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**TRANSFORMATIONAL PROGRAMMING--APPLICATIONS
TO ALGORITHMS AND SYSTEMS**
Summary Paper

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This material is based in part upon work supported by the National Science Foundation under Grant No. MCS7905293. Part of this work was done while the author was visiting Stanford University.

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ABSTRACT

Transformational programming is a nascent software development methodology that promises to reduce programming labor, increase program reliability, and improve program performance. Our research centers around a prototype transformational programming system called RAPTS (Rutgers Abstract Program Transformation System), developed during the past several years at the Laboratory for Computer Science Research. Experiments in RAPTS with algorithm derivations are expected to lead to pragmatic applications to algorithm design, program development, and large system construction.

I. Introduction and Background

Ten years ago Cheatham and Wegbreit [4] proposed a transformational program development methodology based on notions of top-down stepwise program refinement first expressed by Dijkstra [10] and Wirth [40]. A schema describing the process of this methodology is given in fig. 1. To develop a program by transformation, we first specify the program in as high a level of abstraction and as great a degree of clarity as our programming language admits. This high level problem statement program P is proved correct semimechanically according to some standard approach (see Floyd and Hoare [14, 20]). Next, using an interactive system equipped with a library of encoded transformations, each of which maps a correct program into another 'equivalent' program, we select and apply transformations one at a time to successive versions of the program until we obtain a concrete, low level, efficient implementation version P'.

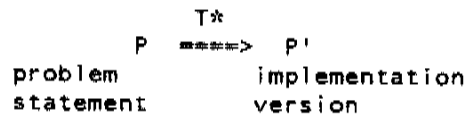


Fig. 1 Transformational Programming Schema

The goals of transformational programming are to reduce programming labor, improve program reliability, and upgrade program performance. In order for labor to be reduced, the effort required to obtain P, prove it correct, and derive P' by transformation should be less than the effort required to code P' from scratch, and also to debug it. Program reliability will be improved if P can be certified correct, and if each transformation preserves program meaning. Finally, program performance will be upgraded if transformations are directed towards increased efficiency.

Experimental transformational systems that emphasize one or more aspects of the methodology outlined above have been implemented by Cheatham [5], Darlington [3], Loveman [24], Standish [37], Feather [13], Huet and Lang [11], and others. However, all of these systems fall short of the goals, because of a number of reasons that include,

1. inability to mechanize the checking of transformation applicability conditions
2. reliance on large, unmanageable collections of low level transformations, and long arduous derivation sequences
3. dependency on transformations whose potential for improving program performance is unpredictable
4. use of source languages insufficiently high level to accomodate perspicuous initial program specifications and powerful algorithmic transformations

Yet, convincing evidence that this new methodology will succeed has come from recent advances in verification, program transformations, syntax directed editing systems, and high level languages. These advances, discussed below, represent partial solutions to the problems stated above, and could eventually be integrated into a single system.

1. The transformational approach to verification was pioneered by Gerhart [18] and strengthened by the results of Schwartz [35], Scherlis [32], Broy et al. [2], Koenig and Paige [23, 28], Blaustein [1], and others. Due mainly to improved technology for the mechanization of proofs of enabling conditions that justify application of transformations, this approach is now at a point where it can be effectively used in a system. Such mechanization depends strongly on program analysis, and, in particular, on reanalysis after a

program is modified. Attribute grammars [22] have been shown to be especially useful in facilitating program analysis [21]. Moreover, Reps [30] has discovered an algorithm that reevaluates attributes in optimal time after a program undergoes syntax directed editing changes (as are allowed on the Cornell Synthesizer [39]). He has implemented his algorithm recently, and has reported initial success.

2. There are encouraging indications that a transformational system can be made to depend mainly on a small but powerful collection of transformations applied top-down fashion to programs specified at various levels of abstraction from logic down to assembler. We envision such a system as a fairly conventional semiautomatic compiler in which classes of transformations are selected semimechanically in a predetermined order, and are justified by predicates supplied mechanically but proved semimanually. Of particular importance is nondeterminism removal, which, as formulated by Sharir [36], could lead to a technique for turning naive, nondeterministic programs into deterministic programs with emergent strategies. Such programs could then be transformed automatically by finite differencing [12, 15, 16, 17, 26, 27, 28] and jamming [25, 28, 19] (which we have implemented) into programs whose data access paths are fully determined. The SETL optimizer could improve these programs further by automatically choosing efficient data structure representations and aggregations.

3. Of fundamental importance to the transformations just mentioned is the fact that they can be associated with speedup predictions. Fong and Ullman [15] were the first to characterize an important class of algorithmic differencing transformations in terms of accurate asymptotic speedup predictions; e.g., they gave conditions under which repeated calculation of a set former $\{x \text{ in } s | k(x)\}$ could be computed in $O(\#s) + \text{cost}(k)$ steps. By considering stronger conditions and special cases for the boolean valued subpart k , Paige [28] later gave sharper speedup predictions (e.g., either $O(1)$ steps for each encounter of the set former or a cumulative cost of $O(\#s)$ steps for every encounter) associated with another differencing method. Both Morgenstern [25] and Paige [28] prove constant factor improvements due to their jamming transformations (implemented by Morgenstern for the improvement of file processing, and by Paige for the optimization of programs). Constant factor speedup has also been observed for data structure selection by the method of basings, but a supporting analytic study has not been presented [8, 33].

4. Essential to the whole transformational process is a wide spectrum programming language (or set of languages) that can express a program at every stage of development from the initial abstract specification down to its concrete implementation realization. Since transformations applied to programs written at the highest levels of abstraction are likely to make the most fundamental algorithmic changes, it is important to stress abstract features in our language. In addition to supporting transformations, the highest level language should support lucid initial specifications, verification, and even program analysis. Of special importance is SETL [34, 9], because its abstract set theoretic diction can model data structures and algorithms easily, because its philosophy of avoiding hidden asymptotic costs facilitates program analysis, because its semantics conforms to finite set theory and can accommodate a set theoretic program logic, and because it is wide spectrum. As is evidenced by the work of Schwartz, Fong, Paige, and Sharir, SETL is also a rich medium for transformation.

II. Main Results

The original contributions of our work are listed below:

i. Our main result is the implementation of a prototype transformational programming system that incorporates several of the ideas mentioned above. This system, called RAPTS (Rutgers Abstract Program Transformation System) [29], supports the semiautomatic development of reliable and efficient software using source-to-source program

transformations for an abstract variant of the SETL language. Like the prior transformational systems of Cheatham [4], Standish [37], Loveman [24], Darlington [3], and Feather [13], RAPTS has modules to perform parsing, unparsing (i.e., prettyprinting), search, and transformation application; it can manipulate libraries of transformations, source programs, and program development states; RAPTS provides a variety of user aids; also, global control flow and data flow analysis are used to prove the applicability conditions of our transformations automatically. However, our system emphasizes the strict stepwise refinement of programs by successive applications of powerful correctness preserving transformations that can be selected, justified, and applied with a much greater degree of mechanization than other systems.

We have used RAPTS for experimenting with algorithm derivation, system construction, and automated database processing. An important conceptual advantage in using SETL as both system implementation language and source language is that RAPTS can be used to improve itself, as was done for its dead code elimination procedure. In Appendix III we show how an inefficient but clear abstract specification of this procedure is transformed into a lower level SETL variant that runs in linear time with respect to the use-to-def links. Moreover, the transformational approach to verification together with appropriate assertion control could be used within RAPTS to prove itself correct.

ii. RAPTS uses a finite differencing method that generalizes John Cocke's strength reduction [6], and provides an efficient implementation of a host of transformations including Jay Earley's 'iterator inversion' [12]. Our differencing algorithm is an outgrowth of less efficient and less general algorithms due to Cocke, Schwartz, and Kennedy [6, 7]. The reduction in strength algorithms found in [6, 7] execute in $O(n)$ steps (where n is the number of nodes in the flow graph of a program loop) for a single pass. However, their algorithm and also the algorithm used by Paige and Koenig [28] takes $O(n^2)$ steps in the worst case to compute, due to successive linear time passes over programs growing successively larger. Our new algorithm only requires a single pass, and executes in $O(n)$ steps overall. We obtain this improvement by detection of all reducible expressions (that would be detected within multiple passes of the classical algorithms) in advance of any transformational steps. Our algorithm gains greater generality by accepting differencing categories of variable modifications upon which we can detect expressions amenable to reduction. Thus, differencing transformations can be applied over a wider range of data types than previously possible. However, in this paper we will stress the important application to set theoretic expressions, first observed by Earley [12].

Fong [16] first presented an algorithm to implement a subset of Earley's transformations, and her approach varied from his and Cocke's approach by using a deferred update strategy. She also gained more information by analyzing program paths instead of loops. However, her algorithm ran in time proportional to $e \log e$ bit vector operations, where e is the number of edges in the program flow graph. Furthermore, like the classical strength reduction algorithms, her algorithm must be reapplied over programs growing successively larger. (As in the case of the classical algorithms, this problem is due to the fact that reduction of one expression f can make another expression $g(f)$, which depends on f , reducible; as was first noted by Cocke and Schwartz [6] and solved by Cocke and Kennedy [7], reduction of f can also introduce new auxiliary expressions that must be further reduced.) Even the improvement of Fong's algorithm by Tarjan [38] and Rosen [31] to almost linear time in e bit vector operations per pass fails to make it a viable competitor to the classical approach or our new improvement. It remains an interesting open problem whether the path analysis approach introduced by Fong can be modified into a single pass algorithm without losing asymptotic efficiency. We conjecture that this problem can be solved affirmatively. Further comparison of her work with ours can be found in [28].

Like Fong and Ullman [15] our application of set theoretic differencing is based

upon reasonable conditions for ensuring asymptotic speedup. Although our current implementation includes about 50 groups of concrete differencing rules [28], recent theoretical improvements provide for a much more compact collection of meta-rules (see Appendix I) that are as easy to specify and implement as our current rules, and even more general than those proposed in [27].

iii. RAPTS can perform new and powerful set expression jamming transformations implemented by a linear time algorithm [27, 28]. A much more powerful algorithm than what is currently implemented and that yields 'optimal expression jamming' is found in [19].

iv. We have designed (but not yet implemented) a new way to mechanically estimate the asymptotic speed of an algorithm derived by transformation within RAPTS.

v. We specify our initial abstract program at an unusually high level of abstraction beyond current standard SETL. Illustrations will be included in the next section and Appendix III.

vi. RAPTS incorporates an implementation of a class of abstract static to dynamic expression transformations that generalize Earley's iterator inversion. (See Appendix II for a sampling of these transformations.)

2. RAPTS Illustrations

It is, perhaps, most convenient to explain the transformational capabilities of RAPTS by example, using 'photo' generated excerpts of an actual RAPTS derivation of topological sorting (an example first considered by Earley [12]).

Before proceeding, the reader may find it helpful to consult the brief description of SETL operations and their estimated computational costs (based on obvious hash table implementations for sets and maps) given in table 1.

Operation	Remarks	Estimated Cost
<code>s with:= x</code>	element addition	$O(1)$
<code>s less:= x</code>	element deletion	$O(1)$
<code>x in s</code>	set membership	$O(1)$
<code>s += delta</code>	set addition	$O(\#delta)$
<code>s -= delta</code>	set deletion	$O(\#delta)$
<code>f(x) := y</code>	indexed map assignment	$O(1)$
<code>f(x1,...,xn)</code>	function retrieval	$O(n)$
<code>(forall x in s) Block(x)</code>	forall loop	$O(\#s \times \text{cost}(\text{Block}))$
<code>end forall</code>		
<code>{x in s k(x)}</code>	set former	$O(\#s \times \text{cost}(k))$
<code>exists x in s k(x)</code>	existential quantifier	$O(\#s \times \text{cost}(k))$
<code>forall x in s k(x)</code>	universal quantifier	$O(\#s \times \text{cost}(k))$
<code>s + t</code>	set union	$O(\#s + \#t)$
<code>s * t</code>	set intersection	$\min(O(\#s), O(\#t))$
<code>s - t</code>	set difference	$O(\#s)$
<code>f[s]</code>	image set	$O(\#f)$

TABLE 1. Complexity Estimates of Setl Operations

Our initial algorithm specification inputs a set s and a set of pairs sp representing an irreflexive transitive predecessor relation defined on s ; as output, it produces a tuple t in which the elements of s are arranged in a total order consistent with the partial order sp ; sp maps each element x of s into the set $sp\{x\}$ of predecessor elements. The algorithm

proceeds by repeatedly searching for the minimal elements of the partially ordered set s , adding such elements to the end of t , and then removing them from s . Prettyprinted in RAPTS, our initial program is,

```

program topsort ;
1  read ( sp ) ;
2  print ( sp ) ;
3  t := [ ] ;
4  s := domain sp + range sp ;
5  ( while exists a in s | ( ( sp { a } ) * s ) = { } )
6    t with := a ;
7    s less := a ;
   end while ;
8  if s = { } then
9    print ( t ) ;
   else
10   print ( 0 ) ;
   end if ;
end program ;

```

The running time of the initial program is slow, essentially $O(E^2)$, which is due to repeated search for the minimal elements of s each cycle through the while loop. Speeding up this program entails searching for the minimal elements only once, and maintaining the set of minimal elements by inexpensive 'differential' computations within the while loop as s decreases. This strategy for program improvement is captured by a basic program optimization method we call finite differencing.

In order to facilitate finite differencing, we must first turn the code above into a normal form in which set update operations are implemented in terms of element additions and deletions, set intersections and deletions are rewritten as set formers, etc. For our example, the system will carry out several local transformations (selected from a production system of simple rewrite rules). Application of each transformation is justified by an assertion specified by a SETL predicate. If the system can simplify the predicate to 'true', the transformation is applied automatically. Otherwise, the system asks the user to confirm the partly simplified predicate. (In all of our examples presented here, the system has proved these predicates.)

The essential fragment of the normal form of the topological sorting procedure appears just below,

```

( while exists a in { set141 in s | # { set142 in sp { set141 } |
  set142 in s } = 0 } )
  t with := a ;
  s less := a ;
end while ;

```

Finite differencing will automatically transform the normal form algorithm into an equivalent but more efficient algorithm that uses the speedup strategy stated earlier. In rough terms, differencing will perform the following three steps based on Cocke's reduction in strength schema [6]:

- i. Just before the while loop, insert code that evaluates the set of minimal elements $\{set141 \text{ in } s \mid \#\{set142 \text{ in } sp\{set141\} \mid set142 \text{ in } s\} = 0\}$ (1) and stores it into the variable 'minset'. We call this code the *initialization* for minset.
- ii. Within the while loop where s is modified, insert code that recalculates minset from its old value so that it always stores the value of the set of minimal elements at the point (line 5) where it is computed. We call the code that updates minset the *difference* of minset with respect to the modification $s \text{ less} := a$. When the difference code is executed just prior to the modification, it is called *predifference* code.

iii. At line 5 replace the minimal set, which is made redundant by steps (i) and (ii), with the variable `minset`.

For this approach to improve program performance, the overall computational cost of calculating the initialization and difference code in the transformed program must be less than the cost of repeated calculations of the minimal set in the unoptimized program. Our system makes this analysis based on classical code motion assumptions (based on Cocke and Schwartz [6, 28]) and mechanical examination of the minimal set (1) and the while loop within the normal form of our algorithm before differencing is applied. For this example, the system will predict that differencing will yield asymptotic improvement in the cost of computing (1). (Note that Fong and Ullman [15] relied on weaker assumptions; see [28] for a comparison.)

The intuitive ideas behind the analysis are based on a decision procedure for a class of expressions for which the cost of computing difference code relative to certain kinds of parameter modifications is asymptotically less expensive than the cost of full expression evaluations. We say that expressions belonging to this class are *differentiable*. To define the class of differentiable expressions, we first define a finite collection of 'elementary' differentiable expressions and their associated difference code blocks whose computational cost is comparatively small. As is shown in [28], the full class of differentiable expressions is formed from composition of the elementary expressions and parameter substitution. This extended definition is justified by a formal calculus that constructs inexpensive difference code for a nonelementary differentiable expression by combining difference code for the elementary differentiable expressions out of which it is formed.

We now apply the preceding analysis to the minimal set (1) using the collection of basic set theoretic differentiable expressions found in Appendix I. Examination of the minimal set calculation detects three potentially differentiable subexpressions,

```
newpred{setl41} = {setl42 in sp {setl41} | setl42 in s }
numpred(setl41) = # newpred {setl41 }
minset = {setl41 in s | numpred(setl41) = 0}
```

that might permit efficient differencing for the minimal set. Unfortunately, neither `newpred{setl41}` nor `numpred(setl41)` are differentiable, because we cannot form efficient difference code for them relative to the arbitrary modifications in the free variable `setl41` that occur within the while loop of the normal form.

However, we can overcome this problem using transformations (listed in Appendix II) that handle dynamic expression formation, a generalization of Earley's iterator inversion. Application of transformation (4) of Appendix II converts `newpred{setl41}` into the following differentiable expression,

```
newpred = {[x,y] in sp | y in s}
```

which removes the troublesome free variable `setl41`, and stores values of `newpred{setl41}` for all relevant values of `setl41`. Likewise, transformation (26) turns `numpred(setl41)` into the following expression that can be maintained dynamically at low cost,

```
numpred = {[x, #newpred{x}]: x in domain newpred}
```

Supported by the elementary differentiable expressions, `newpred`, `numpred`, and `minset`, the minimal set (1) is seen to be differentiable, and our system can proceed to carry out the main transformational steps that will speed up the normal form of the topological sort; i.e.,

- i. Store initial values into `newpred`, `numpred`, and `minset` on entry to the while loop.
- ii. Update `newpred`, `numpred`, and `minset` just prior to line 7 where `s` is modified in order to make the computation of the minimal set at line 5 redundant.

Consistent with previous discussion, we refer to the update code involved in task (ii) as the difference of `newpred`, `numpred`, and `minset` with respect to the element deletion `s less:= set11`, and we form this difference code using a kind of 'chain rule' that combines the separate rules for forming difference code first for `newpred`, then `numpred`, and finally `minset` (i.e., from inner to outer subexpression of the minimal set).

We will illustrate the chain rule by proceeding with this example. The predifference of `newpred` relative to the modification `s less:= a` is

```
(forall set148 in { x in domain sp | a in sp { x } })      (2)
  newpred{set148} less:= a;
end forall ;
```

Observe that the predifference code (2) contains a costly embedded expression

```
succ{a} = { x in domain sp | a in sp { x } }
```

that we do not want to compute. However, the system will recognize that this expression can itself be reduced by 'second' differencing. At the same time that the three other differentiable expressions are detected, the system will recognize that differentiation of the dynamic expression

```
succ = {[y,x]: x in domain sp, y in sp{x}}
```

can efficiently eliminate the costly static expression occurring within the difference code (2).

The predifference code for `numpred` relative to the change in `newpred` within (2) is simply

```
numpred(set148) -= 1;                                     (3)
```

The final step of the chain rule involves forming the difference of `minset` relative to modifications in both of its parameters, `s` and `numpred`. These predifference blocks are

```
comment: relative to changes in s                        (4)
  if numpred ( a ) = 0 then
    minset less := a ;
  end if ;
```

and

```
comment: relative to changes in numpred                  (5)
  if set148 in s then
    if numpred ( set148 ) = 0 then
      minset less := set148 ;
    elseif numpred ( set148 ) = 0 + 1 then
      minset with := set148 ;
    end if ;
  end if ;
```

respectively. The chain rule combines the preceding blocks of difference code to form the following collective predifference of `newpred`, `succ`, `numpred`, and `minset` with respect to `s less:= a`:

```

( forall set148 in succ { a } )                                     (6)
  if set148 in s then
    if numpred ( set148 ) = 0 then
      minset less := set148 ;
    elseif numpred ( set148 ) = 0 + 1 then
      minset with := set148 ;
    end if ;
  end if ;
  numpred ( set148 ) - := 1 ;
  newpred { set148 } less := a ;
end forall ;
if numpred ( a ) = 0 then
  minset less := a ;
end if ;

```

Analysis of the overall cost of executing the block (6) rests on three easy observations.

i. Based on the monotonically decreasing set s within the while loop, we estimate that (6) is executed $O(\#s)$ times, where s is the initial value.

ii. Based on the complexity estimates stated in Table 1, the difference blocks (3), (4), and (5) involve only constant factor costs. Such costs are subsumed by the costs of surrounding code, and can be ignored. These examples illustrate the following general property:

Definition: An expression $E = f(s)$ is strongly continuous with respect to modifications of the form ds to s if the cost of the difference code for E with respect to ds is $O(1)$.

Thus, `minset` is strongly continuous with respect to indexed assignments to `numpred` and element deletions to s . Also, `numpred` is strongly continuous with respect to element additions and deletions to `newpred`.

```

iii. Repeated execution of the difference code for newpred,
(Forall set148 in succ{a})
  newpred{set148} less:= a;
end forall ;

```

relative to each distinct element 'a' removed from the monotonically decreasing set s , has an overall asymptotic cost no worse than a single calculation of `newpred = {[x,y] in sp | y in s}` at the initial value of s ; i.e., $O(\#sp)$. This example illustrates the following general property:

Definition: An expression $E = f(s)$ is weakly continuous with respect to modifications ds to s if for every minimal length sequence of operations ds_1, ds_2, \dots, ds_n (of the form ds) that constructs the final value s_2 from the initial value s_1 , the cumulative cost of all difference code for E with respect to all of the operations ds_1, \dots, ds_n is $O(\max(\text{cost}(f(s_1)), \text{cost}(f(s_2))))$. (Note that all of our speed estimates are based on the heuristics given in Table 1.)

Thus, `newpred` is weakly continuous with respect to element additions to s .

Based on the preceding analysis, the asymptotic cumulative cost of (6) is estimated at $O(\#sp)$, which is an order of magnitude better than the overall cost of the minimal set computation (1) in the normal form algorithm.

REMARK: In the final paper observations (i) - (iii) above will be treated in a more general formal framework. Most differentiable expressions in Appendix I are either

strongly or weakly continuous with respect to element additions or deletions to set or map valued parameters. Observe that strong continuity implies weak continuity. It can easily be shown that strongly continuous expressions are closed under arbitrary composition, and that weakly continuous expressions are closed under limited forms of composition. For example, the image set $f[s]$ is weakly continuous with respect to element additions and deletions to s , as are nested image set expressions. Thus, in a loop where s is monotonically increasing (resp. decreasing) and h, g , and f are invariant, the cumulative cost of executing difference code within the loop after differentiation of $h[g[f[s]]]$ is estimated to be $O(\#h+\#g+\#f)$.

Based on Table 1 and the assumption that initialization of `newpred`, `succ`, `numpred`, and `minset` can be achieved by the straightforward assignments

```

newpred := {[x,y] in sp | y in s};
succ := {[y,x]: x in domain sp, y in sp{x}};
numpred := {[x, #newpred{x}]: x in domain newpred};
minset := {set141 in s | numpred(set141) = 0};

```

(7)

we estimate the preprocessing costs to be $O(\#sp)$, which justifies our prediction of asymptotic speedup. However, we gain a constant factor improvement over this naive initialization by jamming the implicit loops within these set formers [28]. The jamming algorithm implemented in RAPTS constructs `newpred` and `numpred` in a single loop. A deeper investigation of this important transformation and an improved algorithm is found in [19].

The speedup prediction just presented is based on analysis of the normal form algorithm, so that it can be determined whether differentiation is profitable prior to any differencing transformations are applied. By analysis of the normal form, it is sometimes also possible to estimate the asymptotic speed of the transformed algorithm. Based on the Table 1. estimates and detection of the presence of monotonic set growth within 'while' loops (which can provide an estimate for the loop repetition frequency), it can be determined that after the minimal set computation is replaced by 'minset', all code other than that which has been introduced by differencing and initialization contributes no more than $O(\#sp)$ in overall cost. Adding in our estimates for cumulative differencing and initialization costs gives us an overall estimate of $O(\#sp)$ in running time for the transformed algorithm.

Further improvement in time and especially space can be realized by performing dead code elimination, which exploits the increase in data independence resulting from differencing and jamming. Based on an algorithm due to Kennedy [21, 28], our dead code elimination procedure detects all assignments to `newpred` as superfluous. The result of this final step is,

```

program topsort ;
1  read ( sp ) ;
2  print ( sp ) ;
3  t := [ ] ;
4  s := domain sp + range sp ;
5  succ := { } ;
6  ( forall set149 in domain sp , set150 in sp { set149 } )
7    succ { set150 } with := set149 ;
   end forall ;
8  numpred := { } ;
9  ( forall [ set143 , set144 ] in sp )
10   if set144 in s then
11     numpred ( set143 ) + := 1 ;
   end if ;
   end forall ;
12  minset := { } ;
13  ( forall set154 in s )
14   if numpred ( set154 ) = 0 then
15     minset with := set154 ;
   end if ;
   end forall ;
16  ( while exists a in minset )
17   t with := a ;
18   ( forall set148 in succ ( a ) )
19   if set148 in s then
20     if numpred ( set148 ) = 0 then
21       minset less := set148 ;
22     elseif numpred ( set148 ) = 0 + 1 then
23       minset with := set148 ;
24     end if ;
25   end if ;
26   numpred ( set148 ) - := 1 ;
   end forall ;
27   if numpred ( a ) = 0 then
28     minset less := a ;
29   end if ;
30   s less := a ;
   end while ;
31  if s = { } then
32    print ( t ) ;
33  else
34    print ( 0 ) ;
35  end if ;
end program ;

```

Our current implementation outputs the code just above after mechanical application of the preparatory transformations, dynamic expression formation, finite differencing, jamming, and dead code elimination. As is evident from our example, these transformations treated together function primarily to automate the formation of data access paths. Since the length of such paths traversed during execution is strongly related to the asymptotic running time of an algorithm, it is not surprising that finite differencing and its ancillary transformations yield asymptotic speedup. In addition to speedup, the process just illustrated supports verification. The soundness of our transformations, along with a standard correctness proof of the initial abstract algorithm, proves the correctness of the less perspicuous but more efficient equivalent algorithm above.

Further automatic improvement by a large constant factor may be achieved by data structure selection and aggregation [8, 33], transformations that should eventually be integrated into RAPTS.

Conclusion

Interactive syntactic editing systems such as the Cornell Synthesizer have successfully demonstrated a program construction methodology that mitigates compile time error. The Synthesizer speeds the process of program construction by dynamically monitoring syntax and, to some extent, semantics while the program is entered interactively. Transformational programming is a proposed methodology that aims to eliminate run time error, so that debugging would be unnecessary. It seeks to speed the programming process by interactively monitoring program correctness and efficiency during program construction.

RAPTS is a novel implementation of a prototype transformational system that represents a synthesis of old and new ideas. It incorporates new algorithms (its differencing algorithm is an improvement over classical strength reduction used in conventional compiling systems) and new transformations. We have used RAPTS to derive many simple algorithms such as the one just presented. For a more complicated example see Appendix III. We have introduced a straightforward mechanism for estimating the speedup that results from finite differencing and the speed of a differentiated algorithm prior to differentiation. Important followup work to this would be to determine conditions under which these initial performance estimates are preserved by a conventional complexity measure after conventional data structures are chosen to implement the sets and maps occurring within the differentiated algorithm.

Appendix I. Differentiable Set Expressions

Listed below is a small, but fairly complete, collection of elementary set-theoretic meta-expressions that can be maintained efficiently by differencing. We assume that each set former in this collection can also be expressed in terms of multi-iterators, which generalize cartesian product; e.g.,

$$\{e(X,Y): X \text{ in } S, Y \text{ in } T(X) \mid K(X,Y)\}$$

Although the difference rules associated with each elementary expression are not shown, they follow easily from standard distributive laws. Out of these meta-expressions and corresponding difference rules, the more concrete and numerous elementary expressions and efficient difference rules found in [28] can be derived.

1. $S + Q$
2. $\{X \text{ in } S \mid B(X)\}$
3. $\{X \text{ in } S \mid F(X) = T\}$ where T is an integer valued constant, F is integer valued.
4. $\{X \text{ in } S \mid F(X) \neq T\}$ where T is an integer valued constant, F is integer valued.
5. $\{X \text{ in } S \mid e(X) \text{ in } Q\}$
6. $\{X \text{ in } S \mid e(X) \text{ not in } Q\}$
7. $\{X \text{ in } S \mid F(X) \text{ relop }^* R\}$ where $F[S]$ is dense on the interval of integers containing the range of R values; F must be integer valued.
8. $\{X \text{ in } S \mid F(X) \text{ relop } R\}$ where $F[S]$ is sparse on the interval of integers containing the range of R values, or when F and R can be real; in this case, we also maintain the following two auxiliary expressions:
 $V = \text{SORTED}(F[S]) \text{ AND}$
 $K = \text{MIN}/\{I \text{ in } [1..#V + 1] \mid \text{NOT}(V(I) < R)\}$
9. $\{e(X): X \text{ in } S\}$
10. $F[S]$
11. $\#S$
12. $+/S$ where $+$ represents arithmetic sum.

(*) relop can be any of the comparisons $<, >, <=, >=$

Appendix II. DYNAMIC EXPRESSION FORMATION

Below we present rules, based on Earley's iterator inversion [12] and Paige's method of discontinuity removal [27], for transforming static set formers and other set theoretic expressions into a form suitable for efficient dynamic modification. Each basic expression f given below depends on free variables q, q_1, q_2, \dots that can undergo such modifications that disallow efficient dynamic maintenance of the value of f . However, f can be profitably maintained dynamically by eliminating its free variables and using a dynamic expression f' associated with f in the table below. Note that f' stores the values of $f\{q\}$ for all useful instantiations of q .

Static Expression	Dynamic Expression
1. $\{X \text{ in } G\{q\} \mid B(X)\}$	$\{[Y, X] \text{ in } G \mid B(X)\}$
2. $\{X \text{ in } S \mid F(X) = q\}$	$\{[F(X), X]: X \text{ in } S\}$
3. $\{X \text{ in } G\{q_2\} \mid F(X) = q_1\}$	$\{[[F(X), Y], X]: [Y, X] \text{ in } G\}$
4. $\{X \text{ in } G\{q\} \mid X \text{ in } Q\}$	$\{[X, Y] \text{ in } G \mid Y \text{ in } Q\}$
5. $\{X \text{ in } S \mid X \text{ in } F\{q\}\}$	$\{[X, Y] \text{ in } F \mid Y \text{ in } S\}$
6. $\{X \text{ in } G\{q_1\} \mid X \text{ in } F\{q_2\}\}$	$\{[[x, y], z]: [x, z] \text{ in } G, y \text{ in domain } F \mid z \text{ in } F\{y\}\}$
7. $\{X \text{ in } G\{q\} \mid F(X) \text{ in } Q\}$	$\{[Y, X] \text{ in } G \mid F(X) \text{ in } Q\}$
8. $\{X \text{ in } S \mid F(X) \text{ in } H\{q\}\}$	$\{[y, x]: y \text{ in domain } H, x \text{ in domain } F \mid F(x) \text{ in } H\{y\}\}$
9. $\{X \text{ in } G\{q_1\} \mid F(X) \text{ in } H\{q_2\}\}$	$\{[[y, z], x]: [y, x] \text{ in } G, z \text{ in domain } H \mid F(x) \text{ in } H\{y\}\}$
10. $\{X \text{ in } G\{q\} \mid X \text{ notin } Q\}$	$\{[X, Y] \text{ in } G \mid Y \text{ notin } Q\}$
11. $\{X \text{ in } S \mid X \text{ notin } F\{q\}\}$	$\{[y, x]: y \text{ in domain } F, x \text{ in } S \mid x \text{ notin } F\{y\}\}$
12. $\{X \text{ in } G\{q_1\} \mid X \text{ notin } F\{q_2\}\}$	$\{[[y, z], x]: [y, x] \text{ in } G, z \text{ in domain } F \mid x \text{ notin } F\{z\}\}$
13. $\{X \text{ in } G\{q\} \mid F(X) \text{ notin } Q\}$	$\{[Y, X] \text{ in } G \mid F(X) \text{ notin } Q\}$
14. $\{X \text{ in } S \mid F(X) \text{ notin } H\{q\}\}$	$\{[y, x]: y \text{ in domain } H, x \text{ in } S \mid F(x) \text{ notin } H\{y\}\}$
15. $\{X \text{ in } G\{q_1\} \mid F(X) \text{ notin } H\{q_2\}\}$	$\{[[y, z], x]: [y, x] \text{ in } G, z \text{ in domain } H \mid F(x) \text{ notin } H\{z\}\}$
16. $\{X \text{ in } S \mid q \text{ in } F\{X\}\}$	$\{[Y, X]: X \text{ in } S, Y \text{ in } F\{X\}\}$
17. $\{X \text{ in } G\{q_2\} \mid q_1 \text{ in } F\{X\}\}$	$\{[[U, X], W]: [W, X] \text{ in } G, U \text{ in } F\{X\}\}$
18. $\{X \text{ in } G\{q\} \mid X < R\}$	$\{[X, Y] \text{ in } G \mid Y < R\}$
19. $\{X \text{ in } S \mid x < F\{q\}\}$	$\{[y, x]: y \text{ in domain } F, x \text{ in } S \mid x < F\{y\}\}$
20. $\{X \text{ in } G\{q_1\} \mid X < F\{q_2\}\}$	$\{[[y, z], x]: [y, x] \text{ in } G, z \text{ in domain } F \mid x < F\{z\}\}$
21. $\{X \text{ in } G\{q\} \mid F(X) < R\}$	$\{[X, Y] \text{ in } G \mid F(Y) < R\}$
22. $\{X \text{ in } S \mid F(X) < H\{q\}\}$	$\{[y, x]: y \text{ in domain } H, x \text{ in } S \mid F(x) < H\{y\}\}$
23. $\{X \text{ in } G\{q_1\} \mid F(X) < H\{q_2\}\}$	$\{[[y, z], x]: [y, x] \text{ in } G, z \text{ in domain } H \mid F(x) < H\{z\}\}$
24. $\{F(X): X \text{ in } G\{q\}\}$	$\{[y, F(x)]: [y, x] \text{ in } G\}$
25. $F\{G\{q\}\}$	$\{[y, x]: [y, z] \text{ in } G, x \text{ in } F\{z\}\}$
26. $\#F\{q\}$	$\{[y, \#F\{y\}]: y \text{ in domain } F\}$
27. $+/F\{q\}$	$\{[X, +/F\{X\}]: X \text{ in DOMAIN } F\}$

where + represents arithmetic sum

Appendix III. Differencing Applied to Dead Code Elimination Within RAPTS

1. Below is an initial abstract algorithm specifying a portion of the dead code elimination procedure used within RAPTS. The set `crit` is the set of critical statements (initially defined to be the print statements of a program). The algorithm works by repeatedly adding to `crit` the set of instructions that can affect the value of variable uses within `crit` until `crit` no longer grows.

`iuses(q)` is the set of variable uses within statement `q`
`usetodef(u)` is the set of all variable definitions that can reach variable use `u`
`instof(d)` is the statement associated with a variable definition `d`
`compound(q)` is the compound statement immediately containing statement `q`

```

program dead ;
1   read ( instof , usetodef , iuses , compound , crit ) ;
2   ( converge )
3   crit + := ( instof [ usetodef [ iuses [ crit ] ] ] + compound [
      crit ] ) ;
      end ;
4   print ( crit ) ;
end ;

```

2. It is within the normal form below that 14 differentiable expressions are detected, including 1st and 2nd difference expressions. Analysis determines that the maps `instof`, `usetodef`, `iuses`, and `compound` are all weakly continuous with respect to element additions in their set valued arguments, that weak continuity is closed for these expressions, and finally, that the cumulative cost of difference code is estimated to be

$$O(\#instof+\#usetodef+\#iuses+\#compound).$$

which is dominated by $O(\#usetodef)$. This estimate is the same for initialization costs. It is easy to see that after differencing, the remaining costs are proportional to the sum of the input and output sizes.

```

program dead ;
1   read ( instof , usetodef , iuses , compound , crit ) ;
2   ( while exists set11 in { set10 in ( instof [ usetodef [ iuses [
      crit ] ] ] + compound [ crit ] ) } set10 notin crit ) )
3   crit with := set11 ;
      end while ;
4   print ( crit ) ;
end ;

```

3. After differencing and dead code elimination, the main loop of the algorithm appears below. Note that 4 out of the 14 differentiable expressions have been eliminated as useless. The passage from step 1 to 3 is done completely automatically within RAPTS.

```

35 ( while exists set11 in newinsts )
36 ( forall set15 in iuses { set11 } | nusepred ( set15 ) = 0 )
37 ( forall set118 in usestodef { set15 } | ndefpred ( set118 ) =
   0 )
38 ( forall set123 in instof { set118 } | ninstpred ( set123 )
   = 0 )
39   if set123 notin comps then
40     if set123 notin crit then
41       newinsts with := set123 ;
42       end if ;
43       instpnts with := set123 ;
44       end if ;
45       insts with := set123 ;
46     end forall ;
47     ( forall set130 in instof { set118 } )
48     ninstpred ( set130 ) + := 1 ;
49     end forall ;
50     ( forall set127 in usestodef { set15 } )
51     ndefpred ( set127 ) + := 1 ;
52     end forall ;
53   end forall ;
54 ( forall set110 in compound { set11 } | ncompred ( set110 ) = 0
   )
55   if set110 notin insts then
56     if set110 notin crit then
57       newinsts with := set110 ;
58       end if ;
59       instpnts with := set110 ;
60       end if ;
61       comps with := set110 ;
62     end forall ;
63     ( forall set114 in compound { set11 } )
64     ncompred ( set114 ) + := 1 ;
65     end forall ;
66     ( forall set139 in iuses { set11 } )
67     nusepred ( set139 ) + := 1 ;
68     end forall ;
69     if set11 in instpnts then
70       newinsts less := set11 ;
71       end if ;
72     crit with := set11 ;
73   end while ;

```

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