MATHEMATICAL SOPHISTICATION AND CONCEPTUAL UNDERSTANDING IN ASTROPHYSICS: IS THERE A LINK?

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ABSTRACT OF THE DISSERTATION

Mathematical Sophistication and Conceptual Understanding in Astrophysics: Is There a Link?

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The purpose of this study was to examine how upper level students in astrophysics connect mathematical equations to concepts. Only a few studies in physics education research (PER) have investigated connections between student understanding of physics equations with physics concepts and none of those were in the field of astrophysics. As in many upper level physics classrooms, problem solving is a main part of astrophysics education. In upper level astrophysics classrooms, learning physics is about learning the ways physics describes, explains and predicts behavior of celestial objects by building mathematical models. This study evolved from a desire to improve student's conceptual understanding in an upper level physics course, which is highly mathematical in nature. The broad scope of this research is to understand how the students connect astrophysics equations to astrophysics concepts.

This study presents a systematic examination on how students who were enrolled in an upper level astrophysics class at Rutgers University understand astrophysics equations using the framework proposed by Domert et al. (2012) as well as how they frame their mathematical use of equations based on examining the symbolic forms of their mathematical arguments (the framework of Sherin, 2001). A symbolic form, according to Sherin, is composed of two components: a conceptual schema - the idea to be expressed in the equation - and a symbol template - how the idea is written in symbols (Sherin, 2001).

The majority of participants in this study were selected from the first of a two-semester sequence called Principles of Astrophysics (additional participants are experts in the field of astrophysics). The data for this dissertation include multiple homework assignments, two exams, a final essay, and video recordings of interviews of astrophysics students as well as experts working on solving problems involving gravitational potential energy and the virial theorem.

Through the systematic examination of the collected data I was able to determine how students connect mathematical equations to concepts within the framework of Domert et al. (understanding of physics equations) and Sherin (symbolic forms). I found that most upper level undergraduate students in astrophysics have the potential to make meaningful connections between concepts and equations but need more purposeful instruction in order to make these connections.

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Chapter 1: Introduction (Purpose and Significance)

1.1 Introduction

In recent decades, a relatively new field of Physics Education Research (PER) has investigated students learning in physics courses. Studies conducted in PER focus on a wide range of topics including conceptual change, epistemological development, problem-solving skills, and development of science practices. Specific content-related topics investigated in PER include a broad range of studies in kinematics, dynamics, electricity and magnetism, light and optics, waves and sound, modern physics, etc. The common purpose of these studies is to determine how students learn physics and how to make physics teaching more effective. We expect students in physics courses to gain an understanding of many physics concepts, to solve a range of physics problems, and to start thinking like physicists.

In a review article McDermott and Redish reported on the many published studies in physics education research (McDermott & Redish, 1999). This article classified these studies as “problem solving”, “the effectiveness of laboratory instruction and lecture demonstrations”, “the ability to apply mathematics in physics”, “students’ attitudes and beliefs”, and a discussion of research in “reasoning in physics.” This article showed that the research on students’ understanding of concepts in physics has focused about 49% on mechanics, followed by 17% of the studies focusing on “electricity and magnetism”. Research on student understanding of “light and optics” and “properties of matter, fluid mechanics, and thermal physics” received almost the same percentage, 13% and 12% respectively. Only about 4% of the research was devoted to “waves and sound”, and concepts in “modern physics” received the most limited attention in the literature with 1%. Since this review article, more research has been made in physics education research; particularly in the areas of “modern physics” (e.g.: Drefus et al., 2017, Modir, Thompson, &

Sayre, 2017, Sayer, Maries, & Singh, 2017, Wilcox & Pollock, 2015) and physics in the laboratory (e.g.: Stanley, Su, & Lewandowski, 2017, Wilcox & Lewandoski, 2017, Wilcox & Lewandoski, 2016, Zwickl, Finkelstein, & Lewandoski, 2014, Zwickl et al., 2013) but there remains no research in physics education for the upper-level astrophysics classrooms.

In addition to many of the studies included in the review article by McDermott and Redish, a more recent study conducted by David Meltzer in physics education research appears to show that mathematical ability is linked to achievement in traditional physics courses that stress quantitative problem solving (McDermott & Redish, 1999, Meltzer, 2002). However, it has also been shown that students have difficulty in developing a good understanding of fundamental physical concepts in their introductory calculus-based physics classes (McDermott, 1995). These students also have difficulty relating these fundamental concepts to the mathematics they have learned in math courses (Steinberg, Wittmann, & Redish, 1997, Brahmia, 2014). It is not sufficient therefore for students to be able to simply “get the correct answer” to a problem, as educators we want students to have the understanding of the concepts that are the inspiration of the particular problem they are attempting to solve. In upper level classes in which a significant escalation in mathematical rigor and complexity is expected, these issues become particularly important. As educators, we do not want meaningless problem solving manipulation; if a student uses mathematical manipulation of equations, we want them to use the equations with understanding.

As in many upper level physics classrooms, problem solving is a main part of astrophysics education. In these courses learning physics is about learning the ways physics describes, explains and predicts behavior of celestial objects by building mathematical models. Although enhancing students’ problem solving ability is a goal of physics teaching, many students solve astrophysics problems using mathematical problem solving strategies and do not understand the astrophysical concepts behind their mathematical manipulations; they have less trouble with the mathematical part of the problem than they do with the conceptual part (Rave, Etkina, Gawiser, & Jha, 2012).

Identifying when students use mathematical problem solving strategies rather than using conceptual understanding can be problematic because teachers may accept a correct numerical answer without examining students’ qualitative understanding of the related concepts. If this occurs, then students who generate the correct numerical answer may be thought to have an understanding of the underlying concepts. In many respects, teachers may find it easier to teach problem solving strategies and formulas, neglecting the conceptual knowledge, and they encourage students to enhance their problem solving skills. For example, students may be capable of solving problems that involve using equations to predict the properties of stars under a variety of conditions; however, their conceptual understanding falls behind this mathematical understanding. Students’ levels of conceptual understanding have a significant effect on their ability to identify ideas more quickly and clearly and to solve problems by understanding them (Bransford, Brown, & Rodney, 2000).

The purpose of this study is to examine how upper level students in astrophysics connect mathematical equations to concepts. This study has evolved from a desire to improve student's conceptual understanding in an upper level physics course (Rave, Etkina, Gawiser, & Jha, 2012).

1.2 Motivation for the Study

Rutgers University offers a two-semester sequence called Principles of Astrophysics, which is designed to provide a junior-level introduction to the concepts of astrophysics for undergraduate students majoring in physics. One specific goal is to give students exposure to GRE physics like questions, at a relatively high level of difficulty. Many engineering students also take one or both of these classes as a science elective, resulting in roughly 60 students per semester. These courses focus on understanding gravity and on stellar structure and evolution.

Traditionally students in the course had been very successful solving quantitative problems on take home homework sets. For instance, in the Spring 2010 semester the average homework score was a 78% (31 out of 40 with a std. dev. of 6.3). However, when in the Spring 2010 semester the students took a midterm exam consisting of 15 conceptual multiple-choice questions, the student average was 52% (with a std. dev. of 2.5).

An explanation for this discrepancy in student performance seemed clear: for the quantitative problems in the homework sets, students could refer back to their lecture notes, identify relevant formulae, and apply them; but conceptual issues were not emphasized in lectures. To address this issue the course was revised in the spring of 2011 by adding approximately 1-2 conceptual questions to each lecture which the students answered working in small groups and reported their answers with colored cards labeled A, B, C or D. (These in class conceptual questions were changed to i-clicker questions in subsequent years.) The same conceptual exam was given to these students as the year before and the new mean was found to be 60% (with a std. dev. of 2.2). Although it was a positive shift, which was even statistically significant (p_value = 0.000148), the change was unsatisfactory.

To improve student performance on the conceptual exam, two astrophysics professors (EG and SJ) teamed up with the faculty (EE) and a graduate student (HR) from the Graduate School of Education to design the intervention that was carried out in the spring of 2012. While the instructional goal of the intervention was to improve student qualitative understanding of astrophysics concepts, the research goal was to investigate whether the matching of the context of student learning with assessment and increasing the richness of representation would result in better learning outcomes. To accomplish this, we incorporated qualitative questions into the homework assignments that were previously completely quantitative. I took care to ensure that the questions were posed in multiple formats. There were three homework assignments due before the exam. The in-class conceptual questions were unchanged with 10 of the 15 exam questions being identical (or nearly identical) to the in-class conceptual questions.

We found that adding multiple-representations based conceptual questions to the homework assignments improves student performance on a conceptual exam. Examining student responses to the midterm conceptual exam showed us that students demonstrate greater conceptual understanding when they are exposed to more instances of qualitative reasoning.

Although conceptual segments were added to both lecture and homework assignments, the class remained highly mathematical in nature. Also, the midterm conceptual exam reflected only half of the topics taught in the class and was comprised completely of multiple-choice questions. While this exam lends evidence of conceptual understanding, it does not show whether the students connected a conceptual understanding of the topics taught in the class to the mathematical problem solving involved. This preliminary study therefore motivated the focus of this dissertation. In addition, the question of how students connect conceptual understanding to mathematics is of a general interest for the field of physics education research. The work of Bruce Sherin that showed that “successful students learn to understand what equations say in a fundamental sense; they have a feel for expressions, and this guides their work” was focused on introductory students (Sherin, 2001). In addition, Sherin showed that these students learn to understand physics equations in terms of a “vocabulary of elements” that he calls symbolic forms. In PER several groups study how students connect physics understanding to mathematics, the main one is led by John Thompson at the University of Maine (Christensen and Thompson, 2010, Smith, Thompson, and Mountcastle, 2010). The work presented here is different from theirs in that, in contrast to Sherin, I focused this research on upper level students and, in contrast to Thompson (whose work involves connecting the understanding of physics in fields of thermodynamics, statistical mechanics, and quantum mechanics to mathematics), my work addresses a void in this research by focusing on the field of astrophysics.

1.3 Organization of the Dissertation

There are several chapters to this dissertation. The first chapter introduces the study and develops the purpose and significance for the goal of the dissertation: determining how upper level students can connect equations to concepts when the course is mainly mathematical in nature. Chapter 2 reviews literature that is relevant to physics conceptual understanding as related to mathematics as well as literature that describes a foundation for the conceptual framework for the study. Chapter 2 therefore establishes the place of the dissertation's work within the larger body of physics education research. Relevant results of prior research in physics and mathematics education research, as well as specific ways that several researchers frame their results, are discussed in more detail in this literature review.

Chapter 3 presents the research methodology that was used to investigate how the students' conceptual understanding of astrophysics is connected to mathematics. Thus, this chapter focuses on describing the data collection procedures. The dissertation's data came from various sources including: homework assignments, two conceptual exams, interview video recordings of astrophysics students and experts solving astrophysics problems, student's responses to a Likert survey, and a final essay.

Chapter 4 presents and discusses the findings of the study while an interpretation of the findings is presented in Chapter 5. Chapter 5 also includes the answers to the research questions posed below. Lastly, the discussion of the findings and implications for instruction which arise from the study are detailed in Chapter 6.

1.4 Research Questions

In order to achieve the goal of this study, to identify how upper level students in astrophysics connect mathematical equations to concepts, the research questions that were addressed in this study are:

1. What do the students think it means to understand astrophysics equations?
2. What does student qualitative understanding of an equation look like?
3. How do student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?
4. What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation?

Chapter 2: Literature Review

Over the past 20 to 30 years, Physics Education Research (PER) has shown strong evidence that students do not learn much from traditionally taught lecture courses in physics (e.g.: Von Korff, et al., 2016, Cummings, et al., 1999, Redish & Steinberg, 1999, McDermott & Redish, 1999, Hake, 1998). PER seeks to understand how students learn physics and identify student difficulties in understanding physics. One of the directions in the field of PER (among many) is to develop research-based approaches to curriculum design that will help students understand physics conceptually. The research conducted has been successful in determining some of the fundamental problems students have with common physics concepts (e.g.: Clement, 1982, Van Hise, 1988, and Galili & Bar, 1992). Furthermore, with the development of both instructional materials and the development of curricula, PER has been able to suggest constructive ways to overcome some of these difficulties (McDermott, Shaffer, et al., 1998, Douglas et al., 2014, Dounas-Frazer & Reinholz, 2015, Guisasola et al., 2017, Leinonen, Asikainen, & Hirvonen, 2017, McPadden & Brewe, 2017, Scott & Schumayer, 2017, Wilcox et al., 2015, Wilcox & Lewandowski, 2016).

In general, one can distinguish two broad directions in physics education research. The first focuses on student learning of physics (it includes conceptual understanding, epistemological development, role of language, and many others) and the second one focuses on the development of instructional instruments (for both learning and assessment). The examples of the former are research papers produced by the University of Washington PER group (e.g.: Hazelton, Stetzer, Heron, & Shaffer, 2013, Stephanik & Shaffer, 2011, Close & Heron, 2011), the work of David Meltzer (e.g.: Meltzer, 2004, Meltzer, 2005, Meltzer & Thornton, 2012) and many others. The examples of the latter are University of Maryland tutorials (e.g.: Elby, 2001, Redish and Steinberg, 1999), Overview Case study physics and ALPS curricula developed by A. Van

Heuvelen (e.g.: Van Heuvelen, 1991a, Van Heuvelen, 1991b); Investigative Science Learning Environment (ISLE, Etkina and Van Heuvelen 2007, Etkina, 2015) and University of Colorado, Boulder PHET computer simulations (Perkins et al., 2006) and concept inventories, such as the Force Concept Inventory (FCI) (Hestenes, Wells, & Swackhamer, 1992), the Concept of Survey on Electricity and Magnetism (CSEM) (Maloney et al., 2001), the Colorado Learning Attitudes about Science Survey (CLASS) (Adams et al., 2006), and many others. The first direction of physics education research - understanding how students learn - is necessary for developing the research-based material for the second. Although PER research has been extensive regarding these two categories in introductory physics courses, and some work has been done in introductory astronomy (Bailey & Slater, 2003, Wallace, Prather, & Duncan, 2011, Prather et al., 2004) still relatively little PER work has been done on student learning in more advanced courses (e.g.: McDermott & Redish, 1999, Wallace & Chasteen, 2010, Chasteen & Pollock, 2009, Pollock, Chasteen, Dubson, & Perkins, 2010, Zwickl, Finkelstein, & Lewandowski, 2014, Wilcox et al., 2015, Sayer & Singh, 2017) and no work has been done in upper-level astrophysical concepts.

2.1 Students' Perceptions of Understanding Physics

To answer research question #1 (What do the students think it means to understand astrophysics equations?), we need to first examine what it means to understand something and how students perceive what it means to understand. In a paper written by Perkins and Blythe, the authors ask the question “What is understanding?” Their answer is that “understanding is a matter of being able to do a variety of thought demanding things with a topic. Like explaining, finding evidence and examples, generalizing, applying, analogizing, and representing the topic in a new way” (Perkins & Blythe, 1994, p. 5-6). This general description of understanding by Perkins and Blythe can be used when considering studies done in PER pertaining to physics students' understanding; specifically, for the purposes of this dissertation, to understanding physics equations.

In research done by Domert, Airey, Linder, & Kung (2012), the authors "were interested in exploring students' epistemological views of what it means to understand physics equations" and "came up with the notion of a mindset which [they] define as perceived critical attributes of a learning, application, or problem-solving situation that are grounded in epistemology" (Domert et al., 2012, p. 17). The main focus of their study was to answer two fundamental questions of what students' perception of understanding physics equations really is, namely: "When students say that they understand an equation, how do they describe what that means to them, and how can these descriptions be characterized in terms of epistemological mindsets?" and "Are similar epistemological mindsets observable for students at various stages in their academic career?" (Domert et al., 2012, p. 18).

In Domert, Airey, Linder, and Kung's study, twenty physics students (Seven first year undergraduate students, nine upper level undergraduate students and four PhD students) were interviewed using a semi-structured interviewing strategy. The goal of the interviews conducted by the authors was to engage students in a comprehensive discussion to examine the students' views on what it means to understand physics equations. During their data analysis process, the authors subsequently identified what they described as "generic components of epistemological mindsets, where an individual student's epistemological mindset towards the understanding of a physics equation could consist of one or more of these components." (Domert et al., 2012, p. 19). The authors describe five components as: "understanding involves being able to recognize the symbols in the equation in terms of the corresponding physics quantities", "understanding an equation involves being able to recognize the underlying physics of the equation", "understanding involves recognizing the structure of the equation", "understanding involves establishing a link between the equation and everyday life", "understanding involves knowing how to use the equation to solve physics problems", and "understanding involves being able to know when to use the equation" (Domert et al., 2012, p. 19-20).

Domert, Airey, Linder, and Kung conclude their study by stating that "all of the components identified in this study are integral parts needed to have an appropriate understanding of physics equations" (Domert et al., 2012, p. 25). It seems clear that, in an ideal world, the epistemological outlooks of students towards what it means to understand an equation would involve all of these components. This student would be considered to have an appropriate understanding of physics equations. However, we can see that this unfortunately may not be the case for many physics students.

2.2 The Use of Equations

There are a number of studies that have explored students' understanding of equations in physics. In a paper by Redish, Saul and Steinberg (1998), the authors looked at students' expectations of understanding equations as part of a study of students' expectations of physics. In their study, they conclude that many students think that it is the mathematical aspects of an equation that are most important and that many introductory physics students "fail to see the deeper physical relationships present in an equation and instead use the math in a pure arithmetic sense – as a way to calculate numbers" (Redish, Saul & Steinberg, 1998, p. 220). Steinberg, Wittmann and Redish (1997) found students had difficulties when equations involve functions of more than one variable and Rozier and Viennot (1991) identified that students found it hard to explain the relationship between variables in multivariable problems.

Additional studies in the field of mathematics have investigated students' understanding of the symbols and variables in equations. For example, Kieran (1981) has investigated students' interpretations of the equal sign. Kieran concluded that although many students view the equal sign as meaning "do something", it is not evident whether this interpretation is harmful. In another example, Clement, Lochhead and Monk (1981) found that students had difficulties in converting from verbal representations to mathematical representations in terms of algebraic symbols.

In a paper by Sherin (2001), the author creates a framework for students' understanding of physics equations. He does this by examining students' understanding of equations in terms of how these students constructed equations by means of using basic mathematical templates. Sherin argues that, when presented with a physics situation and asked to come up with equations that describe this situation, students make use of various fundamental templates for equations in order to find an equation that describes the situation appropriately. Successful physics students will learn to communicate with a "vocabulary of simple ideas in equations and to read these same ideas out of equations" (Sherin, 2001. p. 482). These successful students consider the conceptual interpretation of a physical situation and express that understanding in an appropriate equation. These students can also look at an equation and understand it as a description of a particular physical system (Sherin, 2001).

2.2.a How Students' Understanding of Physics is Different from Understanding in Mathematics

In order to determine how students understand physics equations, we must first understand how students' understanding of equations in physics is different from understanding in mathematics. In a study by Brahmia, she describes the process of "mathematization" which involves "representing ideas symbolically, defining problems quantitatively, producing solutions, and checking for coherence, all in coordinated service of building understanding of how the world works" (Brahmia, 2014, p. 24). She further states that "to mathematize in physics means to go back and forth between the physical and the symbolic world"; a skill that experts in physics develop and use to communicate ideas. In her dissertation, Brahmia argues that "most students use basic arithmetic and algebraic tools in ways quite different than what university and college physics instructors might expect" (Brahmia, 2014, p. 3).

Physicists use mathematics in a different way than is taught by mathematicians. As Brahmia states, "the mathematics preparation of most students enrolling in introductory physics is limited largely to the rehearsal of algorithmic procedures" (Brahmia, 2014, p. 3). Many students, even those who have done well in their mathematics classes, have internalized the mathematics they learned as a set of algorithms. They then see equations as "recipes for obtaining numerical answers rather than as tools for sense-making" (Brahmia, 2014, p. 44). In another study, Tuminaro examined the mathematical errors that were made by students in an introductory algebra-based Physics course. Tuminaro (Tuminaro, 2004) discovered that the errors were not from a lack of mathematical understanding, but were due to a breakdown when attempting to apply their mathematical knowledge appropriately to physical situations.

Mathematics instruction is "sometimes short-circuited because learners are given procedures to memorize rather than opportunities to develop a proceptual understanding of mathematical notions" (the term "proceptual understanding" integrates procedural mastery and conceptual understanding) (Brahmia, 2014, p. 50). Memorizing and applying mathematical procedures would obviously improve the students' efficiency, but would not help the students mathematical understanding. Furthermore, "struggling students are commonly taught mnemonic devices that bypass stages of mathematical learning. These students are trained to become adept at algorithmic performance" (Brahmia, 2014, p. 52).

Reasoning is an essential skill in physics which "requires conceptual understanding of arithmetic", but the "focus of instruction in [mathematics courses] favors efficiency over conceptual understanding" (Brahmia, 2014, p. 57). The consequence of a procedural mathematical approach to physics is that the students have often had very limited experience with "considering the procedures they have learned in light of their potential for solving specific problems." The students' therefore struggle to understand the equations involved in physics, seeing equations as "just for doing calculations" (Brahmia, 2014, p. 74).

2.3 How Students Understand Physics Equations: Symbolic Forms

How are physics equations understood by physics students? In his paper Sherin shows that "it is possible [for students] to understand equations in a relatively deep manner" (Sherin, 2001, p. 527). Sherin believes that students who are successful in physics "learn to express a moderately large vocabulary of simple ideas in equations to read these same ideas out of equations" (Sherin, 2001, p. 482). and introduces the idea of *symbolic forms* to explain this "vocabulary of elements" in which students learn to understand physics equations. As discussed in more detail below, symbolic forms allow students to "take a conceptual understanding of some physical situation and express that understanding in an equation" (Sherin, 2001, p. 482). An example of a symbolic form is the balancing form. In this symbolic form, two "competing influences" are seen as equal and opposite. Additionally, each of these influences is linked with one side of an equation, as indicated in the symbol template balancing (☐ = ☐; the ☐ refers to a term or group of symbols). Sherin furthermore argues that symbolic forms are an important component of physics knowledge because symbolic forms allow students to interpret existing equations as well as construct new ones. Within this framework, Sherin investigates how students understand physics equations and finds that “it is possible, even for third-semester physics students, to understand equations in a relatively deep manner” (Sherin, 2001, p. 527). Another goal of this paper is to bridge the gap between physics problem solving and naive physics (prior knowledge or intuitive physics).

2.3.a Understanding Physics Equations

In his study, Sherin’s data analysis of videotapes of pairs of students solving physics problems advances the idea that the understanding of physics equations can surpass that of “problem categories” where physics knowledge is associated with the typical circumstances surrounding the use of equations (Chi, Feltovich, & Glaser, 1981). Sherin argues that the details of the equation being used in solving physics problems have meaning for the expert in the arrangements of the symbols it contains and that successful physics students “learn to express a moderately large vocabulary of simple ideas in equations” and that they read physical meaning out of these equations (Sherin, 2001, p. 482). As stated above, these symbolic forms that Sherin introduces for the elements of this vocabulary allow the student to make use of their conceptual understanding of a physics problem and express that understanding in an equation. This description of understanding physics equations is more flexible than in previous research where, for example, other authors argue that “knowledge must be indexed by large numbers of patterns that, on recognition, guide [the successful student] in a fraction of a second to relevant parts of the knowledge store” (Larkin, McDermott, Simon, & Simon, 1980, p. 336).

2.3.b Naïve Physics and P-Prims

Sherin (2001) also argues that there is “an absolutely fundamental connection” between physics problem solving and naïve physics. Naïve physics, or prior knowledge, or intuitive physics, has been studied from several viewpoints. Some research, such as that of Clement (1983) and

McDermott (1984) systematically list the difficulties and “misconceptions” of naïve physics claiming that “detailed knowledge of the problems students have with physics makes passible the design of instruction to meet student needs” (McDermott, 1984, p. 32). Other’s research states that students have their own theories of physics and that “intuitive ideas are difficult to modify” (McCoskey, 1983, p. 72).

As an alternative to the above viewpoints of students’ understanding of physics, diSessa (1993) characterized students’ understanding by description of “knowledge in pieces.” Using this perspective, students’ explanations of the physical world are not understood to be a manifestation of theories or systematic frameworks, but instead are seen as intuitive physics knowledge that diSessa called the *sense-of-mechanism*. According to diSessa, students’ understanding of physical phenomenon is result of the activation of fundamental knowledge elements that diSessa has described as phenomenological primitives (p-prims). diSessa’s p-prims are understood to be knowledge structures that are automatically and unconsciously activated by the student in response to a particular situation. These p-prims are the basis on which the student makes sense of a situation. Thus, the student may construct a number of explanations in response to a single physical phenomenon, based upon the triggered p-prims.

P-prims are a result of the students’ experiences in the world (hence, “phenomenological” primitives). Once p-prims are created on the phenomenological level, they become internalized and develop into the vocabulary later called upon to make sense of later experiences. This sense making process occurs at a very deep cognitive level, which is why the student is for the most part unaware of the source of their understanding. P-prims are fundamental pieces of knowledge that are understood by the student to need no explanation, as they operate as understood assumptions of how the physical phenomenon works. As stated by diSessa, p-prims allow students to understand that “something happens because that’s the way things are” (diSessa, 1993, p. 111-112).

2.3.c Symbolic Forms

The basis of Sherin’s analysis and development of symbolic forms comes from the idea of knowledge analysis research which is exemplified in diSessa’s research on phenomenological primitives (diSessa, 1993). As previously stated, phenomenological primitives, or p-prims, are defined by diSessa as primitive elements of knowledge. “P-prims are rather small knowledge structures, typically involving configurations of only a few parts, that act largely by being recognized in a physical system or in the system’s behavior or hypothesized behavior” (diSessa, 1993, p. 188). diSessa creates a list of 29 individual p-prims and suggest that there are many more. Within these 29 p-prims, diSessa describes “a complex system involving a moderately large number of primitive elements” (as cited in Sherin, 2001, p. 503). Having a “moderately large” number of elements is a feature typical of a knowledge analysis program.

Also relevant to the creation of Sherin’s symbolic forms is the idea that diSessa’s p-prims are activated by relatively simple mechanisms. Sherin (2001) supports the importance of his symbolic forms which are "more specific that the symbolic expressions that they interpret; a symbolic form adds semantics to an equation" with examples from video data of student problem solving.

The participants in Sherin’s study (2001) were students enrolled in a third-semester physics course for engineers - thus at an intermediate level of expertise. The data for this study includes videotapes of five pairs of students solving physics problems while standing at a white board in four to six sessions each. The chosen problems were of intermediate level of difficulty. Analysis of the video data once transcribed, allowed Sherin to create a subset of the total amount of the video data where Sherin observed students using equations with understanding. In the analysis of this data, Sherin hypothesized that "as people develop physics expertise, they acquire knowledge elements" that he calls symbolic forms (Sherin, 2001, p. 490). A symbolic form, according to Sherin, is composed of two components: a conceptual schema (the idea to be expressed in the equation) and a symbol template (how the idea is written in symbols). For example, the symbol template for parts-of-a-whole is two or more terms separated by plus (+) signs ([☐ + ☐ + …]). Sherin defends his idea of symbolic forms using excerpts from the data. From the literature on diSessa’s p-prims and from the analysis of Sherin’s data, Sherin clearly creates a case for these symbolic forms.

Some of the features of the symbolic forms include a moderately large system of elements (about twenty symbolic forms are presented), an intermediate level of abstraction (symbolic forms are “more abstract than a full, rich understanding of the world but less abstract than equations”), ways for a symbolic form to be prompted to an operational level (the mechanism for recalling symbolic forms can be activated by an equation or by the physical situation described), and the genesis of symbolic forms (some come from working with physics equations, some come from early “mathematical experiences”) (Sherin, 2001, p. 504).

Sherin identifies many symbolic forms in the analysis of the video data and illustrates these symbolic forms using brief examples. He then groups these observed symbolic forms, into seven clusters: Competing Terms, Terms are Amounts, Dependence, Coefficient, Multiplication, Proportionality, and Other. The “competing terms” cluster includes forms that relate equations to terms that "conflict and support or that oppose and balance" (Sherin, 2001, p. 506). The “proportionality” cluster includes "seeing individual symbols" as either in the numerator or in the denominator of a ratio (Sherin, 2001, p. 508). In the “terms are amounts” cluster, terms are "treated not as influences but as quantities of generic stuff" (Sherin, 2001, p. 512). The signs in this cluster do not indicate physical directions but as increasing or decreasing the total amount of "stuff". The “dependence” cluster forms include the presence of a specific symbol in the expression. In many cases forms fall into this cluster because the expressions have no dependence in a particular symbol (it does not appear at all in the equation). In the “coefficient” cluster, "a product of factor is seen as broken into two parts" and one of these parts is the coefficient itself which is "distinguished from other forms" (Sherin, 2001, p. 518). The “multiplication” cluster includes two forms: the intensive-extensive form and the extensive-extensive form. (An intensive quantity is a ratio of the amount of a particular quantity to another unit and an extensive quantity is a number of units.) Lastly, the “identity” cluster involves symbolic forms in which a single symbol is separated from an expression by an equal sign. This form is so common it is "nearly invisible in student utterances" (Sherin, 2001, p. 518). Sherin notes in his discussion on symbolic forms that students can and do interpret equations; they are "not just derived and used to obtain numerical results" (Sherin, 2001, p. 510).

In the analysis of his symbolic forms, Sherin admits that his analysis does not explain all aspects of the data and cannot be the entirety of the student's understanding in constructing equations. He understands that "more formal considerations and remembered equations must also play a role" (Sherin, 2001, p. 495). Sherin argues however that the knowledge employed by an expert would include "knowledge of multiple sorts" including knowledge of symbolic forms. He also points out what he calls the "qualitative limits of forms" (Sherin, 2001, p. 495). There are limits to "specificity" in which the forms describe equations. For instance, is the relation linear or quadratic? Forms will only note a relation between two variables and "that is the limit of the specification" (Sherin, 2001, p. 512). This being said, Sherin has clearly established that the students do make use of these symbolic forms with understanding.

2.3.d Forms Framework

After this detailed explanation of symbolic forms, Sherin moves on to discuss two issues: "How can the forms framework inform instruction?" and "Where does knowledge of forms originate?" (Sherin, 2001, p. 519). Firstly, if we can see when students fail to understand equations then these instances may indicate areas for potential learning. As Sherin states, "just being able to recognize and understand the problems to be addressed is important" (Sherin, 2001, p. 522). Secondly, knowing about symbolic forms and pointing them out to the students would potentially be beneficial to instruction. Lastly, to answer the second question, Sherin provides compelling arguments from existing research that prior to the student's physics instruction, the students have learned some mathematical modeling.

Sherin concludes by restating the importance of his study; that his work can be "understood as bridging the gap that has existed between research on naïve physics and research on physics problem solving" (Sherin, 2001, p. 527). Using symbolic forms students were seen to construct their own equations and analyze them, they sometimes use intuitive understanding to construct equations, and construction may be driven by a template-like structure. The forms framework gives researchers a way to talk about student's understanding of equations and the way they interpret them.

For all of the research questions posed in Chapter 1.4, a framework is needed. As Sherin states "to worry about student … qualitative understanding is essentially to worry about the particulars of certain particular models" (Sherin, 2001, p. 523). I believe that to understand the results of my study, I will need to understand the student's (and expert's) responses to solving astrophysics problems using the framework of symbolic forms. In using this “symbolic forms approach”, my study is based, as intended, on understanding astrophysics equations.

2.3.e Negative Gravitational Potential Energy and the Virial Theorem

The focus of the astrophysical interview questions asked as part of this work involve negative gravitational potential energy as well as the virial theorem.

A common expression for gravitational potential energy is derived by integrating Newton's law of gravitational force and is equal to the work done against gravity to bring a mass to a given point in space. Because the gravitational force is an inverse square, the force will approach zero for very large (infinite) distances. It makes sense therefore to choose gravitational potential energy at an infinite distance away to be zero. The gravitational potential energy near a planet is consequently negative, since gravity does positive work as the mass approaches. A negative potential is furthermore suggestive of a "bound state"; once a mass is near a much larger body, it is trapped there (unless something can provide enough energy to allow it to escape).

The expression for the gravitational potential energy of mass m is:

$$U = \int_{\infty}^{r} \frac{GMm}{r^2}\, dr = GMm(\frac{1}{\infty} - \frac{1}{r})$$

$$U = -\frac{GMm}{r}$$

Where G is the gravitational constant, M is the mass of the attracting body, and r is the distance between their centers.

The virial theorem, as used in astrophysics, relates the gravitational potential energy of a system to its kinetic energy as a whole. The virial theorem refers to time averages of the kinetic and potential energy and is generally stated as:

$$\langle K \rangle = -\langle U \rangle / 2$$

where <K> is the time average of the total kinetic energy, and <U> is the time average of the total potential energy.

The simplest case of the virial theorem is used for the purposes of this study, namely a single low mass object (m) in a circular orbit around a much more massive object (M). For an orbit of radius r, the potential energy is (as stated above):

$$U = -\frac{GMm}{r}$$

To get the kinetic energy, we use the gravitational force:

$$F = \frac{GMm}{r^2} = \frac{mv^2}{r}$$

So the kinetic energy is:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} = -\frac{U}{2}$$

Which is the simplest form of the virial theorem.

Chapter 3: Data Sources and Methodology

3.1 Overview

As previously stated, the purpose of this study is to examine how upper level students in astrophysics connect mathematical equations to concepts. Specifically, the study will focus on student understanding of negative gravitational potential energy and the virial theorem. It will present a systematic examination on how students who are taking an upper level astrophysics class at Rutgers University are framing their mathematical use of equations based on examining the symbolic forms of their mathematical arguments. “Upper level” is defined in this study to refer to a physics class that is not intended as an introductory course fulfilling a general education requirement. In other words, an upper level course would not be meant as a student’s first or only physics class. Certain instruments for data collection were used in this study; including Video Research and a Likert Survey. I begin this section with a description of each.

3.1.a Analyzing Video Research

To answer research questions #’s 2, 3, and 4 (What does student qualitative understanding of an equation look like? How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts? What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation?) I videotaped students and experts solving astrophysics problems related to negative potential energy and the virial theorem. This section discusses how one analyzes and interprets video data in educational research.

In the paper *"Conducting Video Research in the Learning Sciences: Guidance on Selection, Analysis, Technology, and Ethics"* by Derry et al., 2010, the authors "address four challenges for scientists who collect and use video records to conduct research in and on learning environments" (Derry et al., 2010, p. 6). Specifically, the sections on Selection, Analysis, and Ethics are significant to my proposed research which entails the use of video records to conduct research on the physics understanding of undergraduates in a particular learning environment – that where students are solving problems concerning astrophysics in an interview setting. "*Video analysis for insight and coding: Examples from tutorials in introductory physics*" by Scheer, 2009, addresses videotaping students in the introductory classroom, yet offers insights at to the data analysis specifically as pertains to physics understanding.

The section in the paper by Derry et al. that focuses on selection describes how researchers can "be systematic in deciding which elements of a complex environment or extensive video corpus to select for study" (Derry et al., 2010, p. 7). If one takes the useful perspective that video segments represent events, the selection process involves selecting which events should be brought into focus for deeper analysis. "The researcher's specific interest will determine which events … a study should select" (Derry et al., 2010, p. 8). Since this research topic involves the understanding of astrophysics of upper level undergraduate students, the events studied from the video would include only those moments in the interviews in which astrophysics understanding is being connected or discussed in relation to equations. Scherr (2009) appears to be of the same opinion, adding that "insight-oriented analysis begins with episode selection and is followed by collaborative viewing" (Scherr, 2009, p. 2). Collaborative viewing (where the analysis of the video is conducted by a group of researchers) is essential to my proposed study in that it ensures the validity of the study.

Derry et al. also indicate that “the inquiry process can be conceptualized as moving among a number of phases; (a) planning a study, (b) shooting original footage, (c) choosing one or more clips from a corpus of such footage, (d) focusing on the selected video clip or clips in appropriate ways depending on the researcher’s goals, (e) developing final products for presentation …” (Derry et al., 2010, p. 8). As such, the video taken of the undergraduate students was taken with consideration of the research goals. This made the selection issue intertwined with technical skills such as knowing where to place cameras and microphones, when to start and end shooting, etc.

Finally, the section in the research paper by Derry et al. deals with analyzing data and how to conduct an in-depth analysis when selected video records are the primary data source (Derry et al. 2010). Derry et al. suggest that when analyzing video data one should always start with a guiding question. That is, “the collection of research video is guided by a plan and a set of research questions that are based on the researcher’s familiarity with the phenomena being studied, although situations also arise in which video that has already been collected and archived is analyzed” (Derry et al., 2010, p. 16). It is therefore necessary to begin this proposed research with the research questions stated above clearly defined and that they originate from observations and from the research literature. Derry et al. also suggest that one should expect unanticipated phenomena – one must remain open to discovering new phenomena – and to develop social practices for viewing. There are many ways in which the researcher can create intermediate representations of video records. Derry et al. describe five general approaches: indexing, macrolevel coding, narrative summaries, diagrams, and transcription. Of these, macrolevel coding (Where topics and themes are coded for comparison, the significant event only is transcribed and analyzed.), transcription, and diagrams, seem most relevant to this research.

Scherr (2009) describes, in great detail, her methodology in coding her video data. Identifying, interpreting, and comparing and correlating patterns were crucial to her research. Reporting results for the research was done by coding as outlined in Scherr, 2009.

3.1.b Likert Surveys

To answer research question #1 (What do the students think it means to understand astrophysics equations?), I administered a Likert survey. Therefore, it is important to take time to describe some of the literature concerning Likert scales and Likert response forms.

According to Likert, a Likert Scale has a series of verbal statements that express a range of positive expressions, views, sentiments, claims, or opinions about the “attitude object” that ranged from mildly positive to strongly positive and then the same relative to a range of negative statements (Likert, 1932). Logically, someone who is positive about the attitude object should agree with the positive statements and disagree with the negative statements so a logical check and validity is built into the construction protocol.

According to Carifio and Perla (2007) “the logical properties and criteria of a Likert Scale are one’s first and foremost concerns and features” (Carifio and Perla, 2007, p. 113). The authors describe these criteria as the factor that produces the differences in the overall scale score. They further note that the questions that comprise a Likert Scale can be scaled to add “further refinements and weighted scoring” to the accumulation of items into scale scores.

In the data for this dissertation, I used a twenty-one item Likert-scale survey using a five-point range from strongly agree to strongly disagree that probes student attitudes, beliefs, and assumptions about using equations in astrophysics.

3.2 Participants

The majority of participants in this study were selected from the first of a two-semester sequence called Principles of Astrophysics offered at Rutgers University. (Additional participants were experts in the field of astrophysics, also at Rutgers University.) This class is designed to provide an upper level introduction to the concepts of astrophysics for undergraduate students majoring in physics. Along with the physics and astrophysics majors taking this class, many engineering students also take one or both of these classes as a science elective, resulting in roughly 60 students per semester. In the Fall 2013 semester 55 students were enrolled in Physics 341: Principles of Astrophysics. The course focuses mainly on gravity as the dominant force in the Universe. The course textbook is *An Introduction to Modern Astrophysics* by Carroll & Ostlie (Benjamin Cummings 2nd Edition, 2006) but lecture notes were drawn from *Principles of Astrophysics* by Charles Keeton (published in 2014). The teaching is done in an interactive mode with students working in small groups in lectures on the problems and questions posed by the instructor.

3.2.a Course Description

The description of this class is taken from the course syllabus[1]:

> *PHYSICS 341: Principles of Astrophysics, Instructor: Dr. Eric Gawiser*
>
> Astrophysics is the application of physical principles to astronomical systems. In Physics 341 and 342 you will learn how to use gravity, electromagnetism, and atomic, nuclear,

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1 https://www.physics.rutgers.edu/~gawiser/341/

> and gas physics to understand planets, stars, galaxies, dark matter, and the Universe as a whole. Gravity is the dominant force in many astronomical systems, and it will be our focus in Physics 341.
>
> Some astrophysical systems are described by equations that are fairly easy to solve, and we will study them. However, many interesting systems cannot be solved exactly. Nevertheless, we can often use physical insight and carefully chosen approximations to understand the key features of a system without sweating the details. One goal of the course is to develop that skill. As you will see, it will take us very far (through the whole Universe, in fact!). Another goal is to learn about recent advances in astrophysics, a very dynamic field of research.
>
> Prerequisites for this class are two semesters of physics and two semesters of calculus. I will briefly review physical principles as we need them, but it is assumed that you have seen them before. I will also assume familiarity with vector calculus. Some of the assignments may involve a bit of computation that can be done with programs like Excel, Google Spreadsheets, Maple, Matlab, or Mathematica.

Throughout the semester, the students submitted eleven (11) homework assignments, took two in-class exams (one at the middle of the semester and one at the end), participated in an online Likert survey, took a take-home final essay, and participated in “clicker questions” in every class. (The course outline can be viewed in Appendix D.)

3.3 Data Collection

The data for this dissertation includes multiple homework assignments, two exams, a final essay, and video recordings of interviews of astrophysics students as well as experts working on solving problems regarding negative potential energy and the virial theorem. The following table includes a list of the data sources and which of the research questions they will contribute to answering:

Table 1: Data Sources and Which Research Questions They Support

Data Source	**Research Question #1: What do the students think it means to understand astrophysics equations?**	**Research Question #2: What does student qualitative understanding of an equation look like?**	**Research Question #3: How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?**	**Research Question #4: What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation?**
Homework Assignments	No	Yes	Yes	No
Interview Video	Yes	Yes	Yes	Yes
Exams	No	Yes	Yes	No
Essay	Yes	Yes	Yes	No
Survey	Yes	No	Yes	No

3.3.a Homework Assignments

Eleven weekly homework assignments were assigned and collected throughout the semester (Appendix C). The students were allowed to work in groups on these homework assignments as long as they listed their collaborators; however, the write up for each assignment was to be done by the individual student. Once graded the students' work was scanned and then returned to the students. Each homework assignment contained both conceptual and problem-solving questions pertaining to the previous week's class topics.

An example of a conceptual homework problem is as follows from Homework Assignment #2, Problem #1 (a.): "Suppose that the Sun were instantaneously replaced by a star with twice as much mass. Would Earth's orbit stay the same? Explain your answer. Now suppose that the Earth doubled in mass instantly but the Sun remained the same. Would Earth's orbit stay the same? Explain your answer."

An example of a problem-solving homework problem is as follows from Homework Assignment #2, Problem #2: "Mars has a mass of 6.4 x 10^{26} g (about one tenth $M_{\oplus}$) and a radius of 3400 km (about half $R_{\oplus}$). Its small moon Phobos has a mass of 1.1 x 10^{19} g and a radius of just 11 km. Phobos orbits Mars with a Semimajor axis of 9380 km. (a.) What are the mean densities of Mars and Phobos, in g cm^{-3}? (b.) What is the Roche limit of the Mars/Phobos system? Is Phobos inside it? (c.) Use Kepler's Third Law to calculate the orbital period of Phobos, in hours. (d.) Recall in class we said that tidal forces are causing the Moon's orbit to recede from the Earth. Because Phobos orbits Mars faster than the rotation period of Mars, Mars is decreasing at a rate of 20 cm yr^{-1}. At that rate, how long is it until Phobos hits the surface of Mars? (e.) However, what is likely to happen to Phobos before then? Think about your answer to part (b.)."

3.3.b Interview Video

Semi-structured interviews of ten students and two experts solving three different, yet related, astrophysics problems were videotaped for analysis. The interviewed students were paid a small amount in compensation for their time.

During these interview sessions the student (and expert) were primarily problem solving where the student was asked to solve a particular problem while vocalizing his or her thoughts. The main problems themselves were preplanned, but the interviews were not scripted at any finer level of detail. Depending on what the students brought up as they worked on the problems, the researcher (Rave) asked various clarifications. Every effort was made to keep the interviewed students talking as continuously as possible, vocalizing their ideas as they attempted to resolve whatever questions came up in real time. Effective prodding, as expected, was very individualized as no two students had the exact same thought processes.

The three preplanned questions that were posed during the interviews are as follows:

1.) (a.) Imagine someone wanted to determine the kinetic energy of Earth moving around the Sun. How would you do that? (b.) What would be the gravitational potential energy of Earth? (c.) Which of these energies is larger in magnitude and what is the ratio of these two energies? (d.) What is the sign of the total energy? Does this make sense? Why?

2.) Propose a question which has the following solution:

$$a_r = F_r / m$$

$$GMm/r^2 = mv^2/r$$

$$v^2 = GM/r$$

$$K = \tfrac{1}{2} mv^2 = GMm/2r = -\tfrac{1}{2} U$$

3.) What would happen to a satellite's orbit if the satellite in orbit about Earth loses total energy due to gradual atmospheric drag? What would happen to the potential energy of the satellite? Do you know of any situation in astrophysics that is similar to this?

With the expected answers:

1.) (a.)

$$K = \left(\frac{1}{2}\right) M_{Earth} v^2$$

$$But: \; \frac{M_{Earth} v^2}{r} = \frac{G M_{Sun} M_{Earth}}{r^2} \; and \; v^2 = (G M_{Sun}/r)$$

$$Therefore: \; K = \left(\frac{1}{2}\right) \frac{G M_{Sun} M_{Earth}}{r}$$

(b.)

$$U = - \, G M_{Sun} M_{Earth}/r$$

(c.)

$$The\ magnitude\ of\ the\ gravitational\ potential\ energy\ is\ twice\ the\ kinetic\ energy.$$

$$The\ ratio\ of\ these\ two\ energies\ is\ negative\ 2.$$

(d.)

$$The\ total\ energy\ will\ be\ negative:$$

$$E = K + U = \left(\frac{1}{2}\right) \frac{G M_{Sun} M_{Earth}}{r} - \frac{G M_{Sun} M_{Earth}}{r} = -\left(\frac{1}{2}\right) \frac{G M_{Sun} M_{Earth}}{r}$$

$$Earth\ is\ in\ a\ bound\ orbit, so\ yes, this\ makes\ sense.$$

2.)

$$What\ is\ the\ relationship\ between\ kinetic\ and\ potential\ energies$$

$$for\ a\ satellite\ of\ mass\ m$$

$$in\ a\ circular\ orbit\ or\ radius\ r\ about\ a\ massive\ object\ of\ mass\ M?$$

3.)

$$E = \frac{U}{2} = -\frac{GM_{Earth}M_{Satellite}}{2r}$$

$$r = -\frac{GM_{Earth}M_{Satellite}}{2E}$$

$$It\ would\ move\ to\ a\ smaller\ radii\ to\ lose\ potential\ energy\ and\ hence\ total\ energy.$$

$$K = -\frac{U}{2} = \frac{1}{2}M_{Satellite}v_{Satellite}^2$$

$$v_{Satellite}^2 = \frac{2K}{M_{Satellite}} = -\frac{2U}{2M_{Satellite}}$$

$$Since\ the\ potential\ energy\ becomes\ more\ negative\ the\ kinetic\ energy\ must\ increase$$

$$according\ to\ the\ virial\ theorem. This\ increase\ in\ K\ is\ only\ one$$

$$half\ the\ loss\ of\ potential\ energy, and\ so\ total\ energy\ indeed\ decreases.$$

$$The\ atmospheric\ drag\ tries\ to\ slow\ the\ satellite,$$

$$but\ instead\ it\ falls\ to\ lower\ orbit\ and\ speeds\ up!$$

3.3.c Exams

Two conceptual exams (Appendix B) were given over the course of the semester, one half-way through the semester and the second at the end of the semester. The first exam consisted of twenty multiple choice questions and five free-response questions. The twenty multiple choice questions were identical to those from previous years so as to compare the results of this year's student's to previous years. The five free response questions were unique to this semester's students for the purpose of this study. The second exam consisted of ten free-response questions on topics related to the second half of the semester. The exams were graded and scanned but were not returned to the students (they could however, view the exams during the instructor's office hours).

3.3.d Essay

A final essay was given at the end of the semester. The essay was to be done individually, without any collaboration. The topic of the essay was as follows: "Which concept from PHY 341 did you find most difficult, and why? Which concept was your favorite, and why? Discuss the steps you took to better understand the difficult concept. Did you find your favorite concept easy or difficult to understand, and did that influence your choice of it as a favorite? If these concepts relate to equations, include discussion of the relevant equations and the meaning of their terms as well as how the equations influenced your attempts to understand the concept." The students' work was scanned then graded and returned to the students.

3.3.e Survey

Finally, an online Likert survey was given to the students at mid-semester. The twenty-one questions in the survey all targeted the theme of what the students think it means to understand astrophysics equations. The survey questions can be found in Appendix A at the end of this paper.

3.4 Analysis Tools

As the main goal of this dissertation is to identify how upper level students in astrophysics connect equations to concepts, the research questions that are addressed in the analysis of the collected data are: #1 What do the students think it means to understand astrophysics equations?, #2 What does student qualitative understanding of an equation look like?, #3 How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?, and #4 What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation? Closely analyzing the students work will answer these questions.

3.4.a Rubric for analyzing Homework and Exams

To answer research questions #2 (What does student qualitative understanding of an equation look like?) and #3 (How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?) the following general rubric was created to analyze the homework assignments and exams.

Table 2: Rubric for Analyzing Homework and Quizzes

0	1	2	3
No attempt made No conceptual connection & No mathematics	Student shows mathematical understanding, but does not connect concepts (or opposite)	Student does mathematics but does not fully explain/makes mistake in one area Connection between mathematics and concepts is clear but one aspect is incorrect	Connects concepts to mathematics clearly and both are correct

Once the student's work was coded according to the rubric, reliability was achieved by discussion and comparison with other researchers. I worked with a second researcher, discussing the scoring of each homework assignment, 20% of the homework assignments were then scored separately and an 83% agreement was achieved on the scores. After discussion on the differences a 95% agreement was achieved. The coded homework and exam data was then analyzed, looking for trends in the data, correlating the qualitative homework question responses to the quantitative homework question responses, comparing homework responses to exam responses, and comparing the results of the coding to the video interview data.

3.4.b Analysis of Video Interviews and Essay

Analysis of the Video interviews helped answer the research questions #2 (What does student qualitative understanding of an equation look like?), #3 (How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?), and in particular #4 (What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation?). As described earlier, twelve interviews were conducted, each lasting about one hour. This resulted in approximately ten hours of raw video of upper level physics students and two hours of experts at work collected for this study. The author was present during all of the tapings and took detailed notes of the students' activity. Transcriptions of these videos were made and the transcriptions were coded and verified by a collaborative outside researcher in the field. The verified coded transcriptions of the data were then used to analyze research question #4, looking for common themes of understanding equations among the students compared to that of the experts. The video interview data for the students was also compared with the data from the students' homework assignments, exams, and essay to investigate research questions #2 and #3.

The student's essays were similarly analyzed, looking for common themes and coding the results. A second researcher and I discussed the codes for the essays and coded 30% of the essays separately. A preliminary 88% agreement was achieved on the essay scores and upon discussion on the differences, a 94% agreement was achieved. The responses of the essays were compared to the other data and helped answer the research questions #1 (What do the students think it means to understand astrophysics equations?), #2 (What does student qualitative understanding of an equation look like?), and #3 (How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?).

3.4.c Analysis of the Survey

To answer the research question #1 (What do the students think it means to understand astrophysics equations?), the results from the Likert Survey were used. The student responses to the survey questions were statistically analyzed and correlated with the other data to answer this question.

Chapter 4: Findings

4.1 What Do Students Believe?

Survey Findings: What do the students think it means to understand astrophysics equations?

4.1.a What Leads to Understanding Astrophysics Ideas?

Table 3: Questions #1, 7, 10, 12, 15, 18: Understanding Astrophysics Ideas

Questions	#/% of Strongly Disagree and Disagree	#/% of Neutral	#/% of Agree and Strongly Agree
Q1: In order to understand the ideas presented in this course, I only need to work through the problem sets and/or pay close attention in class.	9/17%	3/6%	41/77%
Q7: The best way for me to learn astrophysics is by solving the quantitative problems in the problem sets.	11/21%	15/28%	27/51%
Q10: Learning in astrophysics is a matter of developing knowledge that is shown in the equations given in class.	7/13%	18/34%	28/53%
Q12: The derivations and proofs of equations shown in class have little relevance to actually solving problems or understanding the course material.	42/81%	4/8%	6/12%
Q15: The main skill I get out of this course is learning how to solve problems in astrophysics.	11/21%	8/15%	34/64%
Q18: I use the mistakes I make on the problem sets as clues to what I need to do to understand the course better.	2/4%	3/6%	48/91%

The student's responses to these survey questions all indicate that the students believe that in order to understand the ideas in the astrophysics classroom, they need to understand how to solve problems (specifically the quantitative problems – Question 7) in astrophysics. In fact, only 21% of the students **do not** believe that working through the quantitative problems in the problem sets is the best way to learn astrophysics (Question 7). The results of the survey also show that the students believe that the way to understand the material (Question 1) is to work through the problem sets and/or pay attention in class learning how to solve problems in astrophysics (Question 15 – 64%) which agrees with Question 12 where 81% of the students believe that the derivations and proofs are relevant to "actually solving problems or understanding the course material." Since the students clearly show that they believe that understanding in astrophysics is shown by solving problems, it is unsurprising that the students believe that learning astrophysics means that they need to develop knowledge that is shown in the equations given to them in the class (Question 10) and that a vast majority of the students (91%) use the mistakes made in problem sets as indicators of what they need to do to understand the course better (Question 18).

4.1.b Are Derivations Important?

Table 4: Questions #2, 4, 5, 12: Derivations

Questions	#/% of Strongly Disagree and Disagree	#/% of Neutral	#/% of Agree and Strongly Agree
Q2: A derivation or proof of an equation shown in class is useful because I can use the final equation while working on the problem sets without working through the derivation myself.	9/17%	6/11%	38/72%
Q4: I spend a significant amount of time figuring out and understanding at least some of the derivations given in class.	23/43%	15/28%	15/28%

Q5: If I forget an equation or cannot find the right one, there is nothing I can do, I must skip that problem.	48/91%	2/4%	3/6%
Q12: The derivations and proofs of equations shown in class have little relevance to actually solving problems or understanding the course material.	42/81%	4/8%	6/12%

The student's responses to Survey Questions 2, 4, 5, and 12 show that they believe that derivations of equations are important in astrophysics (Question 12) but, once already derived, it is not necessarily important to re-derive the equations (Question 2 and 4); they can simply use the final equation derived (Question 2). It is interesting to note that the students do not all agree that they "spend a significant amount of time figuring out and understanding at least some of the derivations given in class" (Question 4) with the results for this question being divided fairly evenly across the answers. While slightly more students do not spend a significant amount of time working on the derivations in class (23 students), there are almost as many undecided (15 students) as well as students that do spend a significant amount of time working on the derivations (15 students). It is also interesting to note that the students do not all necessarily spend time out of the classroom working through derivations, a vast majority of the students (91%) believe that they would be able to work through a problem without a derived equation (Question 5) implying that they could derive the equations themselves.

4.1.c What is the Role of Equations in Understanding?

Table 5: Questions #2, 3, 5, 6, 10, 12, 14, 19, 21: Equations and Understanding

Questions	#/% of Strongly Disagree and Disagree	#/% of Neutral	#/% of Agree and Strongly Agree

Q2: A derivation or proof of an equation shown in class is useful because I can use the final equation while working on the problem sets without working through the derivation myself.	9/17%	6/11%	38/72%
Q3: To solve a problem in astrophysics I need to match the problem situation with the appropriate equations and then mathematically manipulate and/or substitute values to get an answer.	5/10%	5/10%	42/81%
Q5: If I forget an equation or cannot find the right one, there is nothing I can do, I must skip that problem.	48/91%	2/4%	3/6%
Q6: In astrophysics, I do not need to understand equations in an intuitive sense; they can just be taken as givens.	45/85%	6/11%	2/4%
Q10: Learning in astrophysics is a matter of developing knowledge that is shown in the equations given in class.	7/13%	18/34%	28/53%
Q12: The derivations and proofs of equations shown in class have little relevance to actually solving problems or understanding the course material.	42/81%	4/8%	6/12%
Q14: The most crucial thing I need to do when solving a problem in astrophysics is to find the right equation to use.	12/23%	12/23%	29/55%
Q19: To be able to use an equation in a problem (particularly in a problem that I haven't seen before), I need to know more than what each term in the equation represents.	4/8%	9/17%	40/75%
Q21: When I solve a problem in the problem sets, I go straight to the lecture notes to find an equation that applies to the situation described in the problem.	10/19%	7/13%	36/68%

The students show in their responses to the following eight of the survey questions that they believe that understanding the equations that they use in the problem sets indicates understanding astrophysics (Questions 6, 10, 12, and 19) and that the way to solve problems in the problem sets is to use the correct equations (Questions 2, 3, 5, 12, 14, 19, and 21). In Questions 6 and 12, the students show that they believe that equations do need to be understood conceptually (with only

4% believing that equations can just be taken as givens – Question 6) and that the equations derived in class are relevant to understanding the course material (Question 12). In the survey, over half of the students show that they believe that “learning in astrophysics is a matter of developing knowledge that is shown in the equations given in class” (Question 10). Furthermore, most of the students in the class believe that they need to understand the equations they are using to solve a problem, not just what each term in the equation represents. Only four students believe that they can effectively use an equation in which they only know that the terms represent (Question 19). Interestingly, the students believe that they can match the appropriate final equations that are derived in class (Question 2, 14, and 21) to questions in the problem sets and then “mathematically manipulate and/or substitute values to get an answer” (Question 3). This suggests, with the student’s beliefs that the equations they use need to be understood conceptually (Questions 6, 12, and 19), that they feel they understand the equations that are being derived or proved in class (Question 2) and that as long as that understanding is there (Question 6), they can move forward with mathematical manipulation to arrive at the final answer (Question 3). 91% of the students also believe that if they cannot find the correct equation from the class, that they still would be able to work on the problem (Question 5) showing that they believe that it is possible for them to understand astrophysics enough that they would be able to derive the equation needed to solve the problem.

4.1.d Solving Problems & How Are Multiple Representations Used?

Table 6: Questions #3, 5, 7, 8, 11, 13, 14, 16, 17, 18, 19, 21: Solving Problems

Questions	#/% of Strongly Disagree and Disagree	#/% of Neutral	#/% of Agree and Strongly Agree

Q3: To solve a problem in astrophysics I need to match the problem situation with the appropriate equations and then mathematically manipulate and/or substitute values to get an answer.	5/10%	5/10%	42/81%
Q5: If I forget an equation or cannot find the right one, there is nothing I can do, I must skip that problem.	48/91%	2/4%	3/6%
Q7: The best way for me to learn astrophysics is by solving the quantitative problems in the problem sets.	11/21%	15/28%	27/51%
Q8: After I first read a new problem, I try to visualize the situation and sometimes I draw a sketch before going into mathematics.	8/15%	5/9%	40/75%
Q11: In completing a problem in the problem sets, if my calculations give me a result that differs significantly from what I expect, I would trust the calculation rather than my intuition.	40/75%	10/19%	3/6%
Q13: In completing a problem in the problem sets, I check the units to be sure that my answer is dimensionally accurate.	2/4%	3/6%	48/91%
Q14: The most crucial thing I need to do when solving a problem in astrophysics is to find the right equation to use.	12/23%	12/23%	29/55%
Q16: As long as I have a conceptual understanding of a problem in my mind, I do not need to communicate this understanding through my written work.	38/73%	8/15%	6/12%
Q17: When I solve the quantitative problems in the problem sets, I think about the concepts that lead to the problem.	3/6%	2/4%	48/91%
Q18: I use the mistakes I make on the problem sets as clues to what I need to do to understand the course better.	2/4%	3/6%	48/91%
Q19: To be able to use an equation in a problem (particularly in a problem that I haven't seen before), I need to know more than what each term in the equation represents.	4/8%	9/17%	40/75%
Q21: When I solve a problem in the problem sets, I go straight to the lecture notes to find an equation that applies to the situation described in the problem.	10/19%	7/13%	36/68%

As seen in the data above (Section 4.1.d), the student's responses show that they believe that the way to successfully solve problems in the problem sets is by use of the correct equations (Questions 2, 3, 5, 12, 14, 19, and 21) and that solving the problems from the problem sets helps them learn astrophysics (Question 7 and 18). In addition to these beliefs concerning solving problems in the problem sets, the students also believe that they need to conceptually understand the problem in order to solve it (Questions 8, 11, 16, and 17). The survey shows that 91% of the students believe that they need to think about the concepts behind quantitative problems (Question 17) while solving problems and only 12% do not think that they need to show this conceptual understanding in their written work (Question 16). The students also believe they should "try to visualize the situation" and "draw a sketch" before beginning to mathematically solve the problem (Question 8) and that they would trust their intuition over their calculations if their answer differed significantly from their intuition (Question 11). Finally, 91% of the students will check their units to be sure that the answer they have is dimensionally accurate (Question 13).

As stated above, the students believe that they should show their conceptual understanding in their written work (Question 16). Two ways in which the students can show their understanding of their work is through visualization (a sketch of the problem) and through dimensional analysis. Most of the students (75%) replied in the survey that they prefer to visualize a problem (sometimes with a sketch) before attempting to solve the problem (Question 8) and almost all of the students (91%) check their units to be sure that their results are dimensionally accurate (Question 13).

4.1.e What Do Grades Measure?

Table 7: Questions #9 & #20: Class Grade

Questions	#/% of Strongly Disagree and Disagree	#/% of Neutral	#/% of Agree and Strongly Agree
Q9: My grade in this course is determined by how well I understand the material.	5/9%	4/8%	44/83%
Q20: It is possible to pass this course (get a "C"; or better) without understanding astrophysics very well.	26/49%	21/40%	6/11%

Questions 9 and 20 ask the students if they believe their grade is correlated with how well they understand astrophysics. In Question 9 and in Question 20 only a very small percentage (9% and 11% respectively) of the students feel that their grade is not connected to their understanding of astrophysics. What is interesting is that in Question 9 a great majority of the students (44 of 53) believe that their grade in the class is based on their understanding of the material but in Question 20, when asked about a specific grade value (a "C" or better), the students were more unsure of how their grade correlated to their understanding of astrophysics with 41% of the students claiming that they **do not** believe that they can receive a grade of "C" or better without understanding astrophysics very well while an almost equal amount (40%) is unsure.

4.1.f Summary of the findings from the survey

To summarize students' responses to these survey questions I have grouped several questions together. Questions #1, 7, 10, 12, 15, and 18 show that the students *believe* that understanding in astrophysics is shown by solving problems. Questions #9 & #20 show that the students' beliefs about how their grade is correlated with their understanding of astrophysics are mixed. The results from questions # 2, 4, 5, and 12 show that the students believe that derivations of equations are important in astrophysics but only as a way to get the final equations. In other words, the final equation is the ultimate goal to them and the derivation itself is not important. From the results from questions #2, 3, 5, 6, 10, 12, 14, 19, and 21, it can be concluded that the students believe that the equations they use need to be understood conceptually. They *believe* that they achieve that understanding in the derivations done in class. Additionally, the results from the survey questions #3, 5, 7, 8, 11, 13, 14, 16, 17, 18, 19, and 21 illustrate that, in addition to their beliefs concerning use of the equations in solving problems, the students also believe that they need to conceptually understand the problem in order to solve it. The responses to these survey questions also demonstrate that students *believe* that they show their understanding of their work through visualization (a sketch of the problem) and through dimensional analysis.

4.2 What do students actually do?

4.2.a Essay Findings: How do students understand equations?

Fifty-four students were asked to discuss their favorite concepts and most difficult concepts of the course in an essay at the end of the Spring Semester in 2014. The topic of the essay was as follows: "Which concept from PHY 341 did you find most difficult, and why? Which concept was your favorite, and why? Discuss the steps you took to better understand the difficult concept. Did you find your favorite concept easy or difficult to understand, and did that influence your choice of it as a favorite? If these concepts relate to equations, include discussion of the relevant equations and the meaning of their terms as well as how the equations influenced your attempts to understand the concept." We coded the essays using the criteria for student understanding of equations described in the framework of understanding of physics equations by Domert et al. (2012). The framework discusses several attributes of understanding equations. According to Domert and colleagues a student who understand equations is able to: recognize the symbols in the equation in terms of the corresponding physics quantities, recognize the underlying physics of the equation, recognize the structure of the equation, establish a link between the equation and everyday life. The student knows how to use the equation to solve physics problems, and when to use it. While I could not judge whether student understanding of equations had the last two attributes (knowing how to use the equation to solve physics problems and being able to know when to use the equation) in the essays, I could determine whether the other attributes were present. We used these attributes to develop a coding scheme. Table 8 shows the codes and examples of student writing coded with respective codes.

Table 8: Codes and Examples of Student Writing

Codes:	Example Student Responses:
1: Student Provides Equation? (Yes/No)	"… the gravitational lensing equation and Einstein radios equation shown respectively. $$\beta = \theta - \frac{D_{ls}}{D_s}\hat{\alpha} = \theta - \alpha$$ $$\theta_+ = \frac{1}{2}\left[\beta + (\beta^2 + 4\theta_E^2)^{\frac{1}{2}}\right]$$"
2: Student Provides Equation with Mathematical Understanding? (Understanding/No Understanding)	Understanding: "… I approached the virial theorem as an analogy to conservation of energy, which it is not. This is due to the virial theorem's use of time averages, unlike the instantaneous metrics in conservation of energy." No Understanding: "The mathematical calculations behind black holes were also

	very simple, for example, determining the size of a black hole is a no-brainer. The size of a black hole is given by it's Schwarzschild radius … Where R_S is Schwarzschild Radius (Size), G is our gravitational constant, M is Mass, and c is our Speed of Light constant. This leaves only one variable to find, Mass!"
3: Student Provides Equation in Words? (Yes/No)	"… the square of the period is proportional to the cube of the semi-major axis of the orbit."
4: Does the Student Discuss the Symbols in the Equation? (Yes/No)	"Where M and m are the two masses in the system, with m being the test mass, Δr as the radius of the test mass, and r^3 as the distance between the centers of the masses involved in the tidal force system"
5: Does the Student Discuss the Astrophysics or Physics in the Equation? (0/1/2 scale)	2: "The key is to stay in a simple reference frame, preferably the stationary one relative to the moving object, and everything drops into place. Another way to simplify it is that a yardstick appears shorter the closer it moves to the speed of light, so multiplying the true length by gamma produces a smaller number, thus producing the apparent length to a stationary observer." 1: "Somehow we end up with $\ddot{z} + y^2 z = 0$ and declare it to be an instance of simple harmonic motion." 0: "If I can plug something into an equation I can see the relationships between the terms."
6: Does the Student Discuss the Structure of the Equations? (Yes/No)	"$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

	… By looking at this equation and knowing that it is impossible for anything to exceed the speed of light, I can see that this factor will always be less than 1."
7: Does the Student Understand the Purpose of the Equations? (Yes/No)	"The time dilation equation stated that more time passed in an unprimed/stationary time frame, than did the time in the primed/moving frame, when using the flashes of light clocks as our time frame."
8: Does the Student Show Deep Understanding of Astrophysics Behind the Equations? (Yes/No)	"Given that a small change in r^3, by setting: $r^3 = (r_s)^3 = \left(\frac{GM}{c^2}\right)^3$, … gave a maximum tidal force that, at a certain distance (Schwarzschild Radius) and towards the center of mass of the black hole, was greater than the electrostatic force of the particles that comprise up of our body."
9: Does the Student Talk About Connections Between Equations and the Real World? (Yes/No)	"Learning that stars oscillate throughout the galaxy much like a mass does connected to a spring, astonished me because I never thought to visualize the complexity of galactic star motion, so much so that the vertical motion is of the differential equation, where z is the azimuthal coordinate in cylindrical coordinates: $$\ddot{z} = -\frac{d\varphi}{dz}$$ $$\varphi = \frac{U}{m}$$ U = Potential Energy m = test mass."

The reliability of the scheme was established by two researchers who coded 10% of the essays independently and achieved high levels of agreement. For different codes the Kappa Ranged from: 0.71 (Code 7: “Does the student understand the purpose of the equations?”) to 1.00 (Code 1: “Student Provides Equation?”, Code 2: “Student Provides Equation with Mathematical Understanding?”, and Code 9: “Does the Student Talk About Connections Between Equations and the Real World?”).

Using the above coding scheme, I found the following:

When asked to discuss their favorite concepts and most difficult concepts of the course in an essay, 89% of the students included an equation for these concepts (Fig. 1).

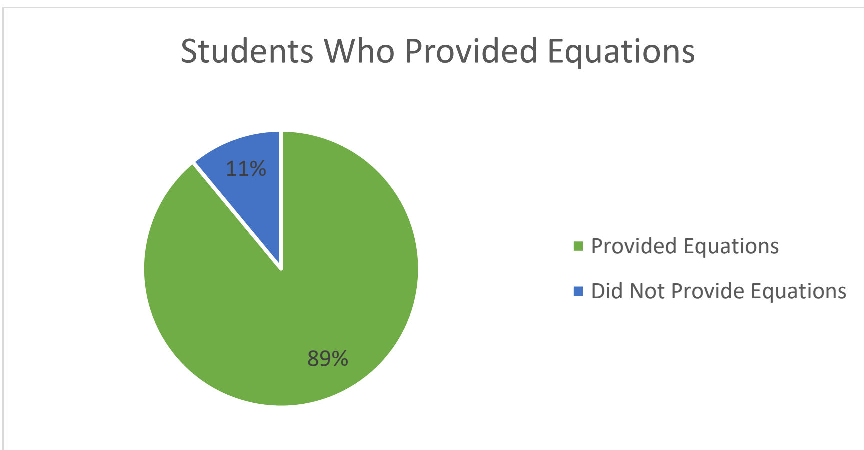


Figure 1: The percentage of students who provided equations in their essays versus those who did not provide an equation.

Of these students two-thirds (67%) showed a mathematical understanding of the equation they provided, but very few of them (19%) described the equations in words. Slightly over half of these students (51%) discussed the symbols in the equations but most did not discuss the structure of the equations (67%). The largest part of these students did an adequate job of discussing the astrophysics or physics in the equations (Fig. 2) with 74% of the students being able to articulate the purpose of the equations but only 30% of the students demonstrated a deep understanding of the astrophysics behind the equation. Most of the students did talk unprompted about connections between their equations and the real world (53%). (Fig. 3)

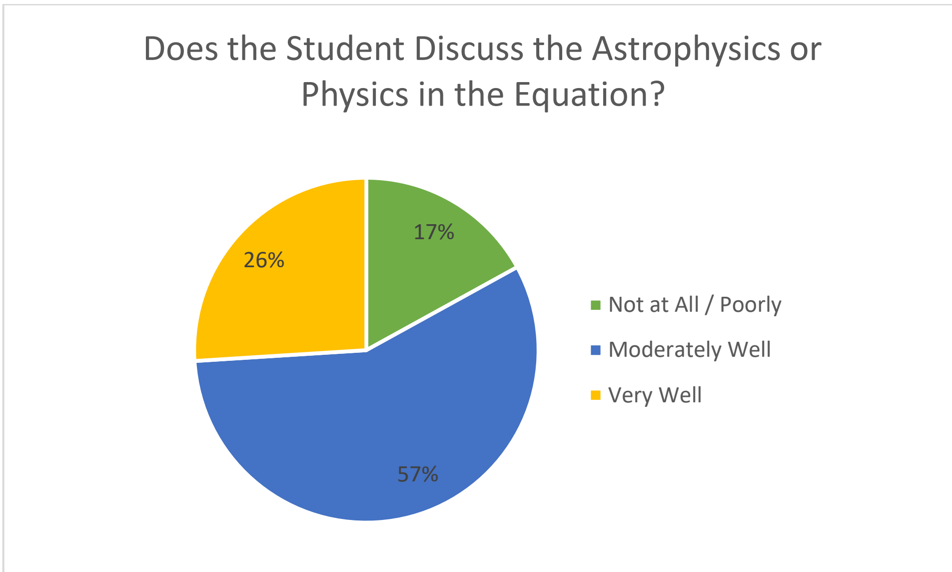


Figure 2: The majority of the students who provided equations in their essays describe the astrophysics or physics in these equations moderately well.

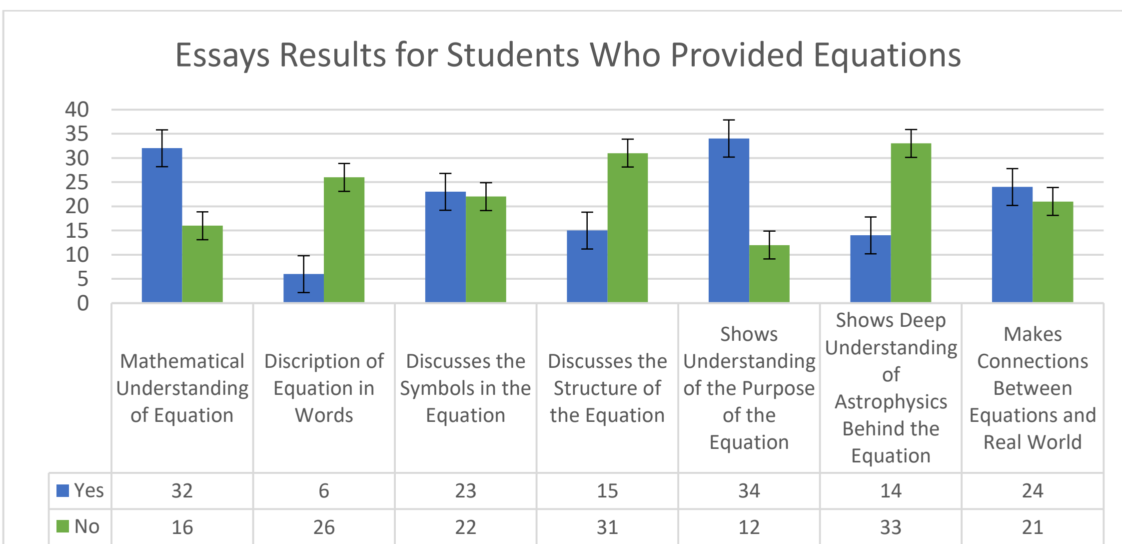


Figure 3: Overall results from students’ essays.

When specifically prompted to do so, the students are able to link conceptual understanding to the mathematical use of the equations; however, they do not show a deep understanding of the equations they use according to the coding scheme developed from the framework of understanding of physics equations by Domert et al. (2012).

4.2.b Homework and Exam Findings: What do student actually do?

Ten homework assignments were collected for fifty-four students throughout the semester and two exams (one at the mid-term and one at the end of term) were collected (for fifty-four and fifty-three students respectively) and analyzed. Exam questions consisted exclusively of conceptual problems whereas homework assignments contained a combination of conceptual questions and quantitative problems. All homework and exam problems were scored on a 0-3 scale reflecting the student's ability to connect concepts to mathematics. In this scale a 0 indicates no conceptual connection and no mathematics shown and 3 indicates that the student successfully connects concepts to mathematics clearly and correctly. The general rubric shown here (Table 9) was created to analyze the both the homework assignments and quizzes. Table 10 shows examples of student responses that were coded using the rubric categories.

Table 9: Rubric for Analyzing Homework and Quizzes (Repeat of Table 2)

0	1	2	3
No attempt made	Student shows mathematical understanding, but does not connect concepts (or opposite)	Student does mathematics but does not fully explain/makes mistake in one area	Connects concepts to mathematics clearly and both are correct

No conceptual connection & No mathematics		Connection between mathematics and concepts is clear but one aspect is incorrect	

Table 10: Examples of Student Responses to Conceptual Questions with Assigned Scores

Rubric:	Source/Conceptual Question:	Example Conceptual Student Responses:
0: No attempt made No conceptual connection & No mathematics	Homework Assignment 6: #1.b) Rank the following black holes based on the magnitude of the tidal forces that they would exert on a spaceship placed near their event horizon. A has mass 10 $M_\odot$; B has mass 100 $M_\odot$; C has mass 10^6 $M_\odot$.	"In order of higher magnitude of tidal forces: 10^6 M_{Sun} > 100 M_{Sun} > 10 M_{Sun}"
1: Student shows mathematical understanding, but does not connect concepts (or opposite)	Homework Assignment 2: #1.b) How do Kepler's laws contradict the idea that all planets are in uniform circular motion around the Sun?	"Kepler's first law says that orbits are elliptical with the central mass (the sun in this case) at one focus, and a circle is just a special form of an ellipse."* * Note that this student's example shows conceptual understanding, but does not connect mathematics.

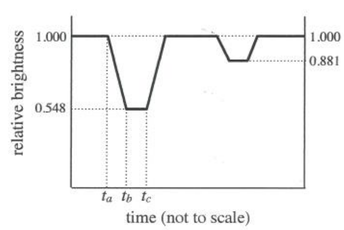
2: Student does mathematics but does not fully explain/makes mistake in one area Connection between mathematics and concepts is clear but one aspect is incorrect	Homework Assignment 3: #1.b) How does the gravitational force that one object exerts on another object change if the distance between them triples? If the distance between them drops by half? Explain how you know.	"If the distance between two objects triples, the gravitational force would decrease because of the relationship $F_{gravity} \propto \frac{m_1 m_2}{d}$ d being the distance between the two. If the distance between them drops by a half, then the force of gravity would increase because of the same relationship as before."
3: Connects concepts to mathematics clearly and both are correct	Exam 2: #2.) You are studying a faraway elliptical galaxy and have been able to measure its distance, size, and velocity dispersion. Unfortunately, images of this galaxy do not show gravitational lensing of background galaxies or quasars. Is it still possible to measure its mass? If so, write a formula for your mass estimate, identify any variables that appear in this formula, and explain any assumptions you choose to make. What physical principle is behind your approach?	"It is still possible to measure its mass. $M = \frac{3\beta}{\eta}\frac{R\sigma^2}{G}$ Where β and η are variables concerning isothermal and isotropic qualities, R is the Radius, σ is the velocity dispersion, and G is the gravitational constant. The formula assumes that this elliptical galaxy has been in a state of near equilibrium, such so that the virial theorem may be applied."

Table 11: Examples of Student Responses to Quantitative Problems with Assigned Scores

Rubric:	Source/Quantitative Problem:	Example Quantitative Student Responses:

0: No attempt made No conceptual connection & No mathematics	Homework Assignment 2: #3.) You are on a space mission to land on various asteroids in the solar system. After many long years in cramped quarters and some "misunderstandings" with your crewmates, you've developed a nagging suspicion that they are planning to leave you behind on the next asteroid. You've decided that you're only going out there again if you can jump off the asteroid under your own power to get back to your spaceship before it can leave without you. Can you do it? Let's find out… (a.) Estimate the velocity you achieve when you jump straight up. Hint: Use the height you reach jumping on Earth to estimate the change in your potential energy, and then use conservation of energy to estimate your initial kinetic energy. (b.) Now estimate the radius of the largest asteroid you could escape from by jumping. (You will need to make an assumption about the mean density of asteroids, which are made of rock and ice; be sure to explain your reasoning.)	(a.) h = .5 m $mgh = \frac{1}{2}mv^2$ $\left(9.8\ {}^{m}/_{s^2}\right).5\ m = \frac{1}{2}v^2$ $v = \sqrt{g}$ (b.) from 1.c): $v_{esc} = \sqrt{\frac{2GM}{r}}$ $v^2 r = 2GM$ $r = \frac{2GM}{v^2}$ where $v^2 = 9.8$ m/s

1: Student shows mathematical understanding, but does not connect concepts (or opposite)	Homework Assignment 8: #2.) The vertical motion of stars in spiral galaxies depends on the gravity exerted by the disk, so it allows us to "weigh" the disk. (a.) Use dimensional analysis to derive an estimate of the mass density ρ of a spiral galaxy disk, in terms of its scale height h_z, its vertical velocity dispersion σ_z, and a relevant physical constant. (b.) In the neighborhood of the Sun, the Milky Way has $h_z \approx 350$ pc and $\sigma_z \approx 16$ km s^{-1} for the thin disk, and $h_z \approx 1$ kpc and $\sigma_z \approx 35$ km s^{-1} for the thick disk. Use these values and your result from part (a.) to estimate the mass density of the Milky Way's disk, in $M_\odot$ pc^{-3}. Do the thin and thick disks give a consistent density estimate to the level of precision we might expect from dimensional analysis?	(a.) Given = h_z, σ_z $\frac{MT^2}{L^3} \cdot \frac{L^2}{T^2} \cdot \frac{1}{L^2} = \frac{\sigma^2}{Gh^2} = \rho$ (b.) $\frac{(16\ {cm}/{s}\ x10^5)^2}{\left(6.67x10^{-8}\ {cm^3}/{gs^2}\right)(1.07x10^{21}cm)^2} = 3.35x10^{-23}\ {g}/{cm^3}$ $\frac{(35\ {cm}/{s}\ x10^5)^2}{\left(6.67x10^{-8}\ {cm^3}/{gs^2}\right)(3.08x10^{12}cm)^2} = 1.936x10^{-17}\ {g}/{cm^3}$ We need a constant to adjust for the inefficiency of the dimensional analysis.
2: Student does mathematics but does not fully explain/makes mistake in one area Connection between	Homework Assignment 5: #2.) From the light and velocity curves of an eclipsing, double-lines spectroscopic binary star system, it is determined that the orbital period is 3.25 yr, and the maximum radial velocities of stars A and B are 5.2 km s^{-1} and 21.6 km s^{-1}, respectively. Furthermore, the time between first contact and minimum light is $t_b - t_a$	(a.) $\frac{k_B}{k_A} = \frac{m_A}{m_B} = \frac{21.6}{5.2} = 4.154$ $M = \frac{P}{2\pi G}(k_1 + k_2)^3 \rightarrow$ $\frac{(9.934x10^7 sec)}{2\pi(6.674x10^{-8})}(520000\ {cm}/{s} + 2160000\ {cm}/{s})^3$

mathematics and concepts is clear but one aspect is incorrect	= 0.45 days, while the length of the primary minimum is $t_c - t_b = 0.52$ days. Relative to the maximum brightness, the primary minimum is only 54.8% as bright, while the secondary minimum is 88.1% as bright (see schematic figure below). You may assume the orbits are circular and seen perfectly edge on. (a.) Find the ratio of the stellar masses (m_A/m_B), the sum of the masses ($M = m_A + m_B$), and the individual masses (m_A and m_B). (b.) Find the radii of the two stars. Hint: Use the speed of one star relative to the other and the eclipse timings given. (c.) During the primary (deeper) eclipse, is the larger star in front of the smaller star or vice versa? Is the larger star brighter or fainter than the smaller star? (Think about the brightness of the two minima relative to the maximum.) 	$M = 4.560x10^{33}g \rightarrow 2.293\ M_\odot$ $m_1 + 4.154m_2 = 2.293$ $m(1 + 4.154) = 2.293$ $m_2 = 0.445M_\odot$ $m_1 = 2.293 - 0.445 = 1.848M_\odot$ (b.) $R_{smaller} = \frac{(v_1 + v_2)(t_b - t_a)}{2}$ $= \frac{(5.2 + 21.6)(.45)}{2}$ $= 6.03\ km$ $R_{larger} = \frac{(v_1 + v_2)(t_c - t_a)}{2}$ $= \frac{(5.2 + 21.6)(.52 + .45)}{2}$ $= 12.998\ km$ (c.) During the primary eclipse, the smaller star is in front. This is evident by the deeper trench which implies the combined light of both stars is visible. The graph also shows that the larger star is somewhat brighter.
3: Connects concepts to mathematics	Homework Assignment 4: #2.) I mentioned that quasars and other active galactic nuclei are	(a.) typical quasar luminosity ~ $10^{12}\ L_\odot$

clearly and both are correct	thought to be powered by supermassive black holes (SMBH). Let's consider one aspect of this idea. (a.) A typical quasar luminosity is about 10^{12} $L_\odot$, where $L_\odot$ = 3.83 x 10^{33} erg s^{-1} is the luminosity of the Sun. If the energy is released by mass falling into a SMBH, estimate the mass accretion rate in solar masses per year. (b.) If the mass accretion rate is roughly constant, how long would it take to build a mass of 10^9 $M_\odot$? Is that long or short compared with the age of the Universe (about 15 Gyr)? Comment on whether the idea that quasars are powered by accretion onto SMBH makes sense or not.	$L_\odot$ = 3.83 x 10^{33} erg s^{-1} If energy is released by mass falling into a supermassive black hole $$L = \varepsilon \dot{M} c^2 \rightarrow \dot{M} = \frac{L}{\varepsilon c^2}$$ $$(\varepsilon = 0.1)$$ $$= \frac{10^{12} L_\odot}{(2.9979x10^{10} cm\ s^{-1})^2} \cdot \frac{3.83 \text{ x } 10^{33}\ erg\ s^{-1}}{L_\odot} \cdot \frac{g\ cm^2 s^{-2}}{erg} \cdot \left(\frac{1}{0.1}\right)$$ $$= 4.27x10^{25} g\ s^{-1} \cdot \frac{M_\odot}{1.989x10^{33} g} \cdot \frac{86400\ s}{d} \cdot \frac{365.25\ d}{yr}$$ $$= 0.678\ M_\odot yr^{-1}$$ (b.) If mass accretion is roughly constant $$\frac{10^9 M_\odot}{0.678\ M_\odot yr^{-1}} = 1.48x10^9\ yr$$ 1.48 x 10^9 yr < 1.38 x 10^{10} yr (age of the universe) So it is reasonable that quasars are powered this way.

The reliability of the scheme was established by two researchers who coded 10% of the data independently and achieved high levels of agreement. The reliability Kappa Ranged from: 0.73 to 0.96 for the questions analyzed between the two researchers.

We found that, for the homework problems collected (all of which contained both conceptual questions and quantitative problems), the students' scores on conceptual problems on the rubric above had an average conceptual mean of 1.92 (standard deviation 0.34), while their scores on the same rubric for the quantitative problems were higher - the average mean of 2.50 (standard deviation 0.36). The homework average scores for conceptual understanding were significantly smaller than the average scores for quantitative understanding using the t-test for paired two sample for means, $t(53) = -14.0$, $p <= 0.001$. In Figure 4, I compare the average scores for the homework assignments for both conceptual questions and quantitative problems. We discovered that the students had more difficulties with the homework's conceptual questions at the beginning of the semester than they did toward the end of the semester; whereas the students' quantitative scores remained fairly constant.

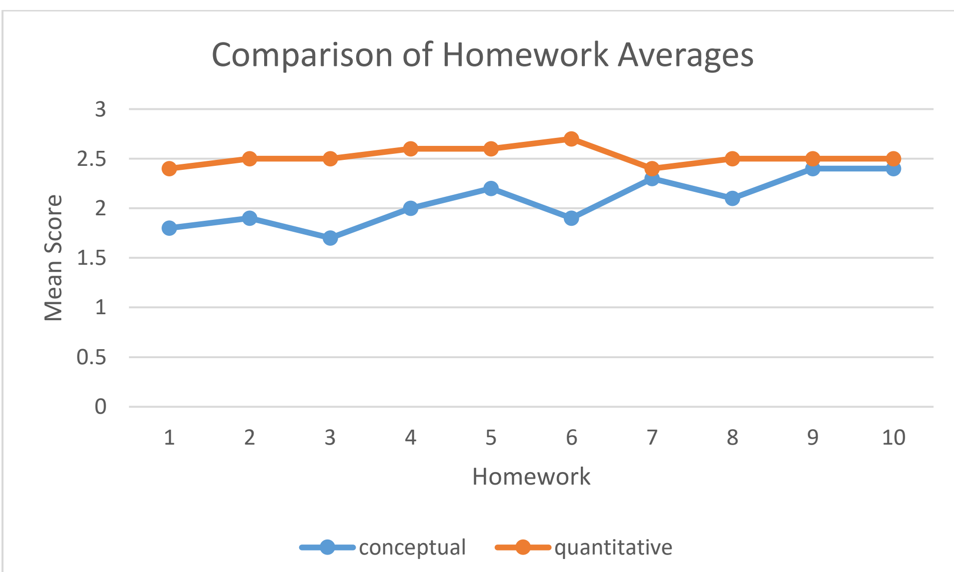
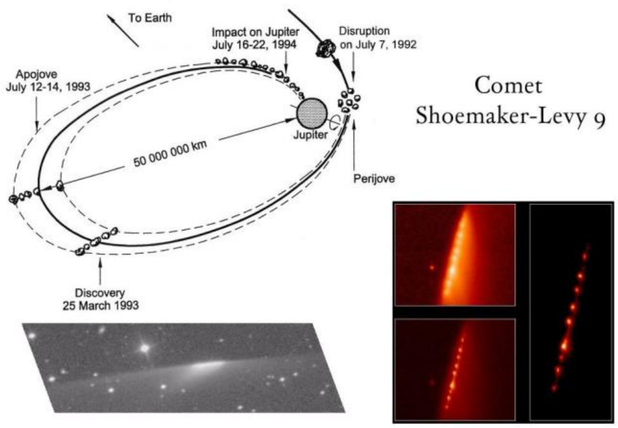


Figure 4: The students' scores for the conceptual questions improved during the semester.

The students also had more difficulties with different types of conceptual questions rather than particular topics. For example, "Jeopardy" style conceptual questions were easier for the students than conceptual questions in which interpretation was required. For instance, from homework assignment 10 (#1a.): "Create a question for which the following equation gives the solution: $\hat{\alpha} = \frac{4(6.67x10^{-8} dyne\ cm^2 g^{-2})(318M_{\oplus})}{(3.00x10^{10} cms^{-1})^2(11.0R_{\oplus})} = 0.0166''$.", the average score was 2.8 whereas in homework assignment 3 (#1a.): "Explain why you might describe the orbital motion of the moon with the statement, 'the moon is falling.'", the average score was only 1.4. Questions that involved concepts only (no equations were necessary to answer these questions) were easier than those that should have involved equations, but they didn't include them. This is illustrated by homework assignment 11 (#1a.) "Suppose you see a friend moving past you at a constant speed. Explain why your friend can equally well say that she is stationary and you are moving past her at a constant speed.", which had an average score of 2.8 compared to homework assignment 6 (#1c.): "The first direct observation of an extraterrestrial collision of Solar System objects was made in July 1994 when comet Shoemaker–Levy 9 (formally designated D/1993 F2) broke apart and collided with Jupiter as shown below. Explain what happened to this comet based on this diagram.", where the average score was only 1.5.



As stated above, both of the exams consisted of conceptual questions only. Exam 1 had a combination of multiple-choice problems and five short answer problems while Exam 2 consisted of ten short answer problems. The short answer problems were scored on the same 0-3 scale as the homework problems. In Exam 1 the average number of correct responses for the multiple choice portion was 10 out of 20. The average conceptual means for the short answer portion of the exams were 1.5 (Exam 1) and 1.8 (Exam 2) (standard deviation 0.64 and 0.48 respectively). This was a significant improvement in conceptual means between the two exams using the test for two-sample assuming equal variances, $t(52) = -2.7$, $p<=0.01$. Similarly to the homework assignments, in the Exams I noticed that the student's had more difficulties with different types of conceptual questions rather than particular topics with one notable exception. Again, "Jeopardy" questions were high scoring questions whereas open ended, interpretation style questions were low scoring (Exam 1/2 Averages: 1.8/2.4 and Exam 1/2 Averages: 1.0/1.3 respectively.); however, in addition to this trend in question style, I noticed that in both Exam 1 and Exam 2 the students' had considerable difficulty with the concept of negative potential energy.

In Exam 1, the conceptual question: "Calculate the kinetic energy of Earth moving around the Sun (assume circular orbits) using the quantities of solar mass, mass of Earth, and the AU. Calculate the gravitational potential energy of Earth - Sun system using the same quantities (with the usual convention of zero potential energy at infinite separation). Which of these energies is larger in magnitude and what is the ratio of these two energies? What is the sign of the total energy? Does this make sense? Why? [Do not plug in the actual numbers.]" had an average score of 1.0. In Exam 2, the conceptual question "Under what circumstances would we consider the gravitational potential energy negative? Are there any circumstances in which we would consider the gravitational potential energy to be positive? Give examples." had an average score of 0.9.

When comparing the mean scores for conceptual understanding to the mean scores for quantitative understanding, I found that the mean scores for conceptual understanding were significantly smaller than the mean scores for quantitative understanding using the t-test for paired two sample for means, t(53) = -15.0, p <= 0.001. We also discovered a strong positive correlation between the mean scores for quantitative understanding and the mean scores for conceptual understanding (r = 0.7296); as the student's conceptual understanding increases, their quantitative understanding also increases and vice versa.

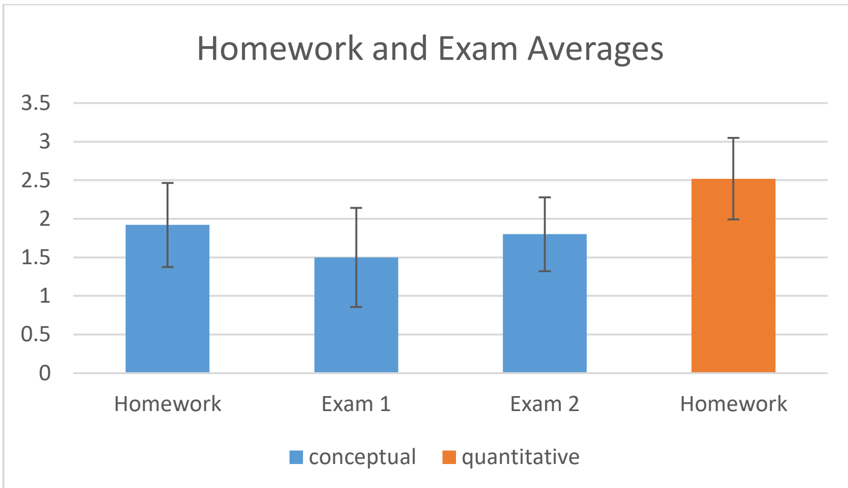


Figure 5: The students' average quantitative scores were significantly higher than the scores for conceptual understanding.

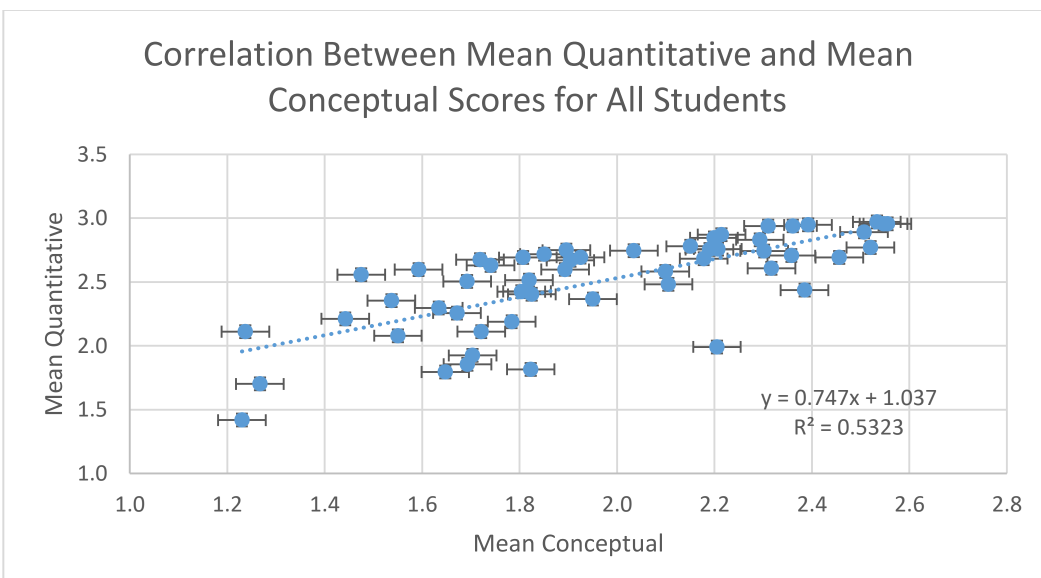


Figure 6: There is a strong positive correlation between Mean Quantitative Scores and Mean Conceptual scores for the students.

Although improving throughout the semester, the students' conceptual understanding was shown to be significantly lower than their quantitative understanding of the material. The style, rather than the topic of the conceptual questions, were indicative of how hard the conceptual question was to the student with "Jeopardy" style questions being the easiest and open ended "interpretation" style questions being the hardest. The students did however find the topic of negative gravitational potential energy to be difficult, earning the lowest scores on each of the two exams. Finally, I did discover a strong positive correlation between the students' mean quantitative scores and mean conceptual scores; indicating that the stronger the student's understanding of conceptual topics, the stronger the students quantitative understanding and vise versa.

4.2.c Interview Findings: How do the students connect the concepts to mathematics?

Ten student volunteers were selected based on their overall class performance. Nine of the students received a final grade of B+ or better; the tenth student selected was unable to attend an interview session and a volunteer with a final grade of C+ was interviewed. The interviews consisted of three problems:

> #1.) (a.) Imagine someone wanted to determine the kinetic energy of Earth moving around the Sun. How would you do that? (b.) What would be the gravitational potential energy of Earth? (c.) Which of these energies is larger in magnitude and what is the ratio of these two energies? (d.) What is the sign of the total energy? Does this make sense? Why?

> #2.) Propose a question which has the following solution:

$$a_r = \frac{F_r}{m}$$

$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

$$v^2 = \frac{GM}{r}$$

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r} = -\frac{1}{2}U$$

#3.) What would happen to a satellite's orbit if the satellite in orbit about Earth loses total energy due to gradual atmospheric drag? What would happen to the potential energy of the satellite? Do you know of any situation in astrophysics that is similar to this?

The videoed interviews were transcribed verbatim and analyzed using the holistic rubric shown below in Table 12. Based on the rubric, out of the ten students interviewed, two students (both of whom were A students) demonstrated no conceptual connections and/or no mathematical understanding of the equations, five students (two A students, two B+ students, and one C+ student) showed some mathematical or conceptual understanding of the equations, but did not grasp or connect this understanding with basic concepts, one student (an A student) showed a definite understanding of the connection between mathematics and concepts but made some mistakes, and two students (both of whom were also A students) showed a complete and correct understanding of connecting concepts to mathematics (See Figure 7). The students' final grades ranged from 104.2 to 78.83 with each grade shown for each of the ten students shown in Figure 8 below.

Table 12: Holistic Rubric

No Understanding	Partial Understanding	Understanding	Complete Understanding
No conceptual connection &/or No mathematical understanding	Student Shows some Mathematical or conceptual understanding, but does not grasp or connect with basic concepts	Student does mathematics but does not fully explain/mistake in one aspect Connection between mathematics and concepts is apparent but one aspect is incorrect	Connects concepts to mathematics clearly and both are correct
Student Example: "So we know the mass of the earth, and we can determine the velocity. The velocity would be the time of a year divided by the radius between the earth. No, it would – yeah. So velocity is the change in position over change in time. So our change in position would be two pi radius, which is one AU, over one year."	Student Example: "Total energy is zero, no it's not zero, it's negative. … so that's just negative ½ Gmm/r." "For [total energy] to be getting more negative, you'd have to make r smaller because the magnitude would have to be bigger, but it's still negative, so it's a smaller number. But if we're just talking about the magnitude of the total energy then the radius would have to get bigger so that the overall…"	Student Example: "Energy is larger so you want to compare [Kinetic Energy] and [Potential Energy] … Well, if you plug in for v squared I mean it would be the kinetic energy is equal to ½ m times v squared. So GM over r so this equals ½ GMm over r equals ½ U."	Student Example: "Since I solved for v squared, I know that kinetic energy is one half m v squared, which is one half G M little m over r. And potential energy is negative GMm over r. So the total energy is just kinetic plus potential. Which when you add them you get negative one half GMm over r. Now, we know that we want it to loose total energy which means that the energy … that E has to go down then the only way that is possible is

			if r gets smaller. As r gets smaller, the denominator gets smaller and this whole term gets … the absolute value of it would get bigger, but since there is a negative sign it's becoming more negative. So your total energy is decreasing that way."
Rational: The student clearly does not understand the basic concepts of rotational motion.	Rational: The student correctly identifies the total energy of the system "so that's just negative ½ Gmm/r."; but fails to understand what changing the size of the radius would do to the total energy.	Rational: The student correctly determines the kinetic energy "kinetic energy is equal to ½ m times v squared. So GM over r so this equals ½ GMm over r" that the student leaves off the negative in the final statement "equals ½ U.".	Rational: The student is correct in all parts of derivation and conceptual thinking.

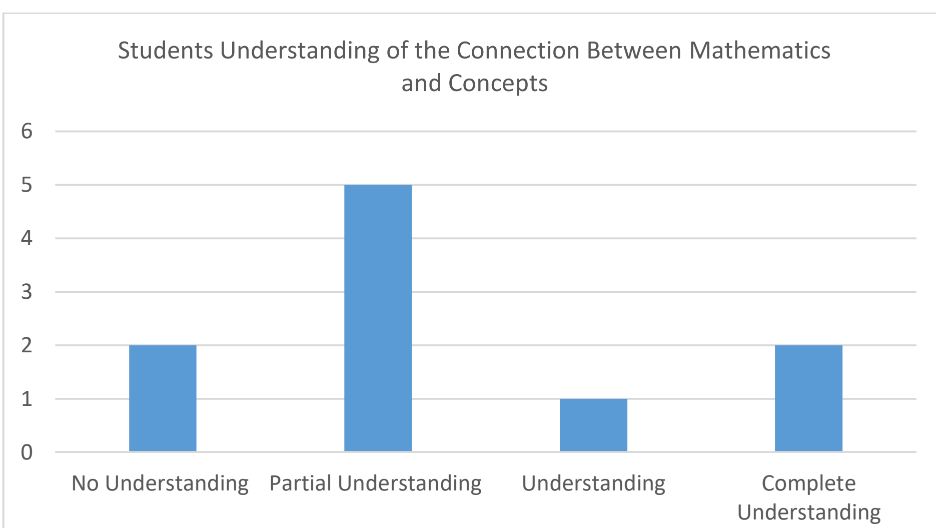


Figure 7: Most of the Students interviewed showed at least a partial understanding of the material.

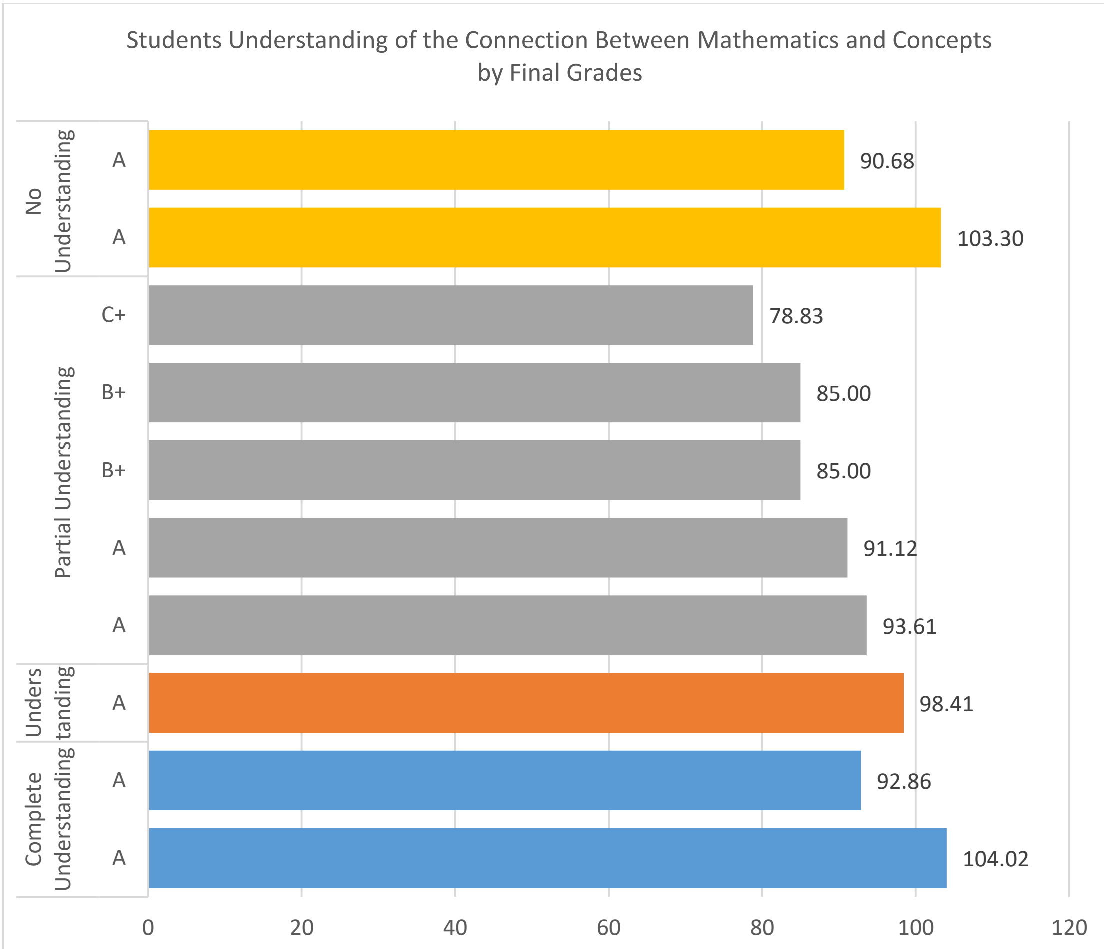


Figure 8: Students' understanding compared to their final grade.

The next analysis I did was the analysis of the "symbolic forms" that students use talking about the equations. A symbolic form is a "knowledge element" that is comprised of two components: a "symbol template" (how the idea is written in symbols) and a "conceptual schema" (the idea to be expressed in the equation). For example, the balancing form is described as "two influences, each associated with a side of the equation, in balance so that the system is in equilibrium" and can be shown by the symbol pattern ☐ = ☐ (Sherin 2001.) Sherin does note that the identity form (a single symbol that appears alone on one side of an equation that has the same properties as the expression on the other side) is so common it is "nearly invisible in student utterances", but does not remark on the variety of symbolic forms used to solve problems. The students in our study used the forms without any prompts, thus I can infer from the analysis that these are the forms they are comfortable with. I found that the students used a variety of forms however the relative proportion of each form was different for the students with different levels of understanding.

Table 13: Student Examples of Symbolic Forms

Symbolic Form	Description of Forms Identified	Symbol Pattern	Example Student Responses
Balancing	Two influences, each associated with a side of the equation, in balance so that the system is in equilibrium.	☐ = ☐	"So we can figure out the velocity by equating the gravitational force between the sun and the earth and the centripetal force of the earth moving around in the circle. So we can equate those two forces to get the velocity …"
Canceling	Two influences that precisely cancel so that there is no net outcome.	0 = ☐ - ☐	"I think it's two, two kinetic energy plus potential is overall […] zero …" (Note that this is incorrect.)
Dependence	A whole depends on a quantity associated with an individual symbol.	[… *x* …]	"…we said that we wanted to make that [velocity] bigger, or more negative. So it would have to bigger in magnitude."
Identity	A single symbol that appears alone on one side of an equation has the	*x* = …	"…we know kinetic energy is just ½ mv^2."

	same properties as the expression on the other side.		
Opposition	Two terms, separated by a minus sign, associates with influences that work against each other.	☐ - ☐	“Because potential energy is bigger, then total energy is going be smaller.”
Parts-of-a-Whole	Amounts of generic substance, associates with terms, that contributes to a whole.	[☐ + ☐ + ☐ …]	“so for that you just add the two of them [Kinetic and Potential Energy] and you get negative one-half G M sun m earth over r.”
Prop -	Indirectly proportional to a quantity, x, which appears as an individual symbol in the denominator.	$\left[\frac{\ldots}{\ldots x \ldots}\right]$	“Which when you add them [Kinetic and Potential Energy] you get negative one half GMm over r. Now, we know that we want it to loose total energy which means that the energy … that E has to go down then the only way that is possible is if r gets smaller.”
Prop +	Directly proportional to a quantity, x, which appears as an individual symbol in the numerator.	$\left[\frac{\ldots x \ldots}{\ldots}\right]$	“So if energy is being lost, then either v^2 would have to get smaller or its radius would have to be smaller as well.”

Ratio	Comparison of a quantity in the numerator and denominator.	$\left[\frac{x}{y}\right]$	"In magnitude, the gravitational potential energy is bigger. And if you take the ratio of gravitational to kinetic, you get … So GMm over r […] Times the reciprocal of that [KE] so you get two r over G M sun m earth. And then canceling everything you get two over one. So the gravitational potential energy is twice as large as the kinetic in magnitude."

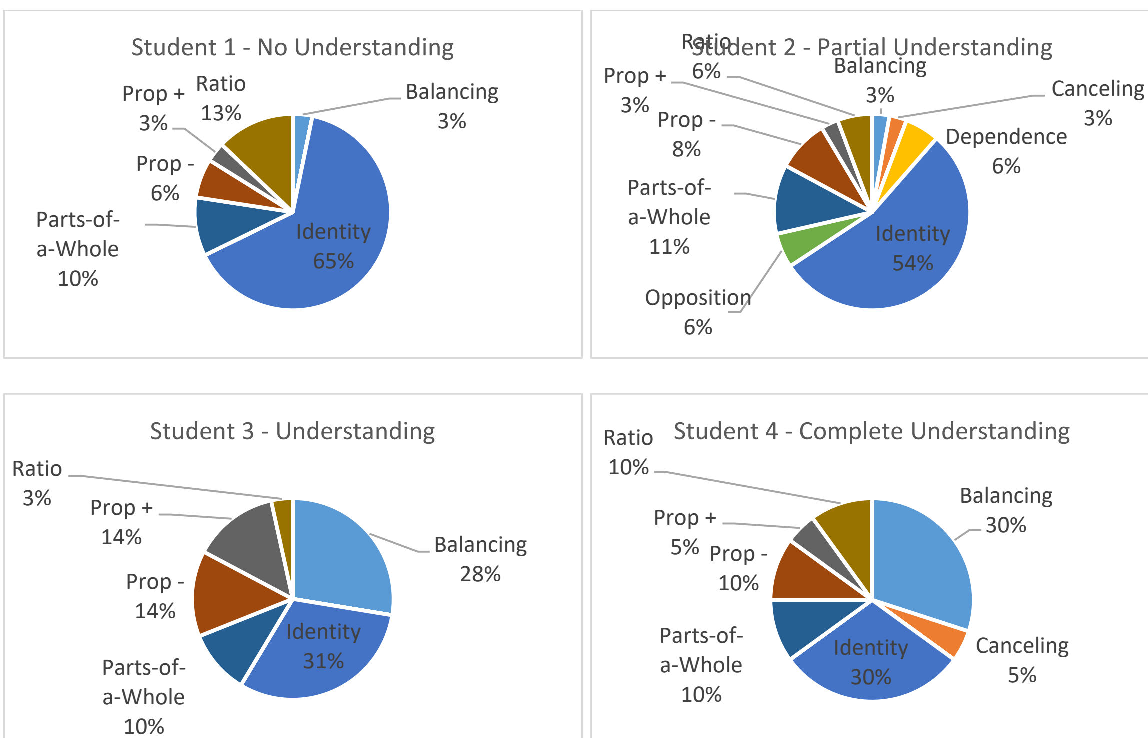


Figure 9: Students' utilization of symbolic forms.

All of the students used about the same variety of symbolic forms; however, the weaker students gave much more weight to the identity symbolic form than the stronger students. The students who used the variety of symbolic forms more evenly were able to set up the problem correctly (usually with a diagram) and had little difficulty understanding all aspects of the problems. The students who used the variety of symbolic forms less evenly, showed little to no understanding of how to approach the problems and in general had to be walked through large portions of the problems. The following discourse is an example of such a moment:

> Student: Right, so in circular motion it's like the velocity is … oh, what was that one?
>
> Investigator: Shall I give you a hint?
>
> Student: Maybe…
>
> Investigator: Draw a force diagram.
>
> Student: A force diagram…
>
> Investigator: A force diagram…
>
> Student: Oh, a force diagram… It's got just one force.
>
> Investigator: What is that force?
>
> Student: The force of gravity.
>
> Investigator: So, it's the force of gravity. Then according to Newton's second law what can we say about that then? Well, first of all, what is Newton's law, second law?
>
> Student: F equals m a. *Identity Form*
>
> Investigator: So can we go from there?
>
> Student: So force equals m a and that's GMm over r.

Investigator: GMm over r…	*Balancing Form (incorrect)*
Student: Squared.	*(correct)*

These students also seemed to be completely thwarted when they were not provided with any formulas. Examples of this include: "I think it's two, two kinetic energy plus potential is overall - or no, zero", "If I had the table I would give a definite answer" and "…we know that is Gm/r, I think. Or r^2, it's one of them." In addition, seven of the ten interviewed students had at least some difficulty understanding the meaning of the negative sign in the gravitational potential energy equation with five of these students having great difficulty. For example: "So the radius has to be getting bigger than … well, if we want to make it more negative, I guess you could say you're losing it that way, then the radius would be getting smaller." "For that to be the case, for it to be getting more negative, you'd have to make r smaller because the magnitude would have to be bigger, but it's still negative, so it's a smaller number. But if we're just talking about the magnitude of the total energy then the radius would have to get bigger so that the overall…" Those who did not conceptually understand negatives did not show any understanding of the virial theorem.

Of the ten students interviewed, those who showed a greater understanding of the conceptual connections to the mathematics and equations involved in solving the problems used a richer variety of symbolic forms, with less emphasis on the identity form, and had an overall greater understanding of the material – they did not need to be given equations, understood the meaning of the negative sign in the gravitational potential energy. For example: "So what you are actually doing is you're deriving it from the force. So you take the integral of F dr or dx depending on what your coordinate system is. And usually you say your reference point as one of the bounds of the integral and the other point being where you are. And in this case you're setting the … one of the bounds as infinity so your second gravitational potential energy term drops to zero”, Note how the same person spoke about the virial theorem: “For some of the previous questions it is [in equilibrium the entire time] because it’s still like a stable orbit. But in this case technically since you’re losing energy through drag, it’s not exactly in equilibrium. But I guess like to an approximation, you can just use that and use the virial theorem.”

Overall, most of the ten interviewed students showed at least a partial understanding of the problems they solved in the course of the interview. Although all of the students showed about the same variety of symbolic forms, the weaker students tended to rely heavily on the “identity” form compared to the stronger students. The weakest students also had a difficult time when they were not presented with any formulas from which to start. Most of the students had at least some difficulty understanding the meaning of the negative sign in the gravitational potential energy equation and did not fully comprehend the virial theorem. Those students who had a better understanding of the material used the “identity” symbolic form less, had a greater understanding of negative gravitational potential energy, and were able to use and understand the virial theorem successfully.

Chapter 5: Interpretation of the Findings

5.1 What do the students think it means to understand astrophysics equations?

To answer Research Question #1, I analyzed data from the online Likert surveys and from the student essays. The Likert survey consisted of twenty-one questions, many of them focusing on what the students think it means to understand astrophysics equations. The student essays focused on the students' beliefs concerning their favorite and least favorite topics from the class; concentrating on the "relevant equations and the meaning of their terms as well as how the equations influenced [their] attempts to understand the concept." From their responses one can infer what they think about understanding equations. Below I present the results of the survey and essays that are relevant to Research Question #1: What do students think it means to understand astrophysics equations.

5.1.a Where do the equations come from?

It is clear, when looking at the data from all of the students' responses from the survey, particularly the results from question # 2, that the majority of the students believe that derivations of equations are important in astrophysics but only as a way to get the final equations. Seventy two percent of the students agreed that "A derivation or proof of an equation shown in class is useful because I can use the final equation while working on the problem sets without working through the derivation myself." It is interesting to note from questions # 5 and 12, however, that although the students believe that the end result (the equation) is the most important aspect of the derivation, the students also believe that they can derive the equations themselves when necessary and that the derivations are essential to understanding the equations that they use. An interesting point to make is that 100% of the top quartile of the class believes that "if I forget an equation or cannot find the right one, they can still work on the problem", implying that they can derive the correct equation on their own. In contrast only 85% of the bottom quartile answered similarly.

This is supported by the students' essays. Only ten students (19%) mentioned derivations in conjunction with the equations they discussed; half of which mentioned only that a "complex" derivation was used to get the equation (9%). Again, this implies that the majority of the students do not think that derivations are very important – at least not important enough to mention in their essays. Each of the five students who mentioned derivations positively in their essays mentioned reviewing the derivations for a better understanding of the equations that they discussed. Comments such as: "The other aspect of difficulty came in the derivation of the correctional forces that must be considered in such a frame. To better understand it, I tried to go through the derivation of the centrifugal, Coriolis, and Euler forces as best I could. I'm not completely comfortable with this concept yet, but I expect to be as I progress in school" and "It always helps to look at multiple ways of deriving an equation, in order to gain other perspectives about the material" were common among these five students. It is interesting to note that of these five students who discussed derivations as a way to understand the equations better, four were in the top 25% of the class. (The other six students were average students; in the middle 50% of the class.) The five students, who mentioned derivations in passing in their essays, mentioned that the derivations were difficult but yielded equations that they used. "Though the derivation is complex, this equation is fairly intuitive and helps clarify what contributes to tidal forces" and "The way we had derived it was unintuitive and rigorous, while seeming very arbitrary and extremely pointless" were common among these five students.

In other words, overall the students believe that the final equation is the ultimate goal and the derivation itself, while useful, is not the most important point of the equations in astrophysics.

This belief is clearly demonstrated by the student's answers to the Likert survey as well as in their discussions (or lack of discussion) of the derivations of the equations they provide in their essays.

5.1.b What does it mean to understand an equation?

From the results from questions #6 and #19 in the student survey, it can be concluded that the students believe that the equations they use need to be understood conceptually. 85% of the class believes that "the equations they use need to be understood conceptually" and 75% of the class believes that "To be able to use an equation in a problem (particularly in a problem that I haven't seen before), I need to know more than what each term in the equation represents." (Matching several of Domert et al.'s arguments that understanding physics equations involve: recognizing the structure of the equation, being able to recognize the symbols in the equation in terms of the corresponding physics quantities, and establishing a link between the equation and everyday life.) This belief is supported in many of the students' essays: "While explanation was given to better understand these terms, I still felt as if I didn't have enough interaction or manipulation with the concepts." The students also show that they think that equations help them understand the concepts better. "This equation definitely helped me understand the concept of tidal forces, and having a strong base understanding is what allowed me to become immersed in the further applications and effects that tidal forces have."

Although the students clearly believe that they need to conceptually understand the equations they use, does their view of what it means to understand an equation match the expert perception of understanding of equations? Using the framework of understanding of physics equations (Domert et al., 2012), we can compare what Domert et al. believe understanding involves versus what the students believe that understanding astrophysics equations involve:

Table 14: Students' Beliefs versus Domert et al. concerning Understanding Equations

<table>
<tr><td>Domert et al.
Maintains that understanding physics equations involve:</td><td>Quotes from Students' Essays
Representative of the Students' Beliefs Concerning Understanding Equations</td><td>Students Beliefs
Students believe that understanding astrophysics equations involve:</td></tr>
<tr><td rowspan="2">• Being able to recognize the underlying physics of the equation.</td><td>• "I had problems understanding the way the winding problem was solved. In my understanding I thought that differential rotation solved the winding problem. After I … understood what the density wave theory physically modeled, I was able to visualize the recycling movement of stars between spiral arms and the rest of the galaxy as time progresses."
• "Though the derivation is complex, this equation is fairly intuitive and helps clarify what contributes to tidal forces"</td><td>• Conceptually understanding the equations that they use.
• Use of derivations of equations, but only as a means to get the equation.</td></tr>
<tr><td colspan="2">These two common student beliefs appear to contradict themselves. On one hand the students agree with Domert et. al and appreciate that they need to conceptually understand the astrophysics equations they use, but on the other hand they do not believe that they need to understand the derivations of these equations and can just use the derived equations to obtain an answer to a problem. 83% of the students discussed the astrophysics in the equation they provided for their essay, but only 30% of the students did so with a deep understanding.</td></tr>
<tr><td>• Being able to recognize the symbols in the equation in terms of the corresponding physics quantities.</td><td>• "$|\Delta F| = (2GMm/r^3)\Delta r$ Where M and m are the two masses of the system, with m being the test mass, Δr as the radius of the test mass, and r^3 as the distance between the centers of the masses involved in the tidal force system."</td><td>• Recognize the symbols in the equation in terms of the corresponding physics quantities.</td></tr>
</table>

<table>
<tr><td></td><td colspan="2">The students’ beliefs agree with Domert et al. in this matter; however, only 51% of the students who provided equations in their essays also voluntarily discussed the symbols in the equation. As the students were not explicitly asked to provide an explanation of the symbols in their equations, this omission could potentially be because the students believed the symbols would be understood by the instructor.</td></tr>
<tr><td rowspan="2">• Recognizing the structure of the equation.</td><td>• “I realized there is a differential force, (ΔF), that changes proportionally to the, Δr of the test mass.”</td><td>• Understanding and recognizing the relationships connecting the variables of astrophysics equations.</td></tr>
<tr><td colspan="2">Again, according to the students’ surveys, the students’ beliefs agree with Domert et al. in this matter; however, only 33% of the students discuss the structure of the equations in their essays. This once again may be due to the fact that the students were not explicitly asked to do so in their essays and not due to their beliefs that it is important in understanding astrophysics equations.</td></tr>
<tr><td rowspan="2">• Establishing a link between the equation and everyday life.</td><td>• “Given that a small change in r^3, by setting $r^3 = (R_s)^3 = (GM/c^2)^3$,
R_s = Schwarzschild Radius
M = Mass of Black Hole
C = speed of light
gave a maximum tidal force that, at a certain distance (Schwarzschild Radius) and towards the center of mass of the black hole, was greater than the electrostatic force of the particles that comprise up of our body.”</td><td>• Connections between equations and real world.</td></tr>
<tr><td colspan="2">Overall, the students’ beliefs agree with Domert et al. in this matter. 53% of the students discuss the connections between the equations they provided in their essays and the real world.</td></tr>
<tr><td>• Knowing how to use the equation to solve problems.</td><td>“… I simply had to relegate to chugging through the equations (specifically the equation for a</td><td>• Finding the “right” equation to use in a problem.</td></tr>
</table>

• Being able to know when to use the equation.	harmonic oscillator), rather being able to trust my intuition …"	
	The students believe that to solve problems they simply need to find the "right" equation to use, not know how or why the equation they found from the class notes is correct.	

From the results of the 48 students who provided equations in their essays, we can infer that the students find importance in describing their mathematical understanding of the equations they mention as well as showing understanding of the purpose of the equation. For example, one student describes time dilation: "The time dilation equation stated that more time passed in an unprimed/stationary time frame, than did the time in the primed/moving frame […] the exact equation was $\Delta t = \Upsilon\Delta t'$, where $\Upsilon=[1-(u/c)^2]^{-.5}$. The unprimed Δt is the time in the frame of the stationary location, $\Delta t'$ is the change in time in the moving reference frame u is the velocity of the moving frame, and c is the speed of light. By observing these equation, you can see that since u can never be equal to c, the Υ is always going to be greater than or equal to 1; therefore, the time that passes in a stationary time frame is going to be greater than the time that passes in the mobbing reference frame." Note that this student also discusses the symbols in the equations he discussed; however, only about half (51%) of the total class does so as well. This aspect of understanding equations is clearly important from the results of the essays. The top quartile discusses the symbols in their equations with a greater percentage than the lowest quartile (67% to 40% respectively).

The students are almost evenly divided concerning making connections between equations and the real world. Those that did make connections however, did so in very tangible ways. "Without the knowledge we have about relativity and time dilation, our GPS systems would not be able to function correctly; this is due to the satellite clocks running slower than ours because they are moving faster than us, at about 14,000 km/hr. This can lead to major time dilation errors after a few minutes of the satellite orbiting Earth, and needs to be corrected by using atomic clocks. Whether we know it or not, time dilation and relativity plays an important role in our everyday lives."

Most of the students (67%) do not discuss the structure of the equations they include in their essays, they do not find significance in describing the equations in words (81%), or show any deep understanding of the astrophysics behind the equation (70%). The top quartile shows similar results to the majority of the class. From this we can see that most of the students are lacking in some of the aspects of understanding of what it means to understand astrophysics equations. This issue becomes much more apparent when we look at the bottom quartile of the class. This group of students lacks this understanding completely; 90% of these students do not discuss the structure of the equations and 90% do not show a deep understanding of the astrophysics behind the equations.

The students clearly believe that they must conceptually understand the equations that they use, but they probably do not know what this means – particularly the bottom quartile.

5.1.c Applying equations to solve problems

One interesting find from the student responses to the Likert Survey is that the students believe that they must find the "right equation" in order to solve problems in astrophysics. 81% of the students believe that "To solve a problem in astrophysics I need to match the problem situation with the appropriate equations and then mathematically manipulate and/or substitute values to get an answer." This is interesting because it implies that the students, while believing that they need to understand the equations in order to understand astrophysics, do not believe that they need to understand the equations when actually using them in the problems. The equations seem to be the start of a mathematical exercise rather than a way to understand an astrophysical phenomenon.

68% of the students believe that "When I solve a problem in the problem sets, I go straight to the lecture notes to find an equation that applies to the situation described in the problem." Again, this belief seems to imply that the students are looking for the "right equation" and not looking for a deeper understanding. Note that only 54% of the top quartile agree with this statement – implying that almost half of the top quartile believe they must understand how the equations relate to the problems they are attempting to solve; while 69% of the bottom quartile agree with the statement.

These results show that the students believe that they need to have a deep understanding of the equations that they use in order to understand astrophysics while simultaneously believing that, when applying the equations to problem solving, they do not need to deeply understand the equations – they can just be taken as a mathematical tool.

5.1.d Summary & Possible Implications

From the results of the Likert survey and the essays, we can see that the majority of students believe that derivations of equations are important in astrophysics but only as a way to get the final equations. The successful students however use derivations to better understand concepts behind the equations. This strongly implies that added emphasis should be given to the *purpose* of derivations throughout the course. This could be in the form of homework assignments by having the students derive some equation on their own and to reflect on what they learned through the derivation.

The students also believe that conceptual understanding of the equations is important. They are not however always successful in doing so. From the framework of understanding of physics equations by Domert et al. (2012), the students are only partially successful in understanding astrophysics equations. It is clear that the students need more practice in recognizing the structure of the equation as well as connecting the equations to a deep understanding of astrophysics. The instructors need to explicitly address all aspects of understanding and include multiple questions on the exams that focus on these aspects. As an example a question such as "Explain why kinetic energy can never be negative, but the potential energy can be positive, negative, or zero." would test that the students understand the structure of these common energy equations as well as understanding their physical meaning.

Lastly, the students appear to have a disconnect when using the equations to solve problems. When discussing the equations, the students see the need to understand the equations in a conceptually meaningful way. However, when in practice, the students simply use the equations as a mathematical platform to solve the problem. Asking the students "What does this [equation, derivation, solution, etc.] mean?" or asking students to do limiting case analysis of the equations throughout the problems might help the students connect the equations to the concepts.

5.2 What does student qualitative understanding of an equation look like?

Research Question #2 was analyzed using the data from the students' homework assignments and exams, the interview videos, and from the student essays. The relevant interview videos consisted of students solving astrophysics problems related to negative potential energy and the virial theorem while the student essays focused on the students' beliefs concerning their favorite and least favorite topics from the class; concentrating on the "relevant equations and the meaning of their terms as well as how the equations influenced [their] attempts to understand the concept."

From the students' responses to these data, we can determine whether the student has a qualitative understanding of astrophysics equations. Below I present the results of the data that are relevant to Research Question #2: What does student qualitative understanding of an equation look like?

5.2.a No Understanding

In order to understand what student qualitative understanding of astrophysics equations looks like, I first needed to designate which students out the students enrolled in the first of the two-semester sequence called Principles of Astrophysics offered at Rutgers University in the Fall 2013 Semester did not have qualitative understanding of an equation. I began the analysis by determining which of the students in this class showed "no" qualitative understanding of the astrophysics equations used throughout the semester. This was accomplished by looking at the student's written work (homework assignments, exams, and essays), followed by examining the student interviews. It became clear, when examining this data through the application of the theoretical framework described in pervious chapters, which students out of the entire class population ought to be classified as having "no understanding" of an equation. These students, those that were shown to have no overall qualitative connection to and/or no mathematical understanding to the equations they use in a problem, were then classified as students with "no understanding" of the astrophysics problems for the particular assessment. The following table illustrates the connection between the theoretical framework and the students' assignments with the students who were classified as those with "no understanding" of astrophysics equations.

Table 15: Indicators of Students with "No Understanding" of Equations

Students with "no understanding" of astrophysics equations …				
	Homework Assignments	Exams	Essays	Interviews

Theoretical Framework				
… recognize the symbols in the equation in terms of the corresponding physics quantities.	No	Yes	No	No
… recognize the underlying physics of the equation.	N/A	N/A	No	No
… recognize the structure of the equation.	N/A	N/A	No	No
… establish a link between the equation and everyday life.	N/A	N/A	No	Yes
…demonstrate knowledge of how to use an equation to solve astrophysics problems.	N/A	N/A	N/A	No
… are able to know when to use an equation.	No	No	N/A	No
Additional Indicators				
... can start the problem without assistance.	Unknown		N/A	No
… made conceptual or mathematical connection to the equations.	No	No	N/A	No

As can be seen in Table 15, the students from the class that were subsequently classified with "no understanding" of astrophysics equations did not meet any of the attributes outlined in the theoretical framework with the exception of: "recognize the symbols in the equation in terms of the corresponding physics quantities" in the Exams and "establish a link between the equation and everyday life" in the Interviews. Additionally, the theoretical framework as outlined in the literature did not cover all of the indicators that were observed that students with "no understanding" share. In addition to the theoretical framework, these students could not start a problem without assistance and made no conceptual or mathematical connection to the equations.

Homework Assignments & Exams

As stated earlier, the students had the option to complete their homework assignments in groups (as long as they stated on their assignment who they collaborated with), while the exams were an individual effort. A formula sheet was provided for both in-class exams and all resources were available for the students during their homework assignments (again, as long as sources were cited on their assignment). Given all of the resources available for these tasks, there was a strong indication that the students who performed poorly did not have a strong qualitative understanding of the equations necessary to execute the problems.

When the scores for the qualitative portion of the homework assignments were averaged, eight (8) students out of 54 (15%) were found to be students with “no understanding” qualitatively of the equations which they used. These data were analyzed using the rubric discussed in detail in the previous section; they either did not attempt the problem or they made no conceptual or mathematical connection to the equations necessary to complete the problem successfully. Although it is impossible to say that the students who did not attempt a problem truly had no understanding of the equations, the fact that they had unlimited resources to complete these homework assignments but still did not attempt a problem in the assignment does suggest that they did not understand the problem or the equation necessary to complete the problem.

The responses given by the students with “no understanding” in homework assignment #3 are indicative of the overall data for these students’ homework assignments. Since gender was not analyzed as part of this study, all students are assigned male names selected from the 100 most popular given names for male babies born during the 100-year period from 1917-2016 based on Social Security card application data as of March 2017. The qualitative problems in homework assignment #3 were as follows:

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise.

(a) Explain why you might describe the orbital motion of the moon with the statement, "the moon is falling."

(b) How does the gravitational force that one object exerts on another object change if the distance between them triples? If the distance between them drops by half? Explain how you know.

(c) Equation Jeopardy: Create a question for which the following equation provides an answer:

$$0.1'' = 1.22\left(\frac{400nm}{1m}\right)\left(\frac{360\ x\ 60\ x\ 60''}{2\pi}\right)$$

With the expected answers to these questions:

(a) If there were no gravity, the moon would travel in a straight line at its instantaneous velocity. Instead, gravity causes it to "fall" towards the Earth with a purely radial acceleration. The initial tangential velocity causes this to lead toward a closed orbit rather than a head-on collision.

(b) Gravitational force is an inverse-square law ($F = G\frac{M_1 M_2}{r^2}$), so if the distance triples, the force decreases by a factor of nine. If the distance drops by half, the force increases by a factor of four.

(c) What is the angular resolution of a 1m diameter telescope at the bluest light that human eyes can see?

Eight (8) out of the 54 students (15%) who submitted this homework assignment showed "no understanding" for part (a), while two (2) students (4%) showed "no understanding" for part (b), and seven (7) students (13%) showed "no understanding" for part (c). Since parts (b) and (c) focus directly on qualitatively understanding equations, I will concentrate on these two parts. Only two students showed "no understanding" in part (b) in this question due to selecting incorrect equations. If the student chose the wrong equation for the problem they were attempting it was considered an indication that the student was not able to know when to use an equation. It is interesting to note that one of these two students (Jonathan) worked alone, while the other (Timothy) worked in a group and had four collaborators.

Jonathan showed "no understanding" of the equation in his answer to this problem because he chose the wrong equation for the problem given and he made no conceptual connection to the problem in his response. Jonathan's answer "If the distance triples, the gravitational force will be $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. If the distance halves, the gravitational force will be $\left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}$. You can find this by assuming that all other variables are constant, changing the distance to its new value, and then finding out how the gravitational force changes." implies that this student incorrectly believed that the gravitational force was proportional to an exponential function ($F \propto \left(\frac{1}{2}\right)^r$). It also shows that he made no conceptual connection to the equations as he only alluded to "other variables" and did not truly explain how the gravitational force would change, just that you could "find out how" it changed.

Timothy also selected an incorrect equation to use in his analysis of the problem. It is clear from his answer: "Gravitational force: $F = \frac{GMm}{r}$ If the distance were tripled then the gravitational force would decrease by a factor of 3. If the distance dropped by half then gravitational force increases by a factor of 2." that this student chose the wrong equation for the problem given; believing that the gravitational force is proportional to the inverse of the distance (instead of the correct inverse-square law). It is particularly interesting that Timothy selected the incorrect equation whereas the rest of the collaborators had correctly identified gravitational force as an inverse square law.

Even if the students are able to choose the correct equation, they do necessarily show understanding of the equation. A third student, Gregory, showed "no understanding" when solving this problem because he was not able to make a mathematical connection to the equation. His solution "Following Newton's Law of Gravitation, one can test to see what happens to the force, thusly: $F = \frac{GMm}{(3r)^2} = \frac{GMm}{9r^2} \Rightarrow 9F = \frac{GMm}{r^2}$, and $F = \frac{GMm}{(\frac{1}{2}r)^2} = \frac{GMm}{\frac{1}{4}r^2} \Rightarrow \frac{1}{4}F = \frac{GMm}{r^2}$. Therefore, if the distance triples, the force changes by a factor if 9, while if the force is only a quarter of what it was originally, the distance drops by half." shows that the student did not make the mathematical connection to the equation necessary to answer the problem correctly.

Jonathan and Timothy both illustrate one aspect of the students that show "no understanding" while answering the homework problems; namely that students cannot complete a problem correctly without identifying the correct equation and any qualitative analysis concerning the equation used for the problem at hand is by default incorrect. However, even when given the correct equation in the assigned homework problem, as in part (c), students with "no understanding" do not make the conceptual connection to the equation given to them. Their answers, such as "What is the diffraction limit in arcseconds for a telescope with a D = .1m, λ = 400 nm.", "How many arc seconds are in .1″?", and "For a telescope with wavelength λ = 400 nm, diameter D = 1m + θ = 60°, what is the resolution?" show that these students do not have a fundamental understanding of the equations that they use in astrophysics problems as they did not recognize the symbols in the equation in terms of the corresponding physics quantities.

After examining the homework assignments, I then looked at the student's exams. Although the students did not have unlimited resources during the exams, they were supplied with a formula sheet with the necessary equations required to complete the exam. The students who showed "no understanding" in the homework problems were not necessarily the same students who showed "no understanding" in the exams, possibly because of this difference. The exams' results however are similar to the homework assignments in terms of not understanding the qualitative aspects of the equations used in problem solving; the students who showed "no understanding" also chose the wrong equation for the problem given and/or made no conceptual connection or mathematical connection to the equations necessary to complete these problems successfully. Unlike the students who showed "no understanding" in the homework assignments however, these students did recognize the symbols in the equation that they used in terms of the corresponding physics quantities but were not always able to know when to use an equation. This last indicator was again included for these students because they were provided with a formula sheet for their exams which included all of the equations necessary to complete the exam successfully. If they chose the wrong equation from the formula sheet for the problem, it can be assumed that they did not know when to use the equations.

To illustrate the similarities and differences of the results from the homework assignments to the exams I will focus on three students' responses to questions #1, #2, and #4 from Exam #2, as these questions most accurately reflect these students' qualitative understanding of equations. Note that these are not all the same students that were the focus of the homework assignments as there was a larger pool of students how showed "no understanding" in the exams. These questions:

Question #1:

The oldest stars and star clusters in the Milky Way appear to be 12 billion years old. Would this produce a conflict if the Hubble constant is 100 km/s/Mpc? Explain why.

Question #2:

You are studying a faraway elliptical galaxy and have been able to measure its distance, size, and velocity dispersion. Unfortunately, images of this galaxy do not show gravitational lensing of background galaxies or quasars. Is it still possible to measure its mass? If so, list the possible methods you could use to determine its mass and choose the best method. Why did you choose this particular method? Now, write a formula for your mass estimate, and explain any assumptions you choose to make. What physical principle is behind your approach?

Question #4:

Jupiter's moon Io has active volcanoes whose energy ultimately comes from the tidal forces exerted on Io by Jupiter.

(a.) If you wanted to estimate the tidal acceleration of a test mass at various points on the surface of the moon Io, which of the following quantities would be most useful?

- Mass of Jupiter
- Radius of Jupiter
- Mass of Io
- Radius of Io
- Distance from Jupiter to Io

(b.) Imagine the (fictitious) moon Galileo has 2 times the mass of Io, 3 times the radius of Io, and is 4 times farther away from Jupiter than Io. How would the maximum tidal acceleration (caused by Jupiter) of a test mass on the surface of Galileo compare with the maximum tidal acceleration of a test mass on Io?

have the following expected answers:

Question #1:

If the Hubble constant is 100 km/s/Mpc, the age of the universe would be:

$$T = \frac{1}{H} \cdot 10^{12}\ years = \frac{1}{100} \cdot 10^{12}\ years = 10^{10}\ years$$

This age is less than the age of the oldest stars and star clusters in the Milky Way.

Question #2:

We have measured the distance to the galaxy, D, its radius, R, and its velocity dispersion, sigma. We could have used gravitational lensing to measure its mass (within the Einstein radius) but nature did not cooperate, so we cannot use the gravitational lensing mass estimate from the formula sheet. We also cannot use the rotation curve mass estimate for spiral galaxies from the formula sheet. Nonetheless, we can use the virial mass estimate (on the formula sheet) to determine $M = \frac{3\beta}{\eta}\frac{R\sigma^2}{G}$. Note that the distance to the galaxy does not appear. If we assume an isothermal density profile, $\eta = 1$ (for a uniform density profile, η=3/5). If we assume isotropic orbits of identical stars, $\beta = 1$. So our best mass estimate is $M = \frac{3R\sigma^2}{G}$. This approach uses the virial theorem to estimate the ratio between kinetic and potential energy of a spheroidal mass distribution and to set that ratio equal to -1/2 to solve for the total mass, which is how we derived the virial mass estimate formula.

Question #4:

(a.) The following quantities would be most useful: Mass of Jupiter, Distance from Jupiter to Io, and Radius of Io.

(b.) The maximum tidal acceleration is proportional to the Mass of Jupiter and the Radius of Io and inversely proportional to the Distance from Jupiter to Io cubed

$a \propto \frac{M_{Jupiter} R_{Io}}{(D_{Io-Jupiter})^3}$ therefore the tidal acceleration would be $\frac{3}{4^3} = \frac{3}{64}$ as much.

However, the students (selected from the students highlighted above as well as some additional students) that had "no understanding" of the equations necessary to complete these problems gave answers such as:

Question #1:

- Timothy – Choses the wrong equation for the problem given, he simply re-wrote the two sequential equations on the formula sheet dealing with the Hubble constant. It is clear that the student makes no mathematical connection to the equations selected and also makes no conceptual connection to the equations.
  - "$v = H_0 d$

    $H_0^{-1} = 10\ Gyr\ x \left(\frac{100\ km\ s^{-1} Mpc^{-1}}{H_0}\right)$

    $H_0 = 100\ km\ s^{-1} Mpc^{-1}$

    No it would not have any conflict with the Hubble constant."
- Edward – Did not know that in order to solve this problem, he needed to evaluate the equation governing the situation and therefore were classified as "not able to know when to use an equation".

- "No because the Hubble constant (which isn't constant) has to do with the expansion of the universe and doesn't affect the inner workings of individual galaxies."

- Joseph – Was not able to know when to use an equation. The student simply preformed some unit conversions.
  - "$12e^{10}yr = 3.156 \cdot 12e^{17}s$

    $100\ km = 100 \cdot \frac{1000\ m}{1\ km} \cdot \frac{100\ cm}{1m} = 1e^{7}$

    $Mpc\ =\ 3.086e^{18}cm \cdot 100{,}000 = 3.086e^{23}\ cm$"

Question #2:

- Timothy – Choses the wrong equation for the problem given; he chose an equation appropriate for gravitational lensing, when the problem explicitly states that there is no observed gravitational lensing. The student was however able to identify the variables in the equation.
  - "$M = \frac{c^2}{4G}\frac{D_l D_s}{D_{ls}}|\theta_+\theta_-|$

    Yes it is possible to measure mass.

    D_l = distance from the image

    D_s = distance from the source

    $D_{ls} = D_l - D_s$

    θ_+ = distance of background galaxy

    θ_- = distance of background quasar
- Joseph - Choses the wrong equation for the problem given and does not make the make the mathematical connection necessary to the equation he provided by also providing a connection to the radius and the given equation.

- "If you know how many stars are in it/the density (galaxies are mostly empty space) you can use the density equation if you can measure its radius: $\rho = \frac{m}{V_{disk}}$"

- Patrick – Was not able to know when to use an equation. The student only acknowledged three variables that were not connected to any equation.
    - "r, R, v" "Yes, it is possible to determine the mass of the galaxy."

Question #4:

- Edward – Was not able to know when to use an equation; he provided three variables in part (a.) which he did not use in part (b.) in an equation to make a final mathematical statement.
    - (a.) Mass of Jupiter, Mass of Io, Distance from Jupiter to Io

      (b.) "$a_{max,galelio} = \frac{1}{5} a_{max,io}$"

- Patrick - Choses the wrong equation for the problem given and does not make the make the conceptual connection necessary to the equation he provided by substituting in for the incorrect variables.
    - (a.) Mass of Jupiter, Mass of Io, Distance from Jupiter to Io

      (b.) "$M_G = 2M_I$, $R_G = 3R_I$, $d_{JG} = 4d_{JI}$

      Max tidal on Io $= -\frac{M_I \cdot G\, M_J}{R_I^2}$

      Max tidal acceleration on Galelio $= \frac{-2M_I \cdot G\, M_J}{9R_I^2}$

      Max tidal on Io is $\frac{9}{2} = 4.5$ times stronger than on Galelio."

- John – Choses the wrong equation for the problem given; the student choses an incorrect equation for force. The student also makes no conceptual connection to the equation (They solve for a force (incorrectly), but makes no connection between force and acceleration.) and the student makes no mathematical connections to the equation (They "cross multiply" a statement of subtraction.)

- (a.) Mass of Jupiter, Radius of Io, Distance from Jupiter to Io

  (b.) "$F_1 = \frac{GM}{(r - r_{Io})} - \frac{GM}{(r + r_{Io})} = GM(r + r_{Io}) - GM(r - r_{Io})$

  $$= GM\big((r + r_{Io}) - (r - r_{Io})\big) = GM(2r_{Io})$$

  $$F_2 = \frac{GM}{(4r - 3r_{Io})} - \frac{GM}{(4r + 3r_{Io})} \qquad = GM(6r_{Io})$$

  $\frac{1}{6}$ max tidal acceleration"

Given the above responses these students gave to the exam questions, it is clear that the students who showed "no understanding" on the exams cannot accurately solve the problems when working on their own. Knowing how to use an equation to solve astrophysics problems and being able to know when to use the equation is an essential aspect of understanding equations. These students therefore showed "no understanding" of the equations supplied.

For these two similar types of assignments, the classifications for students with "no understanding are similar. Students with "no understanding" of astrophysics equations did not recognize the symbols in the equation in terms of the corresponding physics quantities, were not able to know when to use an equation, chose the wrong equation for the problem given, and/or made no conceptual or mathematical connection to the equations. However, one additional interesting finding was noticed while analyzing this data. Although there were relatively few students who showed "no understanding' on the homework assignment (15%) and the exams (15%) there was only one overlap between the two sets of students. From these two sets of data, it appears as if the classification of "no understanding" is dependent on the assignment. Since the students could work together on homework problems and use all available resources, it could be unclear as to whether the students who did well in the homework assignments truly had an understanding of the equations. It could also be that the students who showed no understanding of the astrophysics equations during the exams simply do not do well during high pressure situations; therefore, additional data was examined.

Essays

In addition to the homework assignments and exams, all students were also asked to write an essay discussing their favorite concepts and most difficult concepts of the course at the end of the course. The essays were completed individually and this data was then also used to determine if a student had a qualitative understanding of equations. The theme of the essay was as follows: "Which concept from PHY 341 did you find most difficult, and why? Which concept was your favorite, and why? Discuss the steps you took to better understand the difficult concept. Did you find your favorite concept easy or difficult to understand, and did that influence your choice of it as a favorite? If these concepts relate to equations, include discussion of the relevant equations and the meaning of their terms as well as how the equations influenced your attempts to understand the concept." Using the criteria for student understanding of equations described in the previous chapter (namely: a student who understand equations is able to: recognize the symbols in the equation in terms of the corresponding physics quantities, recognize the underlying physics of the equation, recognize the structure of the equation, establish a link between the equation and everyday life) I was able to determine the level of understanding the students showed concerning the astrophysics equations in their essays.

It was difficult to analyze the students who showed "no understanding" of the equations in the essays, because many of the students who fell into this category did not provide an equation even when their favorite/least favorite concepts clearly had equations that related to their topics. For instance, one student who did not provide an equation spoke about black holes and learning about the astrophysics that govern their behavior: "[The] concepts of black holes led me to study astrophysics … I wanted to know what happens inside them … how did they form, etc." Or another student (John) who did not provide any equations in their essay: "Since I more readily understand concepts than equations … relativity came most easily to me … so I liked putting [the concepts] to use to solve real world problems." As with the previous student, this student chose to speak about a topic (a topic of which they claimed to have a conceptual understanding) without providing the mathematical equations that govern that topic. All of the students who did not provide equations discussed topics that were noticeable choices for including at least one equation.

Out of the students who provided equations in their essays, those that showed "no understanding" fell into one of two categories: either they provided an equation and did not satisfy any of the criteria for student understanding of equations or they provided an equation but only satisfied one of the criteria for student understanding of equations. Clearly, providing the equation but not fulfilling any of the criteria shows no understanding of the equation beyond that it is related to the topic they are discussing. Also, a student that provides an equation and knows only what it's purpose is or can only describe it in words, but has fulfilled no other criteria shows no understanding of the astrophysics equation they provided.

Again, it was discovered that there was little to no overlap between the students who had "no understanding" in the essays when compared to the other assignment data collected. Out of the twelve (12) students who showed "no understanding" of the equations in their essays, six (6) of them did not provide equations; and out of the remaining six (6) students who did provide equations, only two (2) showed "no understanding" – one in the homework assignments only and one in the exams only. It is hard to determine if the students who showed "no understanding" simply did not understand the importance of equations in their topics, or if they simply did not feel that the equations they provided needed any elaboration; but it is clear that these students showed “no understanding” of the relevant equations related to the topics in their essays.

Interviews

In the process of analyzing the student interviews, it was discovered that the students who showed “no understanding” of astrophysics equations displayed all of the criteria identified for students who were classified with “no understanding”. It is noteworthy that for this particular assessment of student understanding all but one of the students who were selected for the interview process were students with high scores in the classroom (obtaining grades of a B+ or better), once interviewed, it was clear that not all of these students truly understood the quantitative equations in a meaningful way. Two of the ten students interviewed showed no understanding of the equations used to solve the astrophysics problems presented even though they were both classified as “understanding” or “complete understanding” in all of the other data sources (homework assignments, exams, and essays) again implying that the classification of ‘no understanding” is dependent upon the assignment.

During the student interviews, the students were asked to solve three astrophysics problems while stating their thought process out loud. The three problems were the same for all students in the interview process as stated in Section 4.2.c. The first problem asked the students to determine both the kinetic energy of Earth as it revolves around the Sun and the potential energy of the Earth-Sun system. To determine which of these energies is larger in magnitude as well as the ratio of these two energies. The second problem gave the students a solution to a problem which was similar to the first and asked them to interpret it. The third problem consisted of a satellite in orbit about Earth loses total energy due to gradual atmospheric drag. The students were asked to determine what would happen to a satellite's orbit and to the potential energy of the satellite-Earth system.

Both of the students who showed "no understanding" of the equations did not recognize the symbols in the equation in terms of the corresponding physics quantities, did not recognize the underlying physics of the equation, did not recognize the structure of the equation, did not demonstrate knowledge of how to use an equation to solve astrophysics problems, were not able to know when to use an equation, could not start the problem without assistance, chose the wrong equation for the problem given, and made no conceptual or mathematical connection to the equations. In addition, these two students memorize a large amount of equations without understanding. The following excerpts from these students' interviews illustrate the student's classification of "no understanding".

Daniel:

In the process of solving the first problem, Daniel correctly identifies the equation for kinetic energy of Earth moving around the Sun - mainly from the student's past experiences with this equation. However, the student's statement "That's just is the general kinetic energy formula … I don't know, I've just used it so many times, I don't really remember how it is derived." shows that the student does not have a conceptual understanding of the equation; and has simply memorized the equation. This is further reinforced when the student proceeds in their thought process "so we know the mass of the Earth, and we can determine the velocity. The velocity would be the time of a year divided by the radius between the Earth. No, it would ... So, velocity is the change in position over change in time. So, our change in position would be two pi radius, which is one AU, over one year." Here the student first recalls an equation for velocity which is incorrect, but then recalls the correct equation to obtain the Earth's average speed in orbit, assuming a circular orbit. Although not incorrect, this student chose the wrong equation for the problem given as it is not a useful equation for this particular problem and furthermore shows that the student is not able to know when to use an equation but is simply trying to remember equations that seem to “work”. The student again is not making a conceptual connection to the equations and does not recognize the underlying physics of the equation.

During the next part of the problem this student once again shows a complete lack of conceptual connections to the equation for gravitational potential energy. The student begins by stating that “for gravitational potential energy, that would be due to gravity, so that would be Newton’s law. So that’s like G M m over r squared is equal to force. So, then energy would be force times distance. Right? So, U would be G M m over r.” Here the student incorrectly equates force and gravitational potential energy, then incorrectly equates work to gravitational potential energy. He almost obtains the correct equation for gravitational potential energy at the end (the student is still missing a negative), not because of a conceptual understanding of the equations, but simply from memorization of the equation for gravitational potential energy. The student also did not recognize the symbols in the equation in terms of the corresponding physics quantities when they imply that the “r” and “d” variables in the two equations mentioned are equivalent.

The next part of this problem asked the students to compare the kinetic energy of Earth moving around the Sun with the gravitational potential energy of the Earth-Sun system. This student did not know how to do this part of the problem “Well, if we compare them then I mean just plugging in the values then you can see.” (Note that no values were given for any of the variables for any of the interview problems.) I had to walk this student through most of the steps to complete this problem including using multiple representations, substitution of like variables, and the correct equations to use for this problem. The student could not start the problem without assistance, and even with continual assistance the student had difficulties completing the problem.

In the process of solving the next two problems, the student showed a similar lack of understanding of the equations, instead simply memorizing multiple equations and struggling to find the ones that "worked" for the problems. To a great extent, a majority of the interview consisted of the student being guided through the problems, rather than solving them on his own, because the student did not have the appropriate equation memorized and did not know how to properly use the equation once it was given.

Christopher:

Christopher showed exceedingly similar tendencies to Daniel through the process of solving the problems given during the interview. Christopher also showed no true conceptual connection to the equations and could not truly remember any equations useful to solving these problems having selected the wrong equation for the problem given. Once the student was given the equations through discussion as before, the student saw them as something to use, but not with any conceptual or mathematical connection to the equation as can be seen from comments such as "You just take the ratio of it, and I think that potential would be greater." or "You [just] set them equal to each other."

This excerpt clearly shows that the student did not understand the equations nor their purpose in this problem; but just as clearly knew that they were tools to be used in some way. Furthermore, at this point Christopher did not have the equations for the kinetic energy and the gravitational potential energy in forms with variables that would cancel when taking a ratio. When asked to compare the two equations the student's answer: "One thing they have to have something in common, which is-- The only thing I see in common is the mass. So… Yeah." shows that the student did not demonstrate knowledge of how to use an equation to solve the problem and furthermore does not recognize the structure of the equation. The student was subsequently guided into making the appropriate substitutions to create a relevant ratio of the two energies.

For Christopher, once again, a great amount of the interview consisted of the student being guided through the problems and could not solve the problems on his own, for the same reasons as Daniel – these students did not recognize the symbols in the equation in terms of the corresponding physics quantities, did not recognize the underlying physics of the equation, did not recognize the structure of the equation, did not demonstrate knowledge of how to use an equation to solve astrophysics problems, were not able to know when to use an equation, could not start the problem without assistance, chose the wrong equation for the problem given, and made no conceptual or mathematical connection to the equations. In addition, these two students memorize a large amount of equations without understanding. They were therefore classified as having "no understanding" of astrophysics equations in this assessment.

Summary - No Understanding

The following two tables show the overall performance of students who were found to have "no understanding" of equations in a particular assessment:

Table 16: Number/Percentage of students showing "no understanding" by assessment

No Understanding			
Homework	Exam	Essay	Interview
8/15%	8/15%	12/22%*	2/20%
		6/11%**	

*Includes students who did not supply an equation in their essay.

**Students who supplied an equation in their essay only.

Table 17: Overlapping of the number of students with “no understanding” by assessments

Number/Percentage of Students Overlapping	
Assessment Overlap	Overlap for Students with "No Understanding"
Homework & Exam	1 of 15/7%
Homework & Essay	1 of 20/5%
Exam & Essay (Includes “No Equation Given”)	2 of 16/13%
Exam & Essay (Equation Given)	1 of 12/8%
Homework & Exam & Essay	None
Interview & "Other"	None
No Overlap	22 of 26/85%

As can be seen in Table 17, there was found to be little to no overlapping of the students who showed no qualitative understanding of astrophysics equations from the different assessments analyzed. This heavily implies that there is no such thing as a student with “no understanding” of the equations in this classroom, but perhaps there is a connection between “no understanding” and context or content. I will explore this in the next section when I answer my third research question “How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?” when I analyze the students’ responses to a specific topic - specifically negative gravitational potential energy and the virial theorem.

5.2.b Partial Understanding

After identifying the students in the class who showed “no understanding” of the equations used in their assessments in their astrophysics course, I went on to identify which of the students in the class showed a partial qualitative understanding of the equations which they used throughout the course and therefore the course’s assignments. I found that these students, the students who showed “partial understanding” of the equations used in the course, generally were able to identify and use the equations; they were able to understand basic equations and partially understood more complex equations at a rudimentary level, but had difficulty fully understanding in a qualitative way all of the equations that were presented in the class. Many of these students therefore had completed problems incorrectly – either with or without assistance. The following table illustrates the connection between the theoretical framework and the students' assignments with the students who were classified as those with "no understanding" of astrophysics equations.

Table 18: Indicators of Students with "Partial Understanding" of Equations

Students with "partial understanding" of astrophysics equations …				
	Homework Assignments	Exams	Essays	Interviews
Theoretical Framework				
… recognize the symbols in the equation in terms of the corresponding physics quantities.	Inconsistent	Inconsistent	No	Yes
… recognize the underlying physics of the equation.	N/A	N/A	Inconsistent	No
… recognize the structure of the equation.	N/A	N/A	No	Inconsistent
… establish a link between the equation and everyday life.	N/A	N/A	No	Yes
…demonstrate knowledge of how to use an equation to solve astrophysics problems.	N/A	N/A	N/A	Inconsistent
… are able to know when to use an equation.	Inconsistent	Inconsistent	N/A	Inconsistent
Additional Indicators				
... can start the problem without assistance.	Unknown		N/A	Inconsistent
… made conceptual or mathematical connection to the equations.	Inconsistent	Inconsistent	N/A	No

As can be seen in Table 18, the students from the class that were subsequently classified with "partial understanding" of astrophysics equations had a combination of the theoretical attributes that students with understanding of equations should have according to the literature. Once again, the theoretical framework as outlined in the literature did not cover all of the indicators that were observed that students with "partial understanding" share. In addition, these students often made no conceptual or mathematical connection to the equations.

Homework Assignments & Exams

When the scores for the qualitative portion of the homework assignments were averaged fifteen (15) students out of 54 (28%) were found to be students with "partial understanding" qualitatively of the equations which they used. I will once again use the responses given by the students in homework assignment #3 in order to illustrate my findings; and will add an additional homework assignment (Question #1b in homework assignment #6) to complete the analysis. As with the responses from the students with "no understanding", the responses for the students with "partial understanding" in these homework assignments are also indicative of the overall data for these students' homework assignments. In question #1 on homework assignment #3, seventeen (17) out of the 54 students (31%) who submitted this homework assignment showed "partial understanding" for part (a), while three (3) students (6%) showed "partial understanding for part (b), and six (6) students (11%) showed "partial understanding for part (c). Similarly, nine (9) out of the 53 students (17%) who submitted homework assignment #6 showed "partial understanding" for question #1, part (b). Once again, since homework assignment #3 parts (b) and (c) and homework assignment #6 part (b) focus directly on qualitatively understanding equations, I will continue to concentrate on these two parts; highlighting five students whose work across all or most of the data analysis was categorized as "partial understanding". Two of these students (Kevin and Nicholas) worked with partners (not each other) and three of them worked alone on the homework assignments.

In response to homework assignment #3 question #1b ("How does the gravitational force that one object exerts on another object change if the distance between them triples? If the distance between them drops by half? Explain how you know.") only one of the five students (Kevin) selected showed "partial understanding" of this particular problem because he chose an incomplete equation for the problem and did not make a full mathematical connection to the equation they selected. All of the five students correctly identified the problem with an equation involving an inverse square, but it is worth noting that while the appropriate equation to evaluate this problem is Newton's law of gravitation ($F = G\frac{M_1M_2}{r^2}$) Kevin supplied the following equation: $F_{grav} = \frac{m_1m_2}{D^2}$, leaving out the gravitational constant, G. Students 2-5 correctly identified that if the distance triples, the force decreases by a factor of nine and if the distance drops by half, the force increases by a factor of four. They did so by using their knowledge of the inverse square nature of the equation and not by substituting in values for the other variables in the equation. For example:

> Denis:
>
> $$F_g = \frac{GMm}{R^2}$$
>
> Using the general formula for gravitational force we can see what happens as R increase & decrease.
>
> $$F_{g2} = \frac{GMm}{(3R)^2} \Rightarrow F_{g2} = \frac{GMm}{R^2} \cdot \frac{1}{9} \Rightarrow F_{g2} = \frac{F_g}{9}$$
>
> When the radius is tripled, F_{g2} becomes a ninth of F_g.

$$F_{g3} = \frac{GMm}{({R}/{2})^2} \Rightarrow F_{g3} = \frac{GMm}{R^2} \cdot \frac{1}{({1}/{4})} \Rightarrow F_{g3} = 4F_g$$

When the radius is halved, F_{g3} becomes 4 times stronger than F_g.

Kevin's response below however shows that while this particular student does understand the nature of an inverse (as the denominator increases in value, the quotient decreases); he was not explicit enough to show a true understanding of the inverse square nature of the force equation:

Kevin:

$$F_{grav} = \frac{m_1 m_2}{D^2}$$

From this formula, if the distance is tripled gravitational force would be smaller and as distance is cut in half gravitational force would be larger.

Most of these students answered the next part of the problem (homework assignment #3, question #1c) with only partial understanding of the equation because they did not make a complete conceptual connection or mathematical connection to the equation given to them. Additionally, they recognized some of the quantities in the equation as corresponding to the symbols in the equation, but not all. The question "Create a question for which the following equation provides an answer: $0.1" = 1.22\left(\frac{400nm}{1m}\right)\left(\frac{360\ x\ 60\ x\ 60"}{2\pi}\right)$" has an expected answer "What is the angular resolution of a 1m diameter telescope at the bluest light that human eyes can see?" The students' responses indicate that without the equation written in variables, they are not completely certain of all of the variables given in the equations when values have been substituted for the variables.

Kevin:

"For a telescope with wavelength λ = 400 nm, Diameter D = 1 m [and] θ = 60°, What is the resolution?"

Henry:

"What is the Diffraction Limited Resolution of a telescope of Diameter = 1 m and captures light at wavelength = 400 nm? m = 1.22."

Nicholas:

"What is the diffraction limit of blue light in a 1 m diameter telescope in arc sec?"

Denis:

"What would be the diffraction limit of a telescope if the diameter of the lense is 1 m and the wavelength is 400 nm. Please find the answer in arc seconds and assume m = 1.22."

Matthew:

"What's the resolution of a telescope in arc seconds with an aperture size of 1m, and the wavelength of the light is 400 nm?"

From the above student answers, we can see that these students do have some understanding of the equation used to determine the angular resolution of a telescope ($\theta = 1.22\frac{\lambda}{D}$) where θ is the angular resolution in radians, λ is the wavelength of light, and D is the diameter of the lens' aperture. The students do correctly link most of these variables to the corresponding given values. However, the knowledge they display is limited. For example, Kevin clearly does not understand that a portion of the equation $\left(\frac{360\ x\ 60\ x\ 60''}{2\pi}\right)$ was given in order to convert the answer obtained for the angular resolution from radians to arc seconds; showing that they did not make a mathematical connection to the equation. This is made clear because the student gives θ = 60° as part of their answer to the question without any reference as to what "θ" is defined in the student's mind. Likewise, Henry and Denis give a quantity "m" (= 1.22) without defining the "variable"; showing that they do not recognize the symbols in the equation in terms of the corresponding physical quantities. From their responses, it is also evident that Kevin, Henry, and Denis did not make a full mathematical connection to the equation because they do not fully recognize that the answer was given in arc seconds, not radians, as they make no mention of this in their answers. Nicholas's response may just be a case of the student being disordered, but was classified as the student not making a conceptual connection to the equation because unambiguously light itself does not have a diffraction limit, a diffraction limit is given for the telescope, not light. Matthew's response was given here for completeness, it is not technically wrong (although it could be improved upon, for example, by adding the word "angular" before "resolution") but was included with these homework assignments as this student does poorly conceptually in other homework questions that do not explicitly have equations and shows partial understanding of equations in other data, so is placed here for completeness and comparison later. Question #1 part b from homework assignment #6 is another example of a problem that illustrates these students' "partial understanding' of the equations. Question #1 part (b) is a "ranking" style problem. The students are asked to rank different black holes based on the magnitude of their tidal forces: "Rank the following black holes based on the magnitude of the tidal forces that they would exert on a spaceship placed near their event horizon. A has mass $10M_{\odot}$; B has mass $100M_{\odot}$; C has mass $10^6 M_{\odot}$." Since the tidal force is proportional to the mass of the body causing the tides and inversely proportional to the separation of the test mass from the center of mass of that body cubed, the ranking of the tidal forces can be determined using proportionalities; namely $F_{tidal} \propto \frac{M_{BH}}{R_S^3}$ where F_{tidal} is the tidal force, M_{BH} is the mass of the black hole, and R_S is the Schwarzschild radius. The expected answer involves knowing that the Schwarzschild radius gives the distance of the event horizon to the center of the black hole ($R_S \propto M_{BH}$) so mass is the determining factor for ranking the given black holes from strongest to weakest tidal forces ($F_{tidal} \propto M_{BH}^{-2}$). The rank from strongest to weakest tidal forces is therefore A, B, C.

The students' answers to this question again show only "partial" understanding of the equations by not recognizing all of the symbols in the full equation as relevant to the problem or that one of the variables (symbols) in the equation they correctly recognize as necessary to solve the problem, is equal to another equation with necessary variables to complete the problem correctly:

Kevin:

$$\Delta F_{max} = \frac{2GMmR}{r^3}$$

C, B, A

Henry:

"The event horizon depends on mass, so all three black holes would exert an equal tidal force on a spaceship on their respective event horizons."

Nicholas:

$$F_{tidal} \propto M$$

C [greatest], B, A [least]

Denis:

"The tidal force is stronger if the mass is larger. This can be shown from

$\Delta F_{spaceship} = \frac{GMmR}{r_{spaceship}^3}$ $\Rightarrow$ where big M is the black hole mass."

"In this Equation, Force and Mass are proportional and Force increases as M increases. C produces the greatest Force, with B in the middle, and A produces the least amount of force."

Here, each of these students correctly identifies that the equation for the tidal force between the black hole and the spaceship is proportional to mass, two of these students (Henry and Nicholas) do not include the "*r*" term of the equation, thus selecting the wrong equation for the problem given, and the other two students (Kevin and Denis) do not recognizes that the "*r*" in the denominator of the equation is the Schwarzschild radius which in turn is proportional to the mass of the black hole, thus not recognizing the symbols in the equation in terms of the corresponding physics. Furthermore, Henry realizes that the mass of the spaceship is not significant in determining the ranking of the tidal force, but fails to recognize that the mass of the black hole is the determining variable in this question, therefore showing that they had no conceptual connection to the equation as well as not recognizing the symbols in the equation in terms of the corresponding astrophysics quantities.

Again, similar results were found when examining the Exam questions in terms of partially understanding the qualitative aspects of the equations used in solving the problems on the Exams. The students who showed “partial understanding” of the equations for the most part did recognize the symbols in the equation in terms of the corresponding astrophysics quantities and were able to know when to use a problem; mostly selecting the correct equation from the supplied formula sheet. However, they only sometimes made conceptual connections or mathematical connections to the equations they selected for the problems.

To illustrate the similarities and differences of the results from the homework assignments to the exams for the students who showed “partial understanding”, as well as to compare these students to students of “no understanding”, I will again focus on the students’ responses to questions #1, #2, and #4 from Exam #2 with the questions and expected answers shown above. The following students are the same students which were highlighted in the homework that had “partial understanding” of the equations necessary to complete the problems:

Question #1:

- Kevin – Shows that he has a mathematical connection to the equation as he uses his knowledge of the Hubble constant which was given to find the age of the universe, but shows no conceptual connection to the equation when he does not explain what the value he obtains means.

- “$H_0^{-1} \approx 10\ Gyr\ x\ \left(\frac{100\ km\ s^{-1} Mpc^{-1}}{H_0}\right)$ $\leftarrow H_0 = 100\ km\ s^{-1} Mpc^{-1}$

  $H_0^{-1} \approx 10 Gyr$ not 12 billion”

- Henry & Dennis – Both of these students did not select an equation to answer this question and therefore were classified as “not able to know when to use an equation” for this particular problem. It is interesting to note that Henry and Dennis developed vastly different answers without using the equation.
  - Henry: “Yes: Everything began much closer together, and was too hot to form elements that created stars. If the Hubble constant were half today what was previously thought, it would imply that matter was not yet far enough dispersed in the universe to cool and create the elements that formed stars 12 BB. The stars would have to be younger.”
  - Dennis: “There would not be a conflict because it is said that the universe id 14 billion years old. This shows that even if the Milkyway is that old, it is still in the correct time frame as the age of the universe.”
- Nicholas – Was able to know what equation to use and recognized the symbols in the equation in terms of the corresponding astrophysics quantities, but did not show a mathematical connection or a conceptual connection to the equation. He states that the different Hubble constant would mean that the age of the Milky Way would change, but he does not state how or if this new age would create a conflict with the quantities given in the question.
  - “$\frac{dx}{dt} \rightarrow\ v = H_0 d\ \leftarrow x$

    If the Hubble constant were different, it would change how old the milkyway is. H_0 is related to velocity which is a function of time.”

Question #2:

- Henry & Matthew – Chose the wrong equation for the problem given; both of these students choose a given equation appropriate for finding the mass of a galaxy when the density is known or can be estimated. Since the appropriate equation $\left(M = \frac{3\beta}{\eta}\frac{R\sigma^2}{G}\right)$ was supplied on the formula sheet these students were not able to know when to use an equation. In addition, Henry did not show that he was able to recognize the symbols in the equations; whereas Matthew was able to identify the variables in the equation but did not make a conceptual connection to the equation when he then gives the formula (given on formula sheet as well) for density instead of mass.
    - Henry: "Given: d, r, v

      $M = 4\pi \int_0^r \rho(u)u^2 du \;\rightarrow\; \rho(u)\frac{1}{4\pi r^2}\frac{dM(r)}{dr}$

      $M = 4\pi V\rho(r)dr$

      We can estimate the galaxy's density and integrate this value using our known galactic radius."
    - Matthew: "Yes it is $M = 4\pi \int_0^r \rho(u)u^2 du$

      where u = mean-squared velocities.

      ρ = density.

      Density can be fund by assuming the average star is a main sequence star. Thus

      $M(r) \approx 1M_{\odot} \times n$ where n = no. of stars

      $\rho(u)\frac{1}{4\pi r^2}\frac{dM(r)}{dr}$,,
- Nicholas & Dennis – Nicholas and Denis both chose the most appropriate equation for the problem given; however Nicholas does not do anything with the equation except write it down showing that he does not make the make the conceptual connection or the mathematical connection necessary to the equation he provided. Dennis on the other

hand, shows an almost perfect answer (dropping constants rather than defining them) all the while, showing that he knows when to use an equation, recognizes the symbols in the equation, and makes some conceptual and mathematical connections to the equation.

- Dennis: "The mass of the object can still be obtained by using

  $M = \frac{3\beta}{\eta}\frac{R\sigma^2}{G}$

  where R = radius σ = velocity dispersion & β, η, and G are constants.

  If the radius and velocity dispersion are known, mass can be found when divided by G. Using dimensional analysis, $\frac{R\sigma^2}{G}$ reduces to mass which means β, η are dimensionless.

Question #4:

- Nicholas & Matthew – Chose the wrong equation for the problem given; these students both chose an equation for force, but make no connection between force and acceleration showing that they are making no conceptual connection to the equation. These students make mathematical connections to the equation; they are able to create a ratio with their equations, canceling out constants. Nicholas however shows that he recognizes the symbols in the equation, substituting correctly for his equation, whereas Matthew does not demonstrate that he can identify the symbols in this equation substituting incorrectly for "R" and "r".
  - Nicholas: (a.) Mass of Jupiter, Mass of Io, Radius of Io, Distance from Jupiter to Io

    (b.) "$\Delta F_{max} = \frac{2GMmR}{r^3}$

    Io: m = 1m, r = 1r, R = 1R

    Galileo: m = 2m, R = 3R, r = 4r

$$\frac{(2)(3)}{(4)^3} = \frac{6}{64} = \frac{3}{32}$$

Tidal forces on Galileo would be $\frac{3}{32}$ x the tidal forces on Io."

- Matthew: (a.) Mass of Jupiter, Radius of Io, Distance from Jupiter to Io

  (b.) "$\frac{2GMmR}{r^3} = \frac{2\times 4}{3^3} = \frac{8}{81}$"

- Kevin – Choses a correct equation for the force and acceleration for problem given; but fails to take a ratio of the two accelerations found and incorrectly substitutes the masses of the moons for the mass of Jupiter. This shows that the student does not identify all of the symbols in the equation correctly and makes no mathematical connection to the equation.
  - (a.) Mass of Jupiter, Radius of Jupiter, Mass of Io, Distance from Jupiter to Io

    (b.) "$M_G = 2M_I \quad R_G = 3MR_I \quad r_G = 4r_I$

    $$F_t = \frac{2GMmR}{r^3} = ma$$

    $$a_{t_I} = \frac{2GM_IR_I}{r^3}$$

    $$a_{t_G} = \frac{2G2M_I3R_I}{4{r_I}^3}$$

    The maximum tidal acceleration on Galileo would be larger."

- Henry – Supplies a correct ratio to determine the maximum tidal acceleration caused by Jupiter of a test mass on the surface of Galileo compared with Io, but does not show how he obtained that equation; the student does not show a mathematical connection to the equations by not showing how the ratio was obtained. The student further makes no conceptual connection to the equation by substituting unknown values into the ratio (thus also showing that he does not correctly identify the symbols) and giving an incorrect answer.
  - (a.) Mass of Jupiter, Mass of Io, Radius of Io, Distance from Jupiter to Io

(b.) "$\frac{ML\hat{}2}{r^3}$ $\rightarrow$

Galileo tidal acceleration $\rightarrow$

$\frac{16}{9}$ that of Io"

For these two similar types of assignments, the homework assignments and the exams, the students are classified as students with "partial understanding" for the same reasons regardless of the assignment. The students with "partial understanding" of astrophysics equations: recognized most, but not all, of the symbols in the equations in terms of the corresponding astrophysics quantities; they inconsistently were able to know when to use an equation; and they showed conceptual connections to most of the equations they use. Additionally, these students occasionally struggled with making mathematical connections to the equations; they were inconsistently able to correctly apply mathematics – for instance, while some of these students created correct ratios, they inaccurately calculated numerical results; others created incorrect ratios; and yet others made mathematical connections to some of the problems, but not all.

Essays

Using the criteria for student understanding of equations described in the previous chapter and stated above, I determine the students who demonstrated "partial understanding" of the astrophysics equations that the students provided in their essays. Out of the above five (5) students highlighted in the homework and essay analysis, three (3) students (Kevin, Henry, and Dennis) also showed "partial understanding" in their essays. (Nicholas did not provide an equation and was thus classified as "no understanding" and Matthew was classified as "complete understanding".) I will therefore concentrate on the essays of these three students which demonstrate some of the commonalities that students with “partial understanding” share in their essays.

All students who showed “partial understanding” of the equations in the essays included an equation in their essay along with at least two more of the criteria for student understanding of equations. The two most common criteria that students with “partial understanding” demonstrated were that these students adequately discussed the astrophysics in the equations and they understood the purpose of the equations they provided. Out of the seventeen (17) students who showed “partial understanding” of equations in their essays, most (14/82%) adequately discuss the astrophysics in the equation:

- Dennis: “[Hubble’s] Law is represented by $v = H_0 d$, which proves that H_0 has units of inverse time. Through this, we were able to figure out the age of the universe which is incredible.”

- Henry: “Scientists have to wait years to extrapolate the period of a star orbiting around [a] Supermassive Black hole. Calculating the angular semi-major axis of this orbital is imperative, not to mention figuring out how far away we are from the Black Hole system.” (In reference to calculating the mass of a supermassive black hole from the supplied equation.)

Eleven (11/65%) of the students who showed “partial understanding” of the equations they provided also demonstrated that they also understood the purpose of the equation:

- Kevin: “I couldn’t imagine going on a trip that for people on earth would have been a thousand years but for me only fifty.” (In reference to special relativity - time dilation equation.)

- Henry: “I loved calculating the mass of Supermassive Black Holes. In the case of SBH the solution comes in the form of an incredible large mass instead of a pretty picture.”

Very few of the seventeen (17) students who showed “partial understanding” discussed the symbols in the equation (4/24%), discussed the structure of the equation (2/12%), or discussed the connections between the equations and the real world (5/29%). None of these students showed a deep understanding of the astrophysics behind the equation they provided in their essays.

Table 19: Summary of criteria of students who had “partial understanding” of equations in their essays.

Students with “Partial Understanding” met the following criteria for understanding equations

Does the student …	“Partial Understanding”	“No Understanding”
... discuss the symbols in the equation?	4/24%	0/0%
... discuss the astrophysics or physics in the equation?	14/82%	1/6%
... discuss the structure of the equations?	2/12%	0/0%
... understand the purpose of the equations?	11/65%	1/6%
... show deep understanding of astrophysics behind the equations?	0/0%	0/0%
... talk about connections between equations and real world?	5/29%	0/0%

It is evident that although these students do meet some of the criteria, they do not meet many of the criteria necessary for understanding equations and therefore have a “partial understanding” of the relevant equations related to the topics in their essays.

Interviews

Five (5) of the ten (10) students interviewed showed “partial understanding” of the equations used to solve the astrophysics problems presented; three (3) of which showed partial understanding in at least one other assessment. These students who showed “partial understanding” of astrophysics equations in the interviews demonstrated the ability to recognize the symbols in the equations they provided in terms of the corresponding physics or astrophysics quantities and were able to establish a link between the equations and everyday life; however, they did not consistently demonstrate that they could start the problem without assistance, display knowledge of how to use an equation to solve the problems, or recognize the structure of the equations. In addition, these students showed no proficiency in recognizing the underlying physics if the equation nor in making conceptual or mathematical connections to the equation. Similarly to the students who showed “no understanding, these students seem to memorize a large amount of equations without understanding. The following excerpts from two of these students’ interviews illustrate the student’s classification of “partial understanding”.

Benjamin:

After reading the first problem, Benjamin immediately determines that the problem “definitely seems like a virial theorem problem”, and attempts to remember the equation related to the virial theorem: “I think it’s two, two kinetic energy plus potential is overall -- or no, zero, I think that’s it.” This student immediately, from the very start of the first problem in the interview process, attempts to remember an equation that relates kinetic and potential energies, but cannot remember the equation correctly from his memory. The student then goes on to recall the correct equation for kinetic energy “we know kinetic energy is just $\frac{1}{2}mv^2$” and correctly identifies the symbols in the equation “you’re going to have to get the mass of the earth [so] we know the mass of the earth, and the velocity which could be trickier [to find].” Although this student can identify the symbols in the equation, he seems to believe that the “values” of the variables must be known in order for the equation to be of use. This indicates that the student does not have a conceptual connection to the equation.

In the next part of the problem, Benjamin attempts to recall the equation for gravitational potential energy. Again, this student begins by incorrectly remembering the equation “we know that is Gm/r, I think. Or r^2, it’s one of them.” Upon being told that the equation was not correct the student then incorrectly attempts to obtain the equation for gravitational potential energy by taking the gradient of the force; showing no mathematical connections to the equations as well as not recognizing the underlying physics of the equations. The student shows no conceptual or mathematical connection to the equation as well as no recognition of the physics of the equation again when discussing the “negative aspect” of the gravitational potential energy. When asked why it is negative, the student replies “because it’s radial and that’s just the rule.” This student’s comment shows that he does not have the knowledge of how to use the equations to solve the problem. After some time (indicating that this student could not start this part of the problem without assistance), the student was guided into obtaining the correct equation for gravitational potential energy.

This student also sometimes struggles with recognizing the structure of the equations that he uses as illustrated by the following two scenarios. While discussing the inverse nature of gravitational potential energy, the student correctly identifies that if the distance between the two objects goes to zero, then the gravitational potential energy of the system would become infinite and vice versa. However, when discussing the inverse nature of the equation with negatives, the student gets confused: “So the radius has to be getting bigger than -- well, if we want to make it more negative, I guess you could say you're losing it that way, then the radius would be getting smaller."

When asked to compare the kinetic energy of Earth moving around the Sun with the gravitational potential energy of the Earth-Sun system. This student shows that they are mostly able to know when to use an equation when he first incorrectly tries to compare the two equations, but then quickly corrects himself and makes the correct comparison: "m a_centripetal is the mass of the earth … v^2/r and then solving for v^2 ... So, if we plug that in here. Oh, that does not help that much does it? Okay, so let's do [this] … v^2 is G times the mass of the sun/r, just by canceling out some variables. So, ½ of that, and then, yeah, so you get -- this is just ½ of U."

Matthew:

Similarly to Benjamin, Matthew also is able to recognize the symbols in the equations used in the interview in terms of the corresponding astrophysics or physics quantities and is also able to establish a link between the equation and everyday life. The latter is demonstrated when discussing the third problem concerning a satellite in orbit which is losing total energy due to atmospheric drag. Matthew correctly assesses the situation using the gravitational potential energy of the Earth-satellite system: "The closer you are to the object [the] more gravitational potential energy you would have... [the] further away you are the less gravitational potential energy you feel. So, [in this case] you are actually [decreasing total] energy." However, Matthew goes on in the next sentence to demonstrate that he does not make any mathematical connection to the equations by stating that he did not know how to show this with the equations and "[I was] just thinking [about the first problem] the closer you are to an object the more gravitational potential energy you have to have" without reference to the equations, showing that he did not have a conceptual connection to the equations; but simply remembered what would happen to the gravitational potential energy of a system if an object was moved closer/further away from another object.

Mathew also showed similar inconsistent trends during the course of the interview when solving the problems that were given during the interview. He showed inconsistency when recognizing the structure of an equation; particularly with the nature of the inverse equations as can be seen from comments such as “gravitational energy does have an inverse square relationship, doesn’t it, with like as you go away from the objects, the inverse square law” or “So energy will dissipate according to like the distance it goes away from it exponentially” and “I’m trying to think if it's linear or whether it's exponential. I have a feeling it's just GM over R because that’s kind of what’s in my head and what I remember.” The latter comment also showing that Matthew memorizes equations, rather than trying to conceptually or mathematically understand them. Like Benjamin, Mathew also demonstrates inconsistency when demonstrating knowledge of how to use an equation to solve astrophysics problems illustrated when he was able to compare the kinetic and potential energies found for the Earth-Sun system, but not when taking a ratio of the two energies. In addition, Mathew was able to successfully start a problem without help for some of the problems, but not most; showing an inconsistency at being able to start a problem without assistance.

Matthew also demonstrates that he does not recognize the underlying astrophysics of the equation for gravitational potential energy as demonstrated when discussing the gravitational potential energy of the Earth-Sun system in the first problem: “I guess that would be the equilibrium where you have like zero would be – where they’re balanced? No. If there was no gravitational potential energy then there would be no force exerted by the sun and the earth.”

The remaining students who showed “partial understanding” in the interviews had the same results – these students were able to recognize the symbols in the equations in terms of the corresponding astrophysics quantities and were able to establish a link between the equations and everyday life. They were inconsistent when recognizing the structure of the equations, demonstrating knowledge of how to use an equation to solve astrophysics problems, knowing when to use an equation, and occasionally they could not start the problem without assistance. However, these students did not recognize the underlying physics of the equation and made no conceptual or mathematical connection to the equations. In addition, these students memorize many equations without understanding. They were therefore classified as having “partial understanding” of astrophysics equations in this assessment.

Summary - Partial Understanding

In comparison to the students with “no understanding” of astrophysics equations, the students with “partial understanding” demonstrated this “partial understanding” in multiple assessments. The following two tables show the overall performance of students who were found to have “partial understanding” of equations in a particular assessment:

Table 20: Number/Percentage of students showing “partial understanding” by assessment

Partial Understanding			
Homework	Exam	Essay	Interview
15/28%	19/36%	17/31%	5/50%

Table 21: Overlapping of the number of students with “no” and “partial understanding” by assessments

Number/Percentage of Students Overlapping		
Assessment Overlap	Overlap for Students with "Partial Understanding"	Overlap for Students with "No Understanding"
Homework & Exam	9 of 25/36%	1 of 15/7%
Homework & Essay	6 of 26/23%	1 of 20/5%
Exam & Essay	9 of 25/36%	1 of 12/8%
Homework & Exam & Essay	4 of 33/12%	None
Interview & "1 Other"	3 of 7/43%	None
Interview & "2 Other"	2 of 7/29%	None
No Overlap	10 of 33/30%	22 of 26/85%

To contrast these students to the students classified with “no understanding”, as can be seen in Table 21, there was found to be much more overlapping of the students who showed “partial understanding” or a partial qualitative understanding of astrophysics equations in at least two of the different assessments analyzed. Whereas results for the students with “no understanding” implied that there is no such thing as a student with “no understanding” of the equations in this classroom, the results for the students with “partial understanding” seem to imply the opposite; since the students with “partial understanding” were classified as such over multiple assignments, this implies that the students did have an overall “partial understanding” of the equations used.

5.2.c Understanding

Out of the students enrolled in Principles of Astrophysics during the Fall 2013 Semester, the students who showed “understanding” of the equations used in the course were able to recognize the symbols in the equations they used in terms of the corresponding physics and astrophysics quantities, establish a link between the equation and everyday life, and demonstrated that they were able to identify and use an equation to solve astrophysics problems without assistance. However, these students did have difficulty deeply recognizing the underlying physics and astrophysics of the equation, struggled with the structure of the equations, and had difficulty with the conceptual or mathematical connection to the equations they used. The following table illustrates the connection between the theoretical framework and the students' assignments with the students who were classified as those with "understanding" of astrophysics equations.

Table 22: Indicators of Students with "Understanding" of Equations

Students with "understanding" of astrophysics equations …				
	Homework Assignments	Exams	Essays	Interviews
Theoretical Framework				
… recognize the symbols in the equation in terms of the corresponding physics quantities.	Yes	Yes	Yes	Yes
… recognize the underlying physics of the equation.	N/A	N/A	Inconsistent	Inconsistent
… recognize the structure of the equation.	N/A	N/A	No	Inconsistent
… establish a link between the equation and everyday life.	N/A	N/A	Yes	Yes
…demonstrate knowledge of how to use an equation to solve astrophysics problems.	N/A	N/A	N/A	Inconsistent
… are able to know when to use an equation.	Yes	Yes	N/A	Yes
Additional Indicators				
... can start the problem without assistance.	Unknown		N/A	Yes
… made conceptual or mathematical connection to the equations.	Inconsistent	Inconsistent	N/A	Inconsistent

As can be seen in Table 22, these students who were consequently classified with "understanding" of astrophysics equations out of the class population, had fewer of the theoretical attributes that students with comprehension of equations should have according to the literature than the students with "partial understanding". In addition to the theoretical framework as outlined in the literature these students, like the students discussed as "no understanding" and "partial understanding" also were identified as making no conceptual or mathematical connection to the equations; however, these students did so more seldomly than the former.

Homework Assignments & Exams

For the qualitative portion of the homework assignments, once averaged, sixteen (16) students out of 54 (30%) were found to be students with "understanding" qualitatively of the equations which they used in the qualitative portion of the homework assignments. The responses given by the students in homework assignment #3 will once again be used in order to illustrate my findings; as well as question from homework assignment #6 (Question #1b) to complete the analysis. Once again the responses selected for the students with "understanding" in these homework assignments are also indicative of the overall data for these students' homework assignments. In question #1 on homework assignment #3, twenty-six (26) out of the 54 students (48%) who submitted this homework assignment showed "understanding" for part (a), while twelve (12) students (22%) showed "understanding for part (b), and thirty-six (36) students (67%) showed "understanding for part (c). Similarly, nine (15) out of the 53 students (28%) who submitted homework assignment #6 showed "understanding" for question #1, part (b). I will continue to concentrate on the parts of the homework assignments which focus directly on qualitatively understanding equations; highlighting several students whose work across two or more of the data analysis was categorized as "understanding". Most of these students worked alone on the homework assignments; however, three of the highlighted students (Michael, Richard, and William) worked with partners (not each other).

In their responses to homework assignment #3 question #1b ("How does the gravitational force that one object exerts on another object change if the distance between them triples? If the distance between them drops by half? Explain how you know.") all of students selected except one were able to successfully answer this particular problem. All of these students correctly identified the problem with the appropriate equation - Newton's law of gravitation ($F = G\frac{M_1 M_2}{r^2}$) – which involves using an inverse square; however, Michael made a mathematical mistake in this solution. The students correctly identified that if the distance triples, the force decreases by a factor of nine and if the distance drops by half, the force increases by a factor of four. In the same way as the students with "partial understanding", they did so by using their knowledge of the nature of the inverse square. For example:

> Samuel:
>
> $$F_{grav} = \frac{GMm}{r^2}$$
>
> So, if $F \propto \frac{1}{r^2}$ and r triples, then the force will be 1/9th of what it was originally. Similarly, if the distance is halved, the force will be $\frac{1}{(0.5)^2}$ or 4 times stronger.

Michael's response however showed that while this particular student did know that the relationship between the force and the distance is that of an inverse square, he made a mathematical mistake showing a misstep in mathematical understanding of the inverse square nature of the equation:

Michael:

Since $F \propto \frac{1}{r^2}$ is an inverse square law, tripling the distance makes the force 1/8 as strong halving the distance makes the force 4 times as strong.

It is clear that Michael does have understanding of the inverse square nature of the equation in the second half of his answer, but makes a mathematical mistake in the first half; calculating: $\frac{1}{2^3} = \frac{1}{8}$ rather than the correct answer: $\frac{1}{3^2} = \frac{1}{9}$.

Most of the students who answered the next part of the problem (homework assignment #3, question #1c) with "understanding" of the equation did so because they did not make a complete conceptual connection to the equation given to them. In other words, they recognized all of the quantities in the equation as corresponding to the symbols in the equation, but not with a full conceptual connection to those symbols. The question "Create a question for which the following equation provides an answer: $0.1'' = 1.22\left(\frac{400nm}{1m}\right)\left(\frac{360\ x\ 60\ x\ 60''}{2\pi}\right)$" has an expected answer "What is the angular resolution of a 1m diameter telescope at the bluest light that human eyes can see?" The students' responses indicate that without the equation written in variables, they do not completely understand what all of the variables given in the equations mean conceptually when values have been substituted for the variables.

Michael:

"What is the diverging limit, in arcseconds, of a telescope 1 meter in diameter measuring 400 nm wavelength?"

Scott:

"Determine the diffraction limit in arcseconds of a telescope with a diameter of 1 m that is observing light of wavelength 400 nm."

Ronald:

“What is the diffraction limit (in arcseconds) for a telescope with diameter D = 1m, observing light with a wavelength of 400 nm?”

Richard:

“What is the resolution of a telescope with diameter = 1 m looking at a wavelength of 400 nm (in arc secs).”

Samuel:

“What is the diffraction limit, in arcseconds, of a telescope with a 1 meter diameter observing visible violet light (400nm)*?”

We can see from the above student answers that the students classified as having “understanding” of the equations are able to determine that they are considering the equation used to determine the angular resolution of a telescope ($\theta = 1.22\frac{\lambda}{D}$) where θ is the angular resolution in radians, λ is the wavelength of light, and D is the diameter of the lens' aperture. The students do correctly link all of these variables to the corresponding given values; showing that they recognize the symbols in the equation in terms of the corresponding physics quantities. The students also all understand that they need to convert the answer obtained for the angular resolution from radians to arc seconds; showing that they do make a mathematical connection to the equation in this particular problem. However, the knowledge that they display concerning the variables is not complete; showing that they do not make a complete conceptual connection to the equation. The first four student’s responses all have the same thing in common; they all do not define what type/color of light is being observed. They all know that “λ = 400 nm” but (with the exception of Samuel) do not go the one step further to “complete understanding” by describing the observed wavelength as blue/violet visible light. This shows the students’ do in fact have “understanding”, but they are showing that they are lacking a complete conceptual connection to the equation. Additionally, Michael's case may be one of simple miswording, but also note that he calls the "angular resolution" or "diffraction limit" the "diverging limit" (clearly not the variable that is being found) and Richard's answer could be improved upon by adding the word "angular" before "resolution". Samuel's response was given here for completeness, it is not wrong in any respect (showing "complete understanding") but was included with these homework assignments as this student shows "understanding" of equations in other data, so is placed here for completeness and comparison later.

Question #1 part b from homework assignment #6 is another problem that helps illustrate these students' "understanding' of the equations. Again, in homework assignment #6, question #1 part (b) the students are asked to rank different black holes based on the magnitude of their tidal forces: "Rank the following black holes based on the magnitude of the tidal forces that they would exert on a spaceship placed near their event horizon. A has mass $10M_{\odot}$; B has mass $100M_{\odot}$; C has mass $10^6M_{\odot}$." Since $F_{tidal} \propto \frac{M_{BH}}{R_S^3}$ and $R_S \propto M_{BH}$ the ranking of the tidal forces can be determined using proportionalities ($F_{tidal} \propto M_{BH}^{-2}$); the rank from strongest to weakest tidal forces is therefore A, B, C.

The students' answers to this question again show only "understanding" of the equations; although they each explicitly correctly identify the tidal force, they do not recognize that the Schwarzschild radius is proportional to the mass of the black hole as well.

Richard & Ronald (not collaborators):

$$\Delta F_{tidal} = \frac{2GMmR}{r^3}$$

Tidal force is proportional to mass. Therefore: C > B > A

William:

"Mass is proportional to the max tidal force, $\Delta F_{max} = \frac{2GMmR}{r^3}$ and since that leaves mass as the only difference between the black holes, the strongest to weakest tidal forces positively correlates with the highest to lowest mass.

A($10M_{\odot}$) [weakest] < B ($100M_{\odot}$) < C ($10^6 M_{\odot}$) [strongest]."

Kenneth:

$$F_{max} = \frac{2GMmR}{r^3}$$

"Because $F_{max} \propto M$, I would rank the masses $10^6 M_{\odot}$, $100M_{\odot}$, $10M_{\odot}$ (greatest to least amount of force exerted on the spaceship."

Here, as in the examples for "partial understanding", each of these students correctly identifies that the equation for the tidal force between the black hole and the spaceship is proportional to mass; but do not recognize that the "r" in the denominator of the equation is the Schwarzschild radius which in turn is proportional to the mass of the black hole. These students therefore do not show a conceptual connection to the equation. These students were classified as "understanding" rather than "partial understanding" because unlike the students with "partial understanding", these students explained the reasoning for their ranking with reference to the correct equation for tidal force.

Similar results were found once again when examining the Exam questions in terms of understanding the qualitative aspects of the equations used in solving the problems on the Exams. The students who showed "understanding" of the equations recognize the symbols in the equation in terms of the corresponding astrophysics quantities and were able to know when to use a problem; overall selecting the correct equation from the supplied formula sheet. However, they only sometimes made conceptual connections or mathematical connections to the equations they selected for the problems.

To illustrate the similarities and differences of the results from the homework assignments to the exams for the students who showed “understanding”, as well as to compare these students to students of “no” or “partial understanding”, I will again focus on the students’ responses to questions #1, #2, and #4 from Exam #2 with the questions and expected answers stated above. Four out of the six students shown below are the same students who were highlighted in the homework that had “understanding” of the equations necessary to complete the problems:

Question #1:

- Richard – Shows that he knows the correct equation to obtain the age of the universe, but does not use it; demonstrating no mathematical connection or conceptual connection to the equation.
    - “No because $H_0^{-1} = 10 \times 10^9 yr\ \left(\frac{100\ km\ s^{-1} Mpc^{-1}}{H_0}\right)$ is not accounting for dark matter.”
- William – Does not select an equation to answer this question; however, obtains the correct answer without using the equation indicating that he has some knowledge of the equation. Since no equation was supplied, but the correct answer was given, there is no way to know if this student has a complete conceptual or a complete mathematical connection to the equation necessary to completely answer this problem.
    - “Yes, because it states that those clusters would be older than the galaxy itself.”
- Frank – Utilized an equation which was not the equation written on his exam. Although he states “$v = H_0 d$”, he uses “$H_0^{-1} = 10 \times 10^9 yr\ \left(\frac{100\ km\ s^{-1} Mpc^{-1}}{H_0}\right)$” which is valid for the problem. Frank did not show a mathematical connection or a conceptual connection

to the equation because he states that a universe which is 10 Gyr would not be a conflict with star clusters with a known age of 10^{12} yr.

- “$v = H_0 d$

  No there would be no conflict because $10\ Gyr \times \frac{100}{100} = 10\ Gyr$ and it would still work.”

Question #2:

- William – Chose the wrong equation for the problem given; this student choose a given equation appropriate for finding the mass of a galaxy when the density is known or can be estimated. This student was classified as “understanding” rather than “partial understanding” because the student identifies that the equation he selected would be “very ineffective”, thus showing some limited conceptual connection to the equation he selected.
  - “The 3-D spherical distribution formulas;

    $$M = 4\pi \int_0^r \rho(u) u^2 du \;\rightarrow\; \rho(r)$$

    Though very ineffective in an elliptical galaxy.”
- Frank –Chose the most appropriate equation for the problem given; however aside from defining some of the relevant variables and identifying that $\beta = 1$ and $\sigma = 1$, he does not make the make the complete conceptual connections or the mathematical connections necessary to the equation he provided.
  - “Yes.

    $$M = \frac{3\beta}{\eta} \frac{R\sigma^2}{G}$$

    R = size σ = velocity dispersion

    $M = \frac{3(1)}{(1)} \frac{R\sigma^2}{G}$ → isothermal / identical stars.

Question #4:

- Richard, William, & Samuel – Each chose the correct equation for the problem given; these students all chose an equation for force and make the connection between force and acceleration showing that they are making a conceptual connection to the equation. However these students are all inconsistent with their conceptual and/or mathematical connections to the equation; although they are able to create a ratio with their equations, canceling out constants (sometimes not shown explicitly), they obtain an incorrect ratio – either through mathematical mistakes or substitution mistakes.
  - Richard: (a.) Mass of Jupiter, Mass of Io, Radius of Io, Distance from Jupiter to Io

    (b.)

    $$"\Delta F_{max} = \frac{2GM_J m_{moon} R_{moon}}{r^3{}_{J+moon}}$$

    $M_G = 2M_{Io}$, $R_G = 3R_{Io}$, $r = 4r_{JI}$

    $$\Delta F = \frac{2GMmR}{r^3} \Rightarrow F \propto \frac{M_J m_{moon} R_{moon}}{r_{dist}^3} \rightarrow \frac{M_J 2M_{Io} 3R_{Io}}{(r_{JI})^3} = \frac{6}{64}$$

    The force would be $\frac{3}{32}$ of what it would be on Io."
  - William: (a.) Mass of Jupiter, Radius of Io, Distance from Jupiter to Io

    $$"a = \frac{\Delta F_{max}}{test\ mass} = \frac{2GM\cancel{m}R}{\cancel{m}r^3}$$

    M = mass of Jupiter, m = test mass, R = radius of Io, r = distance between test mass and Jupiter

    Didn't circle (C.) because the tidal acceleration does not account, at least here, for the gravitational acceleration; the effect of conflicting tidal and gravitational acceleration does.

(b.) Galileo: m = 2m, R = 3R, $r_{JI} = 4r_{JI}$

$$a = \frac{3GM(3R)}{64r^3} = \frac{6GMR}{64r^3} = \frac{3GMR}{32r^3}$$

The maximum tidal acceleration would be less on Galileo compared to Io, if only because it is further away from Jupiter. Even accounting for the increased radius, the closer side of Galileo toward Jupiter has less tidal acceleration that the furthest side of Io."

- Samuel: (a.) Mass of Jupiter, Mass of Io, Radius of Io, Distance from Jupiter to Io

  (b.) "Galileo: 4r, 2M, 3R

$$\frac{\Delta F_{max}}{m} = \frac{2G(2M)(3R)}{(4r)^3}$$

  Io:

$$\frac{\Delta F_{max}}{m} = \frac{2GMR}{r^3}$$

$$\frac{a_{galilleo}}{a_{Io}} = \frac{\frac{2G(2M)(3R)}{(4r)^3}}{\frac{2GMR}{r^3}} = \frac{2 \cdot 3}{64} = \frac{3}{32}$$

  So that the max tidal acceleration on a test mass on galileo is $\frac{3}{32}$ as strong than on Io."

- Scott – Choses a correct equation for the tidal acceleration for problem given; but fails to make a mathematical connection to the equation as well as a conceptual connection. He makes mathematical mistakes in the denominator of the equation and incorrectly substitutes for some of the variables in the equation.
  - (a.) Mass of Jupiter, Radius of Io, Distance from Jupiter to Io

    (b.)

$$" a_T = \frac{2GMR}{r^3} = \frac{2GM_J R_{Io}}{(4r - 3r_{Io})^3} = \frac{1}{64}\frac{2GM_J R_{Io}}{(r - r_{Io})^3}$$

The maximum tidal acceleration on a test mass on [the] surface of Galileo would be approximately 1/64 that of the maximum tidal acceleration on a test mass on Io caused by Jupiter."

- Joshua – Again supplies a correct equation of the force and acceleration and sets up the ratio to determine the maximum tidal acceleration caused by Jupiter of a test mass on the surface of Galileo compared with Io correctly; but does not show a mathematical connection to the equations by making some mathematical errors and obtaining an answer which is not quantitative.
    - (a.) Mass of Jupiter, Distance from Jupiter to Io

      (b.) "In both cases M, R = Jupiter Properties

      Galileo

      $$\Delta F = \frac{2GMm_gR_g}{r_g^3} = \frac{2GM \cdot 2m3R}{(4r)^3} = \frac{12GMmR}{116r^3}$$

      $$a = \frac{12GMR}{116r^3}$$

      Io

      $$\Delta F = \frac{2GMmR}{r^3}$$

      $$a = \frac{GMR}{r^3}$$

      Galileo's acceleration is much slower than Io."

For these two similar types of assignments, the classifications for students with "understanding" are similar. Students with "understanding" of astrophysics equations recognized of the symbols in the equations in terms of the corresponding astrophysics quantities and were able to know when to use an equation. However, these students were inconsistent when it came to making conceptual connections and/or inconsistent then making mathematical connections to the equations.

Essays

Once again using the criteria for student understanding of equations described in the above sections of this chapter, I determine the students who demonstrated “understanding” of the astrophysics equations that they provided in their essays. Out of the above students highlighted in the homework and essay analysis, three (3) students (Samuel, Ronald, and Joshua) also showed “understanding” in their essays. I will therefore concentrate on the essays of these three students which demonstrate some of the commonalities that students with “understanding” share in their essays along with one additional student’s responses.

All students who showed “understanding” of the equations in the essays included an equation in their essay along with a minimum of three more of the criteria for student understanding of equations. The most common criteria that students with “understanding” demonstrated were that these students adequately discussed the astrophysics in the equations and they understood the purpose of the equations they provided. All fourteen (14/100%) of the students who showed “understanding” of the equations discussed the astrophysics in the equation, but not in a deep way:

- Samuel: “The virial theorem is a way of quantitatively describing the equilibrium state of systems with multiple bodies that interact via gravity (or other potential forces).”

- Ronald: “I understand the basics [of special relativity]: the speed of light is consistent in all frames, and length is contracted and time is dilated in frames with relativistic velocity.”

- Thomas: “Using the two postulates that the laws of physics are the same in all inertial reference frames and that the speed of light is constant and universal, I understood that the Lorentz transformation from the Galilean transformation which showed that the time and space was linked and was termed as spacetime. Then I approached the concept of time dilation … and length contraction which were all new to me.”

Eleven (12/86%) of the students who showed “understanding” of the equations they provided also demonstrated that they also understood the purpose of the equation:

- Joshua: “But when the Schwarzschild Radius, defied as $R_s = \frac{2GM}{c^2}$ was introduced I was interested … that there is a radius that is dependent on an objects mass that would be considered a black hole and that from this you can determine if objects will be safe or [pulled] into this black hole.”

- Samuel: “The [virial] theorem states that in a bound system of two or ore bodies/particles, twice the average kinetic energy plus the average potential energy of a system is equal to zero.” “Before knowing this equation, I couldn’t possibly imagine how we would begin to describe systems with millions or billions of bodies, like galaxies. I find it very profound that the average properties of an unimaginably complex system boils down to such a simple statement.”

Additionally, out of the fourteen (14) students who showed “understanding” of equations in their essays, most (10/71%) discuss the symbols in the equation and ten (10/71%) of the students who showed “understanding” of the equations also discussed the connections between the equations they provided and the real world:

- Thomas: “I was always curious about why artists drew a baseball thrown at high speed contracted. I found the solution to my curiosity after learning about the concept of length contraction which stated that moving objects appeared shorter in the direction of motion.” “The [special relativity] equation [for length contraction] explained that the baseball moving at a high speed from our reference frame would appear contracted, whereas from the baseball’s reference frame we would appear contracted because we would appear to move relative to the baseball.”

- Ronald: “The Sun [acts] as a dominate mass to the planets, creating a one-body problem. Black holes work similarly, a black hole in facet acted very simply as a dominate massive object, even one that is measureable using the Schwarzschild radius formula.”

Of the fourteen (14) students who showed “understanding” most discussed the symbols in the equation (10/71%); but few discussed the structure of the equation (4/29%) or showed a deep understanding of the astrophysics behind the equation they provided in their essays (3/21%).

Table 23: Summary of criteria of students who had “understanding” of equations in their essays.

	Students with “Understanding” met the following criteria for understanding equations		
Does the student …	“Understanding”	“Partial Understanding”	“No Understanding”

... discuss the symbols in the equation?	10/71%	4/24%	0/0%
... discuss the astrophysics or physics in the equation?	14/100%	14/82%	1/6%
... discuss the structure of the equations?	4/29%	2/12%	0/0%
... understand the purpose of the equations?	12/86%	11/65%	1/6%
... show deep understanding of astrophysics behind the equations?	3/21%	0/0%	0/0%
... talk about connections between equations and real world?	10/71%	5/29%	0/0%

The students who show "understanding" of the equations meet most of the criteria for understanding equations, but they do not meet all of the criteria necessary. Therefore, these students are classified as having an "understanding" of the relevant equations related to the topics in their essays, but not "complete understanding".

Interviews

One (1) of the ten (10) students interviewed showed "understanding" of the equations used to solve the astrophysics problems presented. This student, Michael, also showed "understanding" in the homework assignments. In the interview, Michael showed "understanding" of astrophysics equations in the problems given during the course of the interview by demonstrating the ability to recognize the symbols in the equations he provided in terms of the corresponding physics or astrophysics quantities, establishing a link between the equations and everyday life, demonstrating knowledge of how to use an equation to solve the problems, and the ability to know when to use an equation. However, although Michael was able to start the problems without assistance, he did not consistently demonstrate that he could recognize the underlying physics of the equations, he did not consistently recognize the structure of the equations, and he did not demonstrate consistency in making conceptual or mathematical connections to the equations. Michael did not seem to memorize a large amount of equations without understanding (unlike the students who showed “no” or “partial” understanding). The following excerpts from this student’s interview illustrate his classification of “understanding”.

Michael:

Unlike the students with “no” or “partial understanding” Michael was able to start the problems given without assistance. He occasionally needed assistance in completing the problems when is train of thought led him in an unproductive path, but even in these cases Michael shows that he has some recognition of the underlying physics of the equation because he recognizes that what he is doing is not helpful to solving the problem. For instance, in the first problem Michael states “kinetic energy us ½ mv squared. I guess you can put [v] in terms of escape velocity, but that would be when potential and kinetic are equal so let’s not do that.” From this small statement, we can see that Michael does recognize the symbols in the equation; he knows that he is looking for the velocity if the earth. He also demonstrates an inconsistent conceptual connection to the equation in this part of the problem, he knows it isn’t correct for this situation, but does not know how to fix it. After guiding Michael through some of the steps, Michael soon realizes that: “Escape velocity is when potential equals kinetic because that means it has enough energy to escape the potential well.” and “[Use] the gravitational force, right. So, Gmm/r^2 and then mv^2/r would be the [centripetal] force. So, you can just start canceling stuff out ... and then v^2 equals Gm over r, so the square root of that.” Showing that he does have an understanding of the underlying physics of the equations, can make mathematical connections to the equations, and recognizes the structure of the equations.

The student shows inconsistent results in terms of mathematical understanding of the equations while discussing the next problem: “So you already know the equations for force, velocity, and energies. You are using the force to derive the energies but with this acceleration … you have to substitute in one of the forces.” Here the student shows a lack of mathematical understanding of the equations given and no recognition of the underlying physics of the equations. The force given in the problem is not being used to “derive energies” as the student claims, but is being used as a step in finding an equation for substitution in one of the variables (velocity) in the kinetic energy equation. Michael further shows no mathematical conceptual understanding of the equations when he states “what it looks like is if you are given [these equations] you can just plug in for [another equation] then it is like you keep deriving new things. So, it is relating the acceleration to the energies.” However, in the same problem, the student demonstrates that he recognizes the underlying physics of the equation when discussing the final answer to the problem – which was to create a question to the situation given – with his final answer: “What are the kinetic and potential energies of a body with mass m and [centripetal] acceleration a for a bound body in a circular orbit.”

In the last problem of the interview, Michael demonstrates that he has a conceptual connection to the equation that he is using. His statement: “And when [the satellite] has atmospheric drag, it is losing velocity, so that means kinetic energy is going to decrease so that means that K would become less than the absolute value of ½ u which means that [the satellite] is not going to be in a bound system anymore. And since the potential energy is going to be greater than – the absolute value of potential energy is going to be greater than 2K the gravitational energy is going to overpower the kinetic energy out of the orbit and it is going to like spiral inward.” shows that, although he starts with an incorrect assumption (that the velocity would decrease) his thought process after that first original misconception shows conceptual understanding, mathematical understanding of the equations as well as recognition of the underlying physics of the equations. When working through the problem, Michael does obtain the final answer that the velocity is increasing in this scenario, recognizing his initial misconception and makes a further connection establishing a link between the equations and another astrophysics scenario: "like if there is something orbiting a black hole and it's like tearing mass off of the star and then it is going to eventually pull the star into the black hole".

Furthermore, Michael did demonstrate an inconsistent knowledge of how to use equations to solve these problems. He was consistently able to start the problem, but in each problem, he a moment when he realized that he was doing the problem incorrectly and did not know how to proceed once he realized that he could not move forward with his way of solving the problem. For instance, in problem one, when Michael realized that he should not use the escape velocity in the kinetic energy equation, I was able to guide him by asking him to draw a picture of the system and use a force diagram. Michael then quickly realized that he could "find the rotational velocity" as described above. With help, Michael was able to move past each of the times he got stuck in the problems and was able to finish the problems on his own.

As this was the only student who showed "understanding" in the interviews the analysis of his interview was taken as overall indicative of "understanding" in this assessment. This student with "understanding" shows that he was consistently able to: recognize the symbols in the equations in terms of the corresponding astrophysics quantities, establish a link between the equations and everyday life, know when to use an equation, and start the problem without assistance. He was inconsistent however when it came to recognizing the structure of the equations, demonstrating knowledge of how to use an equation to solve astrophysics problems, recognizing the underlying physics of the equation, and making a conceptual and/or mathematical connection to the equations. This student was therefore classified as having "understanding" of astrophysics equations in this assessment.

Summary - Understanding

In comparison to the students with “no” or “partial understanding” of astrophysics equations, the students with “understanding” demonstrated this “understanding” in multiple assessments more so than the students that demonstrated “no understanding”, but less than the students who demonstrated “partial understanding”. The following two tables show the overall performance of students who were found to have “understanding” of equations in a particular assessment:

Table 24: Number/Percentage of students showing “understanding” by assessment

Understanding			
Homework	Exam	Essay	Interview
16/30%	17/32%	14/26%	1/10%

Table 25: Overlapping of the number of students with “understanding” by assessments

Number/Percentage of Students Overlapping			
Assessment Overlap	Overlap for Students with "Understanding"	Overlap for Students with "Partial Understanding"	Overlap for Students with "No Understanding"
Homework & Exam	5 of 28/18%	9 of 25/36%	1 of 15/7%
Homework & Essay	3 of 27/11%	6 of 26/23%	1 of 20/5%
Exam & Essay	4 of 19/21%	9 of 25/36%	1 of 12/8%
Homework & Exam & Essay	1 of 35/3%	4 of 33/12%	None
Interview & "Other"	1 of 6/17%	5 of 7/71%	None
No Overlap	23 of 35/66%	10 of 33/30%	22 of 26/85%

As can be seen in Table 25, there was found to be more overlapping of the students who showed “understanding” of astrophysics equations in at least two of the different assessments analyzed than those with “no understanding”, but less than those who showed “partial understanding”. While the results for the students classified with “no understanding” implied that there is no such thing as a student with “no understanding” of the equations in this classroom and the results for the students with “partial understanding” imply that these students did have an overall “partial understanding” of the equations used; the results for students with “understanding” implies that these students more than likely had a better understanding than those with “partial understanding”, but not necessarily.

5.2.d Complete Understanding

The students who were identified with “complete understanding” of the equations used in the course out of the students enrolled in the class fulfilled all of the criteria for understanding astrophysics equations as given in the theoretical framework. These students were able to recognize the symbols in the equations they used in terms of the corresponding physics and astrophysics quantities, recognize the underlying physics and astrophysics of the equation, recognize the structure of the equations, establish a link between the equation and everyday life, demonstrate knowledge of how to identify and use an equation to solve astrophysics problems without assistance, and made a conceptual and/or mathematical connection to the equations they used. The following table illustrates the connection between the theoretical framework and the students’ assignments with the students who were classified as those with “complete understanding” of astrophysics equations.

Table 26: Indicators of Students with “Complete Understanding” of Equations

Students with "complete understanding" of astrophysics equations …				
	Homework Assignments	Exams	Essays	Interviews
Theoretical Framework				

… recognize the symbols in the equation in terms of the corresponding physics quantities.	Yes	Yes	Yes	Yes
… recognize the underlying physics of the equation.	N/A	N/A	Yes	Yes
… recognize the structure of the equation.	N/A	N/A	Yes	Yes
… establish a link between the equation and everyday life.	N/A	N/A	Yes	Yes
…demonstrate knowledge of how to use an equation to solve astrophysics problems.	N/A	N/A	N/A	Yes
… are able to know when to use an equation.	Yes	Yes	N/A	Yes
Additional Indicators				
... can start the problem without assistance.	Unknown		N/A	Yes
… made conceptual or mathematical connection to the equations.	Yes	Yes	N/A	Yes

As can be seen in Table 26, the students from the class that were subsequently classified with "complete understanding" of astrophysics equations had all of the theoretical attributes that students with comprehension of equations should have according to the literature. In addition to the theoretical framework as outlined in the literature these students, unlike the students discussed previously as "no understanding", "partial understanding", and "understanding", these students were identified as making the conceptual and/or mathematical connection to the equations.

Homework Assignments & Exams

When the qualitative portion of the homework assignments was averaged, fifteen (15) students out of 54 (28%) were found to be students with "complete understanding" qualitatively of the equations which they used in the qualitative portion of the homework assignments. The responses given by the students in homework assignment #3 as well as homework assignment #6 will once more be used in order to illustrate my findings in order to complete the analysis of this section. As before, the responses selected for the students with “complete understanding” in these homework assignments are also indicative of the overall student responses for these students’ homework assignments. In question #1 on homework assignment #3, three (3) out of the 54 students (6%) who submitted this homework assignment showed “complete understanding” for part (a), while thirty-seven (37) students (69%) showed “complete understanding for part (b), and five (5) students (9%) showed “complete understanding for part (c). Nineteen (19) out of the 53 students (36%) who submitted homework assignment #6 showed “understanding” for question #1, part (b). As done above, I will continue to concentrate on the parts of the homework assignments which focus directly on qualitatively understanding equations; highlighting the students whose work across two or more of the data analysis was categorized as “complete understanding” for comparison. Most of the students I will consider below worked alone on the homework assignments; however, three of the highlighted students (Alexander, Jack, and Justin) worked with each other as collaborators on the homework assignments.

In the responses made by students with “complete understanding” to homework assignment #3 question #1b; “How does the gravitational force that one object exerts on another object change if the distance between them triples? If the distance between them drops by half? Explain how you know.”, all of students selected were able to successfully answer this particular problem. These students correctly identified the problem with the appropriate equation (Newton's law of gravitation) involving the use of an inverse square. The students correctly identified that if the distance triples, the force decreases by a factor of nine and if the distance drops by half, the force increases by a factor of four. For example:

Steven:

$$F \propto \frac{1}{r^2}; \qquad F \propto \frac{1}{(3r)^2} \propto \frac{1}{9}\left(\frac{1}{r^2}\right)$$

$\frac{1}{9}F$; the gravitational force is 1/9th of the original force if the original distance between object 1 and 2 is tripled.

$$F \propto \frac{1}{r^2}; \qquad F \propto \frac{1}{\left(\frac{1}{2}r\right)^2} \propto 4\left(\frac{1}{r^2}\right)$$

$4(F)$; the gravitational force quadruples if the original distance between the objects drops by half.

Steven's response shows that he recognizes the inverse square nature of Newton's law of gravitation and therefore makes the conceptual connection to the equation. He also recognizes the symbols in the equation in terms of the corresponding astrophysics values and makes a mathematical connection to the equation, correctly solving the problem.

James:

$$F = \frac{Gm_1m_2}{r^2} \implies F \propto \frac{1}{r^2}$$

Distance triples: $F \propto \frac{1}{(3r)^2} \Rightarrow F \propto \frac{1}{9r^2}$

Force decreases by a factor of 9.

Distance drops by ½: $F \propto \frac{1}{\left(\frac{1}{2}r\right)^2} \Rightarrow F \propto \frac{4}{r^2}$

Force increases by a factor of 4.

As the two objects are closer, the gravitational force increases and as they move apart, the gravitational force decreases.

James also demonstrates an understanding of the inverse square nature of the equation in his answer; but starts from first principles (i.e. he writes out Newton's law of gravitation). He shows in his answer that he also recognizes the symbols in the equation in terms of the corresponding astrophysics values and makes a conceptual and a mathematical connection to the equation, correctly solving the problem and being classified as "complete understanding" of the equation in this problem.

Alexander:

If the distance triples:

$F_G = \frac{GMm}{(3r)^2} = \frac{GMm}{9r^2}$ The gravitational force would decrease by a factor of 9.

If the distance drops by half:

$F_G = \frac{GMm}{\left(\frac{1}{2}r\right)^2} = \frac{4GMm}{r^2}$ The gravitational force would increase by a factor of 4.

Jack:

$F_g = \frac{GMm}{r^2}.$

$r \rightarrow 3r \implies F_g \rightarrow \frac{1}{9}F_g.$

$r \rightarrow \frac{r}{2} \implies F_g \rightarrow 4F_g.$

From Newton's Law of Gravitation, we know that F_g is inversely proportional to the square of r. So, when r is triples, F_g becomes 1/9 its original value. Similarly, when r is halved F_g is quadrupled.

Justin:

$F_g = \frac{GM_1M_2}{r^2}$.

This is a scaling relation. The gravitational force is inversely proportional to the square of the radius. So, if distance triples then the force is 1/9 the original and if the distance halves then the force is 4 times the original.

Alexander, Jack, and Justin were collaborators on the homework assignments. It is clear from the above responses to this question however that they all had their own unique “complete understanding” of the inverse square nature of Newton’s law of gravitation. Each of them showed individually that they were able to know when to use an equation (each choosing Newtons law of gravitation, although in different ways), recognized the symbols in terms of the corresponding astrophysics quantities, and made the necessary conceptual and mathematical connection to the inverse square nature of the equation.

All of the students who answered homework assignment #3, question #1c with “complete understanding” of the equation did so because they made what was considered a complete conceptual connection to the equation given to them. In other words, they recognized all of the quantities in the equation as corresponding to the symbols in the equation with a full conceptual connection to those symbols. These students were able to “Create a question for which the following equation provides an answer: $0.1'' = 1.22\left(\frac{400nm}{1m}\right)\left(\frac{360\ x\ 60\ x\ 60''}{2\pi}\right)$” with the expected answer “What is the angular resolution of a 1m diameter telescope at the bluest light that human eyes can see?” The students’ responses indicate that they have “complete understanding” of the equation even when values have been substituted for the variables.

Steven:

“What is the diffraction limit of a telescope with diameter = 1m observing [ultraviolet light] of 400 nm in arcseconds?”

Daniel:

“$\theta = 1.22\frac{\lambda}{D}$ What is the diffraction limited resolution of a telescope with diameter of 1m that observes violet light of wavelength 400 nm. Give your answer in arcseconds.”

Jack:

$$0.1'' = 1.22\left(\frac{400nm}{1m}\right)\left(\frac{360\ x\ 60\ x\ 60''}{2\pi}\right)$$

$$0.1'' \rightarrow \theta$$

$$1.22 \rightarrow m\ for\ round\ aperature$$

$$\left(\frac{400nm}{1m}\right) \rightarrow \frac{\lambda}{D}$$

$$\left(\frac{360\ x\ 60\ x\ 60''}{2\pi}\right) \rightarrow rad\ to\ \mathrm{arcsec}\ conversion$$

What is the diffraction limit for a telescope, in arcseconds, with diameter 1 m focusing on violet light with wavelength 400 nm?”

As can be seen from the above student answers, the students classified as having “complete understanding” of the equations recognized that they are assessing the equation used to determine the angular resolution of a telescope ($\theta = 1.22\frac{\lambda}{D}$) where θ is the angular resolution in radians, λ is the wavelength of light, and D is the diameter of the lens' aperture. Most of these students aslo included the equation in symbol form in their answer. The students do correctly link all of these variables to the corresponding given values (some explicitly); showing that they recognize the symbols in the equation in terms of the corresponding physics quantities. All of these students also all make the connection that part of the given equation is to convert the answer obtained for the angular resolution from radians to arc seconds; showing that they do make a mathematical connection to the equation in this problem. In addition, the understanding that they display concerning the variables is complete. These students demonstrate that they have made a complete conceptual connection to the equation; they all define what type/color of light is being observed. For the students who did not show "complete understanding", most were able to determine that "λ = 400 nm" but, unlike these students, did not go the one step further to "complete understanding" by describing the observed wavelength as blue/violet visible light. The students with "complete understanding" therefore demonstrate, in this particular problem, that they do in fact have a complete conceptual connection to the equation.

Question #1 part b from homework assignment #6 is another problem that helps illustrate these students' "complete understanding" of the equations. In this question the students are asked to rank different black holes based on the magnitude of their tidal forces: "Rank the following black holes based on the magnitude of the tidal forces that they would exert on a spaceship placed near their event horizon. A has mass $10M_{\odot}$; B has mass $100M_{\odot}$; C has mass $10^6 M_{\odot}$." Since is proportional to the mass of the black hole and inversely proportional to the cube or the Schwarzschild radius and the Schwarzschild radius is proportional to the mass of the black hole, the ranking of the tidal forces should be determined using proportionalities ($F_{tidal} \propto M_{BH}^{-2}$); with the rank from strongest to weakest tidal forces being A, B, C. The following students' answers to this question show "complete understanding" of the equations; they each correctly identify the tidal force as well as the Schwarzschild radius thus creating a correct proportionality between force and the mass of the black hole.

Daniel:

"Near the event horizon, $\Delta F \propto \frac{M}{R_S^3}$. Since R_s increases as mass increases, a larger mass means a smaller force. C < B < A"

Jack:

$$\Delta F_R \propto \frac{M}{r^3} \qquad r = \frac{2GM}{c^2} \text{ Schwarzchild radius of black hole}$$

$$\Delta F_R \propto \frac{M}{(M)^3}$$

$$\Delta F_R \propto \frac{1}{M^2}$$

"Tidal force inversely proportional to square of mass."

A: $\Delta F_R \propto \frac{1}{10^2} \rightarrow \Delta F_R \propto 10^{-2}$

B: $\Delta F_R \propto \frac{1}{100^2} \rightarrow \Delta F_R \propto 10^{-4}$

C: $\Delta F_R \propto \frac{1}{(10^3)^2} \rightarrow \Delta F_R \propto 10^{-12}$

"A > B > C"

Justin:

$\Delta F(\theta) = \frac{GMmR}{r^3}(2cos\theta\hat{x} \cdot sin\theta\hat{y})$. $F \propto \frac{M}{r^3} \rightarrow F \propto \frac{M}{R_S^3} \rightarrow F \propto \frac{M}{\left(2GM/c^2\right)^3} \rightarrow \Delta F \propto \frac{1}{M^2}$

> “r is the Schwarzschild radius because the spaceship is at the event horizon. Plugging the formula for R_S in we see that the tidal force is inversely proportional to the square of the mass. So, the most massive black hole has the smallest tidal force. A > B > C”

All of the students with “complete understanding” correctly identifies that the equation for the tidal force between the black hole and the spaceship is proportional to mass as did most of the students with “partial understanding” and “understanding” of the equations in this particular problem. These students alone however recognize that the “r” in the denominator of the equation is the Schwarzschild radius which in turn is proportional to the mass of the black hole. These students therefore show a conceptual connection to the equation. These students are therefore classified as “complete understanding” as they fully grasp all of the aspects of the problem. When examining the Exam questions in terms of understanding the qualitative aspects of the equations used in solving the problems on the Exams, the students who showed “complete understanding” of the equations were able to know when to use an equation, recognized the symbol in the equation in terms of the corresponding astrophysics quantities, and mad conceptual and mathematical connections to the equations.

To illustrate the similarities of the results from the homework assignments to the exams for the students who showed “complete understanding”, as well as to compare these students to students of “no understanding”, “partial understanding”, and “understanding”; I will again focus on the students’ responses to questions #1, #2, and #4 from Exam #2 with the questions and expected answers stated above. Four out of the five students shown below are the same students which were highlighted in the homework that had “complete understanding” of the equations necessary to complete the problems, the fifth student received “understanding” in his homework assignment, but was added here for completeness.

Question #1:

All of the students with “complete understanding” of the equations in this problem, demonstrate that they know to use the correct equation to obtain the age of the universe and have the conceptual and mathematical connection to the equation to obtain the correct answer.

- Steven:

  “$H_0^{-1} = 10Gyr\ \times \left(\frac{100\ km\, s^{-1} Mpc^{-1}}{H_0}\right)$

  $$H_0^{-1} = 10Gyr \times (1) = 10 \times 10^9 years = 10^{10} yrs\ \ \rightarrow \text{10 billion years}$$

  Yes, because if the Hubble constant is 100 km/s/Mpc, then the universes age would be 10 billion years which is < (less than the) claimed age of 12 billion year old stars.”

- Justin:

  $$H_0^{-1} = 10Gyr\ \times \left(\frac{100\ km\, s^{-1} Mpc^{-1}}{H_0}\right)\Bigg|_{H_0 = 100 kms^{-1} Mpc^{-1}} = 10\ Gyr$$

  $$= 10 \text{ billion years}$$

  “Yes, this would produce a conflict because the age of the universe is H_0^{-1} which is approximately 10 Gyr. The oldest stars in our galaxy would be 12 Gyr which is older than the universe. This produces a conflict since stars are older than the universe.”

- Jack:

  “Yes, it does produce a conflict. If H_0 is $100 \frac{km/_s}{Mpc}$, then using the relationship $H_0 = 10\ Gyr \cdot \frac{100}{H_0}$ gives us an age of 10 billion years for the universe. This contradict with the age of the stars and clusters that are apparently older than the universe.”

Question #2:

All of the students with "complete understanding" of this problem are: able to use the correct equation for this problem; can recognize the symbols in the equation in terms of the corresponding physics quantities – identifying and defining the variables and constants; and make conceptual connections to the equation. These students all recognize and identify the underlying physical principle supporting this problem - the viral theorem.

- Steven:

  $$M = \frac{3\beta}{\eta}\frac{R\sigma^2}{G}$$

  R is a variable σ, η, β are constants

  Assumptions: η is based on the object's structural nature, i.e. isothermal sphere, constant density disk [η=1]; β is also assumed [to be 1] - identical stars.

  The physics behind this approach comes from the virial theorem and how the object's mass causes the velocity dispersion.

- Justin:

  "Yes it is.

  We can use $M = \frac{3\beta}{\eta}\frac{R\sigma^2}{G}$ to get a virial mass estimate of the galaxy. The variables are R (radius), σ (velocity dispersion), β and η which are determined by the model we choose for the galaxy (uniform density sphere, isothermal sphere, etc.). G is the gravitational constant. We are assuming that the system is in an equilibrium and the R and σ^2 we observe are time-averaged. We are using the virial theorem."

- Daniel:

  "We can use the virial mass formula $M = \frac{3\beta}{\eta}\frac{R\sigma^2}{G}$, which relies on the physical principle of the virial theorem. R is the radius of the galaxy, σ is the velocity dispersion, G is the gravitational constant, M is the mass of the galaxy, and β and η are constants which equal 1 if we estimate the galaxy is an isothermal sphere."

Question #4:

The students who showed "complete understanding" for this problem choose the correct equation for the tidal force and make the connection between force and acceleration by stating a correct equation for the tidal acceleration; showing that they are making a conceptual connection to the equation. These students also correctly identify the correct ration with their equations, explicitly determining the ratio of the tidal acceleration on Galileo to that of Io; demonstrating that they recognize the symbols in the equation in terms of the corresponding physics quantities and make conceptual and mathematical connections to the equation.

- Justin:

  (a.) Mass of Jupiter, Radius of Io, Distance from Jupiter to Io

$$\Delta F_{max} = \frac{2GMmR}{r^3}$$

$$a = \frac{\Delta F_{max}}{m} = \frac{2GMR}{r^3}$$

  (b.) $r_{J/Io}$ = distance between Jupiter and Io

  $m_G = 2m_{Io}$

  $R_G = 3R_{Io}$

  $r_{J/G} = 4r_{J/Io}$

$$a_{max,Io} = \frac{2GM_J R_{Io}}{r_{J/Io}^3} \qquad a_{max,G} = \frac{2GM_J R_G}{r_{J/G}^3} = \frac{2GM_J(3R_{Io})}{\left(4r_{J/Io}\right)^3}$$

$$= \frac{2GM_J R_{Io}}{r_{J/Io}^3}\left(\frac{3}{64}\right)$$

"The maximum tidal acceleration on the surface of Galileo would be $\frac{3}{64}$ the maximum tidal acceleration on the surface of Io."

- Daniel:

(a.) Mass of Jupiter, Radius of Io, Distance from Jupiter to Io

$$\Delta F_{max} = \frac{2GMmR}{r^3}$$

$$a = \frac{2GMR}{r^3}$$

M – mass of source, R – radius of target, r - distance

(b.) "<u>Io</u>

$$a_{Io} = \frac{GM_J R_{Io}}{r_{Io}^3}$$

<u>Galileo</u>

$$a = \frac{2GM_J(3R_{Io})}{(4r_{Io})^3} = \frac{3}{64}a_{Io}$$

The max tidal acceleration would be $\frac{3}{64}$ that of Io."

- Jeffrey:

(a.) Mass of Jupiter, Radius of Io, Distance from Jupiter to Io

(b.)

$$a_{Io} = \frac{GM_J R_{Io}}{r_{Io}^3}$$

$$a_{Gal} = \frac{2GM_J(3R_{Io})}{(4r_{Io})^3} = \frac{3}{64}\left(\frac{GM_J R_{Io}}{r_{Io}^3}\right) = \frac{3}{64}a_{Io}$$

"The tidal acceleration on Galileo is 3/64 as strong as the tidal acceleration on Io."

It is clear that for these two similar types of assignments, the classifications for all the types of students are similar; students with "complete understanding" included. Students with "complete understanding" of astrophysics equations recognized of the symbols in the equations in terms of the corresponding astrophysics quantities, were able to know when to use an equation, and were able to make both conceptual connections and mathematical connections to the equations they used in these assessments.

Essays

Using the criteria for student understanding of equations described in the above sections of this chapter, I determined the students who demonstrated "complete understanding" of the astrophysics equations that they provided in their essays. Out of the above students highlighted in the homework and essay analysis, three (3) students (Steven, Justin, and Alexander) also showed "complete understanding" in their essays. I will therefore concentrate on the essays of these three students, which demonstrate the common connections to the theoretical framework that students with "complete understanding" share in their essays, along with two additional student's responses for completeness.

All students who showed "complete understanding" of the equations in the essays included an equation in their essay along with meeting all, or all but one, of the criteria for student understanding of equations. The students with "complete understanding" discussed the symbols in the equation, demonstrated that they were fully able to discuss the astrophysics or physics in the equation, discuss the structure of the equations, they understand the purpose of the equations, show deep understanding of astrophysics behind the equations, and talk about connections between equations and real world.

All eleven students (11/100%) that showed "complete understanding" understood the purpose of the equations they discussed as well as had a deep understanding of the astrophysics in those equations they discussed:

- Steven: "[The tidal force equation] was even more interesting when we applied the concept of tidal forces to black holes. Given that a small change in r^3, by setting $r^3 = (R_s)^3 = \left(\frac{GM^2}{C}\right)^3$, [where] R_s= Schwarzschild radius, M = Mass of Black Hole, c = speed of light, gave a maximum tidal force that, at a certain distance (Schwarzschild radius) and towards the center of mass of the black hole, was greater than the electrostatic force of the particles that comprise up of our body."

- Justin: "I found it difficult to accept that simply being in a gravitational field would produce time dilation. To understand this, I first imagined light leaving the potential well of a massive body. I knew that this implied that the object had to slow down according to the conservation of energy, but light can't slow down. Light somehow had to lose energy and it did this by redshifting. Then in class, we derived the gravitational redshift formula, and from that we imagined a clock with a time interval given by 1/Δv. This ultimately gave us the final gravitational time dilation formula."

- Alexander: "General relativity's version of the equation of motion as it relates to an object nearby a black hole is $\Phi_{GR} = -\frac{GM}{r} + \frac{L^2}{2r^2} + \frac{1}{2} - \frac{GML^2}{r^3}$ (and looks like a jumble of variables at first. Upon inspection, however, the first equation isn't the end of the world (although you might disagree if you happen to unfortunately be at some small distance r.) The first three terms come from Newtonian gravitational potential but the fourth is introduced to explain what happens in the special case of motion around a black hole., We see that large r make the GM term insignificant which makes sense because you shouldn't feel the effects of the black hole when you're very far away from it. We can also see the principles of the one-body problem apply: when you have values of r near the black hole and a certain L, the potential graph has a minimum. Minima in potential energy graphs mean circular orbits so when the object is at a certain r and has a certain L, it can have a circular orbit around the black hole just like some of the one-body problems we studied. The equation also shows that no matter how much angular momentum you have – no matter how massive you are – you will not be able to escape being drawn in by the black hole if your r is small enough."

- Jerry: "The size of the star will determine when it supernovas, collapsing in on itself and creating a singularity because the star's mass will become infinitely dense as time moves on while occupying zero space. In order to achieve this singularity, all the mass of the star would have to collapse into a radius governed by the Schwarzschild radius, $R_s = \frac{2GM}{c^2}$, where G is the gravitational constant, M is its mass, and c is the speed of light. Eventually, a black hole is formed, and within the confines of its reach lays the event horizon, which is the area around the black hole that is the point of no return. Once crossing over the black hole's event horizon, the spaghettification process begins in which tidal forces begin to stretch out any object it comes into contact with."

- George: "What I found interesting about tidal forces is how the Moon's tidal force on the Earth has a stronger effect than the Sun's tidal force on the Earth. This is true because when it comes to the source of gravity, the size of the object does not matter if it is spherically symmetric. However, when it comes to the target of gravity, the size of the object does matter since gravity pulls harder on one side than the other. Although the Moon is [less massive] than the Sun, it is so much closer that it exerts the stronger tidal force."

Ten (10/91%) of the students who showed "complete understanding" of the equations discussed the structure of the equations:

- Justin: "$\frac{\Delta v(\infty)}{\Delta v(r)} = \frac{\Delta t(r)}{\Delta t(\infty)} = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2}$. The left side of this equation is the ratio of the passage of time between observers at a point r and one infinitely far away and the right side is the gravitational time dilation factor. Seeing this equation made it conceptually easier to understand what gravitational time dilation really was doing to a photon of light and hence its effect on time."

- Alexander: "$\theta_{\pm} = \frac{1}{2}\left[\beta \pm (\beta^2 + 4\theta_e^2)^{\frac{1}{2}}\right]$ The positive root corresponds to the positive Einstein radius and ditto for the negative root corresponding to the negative Einstein radius. This equation tells us that the negative Einstein ring will be smaller than the positive Einstein ring because $[\beta^2 + 4(\theta_E)^2]^{\frac{1}{2}} > [\beta^2 - 4(\theta_E)^2]^{\frac{1}{2}}$. The opposite relation is true for the positive Einstein ring so it is bigger than the object."

Furthermore, out of the eleven (11) students who showed "complete understanding" of equations in their essays, ten (10/91%) discuss the symbols in the equation and nine (9/82%) of the students who showed "complete understanding" of the equations also discussed the connections between the equations they provided and the real world:

- Steven: "Learning that stars oscillate throughout the galaxy much like a mass does connected to a spring, astonished me because I never thought to visualize he complexity of galactic star motion, so much so that the vertical motion is of the differential equation, where z is the azimuthal coordinate in cylindrical coordinates: $\ddot{z} = -\frac{\partial \varphi}{\partial z}, \varphi = \frac{U}{m}$ where U = Potential Energy and m = test mass."

- George: "The maximum tidal force of an object is given by $\Delta F_{max} = \frac{2GMmR}{r^3}$. The tidal force given by the Moon is $\Delta F_{moon} = \frac{2GM_{moon}mR}{r_{moon}^3}$ whereas the tidal force given by the Sun is $\Delta F_{sun} = \frac{2GM_{sun}mR}{r_{sun}^3}$. Since 2, G, m, and R are constants present in both equations, they could be eliminated from both equations. To compare the tidal force of the Moon [on Earth] and the Sun [on Earth], the following are needed: $\Delta F_{moon} = \frac{M_{moon}}{r_{moon}^3}$ and $\Delta F_{sun} = \frac{M_{sun}}{r_{sun}^3}$. Plugging in [the] values … for M_{moon} = mass of the Moon, M_{sun} = mass of the Sun, r_{moon} = distance from the Earth to the Moon and r_{sun} = the distance from the Earth to the Sun … $\Delta F_{moon} = \frac{7.35\times10^{25}g}{(3.84\times10^{10}cm)^3} = 1.298 \times 10^{-6}$ and $\Delta F_{sun} = \frac{1.99\times10^{33}g}{(1.496\times10^{13}cm)^3} = 5.944 \times 10^{-7}$ [and] taking the ratio, $\frac{\Delta F_{moon}}{\Delta F_{sun}} = \frac{1.298\times10^{-6}}{5.44\times10^{-7}} = 2.18$. So the tidal force exerted by the Moon is about twice as much as the tidal force exerted by the sun [on Earth]."

Table 27: Summary of criteria of students who had "Complete Understanding" of equations in their essays.

Students with "Complete Understanding" met the following criteria for understanding equations

Does the student …	"Complete Understanding"	"Understanding"	"Partial Understanding"	"No Understanding"
... discuss the symbols in the equation?	10/91%	10/71%	4/24%	0/0%
... discuss the astrophysics or physics in the equation?	11/100%	14/100%	14/82%	1/6%
... discuss the structure of the equations?	10/91%	4/29%	2/12%	0/0%
... understand the purpose of the equations?	11/100%	12/86%	11/65%	1/6%
... show deep understanding of astrophysics behind the equations?	11/100%	3/21%	0/0%	0/0%
... talk about connections between equations and real world?	9/82%	10/71%	5/29%	0/0%

The students who show "complete understanding" of the equations meet all or most of the criteria for understanding equations. Therefore, these students are classified as having an "complete understanding" of the relevant equations related to the topics in their essays.

Interviews

Two (2) of the ten (10) students interviewed showed “complete understanding” of the equations used to solve the astrophysics problems presented. Both of these students, Justin and Steven, also showed “complete understanding” in all of the previous assessments: homework assignments, exams, and essays. In the interview, these students who showed “complete understanding” of astrophysics equations in the interviews did so by demonstrating the ability to recognize the symbols in the equations he provided in terms of the corresponding physics or astrophysics quantities, recognizing the underlying physics of the equations, recognizing the structure of the equations, establishing a link between the equations and everyday life, demonstrating knowledge of how to use an equation to solve the problems, demonstrating the ability to know when to use an equation and consistently demonstrating that they make conceptual and/or mathematical connections to the equations. In addition, these students did not memorize a large amount of equations without understanding and were able to start the problems without assistance. The following excerpts from these two students’ interviews illustrate the student’s classification of “complete understanding”.

Justin:

Justin was able to immediately and correctly start each of the problems given without any assistance, indicating that he knew when to use an equation and recognizes the underlying physics of the equation since he was able to start the problem with the correct equations. For instance, his first words upon hearing the first problem were: “So we can figure out the velocity by equating the gravitational force between the sun and the earth and the centripetal force of the earth moving around in the circle. So, we can equate those two forces to get the velocity and you can plug that into one-half m v squared to get the kinetic energy of the earth." Quickly and efficiently summarizing how to solve the first part of the problem and making a conceptual connection to the equations necessary to solve this part. While writing down his solution to this part of the problem he demonstrates that he recognizes the symbols in the equation in terms of the corresponding astrophysics quantities by defining each of the variables in the equations needed to solve the problem. "the formula for gravitational potential energy is G m m over r squared and you set that equal to m v squared over r which is the centripetal acceleration. Then you can cancel the m's and then solve for v squared to get v squared equals G big M, which equals the mass of the sun over r."

In the next part of the problem, when determining the gravitational potential energy of the Earth-Sun system, Justin shows "complete understanding" when describing the underlying astrophysics of the equation. "we know that the formula for the gravitational potential energy between any two objects is G M m over r. It's negative because … you think of it as a potential well. Since you're in a well, you're negative, therefore you have negative energy." When asked where the gravitational potential energy of the Earth-Sun system would be zero, Justin correctly answers that it would be "at infinity" because "basically whenever you're infinite, like you're really really far away from the object, you basically don't feel the effect of that gravitational potential well anymore." He also shows a correct mathematical connection to the equations in the course of solving this problem. "So, what you are actually doing is you're deriving it from the force. So, you take the integral of F dr or dx depending on what your coordinate system is. And usually you say your reference point as one of the bounds of the integral and the other point being where you are. And in this case, you're setting the … one of the bounds as infinity so your second gravitational potential energy term drops to zero." Justin further shows that he can establish a link between the equations and the underling physics of the equation in the following statement: "[To find the sign of the total energy] you just add the two [energies] and you get negative one-half G

M sun m earth over r. So, the sign of the total energy is negative which makes sense because since the earth is in a bound orbit and whenever you have a bound system you need to have a negative total energy. Which is why it makes sense for it to be less than zero."

In the last problem of the interview, Justin demonstrates that he recognizes the structure of the equations he is using "we know that the kinetic and the potential energy is the total energy of the system. So, if it's losing total energy, that is equivalent to its energy becoming more negative" and "the velocity is increasing because if you look here where we solved for v squared, where you get v squared equals G M over r, and so since we just say that r is getting smaller, as r gets smaller v … Well this whole term G M over r gets bigger and so since r and v are inversely proportional, as r gets smaller, v is going to get bigger" These statements also demonstrate conceptual and mathematical connections to the equations as well as establishing a link between the equations and everyday life; he recognizes that the satellite, loosing energy from drag, will actually increase in velocity as "r gets smaller". Justin further demonstrates his "complete understanding" of the equations when he once again shows that he is recognizing the underlying astrophysics of the equation in his statement: "if the energy is becoming more negative, the only way that is possible is if it sort of falls deeper into a potential well. And that's why I reasoned in the beginning that the radius is decreasing. Because as it gets closer and closer to the object it's falling deeper into a potential well."

Justin was able to demonstrate his "complete understanding" by using his knowledge of how to use equations to solve these astrophysics problems. He was consistently able to start the problem, discuss the problem in a way that showed his understanding, and was able to completely solve each of the parts of the problem without assistance.

Steven:

Similarly to Justin, Steven is able to recognize the symbols in the equations he provided in terms of the corresponding physics or astrophysics quantities, recognize the underlying physics of the equations, recognize the structure of the equations, establish a link between the equations and everyday life, demonstrate knowledge of how to use an equation to solve the problems, demonstrate the ability to know when to use an equation and is consistently demonstrating that he makes conceptual and/or mathematical connections to the equations. The following statement from question two as he goes through the given steps of a solved problem and then creates a real life scenario for the problem, demonstrates these abilities:

> "So this is just sigma F = ma and I agree with this. And so, you have here your GMm over r squared should equal mv squared r. I agree with that because you have acceleration which should equal v squared over r. And then you are solving for v squared okay so you just bring it over v squared should equal GM over r and you know that the kinetic energy should equal 1/2 mv squared which is just 1/2 times the mass times what we just figured out here for velocity squared (GM over r) should equal GMm over 2r and you get -1/2 U. They have the negative here given that U is -Gm1 m2 over r you plug it in then you can get the negative here. So, the question would be very similar to the [first] question, what is the relationship between kinetic energy and potential energy of a - of say a planet orbiting around the sun. Assume a circular orbit. Well, it doesn't have to be a planet and a sun it can be any two objects with any two masses as long as we are consistent between [whichever variable the little m is, the one in orbit that is actually rotating around the object] that mass and that mass it is fine. But luckily they do both cancel out so we only need to care about the mass in the center that is actually causing the gravitation."

These students who showed “complete understanding” in the interviews had the same results – these students were able to recognize the symbols in the equations he provided in terms of the corresponding physics or astrophysics quantities, recognize the underlying physics of the equations, recognize the structure of the equations, establish a link between the equations and everyday life, demonstrate knowledge of how to use an equation to solve the problems, demonstrate the ability to know when to use an equation and consistently demonstrate that they could make conceptual and/or mathematical connections to the equations. Since these students fulfilled all of the criterial for understanding equations, they were classified as having “complete understanding” of astrophysics equations in this assessment.

Summary - Complete Understanding

In comparison to the students with “no understanding”, “partial understanding”, or “understanding” of astrophysics equations, the students with “complete understanding” demonstrated this “complete understanding” in multiple assessments similarly to the students that demonstrated “no understanding”, or demonstrated “partial understanding”. There was more overlapping of students with “complete understanding” than there was by students with “no understanding”, particularly with the homework assignment and either exams or the essays. Two students with “complete understanding” showed this understanding across all assessments. The following two tables show the overall performance of students who were found to have “complete understanding” of equations in a particular assessment:

Table 28: Number/Percentage of students showing “Complete Understanding” by assessment

Complete Understanding			
Homework	Exam	Essay	Interview

15/28%	9/17%	11/20%	2/20%

Table 29: Overlapping of the number of students with "Complete Understanding" by assessments

Number/Percentage of Students Overlapping				
Assessment Overlap	Overlap for Students with "Complete Understanding"	Overlap for Students with "Understanding"	Overlap for Students with "Partial Understanding"	Overlap for Students with "No Understanding"
Homework & Exam	6 of 19/32%	5 of 28/18%	9 of 25/36%	1 of 15/7%
Homework & Essay	5 of 21/24%	3 of 27/11%	6 of 26/23%	1 of 20/5%
Exam & Essay	2 of 18/11%	4 of 19/21%	9 of 25/36%	1 of 12/8%
Homework & Exam & Essay	2 of 24/8%	1 of 35/3%	4 of 33/12%	None
Interview & "3 Other"	2 of 8/25%	None	None	None
No Overlap	15 of 24/63%	23 of 35/66%	10 of 33/30%	22 of 26/85%

As can be seen in Table 29, there was found to be more overlapping of the students who showed "complete understanding" of astrophysics equations in at least two of the different assessments analyzed than those with "no understanding"; additionally, it is worth noting that both of the students who showed "complete understanding" in the interviews, also showed "complete understanding" in all of the other forms of assessment.

5.2.e Summary & Possible Implications

From the results of the analysis of the students' assessments as shown above, we can determine several factors concerning a student's qualitative understanding of astrophysics equations.

The vast majority of the students who were found to have “no understanding” of equations in a particular assessment had little to no overlapping with respect to having “no understanding” in other assessments. In other words, these students may have shown “no understanding” in one of the assessments studied in this work, but for most of the students who did show “no understanding” in an assessment, it was only in that assessment, showing at least partial understanding in others. This heavily implies that there is no such thing as a student with “no understanding” of the equations in this classroom. From this it is clear that multiple assessments are necessary in the classroom. If a student does poorly in one assessment, it does not necessarily mean a complete lack of understanding. Whether or not there may be a connection between “no understanding” and context or content is a topic that will be explored in the next section when I answer my third research question “How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysical concepts?”

In comparison to the students with “no understanding”, the students with “partial understanding”, “understanding”, and “complete understanding” of astrophysics equations demonstrated their understanding in multiple assessments - to various degrees of overlapping as shown in Table 29 above. Whereas results for the students with “no understanding” implied that there is no such thing as a student with “no understanding” of the equations in this classroom, the results for the students with “partial understanding”, “understanding, and “complete understanding” seem to imply the opposite; since these students were classified as such over multiple assignments, this implies that these students do have an overall consistent understanding of the astrophysics equations used. Although clearly multiple assessments would still be necessary to determine this, a student showing the same classification over multiple assessments does seem to show their level of understanding and can be guided to better understanding once that level is known. As stated in the theoretical framework, the attributes for student understanding of equations are: recognizing the symbols in the equation in terms of the corresponding physics quantities, recognizing the underlying physics of the equation, recognizing the structure of the equation, establishing a link between the equation and everyday life, demonstrating knowledge of how to use an equation to solve astrophysics problems, and the ability to know when to use an equation. As can be seen in the essays and shown in Table 27, the biggest differences between attributes for “complete understanding”, “understanding”, and “partial understanding” are: discussing the symbols in the equations, discussing the structure of the equations, understanding the purpose of the equations showing deep understanding of the astrophysics behind the equations, and taking about connections between equations and the real world. Out of these three classifications, the students classified with “complete understanding” showed these attributes the most (82%-100%), “understanding” less (21%-86%), and “partial understanding” the least (0%-65%).

Lastly, in addition to the theoretical framework described above for student understanding of equations, instructors should also look for some additional indicators; namely: “Can the student start the problem without assistance” and “does the student make conceptual and/or mathematical connection to the equations?” For example, one of the students with “no understanding” in the interview, Daniel, could not start the problem without assistance and his comment “for gravitational potential energy, that would be due to gravity, so that would be Newton’s law. So that’s like G M m over r squared is equal to force. So, then energy would be force times distance. Right?” shows that he does not have a conceptual understanding of the equations and has simply memorized them. Additionally, Christopher, the other student with no understanding in the interview assessment, could not start the problem without assistance and was given the equations through discussion. He saw the equations as something to use, but not with any conceptual or mathematical connection to the equation as can be seen from comments such as “You just take the ratio of it, and I think that potential would be greater.” or “You [just] set them equal to each other.” In contrast the students with “complete understanding” could always start the problem without assistance and show a conceptual connection to the equations necessary to solve the problem as can be seen in Justin's comment: "So we can figure out the velocity by equating the gravitational force between the sun and the earth and the centripetal force of the earth moving around in the circle. So, we can equate those two forces to get the velocity and you can plug that into one-half m v squared to get the kinetic energy of the earth." Where he quickly and efficiently summarizing how to solve the problem. Students with "complete understanding" were also always able to show a mathematical connection to the equations. Justin's comment "the velocity is increasing because if you look here where we solved for v squared, where you get v squared equals G M over r, and so since we just say that r is getting smaller, as r gets smaller v … Well this whole term G M over r gets bigger and so since r and v are inversely proportional, as r gets smaller, v is going to get bigger" demonstrate this mathematical connection to the equations. Students with "partial understanding" and "understanding" were able to make these connections inconsistently. These additional identifiers were observed while analyzing the student's responses to their multiple assessments.

5.3 How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysics concepts?

Research Question #3 was analyzed using the data from the students' homework assignments and exams, the student surveys, the interview videos, and from the student essays. The relevant interview videos consisted of students solving astrophysics problems related to negative potential energy and the virial theorem; the student surveys consisted of twenty-one questions, a great deal of which focused on what the students' think it means to understand astrophysics equations; and the student essays focused on the students' beliefs concerning their favorite and least favorite topics from the class; concentrating on the "relevant equations and the meaning of their terms as well as how the equations influenced [their] attempts to understand the concept." From the students' responses to these data, particularly on the topic of negative gravitational potential energy and the virial theorem, we can determine how the student's conceptions of understanding of astrophysics equations relates to their qualitative understanding of an astrophysics topic. Below I present the results of the data that are relevant to Research Question #3: How do the student's conception of understanding equations relate to their qualitative understanding of astrophysics concepts?

Negative Gravitational Potential Energy and the Virial Theorem

As discussed in Chapter 2.3.e, the focus of the interview questions involved the understanding and use of negative gravitational potential energy and the virial theorem:

During the interviews I examined the following aspect of negative gravitational potential energy: the change in potential energy of a system consisting of a mass M (assumed to be at rest at the origin) exerting a gravitational force on a particle of mass m as m moves between points a and b is: $\Delta U = U_b - U_a = GMm\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$. If the reference configuration is an infinite separation of the particles, and the potential energy is defined to be zero in that configuration, then $r_b = \infty$ and $U_b = 0$ and $U(\infty) - U(r) = GMm\left(\frac{1}{r} - 0\right)$, or $U(r) = -\frac{GMm}{r}$. Based on the students' comprehension of this equation, specifically how the equation is derived and what the equation means conceptually, I assigned the students to different "levels" of understanding.

I also assigned students to different "levels" of understanding of the topic of the virial theorem based on their comprehension of the virial theorem, which is used in astrophysics to relate the gravitational potential energy of a system to its kinetic energy as a whole. The virial theorem refers to time averages of the kinetic and potential energy and is generally stated as $\langle K\rangle = -\langle U\rangle/2$ where $\langle K\rangle$ is the time average of the kinetic energy and $\langle U\rangle$ is the time average of the total potential energy. In the simplest form of the virial theorem, as used in this study, $K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r} = -\frac{U}{2}$, where $v^2 = \frac{GM}{r}$ as derived from the gravitational force $F = \frac{GMm}{r^2} = \frac{mv^2}{r}$.

5.3.a Negative Gravitational Potential Energy/Virial Theorem - No Understanding

To determine how student qualitative understanding of astrophysics equations relates to their qualitative understanding of astrophysics concepts, I first needed to designate which students out the students in the course had qualitative understanding of the negative sign of the gravitational potential energy and the virial theorem. Therefore, I began the analysis of this research topic by studying the interviews and determining which of the students interviewed showed “no” qualitative understanding of negative gravitational potential energy and/or the virial theorem and then connecting this level of understanding to the students’ beliefs and actual overall understanding of equations. I chose one of the two students who showed “no understanding” of this topic and continued the analysis, by examining the results from the interview and comparing the results to the results in other assessments for the same topic as well as comparing the results to the student’s beliefs in the survey.

Daniel

Daniel was one of two students who was classified as “no understanding” of negative gravitational potential energy during the interview process. As discussed in the previous section, Daniel had trouble starting the problems given as part of the interview and had to be walked through most of the solutions. As part of the first question in the interview, the students were asked to determine the gravitational potential energy of the Earth-Sun system. Daniel had difficulties with this topic from the beginning, incorrectly equating potential energy to force: “For gravitational potential energy, that would be due to gravity, so that would be Newton's law." Then incorrectly equating work to gravitational potential energy: "So then [gravitational potential] energy would be force times distance." He went on to obtain an equation for the gravitational potential energy of the Earth-Sun system which does not include the negative in the equation. After some leading discussion on the nature of potential energy the student realized that gravitational potential energy could only be determined as a change in potential energies, but still did not realize that the equation he had provided was not complete. I had to remind the student that the potential energy needed to be defined as zero at a particular reference point in order to consider the value of the potential energy at another point. Upon this reminder, the student remembered that "So you have to compare it at like infinity" and that the gravitational potential energy at infinity is zero. Even with this knowledge the student still did not realize that the equation required a negative sign until I outright asked him if the equation should have one or not.

Even after obtaining the correct equation for gravitational potential energy, Daniel continued to forget the negative sign many times when using the equation (which indicated that he did not consider the sign important). Furthermore, even when reminded, he showed "no understanding" of the consequence of having the negative in the equation. For instance, the following excerpt from the interview shows that even when the negative nature of the equation is emphasized, the student still does not conceptually understand the nature of negative values:

Daniel: U is GmM/R. so if [U] is decreasing, then the radius is increasing.

Interviewer: What is the equation for U again?

Daniel: Minus

Interviewer: So, if potential [energy] is decreasing, what is happening to the radius?

Daniel: Then the radius is increasing.

In addition, it became clear when examining the interview data that Daniel, who did not conceptually understand negative gravitational energy, also did not understand the virial theorem. In the second and third questions given as part of the interview, the students were asked to solve a problem where the solution involved use of kinetic and potential energies which lead to the virial theorem. In the second question, the students were given the derivation of the simple form of the virial theorem as discussed above and asked to propose as question that had the given derivation and solution. When asked in the third question “What would happen to a satellite’s orbit if the satellite in orbit about Earth loses total energy due to gradual atmospheric drag? What would happen to the potential energy of the satellite? Do you know of any situation in astrophysics that is similar to this?” The students were expected to answer the question using their knowledge of kinetic and potential energies determining that the satellite would move to a smaller radius in order to lose potential energy and hence total energy. Since the potential energy becomes more negative, the kinetic energy must increase according to the virial theorem and this increase in kinetic energy is one-half the loss of potential energy; therefore, total energy indeed decreases. The atmospheric drag tries to slow the satellite, but instead it falls to lower orbit and speeds up.

Daniel had recognized the virial theorem after some leading in the second question during the interview when asked if the derived equation looked familiar. He recognized the virial theorem was used as averages in class: “We talked [about the] virial theorem, specifically like [an] average.” This shows that Daniel does not necessarily understand the virial theorem, but memorized the equation as presented in class. In the third question, Daniel again had to be walked through most of the solution to this problem, but without mention of the virial theorem. When asked outright if the virial theorem applied to this situation, Daniel did not recognize that the virial theorem was used in the solution stating that the virial theorem did “not necessarily” apply “because we don’t know if it’s a circular orbit and … rather than like an average.” Here he stopped his thought after he failed to recognize the assumptions made in the solution to the problem that allowed the application of the virial theorem and then reverted back to the incorrect belief that the virial theorem did not apply to this problem because it could only be applied to averages.

Daniel shows similar results in the Homework and Exam questions that specifically addressed student understanding of gravitational potential energy. In homework assignment #9, Question 1, part a; the students were asked to solve a "jeopardy equation" style problem: "Create a question for which the following equation gives the solution: $U = -\frac{3(6.67x10^{-8}cm^3g^{-1}s^{-2})(0.055\ M_\oplus)^2}{5(0.382\ R_\oplus)}$".

With the expected answer: What is the total gravitational potential energy of Mercury assuming that it has uniform density. In Daniel's case, in his answer he was able to demonstrate that he recognized the derived equation for the total potential energy for a sphere of mass M and radius R from the lecture notes, i.e. $\langle U \rangle = -\eta \frac{GM^2}{R}$, where $\eta = 3/5$ for a constant density sphere.

However, Daniel showed "no understanding" because he was not able to conceptually understand the equation and therefore the topic in his answer:

"For uniform density sphere:

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

What is the total potential energy of a uniform density sphere with radius $0.382\ R_\oplus$ and mass $0.055\ M_\oplus$?"

Daniel's answer clearly shows recognition of the equation given in the lecture notes for the "total potential energy" of a "uniform density sphere with radius R", but does not show any deep understanding as he does not specify that the mass and radius are specific values for the planet Mercury.

Similarly, in Exam #2, Daniel shows “no understanding” of negative gravitational potential energy when answering question #7: “Under what circumstances would we consider the gravitational potential energy negative? Are there any circumstances in which we would consider the gravitational potential energy to be positive? Give examples.” This problem has an expected solution which indicates that the gravitational potential energy that a system has is due to the separation of two objects – at least one with large mass. (A single point-like object does not have gravitational potential energy.) The gravitational potential energy of a system is defined as negative because positive work must be done on the system to move one of the objects further away from the other. When the object is moved infinitely far away from the other, the gravitational potential energy is zero. Since positive energy was added to the system and the final energy became zero, the initial energy must have been negative. Therefore, if the energy is zero at infinity, the gravitational potential energy is negative when the objects are closer together. The negative sign is indicative of a "bound state" and thus any example involving a bound state was considered acceptable.

Gravitational potential energy can be considered positive for an object near the surface of a massive object, such as a planet, where the gravitational acceleration can be assumed to be constant and zero potential energy is defined as zero on the surface of the planet (or some other level close to it). Again, since the zero of gravitational potential energy can be chosen at any point, the gravitational potential energy at a height h above that point will be positive. Any example involving an object at different heights near the surface of a planet was expected here. Daniel’s answer however, shows that he does not understand the concept: “As long as we define radially outward as the positive direction, gravitational potential energy will always be considered negative because it acts in the radially inward direction. If we change sign convention and define radially inward as positive, then we can consider U to be positive." This answer shows that Daniel has "no understanding" of the concept of negative gravitational potential energy. Similarly to defining motion to the right as "positive" and motion to the left as "negative" in a two-dimensional coordinate system without reference to the nature of the motion itself, Daniel describes a sign convention for a radial coordinate system, defining radially outward as "positive" and radially inward as "negative" or vice versa to get a "negative" or "positive" gravitational potential energy rather the concepts make the potential "negative" or "positive". Furthermore, he cannot provide examples for either "positive" or "negative" gravitational potential energy.

As Daniel did not refer to either negative gravitational energy or the virial theorem in his essay, it cannot be added to the data for understanding of these topics. However, it is most interesting to note that Daniel showed "no understanding" of equations in the interview as discussed in Section 5.2.a, but showed more understanding of other astrophysics equations. For example, Daniel shows understanding of the tidal force equation in multiple assignments: homework assignment #6, exam #2, and in his essay. This seems to imply that the students' level of understanding of equations depends on a topic not on the type of assessment as Daniel showed "no understanding" in all assessments for this particular astrophysics topic.

Daniel's beliefs of understanding of astrophysics equations was determined by the student survey concerning students' beliefs of astrophysics equations. Table 30 compares these beliefs with his qualitative understanding of negative gravitational potential energy and the virial theorem.

Table 30: Student's conceptions of understanding astrophysics equations compared to qualitative understanding of an astrophysics topic.

Daniel's Beliefs	**Daniel's Actual** **(For the Topic of Negative Gravitational Potential Energy and the Virial Theorem.)**
He needs to conceptually understand the equations that he uses.	Does not demonstrate conceptual understanding of the equations.
The use of derivations of equations is important, but only as a means to get the equation.	Does not show understanding of derivations but memorizes equations.
He needs to recognize the symbols in the equation in terms of the corresponding physics quantities.	Does not demonstrate a recognition of the symbols in the equation in terms of the corresponding physics quantities.
He needs to understand and recognize the relationships connecting the variables of astrophysics equations.	Does not demonstrate understanding of the relationships connecting the variables of the equations.
He needs to make connections between equations and real world.	Cannot make connections between the equations and the real world.
He is undecided if he needs to find the “right” equation to use in a particular problem.	Cannot determine the equations necessary to solve the problems.

He believes that he needs to conceptually understand the equations that he uses, however; when relating this belief to his qualitative understanding of negative gravitational potential energy and the virial theorem, it has been shown above that Daniel does not actually conceptually understand the negative gravitational potential energy or the virial theorem equations. He believes that the use of derivations is important, but only as a means to get the equation. He demonstrates this belief in the interview when he is unable to derive formulas necessary to solve the problems, but instead attempts to recall the appropriate equations from memory. Daniel believes that he needs to recognize the symbols in the equation in terms of the corresponding astrophysics quantities, but does not specify the variables of mass and radius for the total potential energy of a uniform density sphere in homework assignment #9. He furthermore believes that he needs to make connections between the equations and the real world, but he could not accurately explain when the gravitational potential energy could be negative or positive with real world examples in his answer to question #7 in exam #2 as shown above. Finally, Daniel is undecided when it comes to the necessity of finding the “right” equation to use in a particular problem, but, when solving a problem, he does try to find the “right” equation through memorization of equations and cannot start or complete the problems without the “right” equation. Daniel’s beliefs therefore are largely in contradiction to his actual demonstration of understanding of the equations for the topic of negative gravitational potential energy and the virial theorem.

5.3.b Negative Gravitational Potential Energy/Virial Theorem - Partial Understanding

After examining the attributes of “no” qualitative understanding of negative gravitational potential energy and/or the virial theorem and then connecting this level of understanding to the students’ beliefs and actual overall understanding of equations; I began to look at “partial understanding” of students’ qualitative understanding of astrophysics equations as it relates to their qualitative understanding of astrophysics concepts. Out the ten student interviews, five students showed “partial” qualitative understanding of the negative sign of the gravitational potential energy and of the virial theorem. I chose one of the students who showed “partial understanding” of this topic and continued the analysis, by examining the results from the interview and comparing the results to the results in other assessments for the same topic as well as comparing the results to the student’s beliefs in the survey.

Benjamin

Benjamin was one of five students who was classified as “partial understanding” of negative gravitational potential energy and the virial theorem during the interview process; he is representative of the group as he displayed the same or extremely similar attributes as the remaining four students who showed “partial understanding”. During the interview, Benjamin displays in multiple occasions, that he has difficulties with the concept of negatives while working with negative gravitational potential energy.

When asked, as part of the first question in the interview, to determine the gravitational potential energy of the Earth-Sun system, Benjamin had difficulties deriving the correct equation. He begins by attempting to recall the equation from memory; which he does incorrectly: “We know that is Gm/r, I think. Or r^2, it’s one of them.” When prompted to attempt to determine the correct the equation, Benjamin again reverts to an incorrect memory for the gravitational potential energy: “Force is GMm/r^2. So, U is just G m of the earth, or rather the sun in this case actually, over r.” Upon further prompting, Benjamin does recall that the force and the gravitational potential energy are related by the gradient, first incorrectly “U is the ***gradient of force***” and then almost correctly “No, force [is the] gradient of U” missing the negative ($F = -\nabla U$). When trying to use this knowledge, Benjamin does know how to manipulate the equation mathematically “So U is the integral of F …” but does not show any further knowledge and cannot proceed. Eventually, Benjamin does achieve an almost correct equation for the gravitational potential energy by recalling that the “potential energy is mgh” and “g in this case is GM/r^2” and “h is r”. Although missing the negative, this method does give the magnitude of the equation desired; however, the way the student arrives at this equation shows that he does not have a complete understanding of the topic. Benjamin knows some of the equations necessary, as well as the outcome desired; thus, showing a “partial understanding” of the topic. Furthermore, although Benjamin does not include the negative in the equation at any point in his discussion, after a short leading discussion on the nature of potential energy the student realized that the equation for gravitational potential energy has a negative since for a radius of zero the gravitational potential energy is considered infinite and a zero gravitational potential energy is defined at a radius of infinity.

Benjamin shows some understanding of negatives within the context of the astrophysics topic of gravitational potential energy than he shows understanding of negative values as a mathematical topic. He believes that just because a quantity is negative that means that it will always be a smaller than a positive quantity, however; this is not always true. For instance, an object moving with a velocity of "-10 m/s" is not moving with a smaller speed than an object moving with a velocity of "10 m/s"; it is just moving in the opposite direction. That being said, Benjamin does understand that when the total energy of the Earth-Sun system is negative in question one that means that Earth is "in a bound orbit. So potential energy has to be dominating." This shows an excellent understanding of total energy of the system because he is thinking of the total energy as the sum of kinetic and potential. With the kinetic always positive means that the negative component due the gravitational potential energy should be large in magnitude. Later in question #3, Benjamin adds to this understanding stating: "If you just alter [the energy] a little bit, then potential energy could dominate, and you end up getting sucked towards whatever the main body is." In addition, when using the equation for gravitational potential energy, Benjamin only forgets to include the negative in his solutions once after obtaining the correct equation in problem #1, but quickly recalls that the equation requires the negative and why. He is also able to competently use it to obtain desired answers: "Total energy is KE plus U, which we said U plus KE is just negative GMm/r plus half GMm/r. So, adding those we just get negative ½ GMm /r." However, when discussing the consequences of having the negative in the equation, Benjamin states that since "U is negative … technically, it's always smaller." And in a later part of the interview: "You can say that the magnitude is getting smaller, or that it is just getting more negative." The student confuses the concepts of magnitude or "size" and the concepts of negative numbers, in this case the negative value indicates that since there is a positive work that would need to be done in order to bring a mass infinitely far away, the gravitational potential energy must be a negative number when the objects are closer together. In total, the comments that Benjamin made during his interview and highlighted above demonstrate that this student has a "partial understanding" of how to use the negatives in the concept of gravitational potential energy, but does not have a deep understanding of the nature of negative values.

Benjamin also displayed a "partial" conceptual understanding of the virial theorem throughout the interview. After reading through the first question Benjamin immediately states "Well, this definitely seems like a virial theorem problem" recognizing that the given scenario could be solved by utilizing the virial theorem. However, when asked to explain the virial theorem, Benjamin does not show further understanding of the topic but instead reverts to trying to remember the equation associated with the virial theorem; which he recalls incorrectly: "I think it's two, two kinetic energy plus potential is overall -- or no, zero, I think that's it." Later in this same problem, after deriving expressions for the kinetic and gravitational potential energies of the system, when asked which was larger in magnitude, Benjamin again shows a partial understanding of the virial theorem by recognizing that "that's what the Virial Theorem stuff comes in. So, potential is equal to two times kinetic. So, it would have to be bigger." Again, he is recalling an equation and recognizing that the virial theorem applies to this situation, but does not show any deeper recognition when asked to show that the gravitational potential energy of the system would be larger in magnitude.; continually referring back to the known outcome ("We somehow have to show that GMm/r is bigger than ½mv^{2}"), but having to be lead through the derivation. Similarly, in the third question, Benjamin correctly identifies the problem is one in which the virial theorem applies after discussing the assumptions made in the problem because "that's basically what the virial theorem's for." Throughout the problems in the interview, Benjamin shows a "partial understanding" of the virial theorem by recognizing the astrophysical situations where the theorem is applicable as well as recognizing the correct form of the equation once it was derived; however, he does not show a deeper understanding in that he was not able to derive the equation without assistance and was only able to demonstrate an understanding of the theorem at the most basic level.

As stated above, the Homework and Exam questions that specifically addressed student understanding of gravitational potential energy were found in homework assignment #9, Question 1, part a where the students were asked to solve a "jeopardy equation" style problem and in Exam #2, Question #7 where the students were asked to describe the circumstances where one could consider the gravitational potential energy negative and circumstances where one could consider it positive. Again, Benjamin shows a "partial understanding" of the topic of gravitational potential energy in both of these assessments.

For the homework assignment, when asked to "Create a question for which the following equation gives the solution: $U = -\frac{3\left(6.67x10^{-8}cm^3g^{-1}s^{-2}\right)\left(0.055\,M_\oplus\right)^2}{5(0.382\,R_\oplus)}$", Benjamin's answer demonstrates this "partial understanding":

"$U = -\frac{3\left(6.67x10^{-8}cm^3g^{-1}s^{-2}\right)\left(0.055\,M_\oplus\right)^2}{5(0.382\,R_\oplus)}$

$0.055\,M_\oplus$ = mass of Mercury

$0.382\,R_\oplus$ = radius of Mercury

$U = -\frac{3GM_M^2}{5R_M}$

Question: What is the total potential energy of Mercury, assuming is density is uniform?"

Although this answer is extremely close to the expected answer, it is missing the essential demonstration of understanding that this is for the *gravitational* potential energy of Mercury. By not stating this, Benjamin shows that he has recognized that the given equation is the equation derived in the lecture notes for the “total potential energy” of a “uniform density sphere with radius R”, but does not show that he understands that this equation is for the gravitational potential energy specifically, nor does he indicate that the object is spherical (although since he does recognize that the variables were given for the planet Mercury, this may be assumed).

Similarly, in Exam #2, Benjamin shows “partial understanding” of negative gravitational potential energy when answering question #7: “Under what circumstances would we consider the gravitational potential energy negative? Are there any circumstances in which we would consider the gravitational potential energy to be positive? Give examples.” As stated above, this problem has an expected solution which indicates that the gravitational potential energy of a system can be negative if one is determining the work done against gravity to bring a mass to a given point in space from an infinite distance; which means the choice for the zero gravitational potential energy is set at an infinite distance. Gravitational potential energy can be considered positive for an object near the surface of a massive object, such as a planet, where the gravitational acceleration can be assumed to be constant and zero potential energy is defined as zero on the surface of the massive object (or some other level close to it). Again, since the zero of gravitational potential energy can be chosen at any point, the gravitational potential energy at a height h above that point will be positive. Benjamin’s answer “We always consider the gravitational potential energy to be negative because it is zero at an infinite distance away” shows a partial understanding of this topic. He can clearly identify that the gravitational potential energy, when set at zero at an infinite distance, will be negative; but mentions nothing about it being a system of objects, nor does he give an example of this scenario. Furthermore, Benjamin states that he does not believe that there is such a thing as a positive gravitational potential energy in this problem demonstrating that he only has a "partial understanding" of the topic of gravitational potential energy.

Benjamin was not one of the two students who were interviewed and gave reference to either negative gravitational energy or the virial theorem in their essays, so his essay could not be added to the data for his understanding of these topics. However, even without this data, similarly to Daniel's demonstration of "no understanding" of the topics as shown above, it is again apparent that in Benjamin's case although he showed a "partial understanding" of the negative gravitational potential energy and virial theorem topics in the interview as well as in other assignments, he also showed understanding or complete understanding of other astrophysics equations. For example, Benjamin shows understanding of the gravitational lensing and relativity equations in his essay as well as in other assignments. This once again adds strong evidence that seems to imply that the students' level of understanding of equations is topic driven, and not assessment driven as Benjamin showed "partial understanding" in all assessments for this particular astrophysics topic, negative gravitational potential energy and the virial theorem, but shows greater consistent understanding for other astrophysics topics. It is interesting to note however that one of the students who exhibited a "partial understanding" of the topic in their interview also discussed the virial theorem in their essay. Of the data analyzed, Samuel's interview, homework, and exam results for this topic were very similar to Benjamin's; however, in his essay, as discussed in Section 5.2.c, Samuel demonstrates "understanding" of the virial theorem. This is interesting to note because it shows that this student shows "partial understanding" of the astrophysics concept of negative gravitational potential energy and the virial theorem in all assessments except one; demonstrating that utilizing only one assessment for a particular topic may not be indicative of what the student truly understands.

Benjamin's beliefs of understanding of astrophysics equations was determined by the student survey concerning students' beliefs of astrophysics equations. Table 31 compares these beliefs with his qualitative understanding of negative gravitational potential energy and the virial theorem.

Table 31: Student's conceptions of understanding astrophysics equations compared to qualitative understanding of an astrophysics topic.

Benjamin's Beliefs	**Benjamin's Actual** **(For the Topic of Negative Gravitational Potential Energy and the Virial Theorem.)**
He needs to conceptually understand the equations that they use.	Demonstrates a partial conceptual understanding of the equations.
The use of derivations of equations is important, but only as a means to get the equation.	Shows partial understanding of derivations but memorizes most equations.
He needs to recognize the symbols in the equation in terms of the corresponding physics quantities.	Demonstrates recognition of the symbols in the equation in terms of the corresponding physics quantities.
He is undecided if he needs to understand and recognize the relationships connecting the variables of astrophysics equations.	Does not demonstrate a good understanding of the relationships connecting the variables of the equations.
He needs to make connections between equations and real world.	Can occasionally make connections between the equations and the real world.
He needs to find the "right" equation to use in a problem.	Partially successful in determining the equations necessary to solve the problems.

Benjamin believes that he needs to conceptually understand the equations that he uses, however; when relating this belief to his qualitative understanding of negative gravitational potential energy and the virial theorem, Benjamin has only a partial conceptual understanding of the negative gravitational potential energy and/or the virial theorem equations. He believes that the use of derivations is important, but only as a means to get the equation. He demonstrates this belief in the interview when he is only partially able to derive formulas necessary to solve the problems, and attempts to recall the appropriate virial theorem equation from memory. Benjamin believes that he needs to recognize the symbols in the equation in terms of the corresponding astrophysics quantities, and does this successfully in the interview and in homework assignment #9. He is undecided in his belief that he needs to understand and recognize the relationships connecting the variables of astrophysics equations; this belief is expressed throughout all assessments as he demonstrates that he only has a partial understanding of the relationships in the negative gravitational potential energy and virial theorem equations. He believes he needs to make connections between the equations and the real world, but he could not completely explain when the gravitational potential energy could be negative he did not give examples with reference to real world applications in his answer to question #7 in exam #2 and did not believe that the gravitational potential energy could ever be considered positive in this question as shown above. Finally, Daniel believes in the necessity of finding the "right" equation to use in a particular problem, but, when solving a problem, he does try to find the "right" equation through memorization of equations and has difficulty within the problems obtaining the "right" equation. Daniel's beliefs therefore are partially in contradiction to his actual demonstration of understanding of the equations for the topic of negative gravitational potential energy and the virial theorem.

5.3.c Negative Gravitational Potential Energy/Virial Theorem - Understanding

Next, I examined the work of the student who demonstrated qualitative "understanding" of astrophysics equations as it relates to their qualitative understanding of astrophysics concepts.

Out the ten student interviews, only one students showed “understanding” with respect to the astrophysical concept of the negative gravitational potential energy and of the virial theorem. Again, I connect this level of understanding to the student’s other work as well as their beliefs and actual overall understanding of equations.

Michael

Michael was the only student who was classified as having an “understanding” of negative gravitational potential energy and the virial theorem during the interview process and thus is the focus of this subsection.

During the interview, Michael displays “understanding” with the concept of negatives while working with negative gravitational potential energy. Michael begins the interview by immediately demonstrating this knowledge in question #1 when he gives the correct equation for the gravitational potential energy of the Earth-Sun system “So potential energy is –GMm over r.” And although Michael is uncomfortable with not having values to substitute for the variables in the equation, he demonstrates that he has an understanding of the topic when asked why the equation he gave was negative. His statement “Because the way potential energy works is [that] there’s two different points” and “this would be the [gravitational potential] energy [with] the radius [being] the distance [from] the Sun” shows that he understands that the gravitational potential energy is defined as the energy that a system has due to the position of one object relative to some other object and correctly identifies that the point given is for when the second distance is the radius of the Earth-Sun system. Michael does not show complete understanding however, because he does not state that the gravitational potential energy is zero at an infinite distance away. Michael does on one occasion drop the negative while making a comparison of the two energies determined previously in this question (the kinetic energy versus the gravitational potential energy of the Earth-Sun system) but immediately corrects this mistake once it is pointed out and continues to show "understanding" of the negative gravitational potential energy equation when using it in reference to obtain the sign of the total energy stating that the total energy would be "negative, because the negative [term] is larger than the positive [term]."

Michael shows further "understanding" of negative gravitational potential energy while solving the next two questions in the interview. In the second question, a "jeopardy" style question, the last line shows

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r} = -\frac{1}{2}U$$

where v^2 is identified as GM/r in the line above. When asked what precisely is happening in this line, Michael correctly responds "You are taking the kinetic energy, you're plugging in what you found for v in the previous line and then the last equation is just the equals negative potential." He then goes further, showing an "understanding" of the scenario presented in the question by stating that it is negative because the object is in a bound orbit. In question #3, although Michael starts the problem regarding a satellite in orbit which is losing total energy with an incorrect assumption (he incorrectly assumes that the satellite is losing velocity due to atmospheric drag) he begins to use his knowledge of negative gravitational potential energy correctly based of the incorrect assumption. "So that's where kinetic energy is equal to negative ½ U. And when it has atmospheric drag it is losing velocity, so that means kinetic energy is going to decrease so that means that K would become less than the absolute value of ½ U which means that is not going to be in a bound system anymore. And since the absolute value of potential energy is going to be greater than 2K the gravitational potential energy is going to overpower the kinetic energy out of the orbit and it is going to spiral inward." Unfortunately, Michael also shows here, that although he knows and understands the relevance of the negative in the equation, he considers it unimportant when comparing the energies, taking the absolute value of the gravitational potential energy instead. Additionally, Michael seems to confuse a “bound” system with a stable circular or elliptical orbit. Since the gravitational potential energy is more negative in this example, the satellite is more bound; Michael however states that he believes that since the energy is more negative, it is no longer in a stable orbit and is thus no longer “in a bound system anymore”. After a short discussion on the relevance of the virial theorem to this problem, Michael goes on to determine that the gravitational “potential energy is –GMm over r so [since] r decreases [that] means potential energy has to increase in the negative” and that means “that the kinetic has to increase in the positive” further showing that he has an “understanding” of negative gravitational potential energy, even if it is not a complete understanding of the topic.

In addition to demonstrating conceptual “understanding” of negative gravitational potential energy, Michael also demonstrates “understanding” of the virial theorem. Although he does not state it as the virial theorem until the end of the interview, Michael uses a simplified equation of the virial theorem ($-2K = U$) throughout his answers for all of the questions in the interview process. In the first question, he correctly identifies that the total energy of the Earth-Sun system is “negative because, the negative [term] is larger than the positive [term]” since the total energy is “kinetic plus potential” after he had previously derived that the kinetic energy of the system “equals negative ½U.” And in the third problem he again recognizes that the gravitational potential energy is “going to be equal to negative 2K” so the total energy is equal to “negative K.” Michael shows “understanding” of the virial theorem because he is able to correctly use the equation to get his results, but not a complete understanding because he does not recognize the theorem until it is explicitly stated. Once stated, however, Michael again shows “understanding” of the virial theorem when asked to describe the assumptions made when using the virial theorem: “On average it is $2\langle K\rangle + \langle U\rangle = 0$” where $\langle K\rangle$ and $\langle U\rangle$ are “the mean kinetic and mean potential energies” and we are assuming “that it’s a bound orbit”.

After looking at Michael's answer to question #1, part a on homework assignment #9 as well as his answer to question #7 on exam #2 (the questions that specifically addressed student understanding of gravitational potential energy) I found an interesting pattern. Although Michael shows "understanding" of this astrophysics topic in his interview and, as we will see, in the exam question and essay; Michael did not show understanding of the topic in his answer to the homework question. Perhaps Michael simply needed more time to completely comprehend the topic which should be expected since understanding should develop as time passes; however, this pattern is noted here for completeness.

In homework assignment #9, Question 1, part a, the students were asked to solve a "jeopardy equation" style problem, Michael's answer is similar to Daniel's above and shows "no understanding" of the topic of gravitational potential energy in this particular assessment. For this assessment, when asked to "Create a question for which the following equation gives the solution: $U = -\frac{3(6.67x10^{-8}cm^3g^{-1}s^{-2})(0.055\,M_{\oplus})^2}{5(0.382\,R_{\oplus})}$", Michael's answer "What is the total potential energy enclosed in a sphere of radius $0.382\,R_{\oplus}$ and an enclosed mass of $0.055\,M_{\oplus}$, where it is uniform?" shows that he does not have a complete understanding of the equation. His answer clearly shows a partial recognition of the equation given in the lecture notes for the "total potential energy" (as it was called in the written lecture notes) of a uniform density sphere with radius R, but does not show any true understanding as he does not specify that the mass and radius are specific values for the planet Mercury nor does he mention that the equation is for uniform density.

However, in Exam #2, which was three weeks after the homework Michael again shows "understanding" of negative gravitational potential energy when answering question #7: "Under what circumstances would we consider the gravitational potential energy negative? Are there any circumstances in which we would consider the gravitational potential energy to be positive? Give examples." Once again, since the gravitational potential energy is equal to the work done against gravity to bring a mass to a given point in space, this problem has an expected solution which indicates that because of the inverse square nature of the gravitational force; the force approaches zero for large distances and therefore it is appropriate to choose the zero of the gravitational potential energy at an infinite distance which yields a negative gravitational potential energy. Gravitational potential energy can be considered positive for an object near the surface of a massive object, such as a planet, where the gravitational acceleration can be assumed to be constant and zero potential energy is defined as zero on the surface of the massive object (or some other level close to it). Again, since the zero of gravitational potential energy can be chosen at any point, the gravitational potential energy at a height h above that point will be positive. These two scenarios are consistent with each other as can be seen when considering a height h above the surface of the Earth:

$$\Delta U = \left(-G\frac{mM_{Earth}}{R_{Earth}+h}\right) - \left(-G\frac{mM_{Earth}}{R_{Earth}}\right)$$

$$= \left(-G\frac{mM_{Earth}}{R_{Earth}}\right)\left(\frac{1}{1+h/R_{Earth}}\right) - \left(-G\frac{mM_{Earth}}{R_{Earth}}\right)$$

Since h is much smaller than the radius of the Earth, h/R_{Earth} must be a small number and we can use the mathematical approximation $1/(1+x) \approx 1-x$:

$$\Delta U = \left(-G\frac{mM_{Earth}}{R_{Earth}}\right)(1-h/R_{Earth}) - \left(-G\frac{mM_{Earth}}{R_{Earth}}\right) = m\left(\frac{GM_{Earth}}{{R_{Earth}}^2}\right)h = mgh$$

Michael answers this question as follows:

$$"U(r) = -\frac{GMm}{r}$$

Gravitational potential energy is normally defined as negative, such as when object orbit one another in ellipses, making total energy negative. Potential energy cannot be positive unless there was a negative mass from some exotic particle. If potential were positive, gravity would be a repelling force."

This answer shows that Michael has "understanding" of negative gravitational potential energy because he clearly and accurately states the equation that can be used for negative gravitational potential energy and since negative potential is indicative of a "bound state" his example involving a bound state for a system of two objects was considered acceptable. Although not explicitly, Michael even touches on the virial theorem in his answer when he states that the total energy would be negative. He does not show a complete understanding however, because he does not mention that that the gravitational potential energy will be negative only when set at zero at an infinite distance. Michael's explanation of "positive" gravitational potential energy is only valid theoretically, and he misses the simple solution of gravitational potential energy of a system of two objects when a smaller object is near the surface of a massive object.

Michael was one of two students who were interviewed who also mentioned the virial theorem in his essay; but did not go into great detail about the topic, only mentioning it in passing when discussing several different topics in his essay. "Simplifying the concept [of the virial theorem] to $\langle E\rangle = -\langle K\rangle = \frac{\langle U\rangle}{2}$ helped immensely" when "thinking about the topic."

Michael shows "understanding" of the equations in the interview as discussed in this section, but as can be seen in section 5.2c, Michael also shows "understanding" for additional topics in other assessments. For example, Michael shows "understanding" of Hubble's Law and the Schwarzschild radius in both the homework assignments and the exam questions. This once again gives credence to the implication that the students' level of understanding of equations is topic driven, and not assessment driven.

Michael's beliefs of understanding of astrophysics equations was determined by the student survey concerning students' beliefs of astrophysics equations. Table 32 compares these beliefs with his qualitative understanding of negative gravitational potential energy and the virial theorem.

Table 32: Student's conceptions of understanding astrophysics equations compared to qualitative understanding of an astrophysics topic.

Michael's Beliefs	**Michael's Actual** **(For the Topic of Negative Gravitational Potential Energy and the Virial Theorem.)**
He needs to conceptually understand the equations that they use.	Demonstrates an overall but not complete conceptual understanding of the equations.
The use of derivations of equations is important, but only as a means to get the equation.	Shows partial understanding of derivations and is able to understand and utilize the results.
He is undecided if he needs to recognize the symbols in the equation in terms of the corresponding physics quantities.	Demonstrates recognition of the symbols in the equation in terms of the corresponding physics quantities most of the time.
He needs to understand and recognize the relationships connecting the variables of astrophysics equations.	Demonstrate an understanding of the relationships connecting the variables of the equations.
He needs to make connections between equations and real world.	Can mostly make connections between the equations and the real world.
He is uncertain if he needs to find the "right" equation to use in a problem.	Mostly successful in determining the equations necessary to solve the problems.

As shown in the above table, Michael believes that he needs to conceptually understand the equations that he uses. When relating this belief to his qualitative understanding of negative gravitational potential energy and the virial theorem, as can be seen in his work shown, Michael has a conceptual understanding of the negative gravitational potential energy and/or the virial theorem equations, but it is not complete. Michael also believes that the use of derivations is important, but only as a means to get the equation; which is demonstrated when he has difficulties deriving the equations necessary to solve the problems given. However, once the equation is derived, Michael is able to use the formula competently. Michael is undecided in his belief that he needs to recognize the symbols in the equation in terms of the corresponding astrophysics quantities. This belief is reflected in his work; he successfully recognizes the symbols in the equations used in the interview but is not successful in recognizing the symbols in the equation given in homework assignment #9. He believes that he needs to understand and recognize the relationships connecting the variables of astrophysics equations; this belief is expressed throughout his assessments as he demonstrates that he only has an understanding of the relationships in the negative gravitational potential energy and in particular the virial theorem equations. Michael believes he needs to make connections between the equations and the real world, and he was mostly successful when explaining when the gravitational potential energy could be negative with real world examples in his answer to question #7 in exam #2; however, he was not as successful, giving an answer was theoretical and not reflective of the real world when describing if the gravitational potential energy could ever be considered positive in this question as shown above. Finally, Michael is uncertain in his belief of the necessity of finding the “right” equation to use in a particular problem; however, when solving a problem, he recognizes if he has the “right” equation and if he does not, he works toward obtaining the “right” equation. Michael’s beliefs therefore are only slightly in contradiction to his actual demonstration of understanding of the equations for the topic of negative gravitational potential energy and the virial theorem.

5.3.d Negative Gravitational Potential Energy/Virial Theorem - Complete Understanding

Finally, after examining the attributes of qualitatively having “no understanding”, “partial understanding”, and “understanding” of negative gravitational potential energy and/or the virial theorem and then connecting the level of understanding to the students’ beliefs and actual overall understanding of equations; I was able to examine the students who demonstrated a “complete” qualitative understanding of astrophysics equations as it relates to their qualitative understanding of astrophysics concepts. Out the ten student interviews, two students showed “complete” qualitative understanding of the negative sign of the gravitational potential energy and of the virial theorem. I chose one of the students who showed “complete understanding” of this topic and continued the analysis, by examining the results from the interview and comparing the results to the results in other assessments for the same topic as well as comparing the results to the student’s beliefs in the survey.

Justin

Justin was one of two students who was classified as “complete understanding” of negative gravitational potential energy and the virial theorem during the interview process; he is representative of the two students as he displayed the same or extremely similar attributes as the other students who showed “complete understanding”. During the interview, Justin displays in multiple occasions, that he has no difficulties with the concept of negatives as well as the topic of negative gravitational potential energy and the virial theorem.

Throughout the interview Justin solves each of the three problems without any hesitation or need for assistance. When asked, as part of the first question in the interview, to determine the gravitational potential energy of the Earth-Sun system, Justin’s reply “we know that gravitational potential energy is negative GM_1M_2 over r. And in this case M_1 is M_{Sun} and M_2 is M_{Earth}" shows that he clearly remembers the correct equation for negative gravitational potential energy. When asked to describe why the equation was negative, Justin's replies showed "complete" qualitative understanding of the topic:

- "It's negative because … it's a well. The gravitational potential … you think of it as a potential well." "In which case, since you're in a well, you're negative, therefore you have negative energy."
- "Basically whenever you're infinite, like you're really really far away from the object, you basically don't feel the effect of that gravitational potential well anymore."
- "So what you are actually doing is you're deriving it [gravitational potential energy] from the force. So you take the integral of F dr or dx depending on what your coordinate system is. And usually you say your reference point as one of the bounds of the integral and the other point being where you are. And in this case you're setting the … one of the bounds as infinity so your second gravitational potential energy term drops to zero."

He also shows complete understanding with respect to the negative aspect of the gravitational potential energy as demonstrated when determining the total energy of the Earth-Sun system:

"You just add the two [energies] and you get negative one-half G M_{Sun} m_{Earth} over r. So the sign of the total energy is negative which makes sense since the earth is in a bound orbit. So, it's in a bound orbit and whenever you have a bound system you need to have a negative total energy. Which is why it makes sense for it to be less than zero."

And again, later in the interview during the third question Justin states:

"we know that the kinetic and the potential energy is the total energy of the system. So, if it's losing total energy, that is equivalent to its energy becoming more negative. So, if the energy is becoming more negative, the only way that is possible is if it sort of falls deeper into a potential well. And that's why I reasoned in the beginning that the radius is decreasing. Because as it gets closer and closer to the object it's falling deeper into a potential well."

"Now, we know that we want it to lose total energy which means that the energy … that E has to go down then the only way that is possible is if r gets smaller. As r gets smaller, the denominator gets smaller and this whole term gets … the absolute value of it would get bigger, but since there is a negative sign it's becoming more negative. So, your total energy is decreasing that way."

All of which demonstrate that Justin shows a "complete understanding" of negative gravitational potential energy; he clearly knows when to use the equation and what the negative means both mathematically and conceptually.

Justin shows the same level of "complete understanding" of the virial theorem throughout the interview, although he does not call it by name until the end of the interview. When discussing the assumptions made when using the virial theorem, Justin's answer demonstrates his "complete understanding" of the topic: The virial theorem assumes "that the [object] is in a circular, bound orbit" and that "it has to be in equilibrium." He continues to demonstrate "complete understanding" of the theorem by mentioning the relevance of the virial theorem to the questions given in the interview: "For some of the previous questions it is [in equilibrium the entire time] because it's still like a stable orbit. But in this case technically since you're losing energy through drag, it's not exactly in equilibrium. But I guess like to an approximation, you can just use that and use the Virial Theorem."

As stated in the sections above, the Homework and Exam questions that specifically addressed student understanding of gravitational potential energy were found in homework assignment #9, Question 1, part a where the students were asked to solve a “jeopardy equation” style problem and in Exam #2, Question #7 where the students were asked to describe the circumstances where one could consider the gravitational potential energy negative and circumstances where one could consider it positive. Again, Justin shows “complete understanding” of the topic of gravitational potential energy in both of these assessments.

For the homework assignment, when asked to “Create a question for which the following equation gives the solution: $U = -\frac{3(6.67x10^{-8}cm^3g^{-1}s^{-2})(0.055\ M_{\oplus})^2}{5(0.382\ R_{\oplus})}$”, Justin’s answer demonstrates this “complete understanding”:

“$U = -\frac{3}{5}\frac{(6.67x10^{-8}cm^3g^{-1}s^{-2})(0.055\ M_{\oplus})^2}{(0.382\ R_{\oplus})} = -\frac{3}{5}\frac{GM^2}{R}$

What is the total gravitational potential energy of Mercury, assuming a uniform density sphere of radius $R = 0.382\ R_{\oplus}$ and mass $M = 0.055\ M_{\oplus}$?”

This answer is the expected answer addressing that the equation given is for the *gravitational* potential energy of Mercury. Justin shows that he has recognized that the given equation is the equation derived in the lecture notes for the “total potential energy” of a “uniform density sphere with radius R”, but additionally shows that he understands that this equation is for the gravitational potential energy – specifically for a spherical object.

In his answer to Exam #2, question #7: “Under what circumstances would we consider the gravitational potential energy negative? Are there any circumstances in which we would consider the gravitational potential energy to be positive? Give examples." Justin shows "complete understanding" of negative gravitational potential energy; but in an unexpected way.

"When we deal only with Newtonian gravity in an inertial frame we have $U_{grav} = -\frac{GMm}{r}$ and since $M, m, r > 0$, U_{grav} is negative. This corresponds to being in a gravitational potential well such as the Earth in the Sun's grav. potential well. When we start working in accelerating reference frames, the angular momentum plays a role and we define effective gravitational potentials which can be positive. Furthermore, in GR $\Phi_{GR} = \left(\frac{1}{2} - \frac{GM}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$ can be positive for large enough r. In general, true gravitational Newtonian potential energy will always be negative $\left(-\frac{GMm}{r}\right)$ whereas Φ_{GR} can be positive."

Although Justin did not give the expected solution which indicates that the gravitational potential energy of a system can be negative if one is determining the work done against gravity to bring a mass to a given point in space from an infinite distance; which means the choice for the zero gravitational potential energy is set at an infinite distance. Gravitational potential energy can be considered positive for a system of an object near the surface of a massive object, such as a planet, where the gravitational acceleration can be assumed to be constant and zero potential energy is defined as zero on the surface of the massive object (or some other level close to it). Again, since the zero of gravitational potential energy can be chosen at any point, the gravitational potential energy at a height h above that point will be positive; Justin's answer is accurate and describes the nature of gravitational potential energy "complete understanding" of the topic.

Justin was not one of the two students who were interviewed and gave reference to either negative gravitational energy or the virial theorem in their essays, so his essay could not be added to the data for his understanding of these topics. However, even without this data it is clearly apparent that in Justin's case he shows a "complete understanding" of the astrophysics topic of negative gravitational potential energy and the virial theorem in all assessments. Justin demonstrates this "complete understanding" in multiple astrophysics topics through multiple assessments throughout the semester such as black holes and special and general relativity.

Justin's beliefs of understanding of astrophysics equations was determined by the student survey concerning students' beliefs of astrophysics equations. Table 33 compares these beliefs with his qualitative understanding of negative gravitational potential energy and the virial theorem.

Table 33: Student's conceptions of understanding astrophysics equations compared to qualitative understanding of an astrophysics topic.

Justin's Beliefs	**Justin's Actual** **(For the Topic of Negative Gravitational Potential Energy and the Virial Theorem.)**
He needs to conceptually understand the equations that they use.	Demonstrates a complete conceptual understanding of the equations.
The use of derivations of equations is important, NOT only as a means to get the equation.	Shows complete understanding of derivations and is able to use the derived equations successfully.
He needs to recognize the symbols in the equation in terms of the corresponding physics quantities.	Demonstrates recognition of the symbols in the equation in terms of the corresponding physics quantities.
He needs to understand and recognize the relationships connecting the variables of astrophysics equations.	Demonstrates a complete understanding of the relationships connecting the variables of the equations.
He needs to make connections between equations and real world.	Can make insightful connections between the equations and the real world.

He is unsure about the necessity to find the “right” equation to use in a problem.	Successful in determining the equations necessary to solve the problems.

Justin’s beliefs are in harmony with his actual demonstrations with respect to his understanding of the topic of negative gravitational potential energy and the virial theorem. Justin believes that he needs to: conceptually understand the equations that he uses when solving problems, recognize the symbols in the equation in terms of the corresponding physics quantities, understand and recognize the relationships connecting the variables of astrophysics equations, and make connections between equations and real world; all of which he does with “complete understanding” as shown above. It is of particular not however, that Justin believes that the use of derivations of equations is important, NOT only as a means to get the equation. Justin in in the minority in this belief, but shows this belief when successfully deriving equations without prompting and recognizing when an equation should be derived, not just stated. Justin is also unsure about the necessity to find the “right” equation to use in a problem; but successfully determines a correct equation to answer the problems, even when it is unexpected, such as in his answer to the exam problem shown above.

5.3.e Summary & Possible Implications

Based on the results of the analysis of the students’ understanding of the topic of negative gravitational potential energy and the virial theorem across multiple assessments as shown above, we can determine how student qualitative understanding of astrophysics equations relates to their qualitative understanding of astrophysics concepts. Students that do not show an understanding of the topic of negative gravitational potential energy and the virial theorem across multiple assessments repeatedly do not demonstrate an understanding because they do not recognize the underlying physics of the equations. For example, Daniel who, as stated above, equates potential energy to net force "So, for gravitational potential energy, that would be due to gravity, so that would be Newton's law." These students also do not demonstrate knowledge of how to use the equations to solve the astrophysics problems. For example, Benjamin who states: "We somehow have to show that GMm/r is bigger than $1/2mv^2$", but has to be lead through the derivations. Furthermore, the students that do not show understanding of the topic of negative gravitational potential energy and the virial theorem cannot establish a link between the equations and everyday life as seen in many examples above. This is consistent with the theoretical framework described in the previous section agreeing with several of Domert et al.'s arguments concerning understanding physics equations. In addition to this theoretical framework, the students who do not show understanding of the topic cannot start the problems without assistance and do not make the mathematical connections to the equations necessary for full understanding; in particular they do not demonstrate an understanding of negatives in the equations as, for example, demonstrated by Daniel when he constantly forgets the negative sign throughout the progression of the interview indicating that he does not consider the negative sign important to the equation.

In comparison, the students who show an understanding of the topic of negative gravitational potential energy and the virial theorem across multiple assessments demonstrate this understanding consistently as these students show that they recognize the underlying physics of the equations, demonstrate knowledge of how to use the equations to solve the astrophysics problems, and can establish a link between the equations and everyday life; again consistent with the theoretical framework described in the previous section and agreeing with several of Domert et al.'s arguments concerning understanding physics equations.

In addition to Domert's theoretical framework, the students who show understanding of the topic can easily start the problems without assistance and make the mathematical connections to the equations necessary for full understanding; in particular they demonstrate an understanding of negatives in their solutions. An example that was shown above that meets these criteria is Justin when he determines the total energy of the Earth-Sun system: “You just add the two [energies] and you get negative one-half G M_{Sun} m_{Earth} over r. So, the sign of the total energy is negative which makes sense since the earth is in a bound orbit. So, it’s in a bound orbit and whenever you have a bound system you need to have a negative total energy. Which is why it makes sense for it to be less than zero.”

The majority of the students who were interviewed were consistent in their understanding of the concept of negative gravitational potential energy and the virial theorem across multiple assessments. In other words, those students who showed “no understanding” of this astrophysics topic, such as Daniel above, showed “no understanding’ of the topic in all of the relevant assessments related to the topic; and likewise for the students who showed “partial understanding”, “understanding”, and “complete understanding”. Although this strongly implies that the students’ level of understanding of equations is topic driven and not assessment driven as each of the students discussed above showed overall the same level of understanding in all assessments for this particular astrophysics topic, not all students showed this level of consistency across multiple assessments. The occasional student did show a different level of understanding in one assessment piece, such as Samuel above who demonstrated “understanding” of the virial theorem in his essay but showed “partial understanding” of the astrophysics concept of negative gravitational potential in all other assessments. This is consistent with the finding of Bao and colleagues (Bao et al., 2002) (Page 5): “Students can use mixed ideas in their reasoning. When multiple questions related to a single physical feature are presented to students, they may respond with different models on different questions. This indicates that using these questions we can obtain measurement on students’ mixed model states and the significance of different contexts in triggering students’ use of models.” Therefore, having multiple assessments with different types of questions is beneficial for all students.

5.4 What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation?

Research Question #4 was analyzed using the data from the student interview videos as well as the interviews with experts. The relevant interview videos consisted of students or experts solving astrophysics problems related to negative potential energy and the virial theorem. Comparing the students' responses to these data, we can determine how student understanding of astrophysical equations is similar or different with an expert understanding of the equations. Below I present the findings that that are relevant to Research Question #4: What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation?

Symbolic Forms

In order to discuss student versus expert understanding of astrophysics equations, I focused on the idea of "symbolic forms" from Sherin (2001). Sherin argues that the details of the equation being used in solving physics problems have meaning for the expert in the arrangements of the symbols it contains and that successful physics students "learn to express a moderately large vocabulary of simple ideas in equations" and that they read physical meaning out of these equations. The term that Sherin introduces for the elements of this vocabulary is "symbolic forms" and, as is discussed in more detail in the theoretical framework above, a symbolic form is a "knowledge element" that is comprised of two components: a "symbol template" (how the idea is written in symbols) and a "conceptual schema" (the idea to be expressed in the equation). According to Sherin, these symbolic forms allow students to "take a conceptual understanding of some physical situation and express that understanding in an equation." By studding these symbolic forms as they relate to experts and students we can determine the differences between an expert qualitative understanding of an equation and a student qualitative understanding of an equation.

Multiple Representations

There are many differences between expert qualitative understanding of an equation and a student qualitative understanding of an equation. One difference noted in this study is the differences between experts and students in their practice of using multiple representations when solving problems. Solving problems in astrophysics often involves the expressing the situation discussed in the problem using different representations such as: pictures, diagrams, graphs, and equations and the vast majority of current research suggests that multiple representations are an important tool for student understanding. (Rosengrant, et al, 2007) As stated in a later paper by Rosengrant and colleagues: Experts use multiple "representations more frequently" in order to "explore the problem" and "develop a better understanding of the situation, and to help solve the problem." (Rosengrant, et al, 2009) (Page 2) They also note that students on the other hand, do not necessarily use visual representations to help solve their problems; however, the students that do use multiple representations are the students that correctly solve the majority of the problems.

Evaluation of Equations

While there are many evaluation strategies for different types of information in physics, I will focus on two of the three equation evaluation strategies outlined in Warren's dissertation (2006); namely, "Unit Analysis" and "Special-Case Analysis". Unit analysis is used "to evaluate equations to determine whether they are physically coherent" and special case analysis is used "to evaluate an equation, model, or conceptual claim to determine whether it is consistent with prior knowledge and experience." (Warren, 2006) (Pages 9-10) (The third evaluation strategy, "Quantity Analysis", deals with the coordinate transformation properties of the equations and is thus not applicable to the equations used in this analysis.) When using unit analysis to evaluate an equation it is expected that the units for each term in the equation will be identical in order for the equation to be physically self-consistent. When using special case analysis to evaluate an equation, prior knowledge is used to compare with the results of the equation; in other words, prior knowledge should match up with the results obtained from using the equation.

5.4.a Expert Understanding of Astrophysics Equations

First, I analyzed the use of symbolic forms of the expert interviews. The interviews were conducted with two experts in the Fall 2013 semester and consisted of the same questions that were given to the students in the previous sections. Throughout the interview process, the experts used eleven (11) different symbolic forms as described by Sherin. The symbolic forms that were used by the experts included: Balancing (23%), Canceling (4%), Coefficient (4%), Dependence (3%), Identity (27%), Opposition (3%), Parts-of-a-Whole (3%), Prop – (10%), Prop + (10%), Ratio (10%), and Whole-Part (3%). For both the expert and the student interviews, the interviews were first transcribed and searched for the symbolic forms defined by Sherin, the identified forms were then discussed with another coder until a consensus was established. The following table illustrates these forms.

Table 34: Expert Examples of Symbolic Forms

Symbolic Form	Description of Forms Identified	Symbol Pattern	Example Expert Responses
Balancing	Two influences, each associated with a side of the equation, in balance so that the system is in equilibrium.	□ = □	"The gravitational force between two objects is G M_{Sun} m_{Earth} over big R squared that has to be the same as the centripetal force of the earth at every instant in its orbit, which is m_{Earth} v squared over big R."
Canceling	Two influences that precisely cancel so that there is no net outcome.	0 = □ - □	"escape velocity [is] used for a total energy of zero … between KE and PE for a radius like this"
Coefficient	A product of factors is broken into two parts and one part is identified with an individual symbol, the coefficient.	[x□]	"[then] kinetic energy is 1/2 M_{Sun} m_{Earth} over R and of course that is [multiplied by] G."
Dependence	A whole depends on a quantity associates with an individual symbol.	[… *x* …]	"If we are losing total energy, we are actually getting a larger negative gravitational potential energy because ... Its total energy [is] equal to the [negative] potential energy over 2"
Identity	A single symbol that appears alone on one side of an equation has the same properties as the expression on the other side.	*x* = …	"What I want is to use the formula for KE, which is 1/2mv^2."

Opposition	Two terms, separated by a minus sign, associates with influences that work against each other.	☐ - ☐	"… and that will equal minus U over 2 plus U …"
Parts-of-a-Whole	Amounts of generic substance, associates with terms, that contributes to a whole.	[☐ + ☐ + ☐ …]	"It's KE plus U …"
Prop -	Indirectly proportional to a quantity, x, which appears as an individual symbol in the denominator.	$\left[\frac{\dots}{\dots x \dots}\right]$	"R is in the denominator. This U equals minus G M_{Earth} $m_{satellite}$ over R. R is getting smaller, and so the amplitude is bigger but still negative."
Prop +	Directly proportional to a quantity, x, which appears as an individual symbol in the numerator.	$\left[\frac{\dots x \dots}{\dots}\right]$	"as kinetic energy gets larger amplitude, the velocity gets larger"
Ratio	Comparison of a quantity in the numerator and denominator.	$\left[\frac{x}{y}\right]$	"…the ratio of the gravitational potential energy to the kinetic energy is a half or two. Um, so the gravitational potential energy is twice that of the kinetic energy."
Whole-Part	A new net amount is produced by taking away a piece of an original whole.	[☐ - ☐]	"If you were talking about total energy, this [energy] would just be a part of the total energy."

As can be seen in the table above, an expert will use a wide verity of forms when solving an astrophysics problem. Some of the symbolic forms used to solve the problems during the interview were used more often than others (see figure 9 below) such as the identity, balancing, ratio, prop+, and prop- forms; but were relatively evenly distributed among these forms.

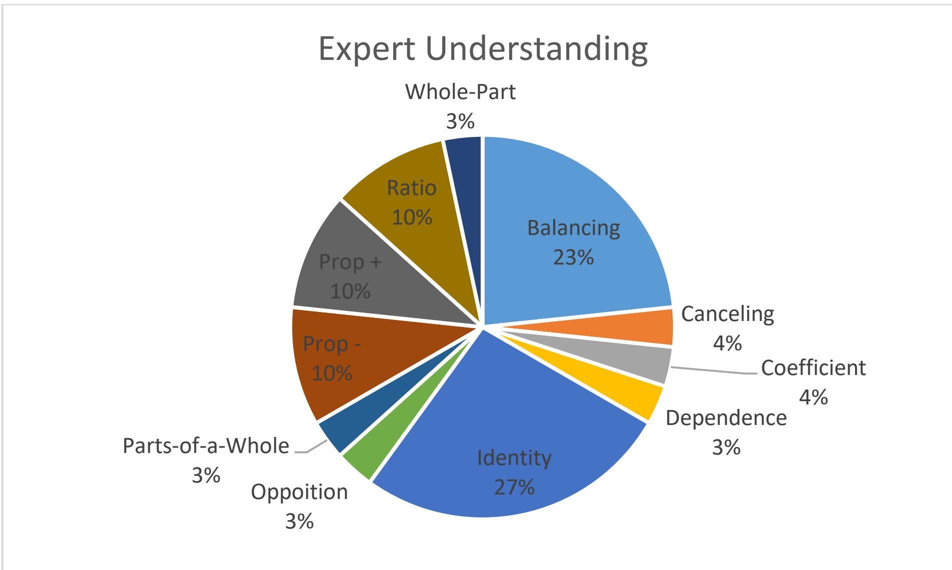


Figure 10: Expert utilization of symbolic forms.

In addition to the symbolic forms, experts tended to use multiple representations often in their work. (The students, as discussed in more detail below, were mixed with their use of multiple representations.) The multiple representations used most commonly were drawing a picture or diagram of the system "By drawing a picture … it helps remind me how the earth is moving and helps remind me what the variables in the problem are." The experts also distinctly used unit analysis while discussing the problem noting that they had "comparable units" for different quantities such as kinetic and potential energies and made sure that their solutions "made sense" when unprompted: "I know that also has been true for astronomical satellites and etcetera. I expect that is the correct physics, but let's make sure it makes sense."

5.4.b Student Understanding of Astrophysics Equations

The next analysis I did was the analysis of the “symbolic forms” that students use when discussing the equations. A detailed account of the data collected for the students is given in Section 4.2.c of this paper. In Section 4.2.c, we found that the students used a variety of forms - between six (6) and nine (9) forms, and while the forms used were similar among the students, the relative proportion of each form was different for the students with different levels of understanding.

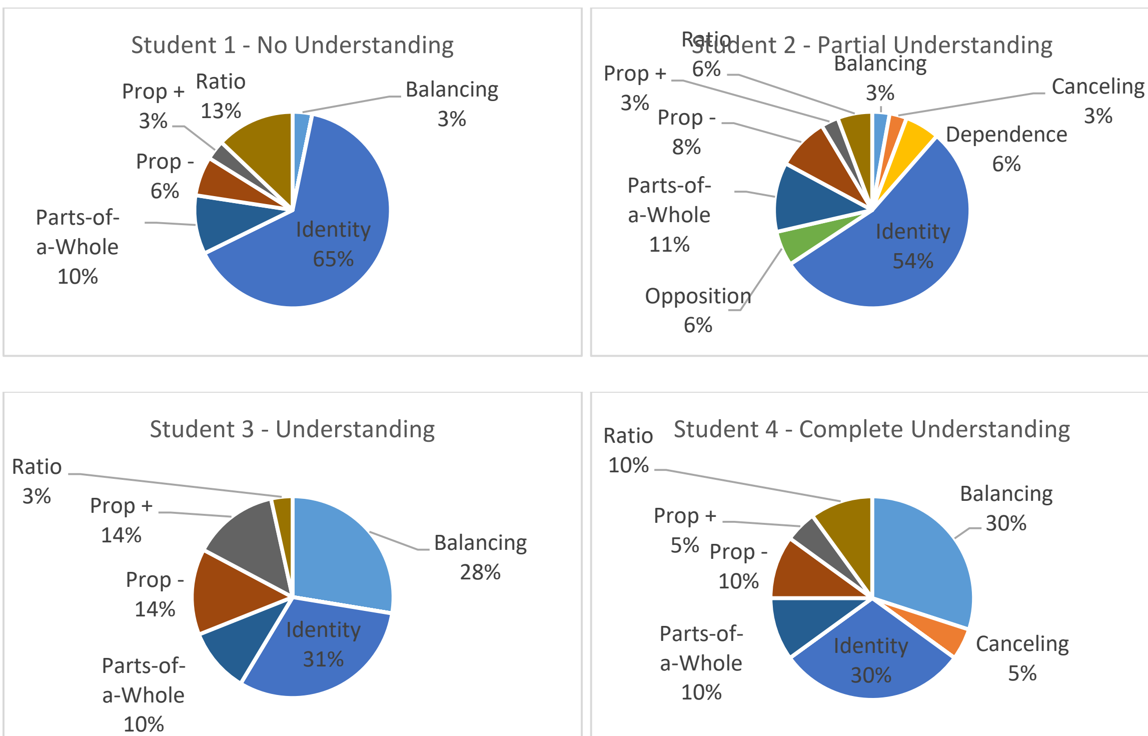


Figure 11: Students’ utilization of symbolic forms. (Repeat of Figure 9)

As can be seen, all of the students used approximately the same variety of symbolic forms; however, the weaker students gave much more weight to the identity symbolic form than the stronger students. The students who used the selection of symbolic forms more evenly were able to set up the problem correctly (usually with a diagram) and had little difficulty understanding all aspects of the problems. The students who used the variety of symbolic forms less evenly, showed little to no understanding of how to approach the problems and in general had to be walked through large portions of the problems.

When using multiple representations, only two of the students did not use a picture or diagram in their discussion (one student with complete understanding and one with partial understanding) and out of the remaining eight students, two of the students (again, one student with complete understanding and one with partial understanding) started one of the three interview problems with a picture or diagram; the remaining students had to be prompted to include a sketch with the poorer students not able to use the sketch to successfully understand the problem or equations they used.

Only one of the ten students used unit analysis while solving the problems during the interview process. This student who was identified as having partial understanding was trying to determine if he had the correct equation for the gravitational potential energy of the Earth-Sun system and attempted to work through the equation using unit analysis: "the next best thing would be to consider the units for the gravitational constant." The student was unsuccessful in his attempt to use unit analysis to determine if the equation was correct, but the attempt was made. Additionally, four students (one student identified with complete understanding, two with partial understanding, and one with no understanding) were able to identity when an equation made sense without prompting; the more successful student to confirm his result "So if [the satellite] loses some [total] energy it would go down to a lower orbit which makes sense. So potential energy initial will be less than the potential energy after drag, which makes sense." and for the less successful students this occurred when the students realized that their equations were not moving forward with the problem or getting the results that they were expecting: "the mass of the earth times the acceleration of the sun … wait, that doesn't make any sense." As the problems in the interview asked the students if their results "made sense and why", only unprompted instances of special case analysis were noted.

5.4.c Expert vs. Student Understanding of Astrophysics Equations

Although the students and the experts used a similar variety of symbolic forms, the students relied much more heavily on particular forms, such as the identity and balancing forms, than did the experts. This is predominantly evident in the weaker students who displayed "no understanding" or "partial understanding" of astrophysics equations as demonstrated in the previous sections of this chapter.

This finding strongly suggests that the reliance of one particular form, particularly the identity form, decreases with increased understanding of astrophysics equations. The experts and students that show "complete understanding" or "understanding" of astrophysics equations rely less on memorization of the equations than the students who show "partial understanding" or "no understanding".

Additionally, the experts tended to use multiple representations (in the form of a picture or diagram) more than the students. Both experts used pictures to analyze the problem in two out of the three problems in the interview while eight students only used pictures or diagrams in at most one of the problems. Of these eight students, only two did not need prompting to include a picture or diagram and the least successful students were not able to use the diagram to help them understand the problem or equations necessary to solve the questions. Additionally, although the more successful students use multiple representations in the form of pictures of the problem or free body diagrams, they do not refer back to the diagrams as much as the experts. The least successful students need to be prompted into using pictures or diagrams in their work and had to be continuously urged to use the picture or diagram to help them with the problems with little success.

It was noted that the experts also made use of unit analysis to help them solve the problems and make sure that their equations were correct, whereas the students rarely do the same with only one student making use of unit analysis in the interviews. Furthermore, both experts made sure that their solutions were consistent with their prior knowledge and experience without prompting, whereas most of the students (six out of ten) did not do so. Of the students that did make use of special case analysis, the more successful students used this analysis to confirm their results, whereas the less successful students used it when they got stuck and could not move forward in the problem.

5.4.d Summary & Possible Implications

From the results of the analysis of the students' interviews as well as the expert interviews as viewed through the lenses of Sherin's "symbolic forms", Warren's "unit analysis" and "special case analysis" equation evaluations, as well as the use of multiple representations, we can determine what the difference is between an expert qualitative understanding of an equation and a student qualitative understanding of an astrophysics equation. With respect to Sherin's symbolic forms, we can see that although the experts and students both use a similar number of forms, the experts apply a more even distribution of the used symbolic forms than the students and that the more successful students use a more even distribution of symbolic forms than the less successful students. This indicates that students are still in the process of learning how to take the conceptual understanding of the astrophysical situation presented in the problems and express that understanding in the equations necessary to solve the problem. (Sherin, 2001) Therefore, this suggests that the students should be exposed to a wider variety of forms and the use of forms in the classroom, or be guided into using more forms in their written work. Additionally, when looking at the types of Sherin's forms used, the experts and more successful students use the identity form much less than the less successful students implying that memorization of multiple formulas should be discouraged especially as a starting point of problem solving. As Sherin states: "It is absolutely critical to acknowledge that physics expertise involves this more flexible and generative understanding of equations, and our instruction should be geared toward helping students to acquire this understanding" (Sherin, 2001,p. 1)

The experts also consistently made use of unit analysis and special case analysis as means of evaluating their solutions to the problems presented while the students seldom used either of these evaluations strategies. Both experts used unit analysis as well as special case analysis to check and confirm their equations and solutions; whereas most of the students did not use any evaluation strategy. Of the few students that did use an evaluation strategy, the successful students also used the evaluation strategy to check and confirm their equations or results whereas the less successful students used the evaluation strategy to attempt to correct their equations while solving the problems. According to Warren, the use of evaluation strategies "can lead the student to recognize that the problem solving [strategy] is incoherently structured or gives results that are inconsistent with" their prior knowledge or experience and that "upon such recognition, the student may correct [their] own mistake" (Warren, 2006, p. 13). It is clear that both the successful and less successful students that used an evaluation strategy were doing so for this purpose. Therefore, it is clear that students should be highly encouraged and taught how to self-evaluate in order to develop an understanding of astrophysics equations.

Lastly, the experts use multiple representations with a much greater frequency than the students; again, with the more successful students using multiple representations more than the less successful students, mimicking the experts. These results agree with Rosengrant and colleagues when they note that there are "differences between experts and novices when they construct representations to help them solve problems" (Rosengrant et al, 2007, p.2) They note that while experts display a variety of diagram-related reasoning behaviors, novices show little to no evidence of these abilities. This research again agrees with the results found above as the experts not only started the problems with a picture or diagram but also referred back to the diagram while solving the problem. The majority of the students, on the other hand, had to be encouraged to create a picture or diagram and of those that did use a picture or diagram, they rarely referred to it further into the problem. This suggests that the use of multiple representations needs to be stressed as an integral part of understanding astrophysics equations; in order to advance the students to a more expert level of thinking.

Chapter 6: Summary of Findings and Instructional Implications

The goal of this dissertation is the study of how students in advanced courses understand astrophysics equations. In this section, I will highlight the most significant findings as they relate to the research questions posed at the beginning of the thesis and discuss the implications of these findings for teaching advanced undergraduate course in astrophysics.

6.1 Student Understanding of Astrophysics Equations with Astrophysics Concepts

6.1.a Research Question #1: What do the students think it means to understand astrophysics equations?

To answer research question #1, I analyzed data from an online Likert survey (consisting of questions including what the students think it means to understand astrophysics equations) and student essays (concentrating on the equations relevant to their favorite and least favorite topic in the class). From the student's responses to both of these data sources as discussed in section 5.1, I have come to a conclusion concerning what students believe when they think about understanding equations.

First, I have concluded that the majority of students believe that derivations of equations are important in astrophysics but only as a way to get the final equations. This conclusion stemmed from the students' agreement from the Likert survey questions, primarily question #2 "A derivation or proof of an equation shown in class is useful because I can use the final equation while working on the problem sets without working through the derivation myself"; as well as the noticeable lack of derivations or mention of derivations in the student's essays. However, I also found that successful students do use derivations to better understand the astrophysics concepts underlying the equations.

Secondly, I have found that while students believe that conceptual understanding of the equations is important in astrophysics; they are not always successful in developing this understanding. Although the students' responses in both the Likert survey and essays indicated that they clearly understood the importance of understanding the astrophysics equations that they use, I found (see section 5.1) that within the framework of understanding of physics equations created by Domert et al. (2012), the students are only partially successful in understanding astrophysics equations. Specifically, most of the students do not discusses the symbols in their equations with great frequency, the students are almost evenly divided concerning making connections between equations and the real world, they do not find significance in describing the equations in any meaningful way, or show any deep understanding of the astrophysics behind the equation.

Lastly, I have found that the students appear to have a disconnect between their belief that they need to have a conceptual understanding of the equation and the use of the equations when they solve problems. From the student responses to the Likert Survey, the students show that they believe that they must find the "right equation" in order to solve problems in astrophysics; they believe that "To solve a problem in astrophysics I need to match the problem situation with the appropriate equations and then mathematically manipulate and/or substitute values to get an answer." This finding shows that the students, while believing that they need to understand the equations in order to understand astrophysics, do not believe that they need to understand the equations when actually using them in the problems. Since the students believe that they need to have a deep understanding of the equations that they use in order to understand astrophysics while simultaneously believing that, when applying the equations to problem solving, they do not need to deeply understand the equations; we might hypothesize that the students ultimately think that the astrophysics equations can just be taken as a mathematical tool.

6.1.b Research Question #2: What does student qualitative understanding of an equation look like?

To answer this research question, I used data from the students' homework assignments and exams, interview videos, and essays. Using the data in section 5.2, I have been able to determine whether a student has a qualitative understanding of astrophysics equations and specifically what student qualitative understanding of an astrophysics equation looks like. Again, using the theoretical framework of understanding of physics equations created by Domert et al. (2012), I analyzed collected artifacts and using this analysis I classified students as having "no understanding", "partial understanding", "understanding", or "complete understanding" of the astrophysics equations which were used in the class assessments. As stated in the theoretical framework by Domert et al. (2012), the characteristics for student understanding of equations are: recognizing the symbols in the equation in terms of the corresponding physics quantities, recognizing the underlying physics of the equation, recognizing the structure of the equation, establishing a link between the equation and everyday life, demonstrating knowledge of how to use an equation to solve astrophysics problems, and the ability to know when to use an equation.

I have found that the groups of students who were classified as having "no understanding" of the astrophysics equations based on a particular assessment could demonstrate understanding on other assessments. In other words, a student may have demonstrated none of the characteristics for student understanding of equations as established by Domert et al. (2012) in one context and thus belonged to a group of "no understanding" in that particular assessment, but the same students would demonstrate partial understanding in a different context. I conclude from this that there is no such thing as a student with “no understanding” of the equations in the astrophysics class from which I obtained my data.

In comparison to the students with “no understanding”, the students with “partial understanding”, “understanding”, and “complete understanding” of astrophysics equations demonstrated the same level of understanding astrophysics equations across multiple assessments - to various degrees of overlapping. I can therefore conclude that whereas the findings for students with “no understanding” showed that there is no such thing as a student with “no understanding” of the astrophysics equations, the findings for the students with “partial understanding”, “understanding, and “complete understanding” show the opposite. Since these students were classified with the same designation over multiple assignments, I conclude that these students do have an overall consistent understanding of the astrophysics equations used. I showed in section 5.2 that the biggest differences between the characteristics for student understanding of equations as established by Domert et al. for “complete understanding”, “understanding”, and “partial understanding” are: being able to discuss the symbols in the equations and the structure of the equations, understanding the purpose of the equations their connections to the astrophysics concepts behind the equations, and seeing connections between equations and the real world. Out of these three classifications, the students classified with “complete understanding” showed the characteristics for student understanding of equations as established by Domert et al. the most, “understanding” less, and “partial understanding” the least.

Lastly, I found some other indicators of student understanding of equations in addition to the indicators in the theoretical framework established by Domert et al., namely: ability to start a problem without assistance” and making conceptual and/or mathematical connections to the equations. I found these indicators while analyzing the student’s responses to their multiple assessments. Students with “no understanding” could not start problems without assistance and furthermore. After they were given an equation they saw the equation as something to use, but not with any conceptual or mathematical connection to the equation; in other words, they did not recognize the equation within the context of the problem and/or could not mathematically use the equation in an appropriate way to solve the problem. In contrast the students with “complete understanding” could always start the problem without assistance and show a conceptual connection to the equations necessary to solve the problem. Students with “partial understanding” and “understanding” were able to make these connections inconsistently.

6.1.c Research Question #3: How do the student's conceptions of understanding equations relate to their qualitative understanding of astrophysics concepts?

To answer research question #3, I used data from the students’ homework assignments and exams, surveys, interview videos, and essays. Specifically, I focused on the topic of negative gravitational potential energy and the virial theorem to determine how student qualitative understanding of astrophysics equations related to their qualitative understanding of astrophysics concepts.

I found that the students who did not show an understanding of the topic of negative gravitational potential energy and the virial theorem across multiple assessments consistently did not recognize the underlying physics of the equations. These students also did not demonstrate knowledge of how to use the equations to solve the astrophysics problems and moreover, the students that did not show understanding of the topic of negative gravitational potential energy and the virial theorem could not establish a link between the equations and everyday life. These findings are consistent with the theoretical framework described by Domert et al. (2012) agreeing with several of his arguments concerning understanding physics equations. In addition to this theoretical framework, the students who did not show understanding of the topic also could not start the problems without assistance and did not make the mathematical connections to the equations necessary for full understanding; in particular these students did not demonstrate an understanding of the negative signs in the equations. This shows that the students who do not understand the equations for the topic, do not understand the concept and vice versa.

In comparison, I showed that the students who showed an understanding of the topic of negative gravitational potential energy and the virial theorem across multiple assessments demonstrated this understanding consistently. I found that these students show that they recognize the underlying physics of the equations, demonstrate knowledge of how to use the equations to solve the astrophysics problems, and can establish a link between the equations and everyday life; again, consistent with the theoretical framework agreeing with several of Domert et al.'s arguments concerning understanding physics equations. I also found that in addition to Domert's theoretical framework, the students who show understanding of the topic can easily start the problems without assistance and make the mathematical connections to the equations necessary for full understanding; in particular they demonstrate an understanding of negatives in their solutions. Again showing a relationship between understanding astrophysics concepts and equations: the students who understand the equations for the topic also understand the concept and vice versa.

Finally, I determined that of the students who were interviewed, almost all were consistent in their understanding of the concept of negative gravitational potential energy and the virial theorem across multiple assessments. I found that most students who showed "no understanding" of this astrophysics topic, showed "no understanding' of the topic in all of the relevant assessments related to the topic; and likewise for the students who showed "partial understanding", "understanding", and "complete understanding". However, not all students showed this level of consistency across multiple assessments. A few did show one level of understanding in one assessment piece and a different level of understanding in other class assessments. So, although the majority of these students were consistent in their level of understanding of the same equation which would suggest that the student understanding of astrophysics equations is topic driven and not assessment driven as each of the students discussed above showed overall the same level of understanding of a specific equation in all assessments for this particular astrophysics topic, a few students did show a different level of understanding in at least one assessment piece. I find that this finding agrees with the findings of Bao et al. (2002) in that these students respond to the problems in different assessments with different mental models.

6.1.d Research Question #4: What is the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation?

Finally, to answer research question #4, I used data from the student interviews and expert interviews. By comparing the students' responses to the interview problems to the experts' responses, I found several differences between an expert qualitative understanding of an equation and a student qualitative understanding of an equation.

I found that, when the data was explored with respect to Sherin's symbolic forms (Sherin, 2001), although the experts and students both use a similar number of symbolic forms to solve the problems presented in the interviews, the forms used by the experts were more evenly distributed across different categories of symbolic forms than forms used by the students. Furthermore, although less evenly distributed than the experts, the more successful students used a more even distribution of symbolic forms than the less successful students. Due to this difference between the experts' and students' use of forms, I concluded that all of the students are still in the process of learning how to take a conceptual understanding of the situation presented in the problem and express that understanding in the astrophysics equations necessary to solve the problem. Additionally, I found that, when looking at the types of Sherin's forms used, the experts and more successful students use the identity form much less often than the students who were less successful.

I also found that, when evaluating their solutions to the problems presented, the experts consistently made use of unit analysis and special case analysis (Warren, 2006), whereas the students rarely if ever used either of these evaluations strategies. When examining the few students that did use an evaluation strategy, I found that the more successful students used the evaluation strategy to check and confirm their equations or results in a similar manner to the experts, in contrast to the less successful students who used the evaluation strategy only when attempting to correct their incorrectly recalled equations while solving the problems, but were unsuccessful in doing so. These results are consistent with the work of A. Warren (2006), who suggests the use of evaluation strategies for the purpose of checking the problem or equation for inconsistent results; and, upon the recognition of an inconsistent result, an attempt can then be made to fix the mistake. I hypothesize that both the successful and less successful students that used an evaluation strategy were doing so for this purpose.

Lastly, I found that the experts use multiple representations with a much greater frequency than the students; again, with the more successful students using multiple representations more than the less successful students, mimicking the experts. These findings match the findings of Rosengrant and colleagues (Rosengrant et al., 2006) who found dissimilarities between experts and novices when it comes to creating representations in order to help solve problems. While experts display a variety of diagram-related reasoning behaviors, novices show little to no evidence of these abilities (Rosengrant et al., 2007). I found similar results as the experts not only started the problems with a picture or diagram but also referred to the diagram while solving the problem. The students however, mostly had to be encouraged to create a picture or diagram. Even for the students that did use a picture or diagram, they rarely referred to it once they created it.

6.2 Instructional Implications

6.2.a Instructional Implications concerning what the students think it means to understand astrophysics equations.

Since the majority of students were found to believe that derivations of equations are important in astrophysics but only as a way to get the final equations and the more successful students use derivations to better understand the astrophysics concepts underlying the equations, I believe that added emphasis needs be given to the *purpose* of derivations throughout a course. It is not enough to simply demonstrate or show a derivation, the students need to understand how critical a derivation is to the understanding of a particular equation. This could be done in class work or homework assignments: first students derive an equation with the guidance of an instructor or on their own and then they *reflect* on what they learned through the derivation.

The students also believe that conceptual understanding of the equations is important, but they are not always successful in doing so. I therefore suggest that the students need more practice in recognizing the structure of an equation as well as connecting the equation to their understanding of astrophysics. It would be helpful if the instructors openly and clearly address all aspects of understanding (as described in the framework by Domert and colleagues) and include multiple questions on the exams that focus on these aspects. For example, a question such as “Explain why kinetic energy can never be negative, but the potential energy can be positive, negative, or zero” would assess whether the students truly understand the structure of common energy equations as well as understanding their physical meaning.

Lastly, I found that students appear to have a disconnect when using equations to solve problems: the students believe that they need to have a deep understanding of the equations that they use in order to understand astrophysics while simultaneously believing that, when applying the equations to problem solving, they do not need to deeply understand the equations – they can just be taken as a mathematical tool. Students see the need to understand the equations in a conceptually meaningful way but, when in practice, the students merely use the equations as a mathematical platform to solve the problem. Asking students “What does this [equation, derivation, solution, etc.] mean?” or asking students to do limiting case analysis of the equations throughout the problems could potentially help them connect the equations to the concepts.

6.2.b Instructional Implications concerning what student qualitative understanding of an equation looks like.

While investigating what student qualitative understanding of an equation looks like, I found that the students who were found to have “no understanding” of the astrophysics equations used in a particular assessment could have shown partial understanding of the equations in their other assessments. I concluded that there is no such thing as a student with “no understanding” of astrophysics equations used in the classroom. This shows that the use of multiple assessments is a critical tool to assess these students, otherwise a false “no understanding” may be registered for a student who can demonstrate a greater understanding in another type of assessment. It would also be prudent to not weigh one particular assessment too highly for the same reasons. In addition, although a student who does show the same level of understanding over multiple assessments (as found with the students who demonstrated “partial understanding”, “understanding”, and “complete understanding”), the multiple assessments would clearly and consistently show their level of understanding and therefore once that level is known, the students could be guided a higher level of understanding through in class or homework assignments targeted at their current level of understanding.

I also determined that the biggest differences between the characteristics for student understanding of equations as established by Domert et al. (2012) for “complete understanding”, “understanding”, and “partial understanding” are: discussing the symbols in the equations, discussing the structure of the equations, understanding the purpose of the equations showing deep understanding of the astrophysics behind the equations, and taking about connections between equations and the real world. Since the students classified to have “complete understanding” showed these characteristics the most and “partial understanding” the least, the instructional implication is clear. Whenever deriving or demonstrating the use of these equations, the emphasis should be given to developing a greater appreciation of all of the characteristics established by Domert et al. (2012) in the classroom. Perhaps a question on an assessment could include writing an equation out in words, or asking the students to describe what they think the purpose of a particular equation is.

Lastly, I found that students with “no understanding” could not start problems without assistance and furthermore, if given an equation, that they see the equations as something to use, but do not have any conceptual or mathematical connection to the equation while the students with “complete understanding” could always start the problem without assistance and show a conceptual connection to the equations necessary to solve the problem. Students with “partial understanding” and “understanding” were able to make these connections inconsistently. Based on this finding, I would suggest that emphasis should not be given to memorizing equations, and in fact memorization should be discouraged. To avoid this, a formula sheet could be given to the students for use during class work and during assessments. In order to help students develop a conceptual or mathematical connection to the equations, it should be made clear that the objective is not to find the “correct equation” in order to “plug in” the given values, but to use a an equation that conceptually connects to the situation or the process in the problem.

6.2.c Instructional Implications concerning how the student's conceptions of understanding equations relate to their qualitative understanding of astrophysics concepts.

The findings from the analysis of the students’ understanding of the topic of negative gravitational potential energy and the virial theorem across multiple assessments showed how student qualitative understanding of astrophysics equations relates to their qualitative understanding of astrophysics concepts.

Students that did not show an understanding of this topic across multiple assessments consistently did not recognize the underlying physics of the equations, did not demonstrate knowledge of how to use the equations to solve the astrophysics problems, and did not establish a link between the equations and everyday life. In addition, they could not start the problems without assistance and did not make the mathematical connections to the equations necessary for full understanding; in particular they do not demonstrate an understanding of negatives in the equations. In order to help these students, I believe it would be helpful to start from the basics. If the students do not understand negatives, they cannot understand the negative gravitational potential energy. And if they cannot understand negative gravitational potential energy, they cannot understand the virial theorem. The use of work-energy bar charts (Van Heuvelen and Zou, 2001) can be extremely helpful for these students. An important aspect of using this representation is choosing a system and recognizing that only external forces can do work on the system and due to internal forces, the system possesses potential energies. Perhaps the classroom discussion can start with the discussion of the initial and final gravitational potential energy of a system consisting of Earth and an object close to it (initial) and when infinitely far away (final). To bring the object to infinity we can use an imaginary space elevator that will be doing work on the system by slowly pulling the object away from Earth. The elevator does a positive work on the system only to bring the energy of the system to zero. On the bar chart we draw the positive bar for work and zero bar for the final energy. The question is what bar should we draw for the initial energy to satisfy energy conservation? The students should find that the bar should be negative (X + positive number = 0, what is the sign of the X number?). This method was found to be an effective bridge between the phenomenon and the mathematical representation of it. The next step would be to write the equation using the bar chart and then have a discussion of the meaning of the negative sign in the equation. From there a more advanced discussion on the virial theorem can take place.

Since the majority of the students who were interviewed were consistent in their lack of understanding of the concept of negative gravitational potential energy and the virial theorem across multiple assessments, this strongly implies that the students' level of understanding of equations is topic driven and not assessment driven; however, not all students showed this level of consistency across multiple assessments. The occasional student did show a different level of understanding in one assessment piece. This once again indicates that having multiple assessments with different types of questions is beneficial for all students.

6.2.d Instructional Implications concerning the difference between an expert qualitative understanding of an equation and a student qualitative understanding of an equation.

To help a student become more "expert like" with respect to qualitative understanding of an astrophysics equation, three things should be considered: Sherin's symbolic forms, Warren's evaluation strategies, and multiple representations. With respect to Sherin's symbolic forms, I found that while the experts and students both use a similar number of forms, the distribution of the use of those symbolic forms is not equal. The experts apply a more even distribution of the symbolic forms than the students and the more successful students use a more even distribution of symbolic forms than the less successful students revealing that students are still in the process of learning how to take the conceptual understanding of the astrophysical situation presented in the problems and express that understanding in the equations necessary to solve the problem. (Sherin, 2001) Therefore, I believe that students need to be exposed to a wider variety of forms and need to be exposed to the use of forms in the classroom, and/or to be guided into using more forms in their written work. For example, the students could be asked to construct expressions for a particular situation and judge the reasonableness of the expressions they derived as done in the interview problems. Additionally, since the experts and more successful students use the identity form to a much smaller degree than the less successful students, I believe that an effective classroom strategy would include the discouragement of memorization of multiple formulas; especially as a starting point of problem solving.

Since the experts also consistently made use of unit analysis and special case analysis as means of evaluating their solutions to the problems presented while the students seldom used either of these evaluations strategies, it is clear that students should be highly encouraged and taught how to self-evaluate in order to develop an understanding of astrophysics equations. When the students learn to self-evaluate using the evaluations strategies presented in Warren's paper, they become more "expert-like" in their thinking and become able to recognize when problem solving whether their results are consistent or inconsistent with their prior knowledge or experience. One way of doing this is be allowing the students to critique their own or other manufactured work and determine where (if at all) the inconsistencies lie in the solution.

Lastly, I determined that the experts use multiple representations with a much greater frequency than the students. Again, mimicking the experts, the more successful students did use multiple representations more than the less successful students. The experts not only started the problems with a picture or diagram but also referred back to the diagram while solving the problem (the finding consistent with the finding of Rosengrant et al. (2006) whereas the majority of the students, had to be encouraged to create a picture or diagram and of those that did use a picture or diagram, they rarely referred to it further into the problem. I believe this suggests that the use of multiple representations needs to have a greater role as an integral part of understanding astrophysics equations; in order to advance the students to a more expert level of thinking. Demonstrations in the classroom with multiple representations as well as more assignments where multiple representations are required would lead to a more “expert-like” thinking when solving problems. For example, in a paper by Van Heuvelen (1991), he remarks on how students learned to use multiple representations in Newtonian physics with the use of a format sheet that provides space for a pictorial representation (including a list of known information and unknown), physical representation (including object description, motion diagram, and free-body diagram and/or force diagram), math representation and solution, and evaluation (including sign, magnitude, and unit). (Van Heuvelen, 1991) A similar approach may be used for the astrophysics classroom for various topics, including negative gravitational potential energy.

6.3 Concluding Remarks

This work is the first detailed investigation of how upper level undergraduate students in astrophysics connect mathematical equations to concepts; while similar work was done in upper level undergraduate physics classes, there was none in astrophysics. The results indicate that it is possible for the students to have a deep understanding of the equations; however, most students need more purposeful instruction to achieve this level. By including the instructional implications that target the findings in this study, it is my hope that more students can obtain “complete understanding” of the concepts and equations they use in the astrophysics classroom and become more “expert-like” when connecting astrophysics concepts to equations.

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Appendix A
Survey Results

Table 35: Likert Survey Questions with Results

	Strongly Disagree	Disagree	Undecided	Agree	Strongly Agree	No Answer	Total Responses
Q1: In order to understand the ideas presented in this course, I only need to work through the problem sets and/or pay close attention in class.	0	9	3	29	12	0	53
Q2: A derivation or proof of an equation shown in class is useful because I can use the final equation while working on the problem sets without working through the derivation myself.	2	7	6	22	16	0	53
Q3: To solve a problem in astrophysics I need to match the problem	0	5	5	26	16	1	52

situation with the appropriate equations and then mathematically manipulate and/or substitute values to get an answer.							
Q4: I spend a significant amount of time figuring out and understanding at least some of the derivations given in class.	3	20	15	12	3	0	53
Q5: If I forget an equation or cannot find the right one, there is nothing I can do, I must skip that problem.	26	22	2	3	0	0	53
Q6: In astrophysics, I do not need to understand equations in an intuitive sense; they can just be taken as givens.	19	26	6	2	0	0	53
Q7: The best way for me to learn astrophysics is	1	10	15	23	4	0	53

by solving the quantitative problems in the problem sets.							
Q8: After I first read a new problem, I try to visualize the situation and sometimes I draw a sketch before going into mathematics.	1	7	5	24	16	0	53
Q9: My grade in this course is determined by how well I understand the material.	1	4	4	34	10	0	53
Q10: Learning in astrophysics is a matter of developing knowledge that is shown in the equations given in class.	1	6	18	22	6	0	53
Q11: In completing a problem in the problem sets, if my calculations give me a result that differs significantly	8	32	10	3	0	0	53

from what I expect, I would trust the calculation rather than my intuition.							
Q12: The derivations and proofs of equations shown in class have little relevance to actually solving problems or understanding the course material.	13	29	4	5	1	1	52
Q13: In completing a problem in the problem sets, I check the units to be sure that my answer is dimensionally accurate.	0	2	3	19	29	0	53
Q14: The most crucial thing I need to do when solving a problem in astrophysics is to find the right equation to use.	0	12	12	19	10	0	53

Q15: The main skill I get out of this course is learning how to solve problems in astrophysics.	1	10	8	27	7	0	53
Q16: As long as I have a conceptual understanding of a problem in my mind, I do not need to communicate this understanding through my written work.	9	29	8	6	0	1	52
Q17: When I solve the quantitative problems in the problem sets, I think about the concepts that lead to the problem.	0	3	2	36	12	0	53
Q18: I use the mistakes I make on the problem sets as clues to what I need to do to understand the course better.	0	2	3	32	16	0	53
Q19: To be able to use an	0	4	9	26	14	0	53

equation in a problem (particularly in a problem that I haven't seen before), I need to know more than what each term in the equation represents.							
Q20: It is possible to pass this course (get a "C"; or better) without understanding astrophysics very well.	4	22	21	5	1	0	53
Q21: When I solve a problem in the problem sets, I go straight to the lecture notes to find an equation that applies to the situation described in the problem.	1	9	7	30	6	0	53

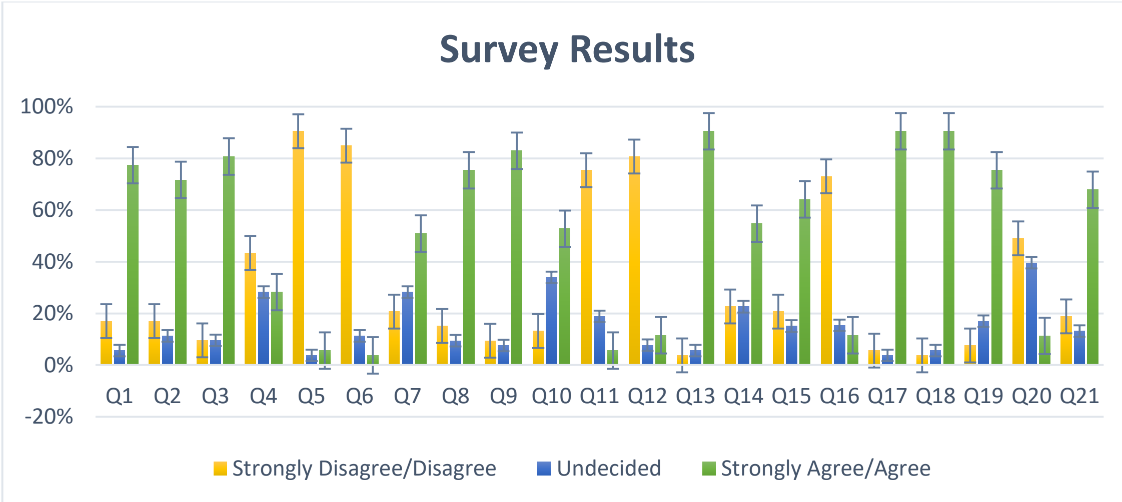


Figure 12: Likert Survey Results

Table 36: Likert Survey Questions with Overall Percentage Results

	Strongly Disagree/ Disagree	Undecided	Strongly Agree/ Agree
Q1: In order to understand the ideas presented in this course, I only need to work through the problem sets and/or pay close attention in class.	17%	6%	77%
Q2: A derivation or proof of an equation shown in class is useful because I can use the final equation while working on the problem sets without working through the derivation myself.	17%	11%	72%
Q3: To solve a problem in astrophysics I need to match the problem situation with the appropriate equations and then mathematically manipulate and/or substitute values to get an answer.	10%	10%	81%
Q4: I spend a significant amount of time figuring out and understanding at least some of the derivations given in class.	43%	28%	28%

Q5: If I forget an equation or cannot find the right one, there is nothing I can do, I must skip that problem.	91%	4%	6%
Q6: In astrophysics, I do not need to understand equations in an intuitive sense; they can just be taken as givens.	85%	11%	4%
Q7: The best way for me to learn astrophysics is by solving the quantitative problems in the problem sets.	21%	28%	51%
Q8: After I first read a new problem, I try to visualize the situation and sometimes I draw a sketch before going into mathematics.	15%	9%	75%
Q9: My grade in this course is determined by how well I understand the material.	9%	8%	83%
Q10: Learning in astrophysics is a matter of developing knowledge that is shown in the equations given in class.	13%	34%	53%
Q11: In completing a problem in the problem sets, if my calculations give me a result that differs significantly from what I expect, I would trust the calculation rather than my intuition.	75%	19%	6%
Q12: The derivations and proofs of equations shown in class have little relevance to actually solving problems or understanding the course material.	81%	8%	12%
Q13: In completing a problem in the problem sets, I check the units to be sure that my answer is dimensionally accurate.	4%	6%	91%
Q14: The most crucial thing I need to do when solving a problem in astrophysics is to find the right equation to use.	23%	23%	55%
Q15: The main skill I get out of this course is learning how to solve problems in astrophysics.	21%	15%	64%
Q16: As long as I have a conceptual understanding of a problem in my mind, I do	73%	15%	12%

not need to communicate this understanding through my written work.			
Q17: When I solve the quantitative problems in the problem sets, I think about the concepts that lead to the problem.	6%	4%	91%
Q18: I use the mistakes I make on the problem sets as clues to what I need to do to understand the course better.	4%	6%	91%
Q19: To be able to use an equation in a problem (particularly in a problem that I haven't seen before), I need to know more than what each term in the equation represents.	8%	17%	75%
Q20: It is possible to pass this course (get a "C"; or better) without understanding astrophysics very well.	49%	40%	11%
Q21: When I solve a problem in the problem sets, I go straight to the lecture notes to find an equation that applies to the situation described in the problem.	19%	13%	68%

Appendix B
Exam Questions

Exam 1

Table 37: Exam 1 - Multiple Choice Questions with Students' Percentage Results

Question 1		
If the force of gravity between two objects was proportional to 1/r3, rather than 1/r2, which of the following would still be true of bound orbits? I. Kepler's 1st law: the orbits would be ellipses II. Kepler's 2nd law: orbits would sweep out equal areas in equal times III. Kepler's 3rd law: the square of the orbital period would be proportional to the cube of the size of the orbit	Answers	Percentage Results
	A. I only	40%
	B. II only	10%
	C. III only	6%
	D. I, II, and III	21%
	E. none would be true	19%
Question 2		
Consider the Sun/Earth system in isolation (i.e., ignore all other objects). Which of the following is true?	Answers	Percentage Results
	A. The magnitude of the gravitational force on the Earth from the Sun is less than the magnitude of the gravitational force on the Sun from the Earth.	0%
	B. The magnitude of the gravitational force on the Earth from the Sun is greater than the magnitude of the gravitational force on the Sun from the Earth.	22%
	C. The magnitude of the gravitational acceleration of the Earth from the Sun is less than the magnitude of the	0%

	gravitational acceleration of the Sun from the Earth	
	D. The magnitude of the gravitational acceleration of the Earth from the Sun is greater than the magnitude of the gravitational acceleration of the Sun from the Earth	61%
	E. the forces and the accelerations are equal	16%
Question 3		
If we choose the gravitational potential energy to be zero when two objects are infinitely far apart, which of the following is true about bound orbits? I. The potential energy is negative II. The kinetic energy is negative III. The total energy is negative	Answers	Percentage Results
	A. I only	33%
	B. II only	6%
	C. I and III	51%
	D. III only	4%
	E. I, II, and III	6%
Question 4		
In which of the following calculations would a Taylor expansion not be appropriate?	Answers	Percentage Results
	A. The gravitational potential energy near the surface of the Earth, to linear order in $h/R_\oplus$, where h is the height.	4%
	B. The gravitational acceleration of the Moon around the Earth, to linear order in $r/R_\oplus$, where r is the distance from the Earth to the Moon.	43%

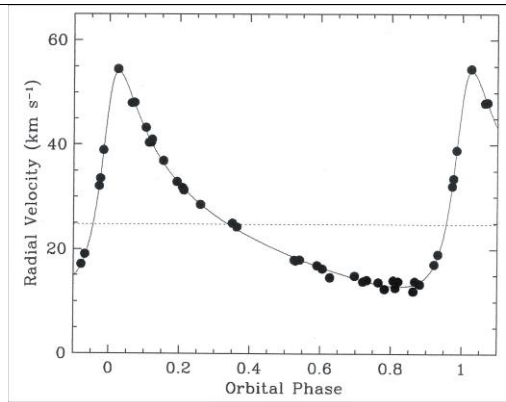
	C. The Doppler shift in the wavelength of light emitted by a slowly moving source, to linear order in the speed v/c.	14%
	D. The specific angular momentum of a nearly circular orbit, to linear order in the eccentricity e.	37%
Question 5		
Consider a free particle moving in a plane with a constant velocity. Let the position of the particle (relative to some fixed origin) be $\hat{r}(t)$, with Cartesian coordinates x(t) and y(t), and polar coordinates r(t) and θ(t). Which of the following is true? I. The Cartesian velocity components v_x and v_y are constant II. The polar velocity components v_r and v_θ are constant III. The Cartesian acceleration components ax and ay are zero IV. The polar acceleration components a_r and a_θ are zero	Answers	Percentage Results
	A. I only	6%
	B. I and III	38%
	C. I, III, and IV	17%
	D. I, II, III, and IV	35%
	E. III and IV	4%
Question 6		
A planet orbits a star with an eccentricity e = 0.5. If the planet is not at apocenter or pericenter, which of the following is true in the reference frame of the star? I. The planet's radial velocity v_r is zero II. The planet's tangential velocity v_θ is zero	Answers	Percentage Results
	A. I only	12%
	B. II only	4%
	C. III only	26%
	D. I and III	10%
	E. IV only	48%

III. The planet's radial acceleration a_r is zero IV. The planet's tangential acceleration a_θ is zero		
Question 7		
At a wavelength of λ = 500 nm the Hubble Space Telescope's resolution is 0.052". The angular diameter of the star α Centauri is 0.009", smaller than HST's resolution limit. What does this mean?	Answers	Percentage Results
	A. HST cannot see any light from α Centauri at this wavelength	12%
	B. HST can observe light from α Centauri, but it cannot take a spectrum	8%
	C. HST can observe light from α Centauri, but it cannot image details on the surface	69%
	D. HST cannot separate light from α Centauri and its binary companion 7" away.	10%
Question 8		
An astronomer observes a galaxy at a wavelength of 4 cm using a 20 m radio telescope, but the observations are inconclusive. She decides she needs at least 3 times better angular resolution. How can she obtain such data?	Answers	Percentage Results
	A. Observe the galaxy at 2 cm with the 20 m telescope	4%
	B. Observe the galaxy at 2 cm with a 40 m telescope	65%
	C. Observe the galaxy at 4 cm with a 50 m telescope	10%
	D. Observe the galaxy at 15 cm with the 20 m telescope	21%
Question 9		

Imagine a planet is orbiting a massive star. If all of the star's mass suddenly collapses down to form a black hole, what will happen to the planet?	Answers	Percentage Results
	A. It will be inexorably sucked into the black hole due to its tremendous gravity	18%
	B. It will orbit the black hole, but at a smaller semi-major axis than it had before	10%
	C. It will orbit the black hole, but at a larger semi-major axis than it had before	10%
	D. It will orbit the black hole at its original semi-major axis as if nothing had happened	61%
Question 10		
If the mass of Sgr A* suddenly doubled, but the orbital semi-major axis and eccentricity of star SO-2 stayed the same, which of the following would occur?	Answers	Percentage Results
	A. The orbital speed of SO-2 at apocenter would increase	62%
	B. The orbital period of SO-2 would increase	8%
	C. The orbital semi-minor axis of SO-2 would decrease	21%
	D. The orbital angular momentum of SO-2 would decrease	6%
	E. none of the above	4%
Question 11		
Why is it surprising that the mass of the central supermassive black hole in a galaxy	Answers	Percentage Results

is correlated to the average random velocity of stars in the galaxy?	A. The stars' velocities should only depend on how far away they are from the black hole	26%
	B. The central black hole does not significantly affect the motion of most of the stars	58%
	C. The black hole should have swallowed up most of the stars	4%
	D. The stars are individually much less massive than the black hole	11%
Question 12		
Which of the following contributes most to the uncertainty in the mass of Sgr A*?	Answers	Percentage Results
	A. The uncertainty in the mass of star SO-2	24%
	B. The uncertainty in the orbital period of star SO-2	16%
	C. The uncertainty in the orbital eccentricity of star SO-2	16%
	D. The uncertainty in the distance to the Galactic Center	45%
Question 13		
Which of these is a true statement about typical visual binaries and typical spectroscopic binaries that we observe?	Answers	Percentage Results
	A. The typical spectroscopic binaries have higher eccentricities	19%
	B. The typical spectroscopic binaries have lower inclinations (closer to face-on)	25%

	C. The typical visual binaries have longer periods	42%
	D. The typical visual binaries have faster orbital speeds	13%
Question 14		
We observe a double-lined spectroscopic binary and measure the radial velocity amplitudes, K_1 and K_2, and the orbital period P but **not** the orbital inclination *i*. Which of the following can we determine? I. The mass ratio, m_1 / m_2 II. The total mass, $M = m_1 + m_2$ III. The individual masses, m_1 and m_2 IV. The reduced mass, $\mu = m_1 m_2 / (m_1 + m_2)$	Answers	Percentage Results
	A. I only	58%
	B. I and IV	6%
	C. II only	2%
	D. I and II	8%
	E. I, II, III, and IV	26%
Question 15		
In a single-lined spectroscopic binary, where we only measure the period P and one radial velocity amplitude K_1, what can we determine about the unseen companion?	Answers	Percentage Results
	A. Its minimum mass	47%
	B. Its maximum mass	18%
	C. Its minimum radius	0%
	D. Its maximum radius	12%
	E. none of the above	22%
Figure 1: Radial velocities of the star HIP 50796, from Torres et al. (2006, AJ, 131, 1022).		



Question 16

Figure 1 shows the observed radial velocities for the single-lined spectroscopic binary HIP 50796. The orbital period is 1.56 yr. Which of the following can we conclude about the system?

I. The unseen companion is likely a planet

II. The center of mass of the system is moving away from us

III. The orbital eccentricity is measurably different than zero

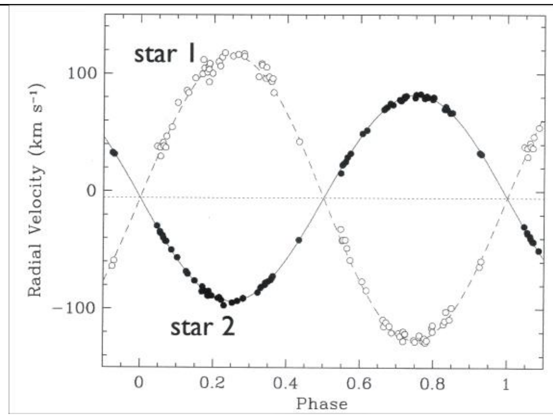
Answers	Percentage Results
A. I only	4%
B. II only	4%
C. III only	8%
D. I and III	16%
E. II and III	67%

Question 17

Based on the data in Figure 1, at which of these orbital phases is the star the closest to its companion?

Answers	Percentage Results
A. 0.0	58%
B. 0.3	26%
C. 0.5	8%
D. the orbits are circular, so the separation is constant	6%

Figure 2: Radial velocities of the double-lined binary system V1061 Cygni, from Torres et al. (2006, ApJ, 640, 1018).

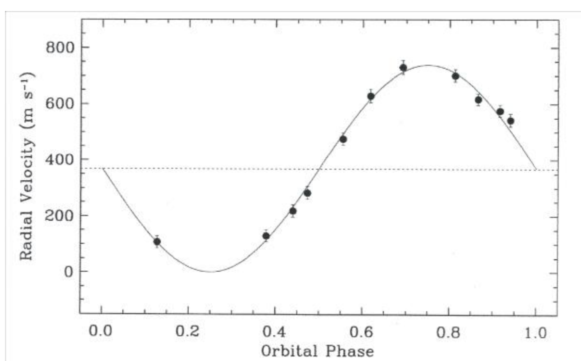


Question 18

The hydrogen Hα line has a rest wavelength λ_{rest} = 656.28 nm. If we see H α from both stars in V1061 Cygni at orbital phase = 0.25 (see Figure 2), at what wavelengths would we observe the lines (λ_1 for star 1, and λ_2 for star 2)? Hint: Think about the possibilities; you won’t need to calculate anything.

Answers	Percentage Results
A. λ_1 = 656.02 nm, λ_2 = 656.06 nm	2%
B. λ_1 = 656.54 nm, λ_2 = 656.50 nm	10%
C. λ_1 = 656.54 nm, λ_2 = 656.06 nm	53%
D. λ_1 = 656.02 nm, λ_2 = 656.50 nm	27%
E. impossible to tell without knowing the inclination angle *i*	6%

Figure 3: Radial velocities of the star TrES-3, from Sozzetti et al. (2009, ApJ, 691, 1145).



Question 19

TrES-3, shown in Figure 3, is a transiting extrasolar planetary system, with an edge-on circular orbit. At what orbital phase does the planet go in front of the star?

Answers	Percentage Results
A. 0.25	12%
B. 0.50	33%

	C. 0.75	10%
	D. 1.00	31%
	E. either 0.50 or 1.00; can't tell which with just the radial velocity curve	15%
Question 20		
Given the radial velocity curve and transit light curve for TrES-3 and estimates for the mass and radius of the parent star, which of the following could we measure? I. The radius of the planet II. The mass of the planet III. The average density of the planet	Answers	Percentage Results
	A. I only	14%
	B. II only	12%
	C. III only	2%
	D. I and II	20%
	E. I, II, and III	51%

Exam 1 – Short Answer Questions

21. Match the following dimensions with the answers to the following problems. Do not solve the problems.

_______ [L][T^{-2}]

A.) Every few hundred years most of the planets line up on the same side of the Sun. Calculate the total force that Venus, Jupiter and Saturn exert on Earth. Compare the force that the Sun's exerts on Earth to the total force exerted on Earth by the planets.

________	$[M][L][T^{-2}]$	B.) The power emitted from each square meter of a hot star with a temperature of 15,000 degrees Kelvin is:
________	$[M][L^2][T^{-2}]$	C.) You are explaining why astronauts feel weightless while orbiting in the international space station. Your friends respond that they thought “gravity” was just a lot weaker up there. What would you do if you wish to convince them (and yourself) that it isn’t so by calculating the acceleration of the satellite 250 km above the earth’s surface?
________	$[M][L^2][T^{-3}]$	D.) An asteroid of mass 1.0×10^5 kg, traveling at a speed of 30 km/s relative to Earth, hits Earth at the equator tangentially, and in the direction of Earth’s rotation. Estimate the change in the angular momentum of Earth as a result of the collision.
________	$[M][L^2][T^{-1}]$	E.) In a massive star supernova explosion, a stellar core about the mass of the Sun and the radius of the Earth collapses to form a neutron star roughly 10 kilometers in radius. Estimate the amount of gravitational potential energy released in a massive star supernova explosion.

22. Equation Jeopardy Question: Create a question for which the following equation gives the solution:

$$P = 2\pi\sqrt{\frac{(10^{11}cm)^3}{G \cdot 318.83M_{\oplus}}}$$

23. A planet is in a circular orbit of radius 3 x 10^{11} meters with an orbital period of 3 years. What can you learn about the star from this information? No calculation is needed.

24. [Do not plug in the actual numbers.] Calculate the kinetic energy of Earth moving around the Sun (assume circular orbits) using the quantities of solar mass, mass of Earth, and the AU. Calculate the gravitational potential energy of Earth - Sun system using the same quantities (with the usual convention of zero potential energy at infinite separation). Which of these energies is larger in magnitude and what is the ratio of these two energies? What is the sign of the total energy? Does this make sense? Why?

25. Below you see five questions that can be answered mathematically.

A: Calculate the ratio of the escape velocities from the Moon and Earth.

B: Determine the mass of the Sun. (The Earth-Sun distance is 1.5 x 10^{11} m.)

C: A planet is discovered orbiting a nearby star once every 125 years. If the star is identical to the Sun and the planet's mass is identical to Mars and its orbit is perfectly circular, how far from the star is the planet?

D: What is the Sun's orbital speed around the center of the galaxy?

E: The Moon's mass is 7.3×10^{22} kg and its radius is 1700 km. What is the speed of a spacecraft moving in a circular orbit just above the lunar surface?

Imagine you answered every question above and wrote the answer in the form of an equation. How relevant would equation (1), written below, be to each of the answers that you wrote?

$$\frac{1}{2} mv^2 + (- GmM/r) = 0 \qquad (1)$$

Circle your choice about how relevant equation (1) is and briefly explain your reasoning.

Answer to A: Very Relevant Somewhat Relevant Less Relevant Even Less Relevant

Answer to B: Very Relevant Somewhat Relevant Less Relevant Even Less Relevant

Answer to C: Very Relevant Somewhat Relevant Less Relevant Even Less Relevant

Answer to D: Very Relevant Somewhat Relevant Less Relevant Even Less Relevant

Answer to E: Very Relevant Somewhat Relevant Less Relevant Even Less Relevant

Exam 2

Exam 2 – Short Answer Questions

1. The oldest stars and star clusters in the Milky Way appear to be 12 billion years old. Would this produce a conflict if the Hubble constant is 100 km/s/Mpc? Explain why.

2. You are studying a faraway elliptical galaxy and have been able to measure its distance, size, and velocity dispersion. Unfortunately, images of this galaxy do not show gravitational lensing of background galaxies or quasars. Is it still possible to measure its mass? If so, list the possible methods you could use to determine its mass and choose the best method. Why did you choose this particular method? Now, write a formula for your mass estimate, and explain any assumptions you choose to make. What physical principle is behind your approach?

3. Equation Jeopardy Question: Create a question for which the following equation gives the solution:

$$R = \frac{2(6.67\ x\ 10^{-8}\ cm^3 g^{-1} s^{-2})(5.97\ x\ 10^{27}\ g)}{(3.00\ x\ 10^{10}\ cm\ s^{-1})^2}$$

4. Jupiter's moon Io has active volcanoes whose energy ultimately comes from the tidal forces exerted on Io by Jupiter.

(a.) If you wanted to estimate the tidal acceleration of a test mass at various points on the surface of the moon Io, which of the following quantities would be most useful?

Mass of Jupiter

Radius of Jupiter

Mass of Io

Radius of Io

Distance from Jupiter to Io

(b.) Imagine the (fictitious) moon Galileo has 2 times the mass of Io, 3 times the radius of Io, and is 4 times farther away from Jupiter than Io. How would the maximum tidal acceleration (caused by Jupiter) of a test mass on the surface of Galileo compare with the maximum tidal acceleration of a test mass on Io?

5. Decide whether the following statements are sensible or not. Explain your decision.

(a.) If a 1 M_moon black hole were orbiting Earth with an orbital radius inside our own Moon's orbit, we'd have no way of knowing it was there.

(b.) If the Sun suddenly became a 1 M_Sun black hole, the orbits of the planets would not change.

6. Under what circumstances would we consider the gravitational potential energy negative? Are there any circumstances in which we would consider the gravitational potential energy to be positive? Give examples.

7. State which, if any, of the following observers have an inertial reference frame, and explain your reasoning: [Standing at the Earth's North pole; sky-diving above the Earth's equator; floating on the International Space Station; moving on a rocket ship at constant speed far beyond the Sun's gravitational reach].

Now state which, if any, of those observers have a reference frame in which the laws of physics behave equivalently to a non-accelerating, non-rotating frame in the absence of a gravitational field, and explain your reasoning.

8. Sketch a rotation curve (speed vs. location graph) for a merry-go-round, the solar system, and the Milky Way. How is the rotation curve for the merry-go-round different from the shape of the rotation curve for the solar system? For the Milky Way? Suggest possible reasons for the differences.

9. Two students attempt to derive an equation for escape velocity from first principles as shown.

Student 1:

$$F = GMm/r^2 = ma$$

$$a = GM/r^2$$

$$but, a = v^2/r$$

$$so: v^2/r = GM/r^2$$

$$and\ v = sqrt(GM/r)$$

Student 2:

$$E = ½\, mv^2 - GMm/r = 0$$

$$½\, mv^2 = GMm/r$$

$$½\, v^2 = GM/r$$

$$v^2 = 2GM/r$$

$$v = sqrt(2GM/r)$$

What is the difference between the two answers? Which student is right and where is the mistake of the student who is wrong?

10. Light that goes through a telescope lens is bent, and the amount of bending is different for different wavelengths of light. Does the bending of light by a gravitational lens depend on the wavelength of the light? Explain why or why not.

Appendix C

Homework Assignments

Homework Assignment #1

Question #1:

(a) Use dimensional analysis to derive a relationship between the total mass of a gravitationally bound system M, its typical size R, and the typical speed v of its components.

(b) Using your formula, at what speed does the Earth orbit the Sun?

(c) Now estimate the mass of a galaxy like the Milky Way, based on the fact that stars at a radius of 10 kpc travel at 200 km s^{-1}. Galaxy masses should be reported in units of solar masses, $M_\odot$.

(d) A cluster of galaxies has a mass of 10^{14} $M_\odot$, and the galaxies typically move around at 600 km s^{-1}. What is the size of the cluster, in Mpc?

Question #2:

One of the most important objects in astronomy are exploding stars called Type Ia supernovae. You can estimate the energy involved in these explosions.

(a) First, use dimensional analysis to estimate the gravitational binding energy of an object with mass M and radius R.

(b) Type Ia supernovae are the explosions of white dwarf stars which have a mass of 1.4 $M_\odot$ but a radius of just 5000 km - that's a little less than the size of the Earth! What is the gravitational binding energy (in ergs) of these white dwarfs?

(c) These supernova explosions are powered by nuclear fusion. How much mass is converted to energy ($E = mc^2$) in order to overcome the binding energy and explode the star?

Question #3:

Your friend sends you a letter deriving an important equation that describes the expansion of the universe:

$$\left(\frac{dR}{dt}\right)^2 - \frac{8\pi}{3}GR^2\alpha = -kc^2$$

She tells you that R is the relative size, t is time, G is Newton's gravitational constant, and c is the speed of light, but forgets to tell you what α and k are.

(a) What are the dimensions of α?

(b) What are the dimensions of k?

(c) Can you infer what quantity they each represent?

Homework Assignment #2

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise.

(a) Suppose that the Sun were instantaneously replaced by a star with twice as much mass. Would Earth's orbit stay the same? Explain your answer. Now suppose that Earth doubled in mass instantly but the Sun remained the same. Would Earth's orbit stay the same? Explain your answer.

(b) How do Kepler's laws contradict the idea that all planets are in uniform circular motion around the Sun?

(c) Equation Jeopardy: Create a question for which the following equation gives the answer:

$$\sqrt{\frac{2 * 6.67x10^{-8}cm^3g^{-1}s^{-2} * 1.99x10^{33}g}{6.96x10^{10}cm}}$$

Question #2:

We showed that for circular orbits, Kepler's Third Law can be written as

$$P^2 = \frac{4\pi^2}{GM} r^3$$

where M is the mass of the central object.

(a) Use this expression and a Taylor expansion to derive the following approximation for the orbital period of a satellite in \low Earth orbit", with a constant height h above the surface of the Earth, assuming that $h \ll R_\oplus$:

$$P \approx P_0 \left(1 + \frac{3h}{2R_\oplus}\right)$$

(b) What is the constant P_0 (in symbols), and what is its value (in minutes)?

(c) The Hubble Space Telescope orbits the Earth at an altitude of h = 600 km. Compare its exact orbital period based on Kepler's Law with your approximation from part (a). Is the approximation as accurate as you would have expected? Why or why not?

Question #3:

You are on a space mission to land on various asteroids in the solar system. After many long years in cramped quarters and some \misunderstandings" with your crewmates, you've developed a nagging suspicion that they are planning to leave you behind on the next asteroid. You've decided that you're only going out there again if you can jump o the asteroid under your own power to get back to your spaceship before it can leave without you. Can you do it? Let's find out …

(a) Estimate the velocity you achieve when you jump straight up. Hint: Use the height you reach jumping on Earth to estimate the change in your potential energy, and then use conservation of energy to estimate your initial kinetic energy.

(b) Now estimate the radius of the largest asteroid you could escape from by jumping. (You will need to make an assumption about the mean density of asteroids, which are made of rock and ice; be sure to explain your reasoning.)

Homework Assignment #3

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise.

(d) Explain why you might describe the orbital motion of the moon with the statement, "the moon is falling."

(e) How does the gravitational force that one object exerts on another object change if the distance between them triples? If the distance between them drops by half? Explain how you know.

(f) Equation Jeopardy: Create a question for which the following equation provides an answer:

$$0.1'' = 1.22\left(\frac{400nm}{1m}\right)\left(\frac{360\ x\ 60\ x\ 60''}{2\pi}\right)$$

Question #2:

For this problem you can use the orbit-ps03 Excel spreadsheet, or you can write your own program. You can download the spreadsheet from our Sakai Resources page. To use the spreadsheet, you will need to edit some of the fields that are shaded yellow.

(a) For this problem we will use natural units for the solar system, measuring lengths in AU and time in years. Use Newton's precise version of Kepler's Third Law to show that GM = $4\pi^2$ AU^3 yr^{-2}. This value is fixed in the spreadsheet (cell B3).

(b) We will compute the orbit of Eris, the infamous "tenth planet" that caused Pluto to be demoted to "dwarf planet" status. Eris (also called 2003 UB_{313} in the textbook) has a semimajor axis a = 68:048 AU and eccentricity e = 0.4336 based on the best current observations. Calculate the orbital period P of Eris (in years).

(c) We can choose the initial conditions to have time t = 0 and angle θ = 0, with coordinates centered on the Sun. Let's start at aphelion (furthest distance from the Sun), so that $v_r = 0$ at t = 0. What are the remaining initial conditions (for *l* and r_0) that you need to reproduce the orbit? Hint: Be sure not to start at perihelion, which is when Eris is closest to the Sun.

(d) Plug these initial conditions into the spreadsheet (or your own program) and plot Eris's orbit. You will need to adjust the time step Δt, to cover one full period. Attach your orbital plot (but not the whole spreadsheet, please!) to your homework submission.

(e) Write down formulas for the perihelion distance, aphelion distance, and semiminor axis in terms of just a and e. Calculate these quantities for Eris and compare them to the values in your spreadsheet orbit. How well do they agree?

(f) The spreadsheet has four additional columns for the tangential velocity (v_θ), the specific kinetic energy (KE/m = $|v|^2/2$), the specific potential energy (PE/m), and the specific total energy (E_{total}/m). Write down formulas for these quantities in terms of quantities you have computed already (GM, *l*, r, v_r, or ω). (Because there are relations between these quantities (for example, $l = r^2\omega$), there is more than one right answer. Use any correct formula that is convenient.) Hint: $|v|^2 = v_r^2 + v^2$.

(g) Fill in the empty columns in the spreadsheet with the appropriate formulas (or compute these quantities in your own program). Make one plot showing the specific potential, specific kinetic, and specific total energy as a function of time. Attach this plot to your homework submission; be sure to label the curves and axes correctly (including units). Verify that the specific total energy is conserved (to within some small numerical error). *Hint: Check the existing spreadsheet, rows 3 and 4, for examples of how formulas work in Excel. Then fill in the formulas that you calculated above in the 3rd row. Highlight those four cells, use Ctrl-C to copy them, then highlight the full set of columns underneath, and then use Ctrl-V to copy those formulas down. Excel automatically permutes row numbers as you do so, e.g., G3 becomes G# in row #. To keep a cell value fixed like B3, use B3 in your formula. You may also find it helpful to copy the time column over to become a new column N. To start a new plot in Excel, try going to Insert Chart and choose XY(Scatter) and Smooth Lines.*

(h) Prove that the specific total energy of the orbit is given by

$$\frac{E_{total}}{m} = -\frac{GM}{2a}$$

Verify that your computed orbit matches this value. Hint: Because the total energy is conserved, to derive the formula you can pick any point on the orbit. Try looking at pericenter or apocenter, and take the sum of the specific kinetic and potential energy at one of those points.

Question #3:

Recent observations of the stars orbiting the black hole at the Galactic Center (Sgr A*) have improved the measurements. Here are the latest results from Gillessen et al. (2009) for star S0-2: period P = 15.8 yr, semimajor axis a = 1025 AU, and eccentricity e = 0:880; and for star S0-16: P = 47.3 yr, a = 2130 AU, and e = 0.963.

(a) Compute the mass (in units of solar masses) of Sgr A* implied by the new results. Do the two stars give a consistent answer?

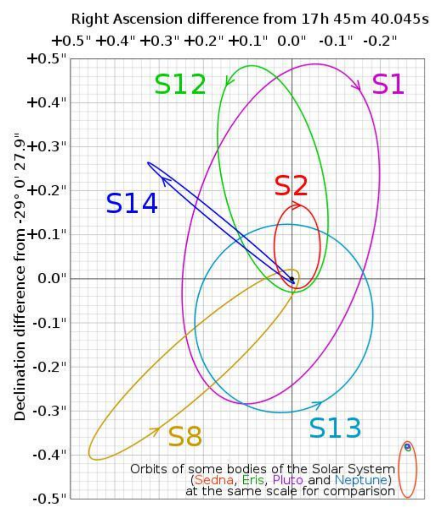
(b) Compute the pericenter and apocenter distances (in AU) for S0-2 and S0-16.

(c) Compute the speeds (in km s^{-1}) of S0-2 and S0-16 at pericenter and apocenter.

Homework Assignment #4

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise.



(a) The following image from Wikipedia shows the "inferred orbits of 6 stars around supermassive black hole candidate Sagittarius A* at the Milky Way galactic centre". Are the orbits of these stars stable? Explain your reasoning.

(b) How would you describe the supporting evidence for supermassive black holes at the center of galaxies to a non-science major?

(c) In a later question, you will be asked to calculate the fractional uncertainty for various astronomical quantities. Describe how you could determine if your answers are reasonable, and give an example from a more familiar measurement you've made yourself.

Question #2:

I mentioned that quasars and other active galactic nuclei are thought to be powered by supermassive black holes (SMBH). Let's consider one aspect of this idea.

(a) A typical quasar luminosity is about 10^{12} $L_\odot$, where $L_\odot = 3.83 \times 10^{33}$ erg s^{-1} is the luminosity of the Sun. If the energy is released by mass falling into a SMBH, estimate the mass accretion rate in solar masses per year.

(b) If the mass accretion rate is roughly constant, how long would it take to build a mass of 10^9 $M_\odot$? Is that long or short compared with the age of the Universe (about 14 Gyr)? Comment on whether the idea that quasars are powered by accretion onto SMBH makes sense or not.

Question #3:

We showed in class (see Lecture Notes 8) that we can derive the total mass from a visual binary orbit,

$$M = m_1 + m_2 = \frac{4\pi^2 d^3 \bar{\alpha}^3}{GP^2}$$

where M is the total mass, α is the angular semi-major axis in radians ($\alpha = \alpha_1 + \alpha_2$), P is the orbital period, and d is the distance to the system.

Often for visual orbits of stars, we know α and P very precisely, but the distance d is much more uncertain. Let's call the fractional uncertainty on the distance f_d, meaning that we think the true distance is within $d = d_0 \pm f_d d_0$ (where d_0 is our best guess and $f_d \ll 1$). For example, if d = 100 ± 3 pc, the fractional uncertainty would be $f_d = 3/100 = 0.03$ (or in other words, a 3% uncertainty).

(a) If f_d is the fractional uncertainty on the distance, what is the fractional uncertainty on the total mass f_M? Hint: All you need to know is $M \alpha d^3$ and then use our favorite Taylor expansion approximation that $(1 \pm x)^\alpha \approx 1 \pm \alpha x$ when $x \ll 1$. The fractional uncertainty in the mass f_M will be related to f_d in a simple way.

(b) Our best estimate for the distance to the center of the Galaxy is d = 8.1± 0.5 kpc. What is the resulting fractional uncertainty in the distance, f_d? Using your result from part (a), what is the fractional uncertainty f_M in our mass estimate for Sgr A*?

(c) Sirius is a visual binary with an orbital period P = 49.94 yr. Sirius A (the bright star) has an angular semimajor axis of $\alpha_A = 2.419''$, while Sirius B (the fainter star) has $\alpha_B = 5.191''$. The distance to the Sirius system is 2.64 ± 0.01 pc. What are the masses of Sirius

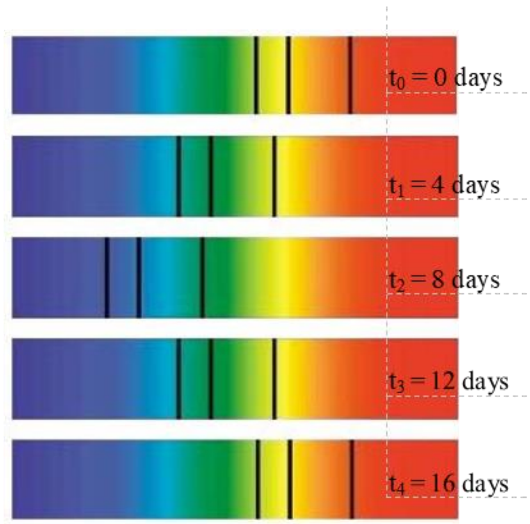
A and B along with their uncertainties? Hint: First calculate your best estimate of the masses and then propagate the fractional uncertainties to report masses as e.g., $m_A = 10.0 \pm 0.1\ M_\odot$.

Homework Assignment #5

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise; unnecessarily long-winded answers will lose credit.

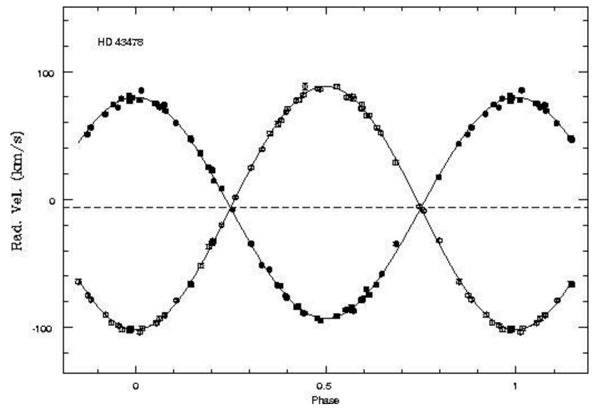
(a) The following diagram shows the spectral lines at different times for what appears to be a single star in the sky. What can we conclude based on these spectra? Explain.



(b) Rank the following binary star systems in decreasing order of orbital period. Hint: this does not require a full calculation; think about proportionalities!

System A	Masses: 5, 10 $M_\odot$	Semi-major axis: 1 A.U.
System B	Masses: 0.5, 1 $M_\odot$	Semi-major axis: 1 A.U.
System C	Masses: 5, 10 $M_\odot$	Semi-major axis: 2 A.U.
System D	Masses: 0.5, 10 $M_\odot$	Semi-major axis: 2 A.U.

(c) The following image shows the radial velocities of a binary star system. What properties of the system can we determine based on the diagram?



Question #2:

(Based on Carroll & Ostlie problem 7.6, but some of the numbers are different.)

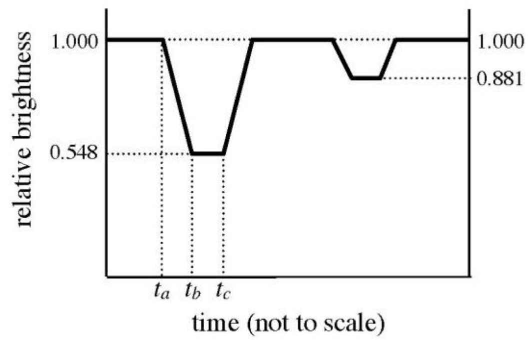
From the light and velocity curves of an eclipsing, double-lined spectroscopic binary star system, it is determined that the orbital period is 3.15 yr, and the maximum radial velocities of stars A and B are 5.2 km s^{-1} and 21.6 km s^{-1}, respectively. Furthermore, the time between first contact and minimum light is $t_b - t_a = 0.45$ days, while the length of the primary minimum is $t_c - t_b = 0.52$ days. Relative to the maximum brightness, the primary minimum is only 54.8% as bright, while the secondary minimum is 88.1% as bright (see schematic figure below).

You may assume the orbits are circular and seen perfectly edge on.

(a) Find the ratio of the stellar masses (m_A/m_B), the sum of the masses ($M = m_A + m_B$), and the individual masses (m_A and m_B).

(b) Find the radii of the two stars. Hint: Use the speed of one star relative to the other and the eclipse timings given.

(c) During the primary (deeper) eclipse, is the larger star in front of the smaller star or vice versa? Is the larger star brighter or fainter than the smaller star? (Think about the brightnesses of the two minima relative to the maximum.)



Question #3:

The Kepler space mission was launched by NASA in March 2009 to look for transiting Earth-mass planets.

(a) The geometric probability of having a system oriented just right so that we see a transiting planet is $p \approx R_*/a$, where R_* is the radius of the star and a is the orbital separation. Kepler is observing 100,000 stars; if all of them were just like the Sun, with an Earth orbiting at 1 AU, how many would show transits? (Assume that the orbits are all randomly oriented, so you can just use the probability as given.)

(b) Imagine that Kepler discovers the planet "New Earth" orbiting the star "New Sol" - transits in a nearly exact analogue of the Sun/Earth system ($M_* = 1\ M_\odot$, $R_* = 1\ R_\odot$, $M_{planet} = 1\ M_\oplus$, $R_{planet} = 1\ R_\oplus$, $e = 0$, $a = 1$ AU). What fraction of New Sol's light is blocked during a transit by New Earth?

(c) How much time passes between subsequent transits of New Sol by New Earth?

(d) How long does each transit last? Assume that the transit is central, i.e. the projected path of the planet goes right over the center of the star ($i = 90°$ exactly).

(e) The reactionary group Just One Earth isn't happy about the discovery of New Earth. They argue that "New Earth" is not a planet, but a white dwarf star instead. White dwarfs can also have a radius like the Earth, $R_{WD} = R_\oplus$, and so their transit light curve would look very similar to a planet, but the typical mass of a white dwarf is much higher, $M_{WD} = 0.6$

$M_{\odot}$. Calculate New Sol's radial velocity amplitude in the two cases when the companion is (1) a planet or (2) a white dwarf (keep the orbital period the same).

(f) Assuming we can make radial velocity measurements of New Sol with the best precision we have today (approaching the level of 40 cm s^{-1}), will we be able to tell if New Earth is a white dwarf or not?

Homework Assignment #6

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise; unnecessarily long-winded answers will lose credit.

(a) State whether the following statements make sense or not, and explain why.

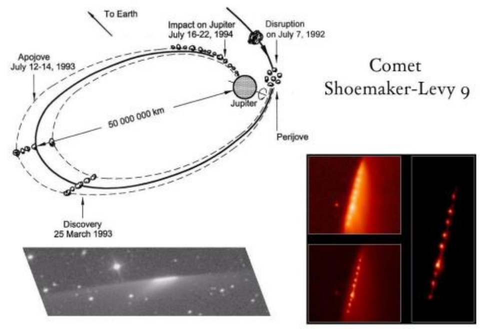
i. It's the year 2025, and scientists have just learned that there is a 10 solar-mass black hole lurking near Pluto's orbit.

ii. The merger of two black holes forms a black hole with a larger Schwarzschild radius than either of the original black holes.

iii. If the moon suddenly collapsed into a black hole without changing in mass, the tides on Earth's oceans would become significantly larger.

(b) Rank the following black holes based on the magnitude of the tidal forces that they would exert on a spaceship placed near their event horizon. A has mass 10 $M_{\odot}$; B has mass 100 $M_{\odot}$; C has mass 10^6 $M_{\odot}$.

(c) The first direct observation of an extraterrestrial collision of Solar System objects was made in July 1994 when comet Shoemaker-Levy 9 (formally designated D/1993 F2) broke apart and collided with Jupiter. Explain what happened to this comet based on the diagram and corresponding images below.



Question #2:

Mars has a mass of 6.4 x 10^{26} g (about one tenth $M_\oplus$) and a radius of 3400 km (about half $R_\oplus$). Its small moon Phobos has a mass of 1.1 x 10^{19} g and a radius of just 11 km. Phobos orbits Mars with a semimajor axis of 9380 km.

(a) What are the mean densities of Mars and Phobos, in g cm^{-3}?

(b) What is the Roche limit for the Mars/Phobos system? Is Phobos inside it?

(c) Use Kepler's Third Law to calculate the orbital period of Phobos, in hours.

(d) Recall in class we said that tidal forces are causing the Moon's orbit to recede from the Earth. Because Phobos orbits Mars faster than the rotation period of Mars, unlike the Moon, tidal forces cause Phobos' orbit to shrink. The semimajor axis is decreasing at a rate of 20 cm yr^{-1}. At that rate, how long is it until Phobos hits the surface of Mars?

(e) However, what is likely to happen to Phobos before then? Think about your answer to part (b).

Question #3:

You may have heard that a person falling feet-first into a black hole would be stretched out by the tidal force, in a process affectionately called “spaghetti cation.” But would the effect actually be dramatic? Let's consider:

(a) What is the tidal force on a person of height h at the event horizon (Schwarzschild radius, $R_s = 2GM/c^2$) of a black hole with mass M? (Use Newtonian gravity.)

(b) It seems reasonable to say that a person would “feel” the stretching only if the tidal acceleration exceeds 1 g (= 980 cm s^{-2}). Find the black hole mass that would produce such a tidal acceleration at the event horizon (you may assume an average human height of 1.8m).

(c) Use your results from (a) and (b) to say whether you would be spaghetti ed by the black hole at the center of the Milky Way ($M = 3.5 \times 10^6\ M_{\odot}$).

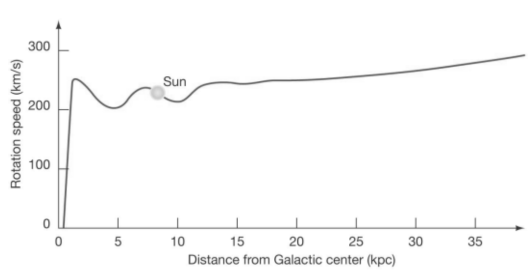
(d) What about by the black hole in the binary system M33-X7 ($M \approx 16\ M_{\odot}$)?

Homework Assignment #7

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise; unnecessarily long-winded answers will lose credit.

(a) Create a rotation curve for the planets in our solar system as an addition to the figure below. Compare the shape of the rotation curve of the Solar System with that of the Milky Way. Describe and explain the major features of each graph. Hint: re-label the x-axis in A.U. of the planets’ semi-major axes for your added curve; then you will not need to change the numerical labels.



(b) Astrophysicists have found strong evidence that dark matter exists and that it makes up about 35% of the universe (compared to 5% which is made of visible matter i.e., atoms). How would you explain the evidence of dark matter to a non-science major?

Question #2:

Recall that the surface brightness of an exponential disk has the form

$$I(R) = I_0 e^{-R/h_R}$$

where I_0 is the central surface brightness and h_R is the disk scale length. The total brightness is given by integrating this profile from R = 0 to R = ∞:

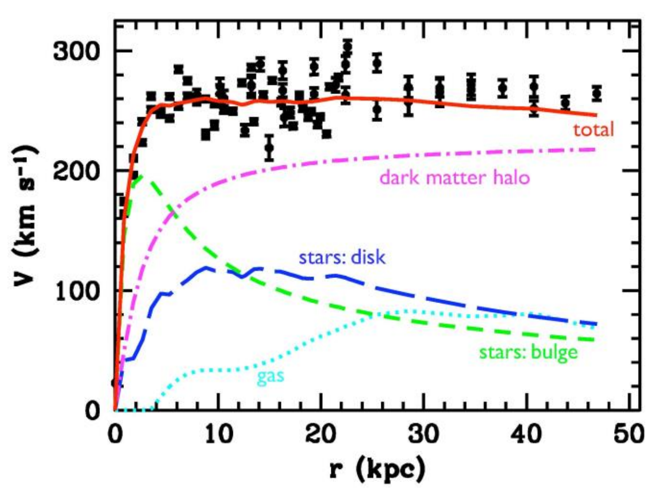
$$I_{total} = \int_0^\infty I(R) 2\pi R \, dR$$

(a) Show that the total brightness of the exponential disk is

$$I_{total} = 2\pi I_0 h_R^2$$

Show your work! Hint: Let x = R/hR and rewrite the integral in terms of x and dx. Note that from integration by parts, $\int xe^{-x}\, dx = -(x+1)e^{-x}$ plus a constant.

(b) What fraction of the total light is within one disk scale length ($R \leq h_R$)? What fraction of the total light is within three disk scale lengths ($R \leq 3h_R$)?



Question #3:

The figure above shows the rotation curve data and model for UGC 5166 that we saw in class. The different components are as follows: short-dash (green) = bulge, long-dash (dark blue) = stellar disk, dotted (cyan) = gas, dot-dash (magenta) = dark matter, solid (red) = total. Total represents the sum of the masses of the various components, and the masses are not directly propotional to the velocities (see Lecture Notes 14). This is why the velocities of the various model components sum to more than the total velocity. The velocity plotted for each component is what would be generated by the mass of that component alone.

(a) What is the mass of dark matter within 10 kpc?

(b) What fraction of the total mass within 45 kpc is dark matter?

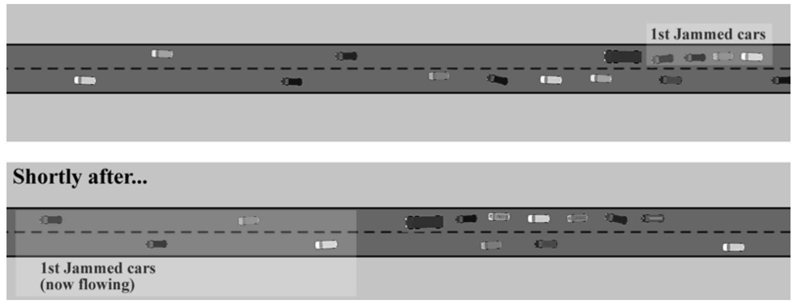
(c) What is the total mass of the bulge? Hint: check several values of the radius to make sure you've gotten all the enclosed bulge mass.

Homework Assignment #8

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise; unnecessarily long-winded answers will lose credit.

(a) The density wave model is one of the more successful models developed to explain the formation of spiral structure in the arms of spiral galaxies. For an astronomy novice, a density wave in a spiral galaxy can be visualized as a traffic jam behind a slow-moving truck (see diagram below). The density moves with the truck over time. What are the strengths and weakness of the "moving truck" analogy?



(b) State whether the following statements make sense or not, and explain why.

i. We did not understand the true size and shape of our galaxy until NASA launched satellites into the galactic halo, enabling us to see what the Milky Way looks like from the outside.

ii. The spiral arms in spiral galaxies rotate about the center of the galaxy like giant pinwheels.

iii. The rotational velocity of most disk stars is so great that they complete several orbits about the galaxy during their lifetime.

iv. Stellar collisions happen frequently when a star passes through a density wave.

Question #2:

The vertical motion of stars in spiral galaxies depends on the gravity exerted by the disk, so it allows us to “weigh” the disk.

(a) Use dimensional analysis to derive an estimate of the mass density ρ of a spiral galaxy disk, in terms of its scale height h_z, its vertical velocity dispersion σ_z, and a relevant physical constant.

(b) In the neighborhood of the Sun, the Milky Way has $h_z \approx 350$ pc and $\sigma_z \approx 16$ km s^{-1} for the thin disk, and $h_z \approx 1$ kpc and $\sigma_z \approx 35$ km s^{-1} for the thick disk. Use these values and your result from part (a) to estimate the mass density of the Milky Way’s disk, in $M_\odot$ pc^{-3}. Do the thin and thick disks give a consistent density estimate to the level of precision we might expect from dimensional analysis?

Question #3:

In class, we showed that the vertical equation of motion for a uniform density disk is

$$\frac{d^2z}{dt^2} + \upsilon^2 z = 0$$

which corresponds to a simple harmonic oscillator, with an angular frequency $\upsilon = \sqrt{4\pi G\rho}$. The solution of this differential equation can be written as

$$z(t) = Asin(\upsilon t)$$

where z(t) is the vertical position of a test particle, A is the amplitude of its motion, and t = 0 is the time when the particle is at the midplane (z = 0).

(a) Given z(t) as above, write out the expression for the vertical velocity, $v_z(t)$.

(b) Let's assume the Milky Way disk in the Solar neighborhood has a constant density, $\rho = 0.15\ M_\odot\ pc^{-3}$. In that case, what is the vertical oscillation frequency in the Solar neighborhood? What is the oscillation period?

(c) The Sun is presently about 20 pc out of the midplane of the disk, and moving away from the midplane at about 7.4 km s^{-1}. Using these as the current position, $z(t_{now})$, and velocity, $v_z(t_{now})$, what is the maximum height z_{max} (in pc) the Sun will reach before turning around? Hint: You might find it useful to apply the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

(d) Some people have suggested that mass extinction events of life on Earth are connected to the Solar system passing through the midplane of the Galactic disk. When did the Sun last cross the midplane? How many years do we have before the Sun passes through the midplane again?

(e) For this problem we used the uniform density disk approximation, but we know the thin disk of the Milky Way has an exponential disk profile with h_z = 350 pc. For the maximum height the Sun reaches (that you calculated in part c), what is z_{max}/h_z? What is

$\rho(z_{max})/\rho(z = 0)$ in the exponential disk model? Comment on whether the uniform density disk is a good approximation for the Sun's motion.

Homework Assignment #9

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise; unnecessarily long-winded answers will lose credit.

(a) Equation Jeopardy: Create a question for which the following equation gives the solution:

$$U = -\frac{3(6.67\ x\ 10^{-8}\ cm^3 g^{-1} s^{-2})\left(\ 0.055\ M_{\oplus}\right)^2}{5(0.382\ R_{\oplus})}$$

(b) For planetary orbits, we can take the Sun to be fixed at position zero, and a planet to be a distance *r* away from it. Create a bar graph of the potential and kinetic energies of a planet in an elliptical orbit at its perihelion and its aphelion. Hint: no calculations are necessary. Be careful with the signs of each bar and their sum.

(c) In an introductory astronomy class, the students learn that collisions between galaxies are relatively common, while collisions between stars are extremely rare. What arguments and evidence support this statement?

Question #2:

The Plummer model for a spherical star cluster is given by the density profile

$$\rho(r) = \frac{3M}{4\pi}\frac{a^2}{(r^2 + a^2)^{5/2}}$$

where M is the total mass, and a is a core radius.

(a) Show that the enclosed mass in the Plummer model is

$$M(r) = \frac{Mr^3}{(r^2 + a^2)^{3/2}}$$

You may need to make an appropriate substitution and use the fact that

$$\int \frac{x^2 dx}{(x^2 + 1)^{5/2}} = \frac{x^3}{3(x^2 + 1)^{3/2}} + \text{constant}$$

(b) Now calculate the total potential energy of the Plummer mass distribution. Hint: The answer should depend only on G, M, and a. After a substitution, you should find the following integral useful: $\int_0^\infty x^4 dx/(x^2 + 1)^4 = \pi/32$.

(c) If the mass distribution is in equilibrium, what is the total kinetic energy? What is the total energy? Again, your answers should depend only on G, M, and a.

(d) The globular cluster ω Centauri can be described by a Plummer model with a total mass $M = 5 \times 10^6\ M_\odot$ and core radius a = 4.5 pc. Use your derived total kinetic energy K from part (c) along with the fact that $K = 3/2M\sigma^2$ (assuming isotropic orbits of identical stars) to estimate the cluster's radial velocity dispersion σ in units of km s^{-1}.

Question #3:

Some time in the future, the Milky Way and Andromeda galaxies will collide.

(a) For a finite isothermal sphere with a radius R and circular velocity v (remember, the rotation curve is constant), we derived the total mass and potential energy as

$$M = \frac{v^2 R}{G} \qquad U = -\frac{GM^2}{R}$$

Express the potential energy U in terms of M and v. Use the virial theorem to find the kinetic energy K and total energy E (again in terms of M and v).

(b) Suppose you start with two identical finite isothermal spheres, each with initial mass M_i and initial circular velocity v_i, that are at rest a distance d apart. What is the total energy of this system? Hint: Consider the total energy for each one in isolation from part (a), and then the potential energy between the two.

(c) Now imagine the two spheres fall toward each other and merge, and that after some time, they equilibrate and end up as a single isothermal sphere. Use conservation of mass, energy, and the virial theorem, to derive the following (and explain your results in words):

i. the final mass M_f (in terms of the initial mass M_i),

ii. the final circular velocity v_f (in terms of v_i, R_i, and d), and

iii. the final radius R_f (in terms of R_i and d).

(d) Now apply your results to a system like the Milky Way and Andromeda — consider two isothermal spheres with circular velocities of 250 km s^{-1} and radii of 150 kpc, which fall from rest at an initial separation of 780 kpc. What are the mass (in $M_\odot$), circular velocity (in km s^{-1}), and radius (in kpc) of the final galaxy?

Homework Assignment #10

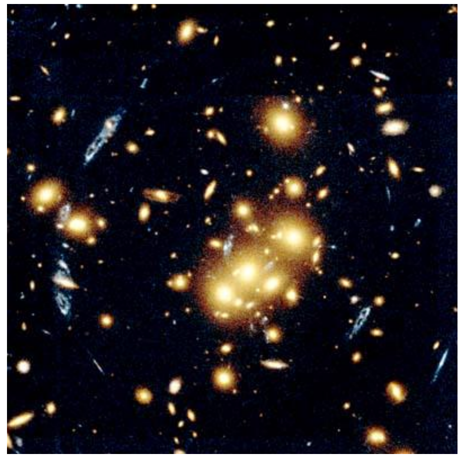
Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise; unnecessarily long-winded answers will lose credit.

(a) Equation Jeopardy: Create a question for which the following equation gives the solution:

$$\hat{\alpha} = \frac{4(6.67\ x\ 10^{-8}\ dyne\ cm^2\ g^2)(318\ M_\oplus)}{(3x10^{10}\ cm\ s^{-1})^2(11.0\ R_\oplus)} = 0.0166''$$

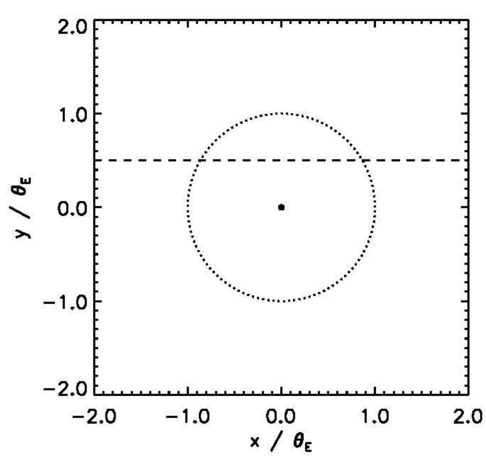
(b) The following image was released to commemorate the sixth anniversary of the Hubble Space Telescope. It shows several blue, loop-shaped objects that are actually multiple images of the same background galaxy. These images surround a cluster of yellow elliptical and spiral galaxies near the image's center. Assuming that the distances from Earth to the cluster and background galaxy are known, what properties of the cluster and the background galaxy can we determine based on the image?



(c) What evidence suggests that MACHOs comprise at most a small fraction of the dark matter?

Question #2:

In this problem you will calculate a microlensing light curve. In the figure below, the dashed straight line represents the trajectory of a point source passing behind a point mass lens (the solid dot in the center). The circle indicates the Einstein radius. The x and y axes show the angular distance from the lens, measured in units of the Einstein radius: x/θ_E and y/θ_E. We will assume that the motion is in the x direction, so that y is constant and equal to the (angular) impact parameter. The figure shows the case of an impact parameter $y = 0.5\ \theta_E$, but we will consider a range of values.



(a) Use the Pythagorean theorem to write down an expression for β/θ_E (the angular separation between the source and lens), in terms of x/θ_E and y/θ_E.

(b) In class we showed that this configuration produces two images, on opposite sides of the lens. The images, source, and lens all lie along the same line. The angular separation between the lens and each image (θ_+ and θ_-) is given by:

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right) \quad \Longrightarrow \quad \frac{\theta_\pm}{\theta_E} = \frac{1}{2}\left(\left[\frac{\beta}{\theta_E}\right] \pm \sqrt{\left[\frac{\beta}{\theta_E}\right]^2 + 4}\right)$$

The magnification of each image μ and the total magnification μ_{total} are given by

$$\mu_\pm = \frac{\theta_\pm^4}{\theta_\pm^4 - \theta_E^4} = \frac{(\theta_\pm/\theta_E)^4}{(\theta_\pm/\theta_E)^4 - 1} \qquad \mu_{\text{total}} = |\mu_+| + |\mu_-|$$

(Negative magnifications correspond to flipped images, so the total magnification is the sum of the absolute values.) With these equations you are ready to proceed. Calculate the six quantities β/θ_E, θ_+/θ_E, θ_-/θ_E, μ_+, μ_-, and μ_{total} for each of the following three source positions:

$(x/\theta_E, y/\theta_E) = (0.5, 0.5)$ and $(0.0, 0.5)$

(c) Doing this calculation a bunch of times seems tedious... but computers never complain! Grab the microlensing spreadsheet from Sakai Resources. You will need to edit a few cells/columns to make this spreadsheet work. The impact parameter (y/θ_E) goes in cell H1.

Put the formulas for β/θ_E, θ_+/θ_E, θ_-/θ_E, μ_+, μ_-, and μ_{total} in columns C through H (in rows 5 to 65). If all goes well, you should see a plot of the total magnification as a function of the source x position, and a plot of the source and image positions!

Change the impact parameter to y/θ_E =0.5, and check the values you calculated by hand for the three positions above.

For the first source position above, $(x/\theta_E, y/\theta_E) = (0.5, 0.5)$, what are the positions of the two images (from columns Q through T)? Which image is brighter?

(d) Using the spreadsheet, what is the maximum magnification for these 5 different values of the impact parameter: y/θ_E = 0.1, 0.2, 0.5, 1.0, and 1.5?

(e) Print out the total magnification and source/image positions plots for one of these cases and hand it in with your solutions. Make sure to put the right value of the impact parameter in the title of the magnification plot!

Question #3:

Gravitational lensing allows us to “weigh” galaxies. For simplicity, you may assume the lensing galaxies behave like point masses.

(a) The Einstein ring B1938+666 has a diameter of 0.95 arcseconds. The distance to the source is 2350 Mpc, while the distance to the lens is 1520 Mpc. Estimate the mass of the lensing galaxy in solar masses. Hint: you should convert angles to radians before applying the formula for mass that appears in the lecture notes.

(b) Two gravitationally lensed images of the quasar Q0957 are on opposite sides of a lensing galaxy. The images are 5.2” and 1.0” away from the galaxy. The distance to the quasar is 1700 Mpc and the distance to the lens galaxy is 1060 Mpc. What is the mass of the lensing galaxy in solar masses? Hint: See hint above.

Homework Assignment #11

Question #1:

Each part of this question covers a key concept. Each requires at most a few sentences to answer; some are much shorter. Please be concise; unnecessarily long-winded answers will lose credit.

(a) Suppose you see a friend moving past you some constant speed. Explain why your friend can equally well say that she is stationary and you are moving past her at a constant speed.

(b) If your friend is moving past you at a high constant speed, you will notice that her time appears to run slowly and her length is contracted in the direction of motion. How will she perceive her own time and length? How will she perceive your time and length?

(c) Decide whether the following statements make sense or not. Explain your reasoning.

   i. You and a friend are standing at opposite sides of a room, and you each eat a peanut at the same instant. According to the theory of relativity, it is possible for a person moving past you at a constant speed to observe that you ate your peanut before your friend at his.

   ii. An object moving by you at very high speed will appear to have a higher density than it has at rest.

   iii. If you could travel away from the Earth at a speed close to the speed of light, you would find yourself feeling uncomfortably heavy because of your increased mass.

   iv. If you had a sufficiently fast spaceship, you could leave today, make a round trip to a star 500 light-years away, and return home to Earth in the year 2050.

Question #2:

Consider a stationary inertial frame (the "unprimed" frame) and an inertial frame moving at a constant velocity u (the "primed" frame) along the x-axis with respect to the stationary frame. Now consider two events, with event 1 occurring at the origin of both frames, and event 2 slightly offset in space and time, with coordinates:

$$\text{event 1: } (t_1, x_1, y_1, z_1) = (0, 0, 0, 0) \quad \text{and} \quad (t'_1, x'_1, y'_1, z'_1) = (0, 0, 0, 0)$$

$$\text{event 2: } (t_2, x_2, y_2, z_2)=(dt, dx, dy, dz) \quad \text{and} \quad (t'_2, x'_2, y'_2, z'_2)=(dt', dx', dy', dz')$$

(a) Use the Lorentz transformation to write dt, dx, dy, and dz in terms of dt', dx', dy', and dz'. Use relativistic units, with time measured as a length ($c = 1$).

(b) In relativistic units, the spacetime intervals are

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \text{ and } ds'^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2$$

Use your results from part (a) to show explicitly that $ds^2 = ds'^2$, i.e. the spacetime interval is invariant under the Lorentz transformation. Note: The terms dt, dx, etc. are just labels for the change in t, x, etc. (think of them as meaning Δt, Δx, etc.) — you won't need to take any derivatives.

Question #3:

(Based on Carroll & Ostlie problems 4.6 and 4.7.)

A starship travels to Barnard's Star, a distance of approximately 6 light-years as measured from Earth, at a speed of $u = v/c = 0.6$.

(a) How long does the trip to Barnard's Star take, as measured by a clock on Earth?

(b) How long does the trip to Barnard's Star take, as measured by the starship pilot?

(c) What is the distance between Earth and Barnard's Star, as measured by the starship pilot?

(d) A radio signal is sent from the starship to Earth every year, as measured by a clock aboard the starship. What is the time interval between the reception of the signals on Earth?

(e) Upon reaching Barnard's Star, the starship immediately reverses direction and travels back to Earth at a speed of $u = 0.6$. (Assume that the turnaround itself takes zero time.) Make a table for the entire trip showing at what times Earth receives the yearly signals sent by the starship. Hint: Remember to change the sign of the velocity on the return trip!

Appendix D

Course Outline

Physics 341

Principles of Astrophysics

Fall 2013

Tuesdays and Thursdays

3:20-4:40 pm

ARC 105, Busch campus

Instructor: Prof. Eric Gawiser

Description

Astrophysics is the application of physical principles to astronomical systems. In Physics 341 and 342 you will learn how to use gravity, electromagnetism, and atomic, nuclear, and gas physics to understand planets, stars, galaxies, dark matter, and the Universe as a whole. Gravity is the dominant force in many astronomical systems, and it will be our focus in Physics 341.

Some astrophysical systems are described by equations that are fairly easy to solve, and we will study them. However, many interesting systems cannot be solved exactly. Nevertheless, we can often use physical insight and carefully chosen approximations to understand the key features of a system without sweating the details. One goal of the course is to develop that skill. As you will see, it will take us very far (through the whole Universe, in fact!). Another goal is to learn about recent advances in astrophysics, a very dynamic field of research.

Prerequisites for this class are two semesters of physics and two semesters of calculus. I will briefly review physical principles as we need them, but it is assumed that you have seen them before. I will also assume familiarity with vector calculus. Some of the assignments may involve a bit of computation that can be done with programs like Excel, Google Spreadsheets, Maple, Matlab, or Mathematica.

The recommended textbook for Physics 341 (and 342) is *An Introduction to Modern Astrophysics (2nd edition)* by Bradley W. Carroll and Dale A. Ostlie (affectionately known as the Big Orange Book). It provides a broad survey of astrophysics and covers the basics thoroughly. However, we will not follow this textbook in sequence, but rather will primarily reference the excellent series of lecture notes written by Prof. Chuck Keeton and updated by Prof. Saurabh Jha and myself. I will draw from other sources as well, letting you know when I do.

Contact Information

Prof. Eric Gawiser

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Email: gawiser[at]physics.rutgers.edu

Phone: 732-445-5500 ext. 2733

Office hours: Thursday 10-11AM

Grading Policy

Grading will be based on weekly problem sets (50%), two in-class midterms (10% each), a final take-home essay (10%), and iClicker scores (20%, with a bonus for active class participation).

Weekly problem sets will be handed out on Thursdays, and will be due the following Thursday at the beginning of class. When necessary, problem sets can also be turned in via our Sakai website in PDF format. It is your responsibility to meet the deadline! No late assignments will be accepted.

You are encouraged to work in groups on the weekly problem sets, but your write-up of the solutions must be your own. You *must* write down the names of your collaborators on your write-up. You *must also* cite any external sources you use (other than the class notes I post or the textbook). You may not refer to notes, assignments, or solutions from previous years of Physics 341 or 342.

The final essay must be entirely your own work, without any collaboration with your peers or usage of materials beyond those provided with the course. It will be due at 3:20pm on Tuesday, December 10.

Schedule: Topics and Assignments

This syllabus may be modified as the semester progresses.

Date	**General Concept**	**Topics**	**Text**	**Assignment**
Sep 3, 5	introduction	gravity; estimation; dimensional analysis		
Sep 10, 12	1-body problem	Newton's laws of motion and gravitation; conservation laws	1.1-1.2, 2.1-2.3	HW 1
Sep 17		deriving Kepler's Laws	2.1-2.3	
Sep 19		Galactic center	6.1, 24.4	HW 2
Sep 24		Doppler effect; supermassive black holes	4.3, 25.2, 28.2	
Sep 26	2-body problem	theory; equivalent 1-body problem	2.3	HW 3
Oct 1		binary stars	7.1-7.3	
Oct 3		binary stars; extrasolar planets	7.4, 23.1	HW 4
Oct 8		transiting planets	7.4, 23.1	
Oct 10		tidal forces	19.2, 21.2-21.3	HW 5
Oct 15	3-body problem	Lagrange points; asteroids; close binaries	18, 22.3	
Oct 17	N-body problem and galaxies	in-class exam basic properties of galaxies	24.2-24.3, 25.1-25.4	Exam 1

Oct 22		spiral galaxy rotation curves; dark matter	24.3, 25.2	
Oct 24, 29		galactic structure beyond rotation	24.2, 25.3	HW 6 due Oct 24
Oct 31		virial theorem; elliptical galaxies; galaxy interactions	2.4, 25.4, 26.1	HW 7
Nov 5, 7	gravitational lensing	basic principles; microlensing; galaxy and cluster lensing	28.4, 24.2	HW 8
Nov 12	relativity	special relativity	4 (all)	
Nov 14		introduction to general relativity	17 (all)	HW 9
Nov 19		applications of general relativity		
Nov 21		black holes		HW 10
Nov 26	cosmology	expanding Universe; geometry and dynamics	27.1-2, 29 (all)	
Nov 28		NO CLASS; Happy Thanksgiving!		
Dec 3		dark energy; future of the Universe		HW 11
Dec 5		in-class exam		Exam 2
Dec 10		final essay due 3:20pm class discussion		Final Essay