A CONSTRUCTION OF THE STONE–ČECH COMPACTIFICATION AND DYNAMICAL APPLICATIONS

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Thesis Abstract

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The ultimate goal of this thesis is to present how the Stone–Čech Compactification can be used to capture the asymptotic behavior of dynamical systems.

We will first cover preliminary definitions and properties of dynamical systems and ε-semigroups. Then we will construct the Stone-Cech compactification using ultrafilters. The final section will recharacterize the dynamical structures in the first section in terms of the Stone-Cech Compactification and contain a proof of the Auslander Ellis lemma.

Contents

1 Introduction

The ultimate goal of this thesis is to present how the Stone–Čech Compactification can be used to capture the asymptotic behavior of dynamical systems.

We will first cover preliminary definitions and properties of dynamical systems and ε-semigroups. Then we will construct the Stone-Cech compactification using ultrafilters. The final section will recharacterize the dynamical structures in the first section in terms of the Stone-Cech Compactification and contain a proof of the Auslander Ellis lemma.

2 Basic Notions and Minimality

We begin with the basic setting, the topological dynamical system.

2.1 Definitions and basic lemmas

Def 2.1 Topological Dynamical System:

Let T be a group/ semigroup, X be a locally compact Hausdorff topological space π be a continuous function from $X \times T$ to X. We use the notation $\pi(x,t) = xt$ and observe the following properties:

(i) If T has identity e, $xe = x$.

(ii) for $s, t \in T$ and $x \in X$, $(xs)t = x(st)$

The ordered triple (X, π, T) is defined as a topological dynamical system. In most situations we will be able to suppress the π without loss of generality and simply write (X, T).

We will henceforth assume that T is a group unless otherwise stated. We may note that if T is a group, Xt is a homeomorphism. This follows from the fact that $(xt)t^{-1} = x(tt^{-1}) = x$ for each $x \in X$ and $t \in T$. We will now observe an example.

Example 2.2 The Shift System:

Here we introduce an example which we will develop through the section and will become very fruitful later. Let $X = \mathbb{N}^C$ where $C = \{c_1, ...c_n\}$ is a finite collection of elements. That is to say, X is the collection of all sequences of C. In addition, let X have the metric $d(x,y) = k$ if for all $n < k$, $x_n = y_n$, $x_k \neq y_k$. It is clear that X is compact and T_2. Finally, we will assign $\mathbb{N}$ the discrete topology.

Now, let $T : X \to X$ where $T(x) = y$ if and only if $y_n = x_{n+1}$. We call T the shift function. To show that T is continuous, let $y' \in X$ and V be the basis set $\{y|d(y', y) < \frac{1}{k}\}$. We can first observe that $T^{-1}(y') = \{x_{c_i}|x_1 = c_i,\ x_n = y_{n-1}\}$. Thus, for each $y \in V$, $T^{-1}(y) \in \cup\{x|d(x, x_{c_i}) < \frac{1}{k}\} = \cup U_{c_i}$ and so $T^{-1}(V) \subset \cup U_{c_i}$. Further, if $x \in \cup U_{c_i}$, then $T_x \in V$. So, $T^{-1}(V)$ is the finite union of open basis sets, hence T is continuous.

Finally, let $T(T(x)) = T^2(x)$. We can observe that $T^2(x)_n = x_{n+2}$, so inductively

$T^i(x)_n = x_{n+i}$. That $(X, T, \mathbb{N})$ is a topological dynamical system follows.

Our concern in this section is to examine the existence and properties of invariant structures that arise in (X, T).

Def 2.3 Orbit:

The orbit of $x \in X$, denoted $\mathrm{Orb}(x)$ or $\mathcal{O}(x)$ is given by $xT = \{xt | t \in T\}$.

Def 2.4 Invariant Set:

A set $A \subseteq X$ is invariant if for all $t \in T$, $At \subseteq A$.

We will sometimes refer to invariant sets as subsystems.

Lemma 2.5 :

i) M is invariant implies $\overline{M}$ is invariant

ii) $\mathcal{O}(x)$ is invariant

iii) $\overline{\mathcal{O}(x)}$ is invariant

iv) if M and N are invariant, $M \cap N$ is invariant.

i) Let M be invariant. Let $t \in T$, $x \in \sigma M$ and $V \subseteq X$ be any open set containing xt. Since $Xt : X \to X$ is continuous, $(Xt)^{-1}(V) = U$ is an open set containing x. Since $x \in \sigma M$, there is a point $y \in U$ which is in the interior of M. Of course, this means $yt \in V$. Thus, for any open set V containing xt, we can find a $yt \in \mathring{M}$ also contained in V. Thus, $xt \in \overline{M}$ and so, $\overline{M}$ is invariant.

ii) and iv) are clear from the definitions.

iii) follows from i) and ii).

We now define minimal sets. Minimal sets are of interest because they have very nice asymptotic behavior, which shall be seen later in this section.

Def 2.6 Minimal Set:

$M \subseteq X$ is said to be minimal if it is a closed, invariant set which contains no proper closed invariant subsets.

Lemma 2.7 :

Let (X, T) be a tds. X has a nonempty minimal set.

First, observe that (X, T) itself is an invariant set. Either X is minimal, or it has an invariant subset. Assuming the latter, we name this set M. By the previous lemma, we know that $\overline{M}$ is also invariant. Now, let I be a collection of invariant sets ordered by inclusion. We can leverage the previous lemma which states that the intersection of invariant sets is invariant to construct a set ordered by inclusion from the intersections of I. Since X is locally compact, this set has a minimal element, namely $\cap I$. Thus, by Zorn's Lemma there exists minimal sets.

Let us note that in general, X cannot be found to be the disjoint union of minimal sets. We will now begin to characterize minimal sets in more useful ways.

Lemma 2.8 :

M is minimal iff $\overline{\mathcal{O}(x)} = M \ \forall x \in M$.

The first direction is trivial as M could not be minimal if $\overline{\mathcal{O}(x)} \neq M$, as $\overline{\mathcal{O}(x)}$ is itself a closed invariant subset of M.

Now, given $M \subset X$, let $\overline{\mathcal{O}(x)} = M$ for all $x \in M$. First, it is clear M is closed and invariant, we need only observe minimality. Let $N \subseteq M$ be closed and invariant. We know that for any $x \in N$, $\overline{\mathcal{O}(x)} = M$. Thus, $M \subseteq \overline{\mathcal{O}(x)} \subseteq N$. Hence, $M = N$ and M is minimal.

2.2 Recurrence and Minimality

We are now going to use a family of ideas surrounding a point being recurrent to characterize minimality. The most basic form of recurrence is periodicity of a point x, where given a tds there exists some $t \in T$ such that $x = xt$. From here we obtain the concept of almost recurrence, which is a point getting close to itself often. To make this idea rigorous, we have a few definitions.

Def 2.9 Syndetic:

Let T be a topological group. $S \subset T$ is said to be right syndetic if there is a compact $F \subset T$ such that $SF = T$.

We can note that on $\mathbb{N}$ with the discrete topology syndetic sets are those with "bounded gaps", that is to say F is finite in addition to being compact. (This is, to the best of my knowledge the notion that was made rigorous by the above definition). We will later show for our purposes, the discrete topology is enough. But for now we work in a general topological semigroup.

Def 2.10 Almost Periodic Orbit:

x is said to be almost periodic if for every open U which contains x, the set $R(x, U) = \{t \in T : xt \in U\}$ is right syndetic.

Example 2.11 The shift system:

We will return to the shift system for an example of almost periodic points. Let X be the shift system, and let $x \in X$ be almost periodic. Then for any basis set $U = \{y | d(x, y) < \frac{1}{k}\}$, $R(x, U)$ is syndetic. Let $n \in R(x, U)$. Then $d(x, xn) < \frac{1}{k}$ which is to say, $x_i = xn_i$ for each $i < k$. We can observe from this that the first k terms of the sequence x repeat very often, and using the definition of syndetic we can figure out how often. Let $F = \{f_1, ..., f_n\}$ be increasing, and $R(x, U)F = \mathbb{N}$ and let $n \in R(x, U)$. Then $x_i = x_{i+n}$ for all $i < k$. Since $R(x, U)$ is syndetic, there is some n' such that $k + n < n' < n + k + f_n$ and $d(xn', x) < \frac{1}{k}$.

We now arrive at the first characterization of minimality, and our first important theorem.

Theorem 2.12 :

Let (X, T) be a tds, with $x_0 \in X$, T a discrete topological group, and X a locally compact Hausdorff space.

$\overline{\mathcal{O}(x_0)}$ is a compact, minimal set if and only if x_0 is almost periodic.

We first note that since T is discrete, S is a syndetic set implies there is a finite $F \subset T$ such that $SF = T$.

[I] Let x_0 be an almost periodic point. That is to say, given any open U that contains x_0, $R(x_0, U)$, the set of return times to U is syndetic. We will first show that $\mathcal{O}(x_0)$ is compact, then that it is minimal.

To show compactness, we first show that if F_U is the finite set corresponding to $R(x_0, U)$, $\mathcal{O}(x_0) \subset UF_U = T$. Let $y \in \mathcal{O}(x_0)$. Then $y = x_0 t_y$. By the definition of syndetic, there is some $t_0 \in R(x_0, U)$ and $f_0 \in F_U$ such that $t_0 f_0 = t_y$. Hence, $(xt_0)f_0 = y$ and since $xt_0 \in U$, $y = (xt_0)f_0 \in Uf_0$. So, for any $y \in \mathcal{O}(x_0)$ there is $f_0 \in F_U$ such that $y \in Uf_0$.

Finally, since X is locally compact Hausdorff, given open U containing x_0, we can find open V containing x_0 such that $x_0 \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. By the previous argument, VF_V covers $\mathcal{O}(x_0)$, so $\overline{\mathcal{O}(x_0)} \subseteq VF_V$. Since VF_V is the finite union of compact sets, $\overline{\mathcal{O}(x_0)}$ is a compact set.

To show that $\mathcal{O}(x_0)$ is minimal, we use the lemma above that states M is an minimal set if and only if $\overline{\mathcal{O}(x)} = M$ for every $x \in M$. So all we need to do is show that for any $x \in \mathcal{O}(x_0)$, $\mathcal{O}(x)$ is dense in $\mathcal{O}(x_0)$. Let $x, y \in \mathcal{O}(x_0)$. We will show that there is $t \in T$ such that for open V containing y, $xt \in V$. This can be achieved almost trivially in the group setting as $x = x_0 t_x$ and $y = x_0 t_y$. Hence, the statement holds if $t = t_x^{-1} t_y$. So, $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(x_0)}$ for every $x \in \mathcal{O}(x_0)$ and by the above lemma also in $\overline{\mathcal{O}(x_0)}$.

[II] Now, let $\overline{\mathcal{O}(x_0)}$ be compact and minimal. We shall show that it is almost periodic or equivalently, $R(x_0, U)$ is syndetic for all open U containing x_0.

We will first observe that for any open U containing x_0, UT is an open cover of $\overline{\mathcal{O}(x_0)}$ indexed by T. By compactness, we may choose a finite subcover indexed by a finite $F_U \subset T$, where $F_U = \{f_1, ..., f_n\}$.

For syndecity, we will show that given $t \in T$, there is $s \in R(x_0), U)$ and $f \in F$ such that $t = sf$. Let t be given. We know that there exists $f \in F_U$ such that $V = Uf$ contains $x_0 t$. Now, let $\hat{V}$ be the open neighborhoods of xt intersected with V. Also, let $\hat{U} = \{vf^{-1} | v \in \hat{V}\}$.

Now, by lemma 1.7, $\mathcal{O}(x_0)$ is dense in itself since it is minimal. Thus, we can choose a net $\{x_u\}_{\hat{U}} \subset U$ where $x_u \in u \in \hat{U}$. Now, we can observe that $\{x_u\}_{\hat{U}} f$ is a net such that we may find an element in every neighborhood of xt by construction. Thus, it converges uniquely (Hausdorff) to xt. Finally, we note that each $x_u f$ can be written as $x_0 s_u f$, where $s_u \in T$. By continuity, this means $s_u \to s$ where $s \in R(x_0, U)$ and $t = sf$. Thus, $R(x_0, U)$ is syndetic.

Our goal in all of this is to capture the behavior of certain points, namely to capture efficiently the concept of a proximal point.

Def 2.13 Proximal pairs:

Let $x, y \in X$. Then $(x, y) \in P(X)$ (the set of proximal pairs in X) if and only if there exists a net $t_\alpha \in T$ such that $\lim xt_\alpha = \lim yt_\alpha$.

In order to do this, we must construct a set in which we can embed T to capture limits in general, which is the purpose of section 3. We also need a relationship between semigroups and idempotents, elements u such that $u^2 = u$. This will be explored in the next brief section.

3 The ε semigroup

In this section, we will set up our second foundational structure, the ε-semigroup. This section will cover the basic definition of the ε-semigroup, and end with the introduction of minimal idempotents, which we will ultimately use to re-characterize recurrence.

Def 3.1 :

i) E is an ε-semigroup if it is compact, Hausdorff, and the left multiplication $L_p(q) = pq$ is continuous for every $p \in E$.

ii) $w \in E$ is an idempotent if and only if $w^2 = w$

The importance of the ε-semigroup to us is its relationship to the idempotent. We will first show that the collection of idempotents is nonempty.

Lemma 3.2 :

Let E be an ε-semigroup, then E contains an idempotent. We refer to the collection of idempotents in E as $J(E)$.

Let us consider the family $\mathbb{F} = \{\emptyset \neq N \subset E | N \cdot N \subseteq N,\ N \text{ closed}\}$. We can see that this set has minimal elements as we can construct nested families by intersection whose total intersection is nonempty by compactness. Let $M \in \mathbb{F}$ be minimal. Then, if $w \in M$, $wM \subset M$ by definition, and also a subset of $\mathbb{F}$. Hence by minimality $wM = M$. Then we can observe that $Q = \{q \in M | wq = w\} = L_w^{-1}(w)$ is nonempty, and closed by continuity of left multiplication and in $\mathbb{F}$. Thus, $Q = M$ and since $w \in M$, $w^2 = w$.

With the existence of idempotents, we are free to develop a further structure for the ε-semigroup. We will find that there is a very fundamental connection between idempotents and right ideals.

Def 3.3 :

Let E be a ε-semigroup. $I \subset E$ is a right ideal if it is closed and $IE \subseteq I$. I is said to be a minimal right ideal if it does not contain any proper right ideals of E. The definitions of left ideals and two sided ideals are analogous.

The existence of minimal ideals in E follows from Zorn's lemma and compactess.

Proposition 3.4 :

Let E be an ε-semigroup and I be a minimal right ideal. Then:

i) $J(I)$ is nonempty

ii) $vp = p$ for all $v \in J(I)$ and $p \in I$

iii) $I = \{Iv | v \in J(I)\}$ where the union is a disjoint one and each Iv is a group with identity v.

iv) qI is a minimal right ideal of E

i) $J(I)$ is nonempty as I is a closed subsemigroup of E and is hence itself an ε-semigroup.

ii) If $v \in J(I)$, by minimality $vI = I$ and there is $q \in I$ such that $vq = p$. This means that $v(vq) = vp$, so $v^2q = vq = p = vp$.

iii) First, we will show that v is an identity for I. As above, let $pv = q$. Then $q = pv = (pv)v = qv$. Thus, v is an identity for I. Now, let $q \in I$. Then $qI = I$ by minimality and so there is $r \in I$ such that $qr = v$ and so $q(rv) = v^2 = v$ and $rv \in Iv$ is a right inverse for q. Now, let $x \in Iv$ be the right inverse of rv. Then $q = qv = q(rvx) = (qrv)x = vx = x$, which makes rv the left inverse of q as well and Iv is a group with identity v.

Now, let $p \in I$. Since $pI = I$, $\{q \in I | pq = p\}$ is a closed subsemigroup (and therefore a ε-semigroup) and hence contains an idempotent v such that $pv = p$, so $p \in Iv$. Thus, $I = \cup_{J(I)} Iv$.

Finally, we show that the Iv's are disjoint. Let $u, v \in J(I)$, and let $p \in Iu \cap Iv$. Since Iu is a group, there is $r \in Iu$ such that $rp = u$. This of course means that $u \in Iv$. However, since $vv = v$, v must be the identity, and so $v = u$. This completes the proof of ii).

iii) Let $q \in E$. It is clear that qI is a right ideal. In addition, if K is a right ideal, $L_q^{-1}(K)$ is also a right ideal. The minimality of q follows from the minimality of I.

Finally, we will define and examine the concept of a minimal idempotent. To do this, we define a quasi order $<$ on $J(E)$, where if $u, v \in J(E)$, $v < u$ if and only if $vu = v$. This leads to a natural equivalence relation, where we would say that u v if and only if $u < v$ and $v < u$.

The notion of a minimal idempotent follows as one would expect it to. Now, we will attempt to show the existence and nature of minimal idempotents in ε-semigroups.

Lemma 3.5 :

Let $u \in J(E)$ and $I \subset E$ be a right ideal. Then uI contains an idempotent θ such that $\theta < u$.

Let $u \in J(E)$ and I be a right ideal. Then uI is a right ideal and must contain some ideal r. Let $r = uv$ where $v \in I$. Finally, set $\theta = ru = uvu \in uI$. Then $\theta^2 = (uvu)(uvu) = (uv)u^2(vu) = (uv)(uv)u = r^2u = ru = \theta$. Then $\theta u = (ru)u = ru^2 = ru = \theta$. Thus $\theta < u$.

It is natural at this point to inquire about the connection between minimal ideals and minimal idempotents. The next proposition will tie the two together.

Proposition 3.6 :

An idempotent is minimal if and only if it is contained in a minimal right ideal.

Let u be a minimal idempotent in E and I be a minimal right ideal. Then uI is a minimal right ideal which contains an idempotent θ such that $\theta < u$. By minimality, $u < \theta$. Hence, $u\theta = u \in uI$.

Now let I be a minimal right ideal. Then I contains an idempotent, say u. Let θ be some idempotent such that $\theta < u$. Then $(u\theta)(u\theta) = u(\theta u)\theta = u\theta^2 = u\theta$. So $u\theta$ is an idempotent in uI. Since I is minimal, u acts as a left ideal for elements of I, so $u = u\theta u = u\theta$. Thus, $u < \theta$.

This is enough of the theory of ε-semigroups for our purposes.

4 The Stone-Čech Compactification and its action on X

This section will first detail the construction of the Stone-Čech (βT) compactification. We will accomplish this via the set of ultrafilters on T. Following this, we will examine the unique extensions of functions of T to functions of βT and we will culminate the section by examining the necessary semigroup structure induced by the operation on T.

4.1 βT As constructed By Ultrafilters

For our purposes, we need only use T with the discrete topology.

Let T be a topological space, Y be a compact Hausdorff space, $f : T \to Y$ be continuous and i be the inclusion map. We define βT to be a Hausdorff compactification such that there is a unique continuous function ϕ such that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "i"] \arrow[dr, "f"'] & \beta T \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$

The first thing to note is that βT is unique up to homeomorphism. This is simply a result of the fact that we may find a unique extension of the inclusion map from T to βT to any other construction of the Stone-Čech compactification.

We will use ultrafilters to construct βT. The following two sections will establish the basics of maximal filters. Following this, we will construct the inclusion map from T to βT and show that our construction is indeed the Stone-Čech compactification.

4.1.1 Filters

Def 4.1 Filter:

Given a set T, (usually assumed to be infinite), a filter $\mathcal{F}$ on T is a collection of subsets which satisfy the following conditions:

i) $\emptyset \notin \mathcal{F}$

ii) for $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$

iii) If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$.

Example 4.2 Fréchet Filter:

Let us look at an example that will prove quite fruitful later. The Fréchet filter, or cofinite filter is given by $C = \{A \subset T : |A^c| < \infty\}$.

Def 4.3 Principal Filter:

We say $\mathcal{F}x_0 = \{A \subseteq T | x_0 \in A\}$ is the principal filter corresponding to $x_0 \in T$.

Def 4.4 Filter Base:

Let β be a collection of subsets of T. We say β is a filter base if

i) $\emptyset \notin \beta$

ii) If $A, B \in \beta$, then we can find $C \in \beta$ such that $C \in A \cap B$.

Lemma 4.5 :

Let β be a filter base. Then

i)$\hat{\beta} = \{A | C \subseteq A \text{ for } B \in \beta\}$ is a filter

ii) $\beta \subset \hat{\beta}$

iii) If C is a filter, that contains β, $\hat{\beta} \subseteq C$

i) To show $\hat{\beta}$ is a filter, first observe that $\emptyset \notin \hat{\beta}$ by definition.

Then, let A, B be elements of $\hat{\beta}$.

Since for each $Y \in \hat{\beta}$ there is an element $Y^* \in \beta$ such that $Y^* \subset Y$, we can choose two such sets $A^* \subset A$ and $B^* \subset B$ where $A^*, B^* \in \beta$. Then by definition, we can choose $E^* \in \beta$ so that $E \subset A^* \cap B^*$ Then, we observe that as $E^* \subset A^* \cap B^* \subset A \cap B$, and by definition, $A \cap B \in \hat{\beta}$.

the fact that if $A \in \hat{\beta}$ then any B such that $A \subset B$ is also in β is trivial from the definition of $\hat{\beta}$.

ii) This is trivial.

iii) Let C be a filter which contains β. If $A \in \hat{\beta}$, there is some $A^* \in \beta$ that is a subset of A. Since $\beta \subseteq C$, $A \in C$. Then by the definition of filter, A must also be in C as it contains A^*. So, $\hat{\beta} \subset C$.

Def 4.6 Finite Intersection Property:

A family of sets $\mathcal{S}$ is said to have the FIP if any intersection over a finite set of members of $\mathcal{S}$ is nonempty.

Corollary 4.7 :

If $\mathcal{S}$ has FIP, then $\hat{B} = \{\cap_{i \in F} B_i | B_i \in S; |F| < \infty\}$ is a filter base.

To show that B is a filter base, we return to the definition. i) is obvious as the FIP is given.for ii), let $A, B \in \hat{B}$. Then $A = \cap B$ must itself be an element of $\hat{B}$ by definition.

Corollary 4.8 :

The FAE:

i) $\mathcal{S}$ has FIP

ii) There exists a filter $\mathcal{F}$ containing every set in $\mathcal{S}$.

This is almost trivial due to the previous corollary.

4.1.2 Ultrafilters

Def 4.9 Ultrafilter:

An ultrafilter u on T is a maximal element of the collection of filters on T. In other words, u is an ultrafilter iff $u \leq v \implies u = v$ where $u \leq v$ iff $A \in u \implies A \in v$.

The first order of business is to show that ultrafilters must exist on any set T.

Lemma 4.10 :

The family of filters on T contains maximal elements.

We wish to use Zorn's lemma to show there exists maximal filters. Let $\mathcal{F} = \{f_\alpha | \alpha \in \Sigma,$ where Σ is ordered and the index corresponds with the ordering on $\mathcal{F}\}$. be a linearly ordered collection of filters on T. To use Zorn's lemma, we must show that $\mathcal{F}$ has a maximal element. We use the set $F = \cup_{\Sigma}\mathcal{F}_{\alpha}$ to construct it. Since F has FIP, it is a filter base for some $\hat{F}$ which contains it. However, we can also note that if $A \in \mathcal{F}_{\alpha}$ then $A \in \hat{F}$ by a previous lemma. Thus, $\hat{F}$ is an upper bound of $\mathcal{F}$ and by Zorn's lemma, there must exist at least one maximal element for the collection of filters on T.

We will hence call the collection of ultrafilters on T βT For the sake of notation. The lack of proof that the collection of ultrafilters is the Stone-Céch compactification will soon be rectified.

First, a Lemma:

Lemma 4.11 :

For ever filter f on T, there is an ultrafilter u that contains f.

We have already shown that βT is nonempty. Now, let f be a filter, and $\mathcal{F}$ be the collection of filters that contain f. Since this set is partially ordered by the previously defined relation $\leq$, we may choose a maximal linearly ordered subset $\mathcal{T}$ of $\mathcal{F}$ by HMP. In addition, let $g = \cup\mathcal{T}$. We may observe that g is a filter.

i) $\emptyset \notin g$ by construction. ii) If $A, B \in g$, there are filters f_1 and f_2 in $\mathcal{T}$ such that $A \in f_1$ and $B \in f_2$. Since $\mathcal{T}$ is linearly ordered, then WLOG we can say that $f_1 \subset f_2$, which means $A \in f_2$ and so, $A \cap B \in f_2$. iii) Since $A \in g$ implies there is a filter $f_0 \in \mathcal{T}$ that contains A, then any B containing A must also be a subset of f_0 by its definition.

g is an upper bound of $\mathcal{T}$ trivially, so all that remains is to show that $g \in \beta T$. Let v be a filter on T such that $u \leq v$. We can observe that the set $\mathcal{T}\{v\}$ is a totally ordered set, and that v is an upper bound. However since $\mathcal{T}$ is maximal, This cannot be unless $v = u$. Thus, u is an ultrafilter.

Corollary 4.12 :

If $u \in \beta T$ and there is some non empty $A \in T$ such that $\cap u = A$, then u is a principal ultrafilter.

Let $\cap u = A$. Then for all $a \in A$, $\mathcal{N}a$ must contain A by the definition of filter. By maximality, $u = \mathcal{N}_{\dashv}$.

Lemma 4.13 :

Let $u \in \beta T$, $E \subseteq T$. Then $E \in u$ if and only if $E \cap F \neq \emptyset$ for all $F \in U$.

Assuming $E \in u$, the statement is trivial.

Now let $E \cap F \neq \emptyset$ for all $F \in u$. First, let $S = \{E \cap F | F \in u\}$. We can observe that S has FIP as $A, B \in S$ implies $A = E \cap F_1$ and $B = E \cap F_2$, where $F_1, F_2 \in u$, and $A \cap B = (E \cap F_1) \cap (E \cap F_2) = E \cap (F_1 \cap F_2) \neq \emptyset$ by construction.

Thus, S is a filter base and there is a filter f that contains it. Finally, we will show that $u \leq f$. Let $A \in u$. Then $E \cap A \in S$ and $E \cap A \in f$ so $A \in f$. Since u is an ultrafilter this means $u = f$ and so, $E \in u$.

The following theorem relates two useful characterizations of ultrafilters.

Theorem 4.14 :

The following are equivalent:

i) u is an ultrafilter on T

ii) If $A \cup B \in u$, then $A \in u$ or $B \in u$

iii) For any $A \subset T$, either A or A^c is an element of u

Let u be an ultrafilter, $A, B \subset T$ and $A \cup B \in u$. For contradiction, assume $A, B \notin u$. This would imply that there was some $C, D \in u$ such that $A \cap C = B \cap D = \emptyset$ However, this means $(A \cap C) \cup (B \cap D) = (A \cup B) \cap (A \cup D) \cap (B \cup C) \cap (B \cup D) = \emptyset$. This is of course a contradiction because each of the components on the lhs is in u. So, i) $\implies$ ii).

Now, let $A \subset T$. Since $A \cup A^c = T \in u$, then by ii) A or A^c is an element of u. Since $A \cap A^c = \emptyset$, only one of the sets can be in u.

Finally, let u be a filter with the property for all $A \subset T$, either A or A^c is in u. In addition, let f be an ultrafilter such that $f \geq u$. Now, let $F \in f$. We know that either F or F^c is an element of u. If F^c were an element of u, then F^c would also have to be an element of f. This is impossible however since f is an ultrafilter and this implies iii). So, $u \geq f$ and so $u = f$. Thus, iii) $\implies$ i).

4.1.3 A topology on βT

We now wish to supply βT with a topology.

Given $A \subseteq T$, $h(A)$ is defined as the collection of ultrafilters containing A. We will observe that $T = \{h(A)|A \subseteq T\}$ is a topological base. If $Y, Z \in T$, and $Y = h(A); Z = h(B)$, it follows $Y \cap Z = h(A) \cap h(B) = h(A \cap B)$, as $f \in h(A) \cap h(B) \implies A, B \in f \implies A \cap B \in f$ and $f \in h(A \cap B) \implies f \in h(A) \cap h(B)$ trivially.

The topology generated by the above base is an example of the hull-kernel topology. We will employ the mapping h as our inclusion map $T \hookrightarrow \beta T$ We highlight three properties:

Lemma 4.15 :

i) $h(A) \cap h(B) = \emptyset$ iff $A \cap B = \emptyset$.

ii) $h(A)^c = h(A^c)$ (Each h(A) is open and closed)

iii) $h(A) \cup h(B) = h(A \cup B)$

We can now proceed to the main properties, that βT is both Hausdorff and compact.

Proposition 4.16 :

βT is Hausdorff.

Let $p, q \in \beta T$, $p \neq q$. That is, there is some $A \in p$ that is not in q. By theorem 2.12, $A^c \in q$. Hence, $p \in h(A)$ and $q \in h(A^c)$. By property ii) above, $q \in h(A)^c$, which is an open set. Hence, we can separate p and q by disjoint open sets.

Proposition 4.17 :

βT is compact.

We will use a proof by contradiction to prove compactness. Let Σ be an indexing set, and $C = \{U\}_\Sigma$ be an open basic cover of βT. Also, let $\mathcal{F}$ be the collection of all finite subsets of Σ. We assume for contradiction that there is no finite subcover of C covering βT, that is to say for $F \in \mathcal{F}$, $\beta T \not\subset \cup\{U\}_F$. Since each U_f (with $f \in \mathcal{F}$) is a basic set, there is a set $A_f \in T$ such that $U_f = h(A_f)$. Hence, $\{U\}_F = \{h(\cup A_f)\}_F$.

Since $\{U\}_F$ is not a cover of βT, $\{U\}_F^c = \{h(A_f^c)\}_F \neq \emptyset$ by Lemma 2.18. We may push this further and observe that given $F_1, F_2 \in \mathcal{F}$, $((\cup_{F_1} U) \cup (\cup_{F_2} U))^c \neq \emptyset$. Otherwise, $\{U\}_{F_1 \cup F_2}$ is a finite open cover of βT which is false by assumption. Finally, we might observe the implication, that $(\cup_{F_1} h(A_f^c)) \cap (\cup_{F_2} h(A_f^c)) = h(\cup_{F_1} A_f^c) \cap h(\cup_{F_2} A_f^c) \neq \emptyset$. We know by lemma 2.18 that this implies $(\cup_{F_1} A_f^c) \cap (\cup_{F_2} A_f^c) \neq \emptyset$. Thus, $\{\cup_F A_f^c\}_{\mathcal{F}}$ has FIP and is a filter base. Thus, there is an ultrafilter v that contains it. However, by construction $v \notin \cup_\Sigma U$. Thus, C cannot be a cover of βT. This is a contradiction drawn from assuming βT is not compact. Therefore, βT is compact.

finally, we have a compact Hausdorff space in which to embed our group T.

4.2 βT as the Stone-Čech Compactification

We have observed that βT is a compact Hausdorff space that can be constructed from the collection of ultrafilters on a set T. We will now observe that βT is a construction of the Stone-Čech compactification. From here on, we define $i(x) = h(x)$, as we will be considering the map from elements of the underlying space to their principal ultrafilters our inclusion map.

To begin, we prove that i is continuous, and that $i(T)$ is a dense subset of βT. Unless otherwise stated, T will be equipped with the discrete topology.

Proposition 4.18 :

i) $i : T \to \beta T$ is injective, continuous and $i(T)$ is dense in βT.

i is 1 to 1 as the principal ultrafilter generated by a point is unique. Likewise the function is continuous because we assumed the discrete topology. For the density of $i(T)$, let $p \in \beta T$ and U be an open set containing p. We can choose $A \subset T$ such that $p \in h(A) \subset U$, and observe that for any $t \in A$, $i(t) \in h(A)$.

Finally, we prove the universal property.

Theorem 4.19 The Universal Property:

Let T be a set with the discrete topology, Y be compact Hausdorff, and $f : T \to Y$ be a continuous function. Then, there exists a unique continuous function $\phi : \beta T \to Y$ such that $\phi(i(t)) = f(t)$.

Let T be discrete, Y be compact Hausdorff, and $f : T \to Y$. We will leverage the properties of ultrafilters found in Appendix A. Let $u \in \beta T$. We will define $\phi(u)$ to be $\lim f(u)$.

First, we will show that the diagram commutes. Let $t \in T$. Since f is trivially continuous, $\lim f(i(t)) = f(t)$ by the filter characterization of continuity. Hence, the diagram commutes.

Now, we wish to observe that the function as extended to βT is also continuous. The first thing we can note is that on all of βT the function $\phi(u) = \lim f(u)$ is well defined, as Y is a compact Hausdorff space, so each ultrafilter $f(u)$ converges to a single point in Y.

Now, let $V \subset Y$ be an open set. To prove continuity, we will use the filter definition: ϕ is continuous if and only if for any convergent filter $F \to u$, $\phi(F) \to \phi(u)$.

Let $F \to u$ be a filter on βT. By definition of continuity by filters, for any basis set U containing u, there is a set $f \in F$ such that $f \subset U$. Now, let V be an open set in Y containing $\phi(u) = \lim f(u)$. Since $\lim f(u) \in Y$, there is a $\hat{V} \in f(u)$ such that $\hat{V} \subset V$. As $\hat{V} \in f(u)$, there is $A \subset T$ such that $f(A) = \hat{V}$. Since $f(A) \in u$, $u \in h(A)$. We can see that since $F \to u$, there is $B \in F$ such that $B \subset h(A)$. Finally, $\phi(B) \subset V$, since $\phi(B) \subset \phi(h(A)) = \hat{V} \subset V$. Hence, $\lim f(F) = \lim f(u)$

4.3 Extending Multiplication to βT

In order to capture dynamics through βT, we require another large piece of machinery. This is a semigroup structure on βT. We desire it to reflect the multiplication on T in the multiplication on βT.

To accomplish this, we apply the universal property. Let $t \in T$, and $\hat{L}_t : T \to \beta T$ be the map $s \to i(ts)$ for $s \in T$. The function is trivially continuous as T has the discrete topology. Therefore by the Universal Property there exists a continuous function L_t such that $L_t(i(s)) = i(ts)$ and $L_t(u) = \lim \hat{L}_t(u)$. We examine $\hat{L}_t(u)$.

$\hat{L}_t(u) = \{tA | A \in u\} = tu$. Let us examine the structure of this set. We will first prove that tu is itself an ultrafilter, and provide a more useful way of expressing it.

Remark 4.20

We use the fact that T is a group freely here by exploiting the inverse; however, the the multiplication can be extended for a semigroup just as well by using the inverse of the left multiplication.

Let $\hat{A}, \hat{B} \in tu$. Then as a result of T being a group there exists $A, B \in u$ such that $t^{-1}\hat{A}$ and $t^{-1}\hat{B} = B$. Since u is a filter, $A \cap B \in U$. Therefore, $t(A \cap B) \in tu$ and since $t(A \cap B) \subset tA \cap tB$,

$\emptyset \neq \hat{A} \cap \hat{B} \in tu$.

Now, let $\hat{C} \subset T$ such that $\hat{A} \subset \hat{C}$. Then $A = t^{-1}\hat{A} \subset t^{-1}\hat{C}$. Thus, $C \in tu$. This shows that tu is a filter. To prove that it is an ultrafilter, Let $D \subset T$. Then, either $(t^{-1}D)^c = t^{-1}D^c$ or $t^{-1}(D)$ is an element of u, which implies that either D or D^c is an element of tu, which implies that tu is an ultrafilter.

Since $\hat{L}_t(u)$ is an ultrafilter, we may simply say that $\lim \hat{L}_t(u) = \{tA | A \in u\} = tu$. In general then, we can refer to tu as the set $\{A \subset T | t^{-1}A \in u\}$.

We almost have a left multiplication on βT. To complete the concept, we will have to define left multiplication by an ultrafilter.

Let $u, v \in \beta T$. We wish to extend the above multiplication to these generic members. To do this, we will will need a more sophisticated method than simply examining the sets $t^{-1}A$. To do this, we will define the operation $Ap = \{t | \exists B \in p \text{ s.t. } Bt \subset A\}$. We can note that when $u = i(s)$ for some $s \in T$, $Au = \{t | \exists B \ni s \text{ s.t. } Bt \subset A\}$, which in context is simply $s^{-1}A$ as above.

Our natural extension then would be to say that $uv = \{A \subset T | Au \in v\}$. It is clear that this is the operation above when u is a principal ultrafilter, we will now show that uv is itself an ultrafilter. First, some properties:

Lemma 4.21 :

i) $A \subset B$ implies $Ap \subset Bp$

ii)$(A \cup B)u = Au \cup Bu$

iii)$(A \cap B)u = Au \cap Bu$

iv) $A^c u = (Au)^c$

i) is clear

ii) If $t \in (A \cup B)u$, then there is some $C \in u$ such that $Ct \subset A \cup B$. We can reconstitute C as $C_1 \cup C_2$, where $C_1 = \{s \in C | st \in A\}$ and likewise $C_2 = \{s \in C | st \in B\}$. By the properties of ultrafilters, either C_1 or C_2 is in u. WLOG let $C_1 \in u$, then $t \in Au$ and so $t \in Au \cup Bu$.

We can directly observe that $Au \cup Bu \subset (A \cup B)u$ by i).

iii) Let $t \in Au \cap Bu$. Then there is $C_1, C_2 \in u$ such that $C_1 t \subset A$ and $C_2 t \subset B$. As $C_3 = C_1 \cap C_2$, and $C_3 t \in A \cap B$, $t \in (A \cap B)u$ as well.

Once again, we can directly observe that $(A \cap B)u \subset Au \cap Bu$ by ii).

iv) By iii), $A^c u \cap Au = (A \cap A^c)u = \emptyset u = \emptyset$ and by ii) $A^c u \cup Au = (A^c \cup A)u = Tu = T$. Thus $A^c u = (Au)^c$.

Let uv be defined as above. First, definitionally it is impossible for the empty set to be in uv.

Next, let $U, V \in uv$. By definition, $Uu, Vu \in v$ and therefore $Uu \cap Vu \in v$. By iii), $Uu \cap Vu = (U \cap V)u \in v$ and so $U \cap V \in uv$.

Now, let $U \in uv$ and $A \subset T$ such that $U \subset A$. Then $Uu \subset Au$ by i), so $U \in uv$.

Finally, let $A \subset T$. We know by definition that either Au or $(Au)^c$ is in v. By property iv), this means either Au or $A^c u$ is in v, which means either A or A^c is in uv. This completes the proof that uv is an ultrafilter.

To make βT a topological semigroup, we must now observe that the left multiplication above is continuous.

Proposition 4.22 :

i) $L_u(v) = uv$ is continuous where $L_u(v) = \{A \subset T | Au \in v\}$

ii) $R_t(p)$ is continuous for all $t \in T$.

We have already showed that when $s \in T$, $u \in \beta T$, $u \to i(s)u$ is the unique continuous extension of left multiplication on T. Therefore by the density of $T \subset \beta T$, it is enough now to show that $i(s) \to ui(s)$ is also continuous. Let $\mathcal{F} \to u$ be a convergent filter on βT and $s \in T$; also, let $h(A') \ni ui(s)$ where $A' \subset T$. By definition $A' \in ui(s) = \{A \subset T | s \in Au\}$, that is to say $s \in \{t \in T | \exists B \in u \text{ s.t. } Bt \subset A'\}$. So, there is $B \in u$ such that $Bs \subset A'$. Finally, we observe that $u \in \overline{B}$; hence there is $F \in \mathcal{F}$ such that $F \subseteq \overline{B}$. Let $f \in F$. Then $B \in f$. Hence B is a member of f such that $Bs \in A'$. That means that $s \in \{t \in T | \exists B \in f \text{ s.t. } Bt \subset A'\}$, so $fi(s) \in \overline{A'}$ and $Fi(s) \subset \overline{A'}$. We have thus found $F \in \mathcal{F}$ such that $F \to Fi(s) \subset \overline{A'}$. So, for every basis open set containing $ui(s)$ there is a member of $\mathcal{F}$ mapped into it, therefore $Fi(s) \to ui(s)$, completing the proof.

ii) To show that $R_t(p)$ is continuous, it is enough to show that it is continuous for $i(s)$, $s \in T$ as the rest follows from the continuity of left multiplication. let $\mathcal{F} \to i(s)$ be a filter and $h(B) = V \ni i(s)i(t)$ be an open set. We have already shown that $i(s)i(t)$ is the principal ultrafilter $i(st)$, So $\{st\} \in V$ and so $s \in Vt^{-1}$. Now, we want to show that $h(Vt^{-1}) \subset R^{-1}(h(B))$. Let $p \in h(s^{-1}V)$. Then $pt = \{A | t \in Ap\}$. To show that $pt \in h(B)$, we have to show $B \in pt$. Note that $Bp = \{r | \exists b' \in p \text{ st } B'r \subset B\}$. Since $Bt^{-1} \in p$, $(Bt^{-1})t = B$ so $t \in Bp$ and so $B \in pt$ and finally, $p \in R_t^{-1}(V)$.

Now, for any open set containing $i(t)$ such as $h(s^{-1}V)$ there is $F \in \mathcal{F}$ such that $F \subset h(V)$. Since $F \subset R_t^{-1}(V)$, $Ft \in V$. This completes the proof.

Remark 4.23

We now briefly note that $i(ts) = i(t)i(s)$ for $s, t \in T$, if $A \in i(t)i(s)$, then $\exists B \in i(t)$ such that $Bs \in A$. But since $t \in B$, $ts \in A$ and so $A \in i(ts)$. Now, let $A \in i(ts)$. Then $ts \in A$, so $t \in As^{-1}$. Hence, there is a member B of $i(t)$ such that $Bs \in i(t)$. Thus, $A \in i(t)i(s)$.

Now we may prove that the left multiplication on βT is indeed an operation and so βT is a semigroup.

Proposition 4.24 :

For $p, q, r \in \beta T$, $(pq)r = p(qr)$.

The proposition follows from the remark above and the continuity of left multiplication.

Thus, βT is a ε-semigroup, and we may now eap the benefits.

5 Dynamical applications of βT

in order to proceed, we can now begin to describe an action of βT on X, and begin to characterize minimality and other dynamical properties in terms of idempotents of βT. We will use two different actions of βT; one of them on X and the other on subsets of βT itself.

Def 5.1 :

i) Let $A \subset \beta T$, $p \in \beta T$. Then $A * p = L_p^{-1}(\overline{A})$

ii) Let $x \in X$ and $u \in \beta T$. Then $xu = \lim L_x(u)$ in X.

Let us observe that these proposed operations are right actions.

i) is clearly continuous, as $\overline{A}$ is both open and closed. Now, let $p, q \in \beta T$. Then $(A * p) * q = (L_q^{-1}(\overline{A * p})) = L_q(\overline{L_p(\overline{A})}) = L_q(L_p(\overline{A})) = L_{pq}(\overline{A}) = (A) * (pq)$

ii) Let $x \in X$, $t \in T$, and $p, q \in \beta T$. The first thing to note, as that $\lim(xt)p$ is the unique extension of

Equipped with these tools, we can approach recurrence. The first order of business is to find a βT definition for syndecity. We will let $\mathcal{S}$ denote the collection of syndetic sets.

Proposition 5.2 Characterization of Syndetic Sets:

Let $A \subseteq T$, then the following are equivalent:

i)$A \in \mathcal{S}$

ii) $A * p \neq \emptyset$ for all $p \in \beta T$

iii)$A * p \neq \emptyset$ for all minimal idempotents $p \in \beta T$

Let $A \in \mathcal{S}$, and $p \in \beta T$; we will first show that $A * p$ is nonempty. First, by definition there exists finite $F \subset T$ such that $AF = T$. This implies that for all $t \in T$, there is $f \in F$ and $a \in A$ such that $af = t$ or $tf^{-1} = a \in A$. Thus, $T = \cup_F Af^{-1}$ and so $\beta T = \cup_F \overline{A}i(f^{-1})$. Hence, there is $f \in F$ such that $pi(f^{-1}) \in \overline{A}$ making $A{*}p$ nonempty.

i)$\rightarrow$ ii) clearly.

Now, let $A * u$ be nonempty for all minimal idempotents u in βT. Assume for contradiction that $\cup_F A f^{-1} \neq T$ for all finite $F \subset T$. Let $B_F = \cap_F \overline{(T \backslash A)} f^{-1}$. Then $\{B_F | F \text{ is finite}\}$ has FIP. By the compactness of βT, there is a $p \in \beta T$ such that $pt \in \overline{T \backslash A} = \beta T \backslash \overline{A}$ and so $\overline{pT} \subset \beta T \backslash \overline{A}$. Since $\overline{pT}$ is invariant, it contains a minimal subset M which in turn contains a minimal idempotent u such that $ut \in \beta T \backslash \overline{A}$ for all $t \in T$ and so $A * u = \emptyset$ which is a contradiction. Thus, $A \in \mathcal{S}$.

Lemma 5.3 :

The following are equivalent:

i) For every $U \ni x$, $R(x, U)$ is infinite

ii) There exists $u \in \beta T \backslash T$ such that $xu = x$.

Assume that $R(x, U)$ is infinite for every $U \ni x$. Consider $R = \{\overline{R(x, U)} \backslash T | U \ni x\}$.

Proposition 5.4 Chracterization of Almost Periodicity:

x is almost periodic if and only if every minimal set $M \subset \beta T$ contains an idempotent such u such that $xu = x$

Let x be almost periodic. Then $\overline{\mathcal{O}(x)}$ is minimal by Theorem 2.12. Now, let M be any minimal subset of βT. Then since $MT = M$, xM is a minimal subset of xM and thus $xM = \overline{\mathcal{O}(x)}$. It follows that $x \in xM$, so $L = \{p | xp = x\}$ is non-empty. Further, it is clearly closed under left multiplication so it is a subsemigroup, which is in turn topologically closed (as left multiplication is continuous, and open sets are closed in βT). Hence, L is an ε-semigroup and contains an idempotent.

Now, let $x \in X$, and any minimal set $M \subset \beta T$ contain an idempotent u such that $xu = x$. We must show that given open $U \ni x$, $R(x, U) * p \neq \emptyset$ for any minimal idempotent $p \in \beta T$. Let v be any minimal idempotent in βT. Then v is contained in some minimal right ideal $M \subset \beta T$, in which by hypothesis is an idempotent u such that $xu = x$. So, for any open $U \ni x$, there must be $A \in u$ such that $xA \subset U$. Hence, $u \in \overline{R(x, U)}$. Since this means $u^2 \in \overline{R(x, U)}$, $u \in L_u^{-1}(\overline{R(x, U)} = R(x, U) * u = R(x, U) * (vu) = (R(x, U) * v) * u$. The fact the left hand side is nonempty requires that $R(x, U) * v$ is nonempty as well. This completes the proof.

Now, we will capture proximality in terms of βT so that we may prove the Ellis Auslander lemma and press forward to a combinatorial result.

Proposition 5.5 Characterization of Proximality:

Let $x, y \in X$. Then the following are equivalent:

i) $(x, y) \in P(X)$

ii) There exists an idempotent $u \in \beta T$ such that $xu = yu$

iii) There exists a minimal idempotent $u \in \beta T$ such that $xu = yu$

Let (x, y) be a proximal pair. That is to say, There exists a net $t_\alpha \subset T$ such that $\lim xt_\alpha = \lim yt_\alpha$. By compact Hausdorff, there is some subnet t'_α of t_α which converges uniquely, say to $p \in \beta T$. Without loss of generality, we let $t_\alpha = t'_\alpha$. Thus, $Q = \{p | xp = yp\}$ is nonempty and A closed right ideal. Thus, it contains a minimal right ideal and a minimal idempotent.

That iii)$\rightarrow$ii) is immediate by minimality, and ii) $\implies$ i) is clear as we can construct a net corresponding to u.

All of this work leads us to a deceptively simple proof of a very powerful result.

Lemma 5.6 Ellis-Auslander:

Let (X, T) be a dynamical system. For every $x \in X$, there is a $y \in \overline{\mathcal{O}(x)}$ such that y is almost periodic and $(x, y) \in P(X)$.

Let $x \in X$,$u \in \beta T$ be a minimal idempotent, and $y = xu$. First, we can observe that y is almost periodic as for any idempotent $v \in \beta T$, $yv = (xu)v = x(uv) = xu = y$ by minimality of u. In addition, proximality is achieved trivially as $yu = (xu)u = xu^2 = xu$. Hence, we have chosen $y \in \overline{\mathcal{O}(x)}$ such that y is almost periodic and $(x, y) \in P(X)$.

The Ellis Auslander Lemma is a very powerful and deep tool for capturing the structure of a dynamical system. One of its corollaries is Schur's Coloring Theorem.

6 Appendix

6.1 A: Convergence By Filters

It will be Very useful to work out the elements of the theory of convergence by filters here. The goal in this section is to establish the filter analogue to sequential compactness in a Hausdorff space. All unstated definitions for this section are taken from Section 2 where the basic theory of filters is established.

Def 6.1 Convergence of a Filter:

Let X be a topological space and F be a filter. F converges to $x \in X$ if and only if for every open U containing x, There is $A \in F$ such that $A \subset U$. we notate this $\lim F = x$

Let us now derive two analogous properties of filters to those of nets. The first, is that we may define a Hausdorff space as one in which all convergent filters converge to unique points. The second is that a space is compact if and only if all ultrafilters are convergent.

Proposition 6.2 :

i) X is Hausdorff if and only if the limit of every convergent filter is unique.

ii) X is compact if and only if every ultrafilter on X converges.

i) Let F be a filter on X and $\lim F = x \in X$. Suppose X is Hausdorff. If there were another point, say $y \in X$ such that $\lim F = y$. Let $U, V \subset X$ be disjoint sets which contain x and y respectively. Then by the definition of convergence there are sets $A, B \in F$ such that $A \subset U$ and $B \subset V$. This cannot be however as $A \cap B = \emptyset$ and thus violate the finite intersection property of filter elements. Therefore, F converges to only x or y.

Now, let every convergent filter have a unique limit. For contradiction, let X not be Hausdorff. That is to say, there exists $x, y \in X$ such that every open set $U \ni x$ contains y, and the converse. Now, let u be the principal ultrafilter generated by x. Thus, $\lim u = x$. Now, let $U \ni y$ be open. Since $x \in V$ by assumption, there is some $A \in u$ such that $A \subset U$. This of course means that $y \in \lim u$. This cannot be, as we assumed u had a unique limit. Therefore, X must be Hausdorff.

ii) Let X be compact and let u be an ultrafilter on X. Recall if X is compact, any family of subsets of X having the FIP has non-empty intersection. Let $F = \{\overline{A} | A \in u\}$. Then

F is a filter, and since it has FIP, $\cap F \neq \emptyset$. Let $x \in \cap F$ and $U \ni x$ be open. Then it is clear that $U \cap A$ is nonempty for any $A \in F$. Therefore, $F \cup \{U\}$ is a filter containing F. Since u contains any refinement of F by the ultrafilter lemma, $U \in u$. Hence, any open set containing $x \in \cap F$ is an element of u and $x \in \lim u$. Thus we have shown that u converges to at least one point if X is compact.

Now let X be a topological space on which every ultrafilter converges. We will prove the compactness of X using the property that X is compact if and only if every collection of closed sets with FIP has nonempty intersection.

Let $A = \{A_i\}_I$ be a collection of closed sets with FIP. Then A is a filter base, and there is an ultrafilter u that contains A. By hypothesis, u converges. That is to say, there is at least one $x \in X$ such that $x \in \lim u$. Hence, if $V \ni$ is an open set there is a subset of V in u, which means $V \in u$. So, for any $U \in u$, $V \cap U$ is nonempty. Therefore, for any such U, $x \in \overline{U}$. So, every closed set in u contains x. Finally, this means that for each $A_i \in A$, $x \in A_i$ and so $\cap A \neq \emptyset$ making X compact.

To make our useful analogues to sequences complete, we will now define continuity in terms of ultrafilters. To do this, we must first note that if $f : X \to Y$, and f is a filter on X, then $f(F)$ is also a filter. The finite intersection property is clear as $A \cap B$ is nonempty implies $f(A) \cap f(B)$ is nonempty. In addition, if we let $C = f^{-1}(f(A) \cap f(B))$ we may note that C must include $A \cap B$. Therefore, $C \in F$ and $f(A) \cap f(B) = f(C)$, giving $f(F)$ property ii of the definition of a filter. For property iii, we can let $D \subset Y$ contain $f(A)$. Then $f^{-1}(D)$ contains A and is therefore an element of F. Thus, $f(F)$ has property iii of the filter definition.

The extension of this idea to ultrafilters follows from Theorem 2.14. Let $u \in \beta X$, and $f : X \to Y$ as before. If $A \subset Y$, then $B = f^{-1}(A)$ or $B^c = f^{-1}(A)^c$ is in u.

Now we are ready to establish continuity.

Def 6.3 Continuity of a function in terms of filters:

We say a function $f : X \to Y$ is continuous if and only if for every filter $F \to x \in X$, $f(F) \to f(x)$.

Proposition 6.4 :

The definition of continuity in terms of filters is equivalent to the standard definition.

Let $f : X \to Y$ be continuous and F be a filter on X such that $\lim F = x$. Since we have already established that $f(F)$ is a filter, to show $\lim f(F) = f(x)$, we need only show that for any open $V \ni f(x)$, there is $B \in f(F)$ such that $B \subset V$. This is clear however, as $f^{-1}(B)$ is an open set containing x, and therefore there is $A \in F$ which is a subset of $f^{-1}(B)$. The desired property follows, as $f(A) \in f(F)$, and $f(A) \subset B$.

Now, let $f : X \to Y$ and assume that for any filter F on X such that $\lim F = x$, $\lim f(F) = f(x)$.

WLOG, we will restrict the codomain to the range and assume that f is onto and that Y has the relevant relative topology. Let V be an open set in Y and $x \in f^{-1}(V) = U$ and let $y = f(x)$. We will show that U is open.

First, we will construct a specific filter, F_x. Let F_x to be the filter $\{A|$A contains an open set containing We will quickly verify that this family is indeed a filter. Let $A, B \in F_x$, and let $\hat{A}, \hat{B}$ be open sets containing x in A and B respectively. Then $A \cap B$ are nonempty as $A\hat{A} \cap \hat{B} \neq \emptyset$. In addition, $A \cap B \in F_x$ as $\hat{A} \cap \hat{B}$ is open. Finally, if $A \subset C$, $C \in F_x$ as $\cap A \subset C$. We may also observe that $F_x \to x$ trivially since every open set containing x is in F_x.

By hypothesis, $\lim(F_x) = f(x) = y$. Thus, there is some $A \in F_x$ such that $f(A) \subset V$ and $A \subset U$. In addition, There is an open $\hat{A} \subset A$ which contains x. Hence, since $f(A) \subset V$, $\hat{A} \subset A \subset U$. Hence, given $x \in f{-}1(V)$, there is an open set $\hat{A} \subset U$ which contains x. Thus, U is open and the continuity of f follows by definition.

This completes the needed theory of convergence by ultrafilters.

References

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