PRODUCTIVE FAILURE IN THE FLIPPED MATHEMATICS CLASSROOM

By

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ABSTRACT OF THE DISSERTATION
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ABSTRACT

The flipped classroom method of instruction involves a shift in how learning and teaching take place. Students in a flipped classroom access video tutorials at home as their primary means of instruction, while they work on rich problems requiring procedural and conceptual knowledge in class. The flipped classroom approach has been gaining popularity in higher education (Abeysekera & Dawson, 2015) and in mathematics (Muir & Geiger, 2015). Although studies have addressed differences between flipped and traditional methods of instruction, few have closely examined how to design activities in a flipped classroom that develop students' higher-order thinking skills (O'Flaherty & Phillips, 2015; Song & Kapur, 2017).

Kapur's (2008) theory of productive failure states when students have an opportunity to generate and explore solutions to a challenging task prior to being instructed on it, they are better positioned to consolidate their knowledge during and after instruction. The purpose of this mixed methods study was to determine whether repurposing the flipped classroom to include productive failure results in students' improved understanding of targeted mathematical content. Through work samples and a survey, it was determined that there was no statistical significance between groups in terms of students' performance on course assessments and video-watching behaviors. Correlations among various course assessments revealed a significant correlation between students' work on one productive failure invention task and its corresponding conceptual knowledge posttest item.

Through video footage of in-class problem solving and a focus-group interview, qualitative information about students' knowledge development helped explain differences between the two learning environments. Students who had the productive failure experience described their peers as critical to their learning, whereas students who did not described their instructor as critical to their learning. A detailed look at individual groups in the productive failure condition revealed that students who generated more solutions and had routines for working in a small group on the invention tasks each week realized greater success, as evidenced by work samples and posttest items.

The results of this study will be used to guide the design of undergraduate mathematics flipped classrooms so that online and in-person class activities promote deep conceptual understanding.

Keywords: flipped classroom, undergraduate mathematics, productive failure, educational technology, higher education, conceptual knowledge, procedural knowledge

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Table of Contents

List of Tables

List of Figures

CHAPTER 1: INTRODUCTION

Mathematics in the undergraduate setting has traditionally been taught in lecture format with a heavy focus on procedures (Goldsmith & Mark, 1999). Instructional strategies that are limited to procedures often prohibit students from using their own reasoning and intuition as they learn mathematical concepts (Boaler, 2002; Bransford, Brown, & Cocking, 1999). Meaningful mathematics instruction also involves a focus on enhancing students' conceptual understanding and problem solving skills as they engage in classroom activities (Song & Kapur, 2017).

Statement of the Problem

Balancing the instructional focus to include students' active development of conceptual knowledge has anecdotally been challenging for math instructors far and wide as they struggle to cover large curricula within prescribed time frames (Muir & Geiger, 2015). As a result, many instructors have started to blend their classes with face-to-face and online interaction in order to offer students multiple opportunities for concept development and exploration (Bergmann & Sams, 2012). Kapur and Song (2017) have also noted the difficulty involved with teaching conceptually and recognized how changing the modality of the instruction is not the solution to this problem, but rather a closer look at the instruction itself is needed to make improvements. In response to advances in online course capabilities and the prevalence of online learning tools and resources (Bergman & Sams, 2012), the Rutgers University Mathematics Department started converting traditional lecture-based mathematics courses to hybrid courses in order to afford students with opportunities to actively engage with course content while also developing a stronger conceptual knowledge base (Prince, 2004). The design of learning activities within these non-traditional learning environments has been of interest to instructors as they continue to develop opportunities for students to actively construct their mathematical knowledge.

One such course that has shifted away from lecture-based instructional methods is the "Topics in Mathematics for Liberal Arts" (Math 103) course, which serves as the context for this study. Math 103 is a popular course for non-majors to take to fulfill their quantitative requirement with classes often running at full capacity, which is approximately 35 students per section. Since its creation in the 1990s, the course has only been taught traditionally with two face-to-face lectures each week for 16 weeks. Beginning in the Spring 2015 semester, several hybrid versions of the course have been offered in such a way that students attend one class meeting a week and learn online asynchronously in lieu of the other weekly face-to-face meeting. Instructional time in this hybrid course is repurposed so that the initial learning takes place outside of class, while active problem solving and application take place inside of class.

The hybrid sections of Math 103 have been taught using the *flipped classroom* model. Despite the fact that there is no uniform definition of flipped classroom (Song & Kapur, 2017), conventional understanding of the flipped classroom method of instruction is that the sequence of classwork and homework is inverted in order to allow students to complete homework during class under a teacher's guidance. The process of learning in a flipped classroom is supported by technology as students first learn by working through asynchronous video tutorials and then have the ability to access electronic resources as they solve problems during the in-person class meeting and after class (Baepler, Walker, & Driessen, 2014). The advantage of inverting the order of learning activities in a flipped classroom is that students are free to learn through the videos at their own pace while also having the support of the instructor and peers in class to actively construct knowledge. In fact, several studies have shown that students in flipped classes benefit from working on problems with peer tutoring and instructor coaching (Bergmann & Sams, 2012; Moraros, Islam, Yu, Banow, & Schindelka, 2015).

However, Song and Kapur (2017) noted that a flipped classroom could essentially be viewed as a form of direct instruction if learning activities are not properly designed, especially if a flipped classroom involves just a switch of where lecture and practice occur. As Math 103 continues to be offered in a hybrid flipped classroom format, instructors continue to question how to design an effective flipped classroom learning environment that focuses on improving students' learning outcomes, rather than being a mere reorganization of the activities that take place in a lecture-based course. Since the Spring 2015 semester, several adjustments to the course have been made toward that effort including a student-led review at the start of each in-person class session and an increased effort by the instructors to use the reflective toss questioning strategy to have students attempt to answer one another's questions (Zhang, Lundeberg, McConnell, Koehler, & Eberhardt, 2010). In addition, instructors have also encouraged students to think aloud and make sense of their understanding as they worked through examples in the video tutorials in an effort to build metacognitive thinking in the classroom (Jonassen, 2006; Vamvakoussi & Vosniadou, 2004). Whereas these efforts might have been helpful in the short-term to increase active learning opportunities for students, work samples and conversations with students have revealed that they still struggle with problem solving on class assignments and assessments. In particular, my own observations from facilitating the in-class session have revealed that the main cause of their difficulties seems to lie in initiating the problem-solving process.

Despite the availability of online resources and a room full of peers, students often immediately seek my help in initiating the problem-solving process, particularly those that require deeper conceptual knowledge to solve. My students have also experienced the same difficulties on exam questions involving the use of conceptual knowledge. In trying to determine why this may be the case, it seems that students have difficulty drawing on prior knowledge and also consolidating what they've learned from video tutorials for future use during the in-person class session. This has been especially apparent during my in-class problem-solving sessions where students have to utilize their resources to solve different types of problems in small groups. Research in the area of flipped classroom supports the notion that some students do perceive a disconnect between in-class and out-of-class learning activities (Bowers & Zazkis, 2012). Reasons for the disconnect could include in-class activities that fail to address students' misconceptions or out-of-class activities that rely on low-level factual recall (Andrews, Leonard, Colgrove, & Kalinowski, 2011). Reflecting back on previous interventions I've tried in the course, I only focused on one aspect of learning in the flipped classroom: either the online portion or in-person portion of the class. A better intervention would position students to learn both at home through instructional videos then in-class through the collaborative application of content.

O'Flaherty and Phillips (2015) recognized the lack of conceptual frameworks available in the flipped classroom literature to guide the development of flipped classroom learning activities. They found that whereas instructors do recognize the value of using instructional approaches that lend themselves to active learning, they seem to struggle with having students build conceptual thinking through a planned sequence of learning activities (O'Flaherty & Phillips, 2015). In mathematics specifically, building students' ability to think conceptually involves having them activate and differentiate prior knowledge when engaging in problem-solving (Kapur & Bielaczyc, 2012). This requires mathematics teachers to create opportunities for students to communicate their conceptual understanding, which is achieved through designing a safe and supportive social surround where students feel comfortable generating solutions to various types of problems (Kapur & Bielaczyc, 2012). These design provisions are most associated with in-class activities; in a flipped classroom, the challenge comes with building opportunities for students to learn concepts both inside and outside of class.

O'Flaherty and Phillips (2015) posited that successful flipped classroom implementation outcomes should involve effective student learning that fosters students' problem-solving skills and engagement inside and outside of the class. They also called for a better connection of pre-class and in-person activities in the flipped classroom, particularly with a focus on making the pre-class activities more active. One such process that offers a way to make initial learning more active for the purposes described by O'Flaherty and Phillips (2015) is delaying instruction, or allowing students to investigate on their own before receiving formal instruction on a topic. Research on delaying instruction (Kapur, 2008; Schwartz & Martin, 2004) indicates that when students are allowed to invent concepts prior to receiving direct instruction, and subsequently experience failure, they are better positioned to understand concepts and apply them to new settings. Kapur (2015, p. 52) named this process *productive failure* (PF), which he defined as "a learning design that affords students opportunities to generate representations and solutions to a novel problem that targets a concept they have not learned yet, followed by consolidation and knowledge assembly where they learn the targeted concept." The PF process can easily be incorporated into the instructional design of a flipped classroom in order to position students to come to class prepared to solve problems that primarily involve deep conceptual knowledge.

Students engaged in a flipped classroom model that employs PF could invent solutions to a novel problem in a small group setting at the end of an in-person class setting. This additional measure addresses O'Flaherty and Phillips' (2015) research-based recommendation to make the pre-class activities more active in a flipped classroom. Inventing prior to learning also prepares students to learn better from subsequent instruction (Kapur, 2014), a strategy that has been absent from the traditional flipped classroom design. Granberg (2016) also found that the process of correcting and reconstructing prior knowledge through a failure process ultimately helps students develop new conceptual knowledge in math class. After the failure process, students consolidate and assemble knowledge through video tutorials outside of the classroom. As a result of the activation of prior knowledge and consolidation of new knowledge, students enter the next in-person class prepared to work on activities that deepen their conceptual understanding (Roehl, Reddy, & Shannon, 2013), making their in-class experience move beyond the rote practice of procedures. As Bahr and DeGarcia (2008) found, students' procedural knowledge is inherently limited by their conceptual knowledge, thereby making the case stronger for designing activities in the flipped classroom that require students to use their conceptual knowledge. In fact, Kapur and Song (2017) found that employing a PF design within the flipped classroom such as the one described allows students to gain a deeper conceptual understanding of mathematics.

Purpose of the Study

The purpose of this quasi-experimental mixed-methods study was to determine if restructuring the flipped classroom to include PF would result in students' improved understanding of targeted content (Song & Kapur, 2017). Song and Kapur (2017, p. 295) referred to a blended learning environment that employs PF as a "productive failure-based flipped classroom" (PFFC), which served as one of the conditions in the study. The second condition was a direct instruction-based flipped classroom (DIFC). The DIFC learning environment resembled a flipped classroom design in which students' first interaction with new content was when they learned through instructional videos. One section of Math 103 was randomly assigned to each treatment condition during the Fall 2017 semester.

In this study, quantitative data from a video-watching survey and students' work samples were used to test the theory of PF, which predicts that a delay of instruction by means of an invention task will positively influence learning and transfer across contexts (Kapur, 2008). Of particular interest was the quality of students' work on posttest items involving the use of conceptual and procedural knowledge. In tracing how students performed on their posttests, video-watching data, video footage of in-class problem-solving sessions, and focus group interview responses were used to carefully examine the mediating processes that resulted in learning mathematics. Video-watching data revealed how and when students accessed their "instruction," and whether or not students who experienced PF accessed videos more or less often than students who did not. Additionally, qualitative focus group interview data illuminated students' experiences with PF tasks and their subsequent work quality. Video footage of the in-class problem-solving sessions also provided further evidence for the efficacy of PF as a learning intervention and was used to complement students' work samples and responses to the focus group interview questions.

Embodied Conjecture

As Sandoval (2014, p. 22) discussed, "design research typically aims to create novel conditions for learning that theory suggests might be productive but are not common or well understood." The PFFC learning environment is one such novel environment that has been shown to be productive for students (Song & Kapur, 2017), but also needs to be broken down into the mechanisms and structures that support learning. As the purpose of this study was to examine whether restructuring the flipped classroom resulted in students' improved understanding of targeted learning outcomes, a careful look at mediating processes (video watching and in-class problem-solving) and final outcomes (posttest results on procedural and conceptual understanding) is necessary to understand the design components of the intervention. Figure 1 shows the conjecture mapping for the PFFC learning environment.

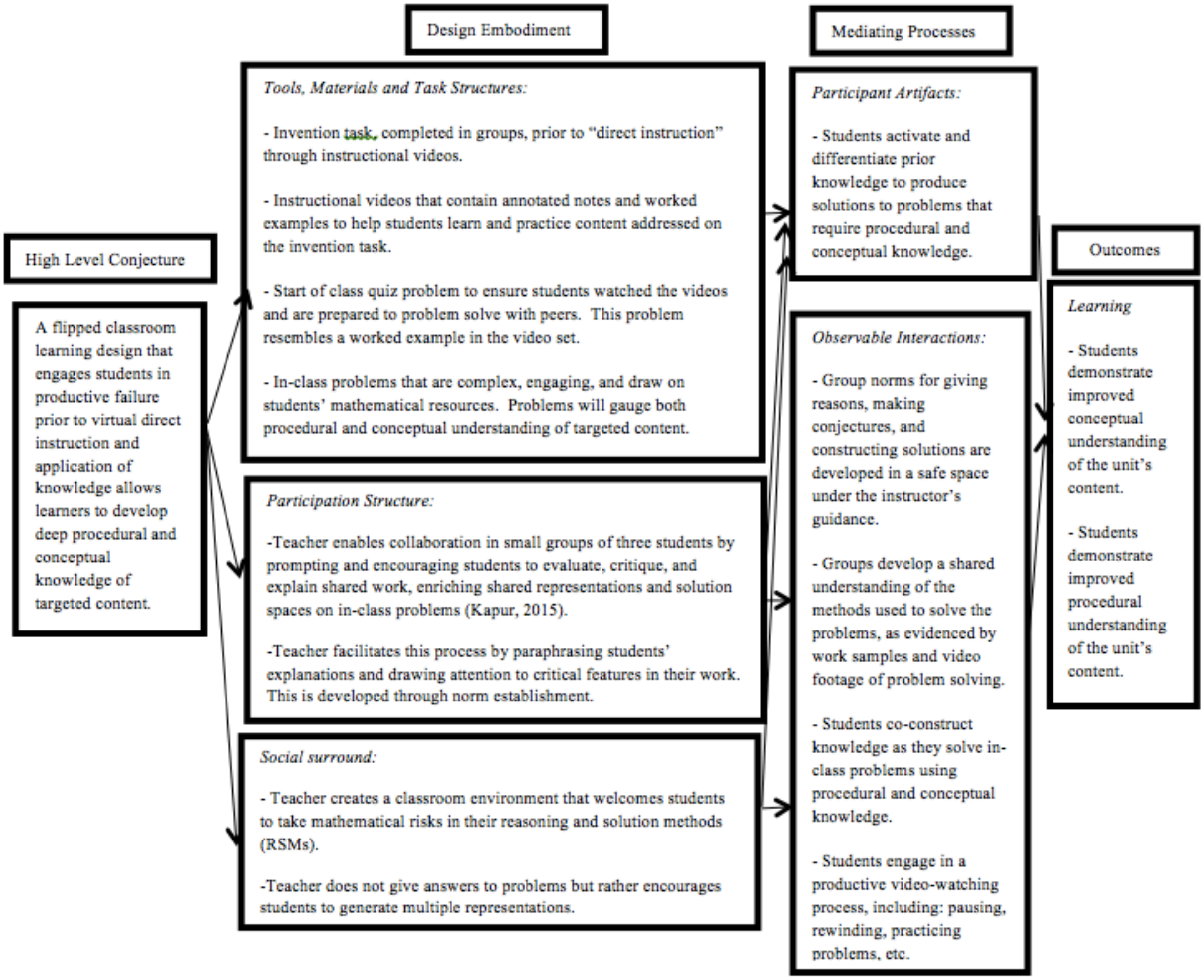


Figure 1. Embodied conjecture for PFFC learning environment.

The high-level conjecture draws from diSessa, Hammer, Sherin, and Kolpakowski's work (1991), which suggests that students' prior knowledge is a critical element in a pedagogical design. Students in a DIFC may not have the opportunity to draw on prior knowledge outside of watching the instructional videos, whereas students in the PFFC group have the additional opportunity to do so through an invention task. Each of the critical design features in the PFFC is described in detail below to provide insight into the learning sequence in a PFFC.

Participation structure. As the facilitator of both sections of the course, I enabled collaboration in small groups so that students could evaluate, critique, and explain shared work, thereby enriching shared solution spaces on both problems that required procedural and conceptual knowledge to solve (Chi, Glaser, & Farr, 1988; diSessa et al., 1991; Kapur, 2015; Scardamalia & Bereiter, 2003). Groups were pre-arranged during the first few weeks of the semester so that students could be matched with peers who were productive in their problem-solving. This determination was made based on my observations of group work during the first few weeks of class. Having pre-arranged groups allowed me to leverage the social dynamics in the classroom to ensure students were working with peers who would be efficient in helping one another learn (Cohen, Lotan, Abram, Scarloss, & Schultz, 2002). This also involved group norm setting for problem-solving and sharing of ideas, which I facilitated once the semester started through the use of a suite of scaffolds. I also facilitated the group problem-solving process by paraphrasing students' explanations if they were in need of assistance and drew their attention to critical features in their work as opportunities arose. For example, I had students try to re-explain their understanding of main concepts from the videos and had them link the concepts to the problems they were solving.

Social surround. Reflecting back on the need for this intervention, I have found that many students ask for help on problems before they even attempt to try the problems themselves. To help address this issue, my design embodiment included the provision of a mathematical safe space (Kapur & Bielaczyc, 2012) where students were encouraged to take mathematical risks. To foster the development of this safe space, I modeled strategies for working in groups with the students during the first class session, with an emphasis on free sharing of ideas. I also assisted students in judging the constraints and affordances of their representations and solutions methods (RSMs), instead of merely classifying their work as “right” or “wrong.” Last, I also provided affective support to individual students and groups of students by providing students with epistemic strategies for thinking like mathematicians. These measures were taken to allow students to actively engage in problem-solving with support from both myself and their peers.

Invention tasks. Schwartz and Martin (2004) found that student-centered activities could effectively prepare students for future learning. Specifically, they found that having students invent a concept prior to receiving formal instruction on the topic was superior to the traditional “tell and practice” method of instruction (Schwartz & Martin, 2004). As a result, my PFFC instructional design included invention tasks that prepared students for future learning through video tutorials. One task was given each week for four weeks as students learned four brand new concepts. Following Kapur and Bielaczyc’s (2012) recommendations for designing a PF learning environment, each invention task was pitched at a level of difficulty that was meant to challenge students but not frustrate them. Kapur and Bielaczyc (2012, p. 51) have referred to this as “sweet spot calibration,” where problem complexity, affective draw, and students’ prior mathematical resources are taken into account when designing the problem. Because Math 103 is a course on math applications in the liberal arts, all problems had an affective draw as they were grounded in actual situations (fair division of goods, apportionment of states, etc.). Students were also able to rely on their constructive resources to generate multiple RSMs to the invention tasks (diSessa & Sherin, 2000), as all of the problems required the use of basic mathematical operations like division, multiplication, ratios, to name a few. The invention tasks served to better position students to activate and differentiate prior knowledge before receiving direct instruction.

Video tutorials. Each week, students accessed a set of brief instructional videos on the course's learning management system. These videos served as the "direct instruction" component of the flipped classroom and were designed in a way that allowed students to learn one concept at a time in each video tutorial. Each set consisted of approximately 13-15 short videos each week. Each video was narrated and annotated with worked examples and relevant definitions and ended with a problem for students to try before they moved on to the next video. Once the students moved on to the next video, they were brought to a fully guided and annotated solution to the problem they just worked out independently. These end-of-video problems served as opportunities for students to practice and understand the material they just learned, as the primary purpose of the video tutorials was for students to consolidate their knowledge from the PF experience (Song & Kapur, 2017). Students were encouraged to use productive video watching habits, which included rewinding and pausing content, working on embedded practice problems, thinking aloud, and taking notes as part of the instructional design. As a result of watching the videos, students were able to consolidate their knowledge from the invention task and the concepts presented in the videos.

In-class problems. Sandoval (2014) described how observable interactions between participants and their artifacts can serve as a way to understand the mediating processes that emerge from a design. After the mathematical safe space and group norms were established in the class, students transitioned into structured collaborative groups of approximately three to four students each where they solved problems related to the content of the invention task and corresponding video set. These problems were adequately complex, engaging, and drew on students' mathematical resources, just as the invention task did. There were two types of problems for each in-person class session: well-structured application problems which required students to work out problems similar to the worked examples in the videos and involved more procedural knowledge, and more complex problems that required students to think very conceptually to invent cases and counterexamples to develop a solution. In this context, a problem requiring conceptual knowledge would involve students trying to generate an example or at least outline the conditions necessary for a mathematically fair division of an item to occur. Staying true to Kapur and Bielaczyc's (2012) recommendations, both problem types were open to multiple RSMs. This means that the problems could have been approached with multiple representation systems, such as ratios, trial and error, and algebra.

Summary. Each of the preceding sections has presented a description of how each of the design features was expected to shape learning outcomes in the course. These effects were measured through posttests that assessed conceptual and procedural mastery of targeted concepts in the course. Some questions primarily assessed procedural knowledge of topics covered in the unit, including execution of fair division, fair distribution, and apportionment methods. Another group of questions assessed students' conceptual knowledge of apportionment, flaws of apportionment, and ways of gaming a fair division settlement to result in desired outcomes. The key distinction between the question types on the posttest is that the conceptual knowledge questions were unfamiliar to students and required them to draw on the invention tasks, videos, and in-class problems to develop solutions.

Research Questions

The research questions involved a careful examination of two flipped classroom designs: the DIFC, which involved students watching video clips for homework followed by solving more challenging problems during class, and the PFFC, which involved invention prior to learning, PF, followed by consolidation through video watching (Schwartz & Martin, 2004; Song &

Kapur, 2017). Each question aimed at addressing various parts of the embodied conjecture, including the in-class inquiry process, video-watching behaviors, and conceptual and procedural knowledge gains. The following research questions are focused on the contrast between the PFFC and DIFC:

1) What is the effectiveness of flipped classroom pedagogical design interventions on the development of students' procedural knowledge, particularly on their written work samples?
2) What is the effectiveness of flipped classroom pedagogical design interventions on the development of students' conceptual knowledge, particularly on their written work samples?
3) How do video watching behaviors influence students' learning performance in the DIFC and PFFC learning environments?
4) How do students in both the DIFC and PFFC learning environments describe their in-class mathematical inquiry process?
5) How do students' problem solutions to in-class problems and their interactions as they discuss these problems mediate conceptual and procedural learning as measured on the posttest?

This study provided evidence on whether a productive-failure-based flipped classroom differed from a traditional flipped classroom in promoting student learning. The quantitative and qualitative data gathered in this study will be used improve the design of undergraduate flipped classroom mathematics learning environments so that both the online and in-person class activities promote deep conceptual understanding.

CHAPTER 2: LITERATURE REVIEW

Introduction

As the purpose of this study was to examine how restructuring the flipped classroom to include productive failure could result in students' improved understanding of targeted content, I drew on four bodies of literature: hybrid learning environments, the flipped classroom model, conceptual and procedural understanding in mathematics, and delaying instruction. Because this study examined what learning looked like specifically in a hybrid mathematics course, the first part of the literature review will focus on the components of hybrid learning environments. I will use the term "hybrid" in place of "blended" where applicable, as many studies interchange the two words. Second, I will draw on flipped classroom studies in order to provide a more nuanced understanding of the merits and drawbacks of the type of pedagogy employed in the research site. Third, the roles of conceptual and procedural knowledge in mathematics will be discussed as they relate to one another and the research context. Finally, relevant research in the area of delaying instruction will then be explored in a way that provides a theoretical framework for the study. Taken as a whole, these bodies of literature will contribute to a better understanding of what the learning environment and learning process should look like when productive failure is added into the design of a hybrid flipped classroom learning environment.

Hybrid Learning Environments

A common course delivery type in the undergraduate setting is the hybrid classroom**.** Hybrid courses involve "an integration of face-to-face and online learning experiences—not a layering of one on top of the other" (Garrison & Kanuka, 2004, p. 99). This means that learning activities in a hybrid course should move beyond just a mix of virtual and in-person lectures. Instead, activities should be purposefully designed for both the online and in-person sessions so that they stimulate students' thinking. Hybrid course formats have gained attention for their potential to change the landscape of learning. In 2010, the United States Department of Education released a meta-analysis that focused on online learning in both the K-12 and higher education arenas. One of the key findings that emerged from this analysis was that students by and large saw stronger learning outcomes in instructional formats that blended online and face-to-face instruction compared to students who received pure online or pure face-to-face instruction (Means, Toyama, Murphy, Bakia, & Jones, 2009). In fact, the researchers found the average effect size for blended learning to be .35 ($p < .001$) when compared to purely face-to-face instruction. Despite the fact that curricular materials, aspects of pedagogy, and learning time in both conditions were not all equalized, this finding suggests that blended learning is advantageous for students for reasons beyond the media used in the course (Means et al., 2009).

The varied instructional delivery methods in a hybrid course format do have appeal for both students and teachers for various reasons; however, the delivery method itself is less critical than the actual design and management of a hybrid course (Bourdeau & Bates, 1996; Mortera-Gutiérrez, 2006). The next section examines the design and integration of the online and face-to-face components of a hybrid course along with the implications they have on students' learning. As previously suggested by Means et al. (2009), hybrid course design must involve a careful look at the non-media components of a course, including the establishment of a learning community and instructor-student interactions.

Online and face-to-face components. Several studies point to the importance of the learning community in creating an effective hybrid learning environment (Chen & Chiou, 2014; Collopy & Arnold, 2009; Helyer & Corkill, 2015; McGee & Reis, 2012). In a hybrid course such as the one in the research site, designing opportunities for learners to work together during the in-class problem-solving session is a top priority for ensuring the success of all learners. Looking across studies on hybrid learning environments, McGee and Reis (2012) conducted a meta-analysis to determine what factors contribute to a well-designed hybrid course and if common features in hybrid course design exist. The researchers found common principles regarding the design process, classroom and online technology utilization, and assessment practices. McGee and Reis (2012) noted that there was a consistent recommendation for interactive group activities in hybrid courses across the studies they analyzed. However, despite these "best practices" for course design, McGee and Reis (2012) found a lack of clearly articulated pedagogical strategies to inform the design of a hybrid course. They reminded readers that although there is a lack of pedagogical strategies available, it is important for course designers and instructors to be flexible in educating students with a variety of learning styles. This could take the form of diverse group arrangements or different amounts of online and in-person instruction. McGee and Reis (2012, p. 13) ended their work with a strong, reaffirming message about hybrid course design: "there is a consistent belief that both varied interactivity and prompt feedback are key to student engagement in blended courses."

Building on the work done by McGee and Reis (2012), Chen and Chiou (2014) investigated how hybrid instruction affects undergraduate students' learning outcomes, satisfaction and sense of community in the course. They sought to find a relationship between students' learning style and learning conditions in a hybrid course. To that end, they implemented a quasi-experimental design with 140 college sophomores to measure students' learning outcomes, satisfaction, sense of community and learning styles in a hybrid course. Their results indicated that students had both higher learning scores and overall satisfaction in the hybrid course than they did in the traditional course. Students in the hybrid course also felt a greater sense of community than their counterparts did in a traditional course. Analysis of learning style showed that learning style had a significant effect on learning outcomes in the study group. The results of this study provide some additional evidence for the importance of interactivity and a learning community in a hybrid course.

Similarly, Collopy and Arnold (2009) studied 80 undergraduate teacher education candidates to better understand students' comfort while working in both online and hybrid learning environments. Their 19 question, Likert-type survey asked a variety of questions, with an emphasis on interactions and working in teams. The researchers found a significantly higher level of learning reported by students in the hybrid format versus in the purely online format. Reasons for this included the fact that blending face-to-face and online learning environments in a hybrid class allowed for a reciprocal structure for student learning. The face-to-face learning environment supported group development, group member accountability, and commitment, as well as effective processing of content with the instructor and class members. The online learning environment enhanced the face-to-face learning environment by giving students time to think, process the material, and discuss the content online outside of class time. The online environment also increased individual accountability. One limitation of this study is that the researchers taught the courses they collected data in, so there may be threats to internal validity (Brown, 1992), which could put the findings in question.

The literature on the hybrid learning model suggests a careful balance of both in-class and online activities in an effectively designed hybrid course, supported by a learning community that fosters the students' and instructor's willingness to participate. Achieving this balance requires instructors to move beyond the lecture and incorporate strategies that get students actively involved in learning. Having students actively involved in learning necessitates pedagogical techniques that lend themselves to the development of a learning community, frequent interaction between the instructor and students, and a careful integration of activities both in person and online (Chen & Chiou, 2014; Collopy & Arnold 2009; Helyer, & Corkill, 2015; McGee & Reis, 2012). The next section of the literature review describes a particular type of hybrid pedagogy that embodies these characteristics and was also employed in the research site.

The Flipped Classroom Model

Hybrid course formats in the college setting can vary greatly with lecture used in different proportions both inside and outside of class. As a result, hybrid courses are usually associated with some type of "digital habitat" that accompanies face-to-face learning, where technology tools allow students and the instructor to work collaboratively on a topic of interest (Wenger, White, & Smith, 2009). Undergraduate learners typically gravitate toward media-rich hybrid learning environments as they view learning as a "plug-and-play" experience, one where they can participate and experiment in learning activities with peer cooperation (Duderstadt, 2001). One example of a technology-driven pedagogy that has gained popularity because of its balance of online and face-to-face learning is the flipped classroom.

The evolution of flipped classroom. Research on the flipped classroom took hold at the beginning of the millennium (Lage, Platt, & Treglia, 2000). Within the last ten years, instructors who have created flipped classrooms have been gravitating away from pre-class readings and have been instead using pre-class streaming lectures (Day & Foley, 2006; Moravec, Williams, Aguilar-Roca, & O'Dowd, 2010; Naccarato & Karakok, 2015). The flipped classroom model has come a long way and continues to evolve as more educators embrace using it.

Lage et al. (2000) provided one of the earliest definitions of what a flipped classroom is.

Lage et al. (2000, p. 32) used the phrase "inverted classroom" to describe how "events that have traditionally taken place inside the classroom now take place outside the classroom and vice versa." Lage et al. (2000) performed qualitative research on an inverted undergraduate economics course to better understand the implications of inverting class activities. Their findings indicated an inverted learning model bred inclusivity in the class, increased motivation of both faculty and students, and left class members feeling more positive about learning. This study also raised additional questions about the flipped classroom model, including the roles of group work, motivation, questioning, and faculty preparation.

Day and Foley (2006) later answered some of these questions, as they were early pioneers of utilizing streaming multimedia lessons in the flipped classroom. They employed a quasi-experimental design to see if a flipped course taught at Georgia Institute of Technology would be just as, or more, educationally effective and enjoyable than a traditional lecture-style version of the same course. Day and Foley (2006) conducted their research over a 15-week semester with a total of 46 students in two sections of the same course. The strength of this study was that the same professor taught both sections of the course and blind grading was used for the entire semester. After examining grades and self-reported attitudes, the researchers found that students in the flipped classroom section earned significantly higher grades than students did in the traditional lecture section. Students in the flipped classroom section also had increasingly strong positive attitudes about the method of instruction.

Along the same line, Moravec et al. (2010, p. 473) determined that "there is mounting evidence that a variety of pre-class activities that introduce new material can increase student performance compared with traditional lectures." Moravec et al. (2010) called their approach "learn before lecture (LBL)," which is analogous to the flipped classroom approach. In this approach, the researchers introduced new content to a large introductory biology lecture hall by removing four to five slides from their 2007-2008 class lecture presentations and turned them into streaming PowerPoint lectures in 2009. This was done for three out of 30 lectures. Class time was then spent applying new knowledge learned from LBL materials. Learning was evaluated by comparing student performance in 2009 versus 2007-2008 on LBL questions. Overall findings indicated that students saw a 21% mean performance increase across the LBL questions. Based on this information, the researchers determined that LBLs along with interactive class exercises can be administered incrementally and result in modest learning gains in large introductory biology classes. One weakness of the study is that the researchers employed this strategy on only a few lectures and not the entire course. It would be interesting to see if doing this for every lecture would produce the same results.

The flipped classroom model is still evolving as teachers and researchers work to design courses that optimize students' learning. In contexts similar to the research site, Naccarato and Karakok (2015) explored trends in the implementation of the flipped classroom model in undergraduate mathematics classes. They conducted a qualitative study to explore different implementations of the flipped classroom model by interviewing 19 faculty members with flipped classroom experience at 14 different higher education institutes. Results indicated that instructors had similar motivations for using the flipped classroom: more student practice through exercises and application problems, increased collaborative learning opportunities, and more opportunities for higher-order thinking. These are some of the exact same reasons for using flipped classroom approach at the research site; problems offered during the in-class meeting offer students opportunities to work together to solve problems that require both procedural and conceptual knowledge. The next section examines performance differences between students in flipped and traditional classes as a result of their work in class.

Comparison of flipped and traditional classes. In tracing the evolution of the flipped classroom, many studies seemed to focus on students' performance differences in traditional learning environments versus flipped classroom environments (Eichler & Peeples, 2016; Gross, Pietri, Anderson, Moyano-Camihort, & Graham, 2015; Maciejewski, 2015; Stone, 2012; Wasserman, Quint, Norris, & Carr, 2017). A strength of this line of inquiry is that there is a lot of quantitative evidence to support increased exam performance when using a flipped classroom approach over a traditional lecture-based approach.

Eichler and Peeples (2016) used flipped classroom modules in large general chemistry lecture halls in an effort to increase active learning opportunities and student performance. By flipping the large lecture halls (250+ students), the researchers were able to provide their students with collaborative group inquiry-based activities instead of a lecture during class time. Quantitative findings showed students in a flipped version of general chemistry had a higher course-wide GPA (2.923) compared to students in the non-flipped version of the same course (2.807). In addition to overall GPA, overall exam performance and scores on clicker formative assessments were higher in the flipped chemistry course than in the non-flipped course. Stone (2012) also observed similar positive results in his flipped college Genetic Diseases course, where students performed almost 13 points higher on midterm examinations than their peers did who were in traditional lecture versions of the same course.

Gross et al. (2015) spent years investigating how college students prepare to learn in a flipped classroom and what the results of their preparation look like. They conducted an experiment in which they took an undergraduate physical chemistry class and observed student outcomes over the course of five years in both a flipped version and traditional version of the same course. Gross et al. (2015) found that exam performance significantly improved by about 12% in the flipped course. Online homework completion and accuracy were also greater in the flipped version of the course. Gross et al. (2015) largely attributed these results to the flipped classroom method of instruction. Despite the substantial quantitative evidence over a long time period, a limitation of the study that must be taken into account is the variability in college students' motivations and study habits when it comes to examinations. In addition, the quality of the instruction in both treatment conditions must also be further examined, as the processes that lead to posttest results are important parts of the learning environment design.

In terms of college-level mathematics courses, findings from a larger study of 690 first-year calculus students (Maciejewski, 2015) also indicated that students in a flipped class outperform students in an equivalent lecture-based class on posttest items. However, the same study also revealed that students with little prior knowledge in calculus and good mathematics skills thrived the most in the flipped setting. Wasserman et al. (2017) noticed similar findings in their two-semester study of students' learning in a flipped Calculus III course; students in the flipped version of Calculus III showed similar performance on procedural problems and moderate gains on conceptual problems compared to their counterparts in a traditionally taught Calculus III section. Both studies pointed to the need to closely examine the learning processes beyond the posttest to gain a more nuanced understanding of learning in the flipped classroom. In particular, the reasons for why certain students do better in this type of environment and how flipped classrooms support procedural and conceptual knowledge gains warrant further exploration.

Concerns about flipped classrooms. The aforementioned studies described mostly positive views and performance increases associated with the flipped classroom. Another series of studies on flipped classroom suggests some students have difficulty with aspects of the model or feel that some parts of the model are less satisfying than others (Frederickson, Reed, & Clifford, 2005; Strayer, 2012; Tague & Czocher, 2016; Toto & Nguyen, 2009). Some of these problems are the same problems students have been facing in the research site.

In a study on learner satisfaction in a flipped classroom, Strayer (2012) found that the flipped classroom strategy might not be the best for introductory college courses. Strayer taught both a flipped and a traditional lecture version of a statistics course and found that students in the flipped course were not satisfied with the way they were prepared for the tasks they had to complete. Despite the fact that students in the flipped classroom very much enjoyed the collaboration within the learning community, their lack of prior knowledge on the topic made it difficult for them to keep up with the intensive nature of the course. In addition, students' performance likely depended on the details of the instruction in both settings. Planned learning activities could have had slight variation between the flipped version and traditional version of the same course.

Frederickson et al. (2005) found similar results in their experimental study of students' learning in a flipped research methods and statistics course. Using cognitive and affective domain data from 16 first-year college students, they found that students thought the online learning component of the flipped course was less than satisfying. However, students rated the in-class collaboration component of the course very highly. Despite the small number of students in the sample, a key result that emerged from this study was the importance of building a learning community both online and in person so that students could value collaboration and social interaction in both settings.

Toto and Nguyen (2009) studied the relationship between learning style and the flipped classroom approach in an undergraduate industrial engineering course. Their study involved 74 undergraduate juniors who completed three survey items: the Soloman and Felder's Index of Learning Styles Questionnaire, a quiz at the beginning of class, and a survey at the end of the semester. Their results indicated that visual-verbal learners and sequential-global learners found it difficult to sit through instructional videos and focus on the material. In addition, active learners found the intensive in-class component to be beneficial to their understanding of concepts, whereas reflective and sensing-intuitive students needed more clarification and review before starting the in-class work. Taken all together, the results of this study indicate different aspects of the flipped classroom approach appeal to different learners.

Building on all of these studies, Tague and Czocher (2016) explored the disconnect between in-class and out-of-class activities in a flipped classroom that seemed to surface in the earlier studies. Their hypothesis was that flipped classroom activities might orient themselves to low-level recall by students and may fail to address students' misconceptions. They described the issue to be rooted in of a lack of *curricular coherence*, which they defined as the extent to which mathematical content is organized logically, cognitively, and epistemologically sequenced (Tague & Czocher, 2016), and a lack of *instructional coherence,* which they defined as the connection of in-class material, out-of-class material, and targeted content. Using 80 undergraduate students in a Differential Equations course, Tague and Czocher examined students' perceptions of curricular and instructional coherence by designing their course to reduce cognitive obstacles and curricular/instructional incoherence. Their results of administering four small surveys to students during their course indicated that over 90% of students found that in-class and out-of-class activities were directly connected. These students were also able to articulate their reasons why they felt class activities were connected, citing things such as useful "refresher" activities, and "review of concepts from past classes" (Tague & Czocher, 2016). This study filled a void in the literature on flipped classroom on the disconnect between in-class and out-of-class activities and how to purposefully design the two for coherent learning.

The aforementioned studies on flipped classroom suggest that great learning gains are possible in this type of learning environment, especially when it is designed to include opportunities for learners to collaboratively work in small learning communities. However, some studies revealed that this approach does not work for every student. Song and Kapur (2017) cautioned us that the real question to consider is how to design the flipped classroom learning environment so that all students have opportunities to engage in learning activities that enhance their higher-order thinking skills. Given the complex nature of both the online and in-person class sessions in a hybrid flipped classroom, it is important to consider how to design activities that lead to students' successful procedural and conceptual knowledge development. The next section considers the roles of procedural and conceptual knowledge in mathematics and their importance in designing learning activities for students. Findings from the literature on conceptual and procedural understanding will be used to guide the development of hybrid flipped classroom learning activities at the research site.

Procedural and Conceptual Knowledge

As Roehl et al. (2013) pointed out, millennials have a preference for learning in environments that include an easy connection to information, multitasking, and group collaboration. The previously described literature on hybrid and flipped classroom learning environments support the use of learning activities to encourage active information access and peer collaboration. However, instructors in a flipped classroom not only face the task of motivating students to learn, but they also have to cover the curriculum in a way that promotes learning (Muir & Geiger, 2015). In mathematics, this includes working with students to develop the procedural and conceptual knowledge necessary to solve problems. As definitions of both procedural and conceptual knowledge continue to evolve, researchers agree that both conceptual and procedural knowledge are intricately related and important to develop in the mathematics classroom (Ambrose, Clement, Philipp, & Chauvot, 2004; Bahr & DeGarcia, 2008; Baroody, Fiel, & Johnson, 2007; Hiebert & Lefevre, 1986; Star, 2005).

Definitions. Conceptual knowledge has been defined in numerous ways. One of the earliest definitions of conceptual knowledge offered by Hiebert and Lefevre (1986, pp. 3-4) indicates that conceptual knowledge is "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information." More recent definitions of conceptual knowledge break down the definition into more specific details, ranging from "an integrated and functional grasp of mathematical ideas" (National Research Council, 2001, p. 118), to "the quality of one's knowledge of concepts" (Star, 2005, p. 407), to "knowledge about facts, [generalizations], and principles" (Baroody, et al., 2007, p. 107). Support for all of these definitions comes from research on conceptual change, which indicates that novices need their conceptual knowledge integrated over the course of their learning, whereas experts' conceptual knowledge expands and becomes better organized over time (diSessa, Gillespie, & Esterly, 2004; Schneider & Stern, 2009). Looking across definitions, several key features of conceptual knowledge emerge, including relationships, knowledge of how to use mathematics, and a repertoire of strategies. These commonalities seem to hint at a knowledge of procedures or strategies for solving mathematical problems.

Procedural knowledge is commonly viewed among mathematics educators as a knowledge of procedures or rote memorization. Indeed, researchers have defined procedural knowledge as knowledge of the specific rules and procedures for solving mathematics problems (Hiebert & Lefevre, 1986; Rittle-Johnson, Siegler, & Alibali, 2001; Star, 2005). Baroody et al. (2007) expanded on this definition by adding in the mental actions, manipulations, strategies, and algorithms needed to solve problems. Many educators often equate procedural knowledge with a naïve or superficial understanding of a topic; however, procedural knowledge is important in mathematics as it gives students a method for navigating a problem. In sum, procedural knowledge involves knowing how to do something, whereas conceptual knowledge involves understanding how the features of a problem allow one to understand why a certain procedure is appropriate for a task (Booth, 2011).

The relationship between conceptual and procedural knowledge. Despite the fact that conceptual knowledge and procedural knowledge are defined separately, both types of knowledge are interrelated (Bahr & DeGarcia, 2008; Baroody et al., 2007; Rittle-Johnson et al., 2001). Looking at definitions of procedural and conceptual knowledge, there appear to be important similarities and differences between the two types of knowledge. Star (2005) and Baroody et al. (2007) described the key difference between procedural and conceptual knowledge to be the type of connection; conceptual knowledge is related to knowledge that has rich connections, whereas procedural knowledge is related to knowledge that is not richly connected. Additionally, Baroody et al. (2007, p. 23) noted that “depth of understanding entails both the degree to which procedural and conceptual knowledge are interconnected and the extent to which that knowledge is otherwise complete, well structured, abstract, and accurate." These findings suggest that although the two types of knowledge are different, there appears to be a relationship between the depth of both procedural and conceptual understanding in mathematics.

In terms of similarities, both types of knowledge can be viewed along a continuum from superficial to deep (Bahr & DeGarcia, 2008; Baroody et al., 2007; Star, 2005). Deep procedural knowledge entails flexibility and critical judgment, in addition to efficiency and accuracy in manipulating procedures (Bahr & DeGarcia, 2008; Baroody et al., 2007). Deep conceptual knowledge involves being able to consistently apply mathematical logic, make generalizations, and apply a variety of principles to solve problems. Superficial types of both knowledges involve only being able to make local generalizations and/or heavily rely on procedures (Baroody et al., 2007; Rittle-Johnson et al., 2001). These definitions seem to suggest a symbiotic relationship between procedural and conceptual knowledge; students' learning may be inhibited by a deep procedural instructional focus that doesn't afford opportunities for conceptual knowledge development. As a result, mathematics instructors need to cultivate both types of knowledge in their courses; course formats like flipped classrooms offer ways for instructors to provide additional opportunities for students to think and learn procedurally and conceptually both at home and in class.

Research on the development of conceptual and procedural knowledge in young children also indicates that children's procedural knowledge is inherently limited by their conceptual knowledge (Bahr & DeGarcia, 2008). For example, whereas students demonstrate ease of working with numbers, a lack of conceptual understanding limits what children can do with those numbers (Bahr & DeGarcia, 2008). Rittle-Johnson et al. (2001) produced similar findings;

children's initial conceptual knowledge gains later predicted procedural knowledge gains. In addition, in a study of teacher beliefs about mathematics, Ambrose et al. (2004) found that teachers whose perspectives about teaching mathematics by promoting understanding often believed that mathematics is a web of interrelated concepts and procedures and that students who learn concepts first are subsequently more likely to understand procedures. These results show the importance of building a conceptual knowledge base in students so that they are positioned to develop additional conceptual knowledge and procedural knowledge. This has implications for the research site because the learning activities in the flipped classroom were aimed at promoting procedural and conceptual understanding with an emphasis on cultivating conceptual understanding during the in-person class meeting.

Instructional focus. In light of the previous discussion of procedural and conceptual understanding, it is also important to examine the instructional practices that lead to both types of understanding. A balance between focusing instruction on procedural and conceptual learning is a key part of developing students' mathematical fluency, especially before they get to college (National Council of Teachers of Mathematics, 2000). Focusing on computational methods without understanding could result in students forgetting or incorrectly remembering content; in fact, students must have conceptual understanding in order to be fluent in mathematical procedures. To take a closer look at instructional focuses, Rittle-Johnson and Alibali (1999) investigated instructional strategies that were primarily conceptual in nature and strategies that were primarily procedural in nature. Using 60 fourth and fifth-grade students and a pretest/posttest design, Rittle-Johnson and Alibali (1999) found that conceptual instruction led to increased conceptual understanding, invention of procedures, and transfer of procedures. They also found that procedural instruction led to some increased conceptual understanding, but only adoption and very limited transfer of the procedure they were instructed on. The key takeaway from their study was that the relationship between procedural and conceptual knowledge is not unidirectional, but the development of conceptual knowledge has a greater effect on procedural knowledge than the reverse. The only known advantages of teaching procedurally seem to be that instructors who teach students multiple procedures set their students up to better problem solvers than students who have only been taught to use one single procedure (Alibali & Goldin-Meadow, 1993; Siegler, 1995).

Flipping the classroom involves a shift in when and how direct instruction takes place in a class. This has implications for how conceptual and procedural knowledge are developed both with and without the instructor present. Whereas many flipped classroom studies have examined students' performance differences and qualitative aspects of learning in a flipped classroom, there are few studies available on how to effectively design learning activities in a flipped classroom that encourage higher-order conceptual thinking (Song & Kapur, 2017). In the traditional classroom setting, delaying instruction has often been used with great success because it allows students to mobilize their prior knowledge in order to become prepared for the future learning of a topic (e.g. Kapur, 2008; Schwartz & Martin, 2004). In addition, the process of failing while trying to generate new knowledge has been shown to be productive for learning (Kapur, 2008; Kapur & Bielaczyc, 2012). These findings, coupled with the structure of a hybrid flipped classroom, suggest that delaying instruction could be a way to improve the design of a flipped classroom so that students can transition in and out of online and in-person learning activities more prepared to learn. Studies on failure, invention, and productive failure are discussed in the next section as a way to provide a theoretical framework for the intervention used in this study.

Delaying Instruction

Each of the subsections below outlines the development of delaying instruction as a learning intervention, leading up to a recent study on delaying instruction specifically in the flipped classroom. By providing both the history and empirical support for delaying instruction, its utility in developing students' conceptual and procedural knowledge in the hybrid flipped classroom can be better understood.

Failure. Research on experiencing failure dates back to work done as early as 1984 when Clifford first discovered the benefits of "constructive failure." The key behind constructive failure is that students turn failed attempts into problem-solving opportunities, where they develop "strategy explanations" to explain their failure, rather than attributing their failure to effort or ability. Clifford (1984) postulated that success is often associated with initial failure and at times guaranteed by failure experiences. Schmidt and Bjork (1992) expanded on this idea by demonstrating how "desirable difficulties" during the initial learning process can set the stage for deeper learning experiences in the long term, despite short-term performance losses. In fact, similar to Clifford (1984), Schmidt and Bjork (1992) went as far as to say conditions that do not seem to support initial learning actually result in better long-term learning.

More recent empirical studies have bolstered the claims made by Clifford (1984) and Schmidt and Bjork (1992) by supporting the role of failure in the learning process (Schwartz & Bransford, 1998; VanLehn, Siler, Murray, Yamauchi, & Baggett, 2003). VanLehn et al. (2003) explored the idea of "impasse-driven learning" with 42 college students who worked through physics problems under the guidance of expert tutors. Through an analysis of over 125 hours of dialogue between tutors and students to see which features of the dialogue were associated with learning, VanLehn et al. (2003) found that successful learning of physics content was associated with the instance of impasse during the problem-solving process. When students did not reach an impasse, students often did not successfully learn the content, even with the assistance of an expert tutor. The major takeaway from VanLehn et al.'s (2003) work is the power of the impasse; delaying instruction and explanations until students reach a dead end is a viable strategy for helping students experience deep learning. The next section takes a closer look at the ways impasse and delaying instruction prepares students for future learning.

Preparing for learning. Building on the idea of delaying instruction with college students, Schwartz and Bransford (1998) investigated the role of "preparation for future learning" (PFL) with their undergraduate students. Schwartz and Bransford (1998) conducted an experiment in which one group of college students had to create a summary of a chapter on memory studies, whereas another group had to analyze data sets from the studies to create a graph representing important patterns in the data. The students who created the graph did not see the chapter on memory studies. On a follow-up true-false assessment, students who created a summary did better than students who had to create a graph from the data sets. A more detailed look at these results revealed that although these students performed better on the assessment, they seemed to be better at retrieval of facts instead of learning for the purpose of transfer.

In the second part of the same study, students in both conditions were able to view a lecture on the memory studies. When a posttest was given a week later, students were given a task in which they had to predict the results of a new experiment. Students who were in the graphing condition were able to produce twice as many correct possible results to the new experiment. Schwartz and Bransford (1998) concluded that these students went into the lecture more prepared to learn as evidenced by their ability to generate many correct solutions. To be able to assign credit to the design features, Schwartz and Bransford (1998) also gave this test to students who did the graphs but did not have the lecture; those students performed the worst on the posttest. A major implication from this study is that whereas the graphing activity did not have immediate payout on the true-false assessment, students were still able to develop new forms of knowledge that helped them to prepare for the lecture and subsequently apply the information they learned. Advocates of the direct instruction approach have criticized the use of PFL as they feel there is a lack of control and manipulation of experimental variables, making it difficult to assign credit to particular design features (Kirschner, Sweller, & Clark, 2006).

The efficacy of PFL was later affirmed by research by Schwartz and Martin (2004). Schwartz and Martin (2004) conducted two studies on teaching statistics to students to investigate whether invention activities helped students prepare to learn. Schwartz and Martin (2004) described this approach as "inventing to prepare for learning" (IPL), which involves giving students tasks related to a new topic where they must attempt to invent a canonical solution to a problem. This activity precedes direct instruction and application of the content covered in the IPL task. In their experiment, Schwartz and Martin (2004) used a 2x2 between-subject experimental design where some students were assigned to an IPL group, whereas others were assigned to a direct instruction (DI) group. Then, half of the students in each group received a resource item on their posttest, which gave explicit steps for solving the mathematics problem. Students in the IPL group were given data that they had to develop standardized test scores for (invention phase). After the invention stage, students received a worked example on how to standardize scores (instruction phase). Students in the control group were given direct instruction on standardizing scores prior to working on a sample problem.

Schwartz and Martin (2004) found that students in the experimental group outperformed the students in the control group on a transfer test, which involved applying the concept of standardized scores to a new context. Despite the fact that students failed to come up with valid solutions when engaging with the invention tasks, students ultimately did better on future learning assessments. In addition, students with the embedded resource in their posttest were more prepared to learn during the assessment. Schwartz and Martin (2004) were able to add additional credibility to earlier studies on the efficacy of delayed instruction and failure (Clifford, 1984; Schmidt & Bjork, 1992; Schwartz & Bransford, 1998; VanLehn et al., 2003). In fact, the idea of delaying instruction by means of students working through an IPL task, and subsequently experiencing short-term failure, forms the basis for my research questions and design. This type of intervention is called *productive failure* (PF) and has been used in mathematics classroom for a number of years (Kapur, 2016). The next section describes the history of PF along with its uses in the mathematics classroom.

Productive failure. There is a widely held view that learning experiences should be appropriately scaffolded in order for learners to avoid failing (Kapur, 2008). Indeed, advocates of learner-centered design argue that scaffolds should be provided to students in order for the conceptual distance between novice and expert, or "gulf of expertise," to be minimized (Quintana, Shin, Norris, & Soloway, 2005). To challenge this viewpoint, Kapur (2008) investigated whether there is some type of efficacy in the failure process. To do so, Kapur has tested a PF model in numerous contexts (2008, 2010, 2011, 2012, 2014, 2015, 2016) to illustrate how novice learners could attempt to generate solutions to highly ill-structured problems without having prior instruction on the task's content. Kapur has observed that when novices work on ill-structured tasks, their inability to generate a canonical solution to a problem could ultimately be productive in helping themselves achieve a deep understanding of the topic. Kapur's large body of work adds additional evidence to support the role of failure and delaying instruction in the learning process.

One such study that highlights the power of failure and delaying instruction is Kapur's 2010 study on PF in mathematical problem-solving. In this study, Kapur (2010) compared a PF design with a "lecture and practice" (LP) design. A total of 75 7th grade math students from Singapore participated in this study during a unit on speed. Students in the PF group were given more complicated problems involving speed to work on in small groups. During this process, the teacher did not provide any support or scaffolding until the end of the unit when a consolidation lecture was given. Kapur (2010) employed process and outcome measures to determine the efficacy of the PF experience, including a qualitative look at work samples and group discourse, in addition to a quantitative look at work samples and students' confidence ratings. Unsurprisingly, students in the PF group reported low confidence while they initially attempted to solve the problems (1.22 out of four points on a scale of zero to four points). However, they produced an array of linked problem representations and problem-solving methods as a result of their attempts to solve the problem. Even though they were unsuccessful during the problem-solving process, students in the PF group managed to outperform their peers in the control group on both well-structured and complex posttest questions. In addition, students in the PF group were able to perform better on problems involving a high-level concept that was not covered during class.

Kapur and Bielaczyc (2012) elaborated further on the PF instructional approach by noting how PF experiences may seem counterproductive at first because learners are not being "told" how to do a problem right away; however, PF supports long-term learning. Kapur and Bielaczyc (2012) described the PF design as one that involves a problem-solving phase, followed by a consolidation phase. The problem-solving phase is where students explore the constraints and affordances of different solutions to complex problems (Kapur & Bielaczyc, 2012). The consolidation phase allows students to compare, contrast, organize, and generate ideas that will lead to canonical solutions (Kapur & Bielaczyc, 2012). Taken altogether, Kapur and Bielaczyc (2012) determined the phases of the PF learning design not only equip students with the same level of procedural fluency they would receive from a lecture, but the PF design empowers students to make deeper conceptual and transfer gains. This design idea easily lends itself to the hybrid flipped classroom setting, as students at the research site historically have had the most trouble problem solving and consolidating their information during class time.

In fact, Song and Kapur (2017) tested the idea of restructuring the flipped classroom to include PF and found that doing so results in students' improved conceptual knowledge of targeted mathematics content. They attributed part of their success to the role of the instructional videos, one of the design features in the research site. By using a PF approach in a flipped classroom, students were able to question, think, and discover before watching video tutorials (Song & Kapur, 2017). Then, when students did watch videos, they had a chance to consolidate what they learned from both the failure experience and the instructional videos (Song & Kapur, 2017). Song and Kapur (2017) also discussed how this process allowed students to actively question the unknown and explore solutions on their own, which leads to part of the social surround to be established at the research site. Last, Song and Kapur (2017) found that using PF freed up students to think and discover in class, which both led to improved problem solving and conceptual knowledge. These are exactly the same mediating processes that were examined in the research study.

Conclusion

The literature on delaying instruction and failure suggest that the combination of using an invention task, having students experience failure, and delaying instruction cannot only better position students to learn, but also to help them make conceptual knowledge gains. The hybrid flipped classroom design easily allows for students to try an invention task at the end of an in-person class with a small learning community, experience failure before watching instructional videos outside of class, watch the videos as “direct instruction,” then come to class having watched the videos ready to work with classmates on a set of problems.

CHAPTER 3: METHODOLOGY

The following sections outline the research design, materials, data collection, and data analysis measures for this study. The actual materials used are included as appendices at the end of the document.

Research Design

A mixed methods quasi-experimental research design was used with two flipped hybrid "Topics in Mathematics for Liberal Arts" (Math 103) classes to test the hypothesis that having students experience PF in a flipped classroom would result in students' improved procedural and conceptual understanding. A quasi-experimental approach was a practical necessity for this study because this study sought to determine if there was a cause-and-effect relationship between the PF experience and learning outcomes in Math 103 (Creswell, 2014). Quasi-experimental designs are also used when individuals are not randomly assigned, as was the case in this context (Creswell, 2014). Students involved in this study deliberately chose to take a flipped section of Math 103 instead of a traditional lecture-based section. In addition, a mixed-methods study was chosen because the research questions involved gathering both quantitative and qualitative data to better understand learning in each of the flipped classroom designs (Creswell, 2014).

One section of Math 103 (N=31) was the control group taught using the direct instruction-based flipped classroom model (DIFC), whereas the other section (N=22) was taught using the PF-based flipped classroom model (PFFC) during the sixth through ninth weeks of the semester. It should be noted that normally sections of Math 103 run at almost 35 students each, however, enrollment seemed to drop this semester and one section had a lower enrollment than the other. The treatment groups were randomly assigned by using a random number generator.

The researcher was the instructor for both sections and had 10 years of experience teaching this particular course, with four semesters teaching it in the hybrid flipped classroom format.

Learning Environments

Two different versions of the learning environment for Math 103 were used as the contexts for this study. Math 103 is a very popular course at Rutgers University for non-mathematics majors to take to fulfill a School of Arts and Sciences (SAS) quantitative requirement. In fact, the course attracts students from a wide range of liberal arts and social science disciplines including journalism, political science, education, and world language, to name a few. Math 103 investigates a variety of areas in which mathematics is concretely applied, including voting systems, the measurement of power in weighted voting systems, apportionment, fair division, fair distribution, and exponential growth in nature and finance. These topics were selected because (a) they are relevant for students who are majoring in areas outside of the physical sciences, and (b) mathematical research into these areas is ongoing, which allows students to engage in additional research and writing opportunities as they relate to their areas of interest. The overarching learning goals for the course are (a) to formulate, evaluate, and communicate conclusions and inferences from quantitative information, and (b) to apply effective and efficient mathematical or other formal processes to reason and to solve problems. Instructors design learning opportunities within Math 103 so that students find the content engaging and accessible, even if students are convinced that mathematics is not their strongest subject.

Both the DIFC and PFFC learning environments in Math 103 involved the combination of having students learn from video tutorials, followed by active learning opportunities for students to deepen their conceptual understanding under the facilitation of the instructor (Prince, 2004; Roehl et al., 2013). The main difference between the two environments was the purpose of the video tutorials. In the PFFC model, students first worked on an invention task at the end of each in-person class session that adopted the PF strategy (Kapur, 2008; Schwartz & Martin, 2004). This means that students worked on an engaging task they had not yet received formal instruction on, but the task itself was sufficiently difficult that students would have trouble finding a completely correct solution to it (Kapur & Bielaczyc, 2012). Then, students watched video tutorials on the topic of the invention task in an effort to consolidate its concepts and associated procedures (Song & Kapur, 2017). The process ended with students coming to class to work on problems that required both procedural and conceptual knowledge in a small group setting. In the DIFC model, students watched videos first in order to gain initial exposure to the content. Then, they worked on the same problems in groups under the facilitation of peers and the instructor. These students continued to work on problems until the end of class, whereas their PFFC counterparts engaged in the next invention task for the last 15 minutes of class. This cycle repeated for four weeks during the unit on apportionment, fair division, and fair distribution, leading up to the summative assessment on these topics. Table 1 outlines the sequence of events for one lesson in each learning environment.

Table 1

Sequence of Activities in the DIFC and PFFC Learning Environments

Learning Environment	**End of in-person class #1 (15 min)**	**Outside of class preparation**	**Start of in-person class #2 (10 min)**	**Middle of in-person class #2 (55 min)**	**End of in-person class #2 (15 min)**
DIFC	Additional practice problem on current topic	Work through a set of 13-15 videos	Quiz on videos	Solve five or six problems facilitated by teachers and peers	Additional practice problem on current topic
PFFC	Invention task on new topic	Work through a set of 13-15 videos	Quiz on videos	Solve five or six problems facilitated by teachers and peers	Invention task on new topic

Sample

The sample for this study was drawn from the undergraduate population of the Rutgers University-New Brunswick campus. Rutgers-New Brunswick currently has an undergraduate population of approximately 36,168 students, of which 53 took part in this study (Rutgers University, 2017). The students were undergraduate liberal arts majors who deliberately opted to take a flipped hybrid course upon registration. The student population of Rutgers University is ethnically diverse, with approximately 7.6% of students being African American, 24.4% Asian, 0.2% Hawaiian, 12.3% Latino, 0.1% Native American, 11.4% International, 39.4% White, 2.9% two or more ethnicities, and 1.8% unknown (Rutgers University, 2017).

Based on undergraduate enrollment figures from the 2016-2017 academic year, approximately 14.5% of the 674 students in all sections of Math 103 were seniors, 23.2% were juniors, 28.4% were sophomores, and 33.9% were freshmen. The students in each section did not exactly follow this distribution; the PFFC group had more freshmen students, whereas the DIFC group had a disproportionate number of seniors. In addition, two students in the DIFC group had advanced mathematical training prior to this course, which is unusual since Math 103 does not count toward any math or computer science major. Both of these students took the course as an elective out of general interest. Table 2 shows additional characteristics of the sample broken down by group.

Table 2

Sample Characteristics

	Experimental group (PFFC)	**Control group (DIFC)**
Gender breakdown	36.4% male 63.6% female	32.3% male 67.7% female
Year breakdown	45.4% freshman (N=10) 27.3% sophomore (N=6) 9.1% junior (N=2) 18.2% senior (N=4)	22.6% freshman (N=7) 22.6% sophomore (N=7) 19.4% junior (N=6) 35.4% senior (N=11)
English as first language	90.9%	96.8%
GPA	Range: 2.8 to 4.0 Mean: 3.4 Standard deviation: 0.4	Range: 1.1 to 3.9 Mean: 3.1 Standard deviation: 0.5
Math background	Only one had calculus background, the rest Algebra 2 or Pre-Calc.	Five students with calculus background, one student took the entire math sequence at Rutgers, two took statistics
Common majors	psychology, sociology, English, history	psychology, sociology, English, communication, computer science (N=1), and mathematics (N=1)

It was expected that students would have no instructional experience with the targeted concepts for this study, because the topics are not traditionally taught in high schools. Based on pretest findings, this was the case for all students.

Materials

The materials used in this study have been developed to allow students to engage, explore, explain, extend, and reflect on mathematics (Krajcik, Blumenfeld, Marx, & Soloway, 2000; Song & Kapur, 2017). The subsections below outline the use of invention tasks, instructional videos, pretests and posttests, in-class problems, a survey, and a focus group interview as they relate to the research context. Table 3 summarizes the alignment of materials used in the research study.

Table 3

Alignment of Materials

Week #	Topic	Pretest/Posttest	Invention Task Topic	Video Set	In-Class Problems
1	Mathematical apportionment	Problem on apportioning buses to routes, problem on flaws of apportionment (Quota Rule and Alabama Paradox)	Problem on apportioning legislative seats to a new country (Hamilton's Method)	15 videos on apportionment methods including Hamilton's and Jefferson's methods, Quota Rule, and Alabama Paradox	Problems on apportioning nurses to hospital shifts using both methods from videos, Alabama Paradox and Quota Rule
2	Mathematical apportionment and introduction to fair division (for two players)	Problem on dividing a business between two people (Divider Chooser) *Posttest only: problem on connecting quotas to apportionments	Problem on fairly dividing a retail space for a year between two people (Divider Chooser Method)	15 videos split between two topics: some on Huntington-Hill and Webster apportionments and some on Divider Chooser method	Problems on apportioning nurses to hospital shifts using methods in videos and problems on dividing retail space over a year

3	Fair division (for three or more players)	Problems on vacation timeshare (Lone Chooser) and sharing a store (Lone Divider) *Posttest only: dishonest bidding while using Lone Divider	Problem on fairly dividing a piece of land among four people (Lone Divider Method)	14 videos on fair division including the Lone Chooser and Lone Divider methods	Problems on dividing land into subshares (Lone Divider) and retail space (Lone Chooser) over the course of the year.
4	Fair distribution	Problems on distributing toys (Method of Markers) settling an inheritance (Method of Sealed Bids)	Problem on fairly distributing jointly owned dorm items among three roommates (Method of Sealed Bids)	14 videos on fair distribution including the Method of Markers and Method of Sealed Bids	Problems on distributing jointly owned business partnership items, house settlement, distributing CDs and distributing fruit among people

Invention task for the PFFC group. First, the PFFC group worked through one invention task during each week of the study. The invention tasks (see Appendix A) were engaging problems that students found sufficiently difficult to solve, but were able to at least draw on their prior knowledge to try and develop a solution (Schwartz & Martin, 2004). Having students complete these tasks was a key component of the PFFC instructional design and as a result served as the learning intervention in this study. In addition, the topics of the invention tasks addressed the topics covered in the videos and in-class problems, but used a slightly different context to avoid surface similarity. For example, the invention task used during the 4th week of the study involved distributing jointly owned items in a college dormitory among college roommates. This required the use of the Method of Sealed Bids, which was formally covered in the videos and in-class problems. The problems on the Method of Sealed Bids completed in class and on the pretest and posttest covered the same skill but instead used inheritances and dissolving business partnerships as contexts.

The criteria used to develop the set of invention tasks were drawn from earlier works on PF (Kapur, 2016; Kapur & Bielaczyc, 2012). First, all tasks were developed so that they would be challenging enough for the learner to want to engage with, but not too difficult that the learner would want to give up on them. This was accomplished by designing tasks that required elementary mathematics operations to solve, yet were challenging enough so that the solution was not readily discoverable. All four tasks involved real-life scenarios where mathematics was needed, including dividing shared dorm items fairly amongst departing roommates, dividing a jointly-owned piece of land fairly, apportioning legislative seats to states in a newly formed country, and dividing retail space for tenants so that they could maximize their revenue. The relatable problem contexts were designed to engage students in solving the problems while also activating their prior knowledge of division. In other words, all four problems were calibrated to a "sweet-spot" (Kapur & Bielaczyc, 2012) where the complexity of the problem, mathematical resources of students, and affective draw were all taken into account as the problems were designed.

Second, all tasks allowed for multiple RSMs. Students had the ability to use algebra, ratios, guess and check, and any other solution strategies they wanted. There was also not one pathway to any particular solution; students could develop their solution using a variety of approaches and some of the aforementioned representations. Third, the problems all activated the learner's prior knowledge, both formal and intuitive. Students were able to draw on prior knowledge of multiplication, division, and ratios, while also trying to develop ways to settle discrepancies that arose with trying to divide items fairly. Last, as the instructor, I built upon students' solutions by drawing their attention to key features of the solution. This was achieved through video instruction as well as in-person instruction after students have completed the task.

For example, the very first invention task was about the concept of mathematical apportionment. In this task, students were asked to apportion 250 legislative seats to six states within a newly formed country. The problem is each state had a very large population, and all states varied in their population size. Due to the size of the numbers, students could not guess or find an answer right away, but could certainly draw on their prior knowledge of ratios and proportionality to attempt to solve the problem. This helped some students make some progress, however, the problem was designed so that students would face issues with decimals. Even if a student was able to calculate the exact number of seats each state should be allocated if every state could receive an exact fair share, he or she would be left with an unrealistic apportionment that involves decimals. This is where the "failure" aspect came in as students were left trying to figure out what to do with messy numbers. These measures were taken to ensure a PF experience where students could attempt to solve the problem, but have difficulty arriving at the correct answer.

The invention task on apportionment embodied all of the PF design elements as suggested by Kapur and Bielaczyc (2012) and Kapur (2016). First, the problem's affective draw was getting students involved in giving out legislative seats to sates based on populations. Second, students could enter this problem using a variety of ways, including proportions, ratios, and estimating. The apportionment problem became challenging when students had to decide what to do with the fractional parts of seats; this is precisely where the values used in the problem were made larger so as to not overwhelm the students, but rather challenge them. During this process, students had to draw on prior knowledge and mathematical resources to push forward and not give up on the problem. Finally, as the instructor, I ensured the instructional video that showed the solution to this task emphasized key features of the solution, such as the concept of a standard divisor to use when apportioning seats and the logistics of using Hamilton's method to apportion seats. The process of identifying key features also continued during the in-person class meeting as I circulated the room to provide feedback and assistance to students.

In-class problems. When students arrived to the in-class problem-solving session, they worked on a set of approximately five to six problems with guidance from both the instructor and peers. Each problem typically had multiple sub-questions attached to it. There were two types of sub-questions asked: those that primarily focused on procedural knowledge to solve, and those that primarily focused on conceptual knowledge to solve. Table 4 shows the content of each week's problem-solving session along with its corresponding problem set (see Appendix B for the full set of problems).

Table 4

Content of Each Weekly In-Person Problem-Solving Session

Problem Type	**Week #1** **Apportionment**	**Week #2** **Apportionment and Fair Division**	**Week #3** **Fair Division**	**Week #4** **Fair Distribution**
Procedural Knowledge	HW #2ab, 3ab, 4, 7 (Hamilton and Jefferson apportionments)	HW #1, 5, 8 (A) HW #2, 4 (FD) (Webster and Huntington-Hill apportionments and Divider Chooser method)	HW #1, 5, 6, 7, 9 (Lone Chooser and Lone Divider methods)	HW #1, 3, 4, 5 (Method of Sealed Bids and Method of Markers)
Conceptual Knowledge	HW #2c, 3c (Application of Hamilton and Jefferson apportionments)	HW #6 (A) HW #3 (FD) (Extension and application of apportionment methods and Divider Chooser method)	HW #8, 10 (Extension and application of Lone Chooser and Divider Chooser methods)	HW #2, 6 (Extension and application of Method of Sealed Bids and Method of Markers)

Procedural knowledge problems. Problems that primarily involved using procedural knowledge to solve required students to demonstrate mastery of the methods they have learned through the video tutorials. This included being able to work through a mathematical division method, as well as being able to explain all steps within the context of the problem. For example, a typical procedural knowledge problem on apportionment looked like the following:

> "HW7. This problem pertains to the Alabama paradox.
> a. Suppose that the Gesundheit Hospital described in problem HW1 above hires one more nurse, for a total of 401, but the number of patients on each shift remains the same. Find the Hamilton apportionment.
> b. Does increasing the number of nurses from 400 to 401 create an instance of the Alabama paradox? Explain how you can tell.

> c. Now suppose that the same hospital hires yet another nurse, for a total of 402, but the number of patients on each shift again remains the same. Find the Hamilton apportionment.
> d. Does increasing the number of nurses from 401 to 402 create an instance of the Alabama paradox? Explain how you can tell" (Weingart & Seneres, 2013).

As one of the learning goals for the course is for students to develop a working knowledge of mathematics and being able to explain and write up their solutions, the problems we ask always require students to analyze their results and explain them in terms of the original problem. In the problem above, students were asked to practice the procedure of the Hamilton apportionment in order to see if an Alabama Paradox has occurred. An Alabama Paradox occurs when adding an additional seat to an apportionment (part c) results in a state losing a seat they originally had (part b). Students were then asked to clearly identify whether or not an Alabama Paradox occurred (parts b and d) using complete sentences, which achieves the course goal of justifying and explaining solutions. The invention task associated with this topic had students assign seats to states in logical and fair fashion; this problem had students take that procedure one step further to realize a flaw of apportionment.

Over the course of the four weeks, students solved problems on apportionment, fair division, and fair division in which they had to decide which procedures to use to solve each problem (Booth, 2011). As such, a solid knowledge of procedures and features of problems that lend themselves to certain procedures was needed to be successful on these problems.

Conceptual knowledge problems. Problems that required conceptual knowledge to solve took students beyond rote application of a procedure learned through the video tutorials. For example, in the set of problems on fair division, students leaned about two methods for dividing discrete goods: the Method of Sealed Bids and the Method of Markers. The two conceptual knowledge questions gave students the opportunity to think beyond application of each method and carefully consider different cases. For example, HW #2 had students carry out the Method of Sealed Bids in order to give out items to business partners from a dissolving partnership. Part of the Method of Sealed Bids involves a surplus of money, which is evenly distributed among players. HW #2 probed students to consider, "Suppose that after the first settlement, the banker runs off with all the surplus money. Do the players still end up with fair shares in spite of not receiving the surplus money? Explain briefly." If students truly understood the concept of the Method of Sealed Bids, they would be able to develop a solution that shows that fairness was achieved even without dividing up surplus money. The invention task on this topic also was designed in a way that revealed surplus money when initial bids are made, thereby drawing attention to this critical feature of the settlement. This was deliberately done so that students would be better positioned to understand the concept of a surplus and how it gets resolved.

For the same week, HW #6 was a problem that required students to explore what happens when people try to game the system. In this problem, students were asked to:

> "Suppose that Wendy knows Xavier's valuation of the house, and writes down $340,000 as her bid (even through the house is actually worth $300,000 to her). Xavier still writes down $350,000 as his bid. When they carry out the Method of Sealed Bids, how much is the total value of Wendy's share, and how does it compare to what she receives when she bids honestly as in HW1? How much is the total value of Xavier's share here, and is it a fair share? Explain" (Weingart & Seneres, 2013).

As a follow up to this part of the question, students were then asked to consider what happens when both Wendy and Xavier try to game the system. After working out both scenarios, students were asked to describe the moral of the story in light of their results to both parts of the problem. Students who were able to generate cases and develop a rationale for why it is not a good idea to lie in the Method of Sealed Bids truly understood the premise of the method and how it relies on honesty in order to give everyone a fair share. All conceptual knowledge problems in this study were of a similar type where students are asked to think beyond the usual execution of the problem and instead take a closer look at the mathematical processes behind a method. As a researcher, having problems like these is important in my class as active discovery and application of mathematical concepts contribute to students' conceptual understanding of targeted content and also bolsters students' procedural understanding of the same content (Baroody et al., 2007).

Video sets. Both groups worked through a weekly set of approximately 13-15 short instructional videos on the topics of apportionment, fair division, and fair distribution. Table 5 outlines the topics and nature of each video set over the four weeks of the study.

Table 5

Video Set Schedule and Content

Class Sessions	Week #1	Week #2	Week #3	Week #4
Topic	Hamilton and Jefferson apportionment methods	Webster and Huntington-Hill apportionment methods and Divider Chooser method	Fair Division: the Lone Chooser and Lone Divider methods	Fair Distribution: the Method of Markers and Method of Sealed Bids
Videos	15 videos ranging from three to 10 minutes each	15 videos ranging from three to 13 minutes each	14 videos ranging from three to eight minutes each	14 videos ranging from two to nine minutes each

These videos featured direct instruction and worked examples on the aforementioned topics. The intent of the videos was to provide all students with the background and motivation for using each method, followed by a detailed explication of how to use each method to achieve mathematical fairness in a variety of settings. Students in both research designs watched the same video set, with the exception of two additional videos for the PFFC group, which consisted of two annotated solutions to two invention tasks. Each video set followed more or less the same layout which included an introductory video, videos that illustrated specific examples of the week's topic, videos that illustrated a mathematical paradox or shortcoming of the methods used in previous videos, and a summary video that concluded each set.

For example, the video set on the topic of fair distribution began with a video that transitioned the students from fair division (the previous topic) to fair distribution by highlighting a few key similarities and differences between the two topics. This video, along with the following two videos, contrasted fair division of continuous goods with fair distribution of discrete goods and had students understand why they could not use fair division to "divide" items that cannot be shared like household objects. Mathematical vocabulary associated with fair distribution was also clearly spelled out on the screen and annotated by the presenter in the first three videos so that students could articulate solutions to solve problems on fair distribution.

Then, the main ideas were presented by way of worked examples. In the case of fair distribution, four videos covered two cases of the Method of Sealed Bids (two players and more than two players). The examples used to demonstrate the methods were always grounded in real-life contexts; for these four videos, the two contexts used were dissolving a law partnership and roommates moving out of a college dormitory. For every worked example, students had the opportunity to try a similar problem on their own before moving on to the next video. Each video in the set began with a solution the end-of-video problem from the video before it so that students had the opportunity to self-diagnose and reflect on their work.

The second half of this video set consisted of a similar sequence of videos on the second fair distribution method, the Method of Markers. Similar to the progression of videos on the

Method of Sealed Bids, the final seven videos began with an introductory video on what the Method of Markers is, followed by four videos that detailed how the method is executed in context (friends sharing a set of fruit). The usual end-of-video questions appeared in order for students to try similar problems to ensure they understood the method. The set ended with two videos: one that summarized the method, and one additional problem where students are asked to work on a more difficult version of the method where they have to place the markers themselves using mathematics. For students in the PFFC group only, the final video in two of the sets was a detailed solution to the invention task for the week, which helped serve to consolidate the knowledge students accessed through their PF experience on the invention task while highlighting key features of the content (Kapur, 2008). For the other two sets of videos, the solutions to the invention tasks were embedded in existing videos, as the invention tasks were taken directly from existing course examples.

Pretest and posttest. The same pretest (see Appendix C) and posttest (see Appendix D) were given to students in both treatment groups included problems similar to those addressed in the invention tasks, instructional videos, and in-class problem-solving assignments. As previously noted, the problems on both assessments had slightly different storylines as they did on the invention tasks and in-class problems to avoid surface similarity. All problems on both assessments addressed specific concepts listed in the specific learning objectives for Math 103. The results of these tests were used to answer the research questions on gains in students' procedural and conceptual understanding.

Pretest. The pretest was used as a measure of prior knowledge of targeted concepts in the unit of study. As a result, all six questions gauged whether or not students already know how to execute the methods described in the unit. These questions were largely procedural in nature;

for example, the following question had students demarcate items for people using the Method of Markers:

> **6.** Jason, Keith, and Clark have are dividing a set of 15 toys – 3 fidget spinners (F), 6 pogs (P), and 6 containers of gak (G) using the Method of Markers.
> - Jason loves fidget spinners but hates pogs and gak.
> - Keith loves fidget spinners and pogs equally well but hates gak.
> - Clark loves pogs and gak equally well but hates fidget spinners.
>
> The toys are lined up in an array as follows:
> G G G P P P F F F P P P G G G
> **6a.** Place markers for each person above based on their value systems.
> **6b.** Describe the allocation of toys to each player and describe what toys are left over.
> **6c**. Suppose that the players agree that each one can pick an extra toy from the leftovers. Suppose Jason picks first, Clark picks second, and Keith picks third. Describe which leftover toys each one would pick.

In order to solve this problem, students needed to know the Method of Markers procedure and how to establish fair shares for each player. The pretest contained five additional questions that asked students to demonstrate knowledge of other methods discussed in the unit, including Hamilton's Method, Webster's Method, the Lone Chooser Method, Method of Sealed Bids, the Lone Divider Method, the Divider Chooser Method, and the Huntington-Hill method. The topics addressed on the pretest also addressed course-wide learning goals of being able to apply effective and efficient mathematical procedures to solve problems.

Posttest. The posttest was designed to test students' understanding of the content immediately after they learned the whole unit. The posttest was comparatively more difficult than the pretest. It included six questions that primarily drew on procedural knowledge (similar to those on the pretest) but also included two additional questions that primarily drew on conceptual knowledge. Similar to those described by Kapur and Bielaczyc (2012), the two conceptual understanding problems were more complex and unfamiliar to students. As a result, students had to be more flexible with their solution methods to solve these problems (Kapur & Bielaczyc, 2012). An example of such a problem on apportionment is as follows:

> **“8.** Consider the problem of apportioning M seats between two states, A and B. Let q_A and q_B denote the standard quotas of A and B, respectively, and assume that these quotas have decimal parts that are not equal to 0.5. Explain why in this case
> **8a**. Hamilton’s and Webster’s methods must give the same apportionment.
> **8b.** the Alabama or population paradoxes cannot occur under Hamilton’s method.
> **8c**. violations of the quota rule cannot occur under Webster’s method”
> (Tannenbaum, 2013).

What makes this problem conceptual is that students needed to generate their own values to solve the problem and consider various cases. In the in-class problem-solving session and video sets, complete tables were provided and students had to follow procedures to produce a result. In this setting, scaffolds were removed and students had to use very limited information to draw conclusions. They had to think conceptually across methods in order to develop examples and counterexamples for all three parts of the problem. This inherently required a procedural understanding of the various apportionment methods as well. Students who successfully solved this problem truly demonstrated conceptual mastery of the concept of apportionment and the flexibility to apply concepts to unfamiliar settings. In addition, the posttest addressed the course learning goal of having students formulate, evaluate, and communicate conclusions from quantitative information.

Survey. Students in both groups also completed a short, four-question survey on their self-reported video watching behaviors (see Appendix E). Students were asked to acknowledge whether they watched all of the instructional videos for the unit, how many times they watched the videos, where they watched the videos, and the device they used to access the videos. As a result, the survey measured frequency of video-watching and the location in which it happened. Additional video-watching information was obtained from Kaltura, the analytic platform on the course’s learning management system, which allowed for a deeper analysis of students’ video-watching behaviors.

One purpose of the survey was to gain insight into students' video watching behavior in order to ensure students in both treatment groups completed the work required in a flipped classroom. A secondary purpose was to examine video watching patterns between groups to see if there was a difference in video watching frequency due to the instructional design. The results of this survey were used to answer the third research question and also provided additional quantitative information helped explain the first two research questions.

Focus group interview. Last, students had the opportunity to reflect on their experiences learning in the flipped classroom format by taking part in a focus group interview. The five questions in the focus group interview were designed in such a way to allow students to reflect on their learning process in the flipped classroom while simultaneously offering suggestions for course design improvement (see Appendix F). The questions included: (a) Can you describe the learning activities in this past unit? (b) How do you find the learning activities? Why? (c) What do you think is most important in your mathematics inquiry process? (d) What aspects of the course design would you like to see strengthened? (e) Is there anything else you'd like to tell me about the course design that might not have been addressed by any of the previous questions? Of utmost importance to the study were the responses to questions (a) and (b), as they gave students an opportunity to comment on the roles of the invention tasks, videos, and in-person supports in their learning. These responses formed the basis for an analysis of what learning looked like in each of the two learning environments.

Creswell (2014) listed several advantages of using focus group interviews to obtain data: they are useful when participants cannot be directly observed, they allow participants to provide historical information, and researchers can have complete control over the line of questioning. Denzin and Lincoln (2011) also added that focus group interviews allow the researcher to observe the interactions among participants. These spontaneous interactions have the effect of decreasing the amount of interaction between the researcher and participants, thereby giving more weight to the participants' opinions and less weight to researcher influence (Denzin & Linclon, 2011). The results of the focus group interview were used to answer the fourth research question.

Procedure

Mixed method data collection strategies are used to validate one form of data with the other form, transform the data for comparison, and address different types of questions (Creswell & Plano Clark, 2007). This was an appropriate data collection strategy for this study as qualitative findings from the focus group interview and video helped validate quantitative findings from surveys and course assessments. Table 6 describes each data source and how it was used in this study.

Table 6

Data Sources and Analysis

Data source	Data analysis	Effectiveness of pedagogical design intervention on the development of	
		(a) procedural knowledge	(b) conceptual knowledge
Survey	Quantitative	**x**	**x**
Pretest	Quantitative	**x**	
Posttest	Quantitative	**x**	**x**
Focus group interview	Qualitative	**x**	**x**
Video data	Qualitative	**x**	**x**
In-class problems	Qualitative/Quantitative	**x**	**x**

After participants in the DIFC and PFFC groups signed consent forms to participate in the study, the procedures below were used to guide the data collection process.

Pretest and posttest. Similar to the Song and Kapur (2017) study, this study employed a pretest/posttest design. A 60-minute pretest was given to participants in both groups to test students' prior knowledge of apportionment, fair division, and fair distribution before they started learning the topics. The pretest was administered during the first class session of the semester. An 80-minute posttest was also given to participants in both groups upon conclusion of the unit and tested the same topics as the pretest. Students were allowed to use a calculator for both assessments.

In-class problems. Students in the DIFC had approximately 55 minutes during each in-person class session to solve problem sets in groups. Students in the PFFC group had approximately 40 minutes during each in-person class session to solve problem sets in groups. These students spent the last 15 minutes of class time working on the invention task for the week. Both groups were able to finish their in-class problems for homework and submit their solutions online within two days.

Survey. All participants completed the anonymous video-watching survey after the posttest. Participants were given five minutes complete the pencil-and-paper survey during class time. In addition, quantitative video-watching data were obtained from the Kaltura program on the learning management system after the four weeks of the study were over.

Focus group interviews. Focus group interviews for each group were administered in the class meeting following the posttest for each group. In order to represent as many participants as possible within each focus group, approximately three participants were randomly chosen from each third of the class, as determined by grades in the learning management system grade book after the posttest. These participants sat in a circular arrangement with the instructor as part of the circle, creating a comfortable dynamic among participants. The interview protocol template was read with emphasis on preserving the participants' anonymity. As stated in the interview protocol, participants referred to themselves by a number given to them at the start of the interview. The focus group interviews were audio-recorded using two devices and were subsequently sent for professional transcription. The focus group interviews took approximately 20 minutes each.

Video data. Collins, Joseph, and Bielaczyc (2004) recommended the use of video records to understand what is happening in an intervention in detail. Chi (1997) also noted the importance of collecting and analyzing data in messy contexts so that researchers can better understand complex activities in the context in which they occur. As a result, video observations of students' in-class problem solving took place during the four weeks of study for two groups of students in each condition. One stronger group and one weaker group were chosen in each condition in order to capture the range of abilities in each group. Video footage of students working in their groups, exchanging ideas, accessing resources, and constructing solutions to problems was taken and subsequently analyzed to understand how students mobilized their procedural and conceptual knowledge to solve problems. Another purpose of the video was to have an additional source of data that could be used to help explain gains in conceptual knowledge for students based on their written work samples. Having video data available could be helpful in determining the precise nature of students' conceptual and procedural knowledge gains and could also explain lack of gains or anything puzzling about students' performance.

Data Analysis Plan

Because this was a mixed-methods study, both quantitative and qualitative methods were used in the data analysis to address the effectiveness of the pedagogical design intervention on both procedural and conceptual understanding. Each data source analysis is described below along with how it was used to answer each research question.

Pretest and posttest. To understand the effectiveness of the PFFC design on procedural and conceptual understanding, the results of the pretests and posttests were analyzed quantitatively to determine if the design features in the embodied conjecture contributed to students' improved procedural and conceptual knowledge. The posttest was designed to measure both procedural and conceptual knowledge gains made in both treatment conditions over the course of the unit. The posttest served as the students' chance to demonstrate the knowledge they've accumulated over the course of working through the video sets, completing problems in class, and studying in between.

As the embodied conjecture for the learning environment involves tracing the development of procedural and conceptual knowledge in Math 103, the questions on the posttest that specifically related to each type of knowledge were scored blindly using a rubric that allowed for partial credit. For questions that largely required procedural knowledge to solve, students earned points for each correct step they took in solving the problems. For questions that largely require conceptual knowledge to solve, students earned points for work and ideas that led to correct explanations. The data related to the two conceptual understanding questions on the posttest were analyzed separately from the rest of the posttest questions that primarily addressed procedural understanding. The posttest questions primarily drawing on procedural knowledge accounted for 76 points out of 100 available points, while the two additional questions requiring conceptual knowledge accounted for approximately 24 points out of 100 available points. To ensure the rubric was applied fairly, both the researcher and an independent grader scored a few work samples together to understand the grading scheme. Then, they continued to grade all papers together and reached 100% consensus before assigning final scores.

The scores for each assessment were recorded in SPSS. A two-sample t-test was used on each group's performance data at the 5% significance level to determine whether students in the PFFC group were able to outperform their DIFC peers on questions involving procedural understanding. Similar to Song and Kapur (2017), the results of the two conceptual understanding questions in the DIFC and PFFC groups were also tested by a t-test (at a 5% level of significance) to determine if conceptual learning performance was statistically different between the two groups. Additional correlational analysis on specific posttest items and other course assessments was also performed. Taken together, the results of facilitating this comparison between pretest and posttest scores were used to answer the first and second research questions. Additional data sources, such as the video footage and video-watching survey described below, were used to help explain quantitative findings in the posttest data.

In-class problems. The results of the in-class problem sets were also compiled and analyzed by means of t-tests for each of the four weeks of the study. Descriptive statistics were also calculated for the in-class problems for both groups for all four weeks of the study. Correlational analysis was used to determine if there were any significant correlations between in-class problems and other courses assessments. In addition, the results of the invention tasks were also compiled and reported using descriptive statistics. This formed the basis for a more detailed look at the in-class work of two groups of students in the PFFC group.

Survey. The results of the video-watching survey were compiled and presented in a table with corresponding percentages for each group in the study. Then, the results were analyzed using a one-way ANOVA test to determine whether students' video watching behaviors influenced their learning performance. In addition, t-tests were performed to see whether the DIFC and PFFC groups had a significant difference in the number of times they've watched the instructional videos, location of where they watched the videos, and what platform they used to watch the videos. Additional detailed information from Kaltura analytics was also used to take a closer look at students' self reported video-watching behaviors and its results were compiled and presented in a table. This information was used to answer the third research question on what students' self-reported video watching behaviors were.

Focus group interview. The focus group interview transcripts were professionally transcribed and coded. Hypothesis coding was used to test the researcher-generated hypothesis that students who experienced PF would view their learning experience differently compared to students who had a direct video instruction experience. In hypothesis coding, codes are developed from a theory or prediction about what will be found before the data is collected and analyzed (Saldaña, 2009). Using the high-level conjecture from the study's conjecture mapping, I developed codes based on the PF design embodiment and mediating processes involved in the learning intervention. For example, because part of the design embodiment involved having students work in groups to solve problems related to the video tutorial content, I hypothesized that students would collaborate and peer teach. Subsequently, I used the code "peer teach" when students described educating a classmate in their groups. An example of a statement that was coded with "peer teach" was when a student commented how the classroom design, specifically work with peers, allowed him to "learn how to do things from one another."

The coding system was refined in an ongoing matter as the transcripts were initially coded then coded again after a second read. For example, the statement, "group chat or group meet, we use so that if we have questions about homework or a test, we just go over them" was initially coded as “peer teach,” but since this was a very specific type of virtual group collaboration outside of class, I ended up building a new subcode, “mobile teaching,” to be more specific about the type of teaching that actually took place. A total of 12 codes and three subcodes were developed for the study. Examples of codes and their subcodes are shown in Table 7; a full table will be provided in Chapter 4.

Table 7

Codes

Code	Definition	Data Example
Understanding (U)	U: any loose reference to comprehension	U: “it's like when you understand something in a video”
Subcodes:		
Conceptual understanding (CU)	CU: students describe development of key ideas	CU: “understand core concepts” PU: “understanding how to do it and the process behind”
Procedural understanding (PU)	PU: knowledge of steps or how to complete a problem	PU: “memorizing certain ways of figuring out a certain problem” PU: “And it's easier to learn in the same order that you did on your own when you come to back to the classroom.”
Peer teaching (PT)	PT: students working in a small group or pairs to help one another understand content	PT: “learn how to do things from one another”
Subcode: mobile teaching (MT)	MT: when students help one another understand through a mobile device	MT: “group chat or group meet we use so that if we have questions about homework or a test, we just go over them”

While coding the transcript, I used search strategies for focus group interview transcripts from Krueger and Casey (2002). First, they recommend scrutinizing words and context as interview respondents will likely use a variety of words, and it is up to the researcher to determine the degree of similarity between responses. In addition, Krueger and Casey (2002) also discussed how context is important to consider as respondents are often triggered by a stimulus or comment from another respondent. These are among the most important times to look for important responses. As a result, I paid careful attention to those instances when coding. The purpose of the interview analysis was to develop findings to answer the fourth research question on students' explanation of their mathematical inquiry process during class. In addition, findings from the interview analysis were used help explain quantitative data findings from the other research questions.

Video data. All video footage of in-class problem solving was transcribed. In this study, video footage of the invention task solving processes for groups in the PFFC was the most useful data to code as it provided an additional lens into the PFFC learning environment. Frequency counts of solution proposals, whether correct or incorrect, were tracked to help answer the fifth research question. This information was also used to help describe what successful group work looked like in the PFFC learning environment.

Limitations, reliability, and validity. Due to my involvement as both the researcher and the instructor of the course, threats to validity surfaced in this study. To ensure fair treatment of both the control and experimental groups in this study, I created a detailed protocol that listed the steps I took to ensure comparable learning environments in both conditions (see Appendix G). The protocol included specific prompts for responding to students' questions depending on the level of help they needed. The questions were largely the same for each group, but the PFFC group had additional questions that required students to reflect on their experiences with the invention tasks. To ensure I was following the protocol, I kept field notes (see

Appendix H) for each class session as part of a reflective analysis of my teaching process (Cobb, 2000). Cobb's work (2000) focused on how the ongoing analysis of classroom events and retrospective analyses of data gathered must follow the initial instructional design process. To that end, I made some adjustments to my instruction in light of my analyses. For example, after the first week of the unit, I began starting both classes with a bulleted list of critical features of each concept along with visual aids for students to better comprehend processes they were learning in the videos and in class.

In terms of data reliability, I shifted gears as often as possible to be both immersed in the research context and critical of the research process (Design-Based Research Collective, 2003). I triangulated my data sources by combining work samples, survey results, focus group interview responses, and video data to see if patterns emerged among the data sources. Since themes became established based on bringing several sources of data together, this process can be said to add validity to the study (Creswell, 2014. Finally, the focus group interview contained an additional question that asked students whether or not they conversed with Math 103 students in other sections; this was used to help understand whether or not students discussed their thoughts about being in one group or another.

CHAPTER 4: RESULTS

The results are organized into three sections. The first section presents a detailed description of the scoring and coding processes used to analyze students' work samples, focus group interview, and video recordings in the course. The second section lays out all quantitative results, including descriptive statistics and the results of statistical tests. Finally, qualitative data will be presented as it relates to work samples, the focus group interview, and videos of students' in-class problem-solving. The qualitative data is meant to help explain the quantitative data and will be used in conjunction with the quantitative data to develop the final conclusions.

Coding

I will begin by explaining the coding and scoring processes for all work samples in both groups, followed by an explanation of how the solutions to the invention tasks were coded for the PFFC group. I will also describe how the focus group interview was coded.

Scoring of in-class problems. Undergraduate graders were assigned to all sections of Math 103 to score in-class problems. The mathematics department employs upperclassmen as graders provided they have earned a final grade of "A" in the course they want to score for, have their professor's recommendation, and have an overall GPA above 3.5. In this study, I had the same grader for both sections of the course. The grader was unaware of which group was which in this study. To ensure that students were given the proper score for each assignment, I went back and scored all of the assignments using the same rubric the grader was supplied with. In the very few instances there was a discrepancy between the score the student assigned and the score I assigned, I re-read the assignment and corresponding rubric one more time and chose the most appropriate score based on the rubric and my superior understanding of the solutions to these problems.

To ensure accuracy and consistency of scoring across all sections of Math 103, all graders were supplied with the same detailed solutions to all in-class problems. These solutions were fully explained and annotated so that the grader would always be fully aware of what to look for in a solution. The standard procedure in Math 103 for scoring in-class problems is that each is worth four points and is scored on accuracy. Students earn all four points when their work is fully correct and completely and accurately justified using English sentences. Students earn three points when their work is mostly correct and justified. Typically, a student earns three points when he or she gets one part of a multi-step question wrong, or if his or her response to a prompt left out an important detail. For example, a student could earn three out of four points if he or she performed all procedures in a problem correctly, but was unable to fully explain or apply his or her findings in the context of the problem. Or, if the problem had multiple parts and a student answered most of the parts correctly, he or she would earn three out of four available points. Students earn two points when their work is half correct. A score of two is given out when students answer only half of a question, or if their reasoning and/or mathematics are on the right track but not fully developed or correct. In rare instances, students earn one point if they have a small part of the problem correct. Zero points are assigned for missing solutions or responses that are completely incorrect. Graders are given the flexibility to give out half points in cases where students make rounding mistakes, forget a small part of a solution, or if they cannot decide between two integer scores. For example, consider the following in-class problem:

> Four siblings, Wendy, Xavier, Yolanda, and Zachary, inherit a house. Suppose that Wendy considers the house to be worth $400,000, Xavier considers it worth $450,000, Yolanda considers it worth $450,000, and Zachary considers it worth $380,000. Which of the following (if any) is a fair division, according to our definition of that term, and why or why not?

a. No one gets the house. Instead it is sold for $400,000 so each sibling receives $100,000 in cash.
b. Xavier gets the house. He pays each of his siblings $110,000 in cash.

The solution to the problem is as follows:

a. This is not a fair division, because both Xavier and Yolanda receive less than a minimum fair share. For each of these two siblings a minimum fair share is $450,000/4 = $112,500, but the proposed settlement gives each one only $100,000. Note that if even one sibling receives less than a minimum fair share, the entire division is unfair, according to the definition of "fair division" we are working with.
b. This is very close to being a fair division, but doesn't quite give Yolanda a minimum fair share; it gives her $110,000, just short of the required $112,500 to be a minimum fair share. Note that this division does give more than a minimum fair share to Xavier (who gets a net value of $450,000 - $330,000 = $120,000, which is more than his minimum fair share amount of $112,500); to Wendy (who receives $110,000 although her minimum fair share amount was only $100,000); and to Zachary (who receives $110,000 although her minimum fair share amount was only $95,000).

Table 8 shows how points were assigned to solutions to this problem.

Table 8

Scoring Rubric for Homework Problems

Score (out of 4)	Solution
4	A score of 4 out of 4 would be assigned if the student provides correct responses to each prompt (each is not a fair division) and has a correct mathematical justification in each case.
3	A score of 3 out of 4 would be assigned if the student correctly responded to most of the problem but got one part wrong (either one justification or one determination of whether or not the share was fair).
2	A score of 2 out of 4 would be assigned if the student correctly responded to half of the problem but got two parts wrong. This could be all of part a, all of part b, or a component of each part.
1	A score of 1 out of 4 would be assigned if the student only answered one component of one part correctly (half of part a or half of part b). For example, the student could acknowledge the division is not fair in one part but have an incorrect justification to follow.
0	A score of 0 out of 4 would be assigned if the student handed in a blank solution or answered both questions incorrectly (yes/no and mathematical justification).

Scoring of pretests and posttests. I blindly scored the pretest and posttest items with another experienced Math 103 professor until consensus was reached in the total number of points awarded for each question to ensure a high degree of reliability. To begin this process, we both reviewed the exams together before they were administered to ensure all questions were clear and that the questions required procedural and conceptual knowledge as appropriate. Then, we developed a scheme for scoring each problem that allowed for partial credit based on how far a student got. The rubric also took into consideration what would happen if a student made a mathematical error early on in a problem, but executed the rest of it correctly with his or her wrong value. If a response did not quite fit the rubric, we put the papers off to the side and then came back to them and jointly decided the most appropriate score to give. Since we graded all papers together, we reached 100% agreement in assigning final scores to students. Table 9 shows how points were distributed on each pretest/posttest item, along with deductions made for different types of errors on each problem. Note that the pretest included questions #1-6, whereas the posttest included the same questions, in addition to questions #7 and #8.

Table 9

Pretest and Posttest Scoring Rubric

Question Number, Topic, and Total Points Available	Point Breakdown
#1, Apportionment, 14 points	3 points for part A (1 point for identifying each state, seat, and population), 3 points for part B (1 point for the standard divisor, 2 points for the explanation), 4 points for part C (2 points for the apportionment, 2 for adjusting it), 4 points for part D (2 points for the Huntington-Hill cutoff, 2 points for correct rounding)
#2, Alabama Paradox of Apportionment, 12 points	4 points each for parts A, B, and C. In parts A and B, students earned 1 point for each correct apportionment. In part C, to receive all 4 points, students had to correctly identify and explain the Alabama Paradox. 1 point was deducted for each missing aspect (adding a seat, loss of a seat to a state that already had one).
#3, Lone Chooser, 13 points	2 points for each part A through E, and 3 points for part F, broken down into 1 point for each player's earnings. If the cut in part A was incorrect but subsequent work was correct based on that result, students would only lose 4 points.
#4, Lone Divider, 12 points	6 points for part A, broken down into 3 points for a correct table and 1 point for each of three players' bid list. 6 points for part B, broken down into 3 points for each of three players' final shares.
#5, Method of Sealed Bids, 13 points	3 points for part A (1 point for identifying total and fair share for each of three players), 6 points for part b (2 points for each players' items and cash in/out), and 4 points for part c (1 point for items, 3 points for final cash settlements for each player).
#6, Method of Markers, 12 points	6 points for part A (2 points for each of three players' markers), 3 points for part B (1 point for each player's final demarcation), 3 points for part C (1 points for each player's leftover). If markers were incorrectly placed in A, but students executed the correct procedure, they only lost 4 points.
#7, Lone Divider with Standoff, 12 points	6 points for part A (1 point for fair share, 2 points for identifying bid lists, 3 points for final list/standoff) 6 points for part B (2 points for identifying correct two shares leftover, 2 points for identifying standoff/recombining, 1 point for showing the math leading to $220,000, 1 point for explaining how Greedy would end up with exactly $220,000 only if he was the divider and not the chooser).
#8, Apportionment Conceptualization, 12 points	4 points for each of parts A, B, and C. In part A, 2 points for identifying each of lower and upper quota rounding, and full credit for mentioning no modified divisors. In parts B and C, students can earn full credit for using mutual exclusivity of Alabama Paradox and Quota Rule violations, or partial credit for giving an example of a case (2 points for each part), or discussing modified divisors (3 points for each part)

An example of a scored posttest problem is given below to help the reader understand how partial credit was assigned. As the problem below was part of one of the two problems that primarily required conceptual knowledge to solve, careful scoring was used to ensure all critical features of the solution were identified. Figure 2 shows the problem exactly as it was presented to students on the posttest.

7. Every fair division method has as built-in disincentive for dishonest play. This exercise illustrates the disincentive for dishonest bidding in the lone-divider method. Four partners (Burly, Curly, Greedy, and Dandy) are dividing a million dollar property using the lone-divider method. Using a map, Dandy divides the property into four parcels, s_1, s_2, s_3, and s_4. The table below shows the value of the four parcels in the eyes of each partner.

	s_1	s_2	s_3	s_4
Dandy	$250,000	$250,000	$250,000	$250,000
Burly	$400,000	$200,000	$200,000	$200,000
Curly	$280,000	$320,000	$200,000	$200,000
Greedy	$320,000	$280,000	$280,000	$120,000

7a. Describe the outcome of the fair division assuming that all players make honest bids.

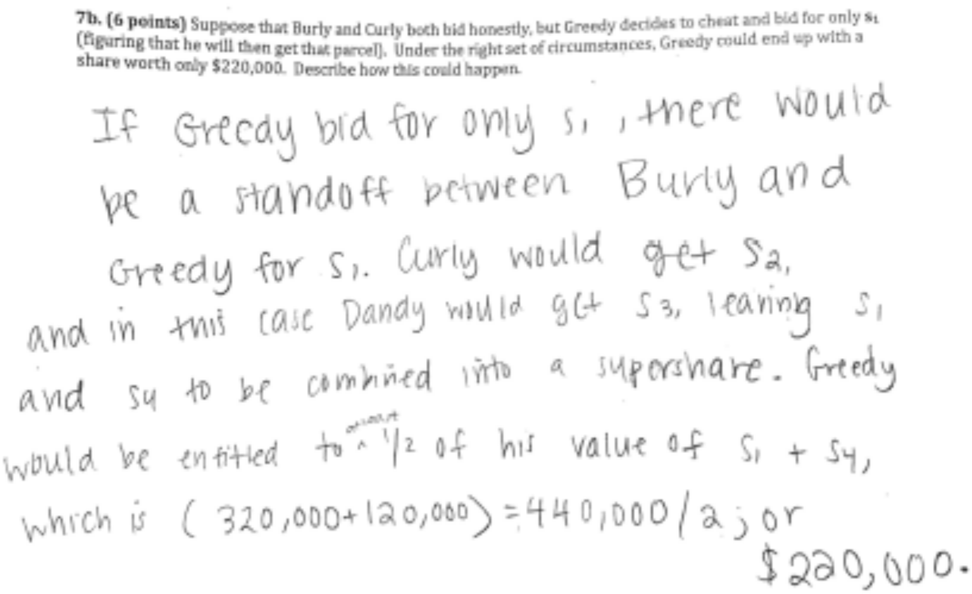
7b. Suppose that Burly and Curly both bid honestly, but Greedy decides to cheat and bid for only s_1 (figuring that he will then get that parcel). Under the right set of circumstances, Greedy could end up with a share worth only $220,000. Describe how this could happen.

Figure 2. Posttest item on the Lone Divider method.

The storyline of this problem on the Lone Divider method involved four players trying to fairly divide a giant piece of land that was already subdivided into four shares by one of the players. Each player was responsible for independently bidding on how much he or she felt each piece was worth. The correct way to solve this problem of this type is to award each player a share of land that was worth at least his/her fair share, which in this case would be at least 25% of the total value each person was willing to put out (or $250,000) since there were four players in total. In cases where two players both find only one share fair and it is the same piece, they are forced to take the desired share and an unassigned share, recombine them, then propose a new division of the recombined share. This process is called standoff. If all parties are honest, the standoff process will result in fair shares for each person.

In this problem, part (b) required students to generate an exact sequence of events for a standoff to occur so that Greedy would only end up with a share worth only $220,000, which is an unfair share. Greedy deliberately chose to inflate his bid on share #1 in an attempt to game the system and guarantee that he would earn that share. However, his plan backfired, since Burly's only possible fair share could be share #1 (valued at $400,000, the only piece he has worth at least $250,000 to himself). To engineer the final outcome for Greedy to walk away with $220,000, students had to think about how to settle the standoff between Greedy and Burly both fighting for share #1. The only way to settle a standoff is to take a piece both players are fighting over and recombine it with a piece they think is not fair, then have one person divide the new piece and have the other person choose a new share. If share #1 was recombined with share #4 to create a supershare, Greedy could walk away with at least $220,000. If Greedy had to divide the supershare into two shares and have Burly choose one, Greedy would definitely play it safe and cut each share into $220,000 shares. However, if Burly cuts the supershare, Greedy would have the advantage of choosing one of Burly's shares, which may be worth more than $220,000 to him.

The work sample in Figure 3 earned five out of six available points on this test question as the student identified the standoff, resolved it using the correct pieces and the correct method. However, she did not earn the sixth point as she did not specify that Greedy had to be the divider during the standoff resolution process in order to be guaranteed exactly $220,000. If Greedy was the chooser, he may have earned chosen a new piece worth more than $220,000, which would be incorrect.



7b. (6 points) Suppose that Burly and Curly both bid honestly, but Greedy decides to cheat and bid for only s_1 (figuring that he will then get that parcel). Under the right set of circumstances, Greedy could end up with a share worth only $220,000. Describe how this could happen.

If Greedy bid for only s_1, there would be a standoff between Burly and Greedy for s_1. Curly would get s_2, and in this case Dandy would get s_3, leaving s_1 and s_4 to be combined into a supershare. Greedy would be entitled to about 1/2 of his value of $s_1 + s_4$, which is (320,000+120,000) = 440,000/2, or $220,000.

Figure 3. Example of coded and scored posttest item.

As this only shows how one specific problem was graded in detail, it would also be informative to see several scored solutions to the same problem to see how the rubric was applied to students. Similar to the scored problem above, posttest question #4 also involved the Lone Divider method but was more procedural in nature in that students had to calculate fair shares and determine a fair division of land among four people. There was no standoff or dishonest bidding, making the division process less arduous. This problem was worth a total of 12 points, with each part of the problem worth six points. In part (a), students had to complete the table for three points and provide each player's list of fair shares for another three points. In part (b), were awarded two points for each player's final fair share and acknowledging there was not a standoff. Posttest problem #4 as it appeared to students is shown in Figure 4.

4. Alex, Blair, and Chris have equal claims to a store location, and are using the Lone Divider method to find a fair division of access to the location over the calendar year. We assume as usual that January 1 has to be one of the start/end dates when dividing the year into shares.

- Alex values each of the months April through December twice as much as each of the months January through March.
- Blair values each of July and August four times as much as each of the other months of the year.
- Chris values each of the months from January through June 1.5 times as much as each of the months from July through December.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Alex													
Blair													
Chris													

4a. Suppose that Blair is the Divider. What are the shares which Blair produces, described in terms of which months (or parts of months) are contained in each share?

4b. Does a standoff occur here? Whether there is a standoff or not, find a fair division of the year, i.e. indicate which months (or parts of months) each player receives in the end.

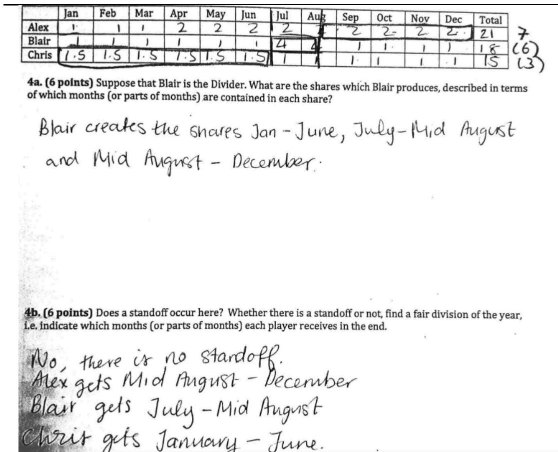
Figure 4. Posttest question #4 as it appeared to students.

Next, examples of students' work are shown to illustrate the scoring process in Table 10. I included a solution that earned all 12 out of 12 points, one that earned eight out 12 points, and one that earned three out of 12 points. The purpose of the table is to show the range of partial credit that was offered for answering parts of the problem correctly.

Table 10

Examples of Graded Responses to Posttest Question #4

Students' Work	Points Earned and Commentary



	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total	
Alex	1	1	1	2	2	2	2	[illegible]	2	2	2	2	21	7
Blair	1	1	1	1	1	1	4	4	1	1	1	1	18	(6)
Chris	1.5	1.5	1.5	1.5	1.5	1.5	1	1	1	1	1	1	15	(3)

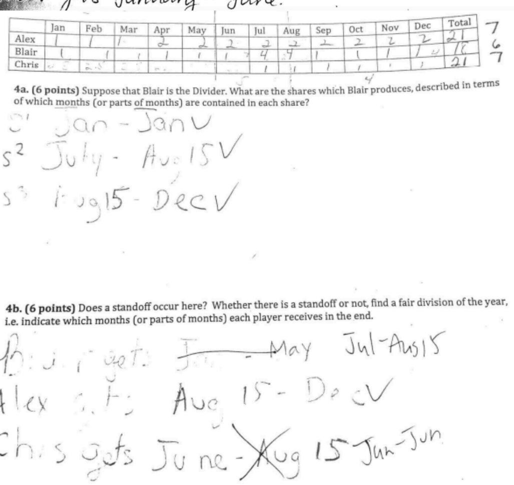
4a. (6 points) Suppose that Blair is the Divider. What are the shares which Blair produces, described in terms of which months (or parts of months) are contained in each share?

Blair creates the shares Jan - June, July - Mid August and Mid August - December.

4b. (6 points) Does a standoff occur here? Whether there is a standoff or not, find a fair division of the year, i.e. indicate which months (or parts of months) each player receives in the end.

No, there is no standoff.
Alex gets Mid August - December
Blair gets July - Mid August
Chris gets January - June.

This student earned 12 out of 12 points for a completely correct solution.



	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total	
Alex	1	1	1	2	2	2	2	2	2	2	2	2	21	7
Blair	1	1	1	1	1	1	4	4	1	1	1	1	18	6
Chris	[illegible]	2.5	[illegible]	[illegible]	[illegible]	[illegible]	1	1	1	1	1	1	21	7

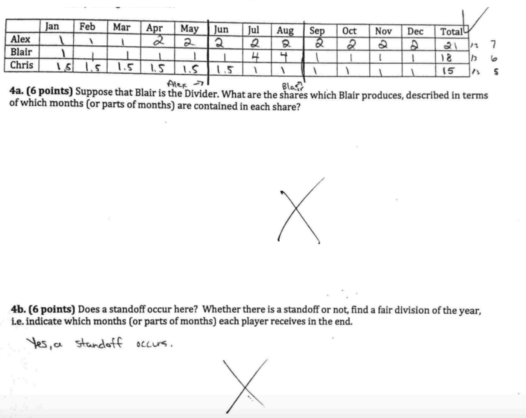
4a. (6 points) Suppose that Blair is the Divider. What are the shares which Blair produces, described in terms of which months (or parts of months) are contained in each share?

S1 Jan - Jan
S2 July - Aug 15
S3 Aug 15 - Dec

4b. (6 points) Does a standoff occur here? Whether there is a standoff or not, find a fair division of the year, i.e. indicate which months (or parts of months) each player receives in the end.

Blair gets: ~~Jan~~ - May Jul-Aug15
Alex gets Aug 15 - Dec
Chris gets June - ~~Aug~~ 15 Jun-Jun

This student earned eight out of 12 points. The student earned all six points for part (a) but only two points in part (b) for identifying only Alex's fair share correctly.



	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total	
Alex	1	1	1	2	2	2	2	2	2	2	2	2	21	7
Blair	1	1	1	1	1	1	4	4	1	1	1	1	18	6
Chris	1.5	1.5	1.5	1.5	1.5	1.5	1	1	1	1	1	1	15	5

Alex → Blair

4a. (6 points) Suppose that Blair is the Divider. What are the shares which Blair produces, described in terms of which months (or parts of months) are contained in each share?

4b. (6 points) Does a standoff occur here? Whether there is a standoff or not, find a fair division of the year, i.e. indicate which months (or parts of months) each player receives in the end.

Yes, a standoff occurs.

This student earned three out of 12 points. The student earned three points in part (a) for correctly filling out the table. The student earned zero points in part (b) for showing no work and for incorrectly stating there should be a standoff.

Coding and scoring invention tasks. Upon conclusion of the data collection process, I went back and scored each student's invention task in the PFFC group. The score I assigned was based on the number of critical features each student identified in each task. Each task's critical features were determined by breaking down each method and identifying the important underlying mathematical processes and concepts that corresponded to a correct solution. Each task was broken down into approximately five or six critical features. Table 11 shows the total number of points available for each invention task and the critical features for each task in the study.

Table 11

Scoring of Invention Tasks

Invention Task Number and Topic	Critical Features
#1 – Apportionment methods	Proportional share of population Standard divisor Standard quota Rounding of quotas Surplus seats Final correct apportionment with integer values
#2 – Divider Chooser method	Role of divider Mathematical fairness (1/N) Per month and per share totals Role of chooser Final correct fair division
#3 – Lone Divider method	Individual fair shares Mathematical fairness (1/N) Order of receiving fair share Desired and unwanted piece lead to standoff Correct resolution of standoff
#4 – Method of Sealed Bids	Mathematical fairness (1/N) Allocation of items Discrepancy between items and fair share Monetary trade-off Correct final allocation

To illustrate how the tasks were scored, I will use invention task #3 and its solution to explain how credit was offered based on identification of critical features. This task is shown in Figure 5.

Invention Task #3

Suppose Andrea, Beatrice, Cleo and David are dividing a plot of land worth $1,000. The land comes in four sections, call them piece 1, piece 2, piece 3, and piece 4. Andrea values piece 1 at $150, piece 2 at $300, piece 3 at $200, and piece 4 at $350. Beatrice values piece 1 at $200, piece 2 at $350, piece 3 at $100, and piece 4 at $350. Cleo values piece 1 at $200, piece 2 at $450, piece 3 at $200, and piece 4 at $150. David values piece 1 at $250, piece 2 at $250, piece 3 at $250, and piece 4 at $250. Describe a way for the players to fairly divide the land.

Figure 5. Invention task #3 prompt given to students.

A normative solution to this problem is as follows: First, a fair share has to be calculated for each player based on his or her bids. This is achieved by adding up each person's bids and dividing by the total number of people in the problem. For example, Andrea's fair share would be found by adding $150, $300, $200, and $350 together, then dividing by four to give $250. Or, another way of looking at is since the plot of land is worth $1,000 in total and has to be split among four people, each person should walk away with $250 worth of the land in order for it to be fair. This is the first critical feature of the problem, as establishing a baseline fair share is needed to make any allocations of pieces to people. The second critical feature is identifying which pieces are fair to each player. In this case, any piece worth $250 or more would be considered mathematically fair to each player. Thus, Andrea would find piece #2 and piece #4 fair, Beatrice would find piece #2 and piece #4 fair, Cleo would only find piece #2 fair, and David would find all pieces fair.

Once fair shares have been established, the third critical feature of the solution involves giving out pieces to players. With one player accepting any of the four pieces, two accepting either of two pieces, and one player only accepting one piece, it is customary to give the player who has the shortest list of fair share pieces first, as they are the most exclusive and cannot settle for many alternatives. Thus, Cleo would be awarded piece #2. This takes piece #2 out of Andrea's and Beatrice's list of fair shares, resulting in a standoff: both players only now could receive piece #4 in order to walk away with a mathematical fair share. To resolve the standoff in a way that is mathematically fair to both players (fourth critical feature), David would be given either piece #1 or piece #3, and Andrea and Beatrice would be forced to recombine piece #4 with one of the remaining pieces (piece #1 or piece #3). By recombining a piece they both want with a piece they both do not want, it is possible for both players to walk away with a fair share. This can be done if the divider-chooser method is employed (fifth critical feature). If the Divider-Chooser method is not employed, there is a risk that a division could be made that would not be fair to one of the players. Either Andrea or Beatrice would have to take this new recombined piece, have one person make a new cut, then have the other person choose which piece she wants.

For example, by recombining piece #4 (wanted piece) and piece #3 (unwanted piece), the new piece would be worth a total of $550 to Andrea. If she is the person left to divide this piece and have Beatrice choose one of the two pieces, Andrea would cut it in a way that guarantees her at least half of the value of the recombined piece, which in this case would be $275 for her. Because this value is over the baseline fair share of $250, she would walk away with a fair share. A similar argument could be used if piece #1 (unwanted piece) was recombined with piece #4 (wanted piece) during the standoff process.

The work sample that follows was given a score of "1," as it correctly established individual fair shares but fell short of assigning correct shares to players and resolving the standoff. Although an excellent attempt was made to organize the information in columns, the final solution submission did not contain any evidence of progress toward a solution beyond just calculating the fair share for each person.

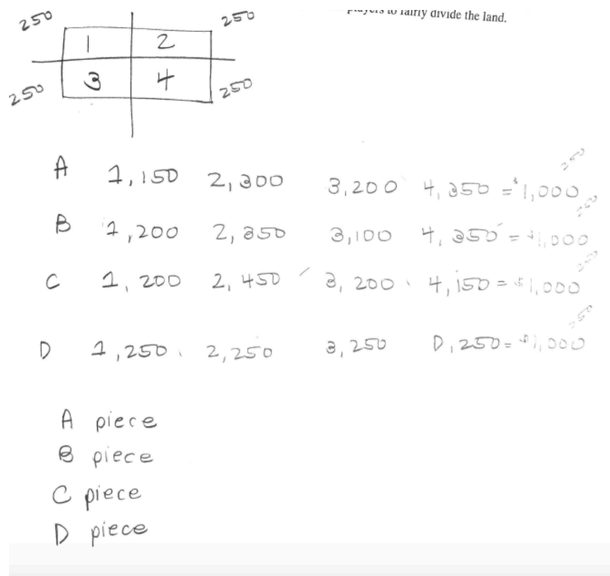


Figure 6. Example of invention task that received a score of "1."

Coding the focus group interview. Hypothesis coding was used to test the researcher-generated hypothesis that students who experienced PF would view their learning experience differently compared to students who did not have the same experience. In hypothesis coding, codes are developed from a theory or prediction about what will be found before the data is collected and analyzed (Saldaña, 2009). Using the high-level conjecture from the study's conjecture mapping, I developed codes based on the PF design embodiment and mediating processes involved in the learning intervention. For example, because part of the design embodiment involved having students work in groups to solve problems related to the video tutorial content, I hypothesized that students would collaborate and peer teach. Subsequently, I used the code "peer teach" when students described educating a classmate in their groups. An example of a statement that was coded with "peer teach" was when a student commented how the classroom design, specifically work with peers, allowed him to "learn how to do things from one another."

The coding system was refined in an ongoing matter as the transcripts were initially coded then coded again after a second read. For example, the statement, "group chat or group meet, we use so that if we have questions about homework or a test, we just go over them" was initially coded as "peer teach," but since this was a very specific type of virtual group collaboration outside of class, I ended up building a new subcode, "mobile teaching," to be more specific about the type of teaching that actually took place. A total of 12 codes and three subcodes were developed for the study and can be found in Table 12. In addition, counts of references to seeking help from the instructor and other students were taken in both groups and analyzed through a Chi-square test to test the hypothesis that students in the PFFC group found their peers to be more important to their learning process than the instructor.

Table 12

Coding Scheme

Code	Definition	Data Example
Understanding (U)	U: any loose reference to comprehension	U: "it's like when you understand something in a video"
Subcodes: Conceptual understanding (CU)	CU: students describe development of key ideas	CU: "understand core concepts" PU: "understanding how to do it and the process behind"
Procedural understanding (PU)	PU: knowledge of steps or how to complete a problem	PU: "memorizing certain ways of figuring out a certain problem" PU: "And it's easier to learn in the same order that you did on your own when you come to back to the classroom."
Video (V)	V: Multimedia tutorial	V: "the videos are really helpful" V: "structure of the video is probably the most important"
Peer teaching (PT) Subcode: mobile teaching (MT)	PT: students working in a small group or pairs to help one another understand content MT: when students help one another understand through a mobile device	PT: "learn how to do things from one another" MT: "group chat or group meet we use so that if we have questions about homework or a test, we just go over them"
Pacing (P)	P: how students progressed through learning activities	P: "You can do it at your own speed, and that was enjoyable."
Quiz (Q)	Q: start-of-class question set given to students to gauge knowledge of the video set	Q: "that can be a little difficult when you have to take a quiz first" Q: "having a quiz at the beginning of class is also really helpful because it hold you accountable for all of the videos"
Direct instruction (DI)	D: teacher disseminates information	DI: "a summary at the beginning of class and you're allowed to ask questions" DI: "Also, when you come into class after you're done watching the videos, the professor goes over all of the main points."
Application (A)	A: using information learned from videos and/or peers	A: "actually apply to something you might use in your life" A: "like actually applying them and using the examples"

		A: "application of the math is most important because doing it physically allows you really understand what it means"
Time (T)	T: any reference to time spent on class	T: "So maybe if the class is a little longer then we could have quiz, reviews, exam reviews, so we can know what we got wrong" T: "I think it would be more helpful if we had more time to work on the problems during class so that we could do it with our classmates, because that helps a lot"
Resources (R)	R: instructional material made available to students (textbook, videos, review guides, etc.)	R: "I think the class, just the way it's set up on Sakai in class, it gives you a lot of resources." R: "I think being more active maximizes how you can use your online tools." R: "And I think that was really helpful because the book was ... the part that you printed out was really good because it could do reviews, and reviews aren't that long."
Teacher-student questioning (TSQ)	TSQ: students seeking feedback from teacher	TSQ: "you're allowed to ask questions, it does offer that opportunity for students who are unsure about certain things" TSQ: "ask about questions when I had them just because of my personal preference with learning through a device rather than person to person."
Group (G)	G: students talk about working with one another	G: "Working in groups to do the homework problems" G: "We also did a lot of group work, which really helped all of us work together and learn"
Productive failure (PF)	PF: any mention made to invention task/failure experience	PF: "we have like a test at the end of every class, and then you go home and you do the videos" PF: "At the end doing the invention task helped with later when I went back to watch the videos. They would connect to ... This is why we learned that, and it made it easier,"

Coding the video data. Due to the fact that written solutions to problems only tell us so much, a closer look at the discussions that went into solving invention tasks was needed to get a complete picture of learning in the PFFC learning environment. Therefore, video footage of in-class problem solving for students in the PFFC group was coded by recording the frequency of solution proposals students made during the invention task. Drawing on research from Kapur and Bielaczyc (2012), a critical part of the PF experience is when learners have the opportunity to generate conceptions and understandings, even though they may not be initially correct. Accordingly, I kept counts for each group as they suggested solution proposals, developed counterarguments for a group member's contribution, and advanced a group member's response by adding to it.

Quantitative Results

I will first present descriptive statistics and *t*-test results for scored work for both groups. This will also include invention task results for students in the PFFC group only. Video watching data will be then be presented, followed a brief analysis. Finally, a more detailed quantitative analysis of PFFC students' work in the class will be presented by means of a correlation table. Taken all together, the quantitative data will serve as a basis for further examination of qualitative data.

Results of in-class problems. Descriptive statistics were obtained for students' in-class problems for all four weeks of the study in order to understand how working on these intermediate assessments may have impacted learning performance in each condition. The mean scores on the weekly in-class problems did not differ significantly each week, as evidenced by a series of *t*-tests: in week #1, $t(45) = -.305$, $p = .762$; week #2, $t(45) = -.545$, $p = .588$; week #3, $t(44) = .607$, $p = .547$; week #4, $t(46) = -.378$, $p = .707$. Table 13 provides a complete summary of all descriptive statistics for all in-class assignments in this study.

Table 13

Descriptive Statistics for Weekly In-Class Problems

Group	PFFC		DIFC		
	M	*SD*	*M*	*SD*	*t*-test
Week #1	17.0	3.1	17.3	3.3	$t = -.305$
Week #2	22.5	5.8	23.3	4.3	$t = -.545$
Week #3	20.3	4.2	19.5	4.3	$t = .607$
Week #4	19.8	2.5	20.0	4.6	$t = -.378$

Results of pretests and posttests. After analyzing the results of students' in-class problems, I analyzed overall performance on the pretest and the posttest for each group to see if there was a significant difference in performance. Similar to the in-class problems, the mean scores on the pretest and posttest did not differ significantly. For the pretest, $t(50) = .397$, $p = .693$, whereas for the posttest, $t(50) = -.538$, $p = .593$. Table 14 lists descriptive statistics for the pretest and the posttest for both groups.

Table 14

Descriptive Statistics for the Pretest and the Posttest

Group	PFFC		DIFC		
	M	*SD*	*M*	*SD*	*t*-test
Pretest	2.3	3.6	2.0	1.6	$t = .397$
Posttest	73.1	17.3	75.6	15.7	$t = -.538$

In order to better understand how the learning intervention affected the development of students' conceptual and procedural knowledge, I also analyzed the four posttest questions that directly corresponded to each of the invention tasks. In addition, I broke the posttest down score further into a total score for the six questions that mainly required procedural knowledge to solve and a total score for the two questions that mainly required conceptual knowledge to solve. Once again, no significant differences emerged when means were compared on individual questions and on aggregated conceptual and procedural knowledge questions. Results of the *t*-test for the procedural knowledge questions were $t(50) = -.935$, $p = .354$, whereas results for conceptual knowledge questions indicated $t(50) = -.888$, $p = .379$. Moving to individual posttest questions that corresponded to invention task problems for the PFFC group, the t-test for question #1 revealed $t(50) = -.351$, $p = .727$. The results for questions #3, #5, and #7 were respectively $t(50) = -1.013$, $p = .315$, $t(50) = .183$, $p = 856$ and $t(50) = .775$, $p = .442$. Table 15 details the complete set of descriptive statistics for the comparisons made between groups for each of the specific question types.

Table 15

Descriptive Statistics for Specific Posttest Items

Group	PFFC		DIFC		
	M	*SD*	*M*	*SD*	*t*-test
Posttest #1	12.3	2.1	12.5	1.9	$t = -.351$
Posttest #3	9.3	4.1	10.4	3.9	$t = -1.103$
Posttest #5	10.8	3.0	10.6	2.3	$t = .183$
Posttest #7	9.1	2.3	8.6	2.4	$t = .775$
Procedural only	67.0	14.1	70.5	12.7	$t = -.935$
Conceptual only	6.1	4.1	5.1	4.2	$t = .888$

Results of the invention tasks. For the PFFC group only, I calculated descriptive statistics so that I could better understand the average number of critical features students identified on invention tasks during each week of the study. Over the course of the four weeks, the tasks themselves became more difficult, as evidenced by the decreasing average number of critical features identified by students. Table 16 shows a concise summary of these data.

Table 16

Descriptive Statistics for the Invention Tasks

	M	*SD*
Week #1	4.0	1.2
Week #2	3.1	1.6
Week #3	1.8	.6
Week #4	0.5	.7

Video-watching survey and analytics. The third research question sought to determine whether or not students' video-watching behaviors influenced their learning performance in each of the learning environments. First, results from the video-watching survey for both groups are shown below in Table 17.

Table 17

Results of Students' Video Watching Activities (PFFC and DIFC)

	Have you watched all of the videos in this unit?		How many times have you watched the videos?			Where do you watch the videos?			On what device do you watch the videos?		
	Yes	No	1	2	3 or more	Home	Bus	Other	Computer	Smart Phone	Tablet
PFFC group	77.3%	22.7%	68.2%	22.7%	9.1%	90%	0%	10%	100%	0%	0%
DIFC group	80%	20%	70%	26.7%	3.3%	86.7%	0%	13.3%	100%	0%	0%

The table above shows that 77.3% of students in the PFFC group and 80% of students in the DIFC group watched all of the videos in the instructional unit; students in both groups by and large reported watching most of the tutorials. In their study on PF in the flipped classroom, Song and Kapur (2017) suggested further examination of how the device used to access videos in a flipped classroom could impact learning; however, in this study, all students in both conditions watched the video tutorials using a computer instead of a phone or tablet. In terms of video-watching frequency, 31.8% of students in the PFFC group and 30% of students in the DIFC group reported watching the video tutorials two or more times. The Chi-square test shows that $\chi2=0.814$ ($df = 2$), $p > .05$ for the distribution of video watching frequencies at a 5% level of significance. The results indicate that the two groups do not have a significant difference in the distribution of times in watching the video tutorials.

In order to gain a deeper understanding of video-watching behavior beyond the self-reports for each group, I downloaded individual and group engagement analytics from Kaltura on the Sakai learning management system. Table 18 lists summary averages for each group over the course of the study.

Table 18

Video analytics from Kaltura (PFFC and DIFC)

	Percent of unique videos watched	Watched videos	Total view time	Average view time	Average drop-off	Loads to plays ratio
PFFC group	68.0	67.0	4:43:43	0:04:17	73%	0.93
DIFC group	77.2	70.6	4:40:02	0:04:07	72%	0.78

The data above are based on a total of 58 videos for the DIFC group and a total of 60 videos for the PFFC, which includes an additional two invention task solution videos for the PFFC group. This information could help explain the slightly larger "total view time" and "average view time" for the PFFC group since they had more videos to watch. In addition to the basic count of numbers of videos watched, Kaltura also reported on more detailed aspects of students' video-watching experience, including average drop-off and loads to plays ratio. For the purposes of this study, none of these additional features provided any useful information about the learning intervention. Finally, in order to understand whether students' self-reported video-watching behaviors influenced their learning performance on the posttest, I conducted a one-way ANOVA test. The results indicated that there was no significant effect of video-watching times on students' learning performance on the posttest ($p > .05$).

Correlational table. Due to the fact that no mean differences were statistically significant on any of the course assignments and activities, I next turned to correlations to examine the relationship between scores on various course assignments and activities. Of 78 correlations, 22 were statistically significant. Some of the significant correlations were not surprising: all four of the posttest items positively and significantly correlated with the overall posttest score. In addition, posttest problems themselves significantly correlated with one another in a few instances. Keeping the embodied conjecture at the forefront of the analysis, I was more interested in determining whether or not any relationships existed across the weeks of the course, from the invention tasks, to in-class problems, to the posttest.

The most interesting observation was the correlation between the third invention task in the course and performance on posttest question #7, which was one of the two conceptual knowledge questions on the posttest. Despite the fact that invention task #3 did not significantly correlate with the in-class problems for that week, it did significantly correlate with the posttest item weeks later ($r = .518$, $p = .023$, N = 19). In all other weeks, the results of the classwork problems, not the invention task, significantly correlated with performance on the corresponding posttest question. For week #1 of the study, the classwork set and posttest question #1 had a significant positive correlation ($r = .560$, $p = .013$, N = 19). For week #2 of the study, the classwork set and posttest question #3 had a significant positive correlation ($r = .476$, $p = .034$, N = 20). For week #4 of the study, the classwork set and posttest question #5 had a significant positive correlation ($r = .508$, $p = .026$, N = 19). These findings make sense in light of the posttest design. Questions #1, #3, and #5 were all similar to problems student solved in class, but with different storylines and subquestions to avoid surface similarity. Table 19 shows all of the correlations among course assessments for the PFFC group.

Table 19

Correlations of Scores on Assessments in the PFFC Group

	Inv. #1	Inv. #2	Inv. #3	Inv. #4	CW #1	CW #2	CW #3	CW #4	Post #1	Post #3	Post #5	Post #7	Post Total
Inv. #1	_												
Inv. #2	.385	_											
Inv. #3	-.178	.139	_										
Inv. #4	-.430	-.208	.385	_									
CW #1	-.261	.269	-.008	-.007	_								
CW #2	-.033	.286	.092	.194	.509*	_							
CW #3	-.251	.383	.204	-.048	.746**	,615**	_						
CW #4	.388	.147	.493*	.330	.026	.226	.212	_					
Post #1	-.331	.041	.579**	-.008	.560*	.443	.758**	.349	_				
Post #3	-.155	.192	.403	-.102	.644**	.476*	.639**	.272	.749**	_			
Post #5	.166	.193	.354	.290	.271	.541*	.121	.508*	.371	.358	_		
Post #7	-.261	-.104	.518*	.444	.379	.440	.183	.329	.542**	.306	.520*	_	
Post Total	-.139	.145	.376	.193	.534*	.676**	.433	.325	.697**	.728**	.775**	.721**	_

Notes: **p* < .05 (2-taied), ***p* < .01 (2-tailed).

Conclusion. The first two research questions sought to determine the impact of the learning environments on students' conceptual and procedural knowledge. Despite the fact that there was no significant difference in means for any of the course assessments between groups, a closer look at the learning intervention in the PFFC group revealed an interesting significant correlation between the week #3 invention task and its corresponding conceptual knowledge posttest question. Further analysis of other correlations between course assessments revealed significant correlations between classwork and corresponding posttest items in other weeks.

In addition, the third research question sought to understand how video watching behaviors influence learning performance in the PFFC and DIFC learning environments. Based on the results of self-reports and additional data obtained from Kaltura, there was no significant effect of video-watching times on students' learning performance on the posttest. Additionally, there was no significant difference in video watching frequency or location between the two groups. Additional qualitative information from the focus group interviews will provide more insight into the video-watching behaviors of each group that the survey responses cannot provide. Further qualitative analysis is also needed to better understand which design features produced critical learning processes that led to the correlational results. The next section will explore qualitative evidence that can help us better understand and expand on the previous quantitative findings.

Qualitative Results

The primary purpose of this section is to expand on the quantitative findings in this study. This section will be organized into three parts. First, I will continue the discussion of the significant correlation found during week #3 between the invention task and posttest item by examining work samples and video footage of groups working on the invention task. Second, I will provide additional information about the invention task solving process over the other three weeks of the study in order to develop characteristics of effective groups in a flipped classroom PF learning environment. Finally, I will discuss the results of the coded focus group interviews for both groups. The results of the focus group interview will provide details about each learning environment that could not be observed through an invention task solution or posttest score. These findings will help paint a complete picture of what learning looked like in each environment and which features of the design embodiment were the most salient in producing mathematics learning in the PFFC.

Week #3 performance by group. In trying to understand why there was a significant positive correlation between the invention task #3 score and performance on its corresponding conceptual knowledge posttest question, I went back to the embodied conjecture to trace what aspects of the design might have led to this outcome. As research suggests, collaboration does not immediately ensue once students are placed into groups (Barron, 2003). A lot of work went into organizing groups and creating a social surround where students felt safe generating mathematical ideas and working with one another. To that end, I took a closer look at the video footage of in-person class sessions and focused on how two recorded groups, “Group A” and “Group B,” worked on the invention task for the topic that week. It became clear that the conversations looked different for the two groups, as one group seemed to generate more solution proposals than the other. In addition, one of the groups seemed to develop a procedure to solve the invention tasks over time, which ended up helping them on scored class assessments. The subsequent paragraphs describe the analysis of group problem-solving processes for students in Group A (“Ron,” “Melanie,” and “Maryann”) and in Group B (“Bill,” “Jaidan,” and “Nicolette”) during week #3.

Prior to video analysis, I compiled data on how students in both groups performed on the invention task, in-class problem set, and posttest question on week #3’s topics. At the outset, it appeared that both groups identified around the same number of critical features on their invention tasks. This was on the basis of their written work on the invention tasks. However, after going back to the videos, Group A had an intense conversation that scratched the surface of many critical features during week #3, but none of them were submitted on their final write-ups. Students in Group A had higher scores on the in-class problems for that week than Group B did.

In addition, students in Group A earned more points on the conceptual knowledge question on the posttest on week #3's topic. Table 20 shows the results for both groups.

Table 20

Week #3 Performance in Two PFFC Groups

	Number of critical features identified in written solution to invention task	In-class problems score (out of 24)	Posttest problem #7 score (out of 12)
Group A			
Ron	2	24	12
Maryann	2	17	12
Alice	3	23	11
Group B			
Nicolette	2	22.5	8
Bill	2	22	9
Jaidan	1	7	6

Week #3 video analysis. This data begged the question, what did students in Group A do differently from students in Group B during their invention task process in order to earn higher scores on their in-class problems and conceptual knowledge posttest question? This is where I turned to the video footage of the invention task solving process for each group. This was important to me as a researcher because the design embodiment included the development of a mathematical safe space, where students' risk-taking and generating ideas and solutions was valued. Due to the difficult nature of the invention tasks, successful groups would have needed to generate many proposals to move toward a solution, even if it was an incorrect one (Kapur & Bielaczyc, 2012). Based on analyzing the video footage of the invention task sessions, Group A generated 14 proposals toward a solution in week #3, whereas Group B generated seven proposals toward a solution. In the case of Group A, students frequently agreed and discussed key ideas as they came up during the solution process.

What made Group A's work on this invention task on the video recordings stand out was not only the number of solution proposals they made but also the quality of their proposals. After all group members had time to read the problem previously described in Figure 5, Maryann initially gave an incorrect proposal to the group. After brief consideration, Alice dismissed her proposal as being unfair to at least one member in the group of four in the invention task, which was one of the critical features of the problem. Ultimately, after some back and forth conversation, Alice described one of the critical features of the problem, the standoff, referring to it as, "that's where the problem is." This occurred because two people in the invention task scenario considered only one piece of land fair, so it would not be right to only give it to one person and not the other. Additional mathematics was required to solve this problem that I did not anticipate students would be cognizant of. Most students continued to suggest additional incorrect solutions at this point; however, Alice identified that this was the "problem" with the task and proposed division of current pieces of land as a way to resolve the standoff. At this point, Ron also added to the conversation in a way that furthered Alice's thoughts. This was very significant progress toward a possible (incorrect) solution that drew extensively on prior knowledge and contributions from all group members. A sample transcript of the most salient aspects of conversation between group members is shown in Table 21.

Table 21

Video Analysis of Group B's Invention Task #3 Solution

Transcript	Commentary
Maryann: "Give A #3, B #1, C #4, and give D #2…I don't know how I did that, but that makes sense in my mind"	Solution proposal
Alice: "OK, so minimum fair share is $250, so Cleo can't get #1, can't get #3, or #4, has to get #2, that's where the problem comes in."	Identified critical feature of fair share and fair shares for one player, alluded to critical feature of standoff
Maryann: "Why not #4?"	
Alice: "That's only $150"	
Alice: "Wait for Cleo, piece #4 is at $150"	
Maryann: "But wait if each person is supposed to get $250"	Referenced critical feature of fair share
Ron: "She's saying, piece #4 for C is only worth $150, so it has to be greater than $250, that can't be a fair piece for her."	Referenced fair share and fair pieces for Cleo
Alice: "David is the only one who wants piece 1. So we can just give him piece 1."	Partial solution proposal
Ron: "He's also the only one who wants piece 3 too."	
Alice: "Can we divide pieces?"	Toward a possible correct solution
Ron: "That's what I was thinking, so let's just give David piece 1"	Partial solution proposal
Alice: "David is the only one who wants piece #1, Can we do that? Give Cleo piece 2 because that's the only one she wants. So then Andrea and Beatrice then divide up…"	Toward a possible correct solution
Ron: "Give Beatrice piece 4, give…"	Solution proposal (incorrect)
Alice: "Wait, we can give Andrea piece 3 and part of Cleo's piece 2."	Solution proposal (incorrect)
Ron: "We have to shift $50 away, but I don't know how"	Solution proposal (incorrect)

While Group B did not reach the same point, members did generate some ideas toward a solution. During this entire process, one student (Jaidan) did not contribute to solving the problem, but instead remained passive and wrote down what the other two group members were saying. As evidenced by the transactions below, there were some solutions generated, but no specific course of action was taken regarding the unfairness of proposed solutions. Members of group B did not specifically point out the problem of the standoff and two people having the same list of fair pieces as members of group A did. Group B's members handed in an answer they knew was incorrect but did not attempt to mathematically reconcile the incorrect answer. As seen in Table 22, members of Group B referenced some critical features and attempted to generate some solutions, but then suggested giving out pieces in an unfair manner to David.

Table 22

Video Analysis of Group B's Invention Task #3 Solution

Transcript	Commentary
Nicolette: "So David can basically get $250, which means he can get anything, so we'll do him last"	Identified critical feature of fair share and pieces fair to one player
Bill: "So yeah yeah, yeah….Which one needs what?"	
Nicolette: "So Andrea would want #4"	Partial solution proposal
Bill: "I think Cleo needs #2"	Partial solution proposal
Nicolette: "So if we gave A #4, C #2, there's #1 and #3 left. But B would want #1?	Partial solution proposal (incorrect)
Bill: "If B gets #1 and D gets #3, is that what we do?"	
Nicolette: "Is that fair though? Don't they at least have to get $250? That doesn't work. They all can't get $250"	Referenced critical features of fair share and fair pieces per person
Nicolette: "If we give her C, she will get $450, and D will get only $200, I guess it doesn't matter?"	Partial solution proposal (incorrect) and disregards fair share critical feature for D
Bill: "It's not about being fair, it is about being rational!"	Incorrect, but interesting qualitative approach to solving the problem

Week #3 invention task analysis. To further understand why Group A was more successful than Group B, consider the following example of conceptual knowledge development from a member of Group A through the course of the learning process described in the embodied conjecture. As previously discussed, the critical features of the solution to invention task #3 included establishing individual fair shares, determining a cutoff for mathematical fairness, giving out the fair shares in an order that allows the neediest person to be addressed first, recognizing the standoff that occurs when one piece is the only piece two people consider fair, and resolving the standoff by recombining an unwanted piece with the piece both players want to establish fairness. This invention task was one of the most difficult tasks for students to solve since they struggled with how to resolve the standoff. Unlike Group B, Group A was able to talk through the issue that further action needs to take place to resolve the problem. Below is Maryann's final submission to the invention task:

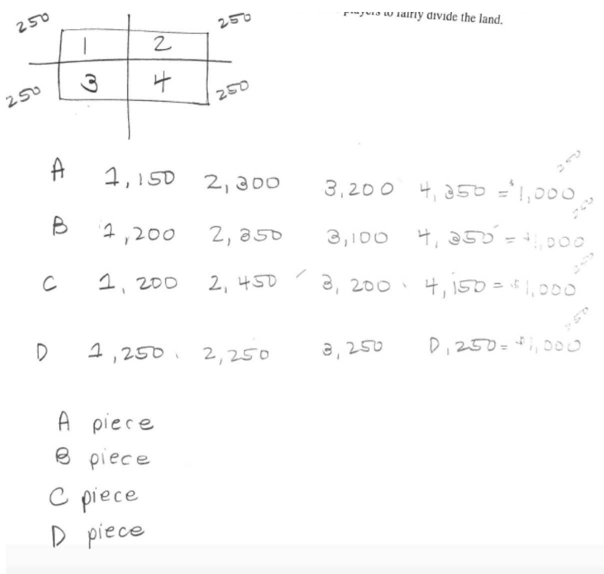
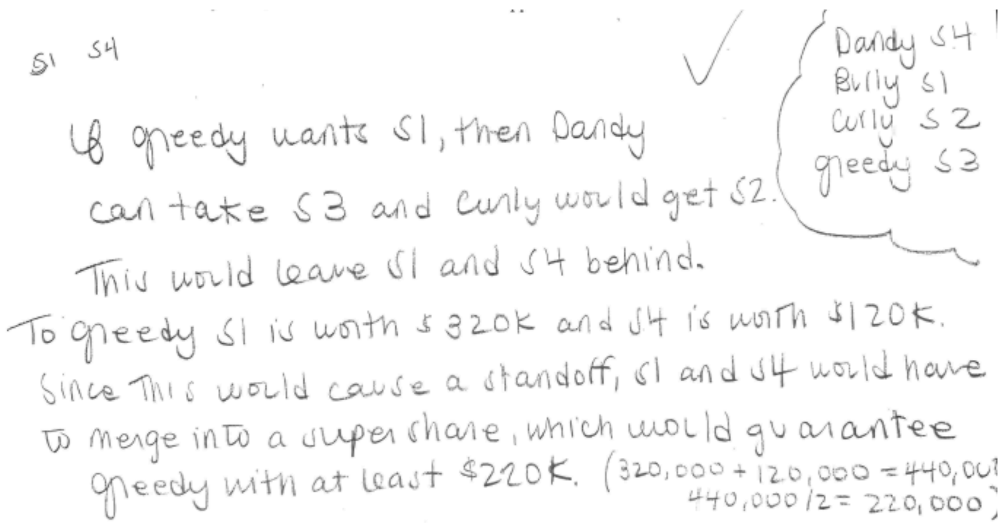


Figure 7. Maryann's solution to the invention task on fair division.

Maryann organized the information from the problem into rows and columns and appropriately added up the total amount each person was willing to pay for the shares. This process of initially organizing information was consistent with how her group began the invention task solving process each week. Maryann also calculated the fair share for each person ($250) and represented this visually using four pieces. As a result, Maryann only identified one critical feature of the problem and did not make any written progress in giving out pieces to people. Upon further examination of the video transcript of this class session, it appears that group members did discuss the inaccuracy of this division and generated new possibilities for how to resolve it, but Maryann did not write any of them down. This active generation of possible solutions was also consistent with how Group A worked through their invention tasks each week.

Corresponding posttest question analysis. On the posttest, students solved a corresponding problem that involved a standoff, but they had to engineer an exact set of circumstances in which a player could end up with an unfair share in the Lone Divider method (see Appendix D). This problem primarily required conceptual knowledge to solve, as students had to not only know the process of resolving a standoff, but they had to think critically about how the critical features of the problem all connected with one another to produce the final result. This is line with the definition of conceptual knowledge offered by Baroody et al. (2007), which refers to knowledge that is rich in connections.

Prior to the posttest, students never solved a problem of this type on the Lone Divider method where they had to develop a scenario using mathematics to justify their answer. Students who provided correct solutions in the PFFC group demonstrated a complete understanding of how the Divider-Chooser method works within the Lone Divider method, a crucial step needed in defending a mathematically sound answer. In particular, these students acknowledged the fact that the player in question could either be a divider or a chooser, and that depending on which player is which, the final outcome would be at least $220,000 for each player. Maryann's work is shown reproduced in Figure 8 as an example of this:


S1 S4

If greedy wants S1, then Dandy can take S3 and Curly would get S2. This wold leave S1 and S4 behind. To greedy S1 is worth $320K and S4 is worth $120K. Since This wold cause a standoff, S1 and S4 wold have to merge into a super share, which wold guarantee greedy with at least $220K. (320,000 + 120,000 = 440,000
440,000/2 = 220,000)

Dandy S4
Billy S1
Curly S2
greedy S3

Figure 8. Maryann's solution to dishonest bidding conceptual understanding question on the posttest.

Not only did Maryann correctly solve the problem, but she also correctly detailed all of the critical features that could result in an unfair share. The role of the divider was a common error for many, as Maryann correctly illustrated the fact that Greedy would get "at least" $220,000, with the underlying understanding that it could be more if Greedy was the chooser. This was a crucial distinction to make in this method as only the divider was guaranteed to get exactly half of what the item in question is worth, whereas the chooser could earn more than that. Maryann's work on this question was comparable to that of other members of her group and was among the highest quality in both the DIFC and PFFC groups.

Outside of week #3. Aside from this detailed analysis of week #3's learning activities, further evidence supports the identification of Group A as a "stronger group" and Group B as a "weaker group" in this study. Over the course of the four weeks, Group A and Group B differed in the number of critical features they identified each week. Group B started out strong by identifying five critical features for the first week, but as weeks went on, members started identifying fewer and fewer features. In comparison, Group A continued to identify approximately two to three critical features each week, despite the fact the tasks themselves gradually became more difficult, as evidenced by the class averages presented earlier. In addition, the video transcripts revealed that Group A often came close to discussing critical features, but sometimes did not include them in their invention task submissions. Table 23 lists the number of critical features identified by members of each group over the course of the four weeks period of the study.

Table 23

Identification of Critical Features in Written Solutions to Invention Tasks

	Task #1	Task #2	Task #3	Task #4
Group A				
Ron	2	2	2	2
Maryann	2	2	1	Absent
Alice	2	4	3	2
Group B				
Nicolette	5	2	2	0
Bill	5	2	2	0
Jaidan	5	1	1	1

As in week #3, I went back to all of the video recordings of the invention task solving process to see how many times members in each group gave a solution proposal during the invention task process. I discovered that as time went on, members of Group A increasingly offered solution proposals from week #1 through week #3, whereas members of Group B decreased their contributions from week #2 through week #4. Group A seemed to develop a pattern of organizing given information into tables and charts, then recalling ways they have worked with data tables and charts before, followed by active generation of possible solutions. This was particularly noticeable in weeks #3 and #4 when Group A provided twice as many solution proposals as Group B. Table 24 lists the number of proposals for each week over the course of the study.

Table 24

Solution Proposals from Group Members

	Task #1	Task #2	Task #3	Task #4
Group A	9	11	14	9
Group B	7	8	7	4

With the hypothesis of stronger groups generating more solution proposals and identifying more critical features in mind, I took an even closer look at the invention task solving process during week #4, the final week of the study. I chose this week as it was the point in time where group members had spent the most time with one another and had likely developed norms and working relationships, as per the embodied conjecture. The invention task for week #4 was based on the premise of the Method of Sealed Bids. This involves people bidding on discrete items that cannot be divided; they have to be given out in whole or not at all. The canonical solution requires the highest bidder to earn each item, and then bidders who do not earn an item, or earn items less than their "fair share," have to receive a cash settlement from the people who earned items that exceeded their fair share. Sometimes there are more items available than people in the problem, which complicates the allocation of items to people and the cash settlement. I chose this scenario for the invention task to see how students would handle giving out items and making sure everyone walked away with something fair. This proved to be the most difficult task as students became confused with establishing fair shares when there were more items than people and how to handle people who did not earn an item.

For Group A, Ron and Alice continued in their usual fashion by immediately setting up a table to organize the given information. This was a step they usually took to try and visualize a solution path. They also recalled their knowledge of fair share to solve the invention task on fair distribution, which was helpful and consistent with the mediating process of activating and differentiating prior knowledge during the PF experience. One of their solution proposals included the concept of the standoff from the week before, which was logical, but not immediately helpful. Both Ron and Alice proposed ways to resolve the issue at hand, which was when multiple people bid the highest for the same item. They also acknowledged a critical feature that there were more items than people, which posed an issue with fair distribution. Once again, thinking outside the box and toward a correct solution, they attempted to bring cash into the problem, but their procedure for doing so was flawed. Despite the fact that their final solution was incorrect, their solution and progress on video followed a routine that they established over four weeks: organization of information, recall of prior knowledge, analysis of the major issue at hand, and possible ways to address it (often with incorrect, but logical attempts). These behaviors seemed to ultimately help them succeed on in-class problems and posttest items.

In Group B, Nicolette took charge by telling her groupmates to make a chart. Nicolette labeled the top row with fair shares per person, which was a step in the right direction, but her conception of fair in this setting was not correct. Nicolette found the fair shares based on the number of items, not the number of people. Nicolette recognized there were more items than people and insisted that someone had to get two items, which was not necessarily true but important to the overall solution. Bill and Jaidan attempted to link the problem to the bid list problem in invention task #3, which was also logical but not helpful. Similar to their solutions from the previous week, Bill and Nicolette employed a rational and humanistic approach to their final solution proposal; they gave each person exactly one item and the final item was given out to a person who the group members perceived deserved a little bit more. This approach, unfortunately, overlooked mathematical fair shares, which was critical to the solution of the problem. Looking across weeks, Group B never developed a set procedure for working through invention tasks. Jaidan's participation was spotty, whereas Nicolette usually led the conversations.

Summary. Digging deeper beyond the significant correlation observed between the invention task and the corresponding posttest item observed in week #3, it was determined that more successful groups of students generated more solution proposals while working through invention tasks. More importantly, these proposals contained critical features of the invention tasks that group members had the opportunity to discuss with one in another in order to advance the solution.

In addition, the stronger group developed a working procedure for working through invention tasks by the fourth week of class. This included accessing prior knowledge, organizing given information, then generating and critiquing solution proposals. The design embodiment's provision of a participation structure that allowed for students to critique, evaluate, and explain work to one another, certainly enabled these rich transactions to happen. Students were also able to make these solution proposals based on their prior knowledge and did so in a way that allowed co-construction of knowledge in the group setting. Had the students worked on the invention tasks themselves, they may not have been able to get as far with them, or have been able to handle more difficult posttest items primarily requiring conceptual knowledge to solve.

Focus group interview results. The goal of the focus group was to have students reflect on their experiences in each of the learning environments in order to better understand students' mathematical inquiry process in each setting. Eight students from the PFFC group volunteered to take part in the focus group interview, including Bill from Group B and Maryann and Alice from Group A. These students represented a range of mathematical abilities in the course as well as year in college, ranging from freshman to senior. Similarly, nine students from the DIFC group volunteered to take part in the focus group interview, and also represented a range of mathematical abilities and year in college. A total of six questions were asked, which included one question on whether or not the students in each section consulted with one another over the course of the semester. Students responded to that question with a unanimous “no” in both groups. The results of the coded interviews for the remaining five interview questions are summarized in the subsequent paragraphs.

Question 1. For the first interview question, students were asked very generally to discuss the learning activities for the unit on apportionment and mathematical fairness. All three students in the PFFC group who responded to this prompt referenced the PF experience and how it helped them make connections in the course. One student gave the following overview of the PF process, thinking of the invention task as a test: “For instance … we have like a test at the end of every class, and then you go home and you do the videos. You come back and you usually have ... you have your answer for the question that you didn't know before, and then you do homework…” A second student identified the experience more clearly by stating, “At the end doing the invention task helped with later when I went back to watch the videos. They would connect to ... this is why we learned that, and it made it easier.” The third student likened the PF process to trial and error, as evidenced by the following quote: “There's a lot of trial and error at first to understand core concepts. And then, as you understood the formula of how to work something out it is a lot easier.” All of these students realized the importance of the invention tasks and how working through them helped them develop both a conceptual and procedural understanding of mathematics in the flipped classroom.

In the DIFC group, the four students who responded to this question described their learning activities in terms of topics covered, assessments, and group activities. The first student gave a brief overview of the two topics of study, summing it up as a unit devoted to learning “specific methods.” The second student cited the routine of videos outside of class, followed by quizzes, then problem solving during the in-person class session. The third student discussed how the class worked in small groups and how advantageous it was to ask a peer for help instead of asking the instructor all of the time. Last, the fourth student built off the third student’s response by adding in how her group collaborated outside of class using mobile devices: “We have a good chat group and group meet we use so that if we have questions about homework or a test, we just go over them while we're at home, too.” As evidenced by frequency counts, students in the DIFC group mostly described their learning in terms of groupwork and in-class assessments like quizzes and homework problems. This is in contrast to the students in the PFFC group, who unanimously spoke about PF and the role of understanding in the flipped classroom.

Question 2. Moving beyond a description of the learning activities, students were next asked to describe how they found the learning activities. In response, students in the PFFC group largely described their satisfaction with their learning activities in the flipped classroom. Most of their descriptions involved pacing and working through video sets. Of the 12 responses to this question, 10 referenced the aspect of time. The first student, Bill from Group B, brought up how the layout of the flipped classroom allowed for flexibility in completing course requirements. Whereas two students commented on liking the ability to re-watch portions of videos, four students commented on how they skipped through parts of videos once they were able to understand and consolidate the main concepts, including Maryann and Alice from Group

A. This information was also valuable as the quantitative survey and additional Kaltura video analytics did not pick up instances of students skipping through segments of video. This could be a result of the PF design, which allowed for students' consolidation of content during the video-watching process, in contrast to the video-watching process being initial exposure for students in the DIFC group. The final two students brought the conversation back to PF and expressed their satisfaction with the invention tasks. One of these students, Alice from Group A, said, "I liked the invention tasks. I thought they were kind of fun, actually, like little games or like puzzles to figure out. Those were just a fun way to learn how to do something because then when you look at the actual answer, it's kind of like, Wow! That's how you do it." The second student agreed and talked about how it was helpful to work on the invention tasks with a group and bounce ideas off of one another to learn the content.

Students in the DIFC group also largely reported positive feelings toward the learning activities. Once again, their focus was on the video tutorials. All eight students who responded to this question based their responses on the videos. In contrast to the PFFC group, no students in the DIFC group reported skipping through any portions of the video. Rather, three students commented on how they liked being able to re-watch and rewind videos as needed, while another three responded favorably to the availability of additional supplemental resources to learn the content of the unit. Despite the fact that students had access to e-mail and weekly office hours before each class session, three students talked about how they did not like videos because they could not ask the instructor questions at the exact time they were watching the videos. Interestingly, two students discussed the importance of having a teacher-led start-of-class review and quiz as ways to help them understand and consolidate the material. One student even said how the quiz "generally reinforces the information and I'm able to understand it for the major exams." This runs in contrast to the PFFC students' responses, where some students cited the initial video watching as being their consolidation process.

Question 3. For the third interview question, students were asked what they found to be most important in their mathematical inquiry process. The six students' responses in the PFFC were split: three students discussed the importance of conceptual understanding, while three students discussed the need to apply classroom content to real-life. An example response from a student in the group was, "For me it's concepts. If you don't understand the concept of doing it you really won't get it. So I think applying the concept and really trying to understand what you're doing so you can be able to apply any kind of way, and doing it as a group together, it helps each other understand our concept." Another student discussed how "understanding the underlying logic" helped him navigate problems in the course. The three students in the PFFC group who described applicability as important to their inquiry made general comments about how authentic contexts and immediate application made them better prepared to work through problems. For example, Alice from Group A commented how, "When the problem is useful, it is easier to get through it."

In contrast, students' responses in the DIFC group were more diverse and did not discuss conceptual understanding. In fact, the five students who responded all valued something different in their inquiry process. The first student spoke about the active application of content and how "doing it physically allows you really understand what it means." The next two students spoke about the consistent wording of prompts across the course and applicability of material, respectively. The final two students discussed successful preparation for class and the design of each video tutorial. These responses were very diverse and difficult to generalize, but on the whole, it seemed the DIFC group commented more about course design features as important to their inquiry process, whereas students in the DIFC group commented more about conceptual understanding and applicability of content.

Question 4. The fourth interview question had students reflect on which aspects of the course design they would like to see strengthened. The theme of time came across many of the students' responses in both groups, but the suggestions for what to do with the added time were different. Some minor suggestions from the PFFC group included moving due dates for class assignments later and opening up video sets earlier, while some of the more common suggestions including lengthening the in-person class session to allow for more time for classwork and review of assessments. Of the seven students who responded to this prompt in the PFFC group, four of them suggested lengthening the in-person class session to allow for more time to complete classwork, review quizzes, and work in groups. Students in the DIFC group also proposed having more opportunities to ask the professor questions as part of the course design improvement. All four students who responded to this question specifically suggested more direct instruction from the instructor. The first student suggested that this come in the form of online forums where the professor could answer questions, whereas the other three students suggested more quiz review, more exam review, and a start-of-class review that mimics the progression of content from the video set.

Question 5. Finally, students in both groups were invited to share general feedback about their experiences in the course in order to garner additional relevant information about the learning environments. Responses from both groups were generally along the same line; students responded favorably to the blend of online and in-person learning activities in the flipped classroom and also appreciated the wealth of resources available to them to be successful. Supporting statements from students in the PFFC group include one where a student felt that the videos "pushed you to do the work," and another from Maryann from Group A who felt that the course offered "so many opportunities that you have as you can actually practice your work." Students in the DIFC talked more about how the videos were important to their success, and how they preferred them over their textbook or used the textbook only as a supplemental resource. One student said, "I think the videos are really important to this class being a hybrid because if there weren't videos, I couldn't imagine myself just reading a textbook and learning like that. That just would not work for me."

Summary. The fourth research question was, "How do students in both the DIFC and PFFC learning environments describe their in-class mathematical inquiry process?" Students' responses to each of the interview questions help form a picture of what learning looks like in each environment. Overall, both groups expressed their appreciation for the flipped method of instruction and how their initial perceptions of blended learning and non-traditional instruction were very different from what they actually experienced after taking the class for many weeks. Both groups cited the importance of their group members, the instructor, and online resources as key features of their success in the course. The most interesting finding from the analysis was that students in the DIFC group cited their instructor more frequently as important their learning process, whereas students in the PFFC group cited their peers more frequently as important to their learning process. This is evidenced by additional quantitative analysis of counts through a Chi-square test, where $\chi 2$ (1, N = 17) = 4.735, $p < .05$.

Additionally, students in the PFFC group seemed to reflect more on understanding concepts over the course of the interview compared to their DIFC counterparts, who mainly viewed their learning online and in class as procedural. Time for concept exploration in class with group members prior to direct instruction was frequently described to be a critical part of the learning process for students in the PFFC group, as evidenced by all students' responses to the second focus group question. In addition, when students in the PFFC described video tutorials, some referenced how they did not feel the need to work all the way through them due to their exposure to the content by means of the invention tasks. On the other hand, students in the DIFC group more frequently described the videos as information they had to master and apply, and would turn back to. None of the students in the DIFC group mentioned skipping or fast forwarding of videos. This different explanation of the videos could be due to the role of the videos within each condition, as they served as the initial "direct instruction phase" for the DIFC group but served as the "consolidation phase" for the DIFC group.

Conclusion

Despite the fact that there was not a significant difference in posttest performance between groups, the data analysis over the five questions reveals that students in the PFFC group benefited from the additional time spent on concept exploration and accessing prior knowledge. Students' responses to the interview prompts made this the most obvious, as students in the PFFC group more often referred to concepts and invention tasks as they discussed their learning in the course, whereas students in the DIFC group usually referred to watching videos to learn procedures as the most important. A more in-depth analysis of in-class video recordings and work samples revealed that more effective groups were able to identify more critical features of problems than their peers did and that over time more effective groups had more productive solution generating habits while solving invention tasks. This ultimately led to stronger work on posttest items involving conceptual knowledge. This study showed no significant difference in video watching between groups, although students in the PFFC did seem to watch the videos less frequently and more purposefully than their DIFC counterparts based on the advanced analytics obtained from Kaltura.

CHAPTER 5: DISCUSSION

Purpose of the Study

The primary purpose of the current study was to redesign a flipped classroom to include productive failure as a learning intervention. This study contributes to the field of research on the design of flipped classrooms through proposing a method for helping students access their prior knowledge inside the classroom before working through instructional videos outside of the classroom (O'Flaherty & Phillips, 2015). In addition, this study extended the findings of previous research in the field of productive failure that have shown the efficacy of delaying instruction by allowing students to work on challenging tasks that require them to access prior knowledge and invent solutions prior to hearing or reading normative explanations (Kapur, 2008; Kapur & Bielaczyc, 2012). This has been accomplished by incorporating a productive failure approach into the flipped classroom as a way to begin learning the next concept before leaving each in-person class meeting. Prior to leaving class, but before watching their weekly video sets, students were able to experience short-term failure that prepared them to work productively through videos and in-class problems. This study also examined qualitative data related to the productive failure experience in the flipped classroom, namely how students solved problems in groups and how they described their inquiry process in the undergraduate mathematics classroom.

This study also built on more recent work completed by Song and Kapur (2017) in the area of productive failure in the flipped classroom. Using this study as a framework for my own, I was able to test their theory and method in my context of higher education. In doing so, I addressed a few limitations in their study in my own study. First, my study included instruction beyond a two-week time period and considered students' knowledge development over the course of a whole four-week unit. Second, I turned to more detailed video analytics in addition to self-reports to better understand students' video-watching behaviors in both treatment conditions. This included the average amount of time students spent watching each video, the percent of unique videos watched, and their average drop-off while watching videos. Third, the Song and Kapur (2017) study involved different instructors; in this study, the same instructor taught both flipped classroom sections using the same procedures to allow for a more equitable comparison between groups. Last, I also ended up relying more on qualitative data to develop my findings as opposed to looking just at post-test scores as measures of students' learning, as they did in the Song and Kapur (2017) study. The results of my focus group interview and video footage provided me with information about the learning environments that could not be captured on a posttest. These modifications to the Kapur and Song (2017) study enabled me to critically examine the productive failure flipped classroom design and understand both the in-person and online components of the course in a more nuanced manner.

Discussion of Findings

In this section, I will summarize findings as they relate to each of the research questions. Each research question is restated below along with a brief synopsis of the findings that answer the question.

Effects on procedural knowledge. The first research question asked, "What is the effectiveness of flipped classroom pedagogical design interventions on the development of students' procedural knowledge, particularly on their written work samples?" A comparison between posttest scores for both treatment conditions demonstrated that there was not a statistically significant difference in performance on procedural knowledge questions. Relatedly, t-tests revealed no statistically significant difference on in-class problems. For the PFFC group only, additional t-tests were run to see if there was a significant relationship between each invention task and its corresponding procedural knowledge question on the posttest. Once again, there were no statistically significant results. A possible reason for this finding is that all students in both conditions had opportunities to acquire procedural knowledge before coming to each in-person class session. Students in both groups watched the exact same video tutorials, which outlined the procedures they needed to learn for solving problems in class.

Effects on conceptual knowledge. The second research question asked, "What is the effectiveness of flipped classroom pedagogical design interventions on the development of students' conceptual knowledge, particularly on their written work samples?" A comparison between posttest scores for both treatment conditions demonstrated that there was not a statistically significant difference in performance on conceptual knowledge questions. Similarly, t-tests revealed no statistically significant difference on in-class problems. For the PFFC group only, additional t-tests were run to see if there was a significant relationship between each invention task and its corresponding conceptual knowledge question on the posttest. Once again, there was no statistically significant result.

However, additional correlation analysis revealed that there was a significant positive correlation between invention task #3 and its corresponding conceptual knowledge posttest item. In addition, students' scores on in-class problems positively correlated to similar problems on the posttest. Further qualitative analysis revealed that students in the PFFC group largely discussed their learning experience in the flipped classroom in terms of concepts and understanding, whereas students in the DIFC largely described their learning experience in terms of procedures and applications.

Research in productive failure literature further demonstrates that students who attempt numerous solution approaches, even if they are failed approaches, tend to learn more than students who attempt fewer solution approaches (Kapur, 2012; Wiedmann, Leach, Rummel, & Wiley, 2012). Despite the fact that posttest scores did not reveal this outcome, correlations between assessments in the course showed a positive relationship between students' invention task solutions and their corresponding conceptual knowledge posttest question. This could explain the interesting correlation observed between the invention task and its corresponding conceptual knowledge posttest item.

Video-watching behaviors. The third research question asked, "How do video watching behaviors influence students' learning performance in the DIFC and PFFC learning environments?" The results of students' self-reported surveys and an additional video-watching report from the learning management system indicate that there were no significant differences in video watching frequency or behaviors between groups. Most students watched the videos at least once; 77.3% of students in the PFFC group and 80% of students in the DIFC group reported watching all of the videos in the unit. In addition, 31.8% of students in the PFFC group and 30% of students in the DIFC group reported watching the video tutorials two or more times. During the focus group interview, students in each group discussed their behaviors as they progressed through the video tutorials. Students in the PFFC group referenced fast forwarding and skipping portions of videos, whereas students in the DIFC group did not make these comments. Additionally, all students in both groups reported using a computer to work through the videos rather than any type of mobile device. Further analysis revealed that there was no significant effect of video-watching times on students' learning performance on the posttest.

This finding was unexpected, as it was hypothesized that students in the PFFC would watch the instructional videos less frequently due to the time they spent on the invention tasks. The purpose of the invention task was to give students time to access prior knowledge and start developing concepts before the video consolidation phase of their learning (Song & Kapur, 2017). With the concept exploration opportunity that the DIFC did not have, it was surprising to see that there was no significant difference between groups in terms of video watching. This could also suggest a need to look further into the content of the videos so that both groups receive the appropriate level of instruction based on their experiences prior to watching the tutorials.

Mathematical inquiry process. The fourth research question asked, “How do students in both the DIFC and PFFC learning environments describe their in-class mathematical inquiry process?” Overall, students in both groups positively described the flipped classroom learning environment and provided valuable feedback on the model. Both groups discussed how important it was to work with other group members, the instructor, and online resources both online and in class to learn the content. The most interesting finding from the analysis was that students in the DIFC group cited their instructor more frequently as important their learning process, whereas students in the PFFC group cited their peers more frequently as important to their learning process.

A possible explanation for this finding is the fact that the PFFC students had more time to work with their peers due to the collaborative nature of the invention tasks. The design embodiment for both groups allowed for collaboration on in-class problems, but students in the PFFC group had the additional invention task problem to collaborate on in a small group setting. All students went into the invention tasks without having had any instructions on the topics, so it could have been the case that students felt more comfortable exchanging ideas on problems they all were not expected to already know how to solve.

Another interesting finding from the focus group interviews was that students in the PFFC group more frequently summarized their experiences as learning concepts, whereas students in the DIFC group more frequently summarized their learning as mastering steps and procedures. This was most salient in their discussions of how they approached watching the instructional videos; students in the PFFC group made it a point to discuss how they fast-forwarded and skipped parts of videos, whereas students in the DIFC group discussed how they watched the videos to learn steps and procedures.

A possible explanation for this finding could be the order of learning activities that students engaged in while in the PFFC group. Students in the PFFC group had the opportunity to mobilize prior knowledge in class and then used the videos to consolidate their understanding from the invention task (Song & Kapur, 2017). Students in the DIFC group did not have this opportunity; rather, the instructional videos were their first exposure to the new content. When reflecting on their inquiry processes, students in the PFFC might have recalled concept development more easily since they had the additional opportunity to develop knowledge of concepts of fair division, apportionment, and fair distribution.

Video footage of in-class sessions. The fifth research question asked, “How do students’ problem solutions to in-class problems and their interactions as they discuss these problems mediate conceptual and procedural learning as measured on the posttest?” Despite the lack of statistical significance on in-class and posttest items between the two groups, a closer look at the mediating processes in the design embodiment through the video footage of in-class sessions revealed that the productive failure experience did lead to identification of critical features of the problems before students were instructed on them. This was particularly salient in groups who generated many solution proposals for each invention task and who also developed a set approach for solving the invention tasks over the four weeks of the study. By tracing the evolution of Group A and Group B over the four weeks of the course, it became clear that a group like Group A had a set strategy of pattern of organizing given information into tables and charts, then recalling ways they have worked with data tables and charts before, followed by active generation of possible solutions.

These processes of immediately organizing information, retrieving prior knowledge, and verbally generating solutions in real time helped members of Group A solve the problems they were faced with throughout the study. This is consistent with Kapur and Bielaczyc's (2012) theory of designing for productive failure, as they posited how a true productive failure experience involves having students access their prior mathematical resources. Kapur and Bielaczyc (2012) also identified collaboration as important in facilitating attention to and elaboration of critical features. In this study, because students had the ability to collaborate on the invention tasks in groups, they were able to generate ideas and critical features in front of their peers. In Group A, students responded to these ideas and critical features with additional questions, clarification, and agreements/disagreements that advanced their invention task solution process (Kapur & Bielaczyc, 2012).

Interpretation of Findings

This quasi-experimental mixed methods study reported on the impact of two learning environments, the PFFC and DIFC, on students' procedural and conceptual knowledge in a flipped undergraduate mathematics course. Findings indicated that although both designs could improve students' procedural knowledge, no statistically significant results were obtained between the DIFC group in terms of procedural and conceptual knowledge. Students in both groups also responded very favorably to the flipped classroom approach on their learning. This was amplified in the PFFC group, where students commented on how their experiences in the course actively working with the content and with one another.

Even though the productive failure setting was not superior in this particular experiment, some aspects of group work during productive failure were more productive for future learning than others. First, having students work on invention tasks in groups resulted in the generation of many ideas that may not have come about if students had to work on the tasks by themselves. In addition, having a safe and supportive social surround developed from the beginning of the course allowed for students to feel comfortable with both the instructor and other peers (Kapur & Bielaczyc, 2012). This was fostered by having well-established groups for weeks prior to the beginning of the study. By the time students worked on invention tasks with one another, they were already used to engaging in mathematical inquiry with one another and felt comfortable discussing mathematics with one another.

Kapur (2011) also unpacked the role of the teachers in the productive failure learning environment and noted how teachers should not immediately provide help during the productive failure process. The design of the productive failure learning environment for this study included a carefully developed set of questions and protocols that ensured this would not happen (see Appendix G). By allowing students to struggle with developing a canonical solution to each invention task, they were forced to try harder and develop a solution by working with one another (Kapur, 2011). These are some of the features of productive failure that were effective in this study and can be engineered into new and future productive failure designs.

To better understand the lack of a significant difference in course assessments between groups, I revisited the mediating processes in the embodied conjecture. If the design worked as it should, the high-level conjecture predicted that the intervention would result in students' improved conceptual and procedural understanding of targeted content, through the operation of the hypothesized mediating processes. Because this was not the case for students in the PFFC, this could suggest some of the mediating processes may not have occurred as planned, or that the mediating processes did produce the expected outcomes. Video footage of group problem-solving processes did reveal students activating and differentiating prior knowledge as planned. More successful groups did show significant evidence of giving reasons, making conjectures, and constructing solutions to invention tasks. Less successful groups were not as forthcoming with their solutions. In very few cases, video footage revealed that students had a shared understanding of how to solve invention tasks, but the actual written submissions did not reflect the productive conversations on videotape. These findings suggest that sometimes various groups did not fully engage in the mediating process of actively making conjectures and supporting them in their written work, which may have prevented them from achieving the desired outcomes in the embodied conjecture.

Last, it was hypothesized that students would engage in productive video-watching habits, including pausing, rewinding, practicing problems, etc. It is certainly a possibility that students did not feel they needed to pause, rewind, or engage with the videos beyond watching them. In fact, students in the PFFC group admitted to fast forwarding and skipping portions of videos in the focus group interview. The process of working through a video was modeled for all students, and the content of the video encouraged students to work on a problem to assess their understanding. Since video watching was an important consolidation phase for PFFC students, perhaps additional measures need to be taken to ensure students have a productive video-watching experience. This could include embedded questions students have to answer throughout the videos before they proceed to each new topic or example.

Limitations and Directions for Future Research

The results of this study cannot be generalized due to sample representativeness and the number of classes per condition. Only two classes were used in this study, one control group and one experimental group. Future studies could include an examination of more sections of Math 103 taught with a productive failure learning intervention. In addition, the composition of each class was very different, as described in the sample section. The control group had two students with advanced mathematical training, whereas the experimental group had many first-year, first-semester students with less experience as university students.

Also, individual pretest data, posttest data, and video watching data were aggregated at the group level to address the efficacy of each group as a whole; however, aggregated data cannot necessarily predict individual student's learning (Cress, 2008; Song & Kapur, 2017). I turned to individual students' work sample data as appropriate to illustrate cases of knowledge development to compensate for this. In addition, because the researcher taught both sections of the class, there is the threat of researcher bias. However, every effort was made to ensure consistent treatment of groups, such as scripted responses to questions in both groups and field notes (see Appendices F and G).

This study raises additional questions and avenues for exploration in the areas of flipped classroom and productive failure. Future research in flipped classroom design could involve a more careful examination of the "individual space" and "group space" aspects of a flipped classroom and how the amount of time spent in each could impact learning (Hamdan, McKnight,

McKnight, & Arfstrom, 2013). This sentiment was echoed in some students' focus group responses in both groups, as they suggested more class time could have afforded additional opportunities for concept exploration and group problem-solving. Students in the PFFC group transitioned from a group space at the end of a class session, to an individual space at home, then back to a group space during the next class session. Students in the DIFC group only went from the individual space at home to the group space in the classroom. The design and evaluation of these spaces could form the basis for future study on the flipped classroom learning environment, particularly what the ideal balance of learning spaces is in a hybrid course to facilitate deep learning, and what the learning advantages are in each type of space.

Another avenue to explore in future work is the type of video content available to students in each treatment condition. Numerous PFFC students in the focus group interview in claimed to have skipped or fast-forwarded through some of the video content. Due to the fact that students in the PFFC group had time in class to mobilize prior knowledge and generate solutions to problems, questions remain about whether their video experience should look different from that of students in the DIFC group. For example, should these students have less introductory information or review in their videos? Should the videos include additional problem-solving and solution generation opportunities for DIFC students? Will future innovations in technology result in personalized learning opportunities for all students? These questions, coupled with the fact that the optimal length for instructional videos should be around six minutes for students' maximum engagement (Guo, Kim, & Rubin, 2006), suggest that instructors who teach flipped classrooms need to make important pedagogical decisions about what should and should not be included in their video lessons.

Considering the diverse sample in this study, additional research is also warranted in the area of how students embrace their learning in a flipped classroom. Generational differences may exist in how students watch videos, approach their classwork, and work with one another in both the in-class and out-of-class learning environments. In this study, the DIFC group had many more third- and fourth-year students, whereas the PFFC group had many more first- and second-year students. This might have had an impact on the results of the course assessments and group work because the third- and fourth-year students may have had a different approach to working through videos and completing tasks outside of class. The first-and second-year students have also had more extensive experiences with blended learning, as many K-12 schools have provided students with their own personal laptops for the past three or four years. A more nuanced investigation of students' work habits inside and outside of class may provide insight on which generation of student would be more successful in a digital learning environment like a flipped classroom.

While analyzing the video and qualitative data, the more time I spent analyzing Group A's work over the four weeks of the study, the more I became intrigued by how their group evolved over their time spent solving invention tasks. They naturally started to follow this route of assembling information, accessing prior knowledge, and generating many solution proposals. Additional research on effective group problem-solving processes, particularly when the problems are designed to generate failure, would be an interesting avenue to explore. Of particular interest in this area is norm setting; would groups' problem-solving processes be positively impacted by additional time spent on norm setting and establishing roles within groups? It is possible that supplementing the course with additional training for students on how to work in groups could enhance the productive failure experience.

Last, designing the productive failure experience for non-algebra students deserves some more attention. Many of the seminal studies on productive failure have taken place in secondary schools on topics from the beginning algebra curriculum (Kapur, 2008; Kapur, 2009; Kapur, 2011; Song & Kapur, 211). These topics, such as standard deviation, polynomial operations, and average speed, are rudimentary enough that any student can approach the topics, yet challenging enough that without prior instruction on the topics, students would have to think critically to solve problems on the topics. This study took place in an undergraduate liberal arts mathematics classroom where the topics were not in the standard algebra curriculum. Based on students' scores on invention tasks and posttest items in this study, some invention tasks proved more challenging than others, and developing test items that challenged students but did not overwhelm them, was also challenging. Kapur and Bielaczyc (2012) spelled out their design recommendations for productive failure in great detail, which were employed in the design of this study. It would be interesting to see if the design recommendations would change for different mathematics courses and different levels of students.

Implications for Practice

The results of this study suggest that a flipped classroom consists of much more than videos and group work. Instructors often naively view a flipped classroom as an inversion of lecturing and applied practice, with lectures replaced by video tutorials. Instructors need to carefully consider ways to design tasks that allow students to learn and build conceptual understanding both inside and outside of in-person class sessions. Whether an instructor plans on using productive failure during class or not, students need to understand the purpose of the videos before they go off watching them. Designing an in-class learning environment that allows for students to freely generate solutions, learn from their mistakes, and seek assistance from peers and the instructor as needed are also crucial elements in designing a flipped classroom.

The design of in-person learning activities in a flipped classroom is also an essential part of creating the flipped classroom learning environment. Song and Kapur (2017) found that the role of the in-class problems changes depending on whether or not students have had opportunities to access and mobilize prior knowledge before coming to class. In particular, students in a flipped classroom who experience productive failure have the opportunity to assemble their knowledge prior to solving in-class problems, and thus have a different level of preparation coming into solving the problems. Similar to Song and Kapur (2017), this was not the case for students in the DIFC in this study, where students were expected to apply what they learned from the videos to the in-class problems without any additional recall of prior knowledge. This difference in how the problems were approached suggests the need for instructors to provide students in any type of flipped classroom with opportunities to activate their prior knowledge before entering into in-class activities. Doing so also allows for the instructor to provide assistance to each student at his or her zone of proximal development (Bruner, 1986), which is imperative for learning.

As a teacher, running a design-based research study made me reflect on my entire teaching practice and examine every aspect of the course from the bottom up. Designing and teaching a flipped classroom can be viewed as testing teaching and learning theories in a messy, complex environment with the goal of producing instructional strategies that can withstand the challenges of ongoing practice (Shavelson, Phillips, Towne, & Feuer 2003). This study made me analyze every question, every video, every consideration necessary for creating a problem-solving environment conducive to students' active participation, and how I would respond to students in both groups. Prior to starting the study, I created a list of questions and responses that I used in my classes to ensure equity and meaningful feedback. In addition, based on students' feedback, I made in-the-moment design changes in the course, such as a pre-typed bulleted list of key ideas to begin each flipped class with. This also helped to ensure that both groups were receiving the same review and coverage of important mathematical content. Overall, these practices have taught me that it is possible to use research, theory, and experiences to improve the effectiveness of a learning environment design (Joseph, 2004).

Last, the video data I obtained from this study taught me about productive processes and strategies for having students work in groups. Group A in my PFFC group developed a very successful working relationship over the four weeks of my study, as evidenced by their written work samples and their ability to generate solutions to very difficult tasks. They were able to successfully think out loud and help one another verify whether or not a solution proposal was correct. They also made a deliberate effort to try and connect what they were working on back to what they learned in previous mathematics classes. As I move forward designing learning environments that include group collaboration of any kind, I will encourage students to use these strategies starting at the beginning of the course.

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Appendix A

Invention Tasks

Invention Task #1 (adapted from Tannenbaum, 2013)

A new country has recently been founded. The country is split into six states, call them A, B, C, D, E, and F. The population of state A is 1,646,000 people, the population of state B is 6,936,000 people, the population of state C is 154,000 people, the population of state D is 2,091,000 people, the population of state E is 685,000 people, and the population of state F is 988,000 people. There are 250 seats available on a legislative body to govern the new country. How many seats should be assigned to each state so that each state would receive a fair representation? Show your work and justify why you think your method is correct.

Invention Task #2

Ann and Bob share the rights to use a certain store location in the mall, but they have separate businesses, and only one can use the space at a time. They agree that each year, one should get the space from January 1 until a certain date, and the other should get the space from that date through the end of December.

Ann estimates that she can make $10,000 per month from January through July, and $14,000 per month from August through December. Bob estimates that he can make $6,000 per month from January through June, and $12,000 per month from July through December.

Describe a way in which Ann and Bob could each receive a fair share of the year, if it is left up to Ann to make the first move.

Invention Task #3

Suppose Andrea, Beatrice, Cleo and David are dividing a plot of land worth $1,000. The land comes in four sections, call them piece 1, piece 2, piece 3, and piece 4. Andrea values piece 1 at $150, piece 2 at $300, piece 3 at $200, and piece 4 at $350. Beatrice values piece 1 at $200, piece 2 at $350, piece 3 at $100, and piece 4 at $350. Cleo values piece 1 at $200, piece 2 at $450, piece 3 at $200, and piece 4 at $150. David values piece 1 at $250, piece 2 at $250, piece 3 at $250, and piece 4 at $250. Describe a way for the players to fairly divide the land.

Invention Task #4

Ashley, Bill, and Cliff have been roommates through college and now are getting ready to graduate and go separate ways. They have jointly owned a table, dresser, chair, and couch among the 3 of them. Ashley values the table at $150, the dresser at $150, the chair at $175, and the couch at $275. Bill values the table at $250, the dresser at $200, the chair at $150, and the couch at $300. Cliff values the table at $175, the dresser at $125, the chair at $125, and the couch at $250. Describe a way for all three people to walk away with a "fair share." Justify your answer using dollar amounts and explanations.

Appendix B

In-Class Problems

I. Apportionment

HW1. The Gesundheit Hospital has a nursing staff of 400 nurses working in four shifts:

- Morning (6am – 12pm)
- Afternoon (12pm – 6pm)
- Evening (6pm – 12am)
- Late night (12am – 6am)

The number of nurses apportioned to each shift is based on the number of patients treated in that shift, given in the following table:

Shift	# of patients
Morning	1742
Afternoon	2067
Evening	1220
Late night	371

a. Which are the seats, which are the states, and what plays the role of population in this apportionment problem?
You are not asked here for numbers; instead answer in terms of shifts, patients, and nurses (full sentences are not required here).

seats:_________________ states:___________________ population: _____________________

b. Find the standard divisor for this apportionment problem.

c. Explain in one or two sentences what the standard divisor represents in this problem (not how to calculate it); use the language of nurses, patients, and shifts to explain.

d. Find the standard quota for each shift.

HW2. This problem refers to the Gesundheit Hospital described in problem HW1 above.

a. Find the Hamilton apportionment of nurses to shifts based on number of patients in the hospital (as presented in problem HW1 above).

b. Based on the number of nurses each shift receives in the Hamilton apportionment, compute for each shift the average number of patients per nurse. This need not be a whole number, since it is an average, and will be different for each shift.

c. Which shift has the most patients per nurse? Is this the best shift on which to be a patient, or the worst, and why?

HW3. This problem refers to the Gesundheit Hospital described in problem HW1 above.

a. Find the Jefferson apportionment of nurses to shifts based on number of patients in the hospital.

[Hint: One of the following modified divisors will work: 13.3, 13.36, 13.42, 13.48, 13.54, 13.6. Before trying any of these, first think whether the modified divisor should be greater than 13.5 or less than 13.5.]

b. Based on the number of nurses each shift receives in the Jefferson apportionment, compute for each shift the average number of patients per nurse. This need not be a whole number, since it is an average, and will be different for each shift.

c. Which shift has the most patients per nurse?

HW4. This problem refers to the Gesundheit Hospital described in problem HW1 above.

a. Find the Webster apportionment of nurses to shifts based on number of patients in the hospital.

[Hint: First, determine whether the standard divisor already works, and results in a Webster apportionment. If not, should the modified divisor be greater than the standard divisor or less? In any case, one of the following modified divisors will work: 13.47, 13.48, 13.49, 13.50, 13.51, 13.52, 13.53.]

b. Based on the number of nurses each shift receives in the Webster apportionment, compute for each shift the average number of patients per nurse. This need not be a whole number, since it is an average, and will be different for each shift.

c. Which shift has the most patients per nurse?

HW5. This problem refers to the Gesundheit Hospital described in problem HW1 above.

Find the Huntington-Hill apportionment of nurses to shifts based on number of patients in the hospital.

[Hint: First, determine whether the standard divisor already works, and results in a Huntington-Hill apportionment. If not, should the modified divisor be greater than the standard divisor or less? In any case, one of the following modified divisors will work: 13.47, 13.48, 13.49, 13.50, 13.51, 13.52, 13.53.]

HW6. This problem refers to the Gesundheit Hospital described in problem HW1 above.

(Summary of questions HW1 through HW5 above): In the end, do all the apportionment methods we have considered give the same apportionment of nurses to shifts? If not, how do they differ?

HW7. This problem pertains to the Alabama paradox.

a. Suppose that the Gesundheit Hospital described in problem HW1 above hires one more nurse, for a total of 401, but the number of patients on each shift remains the same. Find the Hamilton apportionment.

b. Does increasing the number of nurses from 400 to 401 create an instance of the Alabama paradox? Explain how you can tell.

c. Now suppose that the same hospital hires yet another nurse, for a total of 402, but the number of patients on each shift again remains the same. Find the Hamilton apportionment.

d. Does increasing the number of nurses from 401 to 402 create an instance of the Alabama paradox? Explain how you can tell.

HW8. Based on a hypothetical apportionment given below, where 1000 seats are being apportioned:

State	Population	Standard Quota	Modified quota (when modified divisor equals 3.443)	Apportionment under Jefferson's method
A	296	85.80	85.97	85
B	400	115.94	116.18	116
C	850	246.38	246.88	246
D	1904	551.88	553.01	553

Do any violations of the Quota Rule occur in this apportionment? Explain briefly how you can tell.

II. Fair Division

HW1. Four siblings, Wendy, Xavier, Yolanda, and Zachary, inherit a house. Suppose that
Wendy considers the house to be worth $400,000
Xavier considers it worth $450,000
Yolanda considers it worth $450,000
Zachary considers it worth $380,000

Which of the following (if any) is a fair division, according to our definition of that term, and why or why not?

a. No one gets the house. Instead it is sold for $400,000 so each sibling receives $100,000 in cash.

b. Xavier gets the house. He pays each of his siblings $110,000 in cash.

HW2. Martha and Nick share the rights to use a certain store location, but they have separate businesses, and only one can use the space at a time. To minimize the costs and hassles of turnover, they agree that each year, one should get the space from January 1 until a certain date, and the other should get the space from that date through the end of the year.

Martha values each month from January through August twice as much as each month from September through December.

Nick values each of March and April three times as much as each month from January through February or May through December.

They agree to use the Divider-Chooser method to make a fair division of the access time to the store location. As always, we assume that neither one knows the other's preferences, i.e. each one makes choices based solely on his/her own system of values. They toss a coin, and the winner of the coin toss gets to be the Chooser.

In parts a and b you will solve the fair division problem in the case that Nick is the Divider.
Then in parts c, d, and e you will solve the problem all over again in the other case, that Martha is the Divider.

Case 1: Suppose that Nick loses the coin toss and is therefore the Divider.

a. At what point in the year does Nick make the division, and what are the resulting shares? Show the work which leads to your conclusion.

b. Which share does Martha choose, and how much is it worth to her as a fraction of what the whole year is worth to her? Show the work which leads to your conclusion.

Case 2: Suppose that Martha loses the coin toss and is therefore the Divider. Note that we are now going back to the beginning of the problem and solving it all over again, but with a different Divider.

c. At what point in the year does Martha make the division, and what are the resulting shares? Show the work which leads to your conclusion.

d. Which share does Nick choose, and how much is it worth to him as a fraction of what the whole year is worth to him? Show the work which leads to your conclusion.

e. Suppose that Martha and Nick get an expert to give each one a financial estimate of the value of having the store location for the whole year. For Martha, having the store location for the whole year is $300,000, while for Nick it is worth $240,000. If Martha is the Divider, what will her share be worth to her in dollars, and what will Nick's share be worth to him in dollars? Which player receives a higher fraction of the total value of the year according to his/her own value system, and who receives a share worth more in dollars?

HW3.

a. In general, is it better to be the Divider or the Chooser in the Divider-Chooser method, and why? Or does it not matter?

b. Look back at the scenario of the fair division problem in HW2c and d. What would happen if Martha didn't follow the rules, and instead divided the year at the calendar midpoint, i.e. at the end of June?

c. Why is it in the Divider's interest to follow the rules, and divide at the value midpoint rather thanhe calendar midpoint of the year?

HW4. Olivia and Paul face the same division problem as Martha and Nick in problem HW2, but have different preferences:

Olivia values each month from January through July twice as much as each month from August through December.

Paul values each of May and June twice as much as each month from January through April or July through December.

Suppose that Olivia is the Divider.

a. At what point in the year does Olivia make the division, and what are the resulting shares? Show the work which leads to your conclusion.

b. Which share does Paul choose, and how much is it worth to him as a fraction of what the whole year is worth to him? Show the work which leads to your conclusion.

HW5. Eve, Fred, Gail, and Hank are four siblings who inherit a large piece of land. The land is not uniform, and has different features in different locations. A Divider is selected at random, and divides the land into 4 parts ("shares"), which as usual we will call s_1, s_2, s_3, and s_4.

a. Eve, Fred, Gail, and Hank each write their assessments of the value of each share, recorded in the following table. For each player, find his/her fair share in dollars (for which you'll need to find his/her total estimate of the value of the land), find his/her bid list, and record these in the table:

	Value of s_1	**Value of s_2**	**Value of s_3**	**Value of s_4**	**Minimum fair share amount in dollars**	**Bid list**
Eve	\$140,000	\$175,000	\$150,000	\$135,000	\$	
Fred	\$200,000	\$200,000	\$200,000	\$200,000	\$	
Gail	\$200,000	\$200,000	\$200,000	\$500,000	\$	
Hank	\$150,000	\$175,000	\$150,000	\$425,000	\$	

b. Who must have been the Divider, and how can you tell?

c. For each of the following divisions, indicate whether it is a fair division or unfair; if it is unfair, briefly explain why it is unfair.

- Eve gets s_1, Fred gets s_2, Gail gets s_3, and Hank gets s_4.
- Eve gets s_2, Fred gets s_1, Gail gets s_4, and Hank gets s_3.

d. There is a "standoff" here between Gail and Hank. Suppose that before the standoff is resolved, Eve is given s_2 and Fred is given s_1. Describe in words, as specifically as possible in light of the information available, what is done at this stage to resolve the standoff.

e. There is not enough information given to find the exact dollar value of the share which Gail will ultimately receive, after the standoff is resolved. Based on the available information, Gail will receive a share worth at least how much? Is this a fair share, in relation to the original problem of dividing the land among the four siblings?

f. There is not enough information given to find the exact dollar value of the share which Hank will ultimately receive, after the standoff is resolved. Based on the available information, Hank will receive a share worth at least how much in dollars? Is this a fair share, in relation to the original problem of dividing the land among the four siblings?

Recall the time-share fair division problem regarding Martha and Nick in problem HW2 above. The next two problems are similar in spirit, but use the Lone Divider method for three players.

HW6. Ann, Bob, and Chris have equal claims to a store location, and are using the Lone Divider method to find a fair division of access to the location over the calendar year. We assume as in problem HW2 that January 1 has to be one of the start/end dates when dividing the year into shares.

Ann values each of the months January through June twice as much as each of the months July through December.

Bob values each of the months January through March three times as much as each of the months April through December.

Chris values each of month of the year equally.

a. Suppose that Bob is the Divider. At what points does Bob divide the year? What are the shares which Bob produces, described in terms of which months (or parts of months) are contained in each share?

b. Which shares are fair shares for Ann? Which shares are fair shares for Chris?

c. Find a fair division of the year; indicate which share each player receives, and what it is worth to that player as a fraction of what the whole year is worth.

d. Find a different fair division of the year than the one you just found in part (c); indicate which share each player receives, and what it is worth to that player as a fraction of what the whole year is worth to him/her.

HW7. Bob, Dave, and Eve have equal claims to a store location, and are using the Lone Divider method to find a fair division of access to the location over the calendar year. We assume as in problem HW2 that January 1 has to be one of the start/end dates when dividing the year into shares.

Bob values each of the months January through March three times as much as each of the months April through December.

Dave values each of the months April through June twice as much as each of the months January through March or July through September, and does not value October through December at all (these last 3 months of the year are worthless to Dave).

Eve values each of the months January through June three times as much as each of the months July through December.

a. Suppose that Bob is the Divider. At what points does Bob divide the year? What are the shares which Bob produces, described in terms of which months (or parts of months) are contained in each share?

b. Explain why a standoff occurs here, and what it means that there is a standoff.

c. Suppose that prior to resolving the standoff, Bob is given the last share in the calendar year. Describe a specific way in which the standoff can now be resolved; there is more than one way to do this, but you are asked to find just one!!! At the end, indicate which share each player receives, and how much that share is worth to the player as a fraction of what the whole year is worth to him/her.

HW8. Recall the scenario of problem HW5, in which Eve, Fred, Gail, and Hank are four siblings who inherit a large piece of land, and use the Lone Divider method to produce a fair division. Suppose that Eve takes a second look at the numbers she originally wrote down, and decides to reevaluate the shares, so that the following is the full set of information before the division is carried out:

	Value of s_1	Value of s_2	Value of s_3	Value of s_4
Eve	\$120,000	\$125,000	\$120,000	\$235,000
Fred	\$200,000	\$200,000	\$200,000	\$200,000
Gail	\$200,000	\$200,000	\$200,000	\$500,000
Hank	\$150,000	\$175,000	\$150,000	\$425,000

a. There is a standoff here among Eve, Gail, and Hank. Suppose that before the standoff is resolved, Fred is given s_1. Describe in words, as specifically as possible in light of the information available, what is done at this stage to resolve the standoff.

b. There is not enough information given to find the exact dollar value of the share which Eve will ultimately receive, after the standoff is resolved. Based on the available information, Eve will receive a share worth at least how much? Is this a fair share, in relation to the original problem of dividing the land among the four siblings?

HW9. Jason, Kelly, and Lauren use the Lone Chooser method to divide use of a retail space over the course of a year. Jason is the Chooser (the overall Chooser in the Lone Chooser method), and Lauren is the first Divider.

Each runs a different type of business. Their individual preferences are as follows:

Jason values the each month of the year equally.

Kelly values only the months in the first half of the year (and values each one equally); she has no use for the retail space in the second half of the year.

Lauren values each month in the first half of the year **twice as much** as each month in the second half.

Division:

a. Where does Lauren make the first division, i.e. at what point in the year?

b. If we call the two shares into which Lauren has divided the year s_1 and s_2, how much are each of s_1 and s_2 worth to Kelly, as a fraction (or percentage) of what the whole year is worth?

c. Describe Kelly's share (i.e. which share does she select) and Lauren's share (which share is she left with) at the end of the division stage.

Subdivision:

d. Describe the second division which Kelly makes, i.e. describe (in terms of specific months or parts of months) the 3 subshares k_1, k_2, k_3 she creates from her share, and indicate how much each subshare is worth to her. You may describe the value to Kelly in terms of "Kelly points", or as a fraction of the value of the whole year.

e. Describe the second division which Lauren makes, i.e. describe (in terms of specific months or parts of months) the 3 subshares l_1, l_2, l_3 she creates from her share, and indicate how much each subshare is worth to her. You may describe the value to Lauren in terms of "Lauren points", or as a fraction of the value of the whole year.

Selection: Now Jason, the Chooser, returns.

f. In the following table, write down how much each subshare of Kelly's is worth to Jason, and how much each subshare of Lauren's is worth to Jason. You may describe the value to Jason in terms of "Jason points", or as a fraction of the value of the whole year.

Subshare						
value to Jason						

g. Which subshares does Jason select?

h. By filling out the following table, describe the final fair division of the year (indicating which months each player receives), and find the value of each player's final share as a fraction (or percentage) of the value of the entire year:

Player	**Subshares received**	**Description (which months or parts of months)**	**Fraction of value of the entire year**
Jason			
Kelly			
Lauren			

HW10. Suppose that Jane, Kyle, and Lisa have just finished using the Lone Chooser method to divide a piece of land they have inherited. Each player's valuation of the original piece of land, and of his/her own share at the end of the Lone Chooser method, is as follows:

	Total value of land	Value of share received in the Lone Chooser method
Jane	\$600,000	\$360,000
Kyle	\$750,000	\$270,000
Lisa	\$750,000	\$250,000

Now a long lost relative, Mark, makes an appearance, and demonstrates that he has the same legal claim to a fair share of the property as the others do. He values the entire plot of land at \$800,000. They decide that rather than start over, they will continue to the next step of the Lone Chooser method (now for 4 players), in which Mark is the Chooser.

a. Describe in words what each of the original 3 players now does with his/her share, and what Mark then does.

b. At the end of this process, all four players will have shares. For each of the original 3 players, how much in dollars will his/her share be worth, at the end of the process?

c. Will the shares received by the original 3 players at the end of the process still be fair shares? Explain briefly.

d. At least how much, in dollars will Mark's share be worth?

III. Fair Distribution

HW1. Wendy and Xavier are siblings who have inherited a house from their parents. They are not willing to share the house (growing up together was enough), so one or the other will receive it when their parents' estate is settled. This is the only item which they have inherited, and they apply the Method of Sealed Bids to obtain a fair distribution. Wendy considers the house worth $300,000, but Xavier considers it worth $350,000.

a. Find the value of each player's minimum fair share, in dollars.

b. Carry out the allocation of the items (of which there is only one), and the first settlement.

First settlement:

	Items received (if any), and their value	$ initially paid to bank account	$ initially received from bank account
Wendy			
Xavier			

c. Find the surplus after the first settlement is over, and how much of the surplus each player receives.

d. Carry out the final settlement (for each player, indicate the items received and net amount of cash paid or received as of the end of the process).

Final settlement:

	Items received (if any), and their value	net $ paid to bank account after the surplus is distributed	net $ received from bank account after the surplus is distributed
Wendy			
Xavier			

HW2. Larry, Curly, and Mo dissolve their business partnership and must carry out a fair distribution of 5 valuable items which they own jointly. They use the Method of Sealed Bids to make the distribution, and write out their bids as follows:

	Painting	office	desk	computer	company car
Larry	$20,000	$230,000	$4,000	$3,000	$28,000
Curly	$15,000	$250,000	$3,500	$2,300	$29,200
Mo	$16,000	$260,000	$3,800	$3,200	$32,000

a. Find the value of each player's minimum fair share, in dollars.

b. Carry out the allocation of the items, and the first settlement.

First settlement:

	Items received (if any), and their value	$ initially paid to bank account	$ initially received from bank account
Larry			
Curly			
Mo			

c. Find the surplus after the first settlement is over, and how much of the surplus each player receives.

d. Carry out the final settlement (for each player, indicate the items received and net amount of cash paid or received as of the end of the process).

Final settlement:

	Items received (if any), and their value	net $ paid to bank account after the surplus is distributed	net $ received from bank account after the surplus is distributed
Larry			
Curly			
Mo			

e. Suppose that after the first settlement, the banker runs off with all the surplus money. Do the players still end up with fair shares in spite of not receiving the surplus money? Explain briefly.

HW3. Ed, Faith, and Greg must distribute 3 oranges (O), 4 pears (P), and 5 apples (A) among themselves. Since they are doing this before a trip, they don't wish to slice up individual pieces of fruit, but instead wish each piece to go to one person. Their individual preferences are as follows:

- Ed likes all types of fruit equally
- Faith likes only Apples and Pears, not Oranges
- Greg likes Oranges twice as much as Pears, and likes Pears and Apples equally

The items are arranged in the following order. Place each player's markers. You do not need to find which share goes to which player! Placing the markers is all you need to do.

O A O P A A A P A O P P

HW4. Ann (A), Bob (B), Claire (C), and Dave (D) use the method of markers to divide a collection of 15 albums, labeled by letters L through Z. The arrangement of the albums, and the placement of the players' markers, is as follows:

			B_1		A_1 D_1		C_1		B_2		D_2		A_2		C_2				A_3				B_3		C_3		D_3	
L		M		N		O		P		Q		R		S		T		U		V		W		X		Y		Z

Which items does each player receive? Which items are leftovers? Explain your reasoning.

HW5. Claire, Dave, Eve, and Fred use the Method of Markers to carry out a fair distribution of a set of 16 CDs. There are 6 Black Eyed Peas CDs (denoted B below), 6 Linkin Park CDs (denoted L below), and 4 Taylor Swift CDs (denoted T below), arranged as follows:

B B L L L T T L B B B B L L T T

- Claire likes all three artists equally well.
- Dave likes the Black Eyed Peas and Linkin Park equally well, but can't stand Taylor Swift.
- Eve likes Taylor Swift twice as much as the Black Eyed Peas, and like the Black Eyed Peas and Linkin Park equally well.
- Fred likes only Taylor Swift, and can't stand the Black Eyed Peas or Linkin Park.

a. Carry out the placement of each player's markers. (Use C_1, C_2, etc. for Claire's markers, …)

b. Carry out the allocation of CDs to each player, and indicate which CDs are left over.

HW6. This problem explores what happens if you try to "game the system" in the Method of Sealed Bids. Recall Wendy and Xavier from problem HW1, who are applying the Method of Sealed Bids to the house they have just inherited. According to their honest valuations of the house, Wendy considers it worth $300,000, while Xavier considers it worth $350,000.

a. Suppose that Wendy knows Xavier's valuation of the house, and writes down $340,000 as her bid (even through the house is actually worth $300,000 to her). Xavier still writes down $350,000 as his bid.

   When they carry out the Method of Sealed Bids, how much is the total value of Wendy's share, and how does it compare to what she receives when she bids honestly as in HW1?

   How much is the total value of Xavier's share here, and is it a fair share?

b. Now suppose that Wendy and Xavier both try to game the system. Wendy writes down $340,000 as her bid, while Xavier writes down $310,000 as his bid (even through the house is actually worth $350,000 to him).

   When they carry out the Method of Sealed Bids, how much is the total value of Wendy's share, and is it a fair share?

   How much is the total value of Xavier's share, and is it a fair share?

c. What appears to be the moral of the story in this problem?

Appendix C

Pretest

1. A large university has four bus routes, called W, X, Y, and Z. The university has 100 buses, and must apportion these among the routes in proportion to the daily number of passengers on each route. The total number of bus passengers each day is 50,000.

1a. Which are the seats, which are the states, and what plays the role of population in this apportionment problem?

seats:__________________states:____________________population: ______________________

1b. Find the standard divisor for this apportionment problem, and explain the meaning of the standard divisor in this context.

1c. Suppose we try to use 480 as a modified divisor in Jefferson's method. Find all the modified quotas:

bus route	Passengers per day	modified quota	Jefferson apportionment of seats
W	20,800		
X	13,400		
Y	9,100		
Z	6,700		
Total			

Does this modified divisor work for Jefferson's method, and if so, what is the apportionment of buses to routes (indicate the apportionment in the table)? If not, should the modified divisor we try next be greater than 480 or less than 480? Explain.

1d. If we were using the Huntington-Hill method to apportion the buses, and we tried using the modified divisor 511, how many buses would be given to route X based on this modified divisor?

[You don't have to work out the full apportionment, just the number of buses which would be apportioned to route X based on the modified divisor 511.]

bus route	Passengers per day	Modified quota	Huntington-Hill rounding cutoff		Huntington-Hill apportionment
X	13,400				

Show enough work to demonstrate that you understand how to apply the method.

2. Consider a small country with a population of 16,500 people and three states, D, E, and F.

State	Population				Hamilton Apportionment
D	814				
E	7,403				
F	8,283				
Total	16,500				

There are 150 seats in the legislature that must be apportioned among the three states by their population.

2a. Use Hamilton's method to apportion the 150 seats to the three states. Use as many of the blank columns as you need to show your work.

2b. Suppose an additional seat is added to the legislature. Use Hamilton's method to apportion the 151 total seats to the three states.

State	Population				Hamilton Apportionment
D	814				
E	7,403				
F	8,283				
Total	16,500				

2c. What flaws, if any, do the results of **2a** and **2b** indicate about Hamilton's method of apportionment? Please explain your reasoning.

3. Jenna, Kendra, and Leanne use the Lone Chooser method to divide the use of a vacation timeshare over the course of a year. Leanne is the divider, Kendra is the first chooser, and Jenna is the lone chooser. Each has a different preference of when the timeshare is useful for vacationing. Their individual preferences are as follows:

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Jenna	1	1	1	1	1	1	1	1	1	1	1	1	12
Kendra	1	1	1	1	1	1	0	0	0	0	0	0	6
Leanne	1	1	1	1	1	1	2	2	2	2	2	2	18

Division:
3a. Where does Leanne make her first division?

3b. Describe Kendra's share (i.e. which months does she receive) and Leanne's share (which months does she receive) at the end of the division stage.

Subdivision:
3c. How does Kendra make her subdivision? (Indicate which months are in each of her subshares).

3d. How does Leanne make her subdivision? (Indicate which months are in each of her subshares).

3e. Now Jenna enters as the lone chooser. Which subshares does she select? (Indicate which months are in each of her subshares).

3f. By filling out the following table, describe the final fair division of the year (indicating which months each player receives), and find the value of each player's final share as a fraction (or percentage) of the value of the entire year:

Player	Description (which months or parts of months)	Fraction of value of the entire year
Jenna		
Kendra		
Leanne		

4. Alex, Blair, and Chris have equal claims to a store location, and are using the Lone Divider method to find a fair division of access to the location over the calendar year. We assume as usual that January 1 has to be one of the start/end dates when dividing the year into shares.

- Alex values each of the months April through December twice as much as each of the months January through March.
- Blair values each of July and August four times as much as each of the other months of the year.
- Chris values each of the months from January through June 1.5 times as much as each of the months from July through December.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Alex													
Blair													
Chris													

4a. Suppose that Blair is the Divider. What are the shares which Blair produces, described in terms of which months (or parts of months) are contained in each share?

4b. Does a standoff occur here? Whether there is a standoff or not, find a fair division of the year, i.e. indicate which months (or parts of months) each player receives in the end.

5. As part of a last will, a grandfather leaves behind a house, a car, and a wedding ban to his three grandchildren: Sara, Simone, and Cheryl. They use the Method of Sealed Bids to distribute the assets, and write out their bids as follows:

	house	car	wedding ban
Sara	$150,000	$6,000	$3,000
Simone	$170,000	$8,000	$5,000
Cheryl	$125,000	$10,000	$3,000

5a. Find each player's minimum fair share amount.

	Total value of goods	Min. fair share amount
Sara		
Simone		
Cheryl		

5b. Describe the first settlement (for each player, indicate the items received and amount of cash initially paid or received as of this first stage).

	Items received	$ initially paid by player	$ initially received by player
Sara			
Simone			
Cheryl			

5c. Describe the final settlement (for each player, indicate the items received and net amount of cash paid or received as of the end of the process).

	Items received	Net $ paid by player	Net $ received by player
Sara			
Simone			
Cheryl			

6. Jason, Keith, and Clark have are dividing a set of 15 toys – 3 fidget spinners (F), 6 pogs (P), and 6 containers of gak (G) using the Method of Markers.

- Jason loves fidget spinners but hates pogs and gak.
- Keith loves fidget spinners and pogs equally well but hates gak.
- Clark loves pogs and gak equally well but hates fidget spinners.

The toys are lined up in an array as follows:

G G G P P P F F F P P P G G G

6a. Place markers for each person above based on their value systems.

6b. Describe the allocation of toys to each player and describe what toys are left over.

6c. Suppose that the players agree that each one can pick an extra toy from the leftovers. Suppose Jason picks first, Clark picks second, and Keith picks third. Describe which leftover toys each one would pick.

Appendix D

Posttest

1. A large university has four bus routes, called W, X, Y, and Z. The university has 100 buses, and must apportion these among the routes in proportion to the daily number of passengers on each route. The total number of bus passengers each day is 50,000.

1a. Which are the seats, which are the states, and what plays the role of population in this apportionment problem?

seats:__________________states:____________________population: ______________________

1b. Find the standard divisor for this apportionment problem, and explain the meaning of the standard divisor in this context.

1c. Suppose we try to use 480 as a modified divisor in Jefferson's method. Find all the modified quotas:

bus route	Passengers per day	modified quota	Jefferson apportionment of seats
W	20,800		
X	13,400		
Y	9,100		
Z	6,700		
Total			

Does this modified divisor work for Jefferson's method, and if so, what is the apportionment of buses to routes (indicate the apportionment in the table)? If not, should the modified divisor we try next be greater than 480 or less than 480? Explain.

1d. If we were using the Huntington-Hill method to apportion the buses, and we tried using the modified divisor 511, how many buses would be given to route X based on this modified divisor?

[You don't have to work out the full apportionment, just the number of buses which would be apportioned to route X based on the modified divisor 511.]

bus route	Passengers per day	Modified quota	Huntington-Hill rounding cutoff		Huntington-Hill apportionment
X	13,400				

Show enough work to demonstrate that you understand how to apply the method.

2. Consider a small country with a population of 16,500 people and three states, D, E, and F.

State	Population				Hamilton Apportionment
D	814				
E	7,403				
F	8,283				
Total	16,500				

There are 150 seats in the legislature that must be apportioned among the three states by their population.

2a. Use Hamilton's method to apportion the 150 seats to the three states. Use as many of the blank columns as you need to show your work.

2b. Suppose an additional seat is added to the legislature. Use Hamilton's method to apportion the 151 total seats to the three states.

State	Population				Hamilton Apportionment
D	814				
E	7,403				
F	8,283				
Total	16,500				

2c. What flaws, if any, do the results of **2a** and **2b** indicate about Hamilton's method of apportionment? Please explain your reasoning.

3. Jenna, Kendra, and Leanne use the Lone Chooser method to divide the use of a vacation timeshare over the course of a year. Leanne is the divider, Kendra is the first chooser, and Jenna is the lone chooser. Each has a different preference of when the timeshare is useful for vacationing. Their individual preferences are as follows:

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Jenna	1	1	1	1	1	1	1	1	1	1	1	1	12
Kendra	1	1	1	1	1	1	0	0	0	0	0	0	6
Leanne	1	1	1	1	1	1	2	2	2	2	2	2	18

Division:
3a. Where does Leanne make her first division?

3b. Describe Kendra's share (i.e. which months does she receive) and Leanne's share (which months does she receive) at the end of the division stage.

Subdivision:

3c. How does Kendra make her subdivision? (Indicate which months are in each of her subshares).

3d. How does Leanne make her subdivision? (Indicate which months are in each of her subshares).

3e. Now Jenna enters as the lone chooser. Which subshares does she select? (Indicate which months are in each of her subshares).

3f. By filling out the following table, describe the final fair division of the year (indicating which months each player receives), and find the value of each player's final share as a fraction (or percentage) of the value of the entire year:

Player	Description (which months or parts of months)	Fraction of value of the entire year
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4. Alex, Blair, and Chris have equal claims to a store location, and are using the Lone Divider method to find a fair division of access to the location over the calendar year. We assume as usual that January 1 has to be one of the start/end dates when dividing the year into shares.

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- Chris values each of the months from January through June 1.5 times as much as each of the months from July through December.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Total
Alex													
Blair													
Chris													

4a. Suppose that Blair is the Divider. What are the shares which Blair produces, described in terms of which months (or parts of months) are contained in each share?

4b. Does a standoff occur here? Whether there is a standoff or not, find a fair division of the year, i.e. indicate which months (or parts of months) each player receives in the end.

5. As part of a last will, a grandfather leaves behind a house, a car, and a wedding ban to his three grandchildren: Sara, Simone, and Cheryl. They use the Method of Sealed Bids to distribute the assets, and write out their bids as follows:

	house	car	wedding ban
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5a. Find each player's minimum fair share amount.

	Total value of goods	Min. fair share amount
Sara		
Simone		
Cheryl		

5b. Describe the first settlement (for each player, indicate the items received and amount of cash initially paid or received as of this first stage).

	Items received	$ initially paid by player	$ initially received by player
Sara			
Simone			
Cheryl			

5c. Describe the final settlement (for each player, indicate the items received and net amount of cash paid or received as of the end of the process).

	Items received	Net $ paid by player	Net $ received by player
Sara			
Simone			
Cheryl			

6. Jason, Keith, and Clark have are dividing a set of 15 toys – 3 fidget spinners (F), 6 pogs (P), and 6 containers of gak (G) using the Method of Markers.

- Jason loves fidget spinners but hates pogs and gak.
- Keith loves fidget spinners and pogs equally well but hates gak.
- Clark loves pogs and gak equally well but hates fidget spinners.

The toys are lined up in an array as follows:

G G G P P P F F F P P P G G G

6a. Place markers for each person above based on their value systems.

6b. Describe the allocation of toys to each player and describe what toys are left over.

6c. Suppose that the players agree that each one can pick an extra toy from the leftovers. Suppose Jason picks first, Clark picks second, and Keith picks third. Describe which leftover toys each one would pick.

7. (Conceptual understanding question) Every fair division method has as built-in disincentive for dishonest play. This exercise illustrates the disincentive for dishonest bidding in the lone-divider method. Four partners (Burly, Curly, Greedy, and Dandy) are dividing a million dollar property using the lone-divider method. Using a map, Dandy divides the property into four parcels, s_1, s_2, s_3, and s_4. The table below shows the value of the four parcels in the eyes of each partner.

	s_1	s_2	s_3	s_4
Dandy	\$250,000	\$250,000	\$250,000	\$250,000
Burly	\$400,000	\$200,000	\$200,000	\$200,000
Curly	\$280,000	\$320,000	\$200,000	\$200,000
Greedy	\$320,000	\$280,000	\$280,000	\$120,000

7a. Describe the outcome of the fair division assuming that all players make honest bids.

7b. Suppose that Burly and Curly both bid honestly, but Greedy decides to cheat and bid for only $\mathbf{s_1}$ (figuring that he will then get that parcel). Under the right set of circumstances, Greedy could end up with a share worth only \$220,000. Describe how this could happen.

8. (Conceptual understanding question) Consider the problem of apportioning M seats between two states, A and B. Let q_A and q_B denote the standard quotas of A and B, respectively, and assume that these quotas have decimal parts that are not equal to 0.5. Explain why in this case

8a. Hamilton's and Webster's methods must give the same apportionment.

8b. the Alabama paradox cannot occur under Hamilton's method.

8c. violations of the quota rule cannot occur under Webster's method.

Appendix E

Survey

1. Have you watched all of the instructional videos for each lesson in the unit?
   ( ) Yes
   ( ) No
2. How many times have you watched the videos?
   ( ) once
   ( ) twice
   ( ) 3 or more times
3. Where do you watch the videos?
   ( ) home
   ( ) bus
   ( ) other
4. What device do you use to access the videos?
   ( ) computer
   ( ) tablet
   ( ) smart phone

Appendix F

Focus Group Interview Protocol

Good evening and welcome to our focus group session. Thanks for taking the time to join me to talk about your experiences in this class. My name is John Kerrigan, from Rutgers University, and I am here to get some information from you about your experiences in a flipped classroom this semester. You were invited because you have participated in a research study in Math 103 this semester.

There are no wrong answers but rather differing points of view. Please feel free to share your point of view even if it differs from what others have said. Keep in mind that I'm just as interested in negative comments as positive comments, and at times the negative comments are the most helpful.

You've probably noticed an electronic device out. I am tape recording the session because I don't want to miss any of your comments. People often say very helpful things in these discussions and I can't write fast enough to get them all down. We will be identifying ourselves by number, as I won't use any names in my reports. You may be assured of complete confidentiality.

Well, let's begin. I've placed number cards on the table in front of you as a way to identify yourself. Let's find out some more about each other by going around the table. Start by saying your number, then tell me your year at Rutgers and your major. As you respond to each question, please be sure to identify yourself by your number and not your name.

(0) Did you collaborate with students in other sections of this course?

(1) Can you describe the learning activities in this past unit?

(2) How do you find the learning activities? Why?

(3) What do you think is most important in your mathematics inquiry process?

(4) What aspects of the course design would you like to see strengthened?

(5) Is there anything else you'd like to tell me about the course design that might not have been addressed by any of the previous questions?

Thank you so much for taking the time to allow me to speak with you about your experiences with the flipped classroom. Your feedback is greatly appreciated.

Appendix G

Classroom Schedule and Question Protocol

Approximately 7:40-7:50 PM: Student-driven review of videos
Students will have the opportunity to submit all of their pre-class questions to me using a Google Form. I will in turn answer their questions prior to their taking the quiz.

Approximately 7:51-8:01 PM: Quiz
Quizzes will be the same for both sections. Students will not receive any assistance from me during the quiz.

Approximately 8:05-8:45: Problem solving in groups of 3-4
I will use the following questioning strategies in class when students ask for help in both treatment conditions:

	Tuesday Class	**Monday Class**
When students have difficulty initiating the problem solving process	I will first direct students to collaborate with a peer on the problem. If this has already happened, I will have students refer back to their video notes, sample problems, and textbook.	I will first direct students to collaborate with a peer on the problem. If this has already happened, I will have students refer back to their video notes, sample problems, **invention task work**, **invention task video solution**, and textbook**. I will also encourage them to reflect back on their productive failure experience and what kinds of prior knowledge they could retrieve to solve the problem.**
When students have made some progress but ask for help arriving at a solution	I will challenge myself to create a student-led discussion where I will not repeat, rephrase, or validate what students say. Examples: “Can anyone build on that?” “Thoughts?” “Do we agree or disagree?”. This might also involve highlighting critical features of students’ work (Kapur & Bielaczyc, 2012), where I will point out aspects of students’ **in-class** work that are important to finding the solution. For example, this could include the standard divisor in an	I will challenge myself to create a student-led discussion where I will not repeat, rephrase, or validate what students say. Examples: “Can anyone build on that?” “Thoughts?” “Do we agree or disagree?”. This might also involve highlighting critical features of students’ work (Kapur & Bielaczyc, 2012), where I will point out aspects of students’ **in-class and invention task** work that are important to finding the solution. For example, this could

	apportionment problem and the fair share in a fair division problem. It will be up to the student to figure out how to use the critical features.	include the standard divisor in an apportionment problem and the fair share in a fair division problem. It will be up to the student to figure out how to use the critical features.
When students need specific factual information or need help executing a process	I will refer students to course materials, such as the open source textbook, **previously solved problems**, and video sets. Examples of statements will include, “Re-read this paragraph together,” “Refer back to the mixed review problem in your textbook,” “Refer back to the problem you previously solved,” and “Re-watch the video on the skill at hand.”	I will refer students to course materials, such as the open source textbook, **invention task,** and video sets. Examples of statements will include, “Re-read this paragraph together,” “Refer back to the mixed review problem in your textbook,” **“Refer back to the invention task problem and its solution,”** and “Re-watch the video on the skill at hand.”
When students ask for confirmation on their proposed solutions	I will push students for explanation. Examples include, “Why?” “How do you know?” and “Explain that to me.”	I will push students for explanation. Examples include, “Why?” “How do you know?” and “Explain that to me.”

Approximately 8:45-9:00 PM: Additional problems

- The control group will continue working on the assignment, and I will continue using the same strategies described above.
- The experimental group will work on the invention task during the four weeks of the study. I will not provide any assistance to students during this time.

Appendix H

Field Notes

Week #5

CONTROL: needed help with setting up ratios. Worked to scaffold responses and turn questions back to students.

EXPERIMENTAL: needed more help with vocabulary (what does standard divisor mean, what does standard quota mean, etc.). Seemed more comfortable with performing calculations related to standard divisor, standard quota, etc. This group had more questions before the quiz.

Week #6

CONTROL: issues with rounding based on H-H (up or down), when to modify a divisior, how using an MD changes other parts of the problem, more students pulling up videos to refer and looking back at notes. Issues explaining what final answers mean in context. For divider-chooser: concept of mathematically fair not crystal clear. Adjustment for next class: list of critical feature summary to begin the class session.

EXPERIMENTAL: Issues with understanding what the modified divisor does and what it represents, how to compare a quota to a rounding cutoff, students did not have clear knowledge of the process for each different method, heavily relying on notes, not clear on what the apportionment represents, trouble connecting different methods.

Week #7

CONTROL: issues with combining into a supershare and dividing so that it is a fair share, more issues with lone chooser than lone divider potentially because there are more steps involved, issues with grasping the idea of having more players so you divide by n amount of players, not just by 2...Difficulty with supershare divider/chooser and fair share for people

EXPERIMENTAL: Confusion about the methods and how to distinguish which ones to use. This week there were less questions about how to execute each method once they knew which one to use. Group work was better. Less questions about the getting the right answer, more about making sure they were doing the correct steps. Some questions on supershare division.

Week #8

CONTROL- problems with allocating the surplus (when to add it to fair share and when to subtract), some minor confusions with adding the total amounts together and dividing by the number of players…. Subtracting order for earned minus paid.

EXPERIMENTAL - fair share dividing by # of items not number of players, reviewed invention task solution and how fair share there was based on number of players.