 3.2.2 *hold. Then the posterior probabilities* $\pi_i(t)$, $i = 1, \ldots, n$, *satisfy for* $t \in [0, T]$ *the following system of n SDE:*

$$\pi_i(t) = p_i + \int_0^t (\Lambda^* \pi)_i(\tau)\, d\tau + \int_0^t \pi_i(\tau) \frac{A(\alpha_i, \tau) - \bar{A}(\tau)}{B(\tau)}\, d\overline{W}_\tau, \quad \pi_i(0) = p_i, \tag{3.4}$$

where

$$(\Lambda^* \pi)_i(\tau) = \sum_{j=1}^n \lambda_{\alpha_j \alpha_i}(\tau)\, \pi_j(\tau), \qquad \bar{A}(\tau) = \sum_{j=1}^n A(\alpha_j, \tau)\, \pi_j(\tau), \tag{3.5}$$

and $\{\overline{W}_t\}_{0 \le t \le T}$ *is a Wiener process given by the formula*

$$\overline{W}_t = \int_0^t \frac{d\xi_\tau - \bar{A}(\tau)\, d\tau}{B(\tau)}. \tag{3.6}$$

The proof of the above theorem based on the following two lemmas.

Lemma 3.2.4. *If we denote:*

$$p_{\alpha\beta}(s,t) = P(\theta_t = \beta | \theta_s = \alpha), \quad 0 \leq s \leq t \leq T$$

$$p_\beta = P(\theta_t = \beta), \quad 0 \leq t \leq T$$

Then, $p_{\alpha\beta}(s,t)$ satisfies the forward Kolmogorov equation:

$$p_{\alpha\beta}(s,t) = \delta(\beta,\alpha) + \int_s^t \mathcal{L}^* p_{\alpha\beta}(s,u)du$$

where

$$\mathcal{L}^* p_{\alpha\beta}(s,u) = \sum_{\gamma\in E} \lambda_{\gamma\beta}(u) p_{\alpha\gamma}(s,u)$$

The probability $p_\beta(t)$ satisfies the equation:

$$p_\beta(t) = p_\beta(0) + \int_0^t \mathcal{L}^* p_\beta(u)du$$

where

$$\mathcal{L}^* p_\beta(u)du = \sum_{\gamma\in E} \lambda_{\gamma\beta}(u) p_\gamma(u)$$

Lemma 3.2.5. *For each $\beta \in E$, we set the process x_t^β to be the following:*

$$x_t^\beta = \delta(\beta,\theta_t) - \delta(\beta,\theta_0) - \int_0^t \lambda_{\theta_s\beta}(s)ds$$

Then, the process x_t^β is a square integrable martingale with right continuous trajectories, having limits to the left.

Proof. First, x_t^β is bounded by the uniform bound of $\lambda_{\theta_s\beta}(s)$. Second, right continuity is due to the right continuity of the trajectories of the process θ_t. So the only thing left is the martingale property of x_t^β.

$$\begin{aligned}
E(x_t^\beta|\mathcal{F}_s) &= x_s^\beta + E(\delta(\beta,\theta_t) - \delta(\beta,\theta_s) - \int_s^t \lambda_{\theta_u\beta}(u)du|\mathcal{F}_s) \\
&= x_s^\beta + E(\delta(\beta,\theta_t) - \delta(\beta,\theta_s) - \int_s^t \lambda_{\theta_u\beta}(u)du|\theta_s) \\
&= x_s^\beta + p_{\theta_s\beta}(s,t) - \delta(\beta,\theta_s) - \int_s^t \sum_{\gamma\in E} \lambda_{\gamma\beta}(u) p_{\theta_s\gamma}(s,u) \\
&= x_s^\beta
\end{aligned}$$

where the second equation is due to the Markov property of the process θ and the third equation holds by the forward Kolmogorov equation in the above lemma. □

Now based on the above lemma, we can move on to the proof of the Theorem 3.2.3.

Proof. We prove the result based on the Theorem 3.1.1. We apply the result of the main Theorem to the process x_t^β and it is easy to check that the conditions are all satisfied, and since x_t^β is independent of the Wiener process W_t, we have $\left\langle x^\beta, W \right\rangle_t \equiv 0$.

Therefore, with all the assumptions satisfied, the following equation follows:

$$\pi_t(\delta(\beta,\theta_t)) = \pi_0(\delta(\beta,\theta_0)) + \int_0^t \pi_s(\lambda_{\theta_s\beta}(s))ds + \int_0^t \frac{\pi_s(\delta(\beta,\theta_s)A(\theta_s,s)) - \pi_s(\delta(\beta,\theta_s))\pi_s(A(\theta_s,s))}{B(s)} d\overline{W}_s \tag{3.7}$$

where

$$\pi_t(\cdot) = E(\cdot|\mathcal{F}_t^\xi)$$

and

$$\overline{W}_t = \int_0^t \frac{d\xi_r - \pi_r(A)dr}{B(r)} \tag{3.8}$$

where $\{\overline{W}_t\}, 0 \le t \le T$ is a Wiener Process.

Finally, we simplify (3.7) for $\beta = \alpha_i$ with $i \in \{1, 2, ..., n\}$:

$$\begin{aligned}
\pi_t(\delta(\alpha_i,\theta_t)) &= E(\delta(\alpha_i,\theta_0)|\mathcal{F}_0^\xi) + \int_0^t E(\lambda_{\theta_s\alpha_i}(s)|\mathcal{F}_s^\xi)ds \\
&\quad + \int_0^t \frac{E(\delta(\alpha_i,\theta_s)A(\theta_s,s)|\mathcal{F}_s^\xi) - E(\delta(\alpha_i,\theta_s)|\mathcal{F}_s^\xi)E(A(\theta_s,s)|\mathcal{F}_s^\xi)}{B(s)} d\overline{W}_s \\
&= P(\theta_0 = \alpha_i) + \int_0^t \sum_{j=1}^n \lambda_{\alpha_j\alpha_i}(s)\pi_j(s)ds \\
&\quad + \int_0^t \frac{A(\alpha_i,s)\pi_i(s) - \pi_i(s)\sum_{j=1}^n A(\alpha_j,s)\pi_j(s)}{B(s)} d\overline{W}_s \\
&= p_i + \int_0^t (\Lambda^*\pi)_i(s)\,ds + \int_0^t \pi_i(s)\frac{A(\alpha_i,s) - \overline{A}(s)}{B(s)} d\overline{W}_s
\end{aligned}$$

By noticing that $\pi_t(\delta(\alpha_i,\theta_t)) = E(\delta(\alpha_i,\theta_t)|\mathcal{F}_t^\xi) = \pi_i(t)$, we conclude that equation (3.4) holds. □

Remark 3.2.6. *From equation* (3.6)*, it is obvious that* $\mathcal{F}_t^{\overline{W}} \subseteq \mathcal{F}_t^{\xi}, 0 \leq t \leq T$. *It can also be concluded that* $\mathcal{F}_t^{\xi} \subseteq \mathcal{F}_t^{\overline{W}}, 0 \leq t \leq T$ *by noticing that both* $B(\tau)$ *and* $A(\alpha_i, \tau)$*,* $i = 1, \ldots, n$ *are deterministic functions. Therefore,* $\mathcal{F}_t^{\overline{W}} = \mathcal{F}_t^{\xi}$ *for all* $0 \leq t \leq T$.

Example 3.2.7. *Let* θ *be a random variable taking values 0 and 1 with probability* $1 - p$ *and* p*, respectively. The random process* ξ_t *is defined as follows:*

$$d\xi_t = \theta\, dt + dW_t, \quad \xi_0 = 0. \tag{3.9}$$

Since $\pi_0(t) + \pi_1(t) = 1$*, it is sufficient to characterize* $\pi_1(t)$*. Theorem* 3.2.3 *yields:*

$$d\pi_1(t) = \pi_1(t)(1 - \pi_1(t))\, d\overline{W}_t, \tag{3.10}$$

where $\overline{W}_t = \int_0^t (d\xi_\tau - \pi_1(\tau)\, d\tau)$ *is a Wiener process.*

Example 3.2.8. *Let* θ_t*,* $t \geq 0$ *be a Markov process with two states* 0 *and* 1 *with* $P(\theta_0 = 1) = p$*,* $P(\theta_0 = 0) = 1 - p$*. Let the observable process* ξ_t *have the following dynamics:*

$$d\xi_t = \theta_t dt + dW_t, \quad \xi_0 = 0$$

where W_t *is a Wiener process as usual. Also, let the transition kernel of* θ_t *is given by the following matrix:*

$$Q(t) = \begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}$$

where the i-row and j-column entry means the transition rate from state $i - 1$ *to* $j - 1$*. For example,* 1*-row and* 2*-column entry is* λ*, meaning the transition rate from state* 0 *to state* 1 *is always* λ*, and for here, the transition rate is time independent.*

Then, if we denote by

$$\pi(t) = P(\theta_t = 1 | \mathcal{F}_t^{\xi})$$

According to Theorem 3.2.3*, we have the following:*

$$d\pi(t) = \lambda(1 - \pi(t))dt + \pi(t)(1 - \pi(t))d\overline{W}_t \quad \pi(0) = p$$

where $\overline{W}_t = \int_0^t (d\xi_\tau - \pi(\tau)\, d\tau)$ *is a Wiener process.*

3.3 Equations of Optimal Nonlinear Interpolation and Extrapolation

Before we move on to our framework of risk filtering, let's introduce interpolation and extrapolation in the optimal filter problems. As for the interpolation, we are concerned with $\pi_{s,t}(h) = E(h_s|\mathcal{F}_t^\xi)$ for $0 \leq s \leq t \leq T$, and specifically, when $s = t$, $\pi_{t,t}(h) = \pi_t(h)$. As for the extrapolation, we consider $\pi_{t,s}(h) = E(h_t|\mathcal{F}_s^\xi)$ for $0 \leq s \leq t \leq T$. For this section, all the assumption for h_t and ξ_t are exactly the same as the first section in this chapter.

3.3.1 Interpolation

Similar to Theorem 3.1.1, we also have a general theorem for interpolation as follows. Notice that here we consider the 'forward' equations (over t for fixed s).

Theorem 3.3.1. *Let the assumption of Theorem* 3.1.1 *be satisfied. Then for* $0 \leq s \leq t \leq T$*, we have*

$$\pi_{s,t}(h) = \pi_s(h) + \int_s^t \frac{E(h_s A_u|\mathcal{F}_u^\xi) - E(h_s|\mathcal{F}_u^\xi)E(A_u|\mathcal{F}_u^\xi)}{B_u(\xi)} d\overline{W}_u$$

where $\overline{W}_t$ is a $\mathcal{F}_t^\xi$-measurable Wiener process of the following form:

$$\overline{W}_t = W_t + \int_0^t \frac{A_u - \pi_u(A)}{B_u(\xi)} du$$

3.3.2 Extrapolation

We deduce a similar forward equation (over t for fixed s) for the extrapolation $\pi_{t,s}(h) = E(h_t|\mathcal{F}_s^\xi)$ for $0 \leq s \leq t \leq T$.

Theorem 3.3.2. *Let the assumptions for Theorem* 3.1.1 *be satisfied. Then, for fixed t and s such that $s \leq t$, we have:*

$$\pi_{t,s}(h) = \pi_{t,0}(h) + \int_0^s \{\pi_u(D) + \frac{E[E(h_t|\mathcal{F}_u)(A_u - \pi_u(A))|\mathcal{F}_u^\xi]}{B_u(\xi)}\} d\overline{W}_u$$

where $D_s = \frac{d\langle \tilde{x}, W\rangle_s}{ds}$ and $\tilde{x}_s = E(h_t|\mathcal{F}_s^\xi)$ is a square integrable martingale and is measurable with respect to $\mathcal{F}_s$ for $0 \leq s \leq t$. Also, $\overline{W}_t$ is a $\mathcal{F}_t^\xi$-measurable Wiener process of the following form:

$$\overline{W}_t = W_t + \int_0^t \frac{A_u - \pi_u(A)}{B_u(\xi)} du$$

3.3.3 Example

The interpolation and extrapolation have specific forms for specific filters based the above geneal theorems, for example, for the Wonham Filter. I do not present the detail about this, but the reader can refer to [47] for details. So here I give a concrete and simple example for the interpolation as well as extrapolation based on a special case of the Wonham Filter.

Example 3.3.3. *Let $\theta_t,\ t \geq 0$, be a Markov jump process with two states 0 and 1 with $P(\theta_0 = 1) = P(\theta_0 = 0) = \frac{1}{2}$. Let the observable process ξ_t have the following dynamics:*

$$d\xi_t = \theta_t dt + dW_t, \quad \xi_0 = 0$$

where W_t is a Wiener process as usual. Also, let the transition kernel of θ_t be given by the following matrix:

$$Q(t) = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$$

where the i-row and j-column entry means the transition rate from state $i-1$ to $j-1$. For example, 1-row and 2-column is λ, meaning the transition rate from state 0 to state 1 is always λ and for here, the transition rate is no time dependent.

Then, if we let

$$\pi(t) = P(\theta_t = 1|\mathcal{F}_t^{\xi})$$

According to Theorem 3.2.3, we have the following:

$$d\pi(t) = \lambda(1 - 2\pi(t))dt + \pi(t)(1 - \pi(t))d\overline{W}_t \quad \pi(0) = \frac{1}{2}$$

where $\overline{W}_t = \int_0^t (d\xi_\tau - \pi(\tau)\, d\tau)$ is a Wiener process.

As for the forward interpolation, given $0 \leq s \leq t \leq T$, if we denote $\pi(s,t) = P(\theta_s = 1|\mathcal{F}_t^{\xi})$, then

$$\pi(s,t) = \pi(s) + \int_s^t \pi(s,u)(\omega_{11}(u,s) - \pi(u))d\overline{W}_u \tag{3.11}$$

where $\overline{W}_t$ is a Wiener process defined by the follow:

$$d\overline{W}_t = d\xi_t - \pi(t)dt$$

and $\omega_{11} = P(\theta_t = 1|\theta_s = 1, \mathcal{F}_t^\xi)$ *satisfies the equation:*

$$\omega_{11}(t,s) = 1 + \lambda \int_s^t [1 - 2\omega_{11}(u,s)]du + \int_s^t \omega_{11}(u,s)[1 - \omega_{11}(u,s)][d\xi_u - \omega_{11}(u,s)du]$$

Actually, we can further represent $\pi(s,t)$ *according to* (3.11)*:*

$$\pi(s,t) = \pi(s)\exp\{\int_s^t [\omega_{11} - \pi(u)]d\xi_u - \frac{1}{2}\int_s^t [\omega_{11}^2(u,s) - \pi^2(u)]du\}$$

Finally, we consider forward extrapolation. Given $0 \leq s \leq t \leq T$*, denote* $\pi(t,s) = P(\theta_t = 1|\mathcal{F}_s^\xi)$*. Then we have:*

$$\pi(t,s) = \pi(s) + \lambda \int_s^t [1 - 2\pi(u,s)]du$$

which, we can represent $\pi(t,s)$ *as follows:*

$$\pi(t,s) = \pi(s)e^{-2\lambda(t-s)} + \frac{1}{2}(1 - e^{-2\lambda(t-s)})$$

Chapter 4

Risk Filtering

4.1 Filtration Inconsistency Problem

We now consider the problem of estimating a cost functional $\phi(\theta_T)$, where $\phi : E \to \mathbb{R}$, but instead of using its expectation, we intend to apply a risk measure. The corresponding evaluation should be available at each time t, and thus we need a dynamic risk measure $\{\rho_{t,T}\}_{0\le t\le T}$, which is monotonic, time-consistent, and has the local property, at least. However, we cannot evaluate $\phi(\theta_T)$ by $\rho_{t,T}[\phi(\theta_T)]$ directly, because all the information we can observe up to time t is in the σ-subalgebra $\mathcal{F}_t^\xi$. Even at $t = T$, the random variable θ_T is not $\mathcal{F}_T^\xi$-measurable, but only measurable w.r.t. $\mathcal{F}_T$.

In order to overcome this difficulty, let us analyze the classical case first. When we measure the cost $\phi(\theta_T)$ by its conditional expected value $\mathbb{E}[\phi(\theta_T)\,|\mathcal{F}_t^\xi]$, we use the tower property to rewrite it as

$$\mathbb{E}[\mathbb{E}[\phi(\theta_T)|\mathcal{F}_T^\xi]\big|\mathcal{F}_t^\xi] = \mathbb{E}[f(\phi, \pi(T))\,|\,\mathcal{F}_t^\xi]$$

where

$$f(\phi, \pi(T)) = \sum_{i=1}^{n} \phi(\alpha_i)\pi_i(T)$$

Since the function $f(\phi, \pi(T))$ is a linear combination of the posterior estimates $\pi_1(T)$, $\pi_2(T)$, $\ldots$,$\pi_n(T)$, which are $\mathcal{F}_T^\xi$-measurable, it is $\mathcal{F}_T^\xi$-measurable as well. In this way, we can evaluate $\mathbb{E}[f(\phi, \pi(T))]$ without any filtration inconsistency problem.

Hence, in the classical case, we deal with the problem in two stages. First, we construct an unbiased estimator of the unobserved $\phi(\theta_T)$, which is law invariant. Next, since the unbiased estimator is filtration consistent, we can evaluate it by conditional expectation. Therefore, first of all, we can now use a risk measure to reevaluate it by replacing the outside conditional expectation by a conditional risk measure $\rho_{t,T}[\mathbb{E}(\phi(\theta_T)|\mathcal{F}_T^{\xi}]$. This construction makes sense, because we have eliminated the filtration inconsistency issue by the inner conditional expectation and everything inside the risk measure $\rho_{t,T}$ is now measurable w.r.t the sub-filtration $\mathcal{F}_T^{\xi}$.

In this way, we have successfully defined a risk measure in the partially observable problem, but in order to overcome the filtration inconsistency problem, we used the inner conditional expectation, which is law-invariant and linear. A natural question is whether we can also replace the inner conditional expectation by a more general nonlinear law-invariant risk estimator? Moreover, is such a construction general enough, or is is just a convenient trick to deal with the filtration inconsistency problem? The answer to these questions is the theory of risk filters that we introduce in the next section.

4.2 Risk Filter

In order to answer the question in the previous section, we introduce a special nonlinear operator, which will help us overcome the filtration inconsistency problem. After that, we give a formal definition of the risk filter, the core two-stage risk measure structure. Then, the cost function with only a terminal cost and the cost function also with a running cost will be discussed separately, and we will derive a decoupled FBSDE system in both cases.

Consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$, such that $\mathcal{F} = \mathcal{F}_T$. We assume that a subfiltration $\{\mathcal{G}_t\}_{0\leq t\leq T}$ is available, such that $\mathcal{G}_t \subseteq \mathcal{F}_t$ for any $t \in [0, T]$.

Definition 4.2.1. *A* risk filter *is a family of nonlinear operators* $\bar{\rho}_{t,T} : \mathcal{L}^2(\Omega, \mathcal{F}_T, P) \to \mathcal{L}^2(\Omega, \mathcal{G}_t, P)$, $0 \leq t \leq T$, *which satisfies the following conditions:*

(i) ***Normalization****: For any* $t \in [0, T]$ *we have* $\bar{\rho}_{t,T}(0) = 0$;

(ii) ***Translation Equivariance****: For any* $t \in [0, T]$, *any* $V \in \mathcal{L}^2(\Omega, \mathcal{G}_t, P)$, *and any* $Z \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$, $\bar{\rho}_{t,T}(V + Z) = V + \bar{\rho}_{t,T}(Z)$;

(iii) ***Generalized Monotonicity****: For all* $0 \leq t \leq s \leq T$, *and all* Y, $Y' \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$, *if* $\bar{\rho}_{s,T}(Y) \leq \bar{\rho}_{s,T}(Y')$, *then* $\bar{\rho}_{t,T}(Y) \leq \bar{\rho}_{t,T}(Y')$.

It is easy to see that the operators $\{\bar{\rho}_{t,T}\}_{0\leq t\leq T}$ restricted to the spaces $\mathcal{L}^2(\Omega, \mathcal{G}_T, P)$ form a dynamic measure of risk $\{\rho_{t,T}\}_{0\leq t\leq T}$ enjoying the properties (i)–(iii) of Definition 4.2.1. Due to Remark 2.3.2, it is time consistent.

Theorem 4.2.1. *The risk filter defined in Definition* 4.2.1 *is time consistent and allows the following decomposition: for any* $0 \le t \le s \le T$ *and any* $Y \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$*,*

$$\bar{\rho}_{t,T}(Y) = \rho_{t,T}(\bar{\rho}_{s,T}(Y)). \tag{4.1}$$

Proof. The conditions (i) and (ii) of Definition 4.2.1 for any $s \in [0, T]$ and any $Y \in \mathcal{L}^2(\Omega, \mathcal{G}_s, P)$ yield the generalized constant preservation: $\bar{\rho}_{s,T}(Y) = Y + \bar{\rho}_{s,T}(0) = Y$. Thus, for any $X \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$, we obtain $\bar{\rho}_{s,T}(X) = \bar{\rho}_{s,T}(\bar{\rho}_{s,T}(X))$, which means

$$\bar{\rho}_{s,T}(X) \le \bar{\rho}_{s,T}(\bar{\rho}_{s,T}(X))$$

$$\bar{\rho}_{s,T}(X) \ge \bar{\rho}_{s,T}(\bar{\rho}_{s,T}(X))$$

Therefore, by the generalized monotonicity condition (iii) of Definition 4.2.1,

$$\bar{\rho}_{t,T}(X) \le \bar{\rho}_{t,T}(\bar{\rho}_{s,T}(X))$$

$$\bar{\rho}_{t,T}(X) \ge \bar{\rho}_{t,T}(\bar{\rho}_{s,T}(X))$$

Therefore,

$$\bar{\rho}_{t,T}(X) = \bar{\rho}_{t,T}(\bar{\rho}_{s,T}(X)),$$

which is the time consistency of the risk filter $\bar{\rho}$. Since $\bar{\rho}_{s,T}(X) \in \mathcal{L}^2(\Omega, \mathcal{G}_s, P)$, the last equation takes on the form of the postulated decomposition (4.1) of the risk filter. □

Remark 4.2.2. *It is evident that a risk filter defines a wider class of operators* $\bar{\rho}_{t,s} : \mathcal{L}^2(\Omega, \mathcal{F}_s, P) \to \mathcal{L}^2(\Omega, \mathcal{G}_t, P)$*,* $0 \le t \le s \le T$*, as follows: for* $Y \in \mathcal{L}^2(\Omega, \mathcal{F}_s, P)$ *we set* $\bar{\rho}_{t,s}(Y) = \bar{\rho}_{t,T}(Y)$*. In particular, the operator* $\{\rho_{t,s}\}_{0 \le t \le s \le T}$ *is the restriction of the operator* $\{\bar{\rho}_{t,s}\}_{0 \le t \le s \le T}$ *on the spaces* $\mathcal{L}^2(\Omega, \mathcal{G}_s, P)$*.*

We thus obtain the following propositions.

Proposition 4.2.3. *Given* $0 \le t \le r \le s \le T$*,* $Y \in \mathcal{L}^2(\Omega, \mathcal{F}_r, P)$ *and* $Z \in \mathcal{L}^2(\Omega, \mathcal{G}_r, P)$*, we then have:*

$$\bar{\rho}_{t,r}(Y) = \bar{\rho}_{t,s}(Y)$$

$$\rho_{t,r}(Z) = \rho_{t,s}(Z)$$

Proof. Since it is trivial that $Y \in \mathcal{L}^2(\Omega, \mathcal{F}_s, P)$, from Remark 4.2.2

$$\bar{\rho}_{t,r}(Y) = \bar{\rho}_{t,T}(Y) = \bar{\rho}_{t,s}(Y)$$

Since Z is measurable w.r.t. $\mathcal{G}_r$, Z is also measurable w.r.t. $\mathcal{G}_s$. Hence, by Remark 4.2.2 and the first half of proof, we have the following:

$$\rho_{t,r}(Z) = \bar{\rho}_{t,r}(Z) = \bar{\rho}_{t,s}(Z) = \rho_{t,s}(Z)$$

which completes the proof. □

Proposition 4.2.4. *Given $0 \le t \le r \le s \le T$, then for any random variables Y, $Y' \in \mathcal{L}^2(\Omega, \mathcal{F}_s, P)$, if $\bar{\rho}_{r,s}(Y) \le \bar{\rho}_{r,s}(Y')$, then $\bar{\rho}_{t,s}(Y) \le \bar{\rho}_{t,s}(Y')$.*

Proof. From Remark 4.2.2,

$$\bar{\rho}_{r,s}(Y) = \bar{\rho}_{r,T}(Y), \quad \bar{\rho}_{r,s}(Y') = \bar{\rho}_{r,T}(Y')$$

$$\bar{\rho}_{t,s}(Y) = \bar{\rho}_{t,T}(Y), \quad \bar{\rho}_{t,s}(Y') = \bar{\rho}_{t,T}(Y')$$

Therefore,

$$\bar{\rho}_{r,T}(Y) \le \bar{\rho}_{r,T}(Y')$$

By the Monotonicity property, we have

$$\bar{\rho}_{t,T}(Y) \le \bar{\rho}_{t,T}(Y')$$

then, the conclusion follows.

□

Corollary 4.2.5. *Given $0 \le t \le r \le s \le T$ and any $Y \in \mathcal{L}^2(\Omega, \mathcal{F}_s, P)$, we have $\bar{\rho}_{t,T}(Y) = \rho_{t,s}(\bar{\rho}_{r,s}(Y))$. In particular $\bar{\rho}_{t,T}(Y) = \rho_{t,s}(\bar{\rho}_{s,s}(Y))$. Furthermore, $\rho_{t,s}(X) = \rho_{t,s}(\rho_{r,s}(X))$ for any $X \in \mathcal{L}^2(\Omega, \mathcal{G}_s, P)$.*

Proof. By Theorem 4.2.1,

$$\bar{\rho}_{t,T}(Y) = \rho_{t,T}(\bar{\rho}_{r,T}(Y)) = \rho_{t,T}(\bar{\rho}_{r,s}(Y))$$

Then, by Proposition 4.2.3,

$$\rho_{t,T}(\bar{\rho}_{r,s}(Y)) = \rho_{t,r}(\bar{\rho}_{r,s}(Y)) = \rho_{t,s}(\bar{\rho}_{r,s}(Y))$$

which finishes the first part.

As for the second part, again by Theorem 4.2.1 and Proposition 4.2.3,

$$\rho_{t,s}(X) = \rho_{t,T}(X) = \rho_{t,T}(\rho_{r,T}(X)) = \rho_{t,T}(\rho_{r,s}(X)) = \rho_{t,r}(\rho_{r,s}(X)) = \rho_{t,s}(\rho_{r,s}(X))$$

which completes the proof. $\square$

Risk filtering is indeed carried out by a two-stage procedure: for $Y \in \mathcal{L}^2(\Omega, \mathcal{F}_s, P)$ we first calculate $\bar{\rho}_{s,s}(Y) \in \mathcal{L}^2(\Omega, \mathcal{G}_s, P)$, and then evaluate the conditional risk measure $\rho_{t,s}$ on this element.

4.3 Risk Filtering in the Terminal Cost Case

From now on, we will focus on the Wonham filter that we have introduced before. We will introduce our risk filter as an estimation of the terminal cost function first, and then generalize it to the case with running cost function as well.

On the probability space $(\Omega, \mathcal{F}, P)$ with the filtration of the Wiener Process $\{\overline{W}_t, 0 \leq t \leq T\}$ given by equation (3.6), *i.e.*, $\mathcal{F}^{\overline{W}} = \{\sigma(\overline{W}_t), 0 \leq t \leq T\}$, we consider the following 1-dimensional BSDE:

$$-dY_t = g(t, Y_t, Z_t)\,dt - Z_t\,d\overline{W}_t \quad Y_T = \eta, \tag{4.2}$$

where $\eta \in \mathcal{L}^2(\Omega, \mathcal{F}_T^{\overline{W}}, P)$, $g : [0,T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$, and $g(t, Y_t, Z_t)$ is $\mathcal{F}_t^{\overline{W}}$-measurable for all $t \in [0,T]$.

Remark 4.3.1. *Due to Remark* 3.2.6*, we can replace* $\mathcal{F}_t^{\overline{W}}$ *by* $\mathcal{F}_t^{\xi}$ *for any* $t \in [0,T]$*, since* $\mathcal{F}_t^{\overline{W}} = \mathcal{F}_t^{\xi}$ *for any* $t \in [0,T]$*. From now on, we consider the augmentation of* $\mathcal{F}_t^{\overline{W}}$*, but we still use the same notation.*

We introduce a risk filter $\{\bar{\rho}_{t,T}, 0 \leq t \leq T\}$, as defined in Definition 4.2.1, specifying $\{\mathcal{F}_t\}_{0\leq t\leq T}$ to be the whole original filtration, and $\{\mathcal{G}_t\}_{0\leq t\leq T}$ to be the observable filtration $\{\mathcal{F}_t^{\xi}\}_{0\leq t\leq T}$. Then, the decomposition $\bar{\rho}_{t,T}(\cdot) = \rho_{t,T}(\bar{\rho}_{T,T}(\cdot))$ is possible, where the family of operators $\{\rho_{t,T}\}_{0\leq t\leq T}$ is derived in Theorem 4.2.1.

For the final cost function of the form $\phi(\theta_T)$, we have $\bar{\rho}_{t,T}[\phi(\theta_T)] = \rho_{t,T}[\bar{\rho}_{T,T}(\phi(\theta_T))]$. If we assume further that $\bar{\rho}_{T,T}(\cdot)$ is a risk estimator (Definition 2.3.2), then $\bar{\rho}_{t,T}[\phi(\theta_T)]$ can be written as $\rho_{t,T}[\bar{\rho}_{T,T}(\phi,\pi(T))]$. Notice that $\bar{\rho}_{T,T}(\phi,\pi(T)) \in \mathcal{L}^2(\Omega,\mathcal{F}_T^{\xi},P)$, therefore (see Remark 4.3.1) it is natural to specify the operators $\{\rho_{t,T}\}_{0\le t\le T}$ as a g-evaluation w.r.t. BSDE (4.2) for some driver g satisfying Assumptions 2.3.7 and 2.3.17(i). We formalize these considerations below.

Corollary 4.3.2. *Consider a risk filter $\{\bar{\rho}_{t,T}\}_{0\le t\le T}$ whose decomposition component $\rho_{t,T}$ satisfies for all $0 \le t \le T$ the following conditions:*

- $\rho_{t,T}[X+\eta] = \rho_{t,T}[X] + \eta,\ \forall X \in \mathcal{L}^2(\Omega,\mathcal{G}_T,P),\ \forall \eta \in \mathcal{L}^2(\Omega,\mathcal{G}_t,P),$
- $\rho_{0,T}[Y+Z] - \rho_{0,T}[Y] \le \rho_{0,T}^{\mu,\nu}[Z],\ \forall Y,\ Z \in \mathcal{L}^2(\Omega,\mathcal{G}_T,P),$

where $\rho_{0,T}^{\mu,\nu}[\cdot,\cdot]$ is a g-evaluation with $g = \mu|y| + \nu|z|$ for some $\mu,\ \nu > 0$. By Theorem 2.3.16, $\rho_{t,T}$ is a g-evaluation with a unique driver g. If the driver g satisfies Assumption 2.3.17(i), then $\{\rho_{t,T}\}_{0\le t\le T}$ is indeed a dynamic risk measure.

From now on, we shall only consider risk filters satisfying the conditions of Corollary 4.3.2.

Definition 4.3.1. *For the risk filter $\{\bar{\rho}_{t,T}\}_{0\le t\le T}$, we define the value function at time t with initial conditional distribution $p = \{p_1, p_2, \dots, p_n\}$ where $p_i = P(\theta_t = \alpha_i),\ i = 1,2,...,n$, to be $V(t,p) = \bar{\rho}_{t,T}(\phi(\theta_T))$*

Proposition 4.3.3. *For a risk filter $\{\bar{\rho}_{t,T}\}_{0\le t\le T}$, the value function $V(t,p)$ can be represented by Y_t, which is deterministic, where the pair $\{Y_s, Z_s\}_{0\le s\le T}$ is a solution of a BSDE with a driver $g(\cdot,\cdot)$:*

$$Y_t = \bar{\rho}_{T,T}(\phi,\pi(T)) + \int_t^T g(s,Z_s)\,ds - \int_t^T Z_s\,d\overline{W}_s, \tag{4.3}$$

with $\{\pi(s)\}_{t\le s\le T}$ satisfying for $i = 1,\dots,n$ the system of SDE:

$$d\pi_i(s) = (\Lambda^*\pi)_i(s)\,ds + \frac{\pi_i(s)[A(\alpha_i,s) - \overline{A}(s)]}{B(s)}\,d\overline{W}_s, \quad \pi_i(t) = p_i = P(\theta_t = \alpha_i),$$

in which

$$(\Lambda^*\pi)_i(s) = \sum_{j=1}^n \lambda_{\alpha_j\alpha_i}(s)\pi_j(s), \qquad \overline{A}(s) = \sum_{j=1}^n A(\alpha_j,s)\pi_j(s),$$

and $\{\overline{W}_s\}_{t\le s\le T}$ is a Wiener Process given by the equation

$$d\overline{W}_s = \frac{d\xi_s - \overline{A}(s)\,ds}{B(s)}.$$

In this way, we decompose the estimation of the cost function into a two-stage problem: inside is the final risk estimator, and outside is the dynamic risk measure. Mathematically, there are two components of our risk model:

- $\bar{\rho}_{T,T}(\cdot)$, which is the final risk estimator; and
- the driver $g(\cdot,\cdot)$, which determines the convex dynamic risk measure part.

Specifically, if the driver $g \equiv 0$, and $\bar{\rho}_{T,T}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_T^{\xi})$, then by noticing that Y_t is a martingale in this case, we have:

$$\bar{\rho}_{t,T}(\cdot) = Y_t = \mathbb{E}(Y_T|\mathcal{F}_t^{\xi}) = \mathbb{E}(\bar{\rho}_{T,T}(\cdot)|\mathcal{F}_t^{\xi}) = \mathbb{E}(\mathbb{E}(\cdot|\mathcal{F}_T^{\xi})|\mathcal{F}_t^{\xi}) = \mathbb{E}(\cdot|\mathcal{F}_t^{\xi}).$$

It follows that the classical way of evaluating the partially observable problem is a special case of our risk filter.

4.4 Risk Filtering in the Running Cost Case

In this section, we consider the cost functional involving running cost: $\int_0^T \tilde{c}(\theta_s)\ ds + \phi(\theta_T)$, where $\tilde{c} : E \to \mathbb{R}$. In the classical case, the evaluation of the cost to go at time t is via tower property:

$$\mathbb{E}\left[\int_t^T \tilde{c}(\theta_s)\ ds + \phi(\theta_T) \,\middle|\, \mathcal{F}_t^{\xi}\right] = \mathbb{E}\left[\int_t^T \sum_{j=1}^{n} \tilde{c}(\alpha_j)\pi_j(s)\ ds + \phi(\theta_T) \,\middle|\, \mathcal{F}_t^{\xi}\right].$$

We can further rewrite it as $\mathbb{E}\left[\int_t^T c(\pi(s))\, ds + \phi(\theta_T) \,\middle|\, \mathcal{F}_t^{\xi}\right]$ with

$$c(\pi) = \sum_{j=1}^{n} \tilde{c}(\alpha_j)\pi_j. \tag{4.4}$$

A natural generalization of this cost functional is to replace the conditional expectation by a risk filter $\bar{\rho}_{t,T}(\cdot)$ and the function (4.4) by a *state risk evaluator* $c : \mathbb{R}^n \to \mathbb{R}$. We make the following assumption.

Assumption 4.4.1. *The function c is Lipschitz continuous.*

Then, the evaluation of the cost functional can be carried out as follows.

Theorem 4.4.2. *Under Assumption 4.4.1, with* $0 \le t \le T$*, given a cost function of the form* $\int_t^T c(\pi(s))\,ds + \phi(\theta_T)$*, the value function with respect to the risk filter* $\{\bar{\rho}_{t,T}\}_{0\le t\le T}$ *has the following representation:*

$$V(t,p) = \bar{\rho}_{t,T}\left[\int_t^T c(\pi(s))\,ds + \phi(\theta_T)\right] = \rho_{t,T}\left[\int_t^T c(\pi(s))\,ds + \bar{\rho}_{T,T}(\phi,\pi(T))\right], \tag{4.5}$$

where $\rho_{t,T}(\bar{\rho}_{T,T}(\cdot))$ *is the decomposition of* $\bar{\rho}_{t,T}(\cdot)$ *of Theorem 4.2.1.*

Proof. From the decomposition theorem of the nonlinear operator $\bar{\rho}_{t,T}$,

$$\bar{\rho}_{t,T}\left[\int_t^T c(\pi(s))\,ds + \phi(\theta_T)\right] = \rho_{t,T}\left[\bar{\rho}_{T,T}\left[\int_t^T c(\pi(s))\,ds + \phi(\theta_T)\right]\right].$$

Since for any $t \le s \le T$, $c(\pi(s)) \in \mathcal{L}_2(\Omega, \mathcal{F}_s^{\xi}, P)$, then $\int_t^T c(\pi(s))\,ds \in \mathcal{L}_2(\Omega, \mathcal{F}_T^{\xi}, P)$ as well. Therefore, by the translation property of $\bar{\rho}_{T,T}$,

$$\bar{\rho}_{T,T}\left[\int_t^T c(\pi(s))\,ds + \phi(\theta_T)\right] = \int_t^T c(\pi(s))\,ds + \bar{\rho}_{T,T}[\phi(\theta_T)].$$

Also, since $\bar{\rho}_{T,T}$ is law invariant, we can write it as

$$\bar{\rho}_{T,T}[\phi(\theta_T)] = \bar{\rho}_{T,T}(\phi,\pi(T)).$$

Combining all the equations above, we obtain equation (4.5). □

Similar to Proposition 4.3.3, the value function with initial distribution p at time t can be described by Y_t, which is part of the unique solution $\{Y_t, Z_t\}$ of a BSDE.

Proposition 4.4.3. *Under Assumption 4.4.1, with* $0 \le t \le T$*, the value function at time t, with initial distribution* $p_i = P(\theta_t = \alpha_i)$*,* $i = 1,\ldots,n$*, is* $V(t,p) = Y_t$*, which is deterministic, where the triple* $\{\pi(s), Y_s, Z_s\}_{t\le s\le T}$ *is the solution of the FBSDE system:*

$$d\pi_i(s) = (\Lambda^*\pi)_i(s)\,ds + \frac{\pi_i(s)[A(\alpha_i,s) - \overline{A}(s)]}{B(s)}\,d\overline{W}_s,\ \pi_i(t) = p_i,\ i = 1,\ldots,n \tag{4.6}$$

$$-dY_s = [c(\pi(s)) + g(s,Z_s)]\,ds - Z_s\,d\overline{W}_s,\ Y_T = \bar{\rho}_{T,T}(\phi,\pi(T)), \tag{4.7}$$

in which

$$(\Lambda^*\pi)_i(s) = \sum_{k=1}^n \lambda_{\alpha_k\alpha_i}(s)\pi_k(s), \qquad \overline{A}(s) = \sum_{k=1}^n A(\alpha_k,s)\pi_k(s),$$

and $\{\overline{W}_s\}_{t\le s\le T}$ *is a Wiener process given by the equation:*

$$d\overline{W}_s = \frac{d\xi_s - \overline{A}(s)\,ds}{B(s)}.$$

It is evident that under Assumptions 2.3.7, 2.3.17, and 4.4.1, the forward-backward system (4.6)-(4.7) has a unique solution.

In this chapter, we successfully define a two-stage risk filter as a generalization of the classical case and in the following chapter, we will introduce a control into the system and derive the so-called risk averse control problem, and we will discuss the problem with running cost since it's a more general case.

Remark 4.4.4. *The risk measures and risk-averse control problem of partially observable system in discrete time setting has also been explored by J. Fan, A. Ruszczyński in [30].*

Chapter 5

The Risk Averse Control Problem

In this section, we discuss the risk averse control problems for the partially observable system. As in the previous chapter, we will focus on the Wonham filter as well. We put the control parameter into the transition kernel of the unobservable process $\{\theta_t\}_{0\leq t\leq T}$, which means that we can control the rate of the change w.r.t. $\{\theta_t\}_{0\leq t\leq T}$. One thing we have to be careful here is that the control we are considering here is piecewise-constant control for the reason that if we consider the more general continuous control, for any given time t, the transition rate at that time will not be well-defined since it is actually depending on the whole history path up to time t. We derive a similar equation for the optimal nonlinear filter as the one without control on each small interval (same interval as the piecewise-constant control) and then we argue that the innovation process in each of these interval can be patched together as a whole Wiener process starting from time 0 to time T. Finally, we define the risk averse control problem as well as the associated decoupled FBSDE system in this settings.

5.1 Equations of the Optimal Nonlinear Filter

From now on, we discuss the case with running costs and terminal cost, as the most general one.

Definition 5.1.1. *A stochastic process $u(\cdot)$ is called an* admissible control *if $u(\cdot)$ is an element of the set:*

$$\mathcal{U} := \{u : [0,T]\times\Omega \to U \,\big|\, u(\cdot) \text{ is } \{\mathcal{F}_t^{\xi}\}\text{-adapted}\},$$

where $U \in \mathbb{R}^m$ is a compact set.

We focus on piecewise-constant controls. For $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$, and

$i = 0, \ldots, N-1$ we define

$$\mathcal{U}_i^N = \{u \in \mathcal{U} \,\big|\, u(t) = u(t_j),\ \forall\, t \in [t_j, t_{j+1}),\ \forall\, j = i, \ldots, N-1\}. \tag{5.1}$$

In our partially observable problem, the control u impacts the transition rate $\lambda(\cdot)$, hence we have a condition similar to Assumption 3.2.2.

For $j = 0, 1, 2, \ldots, N-1$ and $t_j \le s < t < t_{j+1}$ we define the transition probability

$$P_{\alpha\beta}^{\zeta}(s,t) = P(\theta_t = \beta \,|\, \theta_s = \alpha, u_t = \zeta),$$

where ζ is the value of the control in the interval $[t_j, t_{j+1})$. Based on this, we can further define the controlled transition rates as follows.

Condition 5.1.1. *For every $j = 0, 1, \ldots, N-1$, and for every $s \in [t_j, t_{j+1})$ the following limits are well defined and uniformly bounded:*

$$\lambda_{\alpha\beta}(s,\zeta) = \begin{cases} \lim\limits_{t \downarrow s} \dfrac{P_{\alpha\beta}^{\zeta}(s,t)}{t-s} & \text{if } \beta \neq \alpha, \\ -\sum\limits_{\gamma \in E, \gamma \neq \alpha} \lim\limits_{t \downarrow s} \dfrac{P_{\alpha\gamma}^{\zeta}(s,t)}{t-s} & \text{if } \beta = \alpha. \end{cases} \tag{5.2}$$

First, we recall the forward Kolmogorov equations.

Lemma 5.1.2. *Under Condition 5.1.1, for all $t_j \le s < t < t_{j+1}$ and for all $\alpha, \beta \in E$, the transition probabilities $P_{\alpha\beta}^{\zeta}(s,t)$ satisfy the equations*

$$P_{\alpha\beta}^{\zeta}(s,t) = \delta(\beta,\alpha) + \int_s^t \sum_{\gamma \in E} \lambda_{\gamma\beta}(r,\zeta) P_{\alpha\gamma}^{\zeta}(s,r)\, dr.$$

This allows us to derive the following lemma.

Lemma 5.1.3. *Suppose Condition 5.1.1 holds. For every $j = 0, 1, \ldots, N-1$, every $\beta \in E$, and every $\zeta \in U$, let*

$$m_t^{\beta,\zeta} = \delta(\beta, \theta_t) - \delta(\beta, \theta_{t_j}) - \int_{t_j}^t \lambda_{\theta_r \beta}(r,\zeta)\, dr, \quad t_j \le t < t_{j+1}.$$

Then the process $(m_t^{\beta,\zeta}, \mathcal{F}_t)$ is a square-integrable martingale with right-continuous trajectories on $[t_j, t_{t+1})$, having limits from the left.

Proof. Since θ_t is right continuous and $|\lambda_{\theta_t\beta}(t,\zeta)| \le K$ for some K, then $m_t^{\beta,\zeta}$ is right continuous and uniformly bounded. To verify the martingale property, for $t_j \le s < t < t_{j+1}$ we derive the chain of equalities:

$$\begin{aligned}
\mathbb{E}[m_t^{\beta,\zeta}\big|\mathcal{F}_s] &= \mathbb{E}\Big[m_s^{\beta,\zeta} + \delta(\beta,\theta_t) - \delta(\beta,\theta_s) - \int_s^t \lambda_{\theta_r\beta}(r,\zeta)\,dr\Big|\mathcal{F}_s\Big] \\
&= m_s^{\beta,\zeta} - \delta(\beta,\theta_s) + \mathbb{E}\Big[\delta(\beta,\theta_t) - \int_s^t \lambda_{\theta_r\beta}(r,\zeta)\,dr\Big|\mathcal{F}_s\Big] \\
&= m_s^{\beta,\zeta} - \delta(\beta,\theta_s) + P^{\zeta}_{\theta_s\beta}(s,t) - \int_s^t \sum_{\gamma\in E} \lambda_{\gamma\beta}(r,\zeta)P^{\zeta}_{\theta_s\gamma}(s,r)\,dr.
\end{aligned}$$

Due to Lemma 5.1.2, the last expression is equal to $m_s^{\beta,\zeta}$. □

Consider an interval $[t_j, t_{j+1})$, where $j \in \{0, 1, \dots, N-1\}$. Let

$$p_i = P[\theta_{t_j} = \alpha_i|\mathcal{F}^{\xi}_{t_j}], \quad i = 1, \dots, n.$$

Under the assumption that $u_t = \zeta$ for $t \in [t_j, t_{j+1})$, we define the process

$$\pi_i^{p;\zeta}(t) = P[\theta_t = \alpha_i|\mathcal{F}^{\xi}_t], \quad i = 1, \dots, n.$$

We can now describe the dynamics of the process $\pi^{p;\zeta}$ on this interval.

Theorem 5.1.4. *Let Conditions 3.2.1 and 5.1.1 hold. Then for $t \in [t_j, t_{j+1})$, the posterior probabilities $\pi_i^{p;\zeta}(\cdot)$, $i \in \{1, \dots, n\}$ satisfy the following system of n equations:*

$$\pi_i^{p;\zeta}(t) = p_i + \int_{t_j}^t (\Lambda_j^*\pi^{p;\zeta})_i(r)\,dr + \int_{t_j}^t \pi_i^{p;\zeta}(r)\frac{A(\alpha_i,r) - \overline{A}_j^{\zeta}(r)}{B(r)}\,d\overline{W}_r^j, \tag{5.3}$$

where

$$(\Lambda^*\pi^{p;\zeta})_i(r) = \sum_{k=1}^n \lambda_{\alpha_k\alpha_i}(r,\zeta)\pi_k^{p;\zeta}(r) \tag{5.4}$$

$$\overline{A}^{\zeta}(r) = \sum_{k=1}^n A(\alpha_k,r)\pi_k^{p;\zeta}(r), \tag{5.5}$$

and $\{\overline{W}^j\}_{t_j\le t<t_{j+1}}$ is a Wiener process with

$$\overline{W}_t^j = \int_{t_j}^t \frac{d\xi_r - \overline{A}_j^{\zeta}(r)\,dr}{B(r)}. \tag{5.6}$$

Proof. By Lemma 5.1.3, the process

$$m_t^{\beta,\zeta} = \delta(\beta,\theta_t) - \delta(\beta,\theta_{t_j}) - \int_{t_j}^t \lambda^j_{\theta_r\beta}(r,\zeta)\,dr, \quad t_j \le t < t_{j+1},$$

is a square integrable martingale. First, we claim that the quadratic variation

$$\left\langle m^{\beta,\zeta}, W\right\rangle_t \equiv 0 \quad (P \text{ a.s.}), \quad t_j \le t < t_{j+1}. \tag{5.7}$$

For any partition Γ of the time interval $[t_j,t]$ with $t_j = \Gamma_0 < \Gamma_1 < \Gamma_2 < \cdots < \Gamma_L = t$, we define $\|\Gamma\| = \max_\ell(\Gamma_{\ell+1} - \Gamma_\ell)$. If we denote by

$$Q_\Gamma(D,W) = \sum_{\ell=0}^{L-1}(D_{\Gamma_{\ell+1}} - D_{\Gamma_\ell})(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell}),$$

then the quadratic variation $\langle D,W\rangle_t = \lim_{\|\Gamma\|\to 0} Q_\Gamma(D,W)$.

Let $D_t = \delta(\beta,\theta_t) - \int_{t_j}^t \lambda_{\theta_s\beta}(s,\zeta)\,ds$. For a partition Γ, we calculate $Q_\Gamma(m^{\beta,\zeta},W)$ as follows:

$$\begin{aligned}
Q_\Gamma(m^{\beta,\zeta},W) &= \sum_{\ell=0}^{L-1}(m^{\beta,\zeta}_{\Gamma_{\ell+1}} - m^{\beta,\zeta}_{\Gamma_\ell})(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})\\
&= \sum_{\ell=0}^{L-1}(D_{\Gamma_{\ell+1}} - D_{\Gamma_\ell})(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})\\
&= \sum_{\ell=0}^{L-1}\Big(- \int_{\Gamma_\ell}^{\Gamma_{\ell+1}} \lambda_{\theta_s\beta}(s,\zeta)\,ds + \delta(\beta,\theta_{\Gamma_{\ell+1}}) - \delta(\beta,\theta_{\Gamma_\ell})\Big)(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})\\
&= \sum_{\ell=0}^{L-1}(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})(\delta(\beta,\theta_{\Gamma_{\ell+1}}) - \delta(\beta,\theta_{\Gamma_\ell})) - (W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})\int_{\Gamma_\ell}^{\Gamma_{\ell+1}} \lambda_{\theta_s\beta}(s,\zeta)\,ds.
\end{aligned}$$

On the one hand, we have

$$\begin{aligned}
&\Big|\sum_{\ell=0}^{L-1}(W_{\Gamma_{\ell+1}}) - W_{\Gamma_\ell}))(\delta(\beta,\theta_{\Gamma_{\ell+1}}) - \delta(\beta,\theta_{\Gamma_\ell}))\Big|\\
&\le \max_\ell\Big|(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})\Big|\Big|\sum_{\ell=0}^{L-1}\delta(\beta,\theta_{\Gamma_{\ell+1}}) - \delta(\beta,\theta_{\Gamma_\ell})\Big|.
\end{aligned}$$

The process $\{\theta_t\}$ is right continuous and with probability 1 has only finitely many jumps in any time interval, because the transition rate $\lambda_{\alpha\beta}(t,\zeta)$ is uniformly bounded. Let M be the random number of jumps of $\delta(\beta,\theta_s)$ in the time interval $[t_j,t]$. Since the jump size is 1, we conclude that

$$\Big|\sum_{\ell=0}^{L-1}(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})(\delta(\beta,\theta_{\Gamma_{\ell+1}}) - \delta(\beta,\theta_{\Gamma_\ell}))\Big| \le M\max_\ell\Big|W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell}\Big|. \tag{5.8}$$

This expression converges to zero with probability 1, as $\|\Gamma\| \to 0$.

On the other hand,

$$\begin{aligned}
&\Big|\sum_{\ell=0}^{L-1}(W_{\Gamma_{k+1}} - W_{\Gamma_k})\int_{\Gamma_\ell}^{\Gamma_{\ell+1}} \lambda_{\theta_s,\beta}(s,\zeta)\,ds\Big| \\
&\leq \sum_{\ell=0}^{L-1}\Big|(W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell})\int_{\Gamma_\ell}^{\Gamma_{\ell+1}} \lambda_{\theta_s,\beta}(s,\zeta)\,ds\Big| \\
&\leq \max_\ell \big|W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell}\big| \sum_{\ell=0}^{L-1}\int_{\Gamma_\ell}^{\Gamma_{\ell+1}} \big|\lambda_{\theta_s,\beta}(s,\zeta)\big|\,ds \\
&\leq \max_\ell \big|W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell}\big| \sum_{\ell=0}^{L-1}\int_{\Gamma_\ell}^{\Gamma_{\ell+1}} K\,ds \\
&\leq K(t-t_j)\max_\ell \big|W_{\Gamma_{\ell+1}} - W_{\Gamma_\ell}\big|.
\end{aligned}$$

This expression also goes to 0, as $\|\Gamma\| \to 0$. In view of (5.8), we conclude that (5.7) is true.

Actually, since the control ζ is fixed and known at the time t_j, which means the the transition rate $\lambda_{\alpha\beta}(t,\zeta)$ is deterministic for every $t \in [t_j, t_{j+1})$. Therefore, $m_t^{\beta,\zeta}$ is independent of W on the interval $[t_j, t_{j+1})$. Hence, $\left\langle m^{\beta,\zeta}, W\right\rangle_t \equiv 0 \quad (P \text{ a.s.}), \quad t_j \leq t < t_{j+1}$.

Therefore, with all the assumptions satisfied, we can apply Theorem 8.1 from [47]. The following equation follows:

$$\begin{aligned}
(5.9)\quad \mathbb{E}[\delta(\beta,\theta_t)\big|\mathcal{F}_t^\xi] &= \mathbb{E}[\delta(\beta,\theta_{t_j})\big|\mathcal{F}_{t_j}^\xi] + \int_{t_j}^t \mathbb{E}[\lambda_{\theta_s\beta}(s,\zeta)\big|\mathcal{F}_s^\xi]\,ds \\
&+ \int_{t_j}^t \frac{\mathbb{E}[\delta(\beta,\theta_s)A(\theta_s,s)\big|\mathcal{F}_s^\xi] - \mathbb{E}[\delta(\beta,\theta_s)\big|\mathcal{F}_s^\xi]\mathbb{E}[A(\theta_s,s)\big|\mathcal{F}_s^\xi]}{B(s)}\,d\overline{W}_s^j,
\end{aligned}$$

where

$$(5.10)\qquad \overline{W}_t^j = \int_{t_j}^t \frac{d\xi_r - \mathbb{E}[A(\theta_r,r)\big|\mathcal{F}_r^\xi]\,dr}{B(r)}, \quad t \in [t_j, t_{j+1}),$$

is a Wiener Process.

Finally, we simplify (5.9) for $\beta = \alpha_i$ with $i \in \{1, 2, \ldots, n\}$:

$$\begin{aligned}
\mathbb{E}[\delta(\alpha_i,\theta_t)\big|\mathcal{F}_t^\xi] &= P(\theta_t = \alpha_i\big|\mathcal{F}_t^\xi) + \int_{t_j}^t \sum_{k=1}^n \lambda_{\alpha_k\alpha_i}(s)\pi_k^{p;\zeta}(s,\zeta)\,ds \\
&+ \int_{t_j}^t \frac{A(\alpha_i,s)\pi_i^{p;\zeta}(s) - \pi_i^{p;\zeta}(s)\sum_{k=1}^n A(\alpha_k,s)\pi_k^{p;\zeta}(s)}{B(s)}\,d\overline{W}_s^j \\
&= p_i + \int_{t_j}^t (\Lambda^*\pi)_i^{p;\zeta}(s)\,ds + \int_{t_j}^t \pi_i^{p;u}(s)\frac{A(\alpha_i,s) - \overline{A}_j^\zeta(s)}{B(s)}\,d\overline{W}_s^j.
\end{aligned}$$

By noticing that $\mathbb{E}[\delta(\alpha_i, \theta_t)\big|\mathcal{F}_t^{\xi}] = \pi_i^{p;\zeta}(t)$, we conclude that the equation (5.3) holds. $\square$

In the last result, on each time interval $[t_j, t_{j+1})$, we defined a Wiener process $\{\overline{W}_t^j\}$. Now we patch these Wiener processes together to obtain a Wiener process on the whole interval [0,T]. We set $\widetilde{W}_0 = 0$ and for $j = 0, 1, \ldots, N-1$ we define

$$\widetilde{W_t} = \widetilde{W}_{t_j} + \overline{W}_t^j = \widetilde{W}_{t_j} + \int_{t_j}^t \frac{d\xi_r - \sum_{k=1}^n A(\alpha_k, r)\pi_k^{p;\zeta_j}(r)\,dr}{B(r)}, \quad t_j \le t < t_{j+1},$$

where ζ_j is the control value on the interval $[t_t, t_{j+1})$.

Theorem 5.1.5. *The process $\widetilde{W}_t$ is a Wiener Process on $[0, T]$.*

Proof. For any $0 < s < t < T$, there exists k and j such that $t_k \le s < t_{k+1} \le t_j \le t < t_{j+1}$. Then by Itô formula, with $\imath$ denoting the imaginary unit,

$$\begin{aligned}
e^{\imath z\widetilde{W}_t} &= e^{\imath z\widetilde{W}_s} + \imath z\int_s^t e^{\imath z\widetilde{W}_r}\, d\widetilde{W}_r - \frac{z^2}{2}\int_s^t e^{\imath z\widetilde{W}_r}\, d\langle \widetilde{W}, \widetilde{W}\rangle_r \\
&= e^{\imath z\widetilde{W}_s} + \imath z\int_s^t e^{\imath z\widetilde{W}_r}\, dW_r \\
&\quad + \imath z\Big\{\int_s^{t_{k+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r, r) - \sum_{m=1}^n A(\alpha_m, r)\pi_m^{p;\zeta_k}(r)]\, dr \\
&\quad + \sum_{l=k+1}^{j-1}\int_{t_l}^{t_{l+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r, r) - \sum_{m=1}^n A(\alpha_m, r)\pi_m^{p;\zeta_l}(r)]\, dr \\
&\quad + \int_{t_j}^t e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r, r) - \sum_{m=1}^n A(\alpha_m, r)\pi_m^{p;\zeta_j}(r)]\, dr\Big\} - \frac{z^2}{2}\int_s^t e^{\imath z\widetilde{W}_r}\, dr.
\end{aligned}$$

We take the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_s^{\xi})$ in the above formula, by noticing that

$$\mathbb{E}\Big[\int_s^t e^{\imath z\widetilde{W}_r} dW_r\Big|\mathcal{F}_s^{\xi}\Big] = 0,$$

and

$$
\begin{aligned}
&\mathbb{E}\Big\{\int_s^{t_{k+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_k}(r)]\,dr\\
&\quad+\sum_{l=k+1}^{j-1}\int_{t_l}^{t_{l+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_l}(r)]\,dr\\
&\quad+\int_{t_j}^{t} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_j}(r)]\,dr\Big|\mathcal{F}_s^{\xi}\Big\}\\
&=\mathbb{E}\Big\{\int_s^{t_{k+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_k}(r)]\,dr\\
&\quad+\sum_{l=k+1}^{j-1}\int_{t_l}^{t_{l+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_l}(r)]\,dr\Big|\mathcal{F}_s^{\xi}\Big\}\\
&\quad+\mathbb{E}\Big\{\int_{t_j}^{t} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}\mathbb{E}\Big[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_j}(r)\Big|\mathcal{F}_r^{\xi}\Big]\,dr\Big|\mathcal{F}_s^{\xi}\Big\}\\
&=\mathbb{E}\Big\{\int_s^{t_{k+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_k}(r)]\,dr\\
&\quad+\sum_{l=k+1}^{j-1}\int_{t_l}^{t_{l+1}} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_l}(r)]\,dr\Big|\mathcal{F}_s^{\xi}\Big\}.
\end{aligned}
$$

In the last equation, we removed the following term by conditioning on $\mathcal{F}_r^{\xi}$, $r\in[t_j,t]$, and using the tower property:

$$
\begin{aligned}
&\mathbb{E}\Big\{\int_{t_j}^{t} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}\mathbb{E}\Big[A(\theta_r,r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_j}(r)\Big|\mathcal{F}_r^{\xi}\Big]\,dr\Big|\mathcal{F}_s^{\xi}\Big\}=\\
&\qquad\mathbb{E}\Big\{\int_{t_j}^{t} e^{\imath z\widetilde{W}_r}\frac{1}{B(r)}\Big[\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_j}(r)-\sum_{m=1}^n A(\alpha_m,r)\pi_m^{p;\zeta_j}(r)\Big]\,dr\Big|\mathcal{F}_s^{\xi}\Big\}=0.
\end{aligned}
$$

We use this trick iteratively by conditioning on $\mathcal{F}_r^{\xi}$, $r\in[t_\ell,t_{\ell+1}]$, to remove other terms in the previously displayed equation. Finally, we obtain the equation:

$$
\mathbb{E}\Big\{e^{\imath z(\widetilde{W}_t-\widetilde{W}_s)}\Big|\mathcal{F}_s^{\xi}\Big\}=1-\frac{z^2}{2}\int_s^t E\Big[e^{\imath z(\widetilde{W}_r-\widetilde{W}_s)}\Big|\mathcal{F}_s^{\xi}\Big]\,dr.
$$

From this equation we conclude that:

$$
\mathbb{E}\Big[e^{\imath z(\widetilde{W}_t-\widetilde{W}_s)}\Big|\mathcal{F}_s^{\xi}\Big]=e^{-\frac{z^2}{2}(t-s)}\quad P\text{-a.s.},
$$

which is the characteristic function of the normal distribution with expected value 0 and variance $t-s$. The same equation implies that $\widetilde{W}_t-\widetilde{W}_s$ is independent of $\mathcal{F}_s^{\xi}$. $\square$

Consider a piecewise constant control $u_t = \zeta_j$, for $t \in [t_j, t_{j+1})$, $j = 0, 1, \ldots, N-1$. By virtue of the above theorem, we can now claim that for different time intervals $[t_j, t_{j+1})$, the forward SDE describing the evolution of $\pi_i^{p;\zeta_j}(t)$ is actually driven by one Wiener Process $\{\widetilde{W}_t\}_{0\le t\le T}$. Therefore, we can "patch" the forward SDEs in different time intervals together to obtain the following result.

Proposition 5.1.6. *Let Conditions* 3.2.1 *and* 5.1.1 *hold. Then for any* $0 \le t \le T$*, the posterior probabilities* $\pi_i^{p;u}(t)$*,* $i \in \{1, 2, \ldots, n\}$ *satisfy the following system of n equations:*

$$\pi_i^{p;u}(t) = p_i + \int_0^t (\Lambda^*\pi^{p;u})_i(r)\, dr + \int_0^t \pi_i^{p;u}(r)\frac{A(\alpha_i, r) - \overline{A}^u(r)}{B(r)}\, d\widetilde{W}_r \tag{5.11}$$

where $p_i = P(\theta_0 = \alpha_i)$*,* $\pi_i^{p;u}(t) = \pi_i^{p;\zeta_j}(t)$ *for* $t \in [t_j, t_{j+1})$*, and for* $r \in [t_j, t_{j+1})$

$$(\Lambda^*\pi^{p;u})_i(r) = \sum_{k=1}^n \lambda_{\alpha_k\alpha_i}(r, \zeta_j)\pi_k^{p;u}(r), \tag{5.12}$$

$$\overline{A}^u(r) = \sum_{k=1}^n A(\alpha_k, r)\pi_k^{p;u}(r). \tag{5.13}$$

Our next step is to introduce the risk filter. We consider the running cost case and make the following assumption for the controlled system.

5.2 Risk filter for risk averse control problem

Assumption 5.2.1. *The running cost function* $c(\cdot,\cdot)$ *is Lipschitz continuous with respect to the first argument, and continuous with respect to the second argument.*

We summarize our derivations in the following result.

Proposition 5.2.2. *Suppose Assumption 5.2.1 is satisfied. For a risk filter* $\{\bar{\rho}_{t,T}\}_{0\le t\le T}$ *and a piecewise constant control* $u(\cdot)$*, the value function*

$$V^u(t, p) = \bar{\rho}_{t,T}\Big[\int_t^T c(\pi^{t,p;u}(s), u_s)\, ds + \phi(\theta_T)\Big] \tag{5.14}$$

at time t and with a $\mathcal{F}_t^\xi$*-measurable initial distribution p, is equal to* Y_t*, where* $\{\pi_i^{t,p;u}(s)\}$*,* $\{Y_s, Z_s\}$*, for* $t \le s \le T$*, are given by the following decoupled FBSDE system:*

$$d\pi_i^{t,p;u}(s) = (\Lambda^*\pi^{t,p;u})_i(s)\, ds + \frac{\pi_i^{t,p;u}(s)[A(\alpha_i, s) - \overline{A}^u(s)]}{B(s)} d\widetilde{W}_s \quad \pi_i^{t,p;u}(t) = p_i, \tag{5.15}$$

$$-dY_s = [c(\pi^{t,p;u}(s), u_s) + g(s, Z_s)]\, ds - Z_s\, d\widetilde{W}_s, \quad Y_T = \bar{\rho}_{T,T}(\phi, \pi^{t,p;u}(T)). \tag{5.16}$$

Under Conditions 3.2.1, 5.1.1 and Assumptions 2.3.7, 2.3.17, and 5.2.1, g for any $(t, p) \in [0, T] \times \mathcal{S}$, there exists a unique solution for the FBSDE system (5.15)–(5.16).

It follows that we have successfully transferred a partially observable problem into a fully observable system: Given a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = \{\mathcal{F}_T^{\widetilde{W}}\}$ and $\{\mathcal{F}_t^{\widetilde{W}}, 0 \leq t \leq T\}$ is the augmentation of the filtration generated by the Brownian Motion $\{\widetilde{W}_t, 0 \leq t \leq T\}$, we have the equivalent system (5.15)–(5.16), if the forward SDE starts at time t.

Chapter 6

Dynamic Programming

At the points $t_j,\ j = 0, 1, \ldots, N-1$, the equation (5.14) can be refined.

Theorem 6.0.1. *For any $j = 0, 1, \ldots, N-1$ and $p \in \mathcal{S}$, given a piecewise-constant admissible control u with values ζ_j in the intervals $[t_j, t_{j+1})$, we have:*

$$V^u(t_j, p) = \rho_{t_j,t_{j+1}}\Big[\int_{t_j}^{t_{j+1}} c(\pi^{t_i,p;u}(r), u_r)\, dr + V^u\Big(t_{j+1}, \pi^{t_i,p;u}(t_{j+1})\Big)\Big].$$

Proof. Using the translation property, we obtain:

$$\begin{aligned}
V^u(t_j, p) &= \rho_{t_j,T}\Big(\int_{t_j}^{T} c(\pi^{t_j,p;u}(r), u_r)\, dr + \bar{\rho}_{T,T}(\phi, \pi^{t_j,p;u}(T))\Big)\\
&= \rho_{t_j,t_{j+1}}\Big[\rho_{t_{j+1},T}\Big(\int_{t_j}^{T} c(\pi^{t_j,p;u}(r), u_r)\, dr + \bar{\rho}_{T,T}(\phi, \pi^{t_j,p;u}(T))\Big)\Big]\\
&= \rho_{t_j,t_{j+1}}\Big[\rho_{t_{j+1},T}\Big(\int_{t_j}^{t_{j+1}} c(\pi^{t_j,p;u}(r), u_r)\, dr\\
&\qquad + \int_{t_{j+1}}^{T} c(\pi^{t_j,p;u}(r), u_r)\, dr + \bar{\rho}_{T,T}(\phi, \pi^{t_j,p;u}(T))\Big)\Big]\\
&= \rho_{t_j,t_{j+1}}\Big[\int_{t_j}^{t_{j+1}} c(\pi^{t_j,p;u}(r), u_r)\, dr\\
&\qquad + \rho_{t_{j+1},T}\Big(\int_{t_{j+1}}^{T} c(\pi^{t_j,p;u}(r), u_r)\, dr + \bar{\rho}_{T,T}(\phi, \pi^{t_j,p;u}(T))\Big)\Big].
\end{aligned}$$

In the last expression, by the Markov property of the process $\pi^{t_j,p;u}(r)$, we have

$$\rho_{t_{j+1},T}\Big(\int_{t_{j+1}}^{T} c(\pi^{t_j,p;u}(r), u_r)\, dr + \bar{\rho}_{T,T}(\phi, \pi^{t_j,p;u}(T))\Big)\Big] = V^u\Big(t_{j+1}, \pi^{t_i,p;u}(t_{j+1})\Big),$$

which completes the proof. $\square$

We are now ready to analyze the optimal control problem. Given $j = 0, 1, \ldots, N$ and $p \in \mathcal{S}$, we define the optimal value function:

$$V(t_j, p) = \inf_{u \in \mathcal{U}_j^N} V^u(t_j, p), \tag{6.1}$$

where the inf is taken over all the available piecewise-constant control functions $u(\cdot)$ on the interval $[t_j, T]$, as defined in (5.1).

Theorem 6.0.2. *For any $j = 0, 1, \ldots, N-1$ and $p \in \mathcal{S}$, we have*

$$V(t_j, p) = \inf_{\zeta \in U} \rho_{t_j,t_{j+1}} \Big[\int_{t_j}^{t_{j+1}} c(\pi^{t_i,p;\zeta}(r), \zeta)\, dr + V\Big(t_{j+1}, \pi^{t_i,p;\zeta}(t_{j+1})\Big)\Big], \tag{6.2}$$

with $V(T, p) = \bar{\rho}_{T,T}(\phi, p)$.

Proof. Due to Theorem 6.0.1,

$$\begin{aligned}
\inf_{u \in \mathcal{U}_i^N} V^u(t_j, p) &= \inf_{u \in \mathcal{U}_i^N} \rho_{t_j,t_{j+1}} \Big[\int_{t_j}^{t_{j+1}} c(\pi^{t_i,p;\zeta_j}(r), \zeta_j)\, dr + V^u\Big(t_{j+1}, \pi^{t_i,p;\zeta_j}(t_{j+1})\Big)\Big] \\
&= \inf_{\zeta_j \in U} \rho_{t_j,t_{j+1}} \Big[\int_{t_j}^{t_{j+1}} c(\pi^{t_i,p;\zeta_j}(r), \zeta_j)\, dr + \inf_{u \in \mathcal{U}_{i+1}^N} V^u\Big(t_{j+1}, \pi^{t_i,p;\zeta_j}(t_{j+1})\Big)\Big].
\end{aligned}$$

The last equation follows from the monotonicity of $\rho_{t_j,t_{j+1}}$. Substitution of (6.1) for t_j and t_{j+1} yields (6.2). □

Numerical methods for solving the FBSDE system associated with (6.2) are discussed in [3].

Chapter 7

Conclusion

In this thesis, we mainly focus on the Wonham filter, we generalized the classical way of evaluating a cost function related to the unobservable process by expectation. In our setting, we construct a two-stage structure, in which the inside part is responsible for the estimation of the unobservable process while the outside part is responsible for the evaluation of the risk, which is the solution of a corresponding BSDE. Then we add a control component into the transition rate of the unobservable process and in particular, we consider piecewise-constant control. We first derive a similar equation for the estimator for unosbservable process in each small interval driven by separate innovation process, and then we prove that these innovation process can actually be patched together to formulate a Wiener process from beginning to end. As in the non-controlled problem, we introduce the evaluation of the risk via BSDE in the controlled problem. Finally, we derive the dynamic programming equation under the control problem and further for the optimal control problem.

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