RIGIDITY OF LOCAL HOLOMORPHIC MAPS BETWEEN HERMITIAN SYMMETRIC SPACES

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A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Xiaojun Huang

And approved by

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New Brunswick, New Jersey

May, 2018

ABSTRACT OF THE DISSERTATION

Rigidity of local holomorphic maps between Hermitian Symmetric Spaces

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In this dessertaion, rigidity of local holomorphic maps between Hermitian symmetric spaces has been studied. For local holomorphic maps from an irreducible Hermitian symmetric spaces of compact type to itself, which is equipped with a canonical Kähler-Eisntein metric, we show that every map extends to an isometry of the manifold, provided that the maps satisfy a measure-preserving equation and are generically non-degenerate. To establish the rigidity result, a notion of Serge variety and Segre family in the algebraic setting is introduced. Before obtaining the main theorem, we first prove a basic property for partially degenerate holomorphic maps in a general setting. Then we establish the Nash-algebraicity for one of these maps by applying this basic property. Here the explicit expression of the mimimal embedding of the manifold into a certain projective space is essentially used. Standard monodormy argument is then applied to show the rationality for this Nash-algebraic map. Lastly by a covering trick we show that the map is a birational map and further an isometry. Hence by induction, we conclude the main theorem. This thesis is based on a joint work with Xiaojun Huang and Ming Xiao ([FHX]).

Acknowledgements

Foremost, I would like to express my sincere gratitude to my advisor Prof. Xiaojun Huang. His mathematical taste influences my academic career tremendously. I still remember that the first time I met him is in his several complex variables class during my first week in Rutgers. He enthusiastically told me that the interesting and important thing in mathematics is geometry, is when the general PDE theory does not work but geometric strucure helps. This is the first advice I got from him and probably also one of the priceless treasure I got from him since then. I cannot imagine what my life would be without his warm encouragement, self-giving support, patient guidance and strict training all these years. I remember the days we discuss math in Thomas Sweets, Julliard School, Wuhan, Huzhou, Hong Kong and etc.. These are all my sweet memories. For me, he is not only a great advisor, but a friend and a father as well.

Furthermore, many thanks go in particular to my committee members, not only for their precise advice for my thesis but also for their help through these years. I would like to thank Prof. Shiferaw Berhanu, who invited me to Brazil, taught me CR maximal principle and help with my English patiently. I would like to thank Prof. Sagun Chanillo, who taught me harmonic analysis and added one more lecture on Carleson measure for me. I would like to thank Prof. Jian Song, who shares many great ideas with me both in mathematics and in life and empathizes the key role of Kähler-Einstein metrics in the study of general type varieties.

I would like to thank Prof. Zheng-Chao Han, who gave me the chance to study in Rutgers. I would like to thank Prof. Feng Luo, who taught me Teichimüller theory and financially supported me in Tsinghua in summer 2014. I would like to thank Prof. Abbas Bahri, who taught me Gromov's pseudo-holomorphic curves. I also would like to thank Prof. Xiaochun Rong, who gives many suggestions on my mathematical career

and encourages me to swim.

I am indebted to Prof. Anders Buch, Prof. Peter Ebenfelt, Prof. Siqi Fu, Prof. Xianghong Gong, Prof. Jun-Muk Hwang, Prof. Qingchun Ji, Prof. Song-Ying Li, Prof. Nordine Mir, Prof. Ngaiming Mok, Prof. Yum-Tong Siu, Prof. Xiaodong Wang, Prof. Sai-Kee Yeung and Prof. Fangyang Zheng for their encouragement and help during my graduate study and job search. I am also indebted to my academic brothers and sisters Wanke Yin, Yuan Zhang, Yuan Yuan, Xiaoshan Li, Ming Xiao, Lihua Huang and Chloe Wawrzyniak for their generous support and kind help.

I would like to thank Zhan Li, who leads me to algebraic geometry and supports me in all kinds of way he could afford. I would like to thank Bo Yang, who helps me go through difficulties in life and introduces me to the field of negatively curved spaces. I would like to thank my long time roommate and officemate Liming Sun, with whom we shared many joys and sorrows. I am grateful that he had been driving me back home for three months when we lived far away in different places. I would like to thank my friend and collaborator Xin Fu, who always gives me forthright admonition, pushes me to work harder and warns me to be more responsible for life. I would like to thank my former roommate Bo Zhu, who taught me many lessons in life. I would like to thank my classmate Songyan Xie, who taught me Kobayashi hyperbolicity. I would like to thank my friend Xu Yang, who influences my scientific taste.

I would like to thank Wei Chai, Vernon Chan, Edward Chien, Yu Du, Guosheng Fu, Jin Gong, Purvi Gupta, Jun Li, Han Lu, Siyuan Lu, Jiayin Pan, Jiayu Peng, Fei Qi, Sijian Tang, Matthew Welsh, Jianguo Xiao, Tiancheng Xue, Tian Yang, Xiaoyu Yang, Yishao Yao, Lun Zhang, Zhuohui Zhang, Tiecheng Zhou, Songhao Zhu, Xiaoping Zhu and many others for making my life as a math Ph.D student more enjoyable.

Finally I want to express my love and gratitude to my parents. Their love is the greatest comfort for me. I love you!

Dedication

To my parents Ke Fang, and Xiaoxiao Wang.

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Chapter 1

Introduction

Rigidity properties are among the fundamental phenomena in Complex Analysis and Geometry of several variables, that study the global extension and uniqueness for various holomorphic objects up to certain group actions. The rigidity problem that we consider here was initiated by a celebrated paper of Calabi [Ca]. In [Ca], Calabi studied the global holomorphic extension and uniqueness (up to the action of the holomorphic isometric group of the target space) for a local holomorphic isometric embedding from a Kähler manifold into a complex space form. He established the global extension and the Bonnet type rigidity theorem for a local holomorphic isometric embedding from a complex manifold with a real analytic Kähler metric into a standard complex space form. The phenomenon discovered by Calabi [Ca] has been further explored in the past several decades due to its extensive connection with problems in Analysis and Geometry. (See [U] [DL] [DL1], for instance).

In 2004, motivated by the modularity problem of the algebraic correspondences in algebraic number theory, Clozel and Ullmo [CU] were led to study the rigidity problems for local holomorphic isometric maps and even much more general volume-preserving maps between bounded symmetric domains equipped with their Bergman metrics. By reducing the modularity problem to the rigidity problem for local holomorphic isometries, Clozel-Ullmo proved that an algebraic correspondence in the quotient of a bounded symmetric domain preserving the Bergman metric has to be a modular correspondence in the case of the unit disc in the complex plane and in the case of bounded symmetric domains of rank ≥ 2. Notice that in the one dimensional setting, volume preserving maps are identical to the metric preserving maps. Thus the Clozel-Ullmo result also applies to the volume preserving algebraic correspondences in the lowest dimensional

case.

Motivated by the work in [CU], Mok carried out a systematic study of the rigidity problem for local isometric embeddings in a very general setting. Mok in [Mo2] [Mo3] [Mo4] proved the total geodesy for a local holomorphic isometric embedding between bounded symmetric domains D and Ω when either (i) the rank of each irreducible component of D is at least two or (ii) $D = \mathbb{B}^n$ and $\Omega = (\mathbb{B}^n)^p$ for $n \geq 2$. In a paper of Yuan-Zhang [YZ], the total geodesy is obtained in the case of $D = \mathbb{B}^n$ and $\Omega = \mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_p}$ with $n \geq 2$ and N_l arbitrary for $1 \leq l \leq p$. Earlier, Ng in [Ng2] had established a similar result when $p = 2$ and $2 \leq n \leq N_1, N_2 \leq 2n - 1$. Notice that when $D = \Delta$ is the unit disc in the complex plane, the total geodesy fails due to the existence of the following $p-$th root embedding constructed by Mok in [Mo4]. Let $\mathcal{H}$ be the upper half plane and $\mathcal{H}^p = \mathcal{H} \times \cdots \times \mathcal{H}$ be the Cartesian product of $\mathcal{H}$. Define

$$\theta_p(\tau) = (\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \cdots, \gamma^{p-1}\tau^{\frac{1}{p}}),$$

where $\gamma = e^{\frac{\sqrt{-1}\pi}{p}}$ and $\tau^{\frac{1}{p}} = r^{\frac{1}{p}} e^{\frac{\sqrt{-1}\theta}{p}}$ if $\tau = re^{\sqrt{-1}\theta}$ for $\theta \in (0, \pi)$. Then θ_p is a non-totally geodesic holomorphic isometry. Later certain classification results have been obtained for the holomorphic isometries from Δ to Δ^p by [Ng] and [Ch]. Around 2016, [CM], [UWZ] and [XY] studied the holomorphic isometries from unit ball B^n to type IV classical domains.

In a paper of X. Huang and Y. Yuan [HY1], the rigidity result has been established for local holomorphic isometric embeddings from a Hermitian symmetric space of compact type into the product of Hermitian symmetric spaces of compact type with even negative conformal factors where certain non-cancellation property for the conformal factors holds. (This cancellation condition turns out be the necessary and sufficient condition for the rigidity to hold due to the presence of negative conformal factors.) In a recent paper of Ebenfelt [E], a certain classification, as well as its connection with problems in CR geometry, has been studied for local isometric maps when the cancellation property fails to hold. The recent paper of Yuan [Y] studied the rigidity problem for local holomorphic maps preserving the (p, p)-forms between Hermitian symmetric spaces of non-compact type. We also mention other related studies for the

rigidity of holomorphic mappings, such as the ones by Huang [Hu1] [Hu2], Ji [Ji], Kim-Zaitsev [KZ], Mok [Mo1] [Mo5], Mok-Ng [MN1], Ng [Ng1] [Ng2], and many references therein, to name a few.

The work of Clozel and Ullmo has left open an important question of understanding the modularity problem for volume-preserving correspondences in the quotient of Hermitian symmetric spaces of higher dimension equipped with their Bergman metrics. In 2012, Mok and Ng answered, in the affirmative, the question of Clozel and Ullmo in [MN] by establishing the rigidity property for local holomorphic volume preserving maps from an irreducible Hermitian manifold of non-compact type into its Cartesian products.

In a joint work with X. Huang and M. Xiao [FHX], we continued the above mentioned investigations, especially those in [CU], [MN] and [HY1], and established the following result.

Let M be an irreducible $n-$dimensional Hermitian symmetric space of compact type, equipped with a canonical Kähler-Einstein metric ω. Write ω^n for the associated volume form (up to a positive constant depending only on n). We showed that the following Clozel-Ullmo and Mok-Ng type theorem holds for local measure preserving maps between Hermitian symmetric spaces of compact type:

Theorem 1.1. *([FHX]) Let (M, ω) be an irreducible $n-$dimensional Hermitian symmetric space of compact type as above. Let $F = (F_1, ..., F_m)$ be a holomorphic mapping from a connected open subset $U \subset M$ into the m-Cartesian product $M \times ... \times M$ of M. Assume that each F_j is generically non-degenerate in the sense that $F_j^*(\omega^n) \not\equiv 0$ over U. Assume that F satisfies the following volume-preserving (or measure-preserving) equation:*

$$\omega^n = \sum_{i=1}^{m} \lambda_i F_i^*(\omega^n), \tag{1.1}$$

for certain constants $\lambda_j > 0$. Then for each j with $1 \le j \le m$, F_j extends to a holomorphic isometry of (M, ω). In particular, the conformal factors satisfy the identity: $\sum_{j=1}^{m} \lambda_j = 1$.

Notice that in the Riemann sphere setting, Theorem 1.1 also follows from the isometric rigidity result obtained in [HY1]. However, the basic approach in [FHX] fundamentally differs from that in [HY1]. The method used in [HY1] is to first obtain the result in the simplest projective space setting and then use the minimal rational curves to reduce the general case to the much simpler projective space case. On the other hand, restrictions of volume preserving maps are no longer volume preserving and thus the reduction method in [HY1] can not be applied here. The approach we will present here is first to establish general results under certain geometric and analytic assumptions (i.e., Proposition (I)-(III)) and then verify that these assumptions are automatically satisfied based on a case by case argument in terms of the type of the Hermitian space.

Chapter 2

Irreducible Hermitian symmetric spaces and their Segre varieties in the minimal embeddings

2.1 Segre varieties of projective subvarieties

Write $z = (z_1, \cdots, z_n, z_{n+1})$ for the coordinates of $\mathbb{C}^{n+1}$ and $[z] = [z_1, \cdots, z_n, z_{n+1}]$ for the homogeneous coordinates of $\mathbb{CP}^n$. For a polynomial $p(z)$, we define $\bar{p}(z) := \overline{p(\bar{z})}$. For a connected projective variety $V \subset \mathbb{CP}^n$, write $\mathcal{I}_V$ for the ideal consisting of homogeneous polynomials in z that vanish on V. We define the conjugate variety V^* of V to be the projective variety defined by $\mathcal{I}_V^* := \{\bar{f} : f \in \mathcal{I}_V\}$. Apparently the map $z \mapsto \bar{z}$ defines a diffeomorphism from V to V^*. When $\mathcal{I}_V$ has a basis consisting of polynomials with real coefficients, $V^* = V$. Also if V is irreducible and has a smooth piece parametrized by a neighborhood of the origin of a complex Euclidean space through polynomials with real coefficients, then $V^* = V$.

Next for $[\xi] \in V^*$, we define the Segre variety Q_ξ of V associated with ξ by $Q_\xi = \{[z] \in V : \sum_{j=1}^{n+1} z_j \xi_j = 0\}$ which is a subvariety of codimension one in V. Similarly, for $[z] \in V$, we define the Segre variety Q_z^* of V^* associated with z by $Q_z^* = \{[\xi] \in V^* : \sum_{j=1}^{n+1} z_j \xi_j = 0\}$. It is clear that $[z] \in Q_\xi$ if and only if $[\xi] \in Q_z^*$. The Segre family of V is defined to be the projective variety $\mathcal{M} := \{([z], [\xi]) \in V \times V^*, [z] \in Q_\xi\}$.

Now, we let (M, ω) be an irreducible Hermitian symmetric space of compact type canonically embedded in a certain minimal projective space $\mathbb{CP}^N$, that we will describe in detail later in this chapter. Then under this embedding, its conjugate space M^* is just M itself. Taking ω to be the natural restriction of the Fubini-Study metric to M, the holomorphic isometric group of M is then the restriction of a certain subgroup of the unitary actions of the ambient space. Now, for two points $p_1, p_2 \in M$, let U be an $(N+1) \times (N+1)$ unitary matrix such that $\sigma([z]) = [z] \cdot U$ is an isometry sending p_1

to p_2. Then $\sigma^*([\xi]) = [\xi]\overline{U}$ is an isometry of M^*. By a straightforward verification, we see that σ^* biholomorphically sends $Q^*_{p_1}$ to $Q^*_{p_2}$. Similarly, for any $q_1, q_2 \in M^*$, Q_{q_1} is unitary equivalent to Q_{q_2}. In the canonical embeddings which we will describe later, the hyperplane section at infinity of the manifold is a Segre variety. Since the one at infinity is built up from Schubert cells and all Segre varieties are holomorphically equivalent to each other, one deduces that each Segre variety of M is irreducible. This fact will play a role in the proof of our main theorem.

2.2 Canonical embeddings and explicit coordinate functions

We now describe a special type of canonical embedding of the Hermitian symmetric space M of compact type into $\mathbb{CP}^N$. This embedding will play a crucial role in our computation leading to the proof of Theorem 1.1. See [He] for the classification of the irreducible Hermitian symmetric spaces of compact type. See also [Lo1], [Lo2] on the typical canonical embeddings of the Heritian symmetric spaces of compact type and the related theory of Hermitian positive Jordan triple system.

♣1. Grassmannians (spaces of type I): Write $G(p,q)$ for the Grassmannian space consisting of p planes in $\mathbb{C}^{p+q}$. (Since $G(p,q)$ is biholomorphically equivalent to $G(q,p)$, we will assume $p \leq q$ in what follows).

There is a matrix representation of $G(p,q)$ as the equivalence classes of $p \times (p+q)$ non-degenerate matrices under the matrix multiplication from the left by elements of $GL(p,\mathbb{C})$. A Zariski open affine chart $\mathcal{A}$ for $G(p,q)$ is identified with $\mathbb{C}^{pq}$ with coordinates Z for elements of the form:

$$\begin{pmatrix} I_{p\times p} & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & z_{11} & z_{12} & \cdots & z_{1q} \\ 0 & 1 & 0 & \cdots & 0 & z_{21} & z_{22} & \cdots & z_{2q} \\ & & & \cdots & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & z_{p1} & z_{p2} & \cdots & z_{pq} \end{pmatrix}, \text{ where } Z \text{ is a } p \times q \text{ matrix.}$$

The Plücker embedding $G(p,q) \to \mathbb{CP}(\Lambda^p \mathbb{C}^{p+q})$ is given by mapping the $p-$plane Λ spanned by vectors $v_1, ..., v_p \in \mathbb{C}^{p+q}$ into the wedge product $v_1 \wedge v_2 \wedge ... \wedge v_p \in \wedge^p \mathbb{C}^{p+q}$. The action induced by the multiplication through elements of $SU(p+q)$ from the right

induces a unitary action in the embedded ambient projective space. In homogenous coordinates, the embedding is given by the $p \times p$ minors of the $p \times (p+q)$ matrices (up to a sign). More specifically, in the above local affine chart, we have the following (up to a sign in front of the components):

$$Z \to [1, Z(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}), ...] \tag{2.1}$$

which is denoted for simplicity of notation, in what follows, by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$.

Here and in what follows, $Z(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix})$ is the determinant of the submatrix of Z formed by its $i_1^{\text{th}}, ..., i_k^{\text{th}}$ rows and $j_1^{\text{th}}, ..., j_k^{\text{th}}$ columns, where the indices run through

$$k = 1, 2, ..., p, 1 \le i_1 < i_2 < ... < i_k \le p, 1 \le j_1 < j_2 < ... < j_k \le q.$$

Notice that when $k = 1$, $Z(\begin{smallmatrix} i_1 \\ j_1 \end{smallmatrix}) = z_{i_1 j_1}$.

Notice that under such an embedding into the projective space, $(G(p,q))^* = G(p,q)$. We thus have the same affine coordinates for $(G(p,q))^*$:

$$\begin{pmatrix} I_{p \times p} & \Xi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \xi_{11} & \xi_{12} & \cdots & \xi_{1q} \\ 0 & 1 & 0 & \cdots & 0 & \xi_{21} & \xi_{22} & \cdots & \xi_{2q} \\ & & & \cdots & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & \xi_{p1} & \xi_{p2} & \cdots & \xi_{pq} \end{pmatrix}, \quad \Xi \text{ is a } p \times q \text{ matrix.}$$

By the definition in § 2.1, it follows that the restriction of the Segre family to the product of these Zariski open affine subsets has the following canonical defining function:

$$\rho(z, \xi) = 1 + \sum_{\substack{1 \le i_1 < i_2 < ... < i_k \le p, \\ 1 \le j_1 < j_2 < ... < j_k \le q \\ k=1,...,p}} Z(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}) \Xi(\begin{smallmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{smallmatrix}) \tag{2.2}$$

Here $z = (z_{11}, z_{12}, ..., z_{pq})$, $\xi = (\xi_{11}, \xi_{12}, ..., \xi_{pq})$. For simplicity of notation and terminology, we call this quasi-projective algebraic variety embedded in $\mathbb{C}^{pq} \times \mathbb{C}^{pq}$, which is defined by (2.2), the Segre family of $G(p,q)$. Our defining function $\rho(z, \xi)$ of the Segre

family is closely related to the generic norm of the corresponding Hermitian positive Jordan triple system(cf. [Lo1], [Lo2]).

♣2. Orthogonal Grassmannians (type II): Write $G_{II}(n, n)$ for the submanifold of the Grassmannian $G(n, n)$ consisting of isotropic n-dimensional subspaces of $\mathbb{C}^{2n}$. Then $\tilde{S} \in G_{II}(n, n)$ if and only if

$$\tilde{S} \begin{pmatrix} 0 & I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix} \tilde{S}^T = 0. \tag{2.3}$$

In the aforementioned open affine piece of the Grassmannian $G(n, n)$ with $\tilde{S} = (I, S)$, $\tilde{S} \in G_{II}(n, n)$ if and only if S is an $n \times n$ antisymmetric matrix. We identify this open affine chart $\mathcal{A}$ of $G_{II}(n, n)$ with $\mathbb{C}^{\frac{n(n-1)}{2}}$ through the holomorphic coordinate map:

$$\begin{pmatrix} I_{n\times n} & Z \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -z_{12} & 0 & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -z_{1n} & -z_{2n} & \cdots & 0 \end{pmatrix} \rightarrow (z_{12}, \cdots z_{(n-1)n}). \tag{2.4}$$

Later in the paper we will sometimes use the notation $z_{ji} := -z_{ij}$ if $j > i$ for this type II case. The Plücker embedding of $G(n, n)$ gives a 2-canonical embedding of $G_{II}(n, n)$. Unfortunately this embedding is not good enough for our purposes later. Therefore, we will use a different embedding in this paper, which is given by the spin representation of O_{2n}. This embedding is what is called a one-canonical embedding of $G_{II}(n, n)$. We briefly describe this embedding as following. More details can be found in [Chapter 12; PS].

Let V be a real vector space of dimension $2n$ with a given inner product, and let $\mathcal{K}(V)$ be the space consisting of all orthogonal complex structures on V preserving this inner product. An element of $\mathcal{K}(V)$ is a linear orthogonal transformation $J : V \to V$ such that $J^2 = -1$. Any two choices of J are conjugate in the orthogonal group $O(V) = O_{2n}$, and thus $\mathcal{K}(V)$ can be identified with the homogeneous space O_{2n}/U_n. On the other hand, there is a one-to-one correspondence assigning the complex J to a complex n-dimensional isotropic subspace W of $V_{\mathbb{C}}(= V \otimes \mathbb{C})$. $\mathcal{K}(V)$ has two connected

components $\mathcal{K}_{\pm}(V)$: Noticing that any complex structure defines an orientation on V, these two components correspond to the two possible orientations on V. Write one for $\mathcal{K}_+(V)$, which is actually our $G_{II}(n,n)$.

Now fix an isotropic n-dimensional subspace $W \subset V_{\mathbb{C}}$ with the associated complex structure J of $V_{\mathbb{C}}$ and pick a basis for V: $\{x_1, ..., x_n, y_1, ..., y_n\}$ with $J(x_i) = y_i, J(y_i) = -x_i$. Then W is spanned by $\{x_i - \sqrt{-1}y_i\}_{i=1}^n$. Define $\overline{W}$ to be the space spanned by $\{x_i + \sqrt{-1}y_i\}_{i=1}^n$. As shown in [PS], there is a holomorphic embedding $\mathcal{K}(V) \hookrightarrow \mathbb{CP}(\Lambda(W))$, where $\Lambda(W)$ is the exterior algebra of W. This embedding is equivariant under the action of $O(V)$. Thus $\mathcal{K}_+(V) \hookrightarrow \mathbb{CP}(\Lambda(W))$ is equivariant under $SO(V)$. Choose the open affine cell of $\mathcal{K}_+(V)$ such that $\{Y \in \mathcal{K}_+(V) | Y \cap \overline{W} = \varnothing\}$. Then it can be identified with (2.4).

We next describe the 1-canonical embedding by Pfaffians as following:

Let Π be the set of all partitions of $\{1, 2, ..., 2n\}$ into pairs without regard to order. An element $\alpha \in \Pi$ can be written as

$$\alpha = \{(i_1, j_1), (i_2, j_2), ..., (i_n, j_n)\}$$

with $i_k < j_k$ and $i_1 < i_2 < ... < i_n$. Let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & ... & 2n \\ i_1 & j_1 & i_2 & j_2 & ... & j_n \end{bmatrix}$$

be the corresponding permutation. Given a partition α as above and a $(2n) \times (2n)$ matrix $A = (a_{jk})$, define

$$A_\alpha = \operatorname{sgn}(\pi) a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}.$$

The Pfaffian of A is then given by

$$\operatorname{pf}(A) = \sum_{\alpha \in \Pi} A_\alpha.$$

The Pfaffian of an $m \times m$ skew-symmetric matrix for m odd is defined to be zero.

Therefore in the coordinate system (2.4), the embedding of $\mathcal{A}$ is given by

$$[1, , ..., \operatorname{pf}(Z_\sigma), ...]. \tag{2.5}$$

Write S_k for the collection of all subsets of $\{1, ..., n\}$ with k elements. The σ in (2.5) runs through all elements of S_k with $2 \leq k \leq n$ and k even. For $\sigma = \{i_1 < \cdots < i_k\}$, Z_σ is defined as the submatrix $Z\binom{i_1 \ \cdots \ i_k}{i_1 \ \cdots \ i_k}$. For instance, $(\mathrm{pf}(Z_\sigma))_{\sigma \in S_2} = (z_{12}, ..., z_{(n-1)n})$. We also write (2.5) as $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$ for simplicity of notation. We choose the local coordinates for $(G_{II}(n,n))^*$ in a similar way

$$\left(I_{n \times n} \quad \Xi\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \xi_{12} & \cdots & \xi_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -\xi_{12} & 0 & \cdots & \xi_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -\xi_{1n} & -\xi_{2n} & \cdots & 0 \end{pmatrix}. \tag{2.6}$$

The defining function for the Segre family (in the product of such affine pieces) is given by

$$\rho(z, \xi) = 1 + \sum_{\substack{\sigma \in S_k, \\ 2 \leq k \leq n, 2|k}} \mathrm{Pf}(Z_\sigma)\mathrm{Pf}(\Xi_\sigma). \tag{2.7}$$

♣3. Symplectic Grassmannians (type III): Write $G_{III}(n,n)$ for the submanifold of the Grassmannian space $G(n,n)$ defined as follows: Take the matrix representation of each element of the Grassmannian $G(n,n)$ as an $n \times 2n$ non-degenerate matrix. Then $\widetilde{A} \in G_{III}(n,n)$, if and only if,

$$\widetilde{A} \begin{pmatrix} 0 & I_{n \times n} \\ -I_{n \times n} & 0 \end{pmatrix} \widetilde{A}^T = 0. \tag{2.8}$$

In the Zariski open affine piece of the Grassmannian $G(n,n)$ defined before, we can take a representative matrix of the form: $\widetilde{A} = (I, Z)$. Then we conclude that $\widetilde{A} \in G_{III}(n,n)$ if and only if Z is an $n \times n$ symmetric matrix. We identify this Zariski open affine chart $\mathcal{A}$ of $G_{III}(n,n)$ with $\mathbb{C}^{\frac{n(n+1)}{2}}$ through the holomorphic coordinate map:

$$\tilde{A} = \left(I_{n \times n} \quad Z\right) := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & z_{11} & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & z_{12} & z_{22} & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & z_{1n} & z_{2n} & \cdots & z_{nn} \end{pmatrix} \rightarrow (z_{11}, \cdots, z_{nn}).$$

Later in the paper we will sometimes use the notation $z_{ji} := z_{ij}$ if $j > i$ for this type III case. Through the Plücker embedding of the Grassmannian $G(n,n)$, $G_{III}(n,n)$ is

embedded into $\mathbb{CP}(\Lambda^n\mathbb{C}^{2n})(\cong \mathbb{CP}^{N^*})$. In the above local coordinates, we write down the embedding as (up to a sign)

$$Z \to [1, \cdots, Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}, ...] := [1, \psi_1, \cdots, \psi_{N^*}]. \tag{2.9}$$

Choose the local affine open piece of $(G_{III}(n,n))^*$ consisting of elements in the following form:

$$\left(I_{n\times n} \quad \Xi\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ 0 & 1 & 0 & \cdots & 0 & \xi_{12} & \xi_{22} & \cdots & \xi_{2n} \\ & & & \cdots & & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 & \xi_{1n} & \xi_{2n} & \cdots & \xi_{nn}. \end{pmatrix}.$$

The defining function of Segre family in the product of such affine open pieces is given by

$$\rho(z,\xi) = 1 + \sum_{\substack{1 \le i_1 < i_2 < ... < i_k \le n, \\ 1 \le j_1 < j_2 < ... < j_k \le n \\ k = 1,...,n}} Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \Xi\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \tag{2.10}$$

However the Plücker embedding is not a useful canonical embedding to us for $G_{III}(n,n)$, due to the fact that $\{\psi_j\}$ is not a linearly independent system. For instance, we have the following relation:

$$Z\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + Z\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = Z\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

This embedding can not serve our purposes here. We therefore derive from this embedding a minimal embedding into a certain projective subspace in $\mathbb{CP}(\Lambda^n\mathbb{C}^{2n})(\cong \mathbb{CP}^{N^*})$. We denote this minimal projective subspace by $\mathcal{H} \cong \mathbb{CP}^N$, which is discussed in detail below. We notice that the embedding $G_{III}(n,n) \hookrightarrow \mathbb{CP}^N$ is equivariant under the transitive action of $Sp(n)$.

Following the notations we set up in the Grassmannian case, we write $[1, \psi_1, \cdots \psi_{N^*}]$ for the map of the Plücker embedding into $\mathbb{CP}^{N^*}$. Write $(\psi_{i_1}, ..., \psi_{i_{m_k}})$ for those components of degree k in z among $\{\psi_j\}_{j=1}^{N^*}$. Here $1 \le k \le n$, and $\{i_1, ..., i_{m_k}\}$ depends on k. For instance, if $k = 1$, then

$$(\psi_{i_1}, ..., \psi_{i_{m_1}}) = (z_{11}, ..., z_{nn}),$$

where z_{ij} is repeated twice if $i \neq j$.

Let $\{\psi_1^{(k)}, \cdots, \psi_{m_k^*}^{(k)}\}$ be a maximally linearly independent subset of $\{\psi_{i_1}, ..., \psi_{i_{m_k}}\}$ over $\mathbb{R}$ (and thus also over $\mathbb{C}$). For instance,

$$\{\psi_1^{(1)}, \cdots, \psi_{m_1^*}^{(1)}\} = \{z_{ij}\}_{i \leq j}.$$

Let A_k be the $m_k^* \times m_k$ matrix such that $(\psi_{i_1}, \cdots \psi_{i_{m_k}}) = (\psi_1^{(k)}, \cdots \psi_{m_k^*}^{(k)}) \cdot A_k$. Apparently A_k has real entries and is of full rank. Hence $A_k \cdot A_k^t$ is positive definite.

Then $\{\psi_1^*, \cdots, \psi_N^*\} := \{\psi_1^{(k)}, \cdots, \psi_{m_k^*}^{(k)}\}_{1 \leq k \leq n}$ forms a basis of $\{\psi_1, \cdots \psi_{N^*}\}$, where $N = m_1^* + ... + m_n^*$. Moreover, if we write A as the $(m_1^* + ... + m_n^*) \times (m_1 + ... + m_n)$ matrix:

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix},$$

Then A has full rank and we have a real orthogonal matrix U such that

$$U = \begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_n \end{pmatrix}, \quad U^t(A \cdot A^t)U = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_N \end{pmatrix} \quad \text{with each } \mu_j > 0.$$

Here $U_k, 1 \leq k \leq n$, is an $m_k^* \times m_k^*$ orthogonal matrix. Now we define

$$(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n) := (\psi_1^*, \cdots \psi_N^*) \cdot U \cdot \begin{pmatrix} \sqrt{\mu_1} & & & & \\ & \sqrt{\mu_2} & & & \\ & & \ddots & \\ & & & \sqrt{\mu_N} \end{pmatrix}.$$

Here $N_1 + ... + N_{n-1} + N_n = N^*$, where we set $N_n = 1$. We will also sometimes write $\psi_{N_n}^n = \psi^n$. As a direct consequence,

$$(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n) \cdot (\overline{\psi_1^1}, ..., \overline{\psi_{N_1}^1}, \overline{\psi_1^2}, ..., \overline{\psi_{N_2}^2}, ..., \overline{\psi_1^{n-1}}, ..., \overline{\psi_{N_{n-1}}^{n-1}}, \overline{\psi^n})$$

$$= (\psi_1, \cdots, \psi_{N^*}) \cdot (\overline{\psi_1}, \cdots, \overline{\psi_{N^*}}) = \det(I + Z\bar{Z}^t) = \rho(z, \bar{z}).$$

(2.11)

Moreover $\{\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n\}$ forms a linearly independent system; and $\{\psi_1^k, ..., \psi_{N_k}^k\}$ are polynomials in z of degree k for $k = 1, ..., n$. Now

our canonical embedding of the aforementioned affine piece $\mathcal{A}$ of $G_{III}(n,n)$ is taken as

$$z \in \mathbb{C}^{\frac{n(n+1)}{2}} \to [1, \psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n].$$

For simplicity, we will still denote $(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n)$ by

$$r_z = (\psi_1, \psi_2, ..., \psi_N) = \left(\psi_1^1, ..., \psi_{N_1}^1, \psi_1^2, ..., \psi_{N_2}^2, ..., \psi_1^{n-1}, ..., \psi_{N_{n-1}}^{n-1}, \psi^n \right). \qquad (2.12)$$

Here, for instance, $(\psi_1, ..., \psi_{\frac{n(n+1)}{2}}) = (\psi_1^1, ..., \psi_{N_1}^1) = (a_{ij} z_{ij})_{1 \leq i \leq j \leq n}$, where a_{ij} equals to 1 if $i = j$, equals to $\sqrt{2}$ if $i < j$. Hence the defining function of the Segre family, which is the same as (2.10), is given by $\rho(z, \xi) = 1 + \sum_{i=1}^N \psi_i(z) \psi_i(\xi)$.

♣4. Hyperquadrics (type IV): Let Q^n be the hypersurface in $\mathbb{CP}^{n+1}$ defined by

$$\left\{ [x_0, ..., x_{n+1}] \in \mathbb{CP}^{n+1} : \sum_{i=1}^n x_i^2 - 2x_0 x_{n+1} = 0 \right\},$$

where $[x_1, ..., x_{n+2}]$ are the homogeneous coordinates for $\mathbb{CP}^{n+1}$. It is invariant under the action of the group $SO(n+2)$. We mention that under the present embedding, the action is not the standard $SO(n + 2)$ in $GL(n + 2)$. However it is conjugate to the standard $SO(n+2)$ action by a certain element $g \in U(n+2)$. An Zariski open affine piece $\mathcal{A} \subset Q^n$ identified with $\mathbb{C}^n$ is given by $(z_1, ..., z_n) \mapsto [1, \psi_1, ..., \psi_{n+1}] = [1, z_1, ..., z_n, \frac{1}{2} \sum_{i=1}^n z_i^2]$, which will be denoted by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_{n+1}]$. Choose the same local chart for $(Q^n)^* : (\xi_1, ..., \xi_n) \to [1, \xi_1, ..., \xi_n, \frac{1}{2} \sum_{i=1}^n \xi_i^2]$. Then the defining function of the Segre family restricted to $\mathbb{C}^n \times \mathbb{C}^n \hookrightarrow Q^n \times (Q^n)^*$ is given by

$$\rho(z, \xi) = 1 + \sum_{i=1}^n z_i \xi_i + \frac{1}{4} (\sum_{i=1}^n z_i^2)(\sum_{i=1}^n \xi_i^2) \qquad (2.13)$$

♣5. The exceptional manifold $M_{16} := E_6/SO(10) \times SO(2)$: As shown in [IM1], [IM2], this exceptional Hermitian symmetric space can be realize as the Cayley plane. Take the exceptional 3×3 complex Jordan algebra

$$\mathcal{J}_3(\mathbb{O}) = \left\{ \begin{pmatrix} c_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & c_2 & x_1 \\ x_2 & \bar{x}_1 & c_3 \end{pmatrix} : c_i \in \mathbb{C}, x_i \in \mathbb{O} \right\} \cong \mathbb{C}^{27}. \qquad (2.14)$$

Here $\mathbb{O}$ is the complexified algebra of octonions, which is a complex vector space of dimension 8. Denote a standard basis of $\mathbb{O}$ by $\{e_0, e_1, ..., e_7\}$. The multiplication rule

in terms of this basis is given in the next section. The conjugation operator appeared in (2.14) is for octonions, which is defined as follows: $\bar{x} = x_0 e_1 - x_1 e_1 - ... - x_7 e_7$, if $x = x_0 e_0 + x_1 e_1 + x_2 e_2 + ... + x_7 e_7, x_i \in \mathbb{C}$. Moreover under this basis, the isomorphism $\mathcal{J}_3(\mathbb{O}) \cong \mathbb{C}^{27}$ is realized by identifying each matrix

$$
X = \begin{pmatrix} \xi_1 & \eta & \bar{\kappa} \\ \bar{\eta}_3 & \xi_2 & \tau \\ \kappa & \bar{\tau} & \xi_3 \end{pmatrix} \in \mathcal{J}_3(\mathbb{O})
$$

with the point $(\xi_1, \xi_2, \xi_3, \eta_0, \eta_1, \ldots, \eta_7, \kappa_0, \kappa_2, \ldots, \kappa_7, \tau_0, \tau_1, \ldots, \tau_7) \in \mathbb{C}^{27}$, where $\eta = \sum_{i=0}^{7} \eta_i e_i, \kappa = \sum_{i=0}^{7} \kappa_i e_i$ and $\tau = \sum_{i=0}^{7} \tau_i e_i$.

The Jordan multiplication is defined as $A \circ B = \frac{1}{2}(AB + BA)$ for $A, B \in \mathcal{J}_3(\mathbb{O})$. The subgroup $SL(\mathbb{O})$ of $GL(\mathcal{J}_3(\mathbb{O}))$ consisting of automorphisms preserving the determinant is the adjoint group of type E_6. The action of E_6 on the projectivization $\mathbb{CP}\mathcal{J}_3(\mathbb{O})$ has exactly three orbits: the complement of the determinantal hypersurface, the regular part of this hypersurface, and its singular part which is the closed E_6-orbit. The closed orbit is the Cayley plane or the hermitian symmetric space of compact type corresponding to E_6. It can be defined by the quadratic equation

$$
X^2 = \text{trace}(X)X, \qquad X \in \mathcal{J}_3(\mathbb{O}),
$$

or as the closure of the affine cell $\mathcal{A}$

$$
\mathbb{OP}_1^2 = \left\{ \begin{pmatrix} 1 & x & y \\ \bar{x} & x\bar{x} & y\bar{x} \\ \bar{y} & x\bar{y} & y\bar{y} \end{pmatrix} : x, y \in \mathbb{O} \right\} \cong \mathbb{C}^{16}
$$

in the local coordinates $(x_0, x_1, ..., x_7, y_0, ..., y_7)$. The precise formula for the canonical embedding map is given in§ 2.4. We denote this embedding by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$.

To find the defining function for its Segre family over the product of such standard affine sets, we choose local coordinates for the conjugate Cayley plane to be $(\kappa_0, \kappa_1, ..., \kappa_7, \eta_0, \eta_1, ..., \eta_7)$. Then

$$
\rho(z, \xi) = 1 + \sum_{i=0}^{7} x_i \kappa_i + \sum_{i=0}^{7} y_i \eta_i + \sum_{i=0}^{7} A_i(x, y) A_i(\kappa, \eta) + B_0(x, y) B_0(\kappa, \eta) + B_1(x, y) B_1(\kappa, \eta),
$$

$$
(2.15)
$$

where A_j, B_j are defined as in § 2.4, $z = (x_0, ..., x_7, y_0, ..., y_7)$ and $\xi = (\kappa_0, ..., \kappa_7, \eta_0, ..., \eta_7)$.

♣6. The other exceptional manifold $M_{27} = E_7/E_6 \times SO(2)$: As shown in [CMP], it can be realized as the Freudenthal variety. Consider the Zorn algebra

$$\mathcal{Z}_2(\mathbb{O}) = \mathbb{C} \bigoplus \mathcal{J}_3(\mathbb{O}) \bigoplus \mathcal{J}_3(\mathbb{O}) \bigoplus \mathbb{C}$$

One can prove that there exists an action of E_7 on that 56−dimensional vector space (see [Fr]). The closed E_7−orbit inside $\mathbb{CP}\mathcal{Z}_2(\mathbb{O})$ is the Freudenthal variety $E_7/E_6 \times SO(2)$. An affine cell $\mathcal{A}$ of Freudenthal variety is $[1, X, \text{Com}(X), \det(X)] \in \mathbb{CP}\mathcal{Z}_2(\mathbb{O})$. Here X belongs to $\mathcal{J}_3(\mathbb{O})$; $\text{Com}(X)$ is the comatrix of X such that $X\text{Com}(X) = \det(X)I$ under the usual matrix multiplication rule. Notice that $\text{Com}(X) = X \times X$, where $X \times X$ is the Freudenthal multiplication defined as follows (see [O]):

$$X \times X := X^2 - \text{tr}(X)X + \frac{1}{2}(\text{tr}(X)^2 - \text{tr}(X^2))I.$$

For explicit expressions for $X \times X$ and $\det(X)$ in terms of the entries of X, see § 2.4.

The embedding of $E_7/E_6 \times SO(2) \hookrightarrow \mathbb{CP}^N$ in local coordinates z is given in § 2.4. Choose the local affine open piece for $(E_7/E_6 \times SO(2))^*$ with coordinates

$$\xi = (\xi_1, \xi_2, \xi_3, \eta_0, ..., \eta_7, \kappa_0, ..., \kappa_7, \tau_0, ..., \tau_7).$$

We denote this embedding by $[1, r_z] = [1, \psi_1, \psi_2, ..., \psi_N]$. The defining function for the Segre family is then $\rho(z, \xi) = 1 + r_z \cdot r_\xi$, where

$$
\begin{aligned}
r_z &= (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7, A(z), B(z), C(z), D_0(z), ...D_7(z), \\
&\quad E_0(z), ..., E_7(z), F_0(z), ..., F_7(z), G(z)) \\
r_\xi &= (\psi_1(\xi), \psi_2(\xi), ..., \psi_N(\xi)) = (\xi_1, \xi_2, \xi_3, \eta_0, ..., \eta_7, \kappa_0, ..., \kappa_7, \tau_0, ..., \tau_7, \\
&\quad A(\xi), B(\xi), C(\xi), D_0(\xi), ..., D_7(\xi), E_0(\xi), ..., E_7(\xi), F_0(\xi), ..., F_7(\xi), G(\xi))
\end{aligned}
\tag{2.16}
$$

See § 2.4 for the definition of the functions appeared in the formula.

Summarizing the above, for each irreducible Hermitian symmetric space of compact type M of dimension n, we now have described a canonical embedding from M into a projective space $\mathbb{P}^N$, which restricted to a certain Zariski open affine piece $\mathcal{A}$ holomorphically equivalent to $\mathbb{C}^n$ takes the form: $z(\in \mathbb{C}^n) \mapsto [1, \kappa_1 z_1, \cdots, \kappa_i z_i, \cdots, \kappa_n z_n, O(z^2)]$.

Here $\kappa_i = 1$ for all i except in the case of type III where κ_i can be 1 or $\sqrt{2}$. This is the embedding we will use in later discussions. Notice in our embedding, the conjugate space M^* is the same as M. For simplicity of notation, we will also write $\mathcal{M}$ for the restriction of the Segre family of M restricted to $\mathcal{A} \times \mathcal{A}^* = \mathbb{C}^n \times \mathbb{C}^n$.

From this embedding and the invariant property of Segre varieties, we immediately conclude the following:

Lemma 2.1. *Assume A and B are two distinct points of M. Then their associated Segre varieties are different, namely, $Q_A \neq Q_B$.*

Proof of Lemma 2.1: Since the holomorphic isometric group acts transitively on M, we can assume $A = (0, 0, ..., 0) \in \mathbb{C}^n \cong \mathcal{A} \subset M$. Therefore Q_A is the hyperplane section of $M \hookrightarrow \mathbb{P}^N$ at infinity, namely, $Q_A = M \backslash \mathcal{A}$. Now if $B \in \mathcal{A}$, because $B \neq (0, 0, ..., 0)$, there are non-trivial linear terms in the defining function of Q_B. This leads to the fact that the defining function of Q_B has to be a non-constant polynomial in $\mathbb{C}[\xi_1, ..., \xi_n]$. Therefore $Q_B \cap \mathbb{C}^n \neq \emptyset$ and thus does not coincide with Q_A. If $B \in \mathcal{M} \backslash \mathcal{A}$, by the symmetric property of Segre varieties, we have $(0, ..., 0) \in Q_B$. Therefore $Q_B \neq Q_A$. We then arrive at the conclusion. ∎

Finally, since in our setting, $M^* = M$ and the Segre family on M and M^* are the same. For simplicity of notation, we do not distinguish, in what follows, Q^* and $\mathcal{M}^*$ from Q and $\mathcal{M}$, respectively.

2.3 Explicit expression of the volume forms

From now on, we assume that M is an irreducible Hermitian symmetric space of compact type and we choose the canonical embedding $M \hookrightarrow \mathbb{CP}^N$ as described in § 2.2 according to its type. We denote the metric on M induced from Fubini-Study of $\mathbb{CP}^N$ by ω, and the volume form by $d\mu = \omega^n$ (up to a positive constant). Notice that the metric we obtained is always invariant under the action of a certain transitive subgroup $G \subset \mathrm{Aut}(M)$ (which comes from the restriction of a subgroup of the unitary group of the ambient projective space). Hence by a theorem of Wolf [W], ω is the unique G invariant metric on M up to a scale. We claim ω must be Kähler-Einstein. Indeed, since

the Ricci form $\text{Ric}(\omega)$ of ω is invariant under G, for a small ϵ, $\omega + \epsilon \text{Ric}(\omega)$ is thus also a G invariant metric on M. By [W], it is a multiple of ω, and thus $\text{Ric}(\omega) = \lambda \omega$. Write $d\mu$ as the product of V and the standard Euclidean volume form over the affine subspace $\mathcal{A}$, where V is a positive function in z. Since $\text{Ric}(\omega) = -i\partial\bar{\partial}\log V$, $-i\partial\bar{\partial}\log V = \lambda\omega$. Notice that $\lambda > 0$. In the local affine open piece $\mathcal{A}$ defined before, $\omega = i\partial\bar{\partial}\log\rho(z,\bar{z})$, where $\rho(z,\xi)$ is the defining function for the associated Segre family. As we will see later (§7), $\rho(z,\xi)$ is an irreducible polynomial in (z,ξ). Then we have

$$\partial\bar{\partial}\log(V\rho(z,\bar{z})^{\lambda}) = 0.$$

Hence, $\log(V\rho(z,\bar{z})^{\lambda}) = \phi(z) + \overline{\psi(z)}$, where both ϕ and ψ are holomorphic functions. Therefore $V = \frac{e^{\phi(z)+\overline{\psi(z)}}}{\rho(z,\bar{z})^{\lambda}}$. Because $\rho(z,\xi)$ is an irreducible polynomial, from the way V is defined, V must be a rational function of the form $\frac{p(z,\bar{z})}{\rho(z,\bar{z})^m}$ with p, ρ relatively prime to each other. Since ϕ, ψ are globally defined, by a monodromy argument, it is clear that λ has to be an integer. Also both $e^{\phi(z)}$ and $e^{\overline{\psi(\xi)}}$ must be rational functions. Again, since ϕ, ψ are also globally defined, this forces ϕ, ψ to be constant functions. Therefore, we conclude that

$$V = c\rho(z,\bar{z})^{-\lambda}. \tag{2.17}$$

Here λ is a certain positive integer and c is a positive constant. Next by a well-known result (see [BaMa]), two Kähler-Einstein metrics of M are different by an automorphism of M (up to a positive scalar multiple). Therefore, to prove Theorem 1.1, we can assume, without loss of generality, that the Kähler-Einstein metric in Theorem 1.1 is the metric obtained by restricting the Fubini-Study metric to M through the embedding described in this section.

2.4 Appendix: Affine cell coordinate functions for two exceptional classes of the Hermitian symmetric spaces of compact type

Define the multiplication law of octonions with the standard basis $\{e_0 = 1, e_1, \cdots, e_7\}$ by the following table:

♣1. Case M$_{16}$: Define

$$x = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7),$$

$$y = (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7).$$

Define $A_j(x, y), j = 0, \ldots, 7$, such that

$$x\bar{y} = \sum_{j=0}^{7} A_j(x, y)e_j, \text{ where } x = \sum_{j=0}^{7} x_j e_j \text{ and } y = \sum_{j=0}^{7} y_j e_j.$$

Define $B_j(x, y), j = 0, 1$ such that

$$x\bar{x} = B_0(x, y)e_0 \text{ and } y\bar{y} = B_1(x, y)e_0.$$

Exploiting the multiplication law of octonions, we have the following formulas:

$$A_0 = A_0(x, y) = y_0 x_0 + y_1 x_1 + y_2 x_2 + y_3 x_3 + y_4 x_4 + y_5 x_5 + y_6 x_6 + y_7 x_7,$$

$$A_1 = A_1(x, y) = - y_0 x_1 + y_1 x_0 - y_2 x_4 + y_4 x_2 - y_3 x_7 + y_7 x_3 - y_5 x_6 + y_6 x_5,$$

$$A_2 = A_2(x, y) = - y_0 x_2 + y_2 x_0 - y_4 x_1 + y_1 x_4 - y_3 x_5 + y_5 x_3 - y_6 x_7 + y_7 x_6,$$

$$A_3 = A_3(x, y) = - y_0 x_3 + y_3 x_0 + y_1 x_7 - y_7 x_1 + y_2 x_5 - y_5 x_2 - y_4 x_6 + y_6 x_4,$$

$$A_4 = A_4(x, y) = - y_0 x_4 + y_4 x_0 - y_1 x_2 + y_2 x_1 + y_3 x_6 - y_6 x_3 - y_5 x_7 + y_7 x_5,$$

$$A_5 = A_5(x, y) = - y_0 x_5 + y_5 x_0 + y_1 x_6 - y_6 x_1 - y_2 x_3 + y_3 x_2 + y_4 x_7 - y_7 x_4,$$

$$A_6 = A_6(x, y) = - y_0 x_6 + y_6 x_0 - y_1 x_5 + y_5 x_1 + y_2 x_7 - y_7 x_2 - y_3 x_4 + y_4 x_3,$$

$$A_7 = A_7(x, y) = - y_0 x_7 + y_7 x_0 - y_1 x_3 + y_3 x_1 - y_2 x_6 + y_6 x_2 - y_4 x_5 + y_5 x_4,$$

$$B_0 = B_0(x, y) = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2,$$

$$B_1 = B_1(x, y) = y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2.$$

Then the embedding functions of a Zariski open subset $\mathcal{A}$, which is identified with $\mathbb{C}^{16}$ with coordinates $z := (x_0, \cdots, x_7, y_0, \cdots, y_7)$, of $M_{16} := \frac{E_6}{SO(10) \times SO(2)}$ into $\mathbb{CP}^{26}$ are given by:

$$z \mapsto [1, x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7, B_0, B_1].$$

	e_1	e_2	e_4	e_7	e_3	e_6	e_5
e_1	-1	e_4	$-e_2$	$-e_3$	e_7	$-e_5$	e_6
e_2	$-e_4$	-1	e_1	$-e_6$	e_5	e_7	$-e_3$
e_4	e_2	$-e_1$	-1	$-e_5$	$-e_6$	e_3	e_7
e_7	e_3	e_6	e_5	-1	$-e_1$	$-e_2$	$-e_4$
e_3	$-e_7$	$-e_5$	e_6	e_1	-1	$-e_4$	e_2
e_6	e_5	$-e_7$	$-e_3$	e_2	e_4	-1	$-e_1$
e_5	$-e_6$	e_3	$-e_7$	e_4	$-e_2$	e_1	-1

♣2. Case M$_{27}$: Similarly we define

$$
\begin{aligned}
x &= (x_1, x_2, x_3), \\
y &= (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7), \\
t &= (t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7), \\
\omega &= (\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7).
\end{aligned}
$$

Define functions $A, B, C, D_0, \ldots, D_7, E_0 \ldots, E_7, F_0 \ldots, F_7$ and G such that,

$$
\mathrm{Com}(X) = X \times X = \begin{pmatrix} A & D & \overline{E} \\ \overline{D} & B & F \\ E & \overline{F} & C \end{pmatrix}, \quad G = \det(X),
$$

where $D = \sum_{j=0}^{7} D_j e_j$, $E = \sum_{j=0}^{7} E_j e_j$, $F = \sum_{j=0}^{7} F_j e_j$ and the matrix X corresponding to the point $(x, y, t, w) \in \mathbb{C}^{27}$ is given by

$$
X = \begin{pmatrix} x_1 & y & \overline{t} \\ \overline{y} & x_2 & w \\ t & \overline{w} & x_3 \end{pmatrix} \in \mathcal{J}_3(\mathbb{O}).
$$

Recall the formulas in [O], we have

$$
X \times X = \begin{pmatrix} x_2 x_3 - w\overline{w} & \overline{w}\overline{t} - x_3 y & yw - x_2\overline{t} \\ wt - x_3\overline{y} & x_3 x_1 - t\overline{t} & \overline{t}\overline{y} - x_1 w \\ \overline{y}\overline{w} - x_2 t & ty - x_1\overline{w} & x_1 x_2 - y\overline{y} \end{pmatrix} \in \mathcal{J}_3(\mathbb{O}),
$$

$$
\det(X) = x_1 x_2 x_3 - x_1 w\overline{w} - x_2 t\overline{t} - x_3 y\overline{y} + 2\Re^c(wty),
$$

where $\Re^c(x) = x_0$ for any $x = \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$.

By further computation, we have the explicit expressions as follows:

$$A = A(x, y, t, \omega) = x_2 x_3 - (\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2),$$

$$B = B(x, y, t, \omega) = x_1 x_3 - (t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2),$$

$$C = C(x, y, t, \omega) = x_1 x_2 - (y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2),$$

$$D_0 = D_0(x, y, t, \omega) = t_0\omega_0 + t_1\omega_1 + t_2\omega_2 + t_3\omega_3 + t_4\omega_4 + t_5\omega_5 + t_6\omega_6 + t_7\omega_7 - x_3 y_0,$$

$$D_1 = D_1(x, y, t, \omega) = -\ t_0\omega_1 + t_1\omega_0 - t_2\omega_4 + t_4\omega_2 - t_3\omega_7 + t_7\omega_3 - t_5\omega_6 + t_6\omega_5 - x_3 y_1,$$

$$D_2 = D_2(x, y, t, \omega) = -\ t_0\omega_2 + t_2\omega_0 - t_4\omega_1 + t_1\omega_4 - t_3\omega_5 + t_5\omega_3 - t_6\omega_7 + t_7\omega_6 - x_3 y_2,$$

$$D_3 = D_3(x, y, t, \omega) = -\ t_0\omega_3 + t_3\omega_0 + t_1\omega_7 - t_7\omega_1 + t_2\omega_5 - t_5\omega_2 - t_4\omega_6 + t_6\omega_4 - x_3 y_3,$$

$$D_4 = D_4(x, y, t, \omega) = -\ t_0\omega_4 + t_4\omega_0 - t_1\omega_2 + t_2\omega_1 + t_3\omega_6 - t_6\omega_3 - t_5\omega_7 + t_7\omega_5 - x_3 y_4,$$

$$D_5 = D_5(x, y, t, \omega) = -\ t_0\omega_5 + t_5\omega_0 + t_1\omega_6 - t_6\omega_1 - t_2\omega_3 + t_3\omega_2 + t_4\omega_7 - t_7\omega_4 - x_3 y_5,$$

$$D_6 = D_6(x, y, t, \omega) = -\ t_0\omega_6 + t_6\omega_0 - t_1\omega_5 + t_5\omega_1 + t_2\omega_7 - t_7\omega_2 - t_3\omega_4 + t_4\omega_3 - x_3 y_6,$$

$$D_7 = D_7(x, y, t, \omega) = -\ t_0\omega_7 + t_7\omega_0 - t_1\omega_3 + t_3\omega_1 - t_2\omega_6 + t_6\omega_2 - t_4\omega_5 + t_5\omega_4 - x_3 y_7,$$

$$E_0 = E_0(x, y, t, \omega) = y_0\omega_0 - y_1\omega_1 - y_2\omega_2 - y_3\omega_3 - y_4\omega_4 - y_5\omega_5 - y_6\omega_6 - y_7\omega_7 - x_2 t_0,$$

$$E_1 = E_1(x, y, t, \omega) = y_0\omega_1 + y_1\omega_0 + y_2\omega_4 - y_4\omega_2 + y_3\omega_7 - y_7\omega_3 + y_5\omega_6 - y_6\omega_5 - x_2 t_1,$$

$$E_2 = E_2(x, y, t, \omega) = y_0\omega_2 + y_2\omega_0 + y_4\omega_1 - y_1\omega_4 + y_3\omega_5 - y_5\omega_3 + y_6\omega_7 - y_7\omega_6 - x_2 t_2,$$

$$E_3 = E_3(x, y, t, \omega) = y_0\omega_3 + y_3\omega_0 - y_1\omega_7 + y_7\omega_1 - y_2\omega_5 + y_5\omega_2 + y_4\omega_6 - y_6\omega_4 - x_2 t_3,$$

$$E_4 = E_4(x, y, t, \omega) = y_0\omega_4 + y_4\omega_0 + y_1\omega_2 - y_2\omega_1 - y_3\omega_6 + y_6\omega_3 + y_5\omega_7 - y_7\omega_5 - x_2 t_4,$$

$$E_5 = E_5(x, y, t, \omega) = y_0\omega_5 + y_5\omega_0 - y_1\omega_6 + y_6\omega_1 + y_2\omega_3 - y_3\omega_2 - y_4\omega_7 + y_7\omega_4 - x_2 t_5,$$

$$E_6 = E_6(x, y, t, \omega) = y_0\omega_6 + y_6\omega_0 + y_1\omega_5 - y_5\omega_1 - y_2\omega_7 + y_7\omega_2 + y_3\omega_4 - y_4\omega_3 - x_2 t_6,$$

$$E_7 = E_7(x, y, t, \omega) = y_0\omega_7 + y_7\omega_0 + y_1\omega_3 - y_3\omega_1 + y_2\omega_6 - y_6\omega_2 + y_4\omega_5 - y_5\omega_4 - x_2 t_7,$$

$$F_0 = F_0(x, y, t, \omega) = y_0 t_0 + y_1 t_1 + y_2 t_2 + y_3 t_3 + y_4 t_4 + y_5 t_5 + y_6 t_6 + y_7 t_7 - x_1\omega_0,$$

$$F_1 = F_1(x, y, t, \omega) = y_0 t_1 - y_1 t_0 - y_2 t_4 + y_4 t_2 - y_3 t_7 + y_7 t_3 - y_5 t_6 + y_6 t_5 - x_1\omega_1,$$

$$F_2 = F_2(x, y, t, \omega) = y_0 t_2 - y_2 t_0 - y_4 t_1 + y_1 t_4 - y_3 t_5 + y_5 t_3 - y_6 t_7 + y_7 t_6 - x_1\omega_2,$$

$$F_3 = F_3(x, y, t, \omega) = y_0 t_3 - y_3 t_0 + y_1 t_7 - y_7 t_1 + y_2 t_5 - y_5 t_2 - y_4 t_6 + y_6 t_4 - x_1\omega_3,$$

$$F_4 = F_4(x, y, t, \omega) = y_0 t_4 - y_4 t_0 - y_1 t_2 + y_2 t_1 + y_3 t_6 - y_6 t_3 - y_5 t_7 + y_7 t_5 - x_1\omega_4,$$

$$F_5 = F_5(x, y, t, \omega) = y_0 t_5 - y_5 t_0 + y_1 t_6 - y_6 t_1 - y_2 t_3 + y_3 t_2 + y_4 t_7 - y_7 t_4 - x_1\omega_5,$$

$$F_6 = F_6(x, y, t, \omega) = y_0 t_6 - y_6 t_0 - y_1 t_5 + y_5 t_1 + y_2 t_7 - y_7 t_2 - y_3 t_4 + y_4 t_3 - x_1\omega_6,$$

$$F_7 = F_7(x, y, t, \omega) = y_0 t_7 - y_7 t_0 - y_1 t_3 + y_3 t_1 - y_2 t_6 + y_6 t_2 - y_4 t_5 + y_5 t_4 - x_1\omega_7.$$

$$G = G(x, y, t, \omega) = x_1 x_2 x_3 - x_1(\omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 + \omega_7^2)$$
$$- x_2(t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2 + t_6^2 + t_7^2)$$
$$- x_3(y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2)$$
$$+ 2\{(y_0\omega_0 - y_1\omega_1 - y_2\omega_2 - y_3\omega_3 - y_4\omega_4 - y_5\omega_5 - y_6\omega_6 - y_7\omega_7)t_0$$
$$+ (y_0\omega_1 + y_1\omega_0 + y_2\omega_4 - y_4\omega_2 + y_3\omega_7 - y_7\omega_3 + y_5\omega_6 - y_6\omega_5)t_1$$
$$+ (y_0\omega_2 + y_2\omega_0 + y_4\omega_1 - y_1\omega_4 + y_3\omega_5 - y_5\omega_3 + y_6\omega_7 - y_7\omega_6)t_2$$
$$+ (y_0\omega_3 + y_3\omega_0 - y_1\omega_7 + y_7\omega_1 - y_2\omega_5 + y_5\omega_2 + y_4\omega_6 - y_6\omega_4)t_3$$
$$+ (y_0\omega_4 + y_4\omega_0 + y_1\omega_2 - y_2\omega_1 - y_3\omega_6 + y_6\omega_3 + y_5\omega_7 - y_7\omega_5)t_4$$
$$+ (y_0\omega_5 + y_5\omega_0 - y_1\omega_6 + y_6\omega_1 + y_2\omega_3 - y_3\omega_2 - y_4\omega_7 + y_7\omega_4)t_5$$
$$+ (y_0\omega_6 + y_6\omega_0 + y_1\omega_5 - y_5\omega_1 - y_2\omega_7 + y_7\omega_2 + y_3\omega_4 - y_4\omega_3)t_6$$
$$+ (y_0\omega_7 + y_7\omega_0 + y_1\omega_3 - y_3\omega_1 + y_2\omega_6 - y_6\omega_2 + y_4\omega_5 - y_5\omega_4)t_7\}.$$

Hence the embedding functions of a Zariski open subset $\mathcal{A}$, which is identified with $\mathbb{C}^{27}$ with coordinates $z := (x, y, t, \omega) = (x_1, x_2, x_3, y_0 \cdots, y_7, t_0, \cdots, t_7, \omega_0, \cdots, \omega_7)$, of $M_{27} := \frac{E_7}{E_6 \times SO(2)}$ into $\mathbb{CP}^{55}$ are given by: $z \mapsto [1, x, y, t, \omega, A, B, C, D_0, D_1, D_2, D_3, D_4, D_5, D_6, D_7, E_0, E_1, E_2, E_3, E_4, E_5, E_6, E_7, F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, G]$. The detailed discussions related to this Appendix can be found in [CMP], [Fr] and [O].

Chapter 3

A basic property for partially degenerate holomorphic maps

In this chapter, we introduce a notion of degeneracy for holomorphic maps and derive an important consequence, which will be fundamentally applied in the proof of our main theorem.

Let $\psi(z) := (\psi_1(z), ..., \psi_N(z))$ be a vector-valued holomorphic function from a neighborhood U of 0 in $\mathbb{C}^m, m \geq 2$, into $\mathbb{C}^N, N > m$, with $\psi(0) = 0$. Here we write $z = (z_1, ..., z_m)$ for the coordinates of $\mathbb{C}^m$. In the following, we will write $\widetilde{z} = (z_1, ..., z_{m-1})$, i.e., the vector z with the last component z_m being dropped out. Write $\frac{\partial^{|\alpha|}}{\partial \widetilde{z}^{\alpha}} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} ... \partial z_{m-1}^{\alpha_{m-1}}}$ for an $(m-1)-$multiindex α, where

$$\alpha = (\alpha_1, ..., \alpha_{m-1}).$$

Write

$$\frac{\partial^{|\alpha|}}{\partial \widetilde{z}^{\alpha}} \psi(z) = \left(\frac{\partial^{|\alpha|}}{\partial \widetilde{z}^{\alpha}} \psi_1(z), ..., \frac{\partial^{|\alpha|}}{\partial \widetilde{z}^{\alpha}} \psi_N(z) \right).$$

We introduce the following definition.

Definition 3.1. *Let $k \geq 0$. For a point $p \in U$, write $E_k(p) = \mathrm{Span}_{\mathbb{C}} \{ \frac{\partial^{|\alpha|}}{\partial \widetilde{z}^{\alpha}} \psi(z)|_{z=p} : 0 \leq |\alpha| \leq k \}$. We write r for the greatest number such that for any neighborhood O of 0, there exists $p \in O$ with $\dim_{\mathbb{C}} E_k(p) = r$. r is called the $k-th$ $\widetilde{z}-rank$ of ψ at 0, which is written as $\mathrm{rank}_k(\psi, \widetilde{z})$. F is called $\widetilde{z}-nondegenerate$ if $\mathrm{rank}_{k_0}(\psi, \widetilde{z}) = N$ for some $k_0 \geq 1$.*

Remark 1. *It is easy to see that* $\mathrm{rank}_k(\psi, \widetilde{z}) = r$ *if and only if the following matrix*

$$\begin{pmatrix} \frac{\partial^{|\alpha^0|}}{\partial \widetilde{z}^{\alpha^0}} \psi(z) \\ ... \\ ... \\ \frac{\partial^{|\alpha^s|}}{\partial \widetilde{z}^{\alpha^s}} \psi(z) \end{pmatrix}$$

has an $r \times r$ *submatrix with determinant not identically zero for* $z \in U$ *for some multiindices* $\{\alpha^0, ..., \alpha^s\}$ *with all* $0 \le |\alpha^j| \le k$. *Moreover, any* $l \times l$ $(l > r)$ *submatrix of the matrix has identically zero determinant for any choice of* $\{\alpha^0, ..., \alpha^s\}$ *with* $0 \le |\alpha^j| \le k$.

In particular, ψ *is* $\widetilde{z}-$*nondegenerate if and only if there exist multiindices* $\beta^1, ..., \beta^N$ *such that*

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}}{\partial \widetilde{z}^{\beta^1}} \psi_1(z) & ... & \frac{\partial^{|\beta^1|}}{\partial \widetilde{z}^{\beta^1}} \psi_N(z) \\ ... & ... & ... \\ \frac{\partial^{|\beta^N|}}{\partial \widetilde{z}^{\beta^N}} \psi_1(z) & ... & \frac{\partial^{|\beta^N|}}{\partial \widetilde{z}^{\beta^N}} \psi_N(z) \end{vmatrix}$$

is not identically zero.

Moreover, $\mathrm{rank}_{i+1}(\psi, \widetilde{z}) \ge \mathrm{rank}_i(\psi, \widetilde{z})$ *for any* $i \ge 0$.

For the rest of this chapter, we further assume that the first m components of ψ, i.e.,

$$(\psi_1, ..., \psi_m) : \mathbb{C}^m \to \mathbb{C}^m$$

is a biholomorphic map in a neighborhood of $0 \in \mathbb{C}^m$. Then we have,

Lemma 3.2. *It holds that* $\mathrm{rank}_0(\psi, \widetilde{z}) = 1, \mathrm{rank}_1(\psi, \widetilde{z}) = m$, *and for* $k \ge 1, \mathrm{rank}_k(\psi, \widetilde{z}) \ge m$.

Proof of Lemma 3.2: We first notice that it holds trivially that $\mathrm{rank}_0(\psi, \widetilde{z}) = 1$, for F is not identically zero. We now prove $\mathrm{rank}_1(\psi, \widetilde{z}) = m$. First notice that $\mathrm{rank}_1(\psi, \widetilde{z}) \le m$ as there are only m distinct multiindices β such that $|\beta| \le 1$. On the other hand, since ψ has full rank at 0, we have,

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial z_1} & ... & \frac{\partial \psi_m}{\partial z_1} \\ ... & ... & ... \\ \frac{\partial \psi_1}{\partial z_m} & ... & \frac{\partial \psi_m}{\partial z_m} \end{vmatrix} (0) \ne 0.$$

This together with the fact $\psi(0) = 0$ implies that the z_m derivative of

$$
\begin{vmatrix}
\psi_1 & \cdots & \psi_m \\
\dfrac{\partial \psi_1}{\partial z_1} & \cdots & \dfrac{\partial \psi_m}{\partial z_1} \\
\cdots & \cdots & \cdots \\
\dfrac{\partial \psi_1}{\partial z_{m-1}} & \cdots & \dfrac{\partial \psi_m}{\partial z_{m-1}}
\end{vmatrix}
\tag{3.1}
$$

is nonzero at $p = 0$. Consequently, the quantity in (3.1) is not identically zero in U. By the definition of the $\widetilde{z}$-rank, we then arrive at the conclusion. ∎

We now prove the following degeneracy theorem in terms of its $\widetilde{z}$-rank, which will be used to derive Theorem 3.10.

Theorem 3.3. *Let $\psi = (\psi_1, ..., \psi_m, \psi_{m+1}, ..., \psi_N)$ be a holomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into $\mathbb{C}^N$ with $\psi(0) = 0$. Recall that $\widetilde{z} = (z_1, ..., z_{m-1})$, i.e., the vector z with the last component z_m being dropped out. Assume that $(\psi_1, ..., \psi_m)$ is a biholomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into a neighborhood of $0 \in \mathbb{C}^m$. Suppose*

$$
\mathrm{rank}_{N-m+1}(\psi, \widetilde{z}) < N. \tag{3.2}
$$

Then there exist N holomorphic functions $g_1(z_m), ..., g_N(z_m)$ near 0 in the $z_m - Gauss$ plane with $\{g_1(0), ..., g_N(0)\}$ not all zero such that the following holds for any $(z_1, ..., z_m)$ near 0.

$$
\sum_{i=1}^{N} g_i(z_m) \psi_i(z_1, ..., z_m) \equiv 0. \tag{3.3}
$$

In particular, one can make one of the $\{g_i\}_{i=1}^{N}$ to be identically one.

The geometric intuition for the theorem is as follows: The space of 1-jets has dimension m by Lemma 3.2. We expect that at least one more dimension is increased when we go from the space of k-jets to the space of $(k+1)$-jets until we reach the maximum possible value N. The theorem says that if this process fails, namely, the assumption in (3.2) holds, we then end up with a function relationship as in (3.3).

Proof of Theorem 3.3: We consider the following set,

$$
\mathcal{S} = \{l \geq 1 : \mathrm{rank}_l(\psi, \widetilde{z}) \leq l + m - 2\}.
$$

Note that $1 \notin \mathcal{S}$, for $\mathrm{rank}_1(F) = m$. We claim that $\mathcal{S}$ is not empty. Indeed, we have $1 + N - m \in \mathcal{S}$ by (3.2). Now write t' for the minimum number in $\mathcal{S}$. Then $2 \le t' \le 1 + N - m$. Moreover, by the choice of t',

$$\mathrm{rank}_{t'}(\psi, \widetilde{z}) \le t' + m - 2, \ \ \mathrm{rank}_{t'-1}(\psi, \widetilde{z}) \ge t' + m - 2. \tag{3.4}$$

This yields that

$$\mathrm{rank}_{t'}(\psi, \widetilde{z}) = \mathrm{rank}_{t'-1}(\psi, \widetilde{z}) = t' + m - 2. \tag{3.5}$$

We write $t := t' - 1$, $n := t' + m - 2$. Here we note $t \ge 1, m \le n \le N - 1$. Then there exist multiindices $\{\gamma^1, ..., \gamma^n\}$ with each $|\gamma^i| \le t$ and $j_1, ..., j_n$ such that

$$\Delta(\gamma^1, ..., \gamma^n | j_1, ..., j_n) := \begin{vmatrix} \dfrac{\partial^{|\gamma^1|}\psi_{j_1}}{\partial \widetilde{z}^{\gamma^1}} & \cdots & \dfrac{\partial^{|\gamma^1|}\psi_{j_n}}{\partial \widetilde{z}^{\gamma^1}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial^{|\gamma^n|}\psi_{j_1}}{\partial \widetilde{z}^{\gamma^n}} & \cdots & \dfrac{\partial^{|\gamma^n|}\psi_{j_n}}{\partial \widetilde{z}^{\gamma^n}} \end{vmatrix} \ \text{is not identically zero in } U. \tag{3.6}$$

Since $\mathrm{rank}_1(\psi, \widetilde{z}) = m$, we can choose $(\gamma^1, ..., \gamma^n | j_1, ..., j_n)$ such that

$$\gamma^1 = (0, .., 0), \gamma^2 = (1, 0, ..., 0), ..., \gamma^m = (0, ..., 0, 1).$$

For any $\alpha^1, ..., \alpha^{n+1}$ with $|\alpha^i| \le t + 1$, and $l_1, ..., l_{n+1}$, we have

$$\Delta(\alpha^1, ..., \alpha^{n+1} | l_1, ..., l_{n+1}) = \begin{vmatrix} \dfrac{\partial^{|\alpha^1|}\psi_{l_1}}{\partial \widetilde{z}^{\alpha^1}} & \cdots & \dfrac{\partial^{|\alpha^1|}\psi_{l_n}}{\partial \widetilde{z}^{\alpha^1}} & \dfrac{\partial^{|\alpha^1|}\psi_{l_{n+1}}}{\partial \widetilde{z}^{\alpha^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{|\alpha^{n+1}|}\psi_{l_1}}{\partial \widetilde{z}^{\alpha^{n+1}}} & \cdots & \dfrac{\partial^{|\alpha^{n+1}|}\psi_{l_n}}{\partial \widetilde{z}^{\alpha^{n+1}}} & \dfrac{\partial^{|\alpha^{n+1}|}\psi_{l_{n+1}}}{\partial \widetilde{z}^{\alpha^{n+1}}} \end{vmatrix} \equiv 0 \text{ in } U. \tag{3.7}$$

We write Γ for the collection of $(\gamma^1, ..., \gamma^n | j_1, ..., j_n), j_1 < ... < j_n$, with $\gamma^1 = (0, .., 0)$ and with (3.6) being held. We associate each $(\gamma^1, ..., \gamma^n | j_1, ..., j_n)$ with an integer $s(\gamma^1, ..., \gamma^n | j_1, ..., j_n) := s_0$ where s_0 is the least number $s \ge 0$ such that

$$\frac{\partial^{s_1 + ... + s_{m-1} + s}\Delta(\gamma^1, ..., \gamma^n | j_1, ..., j_n)}{\partial z_1^{s_1} \partial z_2^{s_2} ... \partial z_{m-1}^{s_{m-1}} \partial z_m^{s}}(0) \ne 0.$$

for some integers $s_1, ..., s_{m-1}$. Then $s(\gamma^1, ..., \gamma^n | j_1, ..., j_n) \ge 0$ for any $(\gamma^1, ..., \gamma^n | j_1, ..., j_n) \in \Gamma$.

Let $(\beta^1, ..., \beta^n | i_1, ..., i_n) \in \Gamma, i_1 < ... < i_n$ be indices with the least $s(\gamma^1, ..., \gamma^n | j_1, ..., j_n)$ among all $(\gamma^1, ..., \gamma^n | j_1, ..., j_n) \in \Gamma$.

We will need the following lemma proved in [BX]:

Lemma 3.4. *([BX], Lemma 4.4) For a general $n \times n$ matrix*

$$B = \begin{pmatrix} b_{11} & b_{12} & . & . & . & b_{1n} \\ b_{21} & b_{22} & . & . & . & b_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & . & b_{nn} \end{pmatrix},$$

where $b_{ij} \in \mathbb{C}$ for $1 \leq i, j \leq n, n \geq 3$, we have the following identity:

$$\begin{vmatrix} B\begin{pmatrix} 1 & 2 & . & . & . & n-2 & n-1 \\ 1 & 2 & . & . & . & n-2 & n-1 \end{pmatrix} & B\begin{pmatrix} 1 & 2 & . & . & . & n-2 & n-1 \\ j_1 & j_2 & . & . & . & j_{n-2} & n \end{pmatrix} \\ B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} & n \\ 1 & 2 & . & . & . & n-2 & n-1 \end{pmatrix} & B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} & n \\ j_1 & j_2 & . & . & . & j_{n-2} & n \end{pmatrix} \end{vmatrix} \qquad (*)$$

$$= B\begin{pmatrix} i_1 & i_2 & . & . & . & i_{n-2} \\ j_1 & j_2 & . & . & . & j_{n-2} \end{pmatrix}|B|, \text{ for any } 1 \leq i_1 < i_2 < \cdots < i_{n-2} \leq n-1, 1 \leq j_1 <$$

$j_2 < \cdots < j_{n-2} \leq n-1$. *In particular, if $|B| = 0$, then $(*)$ equals 0. Here we have used*

the notation

$$B\begin{pmatrix} i_1 & i_2 & . & . & . & i_p \\ j_1 & j_2 & . & . & . & j_p \end{pmatrix} = \begin{vmatrix} b_{i_1 j_1} & b_{i_1 j_2} & . & . & . & b_{i_1 j_p} \\ b_{i_2 j_1} & b_{i_2 j_2} & . & . & . & b_{i_2 j_p} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ b_{i_p j_1} & b_{i_p j_2} & . & . & . & b_{i_p j_p} \end{vmatrix} \text{ for } 1 \leq p \leq n.$$

We write $\{i_{n+1}, ..., i_N\} = \{1, ..., N\} \setminus \{i_1, .., i_n\}$, where $i_{n+1} < ... < i_N$. Write $\tilde{U} = \{z \in U : \Delta(\beta^1, ..., \beta^n | i_1, ..., i_n) \neq 0\}$. We then have the following:

Lemma 3.5. *Fix $j \in \{i_{n+1}, ..., i_N\}$. Let $i \in \{i_1, .., i_n\}$. Write $\{i'_1, ..., i'_{n-1}\} = \{i_1, ..., i_n\} \setminus \{i\}$. There exists a holomorphic function $g_i^j(z_m)$ in $\tilde{U}$ which only depends on z_m such*

that the following holds for $z \in \tilde{U}$:

$$
\begin{vmatrix}
\frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}\psi_j}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \tilde{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}\psi_j}{\partial \tilde{z}^{\beta^n}}
\end{vmatrix}(z) = g_i^j(z_m)
\begin{vmatrix}
\frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}\psi_i}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \tilde{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}\psi_i}{\partial \tilde{z}^{\beta^n}}
\end{vmatrix}(z),
$$

$$(3.8)$$

or equivalently,

$$
\begin{vmatrix}
\frac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial^{|\beta^1|}(\psi_j - g_i^j(z_m)\psi_i)}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \tilde{z}^{\beta^n}} & \cdots & \frac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}} & \frac{\partial^{|\beta^n|}(\psi_j - g_i^j(z_m)\psi_i)}{\partial \tilde{z}^{\beta^n}}
\end{vmatrix} \equiv 0. \qquad (3.9)
$$

Proof of Lemma 3.5: For simplicity of notation, we write $\frac{\partial}{\partial \tilde{z}^{\beta^i}}$ for $\frac{\partial^{|\beta^i|}}{\partial \tilde{z}^{\beta^i}}$, and for $\mu = i$ or j, write the matrix

$$
V_\mu := \begin{pmatrix}
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial \psi_\mu}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}} & \frac{\partial \psi_\mu}{\partial \tilde{z}^{\beta^n}}
\end{pmatrix} = \begin{bmatrix} \mathbf{v}_\mu^1 \\ \vdots \\ \mathbf{v}_\mu^n \end{bmatrix},
$$

where $\mathbf{v}_\mu^1, \cdots, \mathbf{v}_\mu^n$ are the row vectors of V_μ. To prove (3.8), one just needs to show that, for each $1 \leq \nu \leq m-1$,

$$
\frac{\partial}{\partial z_\nu} \frac{\det(V_j)}{\det(V_i)} \equiv 0 \text{ in } \tilde{U}. \qquad (3.10)
$$

Indeed, by the quotient rule, the numerator of the left-hand side of (3.10) equals to

$$
\det \begin{pmatrix} \det(V_i) & \det(V_j) \\ \frac{\partial}{\partial z_\nu}\det(V_i) & \frac{\partial}{\partial z_\nu}\det(V_j) \end{pmatrix}
$$

$$
= \det \begin{pmatrix} \det(V_i) & \det(V_j) \\ \det\begin{bmatrix} \frac{\partial}{\partial z_\nu}\mathbf{v}_i^1 \\ \mathbf{v}_i^2 \\ \vdots \\ \mathbf{v}_i^n \end{bmatrix} & \det\begin{bmatrix} \frac{\partial}{\partial z_\nu}\mathbf{v}_j^1 \\ \mathbf{v}_j^2 \\ \vdots \\ \mathbf{v}_j^n \end{bmatrix} \end{pmatrix} + \cdots + \det \begin{pmatrix} \det(V_i) & \det(V_j) \\ \det\begin{bmatrix} \mathbf{v}_i^1 \\ \vdots \\ \mathbf{v}_i^{n-1} \\ \frac{\partial}{\partial z_\nu}\mathbf{v}_i^n \end{bmatrix} & \det\begin{bmatrix} \mathbf{v}_j^1 \\ \vdots \\ \mathbf{v}_j^{n-1} \\ \frac{\partial}{\partial z_\nu}\mathbf{v}_j^n \end{bmatrix} \end{pmatrix}.
$$

By (3.7) and Lemma 3.4, each term on the right-hand side of the equation above equals 0. For instance, the last term above equals to

$$
\left|
\begin{array}{c}
\left|
\begin{array}{cccc}
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial \psi_i}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}} & \frac{\partial \psi_i}{\partial \tilde{z}^{\beta^n}}
\end{array}
\right|
\\
\begin{array}{cccc}
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial \psi_i}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^{n-1}}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^{n-1}}} & \frac{\partial \psi_i}{\partial \tilde{z}^{\beta^{n-1}}} \\
\frac{\partial}{\partial z_\nu}\left(\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^n}}\right) & \cdots & \frac{\partial}{\partial z_\nu}\left(\frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}}\right) & \frac{\partial}{\partial z_\nu}\left(\frac{\partial \psi_i}{\partial \tilde{z}^{\beta^n}}\right)
\end{array}
\end{array}
\right|
\left|
\begin{array}{c}
\left|
\begin{array}{cccc}
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial \psi_j}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}} & \frac{\partial \psi_j}{\partial \tilde{z}^{\beta^n}}
\end{array}
\right|
\\
\begin{array}{cccc}
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial \psi_j}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^{n-1}}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^{n-1}}} & \frac{\partial \psi_j}{\partial \tilde{z}^{\beta^{n-1}}} \\
\frac{\partial}{\partial z_\nu}\left(\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^n}}\right) & \cdots & \frac{\partial}{\partial z_\nu}\left(\frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}}\right) & \frac{\partial}{\partial z_\nu}\left(\frac{\partial \psi_j}{\partial \tilde{z}^{\beta^n}}\right)
\end{array}
\end{array}
\right| .
$$

$$(3.11)$$

It is a multiple of the following determinant (by Lemma 3.4):

$$
\left|
\begin{array}{ccccc}
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^1}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^1}} & \frac{\partial \psi_i}{\partial \tilde{z}^{\beta^1}} & \frac{\partial \psi_j}{\partial \tilde{z}^{\beta^1}} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^n}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^n}} & \frac{\partial \psi_i}{\partial \tilde{Z}^{\beta^n}} & \frac{\partial \psi_j}{\partial \tilde{z}^{\beta^n}} \\
\frac{\partial \psi_{i'_1}}{\partial \tilde{z}^{\beta^{n+1}}} & \cdots & \frac{\partial \psi_{i'_{n-1}}}{\partial \tilde{z}^{\beta^{n+1}}} & \frac{\partial \psi_i}{\partial \tilde{z}^{\beta^{n+1}}} & \frac{\partial \psi_j}{\partial \tilde{z}^{\beta^{n+1}}}
\end{array}
\right| ,
$$

$$(3.12)$$

where $\frac{\partial}{\partial \tilde{z}^{\beta^{n+1}}} = \frac{\partial}{\partial z_\nu}\left(\frac{\partial}{\partial \tilde{z}^{\beta^n}}\right)$, which is identically zero by (3.7). This establishes Lemma 3.5. ∎

The extendability of $g_i^j(z_m)$ will be needed for our later argument, which is proved in the following:

Lemma 3.6. *For any i, j as above, the holomorphic function $g_i^j(z_m)$ can be extended holomorphically to a neighborhood of 0 in the z_m-plane.*

Proof of Lemma 3.6: First, g_i^j is defined on the projection $\pi_m(\tilde{U})$ of $\tilde{U}$, where π_m is the natural projection of $(z_1, ..., z_m)$ to its last component z_m. If $0 \in \pi_m(\tilde{U})$, the claim follows trivially. Now assume that $0 \notin \pi_m(\tilde{U})$. If we write $s = s(\beta_1, ..., \beta_n | i_1, ..., i_n)$, by

its definition, then there exists $(a_1, ..., a_{m-1}) \in \mathbb{C}^{m-1}$ close to 0, such that

$$\begin{vmatrix} \dfrac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \dfrac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \dfrac{\partial^{|\beta^1|}\psi_i}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \dfrac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \dfrac{\partial^{|\beta^n|}\psi_i}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (a_1, ..., a_{m-1}, z_m) = c z_m^s + o(|z_m|^s), \ c \neq 0. \quad (3.13)$$

Then there exists $r > 0$ small enough such that for any $0 < |z_m| < r$, $(a_1, ..., a_{m-1}, z_m) \in \tilde{U}$. That is, at any of such points, equation (3.13) is not zero.

We now substitute $(a_1, ..., a_{m-1}, z_m), 0 < |z_m| < r$, into the equation (3.8), and compare the vanishing order as $z_m \to 0$:

$$c_1 z_m^{s'} + o(|z_m|^{s'}) = g_i^j(z_m)(c z_m^s + o(|z_m|^s)), \ c \neq 0. \quad (3.14)$$

for some $s' \geq 0$. Note that $0 \leq s \leq s'$ by the definition of s and the choice of $(\beta_1, ..., \beta_n | i_1, ..., i_n)$. The holomorphic extendability across 0 of $g_i^j(z_m)$ then follows easily. ∎

We next make the following observation:

Claim 3.7. *For each fixed $j \in \{i_{n+1}, ..., i_N\}$ and any $i'_1 < ... < i'_{n-1}$ with $\{i'_1, ..., i'_{n-1}\} \subset \{i_1, ..., i_n\}$, we have:*

$$\begin{vmatrix} \dfrac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \dfrac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \dfrac{\partial^{|\beta^1|}(\psi_j - \sum_{k=1}^n g_{i_k}^j \psi_{i_k})}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \dfrac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \dfrac{\partial^{|\beta^n|}(\psi_j - \sum_{k=1}^n g_{i_k}^j \psi_{i_k})}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z) \equiv 0, \forall z \in \tilde{U}. \quad (3.15)$$

Proof of Claim 3.7: Note that for each $i'_l, 1 \leq l \leq n-1$, the following trivially holds:

$$\begin{vmatrix} \dfrac{\partial^{|\beta^1|}\psi_{i'_1}}{\partial \bar{z}^{\beta^1}} & \cdots & \dfrac{\partial^{|\beta^1|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^1}} & \dfrac{\partial^{|\beta^1|}(g_{i'_l}^j \psi_{i'_l})}{\partial \bar{z}^{\beta^1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^{|\beta^n|}\psi_{i'_1}}{\partial \bar{z}^{\beta^n}} & \cdots & \dfrac{\partial^{|\beta^n|}\psi_{i'_{n-1}}}{\partial \bar{z}^{\beta^n}} & \dfrac{\partial^{|\beta^n|}(g_{i'_l}^j \psi_{i'_l})}{\partial \bar{z}^{\beta^n}} \end{vmatrix} (z) \equiv 0, \quad (3.16)$$

for the last column in the matrix is a multiple of one of the first $(n-1)$ columns. Then (3.15) is an immediate consequence of (3.9) and (3.16). ∎

Lemma 3.8. *For each fixed $j \in \{i_{n+1}, ..., i_N\}$, we have $\psi_j(z) - \sum_{k=1}^n g_{i_k}^j(z_m)\psi_{i_k}(z) \equiv 0$ for any $z \in \tilde{U}$, and thus it holds also for all $z \in U$.*

Proof of Lemma 3.8: This can be concluded easily from the following Lemma 3.9 and Claim 3.7. Here one needs to use the fact that $\beta^1 = (0, ..., 0)$. ∎

Lemma 3.9. *([BX], Lemma 4.7) Let $\mathbf{b}_1, \cdots, \mathbf{b}_n$ and $\mathbf{a}$ be n-dimensional column vectors with elements in $\mathbb{C}$, and let $B = (\mathbf{b}_1, \cdots, \mathbf{b}_n)$ denote the $n \times n$ matrix. Assume that $\det B \neq 0$ and $\det(\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \cdots, \mathbf{b}_{i_{n-1}}, \mathbf{a}) = 0$ for any $1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n$. Then $\mathbf{a} = \mathbf{0}$.*

Theorem 3.3 now follows easily from Lemma 3.8. ∎

If we further assume that $\psi_i(z), m + 1 \leq i \leq N$, vanishes at least to the second order, then we have the following, which plays a crucial role in our proof of Theorem 1.1.

Theorem 3.10. *Let $\psi = (\psi_1, ..., \psi_m, \psi_{m+1}, ..., \psi_N)$ be a holomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into $\mathbb{C}^N$ with $\psi(0) = 0$. Assume that $(\psi_1, ..., \psi_m)$ is a biholomorphic map from a neighborhood of $0 \in \mathbb{C}^m$ into a neighborhood of $0 \in \mathbb{C}^N$. Assume that $\psi_j(z) = O(|z|^2)$ for $m + 1 \leq j \leq N$. Suppose that $\mathrm{rank}_{N-m+1}(\psi) < N$. Then there exist $a_{m+1}, ..., a_N \in \mathbb{C}$ that are not all zero such that*

$$\sum_{i=m+1}^N a_j \psi_j(z_1, ..., z_{m-1}, 0) \equiv 0, \tag{3.17}$$

for all $(z_1, ..., z_{m-1})$ near 0.

Proof of Theorem 3.10: We first have the following:

Claim 3.11. *For each $1 \leq i \leq m$, $g_i(0) = 0$.*

Proof of Claim 3.11: Suppose not. Write $\mathbf{c} := (g_1(0), ..., g_m(0)) \neq 0$. Then $(g_1(z_m), ..., g_m(z_m)) = \mathbf{c} + O(|z_m|)$. The fact that $\psi_i(z) = O(|z|^2), i \geq m + 1$, implies

$$\sum_{i=1}^m g_i(z_m)\psi_i(z) = O(|z|^2). \tag{3.18}$$

Notice that $(\psi_1, ..., \psi_m)$ is of full rank at 0. That is

$$\begin{vmatrix} \frac{\partial \psi_1}{\partial z_1} & \cdots & \frac{\partial \psi_m}{\partial z_1} \\ \cdots & \cdots & \cdots \\ \frac{\partial \psi_1}{\partial z_m} & \cdots & \frac{\partial \psi_m}{\partial z_m} \end{vmatrix}(0) \neq 0.$$

Hence

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial z_1}(0) & \cdots & \frac{\partial \psi_m}{\partial z_1}(0) \\ \cdots & \cdots & \cdots \\ \frac{\partial \psi_1}{\partial z_m}(0) & \cdots & \frac{\partial \psi_m}{\partial z_m}(0) \end{pmatrix} \mathbf{c}^t \neq 0. \qquad (3.19)$$

This is a contradiction to (3.18). $\blacksquare$

Finally, letting $z_m = 0$ in equation (3.3), we obtain (3.17). Since

$$(g_1(0), ..., g_m(0)) = 0, \quad (g_{m+1}(0), ..., g_N(0)) \neq 0.$$

This establishes Theorem 3.10. $\blacksquare$

Chapter 4

Proof of the main theorem assuming three extra propositions

Let $M \subset \mathbb{CP}^N$ be an irreducible Hermitian symmetric space of compact type, which has been canonically (and isometrically) embedded in the complex projective space through the way described in § 2. In this section, we write n as the complex dimension of M. We also have on M an affine open piece $\mathcal{A}$ that is biholomorphically equivalent to the complex Euclidean space of the same dimension, such that $M \setminus \mathcal{A}$ is a codimension one complex subvariety of M. We identify the coordinates of $\mathcal{A}$ by the parametrization map with $z = (z_1, ..., z_n) \in \mathbb{C}^n$ through what is described in § 2, which we wrote as $[1, \psi_1, ..., \psi_N]$, where $\psi_1, ..., \psi_N$ are polynomial maps in $(z_1, ..., z_n)$ with $\psi_j = \kappa_j z_j$, where $\kappa_j = 1$ or $\sqrt{2}$, for $j = 1, \cdots, n$. We also write $\overline{F}(\xi)$ for $\overline{F(\bar{\xi})}$ for $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$. We still use $\rho(z, \xi)$ for the defining function of the Segre family of M restricted to $\mathcal{A} \times \mathcal{A}^*$, which will be canonically identified with $\mathbb{C}^n \times \mathbb{C}^n$. Since the coefficients of $\psi_1, ..., \psi_N$ are all real, $\overline{\psi} = \psi$ and $\mathcal{A}^* = \mathcal{A}$. Hence, we have

$$\rho(z, \xi) = 1 + \sum_{i=1}^{N} \psi_i(z) \psi_i(\xi). \tag{4.1}$$

Recall the standard metric ω of M on $\mathcal{A}$ is given by

$$\omega = i \partial \overline{\partial} \log(\rho(z, \bar{z})). \tag{4.2}$$

The volume form $d\mu = c_n \omega^n$ associated to ω, by § 2, is now given in $\mathcal{A}$ by the multiplication of V with the standard Euclidean volume form, where

$$V = \frac{c}{(\rho(z, \bar{z}))^\lambda} \tag{4.3}$$

with $c > 0$ and λ a certain positive integer depending on M. For instance, $\lambda = p + q$ when $X = G(p, q)$ [G]. Here c_n is a certain positive constant depending only on n. Recall the main theorem of this thesis:

Theorem 4.1. *Let $\mathcal{A} \subset M$ be as above equipped with the standard metric ω. Let $F_j, j = 1, ..., m$, be a holomorphic map from $U \subset \mathcal{A}$ into M, where U is a connected open neighborhood of $\mathcal{A}$. Assume that $F_j^*(d\mu) \not\equiv 0$ for each j and assume that*

$$d\mu = \sum_{j=1}^{m} \lambda_j F_j^*(d\mu), \tag{4.4}$$

for certain positive constants $\lambda_j > 0$ with $j = 1, \cdots, m$. Then for any $j \in \{1, 2, ..., m\}$, F_j extends to a holomorphic isometry of (M, ω).

For convenience of our discussions, we first fix some notations: In what follows, we identify $\mathcal{A}$ with $\mathbb{C}^n$ having $z = (z_1, \cdots, z_n)$ as its coordinates. On $U \subset \mathcal{A} \subset M$ and after shrinking U if needed, we write the holomorphic map F_j, for $j = 1, ..., m$, from $U \to \mathcal{A} = \mathbb{C}^n$, as follows:

$$F_j = (F_{j,1}, F_{j,2}, ..., F_{j,n}), \ \ j = 1, ..., m. \tag{4.5}$$

Still write the holomorphic embedding from $\mathcal{A}$ into $\mathbb{CP}^N$ as $[1, \psi_1, \cdots, \psi_N]$. We define $\mathcal{F}_j(z) = (\mathcal{F}_{j,1}, ..., \mathcal{F}_{j,N}) = (\psi_1(F_j), \psi_2(F_j), ..., \psi_N(F_j))$ for $j = 1, ..., m$. Finally, all Segre varieties and Segre families are restricted to $\mathcal{A} = \mathbb{C}^n$.

The main purpose of this chapter is to give a proof of Theorem 4.1, assuming the following three propositions hold. These propositions will be separately established in terms of the type of M in § 5, § 6 and § 7. This then completes the proof of our main theorem.

Proposition (I): Write $\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z, \xi)}{\frac{\partial \rho}{\partial z_n}(z, \xi)} \frac{\partial}{\partial z_n}, 1 \leq i \leq n-1$, which are holomorphic vector fields (whenever defined) tangent to the Segre family $\mathcal{M}$ of $M \hookrightarrow \mathbb{CP}^N$ restricted to $\mathcal{A} \times \mathcal{A}^* = \mathbb{C}^n \times \mathbb{C}^n$ defined by $\rho(z, \xi) = 0$. Under the notations we set up above, for any local biholomorphic map $F = (f_1, \cdots, f_n) : U \to \mathbb{C}^n$ with $F(0) = 0$, there are $z^0 \in U, \xi^0 \in Q_{z^0}, \beta^1, ..., \beta^N$, such that

$$\frac{\partial \rho}{\partial z_n}(z^0, \xi^0) \neq 0, \quad \Lambda(\beta^1, ..., \beta^N)(z^0, \xi^0) := \begin{vmatrix} \mathcal{L}^{\beta^1} \mathcal{F}_1 & ... & \mathcal{L}^{\beta^1} \mathcal{F}_N \\ ... & ... & ... \\ \mathcal{L}^{\beta^N} \mathcal{F}_1 & ... & \mathcal{L}^{\beta^N} \mathcal{F}_N \end{vmatrix} (z^0, \xi^0) \neq 0. \tag{4.6}$$

Here $\beta^l = (k_1^l, ..., k_{n-1}^l)$, $k_1^l, ..., k_{n-1}^l$ are non-negative integers, for $l = 1, 2, ..., N$; $\beta^1 = (0, 0, ..., 0)$; $\mathcal{L}^{\beta^l} = \mathcal{L}_1^{k_1^l} \mathcal{L}_2^{k_2^l} \mathcal{L}_3^{k_3^l} ... \mathcal{L}_{n-1}^{k_{n-1}^l}$; $\mathcal{F}(z) = (\mathcal{F}_1, ..., \mathcal{F}_N) = (\psi_1(F), \psi_2(F), ..., \psi_N(F))$.

Moreover, $s_l := \sum_{i=1}^{n-1} k_i^l$ $(l = 1, ..., N)$ is a non-negative integer bounded from above by a universal constant depending only on (M, ω). Also, in what follows, when we like to emphasize the dependence of $\Lambda(\beta^1, ..., \beta^N)$ on F, we also write it as $\Lambda_F(\beta^1, ..., \beta^N)$.

Proposition (II): Suppose that $\xi^0 \in \mathbb{C}^n$ with $\xi^0 \neq (0, 0, ..., 0)$. Then for a generic smooth point z^0 on the Segre variety Q_{ξ^0} and a small neighborhood $U \ni z^0$, there is a $z^1 \in U \cap Q_{\xi^0}$ such that Q_{z^0} and Q_{z^1} both are smooth at ξ^0 and intersect transversally at ξ^0, too. Moreover, we can find a biholomorphic parametrization near ξ^0: $(\xi_1, \xi_2, ..., \xi_n) = \mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n)$ with $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$, where U_1 and U_2 are small neighborhoods of $1 \in \mathbb{C}$, and U_j for $j \geq 3$ are small neighborhoods of $0 \in \mathbb{C}$ such that (i). $\mathcal{G}(1, 1, 0, \cdots, 0) = \xi_0$, (ii). $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}, \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$, and (iii). $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times ... \times U_n)$ or $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times ... \times U_n), s \in U_1, t \in U_2$ is an open piece of a certain Segre variety for each fixed t and s. Moreover $\mathcal{G}$ consists of algebraic functions with total degree bounded by a constant depending only on the manifold M.

Proposition (III): For any $\xi \neq 0 (z \neq 0, \text{ respectively}) \in \mathbb{C}^n$, $\rho(z, \xi)$ is an irreducible polynomial in z (and in ξ, respectively). (In particular, Q_ξ^* and Q_z are irreducible.) Moreover, if U is a connected open set in $\mathbb{C}^n$, then the Segre family $\mathcal{M}$ restricted to $U \times \mathbb{C}^n$ is an irreducible complex subvariety and thus its regular points form a connected complex submanifold. In particular, $\mathcal{M}$ is an irreducible complex subvariety of $\mathbb{C}^n \times \mathbb{C}^n$.

The rest of this chapter is divided into several sections. In § 4.2, we provide a proof of the well-known fact of the modified version of the classical Hurwitz Theorem, which is important for our use in the later sections. In § 4.3, we show the algebraicity and rationality of F_{j_0}. § 4.4 is devoted to proving that F_{j_0} extends to a birational map from M to itself and extends to a holomorphic isometry, which can be used, through an induction argument, to give a proof of Theorem 4.1 assuming Propositions (I)-(III).

4.1 Modified Hurwitz Theorem

Before stating the modified version of a classical result of Hurwitz. We first give the following definition:

Definition 4.2. *Suppose F is an algebraic function in $\xi \in \mathbb{C}^n$. The total degree of F is defined to be the total degree of its minimum polynomial. Namely, let $P(\xi; X)$ be an irreducible minimum polynomial of F, the total degree of F is defined as the degree of $P(\xi; X)$ as a polynomial in (ξ, X).*

We next state some simple facts about algebraic functions and give a short proof for the readers' convenience.

Lemma 4.3. *Suppose ϕ_1, ϕ_2 are algebraic functions defined in variables $\xi \in U \subset \mathbb{C}^n$ with total degree bounded by N. Then $\phi_1 \pm \phi_2, \phi_1 \phi_2, 1/\phi_1$ (if $\phi_1 \not\equiv 0$) are algebraic functions and their degrees are bounded above by a constant depending only on N and n.*

Proof : Suppose $f_1(\xi, X) = \sum_{i=0}^{n_1} a_i(\xi) X^{n_1 - i}$ and $f_1(\xi, \phi_1) \equiv 0$ where $\deg f_1 < N$ and $a_0(\xi) \not\equiv 0$. View f_1 as a polynomial of X and denote n_1 solutions of f_1 by $\phi_1^1 = \phi_1, \phi_1^2, \phi_1^3, \cdots, \phi_1^{n_1}$. Also assume $f_2(\xi, X) = \sum_{j=0}^{n_2} b_j(\xi) X^{n_2 - j}$ and $f_2(\xi, \phi_2) \equiv 0$ where $\deg f_2 < N$ and $b_0(\xi) \not\equiv 0$. Denote n_2 solutions of f_2 by $\phi_2^1 = \phi_2, \phi_2^2, \phi_2^3, ..., \phi_2^{n_2}$. Notice that $n_1 < N$ and $n_2 < N$.

Now define functions in variables ξ and X as follows:

$$f_\pm(\xi, X) = a_0(\xi)^{n_2} b_0(\xi)^{n_1} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (X - \phi_1^i \pm \phi_2^j)$$

$$= a_0(\xi)^{n_2} b_0(\xi)^{n_1} \sum_{k=0}^{n_1 n_2} X^{n_1 n_2 - k} \theta_k(a_0, b_0, \phi_1^1, \phi_1^2, ..., \phi_1^{n_1}, \phi_2^1, \phi_2^2, ..., \phi_2^{n_2}).$$

Notice that θ_k is a symmetric polynomial with respect to ϕ_1^i and ϕ_2^j separately and hence a polynomial in terms of $\frac{a_1(\xi)}{a_0(\xi)}, \frac{a_2(\xi)}{a_0(\xi)}, \cdots, \frac{a_{n_1}(\xi)}{a_0(\xi)}, \frac{b_1(\xi)}{b_0(\xi)}, \cdots, \frac{b_{n_2}(\xi)}{b_0(\xi)}$. After multiplying $a_0(\xi)^{n_2} b_0(\xi)^{n_1}$, $f_\pm(\xi, X)$ is a polynomials in $a_0(\xi), a_1(\xi), \cdots, a_{n_1}(\xi), b_0(\xi), b_1(\xi), \cdots, b_{n_2}(\xi)$ and X. Since $\phi_1 \pm \phi_2$ are annihilated by $f_\pm(\xi, X)$, they are algebraic functions with algebraic total degrees bounded by the degrees of $f_\pm(\xi, X)$. By the explicit construction above it is clear that the degrees of $f_\pm$ depend only on N and n.

For $\phi_1 \phi_2$ we construct the following function:

$$f(\xi, X) = a_0(\xi)^{n_2} b_0(\xi)^{n_1} \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (X - \phi_1^i \phi_2^j) = a_0(\xi)^{n_2} b_0(\xi)^{n_1} \sum_{k=0}^{n_1 n_2} \theta_k(\xi) X^{n_1 n_2 - k},$$

which is an annihilator of $\phi_1\phi_2$. By the same argument we know that $f(\xi, X)$ is a polynomial in terms of $a_0(\xi), a_1(\xi), \cdots, a_{n_1}(\xi), b_0(\xi), b_1(\xi), \cdots, b_{n_2}(\xi)$ and X. Hence the total degree of $\phi_1\phi_2$ is bounded by the degree of $f(\xi, X)$, which depends only on N and n.

Finally denote an annihilator for $\phi_1 \not\equiv 0$ by $f(\xi, X) = \sum_{i=0}^{n_1} a_i(\xi)X^{n_1-i}$, where $1 \leq n_1 \leq N$ and $a_{n_1}(\xi) \not\equiv 0$. Then the polynomial $F(\xi, X) = \sum_{i=0}^{n_1} a_i(\xi)X^i$ annihilates $\frac{1}{\phi_1}$ and hence $\frac{1}{\phi_1}$ is an algebraic function with degree bounded by N and n. ∎

Lemma 4.4. *Suppose $\phi_1(z_1, ..., z_n)$ is an algebraic function of total degree bounded by N, and suppose that $\psi_1(\xi_1, ..., \xi_m), ..., \psi_n(\xi_1, ..., \xi_m)$ are algebraic functions with total degree bounded by N as well. Let*

$$A_0 = (\xi_1^0, \xi_2^0, ..., \xi_m^0) \in \mathbb{C}^m,$$

where $\psi_1, ..., \psi_n$ are holomorphic functions in a neighborhood of A_0 and let ϕ_1 be a holomorphic function in a neighborhood $U \subset \mathbb{C}^n$ of $(\psi_1(A_0), \psi_2(A_0), ..., \psi_m(A_0))$. Then the composition $\Phi(\xi_1, ..., \xi_m) = \phi_1(\psi_1(\xi_1, ..., \xi_m), \psi_2(\xi_1, ..., \xi_m), \psi_3(\xi_1, ..., \xi_m), ..., \psi_n(\xi_1, ..., \xi_m))$ is an algebraic function with total degree bounded by a constant $C(N, n, m)$ depending only on (N, n, m).

Proof : Since ϕ_1 is an algebraic function with total degree bounded by N, the minimal polynomial of it has the following form:

$$f(z_1, ..., z_n, X) = \sum_{I,K,|I|+|K|\leq N} a_I z_1^{i_1} \cdots z_n^{i_n} X^K,$$

which annihilates ϕ_1. For the algebraic function ψ_1 there is a minimal polynomial:

$$\mathcal{F}_1(\xi_1, ..., \xi_m, X) = \sum_{J,K,|J|+|K|\leq N} b_{JK}^1 \xi_1^{j_1} \cdots \xi_m^{j_m} X^K,$$

which annihilates ψ_1. Denote all the solutions of $\mathcal{F}_1$ by $\psi_1^1 = \psi_1, \psi_1^2, ..., \psi_1^{l_1}$, where $l_1 \leq N$. In a similar way, we denote the minimal polynomials for $\psi_2, ..., \psi_m$ respectively by:

$$\mathcal{F}_2(\xi_1, ..., \xi_m, X) = \sum_{J,K,|J|+|K|\leq N} b_{JK}^2 \xi_1^{j_1} \cdots \xi_m^{j_m} X^K \quad \text{with solutions} \quad \psi_2^1 = \psi_2, \psi_2^2, ..., \psi_2^{l_2},$$

$$\mathcal{F}_3(\xi_1, ..., \xi_m, X) = \sum_{J,K,|J|+|K| \leq N} b^3_{JK} \xi_1^{j_1} \cdots \xi_m^{j_m} X^K \text{ with solutions } \psi_3^1 = \psi_3, \psi_3^2, ..., \psi_3^{l_3},$$

$$...$$

$$\text{and } \mathcal{F}_n(\xi_1, ..., \xi_m, X) = \sum_{J,K,|J|+|K| \leq N} b^n_{JK} \xi_1^{j_1} \cdots \xi_m^{j_m} X^K \text{ with solutions } \psi_n^1 = \psi_n, \psi_n^2, ..., \psi_n^{l_n}.$$

Plugging $z_1 = \psi_1(\xi_1, ..., \xi_m), ..., z_n = \psi_n(\xi_1, ..., \xi_m)$ into $\phi_1(z_1, ..., z_n)$, we have

$$0 \equiv f(\psi_1(\xi_1, ..., \xi_m), ..., \psi_n(\xi_1, ..., \xi_n), \phi_1(\psi_1(\xi_1, ..., \xi_m), \psi_2(\xi_1, ..., \xi_m), ..., \psi_n(\xi_1, ..., \xi_m)))$$

$$= \sum_{I,K,|I|+|K| \leq N} a_I \psi_1^{i_1} \cdots \psi_n^{i_n} \phi_1^K.$$

Construct a function as follows:

$$\mathcal{F} = \prod_{v=1}^{m} (b^v_{00})^{l_1 l_2 \cdots l_n N} \prod_{v_1=1}^{l_1} \prod_{v_2=1}^{l_2} \cdots \prod_{v_n=1}^{l_n} \sum_{I,K,|I|+|K| \leq N} a_I (\psi_1^{v_1})^{i_1} (\psi_2^{v_2})^{i_2} \cdots (\psi_n^{v_n})^{i_n} X^K,$$

which annihilates ϕ_1 and is a symmetric polynomial with respect to $\{\psi_1^v\}_{v=1}^{l_1}, ..., \{\psi_n^v\}_{v=1}^{l_n}$ separately. It is easy to see that $\mathcal{F}$ is actually a polynomial in terms of $\{b^v_{JK}(\xi)\}_{v,J,K}$ and X. Since the degrees of b^v_{JK}'s are bounded by N and $l_1, ..., l_n$ are bounded above by N, the degree of $\mathcal{F}$ is bounded by a constant $C(N, n, m)$ depending only on N, n and m. ∎

Let us recall the classical Hurwitz Theorem:

Theorem 4.5. *Suppose G is a holomorphic function defined on a neighborhood $U \times V \times W \subset \mathbb{C} \times \mathbb{C} \times \mathbb{C}^m$ in variables $(s, t, \xi_1, ..., \xi_m)$. (We denote it by (s, t, ξ) for short.) Suppose G is a rational function of $t, \xi_1, ..., \xi_m$ for arbitray $s \in U$ with the degree less then N; and G is a rational function of $s, \xi_1, .., \xi_m$ for arbitray $t \in V$ with the degree less then N. Then G is a rational function on $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^m$ of (s, t, ξ).*

We would like to present a short proof of Hurwitz Theorem in the following, since it also motivates our proof of the modified Hurwitz Theorem.

Proof : (following Bochner-Martin [BM])

If G is identically zero, we are done. Otherwise G is not identically zero and we can choose $(s_0, t_0, \xi_0) \in U \times V \times W$ such that $G(s_0, t_0, \xi_0) \neq 0$. Shrinking the domain if necessary, we assume that $G(s_0, t_0, \xi_0) \neq 0$ holds everywhere in the domain. Without loss of generality, we assume $G \neq 0$ on $U \times V \times W$.

Since G is a rational function of $t, \xi_1, ..., \xi_m$ for arbitrary $s \in U$ with the degree less then N, we write G as $G(s, t, \xi) = \frac{\sum_{|I|+J<N} C_{IJ}(s)\xi^I t^J}{\sum_{|I|+J<N} B_{IJ}(s)\xi^I t^J}$ for arbitray $s \in U$. Here ξ^I is a compact notation for $\xi_1^{i_1} \xi_2^{i_2} ... \xi_m^{i_m}$; I is a notation for $(i_1, ..., i_n)$; $|I|$ is a notation for $i_1 + ... + i_m$; and J is a natural number.

Moreover we view $G(s, t, \xi)$ as a rational function of t,

$$G(s, t, \xi) = \frac{\sum_{J<N}\{\sum_{|I|<N-|J|} C_{IJ}(s)\xi^I\}t^J}{\sum_{J<N}\{\sum_{|I|<N-|J|} B_{IJ}(s)\xi^I\}t^J}, \tag{A1}$$

for arbitrary $s \in U, \xi \in W$.

Since $G \neq 0$ we have $\sum_J \{|\sum_{|I|<N-J} B_{I,J}(s)\xi^I| + |\sum_{|I|<N-J} C_{I,J}(s)\xi^I|\} > 0$ for all $s \in U$ and $\xi \in W$. Rewriting (A1), we get

$$\sum_{|J|<N} \{ \sum_{|I|<N-|J|} B_{IJ}(s)\xi^I \}(t^J G(s, t, \xi)) - \sum_{|J|<N} \{ \sum_{|I|<N-|J|} C_{IJ}(s)\xi^I \}t^J = 0. \tag{A2}$$

For convenience, we put $\{t^J G(s, t, \xi)\}_J$ and $\{t^J\}_J$ together and denote them by $\{\phi_1, ..., \phi_M\}$. In a similar way, we put $\{\sum_{|I|<N-|J|} B_{IJ}(s)\xi^I\}_J$ and $\{\sum_{|I|<N-|J|} C_{IJ}(s)\xi^I\}_J$ together and denote them by $\{\psi_1, ..., \psi_M\}$.

Choosing any $(M-1)$ points $t = t_1, ..., t_{M-1}$ in V, together with $t = t$, and plugging them into A2, we have equations in the the following matrix form:

$$\begin{pmatrix} \phi_1(s, t_1, \xi) & \phi_2(s, t_1, \xi) & \phi_3(s, t_1, \xi) & ... & \phi_M(s, t_1, \xi) \\ ... & ... & ... & ... & ... \\ \phi_1(s, t_{M-1}, \xi) & \phi_2(s, t_{M-1}, \xi) & \phi_3(s, t_{M-1}, \xi) & ... & \phi_M(s, t_{M-1}, \xi) \\ \phi_1(s, t, \xi) & \phi_2(s, t, \xi) & \phi_3(s, t, \xi) & ... & \phi_M(s, t, \xi) \end{pmatrix} \begin{pmatrix} \psi_1(s, \xi) \\ \psi_2(s, \xi) \\ ... \\ \psi_M(s, \xi) \end{pmatrix} = 0.$$

Since $(\psi_1(s, \xi), ... \psi_M(s, \xi))$ is a non zero vector for arbitrary $s \in U, \xi \in W$, we have

$$\det \begin{pmatrix} \phi_1(s, t_1, \xi) & ... & \phi_M(s, t_1, \xi) \\ ... & ... & ... \\ \phi_1(s, t_{M-1}, \xi) & ... & \phi_M(s, t_{M-1}, \xi) \\ \phi_1(s, t, \xi) & ... & \phi_M(s, t, \xi) \end{pmatrix} \equiv 0 \ .$$

By Laplacian expansion of the determinant along the last row, we have

$$\sum_{i=1}^{M} \Phi_i(t_1, ..., t_{M-1}, \xi, s)\phi_i(s, t, \xi) \equiv 0, \tag{B1}$$

where $\{\Phi_i\}$ are the corresponding $(M-1) \times (M-1)$ minors of the $M \times M$ matrix.

If for certain $t_1, t_2, ..., t_{M-1} \in V$ at least one of $\{\Phi_i\}$ is not identically zero, when it is viewed as a function of variables ξ and s, we claim that there exists an index i such that Φ_i is not identically zero and the corresponding $\phi_i(s, t, \xi)$ takes the form $\phi_i(s, t, \xi) = t^J G(s, t, \xi)$ for certain J.

Proof of the claim: If it is not the case, we can reduce Equality B1 to the following equality:

$$0 \equiv \sum_{j=1}^{M} \Phi_j(t_1, ..., t_{M-1}, \xi, s) \phi_j(s, t, \xi) = \sum_{j=1}^{M'} \Phi_j(t_1, ..., t_{M-1}, \xi, s) t^J. \qquad (\text{B2})$$

Notice that at least one of $\{\Phi_j\}$ appeared on the right side of B2 is not identically zero as a function of ξ and s. One the other hand, we should have all $\Phi_j \equiv 0$, if we view the right side of B2 as a polynomial of t. This is a contradiction. ∎

Therefore if for certain $t_1, t_2, ..., t_{M-1} \in V$ at least one of $\{\Phi_i\}$ is not identically zero, we can rewrite Equality B1 as follows:

$$(\sum_{i=1}^{M'} \Phi_i(t_1, ..., t_{M-1}, \xi, s) t^J) G(s, t, \xi) + \sum_{i=M'+1}^{M} \Phi_j(t_1, ..., t_{M-1}, \xi, s) t^J \equiv 0,$$

where the coefficient of G is not identically zero. Notice that for fixed $t = t_1, ..., t_{M-1}$, $G(s, t, \xi)$ is a rational function of s and ξ. Then $\Phi_i(t_1, ..., t_{M-1}, \xi, s)$ are all rational functions of s and ξ, and G is a rational function of s, t and ξ. Hence we proved Theorem 4.5 in this case.

Now if all $\Phi_i(t_1, ..., t_{M-1}, s, \xi)$ are identically zero for any $t_1, ..., t_{M-1} \in V$, we have the consequence that each $(M-1) \times (M-1)$ minor of the determinant is identically zero for any $t_1, ..., t_{M-1}, s$ and ξ. On the other hand, we know that the 1×1 minors are not identically zero. By induction we can reduce it to the previous case by finding the $l \times l$ minor which is identically zero but the $(l-1) \times (l-1)$ minors of it are not. ∎

We now state the following modified version of the classical Hurwitz theorem with a controlled total degree [BM].

Theorem 4.6. *Let $\widetilde{F}(s, t, \xi_1, \xi_2, ..., \xi_m)$ be holomorphic over $U \times V \times W \subset \mathbb{C}^{m+2}$. Suppose that for any fixed $s \in U \subset \mathbb{C}$, $\widetilde{F}$ is an algebraic function in $(t, \xi_1, ..., \xi_m)$ with*

its total degree uniformly bounded by N; and for any fixed $t \in V \subset \mathbb{C}$, $\widetilde{F}$ is an algebraic function of $(s, \xi_1, ..., \xi_m)$ with its total degree uniformly bounded by N. Then $\widetilde{F}$ is an algebraic function with total degree bounded by a constant depending only on (m, N).

Proof : We can assume that $\widetilde{F} \not\equiv 0$ and shrink the domain such that $\widetilde{F} \neq 0$ everywhere in the domain. Without loss of generality we assume $\widetilde{F} \neq 0$ in $U \times V \times W$.

Notice that for each fixed $t \in V$ we have functions $\{C_{IJK}(t)\}$ such that

$$\sum_{I+|J|+K \leq N} C_{IJK}(t) s^I \xi^J (\widetilde{F}(s, t, \xi))^K = 0. \qquad (C1)$$

Since $\widetilde{F} \neq 0$, it is easy to see that $\sum |C_{IJK}(t) \xi^J| > 0$.

As in the proof of Theorem 4.5, we denote $\{s^I \widetilde{F}(s, t, \xi)^K\}_{I,K}$ by $\{\psi_i(s, t, \xi)\}_{i=1}^M$ and $\{C_{IJK}(t)\xi^J\}$ by $\{\phi_1(t, \xi), ..., \phi_M(t, \xi)\}$. Notice that M is an integer bounded by a constant depending on N and m. By taking any $s_1, ..., s_{M-1} \in U$ and plugging them into Equality C1, we have

$$\begin{pmatrix} \psi_1(s_1, t, \xi) & \psi_2(s_1, t, \xi) & ... & \psi_M(s_1, t, \xi) \\ \psi_1(s_2, t, \xi) & \psi_2(s_2, t, \xi) & ... & \psi_M(s_2, t, \xi) \\ ... & ... & ... & ... \\ \psi_1(s_{M-1}, t, \xi) & \psi_2(s_{M-1}, t, \xi) & ... & \psi_M(s_{M-1}, t, \xi) \\ \psi_1(s, t, \xi) & \psi_2(s, t, \xi) & ... & \psi_M(s, t, \xi) \end{pmatrix} \begin{pmatrix} \phi_1(t, \xi) \\ \phi_2(t, \xi) \\ ... \\ \phi_{M-1}(t, \xi) \\ \phi_M(t, \xi) \end{pmatrix} = 0.$$

Noticing that $(\phi_1(t, \xi), ..., \phi_M(t, \xi))^t$ is a non zero vector, we have

$$\begin{vmatrix} \psi_1(s_1, t, \xi) & \psi_2(s_1, t, \xi) & ... & \psi_M(s_1, t, \xi) \\ \psi_1(s_2, t, \xi) & \psi_2(s_2, t, \xi) & ... & \psi_M(s_2, t, \xi) \\ ... & ... & ... & ... \\ \psi_1(s_{(M-1)}, t, \xi) & \psi_2(s_{(M-1)}, t, \xi) & ... & \psi_M(s_{(M-1)}, t, \xi) \\ \psi_1(s, t, \xi) & \psi_2(s, t, \xi) & ... & \psi_M(s, t, \xi) \end{vmatrix} \equiv 0,$$

where $s_1, ..., s_{M-1}$ are arbitrary numbers in U. By the Laplacian expansion of the determinant along the last row, we have

$$\sum_{i=1}^M \Psi_i(s_1, s_2, ..., s_{M-1}, t, \xi) \psi_i(s, t, \xi) \equiv 0,$$

where $\Psi_i(s_1, s_2, ..., s_{M-1}, t, \xi)$ are the $(M-1) \times (M-1)$ minors. Since for any fixed $s_1, ..., s_{M-1} \in U$, $\psi_i(s_j, t, \xi)$ is an algebraic function of t and ξ with the degree bounded by N. By Lemma 4.3 and Lemma 4.4, we conclude that Ψ_i is an algebraic function of t and ξ with degree bounded by a constant depending only on N and m.

Suppose that for certain $s_1, s_2, ..., s_{M-1} \in U$ at least one of Ψ_i is not identically zero as a function of t, ξ. Then by a similar argument as in the proof of the classical Hurwitz Theorem, we will conclude that at least one of $\Psi_i(s_1, s_2, ..., s_{M-1}, t, \xi)$ is not identically zero and the corresponding $\psi_i(s, t, \xi)$ takes the form $\psi_i(s, t, \xi) = s^I \widetilde{F}(s, t, \xi)^K_{|K| \geq 1}$. Because otherwise we get a nontrivial relation

$$\sum_{i=1}^{M'} \Psi_i(s_1, s_2, ..., s_{M-1}, t, \xi) s^I = 0,$$

which is impossible. Hence we have

$$\sum_{i=1}^{M} \Psi_i(s_1, s_2, ..., s_{M-1}, t, \xi) \psi_i(s, t, \xi) = \sum_{|I|+|K| \leq N} f_{IK}(s_1, s_2, ..., s_{M-1}, t, \xi) s^I \widetilde{F}(s, t, \xi)^K = 0.$$

Here $f_{IK}(s_1, s_2, ..., s_{M-1}, t, \xi)$ is another notation for the quantity $\Psi_i(s_1, s_2, ... s_{M-1}, t, \xi)$. Moreover notice that $f_{IK}(s_1, s_2, ..., s_{M-1}, t, \xi)$ is an algebraic function with the degree bounded above by a constant depending only on N and m and at least one of f_{IK} with $|K| \geq 1$ is not identically zero as a function of t and ξ.

Since f_{IK} is an algebraic function with bounded total degree, we can find a minimal polynomials with degree bounded by a constant L depending only on N and m as follows:

$$G_{IK}(t, \xi, Y) = \sum_{Q \leq L} A_{IK,Q}(t, \xi) Y^Q, \quad G_{IK}(t, \xi, f_{IK}(t, \xi)) \equiv 0 \quad \text{and} \quad \text{degree } G < L.$$

Moreover we denote all the solutions of $G_{IK} = 0$ by $f_{IK}^1 = f_{IK}, f_{IK}^2, ..., f_{IK}^{n_{IK}}$ and define

$$\mathcal{F}(s, t, \xi, X) = \prod_{\substack{i_{IK}=1 \\ |I|+K \leq N}}^{n_{IK}} \cdots \prod_{\substack{i_{I'K'}=1 \\ |I'|+K' \leq N}}^{n_{I'K'}} \sum_{|I|+K \leq N} f_{IK}^{i_{IK}}(t, \xi) s^I X^K.$$

Notice that $\mathcal{F}(s, t, \widetilde{F}(s, t, \xi)) \equiv 0$. Since coefficients of X in $\mathcal{F}$ are symmetric polynomials of $\{f_{IK}^i\}_{i=1}^{n_{IK}}$ for each pair IK, $\mathcal{F}$ is a polynomial of s, X and $\frac{A_{IK,Q}}{A_{IK,0}}$ for all IK and Q. Multiplying $\mathcal{F}$ by certain power $\mathcal{P}$ of $A_{IK,0}$, we conclude that $\mathcal{F} \times A_{IK,0}^{\mathcal{P}}$ is a

polynomial of s, t, ξ and X. Moreover noticing that $\mathcal{P}$ is bounded above by a constant depending on N and m, we complete the proof of the theorem. $\blacksquare$

Next we prove the algebraic inverse function theorem with a controlled total degree, which will be use in Chapter 6.

Lemma 4.7. *Suppose* $P_1(z_1, z_2, ..., z_m, \xi_1, \xi_2, ..., \xi_n), ..., P_n(z_1, z_2, ..., z_m, \xi_1, \xi_2, ..., \xi_n)$ *are algebraic functions with total degrees bounded from above by* N *which are holomorphic in a neighborhood* $U \times V \subset \mathbb{C}^m \times \mathbb{C}^n$ *of* $A_0 = (z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_n^0)$. *Suppose that*

$$
\begin{cases}
P_1(z_1, z_2, ..., z_m, \xi_1, ..., \xi_n) = 0 \\
P_2(z_1, z_2, ..., z_m, \xi_1, ..., \xi_n) = 0 \\
... \\
P_n(z_1, z_2, ..., z_m, \xi_1, ..., \xi_n) = 0
\end{cases}
$$

has a solution at $A_0 = (z^0, \xi^0) = (z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_n^0)$ *and* $\frac{\partial(P_1, P_2, ..., P_n)}{\partial(\xi_1, \xi_2, ..., \xi_n)}(z_1^0, z_2^0, ..., z_m^0, \xi_1^0, ..., \xi_n^0) \neq 0$. *Then we can solve* $\xi_1 = \phi_1(z_1, z_2, ..., z_m), \xi_2 = \phi_2(z_1, z_2, ..., z_m), ..., \xi_n = \phi_n(z_1, z_2, ..., z_m)$ *with* $\phi_j(z^0) = \xi^0$ *in a neighborhood of* $z^0 \in \tilde{U} \subset U \subset \mathbb{C}^m$, *where* $\phi_1, ..., \phi_n$ *are algebraic functions with total degree bounded by a constant only depending on* N, n *and* m.

Proof : We will prove it by induction on n. If $n = 1$, we have $P_1(z_1, ..., z_m, \xi_1) = 0$, $\frac{\partial P_1}{\partial \xi_1} \neq 0$ and $P_1(z_1^0, ..., z_m^0, \xi_1^0) = 0$. By holomorphic implicit function theorem, we get a holomorphic function $\xi_1 = \phi_1(z_1, ..., z_m)$ such that $P_1(z_1, ..., z_m, \phi_1(z_1, ..., z_m)) \equiv 0$ in a certain open neighborhood of (z^0, ξ^0). Notice that P_1 is an algebraic function defined on $z_1, ..., z_m \in U \subset \mathbb{C}^m$ and $\xi_1 \in V \subset \mathbb{C}$ with total degree bounded by N. Take a minimal polynomial of P_1 as follows:

$$
\mathcal{F}_1(z_1, ..., z_m, \xi_1, X) = \sum_{I, j, k, |I|+j+k \leq N} A_{Ijk} z^I \xi_1^j X^k.
$$

Since $\mathcal{F}_1$ is minimal, its coefficient of the zero-th order term with respect to X is nonzero. Write this coefficient as $f_1(z, \xi_1) = \sum_{I, j, |I|+j \leq N} A_{Ij0} z^I \xi^j$. Plugging $\xi_1 = \phi_1(z)$ into $\mathcal{F}_1(z, \xi_1, P_1(z, \xi_1)) \equiv 0$, we get

$$
0 \equiv \mathcal{F}_1(z, \phi_1(z), P_1(z, \phi_1(z))) = \sum_{I, j, k, |I|+j+k \leq N} A_{Ijk} z^I \phi_1(z)^j P(z, \phi_1(z))^k = f_1(z, \phi_1(z)).
$$

Hence f_1 is an annihilator of $\phi_1(z)$. Moreover it is easy to see that the total degree of ϕ_1 is bounded by N.

Suppose the theorem is true for $n = 1, ..., k-1, k \geq 2$, we proceed to prove that it is true for $n = k$. Since $\frac{\partial(P_1, P_2, ..., P_k)}{\partial(\xi_1, \xi_2, ..., \xi_k)}(z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_k^0) \neq 0$, we can choose $i, j \in \{1, 2, ..., k\}$ such that $\frac{\partial P_i}{\partial \xi_j}(z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_k^0) \neq 0$. Without loss of generality we can assume $\frac{\partial P_k}{\partial \xi_k}(z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_k^0) \neq 0$. By implicit function theorem for holomorphic functions, we get a holomorphic function $\xi_k = \phi_k(z_1, z_2, ..., z_m, \xi_1, ..., \xi_{k-1})$ such that $P_k(z_1, z_2, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_k(z_1, z_2, ..., z_m, \xi_1, ..., \xi_{k-1})) \equiv 0$ in a certain open neighborhood.

We now proceed to prove that ϕ_k is an algebraic function of $(z_1, z_2, ..., z_m, \xi_1, ..., \xi_{k-1})$ with total degree bounded by N by a similar argument used above.

Noticing that $P_k(z_1, ..., z_m, \xi_1, ..., \xi_k)$ is an algebraic function with total degree bounded by N, we take a minimal polynomial of P_k as follows:

$$\mathcal{F}_k(z_1, ..., z_m, \xi_1, ..., \xi_k, X) = \sum_{I,J,K, |I|+|J|+K \leq N} A_{IJK} z^I \xi^J X^K.$$

Since $\mathcal{F}$ is minimal, its coefficient of the zero-th order term is nonzero. Write the zero-th order term as $f_n(z_1, ..., z_m, \xi_1, ..., \xi_k) = \sum_{I,J, |I|+|J| \leq N} A_{IJ0} z^I \xi^J$. Plugging $\xi_k = \phi_k(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})$ into $\mathcal{F}$, we get

$$0 \equiv \mathcal{F}(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_k(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}), P(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_k(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})))$$

$$= \sum_{|I|+|J|+K \leq N} A_{IJK} z^I \xi_1^{j_1} \cdots \xi_{k-1}^{j_{k-1}} \phi_k(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})^{j_k} P_n(z_1, z_2, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_k)^K$$

$$= \sum_{I,J, |I|+|J| \leq N} A_{IJ0} z^I \xi_1^{j_1} \cdots \xi_{k-1}^{j_{k-1}} \phi_k^{j_k} = f_k(z_1, ..., z_k, \xi_1, ... \xi_{k-1}, \phi_k).$$

Therefore $f_k(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}, X)$ annihilates ϕ_k and its degree is bounded above by N. As a direct consequence ξ_k is an algebraic function with total degree bounded by N.

Now substitute $\xi_k = \phi_n(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})$ into the remaining equations

$$
\begin{cases}
\widetilde{P}_1 = P_1(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_k(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})) = 0 \\
\widetilde{P}_2 = P_2(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_n(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})) = 0 \\
\widetilde{P}_3 = P_3(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_n(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})) = 0 \\
... \\
\widetilde{P}_{k-1} = P_{k-1}(z_1, ..., z_m, \xi_1, ..., \xi_{k-1}, \phi_k(z_1, ..., z_m, \xi_1, ..., \xi_{k-1})) = 0
\end{cases}
$$

By Lemma 4.3 and Lemma 4.4, it is easy to see that $\widetilde{P}_1, ..., \widetilde{P}_{n-1}$ have degrees bounded by a constant $C(N, n, m)$ depending only on N, n and m. Moreover $\widetilde{A}_0 = (z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_{k-1}^0)$ is a solution for the new system of equations. Computing the Jacobian $\frac{\partial(\widetilde{P}_1, ..., \widetilde{P}_{k-1})}{\partial(\xi_1, ..., \xi_{k-1})}$ at the point $\widetilde{A}_0 = (z_1^0, ..., z_m^0, \xi_1^0, ..., \xi_{k-1}^0)$, we have

$$
\begin{vmatrix}
\frac{\partial \widetilde{P}_1}{\partial \xi_1} & \frac{\partial \widetilde{P}_1}{\partial \xi_2} & \cdots & \frac{\partial \widetilde{P}_1}{\partial \xi_{k-1}} \\
\frac{\partial \widetilde{P}_2}{\partial \xi_1} & \frac{\partial \widetilde{P}_2}{\partial \xi_2} & \cdots & \frac{\partial \widetilde{P}_2}{\partial \xi_{k-1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \widetilde{P}_{k-1}}{\partial \xi_1} & \frac{\partial \widetilde{P}_{k-1}}{\partial \xi_2} & \cdots & \frac{\partial \widetilde{P}_{k-1}}{\partial \xi_{k-1}}
\end{vmatrix}
=
\begin{vmatrix}
\frac{\partial P_1}{\partial \xi_1} + \frac{\partial P_1}{\partial \xi_k}\frac{\partial \phi_k}{\partial \xi_1} & \frac{\partial P_1}{\partial \xi_2} + \frac{\partial P_1}{\partial \xi_k}\frac{\partial \phi_k}{\partial \xi_2} & \cdots & \frac{\partial P_1}{\partial \xi_{k-1}} + \frac{\partial P_1}{\xi_k}\frac{\partial \phi_k}{\partial \xi_{k-1}} \\
\frac{\partial P_2}{\partial \xi_1} + \frac{\partial P_2}{\partial \xi_k}\frac{\partial \phi_k}{\partial \xi_1} & \frac{\partial P_2}{\partial \xi_2} + \frac{\partial P_2}{\partial \xi_k}\frac{\partial \phi_k}{\partial \xi_2} & \cdots & \frac{\partial P_2}{\partial \xi_{k-1}} + \frac{\partial P_2}{\partial \xi_{k-1}}\frac{\partial \phi_k}{\partial \xi_{k-1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial P_{k-1}}{\partial \xi_1} + \frac{\partial P_{k-1}}{\partial \xi_k}\frac{\partial \phi_k}{\partial \xi_1} & \frac{\partial P_{k-1}}{\partial \xi_2} + \frac{\partial P_{k-1}}{\partial \xi_k}\frac{\partial \phi_k}{\partial \xi_2} & \cdots & \frac{\partial P_{k-1}}{\partial \xi_{k-1}} + \frac{\partial P_{k-1}}{\partial \xi_{k-1}}\frac{\partial \phi_k}{\partial \xi_{k-1}}
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
\frac{\partial P_1}{\partial \xi_1} - \frac{\partial P_1}{\partial \xi_k}\frac{\frac{\partial P_k}{\partial \xi_1}}{\frac{\partial P_k}{\partial \xi_k}} & \frac{\partial P_1}{\partial \xi_2} - \frac{\partial P_1}{\partial \xi_k}\frac{\frac{\partial P_k}{\partial \xi_2}}{\frac{\partial P_k}{\partial \xi_k}} & \cdots & \frac{\partial P_1}{\partial \xi_{k-1}} - \frac{\partial P_1}{\xi_k}\frac{\frac{\partial P_k}{\partial \xi_{k-1}}}{\frac{\partial P_k}{\partial \xi_k}} \\
\frac{\partial P_2}{\partial \xi_1} - \frac{\partial P_2}{\partial \xi_k}\frac{\frac{\partial P_k}{\partial \xi_1}}{\frac{\partial P_k}{\partial \xi_k}} & \frac{\partial P_2}{\partial \xi_2} - \frac{\partial P_2}{\partial \xi_k}\frac{\frac{\partial P_k}{\partial \xi_2}}{\frac{\partial P_k}{\partial \xi_k}} & \cdots & \frac{\partial P_2}{\partial \xi_{k-1}} - \frac{\partial P_2}{\partial \xi_{k-1}}\frac{\frac{\partial P_k}{\partial \xi_{k-1}}}{\frac{\partial P_k}{\partial \xi_k}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial P_{k-1}}{\partial \xi_1} - \frac{\partial P_{k-1}}{\partial \xi_k}\frac{\frac{\partial P_k}{\partial \xi_1}}{\frac{\partial P_k}{\partial \xi_k}} & \frac{\partial P_{k-1}}{\partial \xi_2} - \frac{\partial P_{k-1}}{\partial \xi_k}\frac{\frac{\partial P_k}{\partial \xi_2}}{\frac{\partial P_k}{\partial \xi_k}} & \cdots & \frac{\partial P_{k-1}}{\partial \xi_{k-1}} - \frac{\partial P_{k-1}}{\partial \xi_{k-1}}\frac{\frac{\partial P_k}{\partial \xi_{k-1}}}{\frac{\partial P_k}{\partial \xi_k}}
\end{vmatrix}
= \frac{1}{\frac{\partial P_k}{\partial \xi_k}}
\begin{vmatrix}
\frac{\partial P_1}{\partial \xi_1} & \frac{\partial P_1}{\partial \xi_2} & \cdots & \frac{\partial P_1}{\partial \xi_{k-1}} \\
\frac{\partial P_2}{\partial \xi_1} & \frac{\partial P_2}{\partial \xi_2} & \cdots & \frac{\partial P_2}{\partial \xi_{k-1}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial P_{k-1}}{\partial \xi_1} & \frac{\partial P_{k-1}}{\partial \xi_2} & \cdots & \frac{\partial P_{k-1}}{\partial \xi_{k-1}} \\
\frac{\partial P_k}{\partial \xi_1} & \frac{\partial P_k}{\partial \xi_2} & \cdots & \frac{\partial P_k}{\partial \xi_k}
\end{vmatrix},
$$

which is nonzero at $\widetilde{A}_0$. By induction, we complete the proof. ∎

4.2   An algebraicity lemma

We use the notations we have set up so far. We now proceed to the proof Theorem 4.1 under the hypothesis that Propositions (I)-(III) hold.

Denote by $J_f(z)$ the determinant of the complex Jacobian matrix of a holomorphic map $f : B \to \mathbb{C}^n$, where $B \subset \mathbb{C}^n$ is an open subset and $z = (z_1, \cdots, z_n) \in B$. For

any holomorphic map $g(\xi)$ from an open subset of $\mathbb{C}^n$ to $\mathbb{C}^m$, where $\xi \in \mathbb{C}^n$, we define $\overline{g}(\xi) := \overline{g(\overline{\xi})}$.

Now from (4.1)(4.2)(4.3)(4.4), we obtain

$$\sum_{j=1}^{m} \lambda_j \frac{|J_{F_j}(z)|^2}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\overline{F}_j(\overline{z})))^\lambda} = \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\overline{z}))^\lambda}, \quad z = (z_1, ..., z_n) \in U. \tag{4.7}$$

Recall that $F_j = (F_{j,1}, F_{j,2}, ..., F_{j,n}), j = 1, ..., n$. Complexifying (4.7), we have

$$\sum_{j=1}^{m} \lambda_j \frac{J_{F_j}(z)\overline{J_{F_j}}(\xi)}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\overline{F}_j(\xi)))^\lambda} = \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda}, \quad (z, \xi) \in U \times \text{conj}(U). \tag{4.8}$$

Here $\text{conj}(U) =: \{z : \overline{z} \in U\}$. Using the transitive action of the holomorphic isometric group of (M, ω) on M, we assume that $0 \in U$, $F_j(0) = 0 \in \mathcal{A}$ and $J_{F_j}(0) \neq 0$ for each j. Also, letting $U = B_r(0)$ for a sufficiently small $r > 0$, we have $\text{conj}(U) = U$. Hence, we will assume that (4.8) holds for $(z, \xi) \in U \times U$.

We will need the following algebraicity lemma.

Lemma 4.8. *Let $F_j's$ be as in Theorem 4.1. Then there exist Nash algebraic maps*

$$\widehat{F}_1(z, X_1, ..., X_m), ..., \widehat{F}_m(z, X_1, ..., X_m)$$

holomorphic in $(z, X_1, ..., X_m)$ near $(0, \overline{J_{F_1}}(0), ..., \overline{J_{F_m}}(0)) \in \mathbb{C}^n \times \mathbb{C}^m$ such that

$$\overline{F}_j(z) = \widehat{F}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_m}}(z)), j = 1, ..., m \tag{4.9}$$

for $z = (z_1, ..., z_n)$ near 0.

Proof of Lemma 4.8: Recall that $\psi_i = \kappa_i z_i$, where $\kappa_i = 1$ or $\sqrt{2}$, for $i = 1, \cdots, n$ and $\psi_i = O(|z|^2)$ is a polynomial of z for each $n + 1 \leq i \leq N$. We obtain from (4.8) the following:

$$\sum_{j=1}^{m} \lambda_j \Big(J_{F_j}(z)\overline{J_{F_j}}(\xi) - \lambda\Big(\sum_{i=1}^{n}(J_{F_j}(z)\kappa_i F_{j,i}(z))(\overline{J_{F_j}}(\xi)\kappa_i \overline{F}_{j,i}(\xi))\Big) + P_j(z, \overline{F}_j(\xi), \overline{J_{F_j}}(\xi))\Big)$$

$$= \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda}. \tag{4.10}$$

Here each

$$P_j(z, \overline{F_j}(\xi), \overline{J_{F_j}}(\xi))$$

is a rational function in $z, \overline{F_j}(\xi)$ and $\overline{J_{F_j}}(\xi)$.

We now set $X_j = J_{F_j}, 1 \leq j \leq m$. Set $Y_j, 1 \leq j \leq m$, to be the vectors:

$$Y_j = (Y_{j1}, ..., Y_{jn}) := (\kappa_1 J_{F_j} F_{j,1}, ..., \kappa_n J_{F_j} F_{j,n}).$$

Then equation (4.10) can be rewritten as

$$\sum_{j=1}^{m} \lambda_j \left(X_j(z)\overline{X}_j(\xi) - \lambda Y_j(z) \cdot \overline{Y}_j(\xi) + Q_j(z, \overline{X}_j(\xi), \overline{Y}_j(\xi)) \right) = \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda} \tag{4.11}$$

over $U \times U$. Here each Q_j with $1 \leq j \leq m$ is rational in $\overline{X}_j, \overline{Y}_j$. Moreover, each $Q_j, 1 \leq j \leq m$, has no terms of the form $\overline{X}_j^k \overline{Y}_{js}^l$ with $l \leq 1$ for any $s \geq 1$ in its Taylor expansion at $(\overline{X_j}(0), \overline{Y_j}(0))$.

We write $D^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}...\partial z_n^{\alpha_n}}$ for an $n-$multiindex $\alpha = (\alpha_1, ..., \alpha_n)$. Taking differentiation in (4.11), we obtain, for each multiindex α, the following:

$$\sum_{j=1}^{m} \left((D^\alpha X_j(z))\overline{X}_j(\xi) - \lambda(D^\alpha Y_j(z)) \cdot \overline{Y}_j(\xi) + D^\alpha Q_j(z, \overline{X}_j(\xi), \overline{Y}_j(\xi)) \right)$$

$$= D^\alpha \left(\frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda} \right).$$

Again each $D^\alpha Q_j, 1 \leq j \leq m$, is rational in $(\overline{X}_j, \overline{Y}_j)$ and has no terms of the form $\overline{X}_j^k \overline{Y}_{js}^l$ with $l \leq 1$ and $s \geq 1$ in its Taylor expansion at $(\overline{X_j}(0), \overline{Y_j}(0))$. Applying a similar argument as in [Proposition 3.1, [HY1]], we can algebraically solve for $\overline{F_j}$ to complete the proof of the lemma. ■

Let $\mathcal{R}$ be the field of rational functions in $z = (z_1, ..., z_n)$. Consider the field extension

$$\mathcal{E} = \mathcal{R}(\overline{J_{F_1}}(z), ..., \overline{J_{F_m}}(z)).$$

Let K be the transcendental degree of the field extension $\mathcal{E}/\mathcal{R}$. If $K = 0$, then each of $\{\overline{J_{F_1}}, ..., \overline{J_{F_m}}\}$ is Nash algebraic. As a consequence of Lemma 4.8, each F_j with $1 \leq j \leq m$ is Nash algebraic. Otherwise, by re-ordering the indices if necessary, we let $\mathcal{G} = \{\overline{J_{F_1}}, ..., \overline{J_{F_K}}\}$ be the maximal algebraic independent subset of $\{\overline{J_{F_1}}, ..., \overline{J_{F_m}}\}$. It

follows that the transcendental degree of $\mathcal{E}/\mathcal{R}(\mathcal{G})$ is zero. For any $l > K$, there exists a minimal polynomial $P_l(z, X_1, ..., X_K, X)$ such that $P_l(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z), \overline{J_{F_l}}(z)) \equiv 0$. Moreover,

$$\frac{\partial P_l(z, X_1, ..., X_K, X)}{\partial X}(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z), \overline{J_{F_l}}(z)) \not\equiv 0$$

in a small neighborhood V of 0, for otherwise, P_l cannot be a minimal polynomial of $\overline{J_{F_l}}(z)$. Now the union of the vanishing set of the partial derivative with respect to X in the above equation for each l forms a proper local complex analytic variety near 0. Applying the algebraic version of the implicit function theorem, there exists a small connected open subset $U_0 \subset U$, with $0 \in \overline{U}_0$ and a holomorphic algebraic function $\widehat{h}_l, l > K$, in a certain neighborhood $\widehat{U}_0$ of $\{(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)) : z \in U_0\}$ in $\mathbb{C}^n \times \mathbb{C}^K$, such that

$$\overline{J_{F_l}}(z) = \widehat{h}_l(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)),$$

for any $z \in U_0$. (We can assume here U_0 is the projection of $\widehat{U}_0$). Substitute this into

$$\widehat{F}_i(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_m}}(z)),$$

and still denote it, for simplicity of notation, by $\widehat{F}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z))$ with

$$\widehat{F}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)) = \widehat{F}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_m}}(z)) \text{ for } z \in U_0.$$

In the following, for simplicity of notation, we also write for $j \leq K$,

$$\widehat{h}_j(z, \overline{J_{F_1}}(z), ..., \overline{J_{F_K}}(z)) = \overline{J_{F_j}}(z) \text{ or } \widehat{h}_j(z, X_1, ..., X_K) = X_j.$$

Now we replace $\overline{F_j}(\xi)$ by $\widehat{F}_j(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi))$, and replace $\overline{J_{F_j}}(\xi)$ by $\widehat{h}_j(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi))$, for $1 \leq j \leq m$, in (4.8). Furthermore, we write $X = (X_1, ..., X_K)$, and replace $\overline{J_{F_j}}(\xi)$ by X_j for $1 \leq j \leq K$ in

$$\widehat{F}_j(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi)), \widehat{h}_j(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi)), 1 \leq j \leq m.$$

We define a new function Φ as follows:

$$\Phi(z, \xi, X) := \sum_{j=1}^{m} \lambda_j \frac{J_{F_j}(z)\widehat{h}_j(\xi, X)}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\widehat{F}_j(\xi, X)))^\lambda} - \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^\lambda}.$$

(4.12)

Lemma 4.9. *Shrinking U if necessary, we have $\Phi(z, \xi, X) \equiv 0$, i.e.,*

$$\sum_{j=1}^{m} \lambda_j \frac{J_{F_j}(z)\widehat{h}_j(\xi, X)}{(1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\widehat{F}_j(\xi, X)))^{\lambda}} = \frac{1}{(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^{\lambda}}. \tag{4.13}$$

or,

$$(1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi))^{\lambda} \sum_{j=1}^{m} \left(\lambda_j J_{F_j}(z)\widehat{h}_j(\xi, X) \prod_{1 \le k \le m, k \ne j} (1 + \sum_{i=1}^{N} \psi_i(F_k(z))\psi_i(\widehat{F}_k(\xi, X)))^{\lambda} \right)$$

$$= \prod_{1 \le j \le m} (1 + \sum_{i=1}^{N} \psi_i(F_j(z))\psi_i(\widehat{F}_j(\xi, X)))^{\lambda} \tag{4.14}$$

for $z \in U$ and $(\xi, X) \in \widehat{U}_0$.

Proof of Lemma 4.9: Suppose not. Notice Φ is Nash algebraic in (ξ, X) for each fixed $z \in U$, by Lemma 4.8. For a generic fixed $z = z_0$ near 0, since $\Phi(z, \xi, X) \not\equiv 0$, there exist polynomials $A_l(\xi, X)$ for $0 \le l \le N$ with $A_0(\xi, X) \not\equiv 0$ such that

$$\sum_{0 \le l \le N} A_l(\xi, X)\Phi^l(z, \xi, X) \equiv 0.$$

As $\Phi(z_0, \xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi)) \equiv 0$ for $\xi \in U_0$, then it follows that $A_0(\xi, \overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi)) \equiv 0$ for $\xi \in U_0$. This is a contradiction to the assumption that $\{\overline{J_{F_1}}(\xi), ..., \overline{J_{F_K}}(\xi)\}$ is an algebraic independent set. ∎

Now that $\widehat{F}_j(\xi, X), 1 \le j \le m$, is algebraic in its variables, if $\widehat{F}_j, 1 \le j \le m$, is independent of X, then F_j is algebraic by Lemma 4.8. This fact motivates the remaining work in this chapter.

4.3 Algebraicity and rationality with uniformly bounded degree

In this section, we prove the algebraicity and rationality for at least one of the $F_j's$. We start with the following:

Lemma 4.10. *Let $F_j(z), j \in \{1, ..., m\}$, be a local holomorphic map defined on a neighborhood of $0 \in U$ as in (4.8). Suppose that there exist $z^0 \in U$ and $\xi^0 \in Q_{z^0}$ such that $\Lambda(\beta^1, ..., \beta^N)(z^0, \xi^0)$ is well defined and non-zero with $\beta^1 = (0, 0, ..., 0)$. Then there is an analytic variety $W \subsetneq U$ such that when $z \in U \backslash W$, $\Lambda(\beta^1, ..., \beta^N)(z, \xi)$ is a rational function in ξ over Q_z and $\Lambda(\beta^1, ..., \beta^N)(z, \xi) \not\equiv 0$ on Q_z.*

Proof of Lemma 4.10: By the assumption, $\frac{\partial \rho}{\partial z_n}(z_0, \xi_0) \neq 0$ and

$$\Lambda(\beta^1, ..., \beta^N)(z^0, \xi^0) = \begin{vmatrix} \mathcal{L}^{\beta^1} \mathcal{F}_{j,1} & ... & \mathcal{L}^{\beta^1} \mathcal{F}_{j,N} \\ ... & ... & ... \\ \mathcal{L}^{\beta^N} \mathcal{F}_{j,1} & ... & \mathcal{L}^{\beta^N} \mathcal{F}_{j,N} \end{vmatrix} (z^0, \xi^0) \qquad (4.15)$$

is non-zero with $\beta^1 = (0, 0, ..., 0)$.

By the definition, $\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z,\xi)}{\frac{\partial \rho}{\partial z_n}(z,\xi)} \frac{\partial}{\partial z_n}$ and $\mathcal{L}^{\beta^l} = \mathcal{L}_1^{k_1^l} \mathcal{L}_2^{k_2^l} \mathcal{L}_3^{k_3^l} ... \mathcal{L}_{n-1}^{k_{n-1}^l}$ for $\beta^l = (k_1^l, ..., k_{n-1}^l)$, $k_1^l, ..., k_{n-1}^l$. Hence $\Lambda(\beta^1, ..., \beta^N)(z, \xi)$ can be written in the form $\Lambda(\beta^1, ..., \beta^N)(z, \xi) = \frac{\mathcal{G}_1(z,\xi)}{\mathcal{G}_2(z,\xi)}$. Here $\mathcal{G}_1(z, \xi) = \sum_{|I|=0}^{M_1} \Phi_I(z)\xi^I$, $\mathcal{G}_2(z, \xi) = \sum_{|J|=0}^{M_2} \Psi_J(z)\xi^J$, with Φ_I and Ψ_J being holomorphic functions defined over $U \subset \mathbb{C}^n$. In fact, $\mathcal{G}_2(z, \xi)$ is simply taken as a certain sufficiently large power of $\rho_{z_n} := \frac{\partial \rho}{\partial z_n}$.

By our assumption, we have $\mathcal{G}_1, \mathcal{G}_2$ not equal to zero at (z^0, ξ^0). Hence, $\mathcal{G}_1, \mathcal{G}_2$ are not zero elements in $\mathcal{O}(U)[\xi_1, ..., \xi_n]$, the polynomial ring of ξ with coefficients from the holomorphic function space over U.

By Proposition (III), the defining function of the Segre family ρ can be written in the form $\rho(z, \xi) = \sum_{|\alpha|=0}^{M_3} \Theta_k(z)\xi^\alpha$, which is an irreducible polynomial in (z, ξ). And for each fixed z, by Proposition (III), we also have $\rho(z, \xi)$ irreducible as a polynomial of ξ only.

Then the set of $z \in U$ where $\Lambda(\beta^1, ..., \beta^N)(z, \xi)$ is undefined over Q_z is a subset of $z \in U$ where $\mathcal{G}_2(z, \xi)$, as a polynomial of ξ, contains the factor $\rho(z, \xi)$ as a polynomial in ξ. We denote the latter set by W_2. Similarly, the set of $z \in U$ with $\Lambda(\beta^1, ..., \beta^N)(z, \xi) \equiv 0$ over Q_z is a subset of $z \in U$ where $\mathcal{G}_1(z, \xi)$, as a polynomial of ξ, contains a factor $\rho(z, \xi)$, which we denote by W_1.

Notice that $\rho(z, \xi) \in \mathcal{O}(U)[\xi_1, ..., \xi_n]$ depends on each ξ_j for $1 \leq j \leq n$. Also notice that $\mathcal{G}_2(z, \xi)$, as a certain power of $\rho_{z_n}(z, \xi)$, depends on ξ_n.

We next characterize W_2 by the resultant R_2 of $\mathcal{G}_2(z, \xi)$ and $\rho(z, \xi)$ as polynomials in ξ_n. We rewrite $\mathcal{G}_2$ and ρ as polynomials of ξ_n as follows:

$$\mathcal{G}_2 = \sum_{i=0}^{k} a_i(z, \xi_1, ..., \xi_{n-1})\xi_n^i, \quad \rho = \sum_{j=0}^{l} b_j(z, \xi_1, ..., \xi_{n-1})\xi_n^j.$$

Here the leading terms $a_k, b_l \not\equiv 0$ with $k, l \geq 1$. We write the resultant as $R_2(z, \xi_1, ...\xi_{n-1}) = \sum_I c_I(z)\xi'^I$, where $c_I's$ are holomorphic functions of $z \in U$.

For those points $z \in W_2$, $R_2(z, \cdot) \equiv 0$ as a polynomial of $\xi_1, ..., \xi_{n-1}$. Then W_2 is contained in the complex analytic set $\widetilde{W}_2 := \{c_I = 0, \forall I\}$. If $\widetilde{W}_2 = U$, then we can find non-zero polynomials $f, g \in \mathcal{O}(U)[\xi_1, ..., \xi_{n-1}][\xi_n]$ such that $f\rho + g\mathcal{G}_2 \equiv 0$, where the degree of g in ξ_n is less than the degree of ρ in ξ_n. Hence $\{\mathcal{G}_2 = 0\} \cup \{g = 0\} \supset \{\rho = 0\} \cap (U \times \mathbb{C}^n)$. Again by the irreducibility of $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$, since $\{g = 0\}$ is a thin set in $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$, $\mathcal{G}_2$ vanishes on $\{\rho = 0\} \cap (U \times \mathbb{C}^n)$. This contradicts $\mathcal{G}_2(z^0, \xi^0) \neq 0$. Hence $W_2 \subset \widetilde{W}_2$ and $\widetilde{W}_2$ is a proper complex analytic subset of U.

By a similar argument, we can prove that W_1 is contained in $\widetilde{W}_1$ that is also a proper analytic set of U. Let $W = \widetilde{W}_1 \cup \widetilde{W}_2$. Then when $z \in U \backslash W$, $\Lambda(\beta^1, ..., \beta^N)(z, \xi)$ is well-defined over Q_z as a rational function in ξ and $\Lambda(\beta^1, ..., \beta^N)(z, \xi) \not\equiv 0$ on Q_z. ∎

Lemma 4.11. *Let $\psi(\xi, X)$ be a non-zero Nash-algebraic function in $(\xi, X) = (\xi_1, ..., \xi_n, X_1, ..., X_m) \in \mathbb{C}^n \times \mathbb{C}^m$. Write E for a proper complex analytic variety of $\mathbb{C}^n \times \mathbb{C}^m$ that contains the branch locus of ψ and the zeros of the leading coefficient in the minimal polynomial of ψ. Then there exists a proper analytic set W_1 in $\mathbb{C}^n$ such that*

$$\{\xi | \exists X^0, (\xi, X^0) \notin E\} \supset \mathbb{C}^n \backslash W_1.$$

Proof of Lemma 4.11: Since ψ is algebraic, there is an irreducible polynomial $\Phi(\xi, X; Y) = \sum_{i=0}^k \phi_i(\xi, X)Y^i$ such that $\Phi(\xi, X, \psi(\xi, X)) \equiv 0$. If $k = 1$ then ψ is a rational function and thus E is just the poles and points of indeterminancy. The proof is then obvious and we hence assume $k \geq 2$.

Define $\Psi(\xi, X, Y) = \frac{\partial \Phi}{\partial Y}$. Since $k \geq 2$, the degree of Ψ in Y is at least one. Consider Φ, Ψ as polynomials in Y, and write $R(\xi, X)$ for their resultant. Then the branch locus is contained in $\{(\xi, X) | R(\xi, X) = 0\}$. Notice that $R \not\equiv 0$, for Φ is irreducible. Write $R = \sum_I r_I(\xi)X^I$ with some $r_I \neq 0$. Write $\phi_k(\xi, X) = \sum \phi_{k,i}(\xi)X^i$ and $W_1 = \{r_I(\xi) = 0, \forall I\} \cup \{\phi_{k,i}(\xi) = 0, \forall i\}$, which is a proper complex analytic set in $\mathbb{C}^n$. Then $\{\xi | \exists X^0, (\xi, X^0) \notin E\} \supset \mathbb{C}^n \backslash W_1$. ∎

Let E be a proper complex analytic variety containing the union of the branch loci of $\widehat{h}_j, \widehat{F}_j$ for $j = 1, \cdots, m$ and the zeros of the leading coefficients in their minimal polynomials. For any point $(z^0, \xi^0, X^0) \in U \times ((\mathbb{C}^n \times \mathbb{C}^K) \backslash E)$, we can find a smooth

Jordan curve γ in $U \times ((\mathbb{C}^n \times \mathbb{C}^K) \backslash E)$ connecting (z^0, ξ^0, X^0) with a certain point in $U \times (\widehat{U}_0 \setminus E)$. We can holomorphically continue the following equation along γ:

$$(\rho(z,\xi))^\lambda \sum_{j=1}^{m} \left(\lambda_j J_{F_j}(z) \widehat{h}_j(\xi, X) \prod_{1 \le k \le m, k \ne j} (1 + \sum_{i=1}^{N} \psi_i(F_k(z)) \psi_i(\widehat{F}_k(\xi, X)))^\lambda \right)$$
$$= \prod_{1 \le j \le m} (1 + \sum_{i=1}^{N} \psi_i(F_j(z)) \psi_i(\widehat{F}_j(\xi, X)))^\lambda, \quad z \in U, \ (\xi, X) \in \widehat{U}_0, \tag{4.16}$$

to a neighborhood of (z^0, ξ^0, X^0).

For our later discussions, we further define

$$\mathcal{M}_{\text{sing,z}} = \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}, \mathcal{M}_{\text{reg,z}} = \mathcal{M} \backslash \mathcal{M}_{\text{sing,z}};$$

$$\mathcal{M}_{\text{SING}} = \{(z, \xi) : \frac{\partial \rho}{\partial \xi_j} = 0, \forall j\} \cup \{(z, \xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}, \quad \mathcal{M}_{\text{REG}} = \mathcal{M} \backslash \mathcal{M}_{\text{SING}};$$

$\text{Pr}_z : \mathbb{C}^{2n} \to \mathbb{C}^n \quad (z, \xi) \mapsto (z)$ and $\text{Pr}_\xi : \mathbb{C}^{2n} \to \mathbb{C}^n \quad (z, \xi) \mapsto (\xi)$.

Notice that $\mathcal{M}_{\text{REG}}$ is a Zariski open subset of $\mathcal{M}$ and the restrictions of $\text{Pr}_z, \text{Pr}_\xi$ to $\mathcal{M}_{\text{REG}}$ are open mappings. Also, for $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}}$, Q_{z^0} is smooth at ξ^0, and Q_{ξ^0} is smooth at z^0. By Proposition (III), $\mathcal{M}_{\text{reg,z}} \cap (Q_{\xi^0}, \xi^0)$ is Zariski open in (Q_{ξ^0}, ξ^0).

Lemma 4.12. *With the notations we have set up so far, there exists a point $(z^0, \xi^0, X^0) \in (U \times \mathbb{C}^n \times \mathbb{C}^K)$ with $(z^0, \xi^0) \in \mathcal{M}_{\text{REG}} \cap (U \times \mathbb{C}^n)$ and $(\xi^0, X^0) \notin E$. Moreover, for each $j = 1, ..., m$, we can find $\beta_j^1, ..., \beta_j^N$ with $\beta_j^1 = (0, ..., 0)$ such that $\Lambda_{F_j}(\beta_j^1, ..., \beta_j^N)(z^0, \xi^0) \ne 0$.*

Proof of Lemma 4.12: This is an easy consequence of Propositions (I) (III), Lemma 4.10 and the Zariski openness of $\mathcal{M}_{\text{REG}}$ in $\mathcal{M}$. ∎

Let (z^0, ξ^0, X^0) be chosen as in Lemma 4.12. We then analytically continue the equation (4.16) to a neighborhood of the point (z^0, ξ^0, X^0) through a Jordan curve γ described above. We denote one of such neighborhoods by $V_1 \times V_2 \times V_3$, where V_1, V_2 and V_3 are chosen to be a small neighborhood of z^0, ξ^0, and X^0, respectively. It is clear, after shrinking V_1, V_2, V_3 if needed, that there exists a $j_0 \in \{1, ..., m\}$ such that

$$1 + \sum_{i=1}^{N} \psi_i(F_{j_0}(z)) \psi_i(\widehat{F_{j_0}}(\xi, X)) = 0, \quad \text{for } (z, \xi) \in \mathcal{M} \cap (V_1 \times V_2), X \in V_3,$$

We next proceed to prove the algbraicity for $F_{j_0}(z)$.

Theorem 4.13. $\widehat{F_{j_0}}(\xi, X)$, *for $\xi \in V_2$, $X \in V_3$, is independent of X and is thus a Nash algebraic function of ξ. Hence F_{j_0} is an algebraic function of z. Moreover, the algebraic total degree of $\widehat{F_{j_0}}(\xi, X) = \overline{F_{j_0}}(\xi)$, and thus of $F_{j_0}(z)$, is uniformly bounded by a constant depending only on the manifold (X, ω) and the described canonical embedding.*

Proof of Theorem 4.13: By the choice of (z^0, ξ^0, X^0), there exist $\beta_{j_0}^1, ..., \beta_{j_0}^N$ such that

$$\Lambda_{F_{j_0}}(\beta_{j_0}^1, ..., \beta_{j_0}^N)(z^0, \xi^0) = \begin{vmatrix} \mathcal{L}^{\beta_{j_0}^1} \mathcal{F}_{j_0,1} & ... & \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,N} \\ ... & ... & ... \\ \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,1} & ... & \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,N} \end{vmatrix} (z^0, \xi^0) \neq 0. \tag{4.17}$$

We can also assume that (z_0, ξ_0) satisfies the assumption in Proposition (II) after a slight perturbation of z_0 inside Q_{ξ_0} if needed. By Proposition (II), we can find $z^1 \in V_1 \cap Q_{\xi^0}$ such that Q_{z^0} intersects Q_{z^1} transversally at ξ^0. Moreover there exists a neighborhood B of ξ^0 and a biholomorphic parametrization of B : $(\xi_1, \xi_2, ..., \xi_n) = \mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n)$ with $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$. Here U_1, U_2 are as in Proposition (II). Moreover, $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}, \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$. Also, for $s \in U_1, t \in U_2$, $\mathcal{G}(\{\tilde{\xi}_0 = t\} \times U_2 \times ... \times U_n), \mathcal{G}(U_1 \times \{\tilde{\xi}_1 = s\} \times U_3 \times ... \times U_n)$ are open pieces of certain Segre varieties. Here $\mathcal{G}$ consists of algebraic functions with total algebraic degree uniformly bounded by M and the canonical embedding.

Consider the equation:

$$1 + \mathcal{F}_{j_0}(z) \cdot \widehat{\mathcal{F}_{j_0}}(\xi, X) = 0, \ (z, \xi, X) \in V_1 \times V_2 \times V_3, (z, \xi) \in \mathcal{M}. \tag{4.18}$$

Since the holomorphic vector fields $\{\mathcal{L}_i\}_{i=1}^{n-1}$ are tangent to the Segre family, we have

$$\begin{pmatrix} \mathcal{L}^{\beta_{j_0}^1} \mathcal{F}_{j_0,1}(z, \xi) & ... & \mathcal{L}^{\beta_{j_0}^1} \mathcal{F}_{j_0,N}(z, \xi) \\ ... & ... & ... \\ \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,1}(z, \xi) & ... & \mathcal{L}^{\beta_{j_0}^N} \mathcal{F}_{j_0,N}(z, \xi) \end{pmatrix} \begin{pmatrix} \widehat{\mathcal{F}_{j_0,1}}(\xi, X) \\ ... \\ \widehat{\mathcal{F}_{j_0,N}}(\xi, X) \end{pmatrix} = \begin{pmatrix} -1 \\ ... \\ 0 \end{pmatrix}, \tag{4.19}$$

where $(z, \xi)(\approx (z^0, \xi^0)) \in \mathcal{M}, \ X \approx X^0$.

By the Cramer's rule, we conclude that $\{\widehat{\mathcal{F}_{j_0,l}}(\xi, X)\}_{l=1}^N$ are rational functions of ξ with a uniformly bounded degree on an open piece of each Segre variety Q_z for $z \approx z^0$.

By the previous modified Hurwitz Theorem (Theorem 4.6), we conclude the algebraicity of $\widehat{\mathcal{F}_{j_0,l}}(\xi, X)$ for $l = 1, ..., N$. Since in (4.19) the matrix $\left(\mathcal{L}^{\beta_{j_0}^\mu} \mathcal{F}_{j_0,\nu}(z,\xi) \right)_{1 \leq \mu, \nu \leq N}$ and the right hand side are independent of X, these functions must also be independent of the X-variables. Moreover, by Lemma 4.3 and Theorem 4.6, the total algebraic degree of $\overline{F}_{j_0,l}(\xi) = \widehat{\mathcal{F}_{j_0,l}}(\xi, X)$, for $l = 1, ..., n$, is uniformly bounded. Since $\overline{F}$ is obtained by holomorphically continuing the conjugation function $\overline{F}$ of F, we conclude the algebraicity of $F_{j_0,l}$ for each $1 \leq l \leq n$. Also the total algebraic degree of each $F_{j_0,l}$ is bounded by a constant depending only on (M, ω). $\blacksquare$

Theorem 4.14. *Under the notations we have just set up, F_{j_0} is a rational map, whose degree depends only on the canonical embedding $M \hookrightarrow \mathbb{CP}^N$.*

For the proof Theorem 4.14, we first recall the following Lemma of [HZ]:

Lemma 4.15. (Lemma 3.7 in [HZ]) *Let $U \subset \mathbb{C}^n$ be a simply connected open subset and $\mathcal{S} \subset U$ be a closed complex analytic subset of codimension one. Then for $p \in U \setminus \mathcal{S}$, the fundamental group $\pi_1(U \setminus \mathcal{S}, p)$ is generated by loops obtained by concatenating (Jordan) paths $\gamma_1, \gamma_2, \gamma_3$, where γ_1 connects p with a point arbitrarily close to a smooth point $q_0 \in \mathcal{S}$, γ_2 is a loop around $\mathcal{S}$ near q_0 and γ_3 is γ_1 reversed.*

Proof of Theorem 4.14: We give a proof for the rationality of F_{j_0}. Once this is done, we then conclude that the degree of F_{j_0} is uniformly bounded, for we know the total algebraic degree of F is uniformly bounded by Theorem 4.13.

Suppose that F_{j_0} and thus $\overline{F_{j_0}}$ is not rational. Write $E \subset \mathbb{C}^n$ for a proper complex analytic variety containing the branch locus of $F_{j_0}, \overline{F_{j_0}}$ and the zeros of the leading coefficients of the minimal polynomials of their components. We first notice that for $A \neq B \in \mathbb{C}^n$, $Q_A^* \neq Q_B^*$, by Lemma 2.1. Hence, for any proper complex analytic variety $V^1, V^2 \subset \mathbb{C}^n$ and any point $(a, b) \in \mathcal{M}$, we can find $(a^1, b^1) \approx (a, b)$ such that $a^1 \in Q_{b^1} \setminus V^1$ and $b^1 \notin V^2$.

We choose (z^0, ξ^0) as above and assume further that $z^0, \xi^0 \notin E$ (after a small perturbation if needed). We choose a sufficiently small neighborhood W of (z^0, ξ^0) in $\mathcal{M}_{\text{REG}}$ such that for each $(z^1, \xi^1) \in W$, we can find, by Lemma 4.15, a loop of the form $\gamma = \gamma_1^{-1} \circ \gamma_2 \circ \gamma_1$ in $\mathbb{C}^n \setminus E$ with $\gamma(0) = \gamma(1) = \xi^1, \gamma_1(1) = q$. Here γ_1 is a simple curve

connecting ξ^1 to q with q in a small ball B_p centered at a certain smooth point p of E such that the fundamental group of $B_p \setminus E$ is generated by γ_2; and γ_1^{-1} is the reverse curve of γ_1. Moreover, when $\overline{F_{j_0}}$ is holomorphically continued along γ, we end up with a different branch $\overline{F_{j_0}}^*$ of $\overline{F_{j_0}}$ near ξ^1. We pick p such that there is an $X_p \notin E$ with $(X_p, p) \in \mathcal{M}_{\text{reg,z}}$. (This follows from Proposition (III) and Lemma 2.1 as mentioned above.) Take a certain small neighborhood $\mathcal{W}$ of (X_p, p) in $\mathcal{M}_{\text{reg,z}}$. We assume, without loss of generality, that the piece $\mathcal{W}$ of $\mathcal{M}_{\text{reg,z}}$ is defined by a holomorphic function of the form $z_1 = \phi(z_2, \cdots, z_n, \xi)$. In particular, writing $X_p = (z_1^p, \cdots, z_n^p)$, we have $z_1^p = \phi(z_2^p, \cdots, z_n^p, p)$. Make B_p sufficiently small such that it is compactly contained in the image of the projection of $\mathcal{W}$ into the ξ-space. Write $X_q = (\phi(z_2^p, \cdots, z_n^p, q), z_2^p, \cdots, z_n^p)$ and define the loop $\gamma_2^*(t) = (\phi(z_2^p, \cdots, z_n^p, \gamma_2(t)), z_2^p, \cdots, z_n^p)$. Then γ_2^* is a loop whose base point is at X_q. Also, we have $(\gamma_2^*(t), \gamma_2(t)) \in \mathcal{M}$.

Notice that $X_p \notin E$. After shrinking B_p if needed, we assume that γ_2^* stays sufficiently close to X_p and is homopotically trivial in $\mathbb{C}^n \setminus E$.

Now we slightly thicken γ_1 to get a simply connected domain U_1 of $\mathbb{C}^n \setminus E$. Since $\mathcal{M}$ is irreducible over $\mathbb{C}^n \times U_1$, we can find a smooth simple curve $\widetilde{\gamma}_1 = (\gamma_1^*, \widehat{\gamma}_1)$ in $\mathcal{M} \setminus ((E \times \mathbb{C}^n) \cup (\mathbb{C}^n \times E))$ connecting (z^1, ξ^1) to (X_q, q). Then $\widehat{\gamma}_1$ is homotopic to γ_1 relatively to $\{\xi^1, q\}$ and $\gamma_1^*(1) = X_q$. Now replace γ by its homopotically equivalent loop $\widehat{\gamma}_1^{-1} \circ \gamma_2 \circ \widehat{\gamma}_1$ and define $\gamma^* = \gamma_1^{*-1} \circ \gamma_2^* \circ \gamma_1^*$. Define $\Gamma = (\gamma^*, \gamma)$. Then the image of Γ lies inside $\mathcal{M} \setminus ((E \times \mathbb{C}^n) \cup (\mathbb{C}^n \times E))$. Continuing Equation (4.18) along Γ and noticing that it is independent of X now, we get both

$$1 + \mathcal{F}_{j_0}(z) \cdot \overline{\mathcal{F}}_{j_0}(\xi) = 0 \text{ and } 1 + \mathcal{F}_{j_0}(z) \cdot \overline{\mathcal{F}}_{j_0}^*(\xi) = 0 \ \forall (z, \xi) \in \mathcal{M} \cap ((V_1 \setminus E) \times (V_2 \setminus E)).$$

Now arguing as before and applying the uniqueness for the solution of the linear system (4.19) (with an invertible coefficient matrix), we then conclude that $\overline{F_{j_0}}^* \equiv \overline{F_{j_0}}$. This is a contradiction. ∎

4.4 Isometric extension of F

For simplicity of notation, in the rest of this section, we denote the map F_{j_0} just by F. Now that all components of F are rational functions, it is easy to verify that F

gives rise to a rational map $M \dashrightarrow M$. By the Hironaka theorem (see [H] and [K]), we have a (connected) complex manifold Y of the same dimension, holomorphic maps $\tau : Y \to M$, $\sigma : Y \to M$, and a proper complex analytic variety E_1 of M such that $\sigma : Y \setminus \sigma^{-1}(E_1) \to M \setminus E_1$ is biholomorphic; $F : M \setminus E_1 \to M$ is well-defined; and for any $p \in Y \setminus \sigma^{-1}(E_1)$, $F(\sigma(p)) = \tau(p)$.

Let E_2 be a proper complex analytic subvariety of M containing E_1, $M \setminus \mathcal{A}$ and let $E_3 \subset Y$ be the proper subvariety where τ fails to be biholomorphic. Write $E^* = \tau(\sigma^{-1}(E_2) \cup E_3) \cup (M \setminus \mathcal{A})$ and $E = \sigma(\tau^{-1}(E^*))$. Then $F : \mathcal{A} \setminus E \to \mathcal{A} \setminus E^*$ is a holomorphic covering map. We first prove

Lemma 4.16. : *Under the above notation, $F : \mathcal{A} \setminus E \to \mathcal{A} \setminus E^*$ is a biholomorphic map.*

Proof of Lemma 4.16: We first notice that since F is biholomorphic near 0 with $F(0) = 0$. We can assume that $0 \notin E$. Consider $F^2 = F \circ F$. Then $\overline{F^2} = \overline{F}^2$. Since $(F, \overline{F})$ maps $\mathcal{M}$ into $\mathcal{M}$ whenever it is defined, it is easy to see that $(F, \overline{F}) \circ (F, \overline{F}) = (F^2, \overline{F}^2)$ also maps $\mathcal{M}$ into $\mathcal{M}$ at the points where it is well-defined. Hence, we can repeat a similar argument for F to conclude that F^2, as a rational map, also has its degree bounded by a constant independent of F^2. Similarly, we can conclude that for any positive integer m, F^m is a rational map with degree bounded by a constant independent of m and F. Now, as for F, we can find complex anaytic subvarieties $E^{(m)}$, $E^{*(m)}$ of $\mathbb{C}^n$ such that F^m is a holomorphic covering map from $\mathcal{A} \setminus E^{(m)} \to \mathcal{A} \setminus E^{*(m)}$. Suppose $F : \mathcal{A} \setminus E \to \mathcal{A} \setminus E^*$ is a k to 1 covering map. It is easy to see that $F^m : \mathcal{A} \setminus E^{(m)} \to \mathcal{A} \setminus E^{*(m)}$ is a k^m to 1 covering map. However, since the degree F^m is independent of m, we conclude that $k = 1$ by the following Bezout theorem:

Theorem 4.17. ([S]) *The number of isolated solutions to a system of polynomial equations*

$$f_1(x_1, ..., x_n) = f_2(x_1, ..., x_n) = ... = f_n(x_1, ..., x_n) = 0$$

is bounded by $d_1 d_2 \cdots d_n$, where $d_i = \deg f_i$.

This proves the lemma. ∎

Now we prove that F extends to a global holomorphic isometry of (M, ω).

Theorem 4.18. $F : (U, \omega|_U) \to (M, \omega)$ *extends to a global holomorphic isometry of* (M, ω).

Proof of Theorem 4.18: By what we just achieved, we then have two proper complex analytic varieties W_1, W_2 of $\mathbb{C}^n$ such that $F : \mathbb{C}^n \backslash W_1 \to \mathbb{C}^n \backslash W_2$ is biholomorphic. Similarly we have two proper complex analytic subvarieties W_1^*, W_2^* of $\mathbb{C}^n$ such that $\overline{F} : \mathbb{C}^n \backslash W_1^* \to \mathbb{C}^n \backslash W_2^*$ is a biholomorphic map. Hence

$$\mathfrak{F} = (F, \overline{F}) : \mathbb{C}^n \backslash W_1 \times \mathbb{C}^n \backslash W_1^* \to \mathbb{C}^n \backslash W_2 \times \mathbb{C}^n \backslash W_2^*$$

is biholomorphic.

Let ρ be the defining function of the Segre family as described before. Since ρ is irreducible as a polynomial in (z, ξ), $\mathcal{M}$ is an irreducible complex analytic variety of $\mathcal{A}$. Since $\mathfrak{F}$ maps a certain open piece of $\mathcal{M}$ into an open piece of $\mathcal{M}$, by the uniqueness of holomorphic functions, we see that $\mathfrak{F} = (F, \overline{F})$ also gives a biholomorphic map from $(\mathbb{C}^n \backslash W_1 \times \mathbb{C}^n \backslash W_1^*) \cap \mathcal{M}$ to $(\mathbb{C}^n \backslash W_2 \times \mathbb{C}^n \backslash W_2^*) \cap \mathcal{M}$. Hence $\rho_F = \rho(F(z), \overline{F}(\xi))$ defines the same subvariety as ρ does over $\mathbb{C}^n \backslash W_1 \times \mathbb{C}^n \backslash W_1^*$. Since ρ_F is a rational function in (z, ξ) with denominator coming from the factors of the denominators of $F(z)$ and $\overline{F}(\xi)$, we can write

$$\rho_F(z, \xi) = (\rho(z, \xi))^l \frac{P_1^{i_1}(z, \xi) P_2^{i_2}(z, \xi) \cdots P_\tau^{i_\tau}(z, \xi)}{Q_1^{j_1}(z) \cdots Q_\mu^{j_\mu}(z) R_1^{k_1}(\xi) \cdots R_\nu^{k_\nu}(\xi)} \tag{4.20}$$

Here the zeros of $Q_j(z)$ and $R_j(\xi)$ stay in W_1 and W_1^*, respectively. All polynomials are irreducible and prime to each other. By what we just mentioned $P_j(z, \xi)$ can not have any zeros in $\mathbb{C}^n \backslash W_1 \times \mathbb{C}^n \backslash W_1^*$, for otherwise it must have ρ as its factor by the irreducibility of ρ. Hence the zeros of $P_j(z, \xi)$ must stay in $(W_1 \times \mathbb{C}^n) \cup (\mathbb{C}^n \times W_1^*)$. From this, it follows easily that $P_j(z, \xi) = P_{j,1}(z)$ or $P_j(z, \xi) = P_{j,2}(\xi)$. Namely, $P_j(z, \xi)$ depends either on z or on ξ. Since $\mathfrak{F}$ is biholomorphic, we see that $l = 1$. Thus replacing ξ by $\bar{z}$ and taking $i\partial\bar{\partial} \log$ to (4.20), we have $i\partial\bar{\partial} \log \rho_F(z, \bar{z}) = i\partial\bar{\partial} \log \rho(z, \bar{z})$. This shows that $F^*(\omega) = \omega$, or F is a local isometry. Now, by the Calabi Theorem (see [Ca]), F extends to a global holomorphic isometry of (M, ω). This proves Theorem 4.18. ∎

We now are ready to give a proof of Theorem 4.1. By what we have obtained, there is a component F_j for F in Theorem 4.1 that extends to a holomorphic isometry to (M, ω). Hence $F_j^*(d\mu) = d\mu$. Notice $\lambda_j < 1$ due to the positivity of all terms in the right hand side of the equation (4.4). After a cancellation, we reduce the theorem to the case with only $(m-1)$- maps. Then by an induction argument, we complete the proof of Theorem 4.1. ■

Chapter 5

Partial non-degeneracy: Proof of Proposition (I)

In this chapter, we prove Proposition (I) for irreducible compact Hermitian spaces of compact type. Since the argument differs as its type varies, we do it on a case by case base.

5.1 Spaces of type I

With the same notations that we have set up in §2, Z is a $p \times q$ matrix $(p \leq q)$; $Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ is the determinant of the submatrix of Z formed by its $i_1^{\text{th}}, ..., i_k^{\text{th}}$ rows and $j_1^{\text{th}}, ..., j_k^{\text{th}}$ columns; $z = (z_{11}, ..., z_{1q}, z_{21}, ..., z_{2q}, ..., z_{p1}, ..., z_{pq})$ is the coordinates of $\mathbb{C}^{pq} \cong \mathcal{A} \subset G(p, q)$.

Let $0 \in U$ be a small neighborhood of 0 in $\mathbb{C}^{pq}$ and F be a biholomorphic map defined over U with $F(0) = 0$. For convenience of our discussions, we represent the map $F : U \to \mathcal{A}$ as a holomorphic matrix-valued map:

$$F = \begin{pmatrix} f_{11} & \cdots & f_{1q} \\ \cdots & \cdots & \cdots \\ f_{p1} & \cdots & f_{pq} \end{pmatrix}.$$

Similar to $Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$, $F\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ denotes the determinant of the submatrix formed by the $i_1^{\text{th}}, ..., i_k^{\text{th}}$ rows and $j_1^{\text{th}}, ..., j_k^{\text{th}}$ columns of the matrix F.

Recall the definition in (2.1):

$$r_z = (\psi_1, \psi_2, ..., \psi_N) = (\cdots, Z\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}, \cdots), 1 \leq i_1 < ... < i_k \leq p,$$

$$1 \le j_1 < ... < j_k \le q, 1 \le k \le p.$$

Similarly, we define:

$$r_F := (\cdots, F\binom{i_1 \quad ... \quad i_k}{j_1 \quad ... \quad j_k}), \cdots), 1 \le i_1 < ... < i_k \le p, 1 \le j_1 < ... < j_k \le q, 1 \le k \le p.$$

Notice that $r_F = (\psi_1(F(z)), ..., \psi_N(F(z)))$. We define

$$\tilde{z} := (z_{11}, ..., z_{1q}, z_{21}, ..., z_{2q}, ..., z_{p1}, ..., z_{p(q-1)}),$$

i.e., $\tilde{z}$ is obtained from z by dropping the last component z_{pq}. Write $\frac{\partial^{|\alpha|}}{\partial \tilde{z}^\alpha} = \frac{\partial^{|\alpha|}}{\partial z_{11}^{\alpha_{11}} ... \partial z_{p(q-1)}^{\alpha_{p(q-1)}}}$ for any $(pq-1)$−multiindex α, where

$$\alpha = (\alpha_{11}, ..., \alpha_{1p}, \alpha_{21}, ..., \alpha_{2q}, ..., \alpha_{p1}, ..., \alpha_{p(q-1)}).$$

We apply the notion of the partial degeneracy defined in Definition 3.1 of §3 by letting $\psi = r_F$ and letting $\tilde{z}$ be as just defined with $m = pq$. We prove in this subsection the following proposition:

Proposition 5.1. r_F *are* $\tilde{z}−$*nondegenerate near* 0. *More precisely,* $\mathrm{rank}_{1+N-pq}(r_F, \tilde{z}) = N.$

Proof of Proposition 5.1: If $p = 1, q = n \ge 1$ i.e., the Hermitian symmetric space $M = \mathbb{P}^n$, then it follows from Lemma 3.2 that $\mathrm{rank}_1(r_F, \tilde{z}) = N = n$. In the following we assume $p \ge 2$.

Suppose the conclusion is not true. Namely, assume that $\mathrm{rank}_{1+N-pq}(r_F, \tilde{z}) < N$. Since the hypothesis of Theorem 3.10 is satisfied, we see that there exist $c_{pq+1}, ..., c_N \in \mathbb{C}$ which are not all zero such that

$$\sum_{i=pq+1}^{N} c_i \psi_i(F)(z_{11}, ..., z_{pq-1}, 0) \equiv 0. \tag{5.1}$$

The next step is to show that (5.1) cannot hold in the setting of Proposition 5.1. This is obvious if we can prove the following:

Lemma 5.2. *Let*

$$H = \begin{pmatrix} h_{11} & ... & h_{1p} \\ ... & ... & ... \\ h_{p1} & ... & h_{pq} \end{pmatrix},$$

be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (z_{11}, ..., z_{p(q-1)}) \in$
$\mathbb{C}^{pq-1}$ with $H(0) = 0$. Assume that H is of full rank at 0. Set

$$(\phi_1, ..., \phi_m) := r_H = \left(\left(H \begin{pmatrix} i_1 & ... & i_k \\ j_1 & ... & j_k \end{pmatrix} \right) \right)_{1 \leq i_1 < ... < i_k \leq p, 1 \leq j_1 < ... < j_k \leq q} \Bigg/_{2 \leq k \leq p} . \quad (5.2)$$

Here

$$m = \begin{pmatrix} p \\ 2 \end{pmatrix} \begin{pmatrix} q \\ 2 \end{pmatrix} + ... + \begin{pmatrix} p \\ p \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

Let $a_1, ..., a_m$ be complex numbers such that

$$\sum_{i=1}^{m} a_i \phi_i(\tilde{z}) \equiv 0 \text{ for all } \tilde{z} \in U. \quad (5.3)$$

Then

$$a_i = 0$$

for each $1 \leq i \leq m$.

Proof of Lemma 5.2: We start with the simple case $p = q = 2$, in which $m = 1$. Then by the assumption (5.3), $a_1 \phi_1 = 0$. Here

$$\phi_1 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix}.$$

Note that $H = (h_{11}, h_{12}, h_{21}, h_{22})$ is of full rank at 0. We assume, without loss of generality, that $\tilde{H} := (h_{11}, h_{12}, h_{21})$ is a local biholomorphic map from $\mathbb{C}^3$ to $\mathbb{C}^3$. After an appropriate biholomorphic change of coordinates preserving 0, we can assume $h_{11} = z_{11}, h_{12} = z_{12}, h_{21} = z_{21}$, and still write the last component as h_{22}. Then we have

$$a_1 \phi_1 = a_1(z_{11}h_{22} - z_{12}z_{21}) \equiv 0,$$

which easily yields that $a_1 = 0$.

We then prove the lemma for the case of $p = 2, q = 3$, in which $m = 3$. As before, without loss of generality, we assume that $\tilde{H} := (h_{11}, h_{12}, h_{13}, h_{21}, h_{22})$ is a local bi-holomorphic map near 0 from $\mathbb{C}^5$ to $\mathbb{C}^5$. After an appropriate biholomorphic change of coordinates, we assume that $\tilde{H} = (z_{11}, ..., z_{22})$. By (5.3), we have

$$a_1\phi_1 + ... + a_3\phi_3 = a_1 \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} + a_2 \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & h_{23} \end{vmatrix} + a_3 \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & h_{23} \end{vmatrix}. \tag{5.4}$$

The conclusion can be easily proved by checking the coefficients in the Taylor expansion at 0. Indeed, the quadratic terms $z_{13}z_{21}, z_{13}z_{22}$ only appear once in the last two determinants. This implies $a_2 = a_3 = 0$. Then trivially $a_1 = 0$.

We also prove the case $p = q = 3$. In this case $m = 10$. As before, without loss of generality, we assume that $\tilde{H} = (h_{11}, ..., h_{32})$ is a biholomorphic map from $\mathbb{C}^8$ to $\mathbb{C}^8$. After an appropariate biholomorphic change of coordinates, we can assume that $\tilde{H} = (z_{11}, ..., z_{32})$. Then by assumption, we have

$a_1\phi_1 + ... + a_{10}\phi_{10} =$

$$a_1 \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} + a_2 \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} + a_3 \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} + a_4 \begin{vmatrix} z_{11} & z_{12} \\ z_{31} & z_{32} \end{vmatrix} + a_5 \begin{vmatrix} z_{11} & z_{13} \\ z_{31} & h_{33} \end{vmatrix} + a_6 \begin{vmatrix} z_{12} & z_{13} \\ z_{32} & h_{33} \end{vmatrix}$$

$$+ a_7 \begin{vmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{vmatrix} + a_8 \begin{vmatrix} z_{21} & z_{23} \\ z_{31} & h_{33} \end{vmatrix} + a_9 \begin{vmatrix} z_{22} & z_{23} \\ z_{32} & h_{33} \end{vmatrix} + a_{10} \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & h_{33} \end{vmatrix} = 0.$$

$$\tag{5.5}$$

We then check the coefficients for each term in its Taylor expansion at 0. First it is easy to note that $a_5 = a_6 = a_8 = a_9 = 0$ by checking the coefficients of quadratic terms

$$z_{13}z_{31}, z_{13}z_{32}, z_{23}z_{31}, z_{23}z_{32},$$

respectively. Then by checking the coefficients of other quadratic terms, we see that $a_1 = a_2 = a_3 = a_4 = a_7 = 0$. Finally we check the coefficient of the cubic term $z_{13}z_{22}z_{31}$ to obtain that $a_{10} = 0$.

We now prove the general case: $q \geq p \geq 2$. As before, we assume without loss of generality that $\tilde{H} = (h_{11}, ..., h_{p(q-1)})$ is a biholomorphic map from $\mathbb{C}^{pq-1}$ to $\mathbb{C}^{pq-1}$.

Furthermore, we have $\tilde{H} = (z_{11}, ..., z_{p(q-1)})$ after an appropriate biholomorphic change of coordinates. We again first consider the coefficients of the quadratic terms in (5.3). For that, we consider the 2×2 submatrix involving h_{pq}, i.e., $H \begin{pmatrix} l & p \\ k & q \end{pmatrix}$, $1 \leq l < p, 1 \leq k < q$. Note that $z_{lq}z_{pk}$ only appears in this 2×2 determinant, which yields that the coefficient a_i associated to this 2×2 determinant is 0, for any $1 \leq i < p, 1 \leq j < q$. Then by checking the coefficients of other quadratic terms, we see that all coefficients $a_i's$ that are associated to 2×2 determinants $H \begin{pmatrix} l_1 & l_2 \\ k_1 & k_2 \end{pmatrix}$, $1 \leq l_1, l_2 \leq p, 1 \leq k_1, k_2 \leq q$, are 0.

We then consider the coefficients of cubic terms in (5.3). We first look at those 3×3 submatrix involving h_{pq}. i.e., $H \begin{pmatrix} l_1 & l_2 & p \\ k_1 & k_2 & q \end{pmatrix}$, $1 \leq l_1 < l_2 < p, 1 \leq k_1 < k_2 < q$. Note that $z_{l_1q}z_{l_2k_2}z_{pk_1}$ only appears in this 3×3 matrix, which yields that the a_i associated to this 3×3 determinant is 0. Furthermore, we see that all coefficients a_i's that are associated to 3×3 determinants are 0.

Now the conclusion can be proved inductively. Indeed, assume that we have proved that all coefficients a_i's that are associated with the determinants of order up to $\mu \times \mu, 3 \leq \mu < p$ are zero. Then we will prove that the coefficients associated with $(\mu+1) \times (\mu+1)$ determinants are also 0. For this we consider all such determinants which involve h_{pq}, i.e., $H \begin{pmatrix} l_1 & ... & l_\mu & p \\ k_1 & ... & k_\mu & q \end{pmatrix}$ where $1 \leq l_1 < ... < l_\mu < p, 1 \leq k_1 < ... < k_\mu < q$. We conclude the a_i associated to it is 0 by noting that $z_{l_1q}z_{l_2k_\mu}...z_{l_\mu k_2}z_{pk_1}$ only appears in this $(\mu+1) \times (\mu+1)$ determinant. Then we can show all coefficients that are associated with other $(\mu + 1) \times (\mu + 1)$ determinants, i.e.,

$$H \begin{pmatrix} l_1 & ... & l_\mu & l_{\mu+1} \\ k_1 & ... & k_\mu & k_{\mu+1} \end{pmatrix}, 1 \leq l_1 < ... < l_{\mu+1} \leq p, 1 \leq k_1 < ... < k_{\mu+1} \leq q, (l_{\mu+1}, k_{\mu+1}) \neq (p, q).$$

are 0 by checking a term of the form $z_{l_1k_1}...z_{l_{\mu+1}k_{\mu+1}}$ that only appears once in the Taylor expansion of the left hand side of (5.1). This proves the lemma. ∎

We thus get a contradiction to the equation (5.1). This establishes Proposition 5.1.

∎

Remark 2. *Let F be as in Proposition 5.1. There exist multiindices $\beta^1, ..., \beta^N$ with $|\beta^j| \leq 1 + N - pq$ and*

$$z^0 = \begin{pmatrix} z_{11}^0 & ... & z_{1q}^0 \\ ... & ... & ... \\ z_{p1}^0 & ... & z_{pq}^0 \end{pmatrix} \neq 0$$

such that z^0 is near 0 and

$$\Delta(\beta^1, ..., \beta^N) := \begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{z}^{\beta^1}} & ... & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \tilde{z}^{\beta^1}} \\ ... & ... & ... \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \tilde{z}^{\beta^N}} & ... & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \tilde{z}^{\beta^N}} \end{vmatrix} (z^0) \neq 0. \tag{5.6}$$

Perturbing z^0 if necessary, we can thus assume that $z_{pq}^0 \neq 0$. Moreover, we can replace one of the $\beta^1, ..., \beta^N$ by $\beta = (0, ..., 0)$, because $(\psi_1(F), ..., \psi_N(F))$ are not identically zero (See also the proof of Theorem 3.3). Without lost of generality, we can assume that $\beta^1 = (0, ..., 0)$.

The defining function of the Segre family now is

$$\rho(z, \xi) = 1 + \sum_{k=1}^{p} \left(\sum_{1 \leq i_1 < i_2 < ... < i_k \leq p, 1 \leq j_1 < j_2 < ... < j_k \leq q} Z\begin{pmatrix} i_1 & ... & i_k \\ j_1 & ... & j_k \end{pmatrix} \Xi\begin{pmatrix} i_1 & ... & i_k \\ j_1 & ... & j_k \end{pmatrix} \right) \tag{5.7}$$

It is a complex manifold for any fixed ξ close enough to the point

$$\xi^0 = \begin{pmatrix} 0 & ... & 0 & 0 \\ 0 & ... & 0 & 0 \\ 0 & ... & 0 & \xi_{pq}^0 \end{pmatrix} \in \mathbb{C}^{pq},$$

where $\xi_{pq}^0 = -\frac{1}{z_{pq}^0}$.

Write for each $1 \leq i \leq p, 1 \leq j \leq q, (i, j) \neq (p, q)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z, \xi)}{\frac{\partial \rho}{\partial z_{pq}}(z, \xi)} \frac{\partial}{\partial z_{pq}}, \tag{5.8}$$

which is a well-defined holomorphic tangent vector field along $\mathcal{M}$ near (z^0, ξ^0). Here we note that $\frac{\partial \rho}{\partial z_{pq}}(z, \xi)$ is nonzero near (z^0, ξ^0). For any $(pq - 1)$-multiindex $\beta = (\beta_{11}, ..., \beta_{p(q-1)})$, we write

$$\mathcal{L}^\beta = \mathcal{L}_{11}^{\beta_{11}} ... \mathcal{L}_{p(q-1)}^{\beta_{p(q-1)}}.$$

Now we define for any N collection of $(pq-1)-$multiindices $\{\beta^1, ..., \beta^N\}$,

$$\Lambda(\beta^1, ..., \beta^N)(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & ... & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix}(z, \xi). \qquad (5.9)$$

We have the following,

Theorem 5.3. *There exists multiindices* $\{\beta^1, ..., \beta^N\}$, *such that*

$$\Lambda(\beta^1, ..., \beta^N)(z, \xi) \neq 0, \qquad (5.10)$$

for (z, ξ) *in a small neighborhood of* (z^0, ξ^0). *Moreover, we can require* $\beta^1 = (0, ..., 0)$.

Proof of Theorem 5.3: First we observe that $\mathcal{L}_{ij}$ evaluating at (z^0, ξ^0) is just $\frac{\partial}{\partial z_{ij}}$. More generally, for any $(pq-1)-$multiindex β, by an easy computation, $\mathcal{L}^\beta$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \tilde{z}^\beta}$. Therefore, we can just choose the same $\beta^1, ..., \beta^N$ as in Remark 2. ∎

5.2 Spaces of type II

In this section, we establish Proposition (I) for the orthogonal Grassmannians $G_{II}(n, n)$. As shown in §2, we have a Zariski open affine chart $\mathcal{A} \subset G_{II}(n, n)$ of elements of the form:

$$\begin{pmatrix} I_{n \times n} & Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & z_{12} & \cdots & z_{1n} \\ 0 & 1 & 0 & \cdots & 0 & -z_{12} & 0 & \cdots & z_{2n} \\ & & & \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -z_{1n} & -z_{2n} & \cdots & 0 \end{pmatrix}$$

Here $z = (z_{12}, z_{13}, ..., z_{(n-1)n})$ is the local coordinates for $\mathcal{A} \cong \mathbb{C}^{\frac{n(n-1)}{2}}$. Its conjugate $\mathcal{A}^* \subset (G_{II}(n, n))^*$ is also a copy of $\mathbb{C}^{\frac{n(n-1)}{2}}$. We write the local coordinates for A^* as $\xi = (\xi_{12}, ..., \xi_{(n-1)n})$.

The canonical embedding is given by

$$(1, , ..., \mathrm{pf}(Z_\sigma), ...).$$

The defining function for the Segre family (in the product of such affine pieces) is given by

$$\rho(z, \xi) = 1 + \sum_{\substack{\sigma \in S_k, \\ 2 \leq k \leq n, 2|k}} \mathrm{Pf}(Z_\sigma)\mathrm{Pf}(\Xi_\sigma)$$

Write

$$r_Z = \Big(\mathrm{Pf}(Z_\sigma)_{\sigma \in S_k}\Big)_{2 \leq k \leq n, 2|k}. \tag{5.11}$$

The local biholomorphic map F defined near $0 \in U$ with $F(0) = 0$ can be represented as a matrix:

$$F = \begin{pmatrix} 0 & f_{12} & ... & f_{1n} \\ -f_{12} & 0 & ... & f_{2n} \\ ... & ... & ... & ... \\ -f_{1n} & ... & ... & 0 \end{pmatrix}.$$

Let r_F be

$$r_F = \Big(\mathrm{pf}((F)_\sigma)_{\sigma \in S_k}\Big)_{2 \leq k \leq n, 2|k}. \tag{5.12}$$

Under the notation of §2, it is easy to see $r_Z = (\psi_1, ..., \psi_N)$, $r_F = (\psi_1(F), ..., \psi_N(F))$.

We write $\widetilde{z}$ for the z with the last component $z_{(n-1)n}$ dropped. More precisely,

$$\widetilde{z} = (z_{12}, ..., z_{1n}, z_{23}, ..., z_{2n}, ..., z_{(n-2)(n-1)}, z_{(n-2)n}), \tag{5.13}$$

Recall z has $n' = n(n-1)/2$ independent variables. Thus $\widetilde{z}$ has $(n'-1)$ components. We define the $\widetilde{z}-$rank and $\widetilde{z}-$ nondegeneracy as in Definition 3.1 using $\psi = r_F$ in (5.12) and $\widetilde{z}$ as in (5.13) with $m = n'$, respectively. We now prove the following:

Proposition 5.4. r_F is $\widetilde{z}-$nondegenerate near 0. More precisely, $\mathrm{rank}_{1+N-n'}(r_F, \widetilde{z}) = N$.

Proof of Proposition 5.4: Suppose not. Without loss of generality, we assume that

$$\mathrm{rank}_{1+N-n'}(r_F, \widetilde{z}) < N.$$

As a consequence of Theorem 3.10, there exist $c_{\sigma,k} \in \mathbb{C}, 4 \leq k \leq n, 2|k, \sigma \in S_k$, which are not all zero, such that

$$\sum_{4 \leq k \leq n, 2|n} \sum_{\sigma \in S_k} c_{\sigma,k} \ \mathrm{pf}((F)_\sigma)(z_{12}, ..., z_{(n-2)n}, 0)) \equiv 0. \tag{5.14}$$

However, (5.14) cannot hold by the following lemma, which gives a contradiction:

Lemma 5.5. *Let*

$$H = \begin{pmatrix} 0 & h_{12} & ... & h_{1n} \\ -h_{12} & 0 & ... & ... \\ ... & ... & ... & ... \\ -h_{1n} & ... & ... & 0 \end{pmatrix}$$

be an anti-symmetric matrix-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (z_{12}, ..., z_{(n-2)n)}) \in \mathbb{C}^{n'-1}$ with $H(0) = 0$. Assume that H is of full rank at 0. Set r_H similar to the definition of r_F,

$$r_H = \left(\text{pf}(H_\sigma)_{\sigma \in S_k} \right)_{2 \le k \le n, 2|k}. \tag{5.15}$$

Assume that $a_{\sigma,k}, \sigma \in S_k, 4 \le k \le n$, are complex numbers such that

$$\sum_{4 \le k \le n, 2|k} \sum_{\sigma \in S_k} a_{\sigma,k} \, \text{pf}(H_\sigma)(z_{12}, ..., z_{(n-2)n)})) \equiv 0 \text{ for all } \tilde{z} \in U. \tag{5.16}$$

Then

$$a_{\sigma,k} = 0$$

for all $\sigma \in S_k, 4 \le k \le n, 2|k$.

Proof of Lemma 5.5: Suppose not. We will prove the lemma by seeking a contradiction. Note that H has full rank at 0. Hence there exist $(n' - 1)$ components $\widehat{H}$ of H that forms a local biholomorphism from $\mathbb{C}^{n'-1}$ to $\mathbb{C}^{n'-1}$. We assume that these $(n' - 1)$ components $\widehat{H}$ are H with $h_{i_0 j_0}$ being dropped, where $i_0 < j_0$. Without loss of generality, we assume $i_0 = n - 1, j_0 = n$. By a local biholomorphic change of coordinates, we assume $\widehat{H} = \tilde{z} = (z_{12}, ..., z_{(n-2)n})$. We still write the missing component as $h_{(n-1)n}$. Now we assume $2(m + 1), m \ge 1$, is the least number k such that $\{a_{\sigma,k}\}_{\sigma \in S_k}$ are not all zero. We then consider $\{a_{\sigma,2(m+1)}\}_{\sigma \in S_{2(m+1)}}$. We first claim that $a_{\sigma,2(m+1)} = 0$ for those $\sigma \in S_{2(m+1)}$ such that $\text{pf}(H_\sigma)$ involves $h_{(n-1)n}$. More precisely, if $\text{pf}(H_\sigma), \sigma \in S_{2(m+1)}$ involves $h_{(n-1)n}$, then $\sigma = \{i_1, ..., i_{2m}, (n - 1), n\}$ for some $1 \le i_1 < ... < i_{2m} \le n - 2$. Suppose its coefficient is not zero. Then $\text{pf}(H_\sigma)$ will produce the monomial $z_{i_1 i_2} z_{i_3 i_4} ... z_{i_{2m-3} i_{2m-2}} z_{i_{2m-1}(n-1)} z_{i_{2m} n}$. This term can only be canceled by

the terms of form: $z_{i_{2m-1}(n-1)}h_{(n-1)n}Q$ or $z_{i_{2m}n}h_{(n-1)n}Q$. But neither of them can appear in any other Pfaffians. This is a contradiction. Once we know there are no $h_{(n-1)n}$ involved, then the remaining Pfaffians have only terms consisting of the product of some of $z_{12}, ..., z_{(n-2)n}$. Their sum cannot be zero unless their coefficients are all zero. This is a contradiction. We thus establishes Lemma 5.5. ∎

We thus also get a contradiction to equation (5.14). This establishes Proposition 5.4. ∎

Remark 3. *By Proposition 5.4, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^N$ with all $|\tilde{\beta}^j| \leq 1 + N - n'$, and there is a point*

$$z^0 = \begin{pmatrix} 0 & z_{12}^0 & \cdots & z_{1(n-1)}^0 & z_{1n}^0 \\ -z_{12}^0 & 0 & \cdots & z_{2(n-1)}^0 & z_{2n}^0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -z_{1(n-1)}^0 & -z_{2(n-1)}^0 & \cdots & 0 & z_{(n-1)n}^0 \\ -z_{1n}^0 & -z_{2n}^0 & \cdots & -z_{(n-1)n}^0 & 0 \end{pmatrix}, z_{(n-1)n}^0 \neq 0;$$

near 0 *such that*

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{z}^{\tilde{\beta}^1}} & \cdots & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \tilde{z}^{\tilde{\beta}^1}} \\ \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \tilde{z}^{\tilde{\beta}^N}} & \cdots & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \tilde{z}^{\tilde{\beta}^N}} \end{vmatrix}(z^0) \neq 0. \tag{5.17}$$

We set

$$\xi^0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \xi_{(n-1)n}^0 \\ 0 & 0 & \cdots & -\xi_{(n-1)n}^0 & 0 \end{pmatrix} \in \mathbb{C}^{n^2}, \xi_{(n-1)n}^0 = -\frac{1}{z_{(n-1)n}^0}.$$

Then it is easy to see that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$.

Write for each $1 \leq i < j \leq n, (i, j) \neq (n-1, n)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z, \xi)}{\frac{\partial \rho}{\partial z_{(n-1)n}}(z, \xi)} \frac{\partial}{\partial z_{(n-1)n}} \tag{5.18}$$

which are holomorphic tangent vector fields along $\mathcal{M}$ near (z^0, ξ^0). Here we note that $\frac{\partial \rho}{\partial z_{(n-1)n}}(z, \xi)$ is nonzero near (z^0, ξ^0). For any $(n'-1)$-multiindex $\beta = (\beta_{12}, ..., \beta_{(n-2)n})$, we write

$$\mathcal{L}^\beta = \mathcal{L}_{12}^{\beta_{12}} ... \mathcal{L}_{(n-2)n}^{\beta_{(n-2)n}}.$$

Now we define for any N collection of $(n'-1)-$multiindices $\{\beta^1, ..., \beta^N\}$,

$$\Lambda(\beta^1, ..., \beta^N)(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & ... & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix}(z, \xi). \qquad (5.19)$$

Note that for any multiindex β, $\mathcal{L}^\beta$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \bar{z}^\beta}$. We thus again have

Theorem 5.6. *There exists multiindices* $\{\beta^1, ..., \beta^N\}$, *such that*

$$\Lambda(\beta^1, ..., \beta^N)(z, \xi) \neq 0,$$

for (z, ξ) *in a small neighborhood of* (z^0, ξ^0) *and* $\beta^1 = (0, ..., 0)$.

5.3   Spaces of type III

Let F be a local biholomorphic map at 0. In this case, both Z and F are $n \times n$ symmetric matrices. We write

$$Z = \begin{pmatrix} z_{11} & z_{12} & ... & z_{1n} \\ z_{12} & z_{22} & ... & z_{2n} \\ ... & ... & ... & ... \\ z_{1n} & z_{2n} & ... & z_{nn} \end{pmatrix}, \quad z = (z_{11}, z_{12}, z_{13}, ..., z_{nn}).$$

Similar notations are used for F.

Recall from (2.12) of ♣3 in § 2:

$$r_z = \left(\psi_1^1(z), ..., \psi_{N_1}^1(z), \psi_1^2(z), ..., \psi_{N_2}^2(z), ..., \psi_1^{n-1}(z), ..., \psi_{N_{n-1}}^{n-1}(z), \psi^n(z) \right), \qquad (5.20)$$

where ψ_j^k is a homogeneous polynomial of degree $k, 1 \leq j \leq N_k$. ψ^n is a homogeneous polynomial of degree n. Moreover, the components of r_z are linearly independent.

We write the number of components in r_z to be $N = N_1 + ... + N_n$, where we set $N_n = 1$. We will also sometimes write $\psi^n_{N_n} = \psi^n$.

We emphasize that for each fixed $k, \psi^k_1, ..., \psi^k_{N_k}$ are linearly independent. Moreover, each ψ^k_j is a certain linear combination of the determinants of $k \times k$ submatrices of Z. This will be crucial for our argument later.

We define r_F as the composition of r_z with the map F:

$$r_F = \left(\psi^1_1(F), ..., \psi^1_{N_1}(F), \psi^2_1(F), ..., \psi^2_{N_2}(F), ..., \psi^{n-1}_1(F), ..., \psi^{n-1}_{N_{n-1}}(F), \psi^n(F) \right).$$

$$(5.21)$$

In what follows, we write also $z_{ij} = z_{ji}$. We write $\det(A)$ as the determinant of A when A is a square matrix.

Let $P, \widetilde{P}$ be monomials in z'_{ij}s, and h a polynomial in z'_{ij}s. Let a, b be two complex numbers. In the following lemmas, when we say h always has the terms $aP, b\widetilde{P}$, we mean h has the term aP if and only if it has the term $b\widetilde{P}$.

Lemma 5.7. *Fixing $1 \leq i, j < n$, let $P = z_{in}z_{nj}Q$ and $\widetilde{P} = z_{ij}z_{nn}Q$ with Q a monomial in z'_{ij}s. The following statements are true.*

- *Let A be a square submatrix of Z. If $z_{ij} \nmid Q$, then $\det(A)$ always has monomials of the form $cP, -c\widetilde{P}$ for some $c \in \mathbb{C}$ depending on the submatrix A. (If $\det(A)$ does not have any multiple of P, it does not have any multiple of $\widetilde{P}$, either; vice versa). If $z_{ij}|Q$, then $\det(A)$ always has monomials $cP, -(c/2)\widetilde{P}$ for some $c \in \mathbb{C}$ depending on A.*

- *Let $k \geq 1$. Let $\psi^k_l(z)$ be as defined in (5.20), $1 \leq l \leq N_k$. If $z_{ij} \nmid Q$, then $\psi^k_l(z)$ always has monomials $\lambda P, -\lambda \widetilde{P}$ for some $\lambda \in \mathbb{C}$, If $z_{ij}|Q$, then $\psi^k_l(z)$ always has monomials $\lambda P, -(\lambda/2)\widetilde{P}$ for some $\lambda \in \mathbb{C}$.*

Proof of Lemma 5.7: The first part is a consequence of the Laplace expansion of a determinant by complementary minors. The second part is due to the fact that ψ^k_j is a linear combination of the determinants of submatrices of Z of order k. ∎

Similarly, one can prove in a similar way Lemmas 5.8-5.10.

Lemma 5.8. *Fixing $1 \leq j < n - 1$, let $P = z_{jn}z_{(n-1)(n-1)}Q$ and $\widetilde{P} = z_{j(n-1)}z_{(n-1)n}Q$ with Q a monomial in z'_{ij}s.*

- *Let A be a square submatrix of Z. If $z_{jn} \nmid Q$, then $\det(A)$ always has monomials $cP, -c\widetilde{P}$ for some $c \in \mathbb{C}$. If $z_{jn}|Q$, then $\det(A)$ always has monomials $cP, -2c\widetilde{P}$ for some $c \in \mathbb{C}$.*

- *Let $k \geq 1$. Let $\psi_l^k(z)$ be as defined in (5.20), $1 \leq l \leq N_k$. If $z_{jn} \nmid Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -\lambda \widetilde{P}$ for some $\lambda \in \mathbb{C}$. If $z_{jn}|Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -2\lambda \widetilde{P}$ for some $\lambda \in \mathbb{C}$.*

Lemma 5.9. *Fixing $1 \leq i < n - 1$, let $P = z_{i(n-1)}z_{ni}Q$ and $\widetilde{P} = z_{ii}z_{(n-1)n}Q$ with Q a monomial in z'_{ij}s.*

- *Let A be a square submatrix of Z. If $z_{(n-1)n} \nmid Q$, then $\det(A)$ always has monomials $cP, -c\widetilde{P}$ for some $c \in \mathbb{C}$. If $z_{(n-1)n}|Q$, then $\det(A)$ always has monomials $cP, -(c/2)\widetilde{P}$ for some $c \in \mathbb{C}$.*

- *Let $k \geq 1$. Let $\psi_l^k(z)$ be as defined in (5.20), $1 \leq l \leq N_k$. If $z_{(n-1)n} \nmid Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -\lambda \widetilde{P}$ for some $\lambda \in \mathbb{C}$. If $z_{(n-1)n}|Q$, then $\psi_l^k(z)$ always has monomials $\lambda P, -(\lambda/2)\widetilde{P}$ for some $\lambda \in \mathbb{C}$.*

Lemma 5.10. *Fixing $1 \leq i < n - 1, 1 \leq j < n - 1, i \neq j$, let $P_1 = z_{i(n-1)}z_{nj}Q, P_2 = z_{in}z_{j(n-1)}Q$, and $\widetilde{P} = z_{ij}z_{(n-1)n}Q$ with Q a monomial in z'_{ij}s.*

- *Let A be a square submatrix of Z. If $z_{ij} \nmid Q, z_{(n-1)n} \nmid Q$, then $\det(A)$ always has terms $c_1 P_1 + c_2 P_2, -(c_1 + c_2)\widetilde{P}$ for some $c_1, c_2 \in \mathbb{C}$. If $z_{ij} \nmid Q, z_{(n-1)n}|Q$, or $z_{ij}|Q, z_{(n-1)n} \nmid Q$, then $\det(A)$ always has terms $c_1 P_1 + c_2 P_2, -\frac{c_1 + c_2}{2}\widetilde{P}$ for some $c_1, c_2 \in \mathbb{C}$. If $z_{ij}|Q, z_{(n-1)n}|Q$, then $\det(A)$ always has terms $c_1 P_1 + c_2 P_2, -\frac{c_1 + c_2}{4}\widetilde{P}$.*

- *Let $k \geq 1$. Let $\psi_l^k(z)$ be as defined in (5.20), $1 \leq l \leq N_k$. If $z_{ij} \nmid Q$ and $z_{(n-1)n} \nmid Q$, then $\psi_l^k(z)$ always has terms $\lambda_1 P_1 + \lambda_2 P_2, -(\lambda_1 + \lambda_2)\widetilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. If $z_{ij} \nmid Q, z_{(n-1)n}|Q$, or $z_{ij}|Q, z_{(n-1)n} \nmid Q$, then $\psi_l^k(z)$ always has terms $\lambda_1 P_1 + \lambda_2 P_2, -\frac{\lambda_1 + \lambda_2}{2}\widetilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. If $z_{ij}|Q, z_{(n-1)n}|Q$, then $\psi_l^k(z)$ always has terms $\lambda_1 P_1 + \lambda_2 P_2, -\frac{\lambda_1 + \lambda_2}{4}\widetilde{P}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$.*

We write $\widetilde{z}$ for z with the last components z_{nn} being dropped. More precisely,

$$\widetilde{z} = (z_{11}, ..., z_{1n}, z_{22}, ..., z_{2n}, ..., z_{(n-1)(n-1)}, z_{(n-1)n}). \tag{5.22}$$

Recall z has $n' = n(n+1)/2$ independent variables. Thus $\widetilde{z}$ has $(n'-1)$ components. We define $\widetilde{z}-$rank and $\widetilde{z}-$nondegeneracy in the same way as before, using r_F in (5.21) and $\widetilde{z}$ in (5.22) with $m = n'$. We now need to prove the following:

Proposition 5.11. r_F is $\widetilde{z}-$nondegenerate at 0. More precisely, $\mathrm{rank}_{1+N-n'}(r_F, \widetilde{z}) = N$.

Proof of Proposition 5.11: Suppose not. Then one easily verifies that the hypothesis of Theorem 3.10 is satisfied. As a consequence of Theorem 3.10, there exist $c_j^k \in \mathbb{C}, 2 \leq k \leq n, 1 \leq j \leq N_k$, which are not all zero such that

$$\sum_{k=2}^{n} \sum_{j=1}^{N_k} c_j^k \psi_j^k(F(z_{11}, ..., z_{(n-1)n}, 0)) \equiv 0. \tag{5.23}$$

Here as before, we write $N_n = 1, \psi_{N_n}^n = \psi^n$.

Then we just need to show it can not happen by the following lemma:

Lemma 5.12. *Let*

$$H = \begin{pmatrix} h_{11} & h_{12} & ... & h_{1n} \\ h_{12} & h_{22} & ... & h_{2n} \\ ... & ... & ... & ... \\ h_{1n} & ... & ... & h_{nn} \end{pmatrix}$$

be a symmetric matrix-valued holomorphic function near 0 in $\widetilde{z} = (z_{11}, ..., z_{1n}, z_{22}, ..., z_{2n}, ..., z_{(n-1)n}) \in \mathbb{C}^{n'-1}$ with $H(0) = 0$. Assume that H is of full rank at 0. Set r_H in a similar way as in (36):

$$r_H = \left(\psi_1^1(H), ..., \psi_{N_1}^1(H), \psi_1^2(H), ..., \psi_{N_2}^2(H), ..., \psi_1^{n-1}(H), ..., \psi_{N_{n-1}}^{n-1}(H), \psi^n(H) \right)$$

Again we write $N_n = 1, \psi^n = \psi_{N_n}^n$. Assume that $a_j^k, 2 \leq k \leq n, 1 \leq j \leq n$ are complex numbers such that

$$\sum_{k=2}^{n} \sum_{j=1}^{N_k} a_j^k \psi_j^k(H(\widetilde{z})) \equiv 0 \quad for \ \ \widetilde{z} \in U. \tag{5.24}$$

Then

$$a_j^k = 0$$

for each $2 \leq k \leq n, 1 \leq j \leq N_k$.

Proof of Lemma 5.12: Suppose not. We will prove the lemma by seeking a contradiction. Notice that H has full rank at 0. Hence there exist $(n'-1)$ components $\widehat{H}$ of H that gives a local biholomorphism from $\mathbb{C}^{n'-1}$ to $\mathbb{C}^{n'-1}$. We assume these $(n'-1)$ components $\widehat{H}$ are H with $h_{i_0 j_0}$ being dropped, where $i_0 \leq j_0$. Then we split our argument into two parts in terms of $i_0 = j_0$ or $i_0 < j_0$.

Case I: Assume that $i_0 = j_0$. Without loss of generality, we assume $i_0 = j_0 = n$. By a local biholomorphic change of coordinates, we assume $\widehat{H} = \widetilde{z} = (z_{11}, ..., z_{n(n-1)})$. We still write the last component as h_{nn}. Now we assume $m \geq 2$ is the least number k such that $\{a_1^k, ..., a_{N_k}^k\}$ are not all zero. For any holomorphic g, we define $T_l(g)$ to be the homogeneous part of degree l in the Taylor expansion of g at 0. Now the assumption in (5.24) yields:

$$T_m \left(\sum_{j=1}^{N_m} a_j^m \psi_j^m(H(\widetilde{z})) \right) \equiv 0. \tag{5.25}$$

We first compute

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(H) = \sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn})$$

formally. Namely, we regard h_{nn} as a formal variable and only conduct formal cancellations. We write formally

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn}) = P_1 + h_{nn} P_2. \tag{5.26}$$

Here $P_1 = P_1(z_{11}, ..., z_{(n-1)n})$ is a homogeneous polynomial of degree m, and $P_2 = P_2(z_{11}, ..., z_{(n-1)n})$ is a homogeneous polynomial of degree $m-1$. We claim $P_2 \neq 0$. Otherwise,

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn}) = P_1.$$

This implies that $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn})$ does not depend on h_{nn} formally. Then we can replace h_{nn} by z_{nn}. That is,

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, z_{nn}) = \sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn}(\widetilde{z})) = P_1. \quad (5.27)$$

By (5.25), we see that (5.27) is identically zero. This is a contradiction to the fact that $\{\psi_1^m, ..., \psi_{N_m}^m\}$ is linearly independent.

Now since $P_2 \neq 0$, thus by (5.26), $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn})$ has a monomial of the form $\mu \widetilde{P} = \mu z_{ij} h_{nn} Q$ of degree m for some $1 \leq i, j < n, \mu \neq 0$ and some monomial Q. By Lemma 5.7, we get that $\sum_{j=1}^{N_m} a_j^m \psi_j^m(z_{11}, ..., z_{(n-1)n}, h_{nn})$ has also the term $-\mu P$ or $-2\mu P$, where $P = z_{in} z_{nj} Q$. This is a contradiction to (5.25). Indeed, P can be only canceled by the terms of the forms: $z_{in} h_{nn} \widetilde{Q}$ or $z_{nj} h_{nn} \widetilde{Q}$, where $\widetilde{Q}$ is of degree $m - 2$. But they cannot appear in determinant of any submatrix of H as z_{in}(or z_{nj}) can not appear with h_{nn}.

Case II: Assume that $i_0 \neq j_0$. Without loss of generality, we assume $i_0 = (n - 1), j_0 = n$. Then $\widehat{H} = (h_{11}, ..., h_{(n-1)(n-1)}, h_{nn})$ is a local biholomorphism. By a local biholomorphic change of coordinates, we assume $\widehat{H} = \widetilde{z} = (z_{11}, ..., z_{(n-1)n})$. We will still write the remaining component as $h_{(n-1)n} = h_{n(n-1)}$. Note that the fact we are using only is that $\{z_{11}, ..., z_{(n-1)n}\}$ are independent variables. Hence, to make our notation easier, we will write

$$\widehat{H} = (z_{11}, ..., z_{(n-1)n}) = (w_{11}, ..., w_{1n}, w_{22}, ..., w_{2n}, ..., w_{(n-1)(n-1)}, w_{nn})$$

such that they have the same indices as h's in $\widehat{H}$. Now we assume m is the least number k such that $\{a_1^k, ..., a_{N_k}^k\}$ are not all zero. Then again assumption (5.24) yields that

$$T_m \left(\sum_{j=1}^{N_m} a_j^m \psi_j^m(H(\widetilde{Z})) \right) \equiv 0. \quad (5.28)$$

Again we formally compute that

$$\sum_{j=1}^{N_m} a_j^m \psi_j^m(w_{11}, ..., h_{(n-1)n}, w_{nn}) = Q_1 + h_{(n-1)n} Q_2. \quad (5.29)$$

Here $Q_1 = Q_1(w_{11}, ..., w_{(n-1)(n-1)}, w_{nn})$ is a homogeneous polynomial of degree m. Similarly, we can show that $Q_2 \neq 0$. We claim that (5.29) does not have a monomial

of the form $h_{(n-1)n}h_{(n-1)n}Q$. Otherwise, by Lemma 5.7, we get that (5.29) has also a monomial of degree m of the form: $w_{(n-1)(n-1)}w_{nn}Q$. But note that in (5.29) it can be canceled only by $h_{(n-1)n}h_{(n-1)n}Q$. Then $h_{(n-1)n}$ will have a linear term $w_{(n-1)(n-1)}$. But then $h_{(n-1)n}h_{(n-1)n}Q$ will produce the term $w_{(n-1)(n-1)}w_{(n-1)(n-1)}Q$. This cannot be canceled out by any other terms.

Now since $Q_2 \neq 0$, (5.29) has a monomial of the form $w_{ij}h_{(n-1)n}Q$, where Q is another monomial in w's. Here $1 \leq i, j \leq n$. Moreover, $(i, j) \neq (n-1, n-1), (n-1, n), (n, n-1)$ or (n, n). We first assume $1 \leq i, j < n - 1, i \neq j$. Then by Lemma 5.10 , (5.29) has either P_1 or P_2, where $P_1 = w_{i(n-1)}w_{nj}Q, P_2 = w_{in}w_{j(n-1)}Q$. Note P_1, P_2 can only be canceled by the terms $w_{i(n-1)}h_{(n-1)n}Q, w_{nj}h_{(n-1)n}Q, w_{in}h_{(n-1)n}Q, w_{j(n-1)}h_{(n-1)n}Q$. So one of them will appear in (5.29). Whichever case it is, by Lemma 5.7, 5.8, (5.29) will have either $P = w_{ln}w_{(n-1)(n-1)}Q$, or $\widehat{P} = w_{l(n-1)}w_{nn}Q$ for some $1 \leq l < n$. We assume, for instance, (5.29) has the monomial P. Then it also has the monomial $\widetilde{P} = w_{l(n-1)}h_{(n-1)n}Q$ by Lemma 5.8. Note that the only term that can cancel P and appear in some determinant is $w_{ln}h_{n(n-1)}Q$. Hence $h_{n(n-1)}$ has a linear $w_{(n-1)(n-1)}$ term. Then $\widetilde{P}$ will have the monomial $w_{l(n-1)}w_{(n-1)(n-1)}Q$, which can not be canceled by any other terms. This is a contradiction. The other cases can be proved similarly. ∎

This establishes Proposition 5.11. ∎

Remark 4. *By Proposition 5.11, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^N$ with $|\tilde{\beta}^j| \leq 1+N-pq$, and there exist*

$$z^0 = \begin{pmatrix} z_{11}^0 & \cdots & z_{1n}^0 \\ \cdots & \cdots & \cdots \\ z_{1n}^0 & \cdots & z_{nn}^0 \end{pmatrix}, \ z_{nn}^0 \neq 0,$$

near 0 such that

$$\begin{vmatrix} \frac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{Z}^{\tilde{\beta}^1}} & \cdots & \frac{\partial^{|\beta^1|}(\psi_N(F))}{\partial \tilde{Z}^{\tilde{\beta}^1}} \\ \cdots & \cdots & \cdots \\ \frac{\partial^{|\beta^N|}(\psi_1(F))}{\partial \tilde{Z}^{\tilde{\beta}^N}} & \cdots & \frac{\partial^{|\beta^N|}(\psi_N(F))}{\partial \tilde{Z}^{\tilde{\beta}^N}} \end{vmatrix} (z^0) \neq 0. \tag{5.30}$$

Here we simply write $r_F = (\psi_1(F), ..., \psi_N(F))$.

We then set

$$\xi^0 = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \xi^0_{nn} \end{pmatrix} \in \mathbb{C}^{n^2}, \xi^0_{nn} = -\frac{1}{z^0_{nn}}.$$

It is easy to verify that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$.

Write for each $1 \le i \le j \le n, (i, j) \ne (n, n)$,

$$\mathcal{L}_{ij} = \frac{\partial}{\partial z_{ij}} - \frac{\frac{\partial \rho}{\partial z_{ij}}(z, \xi)}{\frac{\partial \rho}{\partial z_{nn}}(z, \xi)} \frac{\partial}{\partial z_{nn}}, \tag{5.31}$$

which are holomorphic tangent vector fields along $\mathcal{M}$ near (z^0, ξ^0). Here we note that $\frac{\partial \rho}{\partial z_{nn}}(z, \xi)$ is nonzero near (z^0, ξ^0). For any $(n'-1)$-multiindex $\beta = (\beta_{11}, \dots, \beta_{(n-1)n})$, we write

$$\mathcal{L}^\beta = \mathcal{L}_{11}^{\beta_{11}} \dots \mathcal{L}_{(n-1)n}^{\beta_{(n-1)n}}.$$

Now we define for any N collection of $(n'-1)-$multiindices $\{\beta^1, \dots, \beta^N\}$,

$$\Lambda(\beta^1, \dots, \beta^N)(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^1}(\psi_N(F)) \\ \dots & \dots & \dots \\ \mathcal{L}^{\beta^N}(\psi_1(F)) & \dots & \mathcal{L}^{\beta^N}(\psi_N(F)) \end{vmatrix}(z, \xi). \tag{5.32}$$

Note $\mathcal{L}^\beta$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \bar{Z}^\beta}$. We have

Theorem 5.13. *There exists multiindices $\{\beta^1, \dots, \beta^N\}$ such that $\Lambda(\beta^1, \dots, \beta^N)(z, \xi) \ne 0$ for (z, ξ) in a small neighborhood of (z^0, ξ^0) and $\beta^1 = (0, 0, \dots, 0)$.*

5.4   Spaces of type IV

In this section, we consider the hyperquadric case $M = Q^n$. This case is more subtle because the tangent vector fields of its Segre family are more complicated.

Recall that Q^n is defined by

$$\left\{ [z_0, \dots, z_{n+1}] \in \mathbb{CP}^{n+1} : \sum_{i=1}^n z_i^2 - 2z_0 z_{n+1} = 0 \right\},$$

where $[z_0, \dots, z_{n+1}]$ is the homogeneous coordinates of $\mathbb{CP}^{n+1}$. The previously described minimal embedding $\mathbb{C}^n(\mathcal{A}) \to Q^n$ is given by

$$z := (z_1, \dots, z_n) \mapsto [1, \psi_1(z), \dots, \psi_{n+1}(z)] = [1, z_1, \dots, z_n, \frac{1}{2} \sum_{i=1}^n z_i^2].$$

The defining function of the Segre family over $\mathcal{A} \times \mathcal{A}$ is $\rho(z, \xi) = 1 + r_z \cdot r_\xi$, where

$$r_z = (z_1, ..., z_n, \frac{1}{2} \sum_{i=1}^{n} z_i^2), r_\xi = (\xi_1, ..., \xi_n, \frac{1}{2} \sum_{i=1}^{n} \xi_i^2). \tag{5.33}$$

Let F be a local biholomorphic map at 0 with $F(0) = 0$. We write

$$F = (f_1, ..., f_n), \quad r_F = (f_1, ..., f_n, \frac{1}{2} \sum_{i=1}^{n} f_i^2). \tag{5.34}$$

Notice that

$$r_z = (\psi_1(z), ..., \psi_{n+1}(z)), r_F = (\psi_1(F), ..., \psi_{n+1}(F)).$$

We will need the following lemma:

Lemma 5.14. *For each fixed* $\mu_1, ..., \mu_{n-1}$ *with* $(\sum_{i=1}^{n-1} \mu_i^2) + 1 = 0$ *and each fixed* $(z_1, ..., z_n)$ *with* $(\sum_{i=1}^{n-1} \mu_i z_i) + z_n \neq 0$, *we can find* $(\xi_1, ..., \xi_n)$ *such that*

$$1 + z_1\xi_1 + ... + z_n\xi_n = 0; \quad \sum_{i=1}^{n}(\xi_i)^2 = 0, \quad \xi_j = \mu_j\xi_n, 1 \leq j \leq n-1, \quad \xi_n \neq 0. \tag{5.35}$$

Proof of Lemma 5.14: We just need to set

$$\xi_n = \frac{-1}{(\sum_{i=1}^{n-1} \mu_i z_i) + z_n}, \quad \xi_j = \mu_j\xi_n, 1 \leq j \leq n-1.$$

Then it is easy to verify that (5.35) is satisfied. ∎

Recall that in the type I case, the vector fields $\frac{\partial}{\partial \tilde{z}^\alpha}$ in $\mathbb{C}^{pq}$ are tangent vector fields of the particular hyperplane $\{z_{pq} = 0\}$. We can formulate the result in §3 in a more abstract way and extend it to a more general setting. For instance, it can be generalized to the complex hyperplane case. We briefly discuss this in more details as follows:

First fix $\mu_1, ..., \mu_{n-1}$ with $(\sum_{i=1}^{n-1} \mu_i^2) + 1 = 0$. Take the complex hyperplane $\mathbb{H}$: $z_n + \sum_{i=1}^{n-1} \mu_i z_i = 0$ in $(z_1, ..., z_n) \in \mathbb{C}^n$.

Write

$$L_1 = \frac{\partial}{\partial z_1} - \mu_1 \frac{\partial}{\partial z_n}, ..., L_{n-1} = \frac{\partial}{\partial z_{n-1}} - \mu_{n-1} \frac{\partial}{\partial z_n}.$$

Then $\{L_i\}_{i=1}^{n-1}$ forms a basis of the tangent vector fields of $\mathbb{H}$. For any multiindex $\alpha = (\alpha_1, .., \alpha_{n-1})$, we write $L^\alpha = L_1^{\alpha_1}...L_{n-1}^{\alpha_{n-1}}$. We define L−rank and L−nondegeneracy as in Definition 3.1 by using r_F in (5.34) and by using L^α instead of $\tilde{z}^\alpha$ with $m = n$. We write the kth L-rank defined in this setting as $\text{rank}_k(r_F, L)$ We now need to prove the following

Proposition 5.15. $\text{rank}_2(r_F, L) = n + 1.$

Proof of Proposition 5.15: Suppose not. By applying the same argument as in Section 3 and a linear change of coordinates, we can first obtain a modified version of Theorem 3.10. That is, we have the following:

Lemma 5.16. *There exist $n+1$ holomorphic functions $g_1(w), ..., g_{n+1}(w)$ which are defined near 0 on the $w-$plane with $\{g_1(0), ..., g_{n+1}(0)\}$ not all zero such that the following holds for all $z \in U$.*

$$\sum_{i=1}^{n+1} g_i(z_n + \mu_1 z_1 + ... + \mu_{n-1} z_{n-1})\psi_i(F(z)) \equiv 0. \tag{5.36}$$

Then one shows with a similar argument as in Section 3, by the fact that F has full rank at 0, that $g_1(0) = 0, ..., g_n(0) = 0$. Hence we obtain,

Lemma 5.17. *There exists a non-zero constant $c \in \mathbb{C}$ such that*

$$c\psi_{n+1}(F(z)) = \frac{c}{2}\sum_{i=1}^{n} f_i^2(z) \equiv 0, \tag{5.37}$$

for all $z \in U$ when restricted on $z_n + \sum_{i=1}^{n-1} \mu_i z_i = 0$.

We then just need to show that (5.37) cannot hold by applying the following lemma and a linear change of coordinates.

Lemma 5.18. *Let $H = (h_1, ..., h_n)$ be a vector-valued holomorphic function in a neighborhood U of 0 in $\tilde{z} = (z_1, ..., z_{n-1}) \in \mathbb{C}^{n-1}$ with $H(0) = 0$. Assume that H has full rank at 0. Assume that a is a complex number such that,*

$$a\sum_{i=1}^{n} h_i^2(\tilde{z}) \equiv 0, \tag{5.38}$$

Then $a = 0$.

Proof of Lemma 5.18: Seeking a contradiction, suppose not. Notice that H has full rank at 0. We assume, without loss of generality, that $(h_1, ..., h_{n-1})$ gives a local biholomorphic map near 0 from $\mathbb{C}^{n-1}$ to $\mathbb{C}^{n-1}$. By a local biholomorphic change of coordinates, we assume $(h_1, ..., h_{n-1}) = (z_1, ..., z_{n-1})$, and still write the last component as h_n. Then equation (5.38) is reduced to

$$a(z_1^2 + ... + z_{n-1}^2 + h_n^2) = 0.$$

To cancel the z_1^2, z_2^2 terms, it yields that h_n has linear z_1, z_2 terms. But then h_n^2 would produce a $z_1 z_2$ term, which cannot be canceled out. This is a contradiction. ∎

This also establishes Proposition 5.15. ∎

Remark 5. *By Proposition 5.15, there exist multiindices* $\tilde{\beta}^1, ..., \tilde{\beta}^{n+1}$ *with* $|\tilde{\beta}^j| \leq 2$ *and*

$$z^0 = (z_1^0, ..., z_n^0) \text{ with } \sum_{i=1}^{n-1} \mu_i z_i^0 + z_n^0 \neq 0$$

such that

$$\begin{vmatrix} L^{\tilde{\beta}^1}(\psi_1(F)) & ... & L^{\tilde{\beta}^1}(\psi_{n+1}(F)) \\ ... & ... & ... \\ L^{\tilde{\beta}^{n+1}}(\psi_1(F)) & ... & L^{\tilde{\beta}^{n+1}}(\psi_{n+1}(F)) \end{vmatrix}(z^0) \neq 0. \tag{5.39}$$

We then choose $\xi^0 = (\xi_1^0, ..., \xi_n^0)$ as in Lemma 5.14. That is

$$1 + z_1^0 \xi_1^0 + ... + z_n^0 \xi_n^0 = 0; \quad \sum_{i=1}^{n}(\xi_i^0)^2 = 0, \quad \xi_j^0 = \mu_j \xi_n^0, 1 \leq j \leq n-1, \quad \xi_n^0 \neq 0.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M}$.

We now define

$$\mathcal{L}_i = \frac{\partial}{\partial z_i} - \frac{\frac{\partial \rho}{\partial z_i}(z, \xi)}{\frac{\partial \rho}{\partial z_n}(z, \xi)} \frac{\partial}{\partial z_n}, 1 \leq i \leq n-1 \tag{5.40}$$

for $(z, \xi) \in \mathcal{M}$ near (z^0, ξ^0). They are well-defined holomorphic tangent vector fields along $\mathcal{M}$. Moreover, $\frac{\partial \rho}{\partial z_n}(z, \xi)$ is nonzero near (z^0, ξ^0).

We define for any multiindex $\alpha = (\alpha_1, .., \alpha_{n-1})$, $\mathcal{L}^\alpha = \mathcal{L}_1^{\alpha_1}...\mathcal{L}_{n-1}^{\alpha_{n-1}}$. Then we define, for any $(n+1)$ collection of $(n-1)$-multiindices $\{\beta^1, ..., \beta^N\}$,

$$\Lambda(\beta^1, ..., \beta^{n+1})(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_{n+1}(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^{n+1}}(\psi_1(F)) & ... & \mathcal{L}^{\beta^{n+1}}(\psi_{n+1}(F)) \end{vmatrix}(z, \xi). \tag{5.41}$$

By the fact that $\sum_{i=1}^{n}(\xi_i^0)^2 = 0$, one can check that, for any multiindex $\alpha = (\alpha_1, .., \alpha_n)$, $\mathcal{L}^\alpha = L^\alpha$ when evaluated at (z^0, ξ^0). Then we get the following:

Theorem 5.19. *There exist multiindices* $\{\beta^1, ..., \beta^N\}$ *such that*

$$\Lambda(\beta^1, ..., \beta^N)(z, \xi) \neq 0,$$

for (z, ξ) *in a small neighborhood of* (z^0, ξ^0), *where* $\beta^1 = (0, 0, ..., 0)$.

5.5 The exceptional class M_{16}

This case is very similar to the hyperquadric setting. In this case, we write the coordinates of $\mathbb{C}^{16}$ as

$$z := (x_0, ..., x_7, y_0, ..., y_7).$$

The defining function of the Segre family as described in (2.15) is

$$\rho(z, \xi) = 1 + r_z \cdot r_\xi = 1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi), \quad \text{where } N = 26 \text{ and}$$

$$r_z = (x_0, ..., x_7, y_0, ..., y_7, A_0, ...A_7, B_0, B_1). \tag{5.42}$$

Here $A_i, 0 \le i \le 7, B_0, B_1$ are homogeneous quadratic polynomials in z. For instance,

$$B_0 = \sum_{i=0}^{7} x_i^2, B_1 = \sum_{i=0}^{7} y_i^2.$$

For the expressions for A_i, see Appendix I.

Let F be as before. We write

$$F = (f_0, ..., f_7, \widetilde{f}_0, ...\widetilde{f}_7).$$

And define r_F as the composition of r_z with F :

$$r_F = r_z \circ F = (f_0, ..., f_7, \widetilde{f}_0, ...\widetilde{f}_7, A_0(F), ...A_7(F), B_0(F), B_1(F)). \tag{5.43}$$

Notice that r_F has 26 components.

We will need the following lemma:

Lemma 5.20. *For each fixed $\mu_0, ..., \mu_6$ with $(\sum_{i=0}^{6} \mu_i^2) + 1 = 0$ and fixed $(y_0, ..., y_7)$ with $(\sum_{i=0}^{6} \mu_i y_i) + y_7 \ne 0$, we can always find $(\xi_0, ..., \xi_7)$ such that*

$$1 + y_0\xi_0 + ... + y_7\xi_7 = 0; \quad \sum_{i=0}^{7}(\xi_i)^2 = 0, \quad \xi_j = \mu_j\xi_7, 0 \le j \le 6, \quad \xi_7 \ne 0.$$

Proof of Lemma 5.20: The proof is similar to that as in the hyperquadric case. ∎

Take the complex hyperplane $\mathbb{H} : y_7 + \sum_{j=0}^{6} \mu_j y_j = 0$ in $(x_0, ..., x_7, y_0, ..., y_7) \in \mathbb{C}^{16}$.

Write

$$L_0 = \frac{\partial}{\partial x_0}, ..., L_7 = \frac{\partial}{\partial x_7}.$$

$$L_8 = \frac{\partial}{\partial y_0} - \mu_1 \frac{\partial}{\partial y_7}, ..., L_{14} = \frac{\partial}{\partial y_6} - \mu_6 \frac{\partial}{\partial y_7}.$$

Then $\{L_i\}_{i=0}^{14}$ forms a basis of the tangent vector fields of $\mathbb{H}$. For any multiindex $\alpha = (\alpha_0, .., \alpha_{14})$, we write $L^\alpha = L_0^{\alpha_0}...L_{14}^{\alpha_{14}}$. We define the notion of $L-$rank and $L-$nondegeneracy as in Definition 3.1 using r_F in (5.43) and L^α instead of $\widetilde{z}^\alpha$. We write the kth L-rank defined in this setting as $\mathrm{rank}_k(r_F, L)$. We now need to prove the following:

Proposition 5.21. F is $L-$nondegenerate near 0. More precisely, $\mathrm{rank}_{11}(r_F, L) = 26$.

Proof of Proposition 5.21: Suppose not. As in the hyperquadric case, by a modified version of Theorem 3.10, we have that there exist 26 holomorphic functions $g_0(w), ..., g_{25}(w)$ defined near 0 on the $w-$plane with $\{g_0(0), ..., g_{25}(0)\}$ not all zero such that the following holds for $z \in U$:

$$\sum_{i=0}^{25} g_i(y_7 + \mu_0 y_0 + ... + \mu_6 y_6)\psi_i(F(z)) \equiv 0. \tag{5.44}$$

Then since F has full rank at 0, one can similarly prove that $g_0(0) = 0, ..., g_{15}(0) = 0$. Hence we obtain:

Lemma 5.22. *There exist* $c_0, ..., c_9 \in \mathbb{C}$ *that are not all zero such that*

$$c_0 A_0(F(Z)) + ... + c_7 A_7(F(Z)) + c_8 B_0(F(Z)) + c_9 B_1(F(Z)) \equiv 0, \tag{5.45}$$

for all $Z \in U$ *when restricted on* $y_7 + \sum_{i=0}^{6} \mu_i y_i = 0$.

We then just need to show that (5.45) can not hold by the following lemma after applying a linear change of coordinates.

Lemma 5.23. *Let* $H = (h_0, ..., h_7, g_0, ..., g_7)$ *be a vector-valued holomorphic function in a neighborhood* U *of* 0 *in* $\tilde{z} = (x_0, ..., x_7, y_0, ..., y_6) \in \mathbb{C}^{15}$ *with* $H(0) = 0$. *Assume that* H *has full rank at* 0. *Assume that* $a_0, ..., a_9$ *are complex numbers such that*

$$a_0 A_1(H(\widetilde{z})) + ... + a_7 A_7(H(\widetilde{z})) + a_8 B_0(H(\widetilde{z})) + a_9 B_1(H(\widetilde{z})) = 0 \text{ for all } \widetilde{z} \in U. \tag{5.46}$$

Then $a_i = 0$ *for* $1 \leq i \leq 10$.

Proof of Lemma 5.23: Suppose not. Notice that H has full rank at 0. Hence there exist 15 components $\widehat{H}$ of H that gives a local biholomorphism from $\mathbb{C}^{15}$ to $\mathbb{C}^{15}$. We assume these 15 components $\widehat{H}$ are H with η being dropped, where $\eta \in \{h_0, ..., h_7, g_0, ..., g_7\}$. By a local biholomorphic change of coordinates, we assume $\widehat{H} = (x_0, ..., x_7, y_0, ..., y_6)$. We still write the remaining component as η. Without loss of generality, we assume $\eta = g_7$.

First we claim the coefficient a_9 of B_1 is zero. Suppose not. Note that y_1^2, y_2^2 can be only canceled by g_7^2. Then g_7 will have linear y_1, y_2 terms. Hence g_7^2 will produce a $y_1 y_2$ term. It cannot be canceled by any other terms. This is a contradiction. Now we consider the coefficients of $A_0, ..., A_7$. We claim $a_i = 0, 0 \le i \le 7$. Suppose not. We write

$$y_7(\widetilde{Z}) = \lambda_0 y_0 + ... + \lambda_6 y_6 + \mu_0 x_0 + ... + \mu_7 x_7 + O(2),$$

for some $\lambda_i, \mu_j \in \mathbb{C}, 0 \le i \le 6, 0 \le j \le 7$. By collecting the terms of the form $x_0 y_i$ in the Taylor expansion of (5.46) we get

$$a_i + a_7 \lambda_i = 0, 0 \le i \le 6. \tag{5.47}$$

By collecting the terms of the form $x_1 y_i, 0 \le i \le 6$, we get,

$$a_1 + a_3 \lambda_0 = 0, -a_0 + a_3 \lambda_1 = 0, -a_4 + a_3 \lambda_2 = 0, -a_7 + a_3 \lambda_3 = 0,$$

$$a_2 + a_3 \lambda_4 = 0, -a_6 + a_3 \lambda_5 = 0, a_5 + a_3 \lambda_6 = 0.$$

By collecting the terms of the form $x_2 y_i, 0 \le i \le 6$, we get,

$$a_2 + a_6 \lambda_0 = 0, a_4 + a_6 \lambda_1 = 0, -a_0 + a_6 \lambda_2 = 0, -a_5 + a_6 \lambda_3 = 0.$$

$$-a_1 + a_6 \lambda_4 = 0, a_3 + a_6 \lambda_5 = 0, -a_7 + a_6 \lambda_6 = 0.$$

One can further write down all the coefficients for $x_i y_j, 0 \le i \le 7, 0 \le j \le 6$. Once this is done, one easily sees that $a_i \ne 0$ for any $0 \le i \le 7$. Otherwise, all $a_i = 0, \ 0 \le i \le 7$.

Then by the above equations, we see that the matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_1 & -a_0 & -a_4 & -a_7 & a_2 & -a_6 & a_5 \\ a_2 & a_4 & -a_0 & -a_5 & -a_1 & a_3 & -a_7 \end{pmatrix} \tag{5.48}$$

is of rank one. Then one can get a contradiction by, for instance, carefully checking the determinants of its 2×2 submatrices. Hence $a_i = 0, 0 \leq i \leq 7$. Finally we easily get the coefficient a_8 of B_0 is zero. ∎

This then establishes Proposition 5.21. ∎

Remark 6. *First fix $\mu_0, ..., \mu_6$ with $(\sum_{i=0}^{6} \mu_i^2) + 1 = 0$. By Proposition 5.21, there exists multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^{26}$ with $|\tilde{\beta}^j| \leq 11$, and*

$$Z^0 = (x_0^0, ..., x_7^0, y_0^0, ..., y_7^0) \ \text{with} \ \sum_{i=0}^{6} \mu_i y_i + y_7 \neq 0,$$

such that

$$\begin{vmatrix} L^{\tilde{\beta}^1}(\psi_1(F)) & ... & L^{\tilde{\beta}^1}(\psi_{26}(F)) \\ ... & ... & ... \\ L^{\tilde{\beta}^{26}}(\psi_1(F)) & ... & L^{\tilde{\beta}^{26}}(\psi_{26}(F)) \end{vmatrix} (Z^0) \neq 0.$$

We then let $\xi^0 = (0, ..., 0, \xi_0^0, ..., \xi_7^0)$, where $(\xi_0^0, ..., \xi_7^0)$ is choosen as in Lemma 5.20 associated with $(y_0^0, ..., y_7^0)$. That is

$$1 + y_0^0 \xi_0^0 + ... + y_7^0 \xi_7^0 = 0; \quad \sum_{i=0}^{7} (\xi_i^0)^2 = 0, \quad \xi_j^0 = \mu_j \xi_7^0, 0 \leq j \leq 6, \quad \xi_7^0 \neq 0.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M}$.

We now define

$$\mathcal{L}_i = \frac{\partial}{\partial x_i} - \frac{\frac{\partial \rho}{\partial x_i}(z, \xi)}{\frac{\partial \rho}{\partial y_7}(Z, \xi)} \frac{\partial}{\partial y_7}, 0 \leq i \leq 7; \tag{5.49}$$

$$\mathcal{L}_{8+i} = \frac{\partial}{\partial y_i} - \frac{\frac{\partial \rho}{\partial y_i}(z, \xi)}{\frac{\partial \rho}{\partial y_7}(Z, \xi)} \frac{\partial}{\partial y_7}, 0 \leq i \leq 6; \tag{5.50}$$

for $(z, \xi) \in \mathcal{M}$ near (z^0, ξ^0). They are tangent vector fields along $\mathcal{M}$. Moreover, $\frac{\partial \rho}{\partial y_n}(z, \xi)$ is nonzero near (z^0, ξ^0).

We define for any multiindex $\alpha = (\alpha_0, .., \alpha_{14})$, $\mathcal{L}^\alpha = \mathcal{L}_0^{\alpha_0}...\mathcal{L}_{14}^{\alpha_{14}}$. Define for any 26 collection of 15-multiindices $\{\beta^1, ..., \beta^{26}\}$,

$$\Lambda(\beta^1, ..., \beta^{26})(z, \xi) = \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_{26}(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^{26}}(\psi_1(F)) & ... & \mathcal{L}^{\beta^{26}}(\psi_{26}(F)) \end{vmatrix} (z, \xi). \tag{5.51}$$

By the fact that $\sum_{i=0}^{7} (\xi_i^0)^2 = 0$, one can check that, for any multiindex $\alpha = (\alpha_0, .., \alpha_{14})$, $\mathcal{L}^\alpha = L^\alpha$ when evaluated at (z^0, ξ^0). Then as before, we get the following:

Theorem 5.24. *There exists multiindices* $\{\beta^1, ..., \beta^{26}\}$ *such that*

$$\Lambda(\beta^1, ..., \beta^{26})(z, \xi) \neq 0,$$

for (z, ξ) *in a small neighborhood of* (z^0, ξ^0) *and* $\beta^1 = (0, 0, ..., 0)$.

5.5.1 The exceptional class M_{27}

In this setting, we use the coordinates

$$z = (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7) \in \mathbb{C}^{27}.$$

The defining function of the Segre family described in (2.16) is :

$$\rho(z, \xi) = 1 + r_z \cdot r_\xi = 1 + \sum_{i=1}^{N} \psi_i(z)\psi_i(\xi), \quad \text{where } N = 55 \text{ and}$$

$$r_z = (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7, A, B, C, D_0, ...D_7, E_0, ..., E_7, F_0, ..., F_7, G).$$

$$(5.52)$$

Here A, B, C, D_i, E_i, F_i are homogeneous quadratic polynomials in z and G is a homogeneous cubic polynomial in z:

$$A = x_2 x_3 - \sum_{i=0}^{7} w_i^2, B = x_1 x_3 - \sum_{i=0}^{7} t_i^2, C = x_1 x_2 - \sum_{i=0}^{7} y_i^2. \quad (5.53)$$

For the expressions for D_i, E_i, F_i, G, see Appendix I. Let F be a local biholomorphic map near 0. We write

$$F = (\phi_1, \phi_2, \phi_3, f_{10}, ..., f_{17}, f_{20}, ..., f_{27}, f_{30}, ..., h_{37}).$$

Also define r_F to be the composition of r_z with F:

$$r_F = r_z \circ F = (\phi_1, \phi_2, \phi_3, f_{10}, ..., f_{17}, f_{20}, ..., f_{27}, f_{30}, ..., f_{37}, A(F), B(F), C(F), ..., G(F)).$$

$$(5.54)$$

Notice that r_F has 55 components. We will also write

$$r_F = (\psi_1(F), ..., \psi_{55}(F)).$$

We write $\widetilde{z}$ for z with x_3 being dropped. Namely,

$$\widetilde{z} = (x_1, x_2, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7). \tag{5.55}$$

We define the $\widetilde{z}-$rank and $\psi-$nondegeneracy as in Definition 3.1 using r_F in (5.54) and $\widetilde{z}$ in (5.55) with $m = 27$.

Proposition 5.25. *F is $\widetilde{z}-$nondegenerate near 0. More precisely,* $\mathrm{rank}_{29}(F, \widetilde{z}) = 55$.

Proof of Proposition 5.25: Suppose not. As a consequence of Theorem 3.10, there exist $c_1, ..., c_{28} \in \mathbb{C}$ that are not all zero, such that

$$c_1 A(F(x_1, x_2, 0, y_0, ..., w_7)) + ... + c_{28} G(F(x_1, x_2, 0, y_0, ..., w_7)) \equiv 0. \tag{5.56}$$

We will show that (5.56) cannot hold by the following lemma:

Lemma 5.26. *Let* $H = (\psi_1, \psi_2, \psi_3, h_{10}, ..., h_{17}, h_{20}, ..., h_{27}, h_{30}, ..., h_{37})$ *be a vector-valued holomorphic function in a neighborhood U of 0 in $\widetilde{z} = (x_1, x_2, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7) \in \mathbb{C}^{26}$ with $H(0) = 0$. Assume that H has full rank at 0. Assume that $a_1, ..., a_{28}$ are complex numbers such that*

$$a_1 A(H(\widetilde{z})) + ... + a_{28} G(H(\widetilde{z})) = 0 \ \text{for all } \widetilde{z} \in U. \tag{5.57}$$

Then $a_i = 0$ for all $1 \leq i \leq 28$.

Proof of Lemma 5.26: Suppose not. Notice that H has full rank at 0. Hence there exist 26 components $\widehat{H}$ of H that give a local biholomorphism from $\mathbb{C}^{26}$ to $\mathbb{C}^{26}$. We assume these 26 components $\widehat{H}$ are the H with η dropped, where $\eta \in \{\psi_1, \psi_2, \psi_3, h_{10}, ..., h_{17}, h_{20}, ..., h_{27}, h_{30}, ..., h_{37}\}$. By a local biholomorphic change of coordinates, we assume

$$\widehat{H} = (x_1, x_2, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_7).$$

We still write the remaining components as η.

Case I: If $\eta \in \{\psi_1, \psi_2, \psi_3\}$, without loss of generality, we can assume $\eta = \psi_3$. We first claim that the coefficients of A, B, i.e., a_1, a_2 are zero. This is due to the fact that $t_i^2, w_i^2, 0 \leq i \leq 7$ can only be canceled by $t_i \psi_3, w_i \psi_3$, which do not appear in the expressions of $A(H), ..., G(H)$. We then claim the coefficients of C are zero, for $x_1 x_2$

can not be canceled. Then the coefficients of all D's have to be zero, for each $t_i w_j$ is unique and can not be canceled. Then it follows trivially that all other coefficients are zero.

Case II: If $\eta \in \{h_{10}, ..., h_{17}, h_{20}, ..., h_{27}, h_{30}, ..., h_{37}\}$, without loss of generality, we assume $\eta = h_{37}$. Notice that the only fact we are using about $\widehat{H}$ is that its components are independent variables. For simplicity of notation, we will write

$$\widehat{H} = (x_1, x_2, x_3, y_0, ..., y_7, t_0, ..., t_7, w_0, ..., w_6).$$

We first claim that the coefficient of A is zero. This is due to the fact that $x_2 x_3$ cannot be canceled. We also claim that the coefficient of B is zero. Suppose not. Notice that t_i^2 can only be canceled by $t_i h_{37}$. Then the coefficient of each D_i is non zero for $0 \leq i \leq 7$. Moreover, $x_1 x_3$ can only be canceled by $x_1 h_{37}$. This implies h_{37} has a linear x_3-term. Then, in particular, the $t_7 h_{37}$ term in D_0 will produce a $t_7 x_3$ term. It cannot be canceled by any other terms. This is a contradiction. Similarly, one can show that the coefficient of C is zero. Then we claim the coefficient of D_0 is zero. Otherwise, to cancel the $x_3 y_0$ term, h_{37} needs have a linear x_3 term. Then the term $t_7 h_{37}$ in D_0 will produce a $t_7 x_3$ term, which cannot be canceled by any other term. By the same argument, one can show that the coefficients of all $D_i, 0 \leq i \leq 7$, are zero. Similarly, we can obtain the coefficients of all $E_i, 0 \leq i \leq 7$, are zero. Then we claim the coefficients of all F's have to be zero. This is because each $y_i t_j$ is unique. It can not be canceled out. Finally we get the coefficient of G to be zero. ∎

This also establishes Proposition 5.25. ∎

Remark 7. *By Proposition 5.25, there exist multiindices $\tilde{\beta}^1, ..., \tilde{\beta}^{55}$ with $|\tilde{\beta}^j| \leq 29$, and there exist*

$$z^0 = (x_1^0, x_2^0, x_3^0, y_0^0, ..., y_7^0, t_0^0, .., t_7^0, w_0^0, ..., w_7^0), \quad x_3^0 \neq 0,$$

such that

$$\begin{vmatrix} \dfrac{\partial^{|\beta^1|}(\psi_1(F))}{\partial \tilde{z}^{\tilde{\beta}^1}} & \cdots & \dfrac{\partial^{|\beta^1|}(\psi_{55}(F))}{\partial \tilde{z}^{\tilde{\beta}^1}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial^{|\beta^{55}|}(\psi_1(F))}{\partial \tilde{z}^{\tilde{\beta}^{55}}} & \cdots & \dfrac{\partial^{|\beta^{55}|}(\psi_{55}(F))}{\partial \tilde{z}^{\tilde{\beta}^{55}}} \end{vmatrix} (z^0) \neq 0.$$

Then we set

$$\xi^0 = (0, 0, \xi_3^0, 0, ...0, 0, ..., 0, 0, ..., 0), \xi_3^0 = -\frac{1}{x_3^0}.$$

It is easy to see that $(z^0, \xi^0) \in \mathcal{M} = \{\rho(z, \xi) = 0\}$.

Write

$$\mathcal{L}_i = \frac{\partial}{\partial x_i} - \frac{\frac{\partial \rho}{\partial x_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 1 \le i \le 2;$$

$$\mathcal{L}_{3+i} = \frac{\partial}{\partial y_i} - \frac{\frac{\partial \rho}{\partial y_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 0 \le i \le 7;$$

$$\mathcal{L}_{11+i} = \frac{\partial}{\partial t_i} - \frac{\frac{\partial \rho}{\partial t_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 0 \le i \le 7;$$

$$\mathcal{L}_{19+i} = \frac{\partial}{\partial w_i} - \frac{\frac{\partial \rho}{\partial w_i}(z, \xi)}{\frac{\partial \rho}{\partial x_3}(z, \xi)} \frac{\partial}{\partial x_3}, 0 \le i \le 7.$$

For any 26-multiindex $\beta = (\beta_1, ..., \beta_{26})$, we write $\mathcal{L}^\beta = \mathcal{L}_1^{\beta_1}...\mathcal{L}_{26}^{\beta_{26}}$. Now we define for any 55 collection of $26-$multiindices $\{\beta^1, ..., \beta^{55}\}$,

$$\Lambda(\beta^1, ..., \beta^{55})(z, \xi) := \begin{vmatrix} \mathcal{L}^{\beta^1}(\psi_1(F)) & ... & \mathcal{L}^{\beta^1}(\psi_{55}(F)) \\ ... & ... & ... \\ \mathcal{L}^{\beta^{55}}(\psi_1(F)) & ... & \mathcal{L}^{\beta^{55}}(\psi_{55}(F)) \end{vmatrix}(z, \xi). \qquad (5.58)$$

Note that for any multiindex, $\mathcal{L}^\beta$ evaluating at (z^0, ξ^0) coincides with $\frac{\partial}{\partial \bar{Z}^\beta}$. We have,

Theorem 5.27. *There exists multiindices $\{\beta^1, ..., \beta^{55}\}$, such that*

$$\Lambda(\beta^1, ..., \beta^{55})(z, \xi) \ne 0$$

for (z, ξ) in a small neighborhood of (z^0, ξ^0) and $\beta^1 = (0, ..., 0)$.

Chapter 6

Transversality and flattening of Segre families: Proof of Proposition (II)

We divide the proof in two parts. In the first section, we illustrate the general method to derive a proof and carry it out for Hyperquadric case and Grassmannian case. In the second section, we complete the detailed proof for the remaining cases by the same method.

6.1 Proof of Proposition (II) for Hyperquadrics and Grassmannians

In this section, we prove Proposition (II) for hyperquadrics and Grassmannians. We still use the notations we have set up so far. We equip the space M with the canonical Kähler-Einstein metric ω as described before. We start with the following lemma:

Lemma 6.1. *Let $\widehat{\sigma} : (M, \omega) \to (M, \omega)$ be a holomorphic isometry. In the affine space $\mathcal{A}$, its components consist of rational functions with its degree bounded only by a constant depending on (M, ω).*

Proof of Lemma 6.1: Notice that M has been isometrically embedded into $\mathbb{CP}^N$ through the canonical map defined before. Hence $\widehat{\sigma}$ is the restriction of a unitary transformation. Hence $\widehat{\sigma}$ can be identified with a map of the form:

$$(\tilde{\psi}_0, \tilde{\psi}_1, \tilde{\psi}_2, ..., \tilde{\psi}_N) = (\sum_{j=0}^{N} a_{0j}\psi_j, ..., \sum_{j=0}^{N} a_{ij}\psi_j, ..., \sum_{j=0}^{N} a_{Nj}\psi_j),$$

where $\psi_0 = 1$ and (a_{ij}) is a unitary matrix. Write

$$\Psi(z) : z(\in \mathcal{A}) \mapsto [1, \kappa_1 z_1, \cdots, \kappa_i z_i, \cdots, \kappa_n z_n, o(z^2)] \in \mathbb{CP}^N$$

for the embedding, where $\kappa_i = 1$ or $\sqrt{2}$. $\widehat{\sigma}$ induces a birational self-action σ of $\mathcal{A}$ such that $\Psi(\sigma(z)) = \widehat{\sigma}(\Psi(z))$. Then, from the special form of Ψ, $\sigma(z) = \left(\frac{\tilde{\psi}_1}{\kappa_1 \tilde{\psi}_0}, \frac{\tilde{\psi}_2}{\kappa_2 \tilde{\psi}_0}, ..., \frac{\tilde{\psi}_n}{\kappa_n \tilde{\psi}_0} \right).$

Apparently $\tilde{\psi}_0 \not\equiv 0$. ∎

Theorem 6.2. *Suppose $\xi^0 \in \mathbb{C}^n \setminus \{0\}$. Then for a generic smooth point z^0 on the Segre variety Q_{ξ^0} and a small neighborhood $U \subset \mathbb{C}^n$ of z^0, there is a point $z^1 \in U \cap Q_{\xi^0}$, such that Q_{z^0} and Q_{z^1} are both smooth at ξ^0 and intersect transversally there. Moreover, there is a biholomorphic parametrization $\mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) = (\xi_1, \xi_2, ..., \xi_n)$, with $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$. Here when $1 \leq j \leq 2$, U_j is a small neighborhood of $1 \in \mathbb{C}$. When $3 \leq j \leq n$, U_j is a small neighborhood of $0 \in \mathbb{C}$ with $\mathcal{G}(1, 1, 0, \cdots, 0) = \xi^0$, such that $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}, \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$, and $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times ... \times U_n), \mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times ... \times U_n), s \in U_1, t \in U_2$ are open pieces of Segre varieties. Also, $\mathcal{G}$ consists of algebraic functions with total degree bounded by a constant depending only on (M, ω).*

We first claim that, due to the invariance of the Segre family, we need only to prove the theorem for a special point $0 \neq \xi^0 \in \mathbb{C}^n \subset M$. Indeed, by the invariance property mentioned in § 2, for an isometry σ, $(\sigma, \overline{\sigma})$ preserves the Segre family $\mathcal{M} \subset M \times M$. Here for $p \in \mathbb{CP}^N$, $\overline{\sigma}(p) := \overline{\sigma(\overline{p})}$ as before. Here, we mention that in the statement of the theorem, we assume that z^0 is a generic smooth point because under this transformation, some smooth points on Q_{ξ^0} may be mapped into the hyperplance of M at infinity, which can not be chosen as our z^0.

We now proceed to the proof of Theorem 6.2 by choosing a good point ξ^0. We carry out the proof for the case of hyperquadrics and Grassmannian spaces here. The proof for the remaining cases is similar and will be included in next section.

Proof of Theorem 6.2: **Case 1. Hyperquadrics**: Suppose M is the hyperquadric. Then the defining equation for the Segre family is

$$\rho(z, \xi) = 1 + \sum_{i=1}^n z_i \xi_i + \frac{1}{4}(\sum_{i=1}^n z_i^2)(\sum_{i=1}^n \xi_i^2) = 0.$$

We choose $\xi^0 = (1, 0, 0, ..., 0)$. Hence $Q_{\xi^0} = \{z : \rho(z, \xi^0) = 1 + z_1 + \frac{1}{4}(\sum_{i=1}^n z_i^2) = 0\}$. We compute the gradient of $\rho(z, \xi^0)$ as follows: $\nabla \rho(z, \xi^0) = (1 + \frac{1}{2}z_1, \frac{1}{2}z_2, ..., \frac{1}{2}z_n)$. Notice that Q_{ξ^0} is smooth except at $(-2, 0, ..., 0)$, namely, we have $\nabla \rho(z, \xi^0) \neq 0$ away from $(-2, 0, \cdots, 0)$. For a smooth point $z^0(\neq (-2, 0, \cdots, 0))$ of Q_{ξ^0} , we choose a

neighborhood U of z^0 in $\mathbb{C}^n$ such that $U \cap Q_{\xi^0}$ is a smooth piece of Q_{ξ^0}. Pick also

$z^1(\neq z^0) \in U \cap Q_{\xi_0}$ and compute the gradient of the defining function of Q_{z^0} and Q_{z^1}

at $\xi^0 = (1, 0, ..., 0)$, respectively. Recall

$$Q_{z^s} = \{\xi | \rho(z^s, \xi) = 1 + \sum_{i=1}^{n} z_i^s \xi_i + \frac{1}{4}(\sum_{i=1}^{n}(z_i^s)^2)(\sum_{i=1}^{n} \xi_i^2) = 0\}, \text{ for } s = 0, 1.$$

$$\begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0=(1,0,...,0)} \\ \nabla\rho(z^1, \xi)|_{\xi^0=(1,0,...,0)} \end{pmatrix} = \begin{pmatrix} z_1^0 + \frac{1}{2}\sum_{i=1}^{n}(z_i^0)^2 & z_2^0 & z_3^0 & \cdots & z_n^0 \\ z_1^1 + \frac{1}{2}\sum_{i=1}^{n}(z_i^1)^2 & z_2^1 & z_3^1 & \cdots & z_n^1 \end{pmatrix} = \begin{pmatrix} -2 - z_1^0 & z_2^0 & z_3^0 & \cdots & z_n^0 \\ -2 - z_1^1 & z_2^1 & z_3^1 & \cdots & z_n^1 \end{pmatrix}$$

The second equality is simplified by making use of the fact that $z^0, z^1 \in Q_{\xi^0=(1,0,...,0)}$,

which implies that $0 = 1 + z_1^0 + \frac{1}{4}\sum_{i=1}^{n}(z_i^0)^2 = 1 + z_1^1 + \frac{1}{4}\sum_{i=1}^{n}(z_i^1)^2$. Hence,

$$\text{rank}\begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0=(1,0,...,0)} \\ \nabla\rho(z^1, \xi)|_{\xi^0=(1,0,...,0)} \end{pmatrix} = \text{rank}\begin{pmatrix} -2 - z_1^0 & z_2^0 & \cdots & z_n^0 \\ -2 - z_1^1 & z_2^1 & \cdots & z_n^1 \end{pmatrix} = \text{rank}\begin{pmatrix} -2 - z_1^0 & z_2^0 & \cdots & z_n^0 \\ -\Delta z_1^1 & \Delta z_2^1 & \cdots & \Delta z_n^1 \end{pmatrix}$$

$$= \text{rank}\begin{pmatrix} 2 + z_1^0 & z_2^0 & \cdots & z_n^0 \\ \Delta z_1^1 & \Delta z_2^1 & \cdots & \Delta z_n^1 \end{pmatrix} = \text{rank}\begin{pmatrix} & & \nabla\rho(z, \xi^0)|_{z^0} & \\ \Delta z_1^1 & \Delta z_2^1 & \cdots & \Delta z_n^1 \end{pmatrix},$$

where $\Delta z_i^1 := z_i^1 - z_i^0$. Notice that z^0 is a smooth point on Q_{ξ_0}. Hence $\nabla\rho(z, \xi^0)$ is

transversal to the tangent space of Q_{ξ^0} at z^0. If we choose $z^1 \in Q_{\xi^0}$ close enough to z^0,

which ensures $(\Delta z_1^1, ..., \Delta z_n^1)$ close enough to tangent space of Q_{ξ^0} at z^0, we then get

$$\text{rank}\begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0=(1,0,...,0)} \\ \nabla\rho(z^1, \xi)|_{\xi^0=(1,0,...,0)} \end{pmatrix} = \text{rank}\begin{pmatrix} & & \nabla\rho(z, \xi^0)|_{z^0} & \\ \Delta z_1^1 & \Delta z_2^1 & \cdots & \Delta z_n^1. \end{pmatrix} = 2.$$

We assume, without loss of generality, that $\frac{\partial(\rho(z^0, \xi), \rho(z^1, \xi))}{\partial(\xi_1, \xi_2)} \neq 0$ at ξ^0.

Now we introduce new variables $\tilde{\xi}_1, ..., \tilde{\xi}_n$ and consider the following system of equations:

$$\begin{cases} P_1 : 1 + \sum_{i=1}^{n}(\tilde{\xi}_1 z_i^0)\xi_i + \frac{1}{4}(\sum_{i=1}^{n}(\tilde{\xi}_1)^2(z_i^0)^2)(\sum_{i=1}^{n}\xi_i^2) & = 0 \\ P_2 : 1 + \sum_{i=1}^{n}(\tilde{\xi}_2 z_i^1)\xi_i + \frac{1}{4}(\sum_{i=1}^{n}(\tilde{\xi}_2)^2(z_i^1)^2)(\sum_{i=1}^{n}\xi_i^2) & = 0 \\ P_3 : & \tilde{\xi}_3 - \xi_3 & = 0 \\ ... & ... \\ P_n : & \tilde{\xi}_n - \xi_n & = 0 \end{cases}$$

Then we have $\frac{\partial(P_1,...,P_n)}{\partial(\xi_1,...,\xi_n)}|_A \neq 0$ and $\frac{\partial(P_1,...,P_n)}{\partial(\tilde{\xi}_1,...,\tilde{\xi}_n)}|_A \neq 0$ where

$$A = (\tilde{\xi}_1, ..., \tilde{\xi}_n; \xi_1, ..., \xi_n) = (1, 1, 0, ..., 0; 1, 0, ..., 0).$$

By Lemma 4.3, we get the needed algebraic flattening with total degree bounded only by (M, ω). This completes the proof of Theorem 6.2 in the hyperquadric case.

Case 2. Grassmannians: Pick $\xi^0 = (\xi_{11}^0, \xi_{12}^0, ..., \xi_{pq}^0) = (1, 0, ..., 0)$. The defining function for the Segre family associated with this point is as follows:

$\rho(z, \xi) = 1 + z_{11}\xi_{11} + z_{12}\xi_{12} + ... + z_{1q}\xi_{1q} + z_{21}\xi_{21} + ... + z_{p1}\xi_{p1} + \sum_{i,j \neq 1} z_{ij}\xi_{ij} +$

$\sum_{i,j \geq 2}(z_{11}z_{ij} - z_{i1}z_{1j})(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{(i,j),(k,l)\neq(1,1)}(z_{ij}z_{kl} - z_{il}z_{jk})(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{jk}) +$

higher order terms.

Then $Q_{\xi^0} = \{z | \rho(z, \xi^0) = 1 + z_{11} = 0\}, \nabla\rho(z, \xi^0) = (1, 0, 0, ..., 0)$. Hence Q_{ξ_0} is smooth. For $z \in Q_{\xi^0}$, we have $z = (-1, z_{12}, ..., z_{1q}, z_{21}, ..., z_{p1}, ..., z_{ij}, ..., z_{pq})$. Pick $z^0, z^1 \in Q_{\xi^0}$. Then

$Q_{z^s} = \{\xi | 0 = \rho(z^s, \xi) = 1 + z_{11}^s\xi_{11} + z_{12}^s\xi_{12} + ... + z_{1q}^s\xi_{1q} + z_{21}^s\xi_{21} + ... + z_{p1}^s\xi_{p1} +$

$\sum_{i,j \neq 1} z_{ij}^s\xi_{ij} + \sum_{i,j \geq 2}(z_{11}^s z_{ij}^s - z_{i1}^s z_{1j}^s)(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{(i,j),(k,l)\neq(1,1)}(z_{ij}^s z_{kl}^s - z_{il}^s z_{jk}^s)(\xi_{ij}\xi_{kl} -$

$\xi_{il}\xi_{jk}) + \text{high order terms}\}$,

for $s = 0, 1$.

We then compute their gradients as follows:

$$\begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = \begin{pmatrix} \frac{\partial\rho(z^0,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{12}} & \cdots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{1q}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{21}} & \cdots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{p1}} & \cdots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{pq}} \\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{12}} & \cdots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{1q}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{21}} & \cdots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{p1}} & \cdots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{pq}} \end{pmatrix}\Bigg|_{\xi^0}$$

$$= \begin{pmatrix} -1 & z_{12}^0 & \cdots & z_{1q}^0 & z_{21}^0 & \cdots & z_{p1}^0 & -z_{i1}^0 z_{1j}^0 & \cdots \\ -1 & z_{12}^1 & \cdots & z_{1q}^1 & z_{21}^1 & \cdots & z_{p1}^1 & -z_{i1}^1 z_{1j}^1 & \cdots \end{pmatrix}.$$

Thus, we have

$$\text{rank}\begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = \text{rank}\begin{pmatrix} -1 & z_{12}^0 & \cdots & z_{p1}^0 & -z_{i1}^0 z_{1j}^0 & \cdots \\ 0 & \Delta z_{12}^1 & \cdots & \Delta z_{p1}^1 & (-z_{i1}^0 \Delta z_{1j}^1 - z_{1j}^0 \Delta z_{i1}^1 - \Delta z_{i1}^1 \Delta z_{1j}^1) & \cdots \end{pmatrix},$$

where $\Delta z_{ij}^1 = z_{ij}^1 - z_{ij}^0$. Hence, if we choose z^1 such that $z_{12}^1 \neq z_{12}^0$, Then the rank equals to 2. Hence Q_{z^0} and Q_{z^1} are smooth and intersect transversally at ξ^0.

Without loss of generality, assume $\frac{\partial(\rho(z^0,\xi),\rho(z^1,\xi))}{\partial(\xi_{11},\xi_{12})} \neq 0$ at ξ^0. Now we introduce new variables $\tilde{\xi}_{11},...,\tilde{\xi}_{pq}$ and set up the system:

$$
\begin{cases}
P_{11}: & \rho(z^0, \tilde{\xi}_{11}\xi) = 0 \\[2mm]
P_{12}: & \rho(z^1, \tilde{\xi}_{12}\xi) = 0 \\[2mm]
P_{13}: & \tilde{\xi}_{13} - \xi_{13} = 0 \\[2mm]
... & ... \\[2mm]
P_{pq}: & \tilde{\xi}_{pq} - \xi_{pq} = 0
\end{cases}
$$

Then $\frac{\partial(P_{11},...,P_{pq})}{\partial(\xi_{11},...,\xi_{pq})}|_A$, $\frac{\partial(P_{11},...,P_{pq})}{\partial(\tilde{\xi}_{11},...,\tilde{\xi}_{pq})}|_A \neq 0$, where $A = (\tilde{\xi}_{11},...,\tilde{\xi}_{pq},\xi_{11},...,\xi_{pq}) = (1,1,0,...,0,1,0,...,0)$. By Lemma 4.3, we get the needed algebraic flattening. $\blacksquare$

6.2 Continuation of the proof of Theorem 6.2

We first establish the second part of Theorem 6.2 by assuming the first part of Theorem 6.2 is true. Namely, suppose $\xi^0 \in \mathbb{C}^n \setminus \{0\}$ and z^0 and z^1 are smooth points on the Segre variety Q_{ξ^0} such that Q_{z^0} and Q_{z^1} are both smooth at ξ^0 and intersect transversally there. We shall prove that there is a biholomorphic parametrization $\mathcal{G}(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) = (\xi_1, \xi_2, ..., \xi_n)$ with $(\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \in U_1 \times U_2 \times ... \times U_n \subset \mathbb{C}^n$. Here for $j = 1, 2$, U_j is a small neighborhood of $1 \in \mathbb{C}$; for $3 \leq j \leq n$, U_j is a small neighborhood of $0 \in \mathbb{C}$ with $\mathcal{G}(1, 1, 0, \cdots, 0) = \xi^0$. Moreover $\mathcal{G}(\{\tilde{\xi}_1 = 1\} \times U_2 \times ... \times U_n) \subset Q_{z^0}$, $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = 1\} \times U_3 \times ... \times U_n) \subset Q_{z^1}$ and $\mathcal{G}(\{\tilde{\xi}_1 = t\} \times U_2 \times ... \times U_n)$ and $\mathcal{G}(U_1 \times \{\tilde{\xi}_2 = s\} \times U_3 \times ... \times U_n)$ are open pieces of Segre varieties for $s \in U_1$ and $t \in U_2$. Also $\mathcal{G}$ consists of algebraic functions with total degree bounded by a constant depending only on (M, ω).

Implied by the first part of Theorem 6.2, we have

$$
\text{rank} \begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = 2.
$$

Without loss of generality, we can assume $\frac{\partial(\rho(z^0,\xi),\rho(z^1,\xi))}{\partial(\xi_1,\xi_2)} \neq 0$ at ξ^0. Following the

same method used in § 6.1, we introduce new variables $\tilde{\xi}_1, ..., \tilde{\xi}_n$ and set up the system:

$$\begin{cases} P_1: & \rho(z^0, \tilde{\xi}_1\xi) = 0 \\[2mm] P_2: & \rho(z^1, \tilde{\xi}_2\xi) = 0 \\[2mm] P_3: & \tilde{\xi}_3 - \xi_3 = 0 \\[2mm] ... & ... \\[2mm] P_n: & \tilde{\xi}_n - \xi_n = 0. \end{cases}$$

Then $\frac{\partial(P_1,...,P_n)}{\partial(\xi_1,...,\xi_n)}|_A \neq 0$ and $\frac{\partial(P_1,...,P_n)}{\partial(\tilde{\xi}_1,...,\tilde{\xi}_n)}|_A \neq 0$, where $A = (\tilde{\xi}_1, ..., \tilde{\xi}_n, \xi_1, ..., \xi_n) = (1, 1, 0, ..., 0, 1, 0, ..., 0)$.
By Lemma 4.3, we get the needed algebraic flattening with the bound total degree.

Next, we proceed to prove the first part of Theorem 6.2. It suffices to find a sufficiently close point z^1 to z^0 such that

$$\text{rank}\begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = 2.$$

We establish the above rank estimate case by case as follows:

Case 3. Symplectic Grassmannians: Pick $\xi_0 = (1, 0, 0, ..., 0)$. The defining equation of the Segre family is $\rho = 1 + \sum_{i=1}^{n} z_{ii}\xi_{ii} + 2\sum_{i<j} z_{ij}\xi_{ij} + 2\sum_{2\leq i<j}(z_{11}z_{ij} - z_{1j}z_{i1})(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{i=2}^{n}(z_{11}z_{ii} - z_{1i}^2)(\xi_{11}\xi_{ii} - \xi_{1i}^2) + \sum_{i<k,j<l,(i,j)\neq(1,1)}(z_{ij}z_{kl} - z_{il}z_{kj})(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{kj}) + $ high order terms, where $z_{ji} := z_{ij}$ for $j > i$.

$Q_{\xi^0} = \{z|\rho(z, \xi^0) = 1 + z_{11} = 0\}, \nabla\rho(z, \xi^0) = (1, 0, ..., 0)$. Hence Q_{ξ^0} is smooth, and for $z \in Q_{\xi^0}$ we have $z = (-1, z_{12}, z_{22}, z_{13}, ..., z_{(n-1)n})$. Pick $z^0, z^1 \in Q_{\xi^0}$. Then

$Q_{z^s} = \{\xi|0 = \rho(z^s, \xi) = 1 + \sum_{i=1}^{n} z_{ii}^s\xi_{ii} + 2\sum_{i<j} z_{ij}^s\xi_{ij} + 2\sum_{2\leq i<j}(z_{11}^s z_{ij}^s - z_{1j}^s z_{i1}^s)(\xi_{11}\xi_{ij} - \xi_{i1}\xi_{1j}) + \sum_{i=2}^{n}(z_{11}^s z_{ii}^s - (z_{1i}^s)^2)(\xi_{11}\xi_{ii} - \xi_{1i}^2) + \sum_{i<k,j<l,(i,j)\neq(1,1)}(z_{ij}^s z_{kl}^s - z_{il}^s z_{kj}^s)(\xi_{ij}\xi_{kl} - \xi_{il}\xi_{kj}) + $ high order terms$\}$, for $s = 0, 1$.

$$\begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = \begin{pmatrix} \frac{\partial\rho(z^0,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{12}} & \cdots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{1n}} & \cdots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{ij}} & \cdots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{nn}} \\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_{11}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{12}} & \cdots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{1n}} & \cdots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{ij}} & \cdots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{nn}} \end{pmatrix}\Bigg|_{\xi^0}$$

$$= \begin{pmatrix} -1 & 2z_{12}^0 & 2z_{13}^0 & \cdots & 2z_{1n}^0 & -(z_{12}^0)^2 & -2z_{12}^0 z_{13}^0 & \cdots & -(2 - \delta_{ij})z_{1j}^0 z_{1i}^0 & \cdots \\ -1 & 2z_{12}^1 & 2z_{13}^1 & \cdots & 2z_{1n}^1 & -(z_{12}^1)^2 & -2z_{12}^1 z_{13}^1 & \cdots & -(2 - \delta_{ij})z_{1j}^1 z_{1i}^1 & \cdots \end{pmatrix}.$$

Hence, we have

$$\text{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\\nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix} = \text{rank}\begin{pmatrix}-1 & 2z_{12}^0 & 2z_{13}^0 & \dots & 2z_{1n}^0 & -(z_{12}^0)^2 & -2z_{12}^0 z_{13}^0 & \dots & -(2-\delta_{ij})z_{1j}^0 z_{1i}^0 & \dots \\ -1 & 2z_{12}^1 & 2z_{13}^1 & \dots & 2z_{1n}^1 & -(z_{12}^1)^2 & -2z_{12}^1 z_{13}^1 & \dots & -(2-\delta_{ij})z_{1j}^1 z_{1i}^1 & \dots\end{pmatrix}$$

$$= \text{rank}\begin{pmatrix}-1 & 2z_{12}^0 & 2z_{13}^0 & \dots & 2z_{1n}^0 & -(2-\delta_{ij})z_{1j}^0 z_{1i}^0 & \dots \\ 0 & 2\Delta z_{12}^1 & 2\Delta z_{13}^1 & \dots & 2\Delta z_{1n}^1 & (2-\delta_{ij})\{z_{1j}^1\Delta z_{1i}^1 + \Delta z_{1j}^1 z_{1i}^1 - \Delta z_{1j}^1\Delta z_{1i}^1\} & \dots\end{pmatrix},$$

where $\Delta z_{ij}^1 = z_{ij}^1 - z_{ij}^0$. If we pick $z_{12}^1 \neq z_{12}^0$, then the above rank is 2.

Case 4. Orthogonal Grassmannians: Here we use the Pfaffian embedding stated in §2. Fixing $\xi^0 = (\xi_{12}^0, \xi_{13}^0, \xi_{23}^0, ..., \xi_{(n-1)n}^0) = (1, 0, ..., 0)$, the defining function of the Segre family is given by $\rho = 1 + \sum_{i<j} z_{ij}\xi_{ij} + \sum_{2<i<j}(z_{12}z_{ij} - z_{1i}z_{2j} + z_{1j}z_{2i})(\xi_{12}\xi_{ij} - \xi_{1i}\xi_{2j} + \xi_{1j}\xi_{2i}) + \sum_{i<j<k<l,\{1,2\}\not\subset\{i,j,k,l\}}(z_{ij}z_{kl} - z_{ik}z_{jl} + z_{il}z_{jk})(\xi_{ij}\xi_{kl} - \xi_{ik}\xi_{jl} + \xi_{il}\xi_{jk}) +$ high order terms. Note here we use the notation $z_{ji} := -z_{ij}$ for $j > i$.

Note $Q_{\xi^0} = \{z | 0 = \rho(z, \xi^0) = 1 + z_{12}\}$. Hence it is smooth. Since $z \in Q_{\xi^0}$, we have $z = (-1, z_{13}, ..., z_{(n-1)n})$. Pick z^0, $z^1 \in Q_{\xi^0}$. Then

$Q_{z^s} = \{\xi | 0 = \rho(z^s, \xi) = 1 + \sum_{i<j} z_{ij}^s\xi_{ij} + \sum_{2<i<j}(z_{12}^s z_{ij}^s - z_{1i}^s z_{2j}^s + z_{1j}^s z_{2i}^s)(\xi_{12}\xi_{ij} - \xi_{1i}\xi_{2j} + \xi_{1j}\xi_{2i})$

$+ \sum_{i<j<k<l,\{1,2\}\not\subset\{i,j,k,l\}}(z_{ij}^s z_{kl}^s - z_{ik}^s z_{jl}^s + z_{il}^s z_{jk}^s)(\xi_{ij}\xi_{kl} - \xi_{ik}\xi_{jl} + \xi_{il}\xi_{jk}) + h.o.t.s.\}$, for $s = 0, 1$.

$$\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\\nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix} = \begin{pmatrix}\frac{\partial\rho(z^0,\xi)}{\partial\xi_{12}} & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{13}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{1n}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{ij}} & \dots & \frac{\partial\rho(z^0,\xi)}{\partial\xi_{(n-1)n}} \\ \frac{\partial\rho(z^1,\xi)}{\partial\xi_{12}} & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{13}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{1n}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{ij}} & \dots & \frac{\partial\rho(z^1,\xi)}{\partial\xi_{(n-1)n}}\end{pmatrix}\Big|_{\xi^0}$$

$$= \begin{pmatrix}-1 & z_{13}^0 & \dots & z_{1n}^0 & \dots & z_{2n}^0 & (-z_{13}^0 z_{24}^0 + z_{14}^0 z_{23}^0)a & \dots & (-z_{1i}^0 z_{2j}^0 + z_{1j}^0 z_{2i}^0)a & \dots \\ -1 & z_{13}^1 & \dots & z_{1n}^1 & \dots & z_{2n}^1 & (-z_{13}^1 z_{24}^1 + z_{14}^1 z_{23}^1)a & \dots & (-z_{1i}^1 z_{2j}^1 + z_{1j}^1 z_{2i}^1)a & \dots\end{pmatrix}.$$

Hence,

$$\text{rank}\begin{pmatrix}\nabla\rho(z^0,\xi)|_{\xi^0}\\\nabla\rho(z^1,\xi)|_{\xi^0}\end{pmatrix} = \text{rank}\begin{pmatrix}-1 & z_{13}^0 & \dots & z_{1n}^0 & \dots & z_{2n}^0 & \dots \\ 0 & \Delta z_{13}^1 & \dots & \Delta z_{1n}^1 & \dots & \Delta z_{2n}^1 & \dots\end{pmatrix}.$$

Here $\Delta z_{ij}^1 = z_{ij}^1 - z_{ij}^0$. If we choose $z_{13}^1 \neq z_{13}^0$, then the rank is 2.

Case 5. M_{16}: Pick $\xi^0 = (\kappa_0^0, \kappa_1^0, ..., \kappa_7^0, \eta_0^0, \eta_1^0, ..., \eta_7^0) = (1, 0, ..., 0)$, $z^0 \in Q_{\xi^0}$. The defining equation of the Segre family is

$1 + x_0\kappa_0 + x_1\kappa_1 + ... + x_7\kappa_7 + y_0\eta_0 + y_1\eta_1 + ... + y_7\eta_7 + (x_0y_0 + x_1y_1 + ...)(\kappa_0\eta_0 + \kappa_1\eta_1 + ...) + (-y_0x_1 + y_1x_0 + ...)(-\eta_0\kappa_1 + \eta_1\kappa_0 + ...) + ... + (x_0^2 + x_1^2 + ... + x_7^2)(\kappa_0{}^2 + \kappa_1{}^2... + \kappa_7{}^2) + (y_0^2 + y_1^2 + ... + y_7^2)(\eta_0^2 + \eta_1^2 + ... + \eta_7^2) = 0.$

$Q_{\xi^0} = \{z | \rho(z, \xi^0) = 1 + x_0 + (x_0^2 + x_1^2 + ... + x_7^2) = 0\}$, and $\nabla\rho(z, \xi^0)|_{z^0} = (1 + 2x_0, 2x_1, ..., 2x_7^0, 0, ..., 0)$. Hence Q_{ξ^0} is smooth. Pick $z^0, z^1 \in Q_{\xi^0}$. Then

$Q_{z^s} = \{\xi | 0 = \rho(z^s, \xi) = 1 + x_0^s\kappa_0 + x_1^s\kappa_1 + ... + x_7^s\kappa_7 + y_0^s\eta_0 + y_1^s\eta_1 + ... + y_7^s\eta_7 + (x_0^sy_0^s + x_1^sy_1^s + ...)(\kappa_0\eta_0 + \kappa_1\eta_1 + ...) + (-y_0^sx_1^s + y_1^sx_0^s + ...)(-\eta_0\kappa_1 + \eta_1\kappa_0 + ...) + ... + ((x_0^s)^2 + (x_1^s)^2 + ... + (x_7^s)^2)(\kappa_0{}^2 + \kappa_1{}^2 + ... + \kappa_7{}^2) + ((y_0^s)^2 + (y_1^s)^2 + ... + (y_7^s)^2)(\eta_0^2 + \eta_1^2 + ... + \eta_7^2)\}$,

for $s = 0, 1$.

$$\text{rank} \begin{pmatrix} \nabla\rho(z^0, \xi)|_{\xi^0} \\ \nabla\rho(z^1, \xi)|_{\xi^0} \end{pmatrix} \geq \text{rank} \begin{pmatrix} \frac{\partial\rho(z^0,\xi)}{\partial\kappa_0} & \frac{\partial\rho(z^0,\xi)}{\partial\kappa_1} & \cdots & \frac{\partial\rho(z^0,\xi)}{\partial\kappa_7} \\ \frac{\partial\rho(z^1,\xi)}{\partial\kappa_0} & \frac{\partial\rho(z^1,\xi)}{\partial\kappa_1} & \cdots & \frac{\partial\rho(z^1,\xi)}{\partial y_7} \end{pmatrix}\Big|_{\xi^0}$$

$$= \text{rank} \begin{pmatrix} -2 - x_0^0 & x_1^0 & x_2^0 & \cdots & x_7^0 \\ -2 - x_0^1 & x_1^1 & x_2^1 & \cdots & x_7^1 \end{pmatrix}. \tag{C}$$

Since $(-2 - x_0^0, x_1^0, x_2^0, \cdots, x_7^0) \neq (0, ..., 0)$, we can pick z^1 sufficiently close to z^0, such that the above rank is 2. That is because Q_{ξ^0} is irreducible and the subvarieties, defined by 2×2 minors of the last matrix in (C), are thin subsets of Q_{ξ^0}.

Case 6. M_{27}: Take $\xi^0 = (\xi_1^0, \xi_2^0, \xi_3^0, \eta_0^0, \eta_1^0, ..., \eta_7^0, \kappa_0^0, \kappa_1^0, ..., \kappa_7^0, \tau_0^0, \tau_1^0, ..., \tau_7^0) = (1, 0, ..., 0)$. The defining function of the Segre family is $1 + r_z \cdot r_\xi$ where

$r_z = (x_1, x_2, x_3, y_0, ..., y_7, z_0, ..., z_7, w_0, ..., w_7, A, B, C, D_0, ...D_7, E_0, ..., E_7, F_0, ..., F_7, G)$

$r_\xi = (\xi_1, \xi_2, \xi_3, ..., \eta_7, ..., \kappa_7, ..., \tau_7, A(\xi), B(\xi), C(\xi), ..., D_7(\xi), ..., E_7(\xi), ..., G(\xi))$.

Here A, B, C, D_i, E_i, F_i are homogeneous quadratic polynomials; G is a homogeneous cubic polynomial defined in 2.4.

For our purpose here, we present terms only involving ξ_1, ξ_2, and omit those involving $\xi_3, \eta_0, \eta_1, ..., \eta_7, \kappa_0, \kappa_1, ..., \kappa_7, \tau_0, \tau_1, ..., \tau_7$ as follows: $\rho(z, \xi) = 1 + x_1\xi_1 + x_2\xi_2 + ... + (x_1x_2 - (\sum_{i=0}^7 y_i^2))(\xi_1\xi_2 - (\sum_{i=0}^7(\tau_i)^2)) + \cdots$.

$Q_{\xi^0} = \{z | \rho(z, \xi^0) = 1 + x_1 = 0\}, \nabla \rho(z, \xi^0) = (1, 0, 0, ..., 0)$. Hence Q_{ξ^0} is smooth and

for $z \in Q_{\xi^0}$, we have $z = (-1, x_2, x_3, ...,)$. Pick $z^0, z^1 \in Q_{\xi^0}$. Then

$Q_{z^s} = \{\xi | 0 = \rho(z^s, \xi) = 1 + x_1^s \xi_1 + x_2^s \xi_2 + ... + (x_1^s x_2^s - (\sum_{i=0}^{7}(y_i^s)^2))(\xi_1 \xi_2 - (\sum_{i=0}^{7}(\tau_i)^2)) + ...\}$, for $s = 0, 1$.

$$\text{rank} \begin{pmatrix} \nabla \rho(z^0, \xi)|_{\xi^0} \\ \nabla \rho(z^1, \xi)|_{\xi^0} \end{pmatrix} = \text{rank} \begin{pmatrix} \frac{\partial \rho(z^0, \xi)}{\partial \xi_1} & \frac{\partial \rho(z^0, \xi)}{\partial \xi_2} & \frac{\partial \rho(z^0, \xi)}{\partial \xi_3} & \cdots & \frac{\partial \rho(z^0, \xi)}{\partial \eta_7} & \cdots & \frac{\partial \rho(z^0, \xi)}{\partial \kappa_7} & \cdots & \frac{\partial \rho(z^0, \xi)}{\partial \tau_7} \\ \frac{\partial \rho(z^1, \xi)}{\partial \xi_1} & \frac{\partial \rho(z^1, \xi)}{\partial \xi_2} & \frac{\partial \rho(z^1, \xi)}{\partial \xi_3} & \cdots & \frac{\partial \rho(z^1, \xi)}{\partial \eta_7} & \cdots & \frac{\partial \rho(z^1, \xi)}{\partial \kappa_7} & \cdots & \frac{\partial \rho(z^1, \xi)}{\partial \tau_7} \end{pmatrix} \Bigg|_{\xi^0}$$

$$\geq \text{rank} \begin{pmatrix} \frac{\partial \rho(z^0, \xi)}{\partial \xi_1} & \frac{\partial \rho(z^0, \xi)}{\partial \xi_2} \\ \frac{\partial \rho(z^1, \xi)}{\partial \xi_1} & \frac{\partial \rho(z^1, \xi)}{\partial \xi_2} \end{pmatrix} \Bigg|_{\xi^0} = \text{rank} \begin{pmatrix} -1 & -(\sum_{i=0}^{7}(y_i^0)^2) \\ -1 & -(\sum_{i=0}^{7}(y_i^1)^2) \end{pmatrix} \Bigg|_{\xi^0} \geq 2,$$

for those z^1's such that $\sum_{i=0}^{7}(y_i^1)^2 \neq \sum_{i=0}^{7}(y_i^0)^2$. This can be done in any small neighborhood of z^0; for $\{z | \sum_{i=0}^{7}(y_i)^2 = B\}$ is a thin set in $\{z | 0 = 1 + x_1\}$ for each fixed $B \in \mathbb{C}$.

This completes the proof of the flattening theorem. ∎

Chapter 7

Irreducibility of Segre varieties: Proof of Proposition (III)

In this chapter we will establish Proposition (III). We prove results on the irreducibility of the potential function ρ, Segre varieties and the Segre family. We still adapt the previously used notation and assume that M is an irreducible Hermitian symmetric space of compact type of complex dimension n, which has been minimally embedded into a projective space as described before.

Lemma 7.1. *Each Segre variety is an irreducible algebraic subvariety.*

Proof of lemma 7.1: For a minimally embedded Hermitian symmetric space, since all Segre varieties are unitarily equivalent, it suffices to prove the lemma for a single Segre variety. Without lost of generality, we take $z = (0, ..., 0) \in \mathcal{A} \subset M$. Therefore, the corresponding Segre variety Q_z^* is the hyperplane section $M \setminus \mathcal{A}$, which is of pure dimension. From the classical algebraic geometry [GH], when M is an irreducible Hermitian symmetric space of compact type, the hyperplane section at infinity in the minimal canonical embedding case is a union of Schubert cells. Moreover as shown in [CMP], the top dimensional piece is equivalent to $\mathbb{C}^{n-1}$ and the others are of codimension at least two. Hence, the smooth points of Q_z are connected and thus Q_z is irreducible. ∎

As a corollary of this lemma, we conclude that for each $z \in \mathbb{C}^n$, the defining function $\rho(z, \cdot)$ of Q_z has to be a power of one irreducible factor. However, as in the proof of Theorem 6.2, for some $a(\neq 0) \in \mathbb{C}^n$, $d_\xi \rho(a, \xi)$ is not identically zero along Q_a. Next, we use this property and the symmetric property of M to prove the following:

Proposition 7.2. *For any $b \in \mathcal{A}$ with $b \neq (0, ..., 0)$, $\rho(b, \xi)$ ($\rho(z, b)$, respectively) is irreducible as a polynomial of ξ (as a polynomial in z, respectively).*

Proof of proposition 7.2: Since $\rho(z, \xi) = \rho(\xi, z)$, we need just to verify the first

statement. Let a be as above. For $b \in \mathcal{A}$, there is $\widehat{\sigma} \in \mathrm{Isom}(M, \omega) \cap SU(N+1, \mathbb{C})$ such that $\widehat{\sigma}(a) = b$. (Notice that $\widehat{\sigma}$ is represented by a unitary action.) By Lemma 6.1, let $\sigma = (\frac{l_1}{\kappa_1 l_0}, ..., \frac{l_n}{\kappa_n l_0})$ be the representation of $\widehat{\sigma}$ in $\mathcal{A}$ with $l'_j s$ polynomials in z. Write $\Psi = [1, r_z]$ for the embedding of $\mathcal{A}$ in $\mathbb{P}^N$. Then from the definition of $\rho(z, \overline{z})$, we have

$$\rho(z, \overline{z}) = ||\Psi(z)||^2 = \Psi \cdot \overline{\Psi}^t = (\widehat{\sigma}\Psi) \cdot \overline{(\widehat{\sigma}\Psi)}^t.$$

Lemma 7.3. $(\widehat{\sigma}\Psi) \cdot \overline{(\widehat{\sigma}\Psi)}^t = |l_0(\Psi)|^2 \cdot ||\Psi(\sigma(z))||^2 = |l_0(\Psi)|^2 \cdot \rho(\sigma(z), \overline{\sigma(z)}).$

Proof. Writing $\Psi(z) = [1, r_z] = [1, \psi_1(z), \cdots, \psi_N(z)]$. Then the identity $\Psi(\sigma(z)) = \widehat{\sigma}(\Psi(z))$ obtained in the proof of Lemma 6.1 yields that,

$$(\psi_1(\sigma(z)), \cdots, \psi_N(\sigma(z))) = \left(\frac{\tilde{\psi}_1(\Psi(z))}{\tilde{\psi}_0(\Psi(z))}, \cdots, \frac{\tilde{\psi}_N(\Psi(z))}{\tilde{\psi}_0(\Psi(z))} \right).$$

Here $\tilde{\psi}_j = l_j$ for $0 \leq j \leq n$ and $\widehat{\sigma}(z) = [\tilde{\phi}_0, \cdots, \tilde{\phi}_N]$ as in the proof of Lemma 6.1. Then we have

$$(\hat{\sigma}\Psi) \cdot \overline{(\widehat{\sigma}\Psi)}^t = \sum_{j=0}^{N} |\tilde{\psi}_j(\Psi(z))|^2 = \left(1 + \sum_{j=1}^{N} |\psi_j(\sigma(z))|^2 \right) |\tilde{\psi}_0(\Psi(z))|^2 = |l_0(\Psi)|^2 \cdot ||\Psi(\sigma(z))||^2.$$

This establishes the lemma. $\square$

The Lemma 7.3 yields $\rho(z, \overline{z}) = |l_0(\Psi)|^2 \cdot \rho(\sigma(z), \overline{\sigma(z)})$. Complexifying the identity and substituting z by a, we have:

$$l_0(\Psi)(a) \cdot \overline{l_0(\Psi)}(\xi) \cdot \rho(b, \overline{\sigma}(\xi)) = \rho(a, \xi), \tag{7.1}$$

where $l_0(\Psi)(a) \neq 0$, $l_0(\Psi)(\xi)$, $\rho(a, \xi)$ are polynomials in ξ and $\sigma(\xi)$ is a rational map in ξ. Now supposing $\rho(b, \xi) = f^l(\xi), l \geq 2$, we have $\rho(b, \overline{\sigma}(\xi)) = (f(\overline{\sigma}(\xi)))^l = (\frac{f_1(\xi)}{f_2(\xi)})^l$, where f_1 and f_2 are coprime polynomials. Since $a, b \neq (0, ..., 0)$, f_1 is a non-constant polynomial. Therefore in (7.1), even after cancellation, we still have a factor $f_1^l(\xi)$. However as shown in §6, the right hand side of the identity (7.1) must be an irreducible polynomial, which is a contradiction. ∎

As an immediate application, we have

Proposition 7.4. *$\rho(z, \xi)$ is an irreducible polynomial over $\mathbb{C}^n \times \mathbb{C}^n$. Thus, the Segre family $\mathcal{M}$ restricted to $\mathbb{C}^n \times \mathbb{C}^n = \mathcal{A} \times \mathcal{A} \subset M \times M$ is an irreducible subvariety of dimension $2n - 1$.*

We also have the following slightly strong version of the above proposition, which was used for applying a monodromy argument:

Proposition 7.5. *Suppose U is an connected open set in $\mathbb{C}^n \setminus \{0\}$. Then the Segre family $\mathcal{M}$ restricted to $U \times \mathbb{C}^n$ or restricted to $\mathbb{C}^n \times U$ is an irreducible analytic variety.*

Proof of Proposition 7.5: We need only to prove the first statement. Recall the notations we defined before: $\mathcal{M}_{\mathrm{SING}} = \{(z,\xi) : \frac{\partial \rho}{\partial \xi_j} = 0, \forall j\} \cup \{(z,\xi) : \frac{\partial \rho}{\partial z_j} = 0, \forall j\}$, and $\mathcal{M}_{\mathrm{REG}} = \mathcal{M} \backslash \mathcal{M}_{\mathrm{SING}}$. Since $\rho(z,\xi)$ is an irreducible polynomial and $\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial z_j}, j = 1, ..., n$ are polynomials with lower degrees, $\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial z_j}, j = 1, ..., n$ are not identically zero on $\mathcal{M} = \{\rho(z,\xi) = 0\}$. Each of $\frac{\partial \rho}{\partial \xi_j}, \frac{\partial \rho}{\partial z_j}$ defines a proper subvariety inside $\mathcal{M}$. By Proposition 7.2, for each $\tilde{z}(\neq 0) \in \mathbb{C}^n$, there is a certain point $\tilde{\xi}$ on $Q_{\tilde{z}}$ such that a partial derivative of $\rho(\tilde{z}, \xi)$ in ξ at $(\tilde{z}, \tilde{\xi})$ does not vanish. Hence $\mathcal{M}_{\mathrm{SING}}$ does not contain any Segre variety. Also the standard projection from $\mathcal{M}_{\mathrm{REG}}$ into the z-space is a submersion. Since Q_z is irreducible for $z \in \mathbb{C}^n \backslash (0, ..., 0)$, $Q_z \cap \mathcal{M}_{\mathrm{REG}}$ is connected.

To prove the theorem, we just need to show that $\mathcal{M}_{\mathrm{REG}}|_{U \times \mathbb{C}^n}$ is connected. Write the above projection map to the z-space as $\Phi : \mathcal{M}_{\mathrm{REG}}|_{U \times \mathbb{C}^n} \to U$. Since it is a submersion, it is an open mapping. Suppose z^0 is a point in U. As mentioned above, we know that each fiber of Φ is connected. For any $(z^0, \xi^0) \in \mathcal{M}_{\mathrm{REG}}$ in the fiber above z^0, we can choose a connected neighborhood V of (z^0, ξ^0) on $\mathcal{M}_{\mathrm{REG}}|_{U \times \mathbb{C}^n}$ such that $\Phi(V)$ is neighborhood of z_0. Hence, for any $z \in \Phi(V)$, any point in $Q_z \cap \mathcal{M}_{\mathrm{REG}}$ can be connected by a smooth curve inside $\mathcal{M}_{\mathrm{REG}}|_{V \times \mathbb{C}^n}$ to (z^0, ξ^0). Since U is connected, by a standard open-closeness argument, we see that $\mathcal{M}_{\mathrm{REG}}|_{U \times \mathbb{C}^n}$ is connected. ∎

References

[BaMa] S. Bando and T. Mabuchi, Uniqueness of Einstein Kähler metric modulo connected group actions, *Alegbraic Geometry, Sendai, 1985, Adv. Stud. Pure Math., 10*, North-Holland, Amsterdam,11-40, 1987.

[B] A. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987.

[BM] S. Bochner and W. Martin, *Several Complex Variables*, Princeton Mathematical Series, vol.10, 1948.

[BX] S. Berhanu and M. Xiao, On the C^∞ version of the reflection principle for mappings between CR maifolds, *Amer. J. Math.*, 137 (2015), no. 5, 1365-1400.

[Ca] E. Calabi, Isometric imbedding of complex manifolds, *Ann. of Math.* (2) 58 (1953), 1–23.

[Ch] S. Chan, On global rigidity of the p-th root embedding, *Proc. Amer. Math. Soc.* 144 (2016), no.1, 347-358.

[CM] S. Chan and N. Mok, Holomorphic isometric embeddings of complex hyperbolic space forms into irreducible bounded symmetric domains arise from linear sections of minimal embeddings of their compact duals, preprint.

[CMP] P. E. Chaput, L. Manivel and N. Perrin, Quantum cohomology of minuscule homogeneous spaces, *Transform. Groups* 13 (2008), no.1, 47-89.

[CU] L. Clozel and E. Ullmo, Correspondances modulaires etmesures invariantes, *J. Reine Angew. Math.* 558 (2003), 47–83.

[DL] A. Di Scala and A. Loi, Kähler maps of Hermitian symmetric spaces into complex space forms, *Geom. Dedicata* 125 (2007), 103-113.

[DL1] A. Di Scala and A. Loi, Kähler manifolds and their relatives, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 9 (2010), no. 3, 495-501.

[E] P. Ebenfelt, Local holomorphic isometries of a modified projective space into a standard projective space; Rational Conformal Factors, *Math. Ann.* 363 (2015), no. 3, 1333-1348.

[FHX] H. Fang, X. Huang and M. Xiao, Volume-preserving maps between Hermitian symmetric spaces of compact type, arXiv:1602.01900.

[Fr] H. Freudenthal, Lie groups in the foundations of geometry, *Advances in Mathematics,* 1 (1964), no. 2, 145-190.

[Fu] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.

[G] J. Grivaux, Tian's invariant of the grassmann manifold, *Jour. of Geom. Analysis* 16 (2006), no. 3.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry* , Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.

[He] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces* , Pure and Applied Mathematics 80, Academic Press, Inc.,New York-London, 1978.

[H] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II , *Ann. of Math.* (2) 79 (1964).

[Hu1] X. Huang, On the mapping problem for algebraic real hypersurfaces in the complex spaces of different dimensions, *Ann. Inst. Fourier (Grenoble)* 44 (1994), no. 2, 433–463.

[Hu2] X. Huang, On a linearity problem of proper holomorphic maps between balls in complex spaces ofdifferent dimensions, *J. Differential Geom.* 51 (1999), 13-33.

[HY1] X. Huang and Y. Yuan, Holomorphic isometry from a Kähler manifold into a product of complex projective manifolds, *Geom. Funct. Anal.* 24 (2014), no. 3, 854-886.

[HY2] X. Huang and Y. Yuan, Submanifolds of Hermitian symmetric spaces, Springer Proceedings in Mathematics & Statistics in the memory of Salah Baouendi 127 (2015), 197-206.

[HZ] X. Huang and D. Zaitsev, Non-embeddable real algebraic hypersurface, *Math. Z.*, 275, No. 3-4, 657-671 (2013).

[Ji] S. Ji, Algebraicity of real analytic hypersurfaces with maximal rank, *Amer. J. Math.* 124 (2002), no. 6, 1083-1102.

[IM1] A. Iliev and L. Manivel, The Chow ring of the Cayley plane, *Compos. Math.* 141 (2005), no. 1, 146-160.

[IM2] A. Iliev and L. Manivel, On cubic hypersurfaces of dimensions 7 and 8, *Proc. Lond. Math. Soc.* (3), 108 (2014), no. 2, 517-540.

[K] J. Kollár, *Lectures on Resolution of Singularities* , Annals of Mathematics Studies 166 (2007).

[KZ] S.-Y. Kim and D. Zaitsev, Rigidity of CR maps between Shilov boundaries of bounded symmetric domains, *Invent. Math.* 193 (2013), no. 2, 409-437.

[Lo1] O. Loos, Bounded symmetric domains and Jordan pairs, Math. Lectures, Univ. of California, Irvine, 1977.

[Lo2] O. Loos, Homogeneous algebraic varieties defined by Jordan pairs, Mh. Math. 86 (1978), 107-129.

[Mo1] N. Mok, *Metric rigidity theorems on Hermitian locally symmetric manifolds*, Series in Pure Mathematics. 6. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989. xiv+278 pp.

[Mo2] N. Mok, Local holomorphic isometric embeddings arisingfrom correspondences in the rank-1 case, Contemporary trends inalgebraic geometry and algebraic topology (Tianjin, 2000), 155–165,Nankai Tracts Math., 5, World Sci. Publ., River Edge, NJ, 2002.

[Mo3] N. Mok, Geometry of holomorphic isometries and related maps between bounded domains, Geometry and analysis. No. 2, 225-270, Adv. Lect. Math. (ALM), 18, Int. Press, Somerville, MA, 2011.

[Mo4] N. Mok, Extension of germs of holomorphic isometriesup to normalizing constants with respect to the Bergman metric, *J. Eur. Math. Soc.* 14 (2012), no. 5, 1617-1656.

[Mo5] N. Mok, Holomorphic isometries of the complex unit ball into irreducible bounded symmetric domains, Proc. Amer. Math. Soc. 144 (2016), no. 10, 4515-4525.

[MN1] N. Mok, and S. Ng, Second fundamental forms of holomorphic isometries of the Poincaré disk into bounded symmetric domains and their boundary behavior along the unit circle, *Sci. China Ser. A* 52 (2009), no. 12, 2628-2646.

[MN] N. Mok, and S. Ng, Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products, *J. Reine Angew. Math.* 669 (2012), 47-73.

[Ng] S. Ng, On holomorphic isometric embeddings from the unit disk into polydisks and their generalizations, Ph. D. thesis. 2008.

[Ng1] S. Ng, On holomorphic isometric embeddings of the unit disk into polydisks, *Proc. Amer. Math. Soc.* 138 (2010), 2907-2922.

[Ng2] S. Ng, On holomorphic isometric embeddings of the unit n-ball into products of two unit m-balls, *Math. Z.* 268 (2011) no. 1-2, 347-354.

[O] Y. Ohwashi, E_6 Matrix Model, Progress of Theoretical Physics, Volume 108, Issue 4, 755-782.

[PS] A. Pressley and G. Segal, *Loop groups*, Oxford Mathematical Monographs, 1986.

[R] M. Rios, Jordan C^*-Algebras and Supergravity, arXiv:1005.3514.

[U] M. Umehara, Einstein Kähler submanifolds of a complex linear or hyperbolic space, *Tohoku Math. J.* (2) 39 (1987), no. 3, 385-389.

[S] F. Sottile, From enumerative geometry to solving systems of polynomials equations , Computations in algebraic geometry with Macaulay 2, Algorithms Comput. Math. 8, Springer, Berlin, 2002, 101–129.

[U] M. Umehara, Einstein Kähler submanifolds of a complex linear or hyperbolic space, *Tohoku Math. J.* (2) 39 (1987), no. 3, 385-389.

[UWZ] H. Upmeier, K. Wang and G. Zhang, Holomorphic Isometries from the Unit Ball into Symmetric Domains, *International Mathematics Research Notices. IMRN* (2017), rnx110.

[W] J. A. Wolf, The geometry and structure of isotropy irreducible homogeneous spaces, *Acta Math.* 120 (1968), 59-148.

[XY] M. Xiao and Y. Yuan, Complexity of holomorphic maps from the complex unit ball to classical domains , *Asian Journal of Mathematics* (special issue dedicated to Professor Ngaiming Mok).

[Y] Y. Yuan, On local holomorphic maps preserving invariant $(p, p)-$forms between bounded symmetric domains, *Mathematical Research Letters* 24 (2017),no. 6, 1875-1895.

[YZ] Y. Yuan and Y. Zhang, Rigidity for local holomorphic conformal embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{N_1} \times \cdots \times \mathbb{B}^{N_m}$, *J. Differential Geom.* 90 (2012), no. 2 , 329-349.