# GENERALIZED QUASI POISSON STRUCTURES AND NONCOMMUTATIVE INTEGRABLE SYSTEMS

by

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**ABSTRACT OF THE DISSERTATION**

**Generalized Quasi Poisson Structures and Noncommutative Integrable Systems**

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The dissertation is devoted to the applications of the Noncommutative Geometry Program to the study of Integrable Systems and Cluster Algebras.

In particular, it is shown that cluster algebras introduced by A. Goncharov and R. Kenyon admit a noncommutative generalization. This generalization can be viewed as a family of categories equipped with a double Quasi Poisson bracket and a family of functors between these categories which preserve the double bracket. From this perspective, the commutative cluster algebra appears as the coordinate ring of the moduli space of one dimensional representations of the noncommutative cluster algebra.

It is shown that Noncommutative systems of ODEs, suggested earlier by M. Kontsevich and A. Usnich, admit a formulation as Noncommutative Hamilton flows.

Finally, a non-skew-symmetric generalization of the double Poisson bracket is considered. It is shown that such modified double Poisson brackets inherit major properties of double Poisson brackets.

# Preface

The theory of Integrable Systems was originally developed as an algebraic tool for the study of Hamiltonian Ordinary Differential Equations. The main advantage of the Hamilton formalism is that it reduces the problem of finding the first integrals of ODE (conserved quantities) to the purely algebraic problem of finding the maximal "Poisson-commuting" subalgebra of the algebra of smooth functions $C^\infty(\mathcal{M})$ on some manifold $\mathcal{M}$. When this subalgebra is large enough, the system of ODE is called Integrable in the Liouville sense and can be integrated by quadratures [Arn78]. It appeared later that Integrable Systems have applications well beyond the original framework; in particular Quantum Integrable Systems play a fundamental role in representation theory [Eti07] and low-dimensional topology [RT90].

It seems natural to ask what happens if we follow the concept of noncommutative geometry and replace the commutative algebra of smooth functions on a manifold with some abstract associative algebra, in general noncommutative. My thesis was largely motivated by an idea that Noncommutative Integrable Systems can play the role of a unifying concept in the theory of Hamilton flows on representation varieties, much in the same way that quantum Integrable Systems do in representation theory.

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# Dedication

To my family who supported me on all stages of my education and research. To my wife Sveta, my parents Boris and Marina, and my grandmother Lidya with admiration.

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# Chapter 1

# Introduction

## 1.1 Noncommutative Geometry

The conventional "commutative geometry" was born with a Hilbert Nullstellensatz theorem in 1893 which established the correspondence between algebraic sets and radical ideals of the polynomial ring [Hil93]. It appeared later that this correspondence between commutative algebra and geometry goes much further. In 1939 Gelfand and Kolmogoroff showed that algebraic structure of the ring of continuous functions defines the compact topological space up to a homeomorphism [GK39]. This opened up a broad program of studying the topology and geometry of spaces in terms of the algebraic properties of certain commutative rings associated with these spaces.

In 1985 Alain Connes in his monograph *"Non-commutative differential geometry"* [Con85] suggested to go beyond commutative rings, and announced a series of seven papers intended to extend the familiar notions of commutative geometry for general associative algebras. The program then received a common name — Noncommutative geometry. Or, as it was suggested by V. Ginzburg [Gin05] it should be better referred to as "general associative geometry" vs the "commutative associative geometry".

Shortly after A. Connes' original paper, the noncommutative analogue of the De Rham complex of differential forms was introduced by M. Karoubi [Kar87] followed by a series of papers which extended familiar notions of commutative geometry to the context of general associative algebras. For example, in [CQ95] J. Cuntz and D. Quillen introduced the notion of a smooth noncommutative algebras, in [Lod98] J.-L. Loday introduced noncommutative Lie algebras and their homology.

A remarkable interplay between noncommutative and commutative geometries was suggested in [Kon93]. Following a general philosophy formulated by M. Kontsevich any algebraic property that makes geometric sense is mapped to its commutative counterpart by the functor $\mathrm{Rep}_N$:

$$\mathrm{Rep}_N : \quad \text{fin. gen. Associative algebras} \to \text{Affine schemes},$$

which assigns to a finitely generated associative algebra $\mathcal{A}$ the scheme of its' $N \times N$ matrix representations

$$\mathrm{Rep}_N(\mathcal{A}) = Hom(\mathcal{A}, Mat_N(\mathbf{k})),$$

where $\mathbf{k}$ denotes some field of characteristic zero. This idea has allowed to study the geometry of representation varieties by means of noncommutative geometry. It is worth mentioning here that one of the most recent advances of this kind was the formulation of derived representation schemes in [BCER12, BKR13].

### 1.2 Polyvector fields and Double Geometry

From the algebraic viewpoint, a vector field $d$ on a manifold $\mathcal{M}$ is nothing but the derivation of $C^\infty(\mathcal{M})$, the algebra of smooth functions on $\mathcal{M}$

$$d : C^\infty(\mathcal{M}) \to C^\infty(M), \qquad d(fg) = fd(g) + d(f)g, \tag{1.1}$$

for all $f, g \in C^\infty(\mathcal{M})$. The space $D^1 = Der(C^\infty(\mathcal{M}), C^\infty(\mathcal{M}))$ of all such derivations is naturally a $C^\infty(\mathcal{M})$-module. As a result, one can define the algebra of polyvector fields as a tensor algebra $D^\bullet = T_{C^\infty(\mathcal{M})} D^1$ over the $C^\infty(\mathcal{M})$ generated by the module of vectorfields $D^1$.

The above definitions can be translated literally to the language of affine schemes if we replace $C^\infty(\mathcal{M})$ with some finitely generated commutative algebra over $\mathbf{k}$. However, if we try to naively extend definition (1.1) of a vector field to a general associative algebra $\mathcal{A}$, we would immediately run into the problem that $Der(\mathcal{A}, \mathcal{A})$ is no longer an $\mathcal{A}$-bimodule for a general noncommutative algebra $\mathcal{A}$. To overcome this issue it was suggested independently by M. Van den Bergh in [VdB08] and W. Crawley-Boevey,

P. Etingof, V. Ginzburg in [CBEG07] to define a vector field over $\mathcal{A}$ as a double derivation, a notion introduced earlier in [RSS80]

$$\delta : \quad \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \qquad \delta(ab) = (a \otimes 1)\delta(b) + \delta(a)(1 \otimes b)$$

The space of all biderivations $D_A = Der(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ is then equipped with an $\mathcal{A}$-bimodule structure.

### 1.3 Double Quasi Poisson Brackets

In line with Kontsevich's philosophy, M. Van den Bergh [VdB08] proposed a definition of the double Poisson bracket on an associative algebra which induces a conventional Poisson bracket on the coordinate ring of matrix representations. On the contrary, W. Crawley-Boevey [CB11, CBEG07] suggested yet another related definition of the noncommutative analogue of the Poisson bracket, the so-called $H_0$-Poisson structure. The latter has weaker requirements and in general provides a conventional Poisson bracket only on the moduli space of representations. A double Poisson bracket induces an $H_0$-Poisson structure but not vice versa.

One of the major advantages of the double Poisson bracket as opposed to an $H_0$-Poisson structure is that for a finitely generated associative algebra it is defined completely by its action on generators. This allows one to provide numerous explicit examples of double Poisson brackets [PVdW08, BT16] and even carry out certain partial classification problems [ORS13].

However, double Poisson brackets do not give rise to all $H_0$-Poisson structures, less so to all Poisson brackets on the moduli space of representations. The reason is that it is by no means necessary to have a Poisson bracket on the full coordinate ring of matrix representations in order to induce a Poisson bracket on the $GL(N, \mathbb{C})$-invariant subring i.e., on a coordinate ring of the moduli space. The most interesting example of this kind is the Quasi Poisson bracket introduced in [AKSM02] based on the earlier work [AMM98] about the group-valued moment maps. To describe quasi brackets from the noncommutative geometry point of view, M. Van den Bergh has introduced the notion of the double Quasi Poisson bracket [VdB08].

### 1.4 Poisson brackets on the character varieties of fundamental groups

It was first proved by W. Goldman [Gol86] that the character variety of a fundamental group of a closed oriented surface can be equipped with canonical Poisson bracket. Later this result was extended for Riemann surfaces with nonempty boundary [FR93, FR99], where the authors defined first a Poisson bracket on the full coordinate ring of all matrix representations using $r$-matrix, and then took a quotient with respect to the Poisson action of the gauge group. An expected property of such construction was that the resulting bracket on the coordinate ring of the moduli space of matrix representation depends only on the symmetric part of $r$-matrix, and is the same up to a constant for all proper choices of an $r$-matrix.

Taking this into account it would seem natural to define the bracket on the coordinate ring of all matrix representations on the fundamental group using only the symmetric part of the $r$-matrix, however such a bracket would not satisfy the Jacobi identity. Instead, it would satisfy the quasi Jacobi identity defined in [AKSM02, AKS00]. For a recent proof see [Nie13].

G. Massuyeau and V. Turaev have shown in [MT12] that one can introduce the Quasi Poisson Bracket on the coordinate ring of matrix representations of the fundamental group by means of noncommutative geometry. Namely, one can define a double Quasi Poisson bracket on the group algebra of the fundamental groupoid of a surface with nontrivial boundary.

The double analog of the homotopy intersection form proposed in [MT12] immediately suggests that such brackets have a categorical flavour. This observation is also noted in [MT13].

### 1.5 Cluster algebras

Cluster algebras were originally introduced in [FZ02] largely motivated by the notion of a "canonical basis" in irreducible representations which was suggested earlier in [GZ86]. Each cluster algebra can be described as a commutative ring $\mathcal{B}$ with a distinguished set of generators known as *cluster variables*. It comes equipped with a family of maximally algebraically independent subsets of generators known as *cluster charts.* The relations in $\mathcal{B}$ are of a very special type, such that for each cluster $\{x_1, \ldots x_n\}$ the rest of the generators $\{y_1, \ldots, y_m\}$ can be expressed at a general point as Laurent polynomials in $\{x_1, \ldots, x_n\}$. It was shown by M. Gekhtman, M. Shapiro, and A. Vanstein [GSV03] that cluster charts can be equipped with a Poisson bracket in such a way that any cluster transformation becomes a homomorphism of Poisson algebras. Cluster algebras have found major applications in Teichmüller theory [GSV05, FG06] and its quantization [FC99].

A relation between cluster algebras and topology was suggested in [FST08], where the authors have defined cluster algebras associated to ideal triangulations of surfaces with marked points. Cluster transformations then correspond to a sequence of flips of ideal triangulations. The edge weights of this cluster algebras can be interpreted as holonomies of a rank one connection on the graph associated to the triangulation. This idea was later developed in [GK13], where the authors introduced a Poisson bracket on rank one graph connections and shown that flips of triangulations define Poisson homomorphisms of cluster charts.

This immediately raises a question about the higher rank generalizations of the above construction. One of the goals of the thesis is to show that Noncommutative Geometry allows one to give a positive answer to such question. It was suggested by A. Berenstein and V. Retakh [BR05, BR18] that one can consider noncommutative cluster algebras with edge weights being an elements of some general associative algebra.

## 1.6 Outline

The text of the dissertation is organized as follows:

In chapter 2 we start with a review of Noncommutative Poisson Geometry on linear categories. For a general $\mathbf{k}$-linear category $\mathcal{C}$, we introduce an associated category $\mathcal{V}$ of polyvector fields, a category $\mathcal{K}$ of differential forms, and an evaluation map of $n$-forms on degree $n$ polyvector fields. This material is a straightforward generalization of the corresponding notions for a category with a single object defined in [VdB08, CBEG07].

Similar definitions are also given in the recent preprint [Yeu18] with applications to derived representation schemes, which appeared when the work on the thesis was at the final stage. We conclude the chapter with an alternative definition of the Double Quasi Poisson bracket on a category 2.78. We show that one can avoid the introduction of a quiver path algebra when defining the right hand side of the double Quasi Jacobi Identity (2.79)

$$Y_R = \frac{1}{4} \sum_{V \in Obj\, \mathcal{C}} \overline{\partial_V \star \partial_V \star \partial_V}.$$

This useful technical point allows us to simplify the proofs in the following chapter 3, however it can be interesting on its own.

In chapter 3 we show that cluster algebras introduced by A. Goncharov and R. Kenyon admit a noncommutative generalization. We define a family of categories equipped with a double Quasi Poisson bracket, which are associated to the bipartite ribbon graphs. We then show that noncommutative mutations can be realized as functors between such categories. The main result of the chapter is formulated in Theorem 3.12 where we prove that the above functors preserve the double Quasi Poisson bracket. As an example of such quasi Poisson homomorphism in Section 3.3 we consider a particular case of the Kontsevich map suggested in [Kon11].

Chapter 4 is devoted to the examples of Noncommutative Integrable Systems. We show that noncommutative ODE suggested in [Usn08] and [Kon11] can be presented as a Hamilton flows. Moreover, we show that they belong to the infinite family of pairwise commuting Hamilton flows on the same algebra. This chapter is largely based on a paper [Art15] by the author of the dissertation.

Finally, in chapter 5 a non-skew-symmetric generalization of the double Poisson bracket is considered, following the paper [Art17] by the author of the dissertation. It is defined as the most general biderivation which gives rise to an $H_0$-Poisson bracket. It is shown that such modified double Poisson brackets inherit major properties of double Poisson brackets.

# Chapter 2

# Noncommutative Poisson Geometry

Throughout the chapter we fix a ground field $\mathbf{k}$ of characteristic zero, $\operatorname{char}\mathbf{k} = 0$. All vector spaces, unadorned tensor products etc. are considered over $\mathbf{k}$. We then fix an underlying $\mathbf{k}$-linear category $\mathcal{C}$ and assume that $\mathcal{C}$ is small.

## 2.1 Categorification of Noncommutative Differential Geometry

### 2.1.1 Vector Fields on linear Categories

**Definition 2.1.** Let $\mathcal{C}$ be a $\mathbf{k}$-linear category. For each ordered pair of objects $V, W \in Obj\,\mathcal{C}$ we say that a map

$$\delta: \quad Mor\,\mathcal{C} \to \hom(W, -) \otimes \hom(-, V) \tag{2.2}$$

is a $(V, W)$-*derivation* if

- For all objects $A, B \in Obj\,\mathcal{C}$, the restriction

  $$\delta: \quad \hom(A, B) \to \hom(W, B) \otimes \hom(A, V)$$

  is a $\mathbf{k}$-linear map.

- For all morphisms $f, g \in Mor\,\mathcal{C}$ which are composable, i.e., the source of $f$ coincides with the target of $g$, the map $\delta$ satisfies the double Leibniz identity

  $$\delta(f \circ g) = (f \otimes 1_V) \circ \delta(g) + \delta(f) \circ (1_W \otimes g). \tag{2.3}$$

  Here $1_V$ and $1_W$ are the corresponding identity morphisms on $V$ and $W$.

The collection of all $(V, W)$-derivations associated to a given pair of objects is denoted as $D_{V,W}$ and called a space of $(V, W)$-*vector fields*.

*Remark* 2.4. Note that for all objects $V \in \mathcal{C}$, the collection of endomorphisms $\mathcal{A} = \hom(V,V) = \mathrm{End}(V)$ forms an associative algebra. When $\mathcal{C}$ has a single object, say $V$, Definition 2.1 becomes $D_{V,V} = Der(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$. In such form it was first suggested simultaneously in [VdB08] and [CBEG07].

**Definition 2.5.** Let $\mathcal{C}$ be a **k**-linear category. We call a **k**-linear space $\mathcal{M}$ a *left module* over $\mathcal{C}$ (or left $\mathcal{C}$-module) if

- As a linear space $\mathcal{M}$ is graded by objects of $\mathcal{C}$

$$\mathcal{M} = \bigoplus_{X \in Obj\,\mathcal{C}} \mathcal{M}_X$$

- For each $A, B \in Obj\,\mathcal{C}$ and every $f \in \hom(A,B)$ there is a linear map

$$\varphi(f) : \quad \mathcal{M}_A \to \mathcal{M}_B.$$

- For all $A, B, C \in Obj\,\mathcal{C}$ and all $f \in \hom(A,B),\ g \in \hom(B,C)$ we have

$$\varphi(g) \circ \varphi(f) = \varphi(g \circ f).$$

  In other words, $\varphi$ is a covariant functor.

A homomorphism $\mathcal{F} : (\mathcal{M}, \varphi) \to (\mathcal{M}', \varphi')$ of left $\mathcal{C}$-modules is defined as a natural transformation, i.e., a family of **k**-linear maps $\mathcal{F} : \mathcal{M}_A \to \mathcal{M}'_A$ for all objects of $\mathcal{C}$, which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{M}_A & \xrightarrow{\varphi(f)} & \mathcal{M}_B \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathcal{M}'_A & \xrightarrow{\varphi'(f)} & \mathcal{M}'_B \end{array} \tag{2.6}$$

for all $A, B \in Obj\,\mathcal{C}$ and all $f \in \hom(A,B)$. In what follows we denote the category of left $\mathcal{C}$-modules by $\mathcal{C}$-**mod**.

Similarly, we introduce the category **mod**-$\mathcal{C}$ of right $\mathcal{C}$-modules:

**Definition 2.7.** Let $\mathcal{C}$ be a **k**-linear category we call a **k**-linear space $\mathcal{M}$ a right module over $\mathcal{C}$ if it is a left module over the opposite category $\mathcal{C}^{op}$, i.e.,

- As a linear space $\mathcal{M}$ is graded by objects of $\mathcal{C}$

$$\mathcal{M} = \bigoplus_{X \in Obj\, \mathcal{C}} \mathcal{M}_X$$

- For each $A, B \in Obj\, \mathcal{C}$ and every $f \in \hom(A, B)$ there is a linear map

$$\varphi(f): \quad \mathcal{M}_B \to \mathcal{M}_A.$$

- For all $A, B, C \in Obj\, \mathcal{C}$ and all $f \in \hom(A, B),\ g \in \hom(B, C)$ we have

$$\varphi(f) \circ \varphi(g) = \varphi(g \circ f).$$

  Or, equivalently, one can say that $\varphi$ is a contrvariant functor.

As a corollary, each homogeneous component $\mathcal{M}_X$ is a left (resp. right) module over $\mathrm{End}(X)$.

**Definition 2.8.** We call $\mathcal{M}$ a bimodule over $\mathcal{C}$ (or $\mathcal{C}$-bimodule) if it is simultaneously a left and a right module over $\mathcal{C}$ in the sense of Definitions 2.5 and 2.7, such that the two actions commute with each other.

Throughout the text we denote the category of all $\mathcal{C}$-bimodules as $\mathcal{C}$-**mod**-$\mathcal{C}$. Note, that each $\mathcal{C}$-bimodule $\mathcal{M}$ as a linear space is a direct sum

$$\mathcal{M} = \bigoplus_{X, Y \in Obj\, \mathcal{C}} \mathcal{M}_{X,Y},$$

where each homogeneous component $\mathcal{M}_{X,Y}$ is an $\mathrm{End}(X) - \mathrm{End}(Y)$ bimodule.

*Remark* 2.9. Every **k**-linear bifunctor [Mit65] on a small **k**-linear category $\mathcal{C}$ which is covariant in the first component and contrvariant in the second component defines a $\mathcal{C}$-bimodule, and vice versa.

**Lemma 2.10.** *The space of vector fields*

$$D^1 = \bigoplus_{V, W \in Obj\, \mathcal{C}} D_{V,W}$$

*is equipped with a $\mathcal{C}$-bimodule structure.*

*Proof.* For each homogeneous component $D_{V,W}$ labeled by $V, W \in Obj\,\mathcal{C}$ we define the left action of $\hom(V, \_)$ and right action of $\hom(\_, W)$ as follows. Let $\delta \in D_{V,W}$ and $f \in Mor\,\mathcal{C}$. Then [1]

$$\delta(f) = \delta'(f) \otimes \delta''(f).$$

For all $X, Y \in Obj\,\mathcal{C}$ and $a \in \hom(V, Y)$ and for each $b \in \hom(X, W)$ we set

$$(a \star \delta)(f) = \delta'(f) \otimes (a \circ \delta''(f)), \qquad\qquad (\delta \star b)(f) = (\delta'(f) \circ b) \otimes \delta''(f). \tag{2.11}$$

As a result, $a \star \delta : Mor\,\mathcal{C} \to \hom(W, -) \otimes \hom(-, Y)$ satisfies the Leibnitz identity (2.3), namely

$$\begin{aligned}(a \star \delta)(f \circ g) =& (1_{t(f)} \otimes a) \circ (f \otimes 1_V) \circ \delta(g) + (1_{t(f)} \otimes a) \circ \delta(f) \circ (1_W \otimes g) \\ =& (f \otimes 1_V) \circ (a \star \delta)(g) + (a \star \delta)(f) \circ (1_W \otimes g)\end{aligned}$$

and thus $a \star \delta \in D_{Y,W}$. Similar reasoning shows that $\delta \star b \in D_{V,X}$. Moreover, by (2.11) the two actions commute with each other

$$(a \star \delta) \star b = a \star (\delta \star b) \quad \in D_{Y,X}.$$

Next, from the fact that composition in $\mathcal{C}$ is associative we get

$$(a_1 \circ a_2) \star \delta = a_1 \star (a_2 \star \delta),$$

for all $a_1, a_2$ s.t. $s(a_2) = V$ and $s(a_1) = t(a_2)$. Finally, since the proof holds for arbitrary $V, W \in Obj\,\mathcal{C}$, we conclude that $D^1$ is a left $\mathcal{C}$-module w.r.t. the action (2.11).

Applying similar reasoning to $\delta \star b$ we prove that $D^1$ is also a right $\mathcal{C}$-module. □

**Corollary 2.12.** *The space $D_{V,W}$ of $(V, W)$-vector fields is a left module over* $\mathrm{End}(V)$ *and a right module over* $\mathrm{End}(W)$ *such that the two actions commute with each other*

$$D_{V,W} \quad \in \quad \mathrm{End}(V)\text{-}\mathbf{mod}\text{-}\mathrm{End}(W).$$

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[1] Hereafter we use Sweedler notations to omit the summation index in a tensor product whenever it doesn't lead to any confusion. Thus $\delta(f) = \sum_{i=1}^{k} \delta_i'(f) \otimes \delta_i''(f)$ is written as $\delta'(f) \otimes \delta''(f)$

### 2.1.2 Gerstenhaber Category of Polyvector Fields

Recall that for each pair of right and left $\mathcal{C}$-modules

$$R \in \textbf{mod-}\mathcal{C}, \qquad L \in \mathcal{C}\textbf{-mod}$$

one can define their tensor product over $\mathcal{C}$ as the following quotient

$$R \otimes_{\mathcal{C}} L = \left( \bigoplus_{V \in Obj\, \mathcal{C}} R_V \otimes L_V \right) / J \tag{2.13}$$

where $J$ is the $\mathbf{k}$-span of the elements of the form

$$\rho \circ f \otimes \lambda - \rho \otimes f \circ \lambda,$$

for all objects $A, B \in \mathcal{C}$ and all $\rho \in R_B, f \in \hom(A, B), \lambda \in L_A$.

*Remark* 2.14. Note that the quotient (2.13) is not graded by $V \in Obj\, \mathcal{C}$, because $J$ contains elements which mix different homogeneous components.

Quotient (2.13) allows one to define a composition map

$$R_V \times L_V \longrightarrow R \otimes_{\mathcal{C}} L$$

The $\mathcal{C}$-bimodule structure on the space of vector fields $D^1$ introduced in Lemma 2.10 allows one to define a composition of vector fields. Now, let $V, W, U_0, \ldots, U_{k-1} \in Obj\, \mathcal{C}$ and

$$\delta^{(1)} \in D_{V,U_1}, \quad \delta^{(2)} \in D_{U_1,U_2}, \quad \ldots, \quad \delta^{(k)} \in D_{U_{k-1},W}$$

be a chain of vector fields. One can define the sequential composition of $\delta^{(1)}, \ldots, \delta^{(k)}$ as the following tensor product

$$\delta^{(1)} \star \delta^{(2)} \star \cdots \star \delta^{(k)} = \delta^{(1)} \otimes_{\mathcal{C}} \delta^{(2)} \otimes_{\mathcal{C}} \cdots \otimes_{\mathcal{C}} \delta^{(k)}. \tag{2.15}$$

**Definition 2.16.** Let $V, W \in Obj\, \mathcal{C}$ be a pair of objects in a $\mathbf{k}$-linear category $\mathcal{C}$. We define the space of *$k$-vector fields* associated to $(V, W)$ for all $k \geq 0$ as

$$D^k_{V,W} = \bigoplus_{U_1, \ldots, U_{k-1} \in Obj\, \mathcal{C}} D_{V,U_1} \otimes_{\mathcal{C}} \quad \ldots \quad \otimes_{\mathcal{C}} D_{U_{k-1},W}, \tag{2.17a}$$

where for $k = 0, 1$ we assume that

$$D^0_{V,W} = \hom(W, V), \qquad\qquad D^1_{V,W} = D_{V,W}. \tag{2.17b}$$

Note that the space of $k$-vector fields $D^k_{V,W}$ for $k \geq 1$ is spanned by *elementary $k$-vector fields* of the form (2.15).

**Lemma 2.18.** *The space of all $k$-vector fields on a $\mathbf{k}$-linear category $\mathcal{C}$*

$$D^k = \bigoplus_{V,W \in Obj\,\mathcal{C}} D^k_{V,W} \tag{2.19}$$

*forms a $\mathcal{C}$-bimodule.*

*Proof.* For $k = 0$ the statement is trivial, since the category $\mathcal{C}$ is a bimodule over itself. We have shown in Lemma 2.10 that the statement holds for $k = 1$ as well. For $k > 1$ we have

$$D^k_{V,W} = \bigoplus_{U_1 \in Obj\,\mathcal{C}} D_{V,U_1} \otimes_{\mathcal{C}} D^{k-1}_{U_1,W}. \tag{2.20}$$

where each of the summands is a left $\mathrm{End}\,V$-module by Corollary 2.12. This allows one to define the action of $\hom(V, \_)$ on (2.20) componentwise. Since this action is $\mathrm{End}(V)$-linear, we conclude that $D^k$ is a left $\mathcal{C}$-module. Similar reasoning can be applied to show that $D^k$ is a right $\mathcal{C}$-module. The two actions commute with each other by construction, which finalizes the proof. $\square$

**Definition 2.21.** For each pair of objects $V, W \in Obj\,\mathcal{C}$, we refer to the direct sum

$$D^\bullet_{V,W} = \bigoplus_{k=0}^{\infty} D^k_{V,W} \tag{2.22}$$

as a space of $(V, W)$*-polyvector fiels.*

From the definition above it follows that the space $D^\bullet_{V,W}$ of $(V, W)$-polyvector fields is graded by nonnegative integers. In what follows we say that a $k$-vector field $\delta \in D^k_{V,W} \subset D^\bullet_{V,W}$ has *degree $k$.*

As an immediate corollary of Lemma 2.18 we have

**Lemma 2.23.** *The space of all polyvector fields on a* **k***-linear category* $\mathcal{C}$

$$D^\bullet = \bigoplus_{V,W \in Obj\,\mathcal{C}} D^\bullet_{V,W}$$

*forms a* $\mathcal{C}$*-bimodule.*

**Definition 2.24.** For a **k**-linear category $\mathcal{C}$ we define an associated *category of polyvector fields* $\mathcal{V}$ as a category with the same collection of objects $Obj\,\mathcal{V} = Obj\,\mathcal{C}$ as in category $\mathcal{C}$, morphisms

$$\hom_{\mathcal{V}}(W,V) = D^\bullet_{V,W},$$

and composition $\star$ defined as

$$\star: \quad D^\bullet_{V,W} \times D^\bullet_{W,X} \longrightarrow D^\bullet_{V,X}, \quad \rho_1 \times \rho_2 \;\mapsto\; \rho_1 \otimes_{\mathcal{C}} \rho_2$$

for all objects $V, W, X \in Obj\,\mathcal{V}$.

Associativity of the composition follows immediately from the associativity of the tensor product. Identity morphisms $1_V$ for all $V \in Obj\,\mathcal{V}$ are precisely the identity morphisms in $\mathcal{C}$, using the identification $D^0_{V,W} = \hom(W,V)$.

**Lemma 2.25.** *The category* $\mathcal{V}$ *of polyvector fields on a* $k$*-linear category* $\mathcal{C}$ *is graded by the degrees of vector fields.*

*Proof.* Indeed, for all objects $V, W \in Obj\,\mathcal{V}$ the space of morphisms (2.22) is graded and composition $\star$ is a graded map of degree zero

$$D^k_{V,W} \star D^m_{W,X} \subset D^{k+m}_{V,X}.$$

□

### 2.1.3 Traces of Polyvector Fields and Polyderivations

Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule for a **k**-linear category $\mathcal{C}$. One defines the trace map (over $\mathcal{C}$) as follows

$$\mathrm{tr}_{\mathcal{C}}: \quad \mathcal{M} \to \mathcal{M}_\natural := \left( \bigoplus_{X \in Obj\,\mathcal{C}} \mathcal{M}_{X,X} \right) / [\mathcal{M}, \mathcal{C}], \tag{2.26}$$

where

$$\mathrm{tr}_{\mathcal{C}}\, \mathcal{M}_{X,Y} = 0, \quad \text{for} \quad X \neq Y, \quad X, Y \in Obj\, \mathcal{C},$$

and $[\mathcal{M}, \mathcal{C}]$ is the $\mathbf{k}$-linear space spanned by the elements of the form

$$f \circ m - m \circ f \tag{2.27}$$

for all $A, B \in Obj\, \mathcal{C}$ and all $m \in \mathcal{M}_{A,B}$, $f \in \hom(A, B)$.

*Remark* 2.28. Note that $\mathcal{M}_\natural$ is not graded by objects of the category $\mathcal{C}$ as opposed to $\oplus_{X \in Obj\, \mathcal{C}} \mathcal{M}_{X,X}$. It happens because elements of the form (2.27) mix different homogeneous components.

For each pair of objects $V, W \in Mor\, \mathcal{C}$ we have introduced a space of polyvector fields. When the two objects $V$ and $W$ coincide with each other, the corresponding space of polyvector fields $D^\bullet_{V,V}$ has a nontrivial trace over $\mathcal{C}$

$$\mathrm{tr}_{\mathcal{C}} : \quad D^\bullet_{V,V} \to (D^\bullet)_\natural. \tag{2.29}$$

**Example 2.30.** Let $\delta \in D_{V,V} \subset D^\bullet_{V,V}$ be a vector field. From (2.3) we have

$$\delta : Mor\, \mathcal{C} \to \hom(V, -) \otimes \hom(-, V).$$

Let $f \in Mor\, \mathcal{C}$, in Sweedler notations we have $\delta(f) = \delta'(f) \otimes \delta''(f)$, and taking the trace of both sides we get

$$\mathrm{tr}_{\mathcal{C}}(\delta(f)) = \mathrm{tr}_{\mathcal{C}}\big(\delta'(f) \otimes \delta''(f)\big) = \mathrm{tr}_{\mathcal{C}}\big(\delta'(f) \circ \delta''(f) \otimes 1_{s(f)}\big)$$

So the trace $\mathrm{tr}_{\mathcal{C}}\, \delta$ of a $(V, V)$-vector field $\delta$ is equivalent to the map

$$\overline{\delta} : \quad Mor\, \mathcal{C} \to Mor\, \mathcal{C}, \qquad \overline{\delta}(f) = \delta'(f) \circ \delta''(f).$$

For all objects $A, B \in Obj\, \mathcal{C}$, the restriction

$$\overline{\delta} : \hom(A, B) \to \hom(A, B)$$

is a $\mathbf{k}$-linear map. Moreover, as a corollary of (2.3) we get that $\overline{\delta}$ satisfies the Leibnitz identity

$$\overline{\delta}(f \circ g) = f \circ \overline{\delta}(g) + \overline{\delta}(f) \circ g$$

for all morphisms $f, g \in Mor\, \mathcal{C}$ which are composable, i.e., $f \circ g$ is well defined.

Now, consider a closed chain of vector fields

$$\delta_1 \in D_{V,X_1}, \quad \delta_2 \in D_{X_1,X_2}, \quad \dots, \quad \delta_k \in D_{X_{k-1},V} \tag{2.31}$$

where $V, X_1, \dots, X_{k-1} \in Obj\,\mathcal{C}$ are some objects. Then $(\delta_1 \star \dots \star \delta_k) \in D^k_{V,V}$ is an elementary $(V,V)$-polyvector field of degree $k$. The trace $\mathrm{tr}_{\mathcal{C}}(\delta_1 \star \dots \star \delta_k)$ of this polyvector field is equivalent to the following map

$$\begin{aligned} \overline{\delta_1 \star \dots \star \delta_k} : \quad & (Mor\,\mathcal{C})^{\otimes k} \to (Mor\,\mathcal{C})^{\otimes k}, \\ f_1 \otimes \dots \otimes f_k \quad \mapsto \quad & (\delta'_k(f_k) \circ \delta''_1(f_1)) \otimes (\delta'_1(f_1) \circ \delta''_2(f_2)) \otimes \ \dots \ \otimes (\delta'_{k-1}(f_{k-1}) \circ \delta''_k(f_k)) \end{aligned} \tag{2.32}$$

for all $f_1, \dots f_k \in Mor\,\mathcal{C}$.

*Remark* 2.33. The order of morphisms on the right hand side of (2.32) is conventional. However, our choice guarantees that the sources of morphisms are not permuted. Namely, for all $X_i, Y_i \in Obj\,\mathcal{C}$ we have

$$\begin{aligned} \overline{\delta_1 \star \dots \star \delta_k} : \qquad & \hom(X_1, Y_1) \otimes \dots \otimes \hom(X_k, Y_k) \ \to \\ & \qquad \hom(X_1, Y_k) \otimes \dots \otimes \hom(X_k, Y_{k-1}). \end{aligned}$$

**Definition 2.34.** A map $\Delta : \ (Mor\,\mathcal{C})^{\otimes k} \to (Mor\,\mathcal{C})^{\otimes k}$ is called a *polyderivation* if

$$\begin{aligned} &\Delta(h_1 \otimes \dots \otimes \underset{\substack{\uparrow \\ j}}{f \circ g} \otimes \dots \otimes h_k) \\ &\quad = (1_{t(h_k)} \otimes \dots \otimes \underset{\substack{\uparrow \\ j+1}}{f} \otimes \dots \otimes 1_{t(h_{k-1})}) \circ \Delta(h_1 \otimes \dots \otimes \underset{\substack{\uparrow \\ j}}{g} \otimes \dots \otimes h_k) \\ &\qquad + \Delta(h_1 \otimes \dots \otimes \underset{\substack{\uparrow \\ j}}{f} \otimes \dots \otimes h_k) \circ (1_{s(h_1)} \otimes \dots \otimes \underset{\substack{\uparrow \\ j}}{g} \otimes \dots \otimes 1_{s(h_k)}) \end{aligned} \tag{2.35}$$

for all morphisms $f, g, h_1, \dots h_k \in Mor\,\mathcal{C}$ and all $1 \leq j \leq k$.

**Proposition 2.36.** *Let $\gamma \in D^k_{V,V}$ be a $(V,V)$-derivation of degree $k$, for some object $V \in Obj\,\mathcal{C}$. The map*

$$\overline{\gamma} : \quad (Mor\,\mathcal{C})^{\otimes k} \to (Mor\,\mathcal{C})^{\otimes k}$$

*is a polyderivation in the sense of Definition 2.34.*

*Proof.* First, note that $D^k_{V,V}$ is spanned by elementary polyvector fields of the form $\delta_1 \star \cdots \star \delta_k$, where $\delta_1, \ldots, \delta_k$ is a closed chain of vector fields (2.31). So it would be enough for us to prove the statement only for the case

$$\gamma = \delta_1 \star \cdots \star \delta_k,$$

and the rest will follow by linearity of (2.35).

Recall that each of the vector fields satisfy the double Leibnitz identity (2.3), in particular for $\delta_j \in D_{X_{j-1},X_j}$

$$\delta_j(f \circ g) = (f \otimes 1_{X_{j-1}}) \circ \delta(g) + \delta(f) \circ (1_{X_j} \otimes g). \tag{2.37}$$

Combining (2.37) with (2.32) we get

$$\begin{aligned}
&\overline{\delta_1 \star \cdots \star \delta_k}(h_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{f \circ g} \otimes \cdots \otimes h_k) \\
&\qquad = (1_{t(h_1)} \otimes \cdots \otimes \underset{\substack{\uparrow \\ j+1}}{f} \otimes \cdots \otimes 1_{t(h_k)}) \circ \overline{\delta_1 \star \cdots \star \delta_k}(h_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{g} \otimes \cdots \otimes h_k) \\
&\qquad\quad + \overline{\delta_1 \star \cdots \star \delta_k}(h_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{f} \otimes \cdots \otimes h_k) \circ (1_{s(h_1)} \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{g} \otimes \cdots \otimes 1_{s(h_k)}).
\end{aligned}$$

where $s(h_j)$ and $t(h_j)$ denote the source and target of morphism $h_j$ respectively. □

### 2.1.4 Category of Differential Forms

The $\mathcal{A}$-bimodule of noncommutative 1-forms over a general associative algebra $\mathcal{A}$ originally defined by M. Karoubi [Kar87] can be viewed as kernel of the multiplication map

$$\Omega^1(\mathcal{A}) = \ker\,\mu, \qquad \mu: \quad \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}. \tag{2.38}$$

*Remark* 2.39. The definition above was based on the earlier observation by D. Quillen, who have noted in [Qui70] that in the commutative case $\Omega^1(\mathcal{A})_\natural = \frac{\Omega^1(\mathcal{A})}{[\Omega^1(\mathcal{A}),\mathcal{A}]}$ is isomorphic to the module of Kähler differentials. When $\mathcal{A}$ becomes noncommutative, $\Omega^1(\mathcal{A})_\natural$ is no longer an $\mathcal{A}$-bimodule, so one should use (2.38) instead. Moreover, as noted in [CBEG07], after we have defined noncommutative vector fields as double derivations $Der(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$, it is precisely the $\Omega^1(A)$ which would satisfy the universal property.

The straightforward translation of the definition (2.38) to the context of categories reads:

**Definition 2.40.** For a $\mathbf{k}$-linear category $\mathcal{C}$, let $X, Y \in Obj\,\mathcal{C}$ be a pair objects. We define $\Omega^1_{X,Y}$, the space of *1-forms* associated to $(X, Y)$, as the kernel of the composition map. In other words, we have the following short exact sequence

$$0 \longrightarrow \Omega^1_{X,Y} \hookrightarrow \bigoplus_{V \in Obj\,\mathcal{C}} (\hom(V, X) \otimes \hom(Y, V)) \xrightarrow{\circ} \hom(Y, X) \longrightarrow 0. \quad (2.41)$$

Then the space of all 1-forms $\Omega^1$ and 0-forms $\Omega^0$ are defined as

$$\Omega^1 = \bigoplus_{X,Y \in Obj\,\mathcal{C}} \Omega^1_{X,Y}, \qquad \Omega^0 = \bigoplus_{X,Y \in Obj\,\mathcal{C}} \hom(Y, X). \quad (2.42)$$

**Lemma 2.43.** *The space of 1-forms $\Omega^1$ on a $\mathbf{k}$-linear category $\mathcal{C}$ comes equipped with a $\mathcal{C}$-bimodule structure. Moreover,*

$$0 \longrightarrow \Omega^1 \hookrightarrow \bigoplus_{X,Y,V \in Obj\,\mathcal{C}} (\hom(V, X) \otimes \hom(Y, V)) \xrightarrow{\circ} \Omega^0 \longrightarrow 0 \quad (2.44)$$

*is an exact sequence of $\mathcal{C}$-bimodules.*

*Proof.* Let

$$\beta = \sum_{i=1}^{k} f_i \otimes g_i \quad \subset \quad \bigoplus_{V \in Obj\,\mathcal{C}} (\hom(V, X) \otimes \hom(Y, V))$$

be an arbitrary linear combination of composable morphisms. We set

$$x.\beta.y = \sum_{i=1}^{k} (x \circ f_i) \otimes (g_i \circ y) \quad (2.45)$$

for all $x \in \hom(X, \_)$, $y \in \hom(\_, Y)$. It is easy to see that (2.45) is a $\mathcal{C}$-bimodule action.

Then by associativity of composition we get

$$\circ(x.\beta.y) = \sum_{i=1}^{k} (x \circ f_i \circ g_i \circ y) = x.(\circ(\beta)).y.$$

So the composition map is a homomorphism of $\mathcal{C}$-bimodules, and as a corollary (2.44) is an exact sequence of $\mathcal{C}$-bimodules. $\square$

For $k > 2$ we define *k-forms* associated to $X, Y \in Obj\, \mathcal{C}$ to be

$$\Omega^k_{X,Y} = \bigoplus_{Z_1,\dots,Z_{k-1} \in Obj\, \mathcal{C}} \Omega^1_{X,Z_1} \otimes_{\mathcal{C}} \Omega^1_{Z_1,Z_2} \otimes_{\mathcal{C}} \quad \dots \quad \otimes_{\mathcal{C}} \Omega^1_{Z_{k-1},Y}$$

and denote the space of all forms associated to $(X, Y)$ by

$$\Omega^\bullet_{X,Y} = \bigoplus_{k=0}^{\infty} \Omega^k_{X,Y}.$$

**Lemma 2.46.** *For all $k \geq 0$, the space of all k-forms*

$$\Omega^k = \bigoplus_{X,Y} \Omega^k_{X,Y}$$

*comes equipped with a $\mathcal{C}$-bimodule structure.*

*Proof.* Completely analogous to the proof of Lemma 2.18. □

As an immediate corollary, the space of all forms on a **k**-linear category $\mathcal{C}$

$$\Omega^\bullet = \bigoplus_{k=0}^{\infty} \Omega^k$$

becomes a $\mathcal{C}$-bimodule graded by the degree $k$ of a differential form.

**Definition 2.47.** For a **k**-linear category $\mathcal{C}$ we define an associated *category of poly-vector fields* $\mathcal{K}$ as a category with the same collection of objects $Obj\, \mathcal{K} = Obj\mathcal{C}$ as in category $\mathcal{C}$, morphisms

$$\hom_{\mathcal{K}}(Y, X) = \Omega^\bullet_{X,Y}$$

and composition "." defined as

$$\text{“.”} \;:\quad \Omega^\bullet_{X,Y} \times \Omega^\bullet_{Y,Z} \longrightarrow \Omega^\bullet_{X,Z}, \qquad \omega_1 \times \omega_2 \mapsto \omega_1 \otimes_{\mathcal{C}} \omega_2$$

for all objects $X, Y, Z \in Obj\, \mathcal{K}$.

### 2.1.5 Duality and Evaluation Map

To evaluate differential forms on vector fields we will show that $\Omega^1_{X,Y}$ satisfies a certain universal property which generalizes the universal property of Kähler differentials. For each object $V \in Obj\,\mathcal{C}$ we introduce the *uniderivation* $\partial_V$ as follows:

$$\partial_V : \quad Mor\,\mathcal{C} \to \hom(V,-) \otimes \hom(-,V)$$

$$\partial_V(f) = \begin{cases} f \otimes 1_V - 1_V \otimes f & t(f) = V, \quad s(f) = V, \\ -1_V \otimes f & t(f) = V, \quad s(f) \neq V, \\ f \otimes 1_V & t(f) \neq V, \quad s(f) = V, \\ 0 & t(f) \neq V, \quad s(f) \neq V. \end{cases} \tag{2.48}$$

for all $f \in Mor\,\mathcal{C}$.

**Lemma 2.49.** *The map defined in (2.48) is a* $(V,V)$*-vector field:* $\partial_V \in D_{V,V}$*.*

*Proof.* We have to show that for all composable morphisms $f, g \in Mor\,\mathcal{C}$

$$\partial_V(f \circ g) = (f \otimes 1_V) \circ \partial_V(g) + \partial_V(f) \circ (1_V \otimes g). \tag{2.50}$$

When $t(f) = V$, both sides of (2.50) receive equal contributions $-1_V \otimes (f \circ g)$. When $s(f) = t(g) = V$, the right hand side of (2.50) receives a trivial contribution $f \otimes g - f \otimes g = 0$ while the left hand side remain unchanged. Finally, when $s(g) = V$, both sides of (2.50) receive an equal contribution $(f \circ g) \otimes 1_V$. Altogether, we have taken into account all $2^3 = 8$ possible cases. □

**Corollary 2.51.** *For any object* $V \in Obj\,\mathcal{C}$*, the map* $\overline{\partial_V}$ *associated to the trace of the corresponding uniderivation satisfies the Leibniz identity*

$$\overline{\partial_V}(f \circ g) = f \circ \overline{\partial_V}(g) + \overline{\partial_V}(f) \circ g$$

*for all composable morphisms* $f, g \in Mor\,\mathcal{C}$*.*

**Definition 2.52.** Define $d : Mor\,\mathcal{C} \to Mor\,\mathcal{C} \otimes Mor\,\mathcal{C}$ by

$$d = \sum_{V \in Obj\,\mathcal{C}} \overline{\partial_V}. \tag{2.53}$$

We refer to $d$ as a *universal derivation* on $\mathcal{C}$.

Note that the sum in (2.53) is well defined since for any particular $f \in Mor\,\mathcal{C}$ we have at most two terms with nontrivial contribution. Indeed,

$$d(f) = 1_{t(f)} \otimes f - f \otimes 1_{s(f)} \tag{2.54}$$

for all morphisms $f \in Mor\,\mathcal{C}$.

By Corollary 2.51, the universal derivation on $\mathcal{C}$ satisfies the Leibnitz identity

$$d(f \circ g) = f \circ d(g) + d(f) \circ g$$

for all composable morphisms $f, g \in Mor\,\mathcal{C}$.

Formula (2.54) allows us to describe the image of the universal derivation $d(Mor\,\mathcal{C})$. For all objects $X, Y \in Obj\,\mathcal{C}$ we have

$$d: \ \hom(Y, X) \ \hookrightarrow \ \Omega^1_{X,Y},$$

Moreover, the restriction of a universal derivation $d$ considered above is a $\mathbf{k}$-linear map, so by (2.42) we have

$$d\Omega^0 \subset \Omega^1. \tag{2.55}$$

In other words, the image $d\Omega^0$ can be viewed as a subspace of exact forms in the space $\Omega^1$ of all 1-forms on $\mathcal{C}$.

*Remark* 2.56. Recall that in commutative geometry, the space of Kähler differentials is generated by exact forms *df* as a module over the coordinate ring. The analogous statement holds for (2.55) on a category $\mathcal{C}$.

**Lemma 2.57.** *The subspace $d\Omega^0 \subset \Omega^1$ of exact 1-forms on a $\mathbf{k}$-linear category $\mathcal{C}$ generates the space $\Omega^1$ of all 1-forms as a left $\mathcal{C}$-module (resp. right module or bimodule).*

*Proof.* Below we consider only left action, the other case is essentially equivalent. Let

$$\omega = \sum_{i=1}^{k} f_i \otimes g_i \quad \in \quad \Omega^1_{X,Y}$$

where $f_i, g_i \in Mor\,\mathcal{C}$ for $1 \le i \le k$. By (2.41) we thus have $\sum_{i=1}^{k} f_i \circ g_i = 0 \in \hom(Y, X)$. On the other hand

$$\omega = \sum_{i=1}^{k} f_i \otimes g_i = \sum_{i=1}^{k} (f_i \otimes g_i - (f_i \circ g_i) \otimes 1_Y) = -\sum_{i=1}^{k} f_i \,.\, d(g_i)$$

$\square$

**Proposition 2.58** (Universal property of 1-forms)**.** *Fix arbitrary objects $V, W \in Obj\,\mathcal{C}$. Then for each $(V,W)$-vector field $\delta \in D_{V,W}$ there is a unique homomorphism of $\mathcal{C}$-bimodules $\varphi_\delta$ which makes the following diagram commutative*

$$\begin{tikzcd} D^0 = \Omega^0 = \bigoplus_{X,Y \in Obj\,\mathcal{C}} \hom(Y,X) \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega^1 \arrow[d, "\exists!\,\varphi_\delta"] \\ & \bigoplus_{X,Y \in Obj\,\mathcal{C}} \hom(W,X) \otimes \hom(Y,V) \end{tikzcd} \tag{2.59}$$

*Proof.* Let us first prove the uniqueness. Assume both $\varphi_\delta$ and $\widetilde{\varphi_\delta}$ are $\mathcal{C}$-bimodule homomorphisms which make (2.59) commutative. Then their difference $\psi = \varphi - \widetilde{\varphi_\delta}$ necessarily vanishes on the image of $d$:

$$\psi(d(h)) = \psi(1_{t(h)} \otimes h - h \otimes 1_{s(h)}) = 0$$

for all $h \in Mor\,\mathcal{C}$. On the other hand, by Lemma 2.57 we know that $\Omega^1$ is generated as a $\mathcal{C}$-bimodule by $d\Omega^0$. Since $\varphi_\delta$ is a $\mathcal{C}$-bimodule homomorphism, we conclude that it vanishes on all 1-forms and as a consequence

$$\varphi_\delta(\omega) = \widetilde{\varphi_\delta}(\omega)$$

for all $\omega \in \Omega^1$.

To show the existence, recall that all of the spaces in diagram (2.59) are doubly graded by the objects of $\mathcal{C}$, and both $d$ and $\delta$ are graded maps of degree zero. So it will be enough for us to define $\varphi_\delta$ on each homogeneous component $\Omega^1_{X,Y}$ and then extend it to $\Omega^1$ by linearity. In other words, we have to show that for all objects $X, Y \in Obj\,\mathcal{C}$ there exists $\phi$ which makes the following diagram commutative.

$$\begin{tikzcd} \hom(Y,X) \arrow[r, "d"] \arrow[dr, "\delta"'] & \Omega^1_{X,Y} \arrow[d, "\exists\,\phi"] \\ & \hom(W,X) \otimes \hom(Y,V) \end{tikzcd} \tag{2.60}$$

By the definition of $\Omega^1_{X,Y}$ in (2.41) we can view it as a subspace of the direct sum

$$\Omega^1_{X,Y} \subset \bigoplus_{Z \in Obj\,\mathcal{C}} \hom(Z,X) \otimes \hom(Y,Z).$$

Define $\phi$ on the full space

$$\phi : \quad \bigoplus_{Z \in Obj\,\mathcal{C}} \hom(Z,X) \otimes \hom(Y,Z) \quad \longrightarrow \quad \hom(W,X) \otimes \hom(Y,V)$$

as follows: For each object $Z \in Obj\,\mathcal{C}$, and each pair of morphisms $f \in \hom(Z,X)$, $g \in \hom(Y,Z)$ set

$$\phi(f \otimes g) = (f \otimes 1_V) \circ \delta(g). \tag{2.61}$$

Since we have constructed the map $\phi$ for an arbitrary pair of objects $X, Y \in Obj\,\mathcal{C}$ it can be defined on the direct sum $\Omega^1 = \bigoplus_{X,Y \in Obj\,\mathcal{C}} \Omega^1_{X,Y}$.

We claim that $\phi$ when restricted to the subspace $\Omega^1_{X,Y}$ will make diagram (2.60) commutative. Indeed, for arbitrary $h \in \hom(Y,X)$, we have

$$\phi(d(h)) \overset{(2.54)}{=} \phi(1_X \otimes h - h \otimes 1_Y) \overset{(2.61)}{=} (1_X \otimes 1_V) \circ \delta(h) + (f \otimes 1_V) \circ \delta(1_Y) = \delta(h).$$

Here we have used the fact that $\delta(1_Y) = 0$, which is an immediate consequence of double Leibnitz identity (2.3).

The next thing we prove is that $\phi$ is a homomorphism of $\mathcal{C}$-bimodules. Indeed, let

$$\omega = \sum_{i=1}^{k} f_i \otimes g_i \quad \in \quad \Omega^1_{X,Y}$$

where $f_i, g_i \in Mor\,\mathcal{C}$ for $1 \leq i \leq k$. Then for all morphisms $x \in \hom(X, \_)$ and $y \in \hom(\_, Y)$ we get

$$\begin{aligned}
\phi\big(x.\omega.y\big) &\overset{(2.45)}{=} \phi\Big(\sum_{i=1}^{k}(x \circ f_i) \otimes (g_i \circ y)\Big) = \sum_{i=1}^{k}((x \circ f_i) \otimes 1_V) \circ \delta(g_i \circ y) \\
&\overset{(2.3)}{=} \sum_{i=1}^{k} \big(((x \circ f_i \circ g_i) \otimes 1_V) \circ \delta(y) + ((x \circ f_i) \otimes 1_V) \circ \delta(g_i) \circ (1_W \otimes g_i)\big) \\
&\overset{(2.41)}{=} \sum_{i=1}^{k}((x \circ f_i) \otimes 1_V) \circ \delta(g_i) \circ (1_W \otimes g_i) = x.\phi(\omega).y.
\end{aligned}$$

So $\phi$ is a homomorphism of $\mathcal{C}$-bimodules. Together with uniqueness which we proved earlier this gives the statement of the Proposition. $\square$

Proposition 2.59 allows one to define an evaluation map of 1-forms on vector fields. Indeed, by universal property we have

$$\begin{gathered} \langle , \rangle : \quad D_{V,W} \otimes \Omega^1_{X,Y} \longrightarrow \hom(W,X) \otimes \hom(Y,V), \\ \langle \delta, \omega \rangle = \varphi_\delta(\omega). \end{gathered} \tag{2.62}$$

**Proposition 2.63.** *The evaluation map (2.62) defines a homomorphism of $\mathcal{C}$-quadmodules*

$$\langle , \rangle : \quad D^1 \otimes \Omega^1 \longrightarrow \bigoplus_{V,W,X,Y \in Obj\,\mathcal{C}} \hom(W,X) \otimes \hom(Y,V)$$

*Namely, the following holds*

$$\langle a \star \delta \star b, c.\omega.d \rangle = (c \otimes a) \circ \langle \delta, \omega \rangle \circ (b \otimes d). \tag{2.64}$$

*for all $A, B, C, D, V, W, X, Y \in Obj\,\mathcal{C}$, for all $\omega \in \Omega^1_{X,Y}, \delta \in D^1_{V,W}$, and all* $a \in \hom(V,A), b \in \hom(B,W), c \in \hom(X,C), d \in \hom(D,Y)$.

*Proof.* From (2.62) we have

$$\langle a \star \delta \star b, c.\omega.d \rangle = \varphi_{a\star\delta\star b}(c.\omega.d). \tag{2.65a}$$

Recall that $\varphi$ was defined as a map which makes diagram (2.60) commutative, so by Lemma 2.10 we have further

$$\varphi_{a\star\delta\star b}(c.\omega.d) = (1_C \otimes a) \circ \varphi_\delta(c.\omega.d) \circ (b \otimes 1_D). \tag{2.65b}$$

Next, by Proposition 2.58 we know that $\varphi_\delta$ is a $\mathcal{C}$-bimodule homomorphism. This gives

$$\varphi_\delta(c.\omega.d) = (c \otimes 1_V) \circ \varphi_\delta(\omega) \circ (1_W \otimes d). \tag{2.65c}$$

Combining (2.65a), (2.65b), and (2.65c) we get precisely (2.64). □

From Lemmas 2.18 and 2.43 we know that $D^1 \otimes \Omega^1$ comes equipped with a $\mathcal{C} \times \mathcal{C}$-bimodule structure. Together with Proposition 2.63 this allows one to extend evaluation map on $k$-vector fields and $k$-forms as follows. Pick arbitrary objects $X_0, \ldots, X_k$,

$V_0, \dots, V_k$ in $Obj\,\mathcal{C}$ and let $\omega_1, \dots, \omega_k$ and $\delta_1, \dots, \delta_k$ be some chains of 1-forms and vector fields respectively

$$\omega_1 \in \Omega^1_{X_0,X_1}, \quad \dots, \quad \omega_k \in \Omega^1_{X_{k-1},X_k},$$
$$\delta_1 \in D_{V_0,V_1}, \quad \dots, \quad \delta_k \in D_{V_{k-1},V_k}.$$

We set

$$\Big\langle\, \delta_1 \star \delta_2 \star \dots \star \delta_k,\ \omega_1.\omega_2.\ \cdots\ .\omega_k \Big\rangle = \langle \omega_1, \delta_1 \rangle \otimes_{\mathcal{C}\times\mathcal{C}} \cdots \otimes_{\mathcal{C}\times\mathcal{C}} \langle \omega_k, \delta_k \rangle. \tag{2.66}$$

**Corollary 2.67.** *The evaluation map (2.66) defines a $\mathcal{C}\times\mathcal{C}$-bimodule homomorphism*

$$\langle , \rangle : \quad \Omega^k \otimes D^k \ \to \bigoplus_{X_0,X_k,V_0,V_k \in Obj\,\mathcal{C}} \hom(V_k, X_0) \otimes \hom(X_k, V_0). \tag{2.68}$$

This corollary allows us to introduce three factorizations of the evaluation map:

- Taking the trace (2.26) of the first $\mathcal{C}$-bimodule we define a map $\langle , \rangle_1$, which makes the following diagram commutative.

$$\begin{tikzcd}
D^k \otimes \Omega^k \arrow[r, "\langle,\rangle"] \arrow[d, "\mathrm{tr}_{\mathcal{C}}\otimes \mathrm{Id}"'] & \displaystyle\bigoplus_{V_0,V_k,X_0,X_k \in Obj\,\mathcal{C}} \hom(V_k, X_0) \otimes \hom(X_k, V_0) \arrow[d, "\mathrm{tr}_{\mathcal{C}outer}"] \\
D^k_\natural \otimes \Omega^k \arrow[r, "\langle,\rangle_1"] & \displaystyle\bigoplus_{X_0,X_k \in Obj\,\mathcal{C}} \hom(X_k, X_0)
\end{tikzcd}$$

  As a corollary $\langle , \rangle_1$ is a $\mathcal{C}$-bimodule homomorphism with respect to the remaining $\mathcal{C}$-bimodule structure. In particular, it is bigraded by objects of $\mathcal{C}$, namely

$$\langle , \rangle_1 : \quad D^k_{V_0,V_k} \otimes \Omega^k_\natural \to \hom(V_k, V_0)$$

  for all pairs of objects $V_0, V_k \in Obj\,\mathcal{C}$.

- Similarly, taking the trace of the second $\mathcal{C}$-bimodule we define a $\mathcal{C}$-bimodule homomorphism $\langle,\rangle_2$.

$$\begin{tikzcd}
D^k \otimes \Omega^k \arrow[r, "\langle,\rangle"] \arrow[d, "\mathrm{Id}\otimes\mathrm{tr}_{\mathcal{C}}"] & \bigoplus_{V_0,V_k,X_0,X_k\in Obj\,\mathcal{C}} \hom(V_k,X_0)\otimes\hom(X_k,V_0) \arrow[d, "\mathrm{tr}_{\mathcal{C}inner}"] \\
D^k \otimes \Omega^k_\natural \arrow[r, "\langle,\rangle_2"] \arrow[d, "\mathrm{tr}_{\mathcal{C}}"] & \bigoplus_{V_0,V_k\in Obj\,\mathcal{C}} \hom(V_k,V_0) \arrow[d, "\mathrm{tr}_{\mathcal{C}}"] \\
D^k_\natural \otimes \Omega^k_\natural \arrow[r, "\overline{\langle,\rangle}"] & \mathbf{k}
\end{tikzcd}$$

- Finally, taking the trace in both components we get a $\mathbf{k}$-linear evaluation map $\overline{\langle,\rangle}$

In particular, the evaluation map $\overline{\langle,\rangle}$ allows us to evaluate closed chains of differential forms on closed chains of vector fields.

*Remark* 2.69. When $\mathcal{C}$ is a freely generated category by a double of the quiver, the evaluation map $\overline{\langle,\rangle}$ becomes essentially equivalent to the contraction map on the so-called noncommutative cotangent bundle suggested in [CBEG07].

## 2.2 Double (Quasi) Poisson Brackets on Categories

### 2.2.1 Double Derivations and Skew-Symmetry

Recall that according to Definition 2.34 we call map

$$R: \quad (Mor\,\mathcal{C})^{\otimes 2} \to (Mor\,\mathcal{C})^{\otimes 2} \tag{2.70}$$

a *biderivation* if for all morphisms $f,g,h \in Mor\,\mathcal{C}$

$$\begin{aligned} R((f\circ g)\otimes h) =& (1_{t(h)}\otimes f)\circ R(g\otimes h) + R(f\otimes h)\circ(g\otimes 1_{s(h)}) \quad \text{when} \quad t(g)=s(f), \\ R(f\otimes(g\circ h)) =& (g\otimes 1_{t(f)})\circ R(f\otimes h) + R(f\otimes g)\circ(1_{s(f)}\otimes h) \quad \text{when} \quad t(h)=s(g). \end{aligned} \tag{2.71}$$

In particular, from (2.71) we conclude that the restriction

$$R: \quad \hom(X_1,Y_1)\otimes\hom(X_2,Y_2)) \;\to\; \hom(X_1,Y_2)\otimes\hom(X_2,Y_1) \tag{2.72}$$

is a $\mathbf{k}$-linear map for all objects $W,X,Y,Z \in Obj\,\mathcal{C}$.

**Definition 2.73.** [VdB08] We call a biderivation $R$ *skew-symmetric* if for all morphisms $f, g \in Mor\,\mathcal{C}$ we have

$$R(f \otimes g) = -\big(R(g \otimes f)\big)^{op}. \tag{2.74}$$

Here $(-)^{op}$ is a permutation of components in the tensor product: $(a \otimes b)^{op} = b \otimes a$ for all $a, b \in Mor\,C$.

Note that the skew symmetry condition (2.74) is compatible with property (2.72), i.e. both sides of (2.74) are elements of $\hom(s(f), t(g)) \otimes \hom(s(g), t(f))$.

### 2.2.2 Yang-Baxter Operator and Quasi Brackets

For each $\mathbf{k}$-linear map (2.70) one can associate a family of maps for all $1 \leq i, j \leq n$, defined as

$$\begin{aligned} R_{i,j}: \quad & (Mor\,\mathcal{C})^{\otimes n} \to (Mor\,\mathcal{C})^{\otimes n}, \\ R_{i,j}(f_1 \otimes \cdots \otimes f_n) = & f_1 \otimes \cdots \otimes \underbrace{R'(f_i \otimes f_j)}_{i} \otimes \cdots \otimes \underbrace{R''(f_i \otimes f_j)}_{j} \otimes \cdots \otimes f_n. \end{aligned} \tag{2.75}$$

Here we have used Sweedler notations $R(f_i \otimes f_j) = R'(f_i \otimes f_j) \otimes R''(f_i \otimes f_j)$ to omit summation index in a tensor product.

**Lemma 2.76.** *Let $R$ be a skew-symmetric biderivation, i.e. satisfying (2.71) and (2.74) on a $\mathbf{k}$-linear category $\mathcal{C}$. Then operator $Y_R$ defined as*

$$\begin{aligned} Y_R: \quad & (Mor\,\mathcal{C})^{\otimes 3} \to (Mor\,\mathcal{C})^{\otimes 3} \\ Y_R = & R_{1,2} \circ R_{2,3} + R_{2,3} \circ R_{3,1} + R_{3,1} \circ R_{1,2}. \end{aligned}$$

*becomes a triderivation in the sense of Definition 2.34.*

*Proof.* First, note that by (2.72) for all objects $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in Obj\,\mathcal{C}$ we have

$$\begin{aligned} Y_R: \quad & \hom(X_1, Y_1) \otimes \hom(X_2, Y_2) \otimes \hom(X_3, Y_3) \;\to \\ & \qquad \hom(X_1, Y_3) \otimes \hom(X_2, Y_1) \otimes \hom(X_3, Y_2) \end{aligned}$$

so the right hand side of Leibnitz identity (2.35) is well-defined. From formula (2.75) it follows that $Y_R$ commutes with the cyclic permutations of monomials, so it would be enough for us to prove the Leibnitz identity w.r.t. the first argument only

$$\begin{aligned} Y_R(f_1 \circ f_2 \otimes g \otimes h) - (1_{t(h)} \otimes f_1 \otimes 1_{t(g)}) \circ Y_R(f_2 \otimes g \otimes h) \\ - Y_R(f_1 \otimes g \otimes h) \circ (f_2 \otimes 1_{s(g)} \otimes 1_{s(h)}) \\ = R''(h, f_2) \otimes R'(g, f_1) \otimes R''(g, f_1) \circ R'(h, f_2) \\ + R''(h, f_2) \otimes R''(f_1, g) \otimes R'(f_1, g) \circ R'(h, f_2) \overset{(2.74)}{=} 0. \end{aligned}$$

□

**Definition 2.77.** We refer to a map $R : (Mor\,\mathcal{C})^{\otimes 2} \to (Mor\,\mathcal{C})^{\otimes 2}$ as a **Double Poisson Bracket** on a **k**-linear category $\mathcal{C}$ if it satisfies the following conditions

- Double Leibnitz Identity (2.71)
- Skew-Symmetry (2.74)
- Jacobi Idenity

  $$Y_R(f \otimes g \otimes h) = 0$$

  for all $f, g, h \in Mor\,\mathcal{C}$.

**Definition 2.78.** We refer to a map $R : (Mor\,\mathcal{C})^{\otimes 2} \to (Mor\,\mathcal{C})^{\otimes 2}$ as a **Double Quasi-Poisson Bracket** on a **k**-linear category $\mathcal{C}$ if it satisfies:

- Double Leibnitz Identity (2.71)
- Skew-Symmetry (2.74)
- Quasi Jacobi Identity

$$Y_R = \frac{1}{4} \sum_{V \in Obj\,\mathcal{C}} \overline{\partial_V \star \partial_V \star \partial_V}. \tag{2.79}$$

where $\partial_V$ denotes the uniderivation associated to an object $V \in Obj\,\mathcal{C}$, which and was defined in (2.48).

Note that the sum on the right hand side of (2.79) is well-defined as an operator on $(Mor\,\mathcal{C})^{\otimes 3}$; this follows from (2.48). Indeed, when the sum is applied to any pure tensor product of morphisms, only finitely many terms give a nontrivial contribution.

### 2.2.3 Derivations and Localization

The notion of a localization of a category was first introduced in [GZ67] (for a modern review see [KS05]). We would be interested in a **k**-linear version of the definition

**Definition 2.80.** Let $W \subset Mor\,\mathcal{C}$ be a collection of morphisms of a **k**-linear category $\mathcal{C}$. We refer to the pair $(\mathcal{C}[W^{-1}], \mathcal{F})$ consisting of a **k**-linear category $\mathcal{C}[W^{-1}]$ and a **k**-linear faithful functor $\mathcal{F}:\ \mathcal{C} \to \mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ w.r.t. $W$ if

- Every morphism $f \in \mathcal{F}[W^{-1}]$ in the image of $W$ is an isomorphism in a category $\mathcal{C}[W^{-1}]$.
- For any **k**-linear category $\mathcal{E}$ and any **k**-linear functor $\mathcal{G}:\ \mathcal{C} \to \mathcal{E}$ such that all morphisms in $\mathcal{G}(W)$ are invertible, there is a unique functor $\Phi : \mathcal{C}[W^{-1}] \to \mathcal{E}$ which makes the following diagram commutative

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{F}"] \arrow[dr, "\mathcal{G}"'] & \mathcal{C}[W^{-1}] \arrow[d, "\Phi"] \\ & \mathcal{E} \end{tikzcd}$$

From the definition it immediately follows that when a localization exists, it is unique up to a **k**-linear isomorphism. In the current subsection we would leave aside rather complicated question of existence of $\mathcal{C}[W^{-1}]$ for a given collection of morphisms $W \subset Mor\,\mathcal{C}$ and prove the following

**Lemma 2.81.** *Let $\mathcal{C}[W^{-1}]$ be a localization of a **k**-linear category $\mathcal{C}$ with respect to $W$. For every polyderivation*

$$\Delta : \quad (Mor\,\mathcal{C})^{\otimes n} \to (Mor\,\mathcal{C})^{\otimes n}$$

*satisfying the double Leibnitz identity (2.35) there is a unique extension*

$$\widetilde{\Delta} : \quad \big(Mor\,\mathcal{C}[W^{-1}]\big)^{\otimes n} \to \big(Mor\,\mathcal{C}[W^{-1}]\big)^{\otimes n}$$

*satisfying (2.35), which makes the following diagram commutative*

$$\begin{array}{ccc} (Mor\,\mathcal{C})^{\otimes n} & \xrightarrow{\quad\Delta\quad} & (Mor\,\mathcal{C})^{\otimes n} \\ \Big\downarrow{\scriptstyle\mathcal{F}} & & \Big\downarrow{\scriptstyle\mathcal{F}} \\ \big(Mor\,\mathcal{C}[W^{-1}]\big)^{\otimes 2} & \xrightarrow{\quad\widetilde{\Delta}\quad} & \big(Mor\,\mathcal{C}[W^{-1}]\big)^{\otimes n} \end{array} \tag{2.82}$$

*Proof.* Indeed, if such $\widetilde{\Delta}$ exists, it necessarily satisfies

$$\begin{aligned} &\widetilde{\Delta}(h_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{(\mathcal{F}w)^{-1}} \otimes \cdots \otimes h_n) = -(1_{t(h_n)} \otimes \cdots \otimes \underset{\substack{\uparrow \\ j+1}}{(\mathcal{F}w)^{-1}} \otimes \cdots \otimes 1_{t_{h_{n-1}}}) \\ &\quad \circ \widetilde{\Delta}(h_1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{(\mathcal{F}w)} \otimes \cdots \otimes h_n) \circ (1_{s(h_1)} \otimes \cdots \otimes \underset{\substack{\uparrow \\ j}}{(\mathcal{F}w)^{-1}} \otimes \cdots \otimes 1_{s(h_n)}). \end{aligned} \tag{2.83}$$

On the other hand, $\mathcal{C}[W^{-1}]$ is generated by morphisms of the form $\{\mathcal{F}f \,|\, f \in Mor\,\mathcal{C}\}$ and $\{(\mathcal{F}w)^{-1} \,|\, w \in W\}$, so we conclude that $\widetilde{\Delta}$ exists and unique. □

In what follows we say that $\widetilde{\Delta}$ is an *extension* of $\Delta$ on $\mathcal{C}[W^{-1}]$.

**Proposition 2.84.** *Let $R$ be a Double Quasi Poisson bracket on a $\mathbf{k}$-linear category $\mathcal{C}$ in the sense of Definition 2.78 and let $\mathcal{C}[W^{-1}]$ be a localization of $\mathcal{C}$ with respect to $W$. Then there is a unique Double Quasi Poisson bracket*

$$\widetilde{R}: \quad (Mor\,\mathcal{C}[W^{-1}])^{\otimes 2} \to (Mor\,\mathcal{C}[W^{-1}])^{\otimes 2} \tag{2.85}$$

*extending $R$ on $\mathcal{C}[W^{-1}]$.*

*Proof.* Indeed, by Lemma 2.81 we conclude that there is a unique biderivation (2.85) extending $R$. Next, using Proposition 2.36 and Lemma 2.76 we conclude that

$$Y_{\widetilde{R}} - \frac{1}{4} \sum_{V \in Obj\,\mathcal{C}} \overline{\partial_V \star \partial_V \star \partial_V} \tag{2.86}$$

is a triderivation on $(Mor\,\mathcal{C}[W^{-1}])^{\otimes 3}$ which vanishes on $(Mor\,\mathcal{C})^{\otimes 3}$. As a result (2.86) vanishes identically on $(Mor\,\mathcal{C}[W^{-1}])^{\otimes 3}$ and thus $\widetilde{R}$ is a Double Quasi Poisson bracket.

□

# Chapter 3

# Categorification of Cluster Algebras

## 3.1 Double Quasi Poisson Brackets for Conjugate Surfaces

### 3.1.1 Graph Connections and Cluster Algebras

A *ribbon graph* $\Gamma$ is defined as a graph with an additional structure, namely, for each vertex of $\Gamma$ we fix a cyclic order of edges adjacent to this vertex. Hereinafter we require that $\Gamma$ has only one connected component, assuming that the generalization to the case of several connected components is straightforward.

One can associate an oriented surface with boundary $\mathcal{S}_\Gamma$ to a ribbon graph by replacing each edge of $\Gamma$ by a ribbon and replacing each vertex by a disc with ribbons attached according to the cyclic order.

(a) Ribbon Graph

(b) Disc in $\mathcal{S}_\Gamma$ corresponding to the vertex.

Figure 3.1: Surface with boundary $S_\Gamma$ associated to a ribbon graph.

**Definition 3.1** ([GK13])**.** A *conjugate surface* $\hat{\mathcal{S}}_\Gamma$ associated to the ribbon graph $\Gamma$ is a surface corresponding to the ribbon graph with reversed cyclic order of edges at each vertex.

Both $\hat{\mathcal{S}}_\Gamma$ and $\mathcal{S}_\Gamma$ have the same fundamental group as the underlying graph

$$\pi_1(\hat{\mathcal{S}}_\Gamma) = \pi_1(\mathcal{S}_\Gamma) = \pi_1(\Gamma). \tag{3.2}$$

The identification (3.2) allows one to introduce two different Poisson structures on the character variety of $\pi_1(\Gamma)$.

**Example 3.3.** The coordinate ring of one dimensional representations of $\pi_1(\Gamma)$ is freely generated by finitely many holonomies. Indeed, pick closed loops in general position $M_1, \ \ldots, \ M_n$ on $\hat{\mathcal{S}}_\Gamma$ which freely generate the fundamental group of a graph $\Gamma$. Then each representation

$$\varphi \in Hom(\pi_1(\mathcal{S}_\Gamma), \mathbb{C}^\times)$$

is determined by

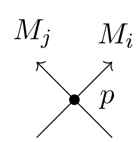
$$x_1 = \varphi(M_1), \quad \ldots, \quad x_n = \varphi(M_n).$$

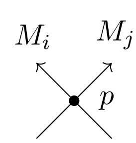
We can equip $\mathbb{C}[x_1, \ldots, x_n]$ with a Poisson bracket as follows

$$\begin{aligned} \{,\}: \quad & \big(\mathbb{C}[x_1, \ldots, x_n]\big)^{\otimes 2} \to \mathbb{C}[x_1, \ldots, x_n] \\ \{x_i, x_j\} = & \sum_p \epsilon_{i,j}(p) x_i x_j \end{aligned}$$

where the sum is taken over all intersection points of $M_i$ and $M_j$ on $\hat{\mathcal{S}}_\Gamma$

$$\epsilon_{ij}(p) = \begin{cases} +1 \\ -1 \end{cases}$$




Now let $\Sigma$ be an oriented surface, possibly with boundary, with a fixed collection $V_1, \ldots, V_n$ of distinct marked points. For each ideal triangulation with vertices precisely at $V_1, \ldots, V_n$ of $\Sigma$ we can associate a bipartite ribbon graph $\Gamma$ as follows:

- Each vertex of the triangulation would correspond to a white vertex of $\Gamma$.
- Each face of the triangulation would correspond to a black vertex of $\Gamma$.
- A white vertex is connected to a black vertex if and only if they belong to the same triangle.

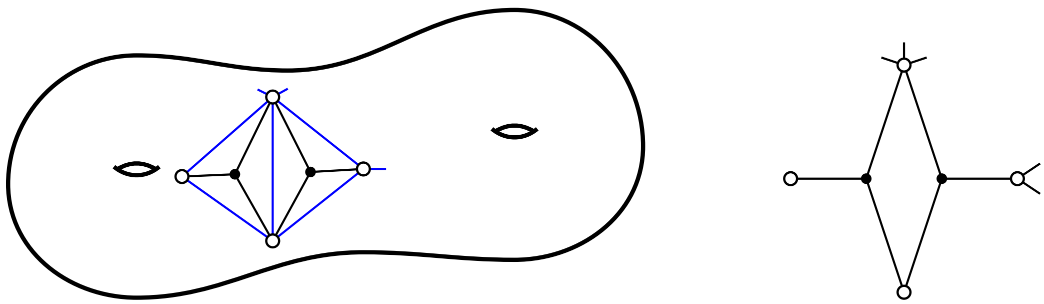


Figure 3.2: Bipartile ribbon graph associated to triangulation of surface $\Sigma$.

As a result we obtain a bipartite ribbon graph with trivalent black vertices. Similarly, each bipartite ribbon graph with trivalent black vertices defines a triangulation of some surface $\Sigma$, in general, with boundary.

Different ideal triangulations of the same surface $\Sigma$ are related by a sequence of rectangular moves, where for a rectangle formed by two adjacent triangles we replace the diagonal. On the level of associated trivalent ribbon graphs it corresponds to the transformation shown on Figure 3.3 and known as a *rectangle move*.


(a) Fragment of graph $\Gamma_1$. (b) Fragment of graph $\Gamma_2$.

Figure 3.3: Rectangle move in one dimensional case.

It was shown by A. Goncharov and R. Kenyon that with each rectangle move one can associate a Poisson homomorphism of the corresponding moduli spaces of one dimensional representations. To describe such homomorphism suppose that $\Gamma_1$ and $\Gamma_2$

are bipartite ribbon graphs corresponding to the ideal triangulation of a surface before and after a rectangular move. As we have mentioned in Example 3.3 the coordinate ring of $Hom(\pi_1(\Gamma_1, \mathbb{C}^\times)$ is freely generated by $n$ monodromies $y_0, \dots, y_n$. Consider the fragment of $\Gamma_1$ shown on Figure 3.3a and assume that all five faces of the graph which present on the figure are distinct. Denote by $y_0, \dots, y_4$ precisely the five monodromies corresponding to the boundary cycles oriented counterclockwise for five faces shown on the figure. Similarly, one picks $z_0, \dots, z_4$ according to the Figure 3.3b. Since away from the fragments displayed on Figure 3.3 the two graphs coincide, one can chose the rest of $y_5, \dots, y_n$ and $z_5, \dots, z_n$ consistently.[1]

**Proposition 3.4** ([GK13]). *The following map*

$$\tau : \quad \begin{cases} z_0 \to y_0^{-1}, & \\ z_i \to y_i(1+y_0), & i = 1, 3, \\ z_i \to y_i(1+y_0^{-1})^{-1}, & i = 2, 4, \\ z_i \to y_i, & i \geq 5. \end{cases} \tag{3.5}$$

*extends to a homomorphism of Poisson algebras*

$$\tau : \quad \mathbb{C}[z_0, \dots, z_n] \to \mathbb{C}[y_0, \dots, y_n], \qquad \tau(\{f, g\}_2) = \{\tau(f), \tau(g)\}_1$$

*for all $f, g \in \mathbb{C}[z_0, \dots, z_n]$. Here $\{,\}_1$ and $\{,\}_2$ are the Poisson brackets on $\hat{\mathcal{S}}_{\Gamma_1}$ and $\hat{\mathcal{S}}_{\Gamma_2}$ respectively.*

The main goal of this chapter is to show that Proposition 3.4 holds in a much more general context. In Section 3.2 we show that Poisson homomorphism (3.5) can be formulated as a "Quasi Poisson" functor between the categories associated to the ribbon graphs $\Gamma_1$ and $\Gamma_2$.

### 3.1.2 Category associated with a ciliated bipartite ribbon graph

In order to associate a category to a bipartite ribbon graph $\Gamma$ we have to endow $\Gamma$ with an additional structure, for each white vertex $V_i$ we pick a distinguished edge which we

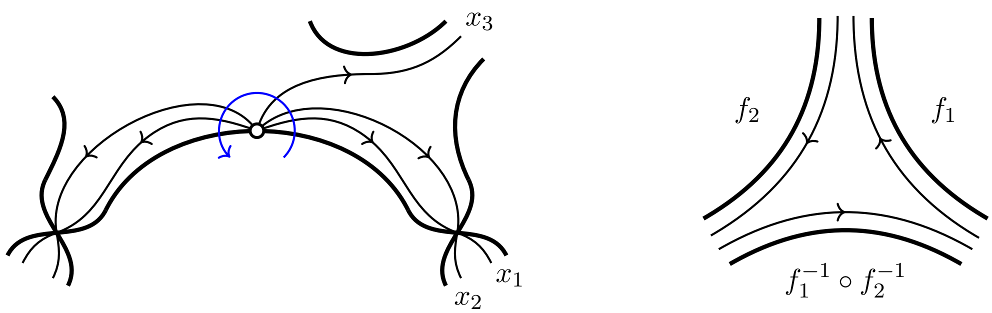
[1] We will not discuss some technical details concerning degenerate cases until Section 3.2 where we develop a theory for general character varieties.

call first. Together with a given cyclic order, this makes the edges adjacent to $V_i$ into an ordered set. Following terminology suggested in [FR99] we call such a graph "ciliated". As we will show further, the induced Poisson brackets on the character variety will not depend on the choice of this additional data.

We define a **k**-linear category $\mathcal{C}$ assoiated to the ciliated bipartite ribbon graph $\Gamma$ as follows. First we draw the corresponding conjugate surface $\hat{\mathcal{S}}_\Gamma$. In the case of a bipartite ribbon graph the conjugate surface can be obtained from $\mathcal{S}_\Gamma$ if we twist once each ribbon corresponding to the edge of graph $\Gamma$. For each disk on $\hat{\mathcal{S}}_\Gamma$ corresponding to white vertex we mark a point $V_i$ on the boundary in between the first and last ribbon attached to the disk. Next, we consider a fundamental groupoid $\pi_1(\hat{\mathcal{S}}_\Gamma, V_1, \ldots, V_n)$ and let the category $\mathcal{C}_0$ be generated by this groupoid as a free **k**-module

$$\mathcal{C}_0 = \mathbf{k}\,\pi_1(\hat{\mathcal{S}}_\Gamma, V_1, \ldots, V_n). \tag{3.6}$$

Then objects of $\mathcal{C}_0$ are precisely the marked points $Obj\,\mathcal{C}_0 = \{V_i\}$ and the morphisms of $\mathcal{C}_0$ are formal linear combinations of homotopy equivalence classes of paths between these points.




(a) Disk corresponding to white vertex

(b) Disc corresponding to black vertex

Figure 3.4: Building blocks for bipartite graph with trivalent black vertices

In the next step we define a category $\mathcal{C}$ associated to the ribbon graph $\Gamma$ as a universal localization of $\mathcal{C}_0$.

### 3.1.3 Double Quasi-Poisson Bivector

Double Quasi Poisson brackets for oriented surfaces were first introduced by G. Massuyeau and V. Turaev in [MT12]. In the original definition they have considered a surface $\Sigma$ with a marked point $p \in \partial\Sigma$ on the boundary and defined a double bracket on a group algebra of the fundamental group $\pi_1(\Sigma, p)$. In a subsequent paper [MT13] the same authors suggested to generalize the notion of a double Quasi Poisson Bracket for a linear category. In this section we calculate the double Quasi Poisson Bracket on a category $\mathcal{C}$ defined in (3.6). We show that the corresponding double Quasi Poisson bivector can be presented as a sum over contributions from each of the marked points associated to white vertices of the ribbon graph (see Figure 3.4a). Such a local formulation will later allow us to prove Theorem 3.12.

Again, let $\Gamma$ be a ciliated bipartite ribbon graph with trivalent black vertices and $\mathcal{C}$ be the associated category defined in (3.6). The fundamental groupoid $\pi_1(\hat{\mathcal{S}}_\Gamma)$ is then freely generated by $2k$ paths $f_1, \dots, f_{2k}$, where $k$ is the number of black vertices, as shown on Figure 3.4b. If we fix some choice of generators, then each marked point $V$ on the boundary carries an information on the order of generating paths starting/ending at $V$. Let $\{x_1, \dots, x_m\}$ be the corresponding ordered set, where each $x_j = f_{l(j)}^{\epsilon(j)}$, $\epsilon(j) = \pm 1$. Then the contribution to the Quasi Poisson bivector reads

$$P_V = \frac{1}{2}\sum_{i<j}\left(x_j \star \frac{\partial}{\partial x_i} \star x_i \star \frac{\partial}{\partial x_j} - x_i \star \frac{\partial}{\partial x_j} \star x_j \star \frac{\partial}{\partial x_i}\right). \tag{3.7a}$$

Here $\frac{\partial}{\partial f_i} \in D_{s(f_i),t(f_i)}$ is a vector field on a category $\mathcal{C}$ defined on generators as

$$\frac{\partial}{\partial f_i}(f_j) = \begin{cases} 1_{t(f_i)} \otimes 1_{s(f_i)}, & i = j, \\ 0, & i \neq j. \end{cases}$$

The derivation w.r.t. the inverses of generators is defined as

$$\frac{\partial}{\partial (f_i^{-1})} = -f_i \star \frac{\partial}{\partial f_i} \star f_i.$$

The double Quasi Poisson Bracket on $\mathcal{C}$ reads

$$\{\!\{,\}\!\} = \sum_{V_m \in Obj\,\mathcal{C}} \mathrm{tr}_{End(V_m)} P_{V_m}. \tag{3.7b}$$

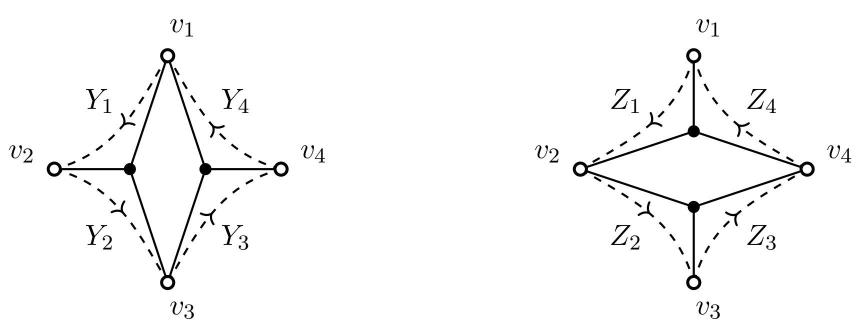
**Proposition 3.8.** *Biderivation (3.7b) is a double Quasi Poisson Bracket, i.e. it satisfies the double Quasi Jacobi Identity (2.79).*

*Proof.* By Lemma 2.76 and Proposition 2.84 it is enough to prove the Quasi jacobi Identity on the generators $f_1, \dots, f_n$ only. The latter is checked by a straightforward calculation. □

## 3.2 Noncommutative Mutations

In this subsection we show that Proposition 3.4 is a corollary of a much more general statement which can be formulated purely in terms of a category $\mathcal{C}$ associated to the bipartite ribbon graph. As in Section 3.1.1, let $\Gamma_1$ and $\Gamma_2$ be two bipartite ribbon graphs with trivalent black vertices which differ precisely in six edges as shown on Figure 3.5.

The rectangle move induces a functor between the associated categories $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows. Let $\mathcal{C}_1^{sub} \subset \mathcal{C}_1$ be a subcategory with four objects $v_1, v_2, v_3, v_4$ and morphisms generated by $Y_1^{\pm 1}, Y_2^{\pm 1}, Y_3^{\pm 1}, Y_4^{\pm 1}$ as shown on Figure 3.5a. Similarly we define a subcategory $\mathcal{C}_2^{sub} \subset \mathcal{C}_2$ as generated by $Z_1^{\pm}, Z_2^{\pm 1}, Z_3^{\pm 1}, Z_4^{\pm}$ shown on Figure 3.5b.




(a) Original morphisms (b) Morphisms after the move

Figure 3.5: Rectangle move

Note that $\mathrm{End}_{\mathcal{C}_1^{sub}}(v_1) = \mathbf{k}(M)$ is a commutative algebra with unit $1_{v_1}$ consisting of rational functions in

$$M = Y_4 \circ Y_3 \circ Y_2 \circ Y_1$$

Now let $\tau^{sub}: \mathcal{C}_2^{sub} \to \mathcal{C}_1^{sub}$ be a functor which is the identity on objects and acts on generators of $\mathcal{C}_2^{sub}$ as

$$
\begin{aligned}
\tau^{sub}(Z_1) &= Y_1 \circ f_1(M), & \tau^{sub}(Z_4) &= f_4(M) \circ Y_4, \\
\tau^{sub}(Z_2) &= Y_2 \circ Y_1 \circ f_2(M) \circ Y_1^{-1}, & \tau^{sub}(Z_3) &= Y_4^{-1} \circ f_3(M) \circ Y_4 \circ Y_3,
\end{aligned} \tag{3.9a}
$$

where $f_1, \ldots, f_4$ are the same as in one-dimensional case:

$$
f_1(M) = f_3(M) = (1_{v_1} + M)^{-1}, \qquad\qquad f_2(M) = f_4(M) = 1_{v_1} + M^{-1}. \tag{3.9b}
$$

*Remark* 3.10. Since the algebra $\mathrm{End}_{\mathcal{C}_1^{sub}} = \mathbf{k}(M)$ is commutative, the form of $f_1, \ldots, f_4$ does not depend on the particular identification, e.g.

$$
Z_3 = Y_3 \circ Y_2 \circ Y_1 \circ f_3(M) \circ Y_1^{-1} \circ Y_2^{-1} = Y_4^{-1} \circ f_3(M) \circ Y_4 \circ Y_3.
$$

The double bracket (3.7b) on $\mathcal{C}_1^{loc}$ and $\mathcal{C}_2^{loc}$ then reads

$$
\begin{aligned}
\{\!\{Y_1 \otimes Y_2\}\!\} &= -\frac{1}{2} Y_2 \circ Y_1 \otimes e_2, & \{\!\{Z_1 \otimes Z_2\}\!\} &= \frac{1}{2} Z_2 \circ Z_1 \otimes e_1, \\
\{\!\{Y_1 \otimes Y_3\}\!\} &= 0, & \{\!\{Z_1 \otimes Z_3\}\!\} &= 0, \\
\{\!\{Y_1 \otimes Y_4\}\!\} &= -\frac{1}{2} e_1 \otimes Y_1 \circ Y_4, & \{\!\{Z_1 \otimes Z_4\}\!\} &= \frac{1}{2} e_1 \otimes Z_1 \circ Z_4, \\
\{\!\{Y_2 \otimes Y_3\}\!\} &= \frac{1}{2} Y_3 \circ Y_2 \otimes e_3, & \{\!\{Z_2 \otimes Z_3\}\!\} &= -\frac{1}{2} Z_3 \circ Z_2 \otimes e_3, \\
\{\!\{Y_2 \otimes Y_4\}\!\} &= 0, & \{\!\{Z_2 \otimes Z_4\}\!\} &= 0, \\
\{\!\{Y_3 \otimes Y_4\}\!\} &= -\frac{1}{2} Y_4 \circ Y_3 \otimes e_4, & \{\!\{Z_3 \otimes Z_4\}\!\} &= \frac{1}{2} Z_4 \circ Z_3 \otimes e_4.
\end{aligned} \tag{3.11}
$$

Graphs $\Gamma_1$ and $\Gamma_2$ are identical away from the subgraphs shown on Figure 3.5, so the remaining generators $Y_5, \ldots, Y_n$ and $Z_5, \ldots, Z_n$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ can be chosen identically. Now let $\tau$ be an extension of $\tau^{loc}$ which fixes all the remaining generators:

$$
\tau: \quad \mathcal{C}_2 \to \mathcal{C}_1, \qquad \tau\big|_{\mathcal{C}_2^{sub}} = \tau^{sub}, \qquad \tau(Z_i) = Y_i \quad i \geq 5.
$$

**Theorem 3.12.** *The functor $\tau$ preserves Double Quasi Poisson Bracket:*

$$
\tau\big(\{\!\{Z_i \otimes Z_j\}\!\}\big) = \{\!\{\tau(Z_i) \otimes \tau(Z_j)\}\!\}, \qquad 1 \leq i, j \leq n. \tag{3.13}
$$

*Proof.* Note that (3.13) holds when both $i, j \geq 5$. Next, when both $i, j \leq 4$ the statement follows from (3.11) by a direct computation. Indeed, for $i = 1, j = 2$ we get

$$
\begin{aligned}
\{\!\{\tau(Z_1 \otimes Z_2)\}\!\} =& \{\!\{\big(Y_1 \circ (1_{v_1} + M)^{-1}\big) \otimes \big(Y_2 \circ Y_1 \circ (1_{v_1} + M^{-1}) \circ Y_1^{-1}\big)\}\!\} \\
=& \frac{1}{2}\big(Y_3^{-1} \circ Y_4^{-1} \circ (1_{v_1} + M)^{-1}\big) \otimes 1_{v_2} \\
& + \big(Y_2 \circ Y_1 \circ (1_{v_1} + M)^{-1}\big) \otimes \big(Y_1 \circ (1_{v_1} + M)^{-1} \circ Y_1^{-1}\big) \\
& + \big(Y_2 \circ Y_1 \circ (1_{v_1} + M)^{-1}\big) \otimes \big(Y_1 \circ (1_{v_1} + M)^{-1}\big) \circ M \circ Y_1^{-1} \\
& - \frac{1}{2} Y_2 \circ Y_1 \circ (1_{v_1} + M)^{-1} \otimes 1_{v_2} \\
=& \frac{1}{2} Y_3^{-1} \circ Y_4^{-1} \circ Y_1 \circ (1_{v_1} + M^{-1}) \otimes 1_{v_2} \\
& + \frac{1}{2} Y_2 \circ Y_1 \circ Y_1 \circ (1_{v_1} + M^{-1}) \otimes 1_{v_2} \\
=& \tau\big(Z_2 \circ Z_1 \otimes e_1\big) = \tau\big(\{\!\{Z_1 \otimes Z_2\}\!\}\big).
\end{aligned}
$$

The calculation for the remaining five pairs is analogous.

The last thing to prove is that (3.9) preserves the brackets with generators $Z_i, i \geq 5$. It follows from (3.7b) that brackets on $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) can be decomposed as a sum of the two terms

$$\{\!\{ , \}\!\}_{\mathcal{C}_1} = \sum_{i=1}^{4} \overline{P_{v_i}^{\mathcal{C}_1}} + \sum_{i=5}^{n} \overline{P_{v_i}^{\mathcal{C}_1}}$$

where the second summand satisfies

$$\tau\left(\sum_{i=5}^{n} \overline{P_{v_i}^{\mathcal{C}_2}}\big(Z_i \otimes Z_j\big)\right) = \sum_{i=5}^{n} \overline{P_{v_i}^{\mathcal{C}_1}}\big(\tau(Z_i) \otimes \tau(Z_j)\big) \qquad 1 \leq i, j \leq n.$$

As a corollary (3.13) holds for the generators with sources and targets away from $v_1, \ldots, v_4$.

It would be enough for us to prove the statement for the four half edges adjacent to $v_1$ and $v_2$ as shown on Figure 3.6. The calculation is essentially based on the following

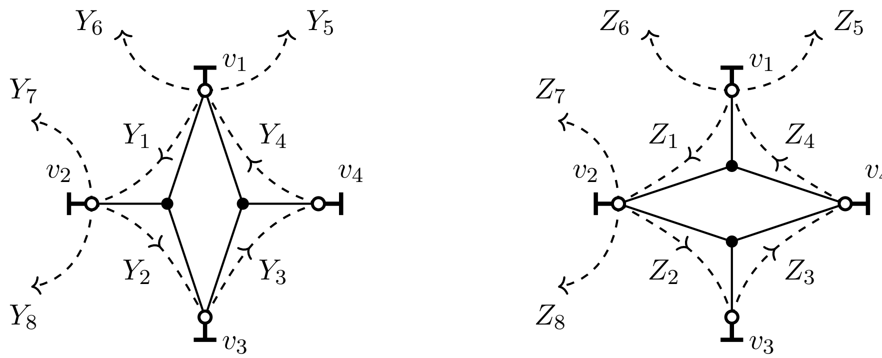



Figure 3.6: Rectangle move with generic half edges

property of $M$:

$$\{\!\{Y_5, M\}\!\} = \frac{-M \otimes Y_5 + 1_{v_1} \otimes Y_5 \circ M}{2}, \tag{3.14a}$$

$$\{\!\{Y_6, M\}\!\} = \frac{M \otimes Y_6 - 1_{v_1} \otimes Y_6 \circ M}{2}, \tag{3.14b}$$

$$\{\!\{Y_7, \widetilde{M}\}\!\} = \frac{-\widetilde{M} \otimes Y_7 + 1_{v_2} \otimes Y_7 \circ \widetilde{M}}{2}, \tag{3.14c}$$

$$\{\!\{Y_8, \widetilde{M}\}\!\} = \frac{\widetilde{M} \otimes Y_8 - 1_{v_2} \otimes Y_8 \circ \widetilde{M}}{2}. \tag{3.14d}$$

Here $\widetilde{M} = Y_1 \circ M \circ Y_1^{-1} = Y_1 \circ Y_4 \circ Y_3 \circ Y_2$.

From the double Leibnitz Identity (2.71) it follows that (3.14a)–(3.14b) holds if we replace $M$ with $f(M)$, where $f(M) \in \mathrm{End}_{C_1^{sub}}(v_1) = \mathbf{k}(M)$ is an arbitrary rational function in $M$. In particular, for $f_1(M) = (1_{v_1} + M)^{-1}$

$$\{\!\{Y_5 \otimes (1_{v_1} + M)^{-1}\}\!\} = \frac{1_{v_1} \otimes Y_5 \circ (1_{v_1} + M)^{-1} - (1_{v_1} + M)^{-1} \otimes Y_5}{2}.$$

And, consequently

$$\begin{aligned}
\{\!\{\tau(Z_5) \otimes \tau(Z_1)\}\!\} = &\{\!\{Y_5 \otimes Y_1 \circ (1_{v_1} + M)^{-1}\}\!\} = \{\!\{Y_5 \otimes Y_1\}\!\} \circ (1_{v_1} \otimes (1_{v_1} + M)^{-1}) \\
&+ (Y_1 \otimes 1_{t(Y_5)}) \circ \{\!\{Y_5 \otimes (1_{v_1} + M)^{-1}\}\!\} \\
= &-\frac{1}{2} Y_1 \otimes Y_5 \circ (1_{v_1} + M)^{-1} \\
&+ \frac{1}{2}(Y_1 \otimes 1_{t(Y_5)}) \circ \big(1_{v_1} \otimes Y_5 \circ (1_{v_1} + M)^{-1} - (1_{v_1} + M)^{-1} \otimes Y_5\big) \\
= &-\frac{1}{2} Y_1 \circ (1_{v_1} + M)^{-1} \otimes Y_5 \\
= &\tau(-Z_1 \otimes Z_5) = \tau\big(\{\!\{Z_5 \otimes Z_1\}\!\}\big).
\end{aligned}$$

Similarly, for $f_4(M) = 1_{v_1} + M^{-1}$

$$\{\!\{Y_5 \otimes f_4(M)\}\!\} = \frac{1_{v_1} \otimes Y_5 \circ f_4(M) - f_4(M) \otimes Y_5}{2}$$

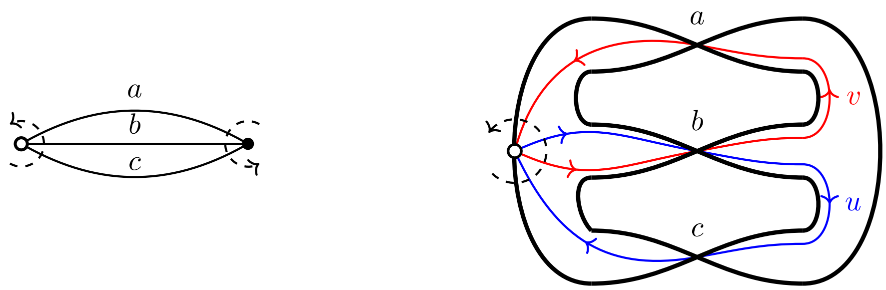
and, as a result

$$\begin{aligned}\{\!\{\tau(Z_5) \otimes \tau(Z_4)\}\!\} =& \{\!\{Y_5 \otimes f_4(M) \circ Y_4\}\!\} \\ =& \{\!\{Y_5 \otimes f_4(M)\}\!\} \circ (1_{v_1} \otimes Y_4) + (f_4(M) \otimes 1_{t(Y_5)}) \circ \{\!\{Y_5 \otimes Y_4\}\!\} \\ =& \frac{1}{2}(1_{v_1} \otimes Y_5 \circ f_4(M) - f_4(M) \otimes Y_5) \circ (1_{v_1} \otimes Y_4) \\ &+ \frac{1}{2}(f_4(M) \otimes 1_{t(Y_5)}) \circ (1_{v_1} \otimes Y_5 \circ Y_4) \\ =& \frac{1}{2} 1_{v_1} \otimes Y_5 \circ f_4(M) \circ Y_4 = \tau(1_{v_1} \otimes Z_5 \circ Z_4) \\ =& \tau\big(\{\!\{Z_5 \otimes Z_4\}\!\}\big)\end{aligned}$$

The calculations for the other six cases are essentially the same. □

## 3.3 Bracket on a torus and Kontsevich map

Consider a bipartite ribbon graph with two vertices and three edges ordered as shown on the Figure 3.7a. The corresponding conjugate surface is precisely the torus with one boundary component depicted on Figure 3.7b. The category $\mathcal{C}_K$ associated to such




(a) Original ribbon graph

(b) Conjugate surface $T \backslash D$

Figure 3.7: Conjugate surface for Kronecker quiver with three vertices

ribbon graph is nothing but a noncommutative fraction field [Coh95] freely generated by $u$ and $v$. The Quasi Poisson bivector (3.7) on $\mathcal{C}_K$ is determined by the ordered set of half edges adjacent to the single object

$$\{u^{-1}, v, u, v^{-1}\}.$$

The resulting double Quasi Poisson Bracket then reads:

$$
\begin{aligned}
&\{\!\{u\otimes u\}\!\}=\frac{1\otimes u^2-u^2\otimes 1}{2}, \qquad \{\!\{v\otimes v\}\!\}=\frac{v^2\otimes 1-1\otimes v^2}{2},\\
&\{\!\{u\otimes v\}\!\}=\frac{u\otimes v-v\otimes u-vu\otimes 1-1\otimes uv}{2}.
\end{aligned} \tag{3.15}
$$

As an immediate corollary of a topological interpretation [MT12] we get the following

**Lemma 3.16.** *Bracket (3.15) is equivariant under the action of the automorphisms $D_a, D_b$ of $\mathcal{C}_K$ defined on generators as*

$$
D_a:\ \begin{cases} u\to u \\ v\to vu \end{cases} \qquad D_b:\ \begin{cases} u\to uv^{-1} \\ v\to v \end{cases} \tag{3.17}
$$

*Proof.* Direct computation on generators. □

*Remark* 3.18. Note that $D_a, D_b$ preserve $\pi_1(T\backslash K)=\langle u,v\rangle\subset\mathcal{C}_K$ embedded in $\mathcal{C}_K$ as a monoid. These automorphisms correspond to Dehn Twists of the underlying surface and satisfy $D_a\circ D_b\circ D_a=D_b\circ D_a\circ D_b$.

However, the automorphisms (3.17) are not the only ones which preserve the bracket (3.15).

**Proposition 3.19.** *Let $K$ be an automorphism of $\mathcal{C}_K$ defined on generators as*

$$
K:\ \begin{cases} u\to uvu^{-1}, \\ v\to u^{-1}+v^{-1}u^{-1}. \end{cases}
$$

*Bracket (3.15) is equivariant under the action of $K$*

$$
K\big(\{\!\{a,b\}\!\}\big)=\{\!\{K(a),K(b)\}\!\} \tag{3.20}
$$

*for all $a,b\in\mathcal{C}_K$*

*Proof.* It is enough for us to check (3.20) on generators. We have:

$$
\begin{aligned}
2\{\!\{K(u)\otimes K(v)\}\!\}=2\{\!\{vuv^{-1}\otimes(u^{-1}+v^{-1}u^{-1})\}\!\}=&-u^{-1}\otimes uvu^{-1}-u^{-1}\otimes 1\\
&-vu^{-1}\otimes 1-v^{-1}u^{-1}\otimes uvu^{-1}+uvu^{-1}\otimes u^{-1}\\
&+uvu^{-1}\otimes v^{-1}u^{-1}-1\otimes uvu^{-2}-1\otimes uvu^{-1}v^{-1}u^{-1}\\
=&K(u\otimes v-v\otimes u-vu\otimes 1-1\otimes uv)=2K\big(\{\!\{u\otimes v\}\!\}\big).
\end{aligned}
$$

Similar calculations show that $\{\!\!\{K(u) \otimes K(u)\}\!\!\}= K\big(\{\!\!\{u \otimes u\}\!\!\}\big)$ and $\{\!\!\{K(v) \otimes K(v)\}\!\!\}= K\big(\{\!\!\{v \otimes v\}\!\!\}\big)$. $\square$

# Chapter 4

# Noncommutative Integrable Systems

The idea to study of Hamiltonian equations over noncommutative algebras and their integrability was originated in the Gelfand school [GD81, Dor87, DF92] around the same time as the first papers on noncommutative geometry appeared [Con85]. The theory of Noncommutative Integrable Systems was later developed by many authors, including [Kri, Kon93, EGR97, OS98, EGR98, HT03, RR10]. On the other hand, a remarkable interplay between the noncommutative geometry and commutative geometry of representation schemes was discovered in [Kon93, KR00]. This opened up a program of studying Hamilton flows on representation algebras by means of noncommutative geometry and was later developed in papers [VdB08, CB11, CBEG07]. In this chapter we show that noncommutative integrable systems suggested earlier in [Usn08] and [Kon11] admit a formulation as noncommutative Hamilton flows on the corresponding associative algebras.

## 4.1 Hamilton flows on associative algebras

A one-dimensional flow on associative algebra $\mathcal{A}$ is defined by a derivation $\frac{\mathrm{d}}{\mathrm{d}t}$ which satisfies the Leibnitz rule

$$\forall a, b \in \mathcal{A}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(ab) = a\left(\frac{\mathrm{d}}{\mathrm{d}t}b\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}a\right)b$$

To present this flow in the Hamilton form

$$\forall a \in \mathcal{A} \quad \frac{\mathrm{d}a}{\mathrm{d}t} = \{h, a\}$$

we must provide a bracket such that, for each hamiltonian $h$, it defines a derivation of $\mathcal{A}$ and thus should satisfy the Leibnitz rule in the second argument

$$\forall a, b, c \in \mathcal{A} \qquad \{a, bc\} = \{a, b\}c + b\{a, c\}. \tag{4.1}$$

However, the Leibnitz rule in the first argument is not required.

It was first suggested in [MS00] that Hamiltonians for integrable system over noncommutative algebra $\mathcal{A}$ should be viewed as elements of the cyclic space

$$\mathcal{A}_\natural = \mathcal{A}/[\mathcal{A}, \mathcal{A}] \tag{4.2}$$

which is the quotient of algebra $\mathcal{A}$ (as linear space) by the commutant

$$[\mathcal{A}, \mathcal{A}] = \operatorname{span}\{ab - ba \mid a, b \in \mathcal{A}\}. \tag{4.3}$$

In general, the resulting quotient space (4.2) is no longer an associative algebra, since the commutant (4.3) is not necessary an associative ideal of $\mathcal{A}$.

In contrast to the commutative case, the arguments of the bracket are elements of the two different spaces

$$\{\_, \_\} : \ \mathcal{A}_\natural \times \mathcal{A} \to \mathcal{A}$$

where the first argument is an element of the cyclic space $\mathcal{A}_\natural = \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ — the natural space for Hamiltonians.

Hamilton flows in classical integrable systems form a representation of a Poisson Lie algebra of functions. This is secured by Jacobi identity, which means that the commutator of the Hamilton vector fields, generated by two different functions $H_1$ and $H_2$, is the Hamilton vector field corresponding to their Poisson bracket $\{H_1, H_2\}$:

$$\{H_1, \{H_2, x\}\} - \{H_2, \{H_1, x\}\} = \{\{H_1, H_2\}, x\}. \tag{4.4}$$

To define a noncommutative analogue of (4.4) we would need a Lie bracket on Hamiltonians

$$\{\_, \_\}_\natural : \ \mathcal{A}_\natural \times \mathcal{A}_\natural \to \mathcal{A}_\natural$$

which is skew-symmetric

$$\{H_1, H_2\}_\natural = -\{H_2, H_1\}_\natural \tag{4.5}$$

for all $H_1, H_2 \in \mathcal{A}_\natural$, and satisfies the Jacobi identity

$$\{H_1, \{H_2, H_3\}_\natural\}_\natural + \{H_2, \{H_3, H_1\}_\natural\}_\natural + \{H_3, \{H_1, H_2\}_\natural\}_\natural = 0 \tag{4.6}$$

for all $H_1, H_2, H_3 \in \mathcal{A}_\natural$. In addition, we would also need a representation of the above Lie bracket in derivations of $\mathcal{A}$, i.e. map $\{,\} : \mathcal{A}_\natural \times \mathcal{A} \to \mathcal{A}$ satisfying the following condition

$$\{H_1, \{H_2, x\}\} - \{H_2, \{H_1, x\}\} = \{\{H_1, H_2\}_\natural, x\} \tag{4.7}$$

for all $H_1, H_2 \in \mathcal{A}_\natural$ and for all $x \in \mathcal{A}$. Note that the order of arguments for brackets above becomes essential. In addition, the inner bracket on the right hand side of (4.7) is of the different type.

**Definition 4.8** ([CB11])**.** A pair of brackets $\{,\}_\natural$ and $\{,\}$ satisfying (4.5), (4.6), and (4.7) is called an $H_0$-Poisson structure.

*Remark* 4.9. Identity (4.7) first appeared in a book by J.-L. Loday [Lod98] in the context of homology of Noncommutative Lie Algebras. It was suggested later by Y. Kosmann-Schwarzbach that (4.7) should be referred to as the left Loday-Jacobi identity.

Definition 4.8 is the most general noncommutative analogue of the Poisson bracket known to this date. However, an apparent disadvantage as compared to the commutative case is that the bracket $\{,\}$ satisfies the Leibnitz rule only in the second argument. This prevents us from defining an $H_0$-Poisson bracket on generators of associative algebra $\mathcal{A}$ and then extending it to the entire algebra. Luckily, the large class of Loday brackets can be constructed by means of the double Quasi-Poisson brackets [VdB08].

Now suppose we have a Hamilton dynamics $\forall f \in \mathcal{A}\ \frac{\mathrm{d}}{\mathrm{d}t} f = \{h, f\}$.

**Definition 4.10.** The space of Hamiltonians (or "trace"-integrals) is the subspace $\mathcal{H} \subset \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ such that

$$x \in \mathcal{H} \qquad \Leftrightarrow \qquad \forall x' \in \mathcal{A} \text{ s.t. } \pi(x') = x \quad \frac{\mathrm{d}}{\mathrm{d}t} x' \equiv 0 \bmod [\mathcal{A}, \mathcal{A}] \tag{4.11}$$

Or, equivalently, one can say that $\{h, x\}_\natural = 0$.

As in the commutative case each Hamiltonian defines a Hamilton flow, such that all other Hamiltonians (as elements of $\mathcal{A}/[\mathcal{A},\mathcal{A}]$) are invariant under this flow. This can be presented in the following way

**Proposition 4.12.** *The $\mathcal{H}$ is a maximal commutative Lie subalgebra in $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ with respect to bracket $\{\_,\_\}_\natural$.*

*Proof.* Since $h \in \mathcal{H}$, maximality follows directly from definition. Next, if $h_1, h_2 \in \mathcal{H}$ then from (4.6) $\{h_1, h_2\}_\natural \in \mathcal{H}$. □

## 4.2 Casimir elements

The analog of the classical Casimir functions is the right Casimir of bracket $\{\_,\_\}$.

**Definition 4.13.** We say that $c \in \mathcal{A}$ is the Casimir element of bracket $\{\_,\_\}$ if $\forall a \in \mathcal{A}/[\mathcal{A},\mathcal{A}] \quad \{a, c\} = 0$.

The latter implies only that any element in $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ defines a derivation of $\mathcal{A}$ which fixes $c$. But $\pi(c)$ doesn't have to define a trivial Hamilton flow, the counterexample was presented in (4.44).

However, we can formulate the following

**Proposition 4.14.** *If $c$ is a Casimir in a sense of Def. 4.13, then its image in the cyclic space $\pi(c)$ necessarily belongs to the center of the Lie bracket $\{\_,\_\}_\natural$ on Hamiltonians.*

## 4.3 Kontsevich system

Let $A = \mathbb{C}\langle u^{\pm 1}, v^{\pm 1}\rangle$ denote the associative group algebra over $\mathbb{C}$ of the free group $G = \langle u, v\rangle$ with two generators. Kontsevich proposed a noncommutative system of ODE's on this algebra

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = uv - uv^{-1} - v^{-1}, \\[2ex] \dfrac{\mathrm{d}v}{\mathrm{d}t} = -vu + vu^{-1} + u^{-1}, \end{cases} \tag{4.15}$$

which admits the following discrete symmetry

$$u \to uvu^{-1}, \qquad v \to u^{-1} + v^{-1}u^{-1}. \tag{4.16}$$

The latter can be viewed as a noncommutative analog of Bäcklund transformations. Based on this data Kontsevich conjectured that (4.15) is integrable.

In paper [EW12] it was proved that system (4.15) admits the Lax representation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, M] \tag{4.17}$$

with the following Lax pair

$$L = \begin{pmatrix} v^{-1} + u & \lambda v + v^{-1}u^{-1} + u^{-1} + 1 \\ v^{-1} + \frac{1}{\lambda}u & v + v^{-1}u^{-1} + u^{-1} + \frac{1}{\lambda} \end{pmatrix}, \qquad M = \begin{pmatrix} v^{-1} - v + u & \lambda v \\ v^{-1} & u \end{pmatrix}. \tag{4.18}$$

This Lax pair gives a rise to an infinite number of Hamiltonians which cover all independent first integrals of (4.15) as was conjectured in [EW12].

### 4.3.1 Classical or commutative counterpart

Note, that in the commutative (classical) case the equations (4.15) are Hamilton

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \{h, u\}, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \{h, v\}$$

with respect to the Hamiltonian

$$h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1} \tag{4.19}$$

and Poisson bracket defined by

$$\{v, u\} = uv. \tag{4.20}$$

This implies that the commutative counterpart of (4.15) is trivially integrable in the Liouville sense.

Bracket (4.20) can be transformed to canonical one via change of variables $u = \mathrm{e}^p,\ v = \mathrm{e}^q$, which gives $\{p, q\} = 1$. Then the Hamiltonian acquires the following form

$$h = \mathrm{e}^p + \mathrm{e}^{-p} + \mathrm{e}^q + \mathrm{e}^{-q} + \mathrm{e}^{-p-q} \tag{4.21}$$

### 4.3.2 Hamiltonians and first integrals

Equations (4.15) preserve the commutator of the underlying free group $\langle u, v\rangle$:

$$uvu^{-1}v^{-1} = c. \tag{4.22}$$

In particular, when $c$ is central, the group algebra $\mathbb{C}\langle u^{\pm 1}, v^{\pm 1}\rangle$ turns into a "quantum group" and this restriction is consistent with equations (4.15). Later in the text we show that this is a noncommutative analog of the Casimir element.

On the classical level, we have another integral of motion, namely Hamiltonian (4.19). Consider its noncommutative analog[1]

$$h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1}. \tag{4.23}$$

It is no longer a first integral of equations of motion. Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t}h(t) = u^{-1} - vuv^{-1} + uv - vu + v^{-1}u^{-1} - u^{-1}v^{-1} + u^{-1}v^{-1}u^{-1} - u^{-2}v^{-1} \neq 0.$$

However, if we consider a matrix representation $\varphi :\ A \to Mat(N, \mathbb{C})$ for any $N$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\varphi(h) = 0. \tag{4.24}$$

Following terminology of [MS00, EW12] we call $h$ a "trace"-integral. Indeed, even more interesting property holds: for any representation $\varphi$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}\varphi(h^k) = 0. \tag{4.25}$$

In other terms for all representations $\varphi(h)(t)$ has adjoint dynamics under (4.15)

$$\varphi(h)(t) = g(t)\varphi(h)(0)g^{-1}(t). \tag{4.26}$$

### 4.3.3 Noncommutative Hamilton equations of motion for Kontsevich system

To present equations of motion (4.15) in the Hamilton form

$$u = \{h, u\}_K, \qquad v = \{h, v\}_K$$

---

[1] Note, that the ordering in the last term doesn't matter, due to the trivial symmetry of equations of motion $u \leftrightarrow v$, $t \to -t$.

with Hamiltonian (4.23) we must provide a bracket s.t. for each Hamiltonian it defines a derivation of $A = \mathbb{C}\langle u, v\rangle$:

$$\forall a \in A \quad \frac{\mathrm{d}a}{\mathrm{d}t} = \{h, a\}_K$$

and thus should satisfy the Leibnitz rule in the second argument. On the other hand we already pointed out that in the case of noncommutative integrable systems the Hamiltonians are not literaly invariant under dynamics (4.26). The invariant in this case is the image of $\pi(h)$ in the cyclic space, so we should require the bracket to be invariant under the cyclic permutations of monomials of the first argument. Or, equivalently we can say that the first argument of the bracket is actually the element of the cyclic space.

Now, the bracket becomes a function of elements in two different spaces and the exact anticommutativity cannot be imposed. Keeping this in mind, we immediately have a lot of inequivalent forms of Jacobi identity, and to restore the proper ordering we should go back to properties we want to be secured by it. From the point of view of Integrable Systems, the Jacobi identity is used to ensure that the commutator of the vector fields generated by two different Hamiltonians is the vector field corresponding to their commutator:

$$\forall H_1, H_2, x \in A \qquad \{H_1, \{H_2, x\}\} - \{H_2, \{H_1, x\}\} = \{\{H_1, H_2\}, x\} \tag{4.27}$$

### 4.3.4 Modified Double Poisson bracket for Kontsevitch system

In this section we construct a bracket on associative algebra $A = \mathbb{C}\langle u^{\pm 1}, v^{\pm 1}\rangle$ which allows us to present equation (4.15) in the Hamilton form. We define $\{\_\}_K : A \times A \to A$ as a composition of modified double quasi-Poisson bracket and multiplication map $\mu$

$$\{a, b\}_K = \mu\left(\{\!\{a \otimes b\}\!\}_K\right) \tag{4.28}$$

Where the modified double quasi-Poisson bracket $\{\!\{\_\}\!\}_K$ is defined by its action on the generators

$$\{\!\{u, v\}\!\}_K = -vu \otimes 1, \qquad \{\!\{v, u\}\!\}_K = uv \otimes 1, \qquad \{\!\{u, u\}\!\}_K = \{\!\{v, v\}\!\}_K = 0. \tag{4.29}$$

along with the following requirements

1. **Bilinearity:** $\{\!\{\_\}\!\}_K$ is bilinear and thus extends to

$$\{\!\{\_\}\!\}_K\colon \quad A\otimes A\to A\otimes A$$

Again, we use further the same notation $\{\!\{\_\}\!\}_K$ for extension of this bracket to $A\otimes A$ as well as for operation defined on $A\times A$.

2. **Leibnitz Identity:**

$$\{\!\{a\otimes bc\}\!\}_K=\{\!\{a\otimes b\}\!\}_K(1\otimes c)+(b\otimes 1)\{\!\{a\otimes c\}\!\}_K \tag{4.30a}$$

$$\{\!\{ab\otimes c\}\!\}_K=\{\!\{a\otimes c\}\!\}_K(b\otimes 1)+(1\otimes a)\{\!\{b\otimes c\}\!\}_K \tag{4.30b}$$

*Remark* 4.31. Brackets (4.29) and (5.27) provide an equivalent $H_0$-Poisson structures (see also Appendix 6.2). Throughout this section we will use biderivation (4.29) following [Art15].

Properties (1)–(2) along with formulae (4.29) define $\{\!\{\_\}\!\}_K$ completely. We employ useful notations following [VdB08]. Let $x,y\in A$ then we define the components of the bracket of their product via $\left(\{\!\{x,y\}\!\}'_K\right)_i$ and $\left(\{\!\{x,y\}\!\}''_K\right)_i$ as below

$$\{\!\{x\otimes y\}\!\}_K=\sum_i\left(\{\!\{x,y\}\!\}'_K\right)_i\otimes\left(\{\!\{x,y\}\!\}''_K\right)_i.$$

In our case the sum is actually redundant. Rewriting (4.29) we immediately get

$$\{\!\{u\otimes v\}\!\}_K=\{\!\{u,v\}\!\}'_K\otimes\{\!\{u,v\}\!\}''_K=-vu\otimes 1,$$

$$\{\!\{v\otimes u\}\!\}_K=\{\!\{v,u\}\!\}'_K\otimes\{\!\{v,u\}\!\}''_K=uv\otimes 1,$$

$$\{\!\{u\otimes u\}\!\}_K=\{\!\{v\otimes v\}\!\}_K=0,$$

and then extend it to $u^{\pm 1}, v^{\pm 1}$ by Leibnitz identity (4.30)

$$\{\!\{u^{-1} \otimes v^{-1}\}\!\}_K = (v^{-1} \otimes u^{-1})\{\!\{u \otimes v\}\!\}_K (u^{-1} \otimes v^{-1}) = -1 \otimes u^{-1}v^{-1},$$
$$\{\!\{v^{-1} \otimes u^{-1}\}\!\}_K = (u^{-1} \otimes v^{-1})\{\!\{v \otimes u\}\!\}_K (v^{-1} \otimes u^{-1}) = 1 \otimes v^{-1}u^{-1},$$

$$\{\!\{u^{-1} \otimes v\}\!\}_K = -(1 \otimes u^{-1})\{\!\{u \otimes v\}\!\}_K (u^{-1} \otimes 1) = v \otimes u^{-1},$$
$$\{\!\{v \otimes u^{-1}\}\!\}_K = -(u^{-1} \otimes 1)\{\!\{v \otimes u\}\!\}_K (1 \otimes u^{-1}) = -v \otimes u^{-1},$$

$$\{\!\{u \otimes v^{-1}\}\!\}_K = -(v^{-1} \otimes 1)\{\!\{u \otimes v\}\!\}_K (1 \otimes v^{-1}) = u \otimes v^{-1},$$
$$\{\!\{v^{-1} \otimes u\}\!\}_K = -(1 \otimes v^{-1})\{\!\{v \otimes u\}\!\}_K (v^{-1} \otimes 1) = -u \otimes v^{-1}.$$

If $a, b \in A$ are monomials, then we can present them in the following form

$$a = a_1 a_2 \dots a_k, \quad a_i = u^{\pm 1}, v^{\pm 1}, \qquad b = b_1 b_2 \dots b_m, \quad b_j = u^{\pm 1}, v^{\pm 1}.$$

With this notation we have

$$\{\!\{a \otimes b\}\!\}_K = \sum_{i,j} \left(b_1 \dots b_{j-1} \{\!\{a_i, b_j\}\!\}'_K a_{i+1} \dots a_k\right) \otimes \left(a_1 a_2 \dots a_{i-1} \{\!\{a_i, b_j\}\!\}''_K b_{j+1} \dots b_m\right) \tag{4.32}$$

and by linearity this formula extends to the full tensor product $A \otimes A$.

Now recall (4.28): $\{x, y\}_K = \mu(\{\!\{x \otimes y\}\!\}_K)$, this defines an operation $\{\_, \_\}_K : A \times A \to A$.

**Proposition 4.33.** *$\{\_, \_\}_K$ satisfies the following properties:*

*(1)* ***Bilinearity:*** *$\{\_, \_\}_K$ is bilinear and thus extends to*

$$\{\_\}_K : \qquad A \otimes A \to A;$$

*(2a)* ***Leibnitz Identity in the second argument:***

$$\{a, bc\}_K = \{a, b\}_K c + b\{a, c\}_K;$$

*(2b)* ***Invariance under cyclic permutations of monomials in the first argument:***

$$\{ab, c\}_K = \{ba, c\}_K;$$

*(3)* ***Skew-symmetricity modulo*** $[A,A]$***:***

$$\{a,b\}_K \equiv -\{b,a\}_K \bmod [A,A];$$

*(4)* ***Jacobi Identity***

$$\forall H_1,H_2,x\in A: \qquad \{H_1,\{H_2,x\}_K\}_K - \{H_2,\{H_1,x\}_K\}_K = \{\{H_1,H_2\}_K,x\}_K.$$

*Proof.* Part (1) is trivial. Part (2a) is provided by the outer bimodule structure of the double bracket. Indeed, apply $\mu$ to both sides of (4.30a), this reads

$$\begin{aligned}\{a,bc\} :=&\mu(\{\!\{a\otimes bc\}\!\}_K) = \mu(\{\!\{a\otimes b\}\!\}_K(1\otimes c)) + \mu((b\otimes 1)\{\!\{a\otimes b\}\!\}_K) = \\ =&\mu(\{\!\{a\otimes b\}\!\}_K)c + b\mu(\{\!\{a\otimes c\}\!\}_K) = \{a,b\}_K c + b\{a,c\}_K.\end{aligned}$$

Part (2b) is provided by the inner bimodule structure of the double bracket. Here

$$\begin{aligned}\{ab,c\}_K :=&\mu(\{\!\{ab\otimes c\}\!\}_K) = \quad \text{by (4.30b)} \\ =&\mu(\{\!\{a\otimes c\}\!\}_K(b\otimes 1)) + \mu((1\otimes c)\{\!\{a\otimes b\}\!\}_K) = \\ =&\mu((1\otimes b)\{\!\{a\otimes c\}\!\}_K) + \mu(\{\!\{a\otimes b\}\!\}_K(c\otimes 1)) = \\ =&\mu(\{\!\{ba,c\}\!\}_K) =: \{ba,c\}_K.\end{aligned}$$

□

Using Proposition 4.33 we conclude that $\{\_,\_\}_K$ is well-defined on $A/[A,A]\times A$ and provide a desired Loday bracket.

### 4.3.5 Lax Matrix, Hamiltonians and Casimir elements.

Note first that

$$\frac{\mathrm{d}}{\mathrm{d}t}h = \{h,h\}_K = [h, v+u^{-1}]. \tag{4.34}$$

This equation was first presented in [EW12]. It is a noncommutative analog of the Lax equation, where the role of the first Lax matrix is played by the element $h$, whereas $M = v+u^{-1}$ plays the role of the second Lax matrix. It was claimed in paper [EW12] that Eq. (4.34) cannot be considered as a Lax equation since this equation doesn't solely define derivatives for generators of associative algebra $A$. However, we point out that it is already enough to define a Loday bracket $\{\_,\_\}_K$ along with $h$ to completely define a derivation of $A$. From this point of view the **Lax equation (4.34) plays the role of the condition that secures $\pi(h^k)$ to be invariant**.

Actually, even stronger statement is true, namely $\pi(h^k)$ is an infinite chain of commuting Hamiltonians in $A/[A,A]$. This is shown by the following proposition.

**Proposition 4.35.** *For all $N, M > 0$ the corresponding hamiltonians $\pi(h^N)$ and $\pi(h^M)$ are in involution: $\{h^N, h^M\}_K \equiv 0 \bmod [A,A]$.*

*Proof.*

$$\{h^N, h^M\}_K = \mu(\{\!\{h^N, h^M\}\!\}_K) = \mu\left(\sum_{j=0}^{N}\sum_{k=0}^{M}(h^k \otimes h^j)\{\!\{h,h\}\!\}_K(h^{N-j-1} \otimes h^{M-k-1})\right). \tag{4.36}$$

On the other hand

$$\begin{aligned} \{\!\{h,h\}\!\}_K =& 1\otimes a - h\otimes b + e\otimes 1, \quad \text{where} \qquad (4.37)\\ a =& u^{-1} + v^{-1} - u^{-1}v^{-1} + v^{-1}u^{-1} + u^{-1}v^{-1}u^{-1} + v^{-1}u^{-1}v^{-1} + u^{-1}v^{-1}u^{-1}v^{-1},\\ b =& u^{-1}v^{-1},\\ e =& uv - vu. \end{aligned}$$

Combining (4.36) with (4.37) we get

$$\begin{aligned} \{h^N, h^M\}_K =& \mu\Big(\sum_{j=0}^{N-1}\sum_{k=0}^{M-1}\Big(h^{N+k-j-1}\otimes h^j a h^{M-k-1} - h^{N+k-j}\otimes h^j b h^{M-k-1} \qquad (4.38)\\ &+ h^k e h^{N-j-1}\otimes h^{M+j-k-1}\Big)\Big)\\ =& \sum_{j=0}^{N-1}\sum_{k=0}^{M-1}\Big(h^{N+k-1}ah^{M-k-1} - h^{N+k}bh^{M-k-1} + h^k e h^{N+M-k-2}\Big) \qquad (4.39)\\ \equiv& MN(a+e-hb)h^{M+N-2} \bmod [A,A]. \end{aligned}$$

But, for $N = M = 1$ we have (4.34), so using the **last but one** line of (4.39) we get

$$(a + e - hb) = [h, v + u^{-1}].$$

And finally

$$\begin{aligned}\{h^N, h^M\}_K \equiv & MN[h, v+u^{-1}]h^{M+N+2} \bmod [A,A] \\ \equiv & 0 \bmod [A,A].\end{aligned}$$

□

**Corollary 4.40.** *For all $k > 0$, $\pi(h^k)$ is integral of a system of equations (4.15).*

*Proof.* $\pi(\frac{\mathrm{d}}{\mathrm{d}t}h^k) = \pi([h^k, v+u^{-1}]) = 0$. □

Here we should point out the fact that $\pi(h^k)$ as elements of the cyclic space are independent, whereas all $h^k$ are generated by a single element. When we come to the quotient space $A/[A,A]$ it is no longer have a natural multiplication. Or in other words given an equivalence class $\pi(h) \in A/[A,A]$ we cannot pick a proper representative $h \in A$ which generates the whole series. This makes $\pi(h^2)$ to be in principle unidentified by $\pi(h)$. Namely, given the equivalence class $\pi(h)$ we don't have Lax equation (4.34) for each representative of each class in $A$.

**Lemma 4.41.** *The infinite series of commuting hamiltonians $\pi(h^k)$ is linearly independent over $\mathbb{C}$.*

*Proof.* It is enough to consider highest term in $u$ in $\pi(h^k)$. □

This brings us to conclusion that $h$ is a noncommutative analog of the Lax matrix. In each representation it has adjoint dynamics (4.26), as well as it generates the infinite series of commuting hamiltonians $\pi(h^k)$. Here $\pi$ is a projection to the cyclic space which can be treated as noncommutative analog of Tr.

Now, we left with the task to understand the meaning of Casimir functions. We already pointed out that (4.22) is invariant under dynamics (4.15). However it appears that even stronger statement is true.

**Proposition 4.42.** *For each $H \in A/[A,A]$ the corresponding Hamilton flow $\{H, \_\}_K$ with respect to bracket (4.28) preserves the group commutator $c = uvu^{-1}v^{-1}$*

*Proof.* Direct computation shows that

$$\begin{aligned}
\{\!\{u \otimes c\}\!\}_K &= uv \otimes v^{-1} - uvu \otimes u^{-1}v^{-1} &&= (1 \otimes u)r - r(u \otimes 1), \\
\{\!\{v \otimes c\}\!\}_K &= &&= (1 \otimes v)r - r(v \otimes 1),
\end{aligned}$$

where $r = uv \otimes u^{-1}v^{-1}$.

Now we use the induction by Leibnitz identity (4.30b) to prove that

$$\forall a \in A \qquad \{\!\{a \otimes c\}\!\}_K = (1 \otimes a)r - r(a \otimes 1) \tag{4.43}$$

Assume that this holds for $a, b$ and prove this for $ab$:

$$\begin{aligned}
\{\!\{ab \otimes c\}\!\}_K =& \{\!\{a \otimes c\}\!\}_K (b \otimes 1) + (1 \otimes a)\{\!\{b \otimes c\}\!\}_K = \\
&= \left((1 \otimes a)r - r(a \otimes 1)\right)(b \otimes 1) + (1 \otimes a)\left((1 \otimes b)r - r(b \otimes 1)\right) = \\
&= (1 \otimes ab)r - r(ab \otimes 1).
\end{aligned}$$

This implies that (4.43) is valid. Note finally that by applying multiplication map $\mu$ to both sides of (4.43) we always get zero. This finalizes the proof. □

In other words, it is a right Casimir function for bracket (4.28) $\{\_,\_\}_K$. But it is not a left Casimir function, which means that $\pi(c)$ doesn't generate the trivial flow, like it was in the commutative case. Say

$$\{c, u\}_K = uvu^{-1}v^{-1}u - u^2vu^{-1}v^{-1} \neq 0. \tag{4.44}$$

However, it satisfies an important property.

**Proposition 4.45.** *For all* $H \in A/[A, A] \quad \{H, c\}_K \equiv 0 \bmod [A, A]$

*Proof.* Combine Proposition 4.42 and property (3) from Proposition 4.33. □

Proposition 4.33 means that Casimir operator belongs to the center of the Lie Algebra on a cyclic space (the natural space for Hamiltonians).

**Discussion on generating set for all "trace" integrals.**

Summarizing we can conclude

**Corollary 4.46.** *If* $x \in \pi(\mathbb{C}\langle h\rangle + \mathbb{C}\langle c, c^{-1}\rangle)$ *for some* $x \in A/[A, A]$*, then* $\frac{d}{dt}x = \{h, x\}_K \equiv 0 \bmod [A, A]$.

In paper [EW12] Efimovskaya and Wolf considered possible "trace"-integrals of equation (4.15) up to degree 12 and conjectured that they are all generated by the usual traces of powers of Lax matrix (4.18).

Another experimental comparison shows that $\pi(\operatorname{Tr} L^k)$, $k \leq 3$, the images of the traces of powers of the Lax matrix in the cyclic space $A/[A, A]$ generate the linear subspace of the image $\pi(\mathbb{C}\langle h\rangle + \mathbb{C}\langle c, c^{-1}\rangle)$. However the $\pi(\operatorname{Tr} L^4)$ is no longer contained in the above space. It provides us an additional $h_2 \in A/[A, A]$ of the form

$$h_2 = uvu^{-1}v^{-1}u + uvvu^{-1}v^{-1} + uvu^{-1}u^{-1}v^{-1} + uvu^{-1}v^{-1}v^{-1} + uvu^{-1}v^{-1}u^{-1}v^{-1}. \tag{4.47}$$

### 4.3.6 Specialization to Quantum and Classical Integrable System

The invariance of the Casimir element $c = uvu^{-1}v^{-1}$ under dynamics (4.22) allows one to construct certain specializations of algebra $A = \mathbb{C}\langle u^{\pm 1}, v^{\pm 1}\rangle$ consistent with equations of motion. One can impose relation of the form $c = \mathrm{e}^{\mathrm{i}\hbar} \in \mathbb{C}$. This reduces the algebra to the so-called "quantum group" (the word group is misleading here, although widely accepted). This is the exponential form of the usual Heisenberg algebra $u = \mathrm{e}^p$, $v = \mathrm{e}^q$, $[p, q] = -\mathrm{i}\hbar$. The latter makes the quantum version naturally embedded in the associative case. However the relation between the natural space for noncommutative Hamiltonians, namely the cyclic space, and quantum Hamiltonians is still vague.

Finally, the particular case $c = 1$ corresponds to commutative algebra, here the cyclic space coincides with algebra itself and the bracket turns into anticommutative. On the other hand the fact that associative algebra coincides with its cyclic space endows the latter with multiplication, which makes all Hamiltonians $h^k$ algebraically dependent.

# Chapter 5

# Modified Double Poisson brackets

## 5.1 Modified double Poisson bracket

Let $\mathcal{A} = \mathbb{C}\langle x^{(1)}, \dots, x^{(k)}\rangle/\mathcal{R}$ be an associative algebra over $\mathbb{C}$, which is finitely generated by $\{x^{(1)}, \dots x^{(k)}\}$, possibly with some finite number of relations $\mathcal{R}$.

**Definition 5.1.** A modified double Poisson bracket on $\mathcal{A}$ is a map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ s.t. for all $a, b, c \in \mathcal{A}$

$$\{\!\{a \otimes bc\}\!\} = (b \otimes 1)\{\!\{a \otimes c\}\!\} + \{\!\{a \otimes b\}\!\}(1 \otimes c) \tag{5.2a}$$

$$\{\!\{ab \otimes c\}\!\} = (1 \otimes a)\{\!\{b \otimes c\}\!\} + \{\!\{a \otimes c\}\!\}(b \otimes 1) \tag{5.2b}$$

$$\{a \otimes \{b \otimes c\}\} - \{b \otimes \{a \otimes c\}\} = \{\{a \otimes b\} \otimes c\} \quad \text{where} \quad \{\_\} := \mu \circ \{\!\{\_\}\!\} \tag{5.2c}$$

$$\{a, b\} + \{b, a\} = 0 \mod [\mathcal{A}, \mathcal{A}] \tag{5.2d}$$

The fact that we do not require skew-symmetry in a sense of Van den Bergh $\{\!\{a, b\}\!\} = -\{\!\{b, a\}\!\}^{op}$ is the major difference with the case studied in [VdB08, ORS12]. To distinguish with definitions introduced in [VdB08] we call this object **modified** double Poisson Bracket. In Section 5.3 we show that there are examples of the modified double Poisson brackets which are non-skew-symmetric in the sense of M. Van den Bergh.

**Corollary 5.3.** *Composition with the multiplication map $\{\_\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defines an $H_0$-Poisson structure, namely for all $a, b, c \in \mathcal{A}$*

$$\{a, bc\} = b\{a, c\} + \{a, b\}c, \tag{5.4a}$$

$$\{ab, c\} = \{ba, c\}, \tag{5.4b}$$

$$\{a, \{b, c\}\} - \{b, \{a, c\}\} = \{\{a, b\}, c\}, \tag{5.4c}$$

$$\{a, b\} + \{b, a\} \equiv 0 \mod [\mathcal{A}, \mathcal{A}]. \tag{5.4d}$$

In particular, the latter implies that $\{\_\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ factors through $\{\_\} : \mathcal{A}/[\mathcal{A},\mathcal{A}] \otimes \mathcal{A} \to \mathcal{A}$ which we denote by the same brackets.

**Corollary 5.5.** *An $H_0$-Poisson structure $\{\_\}$ in turn induces a Lie Algebra structure $\{\_\}^{Lie} : \mathcal{A}/[\mathcal{A},\mathcal{A}] \otimes \mathcal{A}/[\mathcal{A},\mathcal{A}] \to \mathcal{A}/[\mathcal{A},\mathcal{A}]$ on abelianization $\mathcal{A}_\natural = \mathcal{A}/[\mathcal{A},\mathcal{A}]$ of $\mathcal{A}$.*

## 5.2 Poisson brackets on the moduli space of representations

Double derivation property introduced to Definition 5.1 at the same time provides a constructive definition for a certain subclass of $H_0$-Poisson structures and allows one to establish a precise correspondence between $H_0$-Poisson structures and geometry. Throughout this section we will review main ideas of pioneering papers [CB99, CBEG07, VdB08] and apply them to the context of the Modified Double Poisson Bracket.

### 5.2.1 Representation scheme

As before, let $\mathcal{A} = \langle x^{(1)}, \dots, x^{(k)} \rangle / \mathcal{R}$ be a finitely generated associative algebra with a finite set of relations $\mathcal{R}$. Each representation of $\mathcal{A}$ in $Mat_N(\mathbb{C})$ can be defined by the image of the generators, let

$$\varphi(x^{(i)}) = \begin{pmatrix} x^{(i)}_{11} & \dots & x^{(i)}_{1N} \\ \vdots & & \vdots \\ x^{(i)}_{N1} & \dots & x^{(i)}_{NN} \end{pmatrix}. \tag{5.6}$$

Representations of $\mathcal{A}$ then form an affine scheme $\mathcal{V}$ with a coordinate ring $\mathbb{C}[\mathcal{V}] := \mathbb{C}\left[x^{(i)}_{j,k}\right]/\varphi(\mathcal{R})$. Denote as $\mathbb{C}_\mathcal{V}$ — the corresponding sheaf of rational functions. Then $\varphi : \mathcal{V} \times \mathcal{A} \to Mat_N(\mathbb{C})$. For a general point $m \in \mathcal{V}$ map $\varphi(m, \_)$ provides an $N$-dimensional matrix representation of $\mathcal{A}$. Hereinafter, we often omit the first argument of $\varphi$ where it is assumed to be a function on $\mathcal{V}$.

### 5.2.2 Moduli space of representations

There is a natural action of $GL_N(\mathbb{C})$ on $Mat_N(\mathbb{C})$ which corresponds to the change of basis in the underlying finite dimensional module. It induces the $GL_N(\mathbb{C})$ action on the sheaf of rational functions $\mathbb{C}_{\mathcal{V}}$. We denote as $\mathbb{C}[\mathcal{V}]^{inv} \subset \mathbb{C}[\mathcal{V}]$ (respectively $\mathbb{C}_{\mathcal{V}}^{inv} \subset \mathbb{C}_{\mathcal{V}}$) the subalgebra of $GL_N(\mathbb{C})$ invariant elements. We refer to the orbit of the $GL_N(\mathbb{C})$ action as an isomorphism class of representations and thus $\mathbb{C}[\mathcal{V}]^{inv}$ is the coordinate ring of the corresponding moduli space.

One can construct elements of $\mathbb{C}[\mathcal{V}]^{inv}$ by taking traces $\varphi_{ii}(x)$ for different $x \in A$, clearly the image would be invariant under the cyclic permutations of generators in each monomial and thus would depend only on the element of the cyclic space $A_\natural = \mathcal{A}/[\mathcal{A},\mathcal{A}]$. This induces a map $\varphi_0 : \mathcal{A}/[\mathcal{A},\mathcal{A}] \to \mathbb{C}[\mathcal{V}]^{inv}$ from $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ to the invariant subalgebra $\mathbb{C}[\mathcal{V}]^{inv}$. Denote the image of this map by $\mathcal{H} := \varphi_0(\mathcal{A}/[\mathcal{A},\mathcal{A}])$.

**Lemma 5.7.** *[Pro76] $C[\mathcal{V}]^{inv}$ is generated by $\mathcal{H}$ as a commutative algebra.*

**Example 5.8.** If $A$ happens to be commutative, the representation functor $\mathrm{Rep}_1$ for $N = 1$ will map it to itself. Moreover $\mathbb{C}_{\mathcal{V}}^{inv}$ will coincide with $\mathbb{C}_{\mathcal{V}}$ for this case.

**Example 5.9.** The simplest scheme here corresponds to the representations of $A = \mathbb{C}\langle x^{(1)}, \ldots, x^{(k)}\rangle$ — free algebra with $k$ generators. In the absence of relations, the corresponding scheme is birational to $\mathbb{C}^{kN^2}$, a $kN^2$-dimensional vector space. And the corresponding sheaf of rational functions is nothing but the field of rational functions in $kN^2$ variables.

**Example 5.10.** Another interesting special case corresponds to the so-called smooth algebras. Finitely generated algebra $A$ is called smooth if $\Omega^1 := \ker \mu = \{a_1 \otimes a_2 \,|a_1 a_2 = 0,\ a_1, a_2 \in A\}$ is projective as an inner bimodule. This guarantees that the representation scheme is actually a smooth affine variety. This case was in details studied in [CBEG07].

The major advantage of the Poisson formalism as compared to the Symplectic formalism is that it can be easily generalized beyond the smooth case.

### 5.2.3 Bracket

Define induced bracket $\{,\}^{\mathcal{V}} : \mathbb{C}_{\mathcal{V}} \otimes \mathbb{C}_{\mathcal{V}} \to \mathbb{C}_{\mathcal{V}}$ on generators $x_{ij}^{(m)}$ of $\mathbb{C}[\mathcal{V}]$ by

$$\left\{x_{ij}^{(m)}, x_{kl}^{(n)}\right\}^{\mathcal{V}} = \varphi\left(\{\!\{x^{(m)} \otimes x^{(n)}\}\!\}\right)_{(kj),(il)} \tag{5.11a}$$

And then extend it to the entire $\mathbb{C}_\mathcal{V}$ by Leibnitz identities w.r.t. both arguments. Namely, for all $a, b, c \in \mathbb{C}_\mathcal{V}$.

$$\{ab, c\}^\mathcal{V} = a\{b, c\}^\mathcal{V} + b\{a, c\}^\mathcal{V}, \tag{5.11b}$$

$$\{a, bc\}^\mathcal{V} = c\{a, b\}^\mathcal{V} + b\{a, c\}^\mathcal{V}. \tag{5.11c}$$

As opposed to [VdB08], the bracket (5.11) in the context of Definition 5.1 is not necessarily skew-symmetric and thus is not yet a Poisson bracket on $\mathbb{C}_\mathcal{V}$. It is a famous result of W. Crawley-Boevey [CB11] that any $H_0$-Poisson structure induces a conventional Poisson bracket on the moduli space of representations. In addition to that, we show in Proposition 5.18 that it comes with a Lie module action on the coordinate space of representations. In the case of bracket induced by the modified double Poisson bracket both are nothing but restrictions of (5.11) to $\mathbb{C}_\mathcal{V}^{inv} \otimes \mathbb{C}_\mathcal{V}^{inv}$ and $\mathbb{C}_\mathcal{V}^{inv} \otimes \mathbb{C}_\mathcal{V}$ respectively.

It is easy to check that the above extension (5.11b) – (5.11c) is consistent with the double Leibnitz identity and relations $\varphi(\mathcal{R})$ in the coordinate ring $\mathbb{C}[\mathcal{V}]$, namely

**Lemma 5.12.** *Equations (5.11) define a unique linear map $\{,\}^\mathcal{V} : \mathbb{C}[\mathcal{V}] \otimes \mathbb{C}[\mathcal{V}] \to \mathbb{C}[\mathcal{V}]$ given by*

$$\forall x, y \in A : \ \{\varphi(x)_{ij}, \varphi(y)_{kl}\}^\mathcal{V} = \varphi(\{\!\{x, y\}\!\}')_{kj}\varphi(\{\!\{x, y\}\!\}'')_{il} \tag{5.13}$$

*Proof.* Define $X = \varphi(x)$ and $Y = \varphi(y)$. In what follows assume the summation over repeating indexes

$$\{X_{ij}, Y_{kl}Z_{lm}\}^\mathcal{V} = \varphi(\{\!\{x, yz\}\!\}')_{kj}\varphi(\{\!\{x, yz\}\!\}'')_{im}.$$

On the other hand

$$\begin{aligned}\{\!\{x, yz\}\!\} &= (y \otimes 1)\{\!\{x, z\}\!\} + \{\!\{x, y\}\!\}(1 \otimes z) \\ &= y\{\!\{x, z\}\!\}' \otimes \{\!\{x, z\}\!\}'' + \{\!\{x, y\}\!\}' \otimes \{\!\{x, y\}\!\}'' z\end{aligned}$$

Which leads us to

$$\begin{aligned}\{X_{ij}, Y_{kl}Z_{lm}\}^{\mathcal{V}} =& Y_{kl}\varphi(\{\!\{x,z\}\!\}')_{lj}\varphi(\{\!\{x,z\}\!\}'')_{im} + \varphi(\{\!\{x,y\}\!\}')_{kj}\varphi(\{\!\{x,y\}\!\}'')_{il}Z_{lm}\\ =& Y_{kl}\{X_{ij}, Z_{lm}\}^{\mathcal{V}} + \{X_{ij}, Y_{kl}\}^{\mathcal{V}}Z_{lm}\end{aligned}$$

By the same derivation

$$\{X_{il}Y_{lj}, Z_{km}\}^{\mathcal{V}} = X_{il}\{Y_{lj}, Z_{km}\}^{\mathcal{V}} + Y_{lj}\{X_{il}, Z_{km}\}^{\mathcal{V}}.$$

Now, using the fact that $\mathcal{A}$ is finitely generated by $x^{(m)}$ we conclude that (5.11$b$) and (5.11$c$) uniquely extend $\{\_,\_\}^{\mathcal{V}}$ on pairs of monomials. Moreover, defining ideal $\varphi(\mathcal{R}) \subset \mathbb{C}[\mathcal{V}]$ for the coordinate ring $\mathbb{C}[\mathcal{V}]$ is thus within the left and right kernel of $\{\_,\_\}^{\mathcal{V}}$. So $\{\_,\_\}^{\mathcal{V}}$ extends uniquely to $\mathbb{C}[\mathcal{V}]$. □

Equation (5.13) immediately implies

**Corollary 5.14.** *For all $a, b \in A$, $\{\varphi_0(a), \varphi(b)\}^{\mathcal{V}} = \varphi(\{a,b\})$.*

*Proof.*

$$\begin{aligned}\forall x,y \in A:\ \{\varphi(x)_{ii}, \varphi(y)_{kl}\}^{\mathcal{V}} =& \varphi(\{\!\{x,y\}\!\}')_{ki}\varphi(\{\!\{x,y\}\!\}'')_{il} = \varphi(\{\!\{x,y\}\!\}'\{\!\{x,y\}\!\}'')_{kl}\\ =& \varphi(\mu(\{\!\{x,y\}\!\}))_{kl} = \varphi(\{x,y\}^{\mathcal{V}})_{kl}.\end{aligned} \tag{5.15}$$

□

**Lemma 5.16.** *Equations (5.11) define a unique linear map $\{,\}^{\mathcal{V}}:\ \mathbb{C}_{\mathcal{V}} \otimes \mathbb{C}_{\mathcal{V}} \to \mathbb{C}_{\mathcal{V}}$.*

*Proof.* Taking into account Lemma 5.12 it would be enough to prove that the bracket $\{,\}^{\mathcal{V}}$ can be extended to a properly localized ring. Let $R$ be a $\mathbb{C}$-algebra s.t. $\{,\}^{\mathcal{V}}$ : $R \otimes R \to R$ is well defined and satisfies (5.11b) – (5.11c). For any multiplicative subset $S \subset R$ and $a \in S,\ b \in R$ we immediately get $\{a^{-1}, b\}^{\mathcal{V}} = -a^{-2}\{a,b\}^{\mathcal{V}}$ and $\{b, a^{-1}\}^{\mathcal{V}} = -a^{-2}\{b,a\}^{\mathcal{V}}$. This provides a unique extension of $\{,\}^{\mathcal{V}}$ to $S^{-1}R$.

Thus for each distinguished open subset $\mathcal{V}_f \subset \mathcal{V}$ we have a unique extension of $\{,\}^{\mathcal{V}}$ to $\Gamma(\mathcal{V}_f, \mathcal{O}_{\mathcal{V}})$. Now denote by $S(\mathcal{V}_f) \subset \Gamma(X_f, \mathcal{O}_{\mathcal{V}})$ the set of functions which are not a zero divisor on any stalk, we have a unique extension of $\{,\}^{\mathcal{V}}$ to $S(\mathcal{V}_f)^{-1}\Gamma(\mathcal{V}_f, \mathcal{O}_{\mathcal{V}}) = \Gamma(\mathcal{V}_f, \mathbb{C}_{\mathcal{V}})$. □

**Lemma 5.17.** *Following restriction of* $\{,\}^{\mathcal{V}}$

$$\{,\}^{inv}:\mathbb{C}_{\mathcal{V}}^{inv}\otimes\mathbb{C}_{\mathcal{V}}^{inv}\to\mathbb{C}_{\mathcal{V}}^{inv}$$

*is skew-symmetric, namely for all* $f,g\in\mathbb{C}_{\mathcal{V}}^{inv}$ *we have* $\{f,g\}^{\mathcal{V}}\in\mathbb{C}_{\mathcal{V}}^{inv}$ *and* $\{f,g\}^{\mathcal{V}}=-\{g,f\}^{\mathcal{V}}$.

*Proof.* In light of Leibnitz identities (5.11b) and (5.11c) it would be enough for us to show the statement for generators of $\mathbb{C}[\mathcal{V}]^{inv}$. So w.l.o.g we can assume that $f,g\in\mathcal{H}$ (see Lemma 5.7). Under this assumption there exist $x,y\in A$ s.t. $f=\varphi_0(x)$ and $g=\varphi_0(y)$. Denote $X:=\varphi(x)$, $Y:=\varphi(y)$. Using (5.15) we get

$$\{f,g\}^{\mathcal{V}}=\{X_{ii},Y_{kk}\}^{\mathcal{V}}=\varphi(\{x,y\})_{kk}=\varphi_0(\{x,y\})\ \in\mathbb{C}[\mathcal{V}]^{inv}$$

as a result

$$\{f,g\}^{\mathcal{V}}+\{g,f\}^{\mathcal{V}}=\varphi_0(\{x,y\}+\{y,x\})=0.$$

□

**Proposition 5.18.** *The following restriction*

$$\{\_,\_\}^{\mathcal{V}}:\quad\mathbb{C}_{\mathcal{V}}^{inv}\otimes\mathbb{C}_{\mathcal{V}}\to\mathbb{C}_{\mathcal{V}} \tag{5.19}$$

*satisfies the Jacobi identity for the left Loday bracket, for all* $f,g\in\mathbb{C}_{\mathcal{V}}^{inv}$ *and* $h\in\mathbb{C}_{\mathcal{V}}$ *:*

$$\{f,\{g,h\}^{\mathcal{V}}\}^{\mathcal{V}}-\{g,\{f,h\}^{\mathcal{V}}\}^{\mathcal{V}}=\{\{f,g\}^{\mathcal{V}},h\}^{\mathcal{V}}.$$

*Proof.* For $f,g\in\mathcal{H}$ and $h\in\mathbb{C}[\mathcal{V}]$ the statement is a straightforward consequence of Corollary 5.14 and the fact that $\{,\}:\mathcal{A}_\natural\otimes\mathcal{A}\to\mathcal{A}$ is a Loday bracket. Denote

$$\phi(f,g,h):=\{f,\{g,h\}^{\mathcal{V}}\}^{\mathcal{V}}-\{g,\{f,h\}^{\mathcal{V}}\}^{\mathcal{V}}-\{\{f,g\}^{\mathcal{V}},h\}^{\mathcal{V}}.$$

Since $\phi$ is a derivation in its' last argument, left Jacobi identity extends for $h\in\mathbb{C}_{\mathcal{V}}$. Next, we have

$$\begin{aligned}&\phi(f_1f_2,g,h)-f_1\phi(f_2,g,h)-f_2\phi(f_1,g,h)=\\&\quad=-\{g,f_1\}^{\mathcal{V}}\{f_2,h\}^{\mathcal{V}}-\{g,f_2\}^{\mathcal{V}}\{f_1,h\}^{\mathcal{V}}-\{f_2,g\}^{\mathcal{V}}\{f_1,h\}^{\mathcal{V}}-\{f_2,h\}^{\mathcal{V}}\{f_1,g\}^{\mathcal{V}}.\end{aligned}$$

Note, that $\{,\}^{\mathcal{V}}$ is not skew-symmetric in general, however by Lemma 5.17 we have for $f_1, f_2, g \in \mathbb{C}_{\mathcal{V}}^{inv}$

$$\{g, f_1\}^{\mathcal{V}} + \{f_1, g\}^{\mathcal{V}} = 0, \qquad \{f_2, g\}^{\mathcal{V}} + \{g, f_2\}^{\mathcal{V}} = 0$$

which is enough to conclude that for all $f_1, f_2, g \in \mathbb{C}_{\mathcal{V}}^{inv}$ and $h \in \mathbb{C}_{\mathcal{V}}$

$$\phi(f_1 f_2, g, h) = f_1 \phi(f_2, g, h) + f_2 \phi(f_1, g, h).$$

We also get for all $f, g \in \mathbb{C}_{\mathcal{V}}^{inv}$ and $g \in \mathbb{C}_{\mathcal{V}}$ s.t. $f^{-1} \in \mathbb{C}_{\mathcal{V}}^{inv}$

$$\phi(f^{-1}, g, h) = -f^{-2} \phi(f, g, h).$$

Similar reasoning applies for the second argument which finalizes the proof. □

*Remark* 5.20. Proposition 5.18 defines a representation analogue of an $H_0$-Poisson structure. Note the dual properties, once $H_0$-Poisson structure factors through $\{,\} : \mathcal{A}/[\mathcal{A}, \mathcal{A}] \otimes \mathcal{A} \to \mathcal{A}$, the induced bracket defined above has to be restricted on invariant subalgebra $\{,\}^{\mathcal{V}} : \mathbb{C}_{\mathcal{V}}^{inv} \otimes \mathbb{C}_{\mathcal{V}} \to \mathbb{C}_{\mathcal{V}}$ in order to satisfy Jacobi identity. Following ideas of [Tur14] we formulate this duality fundamentally in Section **??** when we show that one can generalize Proposition 5.18 beyond matrix representations.

**Corollary 5.21.** *The following restriction*

$$\{\_,\_\}^{inv} : \quad \mathbb{C}_{\mathcal{V}}^{inv} \otimes \mathbb{C}_{\mathcal{V}}^{inv} \to \mathbb{C}_{\mathcal{V}}^{inv} \tag{5.22}$$

*is a Poisson bracket.*

*Proof.* This statement follows from Corollary 5.3 and results of [CB11], however below we will present a direct proof using Proposition 5.18.

Indeed, by Lemma 5.17 this restriction is skew-symmetric, it satisfies Leibnitz identity in both arguments by definition (5.11b) – (5.11c). As a particular case of Proposition 5.18 it also satisfies Jacobi identity. □

### 5.2.4 Casimir elements

Action of the Poisson bracket (5.22) on the full representation scheme (5.19) provides a convenient way to construct Casimir elements. Recall

**Definition 5.23.** Element $c \in \mathcal{A}$ is a right Casimir of bracket $\{,\}$ if for all $h \in \mathcal{A}/[\mathcal{A},\mathcal{A}]$ we have $\{h, c\} = 0$.

*Remark* 5.24. It is worth noting that right Casimir elements are not necessary within the left kernel of the bracket beyond skew-symmetric case. For a particular example see (5.27) and (5.30).

Since $\{,\}$ is a derivation in the second argument, the set of all right Casimir elements forms a subalgebra $\mathcal{C} \subset \mathcal{A}$. This subalgebra allows one to construct Casimir elements of the bracket on representation scheme.

**Proposition 5.25.** *Assume that $c \in A$ is a right Casimir of bracket $\{,\} : A/[A,A] \otimes A \to A$. Subalgebra $\mathbb{C}[\varphi(c)_{kl}]$ generated by components of the matrix $\varphi(c)$ consist of right Casimirs of the bracket $\{,\}^{\mathcal{V}}$*

*Proof.* For all $h \in \mathcal{H}$ we have

$$\{\varphi_0(h), \varphi(c)_{kl}\}^{\mathcal{V}} = \{\varphi(h)_{ii}, \varphi(c)_{kl}\}^{\mathcal{V}} = \varphi(\{h, c\})_{kl} = 0.$$

Since $\{,\}^{\mathcal{V}}$ satisfies Leibnitz identity w.r.t to both arguments we conclude that for all $f \in \mathbb{C}^{inv}_{\mathcal{V}}$ and $x \in \mathbb{C}[\varphi(c)_{kl}]$

$$\{f, x\}^{\mathcal{V}} = 0.$$

$\square$

Note, that in Proposition 5.25 it is essential to consider a restricted bracket $\{\_,\_\}^{\mathcal{V}}$ on $\mathbb{C}^{inv}_{\mathcal{V}} \otimes \mathbb{C}_{\mathcal{V}}$ as defined in (5.25). If instead of element $\mathbb{C}^{inv}_{\mathcal{V}}$ as a first argument we take arbitrary $f \in \mathbb{C}_{\mathcal{V}}$ the bracket with a Casimir elements do not have to be zero.

**Corollary 5.26.** *Let $\mathcal{C} \subset \mathcal{A}$ be a subalgebra of right Casimir elements of bracket $\{,\}$, then $\varphi_0(\mathcal{C}) \subset \mathbb{C}^{inv}_{\mathcal{V}}$ consist of Casimir elements of bracket $\{,\}^{inv}$.*

This Corollary is especially useful when $\mathcal{C}$ is finitely generated. We illustrate this method in Section 5.3.1.

## 5.3 Examples of the Modified Double Poisson brackets

### 5.3.1 Bracket for Kontsevich system

Here we describe a particular example of modified double bracket on $\mathbb{C}\langle u^{\pm}, v^{\pm}\rangle$ introduced in [Art15]. This double bracket is not skew-symmetric and thus provides an example beyond the case considered in [VdB08].

Let $\mathcal{A}^{+} = \mathbb{C}\langle u, v\rangle$ be a free associative algebra with two generators. Define a biderivation of $\mathcal{A}$ on the generators as

$$\{\!\{u, v\}\!\}_K = -vu \otimes 1, \qquad \{\!\{v, u\}\!\}_K = uv \otimes 1, \qquad \{\!\{u, u\}\!\}_K = \{\!\{v, v\}\!\}_K = 0. \tag{5.27}$$

**Proposition 5.28.** *[Art15] The biderivation $\{\!\{\_\}\!\}_K$ is a modified double Poisson bracket.*

Under the representation functor $\mathrm{Rep_N}$ our algebra is mapped to the commutative algebra $\mathcal{A}_N = \mathbb{C}(u_{i,j}, v_{i,j})$ of rational functions in $2N^2$ variables $\{u_{i,j}, v_{i,j} \,|\, 1 \le i, j \le N\}$. The corresponding affine scheme $\mathcal{V}$ is just a $2N^2$-dimensional vector space over $\mathbb{C}$.

The induced bracket is a biderivation $\{,\}^{\mathcal{V}} : \mathcal{A}_N \otimes \mathcal{A}_N \to \mathcal{A}_N$ defined on generators as

$$\begin{aligned} \{u_{ij}, v_{kl}\}^{\mathcal{V}} &= -\,\delta_{il} \sum_m v_{km} u_{mj} \\ \{v_{kl}, u_{ij}\}^{\mathcal{V}} &= \delta_{kj} \sum_m u_{im} v_{ml}. \end{aligned} \tag{5.29}$$

Proposition 5.21 implies that restriction of $\{,\}^{\mathcal{V}}$ on invariant rational functions $\{\_,\_\}^{inv} : \mathcal{A}_N^{inv} \otimes \mathcal{A}_N^{inv} \to \mathcal{A}_N^{inv}$ is a Poisson bracket.

#### Dimensions of the symplectic leaves

Element

$$c = uvu^{-1}v^{-1} \tag{5.30}$$

is a right Casimir of the $H_0$-Poisson bracket induced by (5.27) (see [Art15] for a proof). One can show that $\operatorname{Tr} \varphi(c^k)$ provide Casimirs for the indeuced bracket $\{\_,\_\}^{inv}$. The Poisson bracket on $\mathbb{C}_{\mathcal{V}}^{inv}$ we defined earlier is degenerate due to existence of Casimirs.

Which means that the Poisson tensor is not invertible at a generic point. In order to make it invertible (and thus induce a symplectic structure) one has to restrict the bracket to the subvariety corresponding to the fixed level of all Casimir functions (See e.g. [Arn78]). The codimension of such variety is, of course, simply the number of algebraically independent Casimir functions.

Based on direct computation of dimensions of symplectic leaves for bracket $\{\_,\_\}^{inv}$ we come to the following

**Conjecture 5.31.** *There are exactly $N-1$ algebraically independent Casimir elements given by $Tr\,\varphi(c^k)$ for the bracket $\{,\}^{inv}$.*

We summarize a computational evidence in favour of this conjecture in the Table 5.1. Here $\dim L$ — dimension of a generic symplectic leaf, $\operatorname{codim}\varphi_0(c^k)$ — number of algebraically independent Casimirs provided by $\varphi_0(c^k)$.

| $N$ | $\dim \mathbb{C}_{\mathcal{V}}^{inv}$ | $\dim L$ | $\operatorname{codim}\varphi_0(c^k)$ |
|---|---|---|---|
| 1 | 2 | 2 | 0 |
| 2 | 5 | 4 | 1 |
| 3 | 10 | 8 | 2 |
| 4 | 17 | 14 | 3 |
| 5 | 26 | 22 | 4 |
| 6 | 37 | 32 | 5 |

Table 5.1: Summary on tests of dimensions of symplectic leaf

### 5.3.2 Other examples Double Poisson Brackets

Below we present a couple of other examples of modified double Poisson brackets on $Free_3 = \mathbb{C}\langle x_1, x_2, x_3\rangle$. Unlike (5.27), examples presented in this subsection are conjectural although very well tested. More examples and partial classification are in progress.

$$\begin{aligned}
&\{\!\{x_1, x_2\}\!\}^I = -x_2 x_1 \otimes 1, && \{\!\{x_2, x_1\}\!\}^I = x_1 x_2 \otimes 1,\\
&\{\!\{x_2, x_3\}\!\}^I = -x_2 \otimes x_3, && \{\!\{x_3, x_2\}\!\}^I = x_2 \otimes x_3,\\
&\{\!\{x_3, x_1\}\!\}^I = -1 \otimes x_3 x_1, && \{\!\{x_1, x_3\}\!\}^I = 1 \otimes x_1 x_3.
\end{aligned} \tag{5.32}$$

Here all omitted brackets of generators are assumed to be zero.

$$\begin{aligned}
&\{\!\{x_1, x_2\}\!\}^{II} = -x_1 \otimes x_2, && \{\!\{x_2, x_1\}\!\}^{II} = x_1 \otimes x_2,\\
&\{\!\{x_2, x_3\}\!\}^{II} = x_3 \otimes x_2, && \{\!\{x_3, x_2\}\!\}^{II} = -x_3 \otimes x_2,\\
&\{\!\{x_3, x_1\}\!\}^{II} = x_1 \otimes x_3 - x_3 \otimes x_1.
\end{aligned} \tag{5.33}$$

**Conjecture 5.34.** *Brackets (5.32) and (5.33) are modified double Poisson brackets on $Free_3$. Namely, corresponding biderivations satisfy (5.2c) and (5.2d).*

We have tested equations (5.2c) and (5.2d) for all monomials up to length 5.

# Chapter 6

# Appendix

## 6.1 Classification of Graded Brackets on $Free_2$

For this section, let $A_+ = \mathbb{C}\langle x_1, x_2\rangle$ be a free algebra with two generators. There is $\mathbb{C}^\times \times \mathbb{C}^\times$ action on $A_+$, s.t. for each $\lambda_1, \lambda_2 \in \mathbb{C}^\times$

$$(\lambda_1, \lambda_2) : \; A_+ \to A_+, \quad \text{s.t.} \quad \begin{array}{l} x_1 \mapsto \lambda_1 x_1 \\ x_2 \mapsto \lambda_2 x_2 \end{array} \tag{6.1}$$

The most general skew-symmetric bi-derivation $\{\!\{,\}\!\}$: $A_+^{\otimes 2} \to A_+^{\otimes 2}$ which is equivariant with respect to $(\mathbb{C}^\times)^2$ action (6.1) is determined by its action on pairs of generators and has the following form

$$\begin{aligned}
\{\!\{x_1, x_1\}\!\} =& b_1(x_1x_1 \otimes 1 - 1 \otimes x_1x_1) \\
\{\!\{x_1, x_2\}\!\} =& b_2\, x_1x_2 \otimes 1 + b_3\, x_1 \otimes x_2 + b_4\, 1 \otimes x_1x_2 \\
& + b_5\, x_2x_1 \otimes 1 + b_6\, x_2 \otimes x_1 + b_7\, 1 \otimes x_2x_1 \\
\{\!\{x_2, x_1\}\!\} =& - b_2\, 1 \otimes x_1x_2 - b_3\, x_2 \otimes x_1 - b_4\, x_1x_2 \otimes 1 \\
& - b_5\, 1 \otimes x_2x_1 - b_6\, x_1 \otimes x_2 - b_7\, x_2x_1 \otimes 1 \\
\{\!\{x_2, x_2\}\!\} =& b_8(x_2x_2 \otimes 1 - 1 \otimes x_2x_2)
\end{aligned} \tag{6.2}$$

where $b_1, \ldots, b_8 \in \mathbb{C}$ are arbitrary parameters. In other words, these are all graded bi-derivations of $A_+$ of degree zero w.r.t. to the grading given by weights of $(\mathbb{C}^\times)^2$-action.

Below we classify all such biderivations which give rise to an $H_0$-Poisson structure on $A_+$. It would be enough for us to classify biderivations modulo the ones which give trivial contribution to an $H_0$-Poisson structure.

**Proposition 6.3.** *Each equivalence class has a unique skew-symmetric representative*

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=-1\otimes x_1x_1+x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=-1\otimes x_2x_2+x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2+x_2x_1\otimes 1+x_1\otimes x_2 \end{aligned} \tag{6.4a}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=-1\otimes x_1x_1+x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=-1\otimes x_2x_2+x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2-x_2x_1\otimes 1+x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4b}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=1\otimes x_1x_1-x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=-1\otimes x_2x_2+x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2-x_2x_1\otimes 1+x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4c}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=1\otimes x_1x_1-x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=-1\otimes x_2x_2+x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2+x_2x_1\otimes 1-x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4d}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=-1\otimes x_1x_1+x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=-1\otimes x_2x_2+x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2+x_2x_1\otimes 1+x_1\otimes x_2-x_2\otimes x_1 \end{aligned} \tag{6.4e}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=-1\otimes x_1x_1+x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=1\otimes x_2x_2-x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=1\otimes x_1x_2-x_2x_1\otimes 1-x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4f}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=1\otimes x_1x_1-x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=-1\otimes x_2x_2+x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=1\otimes x_1x_2+x_2x_1\otimes 1+x_1\otimes x_2-x_2\otimes x_1 \end{aligned} \tag{6.4g}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=-1\otimes x_1x_1+x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=1\otimes x_2x_2-x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=1\otimes x_1x_2+x_2x_1\otimes 1-x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4h}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=-1\otimes x_1x_1+x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=1\otimes x_2x_2-x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2+x_2x_1\otimes 1-x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4i}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=-1\otimes x_1x_1+x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=1\otimes x_2x_2-x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2-x_2x_1\otimes 1+x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4j}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=1\otimes x_1x_1-x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=1\otimes x_2x_2-x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=1\otimes x_1x_2-x_2x_1\otimes 1+x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4k}$$

$$\begin{aligned} &\{\!\!\{x_1,x_1\}\!\!\}=1\otimes x_1x_1-x_1x_1\otimes 1 \qquad \{\!\!\{x_2,x_2\}\!\!\}=1\otimes x_2x_2-x_2x_2\otimes 1\\ &\{\!\!\{x_1,x_2\}\!\!\}=-1\otimes x_1x_2-x_2x_1\otimes 1+x_1\otimes x_2+x_2\otimes x_1 \end{aligned} \tag{6.4l}$$

## 6.2 Equivalence classes of biderivations

Denote by $e \in D_{\mathcal{A}}$ noncommutative vector field s.t. for all $a \in \mathcal{A},\ e(a) = a \otimes 1 - 1 \otimes a$.

**Lemma 6.5.** *For all $\gamma \in D_{\mathcal{A}}$ denote $X : \mathcal{A}^{\otimes 2} \to \mathcal{A}^{\otimes 2}$,*

$$X := tr_{\mathcal{A}}(e \star \gamma). \tag{6.6}$$

*Then for all $a, b \in \mathcal{A}$*

$$\mu(X(a \otimes b)) = 0.$$

*Proof.* Indeed,

$$X(a \otimes b) = \gamma(b)' e(a)'' \otimes e(a)' \gamma(b)'' = \gamma(b)' \otimes a\gamma(a)'' - \gamma(b)' a \otimes \gamma(b)'' \ \in \ker \mu.$$

□

For each $P \in (DA)_2$ define the corresponding double bracket $\{\!\{,\}\!\}_P$: $\mathcal{A}^{\otimes 2} \to \mathcal{A}^{\otimes 2}$ as

$$\{\!\{,\}\!\}_P := \mathrm{tr}_{\mathcal{A}} P.$$

**Corollary 6.7.** *If $P$ is a modified double Poisson bivector, then so is $P + e \star \delta$ for an arbitrary $\delta \in D_{\mathcal{A}}$. Moreover, they induce the same $H_0$-Poisson structure*

$$\{,\}_P = \{,\}_{P + e\star\delta}.$$

*Proof.* Apply Lemma 6.5 to the difference. □

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