COUNTING AND DISCOUNTING SLOWLY OSCILLATING PERIODIC SOLUTIONS TO WRIGHT'S EQUATION

by

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ABSTRACT OF THE DISSERTATION

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A classical example of a nonlinear delay differential equations is Wright's equation: $y'(t) = -\alpha y(t-1)[1 + y(t)]$, considering $\alpha > 0$ and $y(t) > -1$. This thesis proves two conjectures associated with this equation: Wright's conjecture, which states that the origin is the global attractor for all $\alpha \in (0, \frac{\pi}{2}]$; and Jones' conjecture, which states that there is a unique slowly oscillating periodic solution for $\alpha > \frac{\pi}{2}$. Moreover, we prove there are no isolas of periodic solutions to Wright's equation; all periodic orbits arise from Hopf bifurcations.

To prove Wright's conjecture our approach relies on a careful investigation of the neighborhood of the Hopf bifurcation occurring at $\alpha = \frac{\pi}{2}$. Using a rigorous numerical integrator we characterize slowly oscillating periodic solutions and calculate their stability, proving Jones' conjecture for $\alpha \in [1.9, 6.0]$ and thereby all $\alpha \geq 1.9$. We complete the proof of Jones conjecture using global optimization methods, extended to treat infinite dimensional problems.

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I greatly appreciate the mathematical community at the VU Amsterdam for hosting me in 2015. Thank you in particular to Jan Bouwe van den Berg, who acted as my secondary advisor; to Jean-Philippe Lessard, for introducing me to Wright's equation; to J.D. Mireles James, who helped lure me towards dynamical systems and rigorous numerics; and to Roger Nussbaum, for his sage advice on delay differential equations.

Chapter 1, and this thesis as a whole, is drawn from the papers [JLM17, vdBJ18, Jaq18]. More specifically: Chapters 2 and 3 are drawn from [vdBJ18], coauthored with Jan Bouwe van den Berg; Chapter 4 is drawn from [JLM17], coauthored with Jean-Philippe Lessard and Konstantin Mischaikow; Chapters 5 and 6 are drawn from [Jaq18].

Lastly, I acknowledge the Office of Advanced Research Computing (OARC) at Rutgers, The State University of New Jersey for providing access to the Amarel cluster and associated research computing resources that have contributed to the results reported here.

Dedication

For my family.

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Chapter 1

Introduction

1.1 Background

In many biological and physical systems the dependency of future states relies not only on the present situation, but on a broader history of the system. For simplicity, mathematical models often ignore the causal influence of all but the present state. However, in a wide variety of applications delayed feedback loops play an inextricable role in the qualitative dynamics of a system [KM13]. These phenomena can be modeled using delay and integro-differential equations, the theory of which has developed significantly over the past 60 years [Hal06]. An often studied class of delay differential equations are negative feedback systems of the form:

$$x'(t) = -\alpha f(x(t-1)) \tag{1.1}$$

where $xf(x) > 0$ for $x \neq 0$ and $f'(0) > 0$. One particularly well studied example of (1.1) is when $f(x) = e^x - 1$, better known as Wright's equation, which after making the change of variables $y = e^x - 1$ can be written in the following form:

$$y'(t) = -\alpha\, y(t-1)\, [1 + y(t)]\, . \tag{1.2}$$

Here α is considered to be both real and positive. This equation has been a central example considered in the development of much of the theory of functional differential equations. We cite some basic properties of its global dynamics [Wri55]:

- Corresponding to every $y \in C^0([-1, 0])$, there is a unique solution of (1.2) for all $t > 0$.
- Wright's equation has two equilibria $y \equiv -1$ and $y \equiv 0$. Moreover, solutions

cannot cross -1. Any solution with $y(t_0) = -1$ (for some $t_0 \geq 0$) is identically equal to -1 for $t \geq 0$.

- When $y(0) < -1$ then the solution decreases monotonically (for $t > 1$) without bound.

- When $y(0) > -1$ then $y(t)$ is globally bounded as $t \to +\infty$.

Henceforth we restrict our attention to $y(t) > -1$. In Wright's seminal 1955 paper [Wri55], he showed that if $\alpha \leq \frac{3}{2}$, then any solution having $y(t) > -1$ is attracted to 0 as $t \to +\infty$. Wright suggested that $y \equiv 0$ could be the global attractor for a larger range of α. The natural upper limit for this range is $\alpha = \frac{\pi}{2}$, where the equilibrium $y \equiv 0$ changes from asymptotically stable to unstable, a claim which has come to be known as Wright's conjecture:

Conjecture 1.1.1 (Wright's Conjecture)**.** *For every $0 < \alpha \leq \frac{\pi}{2}$, the zero solution to* (1.2) *is globally attractive.*

For $\alpha > \frac{\pi}{2}$, Wright proved the existence of oscillatory solutions to (1.2) which do not tend towards 0, and whose zeros are spaced at distances greater than the delay. Such a periodic solution is said to be *slowly oscillating*, and formally defined as follows:

Definition 1.1.2. *A* slowly oscillating periodic solution (SOPS) *is a periodic solution $y(t)$ which up to a time translation satisfies the following property: there exists some $q, \bar{q} > 1$ and $L = q + \bar{q}$ such that $y(t) > 0$ for $t \in (0, q)$, $y(t) < 0$ for $t \in (-\bar{q}, 0)$, and $y(t + L) = y(t)$ for all t, so that L is the minimal period of $y(t)$.*

In Jones' 1962 paper [Jon62a] he proved that for $\alpha > \frac{\pi}{2}$ there exists a slowly oscillating periodic solution to (1.2). Based on numerical calculations [Jon62b] Jones made the following conjecture:

Conjecture 1.1.3 (Jones' Conjecture)**.** *For every $\alpha > \frac{\pi}{2}$ there exists a unique (up to time translation) slowly oscillating periodic solution to* (1.2)*.*

In this thesis we complete the proofs of both of these conjectures, contributing a capstone to many decades of mathematical work studying Wright's equation. For further details, we refer the reader to [Hal71, Wal14] and the references contained therein.

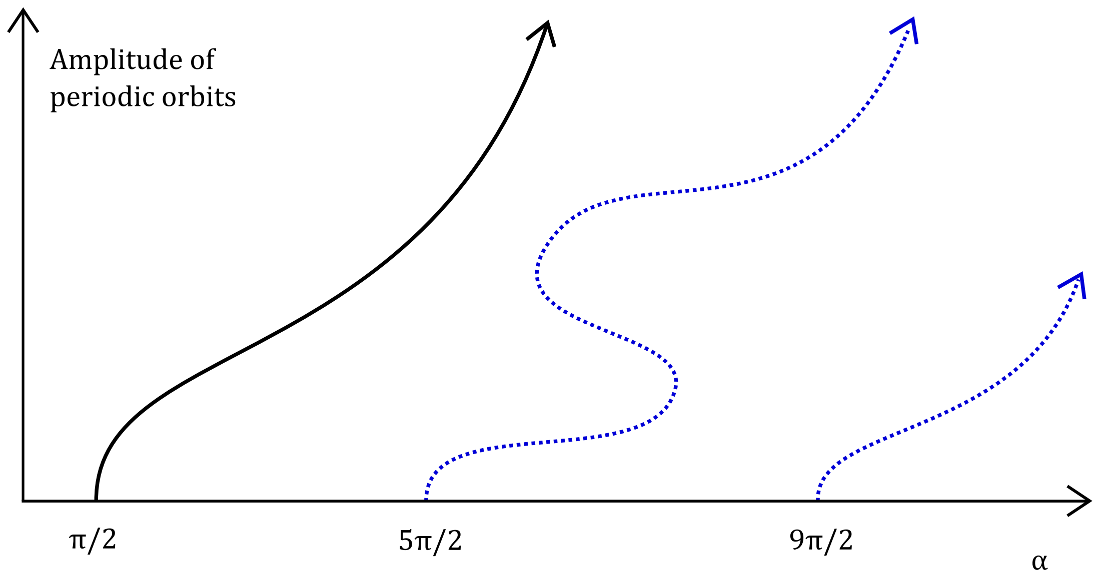


Figure 1.1: A bifurcation diagram for periodic solutions to Wright's equation. There are no folds in the principal branch of slowly oscillating periodic solutions (solid curve). While there may be folds in the branches of rapidly oscillating periodic solutions (dotted curves), it is conjectured that this does not occur. There are no other connected components of periodic solutions (not displayed).

To briefly review, a *principal branch* of slowly oscillating periodic orbits is born at $\alpha = \frac{\pi}{2}$ and continues on for all $\alpha > \frac{\pi}{2}$ [Nus75]. Moreover, Wright's equation has supercritical Hopf bifurcations at $\alpha = \frac{\pi}{2} + 2n\pi$ for integers $n \geq 0$, with slowly oscillating periodic orbits arising when $n = 0$, and rapidly oscillating periodic orbits arising when $n \geq 1$ (see Figure 1.1) [CMP77]. Since a Poincaré-Bendixson type theorem applies to Wright's equation, any initial condition will limit to the zero-equilibrium or a periodic orbit [MPS96]. The rest of the global attractor is built from connecting orbits. Together with the parameter α, the collection of periodic orbits forms a 2-dimensional manifold [Reg89]. A two-part geometric version of Jones' conjecture was proposed in [Les10]:

(i) the principal branch of SOPS does not fold back on itself, and

(ii) there are no other connected components (*isolas*) of SOPS.

If Conjectures 1.1.1 and 1.1.3 were false, then the solid black curve representing SOPS in Figure 1.1 instead could have exhibited all sorts of wild behavior as depicted in Figure 1.2. The papers [Wri55, BCKN14, Les10, Xie91] rule out these types of bad behavior depicted in the blue short-dashed lines in Figure 1.2.

Computer-assisted proofs using interval arithmetic [MKC09] have proved to provide powerful tools for studying Wright's equation [Les10,BCKN14], and nonlinear dynamics more generally (e.g. see [Rum10,Tuc11,KSW96]). By applying and building upon these tools this thesis fills in the missing pieces, depicted in the red long-dashed lines in Figure 1.2, needed to prove Conjecture 1.1.1 and Conjecture 1.1.3.

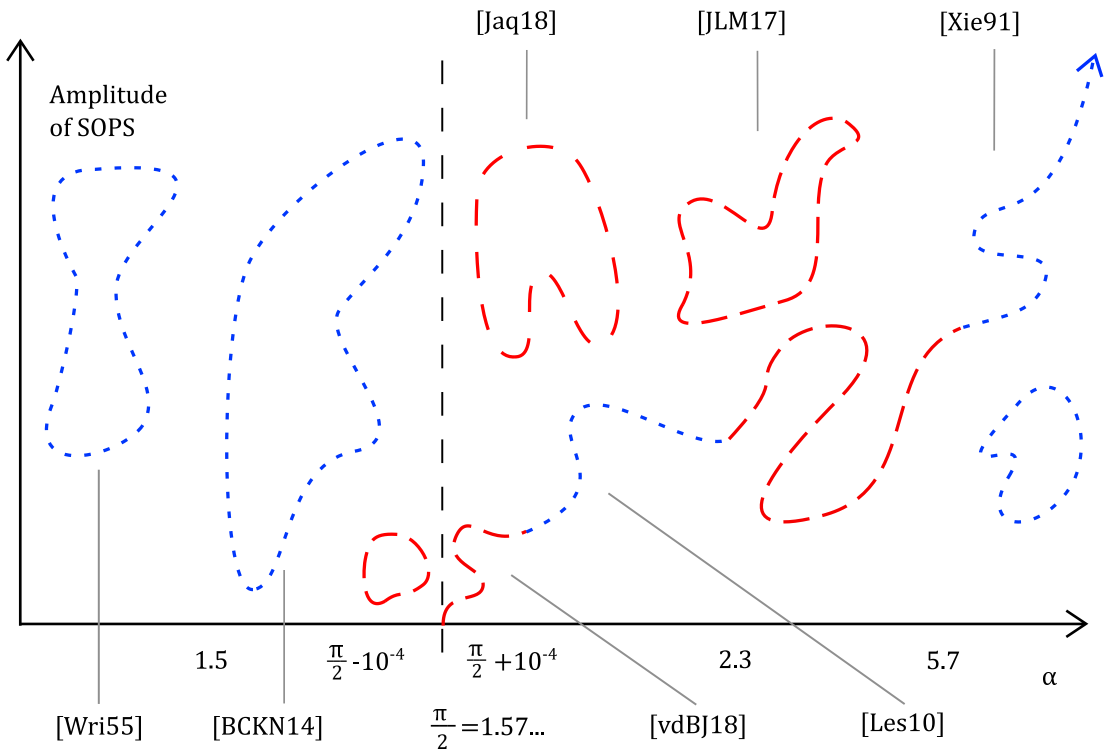


Figure 1.2: Ways the Wright's conjecture or the Jones' conjecture could have failed, and the papers which show these failures do not occur.

1.2 Thesis Outline

1.2.1 Hopf Bifurcation Analysis

To prove Conjecture 1.1.1, it sufficies to rule out existence of SOPS, as shown in [BCKN14] by the following theorem:

Theorem 1.2.1 (Theorem 3.1 in [BCKN14])**.** *The zero solution of* (1.2) *is globally attracting if and only if* (1.2) *has no slowly oscillating periodic solution.*

By using a branch and bound method and rigorous numerical integration, [BCKN14] shows that no SOPS exist for $\alpha \in [1.5, \frac{\pi}{2} - \delta_1]$ where $\delta_1 = 1.9633 \times 10^{-4}$, and that any SOPS for which $\alpha \in [\frac{\pi}{2} - \delta_1, \frac{\pi}{2}]$ would have a small amplitude. However, for every order of magnitude they approached the Hopf bifurcation at $\alpha = \frac{\pi}{2}$ their computation time increased by three orders of magnitude. "In other words," to quote from [BCKN14], "substantial improvement of the theoretical part of the present proof is needed to prove Wright's conjecture fully."

Hopf bifurcations are canonically analyzed with the method of normal forms, which transforms a given equation into a simpler expression having the same qualitative behavior as the original equation [Far06]. By an implicit-function-theorem type argument, this transformation is valid in some neighborhood of the bifurcation. However, the proof does not offer any insight into the size of this neighborhood.

In Chapter 2 we develop an explicit description of a neighborhood wherein the only periodic solutions are those originating from the Hopf bifurcation, basing our analysis around the normal forms derived in [CMP77, HKW81]. The main result of this analysis is the resolution of Wright's conjecture (see Theorem 2.3.7). Furthermore, we show in Theorem 2.3.8 that the branch of slowly oscillating periodic orbits originating from this Hopf bifurcation does not have any subsequent bifurcations (and in particular no folds) for $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta_2]$ where $\delta_2 = 6.830 \times 10^{-3}$. The proofs of many technical estimates used in Chapter 2 are postponed until Chapter 3.

1.2.2 Computationally Characterizing SOPS and their Stability

In [Xie91, Xie93], Xie showed that there is a unique SOPS to Wright's equation for $\alpha \geq 5.67$. He accomplished this by first showing that there is a unique SOPS to (1.2) if and only if every SOPS is asymptotically stable. By using asymptotic estimates of SOPS for large α (see [Nus82]) Xie was able to estimate their Floquet multipliers, and prove that all SOPS had to be stable. However, at $\alpha = 5.67$ these asymptotic estimates break down.

In Chapter 4, we essentially replace the asymptotic estimates in this argument with rigorous numerics, proving the following theorem:

Theorem 1.2.2. *There exists a unique SOPS to Wright's equation for* $\alpha \in [1.9, 6.0]$.

As in [BCKN14], we use bounding functions $X(t) = [\ell(t), u(t)]$ to characterize *all* SOPS to (1.2). Using *a priori* estimates, we construct for a given parameter range $[\alpha_{min}, \alpha_{max}]$ an initial "fat" bounding function. This bounding function is constructed so that if x is a SOPS at parameter $\alpha \in [\alpha_{min}, \alpha_{max}]$ then $x(t) \in X(t)$. We use a branch and bound algorithm to develop tight bounds on all the SOPS to Wright's equation. When the branch and bound algorithm has finished, the end result is a finite collection of relatively "thin" bounding functions.

On this collection of "thin" bounding functions, we bound the Floquet multipliers of all possible SOPS by solving an eigenvalue problem, again using a formulation introduced in [Xie91]. Using these two main steps, we prove for $\alpha \in [1.9, 6.0]$ that all SOPS to Wright's equation are asymptotically stable, and thereby unique. At this point we are able to show that for all $\alpha > \frac{\pi}{2}$ there are no folds in the principal branch of SOPS (see Corollary 4.6.1).

1.2.3 Ruling out Isolas

In [Les10] it is shown that there are no folds in the principal branch of periodic orbits for $\alpha \in [\frac{\pi}{2} + \delta_3, 2.3]$ where $\delta_3 = 7.3165 \times 10^{-4}$. However this does not rule out the possibility of SOPS far away from the principal branch. We address this in Chapter 5, where we prove the following theorem:

Theorem 1.2.3. *There exists a unique SOPS to Wright's equation for* $\alpha \in (\frac{\pi}{2}, 1.9]$.

As in Chapter 2, we recast the problem of studying periodic solutions to (1.2) as the problem of finding zeros of a functional defined on a space of Fourier coefficients. To obtain *a priori* bounds, we translate the bounding functions produced by Chapter 4 into estimates on the Fourier coefficients of a SOPS to (1.2). We then use a branch and bound method to find all of the SOPS in this space of Fourier coefficients. The primary technique we use to determine whether a region contains a SOPS is the Krawczyk method. This numerical method is commonly used to rigorously find all of the zeros in finite dimensional systems of nonlinear equations [Neu90], and more recently infinite dimensional systems [GZ07]. Our branch and bound algorithm produces a collection of "cubes" in Fourier space which contains all SOPS to Wright's equation for $\alpha \in (\frac{\pi}{2}, 1.9]$, and moreover these SOPS are unique with respect to α (see Figure 5.1). Together with Theorem 1.2.2 and [Xie91], this proves the Jones' conjecture:

Theorem 1.2.4. *For every $\alpha > \frac{\pi}{2}$ there exists a unique (up to time translation) slowly oscillating periodic solution to* (1.2)*.*

1.2.4 A look forward

Beyond just the slowly oscillating periodic solutions, Theorem 1.2.4 allows us to deduce that there are no isolas of rapidly oscillating periodic solutions. Since the nonlinearity in (1.1) depends only on $x(t-1)$, in fact any periodic orbit is either a SOPS or rescaling thereof. This rescaling between slowly and rapidly oscillating periodic solutions is given in terms of a solution's lap number [MP88] and its period, as detailed in the following theorem:

Theorem 1.2.5. *Let x_0 be a periodic solution to* (1.1) *at parameter α_0 with period L_0 and lap number N. Then there exists a SOPS $x_1(t) = x_0(rt)$ to* (1.1) *at parameter $\alpha_1 = r\alpha_0$ where $r := 1 - \frac{N-1}{2}L_0$.*

Thus, every periodic orbit is on a branch originating from one of the Hopf bifurcations at $\alpha = \frac{\pi}{2} + 2n\pi$. That is to say, there are no isolas of rapidly oscillating periodic solutions. However, this is not sufficient to show there are no folds in the branches of rapidly oscillating periodic solutions. The proof for Theorem 1.2.5 is presented in Chapter 6, and future directions are discussed.

Chapter 2

Hopf Bifurcation Analysis

2.1 Preliminaries

Many normal form techniques for functional differential equations have been developed to transform a given equation into a simpler expression having the same qualitative behavior as the original equation (see [Far06] and references contained therein). While this transformation is valid in some neighborhood about the bifurcation point, such results usually do not describe the size of this neighborhood explicitly. In this chapter we develop an explicit description of a neighborhood wherein the only periodic solutions are those originating from the Hopf bifurcation. The main result of this analysis is the resolution of Wright's conjecture.

Theorem 2.1.1. *For every* $0 < \alpha \leq \frac{\pi}{2}$*, the zero solution to* (1.2) *is globally attractive.*

By the work in [BCKN14], to prove Wright's conjecture it is sufficient to show that there do not exist any slowly oscillating periodic solutions for $\alpha \in [\frac{\pi}{2} - \delta_2, \frac{\pi}{2}]$, where $\delta_2 = 1.9633 \times 10^{-4}$. Indeed, we construct an explicit neighborhood about $\alpha = \frac{\pi}{2}$ for which the bifurcation branch of periodic orbits are the only periodic orbits. Then we show that throughout this entire neighborhood the solution branch behaves as expected from a supercritical bifurcation branch, i.e., it does not bend back into the parameter region $\alpha \leq \frac{\pi}{2}$.

Rather than trying to resolve all small bounded solutions near the bifurcation point through a center manifold analysis, we focus on periodic orbits only. In particular, we ignore orbits that connect the trivial state to the periodic states, since those are not relevant for our analysis. The advantage is that, by restricting our attention to periodic solution, we can perform our analysis in Fourier space. We first note that all periodic solutions are smooth, as was established in [Wri55] and more generally in [Nus73].

Lemma 2.1.2 ([Nus73])**.** *All periodic solutions of* (1.2) *are real analytic.*

For a periodic function $y : \mathbb{R} \to \mathbb{R}$ with frequency $\omega > 0$ we write

$$y(t) = \sum_{k\in\mathbb{Z}} a_k e^{i\omega kt}, \tag{2.1}$$

where $a_k \in \mathbb{C}$. This transforms the delay equation (1.2) into

$$(i\omega k + \alpha e^{-i\omega k})a_k + \alpha \sum_{k_1+k_2=k} e^{-i\omega k_1} a_{k_1} a_{k_2} = 0 \qquad \text{for all } k \in \mathbb{Z}. \tag{2.2}$$

In effect, the problem of finding periodic solutions to Wright's equation can be reformulated as finding a parameter α, a frequency ω, and a sequence $\{a_k\}$ for which (2.2) is satisfied. In Section 2.1 we define an appropriate sequence space to work in, and define a zero finding problem $F_\epsilon(\alpha, \omega, c) = 0$ equivalent to (2.2). The auxiliary variable ϵ, which represents the dominant Fourier mode, corresponds to the rescaling $y \mapsto \epsilon y$ canonical to the study of Hopf bifurcations.

In Section 2.2 we construct a Newton-like operator T_ϵ whose fixed points correspond to the zeros of $F_\epsilon(\alpha, \omega, c)$. By applying a Newton-Kantorovich like theorem, we identify explicit neighborhoods B_ϵ wherein $T_\epsilon : B_\epsilon \to B_\epsilon$ is a uniform contraction mapping. By the nature of our argument, we have the freedom to construct both large and small balls B_ϵ on which we may apply the Banach fixed point theorem. Using smaller balls will produce tighter approximations of the periodic solutions, while using larger balls will produce a larger region within which the periodic solution is unique.

These results are leveraged in Section 2.3 to derive global results such as the resolution of Conjecture 1.1.1, as well as Theorem 2.3.8 which shows that there do not exist any subsequent bifurcations in the principal branch for $\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} + 6.830 \times 10^{-3}$. While we prove in Proposition 2.2.15 that for $0 < \epsilon \leq 0.1$ there is a locally unique $(\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon)$ which solves $F_\epsilon(\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon) = 0$, this is not sufficient. To show that the branch of periodic solutions does not have any subsequent bifurcations, we prove that $\hat{\alpha}_\epsilon$ is monotonically increasing in ϵ. Since $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon \approx \frac{2\epsilon}{5}(\frac{3\pi}{2} - 1)$, in order to have any hope of proving $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon > 0$, it is imperative that we derive an $\mathcal{O}(\epsilon^2)$ approximation of $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon$, an approach we take from the beginning of our analysis.

For the remainder of this section we systematically recast the Hopf bifurcation problem in Fourier space. We introduce appropriate scalings, sequence spaces of Fourier coefficients and convenient operators on these spaces. To study Equation (2.2) we consider Fourier sequences $\{a_k\}$ and fix a Banach space in which these sequences reside. It is indispensable for our analysis that this space have an algebraic structure. The Wiener algebra of absolutely summable Fourier series is a natural candidate, which we use with minor modifications. In numerical applications, weighted sequence spaces with algebraic and geometric decay have been used to great effect to study periodic solutions which are C^k and analytic, respectively [Les10, HLMJ16]. Although it follows from Lemma 2.1.2 that the Fourier coefficients of any solution decay exponentially, we choose to work in a space of less regularity. The reason is that by working in a space with less regularity, we are better able to connect our results with the global estimates in [BCKN14], see Theorem 2.3.11.

Remark 2.1.3. *There is considerable redundancy in Equation* (2.2)*. First, since we are considering real-valued solutions y, we assume a_{-k} is the complex conjugate of a_k. This symmetry implies it suffices to consider Equation* (2.2) *for $k \geq 0$. Second, we may effectively ignore the zeroth Fourier coefficient of any periodic solution [Jon62a], since it is necessarily equal to* 0*. The self contained argument is as follows. As mentioned in the introduction, any periodic solution to Wright's equation must satisfy $y(t) > -1$ for all t. By dividing Equation* (1.2) *by* $(1 + y(t))$*, which never vanishes, we obtain*

$$\frac{d}{dt}\log(1 + y(t)) = -\alpha y(t-1).$$

Integrating over one period L we derive the condition $0 = \int_0^L y(t)dt$. Hence $a_0 = 0$ for any periodic solution. It will be shown in Theorem 2.1.4 that a related argument implies that we do not need to consider Equation (2.2) *for $k = 0$.*

We define the spaces of absolutely summable Fourier series

$$\ell^1 := \left\{ \{a_k\}_{k\geq 1} : \sum_{k\geq 1} |a_k| < \infty \right\},$$

$$\ell^1_{\mathrm{bi}} := \left\{ \{a_k\}_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} |a_k| < \infty \right\}.$$

We identify any semi-infinite sequence $\{a_k\}_{k\geq 1} \in \ell^1$ with the bi-infinite sequence $\{a_k\}_{k\in\mathbb{Z}} \in \ell^1_{\mathrm{bi}}$ via the conventions (see Remark 2.1.3)

$$a_0 = 0 \qquad \text{and} \qquad a_{-k} = a_k^*, \tag{2.3}$$

where a_k^* denotes the complex conjugate of a_k. In other words, we identify ℓ^1 with the set

$$\ell^1_{\mathrm{sym}} := \left\{a \in \ell^1_{\mathrm{bi}} : a_0 = 0,\ a_{-k} = a_k^*\right\}.$$

On ℓ^1 we introduce the norm

$$\|a\| = \|a\|_{\ell^1} := 2\sum_{k=1}^{\infty} |a_k|. \tag{2.4}$$

The factor 2 in this norm is chosen to have a Banach algebra estimate. Indeed, for $a, \tilde{a} \in \ell^1 \cong \ell^1_{\mathrm{sym}}$ we define the discrete convolution

$$[a * \tilde{a}]_k = \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1+k_2=k}} a_{k_1}\tilde{a}_{k_2}.$$

Although $[a * \tilde{a}]_0$ does not necessarily vanish, we have $\{a * \tilde{a}\}_{k\geq 1} \in \ell^1$ and

$$\|a * \tilde{a}\| \leq \|a\| \cdot \|\tilde{a}\| \qquad \text{for all } a, \tilde{a} \in \ell^1,$$

hence ℓ^1 with norm (2.4) is a Banach algebra.

By Lemma 2.1.2 it is clear that any periodic solution of (1.2) has a well-defined Fourier series $a \in \ell^1_{\mathrm{bi}}$. The next theorem shows that in order to study periodic orbits to Wright's equation we only need to study Equation (2.2) for $k \geq 1$. For convenience we introduce the notation

$$G(\alpha, \omega, a)_k = (i\omega k + \alpha e^{-i\omega k})a_k + \alpha \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ k_1+k_2=k}} e^{-i\omega k_1} a_{k_1} a_{k_2} \qquad \text{for } k \in \mathbb{N}.$$

We note that we may interpret the trivial solution $y(t) \equiv 0$ as a periodic solution of arbitrary period.

Theorem 2.1.4. *Let $\alpha > 0$ and $\omega > 0$. If $a \in \ell^1 \cong \ell^1_{\mathrm{sym}}$ solves $G(\alpha, \omega, a)_k = 0$ for all $k \geq 1$, then $y(t)$ given by (2.1) is a periodic solution of (1.2) with period $2\pi/\omega$. Vice versa, if $y(t)$ is a periodic solution of (1.2) with period $2\pi/\omega$ then its Fourier coefficients $a \in \ell^1_{\mathrm{bi}}$ lie in $\ell^1_{\mathrm{sym}} \cong \ell^1$ and solve $G(\alpha, \omega, a)_k = 0$ for all $k \geq 1$.*

Proof. If $y(t)$ is a periodic solution of (1.2) then it is real analytic by Lemma 2.1.2, hence its Fourier series a is well-defined and $a \in \ell^1_{\text{sym}}$ by Remark 2.1.3. Plugging the Fourier series (2.1) into (1.2) one easily derives that a solves (2.2) for all $k \geq 1$.

To prove the reverse implication, assume that $a \in \ell^1_{\text{sym}}$ solves Equation (2.2) for all $k \geq 1$. Since $a_{-k} = a_k^*$, Equation (2.2) is also satisfied for all $k \leq -1$. It follows from the Banach algebra property and (2.2) that $\{ka_k\}_{k\in\mathbb{Z}} \in \ell^1_{\text{bi}}$. Hence y, given by (2.1), is continuously differentiable and $2\pi/\omega$-periodic. Since (2.2) is satisfied for all $k \in \mathbb{Z} \setminus \{0\}$ (but not necessarily for $k = 0$) one may perform the inverse Fourier transform on (2.2) to conclude that y satisfies the delay equation

$$y'(t) = -\alpha y(t-1)[1 + y(t)] + C \tag{2.5}$$

for some constant $C \in \mathbb{R}$. Finally, to prove that $C = 0$ we argue by contradiction. Suppose $C \neq 0$. Then $y(t) \neq -1$ for all t. Namely, at any point where $y(t_0) = -1$ one would have $y'(t_0) = C$ which has fixed sign, hence it would follow that y is not periodic (y would not be able to cross -1 in the opposite direction, preventing y from being periodic). We may thus divide (2.5) through by $1 + y(t)$ and obtain

$$\frac{d}{dt} \log |1 + y(t)| = -\alpha y(t-1) + \frac{C}{1 + y(t)}.$$

By integrating both sides of the equation over one period L and by using that $a_0 = 0$, we obtain

$$C \int_0^L \frac{1}{1 + y(t)} dt = 0.$$

Since the integrand is either strictly negative or strictly positive, this implies that $C = 0$. Hence (2.5) reduces to (1.2), and y satisfies Wright's equation. □

To efficiently study Equation (2.2), we introduce the following linear operators on ℓ^1:

$$\begin{aligned} [Ka]_k &:= k^{-1} a_k, \\ [U_\omega a]_k &:= e^{-ik\omega} a_k. \end{aligned} \tag{2.6}$$

The map K is a compact operator, and it has a densely defined inverse K^{-1}. The domain of K^{-1} is denoted by

$$\ell^K := \{a \in \ell^1 : K^{-1}a \in \ell^1\}.$$

The map U_ω is a unitary operator, but it is discontinuous in ω. Furthermore, we note that (2.6) also defines U_ω as a linear operator on ℓ^1_{bi}, and since $[U_\omega a]_k^* = [U_\omega a]_{-k}$ if $a_k^* = a_{-k}$ this is compatible with the earlier identification of ℓ^1 as a subspace of ℓ^1_{bi} through (2.3). With this notation, Theorem 2.1.4 implies that our problem of finding a SOPS to (1.2) is equivalent to finding an $a \in \ell^1$ such that

$$G(\alpha, \omega, a) := \left(i\omega K^{-1} + \alpha U_\omega\right) a + \alpha \, [U_\omega \, a] * a = 0. \tag{2.7}$$

In the convolution product both a and $U_\omega a$ are interpreted as elements of ℓ^1_{bi}.

Periodic solutions are invariant under time translation: if $y(t)$ solves Wright's equation, then so does $y(t+\tau)$ for any $\tau \in \mathbb{R}$. We remove this degeneracy by adding a phase condition. For a periodic function $y(t)$ as given in (2.1), the Fourier coefficients of $y(t+\tau)$ are given as $\{a_k e^{i\omega k\tau}\}_{k\in\mathbb{Z}}$. Hence, without loss of generality, if $a \in \ell^1$ solves Equation (2.7) for nonvanishing ω, we may assume that $a_1 = \epsilon$ for some *real non-negative* ϵ:

$$\ell^1_\epsilon := \{a \in \ell^1 : a_1 = \epsilon\} \qquad \text{where } \epsilon \in \mathbb{R}, \epsilon \geq 0.$$

In the rest of our analysis, we will split elements $a \in \ell^1$ into two parts: a_1 and $\{a_k\}_{k\geq 2}$. We define the basis elements $\mathrm{e}_j \in \ell^1$ for $j = 1, 2, \ldots$ as

$$[\mathrm{e}_j]_k = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

We note that $\|\mathrm{e}_j\| = 2$. Then we can decompose any $a \in \ell^1_\epsilon$ uniquely as

$$a = \epsilon \mathrm{e}_1 + \tilde{c} \qquad \text{with} \quad \tilde{c} \in \ell^1_0 := \{\tilde{c} \in \ell^1 : \tilde{c}_1 = 0\}. \tag{2.8}$$

We follow the classical approach in studying Hopf bifurcations and consider $a_1 = \epsilon$ to be a parameter, and then find periodic solutions with Fourier modes in ℓ^1_ϵ. We thus substitute (2.8) into (2.7) and replace the function $G : \mathbb{R}^2 \times \ell^K \to \ell^1$ by a function $\tilde{F}_\epsilon : \mathbb{R}^2 \times \ell^K_0 \to \ell^1$, where we denote

$$\ell^K_0 := \ell^1_0 \cap \ell^K.$$

Definition 2.1.5. *We define the ϵ-parameterized family of functions $\tilde{F}_\epsilon : \mathbb{R}^2 \times \ell_0^K \to \ell^1$ by*

$$\tilde{F}_\epsilon(\alpha, \omega, \tilde{c}) := \epsilon[i\omega + \alpha e^{-i\omega}]\mathrm{e}_1 + (i\omega K^{-1} + \alpha U_\omega)\tilde{c} + \epsilon^2 \alpha e^{-i\omega}\mathrm{e}_2 + \alpha\epsilon L_\omega \tilde{c} + \alpha[U_\omega \tilde{c}] * \tilde{c}, \tag{2.9}$$

where $L_\omega : \ell_0^1 \to \ell^1$ is given by

$$L_\omega := \sigma^+(e^{-i\omega} I + U_\omega) + \sigma^-(e^{i\omega} I + U_\omega),$$

with I the identity and $\sigma^\pm$ the shift operators on ℓ^1:

$$\begin{aligned} \left[\sigma^- a\right]_k &:= a_{k+1}, \\ \left[\sigma^+ a\right]_k &:= a_{k-1} \qquad \textit{with the convention } a_0 = 0. \end{aligned}$$

The operator L_ω is discontinuous in ω and $\|L_\omega\| \leq 4$.

We reformulate Theorem 2.1.4 in terms of the map $\tilde{F}$. We note that it follows from Lemma 2.1.2 and Equation (2.2) that the Fourier coefficients of any periodic solution of (1.2) lie in ℓ^K. These observations are summarized in the following theorem.

Theorem 2.1.6. *Let $\epsilon \geq 0$, $\tilde{c} \in \ell_0^K$, $\alpha > 0$ and $\omega > 0$. Define $y : \mathbb{R} \to \mathbb{R}$ as*

$$y(t) = \epsilon\left(e^{i\omega t} + e^{-i\omega t}\right) + \sum_{k=2}^{\infty} \tilde{c}_k e^{i\omega k t} + \tilde{c}_k^* e^{-i\omega k t}. \tag{2.10}$$

Then $y(t)$ solves (1.2) if and only if $\tilde{F}_\epsilon(\alpha, \omega, \tilde{c}) = 0$. Furthermore, up to time translation, any periodic solution of (1.2) with period $2\pi/\omega$ is described by a Fourier series of the form (2.10) with $\epsilon \geq 0$ and $\tilde{c} \in \ell_0^K$.

Since we want to analyze a Hopf bifurcation, we will want to solve $\tilde{F}_\epsilon = 0$ for small values of ϵ. However, at the bifurcation point, $D\tilde{F}_0(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ is not invertible. In order for our asymptotic analysis to be non-degenerate, we work with a rescaled version of the problem. To this end, for any $\epsilon > 0$, we rescale both $\tilde{c}$ and $\tilde{F}$ as follows. Let $\tilde{c} = \epsilon c$ and

$$\tilde{F}_\epsilon(\alpha, \omega, \epsilon c) = \epsilon F_\epsilon(\alpha, \omega, c). \tag{2.11}$$

For $\epsilon > 0$ the problem then reduces to finding zeros of

$$F_\epsilon(\alpha, \omega, c) := [i\omega + \alpha e^{-i\omega}]\mathrm{e}_1 + (i\omega K^{-1} + \alpha U_\omega)c + \epsilon\alpha e^{-i\omega}\mathrm{e}_2 + \alpha\epsilon L_\omega c + \alpha\epsilon[U_\omega c] * c. \tag{2.12}$$

We denote the triple $(\alpha, \omega, c) \in \mathbb{R}^2 \times \ell_0^1$ by x. To pinpoint the components of x we use the projection operators

$$\pi_\alpha x = \alpha, \quad \pi_\omega x = \omega, \quad \pi_c x = c \qquad \text{for any } x = (\alpha, \omega, c).$$

After the change of variables (2.11) we now have an invertible Jacobian $DF_0(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ at the bifurcation point. On the other hand, for $\epsilon = 0$ the zero finding problems for $\tilde{F}_\epsilon$ and F_ϵ are not equivalent. However, it follows from the following lemma that any nontrivial periodic solution having $\epsilon = 0$ must have a relatively large size when α and ω are close to the bifurcation point.

Lemma 2.1.7. *Fix $\epsilon \geq 0$ and $\alpha, \omega > 0$. Let*

$$b_* := \frac{\omega}{\alpha} - \frac{1}{2} - \epsilon \left(\frac{2}{3} + \frac{1}{2} \sqrt{2 + 2|\omega - \tfrac{\pi}{2}|} \right).$$

Assume that $b_* > \sqrt{2}\epsilon$. Define

$$z_*^\pm := b_* \pm \sqrt{(b_*)^2 - 2\epsilon^2}. \tag{2.13}$$

If there exists a $\tilde{c} \in \ell_0^1$ such that $\tilde{F}_\epsilon(\alpha, \omega, \tilde{c}) = 0$, then both:

(a) *either $\|\tilde{c}\| \leq z_*^-$ or $\|\tilde{c}\| \geq z_*^+$.*

(b) $\|K^{-1}\tilde{c}\| \leq (2\epsilon^2 + \|\tilde{c}\|^2)/b_*$.

Proof. The proof follows from Lemmas 3.5.3 and 3.5.4 in Section 3.5, combined with the observation that $\frac{\omega}{\alpha} - \gamma \geq b_*$, with γ as defined in Lemma 3.5.3. □

Remark 2.1.8. *We note that for $\alpha < 2\omega$ and for small ϵ, then:*

$$z_*^+ \geq \frac{2\omega - \alpha}{\alpha} - \epsilon \left(4/3 + \sqrt{2 + 2|\omega - \tfrac{\pi}{2}|} \right) + \mathcal{O}(\epsilon^2),$$
$$z_*^- \leq \mathcal{O}(\epsilon^2).$$

Hence Lemma 2.1.7 implies that for values of (α, ω) near $(\frac{\pi}{2}, \frac{\pi}{2})$ any solution has either $\|\tilde{c}\|$ of order 1 or $\|\tilde{c}\| = \mathcal{O}(\epsilon^2)$. The asymptotically small term bounding z_*^- is explicitly calculated in Lemma 3.5.5. A related consequence is that for $\epsilon = 0$ there are no nontrivial solutions of $\tilde{F}_0(\alpha, \omega, \tilde{c}) = 0$ with $\|\tilde{c}\| < \frac{2\omega - \alpha}{\alpha}$.

Remark 2.1.9. *In Section 2.2.2 we will work on subsets of* ℓ_0^K *of the form*

$$\ell_\rho := \{c \in \ell_0^K : \|K^{-1}c\| \leq \rho\}.$$

Part (b) of Lemma 2.1.7 will be used in Section 2.3 to guarantee that we are not missing any solutions by considering ℓ_ρ *(for some specific choice of* ρ*) rather than the full space* ℓ_0^K*. In particular, we infer from Remark 2.1.8 that small solutions (meaning roughly that* $\|\tilde{c}\| \to 0$ *as* $\epsilon \to 0$*) satisfy* $\|K^{-1}\tilde{c}\| = \mathcal{O}(\epsilon^2)$*.*

The following theorem guarantees that near the bifurcation point the problem of finding all periodic solutions is equivalent to considering the rescaled problem $F_\epsilon(\alpha, \omega, c) = 0$.

Theorem 2.1.10. (a) *Let* $\epsilon > 0$*,* $c \in \ell_0^K$*,* $\alpha > 0$ *and* $\omega > 0$*. Define* $y : \mathbb{R} \to \mathbb{R}$ *as*

$$y(t) = \epsilon \left(e^{i\omega t} + e^{-i\omega t}\right) + \epsilon \sum_{k=2}^{\infty} c_k e^{i\omega k t} + c_k^* e^{-i\omega k t}. \tag{2.14}$$

Then $y(t)$ *solves* (1.2) *if and only if* $F_\epsilon(\alpha, \omega, c) = 0$*.*
(b) *Let* $y(t) \not\equiv 0$ *be a periodic solution of* (1.2) *of period* $2\pi/\omega$ *with Fourier coefficients* a*. Suppose* $\alpha < 2\omega$ *and* $\|a\| < \frac{2\omega - \alpha}{\alpha}$*. Then, up to time translation,* $y(t)$ *is described by a Fourier series of the form* (2.14) *with* $\epsilon > 0$ *and* $c \in \ell_0^K$*.*

Proof. Part (a) follows directly from Theorem 2.1.6 and the change of variables (2.11). To prove part (b) we need to exclude the possibility that there is a nontrivial solution with $\epsilon = 0$. The asserted bound on the ratio of α and ω guarantees, by Lemma 2.1.7 (see also Remark 2.1.8), that indeed $\epsilon > 0$ for any nontrivial solution. $\square$

We note that in practice (see Section 2.3) a bound on $\|a\|$ is derived from a bound on y or y' using Parseval's identity.

Remark 2.1.11. *It follows from Theorem 2.1.10 and Remark 2.1.8 that for values of* (α, ω) *near* $(\frac{\pi}{2}, \frac{\pi}{2})$ *any reasonably bounded solution satisfies* $\|c\| = O(\epsilon)$ *as well as* $\|K^{-1}c\| = O(\epsilon)$ *asymptotically (as* $\epsilon \to 0$*). These bounds will be made explicit (and non-asymptotic) for specific choices of the parameters in Section 2.3.*

We finish this section by defining a curve of approximate zeros $\bar{x}_\epsilon$ of F_ϵ (see [CMP77, HKW81]).

Definition 2.1.12. *Let*

$$\begin{aligned}\bar{\alpha}_\epsilon &:= \tfrac{\pi}{2} + \tfrac{\epsilon^2}{5}\left(\tfrac{3\pi}{2} - 1\right)\\ \bar{\omega}_\epsilon &:= \tfrac{\pi}{2} - \tfrac{\epsilon^2}{5}\\ \bar{c}_\epsilon &:= \left(\tfrac{2-i}{5}\right)\epsilon\, \mathrm{e}_2\,.\end{aligned}$$

We define the approximate solution $\bar{x}_\epsilon := (\bar{\alpha}_\epsilon, \bar{\omega}_\epsilon, \bar{c}_\epsilon)$ *for all* $\epsilon \geq 0$.

We leave it to the reader to verify that both $F_\epsilon(\frac{\pi}{2}, \frac{\pi}{2}, \bar{c}_\epsilon) = \mathcal{O}(\epsilon^2)$ and $F_\epsilon(\bar{x}_\epsilon) = \mathcal{O}(\epsilon^2)$. We choose to use the more accurate approximation for the α and ω components to improve our final quantitative results.

2.2 Local results

2.2.1 Constructing a Newton-like operator

In this section and in the appendices we often suppress the subscript in $F = F_\epsilon$. We will find solutions to the equation $F(\alpha, \omega, c) = 0$ by constructing a Newton-like operator T such that fixed points of T corresponds precisely to zeros of F. In order to construct the map T we need an operator $A^\dagger$ which is an approximate inverse of $DF(\bar{x}_\epsilon)$. We will use an approximation A of $DF(\bar{x}_\epsilon)$ that is linear in ϵ and correct up to $\mathcal{O}(\epsilon^2)$. Likewise, we define $A^\dagger$ to be linear in ϵ (and again correct up to $\mathcal{O}(\epsilon^2)$).

It will be convenient to use the usual identification $i_{\mathbb{C}} : \mathbb{R}^2 \to \mathbb{C}$ given by $i_{\mathbb{C}}(x, y) = x + iy$. We also use $\omega_0 := \pi/2$.

Definition 2.2.1. *We introduce the linear maps* $A : \mathbb{R}^2 \times \ell^K_0 \to \ell^1$ *and* $A^\dagger : \ell^1 \to \mathbb{R}^2 \times \ell^K_0$ *by*

$$\begin{aligned}A &:= A_0 + \epsilon A_1\,,\\ A^\dagger &:= A_0^{-1} - \epsilon A_0^{-1} A_1 A_0^{-1}\,,\end{aligned}$$

where the linear maps $A_0, A_1 : \mathbb{R}^2 \times \ell_0^K \to \ell^1$ *are defined below. Writing* $x = (\alpha, \omega, c)$, *we set*

$$A_0 x = A_0(\alpha, \omega, c) := i_{\mathbb{C}} A_{0,1} \begin{bmatrix} \alpha \\ \omega \end{bmatrix} \mathrm{e}_1 + A_{0,*} c,$$

$$A_1 x = A_1(\alpha, \omega, c) := i_{\mathbb{C}} A_{1,2} \begin{bmatrix} \alpha \\ \omega \end{bmatrix} \mathrm{e}_2 + A_{1,*} c.$$

Here the matrices $A_{0,1}$ *and* $A_{1,2}$ *are given by*

$$A_{0,1} := \begin{bmatrix} 0 & -\frac{\pi}{2} \\ -1 & 1 \end{bmatrix} \qquad \textit{and} \qquad A_{1,2} := \frac{1}{5} \begin{bmatrix} -2 & 2 - \frac{3\pi}{2} \\ -4 & 2(2+\pi) \end{bmatrix}, \tag{2.15}$$

and the linear maps $A_{0,*} : \ell_0^K \to \ell_0^1$ *and* $A_{1,*} : \ell_0^K \to \ell^1$ *are given by*

$$A_{0,*} := \tfrac{\pi}{2}(iK^{-1} + U_{\omega_0}) \qquad \textit{and} \qquad A_{1,*} := \tfrac{\pi}{2} L_{\omega_0}.$$

Since K and U_{ω_0} both act as diagonal operators, the inverse $A_{0,*}^{-1} : \ell_0^1 \to \ell_0^K$ of $A_{0,*}$ is given by

$$(A_{0,*}^{-1} a)_k = \frac{2}{\pi} \frac{a_k}{ik + e^{-ik\omega_0}} \qquad \text{for all } k \geq 2.$$

An explicit computation, which we leave to the reader, shows that these approximations are indeed correct up to $\mathcal{O}(\epsilon^2)$. In particular, $A^\dagger = [DF(\bar{x}_\epsilon)]^{-1} + \mathcal{O}(\epsilon^2)$. In Appendix 3.1 several additional properties of these operators are derived. The most important one is the following.

Proposition 2.2.2. *For* $0 \leq \epsilon < \frac{\sqrt{10}}{4} \approx 0.790$ *the operator* $A^\dagger$ *is injective.*

Proof. In order to show that $A^\dagger$ is injective we show that it has a left inverse. Note that $AA^\dagger = I - \epsilon^2 (A_1 A_0^{-1})^2$. By Proposition 3.1.2 it follows that $\|A_1 A_0^{-1}\| \leq \frac{2\sqrt{10}}{5}$. By choosing $\epsilon < \frac{\sqrt{10}}{4}$ we obtain $\|\epsilon^2 (A_1 A_0^{-1})^2\| < 1$, whereby $AA^\dagger$ is invertible, and so $A^\dagger$ is injective. □

Definition 2.2.3. *We define the operator* $T : \mathbb{R}^2 \times \ell_0^K \to \mathbb{R}^2 \times \ell_0^K$ *by*

$$T(x) := x - A^\dagger F(x),$$

where F *is defined in Equation* (2.12) *and* $A^\dagger$ *in Definition 2.2.1. We note that* F, $A^\dagger$ *and* T *depend on the parameter* $\epsilon \geq 0$, *although we suppress this in the notation.*

2.2.2 Explicit contraction bounds

The map T is not continuous on all of $\mathbb{R}^2 \times \ell_0^K$, since $U_\omega c$ is not continuous in ω. While continuity is "recovered" for terms of the form $A^\dagger U_\omega c$, this is not the case for the nonlinear part $-\alpha\epsilon A^\dagger[U_\omega c] * c$. We overcome this difficulty by fixing some $\rho > 0$ and restricting the domain of T to sets of the form

$$\mathbb{R}^2 \times \{c \in \ell_0^K : \|K^{-1}c\| \leq \rho\} = \mathbb{R}^2 \times \ell_\rho.$$

Since we wish to center the domain of T about the approximate solution $\bar{x}_\epsilon$, we introduce the following definition, which uses a triple of radii $r \in \mathbb{R}_+^3$, for which it will be convenient to use two different notations:

$$r = (r_\alpha, r_\omega, r_c) = (r_1, r_2, r_3).$$

Definition 2.2.4. *Fix $r \in \mathbb{R}_+^3$ and $\rho > 0$ and let $\bar{x}_\epsilon = (\bar{\alpha}_\epsilon, \bar{\omega}_\epsilon, \bar{c}_\epsilon)$ be as defined in Definition 2.1.12. We define the ρ-ball $B_\epsilon(r,\rho) \subset \mathbb{R}^2 \times \ell_0^1$ of radius r centered at $\bar{x}_\epsilon$ to be the set of points satisfying*

$$\begin{aligned} |\alpha - \bar{\alpha}_\epsilon| &\leq r_\alpha \\ |\omega - \bar{\omega}_\epsilon| &\leq r_\omega \\ \|c - \bar{c}_\epsilon\| &\leq r_c \\ \|K^{-1}c\| &\leq \rho. \end{aligned}$$

We want to show that T is a contraction map on some ρ-ball $B_\epsilon(r,\rho) \subset \mathbb{R}^2 \times \ell_0^1$ using a Newton-Kantorovich argument. This will require us to develop a bound on DT using some norm on $\mathbb{R}^2 \times \ell_0^1$. Unfortunately there is no natural choice of norm on the product space $X := \mathbb{R}^2 \times \ell_0^1$. Furthermore, it will not become apparent if one norm is better than another until after significant calculation. For this reason, we use a notion of an "upper bound" which allows us to delay our choice of norm. We first introduce the operator $\zeta : X \to \mathbb{R}_+^3$ which consists of the norms of the three components:

$$\zeta(x) := (|\pi_\alpha x|, |\pi_\omega x|, \|\pi_c x\|)^T \in \mathbb{R}_+^3 \qquad \text{for any } x \in X.$$

Definition 2.2.5 (upper bound)**.** *We call $\overline{x} \in \mathbb{R}^3_+$ an upper bound on x if $\zeta(x) \leq \overline{x}$, where the inequality is interpreted componentwise in $\mathbb{R}^3$. Let X' be a subspace of X and let X'' be a subset of X'. An upper bound on a linear operator $A' : X' \to X$ over X'' is a 3×3 matrix $\overline{A'} \in \mathrm{Mat}(\mathbb{R}^3, \mathbb{R}^3)$ such that*

$$\zeta(A'x) \leq \overline{A'} \cdot \zeta(x) \qquad \text{for any } x \in X'',$$

where the inequality is again interpreted componentwise in $\mathbb{R}^3$. The notion of upper bound conveniently encapsulates bounds on the different components of the operator A' on the product space X. Clearly the components of the matrix $\overline{A'}$ are nonnegative.

For example, in Proposition 3.1.3 we calculate an upper bound on the map $A_0^{-1}A_1$. As for the domain of definition of T, in practice we use $X' = \mathbb{R}^2 \times \ell_0^K$ and $X'' = \mathbb{R}^2 \times \ell_\rho$. The subset X'' does not always affect the upper bound calculation (such as in Proposition 3.1.3). However, operators such as $U_\omega - U_{\omega_0}$ have upper bounds which contain ρ-terms (see for example Proposition 3.2.3).

Using this terminology, we state a "radii polynomial" theorem, which allows us to check whether T is a contraction map. This technique has been used frequently in a computer-assisted setting in the past decade. Early application include [DLM07, vdBL08], while a previous implementation in the context of Wright's delay equation can be found in [Les10]. Although we use radii polynomials as well, our approach differs significantly from the computer-assisted setting mentioned above. While we do engage a computer (namely the Mathematica file [vdBJ]) to optimize our quantitative results, the analysis is performed essentially in terms of pencil-and-paper mathematics (in particular, our operators do not involve any floating point numbers). In our current setup we employ *three* radii as a priori unknown variables, which builds on an idea introduced in [vdB16]. We note that in most of the papers mentioned above the notation of A and $A^\dagger$ is reversed compared to the current paper.

As preparation, the following lemma (of which the proof can be found in Appendix 3.2) provides an explicit choice for ρ, as a function of ϵ and r, for which we have proper control on the image of $B_\epsilon(r, \rho)$ under T.

Lemma 2.2.6. *For any $\epsilon \geq 0$ and $r \in \mathbb{R}^3_+$, let $C = C(\epsilon, r)$ be given by Equation* (3.11)*. If $C(\epsilon, r) > 0$ then*

$$\|K^{-1}\pi_c T(x)\| \leq \rho \quad \text{whenever } x \in B_\epsilon(r,\rho) \text{ and } \rho \geq C(\epsilon, r). \tag{2.16}$$

Moreover, $C(\epsilon, r)$ is nondecreasing in ϵ and r.

Proof. See Proposition 3.2.4. □

Theorem 2.2.7. *Let $0 \leq \epsilon < \frac{\sqrt{10}}{4}$ and fix $r = (r_\alpha, r_\omega, r_c) \in \mathbb{R}^3_+$. Fix $\rho > 0$ such that $\rho \geq C(\epsilon, r)$, as given by Lemma 2.2.6. Suppose that $Y(\epsilon)$ is an upper bound on $T(\bar{x}_\epsilon) - \bar{x}_\epsilon$ and $Z(\epsilon, r, \rho)$ a (uniform) upper bound on $DT(x)$ for all $x \in B_\epsilon(r, \rho)$. Define the* radii polynomials *$P : \mathbb{R}^5_+ \to \mathbb{R}^3$ by*

$$P(\epsilon, r, \rho) := Y(\epsilon) - [I - Z(\epsilon, r, \rho)] \cdot r\,. \tag{2.17}$$

If each component of $P(\epsilon, r, \rho)$ is negative, then there is a unique $\hat{x}_\epsilon \in B_\epsilon(r, \rho)$ such that $F(\hat{x}_\epsilon) = 0$.

Proof. Let $r \in \mathbb{R}^3_+$ be a triple such that $P(\epsilon, r, \rho) < 0$. By Proposition 2.2.2, if $\epsilon < \frac{\sqrt{10}}{4}$ then $A^\dagger$ is injective. Hence $\hat{x}_\epsilon$ is a fixed point of T if and only if $F(\hat{x}_\epsilon) = 0$. In order to show there is a unique fixed point $\hat{x}_\epsilon$, we show that T maps $B_\epsilon(r, \rho)$ into itself and that T is a contraction mapping.

We first show that $T : B_\epsilon(r, \rho) \to B_\epsilon(r, \rho)$. Since $\rho \geq C(\epsilon, r)$ then by Equation (2.16) it follows that $\|K^{-1}\pi_c T(x)\| \leq \rho$ for all $x \in B_\epsilon(r, \rho)$. In order to show that $T(x) \in B_\epsilon(r, \rho)$, it suffices to show that $r = (r_\alpha, r_\omega, r_c)$ is an upper bound on $T(x) - \bar{x}_\epsilon$ for all $x \in B_\epsilon(r, \rho)$. We decompose

$$T(x) - \bar{x}_\epsilon = [T(\bar{x}_\epsilon) - \bar{x}_\epsilon] + [T(x) - T(\bar{x}_\epsilon)], \tag{2.18}$$

and estimate each part separately. Concerning the first term, by assumption, $Y(\epsilon)$ is an upper bound on $T(\bar{x}_\epsilon) - \bar{x}_\epsilon$. Concerning the second term, we claim that $Z(\epsilon, r, \rho) \cdot r$ is an upper bound on $T(x) - T(\bar{x}_\epsilon)$. Indeed, we have the following somewhat stronger bound:

$$\zeta(T(y) - T(x)) \leq Z(\epsilon, r, \rho) \cdot \zeta(y - x) \qquad \text{for all } x, y \in B_\epsilon(r, \rho). \tag{2.19}$$

The latter follows from the mean value theorem, since T is continuously Fréchet differentiable on $B_\epsilon(r,\rho)$. Since r is an upper bound on $x - \bar{x}_\epsilon$ for all $x \in B_\epsilon(r,\rho)$, we find, by using (2.18), that $Y(\epsilon) + Z(\epsilon,r,\rho)\cdot r \le r$ (with the inequality, interpreted componentwise, following from $P(\epsilon,r,\rho) < 0$) is an upper bound on $T(x) - \bar{x}_\epsilon$ for all $x \in B_\epsilon(r,\rho)$. That is to say, if all of the radii polynomials are negative, then T maps $B_\epsilon(r,\rho)$ into itself.

To finish the proof, we show that T is a contraction mapping. We abbreviate $Z = Z(\epsilon,r,\rho)$ and recall that $r = (r_\alpha, r_\omega, r_c) = (r_1,r_2,r_3) \in \mathbb{R}^3_+$ is such that $Z\cdot r < r$, hence for some $\kappa < 1$ we have

$$\frac{(Z\cdot r)_i}{r_i} \le \kappa \qquad \text{for } i = 1,2,3. \tag{2.20}$$

We now need to choose a norm on X. We define a norm $\|\cdot\|_r$ on elements $x = (\alpha,\omega,c) \in X$ by

$$\|(\alpha,\omega,c)\|_r := \max\left\{\frac{|\alpha|}{r_\alpha}, \frac{|\omega|}{r_\omega}, \frac{\|c\|}{r_c}\right\},$$

or

$$\|x\|_r = \max_{i=1,2,3} \frac{\zeta(x)_i}{r_i} \qquad \text{for all } x \in X.$$

By using the upper bound Z, we bound the Lipschitz constant of T on $B_\epsilon(r,\rho)$ as follows:

$$\begin{aligned}
\|T(y) - T(x)\|_r &= \max_{i=1,2,3} \frac{\zeta(T(y)-T(x))_i}{r_i} \\
&\le \max_{i=1,2,3} \frac{(Z\cdot\zeta(y-x))_i}{r_i} \\
&\le \max_{i=1,2,3}\max_{j=1,2,3} \frac{\zeta(y-x)_j}{r_j}\frac{(Z\cdot r)_i}{r_i} \\
&= \|y-x\|_r \max_{i=1,2,3}\frac{(Z\cdot r)_i}{r_i} \\
&\le \kappa\|y-x\|_r,
\end{aligned}$$

where we have used (2.19) and (2.20) with $\kappa < 1$. Hence $T : B_\epsilon(r,\rho) \to B_\epsilon(r,\rho)$ is a contraction with respect to the $\|\cdot\|_r$ norm.

Since $B_\epsilon(r,\rho)$ with this norm is a complete metric space, by the Banach fixed point theorem T has a unique fixed point $\hat{x}_\epsilon \in B_\epsilon(r,\rho)$. Since $A^\dagger$ is injective, it follows that $\hat{x}_\epsilon$ is the unique point in $B_\epsilon(r,\rho)$ for which $F(\hat{x}_\epsilon) = 0$. $\square$

Remark 2.2.8. *Under the assumptions in Theorem 2.2.7, essentially the same calculation as in the proof above leads to the estimate*

$$\|DT(x)y\|_r \leq \kappa \|y\|_r \qquad \textit{for all } y \in \mathbb{R}^2 \times \ell_0^K,\ x \in B_\epsilon(r,\rho),$$

where $\kappa := \max_{i=1,2,3}(Z \cdot r)_i / r_i$.

In Appendix 3.3 and Appendix 3.4 we construct explicit upper bounds $Y(\epsilon)$ and $Z(\epsilon, r, \rho)$, respectively. These functions are constructed such that their components are (multivariate) polynomials in ϵ, r and ρ with nonnegative coefficients, hence they are increasing in these variables. This construction enables us to make use of the uniform contraction principle.

Corollary 2.2.9. *Let* $0 < \epsilon_0 < \frac{\sqrt{10}}{4}$ *and fix some* $r = (r_\alpha, r_\omega, r_c) \in \mathbb{R}^3_+$. *Fix* $\rho > 0$ *such that* $\rho \geq C(\epsilon_0, r)$, *as given by Lemma 2.2.6. Let* $Y(\epsilon)$ *and* $Z(\epsilon, r, \rho)$ *be the upper bounds as given in Propositions 3.3.2 and 3.4.1. Let the radii polynomials* P *be defined by Equation* (2.17).

If each component of $P(\epsilon_0, r, \rho)$ *is negative, then for all* $0 \leq \epsilon \leq \epsilon_0$ *there exists a unique* $\hat{x}_\epsilon \in B_\epsilon(r, \rho)$ *such that* $F(\hat{x}_\epsilon) = 0$. *The solution* $\hat{x}_\epsilon$ *depends smoothly on* ϵ.

Proof. Let $0 \leq \epsilon \leq \epsilon_0$ be arbitrary. Because $\rho \geq C(\epsilon_0, r) \geq C(\epsilon, r)$ by Lemma 2.2.6, Theorem 2.2.7 implies that it suffices to show that $P(\epsilon, r, \rho) < 0$. Since the bounds $Y(\epsilon)$ and $Z(\epsilon, r, \rho)$ are monotonically increasing in their arguments, it follows that $P(\epsilon, r, \rho) \leq P(\epsilon_0, r, \rho) < 0$. Continuous and smooth dependence on ϵ of the fixed point follows from the uniform contraction principle (see for example [CH82]). □

Given the upper bounds $Y(\epsilon)$ and $Z(\epsilon, r, \rho)$, trying to apply Corollary 2.2.9 amounts to finding values of $\epsilon, r_\alpha, r_\omega, r_c, \rho$ for which the radii polynomials are negative. Selecting a value for ρ is straightforward: all estimates improve with smaller values of ρ, and Proposition 3.2.4 (see also Lemma 2.2.6) explicitly describes the smallest allowable choice of ρ in terms of $\epsilon, r_\alpha, r_\omega, r_c$.

Beyond selecting a value for ρ, it is difficult to pinpoint what constitutes an "optimal" choice of these variables. In general it is interesting to find such viable radii (i.e.

radii such that $P(r) < 0$) which are both large and small. The smaller radius tells us how close the true solution is to our approximate solution. The larger radius tells us in how large a neighborhood our solution is unique. With regard to ϵ, larger values allow us to describe functions whose first Fourier mode is large. However this will "grow" the smallest viable radius and "shrink" the largest viable radius.

Proposition 2.2.10 presents two selections of variables which satisfy the hypothesis of Corollary 2.2.9. We check the hypothesis is indeed satisfied by using interval arithmetic. All details are provided in the Mathematica file [vdBJ]. While the specific numbers used may appear to be somewhat arbitrary (see also the discussion in Remark 2.2.11) they have been chosen to be used later in Theorem 2.3.7 and Theorem 2.3.10.

Proposition 2.2.10. *Fix the constants ϵ_0, $(r_\alpha, r_\omega, r_c)$ and ρ according to one of the following choices:*

(a) $\epsilon_0 = 0.029$ *and* $(r_\alpha, r_\omega, r_c) = (0.13,\, 0.17,\, 0.17)$ *and* $\rho = 1.78$*;*

(b) $\epsilon_0 = 0.09$ *and* $(r_\alpha, r_\omega, r_c) = (0.1753,\, 0.0941,\, 0.3829)$ *and* $\rho = 1.5940$.

For either of the choices (a) and (b) we have the following: for all $0 \le \epsilon \le \epsilon_0$ there exists a unique point $(\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon) \in B_\epsilon(r, \rho)$ satisfying $F_\epsilon(\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon) = 0$ and

$$|\hat{\alpha}_\epsilon - \bar{\alpha}_\epsilon| \le r_\alpha, \quad |\hat{\omega}_\epsilon - \bar{\omega}_\epsilon| \le r_\omega, \quad \|\hat{c}_\epsilon - \bar{c}_\epsilon\| \le r_c, \quad \|K^{-1}\hat{c}_\epsilon\| \le \rho.$$

Proof. In the Mathematica file [vdBJ] we check, using interval arithmetic, that $\rho \ge C(\epsilon_0, r)$ and the radii polynomials $P(\epsilon_0, r, \rho)$ are negative for the choices (a) and (b). The result then follows from Corollary 2.2.9. $\square$

Remark 2.2.11. *In Proposition 2.2.10 we aimed for large balls on which the solution is unique. Even for a fixed value of ϵ, it is not immediately obvious how to find a "largest" viable radius r, since r has three components. In particular, there is a trade-off between the different components of r. On the other hand, as explained in Remark 2.2.14, no such difficulty arises when looking for a "smallest" viable radius.*

We will also need a rescaled version of the radii polynomials, which takes into account the asymptotic behavior of the bound Y on the residue $T(\bar{x}_\epsilon) - \bar{x}_\epsilon = -A^\dagger F(\bar{x}_\epsilon)$

as $\epsilon \to 0$, namely it is of the form $Y(\epsilon) = \epsilon^2 \tilde{Y}(\epsilon)$, see Proposition 3.3.2. The proofs of the following monotonicity properties can be found in Appendices 3.3 and 3.4.

Lemma 2.2.12. *Let $\epsilon \geq 0$, $\rho > 0$ and $r \in \mathbb{R}^3_+$. Then there are upper bounds $Y(\epsilon) = \epsilon^2 \tilde{Y}(\epsilon)$ on $T(\bar{x}_\epsilon) - \bar{x}_\epsilon$ and a (uniform) upper bound $Z(\epsilon, r, \rho)$ on $DT(x)$ for all $x \in B_\epsilon(r, \rho)$. These bounds are given explicitly by Propositions 3.3.2 and 3.4.1, respectively. Moreover, $\tilde{Y}(\epsilon)$ is nondecreasing in ϵ, while $Z(\epsilon, r, \rho)$ is nondecreasing in ϵ, r and ρ.*

This implies, roughly speaking, that if we are able to show that T is a contraction map on $B_{\epsilon_0}(\epsilon_0^2 \check{r}, \rho)$ for a particular choice of ϵ_0, then it will be a contraction map on $B_\epsilon(\epsilon^2 \check{r}, \rho)$ for all $0 \leq \epsilon \leq \epsilon_0$. Here, and in what follows, we use the notation $r = \epsilon^2 \check{r}$ for the ϵ-scaled version of the radii.

Corollary 2.2.13. *Let $0 < \epsilon_0 < \frac{\sqrt{10}}{4}$ and fix some $\check{r} = (\check{r}_\alpha, \check{r}_\omega, \check{r}_c) \in \mathbb{R}^3_+$. Fix $\rho > 0$ such that $\rho \geq C(\epsilon_0, \epsilon_0^2 \check{r})$, as given by Lemma 2.2.6. Let $Y(\epsilon)$ and $Z(\epsilon, r, \rho)$ be the upper bounds as given by Lemma 2.2.12. Let the radii polynomials P be defined by* (2.17).

If each component of $P(\epsilon_0, \epsilon_0^2 \check{r}, \rho)$ is negative, then for all $0 \leq \epsilon \leq \epsilon_0$ there exists a unique $\hat{x}_\epsilon \in B_\epsilon(\epsilon^2 \check{r}, \rho)$ such that $F(\hat{x}_\epsilon) = 0$. Furthermore, $\hat{x}_\epsilon$ depends smoothly on ϵ.

Proof. Let $0 \leq \epsilon < \epsilon_0$ be arbitrary. Because $\rho \geq C(\epsilon_0, \epsilon_0^2 \check{r}) \geq C(\epsilon, \epsilon^2 \check{r})$ by Lemma 2.2.6, Theorem 2.2.7 implies that it suffices to show that $P(\epsilon, \epsilon^2 \check{r}, \rho) < 0$. By using the monotonicity provided by Lemma 2.2.12, we obtain

$$
\begin{aligned}
P(\epsilon, \epsilon^2 \check{r}, \rho) &= Y(\epsilon) - \left[I - Z(\epsilon, \epsilon^2 \check{r}, \rho)\right] \cdot \epsilon^2 \check{r} \\
&= (\epsilon/\epsilon_0)^2 \left[\epsilon_0^2 \tilde{Y}(\epsilon) - \epsilon_0^2 \check{r} + Z(\epsilon, \epsilon^2 \check{r}, \rho) \cdot \epsilon_0^2 \check{r}\right] \\
&\leq (\epsilon/\epsilon_0)^2 \left[\epsilon_0^2 \tilde{Y}(\epsilon_0) - \epsilon_0^2 \check{r} + Z(\epsilon_0, \epsilon_0^2 \check{r}, \rho) \cdot \epsilon_0^2 \check{r}\right] \\
&= (\epsilon/\epsilon_0)^2 P(\epsilon_0, \epsilon_0^2 \check{r}, \rho) \\
&< 0,
\end{aligned}
$$

where inequalities are interpreted componentwise in $\mathbb{R}^3$, as usual. $\square$

These ϵ-rescaled variables are used in Proposition 2.2.15 below to derive *tight* bounds on the solution (in particular, tight enough to conclude that the bifurcation is super-critical). The following remark explains that the monotonicity properties of the bounds

Y and Z imply that looking for small(est) radii which satisfy $P(r) < 0$, is a well-defined problem.

Remark 2.2.14. *The set R of radii for which the radii polynomials are negative is given by*

$$R := \{r \in \mathbb{R}^3_+ : r_j > 0, P_i(r) < 0 \text{ for } i, j = 1, 2, 3\}.$$

This set has the property that if $r, r' \in R$, then $r'' \in R$, where $r''_j = \min\{r_j, r'_j\}$. Namely, the main observation is that we can write $P_i(r) = \tilde{P}_i(r) - r_i$, where $\partial_{r_j} \tilde{P}_i \geq 0$ for all $i, j = 1, 2, 3$. Now fix any i; we want to show that $P_i(r'') < 0$. We have either $r''_i = r_i$ or $r''_i = r'_i$, hence assume $r''_i = r_i$ (otherwise just exchange the roles of r and r'). We infer that $P_i(r'') \leq P_i(r) < 0$, since $\partial_{r_j} P_i \geq 0$ for $j \neq i$. We conclude that there are no trade-offs in looking for minimal/tight radii, as opposed to looking for large radii, see Remark 2.2.11.

Proposition 2.2.15. *Fix $\epsilon_0 = 0.10$ and $(\check{r}_\alpha, \check{r}_\omega, \check{r}_c) = (0.0594, 0.0260, 0.4929)$ and $\rho = 0.3191$. For all $0 < \epsilon \leq \epsilon_0$ there exists a unique point $\hat{x}_\epsilon = (\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon)$ satisfying $F(\hat{x}_\epsilon) = 0$ and*

$$|\hat{\alpha}_\epsilon - \bar{\alpha}_\epsilon| < \check{r}_\alpha \epsilon^2, \qquad |\hat{\omega}_\epsilon - \bar{\omega}_\epsilon| < \check{r}_\omega \epsilon^2, \qquad \|\hat{c}_\epsilon - \bar{c}_\epsilon\| < \check{r}_c \epsilon^2, \qquad \|K^{-1} \hat{c}_\epsilon\| < \rho. \tag{2.21}$$

Furthermore, $\hat{\alpha}_\epsilon > \frac{\pi}{2}$ for $0 < \epsilon < \epsilon_0$.

Proof. In the Mathematica file [vdBJ] we check, using interval arithmetic, that $\rho \geq C(\epsilon_0, \epsilon_0^2 \check{r})$ and the radii polynomials $P(\epsilon_0, \epsilon_0^2 \check{r}, \rho)$ are negative. The inequalities in Equation (2.21) follow from Corollary 2.2.13. Since $\hat{\alpha}_\epsilon \geq \bar{\alpha}_\epsilon - \check{r}_\alpha \epsilon^2 = \frac{\pi}{2} + \frac{1}{5}(\frac{3\pi}{2} - 1)\epsilon^2 - \check{r}_\alpha \epsilon^2$ and $\check{r}_\alpha < \frac{1}{5}(\frac{3\pi}{2} - 1)$, it follows that $\hat{\alpha}_\epsilon > \frac{\pi}{2}$ for all $0 < \epsilon \leq \epsilon_0$. $\square$

Remark 2.2.16. *Since $\epsilon_0^2 \check{r} < r$ for the choices (a) and (b) in Proposition 2.2.10, and the choices of ρ and ϵ_0 are compatible as well, the solutions found in Proposition 2.2.10 are the same as those described by Proposition 2.2.15. While the former proposition provides large isolation/uniqueness neighborhoods for the solutions, the latter provides tight bounds and confirms the supercriticality of the bifurcation suggested in Definition 2.1.12.*

2.3 Global results

When deriving global results from the local results in Section 2.2, we need to take into account that there are some obvious reasons why the branch of periodic solutions, described by $F_\epsilon(\alpha, \omega, c) = 0$, bifurcating from the Hopf bifurcation point at $(\alpha, \omega) = (\frac{\pi}{2}, \frac{\pi}{2})$ does not describe the entire set of periodic solutions for α near $\frac{\pi}{2}$. First, there is the trivial solution. In particular, one needs to quantify in what sense the trivial solution is an isolated invariant set. This is taken care of by Remarks 2.1.8 and 2.1.11, which show there are no "spurious" small solutions in the parameter regime of interest to us (roughly as long as we stay away from the next Hopf bifurcation at $\alpha = \frac{5\pi}{2}$). Second, one can interpret any periodic solution with frequency ω as a periodic solution with frequency ω/N as well, for any $N \in \mathbb{N}$. Since we are working in Fourier space, showing that there are no "spurious" solutions with lower frequency would require us to perform an analysis near $(\alpha, \omega) = (\frac{\pi}{2}, \frac{\pi}{2N})$ for all $N \geq 2$. This obstacle can be avoided by bounding (from below) ω away from $\pi/4$. This is done in Lemma 2.3.4.

For later use, we recall an elementary Fourier analysis bound.

Lemma 2.3.1. *Let $y \in C^1$ be a periodic function of period $2\pi/\omega$ with Fourier coefficients $a \in \ell^1_{\text{sym}}$ (in particular this means $a_0 = 0$), as described by* (2.1)*. Then*

$$\|a\| \leq \sqrt{\frac{\pi}{6\omega}}\, \|y'\|_{L^2([0,2\pi/\omega])} \qquad \text{and} \qquad \|a\| \leq \frac{\pi}{\omega\sqrt{3}}\, \|y'\|_\infty.$$

Proof. From the Cauchy-Schwarz inequality and Parseval's identity it follows that

$$\begin{aligned}
\|a\| &= 2\sum_{k=1}^{\infty} |a_k| \leq 2\left(\sum_{k=1}^{\infty} k^{-2}\right)^{1/2} \left(\sum_{k=1}^{\infty} |k\, a_k|^2\right)^{1/2} \\
&= \frac{\sqrt{2}}{\omega}\left(\frac{\pi^2}{6}\right)^{1/2} \left(2\sum_{k=1}^{\infty} |i\omega k\, a_k|^2\right)^{1/2} = \frac{\pi}{\omega\sqrt{3}} \left(\sum_{k\in\mathbb{Z}} |i\omega k\, a_k|^2\right)^{1/2} \\
&= \frac{\pi}{\omega\sqrt{3}} \left(\frac{\omega}{2\pi}\int_0^{2\pi/\omega} |y'(t)|^2 dt\right)^{1/2} \leq \frac{\pi}{\omega\sqrt{3}}\, \|y'\|_\infty.
\end{aligned}$$

$\square$

2.3.1 A proof of Wright's conjecture

Based on the work in [BCKN14] and [Wri55], in order to prove Wright's conjecture it suffices to prove that there are no slowly oscillating periodic solutions (SOPS) to Wright's equation for $\alpha \in [1.5706, \frac{\pi}{2}]$. Moreover, in [BCKN14] it was shown that no SOPS with $\|y\|_\infty \geq e^{0.04} - 1$ exists for $\alpha \in [1.5706, \frac{\pi}{2}]$. These results are summarized in the following proposition.

Proposition 2.3.2 ([BCKN14,Wri55])**.** *Assume y is a SOPS to Wright's equation for some $\alpha \leq \frac{\pi}{2}$. Then $\alpha \in [1.5706, \frac{\pi}{2}]$ and $\|y\|_\infty \leq e^{0.04} - 1$.*

For convenience we introduce

$$\mu := e^{0.04} - 1 \approx 0.0408.$$

We now derive a lower bound on the frequency ω of the SOPS, part of which is later used in Lemma 4.3.4.

Lemma 2.3.3. *Assume $\alpha > 1$ and y is a SOPS to* (1.2)*. Then q, as given in Definition 1.1.2, satisfies $q < 2 + \frac{1}{\alpha}$.*

Proof. Without loss of generality, assume that $y(0) = 0$ and $y(t) > 0$ for $t \in (0, q)$. To obtain an upper bound on q, assume that $q \geq 2$. Set $q' = \min\{q, 3\}$. Then it follows from (1.2) that $y'(t) < 0$ for $t \in (1, q']$, hence $y(t-1) > y(2)$ for $t \in [2, q']$. We infer that for $t \in [2, q']$ we have $y'(t) = -\alpha y(t-1)[1 + y(t)] < -\alpha y(2)$. Solving the IVP $y'(t) < -\alpha y(2)$ with the initial condition $y(2) = y(2)$, we see that $y(t)$ hits 0 before $t = 2 + \frac{1}{\alpha}$. Since $\alpha > 1$ (hence $2 + \frac{1}{\alpha} < 3$), this implies that $q' = q$ and $q < 2 + \frac{1}{\alpha}$.

$\square$

Lemma 2.3.4. *Let $\alpha \in [1.5706, \frac{\pi}{2}]$. Assume y is a SOPS to Wright's equation with minimal period $2\pi/\omega$, and assume that $\|y\|_\infty \leq \mu$. Then $\omega \in [1.11, 1.93]$.*

Proof. Without loss of generality, we assume in this proof that $y(0) = 0$, that $y(t) < 0$ for $t \in (-\bar{q}, 0)$ and that $y(t) > 0$ for $t \in (0, q)$. We will show that $\bar{q}$ and q are bounded by

$$1 + \frac{1}{\alpha}\frac{\log(1+\mu)}{\mu} < q < 2 + \frac{1}{\alpha},$$
$$1 + \frac{1}{\alpha} < \bar{q} < 3.$$

The upper bound on q follows from Lemma 2.3.3. The lower bounds for both $\bar{q}$ and q follow directly from [Jon62b, Theorem 3.5]. While [Jon62b, Theorem 3.5] assumes $\alpha \geq \frac{\pi}{2}$, this part of the theorem simply relies on [Jon62b, Lemma 2.1], which only requires $\alpha > e^{-1}$.

To obtain the upper bound on $\bar{q}$, assume for the sake of contradiction that $\bar{q} \geq 3$. Then it follows from (1.2) that $y'(t) \geq 0$ for $t \in [-2, 0]$. Hence $y(t) \leq y(-1)$ for $t \in [-2,-1]$, and $y'(t) \geq -\alpha y(-1)[1 + y(t)]$ for $t \in [-1, 0]$. Solving this IVP with the initial condition $y(-1) = -\nu$, we obtain $y(t) \geq (1-\nu)e^{\alpha\nu(t+1)} - 1$ for $t \in [-1, 0]$, and in particular $y(0) \geq (1-\nu)e^{-\alpha\nu} - 1$. By assumption $y(0) = 0$ and $\nu = |y(-1)| \leq \mu$, but $(1-\nu)e^{-\alpha\nu} - 1 > 0$ for $\nu \in (0, \mu]$ and $\alpha \in [1.5706, \frac{\pi}{2}]$, a contradiction. Thereby $\bar{q} < 3$.

The bound on α implies that the minimal period $L = q + \bar{q}$ of the SOPS must lie in $[3.26, 5.64]$. It then follows that $\omega \in [1.11, 1.93]$ □

It turns out that this bound on ω can (and needs to be) sharpened. This is the purpose of the following lemma, which considers solutions in unscaled variables.

Lemma 2.3.5. *Suppose* $\tilde{F}_\epsilon(\alpha, \omega, \tilde{c}) = 0$. *If* $\omega \in [1.1, 2]$ *and* $\alpha \in [1.5, 2.0]$ *then*

$$\frac{\sqrt{(\omega-\alpha)^2 + 2\alpha\omega(1-\sin\omega)}}{2\alpha} \leq 2\epsilon + \|\tilde{c}\|. \tag{2.22}$$

Proof. This follows from Proposition 3.5.1 in Section 3.5, combined with Proposition 3.5.2, which shows that for $\omega \in [1.1, 2.0]$ and $\alpha \in [1.5, 2.0]$, the minimum in Equation (3.35) is attained for $k = 1$. □

Next we derive bounds on ϵ and $\tilde{c}$, which also lead to improved bounds on ω.

Lemma 2.3.6. *Let* $\alpha \in [1.5706, \frac{\pi}{2}]$. *Assume* y *is a SOPS with* $\|y\|_\infty \leq \mu$. *Then* y *corresponds, through the Fourier representation* (2.14), *to a zero of* $F_\epsilon(\alpha, \omega, c)$ *with* $|\omega - \frac{\pi}{2}| \leq 0.1489$ *and*

$$0 < \epsilon \leq \epsilon_* := \mu/\sqrt{2} \leq 0.02886,$$

and $\|c\| \leq 0.0796$ *and* $\|K^{-1}c\| \leq 0.16$.

Proof. First consider the Fourier representation (2.10) of y in unscaled variables. Recall that a_0 vanishes (see Remark 2.1.3). Since $|y'(t)| \leq \alpha|y(t-1)|(1+|y(t)|) \leq \alpha\mu(1+\mu)$ we see from Lemma 2.3.1 that

$$2\epsilon + \|\tilde{c}\| \leq \frac{\pi}{\omega\sqrt{3}}\alpha\mu(1+\mu). \tag{2.23}$$

Combining this with Lemma 2.3.5 leads to the inequality

$$\omega\sqrt{(\omega-\alpha)^2 + 2\alpha\omega(1-\sin\omega)} \leq \tfrac{2\pi}{\sqrt{3}}\alpha^2\mu(1+\mu). \tag{2.24}$$

In the Mathematica file [vdBJ] we show that when $\alpha \in [1.5706, \frac{\pi}{2}]$, then inequality (2.24) is violated for any $\omega \in [1.1, 2.0] \setminus [1.4219, 1.6887]$. From Lemma 2.3.4 we obtain the a priori bound $\omega \in [1.11, 1.93]$, whereby it follows that $\omega \in [1.4219, 1.6887]$, and in particular $|\omega - \frac{\pi}{2}| \leq 0.1489$.

Using this sharper bound on ω as well as $\alpha \in [1.5706, \frac{\pi}{2}]$ we conclude from (2.23) that

$$2\epsilon + \|\tilde{c}\| \leq \frac{\pi}{\omega\sqrt{3}}\alpha\mu(1+\mu) \leq \frac{2\omega-\alpha}{\alpha}. \tag{2.25}$$

Since we also infer that $\alpha < 2\omega$, Theorem 2.1.10(b) shows that the solution corresponds to a zero of $F_\epsilon(\alpha, \omega, c)$, with $\tilde{c} = \epsilon c$. We can improve the bound on ϵ from (2.23) by observing that

$$(\epsilon^2 + \epsilon^2)^{1/2} \leq \left(\sum_{k\in\mathbb{Z}} |a_k|^2\right)^{1/2} = \left(\frac{\omega}{2\pi}\int_0^{2\pi/\omega} |y|^2 dt\right)^{1/2} \leq \mu.$$

Hence $\epsilon \leq \epsilon_* := \mu/\sqrt{2}$.

Finally, we derive the bounds on c. Namely, for $\alpha \in [1.5706, \frac{\pi}{2}]$, $\omega \in [1.4219, 1.6887]$ and $\epsilon \leq \epsilon_*$, we find that b_* and z_*^+, as defined in (2.13), are bounded below by $b_* \geq 0.364$ and $z_*^+ \geq 0.72$. Since it follows from (2.23) that $\|\tilde{c}\| \leq 0.09$ in the same parameter range of α and ω, we infer from Lemma 2.1.7(a) that $\|\tilde{c}\| \leq z_*^-$. Via an interval arithmetic computation, the latter can be bounded above using Lemma 3.5.5, for $\alpha \in [1.5706, \frac{\pi}{2}]$, $\omega \in [1.4219, 1.6887]$ and $\epsilon \leq \epsilon_*$, by $z_*^- \leq 0.0796\epsilon$. Hence $\|c\| \leq z_*^-/\epsilon \leq 0.0796$. Furthermore, Lemma 2.1.7(b) implies the bound $\|K^{-1}c\| \leq (2\epsilon^2 + (z_*^-)^2)/(\epsilon b_*) \leq 5.52\epsilon$. Since $\epsilon \leq \epsilon_*$, it then follows that $\|K^{-1}c\| \leq 0.16$. □

With these tight bounds on the solutions, we are in a position to apply the local bifurcation result formulated in Proposition 2.2.15 to prove the ultimate step of Wright's conjecture.

Theorem 2.3.7. *For $\alpha \in [0, \frac{\pi}{2}]$ there is no SOPS to Wright's equation.*

Proof. By Theorem 1.2.1 and Proposition 2.3.2 it suffices to exclude a slowly oscillating periodic solution y for $\alpha \in [1.5706, \frac{\pi}{2}]$ with $\|y\|_\infty \leq \mu$. By Lemma 2.3.6, if such a solution would exist, it corresponds to a solution of $F_\epsilon(\alpha, \omega, c) = 0$ with $|\omega - \frac{\pi}{2}| \leq 0.1489$, $0 < \epsilon \leq \epsilon_* = \mu/\sqrt{2}$, $\|c\| \leq 0.0796$ and $\|K^{-1}c\| \leq 0.16$. We claim that no such solution exists. Indeed, we define the set

$$S := \{(\alpha, \omega, c) \in X : |\alpha - \tfrac{\pi}{2}| \leq 0.0002;\ |\omega - \tfrac{\pi}{2}| \leq 0.15;\ \|c\| \leq 0.08;\ \|K^{-1}c\| \leq 0.16\}.$$

To show that there is no SOPS for $\alpha \in [1.5706, \frac{\pi}{2}]$, it now suffices to show that all zeros of $F_\epsilon(\alpha, \omega, c)$ in S for any $0 < \epsilon \leq \epsilon_*$ satisfy $\alpha > \frac{\pi}{2}$.

Let us consider $B_\epsilon(r, \rho)$, which is centered at $\bar{x}_\epsilon$ (see Definition 2.1.12) with r and ρ taken as in Proposition 2.2.10(a). In the Mathematica file [vdBJ] we check that the following inequalities are satisfied:

$$\begin{aligned}
r_\alpha &= 0.13 \geq 0.0002 + |\bar{\alpha}_{\epsilon_*} - \tfrac{\pi}{2}|, \\
r_\omega &= 0.17 \geq 0.15 + |\bar{\omega}_{\epsilon_*} - \tfrac{\pi}{2}|, \\
r_c &= 0.17 \geq 0.08 + \|\bar{c}_{\epsilon_*}\|, \\
\rho &= 1.78 \geq 0.16.
\end{aligned}$$

By the triangle inequality we obtain that $S \subset B_\epsilon(r, \rho)$ for all $0 < \epsilon \leq \epsilon_*$. Proposition 2.2.10(a) shows that for each $0 < \epsilon \leq \epsilon_*$ there is a unique zero $\hat{x}_\epsilon = (\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon) \in B_\epsilon(r, \rho)$ of F_ϵ. By Proposition 2.2.15 and Remark 2.2.16 this zero satisfies $\hat{\alpha}_\epsilon > \frac{\pi}{2}$. Hence, for any $0 < \epsilon \leq \epsilon_*$ the only zero of F_ϵ in S (if there is one) satisfies $\alpha > \frac{\pi}{2}$. This completes the proof.

□

2.3.2 Towards Jones' conjecture

Jones' conjecture states that for $\alpha > \frac{\pi}{2}$ there exists a (globally) unique SOPS to Wright's equation. Theorem 2.2.7 shows that for a fixed small ϵ there is a (locally) unique α at which Wright's equation has a SOPS, represented by $(\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon)$. This is not sufficient to prove the local case of Jones conjecture. To accomplish the latter, we show in Theorem 2.3.10 that near the bifurcation point there is, for each fixed $\alpha > \frac{\pi}{2}$, a (locally) unique SOPS to Wright's equation. We begin by showing that on the solution branch emanating from the Hopf bifurcation $\hat{\alpha}_\epsilon$ is monotonically increasing in ϵ, i.e. $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon > 0$. Since $\bar{\alpha}_\epsilon = \frac{\pi}{2} + \frac{1}{5}(\frac{3\pi}{2} - 1)\epsilon^2$, we expect that $\frac{d}{d\epsilon}\hat{\alpha}(\epsilon) = \frac{2}{5}(\frac{3\pi}{2} - 1)\epsilon + \mathcal{O}(\epsilon^2)$. For this reason it is essential that we calculate an approximation of $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon$ which is accurate up to order $\mathcal{O}(\epsilon^2)$.

Theorem 2.3.8. *For $0 < \epsilon \leq 0.1$ we have $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon > 0$. For $\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} + 6.830 \times 10^{-3}$ there are no bifurcations in the branch of SOPS that originates from the Hopf bifurcation.*

Proof. We show that the branch of solutions $\hat{x}_\epsilon = (\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon)$ obtained in Proposition 2.2.15 satisfies $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon > 0$ for $0 < \epsilon \leq 0.1$. This implies that the solution branch is (smoothly) parametrized by α, i.e., there are no secondary nor any saddle-node bifurcations in this branch. We then show that these ϵ-values cover the range $\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} + 6.830 \times 10^{-3}$.

We begin by differentiating the equation $F(\hat{x}_\epsilon) = 0$ with respect to ϵ:

$$\frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon) + DF(\hat{x}_\epsilon)\frac{d}{d\epsilon}\hat{x}_\epsilon = 0. \tag{2.26}$$

In terms of the map T we obtain the relation

$$[I - DT(\hat{x}_\epsilon)]\frac{d}{d\epsilon}\hat{x}_\epsilon = -A^\dagger \frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon).$$

To isolate $\frac{d}{d\epsilon}\hat{x}_\epsilon$, we wish to left-multiply each side of the above equation by $[I - DT(\hat{x}_\epsilon)]^{-1}$. To that end, we define an upper bound on $DT(\hat{x}_\epsilon)$ by the matrix

$$\mathcal{Z}_\epsilon := Z(\epsilon, \epsilon^2 \check{r}, \rho), \tag{2.27}$$

with $\check{r}$ and ρ as in Proposition 2.2.15. We know from Remark 2.2.8 that with respect to the norm $\|\cdot\|_{\check{r}}$ on $\mathbb{R}^2 \times \ell_0^K$

$$\|DT(\hat{x}_\epsilon)\|_{\check{r}} \leq \max_{i=1,2,3} \frac{(\mathcal{Z}_\epsilon \cdot \check{r})_i}{\check{r}_i} < 1, \qquad \text{for all } 0 \leq \epsilon \leq \epsilon_0,$$

with ϵ_0 given in Proposition 2.2.15. Hence $I - DT(\hat{x}_\epsilon)$ is invertible. In particular,

$$\begin{aligned} \frac{d}{d\epsilon}\hat{x}_\epsilon &= -\left[I - DT(\hat{x}_\epsilon)\right]^{-1} A^\dagger \frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon) \\ &= -\left[I + \sum_{n=1}^{\infty} DT(\hat{x}_\epsilon)^n\right] A^\dagger \frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon). \end{aligned}$$

We have an upper bound $\mathcal{Q}_\epsilon \in \mathbb{R}^3_+$ on $A^\dagger \frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon)$, as defined in Definition 2.2.5, given by Lemma 3.6.4. We define I_3 to be the 3×3 identity matrix. For the α-component we then obtain the estimate

$$\begin{aligned} \frac{d}{d\epsilon}\hat{\alpha}_\epsilon &\geq -\pi_\alpha A^\dagger \frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon) - \left(\sum_{n=1}^{\infty} \mathcal{Z}_\epsilon^n \mathcal{Q}_\epsilon\right)_1 \\ &= -\pi_\alpha A^\dagger \frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon) - \left(\mathcal{Z}_\epsilon (I_3 - \mathcal{Z}_\epsilon)^{-1} \mathcal{Q}_\epsilon\right)_1. \end{aligned} \tag{2.28}$$

We approximate $\frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon)$ by

$$\Gamma_\epsilon := \tfrac{\pi}{2} \tfrac{3i-1}{5} \epsilon \, \mathrm{e}_1 - i \tfrac{\pi}{2} \, \mathrm{e}_2 - \tfrac{\pi}{2} \tfrac{3+i}{5} \epsilon \, \mathrm{e}_3,$$

which is accurate up to quadratic terms in ϵ. In Lemma 3.6.1 it is shown that

$$-\pi_\alpha A^\dagger \Gamma_\epsilon = \tfrac{2}{5}(\tfrac{3\pi}{2} - 1)\epsilon. \tag{2.29}$$

It remains to incorporate two explicit bounds for the remaining terms in (2.28). In Lemma 3.6.5 we define M_ϵ and M'_ϵ that satisfy the following inequalities:

$$\left| \pi_\alpha A^\dagger \left(\tfrac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon) - \Gamma_\epsilon \right) \right| \leq \epsilon^2 M_\epsilon, \tag{2.30}$$

$$\left(\mathcal{Z}_\epsilon (I_3 - \mathcal{Z}_\epsilon)^{-1} \mathcal{Q}_\epsilon\right)_1 \leq \epsilon^2 M'_\epsilon. \tag{2.31}$$

Moreover, we infer from Lemma 3.6.5 that M_ϵ and M'_ϵ are positive, increasing in ϵ, and can be obtained explicitly by performing an interval arithmetic computation, using the explicit expressions for the matrix $\mathcal{Z}_\epsilon$ and the vector $\mathcal{Q}_\epsilon$ given by Equation (2.27) and Lemma 3.6.4, respectively (the expression for $Z(\epsilon, r, \rho)$ is provided in Section 3.4).

Finally, we combine (2.28), (2.29), (2.30) and (2.31) to obtain

$$\frac{d}{d\epsilon}\hat{\alpha}_\epsilon \geq \tfrac{2}{5}(\tfrac{3\pi}{2}-1)\epsilon - \epsilon^2(M_\epsilon + M'_\epsilon).$$

From the monotonicity of the bounds M_ϵ and M'_ϵ in terms of ϵ, we infer that in order to conclude that $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon > 0$ for $0 < \epsilon \leq \epsilon_0$ it suffices to check, using interval arithmetic, that

$$\tfrac{2}{5}(\tfrac{3\pi}{2}-1)\epsilon_0 - \epsilon_0^2(M_{\epsilon_0} + M'_{\epsilon_0}) > 0. \tag{2.32}$$

In the Mathematica file [vdBJ] we check that (2.32) is satisfied for $\epsilon_0 = 0.1$. Since $\bar{\alpha}_{\epsilon_0} \geq \frac{\pi}{2} + 7.4247 \times 10^{-3}$, and taking into account the control provided by Proposition 2.2.15 on the distance between $\hat{\alpha}_\epsilon$ and $\bar{\alpha}_\epsilon$ in terms of $\check{r}_\alpha$, we find that $\hat{\alpha}_{\epsilon_0} \geq \bar{\alpha}_{\epsilon_0} - \epsilon_0^2\check{r}_\alpha \geq \frac{\pi}{2} + 6.830 \times 10^{-3}$. Hence there can be no bifurcation on the solution branch for $\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} + 6.830 \times 10^{-3}$. □

The analysis performed in Theorem 2.3.8 can similarly be applied to show that the frequency (period length) of SOPS along the principal branch monotonically decreases (increases) with respect to α when $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 6.830 \times 10^{-3}]$.

Corollary 2.3.9. *For $0 < \epsilon \leq 0.1$ we have $\frac{d}{d\epsilon}\hat{\omega}_\epsilon < 0$. For $\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} + 6.830 \times 10^{-3}$ the frequency (period length) of SOPS along the principal branch decreases (increases) monotonically in α.*

Proof. Analogous to (2.28), we have the following inequality:

$$\frac{d}{d\epsilon}\hat{\omega}_\epsilon \leq -\pi_\omega A^\dagger \frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon) + \big(\mathcal{Z}_\epsilon(I_3 - \mathcal{Z}_\epsilon)^{-1}\mathcal{Q}_\epsilon\big)_2 . \tag{2.33}$$

By Lemma 3.6.1 we have $-\pi_\omega A^\dagger \Gamma_\epsilon = -\frac{2}{5}\epsilon$, so by Corollary 3.6.6 it follows that:

$$\frac{d}{d\epsilon}\hat{\omega}_\epsilon \leq -\tfrac{2}{5}\epsilon + \epsilon^2(\tilde{M}_\epsilon + \tilde{M}'_\epsilon). \tag{2.34}$$

From our monotonicity bounds, to show $\frac{d}{d\epsilon}\hat{\omega}_\epsilon < 0$ for all $0 < \epsilon \leq \epsilon_0$, it suffices to check that the RHS of (2.34) is bounded above by zero at $\epsilon_0 = 0.1$, which we verify using interval arithmetic. Slowly oscillating periodic solutions along the principal branch satisfy $\epsilon \leq 0.1$ for $\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} + 6.830 \times 10^{-3}$. Hence, for $\frac{\pi}{2} < \alpha \leq \frac{\pi}{2} + 6.830 \times 10^{-3}$

the frequency (period length) of SOPS along the principal branch decreases (increases) monotonically in α.

□

To prove Jones' conjecture, it is insufficient to prove only locally that Wright's equation has a unique SOPS. We must be able to connect our local results with global estimates. When we make the change of variables $\tilde{c} = \epsilon c$ in defining the function F_ϵ, we restrict ourselves to proving local results. Theorems 2.3.10 and 2.3.11 connect these local results with a global argument, and construct neighborhoods, independent of any ϵ-scaling, within which the only SOPS to Wright's equation are those originating from the Hopf bifurcation. These results are later used in Chapter 5.

The next theorem uses the large radius calculation from Proposition 2.2.10(b) to show that for $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 5.53 \times 10^{-3}]$ all periodic solutions in a neighborhood of 0 lie on the Hopf bifurcation curve, which has neither folds nor secondary bifurcations.

Theorem 2.3.10. *For each $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 5.53 \times 10^{-3}]$ there is a unique triple (ϵ, ω, c) in the range $0 < \epsilon \leq 0.09$ and $|\omega - \frac{\pi}{2}| < 0.0924$ and $\|c\| \leq 0.30232$ such that $F_\epsilon(\alpha, \omega, c) = 0$.*

Proof. Fix $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 5.53 \times 10^{-3}]$ and let $F_\epsilon(\alpha, \omega, c) = 0$ for some ϵ, ω, c satisfying the assumed bounds. From Lemma 2.1.7(b) it follows that $\|K^{-1}c\| \leq \epsilon^2(2 + \|c\|^2)/(\epsilon b_*) \leq 0.61$ for $\epsilon \leq \epsilon_0$, since $b_* \geq 0.31$. Hence the zeros under consideration all lie in the set

$$\widetilde{S} := \{(\alpha, \omega, c) \in X : |\alpha - \tfrac{\pi}{2}| \leq 0.00553, |\omega - \tfrac{\pi}{2}| \leq 0.0924, \|c\| \leq 0.30232, \|K^{-1}c\| \leq 0.61\}.$$

Proposition 2.2.10(b) shows that for each $0 \leq \epsilon \leq 0.09$ there is a unique zero $\hat{x}_\epsilon = (\hat{\alpha}_\epsilon, \hat{\omega}_\epsilon, \hat{c}_\epsilon) \in B_\epsilon(r, \rho)$ of F_ϵ, with $r = (r_\alpha, r_\omega, r_c) = (0.1753, 0.0941, 0.3829)$ and $\rho = 1.5940$. For each $0 \leq \epsilon \leq 0.09$ it follows from the triangle inequality that $\widetilde{S} \subset B_\epsilon(r, \rho)$. This shows that F_ϵ has at most one zero in $\widetilde{S}$ for each $0 \leq \epsilon \leq \epsilon_0$. By Remark 2.2.16 this solution lies on the branch $\hat{x}_\epsilon$ originating from the Hopf bifurcation, in particular $\hat{x}_0 = (\frac{\pi}{2}, \frac{\pi}{2}, 0) \in \widetilde{S}$. Proposition 2.2.15 gives us tight bounds

$$|\hat{\omega}_\epsilon - \tfrac{\pi}{2}| \leq |\bar{\omega}_\epsilon - \tfrac{\pi}{2}| + \check{r}_\omega \epsilon^2 \leq 0.0924 \qquad \text{and} \qquad \|\hat{c}_\epsilon\| \leq \|\bar{c}_\epsilon\| + \check{r}_c \epsilon^2 \leq 0.30232$$

for all $0 \leq \epsilon \leq \epsilon_0$. Moreover, from similar considerations it follows that $\hat{\alpha}_{\epsilon_0} \geq \bar{\alpha}_{\epsilon_0} - r_\alpha \epsilon_0^2 > 0.00553$. Hence $\hat{x}_{\epsilon_0} \notin \widetilde{S}$ and the solution curve leaves $\widetilde{S}$ through $|\alpha - \frac{\pi}{2}| = 0.00553$

for some $0 < \epsilon < \epsilon_0$. Since $0.00553 < 6.830 \times 10^{-3}$ the assertion now follows directly from Theorem 2.3.8. $\square$

We translate this result to a neighborhood about the Hopf bifurcation without any ϵ-scaling.

Theorem 2.3.11. *For each* $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 5.53 \times 10^{-3}]$ *there is at most one (up to time translation) periodic solution to Wright's equation with Fourier coefficients satisfying* $\|a\| \leq 0.18$ *and having frequency* $|\omega - \frac{\pi}{2}| \leq 0.0924$.

Proof. We show that any such periodic solution y to Wright's equation has Fourier coefficients satisfying the bounds in Theorem 2.3.10. For the parameter range of α and ω under consideration we conclude that $\alpha < 2\omega$ and $\|a\| < \frac{2\omega-\alpha}{\alpha}$. Hence we see from Theorem 2.1.10 that y corresponds to a zero of F_ϵ. The *a priori* bound on $\|a\|$ translates via (2.8) into the bounds

$$\epsilon \leq 0.09 \qquad \text{and} \qquad \|\tilde{c}\| \leq 0.18.$$

We derive bounds on $c = \tilde{c}/\epsilon$ as in the proof of Lemma 2.3.6. Namely, for $|\alpha - \frac{\pi}{2}| \leq 0.00553$, $|\omega - \frac{\pi}{2}| \leq 0.0924$ and $\epsilon \leq 0.09$, we find that z_*^+, as defined in (2.13), is bounded below by $z_*^+ \geq 0.595$. It follows that $\|\tilde{c}\| \leq 0.18 \leq z_*^+$, so we infer from Lemma 2.1.7(a) that $\|\tilde{c}\| \leq z_*^-$. Via Lemma 3.5.5 and an interval arithmetic computation, the latter can be bounded above, for $|\alpha - \frac{\pi}{2}| \leq 0.00553$, $|\omega - \frac{\pi}{2}| \leq 0.0924$ and $\epsilon \leq 0.09$, by $z_*^- \leq 0.30226\epsilon$. Hence $\|c\| \leq z_*^-/\epsilon \leq 0.30232$. We conclude that y corresponds to a zero of $F_\epsilon(\alpha, \omega, c)$ in the parameter set described by Theorem 2.3.10, which implies uniqueness. $\square$

Corollary 2.3.12. *For each* $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 5.53 \times 10^{-3}]$ *there is at most one (up to time translation) periodic solution to Wright's equation satisfying* $\|y'\|_{L^2([0,2\pi/\omega])} \leq 0.302$ *and having frequency* $|\omega - \frac{\pi}{2}| \leq 0.0924$.

Proof. For the Fourier coefficients a of y we infer from Lemma 2.3.1 that $\|a\| \leq \sqrt{\frac{\pi}{6\omega}} \cdot 0.302 \leq 0.18$. Hence any periodic solution y to Wright's equation of period $2\pi/\omega$ that satisfies $\|y'\|_{L^2} \leq 0.302$ has Fourier coefficients satisfying the bounds in Theorem 2.3.11.

□

Chapter 3

Technical Estimates

In this chapter we derive many of the technical estimates used in Chapter 2.

3.1 Operator Norms

We set $\omega_0 = \frac{\pi}{2}$ and recall that

$$\begin{aligned}
[U_\omega a]_k &= e^{-ik\omega} a_k \\
[U_{\omega_0} a]_k &= (-i)^k a_k \\
L_\omega &= \sigma^+(e^{-i\omega} I + U_\omega) + \sigma^-(e^{i\omega} I + U_\omega) \\
L_{\omega_0} &= \sigma^+(-iI + U_{\omega_0}) + \sigma^-(iI + U_{\omega_0}).
\end{aligned}$$

To more efficiently express the inverse of $A_{0,*}$ we define an operator $\hat{U} : \ell_0^1 \to \ell_0^1$ by

$$[\hat{U} c]_{k\geq 2} := (1 - ik^{-1} e^{-ik\pi/2})^{-1} c_k, \tag{3.1}$$

so that $A_{0,*}^{-1} = \frac{2}{i\pi} \hat{U} K$.

The operator norm of $Q \in B(\ell_0^1, \ell^1)$ can be expressed using the basis elements e_k (which have norm $\|\mathrm{e}_k\| = 2$):

$$\|Q\| = \frac{1}{2} \sup_{k \geq 2} \|Q \mathrm{e}_k\|. \tag{3.2}$$

Some of the operators in $B(\ell_0^1, \ell^1)$ considered in these appendices restrict naturally to $B(\ell_0^1)$, with the same expression for the norm. For operators in $B(\ell^1)$ a similar expression for the norm holds (the supremum being over $k \geq 1$). We will abuse the notation $\|Q\|$ by not indicating explicitly which of these operator norms is considered; this will always be clear from the context.

Proposition 3.1.1. *The operators* $\hat{U}, \hat{U}K, L_\omega, A_{0,*}^{-1}$ *and* $A_{1,*}$ *in* $B(\ell_0^1, \ell^1)$ *satisfy the bounds*

$$\begin{aligned} \|\hat{U}\| &= \tfrac{5}{4} & \qquad \|A_{0,*}^{-1}\| &= \tfrac{2}{\pi\sqrt{5}} \\ \|\hat{U}K\| &= \tfrac{1}{\sqrt{5}} & \qquad \|A_{1,*}\| &\le 2\pi \\ \|L_\omega\| &\le 4 \end{aligned}$$

Proof. The value $\|\hat{U}\mathrm{e}_k\|$ is maximized when $k = 5$, whence $\|\hat{U}\| = 5/4$. The value $\|\hat{U}K\mathrm{e}_k\|$ is maximized when $k = 2$, whence $\|\hat{U}K\| = \frac{1}{\sqrt{5}}$ and $\|A_{0,*}^{-1}\| = \frac{2}{\pi\sqrt{5}}$. It follows from the definition of L_ω and the fact that U_ω is unitary that $\|L_\omega\| \le 4$, whereby it follows that $\|A_{1,*}\| = \|\frac{\pi}{2}L_{\omega_0}\| \le 2\pi$. □

We recall, for any $a \in \ell^1$, the splitting $a = a_1\mathrm{e}_1 + \tilde{a}$ with $a_1 \in \mathbb{C}$ and $\tilde{a} \in \ell_0^1$, and as a tool in the estimates below we introduce the projections

$$\pi_1 a = a_1 \in \mathbb{C} \tag{3.3}$$

$$\pi_{\ge 2} a = \tilde{a}. \tag{3.4}$$

Proposition 3.1.2. *We have for the map* $A_1A_0^{-1} : \ell^1 \to \ell^1$ *that*

$$\|A_1A_0^{-1}\| = \frac{2\sqrt{10}}{5}. \tag{3.5}$$

Proof. Expanding $A_1A_0^{-1}$ we see that it splits into two parts: $A_{1,2}A_{0,1}^{-1}$ and $A_{1,*}A_{0,*}^{-1}$, which we estimate separately. To be precise

$$A_1A_0^{-1}a = (i_{\mathbb{C}}A_{1,2}A_{0,1}i_{\mathbb{C}}^{-1}\pi_1 a)\mathrm{e}_2 + A_{1,*}A_{0,*}^{-1}\pi_{\ge 2}a.$$

First, we calculate the matrix

$$A_{1,2}A_{0,1}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ -4 & 4 \end{bmatrix}.$$

Using the identification of $\mathbb{R}^2$ and $\mathbb{C}$, which is an isometry if one uses the 2-norm on $\mathbb{R}^2$, this matrix contributes to $A_1A_0^{-1}$ as an operator mapping the (complex) one-dimensional subspace spanned by e_1 to the (complex) one-dimensional subspace spanned by e_2. To determine its contribution to the estimate of the norm of $A_1 A_0^{-1}$, we thus need to determine the 2-norm of the matrix (as a linear map from $\mathbb{R}^2 \to \mathbb{R}^2$):

$$\|A_{1,2}A_{0,1}^{-1}\| = \frac{1}{5}\sqrt{\frac{45+5\sqrt{17}}{2}}.$$

Next, we calculate a bound on the map $A_{1,*}A_{0,*}^{-1} : \ell_0^1 \to \ell^1$:

$$\|A_{1,*}A_{0,*}^{-1}\| = \|L_{\omega_0}\hat{U}K\|. \tag{3.6}$$

To bound (3.6) we first compute how $L_{\omega_0}K\hat{U}$ operates on basis elements e_k for $k \geq 2$:

$$L_{\omega_0}K\hat{U}e_k = \frac{-i+(-i)^k}{k-i(-i)^k}e_{k+1} + \frac{i+(-i)^k}{k-i(-i)^k}e_{k-1}.$$

Since the norm of this expression is maximized when $k = 2$ and $\|L_{\omega_0}K\hat{U}e_2\| = \frac{4\sqrt{10}}{5}$, we have calculated the $B(\ell_0^1, \ell^1)$ operator norm $\|L_{\omega_0}K\hat{U}\| = \frac{2\sqrt{10}}{5}$. As $\|A_1A_0^{-1}\|$ is equal to the maximum of $\|A_{1,2}A_{0,1}^{-1}\|$ and $\|A_{1,*}A_{0,*}^{-1}\|$, it follows that $\|A_1A_0^{-1}\| = \max\{\frac{1}{5}\sqrt{\frac{45+5\sqrt{17}}{2}}, \frac{2\sqrt{10}}{5}\} = \frac{2\sqrt{10}}{5}$. ☐

Proposition 3.1.3. *Define* $\overline{A_0^{-1}A_1} \in \mathrm{Mat}((\mathbb{R}^3,\mathbb{R}^3)$ *by*

$$\overline{A_0^{-1}A_1} := \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{2+\frac{\pi^2}{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{8}{5\pi} & \frac{2\sqrt{16+8\pi+5\pi^2}}{5\pi} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

Then $\overline{A_0^{-1}A_1}$ *is an upper bound (as defined in Definition 2.2.5) for* $A_0^{-1}A_1$.

Proof. We write $x = (\alpha, \omega, c)$. Let $\pi_{\alpha,\omega}$ be the projection onto $\mathbb{R}^2$, whereas π_c is the projection onto ℓ_0^1. Then we can expand $A_0^{-1}A_1$ as follows:

$$\pi_{\alpha,\omega}A_0^{-1}A_1x = A_{0,1}^{-1}i_{\mathbb{C}}^{-1}\pi_1 A_{1,*}\pi_c x \tag{3.7}$$

$$\pi_c A_0^{-1}A_1 x = A_{0,*}^{-1}((i_{\mathbb{C}}A_{1,2}\pi_{\alpha,\omega}x)e_2) + A_{0,*}^{-1}\pi_{\geq 2}A_{1,*}\pi_c x. \tag{3.8}$$

We estimate the three operators that appear separately.

First, we note that the term $A_{0,*}^{-1}((i_{\mathbb{C}}A_{1,2}\pi_{\alpha,\omega}x)e_2)$ in (3.8) essentially represents an operator from $\mathbb{R}^2$ to the (complex) one-dimensional subspace spanned by e_2. Using the identification of $\mathbb{C}$ with $\mathbb{R}^2$, this map is represented by the matrix

$$\frac{-2}{25\pi}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2-\frac{3\pi}{2} \\ -4 & 2(2+\pi) \end{bmatrix} = \frac{2}{25\pi}\begin{bmatrix} -6 & 6+11\frac{\pi}{2} \\ 8 & \pi-8 \end{bmatrix}.$$

It then follows that

$$\begin{aligned}\|A_{0,*}^{-1}((i_{\mathbb{C}}A_{1,2}\pi_{\alpha,\omega}x)\mathrm{e}_2)\| &\leq \frac{4}{25\pi}\left(|\alpha|\sqrt{(-6)^2+8^2}+|\omega|\sqrt{(6+11\tfrac{\pi}{2})^2+(\pi-8)^2}\right)\\ &= \frac{4}{5\pi}\left(2|\alpha|+\frac{\sqrt{16+8\pi+5\pi^2}}{2}|\omega|\right).\end{aligned}$$

Next, we note that the term $A_{0,1}^{-1}i_{\mathbb{C}}^{-1}\pi_1 A_{1,*}\pi_c x$ in (3.7) essentially represents an operator from the (complex) one-dimensional subspace spanned by e_2 to $\mathbb{R}^2$. Using the identification of $\mathbb{C}$ with $\mathbb{R}^2$, this map is represented by the matrix

$$\frac{\pi}{2}\begin{bmatrix}0 & -\frac{\pi}{2}\\ -1 & 1\end{bmatrix}^{-1}\cdot\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}1-\frac{\pi}{2} & 1+\frac{\pi}{2}\\ 1 & 1\end{bmatrix},$$

because $\pi_1 A_{1,*}\mathrm{e}_2 = \frac{\pi}{2}(i-1)$. Hence

$$\begin{aligned}|\pi_\alpha A_0^{-1}A_1 x| &\leq \tfrac{1}{2}\sqrt{2+\tfrac{\pi^2}{2}}\|c\|\\ |\pi_\omega A_0^{-1}A_1 x| &\leq \tfrac{1}{2}\sqrt{2}\|c\|.\end{aligned}$$

Finally, note that the term $A_{0,*}^{-1}\pi_{\geq 2}A_{1,*}$ appearing in (3.8) maps ℓ_0^1 to itself. It can be expressed as

$$A_{0,*}^{-1}\pi_{\geq 2}A_{1,*} = -iK\hat{U}\pi_{\geq 2}L_{\omega_0}.$$

The operator $K\hat{U}\pi_{\geq 2}L_{\omega_0}$ acts on basis elements $\{\mathrm{e}_k\}_{k\geq 2}$ as follows:

$$\begin{aligned}K\hat{U}\pi_{\geq 2}L_{\omega_0}\mathrm{e}_2 &= -\frac{1+i}{4}\mathrm{e}_3\\ K\hat{U}\pi_{\geq 2}L_{\omega_0}\mathrm{e}_k &= \frac{-i+(-i)^k}{(k+1)-i(-i)^{k+1}}\mathrm{e}_{k+1}+\frac{i+(-i)^k}{(k-1)-i(-i)^{k-1}}\mathrm{e}_{k-1} \qquad \text{for } k\geq 3.\end{aligned}$$

Since $\max_{k\geq 2}\|K\hat{U}\pi_{\geq 2}L_{\omega_0}\mathrm{e}_k\| = \|K\hat{U}L_{\omega_0}\mathrm{e}_3\| = \frac{4}{\sqrt{5}}$, the operator norm of $A_{0,*}^{-1}\pi_{\geq 2}A_{1,*}$ is $\frac{2}{\sqrt{5}}$.

These three bounds on the three operators appearing in (3.7) and (3.8) lead to the asserted upper bound. □

3.2 Endomorphism on a Compact Domain

In order to construct the Newton-like map T we defined operators $A = DF(\bar{x}_\epsilon)+\mathcal{O}(\epsilon^2)$ and $A^\dagger = A^{-1}+\mathcal{O}(\epsilon^2)$. However, as $(\bar{\alpha}_\epsilon,\bar{\omega}_\epsilon,\bar{c}_\epsilon) = (\frac{\pi}{2},\frac{\pi}{2},\bar{c}_\epsilon)+\mathcal{O}(\epsilon^2)$, the map A can be better thought of as an $\mathcal{O}(\epsilon^2)$ approximation of $DF(\frac{\pi}{2}, \frac{\pi}{2}, \bar{c}_\epsilon)$. Thus, when working with the map T and considering points $x \in B_\epsilon(r, \rho)$ in its domain, we will often have to measure the distances of α and ω from $\frac{\pi}{2}$. To that end, we define the following variables which will be used throughout the rest of the appendices.

Definition 3.2.1. *For $\epsilon \geq 0$, and $r_\alpha, r_\omega, r_c > 0$ we define*

$$
\begin{aligned}
\Delta_\alpha^0 &:= \tfrac{\epsilon^2}{5}(3\tfrac{\pi}{2} - 1) & \Delta_\alpha &:= \Delta_\alpha^0 + r_\alpha \\
\Delta_\omega^0 &:= \tfrac{\epsilon^2}{5} & \Delta_\omega &:= \Delta_\omega^0 + r_\omega \\
\delta_c^0 &:= \tfrac{2\epsilon}{\sqrt{5}} & \delta_c &:= \delta_c^0 + r_c.
\end{aligned}
$$

When considering an element (α, ω, c) for our $\mathcal{O}(\epsilon^2)$ analysis, we are often concerned with the distances $|\alpha - \frac{\pi}{2}|$, $|\omega - \frac{\pi}{2}|$ and $\|c - \bar{c}_\epsilon\|$, each of which is of order ϵ^2. To create some notational consistency in these definitions, Δ_α^0 and Δ_ω^0 are of order ϵ^2, whereas δ_c^0 is not capitalized as it is of order ϵ. Using these definitions, it follows that for any $\rho > 0$ and all $(\alpha, \omega, c) \in B_\epsilon(r, \rho)$ we have:

$$
|\alpha - \tfrac{\pi}{2}| \leq \Delta_\alpha, \qquad |\omega - \tfrac{\pi}{2}| \leq \Delta_\omega, \qquad \|c\| \leq \delta_c.
$$

In this interpretation the superscript 0 simply refers to $r = 0$, i.e., the center of the ball $(\alpha, \omega, c) = \bar{x}_\epsilon$.

The following elementary lemma will be used frequently in the estimates.

Lemma 3.2.2. *For all $x \in \mathbb{R}$ we have $|e^{ix} - 1| \leq |x|$. Furthermore, for all $|\omega - \bar{\omega}_\epsilon| \leq r_\omega$ we have $|e^{-i\omega} + i| \leq \Delta_\omega$ and $|e^{-2i\omega} + 1| \leq 2\Delta_\omega$.*

Proof. We start with

$$
|e^{ix} - 1|^2 = (\cos x - 1)^2 + (\sin x)^2 = 2(1 - \cos x) \leq 2 \cdot \tfrac{1}{2}x^2 = x^2.
$$

Let $\theta = \omega - \frac{\pi}{2}$. Then $|\theta| \leq \Delta_\omega$ and, using the previous inequality,

$$
|e^{-i\omega} + i|^2 = |e^{-i(\frac{\pi}{2}+\theta)} + i|^2 = |e^{-i\theta} - 1|^2 \leq \theta^2 \leq \Delta_\omega^2.
$$

The final asserted inequality follows from an analogous argument. $\square$

While the operators U_ω and L_ω are not continuous in ω on all of ℓ^1_0, they are within the compact set $B_\epsilon(r,\rho)$. To denote the derivative of these operators, we define

$$\begin{aligned} U'_\omega &:= -iK^{-1}U_\omega \\ L'_\omega &:= -i\sigma^+(e^{-i\omega}I + K^{-1}U_\omega) + i\sigma^-(e^{i\omega}I - K^{-1}U_\omega), \end{aligned} \tag{3.9}$$

and we derive Lipschitz bounds on U_ω and L_ω in the following proposition.

Proposition 3.2.3. *For the definitions above, $\frac{\partial}{\partial\omega}U_\omega = U'_\omega$ and $\frac{\partial}{\partial\omega}L_\omega = L'_\omega$. Furthermore, for any $(\alpha,\omega,c) \in B_\epsilon(r,\rho)$, we have the norm estimates*

$$\begin{aligned} \|(U_\omega - U_{\omega_0})c\| &\le \Delta_\omega \rho \\ \|(L_\omega - L_{\omega_0})c\| &\le 2\Delta_\omega(\delta_c + \rho). \end{aligned} \tag{3.10}$$

Proof. One easily calculates that $\frac{\partial U_\omega}{\partial\omega} = U'_\omega$, whereby $\|(U_\omega - U_{\omega_0})c\| \le \int_{\omega_0}^{\omega} \|\frac{\partial}{\partial\omega}U_\omega c\| \le \Delta_\omega\rho$. Calculating $\frac{\partial}{\partial\omega}L_\omega$, we obtain the following:

$$\begin{aligned} \frac{\partial}{\partial\omega}L_\omega &= \frac{\partial}{\partial\omega}\left[\sigma^+(e^{-i\omega}I + U_\omega) + \sigma^-(e^{i\omega}I + U_\omega)\right] \\ &= -i\sigma^+(e^{-i\omega}I + K^{-1}U_\omega) + i\sigma^-(e^{i\omega}I - K^{-1}U_\omega), \end{aligned}$$

thus proving $\frac{\partial L_\omega}{\partial\omega} = L'_\omega$, and $\|(L_\omega - L_{\omega_0})c\| \le \int_{\omega_0}^{\omega} \|\frac{\partial}{\partial\omega}L_\omega c\| \le \Delta_\omega(2\delta_c + 2\rho)$. ∎

Proposition 3.2.4. *Let $\epsilon \ge 0$ and $r = (r_\alpha, r_\omega, r_c) \in \mathbb{R}^3_+$. For any $\rho > 0$ the map $T : B_\epsilon(r,\rho) \to \mathbb{R}^2 \times \ell^K_0$ is well defined. We define functions*

$$\begin{aligned} C_0 &:= \frac{2\epsilon^2}{\pi}\left[\frac{8}{5}, \frac{2}{5}\sqrt{16 + 8\pi + 5\pi^2}, \frac{5\pi}{2}\right] \cdot \overline{A_0^{-1}A_1} \cdot [0,0,\delta_c]^T, \\ C_1 &:= \frac{5}{2\pi} + \frac{\epsilon\sqrt{10}}{\pi}, \\ C_2 &:= \Delta_\omega\left[(1 + \tfrac{\pi}{2}) + \epsilon\pi\right], \\ C_3 &:= \Delta_\alpha(2 + \delta_c) + 2\Delta_\omega(1 + \tfrac{\pi}{2}) + \epsilon\left[\pi + 2\Delta_\alpha + 4\delta_c\Delta_\alpha + \pi\Delta_\omega\delta_c + (\tfrac{\pi}{2} + \Delta_\alpha)\delta_c^2\right], \end{aligned}$$

where the expression for C_0 should be read as a product of a row vector, a (3×3) matrix and a column vector. Furthermore we define, for any ϵ, r_ω such that $C_1C_2 < 1$,

$$C(\epsilon, r_\alpha, r_\omega, r_c) := \frac{C_0 + C_1C_3}{1 - C_1C_2}. \tag{3.11}$$

All of the functions C_0, C_1, C_2, C_3 and C are nonnegative and monotonically increasing in their arguments ϵ and r. Furthermore, if $C_1C_2 < 1$ and $C(\epsilon, r_\alpha, r_\omega, r_c) \leq \rho$ then $\|K^{-1}\pi_c T(x)\| \leq \rho$ for $x \in B_\epsilon(r, \rho)$.

Proof. Given their definitions, it is straightforward to check that the functions C_i and C are monotonically increasing in their arguments. To prove the second half of the proposition, we split $K^{-1}\pi_c T(x)$ into several pieces. We define the projection $\pi_c^0 x = (0, 0, \pi_c x)$. We then obtain

$$\begin{aligned}
K^{-1}\pi_c T(x) &= K^{-1}\pi_c[x - A^\dagger F(x)] \\
&= K^{-1}\pi_c[I\pi_c^0 x - A^\dagger(A\pi_c^0 x + F(x) - A\pi_c^0 x)] \\
&= \epsilon^2 K^{-1}\pi_c(A_0^{-1}A_1)^2\pi_c^0 x + K^{-1}\pi_c A^\dagger(F(x) - A\pi_c^0 x) \\
&= \frac{2\epsilon^2}{i\pi}\hat{U}\pi_{\geq 2}A_1A_0^{-1}A_1\pi_c^0 x + \frac{2}{i\pi}\hat{U}\pi_{\geq 2}(I - \epsilon A_1A_0^{-1})(F(x) - A\pi_c^0 x),
\end{aligned}$$

where we have used that $K^{-1}\pi_c A_0^{-1} = \frac{2}{i\pi}\hat{U}\pi_{\geq 2}$, with the projection $\pi_{\geq 2}$ defined in (3.4). By using $\|\hat{U}\| \leq \frac{5}{4}$ (see Proposition 3.1.1) we obtain the estimate

$$\|K^{-1}\pi_c T(x)\| \leq \frac{2\epsilon^2}{\pi}\overline{\hat{U}\pi_{\geq 2}A_1} \cdot \overline{A_0^{-1}A_1} \cdot [0, 0, \delta_c]^T + \frac{5}{2\pi}\left(1 + \epsilon\|A_1A_0^{-1}\|\right)\|F(x) - A\pi_c^0 x\|. \tag{3.12}$$

Here the (1×3) row vector $\overline{\hat{U}\pi_{\geq 2}A_1}$ is an upper bound on $\hat{U}\pi_{\geq 2}A_1$ interpreted as a linear operator from $\mathbb{R}^2 \times \ell_0^1$ to ℓ_0^1, thus extending in a straightforward manner the definition of upper bounds given in Definition 2.2.5.

We have already calculated an expression for $\overline{A_0^{-1}A_1}$ in Proposition 3.1.3, and $\|A_1A_0^{-1}\| = \frac{2\sqrt{10}}{5}$ by Proposition 3.1.2. In order to finish the calculation of the right hand side of equation (3.12), we need to estimate $\|F(x) - A\pi_c^0 x\|$ and $\overline{\hat{U}\pi_{\geq 2}A_1}$. We first calculate a bound on $\hat{U}\pi_{\geq 2}A_1$. We note that $\hat{U}\pi_{\geq 2}A_1 = \hat{U}\mathrm{e}_2(i_{\mathbb{C}}A_{1,2}\pi_{\alpha,\omega}) + \hat{U}\pi_{\geq 2}A_{1,*}\pi_c$. As $\|\hat{U}e_2\| = \|\frac{4-2i}{5}\mathrm{e}_2\|$, it follows from the definition of $A_{1,2}$ that

$$\left| i_{\mathbb{C}}A_{1,2}\begin{pmatrix}\alpha \\ \omega\end{pmatrix}\right| \cdot \|\hat{U}\mathrm{e}_2\| \leq \left(\frac{\sqrt{20}}{5}|\alpha| + \frac{\sqrt{(2 - 3\pi/2)^2 + 4(2+\pi)^2}}{5}|\omega|\right) \cdot \frac{4}{\sqrt{5}}.$$

To calculate $\|\hat{U}\pi_{\geq 2}A_{1,*}\|$ we note that $\|\hat{U}\| \leq \frac{5}{4}$ and $\|A_{1,*}\| = \frac{\pi}{2}\|L_{\omega_0}\| \leq 2\pi$. Hence $\|\hat{U}\pi_{\geq 2}A_{1,*}\| \leq \frac{5\pi}{2}$. Combining these results, we obtain that

$$\overline{\hat{U}\pi_{\geq 2}A_1} = \left[\frac{8}{5}, \frac{2}{5}\sqrt{16 + 8\pi + 5\pi^2}, \frac{5\pi}{2}\right].$$

Thereby, it follows from (3.12) that

$$\|K^{-1}\pi_c T(x)\| \leq C_0 + C_1\|F(x) - A\pi_c^0 x\|. \tag{3.13}$$

We now calculate

$$\begin{aligned}
F(x) - A\pi_c^0 x = {}& (i\omega + \alpha e^{-i\omega})\mathrm{e}_1 + (i\omega K^{-1} + \alpha U_\omega)c + \epsilon\alpha e^{-i\omega}\mathrm{e}_2 + \alpha\epsilon L_\omega c + \alpha\epsilon[U_\omega c] * c \\
& - \tfrac{\pi}{2}(iK^{-1} + U_{\omega_0} + \epsilon L_{\omega_0})c \\
= {}& i(\omega - \tfrac{\pi}{2})K^{-1}c + (\alpha - \tfrac{\pi}{2})U_\omega c + \tfrac{\pi}{2}(U_\omega - U_{\omega_0})c \\
& + \left[i(\omega - \tfrac{\pi}{2}) + (\alpha - \tfrac{\pi}{2})e^{-i\omega} + \tfrac{\pi}{2}(e^{-i\omega} + i)\right]\mathrm{e}_1 \\
& + \epsilon\alpha e^{-i\omega}\mathrm{e}_2 + (\alpha - \tfrac{\pi}{2})\epsilon L_\omega c + \tfrac{\pi}{2}\epsilon(L_\omega - L_{\omega_0})c + \alpha\epsilon[U_\omega c] * c.
\end{aligned}$$

Taking norms and using (3.10) and Lemma 3.2.2, we obtain

$$\begin{aligned}
\|F(x) - A\pi_c^0 x\| \leq {}& \Delta_\omega\rho + \Delta_\alpha\delta_c + \tfrac{\pi}{2}\Delta_\omega\rho + 2(\Delta_\omega + \Delta_\alpha + \tfrac{\pi}{2}\Delta_\omega) \\
& + \epsilon\left[2(\tfrac{\pi}{2} + \Delta_\alpha) + 4\delta_c\Delta_\alpha + \pi\Delta_\omega(\delta_c + \rho) + (\tfrac{\pi}{2} + \Delta_\alpha)\delta_c^2\right] \\
= {}& \Delta_\omega[(1 + \tfrac{\pi}{2}) + \epsilon\pi]\rho \\
& + \Delta_\alpha(2 + \delta_c) + 2\Delta_\omega(1 + \tfrac{\pi}{2}) \\
& + \epsilon\left[\pi + 2\Delta_\alpha + 4\delta_c\Delta_\alpha + \pi\Delta_\omega\delta_c + (\tfrac{\pi}{2} + \Delta_\alpha)\delta_c^2\right].
\end{aligned}$$

We have now computed all of the necessary constants. Thus $\|F(x) - A\pi_c^0 x\| \leq C_2\rho + C_3$, and from (3.13) we obtain

$$\|K^{-1}\pi_c T(c)\| \quad \leq \quad C_0 + C_1(C_2\rho + C_3),$$

with the constants defined in the statement of the proposition. We would like to select values of ρ for which

$$\|K^{-1}\pi_c T(c)\| \leq \rho$$

This is true if $C_0 + C_1(C_2\rho + C_3) \leq \rho$, or equivalently

$$\frac{C_0 + C_1 C_3}{1 - C_1 C_2} \leq \rho.$$

This proves the theorem. □

3.3 The upper bound for $Y(\epsilon)$

We need to define $Y(\epsilon)$ so that it bounds $T(\bar{x}_\epsilon) - \bar{x}_\epsilon = A^\dagger F(\bar{x}_\epsilon)$. We introduce $c_2(\epsilon) := \frac{2-i}{5}\epsilon$. We can explicitly calculate $F(\bar{x}_\epsilon)$ as follows:

$$
\begin{aligned}
F_1(\bar{x}_\epsilon) &= (i\bar{\omega}_\epsilon + \bar{\alpha}_\epsilon e^{-i\bar{\omega}_\epsilon}) + \bar{\alpha}_\epsilon \epsilon (e^{i\bar{\omega}_\epsilon} + e^{-2i\bar{\omega}_\epsilon}) c_2(\epsilon) \\
F_2(\bar{x}_\epsilon) &= (2i\bar{\omega}_\epsilon + \bar{\alpha}_\epsilon e^{-2i\bar{\omega}_\epsilon}) c_2(\epsilon) + \bar{\alpha}_\epsilon \epsilon e^{-i\bar{\omega}_\epsilon} \\
F_3(\bar{x}_\epsilon) &= \bar{\alpha}_\epsilon \epsilon (e^{-i\bar{\omega}_\epsilon} + e^{-2i\bar{\omega}_\epsilon}) c_2(\epsilon) \\
F_4(\bar{x}_\epsilon) &= \bar{\alpha}_\epsilon \epsilon e^{-2i\bar{\omega}_\epsilon} c_2(\epsilon)^2 \\
F_k(\bar{x}_\epsilon) &= 0 \qquad \text{for all } k \geq 5.
\end{aligned}
$$

By using the definition of $A^\dagger = A_0^{-1} - \epsilon A_0^{-1} A_1 A_0^{-1}$ we can calculate $A^\dagger F(\bar{x}_\epsilon)$ explicitly using a finite number of operations. However, proving $\epsilon^{-2} Y(\epsilon)$ is well defined and increasing requires more work. To estimate $A^\dagger F(\bar{x}_\epsilon)$ in Theorem 3.3.2 below, we will take entry-wise absolute values in the constituents of $A^\dagger$, as clarified in the next remark.

Remark 3.3.1. *Since $F(\bar{x}_\epsilon)$ is a finite linear combination of the basis elements e_k, and the operators A_0 and A_1 are diagonal and tridiagonal, respectively, we can represent $A_0^{-1} \cdot F(\bar{x}_\epsilon)$ and $A_0^{-1} A_1 A_0^{-1} \cdot F(\bar{x}_\epsilon)$ by finite dimensional matrix-vector products. By $|A_0^{-1}|$ and $|A_0^{-1} A_1 A_0^{-1}|$ we denote the entry-wise absolute values of these matrices.*

Theorem 3.3.2. *Let $f_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, 3, 4$ be defined as in Propositions 3.3.3, 3.3.4, 3.3.5, and 3.3.6 below. Define $f(\epsilon) = \sum_{i=1}^4 f_i \mathrm{e}_i \in \ell^1$ and define the function $\hat{Y} : \mathbb{R} \to \mathbb{R}^2 \times \ell^1_0$ to be*

$$
\hat{Y}(\epsilon) := \left|A_0^{-1}\right| \cdot f(\epsilon) + \epsilon \left|A_0^{-1} A_1 A_0^{-1}\right| \cdot f(\epsilon). \tag{3.14}
$$

Then the only nonzero components of $\hat{Y} = (\hat{Y}_\alpha, \hat{Y}_\omega, \hat{Y}_c)$ are $\hat{Y}_\alpha$, $\hat{Y}_\omega$ and $(\hat{Y}_c)_k$ for $k = 2, 3, 4, 5$. Furthermore, define

$$
Y_\alpha(\epsilon) := \hat{Y}_\alpha(\epsilon) \qquad Y_\omega(\epsilon) := \hat{Y}_\omega(\epsilon) \qquad Y_c(\epsilon) := 2 \sum_{k=2}^{5} (\hat{Y}_c)_k(\epsilon) \tag{3.15}
$$

Then $[Y_\alpha(\epsilon), Y_\omega(\epsilon), Y_c(\epsilon)]^T$ is an upper bound on $T(\bar{x}_\epsilon) - \bar{x}_\epsilon$, and $\epsilon^{-2}[Y_\alpha(\epsilon), Y_\omega(\epsilon), Y_c(\epsilon)]$ is non-decreasing in ϵ.

Proof. By Propositions 3.3.3, 3.3.4, 3.3.5 and 3.3.6 it follows that $|F_i(\bar{x}_\epsilon)| \leq f_i(\epsilon)$ for $i = 1, 2, 3, 4$. By taking the entry-wise absolute values $\left|A_0^{-1}\right|$ and $\left|A_0^{-1}A_1A_0^{-1}\right|$, it follows that $|T(\bar{x}_\epsilon)-\bar{x}_\epsilon| \leq \hat{Y}$, where the absolute values and inequalities are taken element-wise. We note that in defining Y_c the factor 2 arises from our choice of norm in (2.4). To see that $(\hat{Y}_c)_k$ is non-zero for $k = 2, 3, 4, 5$ only, we note that while A_0^{-1} is a block diagonal operator, A_1 has off-diagonal terms. In particular, $A_{1,*}\mathrm{e}_k = \frac{\pi}{2}(-i+(-i)^k)\mathrm{e}_{k+1} + \frac{\pi}{2}(i+(-i)^k)\mathrm{e}_{k-1}$ for $k \geq 2$, whereby $(\hat{Y})_k = 0$ for $k \geq 6$.

Next we show that $\epsilon^{-2}[Y_\alpha(\epsilon), Y_\omega(\epsilon), Y_c(\epsilon)]^T$ is nondecreasing in ϵ. We note that it follows from Definition 3.2.1 that each function $f_i(\epsilon)$ is a polynomial in ϵ with nonnegative coefficients, and the lowest degree term is at least ϵ^2. Additionally, $\left|A_0^{-1}\right| \cdot f(\epsilon)$ is a positive linear combination of the functions $\{f_i(\epsilon)\}_{i=1}^4$, whereas $\left|A_0^{-1}A_1A_0^{-1}\right| \cdot f(\epsilon)$ is ϵ times a positive linear combination of $\{f_i(\epsilon)\}_{i=1}^4$. It follows that each component of $\hat{Y}$ is a polynomial in ϵ with nonnegative coefficients, and the lowest degree term is at least ϵ^2. Thereby $\epsilon^{-2}[Y_\alpha(\epsilon), Y_\omega(\epsilon), Y_c(\epsilon)]^T$ is nondecreasing in ϵ.

□

Before presenting Propositions 3.3.3, 3.3.4, 3.3.5 and 3.3.6, we recall that the definitions of Δ_α^0, Δ_ω^0 and δ_c^0 are given in Definition 3.2.1.

Proposition 3.3.3. *Define*

$$f_1(\epsilon) := \tfrac{\pi}{2}(\tfrac{1}{2}(\Delta_\omega^0)^2 + \tfrac{1}{6}(\Delta_\omega^0)^3) + \Delta_\alpha^0\Delta_\omega^0 + \Delta_\alpha^0\epsilon\delta_c^0 + \tfrac{3\pi}{4}\Delta_\omega^0\epsilon\delta_c^0. \tag{3.16}$$

Then $|F_1(\bar{x}_\epsilon)| \leq f_1(\epsilon)$.

Proof. Note that

$$F_1(\bar{x}_\epsilon) = i\bar{\omega}_\epsilon + \bar{\alpha}_\epsilon e^{-i\bar{\omega}_\epsilon} + \bar{\alpha}_\epsilon\epsilon c_2(\epsilon)(e^{i\bar{\omega}_\epsilon} + e^{-2i\bar{\omega}_\epsilon}). \tag{3.17}$$

We will show that all the $\mathcal{O}(\epsilon^3)$ terms in $F_1(\bar{x}_\epsilon)$ cancel. We first expand the first summand (3.17):

$$i\bar{\omega}_\epsilon = i\tfrac{\pi}{2} - i\Delta_\omega^0.$$

Next, we expand the second summand in (3.17):

$$\bar{\alpha}_\epsilon e^{-i\bar{\omega}_\epsilon} = -i\bar{\alpha}_\epsilon e^{i\Delta_\omega^0} = -i\left(\tfrac{\pi}{2}e^{i\Delta_\omega^0} + \Delta_\alpha^0 e^{i\Delta_\omega^0}\right)$$
$$= -i\left(\tfrac{\pi}{2}(1+i\Delta_\omega^0) + \Delta_\alpha^0\right) - i\left(\tfrac{\pi}{2}(e^{i\Delta_\omega^0} - 1 - i\Delta_\omega^0) + \Delta_\alpha^0(e^{i\Delta_\omega^0} - 1)\right). \quad (3.18)$$

Finally, we expand the third summand (3.17) as

$$\bar{\alpha}_\epsilon \epsilon^2 \tfrac{2-i}{5}(e^{i\bar{\omega}_\epsilon} + e^{-2i\bar{\omega}_\epsilon}) = \tfrac{\pi}{2}\epsilon^2 \tfrac{2-i}{5}(i-1) + \tfrac{\pi}{2}\epsilon^2\tfrac{2-i}{5}\left(i(e^{-i\Delta_\omega^0} - 1) - (e^{2i\Delta_\omega^0} - 1)\right)$$
$$+ \Delta_\alpha^0 \epsilon^2 \tfrac{2-i}{5}\left(ie^{-i\Delta_\omega^0} - e^{2i\Delta_\omega^0}\right). \quad (3.19)$$

If we now collect the final term from (3.18) and the final two terms from (3.19) in

$$g(\epsilon) := -i\left(\tfrac{\pi}{2}(e^{i\Delta_\omega^0} - 1 - i\Delta_\omega^0) + \Delta_\alpha^0(e^{i\Delta_\omega^0} - 1)\right)$$
$$+ \Delta_\alpha^0\epsilon^2\tfrac{2-i}{5}\left(ie^{-i\Delta_\omega^0} - e^{2i\Delta_\omega^0}\right)$$
$$+ \tfrac{\pi}{2}\epsilon^2\tfrac{2-i}{5}\left(i(e^{-i\Delta_\omega^0} - 1) - (e^{2i\Delta_\omega^0} - 1)\right),$$

then we can write $F_1(\bar{x}_\epsilon)$ as

$$F_1(\bar{x}_\epsilon) = g(\epsilon) + i\tfrac{\pi}{2} - i\Delta_\omega^0 - i\left(\tfrac{\pi}{2}(1+i\Delta_\omega^0) + \Delta_\alpha^0\right) + \tfrac{\pi}{2}\epsilon^2\tfrac{2-i}{5}(i-1)$$
$$= g(\epsilon).$$

Using Lemma 3.2.2 it is not difficult to see that $|g(\epsilon)|$ can be bounded by $f_1(\epsilon)$, as defined in (3.16). □

Proposition 3.3.4. *Define*

$$f_2(\epsilon) := (\tfrac{\pi}{2} + \Delta_\alpha^0)\Delta_\omega^0(\delta_c^0 + \epsilon) + \tfrac{1}{2}\delta_c^0(2\Delta_\omega^0 + \Delta_\alpha^0) + \epsilon\Delta_\alpha^0. \quad (3.20)$$

Then $|F_2(\bar{x}_\epsilon)| \leq f_2(\epsilon)$.

Proof. First note that

$$F_2(\bar{x}_\epsilon) = (2i\bar{\omega}_\epsilon + \bar{\alpha}_\epsilon e^{-2i\bar{\omega}_\epsilon})c_2(\epsilon) + \bar{\alpha}_\epsilon\epsilon e^{-i\bar{\omega}_\epsilon} = \left(2i\bar{\omega}_\epsilon - \bar{\alpha}_\epsilon e^{2i\Delta_\omega^0}\right)\tfrac{2-i}{5}\epsilon - i\bar{\alpha}_\epsilon\epsilon e^{i\Delta_\omega^0}$$
$$= (2i\bar{\omega}_\epsilon - \bar{\alpha}_\epsilon)\tfrac{2-i}{5}\epsilon - i\bar{\alpha}_\epsilon\epsilon - \bar{\alpha}_\epsilon(e^{2i\Delta_\omega^0} - 1)\tfrac{2-i}{5}\epsilon - i\bar{\alpha}_\epsilon\epsilon(e^{i\Delta_\omega^0} - 1). \quad (3.21)$$

We expand the first part of the right hand side in (3.21) as

$$\begin{aligned}(2i\bar{\omega}_\epsilon - \bar{\alpha}_\epsilon)\tfrac{2-i}{5}\epsilon - i\bar{\alpha}_\epsilon\epsilon &= \left(2i\tfrac{\pi}{2} - \tfrac{\pi}{2}\right)\tfrac{2-i}{5}\epsilon - i\tfrac{\pi}{2}\epsilon + \left(-2i\Delta_\omega^0 - \Delta_\alpha^0\right)\tfrac{2-i}{5}\epsilon - i\Delta_\alpha^0\epsilon \\ &= -\left(2i\Delta_\omega^0 + \Delta_\alpha^0\right)\tfrac{2-i}{5}\epsilon - i\Delta_\alpha^0\epsilon.\end{aligned}$$

Hence, we can rewrite $F_2(\epsilon)$ as

$$F_2(\bar{x}_\epsilon) = -\bar{\alpha}_\epsilon(e^{2i\Delta_\omega^0} - 1)\tfrac{2-i}{5}\epsilon - i\bar{\alpha}_\epsilon\epsilon(e^{i\Delta_\omega^0} - 1) - \left(2i\Delta_\omega^0 + \Delta_\alpha^0\right)\tfrac{2-i}{5}\epsilon - i\Delta_\alpha^0\epsilon.$$

Using Lemma 3.2.2 it is then not difficult to see that $|F_2(\bar{x}_\epsilon)|$ can be bounded by $f_2(\epsilon)$, as defined in (3.20). □

Proposition 3.3.5. *Define*

$$f_3(\epsilon) := \tfrac{1}{2}(\tfrac{\pi}{2} + \Delta_\alpha^0)(\sqrt{2} + 3\Delta_\omega^0)\epsilon\delta_c^0. \tag{3.22}$$

Then $|F_3(\bar{x}_\epsilon)| \leq f_3(\epsilon)$.

Proof. Note that

$$F_3(\bar{x}_\epsilon) = \bar{\alpha}_\epsilon\epsilon(e^{-i\bar{\omega}_\epsilon} + e^{-2i\bar{\omega}_\epsilon})c_2(\epsilon).$$

We expand this as

$$\begin{aligned}F_3(\bar{x}_\epsilon) &= -\bar{\alpha}_\epsilon\epsilon^2\tfrac{2-i}{5}(ie^{i\Delta_\omega^0} + e^{2i\Delta_\omega^0}) \\ &= -\bar{\alpha}_\epsilon\epsilon^2\tfrac{2-i}{5}(i+1) - \bar{\alpha}_\epsilon\epsilon^2\tfrac{2-i}{5}\left(i(e^{i\Delta_\omega^0} - 1) + (e^{2i\Delta_\omega^0} - 1)\right).\end{aligned}$$

Using Lemma 3.2.2 it is then not difficult to see that $|F_3(\bar{x}_\epsilon)|$ can be bounded by $f_3(\epsilon)$, as defined in (3.22). □

Proposition 3.3.6. *Define*

$$f_4(\epsilon) := \tfrac{1}{5}(\tfrac{\pi}{2} + \Delta_\alpha^0)\epsilon^3 \tag{3.23}$$

Then $|F_4(\bar{x}_\epsilon)| \leq f_4(\epsilon)$.

Proof. Note that

$$F_4(\bar{x}_\epsilon) = \bar{\alpha}_\epsilon\epsilon e^{-2i\bar{\omega}_\epsilon}\left(\tfrac{2-i}{5}\epsilon\right)^2,$$

from which it follows that $|F_4(\bar{x}_\epsilon)|$ can be bounded by $f_4(\epsilon)$, as defined in (3.23). □

3.4 The upper bound for $Z(\epsilon, r, \rho)$

In this section we calculate an upper bound on DT. To do so we first calculate $DF = [\frac{\partial F}{\partial \alpha}, \frac{\partial F}{\partial \omega}, \frac{\partial F}{\partial c}]$:

$$\frac{\partial F}{\partial \alpha} = e^{-i\omega}\mathrm{e}_1 + U_\omega c + \epsilon e^{-i\omega}\mathrm{e}_2 + \epsilon L_\omega c + \epsilon[U_\omega c] * c, \tag{3.24}$$

$$\frac{\partial F}{\partial \omega} = i(1-\alpha e^{-i\omega})\mathrm{e}_1 + iK^{-1}(I-\alpha U_\omega)c - i\alpha\epsilon e^{-i\omega}\mathrm{e}_2 + \alpha\epsilon L'_\omega c - i\alpha\epsilon[K^{-1}U_\omega c] * c, \tag{3.25}$$

$$\frac{\partial F}{\partial c}\cdot b = (i\omega K^{-1} + \alpha U_\omega)b + \alpha\epsilon\left(L_\omega b + [U_\omega b] * c + [U_\omega c] * b\right), \qquad \text{for all } b \in \ell_0^K, \tag{3.26}$$

where L'_ω is given in (3.9), and $\frac{\partial F}{\partial c}$ is expressed in terms of the directional derivative. Recall that I_3 is used to denote the 3×3 identity matrix.

Theorem 3.4.1. *Define $\overline{A_0^{-1}A_1}$ as in Proposition 3.1.3 and define the matrix*

$$M := \begin{pmatrix} \sqrt{\frac{4}{\pi^2}+1} & 0 \\ \frac{2}{\pi} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_{1,\alpha} & f_{1,\omega} & f_{1,c} \\ f_{*,\alpha} & f_{*,\omega} & f_{*,c} \end{pmatrix}, \tag{3.27}$$

where the functions $f_{1,\cdot}(\epsilon, r, \rho)$ and $f_{*,\cdot}(\epsilon, r, \rho)$ are defined as in Propositions 3.4.2–3.4.7. If we define $Z(\epsilon, r, \rho)$ as

$$Z(\epsilon, r, \rho) := \epsilon^2\left(\overline{A_0^{-1}A_1}\right)^2 + \left(I_3 + \epsilon\overline{A_0^{-1}A_1}\right)\cdot M, \tag{3.28}$$

then $Z(\epsilon, r)$ is an upper bound (in the sense of Definition 2.2.5) on $DT(x)$ for all $x \in B_\epsilon(r, \rho)$. Furthermore, the components of $Z(\epsilon, r, \rho)$ are increasing in ϵ, r and ρ.

Proof. If we fix some $x \in B_\epsilon(r, \rho)$, then we obtain

$$\begin{aligned} DT(x) &= I - A^\dagger DF(x) \\ &= (I - A^\dagger A) - A^\dagger\left[DF(x) - A\right] \\ &= \epsilon^2(A_0^{-1}A_1)^2 - [I - \epsilon(A_0^{-1}A_1)]\cdot A_0^{-1}\cdot\left[DF(x) - A\right], \end{aligned}$$

hence an upper bound on $DT(x)$ is given by

$$\epsilon^2\left(\overline{A_0^{-1}A_1}\right)^2 + \left(I_3 + \epsilon\overline{A_0^{-1}A_1}\right)\cdot\overline{A_0^{-1}\left[DF(x) - A\right]},$$

where $\overline{A_0^{-1}\left[DF(x)-A\right]}$ is a yet to be determined upper bound on $A_0^{-1}\left[DF(x)-A\right]$. To calculate this upper bound, we break it up into two parts:

$$\pi_{\alpha,\omega}A_0^{-1}\left(DF(x)-A\right) = A_{0,1}^{-1}i_{\mathbb{C}}^{-1}\pi_1\left(DF(x)-A\right) \tag{3.29}$$

$$\pi_c A_0^{-1}\left(DF(x)-A\right) = A_{0,*}^{-1}\pi_{\geq 2}\left(DF(x)-A\right). \tag{3.30}$$

To calculate an upper bound on (3.29), we use the explicit expression for $A_{0,1}^{-1}$ to estimate

$$\begin{aligned}
\left|\pi_\alpha A_{0,1}^{-1}\pi_1\left(DF(x)-A\right)\right| &\leq \sqrt{\tfrac{4}{\pi^2}+1}\,\overline{\pi_1(DF(x)-A)} \\
\left|\pi_\omega A_{0,1}^{-1}\pi_1\left(DF(x)-A\right)\right| &\leq \tfrac{2}{\pi}\,\overline{\pi_1(DF(x)-A)},
\end{aligned}$$

where $\overline{\pi_1(DF(x)-A)}$ is an upper bound on $\pi_1(DF(x)-A)$, viewed as an operator from $\mathbb{R}^2\times\ell_0^K$ to $\mathbb{C}$ (a straightforward generalization of Definition 2.2.5). Indeed, in Propositions 3.4.2, 3.4.3 and 3.4.4 we determine functions $f_{1,\cdot}$ such that, for all $x\in B_\epsilon(r,\rho)$,

$$\begin{aligned}
f_{1,\alpha}(\epsilon,r,\rho) &\geq \left|\frac{\partial F_1}{\partial\alpha}(x)+i\right|, \\
f_{1,\omega}(\epsilon,r,\rho) &\geq \left|\frac{\partial F_1}{\partial\omega}(x)-(i-\tfrac{\pi}{2})\right|, \\
f_{1,c}(\epsilon,r,\rho) &\geq \left|\frac{\partial F_1}{\partial c}(x)\cdot b-\tfrac{\pi}{2}\epsilon(i-1)\pi_2 b\right|, \qquad \text{for all } b\in\ell_0^K \text{ with } \|b\|\leq 1.
\end{aligned}$$

Here the projection π_2 is defined as $\pi_2 b := b_2\in\mathbb{C}$ for $b=\{b_k\}_{k=1}^\infty\in\ell^1$. Hence $[f_{1,\alpha},f_{1,\omega},f_{1,c}]$ is an upper bound on $\pi_1(DF(x)-A)$.

For calculating an upper bound on Equation (3.30), in Propositions 3.4.5, 3.4.6 and 3.4.7 we determine functions $f_{*,\cdot}$ such that, for all $x\in B_\epsilon(r,\rho)$,

$$\begin{aligned}
f_{*,\alpha}(\epsilon,r,\rho) &\geq \left\|A_{0,*}^{-1}\pi_{\geq 2}\left(\frac{\partial F}{\partial\alpha}(x)+\epsilon\tfrac{2+4i}{5}\mathrm{e}_2\right)\right\|, \\
f_{*,\omega}(\epsilon,r,\rho) &\geq \left\|A_{0,*}^{-1}\pi_{\geq 2}\left(\frac{\partial F}{\partial\omega}(x)-\epsilon\left[\tfrac{4-3\pi}{10}+\tfrac{2(2+\pi)}{5}i\right]\mathrm{e}_2\right)\right\|, \\
f_{*,c}(\epsilon,r,\rho) &\geq \left\|A_{0,*}^{-1}\pi_{\geq 2}\left(\frac{\partial F}{\partial c}(x)\cdot b-(A_{0,*}+\epsilon A_{1,*})b\right)\right\|, \quad \text{for all } b\in\ell_0^K \text{ with } \|b\|\leq 1.
\end{aligned}$$

Hence $[f_{*,\alpha},f_{*,\omega},f_{*,c}]$ is an upper bound on $A_{0,*}^{-1}\pi_{\geq 2}\left(DF(x)-A\right)$, viewed as an operator from $\mathbb{R}^2\times\ell_0^K$ to ℓ_0^1. We have thereby shown that M, as defined in (3.27), is an upper bound on $\overline{A_0^{-1}\left[DF(x)-A\right]}$, which concludes the proof. $\square$

Proposition 3.4.2. *Define*

$$f_{1,\alpha} := \Delta_\omega + \epsilon \frac{\delta_c(2+\delta_c)}{2}.$$

Then for all $x = (\alpha, \omega, c) \in B_\epsilon(r, \rho)$

$$f_{1,\alpha} \geq \left| \frac{\partial F_1}{\partial \alpha}(x) + i \right|.$$

Proof. We calculate

$$\frac{\partial F_1}{\partial \alpha}(x) + i = e^{-i\omega} + i + \epsilon \left(e^{i\omega} + e^{-2i\omega} \right) \pi_2 c + \epsilon \pi_1([U_\omega c] * c).$$

Hence, using Lemma 3.2.2,

$$\left| \frac{\partial F_1}{\partial \alpha}(x) + i \right| \leq |e^{-i\omega} + i| + 2\epsilon \frac{\delta_c}{2} + \epsilon \frac{1}{2} \delta_c^2 \leq \Delta_\omega + \epsilon \frac{\delta_c(2+\delta_c)}{2}.$$

Here we have used that $|\pi_k a| \leq \frac{1}{2}\|a\|$ for $k = 1, 2$ and all $a \in \ell^1$. □

Proposition 3.4.3. *Define*

$$f_{1,\omega} := \Delta_\alpha + \tfrac{\pi}{2}\Delta_\omega + (\tfrac{\pi}{2} + \Delta_\alpha) \frac{\epsilon \delta_c}{2}(3 + \rho).$$

Then for all $x = (\alpha, \omega, c) \in B_\epsilon(r, \rho)$

$$f_{1,\omega} \geq \left| \frac{\partial F_1}{\partial \omega}(x) - (i - \tfrac{\pi}{2}) \right|.$$

Proof. We calculate

$$\begin{aligned} \frac{\partial F_1}{\partial \omega}(x) - (i - \tfrac{\pi}{2}) = (i - i\alpha e^{-i\omega}) - (i - \tfrac{\pi}{2}) + \alpha\epsilon(ie^{i\omega} - 2e^{-2i\omega})\pi_2 c \\ - i\alpha\epsilon\pi_1([K^{-1}U_\omega c] * c) \\ = -i(\alpha - \tfrac{\pi}{2})e^{-i\omega} - i\tfrac{\pi}{2}(i + e^{-i\omega}) + \alpha\epsilon(ie^{i\omega} - 2e^{-2i\omega})\pi_2 c \\ - i\alpha\epsilon\pi_1([K^{-1}U_\omega c] * c). \end{aligned}$$

Hence, using Lemma 3.2.2 again,

$$\left| \frac{\partial F_1}{\partial \omega}(x) - (i - \tfrac{\pi}{2}) \right| \leq \Delta_\alpha + \tfrac{\pi}{2}\Delta_\omega + \frac{3}{2}\alpha\epsilon\delta_c + \frac{1}{2}\alpha\epsilon\rho\delta_c.$$

□

Proposition 3.4.4. *Define*

$$f_{1,c} := \epsilon \left(\Delta_\alpha + \tfrac{3\pi}{4}\Delta_\omega + (\tfrac{\pi}{2} + \Delta_\alpha)\delta_c \right).$$

Then for all $x = (\alpha, \omega, c) \in B_\epsilon(r, \rho)$

$$f_{1,c} \geq \left| \frac{\partial F_1}{\partial c}(x) \cdot b - \tfrac{\pi}{2}\epsilon(i-1)\pi_2 b \right|, \qquad \textit{for all } b \in \ell_0^K \textit{ with } \|b\| \leq 1.$$

Proof. We calculate

$$\begin{aligned}
\frac{\partial F_1}{\partial c}(x) \cdot b - \tfrac{\pi}{2}\epsilon(i-1)\pi_2 b &= \epsilon[\alpha(e^{i\omega} + e^{-2i\omega}) - \tfrac{\pi}{2}(i-1)]\pi_2 b \\
&\quad + \alpha\epsilon\pi_1\big([U_\omega b] * c + [U_\omega c] * b\big) \\
&= \epsilon[(\alpha - \tfrac{\pi}{2})(e^{i\omega} + e^{-2i\omega})]\pi_2 b + \epsilon\tfrac{\pi}{2}[(e^{i\omega} + e^{-2i\omega}) - (i-1)]\pi_2 b \\
&\qquad + \alpha\epsilon\pi_1\big([U_\omega b] * c + [U_\omega c] * b\big).
\end{aligned}$$

Hence, for $\|b\| \leq 1$,

$$\left| \frac{\partial F_1}{\partial c}(x) \cdot b - \tfrac{\pi}{2}\epsilon(i-1)\pi_2 b \right| \leq \epsilon \left(\Delta_\alpha + \tfrac{\pi}{4}(\Delta_\omega + 2\Delta_\omega) + (\tfrac{\pi}{2} + \Delta_\alpha)\delta_c \right).$$

□

Proposition 3.4.5. *Define*

$$f_{*,\alpha} := \frac{2}{\pi\sqrt{5}} \left(r_c + 2\Delta_\omega(\delta_c^0 + \epsilon) + \epsilon\delta_c(4 + \delta_c) \right).$$

Then for all $x = (\alpha, \omega, c) \in B_\epsilon(r, \rho)$

$$f_{*,\alpha} \geq \left\| A_{0,*}^{-1}\pi_{\geq 2} \left(\frac{\partial F}{\partial \alpha}(x) + \epsilon\tfrac{2+4i}{5}\mathrm{e}_2 \right) \right\|.$$

Proof. We note that $\epsilon\frac{2+4i}{5}\mathrm{e}_2 = \bar{c}_\epsilon + \epsilon i \mathrm{e}_2$ and calculate

$$\pi_{\geq 2}\frac{\partial F}{\partial \alpha}(x) + \epsilon\tfrac{2+4i}{5}\mathrm{e}_2 = U_\omega(c - \bar{c}_\epsilon) + (1 + e^{-2i\omega})\bar{c}_\epsilon + \epsilon(e^{-i\omega} + i)\mathrm{e}_2 + \epsilon\pi_{\geq 2}L_\omega c + \epsilon\pi_{\geq 2}([U_\omega c] * c).$$

By using Proposition 3.1.1 and Lemma 3.2.2, we obtain the estimate

$$\begin{aligned}
\left\| A_{0,*}^{-1}\pi_{\geq 2} \left(\frac{\partial F}{\partial \alpha}(x) + \epsilon\tfrac{2+4i}{5}\mathrm{e}_2 \right) \right\| &\leq \|A_{0,*}^{-1}\| \left(r_c + \delta_c^0|1 + e^{-2i\omega}| + 2\epsilon|e^{-i\omega} + i| + 4\epsilon\delta_c + \epsilon\delta_c^2 \right) \\
&\leq \frac{2}{\pi\sqrt{5}} \left(r_c + 2\Delta_\omega(\delta_c^0 + \epsilon) + \epsilon\delta_c(4 + \delta_c) \right).
\end{aligned}$$

□

Proposition 3.4.6. *Define*

$$f_{*,\omega} := \tfrac{5}{2\pi}(1+\tfrac{\pi}{2})r_c + \tfrac{2}{\sqrt{5}}\epsilon\left((1+\tfrac{4}{\sqrt{5}})\Delta_\omega + \tfrac{2}{\pi}\Delta_\alpha\right) + \tfrac{5}{2\pi}\Delta_\alpha(r_c+\delta_c)$$
$$+ \tfrac{2}{\pi}\epsilon(\tfrac{\pi}{2}+\Delta_\alpha)\left(\frac{1}{\sqrt{5}}(\delta_c+r_c) + \frac{5}{4}\left(\delta_c+\tfrac{3}{2}r_c\right) + \frac{\rho\delta_c}{\sqrt{5}}\right). \tag{3.31}$$

Then for all $x=(\alpha,\omega,c)\in B_\epsilon(r,\rho)$

$$f_{*,\omega} \geq \left\| A_{0,*}^{-1}\pi_{\geq 2}\left(\frac{\partial F}{\partial\omega}(x) - \epsilon\left[\tfrac{4-3\pi}{10}+\tfrac{2(2+\pi)}{5}i\right]\mathrm{e}_2\right)\right\|.$$

Proof. We note that $\epsilon\left[\tfrac{4-3\pi}{10}+\tfrac{2(2+\pi)}{5}i\right]\mathrm{e}_2 = i(2+\pi)\bar{c}_\epsilon - \tfrac{\pi}{2}\epsilon\mathrm{e}_2$ and calculate

$$\begin{aligned}\pi_{\geq 2}\frac{\partial F}{\partial\omega}(x) - \epsilon\left[\tfrac{4-3\pi}{10}+\tfrac{2(2+\pi)}{5}i\right]\mathrm{e}_2 &= iK^{-1}(I-\alpha U_\omega)c - i\alpha\epsilon e^{-i\omega}\mathrm{e}_2 + \alpha\epsilon\pi_{\geq 2}L'_\omega c \\ &\quad - i\alpha\epsilon\pi_{\geq 2}([K^{-1}U_\omega c]*c) \\ &\quad - iK^{-1}(I-\tfrac{\pi}{2}U_{\omega_0})\bar{c}_\epsilon + \tfrac{\pi}{2}\epsilon\mathrm{e}_2 \\ &= iK^{-1}(c-\bar{c}_\epsilon) - \epsilon(i\alpha e^{-i\omega}-\tfrac{\pi}{2})\mathrm{e}_2 \\ &\quad - iK^{-1}\left[U_\omega\left(\tfrac{\pi}{2}(c-\bar{c}_\epsilon) + (\alpha-\tfrac{\pi}{2})c\right)\right] \\ &\quad - iK^{-1}\left[(U_\omega - U_{\omega_0})\tfrac{\pi}{2}\bar{c}_\epsilon\right] \\ &\qquad + \alpha\epsilon\pi_{\geq 2}L'_\omega c - i\alpha\epsilon\pi_{\geq 2}([K^{-1}U_\omega c]*c).\end{aligned}$$

Applying the operator $A_{0,*}^{-1}$ to this expression, we obtain (with $\hat{U}$ defined in (3.1))

$$\begin{aligned}A_{0,*}^{-1}\pi_{\geq 2}\left(\frac{\partial F}{\partial\omega}(x) - \epsilon\left[\tfrac{4-3\pi}{10}+\tfrac{2(2+\pi)}{5}i\right]\mathrm{e}_2\right) &= \frac{2}{\pi}\hat{U}(c-\bar{c}_\epsilon) - \frac{2\epsilon}{i\pi}\hat{U}K(i\alpha e^{-i\omega}-\tfrac{\pi}{2})\mathrm{e}_2 \\ &\quad - \frac{2}{\pi}\hat{U}\left[U_\omega\left(\alpha(c-\bar{c}_\epsilon) + (\alpha-\tfrac{\pi}{2})c\right)\right] \\ &\quad - \frac{2}{\pi}\hat{U}\left(U_\omega - U_{\omega_0}\right)\tfrac{\pi}{2}\bar{c}_\epsilon \\ &\quad + \tfrac{2\alpha\epsilon}{i\pi}\hat{U}K\pi_{\geq 2}\left(L'_\omega c - i[K^{-1}U_\omega c]*c\right).\end{aligned}$$

We use the triangle inequality to estimate its norm, splitting it into the five pieces:

$$\begin{aligned}
\left\|\frac{2}{\pi}\hat{U}(c-\bar{c}_\epsilon)\right\| &\leq \frac{2}{\pi}\frac{5}{4}r_c \\
&= \frac{5}{2\pi}r_c \\
\left\|-\frac{2\epsilon}{i\pi}\hat{U}K(i\alpha e^{-i\omega}-\tfrac{\pi}{2})\mathrm{e}_2\right\| &\leq \frac{4\epsilon}{\pi}\frac{1}{\sqrt{5}}\left(\tfrac{\pi}{2}\Delta_\omega+\Delta_\alpha\right) \\
&= \tfrac{2\epsilon}{\sqrt{5}}\left(\Delta_\omega+\tfrac{2}{\pi}\Delta_\alpha\right) \\
\left\|-\tfrac{2}{\pi}\hat{U}\left[U_\omega\left(\alpha(c-\bar{c}_\epsilon)+(\alpha-\tfrac{\pi}{2})c\right)\right]\right\| &\leq \tfrac{2}{\pi}\tfrac{5}{4}\left((\tfrac{\pi}{2}+\Delta_\alpha)r_c+\Delta_\alpha\delta_c\right) \\
&= \tfrac{5}{2\pi}\left(\tfrac{\pi}{2}r_c+\Delta_\alpha(r_c+\delta_c)\right) \\
\left\|-\tfrac{2}{\pi}\hat{U}\left(U_\omega-U_{\omega_0}\right)\tfrac{\pi}{2}\bar{c}_\epsilon\right\| &\leq \tfrac{2}{\pi}\tfrac{2}{\sqrt{5}}(2\Delta_\omega)\tfrac{\pi}{2}\tfrac{2\epsilon}{\sqrt{5}} \\
&= \tfrac{8\epsilon}{5}\Delta_\omega \\
\left\|\tfrac{2\alpha\epsilon}{i\pi}\hat{U}K\pi_{\geq 2}\left(L'_\omega c-i[K^{-1}U_\omega c]*c\right)\right\| &\leq \frac{2\alpha\epsilon}{\pi}\left(\|\hat{U}K\pi_{\geq 2}L'_\omega c\|+\frac{\rho\delta_c}{\sqrt{5}}\right),
\end{aligned}$$

where we have used Proposition 3.1.1 and Lemma 3.2.2. Finally, we estimate

$$\begin{aligned}
\left\|\hat{U}K\pi_{\geq 2}L'_\omega c\right\| &= \left\|\hat{U}K\pi_{\geq 2}\left(-i\sigma^+(e^{-i\omega}I+K^{-1}U_\omega)+i\sigma^-(e^{i\omega}I-K^{-1}U_\omega)\right)c\right\| \\
&\leq \left\|\hat{U}K\pi_{\geq 2}(\sigma^++\sigma^-)c\right\|+\left\|\hat{U}\pi_{\geq 2}K(\sigma^++\sigma^-)K^{-1}U_\omega c\right\| \\
&\leq \frac{1}{\sqrt{5}}(\|\sigma^+c\|+\|\pi_{\geq 2}\sigma^-c\|)+\frac{5}{4}\left(\|K\sigma^+K^{-1}\|\delta_c+\|\pi_{\geq 2}K\sigma^-K^{-1}\|r_c\right) \\
&\leq \frac{1}{\sqrt{5}}(\delta_c+r_c)+\frac{5}{4}\left(\delta_c+\frac{3}{2}r_c\right). \qquad (3.32)
\end{aligned}$$

Hence, with $f_{*,\omega}$ as defined in (3.31), it follows that

$$\left\|A_{0,*}^{-1}\pi_{\geq 2}\left(\frac{\partial F}{\partial\omega}(x)-\epsilon\left[\tfrac{4-3\pi}{10}+\tfrac{2(2+\pi)}{5}i\right]\mathrm{e}_2\right)\right\| \leq f_{*,\omega}.$$

□

Proposition 3.4.7. *Define*

$$f_{*,c} := \left[\frac{5}{2}\left(\frac{1}{2}+\frac{1}{\pi}\right)\Delta_\omega+\frac{\Delta_\alpha}{\sqrt{5}}\right]+\epsilon\left[\frac{8}{\pi\sqrt{5}}\Delta_\alpha+\left(\frac{2}{\sqrt{5}}+\frac{25}{8}\right)\Delta_\omega+\frac{4(\frac{\pi}{2}+\Delta_\alpha)\delta_c}{\pi\sqrt{5}}\right].$$

Then for all $x=(\alpha,\omega,c)\in B_\epsilon(r,\rho)$

$$f_{*,c} \geq \left\|A_{0,*}^{-1}\pi_{\geq 2}\left(\frac{\partial F}{\partial c}(x)\cdot b-(A_{0,*}+\epsilon A_{1,*})b\right)\right\|, \qquad \textit{for all } b\in\ell_0^K \textit{ with } \|b\|\leq 1.$$

Proof. We write $A_* := A_{0,*} + \epsilon A_{1,*}$ and calculate

$$\begin{aligned}
\frac{\partial F}{\partial c}(x)\cdot b - A_* b &= \big[(i\omega K^{-1} + \alpha U_\omega) - (i\tfrac{\pi}{2}K^{-1} + \tfrac{\pi}{2}U_{\omega_0})\big]b + \alpha\epsilon L_\omega b - \tfrac{\pi}{2}\epsilon L_{\omega_0} b \\
&\qquad + \alpha\epsilon\left[[U_\omega b] * c + [U_\omega c] * b\right] \\
&= \big[i(\omega - \tfrac{\pi}{2})K^{-1} + (\alpha - \tfrac{\pi}{2})U_\omega + \tfrac{\pi}{2}(U_\omega - U_{\omega_0})\big]b \\
&\qquad + \epsilon\big[(\alpha - \tfrac{\pi}{2})L_\omega + \tfrac{\pi}{2}(L_\omega - L_{\omega_0})\big]b + \alpha\epsilon\left([U_\omega b] * c + [U_\omega c] * b\right).
\end{aligned}$$

Hence, for $\|b\| \le 1$,

$$\begin{aligned}
\left\|A_{0,*}^{-1}\pi_{\ge 2}\left(\frac{\partial F}{\partial c}(x)\cdot b - A_* b\right)\right\| &\le \Delta_\omega\|A_{0,*}^{-1}K^{-1}\| + \tfrac{\pi}{2}\Delta_\alpha\|A_{0,*}^{-1}\| + \tfrac{\pi}{2}\|A_{0,*}^{-1}(U_\omega - U_{\omega_0})\| \\
&\qquad + \epsilon\left[4\Delta_\alpha\|A_{0,*}^{-1}\| + \tfrac{\pi}{2}\|A_{0,*}^{-1}\pi_{\ge 2}(L_\omega - L_{\omega_0})\|\right] \\
&\qquad + \epsilon\left[2\alpha\delta_c\|A_{0,*}^{-1}\|\right], \qquad (3.33)
\end{aligned}$$

where all norms should be interpreted as operators on ℓ^1_0. Since $\frac{\partial U_\omega}{\partial \omega} = -iK^{-1}U_\omega$ and $A_{0,*}^{-1} = \frac{2}{i\pi}\hat{U}K$, it follows from Proposition 3.1.1 that

$$\|A_{0,*}^{-1}(U_\omega - U_{\omega_0})\| \le \frac{2}{\pi}\Delta_\omega\|\hat{U}\| = \frac{5}{2\pi}\Delta_\omega. \qquad (3.34)$$

Next, we compute

$$\begin{aligned}
L_\omega - L_{\omega_0} &= \sigma^+\left[(e^{-i\omega} + i)I + (U_\omega - U_{\omega_0})\right] + \sigma^-\left[(e^{i\omega} - i)I + (U_\omega - U_{\omega_0})\right] \\
&= (e^{-i\omega} + i)\sigma^+ - ie^{i\omega}(i + e^{-i\omega})\sigma^- + (\sigma^+ + \sigma^-)(U_\omega - U_{\omega_0}).
\end{aligned}$$

Analogous to (3.32) and (3.34) we infer that

$$\|A_{0,*}^{-1}\pi_{\ge 2}(L_\omega - L_{\omega_0})\| \le \frac{4}{\pi\sqrt{5}}|i + e^{-i\omega}| + \frac{5}{\pi}\|\hat{U}\|\Delta_\omega \le \frac{4}{\pi\sqrt{5}}\Delta_\omega + \frac{25}{4\pi}\Delta_\omega.$$

Finally, by putting all estimates together and once again using Proposition 3.1.1, it follows from (3.33) that

$$\begin{aligned}
\left\|A_{0,*}^{-1}\pi_{\ge 2}\left(\frac{\partial F}{\partial c}(x)\cdot b - A_* b\right)\right\| &\le \left[\frac{5}{2}\left(\frac{1}{2} + \frac{1}{\pi}\right)\Delta_\omega + \frac{\Delta_\alpha}{\sqrt{5}}\right] \\
&\qquad + \epsilon\left[\frac{8}{\pi\sqrt{5}}\Delta_\alpha + \left(\frac{2}{\sqrt{5}} + \frac{25}{8}\right)\Delta_\omega + \frac{4(\frac{\pi}{2} + \Delta_\alpha)\delta_c}{\pi\sqrt{5}}\right].
\end{aligned}$$

□

3.5 A priori bounds on periodic orbits

In order to isolate periodic orbits, we need to separate them from the trivial solution. In this section we prove some lower bounds on the size of periodic orbits. First we work in the original Fourier coordinates. Then we derive refined bounds in rescaled coordinates.

Recall that periodic orbits of Wright's equation correspond to zeros of $G(\alpha, \omega, a) = 0$, as defined in (2.7). Clearly $G(\alpha, \omega, 0) = 0$ for all frequencies $\omega > 0$ and parameter values $\alpha > 0$. There are bifurcations from this trivial solution for $\alpha = \alpha_n := \frac{\pi}{2}(4n+1)$ for all $n \geq 0$. The corresponding natural frequency is $\omega = \alpha_n$, but there are bifurcations for any $\omega = \alpha_n/\tilde{n}$ with $\tilde{n} \in \mathbb{N}$ as well, which are essentially copies of the primary bifurcation. The following proposition quantifies that away from these bifurcation points the trivial solution is isolated.

Proposition 3.5.1. *Suppose $G(\alpha, \omega, a) = 0$ for some $\alpha, \omega > 0$. Then either $a \equiv 0$ or*

$$\|a\| \geq \min_{k \in \mathbb{N}} \sqrt{\left(1 - k\,\frac{\omega}{\alpha}\right)^2 + 2k\,\frac{\omega}{\alpha}\big(1 - \sin k\omega\big)}. \tag{3.35}$$

Proof. We fix $\alpha, \omega > 0$ and define

$$\beta_1 := \min_{k \in \mathbb{N}} \, (\alpha - k\omega)^2 + 2\alpha k\omega(1 - \sin k\omega).$$

If $\beta_1 = 0$ then there is nothing to prove. From now on we assume that $\beta_1 > 0$. We recall that

$$G(\alpha, \omega, a) = (i\omega K^{-1} + \alpha U_\omega)a + \alpha\,[U_\omega\, a] * a.$$

We note that $i\omega K^{-1} + \alpha U_\omega$ is invertible, since for any $k \in \mathbb{N}$

$$\begin{aligned}
|ik\omega + \alpha e^{-ik\omega}|^2 &= (\alpha \cos k\omega)^2 + (\omega - \alpha \sin k\omega)^2 \\
&= (k\omega)^2 + \alpha^2 - 2\alpha k\omega \sin k\omega \\
&= (\alpha - k\omega)^2 + 2\alpha k\omega(1 - \sin k\omega) \\
&\geq \beta_1 > 0.
\end{aligned}$$

We may thus rewrite $G(\alpha, \omega, a) = 0$ as

$$a = -\alpha(i\omega K^{-1} + \alpha U_\omega)^{-1}([U_\omega\, a] * a). \tag{3.36}$$

Since $\|(\omega K^{-1} + \alpha U_\omega)^{-1}\| = \beta_1^{-1/2}$ and $\|\,[U_\omega\, a] * a\| \le \|a\|^2$, we infer from (3.36) that

$$\|a\| \le \alpha \beta_1^{-1/2} \|a\|^2.$$

We conclude that either $a \equiv 0$ or $\|a\| \ge \beta_1^{1/2}/\alpha$. $\square$

Proposition 3.5.2. *Suppose that* $\omega \ge 1.1$ *and* $\alpha \in (0, 2]$. *Define*

$$g_k(\omega, \alpha) = \big(1 - k\,\tfrac{\omega}{\alpha}\big)^2 + 2k\,\tfrac{\omega}{\alpha}\big(1 - \sin k\omega\big). \tag{3.37}$$

Then $g_1 < g_k$ *for all* $k \ge 2$.

Proof. This is equivalent to showing that

$$(1 - \tfrac{\omega}{\alpha})^2 + 2\tfrac{\omega}{\alpha}(1 - \sin\omega) < (1 - k\tfrac{\omega}{\alpha})^2 + 2k\tfrac{\omega}{\alpha}(1 - \sin k\omega) \qquad \text{for } k \ge 2.$$

Making the substitution $x = \frac{\omega}{\alpha}$, we can simplify this to the equivalent inequality

$$(k^2 - 1)x + 2\sin\omega - 2k \sin k\omega > 0.$$

Since $\alpha \le 2$, we have $x \ge \omega/2$. Hence it suffices to prove that

$$h_k(\omega) := \frac{k^2 - 1}{2}\omega + 2\sin\omega - 2k\sin k\omega > 0 \qquad \text{for all } k \ge 2. \tag{3.38}$$

We first consider $k = 2$. It is clear that $h_2(\omega) > 0$ for $\omega > 4$. We note that h_2 has a simple zero at $\omega \approx 1.07146$ and it is easy to check using interval arithmetic that $h_2(\omega)$ is positive for $\omega \in [1.1, 4]$. Hence $h_2(\omega) > 0$ for all $\omega \ge 1.1$.

For $k = 3$ and $k = 4$ we can repeat a similar argument. For $k \ge 5$ it is immediate that $h_k(\omega) > \frac{k^2-1}{2} - 2 - 2k \ge 0$ for $\omega > 1$. $\square$

As discussed in Section 2.1, the function $G(\alpha, \omega, a)$ gets replaced by $\widetilde{F}_\epsilon(\alpha, \omega, \tilde{c})$ in rescaled coordinates. In these coordinates we derive a result analogous to Proposition 3.5.1 below, see Lemma 3.5.4. First we bound the inverse of the operator $\widehat{B} \in B(\ell_0^1)$ defined by

$$\widehat{B} := i\frac{\omega}{\alpha} I + U_\omega K + \epsilon L_\omega K,$$

where K, U_ω and L_ω have been introduced in Section 2.1.

Lemma 3.5.3. *Let $\epsilon \geq 0$ and $\alpha, \omega > 0$. Let*

$$\gamma := \frac{1}{2} + \epsilon \left(\frac{2}{3} + \max \left\{ \frac{\sqrt{2 - 2\sin(\omega - \frac{\pi}{2})}}{2}, \frac{2}{3} \right\} \right).$$

If $\gamma < \omega/\alpha$ then the operator $\widehat{B}$ is invertible and the inverse is bounded by

$$\|\widehat{B}^{-1}\| \leq \frac{1}{\frac{\omega}{\alpha} - \gamma}.$$

Proof. Writing

$$\widehat{B} = i\frac{\omega}{\alpha}\left(I + \frac{\alpha}{i\omega}\left(U_\omega + \epsilon L_\omega\right)K\right)$$

and using a (formal) Neumann series argument, we obtain

$$\|\widehat{B}^{-1}\| \leq \frac{\alpha}{\omega}\sum_{n=0}^{\infty}\left(\frac{\alpha}{\omega}\right)^n \|(U_\omega+\epsilon L_\omega)K\|^n \leq \frac{\frac{\alpha}{\omega}}{1 - \frac{\alpha}{\omega}\|(U_\omega + \epsilon L_\omega)K\|} = \frac{1}{\frac{\omega}{\alpha} - \|(U_\omega + \epsilon L_\omega)K\|}.$$

It remains to prove the estimate $\|(U_\omega + \epsilon L_\omega)K\| \leq \gamma$. Then, in particular, for $\gamma < \omega/\alpha$ the formal argument is rigorous.

Recalling that $L_\omega = \sigma^+(e^{-i\omega}I + U_\omega) + \sigma^-(e^{i\omega}I + U_\omega)$, we use the triangle inequality

$$\|(U_\omega + \epsilon L_\omega)K\| \leq \|U_\omega K\| + \epsilon\|\sigma^+(e^{-i\omega}I + U_\omega)K\| + \epsilon\|\sigma^-(e^{i\omega}I + U_\omega)K\|,$$

and estimate each term separately as an operator on ℓ_0^1. We recall the formula (3.2) for the operator norm. Using that $\|K\tilde{c}\| \leq \frac{1}{2}\|\tilde{c}\|$ for all $\tilde{c} \in \ell_0^1$, the first term is bounded by $\|U_\omega K\| \leq \frac{1}{2}$. Since σ^- shifts the sequence to the left and we consider the operators acting on ℓ_0^1, we obtain $\|\sigma^-(e^{i\omega}I + U_\omega)K\| \leq \frac{2}{3}$. For the final term, $\|\sigma^+(e^{-i\omega}I + U_\omega)K\|$, to obtain a slightly more refined estimate, we first consider the action of $\sigma^+(e^{-i\omega}I + U_\omega)K$ on e_2. We observe that

$$|e^{-i\omega} + e^{-2i\omega}| = \sqrt{2 - 2\sin(\omega - \tfrac{\pi}{2})}.$$

Hence $\|\sigma^+(e^{-i\omega}I + U_\omega)K\mathrm{e}_2\| \leq \sqrt{2 - 2\sin(\omega - \frac{\pi}{2})}$, leading to

$$\|\sigma^+(e^{-i\omega}I + U_\omega)K\| \leq \max\left\{\frac{\sqrt{2 - 2\sin(\omega - \frac{\pi}{2})}}{2}, \frac{2}{3}\right\}.$$

We conclude that

$$\|(U_\omega + \epsilon L_\omega)K\| \leq \frac{1}{2} + \epsilon\left(\frac{2}{3} + \max\left\{\frac{\sqrt{2 - 2\sin(\omega - \frac{\pi}{2})}}{2}, \frac{2}{3}\right\}\right).$$

□

Lemma 3.5.4. *Fix $\epsilon \geq 0$, $\alpha, \omega > 0$. Assume that $\widehat{B}$ is invertible. Let b_0 be a bound on $\|\widehat{B}^{-1}\|$. Define*

$$z^{\pm} = b_0^{-1} \pm \sqrt{b_0^{-2} - 2\epsilon^2}.$$

Let $\tilde{c} \in \ell_0^1$ be such that $\widetilde{F}_\epsilon(\alpha, \omega, \tilde{c}) = 0$. Then either $\|\tilde{c}\| \leq z^-$ or $\|\tilde{c}\| \geq z^+$. Additionally, $\|K^{-1}\tilde{c}\| \leq b_0(2\epsilon^2 + \|\tilde{c}\|^2)$.

Proof. If $\widetilde{F}_\epsilon(\alpha, \omega, \tilde{c}) = 0$, then it follows that the equations $\pi_c \widetilde{F}_\epsilon = 0$ can be rearranged as

$$\tilde{c} = -K\widehat{B}^{-1}(\epsilon^2 e^{-i\omega}\mathrm{e}_2 + \ [U_\omega \tilde{c}] * \tilde{c}). \tag{3.39}$$

Taking norms, and using that $\|K\tilde{c}\| \leq \frac{1}{2}\|\tilde{c}\|$ for all $\tilde{c} \in \ell_0^1$, we obtain

$$\|\tilde{c}\| \leq \frac{1}{2}\|B^{-1}\| \left(\epsilon^2\|\mathrm{e}_2\| + \|[U_\omega \tilde{c}] * \tilde{c}\|\right) \leq \frac{1}{2} b_0 \left(2\epsilon^2 + \|\tilde{c}\|^2\right). \tag{3.40}$$

The quadratic $x^2 - 2b_0^{-1}x + 2\epsilon^2$ has two zeros z^+ and z^- given by

$$z^{\pm} = b_0^{-1} \pm \sqrt{b_0^{-2} - 2\epsilon^2}.$$

The inequality (3.40) thus implies that either $\|\tilde{c}\| \leq z^-$ or $\|\tilde{c}\| \geq z^+$.

Furthermore, it follows from (3.39) that $\|K^{-1}\tilde{c}\| \leq \|\widehat{B}^{-1}\| \,(2\epsilon^2 + \|\tilde{c}\|^2) \leq b_0(2\epsilon^2 + \|\tilde{c}\|^2)$. $\square$

In practice we use the bound $\|\widehat{B}^{-1}\| \leq b_*^{-1}$, where

$$b_*(\epsilon) := \frac{\omega}{\alpha} - \frac{1}{2} - \epsilon\left(\frac{2}{3} + \frac{1}{2}\sqrt{2 + 2|\omega - \tfrac{\pi}{2}|}\right).$$

When doing so, we will refer to $z^{\pm}$ as $z_*^{\pm}$. Additionally, we will need the following monotonicity property.

Lemma 3.5.5. *Fix $\alpha, \omega, \epsilon_0 > 0$ and assume that $\epsilon_0 \leq b_*(\epsilon_0)/\sqrt{2}$. Define*

$$z_*^-(\epsilon) := b_*(\epsilon) - \sqrt{(b_*(\epsilon))^2 - 2\epsilon^2}.$$

Let $C_0 := \frac{z_*^-(\epsilon_0)}{\epsilon_0}$. Then

$$z_*^-(\epsilon) \leq C_0\epsilon \qquad \textit{for all } 0 \leq \epsilon \leq \epsilon_0. \tag{3.41}$$

Proof. Let $x := \sqrt{2}\epsilon/b_*(\epsilon) \geq 0$. Clearly $\frac{d}{d\epsilon}x > 0$. It thus suffices to observe that

$$\frac{z_*^-(\epsilon)}{\epsilon} = \sqrt{2}\,\frac{1 - \sqrt{1 - x^2}}{x}$$

is increasing for $x \in [0, 1]$. $\square$

3.6 Implicit Differentiation

We will approximate

$$\frac{\partial F}{\partial \epsilon}(x) = \alpha e^{-i\omega} \mathrm{e}_2 + \alpha L_\omega c + \alpha [U_\omega c] * c$$

by

$$\Gamma := \tfrac{\pi}{2} \tfrac{3i-1}{5} \epsilon \mathrm{e}_1 - \tfrac{\pi}{2} i \mathrm{e}_2 - \tfrac{\pi}{2} \tfrac{3+i}{5} \epsilon \mathrm{e}_3 \tag{3.42}$$

$$= -\tfrac{\pi}{2} i \mathrm{e}_2 + \tfrac{\pi}{2} L_{\omega_0} \bar{c}_\epsilon, \tag{3.43}$$

which has been chosen so that $\frac{\partial F}{\partial \epsilon}(\frac{\pi}{2}, \frac{\pi}{2}, \bar{c}_\epsilon) - \Gamma = \mathcal{O}(\epsilon^2)$.

Lemma 3.6.1. *When we write $A^\dagger \Gamma = (\alpha', \omega', c') \in \mathbb{R}^2 \times \ell_0^K$, then*

$$\alpha' = -\tfrac{2}{5}(\tfrac{3\pi}{2} - 1)\epsilon,$$

$$\omega' = \tfrac{2}{5}\epsilon,$$

$$c' = \left[(\tfrac{1+2i}{5}) - \epsilon^2 \tfrac{9}{250}(7-i) \right] \mathrm{e}_2 + \epsilon \tfrac{3i-1}{10} \mathrm{e}_3.$$

Proof. First we calculate the α and ω components of the image of $A^\dagger$:

$$\begin{aligned}
\pi_{\alpha,\omega} A^\dagger &= A_{0,1}^{-1} i_{\mathbb{C}}^{-1} \pi_1 [I - \epsilon A_1 A_0^{-1}]] \\
&= A_{0,1}^{-1} i_{\mathbb{C}}^{-1} \pi_1 [I - \epsilon \tfrac{\pi}{2} L_{\omega_0} A_{0,*}^{-1}] \\
&= A_{0,1}^{-1} i_{\mathbb{C}}^{-1} \pi_1 [I - \epsilon \sigma^- (iI + U_{\omega_0})(iK^{-1} + U_{\omega_0})^{-1}] \\
&= A_{0,1}^{-1} i_{\mathbb{C}}^{-1} [\pi_1 - \epsilon (\tfrac{3+i}{5}) \pi_2].
\end{aligned} \tag{3.44}$$

Here we have used projections $\pi_k a = a_k$ for $a = \{a_k\}_{k \geq 1} \in \ell^1$. We now calculate the α and ω components of $A^\dagger \Gamma$. It follows from (3.42) and (3.44) that

$$\begin{aligned}
\pi_{\alpha,\omega} A^\dagger \Gamma &= A_{0,1}^{-1} i_{\mathbb{C}}^{-1} \left[\tfrac{\pi}{2} \tfrac{3i-1}{5} \epsilon + \tfrac{\pi}{2} \tfrac{3+i}{5} i \epsilon \right] \\
&= \tfrac{\pi \epsilon}{5} A_{0,1}^{-1} i_{\mathbb{C}}^{-1} (3i - 1) \\
&= -\frac{2\epsilon}{5} \begin{pmatrix} \frac{3\pi}{2} - 1 \\ -1 \end{pmatrix}.
\end{aligned}$$

We now calculate

$$\pi_c A^\dagger \Gamma = A_{0,*}^{-1} \pi_{\geq 2} [I - \epsilon A_1 A_0^{-1}] \Gamma, \tag{3.45}$$

where $A_1 A_0^{-1}$ decomposes as

$$A_1 A_0^{-1} = \mathrm{e}_2[i_{\mathbb{C}} A_{1,2} A_{0,1}^{-1} i_{\mathbb{C}}^{-1} \pi_1] + A_{1,*} A_{0,*}^{-1} \pi_{\geq 2}. \tag{3.46}$$

We first calculate

$$\begin{aligned} A_{0,*}^{-1} \pi_{\geq 2} \Gamma &= \tfrac{2}{\pi}(iK^{-1} + U_{\omega_0})^{-1}[-\tfrac{\pi}{2} i\mathrm{e}_2 - \tfrac{\pi}{2}\tfrac{3+i}{5}\epsilon \mathrm{e}_3] \\ &= -(2i-1)^{-1}\mathrm{e}_2 - (3i+i)^{-1}\tfrac{3+i}{5}\epsilon \mathrm{e}_3 \\ &= \tfrac{1+2i}{5}\mathrm{e}_2 + \epsilon \tfrac{3i-1}{20}\mathrm{e}_3. \end{aligned} \tag{3.47}$$

Since Γ has three nonzero components only, we next compute the action of $A_{0,*}^{-1}\pi_{\geq 2} A_1 A_0^{-1}$ on each of these. Taking into account the decomposition (3.46), we first compute its action on $\lambda \mathrm{e}_1$ for $\lambda \in \mathbb{C}$. After a straightforward but tedious calculation we obtain

$$\begin{aligned} A_{0,*}^{-1}\pi_{\geq 2} A_1 A_0^{-1} \lambda \mathrm{e}_1 &= [i_{\mathbb{C}} A_{1,2} A_{0,1}^{-1} i_{\mathbb{C}}^{-1} \lambda] A_{0,*}^{-1}\mathrm{e}_2 \\ &= -\tfrac{2}{25\pi}\big[(11+2i)\mathrm{Re}\lambda + (-6+8i)\mathrm{Im}\lambda\big]\mathrm{e}_2. \end{aligned}$$

Next, we compute the action of $A_{0,*}^{-1}\pi_{\geq 2} A_1 A_0^{-1}$ on e_k for $k = 2, 3$:

$$\begin{aligned} A_{0,*}^{-1}\pi_{\geq 2} A_1 A_0^{-1}\mathrm{e}_2 &= A_{0,*}^{-1}[\tfrac{\pi}{2}\sigma^+(e^{-i\omega_0} I + U_{\omega_0})] A_{0,*}^{-1}\mathrm{e}_2 \\ &= \tfrac{2}{\pi}\tfrac{3+i}{20}\mathrm{e}_3, \\ A_{0,*}^{-1}\pi_{\geq 2} A_1 A_0^{-1}\mathrm{e}_3 &= A_{0,*}^{-1}[\tfrac{\pi}{2}\sigma^-(e^{i\omega_0} I + U_{\omega_0})] A_{0,*}^{-1}\mathrm{e}_3 \\ &= -\tfrac{2}{\pi}\tfrac{1+2i}{10}\mathrm{e}_2, \end{aligned}$$

where we have used that $(e^{-i\omega_0} I + U_{\omega_0})\mathrm{e}_3$ vanishes. Hence, by using the explicit expression (3.42) for Γ we obtain

$$-\epsilon A_{0,*}^{-1}\pi_{\geq 2} A_1 A_0^{-1}\Gamma = -\epsilon^2 \frac{29-22i}{125}\mathrm{e}_2 + \epsilon \frac{3i-1}{20}\mathrm{e}_3 - \epsilon^2 \frac{1+7i}{50}\mathrm{e}_2. \tag{3.48}$$

Finally, combining (3.45), (3.47) and (3.48) completes the proof. □

Lemma 3.6.2. *Let*

$$\hat{f}_{\epsilon,1} := \tfrac{1}{2}\delta_c^0 \left(\sqrt{2}\Delta_\alpha + 3\Delta_\omega(\tfrac{\pi}{2} + \Delta_\alpha)\right) + r_c(\tfrac{\pi}{2} + \Delta_\alpha)\left(1 + \delta_c^0 + \tfrac{1}{2} r_c\right), \tag{3.49}$$

$$\hat{f}_{\epsilon,c} := \tfrac{2}{\pi\sqrt{5}}\left[2\left(\Delta_\alpha + \tfrac{\pi}{2}\Delta_\omega\right) + \delta_c^0[\sqrt{2}\Delta_\alpha + 3\Delta_\omega(\tfrac{\pi}{2} + \Delta_\alpha)] + (\tfrac{\pi}{2} + \Delta_\alpha)(4r_c + \delta_c^2)\right]. \tag{3.50}$$

Then the vector $[(1+\frac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \frac{2}{\pi}\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}]^T$ *is an upper bound on* $A_0^{-1}(\frac{\partial F}{\partial \epsilon}(x)-\Gamma)$ *for any* $x \in B_\epsilon(r,\rho)$.

Proof. The α- and ω-component of $A_0^{-1}(\frac{\partial F}{\partial \epsilon}(x)-\Gamma)$ are given by $A_{0,1}^{-1} i_{\mathbb{C}}^{-1}\pi_1[\frac{\partial F}{\partial \epsilon}(x)-\Gamma]$. If we can show that $|\pi_1[\frac{\partial F}{\partial \epsilon}(x)-\Gamma]| \leq \hat{f}_{\epsilon,1}$, then it follows from the explicit expression for $A_{0,1}^{-1}$ that $[(1+\frac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \frac{2}{\pi}\hat{f}_{\epsilon,1}]^T$ is an upper bound on $\pi_{\alpha,\omega}A_0^{-1}(\frac{\partial F}{\partial \epsilon}(x)-\Gamma)$. Let us write $c = \bar{c}_\epsilon + h_c$ for some $h_c \in \ell_0^1$ with $\|h_c\| \leq r_c$. Recalling (3.43), we obtain

$$\begin{aligned}
\pi_1[\tfrac{\partial F}{\partial \epsilon}(x)-\Gamma] &= \pi_1\big[\alpha L_\omega c + \alpha[U_\omega c]*c - \tfrac{\pi}{2}L_{\omega_0}\bar{c}_\epsilon\big] \\
&= \pi_1\big[\alpha\sigma^-(e^{i\omega}+e^{-2i\omega})\bar{c}_\epsilon - \tfrac{\pi}{2}\sigma^-(i-1)\bar{c}_\epsilon\big] \\
&\qquad + \pi_1\big[\alpha\sigma^-(e^{i\omega}+e^{-2i\omega})h_c + \alpha[U_\omega c]*c\big] \\
&= \pi_1\big[(\alpha-\tfrac{\pi}{2})(i-1)\bar{c}_\epsilon + \alpha(e^{i\omega}-i+e^{-2i\omega}+1)\bar{c}_\epsilon\big] \\
&\qquad + \pi_1\big[\alpha\sigma^-(e^{i\omega}+e^{-2i\omega})h_c + \alpha[U_\omega c]*c\big].
\end{aligned}$$

We note that

$$\pi_1([U_\omega c]*c) = \pi_1([U_\omega(\bar{c}_\epsilon+h_c)]*(\bar{c}_\epsilon+h_c)) = \pi_1([U_\omega\bar{c}_\epsilon]*h_c + [U_\omega h_c]*\bar{c}_\epsilon + [U_\omega h_c]*h_c).$$

Hence, using Lemma 3.2.2 we obtain the estimate

$$\big|\pi_1[\tfrac{\partial F}{\partial \epsilon}(x)-\Gamma]\big| \leq \tfrac{1}{2}\delta_c^0\left(\sqrt{2}\Delta_\alpha + 3\Delta_\omega(\tfrac{\pi}{2}+\Delta_\alpha)\right) + r_c(\tfrac{\pi}{2}+\Delta_\alpha)\left(1+\delta_c^0+\tfrac{1}{2}r_c\right).$$

We thus find that $|\pi_1[\frac{\partial F}{\partial \epsilon}(x)-\Gamma]| \leq \hat{f}_{\epsilon,1}$, with $\hat{f}_{\epsilon,1}$ defined in (3.49).

The c-component of $A_0^{-1}(\frac{\partial F}{\partial \epsilon}(x)-\Gamma)$ is given by $A_{0,*}^{-1}\pi_{\geq 2}[\frac{\partial F}{\partial \epsilon}(x)-\Gamma]$. We will use the estimate $\|A_{0,*}^{-1}\| \leq \frac{2}{\pi\sqrt{5}}$, so that it remains to determine a bound on $\|\pi_{\geq 2}[\frac{\partial F}{\partial \epsilon}(x)-\Gamma]\|$. Using (3.43) we compute

$$\pi_{\geq 2}[\tfrac{\partial F}{\partial \epsilon}(x)-\Gamma] = \alpha e^{-i\omega}\mathrm{e}_2 + \tfrac{\pi}{2}i\mathrm{e}_2 + \pi_{\geq 2}\big(\alpha L_\omega\bar{c}_\epsilon - \tfrac{\pi}{2}L_{\omega_0}\bar{c}_\epsilon + \alpha L_\omega h_c + \alpha[U_\omega c]*c\big).$$

We split the right hand side into three parts, which we estimate separately. First

$$\big\|\pi_2\big[\alpha L_\omega h_c + \alpha[U_\omega c]*c\big]\big\| \leq (\tfrac{\pi}{2}+\Delta_\alpha)(4r_c+\delta_c^2).$$

Next, we calculate

$$\begin{aligned}
\pi_{\geq 2}\left[\alpha L_\omega\bar{c}_\epsilon - \tfrac{\pi}{2}L_{\omega_0}\bar{c}_\epsilon\right] &= \alpha\sigma^+(e^{-i\omega}+e^{-2i\omega})\bar{c}_\epsilon - \tfrac{\pi}{2}\sigma^+(-i-1)\bar{c}_\epsilon \\
&= \left[(\alpha-\tfrac{\pi}{2})(-i-1)\tfrac{2-i}{5}\epsilon + \alpha(e^{-i\omega}+e^{-2i\omega}-(i+1))\tfrac{2-i}{5}\epsilon\right]\mathrm{e}_3.
\end{aligned}$$

Hence

$$\left\|\pi_{\geq 2}\left[\alpha L_\omega \bar{c}_\epsilon - \tfrac{\pi}{2} L_{\omega_0}\bar{c}_\epsilon\right]\right\| \leq \delta_c^0[\sqrt{2}\Delta_\alpha + 3\Delta_\omega(\tfrac{\pi}{2}+\Delta_\alpha)].$$

Finally, we estimate

$$\left\|(\alpha e^{-i\omega} + \tfrac{\pi}{2}i)\mathrm{e}_2\right\| = 2\left|(\alpha - \tfrac{\pi}{2})e^{-i\omega} + \tfrac{\pi}{2}(e^{-i\omega}+i)\right| \leq 2\left(\Delta_\alpha + \tfrac{\pi}{2}\Delta_\omega\right).$$

Collecting all estimates, we thus find that $\|\pi_c A_0^{-1}[\frac{\partial F}{\partial \epsilon}(x) - \Gamma]\| \leq \hat{f}_{\epsilon,c}$, with $\hat{f}_{\epsilon,c}$ defined in (3.50). □

Recall that I_3 is used to denote the 3×3 identity matrix.

Corollary 3.6.3. *Let $\overline{A_0^{-1}A_1}$ be defined in Proposition 3.1.3. The vector*

$$(I_3 + \epsilon\overline{A_0^{-1}A_1})\cdot[(1+\tfrac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \tfrac{2}{\pi}\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}]^T$$

is an upper bound on $A^\dagger(\frac{\partial F}{\partial \epsilon}(x) - \Gamma)$ for any $x \in B_\epsilon(r,\rho)$.

Proof. From Lemma 3.6.2 it follows that $[(1+\frac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \frac{2}{\pi}\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}]^T$ is an upper bound on $A_0^{-1}(\frac{\partial F}{\partial \epsilon}(x) - \Gamma)$. Since $A^\dagger = (I - \epsilon A_0^{-1}A_1)A_0^{-1}$ and $I_3 + \epsilon\overline{A_0^{-1}A_1}$ is an upper bound on $I - \epsilon A_0^{-1}A_1$, the result follows from Lemma 3.6.2. □

We combine Lemmas 3.6.1 and 3.6.2 into an upper bound on $A^\dagger\frac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon)$.

Lemma 3.6.4. *Define $\mathcal{Q}_\epsilon^0, \mathcal{Q}_\epsilon \in \mathbb{R}_+^3$ as follows:*

$$\begin{aligned}
\mathcal{Q}_\epsilon^0 &:= \left[\frac{2}{5}\left(\frac{3\pi}{2} - 1\right)\epsilon, \frac{2}{5}\epsilon, \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{10}}\epsilon + \frac{18}{5\sqrt{50}}\epsilon^2\right]^T, \\
\mathcal{Q}_\epsilon &:= \mathcal{Q}_\epsilon^0 + (I_3 + \epsilon\overline{A_0^{-1}A_1})\cdot\left[(1+\tfrac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \tfrac{2}{\pi}\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}\right]^T.
\end{aligned} \tag{3.51}$$

Then the vector $\mathcal{Q}_\epsilon \in \mathbb{R}_+^3$ is an upper bound on $A^\dagger\frac{\partial F}{\partial \epsilon}(x)$ for any $x \in B_\epsilon(r,\rho)$.

Proof. It follows from Lemma 3.6.1 that the vector $\mathcal{Q}_\epsilon^0$ is an upper bound on $A^\dagger\Gamma$ (for example, the third component of $\mathcal{Q}_\epsilon^0$ is a bound on $\|c'\|$). It follows from Corollary 3.6.3 that

$$(I_3 + \epsilon\overline{A_0^{-1}A_1})\cdot[(1+\tfrac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \tfrac{2}{\pi}\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}]^T$$

is an upper bound on $A^\dagger(\frac{\partial F}{\partial \epsilon}(x) - \Gamma)$. We conclude from the triangle inequality that $\mathcal{Q}_\epsilon$ is an upper bound on $A^\dagger\frac{\partial F}{\partial \epsilon}(x)$. □

Finally, we prove the bounds needed to control the derivative $\frac{d}{d\epsilon}\hat{\alpha}_\epsilon$ in Section 2.3.2 (in particular the implicit differentiation argument in Theorem 2.3.8).

Lemma 3.6.5. *Fix $\epsilon_0 > 0, \check{r} \in \mathbb{R}^3_+$ and $\rho > 0$ as in the hypothesis of Proposition 2.2.15. Let $0 < \epsilon \leq \epsilon_0$ and let $\hat{x}_\epsilon \in B_\epsilon(\epsilon^2 \check{r}, \rho)$ denote the unique solution to $F(x) = 0$. Recall the definitions of $\mathcal{Z}_\epsilon \in \mathrm{Mat}(\mathbb{R}^3_+, \mathbb{R}^3_+)$ and $\mathcal{Q}_\epsilon \in \mathbb{R}^3_+$ in Equations (2.27) and (3.51). Define*

$$M_\epsilon := \frac{1}{\epsilon^2}\left((I_3 + \epsilon\overline{A_0^{-1}A_1}) \cdot \left[(1+\tfrac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \tfrac{2}{\pi}\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}\right]^T\right)_1,$$
$$M'_\epsilon := \frac{1}{\epsilon^2}\left(\mathcal{Z}_\epsilon(I_3 - \mathcal{Z}_\epsilon)^{-1}\mathcal{Q}_\epsilon\right)_1,$$

where the subscript denotes the first component of the vector. Then M_ϵ and M'_ϵ are positive, increasing in ϵ, and satisfy the inequalities

$$\left|\pi_\alpha A^\dagger\left(\tfrac{\partial F}{\partial \epsilon}(\hat{x}_\epsilon) - \Gamma_\epsilon\right)\right| \leq \epsilon^2 M_\epsilon, \tag{3.52}$$
$$\left(\mathcal{Z}_\epsilon(I_3 - \mathcal{Z}_\epsilon)^{-1}\mathcal{Q}_\epsilon\right)_1 \leq \epsilon^2 M'_\epsilon. \tag{3.53}$$

Proof. To first show that $(I_3 - \mathcal{Z}_\epsilon)^{-1}$ is well defined, we note that by Proposition 2.2.15 the radii polynomials $P(\epsilon, \epsilon^2\check{r}, \rho)$ are all negative. As was shown in the proof of Theorem 2.2.7, the operator norm of $\mathcal{Z}_\epsilon$ on $\mathbb{R}^3$ equipped with the norm $\|\cdot\|_{\epsilon^2\check{r}}$ is given by some $\kappa < 1$, whereby the Neumann series of $(I_3 - \mathcal{Z}_\epsilon)^{-1}$ converges.

From the definition of M_ϵ and Corollary 3.6.3, inequality (3.52) follows. Inequality (3.53) is a direct consequence of the definition of M'_ϵ. Since the functions $\hat{f}_{\epsilon,1}$ and $\hat{f}_{\epsilon,c}$ are positive, then M_ϵ and $\mathcal{Q}_\epsilon$ are positive. Since the matrix $\mathcal{Z}_\epsilon$ has positive entries only, the Neumann series for $(I_3 - \mathcal{Z}_\epsilon)^{-1}$ has summands with exclusively positive entries, whereby M'_ϵ is positive.

Next we show that the components of $\mathcal{Z}_\epsilon$ and $\mathcal{Q}_\epsilon - \mathcal{Q}^0_\epsilon$ are polynomials in ϵ with positive coefficients and their lowest degree terms are at least quadratic. To do so, it suffices to prove as much for the functions $\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}, f_{1,\alpha}, f_{1,\omega}, f_{1,c}, f_{*,\alpha}, f_{*,\omega}, f_{*,c}$. We note that all of these functions are given as polynomials with positive coefficients in the variables $\epsilon, \Delta_\alpha, \Delta_\omega, \delta_c, r_c, \delta^0_c$ (recall that ρ is fixed and does not vary with ϵ). Since $(r_\alpha, r_\omega, r_c) = \epsilon^2(\check{r}_\alpha, \check{r}_\omega, \check{r}_c)$, then by Definition 3.2.1 the terms $\Delta_\alpha, \Delta_\omega, r_c$ are all $\mathcal{O}(\epsilon^2)$. Furthermore, whenever any of the terms $\epsilon, \delta_c, \delta^0_c$ appears, it is multiplied by another term of order at least $\mathcal{O}(\epsilon)$. It follows that every component of $\mathcal{Z}_\epsilon$ and $\mathcal{Q}_\epsilon - \mathcal{Q}_\epsilon^0$ is a polynomial in ϵ with positive coefficients for which the lowest degree term is at least quadratic.

From these considerations it follows that the components of both $M_\epsilon = \epsilon^{-2}(\mathcal{Q}_\epsilon - \mathcal{Q}_\epsilon^0)_1$ and $\epsilon^{-2}\mathcal{Z}_\epsilon$ are polynomials in ϵ with positive coefficients. It also follows that both $\mathcal{Q}_\epsilon$ and $(I_3 - \mathcal{Z}_\epsilon)^{-1}$ are increasing in ϵ, whereby M'_ϵ is increasing in ϵ. $\square$

As a trivial extension, we have the following corollary:

Corollary 3.6.6. *Fix* $\epsilon_0 > 0, \check{r} \in \mathbb{R}^3_+$ *and* $\rho > 0$ *as in the hypothesis of Proposition 2.2.15. Let* $0 < \epsilon \leq \epsilon_0$ *and let* $\hat{x}_\epsilon \in B_\epsilon(\epsilon^2\check{r}, \rho)$ *denote the unique solution to* $F(x) = 0$. *Recall the definitions of* $\mathcal{Z}_\epsilon \in \mathrm{Mat}(\mathbb{R}^3_+, \mathbb{R}^3_+)$ *and* $\mathcal{Q}_\epsilon \in \mathbb{R}^3_+$ *in Equations* (2.27) *and* (3.51). *Define*

$$\tilde{M}_\epsilon := \frac{1}{\epsilon^2}\left((I_3 + \epsilon\overline{A_0^{-1}A_1}) \cdot \left[(1 + \tfrac{4}{\pi^2})^{1/2}\hat{f}_{\epsilon,1}, \tfrac{2}{\pi}\hat{f}_{\epsilon,1}, \hat{f}_{\epsilon,c}\right]^T\right)_2,$$
$$\tilde{M}'_\epsilon := \frac{1}{\epsilon^2}\left(\mathcal{Z}_\epsilon(I_3 - \mathcal{Z}_\epsilon)^{-1}\mathcal{Q}_\epsilon\right)_2,$$

where the subscript denotes the second component of the vector. Then $\tilde{M}_\epsilon$ *and* $\tilde{M}'_\epsilon$ *are positive, increasing in* ϵ, *and satisfy the inequalities*

$$\left|\pi_\omega A^\dagger\left(\tfrac{\partial F}{\partial\epsilon}(\hat{x}_\epsilon) - \Gamma_\epsilon\right)\right| \leq \epsilon^2\tilde{M}_\epsilon, \tag{3.54}$$

$$\left(\mathcal{Z}_\epsilon(I_3 - \mathcal{Z}_\epsilon)^{-1}\mathcal{Q}_\epsilon\right)_2 \leq \epsilon^2\tilde{M}'_\epsilon. \tag{3.55}$$

Chapter 4

Computationally Characterizing SOPS and their Stability

In Chapters 4 and 5 we use the global optimization technique of *branch and bound* to prove global results in Wright's equation. This technique is analogous to the mathematical method of proof whereby one divides a problem into different cases, and analyzes each case individually. While the branch and bound method is prototypically used for finding the global maximum of a function, it can be applied in a multitude of situations where local analysis needs to be stitched together into a global picture.

4.1 Background

In this chapter, we combine the branch and bound approach with a rigorous numerical integrator to derive pointwise bounds on *all* SOPS to (1.1) at a given parameter range. We then calculate bounds on the Floquet multipliers of SOPS, hoping to prove that any SOPS that could exist is asymptotically stable. If successful, we can use the following theorem to prove uniqueness:

Theorem 4.1.1 (See [Xie91, Xie93])**.** *If $\alpha > \frac{\pi}{2}$ and every SOPS to* (1.1) *is asymptotically stable, then* (1.1) *has a unique SOPS up to a time translation.*

Following Xie's approach, we define the function space

$$\mathcal{X} := \left\{ x \in C^1(\mathbb{R}, \mathbb{R}) \mid x(0) = 0, x'(0) > 0 \text{ and } x(t) < 0 \text{ for } t \in (-1, 0) \right\}.$$

Up to a time translation, the space $\mathcal{X}$ contains all SOPS to Wright's equation. Xie showed that if $x \in \mathcal{X}$ is a SOPS to Wright's equation with period L, then its nontrivial Floquet multipliers $\lambda \in \mathbb{C}$ are given by solutions to the nonautonomous linear DDE:

$$y'(t) = -\alpha f'(x(t-1))y(t-1) \tag{4.1}$$

subject to the boundary condition

$$\lambda y(s) = -y(L)\frac{x'(s+L)}{x'(L)} + y(s+L), \qquad s \in [-1,0]. \tag{4.2}$$

For a SOPS $x \in \mathcal{X}$, showing that $|\lambda| < 1$ for all possible solutions y to (4.1) and (4.2) it suffices to show that x is asymptotically stable. By doing so for all possible SOPS to Wright's equation when $\alpha \geq 5.67$, Xie achieved his proof for uniqueness. Xie's method has two parts: (1) obtain estimates on SOPS to Wright's equation and (2) use these estimates to develop an upper bound on the magnitude of their Floquet multipliers. Xie was only able to obtain a proof for $\alpha \geq 5.67$ because of the difficulty of the first part. In this chapter we continue Xie's method by means of a computer-assisted proof.

Our approach to obtaining bounds on SOPS is based on an algorithmic case-by-case analysis of the locations of the zeros of a function $x \in \mathcal{X}$ and the size of its extrema. In [Wri55, Lemmas 4 and 5] it is shown that if $x \in \mathcal{X}$ and $\alpha > 1$ then the zeros $\{z_i(x)\}_{i=0}^{\infty}$ of x are countably infinite and $z_{i+1}(x) - z_i(x) > 1$. This result implies that we can define the maps $q : \mathcal{X} \to (1,\infty)$ and $\bar{q} : \mathcal{X} \to (1,\infty)$ as follows given $x \in \mathcal{X}$:

$$\begin{aligned} q(x) &:= z_1(x) - z_0(x), \\ \bar{q}(x) &:= z_2(x) - z_1(x). \end{aligned}$$

By construction, if $x \in \mathcal{X}$, then its first zero is $z_0(x) = 0$. Moreover, if x is a SOPS then $q(x) + \bar{q}(x)$ is its period and furthermore if it solves (1.1), then its extrema are given as

$$\begin{aligned} \max_{t\in\mathbb{R}} x(t) &= x(1), \\ \min_{t\in\mathbb{R}} x(t) &= x\left(q(x)+1\right). \end{aligned}$$

In [BCKN14] a branch and bound algorithm is applied to the 2-dimensional domain $\{\max x, \min x\}$ to show that there do not exist any SOPS to Wright's equation for $\alpha \leq 1.5706$, making substantial progress on Wright's conjecture that the origin is the global attractor to (1.2) for $\alpha < \frac{\pi}{2} \approx 1.57079$. Without an exact value for $q(x)$, one cannot pinpoint the location of the minimum of x. To account for this ambiguity, the authors in [BCKN14] use a collection of six different functions to bound x, each defined relative to one of the zeros $\{z_0(x), z_1(x), z_2(x)\}$. We use an alternative approach that allows us to work with just two functions. In particular, we classify the space $\mathcal{X}$ according to the finite dimensional reduction map $\kappa : \mathcal{X} \to \mathbb{R}^3$ defined as follows:

$$\kappa(x) := \{q(x), \bar{q}(x), x(1)\}. \tag{4.3}$$

Relative to a SOPS's image under κ, we formally define bounding functions as follows.

Definition 4.1.2. *Fix an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ and a region $K \subset \mathbb{R}^3$. The functions $\ell_K, u_K : \mathbb{R} \to \mathbb{R}$ are* bounding functions *(associated with K) if*

$$\ell_K(t) \leq x(t) \leq u_K(t), \quad \textit{for all } t \in \mathbb{R},$$

whenever $x \in \mathcal{X}$ is a SOPS to Wright's equation at a parameter $\alpha \in I_\alpha$ satisfying $\kappa(x) \in K$.

In practice, we define the functions u_K, ℓ_K as piecewise constant functions, which are easy to represent and rigorously integrate on a computer. To ensure proper mathematical rigor and computational reliability, we have used interval arithmetic for the execution of our computer-assisted proofs [Rum99, MKC09]. Notably, our algorithms use a rigorous numerical integrator for delay differential equations, about which there is a growing literature [BCKN14, MN10, Szc14, SZ16]. These computational details are discussed further in Section 4.7.

To summarize Theorem 4.1.1, in order to prove that there is a unique SOPS, it is sufficient to show that every SOPS is asymptotically stable. This breaks into two major parts: (i) characterizing SOPS to Wright's equation, and (ii) bounding their Floquet multipliers. To accomplish the first part, we begin by constructing compact regions $K_1, K_2 \subset \mathbb{R}^3$, described in Algorithm 4.3.5 and Algorithm 4.3.8 respectively, for which $K_1 \cup K_2$ contains the κ-image of all SOPS to Wright's equation. We then use a *branch and prune* method, defined in Algorithm 4.5.1, to refine these initial global bounds. This algorithm *branches* by subdividing $K_1 \cup K_2$ into smaller pieces, and *prunes* by using Algorithm 4.2.2 to develop tighter bounding functions. The end result of this process is a collection $\mathcal{A}$ of subsets of $K \subset \mathbb{R}^3$, and in Theorem 4.5.2 we prove for a given parameter range $[\alpha_{min}, \alpha_{max}]$ that if $x \in \mathcal{X}$ is a SOPS then $\kappa(x) \in \bigcup_{K \in \mathcal{A}} K$.

The task then becomes to show that every SOPS is asymptotically stable. For a given region $K \subset \mathbb{R}^3$, we use Algorithm 4.4.2 to derive a bound on the Floquet multipliers of any SOPS with κ-image contained in K. This is then combined with the branch and prune method in Algorithm 4.5.3. Finally, the proof to Theorem 1.2.2 is given in Section 6, where, in addition, we discuss the computational limitations of our approach.

4.2 A computational approach

Theorem 4.1.1 effectively transforms Jones' Conjecture (Conjecture 1.1.3) into the problem of studying the asymptotic dynamics of SOPS, and in turn, their Floquet multipliers. In a neighborhood about a periodic function, one can develop estimates on these Floquet multipliers [CL13, Xie91]. However these bounds rely significantly on this neighborhood about the periodic function being relatively small. In effect, Xie shows that any SOPS to Wright's equation is stable for each $\alpha \geq 5.67$ by first showing that all such solutions reside within a narrow region, and subsequently shows that all periodic orbits in that region are asymptotically stable. This first step is the more difficult part, and the reason Xie restricts his proof to $\alpha \geq 5.67$.

In Xie's thesis [Xie91] a case-by-case analysis is used to obtain a region within which all SOPS must lie. Specifically, if $\bar{q}(x) \geq 3$ then asymptotic analysis [Nus82] precisely describes the approximate form of the SOPS with tight error estimates. For the alternative case, Xie divided the possibility of $\bar{q}(x) < 3$ into several sub-cases and showed that each of these led to a contradiction when $\alpha \geq 5.67$. In our analysis we make similar assumptions by considering a SOPS's image under the map $\kappa(x) = \{q(x), \bar{q}(x), x(1)\}$ and the bounding functions associated with various regions $K \subset \mathbb{R}^3$. For any region $K \subset \mathbb{R}^3$ there is not a unique choice of bounding functions. In fact, we develop techniques which iteratively tighten the bounding functions for a fixed region K. If in our process of tightening bounding functions we derive a contradiction, such as $\ell_K(t) > u_K(t)$, then we may conclude that there does not exist any SOPS x for which $\kappa(x) \in K$.

In performing a case-by-case analysis of SOPS to Wright's equation, we are principally concerned with bounding all possible SOPS, and we find it useful to introduce the notion of an I_α-exhaustive set.

Definition 4.2.1. *Fix an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ and consider a set $K \subset \mathbb{R}^3$. The set K is I_α-*exhaustive *if $\kappa(x) \in K$ for any SOPS $x \in \mathcal{X}$ to Wright's equation at parameter $\alpha \in I_\alpha$.*

To derive a sufficiently small I_α-exhaustive set we employ techniques from global optimization theory. Specifically, we use a branch and prune algorithm which is derived from the classical global optimization technique of branch and bound [Sch11, RR88, HT13]. Our branch and prune is designed so that it will output an I_α-exhaustive set, a result proved in Theorem 4.5.2.

The branch and prune algorithm begins with an initial finite set $\mathcal{S} = \{K_i : K_i \subset \mathbb{R}^3\}$ for which $\bigcup_{K \in \mathcal{S}} K$ is I_α-exhaustive. The construction of this initial set is described in Section 4.3, specifically in Algorithms 4.3.5 and 4.3.8. We then alternate between branching and pruning the elements of $\mathcal{S}$. The branching subroutine divides an element $K \in \mathcal{S}$ into two pieces K_A and K_B for which $K = K_A \cup K_B$, and then replaces K in the set $\mathcal{S}$ by the two smaller regions. The pruning algorithm uses a variety of techniques to derive sharper bounding functions on the region K. Furthermore, if we can prove that the preimage $\kappa^{-1}(K) \subseteq \mathcal{X}$ cannot contain any SOPS, then we remove the region K from the set $\mathcal{S}$. The branch and prune algorithm terminates when the diameter of every region K is less than some preset constant.

In contrast to the prototypical optimization problem of bounding the minimum of an objective function, we are concerned with characterizing SOPS to Wright's equation. In particular, our pruning algorithm is designed to tighten the bounding functions associated with a region K, reduce the size of K, and to discard the region if we can prove that $\kappa^{-1}(K)$ does not contain any SOPS. The algorithm takes as input an interval I_α, a region K, and a pair of bounding functions u_K, ℓ_K. As output the algorithm produces a region $K' \subset K$ and a pair of bounding functions $u_{K'}, \ell_{K'}$. The set K is taken to be rectangular, that is $K = I_q \times I_{\bar{q}} \times I_M$ where $I_q = [q_{min}, q_{max}]$ and

$I_{\bar{q}} = [\bar{q}_{min}, \bar{q}_{max}]$ and $I_M = [M_{min}, M_{max}]$. Additionally, this algorithm takes as input a computational parameter $n_{Time} \in \mathbb{N}$ relating to how we store the bounding functions u_K, ℓ_K on the computer (see Section 4.7).

The six steps in the pruning algorithm (Algorithm 4.2.2) are independent of one another and can be implemented in any order. In Steps 1-4 we describe how to tighten the bounds on K, u_K and ℓ_K (see Figure 4.3). Each step is constructed so that the output does not worsen the existing bounds. That is each step of the algorithm produces an output for which $K' \subseteq K$ and the inequalities $u_{K'} \leq u_K$ and $\ell_{K'} \geq \ell_K$ hold. At the end of each step we update our input so that we use the improved bounds in the next step. That is, we define:

$$K := K' \qquad\qquad u_K := u_{K'} \qquad\qquad \ell_K := \ell_{K'} \tag{4.4}$$

and subsequently modify K', $u_{K'}$ and $\ell_{K'}$ as described in each individual step. In Steps 5-6, we check conditions which would imply that the region K cannot contain the κ-image of SOPS to Wright's equation. If this is the case, the algorithm returns $K = \emptyset$.

Algorithm 4.2.2 (Pruning Algorithm)**.** *This algorithm takes as input* $I_\alpha = [\alpha_{min}, \alpha_{max}]$, $K = [q_{min}, q_{max}] \times [\bar{q}_{min}, \bar{q}_{max}] \times [M_{min}, M_{max}] \subseteq \mathbb{R}^3$ *and associated bounding functions* ℓ_K *and* u_K*, as well as the computational parameter* $n_{Time} \in \mathbb{N}$*. The outputs consist of a region* $K' \subseteq \mathbb{R}^3$ *and associated bounding functions* $\ell_{K'}$ *and* $u_{K'}$*.*

Define $I_q := [q_{min}, q_{max}]$*,* $I_{\bar{q}} := [\bar{q}_{min}, \bar{q}_{max}]$ *and* $I_M := [M_{min}, M_{max}]$ *as well as* $L_{min} := q_{min} + \bar{q}_{min}$*,* $L_{max} := q_{max} + \bar{q}_{max}$ *and* $I_L := [L_{min}, L_{max}]$*.*

1. *We tighten the bounding functions associated with the region K using*

$$u_{K'}(t) := \begin{cases} \min\{M_{max}, u_K(1)\} & \textit{if } t = 1 \\ \min\{0, u_K(t)\} & \textit{if } t \in [-\bar{q}_{min}, 0] \cup [q_{max}, L_{min}] \\ u_K(t) & \textit{otherwise} \end{cases} \tag{4.5}$$

$$\ell_{K'}(t) := \begin{cases} \max\{M_{min}, \ell_K(1)\} & \textit{if } t = 1 \\ \max\{0, \ell_K(t)\} & \textit{if } t \in [-L_{min}, \bar{q}_{max}] \cup [0, q_{min}] \\ & \qquad \cup [L_{max}, L_{min} + q_{min}] \\ \ell_K(t) & \textit{otherwise.} \end{cases} \tag{4.6}$$

Lastly we update our bounds using Line (4.4)*.*

2. *If x satisfies Wright's equation we can use variation of parameters to refine the bounding functions. For our computational parameter $n_{Time} \in \mathbb{N}$, we base our calculation about a collection of points separated by a uniform distance of $1/n_{Time}$. That is, define $\Delta = 1/n_{Time}$ and $I_\Delta := [0, \Delta]$, and fix $t_0 \in \{k \cdot \Delta\}_{k \in \mathbb{Z}}$ and $s \in I_\Delta$. We may refine the values of $u_K(t_0 + s)$, $u_K(t_0 - s)$, $\ell_K(t_0 + s)$, $\ell_K(t_0 - s)$, temporarily defining functions $u_{K''}, \ell_{K''}$ as follows:*

$$\begin{aligned} u_{K''}(t_0 + s) &:= u_K(t_0) + s \cdot \sup_{\alpha \in I_\alpha, r \in I_\Delta} \sup_{\ell_K \le x \le u_K} -\alpha \left(e^{x(t_0 - 1 + r)} - 1 \right) \\ u_{K''}(t_0 - s) &:= u_K(t_0) - s \cdot \inf_{\alpha \in I_\alpha, r \in I_\Delta} \inf_{\ell_K \le x \le u_K} -\alpha \left(e^{x(t_0 - 1 - r)} - 1 \right) \\ \ell_{K''}(t_0 + s) &:= \ell_K(t_0) + s \cdot \inf_{\alpha \in I_\alpha, r \in I_\Delta} \inf_{\ell_K \le x \le u_K} -\alpha \left(e^{x(t_0 - 1 + r)} - 1 \right) \\ \ell_{K''}(t_0 - s) &:= \ell_K(t_0) - s \cdot \sup_{\alpha \in I_\alpha, r \in I_\Delta} \sup_{\ell_K \le x \le u_K} -\alpha \left(e^{x(t_0 - 1 - r)} - 1 \right), \end{aligned} \tag{4.7}$$

and then defining:

$$\begin{aligned} u_{K'}(t_0 + s) &:= \min\{u_K(t_0 + s), u_{K''}(t_0 + s)\} \\ u_{K'}(t_0 - s) &:= \min\{u_K(t_0 - s), u_{K''}(t_0 - s)\} \\ \ell_{K'}(t_0 + s) &:= \max\{\ell_K(t_0 + s), \ell_{K''}(t_0 + s)\} \\ \ell_{K'}(t_0 - s) &:= \max\{\ell_K(t_0 - s), \ell_{K''}(t_0 - s)\}. \end{aligned}$$

Section 4.7 explains in further detail the computational aspects of this step. Lastly we update our bounds using Line (4.4).

3. *In this step we refine our bounds on* I_q *and* I_M *using* u_K *and* ℓ_K*. At* $t = q(x)$ *the function* $x(t)$ *changes sign from positive to negative. We sharpen the bounds on* I_q *by defining:*

$$\begin{aligned} q'_{min} &:= \inf\{t \in I_q : \ell_K(t) \leq 0\} \\ q'_{max} &:= \sup\{t \in I_q : u_K(t) \geq 0\} \\ I_{q'} &:= [q'_{min}, q'_{max}]. \end{aligned} \tag{4.8}$$

Additionally we make the following refinement:

$$I_{M'} := [M_{min}, M_{max}] \cap [\ell_K(1), u_K(1)].$$

Lastly we define $K' := I_{q'} \times I_{\bar{q}} \times I_{M'}$ *and update our bounds using Line* (4.4).

4. *If* $x \in \mathcal{X}$ *is a SOPS with period* $L \in I_L$*, then* $x(t) = x(t + L)$*. Using this relation, we make the following refinement:*

$$\ell_{K'}(t) := \max\left\{\ell_K(t), \min_{L' \in I_L} \ell_K(t + L')\right\} \tag{4.9}$$

$$u_{K'}(t) := \min\left\{u_K(t), \max_{L' \in I_L} u_K(t + L')\right\}. \tag{4.10}$$

Lastly we update our bounds using Line (4.4) *as appropriate.*

5. *If there is some point* $t \in \mathbb{R}$ *for which* $\ell_K(t) > u_K(t)$ *then* RETURN $K' := \emptyset$.

6. *If* $\min_{t \in I_q} \ell_K(t+1) > -\log \frac{\alpha_{min}}{\pi/2}$*, then* RETURN $K' := \emptyset$.

Proposition 4.2.3. *Let* $I_\alpha = [\alpha_{min}, \alpha_{max}]$*,* $\alpha_{min} \geq \frac{\pi}{2}$*,*

$$K = [q_{min}, q_{max}] \times [\bar{q}_{min}, \bar{q}_{max}] \times [M_{min}, M_{max}] \subseteq \mathbb{R}^3$$

and u_K, ℓ_K *be input for Algorithm 4.2.2 with any computational parameter* $n_{Time} \in \mathbb{N}$*. Suppose that* $x \in \mathcal{X}$ *is a SOPS at parameter* $\alpha \in I_\alpha$*, and let* $\{K', u_{K'}, \ell_{K'}\}$ *be the result of Algorithm 4.2.2. If* $\kappa(x) \in K$*, then* $\kappa(x) \in K'$ *and* $\ell_{K'} \leq x \leq u_{K'}$*.*

Proof. We prove that Proposition 4.2.3 holds for each step of the algorithm individually. Given an interval $I_\alpha \subset \mathbb{R}$, let $x \in \mathcal{X}$ be a SOPS at parameter $\alpha \in I_\alpha$.

1. Recall $\kappa(x) = \{q(x), \bar{q}(x), \max(x)\}$. Since $\max_{t\in\mathbb{R}} x = x(1)$ then the refinements in (4.5) and (4.6) for the case in which $t = 1$ are appropriate. For the other two refinements in each equation note that by definition a function $x \in \mathcal{X}$ is non-negative on the interval $[0, q(x)]$ and non-positive on the interval $[q(x), q(x) + \bar{q}(x)]$. Hence, x is non-negative on $[0, q_{min}]$ and non-positive on $[q_{max}, L_{min}]$. If x is a SOPS then it has period $L = q(x) + \bar{q}(x)$ and we may further conclude that it is non-negative on the intervals $[-L, -\bar{q}(x)]$ and $[L, L + q(x)]$, and non-positive on the interval $[-\bar{q}(x), 0]$. Hence, x is non-negative on $[-L_{min}, -\bar{q}_{max}]$ and $[L_{max}, L_{min} + q_{min}]$, and non-positive on $[-\bar{q}_{min}, 0]$. The refinements in (4.5) and (4.6) reflect these restrictions.

2. To estimate an upper bound on $x(t_0 + s)$, we apply variation of parameters to Wright's equation, obtaining

$$x(t_0 + s) = x(t_0) + \int_{t_0}^{t_0+s} -\alpha \left(e^{x(r-1)} - 1 \right) dr. \tag{4.11}$$

   Taking the Riemann upper sum of this integral with step size s, we deduce that $x(t_0+s)$ is bounded above by the RHS of (4.7). As $x(t_0+s) \leq u_K(t_0+s)$ it follows that $x(t_0 + s) \leq \min\{u_K(t_0 + s), u_{K''}(t_0 + s)\}$. The proofs for the refinements of $u_K(t_0 - s), \ell_K(t_0 + s), \ell_K(t_0 - s)$ follow with parity.

3. Let $x \in \mathcal{X}$ be such that $\kappa(x) \in K$. From our definitions of q'_{min} and q'_{max} it follows that

$$\begin{aligned} x(t) \geq \ell_K(t) > 0, &\quad \text{for all } t \in \left(q_{min}, q'_{min}\right), \\ x(t) \leq u_K(t) < 0, &\quad \text{for all } t \in \left(q'_{max}, q_{max}\right). \end{aligned}$$

   Hence it follows that $x(t) \neq 0$ for $t \in (q_{min}, q'_{min}) \cup (q'_{max}, q_{max})$. Since $q(x) \in I_q$, it must follow that $q(x) \in [q'_{min}, q'_{max}]$, thus justifying the refinement in (4.8). Regarding the refinement of I_M, as $\ell_K(1) \leq x(1) \leq u_K(1)$ it clearly follows that $[\kappa(x)]_3 = x(1) \in I_{M'}$.

4. If x is periodic with period L, then $x(t) = x(t+L)$. Since $L \in I_L$ then we may derive upper/lower bounds on $x(t+L)$ as follows:
$$\min_{L' \in I_L} \ell_K(t+L') \leq \min_{L' \in I_L} x(t+L') \leq x(t+L) \leq \max_{L' \in I_L} x(t+L') \leq \max_{L' \in I_L} u_K(t+L').$$
Hence it follows that $\min_{L' \in I_L} \ell_K(t+L') \leq \ell_K(t)$ and $u_K(t) \leq \max_{L' \in I_L} u_K(t+L')$, thus justifying our refinements in (4.9) and (4.10).

5. If $\ell_K(t) > u_K(t)$, then it is impossible for any $x \in \mathcal{X}$ to satisfy $\ell_K(t) \leq x(t) \leq u_K(t)$. Since u_K, and ℓ_K are bounding functions associated with K, this contradiction leads us to conclude that there cannot exist any SOPS $x \in \mathcal{X}$ for which $\kappa(x) \in K$.

6. By [Wal78, Corollary 1], if x is a SOPS to Wright's equation and $\alpha \geq \frac{\pi}{2}$, then
$$\min x \leq -\log \tfrac{\alpha}{\pi/2}. \tag{4.12}$$
If $x \in \mathcal{X}$ is a SOPS then $\min_{t \in \mathbb{R}} x(t) = x(q+1)$, whereby $\min_{t\in\mathbb{R}} x(t) > \min_{t \in I_q} \ell_k(t+1)$. Hence, if $\min_{t \in I_q} \ell_k(t+1) > -\log \frac{\alpha_{min}}{\pi/2}$ then (4.12) is violated, and so there cannot exist any SOPS $x \in \mathcal{X}$ for which $\kappa(x) \in K$. □

In Chapter 5 we apply Algorithm 4.2.2 to study SOPS for $\alpha \in (\frac{\pi}{2}, 1.9]$. However, the algorithm has great difficulty discarding low amplitude solutions near the Hopf bifurcation at $\alpha = \frac{\pi}{2}$. To remedy this, we use our estimates from Chapter 3 to modify Algorithm 4.2.2 with the addition of a seventh step given in Algorithm 4.2.4. This allows for a new way to potentially conclude that a given bounding function cannot contain any SOPS. When $I_\alpha = [\frac{\pi}{2}, 1.6]$ in the proof of Theorem 1.2.3, given in Section 5.5, this additional step reduces the output size of Algorithm 4.5.1 by 50% and reduces the runtime of Algorithm 4.5.1 by 40%. However, when $I_\alpha = [1.8, 1.9]$ it provides no discernible advantage.

Algorithm 4.2.4. *Append the following step to Algorithm 4.2.2.*

7. *Define the following:*
$$M := \sup_{t \in [0, L_{max}], \ell \leq x \leq u} \left| e^{x(t)} - 1 \right|$$
$$g(\alpha, \omega) := \sqrt{\left(1 - \tfrac{\omega}{\alpha}\right)^2 + 2\,\tfrac{\omega}{\alpha}\,(1 - \sin\omega)}.$$

If $M < -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\sqrt{3}\omega}{\pi\alpha} g(\alpha, \omega)}$, *then RETURN* $K' = \emptyset$.

Proposition 4.2.5. *If* y *is a nontrivial periodic solution to* (1.2) *at parameter* $\alpha \in (0, 2]$ *and frequency* $\omega \geq 1.1$, *then:*

$$\sup |y(t)| > -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{4\sqrt{3}\omega}{\pi\alpha} g(\alpha, \omega)}.$$

Proof. Define $M := \sup |y(t)|$. From Lemma 2.3.1, with G defined as in (2.7), we know that if $G(\alpha, \omega, c) = 0$, then:

$$\|c\|_{\ell^1} \leq \frac{\pi}{\omega\sqrt{3}} \|y'\|_\infty \leq \frac{\pi}{\omega\sqrt{3}} \alpha M(1 + M).$$

From Propositions 3.5.1 and 3.5.2 (see also Lemma 5.1.7), the only solutions satisfying $\|c\|_{\ell^1} < g(\alpha, \omega)$ are trivial. Hence (α, ω, c) would only be a trivial solution at best if the following inequality is satisfied:

$$\|c\|_{\ell^1} \leq \frac{\pi}{\omega\sqrt{3}} \alpha M(1 + M) < g(\alpha, \omega).$$

Solving the quadratic equation $M^2 + M - \frac{\omega\sqrt{3}}{\pi\alpha} g(\alpha, \omega) < 0$ produces the desired inequality. □

4.3 Initial Bounds on SOPS to Wright's Equation

In order to apply the branch and prune algorithm, we must first construct an initial I_α-exhaustive set. Due to the sustained interest in Wright's equation, there are considerable *a priori* estimates we can employ to describe slowly oscillating solutions [Jon62b, Wri55, Nus82]. Since considerably sharper estimates are obtained under the assumption $\bar{q} \geq 3$, we will construct two regions K_1 and K_2 corresponding to SOPS $x \in \mathcal{X}$ for which $\bar{q}(x) \leq 3$ and $\bar{q}(x) \geq 3$ respectively. Taken together $K_1 \cup K_2$ will form an I_α-exhaustive set, which we prove in Corollary 4.3.12.

While sharper estimates are available for additional sub-cases [Nus82], we present a collection of these estimates we have found sufficient for our purposes. Note that these lemmas are not a verbatim reproduction. We have translated results applicable to the quadratic form of Wright's equation given in (1.2) so that they apply to the exponential form of Wright's equation given in (1.1).

Lemma 4.3.1 (See [Wri55])**.** *Let $x \in \mathcal{X}$ be a solution to* (1.1) *at parameter $\alpha > 0$. Then*

$$-\alpha(e^{\alpha} - 1) \leq x(t) \leq \alpha$$

for all $t > 0$.

Lemma 4.3.2 (See [Jon62b, Theorem 3.1])**.** *Let $\alpha > e^{-1}$ and suppose that $x \in \mathcal{X}$ is a solution to* (1.1)*. We construct a sequence of functions $p_i : (-\infty, 1] \to \mathbb{R}$ for $i = 1, 2, \cdots$ by setting $p_1(t) = \alpha t$ and recursively defining:*

$$p_{i+1}(t) := -\alpha \int_0^t \left(e^{p_i(s-1)} - 1 \right) ds.$$

For example $p_2(t) = \alpha t + e^{-\alpha} - e^{\alpha(t-1)}$. Then $x(t) > p_i(t)$ for $t < 0$, and $x(t) < p_i(t)$ for $t \in (0, 1]$. Furthermore $x(t) < p_i(1)$ for all $t \geq 0$. Additionally $|p_i(t)|$ is increasing in α.

Lemma 4.3.3 (See [Jon62b, Theorem 3.4])**.** *Suppose $x \in \mathcal{X}$ is a solution to* (1.1)*, suppose that $\bar{q} \geq 3$ and that $\alpha \geq \frac{\pi}{2}$. Define $a_1(\alpha) = -(\alpha - 1)$ and the recursive relation $a_{i+1}(\alpha) = \alpha(e^{a_i(\alpha)} - 1)$. Then $x(t) < -t \cdot a_i(\alpha)$ for $t \in [-1, 0)$ and $i \in \mathbb{N}$.*

Lemma 4.3.4. *Suppose that $x \in \mathcal{X}$ is a SOPS to* (1.1)*. If $\alpha \geq \frac{\pi}{2}$ then*

$$\begin{array}{rcccl} 1 + \frac{1}{\alpha}\left(\frac{\alpha + e^{-\alpha} - 1}{\exp\{\alpha + e^{-\alpha} - 1\} - 1} \right) & < & q & < & 2 + \frac{1}{\alpha} \\ 1 + \frac{1}{\alpha} & < & \bar{q} & < & \max\{3, 2 + |\frac{e^{\alpha} - 1}{e^{a_i(\alpha)} - 1}|\} \end{array}$$

where $a_i(\alpha)$ is taken as in Lemma 4.3.3. Additionally, if $\alpha \geq 2$ then $q < 2$.

Proof. The upper bound on q follows from Lemma 2.3.3, and everything else follows from [Jon62b, Theorem 3.5]. □

Below in Algorithm 4.3.5 we construct the initial bounds for a region $K \subseteq \mathbb{R}^3$ containing the κ-image of SOPS $x \in \mathcal{X}$ for which $\bar{q}(x) \leq 3$.

Algorithm 4.3.5. *The input we take is an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ and computational parameters $i_0, n_{Time} \in \mathbb{N}$. The output is a rectangle $K = I_q \times I_{\bar{q}} \times I_M \subseteq \mathbb{R}^3$ and bounding functions u_K, ℓ_K.*

1. *Make the following definitions:*

$$
\begin{aligned}
q_{min} &:= 1 + \inf_{\alpha \in I_\alpha} \tfrac{1}{\alpha} \left(\tfrac{\alpha + e^{-\alpha} - 1}{\exp\{\alpha + e^{-\alpha} - 1\} - 1} \right) \\
M_{min} &:= \inf_{\alpha \in I_\alpha} \log \left(1 + \alpha^{-1} \log \tfrac{\alpha}{\pi/2} \right) \\
I_q &:= \begin{cases} [q_{min}, 2] & \text{if } \alpha_{min} \geq 2 \\ \left[q_{min}, 2 + \tfrac{1}{\alpha_{min}} \right] & \text{otherwise} \end{cases} \\
I_{\bar{q}} &:= \left[1 + \tfrac{1}{\alpha_{max}}, 3 \right] \\
I_M &:= [M_{min}, p_{i_0}(1)] \,.
\end{aligned}
$$

2. *For p_i given as in Propositions 4.3.2, define bounding functions*

$$
\ell_K(t) := \begin{cases} 0 & \text{if } t = 0 \\ -\alpha_{max}(e^{\alpha_{max}} - 1) & \text{otherwise} \end{cases} \qquad u_K(t) := \begin{cases} 0 & \text{if } t = 0 \\ p_{i_0}(1) & \text{otherwise.} \end{cases}
$$

   These bounding functions are stored on the computer with time resolution n_{Time} as described in Section 4.7.

Proposition 4.3.6. *Fix an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ such that $\alpha_{min} \geq \frac{\pi}{2}$. Let K, u_K, ℓ_K denote the output of Algorithm 4.3.5. If $x \in \mathcal{X}$ is a SOPS to* (1.1) *at parameter $\alpha \in I_\alpha$ and $\bar{q}(x) \leq 3$ then $\kappa(x) \in K$ and $\ell_K \leq x \leq u_K$.*

Proof. We treat the two steps in order.

1. If $x \in \mathcal{X}$ is a SOPS to (1.1) and $\bar{q}(x) \leq 3$, then by Lemma 4.3.4 it follows that $q(x) \in I_q$ and $\bar{q}(x) \in I_{\bar{q}}$. By Proposition 4.3.2 it follows that $x(1) \leq p_{i_0}(1)$. If x is a SOPS to Wright's equation with $\alpha \geq \pi/2$ then $\min x \leq -\log \frac{2\alpha}{\pi}$ [Wal78]. If $\max x < \log\left(1 + \alpha^{-1} \log \frac{\alpha}{\pi/2}\right)$ then by integrating Wright's equation forward from $t = q$ to $t = q+1$ it follows that $x(q+1) = \min x \leq -\log \frac{\alpha}{\pi/2}$, a contradiction. Hence we may assume that $x(1) \geq \log\left(1 + \alpha^{-1} \log \frac{\alpha}{\pi/2}\right) \geq M_{min}$.

2. Since $x \in \mathcal{X}$ then $x(0) = 0$, and by Lemma 4.3.1 and Proposition 4.3.2 it follows that $-\alpha(e^\alpha - 1) \leq x \leq p_{i_0}(1)$ for any SOPS $x \in \mathcal{X}$. Hence ℓ_K and u_K are bounding functions for $K = I_q \times I_{\bar{q}} \times I_M$. $\square$

To construct the initial bounds for the case $\bar{q}(x) \geq 3$, we make greater use of *a priori* bounds. Unfortunately the bounds on $I_{\bar{q}}$ given in Lemma 4.3.4 are not sharp, i.e., the width of this estimate of $I_{\bar{q}}$ is greater than $e^{\alpha} - 2$. Using this estimate would be computational difficult. In [Nus82] Nussbaum estimates the value of $\bar{q}$ up to $\mathcal{O}(\frac{1}{\alpha})$ in the case of $\bar{q}(x) \geq 3$ and $\alpha \geq 3.8$. We derive a similar estimate which only assumes $\bar{q}(x) \geq 2$ and $\alpha > 0$. This estimate is better suited for numerical applications, and only needs bounds $\ell(t) \leq x(t) \leq u(t)$ that are defined over the time domain $t \in [-1, 4]$.

Lemma 4.3.7. *Fix some $\alpha > 0$ and suppose that $x \in \mathcal{X}$ is a SOPS to* (1.1)*, and let $\ell, u : \mathbb{R} \to \mathbb{R}$ be functions for which $\ell(t) \leq x(t) \leq u(t)$. Let $I_q \subset \mathbb{R}$ be an interval for which $q(x) \in I_q$ and suppose that $\bar{q}(x) \geq 2$. Define the following integral bounds:*

$$U^+ := \sup_{q \in I_q} \int_{q-1}^{q} \max\left\{e^{u(t)} - 1, 0\right\} dt \quad U_1^- := \sup_{q \in I_q} \int_{q}^{q+1} -\min\left\{e^{\ell(t)} - 1, 0\right\} dt \tag{4.13}$$

$$L^+ := \inf_{q \in I_q} \int_{q-1}^{q} \max\left\{e^{\ell(t)} - 1, 0\right\} dt \quad L_1^- := \inf_{q \in I_q} \int_{q}^{q+1} -\min\left\{e^{u(t)} - 1, 0\right\} dt \tag{4.14}$$

and define $m := \min_{t \in I_q} \ell(t+1)$. Then $\bar{q}$ is bounded by the inequalities

$$2 + \frac{L^+ - U_1^-}{|e^m - 1|} \leq \bar{q} \leq 2 + \frac{U^+ - L_1^-}{|e^{u(-1)} - 1|}. \tag{4.15}$$

The proof is delayed until the end of this section. The computational details of how we evaluate the integrals in (4.13) and (4.14) are discussed in Section 4.7. Below in Algorithm 4.3.8 we construct the initial bounds for a region $K \subseteq \mathbb{R}^3$ containing the κ-image of SOPS $x \in \mathcal{X}$ for which $\bar{q}(x) \geq 3$.

Algorithm 4.3.8. *The input is an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ and computational parameters i_0, j_0, n_{Time}, $N_{period} \in \mathbb{N}$. The output is a rectangle $K = I_q \times I_{\bar{q}} \times I_M$ and bounding functions u_K, ℓ_K.*

1. *Make the following definitions for $K = I_q \times I_{\bar{q}} \times I_M$:*

$$
\begin{aligned}
q_{min} &:= 1 + \inf_{\alpha \in I_\alpha} \tfrac{1}{\alpha}\left(\tfrac{\alpha + e^{-\alpha} - 1}{\exp\{\alpha + e^{-\alpha} - 1\} - 1}\right) \\
I_q &:= \left[q_{min}, 2 + \tfrac{1}{\alpha_{min}}\right] \\
I_{\bar{q}} &:= \left[3, \sup_{\alpha \in I_\alpha} 2 + \left|\frac{e^\alpha - 1}{e^{a_{j_0}(\alpha)} - 1}\right|\right] \\
I_M &:= [0, p_{i_0}(1)]
\end{aligned}
$$

where $a_i(\alpha)$ is taken as in Lemma 4.3.3.

2. *For p_i and a_j given as in Propositions 4.3.2 and 4.3.3 respectively, define bounding functions ℓ_K and u_K*

$$
\ell_K(t) := \begin{cases} 0 & \text{if } t = 0 \\ p_{i_0}(t) & \text{if } t < 0 \\ \inf_{\alpha \in I_\alpha} -\alpha(e^\alpha - 1) & \text{otherwise} \end{cases}
$$

$$
u_K(t) := \begin{cases} 0 & \text{if } t = 0 \\ \sup_{\alpha \in I_\alpha} -t \cdot a_{j_0}(\alpha) & \text{if } t \in [-1, 0) \\ p_{i_0}(1) & \text{otherwise.} \end{cases}
$$

These bounding functions are stored on the computer with time resolution n_{Time} as described in Appendix 4.7.

3. *Refine u_K and ℓ_K according to Step 1 of Algorithm 4.2.2. For N_{period} iterations, refine u_K and ℓ_K according to Step 2 of Algorithm 4.2.2 for values $t_0 \in [-4, 4]$. Then define I_q and I_M according to Step 3 of Algorithm 4.2.2.*

4. *For values of $m, L^+, L_1^-, U^+, U_1^-$ given as in Proposition 4.3.7, define:*

$$
\bar{q}_{min} := 2 + \frac{L^+ - U_1^-}{|e^m - 1|}, \qquad \bar{q}_{max} := 2 + \frac{U^+ - L_1^-}{|e^{u_K(-1)} - 1|}.
$$

If $\bar{q}_{max} < 3$ then define $K = \emptyset$. Otherwise define $I_{\bar{q}} = [\bar{q}_{min}, \bar{q}_{max}]$ and $K = I_q \times I_{\bar{q}} \times I_M$.

Remark 4.3.9. *In practice we select* $i_0 = 2$ *and* $j_0 = 20$ *in Step 2, which have proved sufficient for our purposes. In [Jon62b] the expressions for* p_i *are given in closed form for* $i = 1, 2, 3, 4$, *each function being increasingly complex. The sequence* $a_j(\alpha)$ *is convergent, and we use* $j_0 = 20$ *because we have found negligible improvements when using a larger index.*

Proposition 4.3.10. *Fix an interval* $I_\alpha = [\alpha_{min}, \alpha_{max}]$ *such that* $\alpha_{min} \geq \frac{\pi}{2}$, *and fix computational parameters* $i_0, j_0, n_{Time}, N_{period} \in \mathbb{N}$. *Let* $\{K, u_K, \ell_K\}$ *denote the output of Algorithm 4.3.8. If* $x \in \mathcal{X}$ *is a SOPS to* (1.1) *and* $\bar{q}(x) \geq 3$ *then* $\kappa(x) \in K$ *and* $\ell_K \leq x \leq u_K$.

Proof. Let x be as described above. We describe the effect of each step of the algorithm in turn.

1. For $I_q, I_{\bar{q}}$ and I_M defined in Step 1, it follows from Lemma 4.3.4 that $q(x) \in I_q$ and $\bar{q}(x) \in I_{\bar{q}}$, and it follows from Lemma 4.3.1 and Lemma 4.3.2 that $x(1) \in I_M$.

2. Since $x \in \mathcal{X}$ then $x(0) = 0$. By Lemma 4.3.1 then any SOPS $x \in \mathcal{X}$ satisfies the inequality $-\alpha(e^\alpha - 1) \leq x(t) \leq p_{i_0}(1)$. The definition of the ℓ_K bound for $t < 0$ follows from Lemma 4.3.2, and the definition of the u_K bound for $t \in [-1, 0)$ follows from Lemma 4.3.3.

3. The results of Steps 1 and 2 produce a region K with bounding functions u_K, ℓ_K for which $\kappa(x) \in K$ whenever there is a SOPS $x \in \mathcal{X}$ satisfying $\bar{q}(x) \geq 3$. By Proposition 4.2.3, implementing Steps 2 and 3 of Algorithm 4.2.2 preserves this property.

4. Since $\bar{q}(x) \geq 3 > 2$ then by Lemma 4.3.7 it follows that $\bar{q}_{min} \leq \bar{q}(x) \leq \bar{q}_{max}$. If $\bar{q}_{max} < 3$, this contradicts our initial assumption that $\bar{q}(x) \geq 3$, whereby there are no SOPS $x \in \mathcal{X}$ to (1.1) at any parameter $\alpha \in I_\alpha$ for which $\bar{q}(x) \geq 3$. Otherwise for our definition of $K = I_q \times I_{\bar{q}} \times I_M$ it follows that $\kappa(x) \in K$ whenever $\bar{q}(x) \geq 3$. □

We present an application of this theorem.

Proposition 4.3.11. *If $x \in \mathcal{X}$ is a SOPS to* (1.2) *and $\alpha \in [\frac{\pi}{2}, 2.07]$ then $\bar{q}(x) < 3$.*

Proof. First we constructed subintervals I_α of $[1.57, 2.07]$ of width 0.01, and for each subinterval I_α we ran Algorithm 4.3.8 with computational parameters $i_0 = 2$, $j_0 = 20$, $n_{time} = 128$, and $N_{period} = 10$ (see [JLM] for associated MATLAB code). In each case the algorithm returned $K = \emptyset$. □

Corollary 4.3.12. *Fix an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ such that $\alpha_{min} > \frac{\pi}{2}$, and fix computational parameters $i_0, j_0, N_{period} \in \mathbb{N}$. Let $\{K_1, u_{K_1},\ \ell_{K_1}\}$ denote the output of Algorithm 4.3.5 and let $\{K_2, u_{K_2},\ \ell_{K_2}\}$ the output of Algorithm 4.3.8. Then $K_1 \cup K_2$ is I_α-exhaustive.*

Proof. Suppose that $x \in \mathcal{X}$ is SOPS to (1.1). If $\bar{q}(x) \leq 3$, then by Proposition 4.3.6 it follows that $\kappa(x) \in K_1$. If $\bar{q}(x) \geq 3$, then by Proposition 4.3.10 it follows that $\kappa(x) \in K_2$. Hence the set $K_1 \cup K_2$ is I_α-exhaustive. □

Proof of Lemma 4.3.7. Let p denote the period of a SOPS $x \in \mathcal{X}$. By assumption $x(p) = x(q) = 0$, so by the fundamental theorem of calculus we have that for any SOPS x,

$$0 = x(p) - x(q) = \int_q^p x'(t)dt = \int_q^p -\alpha(e^{x(t-1)} - 1)dt = \int_{q-1}^{p-1} (e^{x(t)} - 1)dt.$$

Recall that any SOPS $x(t)$ is positive for $t \in (0, q)$ and negative for $t \in (q, p)$. Hence the integrand above is positive on $(q-1, q)$ and negative on $(q, p-1)$, thus producing the following estimate:

$$\int_{q-1}^{q} |e^{x(t)} - 1|dt = \int_q^{p-1} |e^{x(t)} - 1|dt. \tag{4.16}$$

For $t \in (q-1, q)$ the function $x(t)$ is positive, whereby $|e^{x(t)} - 1| = \max\{e^{x(t)} - 1, 0\}$. For the definitions of L^+ and U^+ given in (4.13) and (4.14), it follows that L^+ and U^+ bound the LHS of (4.16) as described below:

$$L^+ \leq \int_{q-1}^{q} \max\{e^{\ell(t)} - 1, 0\}dt \leq \int_{q-1}^{q} |e^{x(t)} - 1|dt \leq \int_{q-1}^{q} \max\{e^{u(t)} - 1, 0\}dt \leq U^+.$$

We estimate the RHS of (4.16) using the two sums below:

$$L_1^- + L_2^- \leq \int_q^{p-1} |e^{x(t)} - 1|dt \leq U_1^- + U_2^-$$

where the constants $L_1^-, L_2^-, U_1^-, U_2^-$ are appropriately defined so that

$$L_1^- \leq \int_q^{q+1} |e^{x(t)} - 1| dt \leq U_1^- \tag{4.17}$$

$$L_2^- \leq \int_{q+1}^{p-1} |e^{x(t)} - 1| dt \leq U_2^-. \tag{4.18}$$

For $t \in (q, q+1)$ the function $x(t)$ is negative, whereby $|e^{x(t)} - 1| = -\min\{e^{x(t)} - 1, 0\}$. It follows from the definitions of L_1^- and U_1^- given in (4.13) and (4.14) that (4.17) is satisfied. To define L_2^- and U_2^- note that for the time period $t \in [q+1, p-1]$ we have that $x'(t) > 0$, whereby

$$\begin{aligned} x(t) &\geq x(q+1) \\ x(t) &\leq x(p-1) = x(-1) \leq u(-1). \end{aligned}$$

By definition $m \leq x(q+1)$, and as $p - q = \bar{q}$ we can then define

$$\begin{aligned} U_2^- &:= \int_{q+1}^{p-1} |e^m - 1| dt & \qquad L_2^- &:= \int_{q+1}^{p-1} |e^{u(-1)} - 1| dt \\ &= (\bar{q} - 2)|e^m - 1| & &= (\bar{q} - 2)|e^{u(-1)} - 1|. \end{aligned}$$

Using these definitions, (4.18) is satisfied. From (4.16), we get the following upper and lower bounds on $\bar{q}$, from which (4.15) follows.

$$\begin{aligned} L_1^- + L_2^- &\leq U^+ & \qquad U_1^- + U_2^- &\geq L^+ \\ (\bar{q} - 2)|e^{u(-1)} - 1| &\leq U^+ - L_1^- & (\bar{q} - 2)|e^m - 1| &\geq L^+ - U_1^- \\ \bar{q} &\leq 2 + \frac{U^+ - L_1^-}{|e^{u(-1)} - 1|} & \bar{q} &\geq 2 + \frac{L^+ - U_1^-}{|e^m - 1|}. \end{aligned} \qquad \square$$

4.4 Bounding the Floquet Multipliers.

In this section we describe how to estimate the Floquet multipliers of SOPS contained within the bounds derived in Sections 4.2 and 4.3. This method follows the approach of [Xie91] with modifications to take advantage of numerical computations. To review this method, we first define a hyperplane in $C[-1, 0]$ as

$$H := \{\varphi \in C[-1, 0] : \varphi(0) = 0\}.$$

For a function y we define $y_0 \in C[-1,0]$ to be the cut-off function of y on $[-1,0]$, and for a constant $L \in \mathbb{R}$ we define $y_L := [y(t+L)]_0$. Locally, one can construct a smooth Poincaré map $\Phi : H \to H$ via the solution operator. If $x \in \mathcal{X}$ is a SOPS, then x_0 is a fixed point of Φ, and the Floquet multipliers of x are the eigenvalues of $D_\varphi \Phi(x_0)$. Of course x_0 is a trivial eigenfunction with associated eigenvalue $\lambda = 1$. The nontrivial and nonzero Floquet multipliers of the SOPS can be calculated by solving the following boundary value problem:

Theorem 4.4.1 (See [Xie91, Theorem 2.2.3])**.** *Suppose that $x \in \mathcal{X}$ is a SOPS to* (1.1) *with period L. Define the linearized DDE below:*

$$y'(t) = -\alpha e^{x(t-1)} y(t-1). \tag{4.19}$$

Then $\lambda \neq 0$ is a nontrivial eigenvalue of $D_\varphi \Phi(x_0)$ if and only if (4.20) *has a nonzero solution $h \in H$ for which*

$$-y(L)\frac{x'_L(t)}{x'(L)} + y_L(t) = \lambda h(t) \tag{4.20}$$

where the function h is then an eigenfunction of $D_\varphi \Phi(x_0)$ associated with λ, and $y(t)$ solves (4.19) *with initial condition $y_0 = h$.*

We are able to bound the Floquet multipliers by studying this boundary value problem defined in (4.19) and (4.20), a calculation which is systematized through Algorithm 4.4.2. If this algorithm outputs a value $\Lambda_{max} < 1$ then all SOPS $x \in \kappa^{-1}(K)$ are asymptotically stable. We are able to improve upon Xie's method in [Xie91, Xie93] by repeating certain steps, somewhat analogous to the recursive bounds defined in Lemma 4.3.2. The great advantage for doing this numerically as opposed to analytically is that these repetitions while tedious and time consuming for the mathematician are "effortless" for the computer.

Algorithm 4.4.2. *Fix $I_\alpha = [\alpha_{min}, \alpha_{max}]$ and $K = I_q \times I_{\bar{q}} \times I_M \subset \mathbb{R}^3$ (where $I_q = [q_{min}, q_{max}]$, $I_{\bar{q}} = [\bar{q}_{min}, \bar{q}_{max}]$ and $I_M = [M_{min}, M_{max}]$) and fix associated bounding functions u_K, ℓ_K. Furthermore, fix computational parameters $n_{Time}, N_{Floquet}, M_{Floquet} \in \mathbb{N}$. The output of the algorithm is $\Lambda_{max} \in \mathbb{R}_+$.*

1. *Define* $L_{min} := q_{min} + \bar{q}_{min}$, $L_{max} := q_{max} + \bar{q}_{max}$ *and* $I_L := [L_{min}, L_{max}]$, *and define the function* $Y : [-1, 0] \to \mathbb{R}$ *by*

$$Y(t) := \begin{cases} 1 & \text{if } t \in [-1, 0) \\ 0 & \text{if } t = 0. \end{cases}$$

2. *Extend the function* $Y : [-1, L_{max}] \to \mathbb{R}$ *by*

$$Y(t) := \alpha_{max} \int_0^t \left(Y(s-1) \sup_{\ell_K \le x \le u_K} e^{x(s-1)} \right) ds \qquad \text{if } t \ge 0, \quad (4.21)$$

evaluating the integral using an upper Riemann sum with a uniform step size of $1/n_{Time}$. *Section 4.7 discusses in further detail how we compute this integral.*

3. *Define* Z *as below:*

$$Z(t) := \left(\max_{L \in [L_{min}, L_{max}]} Y(L) \right) \max_{\ell_K \le x \le u_K} \left| \frac{e^{x(t-1)} - 1}{e^{x(-1)} - 1} \right| + Y(t).$$

4. *For* $t \in [-L_{min}, 0]$ *define* Z_L *as below:*

$$Z_L(t) := \max_{L_{min} \le L \le L_{max}} Z(t + L).$$

5. *For* $t \in [-(L_{min} - 1), 0]$ *refine the function* Z_L *by*

$$\begin{aligned} Z_L'(-t) &:= \alpha_{max} \int_{-t}^0 \left(Z_L(s-1) \sup_{\ell_K \le x \le u_K} e^{x(s-1)} \right) ds \qquad (4.22) \\ Z_L(-t) &:= \min \left\{ Z_L(-t), Z_L'(-t) \right\}, \end{aligned}$$

evaluating the integral using an upper Riemann sum with a uniform step size of $1/n_{Time}$. *Section 4.7 discusses in further detail how we compute this integral.*

Repeat this step $M_{Floquet}$ *number of times.*

6. *Define*

$$\Lambda_{max} := \sup_{t \in [-1, 0]} Z_L(t).$$

7. *If* $\Lambda_{max} < 1$ *then* STOP.

8. *Otherwise define*

$$Y(t) := \min \{1, Z_L(t)\}, \qquad \text{for } t \in [-1, 0] \qquad (4.23)$$

and GOTO Step 2. After reaching this step $N_{Floquet}$ *times, exit the program.*

Theorem 4.4.3. *Fix* $K = [q_{min}, q_{max}] \times [\bar{q}_{min}, \bar{q}_{max}] \times [M_{min}, M_{max}] \subseteq \mathbb{R}^3$ *and* $I_\alpha = [\alpha_{min}, \alpha_{max}]$. *If Algorithm 4.4.2 terminates with* $\Lambda_{max} < 1$, *then all SOPS* $x \in \mathcal{X}$ *satisfying* $\kappa(x) \in K$ *must be asymptotically stable. If the algorithm terminates having never reached Step 8, then the norm of all nontrivial Floquet multiplier are bounded above by* Λ_{max}.

Proof. Fix some $x \in \mathcal{X}$ for which $\kappa(x) \in K$. By the definition made in Step 1, the period of x is some $L \in I_L$. We use Theorem 4.4.1 to estimate the range of Floquet multipliers of x. That is, fix $\lambda \in \mathbb{C}$ and $h \in H$ and suppose that (λ, h) is a solution to (4.20). Define $y(t)$ to be the solution of (4.19) through h, define z as

$$z(t) := -y(L)\frac{x'(t)}{x'(L)} + y(t) \tag{4.24}$$

and define $z_L(t) := z(t + L)$. Hence (λ, h) is a solution to (4.20) if and only if $z_L(t) = \lambda h(t)$ for $t \in [-1, 0]$. As (4.19) is a linear DDE, we may assume without loss of generality that $\sup_{t\in[-1,0]} |h(t)| = 1$. Thereby, it follows that

$$|\lambda| = \sup_{t\in[-1,0]} |z_L(t)|. \tag{4.25}$$

If we can show that the RHS of (4.25) is less than 1 uniformly for $x \in \kappa^{-1}(K)$, then we will have proven that all such SOPS are asymptotically stable. We prove that Steps 1-7 of Algorithm 4.4.2 produce functions Y, Z and Z_L and a bound Λ_{max} which satisfy the following inequalities uniformly for $x \in \kappa^{-1}(K)$

$$|y(t)| \leq Y(t), \qquad |z(t)| \leq Z(t), \qquad |z_L(t)| \leq Z_L(t), \qquad |\lambda| \leq \Lambda_{max}.$$

We describe the results of each step of Algorithm 4.4.2 in order, and then discuss how Step 8 affects what we may deduce about the output Λ_{max}.

1. By definition, if $h \in H$ then $h(0) = 0$, and by assumption $|h(t)| \leq 1$ for $t \in [-1, 0]$. Thereby our definition of $Y(t)$ in Step 1 satisfies $|y(t)| \leq Y(t)$ for $t \in [-1, 0]$.

2. By definition y solves the linear DDE in (4.19). By variation of parameters it follows that

$$y(t) = \int_0^t -\alpha e^{x(s-1)} y(s-1) ds$$

for all $t \geq 0$. Equation (4.21) follows from this by taking a supremum over $\alpha \in I_\alpha$ and $\ell_K \leq x \leq u_K$. Thereby, Step 2 produces a function Y satisfying $|y(t)| \leq Y(t)$ for $t \geq 0$.

3. Step 3 defines a function Z to bound the norm of z defined in (4.24). As $x'(t) = -\alpha(e^{x(t-1)} - 1)$ it follows that

$$|z(t)| \leq \left| y(L) \frac{e^{x(t-1)} - 1}{e^{x(L-1)} - 1} \right| + |y(t)|.$$

   By periodicity, we may replace $x'(L-1)$ with $x'(-1)$. By taking a supremum over $L \in I_L$ and $\ell_K \leq x \leq u_K$, it follows that the function defined in Step 3 satisfies $|z(t)| \leq Z(t)$.

4. Since $L \in I_L$ and $|z(t)| \leq Z(t)$, we obtain the estimate for $t \in [-L_{min}, 0]$ below:

$$|z(t+L)| \leq Z(t+L) \leq \max_{L_{min} \leq L \leq L_{max}} Z(t+L) = Z_L(t).$$

   Since by definition $z_L(t) = [z(t+L)]_0$, then for the definition of Z_L in Step 4, we have $|z_L(t)| \leq Z_L(t)$ for $t \in [-1, 0]$.

5. Note that both y and x' satisfy (4.19), so by linearity z solves (4.19). Since $z_L(0) = 0$, we obtain the following estimate using variation of parameters:

$$z_L(-t) = -\int_{-t}^{0} -\alpha e^{x(s-1)} z_L(s-1) ds.$$

   By taking the suprema over $\alpha \in I_\alpha$ and $\ell_K \leq x \leq u_K$ as in Step 5, we obtain a refinement for which $|z_L(t)| \leq Z_L(t)$. This refinement can be repeated any number of times.

6. If (λ, h) solves (4.19), then by (4.25) we obtain the following:

$$|\lambda| = \sup_{t \in [-1,0]} |z_L(t)| \leq \sup_{t \in [-1,0]} Z_L(t) = \Lambda_{max}.$$

   Hence $|\lambda| < \Lambda_{max}$ uniformly for $x \in \kappa^{-1}(K)$.

7. We have shown that $|\lambda| \leq \Lambda_{max}$ for any Floquet multiplier λ. If $\Lambda_{max} < 1$, then it follows that x is asymptotically stable.

8. If $\Lambda_{max} \geq 1$, then we make the assumption that x is not asymptotically stable for the sake of contradiction. Then the largest Floquet multiplier λ_{max} of x satisfies $|\lambda_{max}| \in [1, \Lambda_{max}]$. If h is an eigenfunction associated with λ_{max}, then $z_L(t) = \lambda_{max} \cdot h(t)$ for $t \in [-1, 0]$ and furthermore $|h(t)| = |\lambda_{max}|^{-1}|z_L(t)| \leq |z_L(t)|$. Hence for all $t \in [-1, 0]$ we may assume that the eigenfunction $h(t)$ satisfies the inequality:

$$|h(t)| \leq \min\{1, |z_L(t)|\}.$$

By definition $y(t) = h(t)$ for $t \in [-1, 0]$. Hence for our refinement of Y in (4.23) it follows that $|y(t)| \leq Y(t)$ for $t \in [-1, 0]$.

If the algorithm terminates having never passed through Step 8, then $|\lambda| \leq \Lambda_{max} < 1$ for all solutions (λ, h) to (4.20) uniformly for all SOPS $x \in \kappa^{-1}(K)$. If the program terminates having passed through Step 8 at least once, then it has shown that every solution (λ, h) to (4.20) satisfies $|\lambda| < 1$ under the assumption that there exists a solution for which $|\lambda| \geq 1$, a contradiction. In this case we have shown that x is asymptotically stable without calculating an explicit bound on its Floquet multipliers.

□

4.5 A Comprehensive Algorithm

We state our branch and prune algorithm in Algorithm 4.5.1, and describe how we use it to prove the uniqueness of SOPS to Wright's equation in Algorithm 4.5.3. Algorithm 4.5.1 takes as input an interval $I_\alpha \subseteq \mathbb{R}$ and constructs an I_α-exhaustive set. Furthermore, this algorithm uses several computational parameters: $\epsilon_1, \epsilon_2 \in \mathbb{R}$ which defines the algorithm's stopping criterion, $n_{Time} \in \mathbb{N}$ which defines the time resolution used in representing bounding functions on the computer, and $N_{prune} \in \mathbb{N}$ which defines the number of times the pruning algorithm is performed before branching. Additionally it requires the computational parameters $i_0, j_0, N_{Period} \in \mathbb{N}$ needed for running Algorithms 4.3.5 and 4.3.8. As we have stated before, this is a canonical algorithm which terminates in finite time (see [Sch11, RR88, HT13]).

Algorithm 4.5.1. *The input is an interval* $I_\alpha = [\alpha_{min}, \alpha_{max}]$ *and computational parameters* $\epsilon_1, \epsilon_2 > 0$ *and* $i_0, j_0, n_{Time}, N_{Period}, N_{Prune} \in \mathbb{N}$. *The output is a set* $\mathcal{A} = \{K_i : K_i \subseteq \mathbb{R}^3\}$ *and an associated collection of bounding functions* $\{u_K, \ell_K\}_{K\in\mathcal{A}}$.

1. *Construct regions* K_1 *and* K_2 *according to Algorithms 4.3.5 and 4.3.8 respectively. Define the sets* $\mathcal{S} = \{K_1, K_2\}$ *and* $\mathcal{A} = \emptyset$.

2. *If* $\mathcal{S} = \emptyset$ *then return* $\mathcal{A}$ *and* STOP.

3. *Define* K *to be an element of* $\mathcal{S}$ *and remove* K *from* $\mathcal{S}$.

4. *Define* $\{K', u_{K'}, \ell_{K'}\}$ *to be the output of Algorithm 4.2.2 using input* K, u_K, ℓ_K *and computational parameter* n_{Time}. *Then redefine* $\{K, u_K, \ell_K\} := \{K', u_{K'}, \ell_{K'}\}$. *Repeat this step* N_{Prune} *times.*

5. *If the diameter of* K *is less than* ϵ_1 *and* $\bar{q} < 3$, *or the diameter of* K *is less than* ϵ_2 *and* $\bar{q} \geq 3$, *then add* K *to* $\mathcal{A}$ *and* GOTO Step 2.

6. *Subdivide* K *along its fattest dimension into two regions* K_A *and* K_B. *That is, write* $K = I_1 \times I_2 \times I_3$ *where each* I_i *is given by the interval* $I_i = [a_i, b_i]$ *and fix some* $j \in \{1, 2, 3\}$ *which maximizes* $|b_j - a_j|$. *The regions* $K_A := I_1' \times I_2' \times I_3'$ *and* $K_B := I_1'' \times I_2'' \times I_3''$ *are defined according to the following formulas*
$$I_i' := \begin{cases} [a_i, b_i] & \text{if } i \neq j \\ [a_i, (a_i + b_i)/2] & \text{if } i = j \end{cases} \qquad I_i'' := \begin{cases} [a_i, b_i] & \text{if } i \neq j \\ [(a_i + b_i)/2, b_i] & \text{if } i = j. \end{cases} \tag{4.26}$$

7. *Add to* $\mathcal{S}$ *the regions* K_A *and* K_B, *each with associated bounding functions* u_K *and* ℓ_K. *Then GOTO Step 2.*

As a notational convention for the next two theorems we define $\bigcup \mathcal{S} := \bigcup_{K\in\mathcal{S}} K$.

Theorem 4.5.2. *Fix an interval* $I_\alpha = [\alpha_{min}, \alpha_{max}]$ *such that* $\alpha_{min} \geq \frac{\pi}{2}$, *and fix any selection of computational parameters* $\epsilon > 0$ *and* $i_0, j_0, n_{Time}, N_{Period}, N_{Prune} \in \mathbb{N}$. *If* $\mathcal{A}$ *is the output of Algorithm 4.5.1 with these inputs, then* $\bigcup \mathcal{A}$ *is* I_α*-exhaustive.*

Proof. We prove by induction that every time the algorithm arrives at Step 2, then $\bigcup \mathcal{S} \cup \bigcup \mathcal{A}$ is I_α-exhaustive. This suffices to prove the theorem, as the only way for the algorithm to exit is on Step 2 when $\mathcal{S} = \emptyset$.

For the initial case, the set $\bigcup \mathcal{S} = K_1 \cup K_2$ produced in Step 1 is I_α-exhaustive by Proposition 4.3.12. The result of Step 3 simply rearranges the collection of regions, after which $\bigcup \mathcal{S} \cup \bigcup \mathcal{A} \cup K$ is I_α-exhaustive. In Step 4, this I_α-exhaustivity is maintained when replacing K with the output of Algorithm 4.2.2 as a direct result of Proposition 4.2.3. If Step 5 adds K to $\mathcal{A}$, then when the algorithm arrives at Step 2 the set $\bigcup \mathcal{S} \cup \bigcup \mathcal{A}$ will be I_α-exhaustive. Otherwise Step 6 will divide K into two regions K_A and K_B for which $K = K_A \cup K_B$. Then in Step 7 both K_A and K_B are then added to $\mathcal{S}$, after which $\bigcup \mathcal{S} \cup \bigcup \mathcal{A}$ is still I_α-exhaustive. □

We are finally able to state our algorithm which can prove that Wright's equation has a unique SOPS over a given range of parameters.

Algorithm 4.5.3. *The input is an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ and computational parameters $\epsilon_1, \epsilon_2 > 0$ and $i_0, j_0, n_{Time}, N_{Period}, N_{Prune}, N_{Floquet}, M_{Floquet} \in \mathbb{N}$. The output is a True or False statement.*

1. *Run Algorithm 4.5.1 with input I_α and computational parameters ϵ_1, ϵ_2, i_0, j_0, n_{Time}, N_{Period} and N_{Prune}. Define $\mathcal{A}$ and $\{u_K, \ell_K\}_{K \in \mathcal{A}}$ to be its output.*

2. *For each $K \in \mathcal{A}$ calculate $\Lambda_{max}(K)$ to be the output of Algorithm 4.4.2, run with input I_α, K, u_K, ℓ_K, and computational parameters $n_{Time}, N_{Floquet}$ and $M_{Floquet}$.*

3. *If $\Lambda_{max}(K) < 1$ for all $K \in \mathcal{A}$, then return* TRUE. *Otherwise return* FALSE.

Theorem 4.5.4. *Fix an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ with $\alpha_{min} > \pi/2$. If Algorithm 4.5.3 returns the output* TRUE *for any selection of computational parameters $\epsilon > 0$, and $i_0, j_0, n_{Time}, N_{Prune}, N_{Floquet}, M_{Floquet} \in \mathbb{N}$, then there exists a unique SOPS to Wright's equation for all $\alpha \in I_\alpha$.*

Proof. By Theorem 4.5.2 it follows that $\bigcup \mathcal{A} = \bigcup_{K \in \mathcal{A}} K$ is an I_α exhaustive set. That is, by Definition 4.2.1, up to a time translation any SOPS to Wright's equation for parameter $\alpha \in I_\alpha$ can be expressed as a function $x \in \mathcal{X}$ for which $\kappa(x) \in \bigcup \mathcal{A}$. If Algorithm 4.4.2 terminates with $\Lambda_{max}(K) < 1$ for all $K \in \mathcal{A}$, then by Theorem 4.4.3 it follows that any SOPS $x \in \mathcal{X}$ satisfying $\kappa(x) \in \bigcup \mathcal{A}$ must be asymptotically stable. Hence, by Theorem 4.1.1 it follows that there must be a unique SOPS to Wright's equation for each $\alpha \in I_\alpha$. $\square$

4.6 Discussion

In Algorithm 4.5.3 we defined an algorithm which, if successful, proves the uniqueness of SOPS to Wright's equation for a finite range of parameters I_α. Below we describe how we applied this algorithm to prove Theorem 1.2.2.

Proof of Theorem 1.2.2. To prove Theorem 1.2.2 we divide the interval $[1.9, 6.0]$ into various subintervals I_α, and then divide each of these intervals into further subintervals of width $\Delta\alpha$. For example, the interval $I_\alpha = [2.1, 6.0]$ with $\Delta\alpha = 0.1$ was divided into subintervals $[2.1, 2.2]$, $[2.2, 2.3]$, $\ldots$, $[5.9, 6.0]$. The various computational parameters we used are given in the table below (see [JLM] for associated MATLAB code).

I_α	$\Delta\alpha$	n_{Time}	ϵ_1	ϵ_2	i_0	j_0	N_{Period}	N_{Prune}	$N_{Floquet}$	$M_{Floquet}$
$[1.90, 1.96]$	0.01	128	0.02	0.25	2	20	10	4	20	5
$[1.96, 2.10]$	0.01	64	0.05	0.25	2	20	10	4	20	5
$[2.10, 6.00]$	0.10	32	0.05	0.25	2	20	10	4	20	5

Table 4.1: For descriptions of how these parameters affect Algorithm 4.5.3, refer to Algorithms 4.3.5 and 4.3.8 for i_0, j_0 and N_{Prune}; refer to Algorithm 4.4.2 for $N_{Floquet}$ and $M_{Floquet}$; and refer to Algorithm 4.5.1 for ϵ_1, ϵ_2 and N_{Prune}.

For each of these parameter values, we ran Algorithm 4.5.3 which returned *TRUE* as its output. By Theorem 4.5.4 it follows that there must be a unique SOPS to Wright's equation for each $\alpha \in [1.9, 6.0]$. $\square$

As described in Theorem 4.4.3, if Algorithm 4.4.2 terminates without having reached Step 8, then it produces explicit bounds on the Floquet multipliers of the SOPS to

Wright's equation. These bounds are summarized in Figure 4.1. In the range $[2.2, 6.0]$ Algorithm 4.4.2 exits on Step 7, so by Theorem 4.4.3 we obtain upper bounds on the Floquet multipliers. In the regime $\alpha \in [1.90, 2.20]$ Algorithm 4.4.2 only terminated after reaching Step 8 at least once, so we are only able to deduce that any non-trivial Floquet multiplier has modulus strictly bounded above by 1. In total, the computation took 115 hours to run using a i7-5500U processor, and Algorithm 4.5.1 accounted for 94% of the computation time.

Running Algorithm 4.5.3 at high values of α is computationally expensive. This is because the period length of SOPS to Wright's equation grows exponentially [Nus82], whereby our algorithm's run time and memory requirements also increase exponentially in α. Nevertheless, proving Theorem 1.2.2 with an upper limit of $\alpha = 6$ is sufficient for our purposes considering the results in [Xie91] proved uniqueness for $\alpha \geq 5.67$.

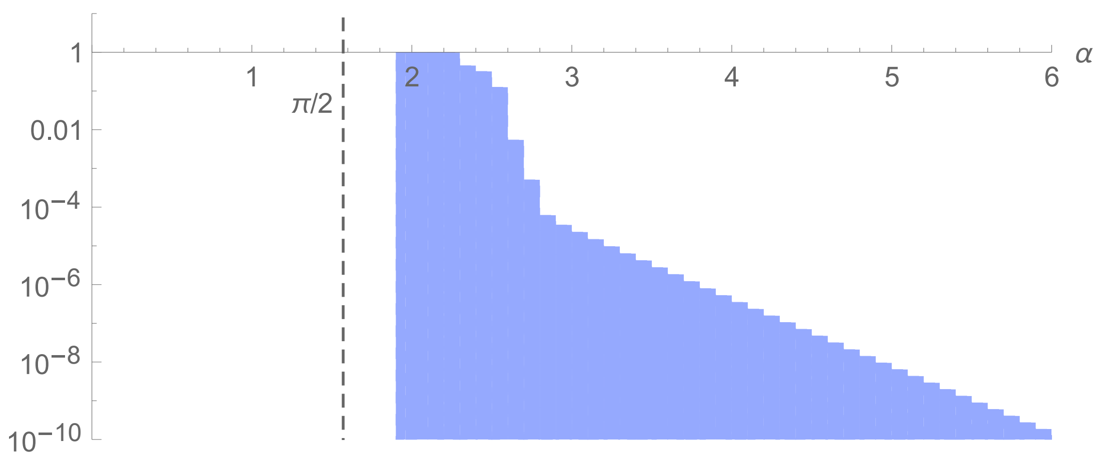


Figure 4.1: An upper bound on the modulus of the Floquet multipliers for SOPS to (1.1) for $\alpha \in [1.9, 6.0]$.

A different challenge presents itself for decreasing the lower limit of $\alpha = 1.9$ in Theorem 1.2.2. Namely, Xie's method for bounding the largest Floquet multiplier is not well suited to weakly attracting SOPS. Even when using precise numerical approximations (from [Les10]) of SOPS to Wright's equation at single values of α, Algorithm 4.4.2 was only able to show that the SOPS was asymptotically stable for values of α no lower than 1.85. By decreasing the parameters $\Delta\alpha$ and ϵ_1, and increasing the other computational parameters, we could expect the uniqueness result for $\alpha \geq 1.9$ could be pushed closer to

$\alpha = 1.85$. However we will need a different approach to finish the proof of Conjecture 1.1.3. Nevertheless, we can now prove that there are no folds in the principal branch of SOPS.

Corollary 4.6.1 (See [vdBJ18])**.** *The branch of SOPS originating from the Hopf bifurcation at $\alpha = \frac{\pi}{2}$ has no folds or secondary bifurcations for any $\alpha > \frac{\pi}{2}$.*

Proof. We prove the corollary by combining results on four overlapping subintervals of $(\frac{\pi}{2}, \infty)$. In Theorem 2.3.8 we show that the (continuous) branch of SOPS originating from the Hopf bifurcation does not have any folds or secondary bifurcations for $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta_2]$ where $\delta_2 = 6.830 \times 10^{-3}$. In Theorem 1.2.2 the same result is proved for $\alpha \in [\frac{\pi}{2} + \delta_3, 2.3]$, where $\delta_3 = 7.3165 \times 10^{-4}$. In [JLM17] it is shown that there is a unique SOPS for α in the interval $[1.9, 6.0]$. Since $1.9 \leq 2.3$, then the SOPS in this interval belong to the branch originating from the Hopf bifurcation, and since they are unique for each α, the branch is continuous and cannot have any folds or secondary bifurcations. In [Xie91] it is shown that there is a unique SOPS for α in the interval $[5.67, +\infty)$, and by a similar argument the branch of SOPS cannot have any folds or secondary bifurcations in this interval either. Since

$$(\tfrac{\pi}{2}, \infty) = (\tfrac{\pi}{2}, \tfrac{\pi}{2} + \delta_2] \cup [\tfrac{\pi}{2} + \delta_3, 2.3] \cup [1.9, 6.0] \cup [5.67, \infty),$$

it follows that the branch of SOPS originating from the Hopf bifurcation at $\alpha = \frac{\pi}{2}$ has no folds or secondary bifurcations for any $\alpha > \frac{\pi}{2}$. $\square$

4.7 Computational Considerations

To implement our algorithm we used *Intlab*: an interval arithmetic package for Matlab [Rum99]. Some of the calculations we performed are a simple application of interval arithmetic, such as defining $I_q, I_{\bar{q}}, I_M$ in Algorithm 4.3.5. However there is a nontrivial degree of complexity in how we store and represent the functions used in the algorithms, such as u_K, ℓ_K in Algorithm 4.2.2 or Y, Z, Z_L in Algorithm 4.4.2. In short, we defined these functions to be piecewise constant.

To explain our methodology, first fix a constant $n_{Time} \in \mathbb{N}$. To define an interval extension of a function $y : \mathbb{R} \to \mathbb{R}$, we define a collection of intervals $I_i^P, I_i^I \subseteq \mathbb{R}$ for $i \in \mathbb{Z}$ and define Y as follows:

$$Y(t) = \begin{cases} I_i^P & \text{if } t = \frac{i}{n_{Time}} \\ I_i^I & \text{if } t \in \left(\frac{i}{n_{Time}}, \frac{i+1}{n_{Time}}\right). \end{cases}$$

Of course any computer has finite memory, and so we would only store the function Y over a finite domain. Furthermore, as the bounding functions u, ℓ are intended to provide upper and lower bounds on a function x, we simply define an interval valued function $X(t) = [\ell(t), u(t)]$. In Figure 4.2 we present a graphical representation of how we store such a function, wherein we have defined the function $X(t)$ for $t \in [-1, 0]$ as follows:

$$\begin{aligned} I_{-4}^P :=& [-2.0, -1.2] & I_{-4}^I :=& [-2.0, -0.9] \\ I_{-3}^P :=& [-1.6, -0.9] & I_{-3}^I :=& [-1.6, -0.6] \\ I_{-2}^P :=& [-1.2, -0.6] & I_{-2}^I :=& [-1.2, -0.3] \\ I_{-1}^P :=& [-0.8, -0.3] & I_{-1}^I :=& [-0.8, -0.0] \\ I_0^P :=& [0.0, 0.0] \end{aligned}$$

For such a function, it is a straightforward procedure to calculate its supremum. To calculate $\sup_{t \in [a,b]} x(t)$ one simply needs to compare the intervals I_i^P for which $a \leq \frac{i}{n_{Time}} \leq b$, and the intervals I_i^I for which $a - n_{Time}^{-1} < \frac{i}{n_{Time}} < b$. Both these collections of intervals are finite. For bounds which are defined to be the integrals of various functions, as in (4.13) and (4.14) of Lemma 4.3.7, we use a Riemann sum of step size $1/n_{Time}$.

Unfortunately there is a loss in fidelity when we numerically integrate these functions, as we do in Step 2 of Algorithm 4.2.2. Therein we refine the values of $u_{K'}(t_0 + s)$, $\ell_{K'}(t_0 + s)$, $u_{K'}(t_0 - s)$ and $\ell_{K'}(t_0 - s)$, where $t_0 = \frac{i_0}{n_{Time}}$ and $s \in [0, \frac{1}{n_{Time}}]$. To just discuss the refinements of $u_{K'}(t_0 + s)$ and $\ell_{K'}(t_0 + s)$, if we choose $s = \frac{1}{n_{Time}}$, then this procedure refines the bound of $[\ell_{K'}(\frac{i_0+1}{n_{Time}}), u_{K'}(\frac{i_0+1}{n_{Time}})]$, a value which is stored in the interval $I_{i_0+1}^P$. However in order to refine $I_{i_0}^I$ this interval must include $[\ell_{K'}(t_0 + s'), u_{K'}(t_0 + s')]$

for all $s' \in (t_0, t_0 + \frac{1}{n_{Time}})$. This is represented in Figure 4.2, where the darker red region represents the sharpest possible bounds able to be derived from in Step 2 of Algorithm 4.2.2 when integrating the initial data given above, and the pink region represents the values we store in the computer. When we define functions as integrals as in Steps 2 and 5 of Algorithm 4.4.2 we use the same procedure.

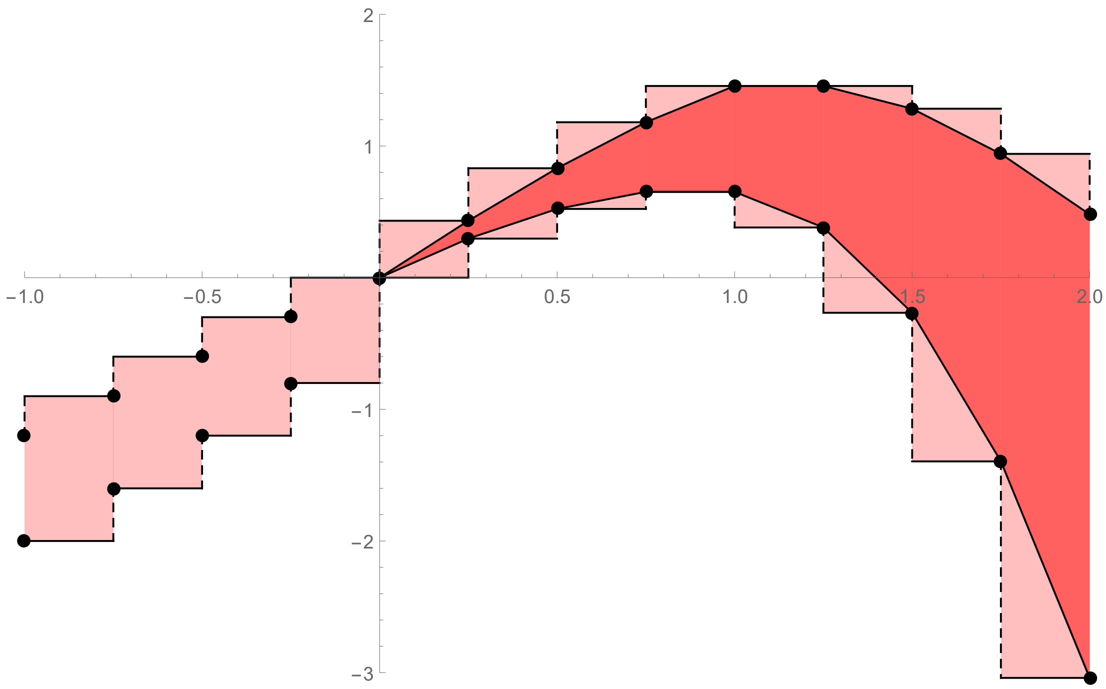
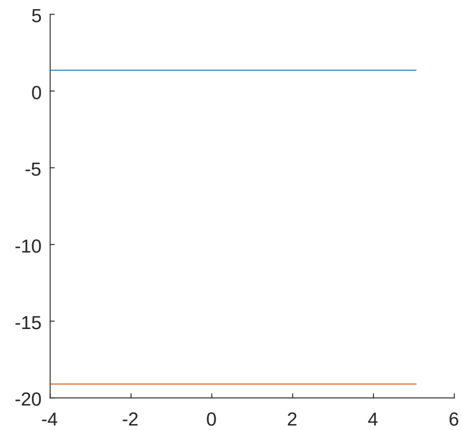
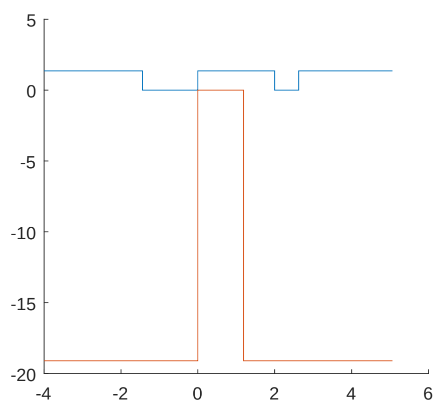


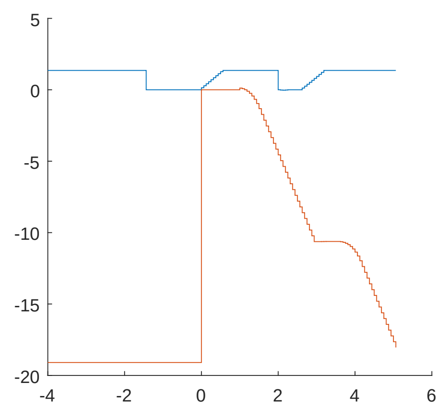
Figure 4.2: An example of how we store an interval valued function $[\ell(t), u(t)]$ in our algorithm.



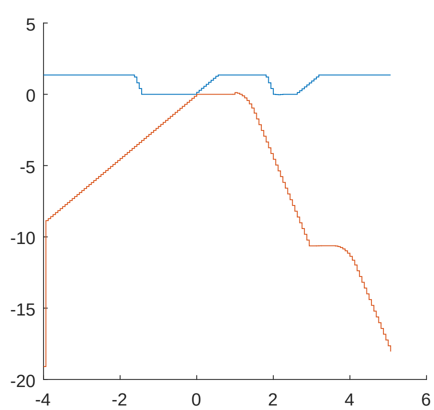
(a) Initial bounding functions for parameters $I_\alpha = [2.2, 2.25]$ associated with a region $K = I_q \times I_{\bar{q}} \times I_M$ where $I_q = [1.20, 2.00]$, $I_{\bar{q}} = \times[1.44, 3.00]$ and $I_M = [0.14, 1.36]$.



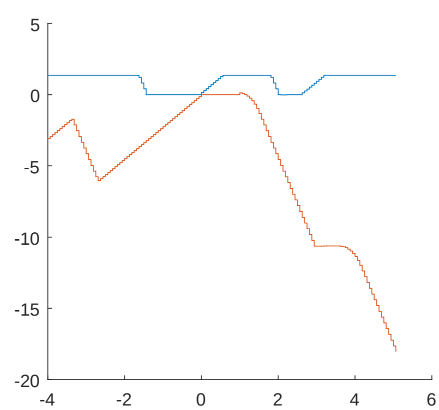
(b) The bounding functions after Step 1.



(c) The bounding functions midway through Step 2 (solving the DDE forward).



(d) The bounding functions after Step 2 (solving the DDE backwards).



(e) The bounding functions after Step 4 (imposing periodicity conditions).

Figure 4.3: An example of how the various steps in Algorithm 4.2.2 tighten the bounding functions associated with a region K.

Chapter 5

Ruling out Isolas

5.1 Outline of Proof

In this chapter we show that there is a unique slowly oscillating periodic orbit to (1.2) for all $\alpha \in (\frac{\pi}{2}, 1.9]$. As in Chapter 2, we recast the problem of studying the periodic orbits of (1.2) as the problem of finding the zeros of a functional F defined in a space of Fourier coefficients. Since periodic solutions to (1.2) must have a high degree of smoothness, in particular real analyticity [Wri55, Nus73], their Fourier coefficients will decay very rapidly. That is to say, the functional we are interested in can be well approximated by a Galerkin projection onto a finite number of Fourier modes.

In finite dimensions, there are efficacious techniques for rigorously locating and enumerating the solutions to a system of nonlinear equations by way of interval arithmetic [Neu90,HW03,MKC09]. We apply these techniques in infinite dimensions, specifically the *branch and bound* method, also referred to as a *branch and prune* method. That is, we first construct a bounded set X of Fourier coefficients which contains all the zeros of F (see Section 5.4). Then we partition X into a finite number of pieces $\{X_n\}$ which we refer to as *cubes* (see Definition 5.1.6). For each cube X_n we are interested to know whether:

(a) there exists a unique point $\hat{x} \in X_n$ for which $F(\hat{x}) = 0$, or

(b) there does not exist any points $\hat{x} \in X_n$ for which $F(\hat{x}) = 0$.

If we can show that (a) holds for one cube, and (b) holds for all the other cubes, then we will have shown that $F = 0$ has a unique solution.

This approach requires some additional preparation. Since periodic orbits to (1.2) form a 2-manifold in phase space [Reg89], the functional F we construct in Section 5.1.2

will not have isolated zeros. The numerical techniques we employ are suited to finding isolated zeros, so it is necessary to reduce the dimension of the kernel by two. Along the principal branch α can be taken as one of the coordinate dimensions. We reduce this dimension by treating α as a parameter and performing our estimates uniformly in α. The other dimension can be attributed to time translation; if $y(t)$ is a periodic orbit, then so is $y(t+\tau)$ for any $\tau \in \mathbb{R}$. We reduce this dimension by imposing a phase condition; we may assume without loss of generality that the first Fourier coefficient is a positive real number (see Proposition 5.4.4).

The central technique we use to determine whether (a) or (b) holds for a given cube is the Krawczyk method [Neu90,MKC09,HW03,Moo77]. For a function $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ the Krawczyk operator takes as input a rectangular set $X \subseteq \mathbb{R}^n$ and produces as output a rectangular set $\mathcal{K}(X) \subseteq \mathbb{R}^n$. This set $\mathcal{K}(X)$ has the properties that, (i) if $\mathcal{K}(X) \subseteq X$, then there exists a unique point $\hat{x} \in X$ for which $f(\hat{x}) = 0$, and (ii) if $\hat{x} \in X$ and $f(\hat{x}) = 0$, then $\hat{x} \in \mathcal{K}(X)$. Clearly (i) implies (a), and if $X \cap \mathcal{K}(X) = \emptyset$ then (b) follows. Additionally, even if we can prove neither (a) nor (b) our situation could still improve; we can replace $X \mapsto X \cap \mathcal{K}(X)$ without losing any solutions.

Adjustments are needed to generalize the Krawczyk operator to infinite dimensional systems. In [GZ07] a Krawczyk operator is defined in Hilbert space to study fixed points and period-2 orbits in an infinite dimensional map. In Section 5.1.1 we present a generalization of the Krawczyk operator to Banach spaces.

To determine whether (a) or (b) holds the Krawczyk operator by itself is not always sufficient, and we combine several additional tests to create a single *pruning operator* (see Section 5.3). One problem is that $y \equiv 0$ is always a trivial periodic solution to (1.2). To avoid this pitfall we use Lemma 5.1.7, a corollary to Propositions 3.5.1 and 3.5.2, which rules out small periodic solutions. A further difficulty is that at the Hopf bifurcation, the principal branch of periodic solutions is pinched to a point as their amplitudes approach zero. To handle this case, we use Theorem 2.3.11 which explicitly gives a neighborhood about the Hopf bifurcation within which the only solutions that could exist are on the principal branch. Lastly, and most simply, if we can directly show that $\|F\|$ is bounded away from zero on a cube X_n, then (b) holds.

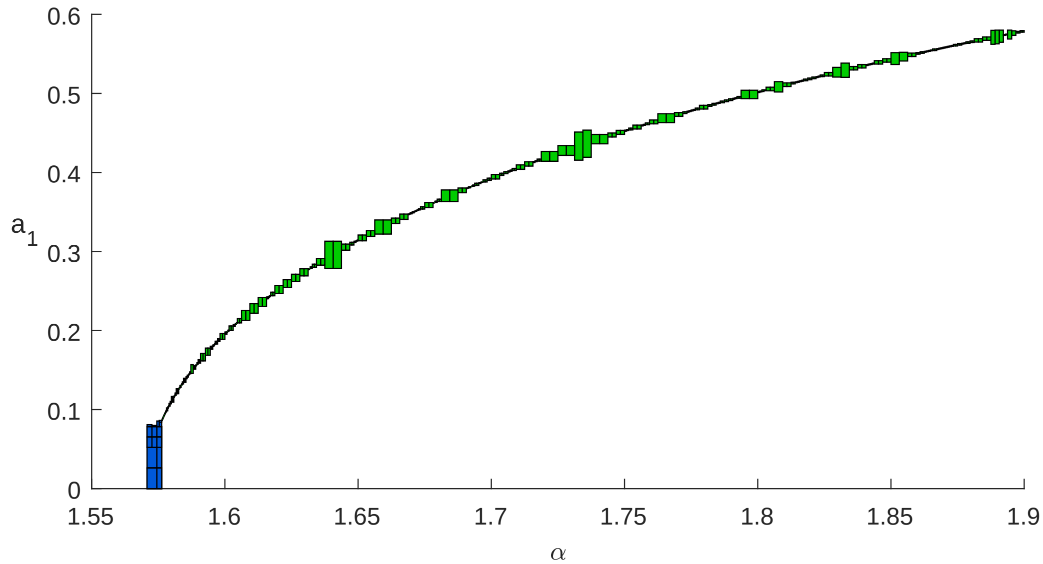


Figure 5.1: The main result of this chapter is a collection of "cubes" in Fourier space which cover the Fourier coefficients of SOPS to (1.2). The first Fourier coefficient of this cover is plotted here with respect to α. Inside each green cube there exists a unique SOPS corresponding to each α, essentially by Theorem 5.1.2. Inside each blue cube the only SOPS that can exist are on the principal branch, by Theorem 2.3.11.

Algorithm 5.5.1 follows the standard format of a global branch and bound method. In short, for a collection of cubes we successively prune each of its cubes. If (a) holds for a given cube, then it is set aside and added to a list of solutions. If (b) holds for a given cube, then that cube is discarded. If the pruning operator significantly reduces the size of a cube, then the pruning operator is applied again. If none of these are the case, then the cube is split in half, and both pieces are added back to the collection of cubes to inspect. This process repeats until all of the cubes have been removed or reduced to a sufficiently small size.

The output of Algorithm 5.5.1 is three collections of cubes: $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{R}$ (see Figure 5.1). In Theorem 5.5.2 we show that these sets have the properties that, (i) each cube in $\mathcal{A}$ has a unique solution with respect to α, (ii) the cubes in $\mathcal{B}$ are near the Hopf bifurcation, with any solutions contained therein residing on the principal branch, and (iii) all solutions to $F = 0$ are contained in $\bigcup \mathcal{A} \cup \mathcal{B} \cup \mathcal{R}$.

Ideally $\mathcal{R} = \emptyset$, and this will often be the case if the zeros of F are simple and the algorithm is allowed to run a sufficiently long time. However we are trying to verify not just simple, isolated solutions, but a 1-parameter family of solutions. As such, sometimes when a cube is split in two this division will bisect the curve of solutions (see Figure 5.4). When this occurs the algorithm will be forced to subdivide many cubes near where the solution curve was bisected, resulting in the variably sized cubes noticeable in Figure 5.1. To address this we recombine the cubes in $\mathcal{R}$ which have the same α values, then subsequently use the Krawczyk operator to show that (a) holds on the recombined cubes (see Algorithm 5.5.3). In this fashion, we prove Theorem 1.2.3.

5.1.1 Krawczyk Operator

In numerical analysis there are many variations on the theme of Newton's method: $x_{n+1} \mapsto x_n - Df(x_n)^{-1}f(x_n)$. As inverting a matrix is computationally expensive, one alternative method is to replace $DF(x_n)^{-1}$ with a fixed matrix $A^\dagger \approx Df(x_0)^{-1}$. If $f(x_0) \approx 0$, then the Newton-Kantorovich theorem gives conditions for when the map $T(x) = x - A^\dagger f(x)$ defines a contraction map in a neighborhood about x_0. The Krawczyk operator may be thought of as a way of bounding the image of T, itself being defined on rectangular sets $X \subseteq \mathbb{R}^n$ and having the property that $T(X) \subseteq \mathcal{K}(X, x_0)$. Rectangular, in the sense that X can be given as the product of intervals in the coordinate directions of $\mathbb{R}^n$. Here we generalize the Krawczyk operator to non-rectangular subsets of Banach spaces.

Definition 5.1.1. *Let Y, Z denote Banach spaces and let $A^\dagger : Z \to Y$ be an injective, bounded linear operator. Fix a convex, closed and bounded set $X \subseteq Y$, a neighborhood $U \supseteq X$, and a Frechet differentiable function $f : U \to Z$. Let*

$$(I - A^\dagger Df(X))(X - \bar{x}) = \overline{conv}\left(\bigcup_{x_1, x_2 \in X} (I - A^\dagger Df(x_1))(x_2 - \bar{x}) \right),$$

where $\overline{conv}$ denotes the closure of the convex hull. For a point $\bar{x} \in X$ we define the Krawczyk operator $\mathcal{K}(X, \bar{x})$ as:

$$\mathcal{K}(X, \bar{x}) := \bar{x} - A^\dagger f(\bar{x}) + (I - A^\dagger Df(X))(X - \bar{x}) \subseteq Y. \tag{5.1}$$

Typically $\bar{x}$ is taken to be the center of X, and $A^\dagger$ is taken to be an approximate inverse of $DF(\bar{x})$. If $\mathcal{K}(X, \bar{x}) \subseteq X$ for a rectangular set $X \subseteq \mathbb{R}^n$, then there exists a unique $\hat{x}$

such that $f(\hat{x}) = 0$. In Theorem 5.1.2 we prove an analogous result. The existence of a fixed point is achieved by the Schauder fixed point theorem. However to prove uniqueness, dropping the rectangular condition causes problems even in finite dimensions; in Theorem 5.1.2 *(iv)* we prescribe a hypothesis sufficient for proving uniquessness in our level of generality.

Theorem 5.1.2. *Suppose $\mathcal{K}$ is a Krawczyk operator as given in Definition 5.1.1 and $T := x - A^\dagger f(x)$.*

(i) If $x \in X$, then $T(x) \in \mathcal{K}(X, \bar{x})$.

(ii) If $\hat{x} \in X$ and $f(\hat{x}) = 0$, then $\hat{x} \in \mathcal{K}(X, \bar{x})$.

(iii) If $\mathcal{K}(X, \bar{x}) \subseteq X$ and X is compact, then there exists a point $\hat{x} \in X$ such that $f(\hat{x}) = 0$.

(iv) If $\mathcal{K}(X, \bar{x}) \subseteq X$ and there exists $0 \leq \lambda < 1$ such that $(I - A^\dagger Df(X))(X - \bar{x}) \subseteq \lambda \cdot (X - \bar{x})$, then there exists a unique point $\hat{x} \in X$ such that $f(\hat{x}) = 0$.

Proof.

(i) Fix a point $x \in X$ and write $h = x - \bar{x}$. By the mean-value theorem for Frechet differentiable functions [AP95], we have:

$$\begin{aligned} T(x) &= \bar{x} - A^\dagger f(\bar{x}) + \int_0^1 DT(\bar{x} + th) \cdot h \, dt \\ &= \bar{x} - A^\dagger f(\bar{x}) + \lim_{N \to \infty} \sum_{i=1}^{N} \tfrac{1}{N} \left(I - A^\dagger Df(\bar{x} + \tfrac{i}{N} h) \right) \cdot h \\ &\in \bar{x} - A^\dagger f(\bar{x}) + \overline{conv} \left(\left(I - A^\dagger Df(X) \right) \cdot (x - \bar{x}) \right) \\ &\subseteq \mathcal{K}(X, \bar{x}). \end{aligned}$$

(ii) If there is some $\hat{x} \in X$ such that $f(\hat{x}) = 0$, then $\hat{x} = T(\hat{x}) \in \mathcal{K}(X, \bar{x})$.

(iii) Since $T(X) \subseteq \mathcal{K}(X, \bar{x})$ by *(i)* and $\mathcal{K}(X, \bar{x}) \subseteq X$ by assumption, therefore $T(X) \subseteq X$. As T is continuous and X is convex and compact, then by the Schauder fixed point theorem there exists some $\hat{x} \in X$ such that $\hat{x} = T(\hat{x})$. Since A is injective, the zeros of f are in bijective correspondence with the fixed points of T, thereby $f(\hat{x}) = 0$.

(iv) Inductively define: $X_0 = X$, $x_0 = \bar{x}$, and $X_{n+1} = T(X_n)$, $x_{n+1} = T(x_n)$. Note that as $T(X) \subseteq X$ then $X_{n+1} \subseteq X_n$ for all n. We show that $X_n \subseteq x_n + \lambda^n(X_0 - x_0)$. This is clearly true for $n = 0$. For $n \geq 1$ then:

$$\begin{aligned} X_{n+1} &\subseteq \mathcal{K}(X_n, x_n) \\ &= x_n - A^\dagger f(x_n) + (I - A^\dagger Df(X_n)) \cdot (X_n - x_n) \\ &\subseteq x_{n+1} + (I - A^\dagger Df(X_0)) \cdot \lambda^n(X_0 - x_0) \\ &\subseteq x_{n+1} + \lambda^{n+1}(X_0 - x_0). \end{aligned}$$

Since $\lambda^n \|X_0 - x_0\|$ can be made arbitrarily small and $\{x_n\}_{n=N}^\infty \subseteq X_N$, it follows that $\{x_n\}$ is a Cauchy sequence. As X is complete, then $\lim x_n = \hat{x}$ and additionally $\bigcap_{n=0}^\infty X_n = \hat{x}$. Thereby $\hat{x}$ is the unique fixed point of T in $X_0 = X$ and the unique zero of f in X.

□

5.1.2 Functions and Domains

As in Chapter 2, we convert Wright's equation into a functional equation on the space of Fourier coefficients. For a continuous periodic function $y : \mathbb{R} \to \mathbb{R}$ with frequency $\omega > 0$, we may write it as:

$$y(t) = \sum_{k \in \mathbb{Z}} c_k e^{i\omega k t} \tag{5.2}$$

where $c_k \in \mathbb{C}$ and $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$. By Theorem 2.1.4 it suffices to work with sequences $\{c_k\}_{k=1}^\infty$ to study periodic solutions to (1.2). Hence we define the following Banach spaces:

$$\ell^1 := \{\{c_k\}_{k=1}^\infty : c_k \in \mathbb{C} \text{ and } \|c\|_{\ell^1} < \infty\} \qquad \|c\|_{\ell^1} = 2\sum_{k=1}^\infty |c_k| \tag{5.3}$$

$$\Omega^s := \{\{c_k\}_{k=1}^\infty : c_k \in \mathbb{C} \text{ and } \|c\|_s < \infty\} \qquad \|c\|_s = \sup_{k\in\mathbb{N}} k^s |c_k|. \tag{5.4}$$

The smoother a function is the faster its Fourier coefficients will decay; if a function is s–times continuously differentiable, then its Fourier coefficients will be in Ω^s. Since periodic solutions to (1.2) are real analytic [Wri55, Nus73], it follows that their Fourier coefficients will be in Ω^s for all $s \geq 0$.

If y is a solution to Wright's equation, then by substituting (5.2) into (1.2) we obtain:

$$\sum_{k\in\mathbb{Z}} i\omega k c_k e^{i\omega kt} = -\alpha \left(\sum_{k\in\mathbb{Z}} c_k e^{-i\omega k} e^{i\omega kt} \right) \left(1 + \sum_{k\in\mathbb{Z}} c_k e^{i\omega kt} \right). \tag{5.5}$$

By matching the $e^{i\omega kt}$ terms, subtracting the RHS, and dividing through by α, we obtain the following sequence of equations for $k \in \mathbb{Z}$ below:

$$[F(\alpha,\omega,c)]_k = \left(i\tfrac{\omega}{\alpha} k + e^{-i\omega k} \right) c_k + \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k}} e^{-i\omega k_1} c_{k_1} c_{k_2} \tag{5.6}$$

$$= \left(i\tfrac{\omega}{\alpha} k + e^{-i\omega k} \right) c_k + \sum_{j=1}^{k-1} e^{-i\omega j} c_j c_{k-j} + \sum_{j=1}^{\infty} \left(e^{-i\omega(j+k)} + e^{i\omega j} \right) c_j^* c_{j+k}. \tag{5.7}$$

Dividing through by α ensures that the parameter dependence in F is solely concentrated in the linear part. Note that $F(\alpha,\omega,c) = \frac{1}{\alpha} G(\alpha,\omega,c)$ for G as defined in (2.7). By Theorem 2.1.4, y is a periodic solution with frequency ω to Wright's equation at parameter α if and only if $[F(\alpha,\omega,c)]_k = 0$ for all $k \in \mathbb{Z}$.

To more succinctly express the functional F we introduce additional notation. For a sequence $c = \{c_k\}_{k=1}^{\infty}$ we denote the projection onto the k-coefficient by $[c]_k := c_k$. We define unnormalized basis elements $e_j \in \ell^1, \Omega^s$ for $j \in \mathbb{N}$ by:

$$[e_j]_k = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

We define the discrete convolution $a * b$ for $a, b \in \ell^1$ component-wise by:

$$[a * b]_k := \sum_{|k_1|+|k_2|=k} a_{k_1} b_{k_2} \qquad = \sum_{j=1}^{k-1} a_j b_{k-j} + \sum_{j=1}^{\infty} a_j^* b_{k+j} + a_{k+j} b_j^*,$$

where $a_{-k} = a_k^*$ and $b_{-k} = b_k^*$, and the sum is taken over $k_1, k_2 \in \mathbb{Z}$. The space ℓ^1 is a Banach algebra, which is to say that $\|a * b\|_{\ell^1} \leq \|a\|_{\ell^1} \|b\|_{\ell^1}$ for all $a, b \in \ell^1$. While Ω^s

is not a Banach algebra *per se*, if $s \geq 2$ then there exists a constant $B \geq 0$ such that $\|a * b\|_s \leq B\|a\|_s\|b\|_s$ for all $a, b \in \Omega^s$ (see [Les10, vdBL08]). Lastly, we define a linear operator $K : \Omega^s \to \Omega^{s+1}$ and a continuous family of linear operators $U_\omega : \Omega^s \to \Omega^{s-1}$ as below:

$$[Kc]_k := c_k/k, \qquad\qquad [U_\omega c]_k := e^{-ik\omega}c_k.$$

The loss of regularity in the range of U_ω is necessary for its continuity, as $\frac{\partial}{\partial\omega}U_\omega = -iK^{-1}U_\omega$. We may extend U_ω to act on bi-infinite sequences $\{c_k\}_{k\in\mathbb{Z}}$ using the same component-wise definition. Additionally, this extension is compatible with our definition of the discrete convolution, as $[U_\omega c]_k^* = [U_\omega c]_{-k}$ whenever $c_k^* = c_{-k}$. In Definition 5.1.3 we rewrite (5.6) in operator notation and list several propositions, the proofs of which are left to the reader.

Definition 5.1.3. *Define the function $F : \mathbb{R}^2 \times \Omega^s \to \Omega^{s-1}$ as:*

$$F(\alpha, \omega, c) := (i\tfrac{\omega}{\alpha}K^{-1} + U_\omega)c + (U_\omega c) * c. \tag{5.8}$$

Proposition 5.1.4. *For each $\alpha > 0$ and $s \geq 2$ the function $F : \mathbb{R}^2 \times \Omega^s \to \Omega^{s-1}$ is Frechet differentiable, with partial derivatives given as:*

$$\frac{\partial}{\partial\omega}F(\alpha, \omega, c) = iK^{-1}(\alpha^{-1}I - U_\omega)c - i(K^{-1}U_\omega c) * c \tag{5.9}$$

$$\frac{\partial}{\partial c}F(\alpha, \omega, c) \cdot h = (i\tfrac{\omega}{\alpha}K^{-1} + U_\omega)h + (U_\omega c) * h + (U_\omega h) * c, \tag{5.10}$$

where $h \in \Omega^s$.

Proposition 5.1.5. *Define $\gamma_1(k, n) := e^{-i\omega(n+k)} + e^{i\omega n}$ and $\gamma_2(k, n) := e^{-i\omega n} + e^{i\omega(n-k)}$.*

Writing $c_k = a_k + ib_k$, the component-wise derivatives of F are given as:

$$\frac{\partial}{\partial \omega}[F(\alpha,\omega,c)]_k = ik(\alpha^{-1} - e^{-i\omega k})c_k - i\sum_{j=1}^{k-1} je^{-i\omega j} c_j c_{k-j}$$
$$- i\sum_{j=1}^{\infty}\left((j+k)e^{-i\omega(j+k)} - je^{i\omega j}\right) c_j^* c_{j+k}.$$

$$\frac{\partial}{\partial a_n}[F(\alpha,\omega,c)]_k = (i\tfrac{\omega}{\alpha}k + e^{-i\omega k}) + \begin{cases} \gamma_1 c_{n+k} + \gamma_2 c_{k-n} & \text{if } 1 \le n < k \\ \gamma_1 c_{n+k} + \gamma_2 c_{n-k}^* & \text{if } k \le n. \end{cases}$$

$$\frac{1}{i}\frac{\partial}{\partial b_n}[F(\alpha,\omega,c)]_k = (i\tfrac{\omega}{\alpha}k + e^{-i\omega k}) + \begin{cases} -\gamma_1 c_{n+k} + \gamma_2 c_{k-n} & \text{if } 1 \le n < k \\ -\gamma_1 c_{n+k} + \gamma_2 c_{n-k}^* & \text{if } k \le n. \end{cases}$$

5.1.3 Decomposition of Phase Space

By working in a space of rapidly decaying Fourier coefficients, we are able to closely approximate the value of F using a Galerkin projection. Since $F : \mathbb{R}^2 \times \Omega^s \to \Omega^{s-1}$ has distinct domain and range, we need to define two sets of projection maps. We define projection maps $\pi_\alpha, \pi_\omega : \mathbb{R}^2 \times \Omega^s \to \mathbb{R}$ and $\pi_c : \mathbb{R}^2 \times \Omega^s \to \Omega^s$ on points $x = (\tilde{\alpha}, \tilde{\omega}, \tilde{c}) \in \mathbb{R}^2 \times \Omega^s$ as:

$$\pi_\alpha(x) := \tilde{\alpha} \qquad \pi_\omega(x) := \tilde{\omega} \qquad \pi_c(x) := \tilde{c}. \tag{5.11}$$

For a fixed integer $M \in \mathbb{N}$, define the projection maps $\pi_M, \pi_\infty : \Omega^s \to \Omega^s$ by:

$$\pi_M(c) := \sum_{k=1}^{M}[c]_k e_k \qquad \pi_\infty(c) := c - \pi_M(c). \tag{5.12}$$

Define the projection maps $\pi'_M, \pi'_\infty : \mathbb{R}^2 \times \Omega^s \to \mathbb{R}^2 \times \Omega^s$ by:

$$\pi'_M(c) := (\pi_\alpha(x), \pi_\omega(x), \pi_M \circ \pi_c(x)) \qquad \pi'_\infty(c) := (0, 0, \pi_\infty \circ \pi_c(x)). \tag{5.13}$$

For any bounded set $X \subseteq \mathbb{R}^2 \times \Omega^s$, define:

$$|X|_k := \sup_{x \in X} |[\pi_c(x)]_k|.$$

We define for F its Galerkin projection and remainder $F_M, F_\infty : \mathbb{R}^2 \times \Omega^s \to \Omega^{s-1}$ as follows:

$$F_M(x) := \pi_M \circ F(\pi'_M(x)), \qquad F_\infty(x) := F(x) - F_M(x). \tag{5.14}$$

By construction $F = F_M + F_\infty$.

To show that there is a unique SOPS to (1.2) we need to evaluate F not just on single points but on voluminous subsets of its domain. The central subset of $\mathbb{R}^2 \times \Omega^s$ we consider in this chapter are *cubes* which we define as follows:

Definition 5.1.6. *For $M \in \mathbb{N}$, $s \geq 0$, $C_0 > 0$ define a cube $X := X_M \times X_\infty \subseteq \mathbb{R}^2 \times \Omega^s$ to be of the following form:*

$$X_M := [\underline{\alpha}, \overline{\alpha}] \times [\underline{\omega}, \overline{\omega}] \times \prod_{k=1}^{M} [\underline{A}_k, \overline{A}_k] \times [\underline{B}_k, \overline{B}_k] \tag{5.15}$$

$$X_\infty := \{c_k \in \mathbb{C} : |c_k| \leq C_0/k^s\}_{k=M+1}^{\infty} . \tag{5.16}$$

To denote the union of a collection of cubes $\mathcal{S} := \{X_i \subseteq \mathbb{R}^2 \times \Omega^s\}$ we define $\bigcup \mathcal{S} := \bigcup_{X \in \mathcal{S}} X \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s$.

There are primarily two reasons we have chosen to consider cubical subsets of $\mathbb{R}^2 \times \Omega^s$. Firstly, cubes are particularly easy to refine into smaller pieces. This is useful because to begin using a branch and bound method, we need to obtain global bounds on the solution space, and then partition these bounds into smaller pieces. In practice, we reduce the size of a cube by either subdividing it along a lower dimension into two cubes, or replacing the cube by its intersection with the Krawczyk operator: $X \mapsto X \cap \mathcal{K}(X, \bar{x})$. In both these cases the resulting object is again a cube. In this manner, we can use cubes to cover the solutions to $F = 0$, and then refine the cover using successively smaller cubes.

Secondly, cubes facilitate explicit computations of F_M and analytical estimates of F_∞. While formally F_M is an infinite dimensional map, computationally, we may consider F_M to be a map $\mathbb{R}^2 \times \mathbb{C}^M \to \mathbb{C}^M$. To calculate F_M, we simply truncate the second sum in (5.7) at $j = M - k$. As the π'_M projection of a cube is given as a finite product of intervals, it is well suited for using interval arithmetic [MKC09] to bound the image of $F_M(X)$. On the other hand, bounding F_∞ requires significantly more analysis. Below is a simple, yet ever recurring estimate in our calculations:

$$\sum_{k=M+1}^{\infty} \frac{1}{k^s} \leq \int_M^\infty \frac{1}{x^s} dx = \frac{1}{(s-1)M^{s-1}}, \tag{5.17}$$

where we take $s > 1$. For example, if a cube $X \subseteq \mathbb{R}^2 \times \Omega^s$ satisfies $s > 1$, then $\|\pi_c x\|_{\ell^1} \leq 2\sum_{k=1}^M |X|_k + \frac{2C_0}{(s-1)M^{s-1}}$ for all $x \in X$. This specific bound on the ℓ^1 norm is later used in Algorithm 5.3.1 to check whether Theorem 2.3.11 or Lemma 5.1.7 apply.

Lemma 5.1.7. *Let $\omega \geq 1.1$, $\alpha \in (0, 2]$, and define*

$$g(\alpha, \omega) := \sqrt{\left(1 - \tfrac{\omega}{\alpha}\right)^2 + 2\,\tfrac{\omega}{\alpha}\,(1 - \sin\omega)}. \tag{5.18}$$

If $F(\alpha, \omega, c) = 0$, then either $c \equiv 0$ or $g(\alpha, \omega) \leq \|c\|_{\ell^1}$.

Proof. See Propositions 3.5.1 and 3.5.2. □

The remainder of this section is dedicated to proving Lemma 5.1.10, which estimates F_∞, its derivatives, and convolution products resulting from points inside of a cube. These estimates are used in Definition 5.2.2 to construct an outer approximation to the Krawczyk operator. The reader is encouraged to skip the proof of Lemma 5.1.10 on a first reading, which is best summarized as bounding various infinite sums by various finite sums and the estimate in (5.17). These bounds are presented in Definition 5.1.9, all of which are given as a finite number of operations, explicitly computable in terms of C_0 and the π'_M-projection of a given cube. In Lemma 5.1.8 we define the constant γ_M which is needed for the definition of (5.24).

Lemma 5.1.8 (Lemma 24 [vdBL08])**.** *Let $s \geq 2$ and let s_* be the largest integer such that $s_* \leq s$ and define:*

$$\gamma_k := 2\left[\frac{k}{k-1}\right]^s + \left[\frac{4\ln(k-2)}{k} + \frac{\pi^2 - 6}{3}\right]\left[\frac{2}{k} + \frac{1}{2}\right]^{s_*-2}.$$

For $k \geq 4$, we have that $\sum_{k_1=1}^{k-1} \frac{k^s}{k_1^s (k-k_1)^s} \leq \gamma_k$. If $6 \leq M \leq k$, then $\gamma_k \leq \gamma_M$.

Definition 5.1.9. *Fix a cube X with $s > 2$, define $C_1 := \sup_{x \in X} \|\pi_c x\|_s$, and select a point $\bar{x} = (\bar{\alpha}, \bar{\omega}, \bar{c}) \in X$ such that $\bar{x} = \pi'_M(\bar{x})$. Define $H = X - \bar{x}$, and define $\Delta_\omega \in \mathbb{R}$ such that $\Delta_\omega \geq \sup_{x \in H} |\pi_\omega(x) - \bar{\omega}|$.*

Define h, g^i_M, g^{ii}_M to be functions of the form $g_M : X \mapsto g_M(X) \in \mathbb{R}^M$ and define $g^i_\infty, g^{ii,a}_\infty, g^{ii,b}_\infty$ to be functions of the form $g_\infty : X \mapsto g_\infty(X) \in \mathbb{R}$ as follows:

$$[h(X)]_k := \frac{2C_0^2}{(s-1)M^{s-1}(M+k+1)^s} + 2C_0 \sum_{j=M-k+1}^{M} \frac{|X|_j}{(j+k)^s} \tag{5.19}$$

$$\begin{aligned}[g_M^i(X)]_k := {} & 2C_0\Delta_\omega \sum_{j=M-k+1}^{M} \frac{|X|_j}{(j+k)^{(s-1)}} \\ & + \frac{C_0^2\Delta_\omega}{(s-2)(M+k+1)^s M^{(s-2)}} + \frac{C_0^2\Delta_\omega}{(s-1)(M+k+1)^{(s-1)}M^{(s-1)}}\end{aligned} \tag{5.20}$$

$$[g_M^{ii}(X)]_k := \frac{4C_0^2}{(s-1)(M+k+1)^s M^{s-1}} + 2C_0 \sum_{j=M-k+1}^{M} \frac{|H|_j}{(j+k)^s} \tag{5.21}$$

$$g_\infty^i(X) := \max_{M+1\le k\le 2M} k^s \sum_{j=k-M}^{M} |\bar{c}_j \bar{c}_{k-j}| \tag{5.22}$$

$$\begin{aligned}g_\infty^{ii,a}(X) := {} & \max_{M+1\le k\le 2M} k^s \sum_{j=k-m}^{M} |H|_j |X|_{k-j} \\ & + \frac{2C_0^2(2^s+1)}{(s-1)M^{s-1}} + C_0 \sum_{j=1}^{M} (|X|_j + |H|_j)\left(\left(\frac{M+j+1}{M+1}\right)^s + 1\right)\end{aligned} \tag{5.23}$$

$$g_\infty^{ii,b}(X) := \frac{C_1^2\gamma_{M+1}}{2} + C_0C_1\left(\frac{s-1}{(M+2)(s-2)} + \frac{s}{s-1}\right). \tag{5.24}$$

Lemma 5.1.10. *Fix a cube X with $M \geq 5$, $s > 2$, a point $\bar{x} \in X$ such that $\bar{x} = \pi'_M(\bar{x})$, and define $H = X - \bar{x}$. Then the following inequalities hold:*

$$\sup_{x\in X} |F_\infty(x)|_k < [h(X)]_k \qquad 1 \le k \le M \tag{5.25}$$

$$\sup_{x\in X, h\in H} \left|\tfrac{\partial}{\partial\omega} F_\infty(x)\cdot \pi_\omega(h)\right|_k \le [g_M^i(X)]_k \qquad 1 \le k \le M \tag{5.26}$$

$$\sup_{x\in X, h\in H} \left|\tfrac{\partial}{\partial c} F_\infty(x)\cdot \pi_c(h)\right|_k \le [g_M^{ii}(X)]_k \qquad 1 \le k \le M \tag{5.27}$$

$$|F_\infty(\bar{x})|_k \le \frac{1}{k^s} g_\infty^i(X) \qquad M+1 \le k \tag{5.28}$$

$$\sup_{x\in X, h\in H} |\pi_c(h) * \pi_c(x)|_k \le \frac{1}{k^s} g_\infty^{ii,a}(X) \qquad M+1 \le k \tag{5.29}$$

$$\sup_{x_1,x_2\in X} \left|(K^{-1}\pi_c(x_1)) * \pi_c(x_2)\right|_k \le \frac{1}{k^{s-1}} g_\infty^{ii,b}(X) \qquad M+1 \le k. \tag{5.30}$$

Throughout, let us write $X_M = \pi'_M(X)$, $H_M = \pi'_M(H)$, and $H_\infty = \pi'_\infty(H)$, noting also that $H_\infty = \pi'_\infty(X)$.

Proof of (5.25). We show that $|F_\infty(x)|_k < [h(X)]_k$ for $1 \le k \le M$ and all $x \in X$. Fix

$x = (\alpha, \omega, c) \in X$, and write $c_M = \pi_M(c)$ and $c_\infty = \pi_\infty(c)$. We compute:

$$\begin{aligned}
\pi_M \circ F_\infty(x) &= \pi_M \circ \big(F(x) - F(\pi_M' x)\big) \\
&= \pi_M \circ ((U_\omega c) * c - (U_\omega c_M) * c_M) \\
&= \pi_M \circ ((U_\omega c_M) * c_\infty + (U_\omega c_\infty) * c_M + (U_\omega c_\infty) * c_\infty)
\end{aligned}$$

Since $|U_\omega c|_k = |c|_k$, it follows that for $1 \leq k \leq M$ we compute the estimate below:

$$\begin{aligned}
|(U_\omega c_M) * c_\infty|_k + |(U_\omega c_\infty) * c_M|_k \leq& 2\sum_{j=1}^{\infty} |c_M^*|_j |c_\infty|_{k+j} + |c_M|_{k+j} |c_\infty^*|_j \\
=& 2 \sum_{j=M-k+1}^{M} |c_M^*|_j \, |c_\infty|_{j+k} \\
\leq& 2 \sum_{j=M-k+1}^{M} |X|_j \frac{C_0}{(j+k)^s}.
\end{aligned}$$

The last estimate uses the property that $|c_j| \leq C_0/j^s$ for $j \geq M+1$.

We calculate $(U_\omega c_\infty) * c_\infty$ as below, again using $|c_j| \leq C_0/j^s$ for $j \geq M+1$.

$$\begin{aligned}
|(U_\omega c_\infty) * c_\infty|_k &\leq \sum_{j=M+1}^{\infty} |c_\infty^*|_j |c_\infty|_{k+j} + |c_\infty|_{j+k} |c_\infty^*|_j \\
&\leq \sum_{j=M+1}^{\infty} \frac{2C_0^2}{j^s (j+k)^s} \leq \frac{2C_0^2}{(s-1)M^{s-1}(M+k+1)^s}.
\end{aligned}$$

Hence for $1 \leq k \leq M$, it follows that:

$$\begin{aligned}
|F_\infty(x)|_k \leq& \frac{2C_0^2}{(s-1)M^{s-1}(M+k+1)^s} + 2C_0 \sum_{j=M-k+1}^{M} \frac{|X|_j}{(j+k)^s} \\
=& [h(X)]_k.
\end{aligned}$$

$\square$

Proof of (5.26). We show that $\left|\frac{\partial}{\partial\omega} F_\infty(x) \cdot \pi_\omega(h)\right|_k \leq [g_M^i(X)]_k$ for $1 \leq k \leq M$ and all $x \in X$ and $h \in H$. Select some $x = (\alpha, \omega, c) \in X$ and write $c_M = \pi_M(c)$ and $c_\infty = \pi_\infty(c)$. From (5.9) we can calculate $\frac{\partial}{\partial\omega} F_\infty(x)$ as follows:

$$\begin{aligned}
\tfrac{\partial}{\partial\omega} F_\infty(x) &= -i(K^{-1} U_\omega c) * c + i\pi_M (K^{-1} U_\omega c_M) * c_M \\
&= -i\pi_\infty \left(K^{-1} U_\omega c_M\right) * c_M - i\left(K^{-1} U_\omega c_M\right) * c_\infty - i\left(K^{-1} U_\omega c_\infty\right)(c_M + c_\infty).
\end{aligned}$$

Hence, for $1 \le k \le M$ we may calculate the following:

$$\left|\tfrac{\partial}{\partial\omega}F_\infty(x)\right|_k \le \sup_{c_M\in X_M;\, c_\infty, c'_\infty \in H_\infty} \left|(K^{-1}c_M) * c_\infty + (K^{-1}c_\infty) * c_M + (K^{-1}c_\infty) * c'_\infty\right|_k . \tag{5.31}$$

For $1 \le k \le M$ and any $c_M \in X_M, c_\infty \in H_\infty$ we can simplify the first two summands in (5.31) as follows:

$$\begin{aligned}(K^{-1}c_M) *_k c_\infty &= \sum_{j=1}^{\infty}[\mathcal{K}^{-1}c_M^*]_j[c_\infty]_{k+j} + [\mathcal{K}^{-1}c_M]_{k+j}[c_\infty^*]_j \\ &= \sum_{j=M+1-k}^{\infty} j[c_M^*]_j[c_\infty]_{k+j} \\ (K^{-1}c_\infty) *_k c_M &= \sum_{j=1}^{\infty}[\mathcal{K}^{-1}c_\infty^*]_j[c_M]_{k+j} + [\mathcal{K}^{-1}c_\infty]_{k+j}[c_M^*]_j \\ &= \sum_{j=M+1-k}^{\infty} (k+j)[c_\infty]_{k+j}[c_M^*]_j.\end{aligned}$$

Hence, we have the following estimate:

$$\begin{aligned}(K^{-1}c_M) *_k c_\infty + (K^{-1}c_\infty) *_k c_M &= \sum_{j=M-k+1}^{M}(2j+k)[c_\infty]_{j+k}[c_M^*]_j \\ \left|(K^{-1}c_M) * c_\infty\right|_k + \left|(K^{-1}c_\infty) * c_M\right|_k &\le \sum_{j=M-k+1}^{M}\frac{(2j+k)C_0}{(j+k)^s}|X|_j \\ &\le 2C_0 \sum_{j=M-k+1}^{M}\frac{|X|_j}{(j+k)^{s-1}}.\end{aligned} \tag{5.32}$$

Again, we used the estimate $|c_j| \le C_0/j^s$ for $j \ge M+1$. We estimate the third summand in (5.31) for $c_\infty, c'_\infty \in H_\infty$ as follows:

$$\begin{aligned}(K^{-1}c_\infty) *_k c'_\infty &= \sum_{j=M+1}^{\infty} j[c_\infty^*]_j[c'_\infty]_{k+j} + (j+k)[c_\infty]_{j+k}[{c'_\infty}^*]_j \\ \left|(K^{-1}c_\infty) * c'_\infty\right|_k &\le \sum_{j=M+1}^{\infty}\frac{C_0^2}{j^{(s-1)}(j+k)^s} + \frac{C_0^2}{j^s(j+k)^{(s-1)}} \\ &\le \frac{C_0^2}{(s-2)(M+k+1)^s M^{(s-2)}} + \frac{C_0^2}{(s-1)(M+k+1)^{(s-1)}M^{(s-1)}}.\end{aligned} \tag{5.33}$$

By combining the estimates from (5.32) and (5.33) into (5.31), and recalling our choice of Δ_ω in Definition 5.1.9, then for $1 \le k \le M$ we obtain the following:

$$\sup_{x\in X, h\in H} \left|\tfrac{\partial}{\partial\omega} F_\infty(x)\pi_\omega(h)\right|_k \le 2C_0\Delta_\omega \sum_{j=M-k+1}^{M} \frac{|X|_j}{(j+k)^{(s-1)}} + \frac{C_0^2\Delta_\omega}{(s-2)(M+k+1)^s M^{(s-2)}}$$
$$+ \frac{C_0^2\Delta_\omega}{(s-1)(M+k+1)^{(s-1)} M^{(s-1)}}$$
$$= [g_M^i(X)]_k.$$

□

Proof of (5.27). We show that $\left|\tfrac{\partial}{\partial c} F_\infty(x)\cdot\pi_c(h)\right|_k \le [g_M^{ii}(X)]_k$ for $1 \le k \le M$ and all $x \in X$ and $h \in H$. Let $(\alpha, \omega, c) \in X$ and $h \in \pi_c(H)$. From (5.10) we calculate $\tfrac{\partial}{\partial c}(F(X) - F_M(X)) \cdot h$ below:

$$\tfrac{\partial}{\partial c}(F(x) - F(\pi'_M x)) \cdot h = ((U_\omega h) * c + (U_\omega c) * h) - ((U_\omega h) * c_M + (U_\omega c_M) * h)$$
$$= (U_\omega h) * (c - c_M) + (U_\omega(c - c_M)) * h.$$

Since $c - c_M \in H_\infty$, it follows that:

$$|\tfrac{\partial}{\partial c}[F(x) - F(\pi'_M x)] \cdot h|_k \le \sup_{h\in H, h'\in H_\infty} 2 \cdot |h * h'_\infty|_k.$$

For $h \in H$ and $h' \in H_\infty$ and for $1 \le k \le M$, we calculate $h *_k h'$ below, using the property that $[h']_j = 0$ for $j \le M$.

$$h *_k h' = \sum_{j=1}^{\infty} [h^*]_j [h']_{k+j} + [h]_{k+j}[h'^*]_j$$
$$= \sum_{j=M-k+1}^{M} [h^*]_j [h']_{k+j} + \sum_{j=M+1}^{\infty} [h^*]_j [h']_{k+j} + [h]_{k+j}[h'^*]_j.$$

By applying the estimates $|h_j| \le |H|_j$ for $j \le M$, and $|h|_j, |h'|_j \le C_0/j^s$ for $j \ge M+1$, we obtain the following:

$$\left|\tfrac{\partial}{\partial c} F_\infty(x)\cdot h\right|_k \le 2\left(\sum_{j=M-k+1}^{M} |H|_j \frac{C_0}{(j+k)^s} + \sum_{j=M+1}^{\infty} \frac{2C_0^2}{j^s (j+k)^s}\right)$$
$$\le 2C_0 \sum_{j=M-k+1}^{M} \frac{|H|_j}{(j+k)^s} + \frac{4C_0^2}{(s-1)(M+k+1)^s M^{s-1}}$$
$$= [g_M^{ii}(X)]_k.$$

□

Proof of (5.28). We show that $|F_\infty(\bar{\alpha},\bar{\omega},\bar{c})|_k \leq \frac{1}{k^s} g^i_\infty(X)$ for $M+1 \leq k$. Since $\pi'_M(\bar{x}) = \bar{x}$ and $[\bar{c}]_k = 0$ for $k \geq M+1$, it follows that:

$$[F_\infty(\bar{\alpha},\bar{\omega},\bar{c})]_k = \begin{cases} 0 & \text{if } k \leq M \\ \sum_{j=1}^{k-1} e^{-i\omega j} \bar{c}_j \bar{c}_{k-j} & \text{otherwise.} \end{cases} \tag{5.34}$$

As $\bar{c}_j \bar{c}_{k-j} = 0$ when either $j > M$ or $k-j > M$, then it follows that:

$$|F_\infty(\bar{\alpha},\bar{\omega},\bar{c})|_k \leq \sum_{j=k-M}^{M} |\bar{c}_j \bar{c}_{k-j}|.$$

Noting that $|F_\infty(\bar{\alpha},\bar{\omega},\bar{c})|_k = 0$ for $k > 2M$, we calculate:

$$\begin{aligned} |F_\infty(\bar{\alpha},\bar{\omega},\bar{c})|_k &\leq k^{-s} \max_{M+1 \leq k_0 \leq 2M} k_0^s \sum_{j=k_0-M}^{M} |\bar{c}_j \bar{c}_{k_0-j}| \\ &= k^{-s} g^i_\infty(X). \end{aligned}$$

□

Proof of (5.29). We show that $|h * c|_k \leq \frac{1}{k^s} g^{ii,a}_\infty(X)$ for $M+1 \leq k$ and all $c \in \pi_c(X)$ and $h \in \pi_c(H)$. Fix $x = (\alpha,\omega,c) \in X$ and $h \in \pi_c(H)$, and write $c_M = \pi_M(c), c_\infty = \pi_\infty(c), h_M = \pi_M(h)$, and $h_\infty = \pi_\infty(h)$. We may expand $h * c$ as follows:

$$h * c = h_M * c_M + h_M * c_\infty + c_M * h_\infty + h_\infty * c_\infty. \tag{5.35}$$

The composition $h_M * c_M$ only has non-zero components for $M+1 \leq k \leq 2M$, thereby it is bounded by the computable value below:

$$\begin{aligned} h_M *_k c_M &\leq \tfrac{1}{k^s} \max\{k_0^s \cdot h_M *_{k_0} c_M : M+1 \leq k_0 \leq 2M\} \\ &\leq \frac{1}{k^s} \max_{M+1 \leq k_0 \leq 2M} k_0^s \sum_{j=k_0-m}^{M} |H|_j |X|_{k_0-j}. \end{aligned} \tag{5.36}$$

We calculate $c_M * h_\infty$ for $k \geq M+1$, noting that $[h_\infty]_{k-j} = 0$ if $k-j \leq M$, as below:

$$\begin{aligned} c_M *_k h_\infty &= \sum_{j=1}^{k-1} [c_M]_j [h_\infty]_{k-j} + \sum_{j=1}^{\infty} [c_M^*]_j [h_\infty]_{k+j} + [c_M]_{k+j} [h_\infty^*]_j \\ &= \sum_{j=k-M-1}^{M} [c_M]_j [h_\infty]_{k-j} + \sum_{j=1}^{M} [c_M^*]_j [h_\infty]_{k+j}. \end{aligned}$$

Using the estimates $|c_j| \leq |X|_j$ for $j \leq M$ and $|h_j| \leq C_0/j^s$ for $j \geq M+1$, we calculate the following:

$$\begin{aligned} |c_M * h_\infty|_k \leq& \sum_{j=k-M-1}^{M} |X|_j \frac{C_0}{(k-j)^s} + \sum_{j=1}^{M} |X|_j \frac{C_0}{(k+j)^s} \\ \leq& \frac{C_0}{k^s} \left(\sum_{j=k-M-1}^{M} |X|_j \left(\frac{k}{k-j} \right)^s + \sum_{j=1}^{M} |X|_j \right). \end{aligned} \tag{5.37}$$

Note that $\frac{k}{k-j}$ is decreasing with k. To maximize the coefficient of $|X|_j$ in the first sum of (5.37), we choose the smallest k such that $j \leq k-M-1$. Hence, for each coefficient, we choose $k = M+j+1$ as an upper bound. We obtain the following:

$$|c_M * h_\infty|_k \leq \frac{C_0}{k^s} \sum_{j=1}^{M} |X|_j \left(\left(\frac{M+j+1}{M+1} \right)^s + 1 \right). \tag{5.38}$$

An analogous calculation produces a bound for $|h_M * c_\infty|$ as given below:

$$|h_M * c_\infty|_k \leq \frac{C_0}{k^s} \sum_{j=1}^{M} |H|_j \left(\left(\frac{M+j+1}{M+1} \right)^s + 1 \right). \tag{5.39}$$

Lastly we estimate $|h_\infty * c_\infty|_k$. For $h_\infty, c_\infty \in H_\infty$ and $k \geq M+1$ we calculate:

$$\begin{aligned} h_\infty * c_\infty =& \sum_{j=1}^{k-1} [h_\infty]_j [c_\infty]_{k-j} + \sum_{j=1}^{\infty} [h_\infty^*]_j [c_\infty]_{k+j} + [h_\infty]_{k+j} [c_\infty^*]_j \\ =& \sum_{j=M+1}^{k-M-1} [h_\infty]_j [c_\infty]_{k-j} + \sum_{j=M+1}^{\infty} [h_\infty^*]_j [c_\infty]_{k+j} + [h_\infty]_{k+j} [c_\infty^*]_j. \end{aligned}$$

Taking norms and using the estimate $|h_j| \leq C_0/j^s$ for $M+1 \leq j$ we obtain:

$$\begin{aligned} |h_\infty * c_\infty|_k \leq& \sum_{j=M+1}^{k-M-1} \frac{C_0^2}{j^s(k-j)^s} + 2 \sum_{j=M+1}^{\infty} \frac{C_0^2}{j^s(k+j)^s} \\ \leq& C_0^2 \left(\sum_{j=M+1}^{k-M-1} \frac{1}{j^s(k-j)^s} \right) + \frac{2}{k^s} \frac{C_0^2}{(s-1)M^{s-1}}. \end{aligned}$$

The remaining sum is only nonzero for $k \geq 2(M+1)$, and we bound it as follows:

$$\begin{aligned} \sum_{j=M+1}^{k-M-1} \frac{1}{j^s(k-j)^s} =& \frac{1}{k^s} \sum_{j=M+1}^{k-M-1} \left(\frac{1}{j} + \frac{1}{k-j} \right)^s \\ \leq& \frac{2}{k^s} \sum_{j=M+1}^{k/2} \left(\frac{2}{j} \right)^s \\ \leq& \frac{2^{s+1}}{k^s(s-1)} \left(\frac{1}{M^{s-1}} - \frac{1}{(k/2)^{s-1}} \right). \end{aligned}$$

This estimate is maximized in the $\|\cdot\|_s$ norm by taking $k \to \infty$. Thereby, we obtain the following estimate:

$$|h_\infty * c_\infty|_k \leq \frac{1}{k^s} \frac{2C_0^2(2^s+1)}{(s-1)M^{s-1}}. \tag{5.40}$$

By combining the results from (5.36 - 5.40) into (5.35), it follows that if $M+1 \leq k$, then $|h * c|_k \leq \frac{1}{k^s} g_\infty^{ii,a}(X)$.

□

Proof of (5.30). We show that $\left|(K^{-1}\pi_c(x_1)) * \pi_c(x_2)\right|_k \leq \frac{1}{k^{s-1}} g_\infty^{ii,b}(X)$ for $M+1 \leq k$ and all $x_1, x_2 \in X$. For $i = 1,2$ let us fix $c_i \in \pi_c(X)$ and recall that $C_1 \geq \|c_i\|_s$ by Definition 5.1.9. We can write $(K^{-1}c_1) *_k c_2$ as below:

$$(K^{-1}c_1) *_k c_2 = \sum_{j=1}^{k-1} j[c_1]_j[c_2]_{k-j} + \sum_{j=k+1}^{\infty} j[c_1^*]_j[c_2]_{k+j} + (k+j)[c_1]_{k+j}[c_2^*]_j. \tag{5.41}$$

Using $|c|_j \leq C_1/j^s$ and $|c|_{k+j} \leq C_0/(k+j)^s$ for $k \geq M+1$, we obtain a bound on $|(K^{-1}c_1) * c_2|_k$ as below:

$$\begin{aligned}
|(K^{-1}c_1) * c_2|_k &\leq \sum_{j=1}^{k-1} \frac{jC_1C_1}{j^s(k-j)^s} + \sum_{j=1}^{\infty} \frac{C_1C_0}{j^{s-1}(k+j)^s} + \sum_{j=1}^{\infty} \frac{C_0C_1}{(k+j)^{s-1}j^s} \\
&\leq C_1^2 \left(\sum_{j=1}^{k-1} \frac{1}{j^{s-1}(k-j)^s} \right) + \frac{C_1C_0}{(k+1)^s}\left(1 + \frac{1}{s-2}\right) \\
&\quad + \frac{C_1C_0}{(k+1)^{s-1}}\left(1 + \frac{1}{s-1}\right).
\end{aligned}$$

Since $5 \leq M$, thereby $6 \leq M+1 \leq k$ and by Lemma 5.1.8 we can simplify the remaining sum as follows:

$$\sum_{j=1}^{k-1} \frac{1}{j^{s-1}(k-j)^s} = \frac{k}{2}\sum_{j=1}^{k-1} \frac{1}{j^s(k-j)^s} \leq \frac{k}{2}\frac{\gamma_k}{k^s} \leq \frac{\gamma_{M+1}}{2k^{s-1}}.$$

Taking $k \geq M+1$, it follows that:

$$\begin{aligned}
|(K^{-1}\pi_c(x_1)) * \pi_c(x_2)|_k &\leq \frac{1}{k^{s-1}}\left(\frac{C_1^2\gamma_{M+1}}{2} + C_1C_0\left(\frac{s-1}{(M+2)(s-2)} + \frac{s}{s-1} \right)\right) \\
&= \frac{1}{k^{s-1}} g_\infty^{ii,b}(X).
\end{aligned}$$

□

5.2 Bounding the Krawczyk Operator

When defining a Krawczyk operator $\mathcal{K}(X, \bar{x})$ for a function $f : Y \to Z$ one must choose a linear operator $A^\dagger : Z \to Y$. The map $A^\dagger$ is typically chosen to approximate $Df(\bar{x})^{-1}$. Even in finite dimensions it may be impossible to exactly calculate the inverse of a matrix using floating point arithmetic. To denote a fixed but numerically approximate definition, we introduce the notation $:\approx$. Since we set up our theorems in an *a posteriori* format, the question of whether our numerical approximation is sufficiently accurate is answered by whether our computer-assisted proof is successful or not.

As with any method relying on a contraction mapping argument, the Krawczyk operator is only truly effective in locating the zeros of a function if they are isolated. Since the non-trivial zeros of F are not isolated, and in fact form a 2-manifold [Reg89], we do not define a Krawczyk operator corresponding directly to $F : \mathbb{R}^2 \times \Omega^s \to \Omega^{s-1}$. We must first reduce the dimensionality of its domain by two.

We reduce one of the dimensions by imposing a phase condition; we may assume without loss of generality that the first Fourier coefficient is a positive real number (see Proposition 5.4.4). To that end, we define a codimension-1 subspace $\tilde{\Omega}^s \subseteq \Omega^s$ as follows:

$$\tilde{\Omega}^s := \{c \in \Omega^s : c_1 = c_1^*\}.$$

To reduce the other dimension, we consider α as a parameter and perform our estimates uniformly in α.

For a cube $X \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s$ we define a Krawczyk operator to find the zeros of functions $F_\alpha : \mathbb{R}^1 \times \tilde{\Omega}^s \to \Omega^{s-1}$ for all $\alpha \in \pi_\alpha(X)$. To that end, we would like to define a map $A^\dagger$ to be an approximate inverse of the derivative $DF_{\bar{\alpha}}(\bar{\omega}, \bar{c}) \in \mathcal{L}(\mathbb{R}^1 \times \tilde{\Omega}^s, \Omega^{s-1})$ for some $(\bar{\alpha}, \bar{\omega}, \bar{c}) \in X$. We construct this approximate inverse by combining $A_M^\dagger$, a $2M \times 2M$ real matrix on the lower Fourier modes, with the operator $-(i\frac{\bar{\alpha}}{\bar{\omega}})\mathcal{K}\pi'_\infty$ on the higher Fourier modes.

As is ever the case, we may only explicitly perform a finite number of operations on fundamentally finite dimensional objects, and because of this we defined Galerkin projections in (5.12) and (5.13). To ensure the sum $F = F_M + F_\infty$ makes sense, the maps π_M, π'_M are defined to be but finite rank maps onto a subspace of an infinite dimensional Banach space. To emphasize this finite dimensional subspace as a space in its own right, as well as the new domain $\mathbb{R}^1 \times \tilde{\Omega}^s$, we define the following projection and inclusion maps:

$$\tilde{\pi}_M : \Omega^s \twoheadrightarrow \mathbb{R}^{2M}, \quad \tilde{\pi}'_M : \mathbb{R}^1 \times \tilde{\Omega}^s \twoheadrightarrow \mathbb{R}^{2M}, \quad \tilde{i}_M : \mathbb{R}^{2M} \hookrightarrow \Omega^s, \quad \tilde{i}'_M : \mathbb{R}^{2M} \hookrightarrow \mathbb{R}^1 \times \tilde{\Omega}^s.$$

$$\tilde{\pi}_M \circ \tilde{i}_M = id_{\mathbb{R}^{2M}}, \quad \tilde{\pi}'_M \circ \tilde{i}'_M = id_{\mathbb{R}^{2M}}, \quad \tilde{i}_M \circ \tilde{\pi}_M = id_{\Omega^s}, \quad \tilde{i}'_M \circ \tilde{\pi}'_M = id_{\mathbb{R}^1 \times \tilde{\Omega}^s}.$$

We define the linear operator $A^\dagger$ below in Definition 5.2.1 as follows: We note that $A^\dagger$ will be injective if the $2M \times 2M$ matrix $A^\dagger_M$ has rank $2M$.

Definition 5.2.1. *Fix a cube $X \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s$. For a point $(\bar{\alpha}, \bar{\omega}, \bar{c}) = \bar{x} \in X$ such that $\bar{x} = \pi'_M(\bar{x})$, define the following linear operators:*

$$\begin{aligned}
A_M &:\approx \tilde{\pi}_M \circ DF_{\bar{\alpha}}(\bar{\omega}, \bar{c}) \circ \tilde{i}'_M & A_M &\in \mathcal{L}(\mathbb{R}^{2M}, \mathbb{R}^{2M}) \\
A^\dagger_M &:\approx A_M^{-1} & A^\dagger_M &\in \mathcal{L}(\mathbb{R}^{2M}, \mathbb{R}^{2M}) \\
A(\bar{x}, M) &:= \tilde{i}_M \circ A_M \circ \tilde{\pi}'_M + i\tfrac{\bar{\omega}}{\bar{\alpha}} K^{-1} \pi'_\infty & A(\bar{x}, M) &\in \mathcal{L}(\mathbb{R}^1 \times \tilde{\Omega}^s, \Omega^{s-1}) \\
A^\dagger(\bar{x}, M) &:= \tilde{i}'_M \circ A^\dagger_M \circ \tilde{\pi}_M - i\tfrac{\bar{\alpha}}{\bar{\omega}} K \pi_\infty & A^\dagger(\bar{x}, M) &\in \mathcal{L}(\Omega^{s-1}, \mathbb{R}^1 \times \tilde{\Omega}^s).
\end{aligned}$$

While a Krawczyk operator $\mathcal{K}(X, \bar{x})$ given as in Definition 5.1.1 is sufficient from a mathematical perspective, from a computational perspective it leaves something to be desired. We address this deficiency in Definition 5.2.2 by defining an explicitly computable operator $\mathcal{K}'(X, \bar{x})$ as an outer approximation to $\mathcal{K}(X, \bar{x})$, which is to say that $\mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'(X, \bar{x})$. In Theorem 5.2.3 we prove this, and in Theorem 5.2.4 we give an analogue of Theorem 5.1.2.

In practice, we use *interval arithmetic* [MKC09] to compute an outer approximations for the arithmetic combination of sets (e.g. $A + B = \bigcup_{a \in A, b \in B} a + b$). This allows us to bound the image of functions over rectangular domains, which is to say domains given as the product of intervals. By employing outward rounding, interval arithmetic can be rigorously implemented on a computer [Rum99]. In every step an outer approximation is constructed as a rectangular domain, and the end result will too be an outer approximation. While obtaining a tight approximation is desirable, it is not required; as long as we have an outer approximation, that is sufficient.

Definition 5.2.2. *Fix a cube $X \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s$ as in Definition 5.1.6 with $M \geq 5$, $s > 2$ and $C_0 > 0$. Fix some $\bar{x} = (\bar{\alpha}, \bar{\omega}, \bar{c}) \in X$ such that $\bar{x} = \pi'_M(\bar{x})$ and $\Delta_\omega \geq \sup_{x \in X} |\pi_\omega(x) - \bar{\omega}|$. Fix $A := A(\bar{x}, M)$ and $A^\dagger := A^\dagger(\bar{x}, M)$ as in Definition 5.2.1. Define the following functions:*

$$\begin{aligned} g^{ii}_\infty(X) :=& \frac{2\bar{\alpha}}{\bar{\omega}(M+1)} g^{ii,a}_\infty(X) + \sup_{\alpha \in \pi_\alpha(X)} \Delta_\omega \tfrac{\bar{\alpha}}{\bar{\omega}} \left((\alpha^{-1} + 1) C_0 + g^{ii,b}_\infty(X) \right) \\ &+ \sup_{\alpha \in \pi_\alpha(X), \omega \in \pi_\omega(X)} \left(|1 - \tfrac{\bar{\alpha}}{\alpha} \tfrac{\omega}{\bar{\omega}}| + \frac{\bar{\alpha}}{\bar{\omega}(M+1)} \right) C_0 \end{aligned} \tag{5.42}$$

$$g_M(X) := g^i_M(X) + g^{ii}_M(X) \tag{5.43}$$

$$g_\infty(X) := \tfrac{\bar{\alpha}/\bar{\omega}}{M+1} g^i_\infty(X) + g^{ii}_\infty(X). \tag{5.44}$$

Define $\mathcal{K}'(X, \bar{x}) := \mathcal{K}'_M(X, \bar{x}) \times \mathcal{K}'_\infty(X, \bar{x})$ by:

$$\begin{aligned} \mathcal{K}'_M(X, \bar{x}) :=& \bar{x} - A^\dagger_M F_M(\bar{x}) + (I_M - A^\dagger_M A_M) \cdot \pi'_M(X - \bar{x}) \\ &+ A^\dagger_M (A_M - DF_M(X))(X - \bar{x}) \pm A^\dagger_M g_M(X) \end{aligned} \tag{5.45}$$

$$\mathcal{K}'_\infty(X, \bar{x}) := \{c_k \in \mathbb{C} : |c_k| < g_\infty(X)/k^s\}_{k=M+1}^\infty, \tag{5.46}$$

where $F_M(\bar{x}) \subseteq \mathbb{R}^{2M}$ is calculated to include the image of $F_M(\bar{x})$ for all $\alpha \in \pi_\alpha(X)$, where $DF_M(X) \subseteq \mathcal{L}(\mathbb{R}^{2M}, \mathbb{R}^{2M})$ is calculated to include the image of $\tilde{\pi}_M \circ DF_\alpha(\omega, c) \circ \tilde{i}'_M$ for all $(\alpha, \omega, c) \in X$, and where $\pm A^\dagger_M g_M(X) \subseteq \mathbb{R}^{2M}$ is calculated to be a set satisfying:

$$\bigcup_{|v|_k \leq |g_M(X)|_k} A^\dagger_M \cdot v \subseteq \pm A^\dagger_M g_M(X).$$

Theorem 5.2.3. *Fix a cube X as in Definition 5.1.6 with $M \geq 5$, $s > 2$ and $C_0 > 0$. Fix a point $\bar{x} \in X$ such that $\bar{x} = \pi'_M(\bar{x})$, and fix $A := A(\bar{x}, M)$, $A^\dagger := A^\dagger(\bar{x}, M)$ as in Definition 5.2.1. Fix some $\alpha \in \pi_\alpha(X)$, and for $f \equiv F_\alpha : \mathbb{R}^1 \times \tilde{\Omega}^s \to \Omega^{s-1}$ let $\mathcal{K}$ be given as in Definition 5.1.1. Then $\mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'(X, \bar{x})$.*

Proof. Let $H := X - \bar{x}$. We begin by proving that $\pi'_M(\mathcal{K}(X, \bar{x})) \subseteq \pi'_M(\mathcal{K}'(X, \bar{x}))$, first showing that:

$$\pi'_M \circ (I - A^\dagger DF(X)) \cdot H \subseteq \mathcal{K}'_M(X, \bar{x}) - \left(\bar{x} - A^\dagger_M F_M(\bar{x}) \right). \tag{5.47}$$

Fix some $x \in X$ and $h = (h_\omega, h_c) \in H$. We start by adding and subtracting $A^\dagger A$, rewriting the LHS of (5.47) as follows:

$$\begin{aligned}\pi'_M(I - A^\dagger DF(x)) \cdot h =& (I_M - A_M^\dagger A_M) \cdot \pi'_M(h) + \pi'_M A^\dagger(A - DF(x)) \cdot h \\ =& (I_M - A_M^\dagger A_M) \cdot \pi'_M(h) \\ &+ A_M^\dagger(A_M - DF_M(x)) \cdot \pi'_M(h) + A_M^\dagger \pi_M DF_\infty(x) \cdot \pi'_M(h).\end{aligned}$$

By (5.26) and (5.27) it follows that $|\pi_M DF_\infty(x) \cdot h|_k \leq [g_M^i(X) + g_M^{ii}(X)]_k$. Thereby, it follows that: $A_M^\dagger \pi_M DF_\infty(x) \cdot h \subseteq \pm|A_M^\dagger| \cdot g_M(X)$ for all $x \in X$ and $h \in H$. Hence from the definition of $\mathcal{K}'(X, \bar{x})$ given in (5.45), then (5.47) follows. From (5.34) we have that $\pi_M F_\infty(\bar{x}) = 0$, hence $\pi'_M(\bar{x} - A^\dagger F(\bar{x})) = \bar{x} - A_M^\dagger F_M(\bar{x})$. It then follows that $\pi_M \circ \mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'_M(X, \bar{x})$.

We now prove that $\pi'_\infty(\mathcal{K}(X, \bar{x})) \subseteq \pi'_\infty(\mathcal{K}'(X, \bar{x}))$, first showing that:

$$\left\| \pi'_\infty \circ (I - A^\dagger DF(X)) \cdot (X - \bar{x}) \right\|_s \leq g_\infty^{ii}(X). \tag{5.48}$$

Fix some $x = (\alpha, \omega, c) \in X$ and $h = (h_\omega, h_c) \in H$. We start by adding and subtracting $A^\dagger A$, rewriting the LHS of (5.48) as follows:

$$\begin{aligned}\pi'_\infty(I - A^\dagger DF(x)) \cdot h &= \pi'_\infty(I - A^\dagger A) \cdot h + \pi'_\infty A^\dagger(A - DF(x)) \cdot h \\ &= \pi'_\infty \circ A^\dagger(A - DF(x)) \cdot h \\ &= \pi'_\infty \circ A^\dagger \left(A - \tfrac{\partial}{\partial c} DF(x)\right) \cdot h_c - \pi'_\infty \circ A^\dagger \tfrac{\partial}{\partial \omega} DF(x) \cdot h_\omega.\end{aligned}$$

We calculate $-\pi_\infty A^\dagger \frac{\partial}{\partial \omega} F(x) \cdot h_\omega$ writing $\frac{\partial}{\partial \omega} F(x)$ as in (5.9) below:

$$\begin{aligned}-\pi_\infty \circ A^\dagger \tfrac{\partial}{\partial \omega} F(X)) \cdot h_\omega =& - i\pi_\infty \frac{\bar{\alpha}}{\bar{\omega}} K \left(iK^{-1}(\alpha^{-1}I - U_\omega)c - i(K^{-1}U_\omega c) * c\right) \cdot h_\omega \\ =& h_\omega \frac{\bar{\alpha}}{\bar{\omega}} \pi_\infty \left((\alpha^{-1}I - U_\omega)c - K(K^{-1}U_\omega c) * c\right).\end{aligned}$$

Using $|c|_j \leq C_0/j^s$ and (5.30) we obtain for $k \geq M + 1$ that:

$$\begin{aligned}\left| \pi_\infty \circ A^\dagger \tfrac{\partial}{\partial \omega} F(x)) \cdot \Delta_\omega \right|_k \leq& \Delta_\omega \frac{\bar{\alpha}}{\bar{\omega}} \left((\alpha^{-1} + 1)\frac{C_0}{k^s} + \frac{1}{k} \frac{g_\infty^{ii,b}(X)}{k^{s-1}} \right) \\ \left\| \pi_\infty \circ A^\dagger \tfrac{\partial}{\partial \omega} F(x)) \cdot \Delta_\omega \right\|_s \leq& \Delta_\omega \frac{\bar{\alpha}}{\bar{\omega}} \left((\alpha^{-1} + 1)C_0 + g_\infty^{ii,b}(X) \right).\end{aligned} \tag{5.49}$$

For $(\alpha, \omega, c) \in X$ we calculate $\pi_\infty A^\dagger(A - \frac{\partial}{\partial c}F) \cdot h_c$ below:

$$\begin{aligned}
\pi_\infty A^\dagger(A - \tfrac{\partial}{\partial c}F(x))h_c &= -i\frac{\bar{\alpha}}{\bar{\omega}} K \left(\left(i\tfrac{\bar{\omega}}{\bar{\alpha}} K^{-1} - (i\tfrac{\omega}{\alpha} K^{-1} + U_\omega) \right) h_c - (U_\omega h_c) * c - (U_\omega c) * h_c \right) \\
&= \pi_\infty \left((1 - \tfrac{\bar{\alpha}}{\alpha}\tfrac{\omega}{\bar{\omega}}) I + i\tfrac{\bar{\alpha}}{\bar{\omega}} K U_\omega \right) h_c \\
&\quad - \pi_\infty i\tfrac{\bar{\alpha}}{\bar{\omega}} K \left((U_\omega c) * h_c + (U_\omega h_c) * c \right).
\end{aligned}$$

Taking norms and using (5.29) we obtain:

$$\left\| \pi_\infty \circ A^\dagger(A - \tfrac{\partial}{\partial c}F(x)) \cdot h_c \right\|_s \leq \left(|1 - \tfrac{\bar{\alpha}}{\alpha}\tfrac{\omega}{\bar{\omega}}| + \frac{\bar{\alpha}}{\bar{\omega}(M+1)} \right) C_0 + \frac{2\bar{\alpha}}{\bar{\omega}(M+1)} g_\infty^{ii,a}(X). \tag{5.50}$$

By combining (5.49) and (5.50) and taking a supremum over α and ω, we obtain the definition of g_∞^{ii} in (5.42), whereby (5.48) follows.

To show that $\pi_\infty \mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'_\infty(X, \bar{x})$ note that from (5.28) it follows that:

$$\|\pi_\infty(\bar{x} - A^\dagger F(\bar{x}))\|_s = \| - i\tfrac{\bar{\alpha}}{\bar{\omega}} K \pi_\infty F(\bar{x})\|_s \leq \frac{\bar{\alpha}/\bar{\omega}}{M+1} g_\infty^i(X).$$

Expanding out $\pi_\infty \mathcal{K}(X, \bar{x})$, it follows that:

$$\begin{aligned}
\|\pi_\infty \mathcal{K}(X, \bar{x})\|_s \leq & \|\pi_\infty(\bar{x} - A^\dagger F(\bar{x}))\|_s + \|\pi_\infty(I - ADF(X)) \cdot (X - \bar{x})\|_s \\
\leq & \frac{\bar{\alpha}/\bar{\omega}}{M+1} g_\infty^i(X) + g_\infty^{ii}(X) = g_\infty(X).
\end{aligned}$$

Thus $\pi_\infty \mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'_\infty(X, \bar{x})$. Thus, we have proved both that $\pi'_M(\mathcal{K}'(X, \bar{x})) \subseteq \pi'_M(\mathcal{K}(X, \bar{x}))$ and $\pi'_\infty(\mathcal{K}'(X, \bar{x})) \subseteq \pi'_\infty(\mathcal{K}(X, \bar{x}))$. Hence it follows that $\mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'(X, \bar{x})$.

□

Theorem 5.2.4. *Fix a cube X as in Definition 5.1.6 with $M \geq 5$, $s > 2$ and $C_0 > 0$. Fix a point $\bar{x} \in X$ such that $\bar{x} = \pi'_M(\bar{x})$. Let $\mathcal{K}(X, \bar{x})$ and $\mathcal{K}'(X, \bar{x})$ be given as in Definition 5.1.1 and 5.2.2 respectively. If $\mathcal{K}'(X, \bar{x}) \subseteq X$, and moreover $g_\infty(X) < C_0$ and:*

$$\tilde{\pi}'_M \left(\mathcal{K}'_M(X, \bar{x}) + A_M^\dagger F_M(\bar{x}) \right) \subseteq int(\tilde{\pi}'_M(X)),$$

then for all $\alpha \in \pi_\alpha(X)$ there exists a unique point $\hat{x}_\alpha = (\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) \in X$ such that $F(\hat{x}_\alpha) = 0$.

Proof. Fix $\alpha \in \pi_\alpha(X)$. By Theorem 5.1.2, in order to show that there exists a unique solution to $F_\alpha = 0$, it suffices to show that there is some $0 \le \lambda < 1$ for which:

$$(I - A^\dagger DF(X))(X - \bar{x}) \subseteq \lambda(X - \bar{x}).$$

We find a λ_M which works for the π'_M-projection and a λ_∞ which works for the π'_∞-projection. Since $\mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'(X, \bar{x})$ by Theorem 5.2.3 and $\tilde{\pi}'_M \left(\mathcal{K}'_M(X, \bar{x}) + A^\dagger_M F_M(\bar{x})\right)$ $\subseteq int(\tilde{\pi}'_M(X))$, it follows from the definition of $\mathcal{K}(X, \bar{x})$ in (5.1) that:

$$\begin{aligned} \tilde{\pi}'_M \left(\mathcal{K}(X, \bar{x}) + A^\dagger F(\bar{x})\right) &\subseteq int(\tilde{\pi}'_M(X)) \\ \tilde{\pi}'_M \left((I - A^\dagger DF(X))(X - \bar{x})\right) &\subseteq int\left(\tilde{\pi}'_M(X - \bar{x})\right) \end{aligned} \tag{5.51}$$

Since $\tilde{\pi}'_M \left((I - A^\dagger DF(X))(X - \bar{x})\right)$ is compactly contained inside of $\tilde{\pi}'_M(X - \bar{x}) \subseteq \mathbb{R}^{2M}$, there is some positive distance separating the LHS of (5.51) away from the boundary of $\tilde{\pi}'_M(X - \bar{x})$. It follows that there must exist some $0 \le \lambda_M < 1$ such that $\tilde{\pi}'_M \left((I - A^\dagger DF(X))(X - \bar{x})\right) \subseteq \lambda_M \cdot \tilde{\pi}'_M(X - \bar{x})$.

Since $\mathcal{K}'_\infty(X, \bar{x}) \subseteq \pi'_\infty X$ it follows that $g_\infty(X) \le C_0$, and by our additional assumption this is in fact a strict inequality. If we define $\lambda_\infty := g^{ii}_\infty(X)/C_0 < 1$, then by (5.48) it follows that:

$$\pi_\infty(I - A^\dagger DF(X)) \cdot (X - \bar{x}) \le \lambda_\infty \pi_\infty(X - \bar{x}).$$

If we define $\lambda := \max\{\lambda_M, \lambda_\infty\} < 1$ then it follows that:

$$(I - A^\dagger DF(X)) \cdot (X - \bar{x}) \le \lambda(X - \bar{x}).$$

By Theorem 5.1.2 there exists a unique point $\hat{x}_\alpha = (\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) \in X$ such that $F_\alpha(\hat{\omega}_\alpha, \hat{c}_\alpha) = 0$. Moreover, this is true for all $\alpha \in \pi_\alpha(X)$. □

5.3 Pruning Operator

For a given cube, we want to know if it contains any solutions to $F = 0$. We try to determine this by combining several different tests into one *pruning* operator described in Algorithm 5.3.1. It is called a pruning operator because even if we cannot determine whether a cube contains a solution, we may still be able to reduce the size of the cube without losing any solutions.

We describe the tests performed in Algorithm 5.3.1. Most simply, if we can prove that $|F(X)|_k > 0$ for some $1 \leq k \leq M$, then F has no zeros in X. From Lemma 5.1.7, we know that if a cube has a small $\|\cdot\|_{\ell^1}$ norm then it cannot contain any nontrivial zeros. Furthermore, if a cube is contained in the neighborhood of the Hopf bifurcation explicitly given by Corollary 2.3.11, then the only solutions that can exist therein are on the principal branch. If none of those situations apply, then we calculate the outer approximation of the Krawczyk operator given in Definition 5.2.2. If the hypothesis of Theorem 5.2.4 is satisfied, then there exists a unique solution. Alternatively, if $X \cap \mathcal{K}(X, \bar{x}) = \emptyset$, then there do not exist any solutions in X. If none of these other situations apply, then we replace X by $X \cap \mathcal{K}(X, \bar{x})$. Algorithm 5.3.1 arranges these steps in order of ease of computation.

Algorithm 5.3.1 (Prune)**.** *Take as input a cube X with $M \geq 5$ and $s > 2$. The output is a pair $\{flag, X'\}$ where $flag \in \mathbb{Z}$ and $X' \subseteq X$ is a cube.*

1. *Compute $\delta := 2\sum_{k=1}^{M} |X|_k + \frac{2C_0}{(s-1)M^{s-1}}$.*
2. *If for all $(\alpha, \omega, \cdot) \in X$ we have $\alpha \in (0, 2]$, $\omega \geq 1.1$, and $\delta < g(\alpha, \omega)$ for g defined in (5.18), then return $\{1, \emptyset\}$.*
3. *If for all $(\alpha, \omega, \cdot) \in X$ we have $|\alpha - \frac{\pi}{2}| \leq 0.00553$, $|\omega - \frac{\pi}{2}| \leq 0.0924$ and $\delta < 0.18$, then return $\{2, X\}$.*
4. *If $\inf_{x \in X} |F_M(x)|_k > h_k(X)$ for h_k defined in (5.19) and some $1 \leq k \leq M$, then return $\{1, \emptyset\}$.*
5. *Fix some $\bar{x} \in X$ such that $\bar{x} = \pi'_M(\bar{x})$ and $\pi'_M(\bar{x})$ is approximately the center of $\pi'_M(X)$. Construct $\mathcal{K}'(X, \bar{x})$ as in Definition 5.2.2.*
6. *If $\mathcal{K}'(X, \bar{x}) \subseteq X$, $g_\infty(X) < C_0$, and $\tilde{\pi}_M\left(\mathcal{K}'_M(X, \bar{x}) + A^\dagger_M F_M(\bar{x})\right) \subseteq int(\tilde{\pi}_M(X))$, then return $\{3, X\}$.*
7. *If $X \cap \mathcal{K}'(X, \bar{x}) = \emptyset$, then return $\{1, \emptyset\}$.*
8. *Else return $\{0, X \cap \mathcal{K}'(X, \bar{x})\}$.*

Theorem 5.3.2. *Let $\{flag, X'\}$ denote the output of Algorithm 5.3.1 with input a cube X.*

(i) If $flag = 1$, then $F(x) \neq 0$ for all nontrivial $x \in X$.

(ii) If $flag = 2$, then the only solutions to $F = 0$ in X are on the principal branch.

(iii) If $flag = 3$, then for all $\alpha \in \pi_\alpha(X)$ there is a unique $\hat{\omega}_\alpha \in \pi_\omega(X)$ and $\hat{c}_\alpha \in \pi_c(X)$ such that $F(\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) = 0$.

(iv) If there are any points $\hat{x} \in X$ for which $F(\hat{x}) = 0$, then $\hat{x} \in X'$.

Proof. To prove (i) we must check the output from Steps 2, 4, and 7. To prove (ii) we must check Step 3. To prove (iii) we must check Step 6. The proof of (iv) follows from (i), (ii), (iii), and Step 8. We organize the proof into the steps of the algorithm.

1. It follows from (5.17) that $\|c\|_{\ell^1} < \delta$ for all $c \in \pi_c(X)$.

2. Since $\alpha \in (0, 2]$ and $\omega \geq 1.1$, Lemma 5.1.7 applies. If $\|c\|_{\ell^1} < \delta < g(\alpha, \omega)$, then by Lemma 5.1.7 the only solutions to $F(\alpha, \omega, c) = 0$ are trivial, which is to say $c = 0$.

3. If Step 3 returns $flag = 2$, then by Corollary 2.3.11 there is at most one SOPS $c \in X$ with frequency ω, and it lies on the branch of SOPS originating from the Hopf bifurcation at $\alpha = \frac{\pi}{2}$.

4. Suppose that $\inf_{x \in X} |F_M(x)|_k > h_k(X)$ for some $1 \leq k \leq M$. Since we have $\sup_{x \in X} |F_\infty(x)|_k < h_k(X)$ by (5.25), it follows from the triangle inequality that for all $x \in X$ we have:
$$|F(x)|_k \geq \inf_{x \in X} |F_M(x)|_k - \sup_{x \in X} |F_\infty(x)|_k > 0.$$
Hence $|F(x)|_k > 0$, and so X cannot contain any zeros of F.

5. Note that $\mathcal{K}(X, \bar{x}) \subseteq \mathcal{K}'(X, \bar{x})$ by Theorem 5.2.4.

6. If Step 6 returns $flag = 3$, then the hypothesis of Theorem 5.2.4 is satisfied. Hence for all $\alpha \in \pi_\alpha(X)$ there is a unique $\hat{\omega}_\alpha \in \pi_\omega(X)$ and $\hat{c}_\alpha \in \pi_c(X)$ such that $F(\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) = 0$.

7. By Theorem 5.1.2 all solutions in X are contained in $\mathcal{K}(X, \bar{x})$. Hence, all of the zeros of F in X are contained in $X \cap \mathcal{K}(X, \bar{x}) \subseteq X \cap \mathcal{K}'(X, \bar{x})$.

   If $X \cap \mathcal{K}'(X, \bar{x}) = \emptyset$ then $X \cap \mathcal{K}(X, \bar{x}) = \emptyset$, whereby there cannot be any solutions in X.

8. As proved in Step 7, all solutions in X are contained in $X \cap \mathcal{K}'(X, \bar{x})$.

$\square$

5.4 Global Bounds on the Fourier Coefficients

The goal of this section is to construct a bounded region in $\mathbb{R}^2 \times \Omega^s$ which contains all of the nontrivial zeros of F. This is ultimately achieved in Algorithm 5.4.7, which is discussed in Section 5.4.2, along with other estimates pertaining specifically to Wright's equation.

In Section 5.4.1, we discuss generic algorithms used to construct bounds in Fourier space. Algorithm 5.4.1 converts pointwise bounds on a periodic function and its derivatives into a cube containing its Fourier coefficients. Algorithm 5.4.3 modifies a cube so that after a time translation, any periodic function contained therein will satisfy the phase condition $c_1 = c_1^*$.

5.4.1 Converting Pointwise Bounds into Fourier Bounds

To translate pointwise bounds on a periodic function into bounds on its Fourier coefficients we use the unnormalized L^2 inner product, which we define for $g, h \in L^2([0, \frac{2\pi}{\omega}], \mathbb{C})$ as:

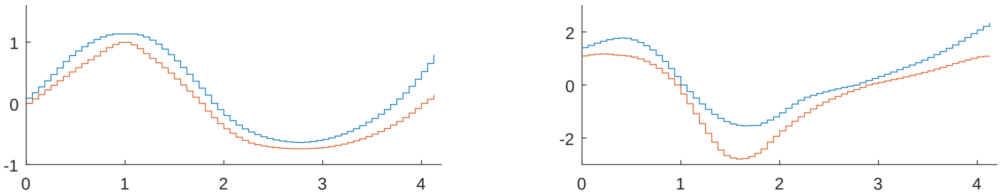
$$\langle g, h \rangle := \int_0^{2\pi/\omega} g(t) h(t)^* \, dt. \tag{5.52}$$

For a function y given as in (5.2), its Fourier coefficients may be calculated as $c_k = \frac{1}{2\pi/\omega} \left\langle y(t), e^{i\omega k t} \right\rangle$. By applying (5.52) to *a priori* estimates on y we are able to derive bounds on its Fourier coefficients. For example, in [Wri55] it is shown that $-1 < y(t) < e^\alpha - 1$ for any global solution to (1.2). Hence, when $e^\alpha \geq 2$ the Fourier coefficients of any periodic solution to (1.2) must satisfy $|c_k| \leq \frac{1}{2\pi/\omega}(e^\alpha - 1)$ for all $k \in \mathbb{Z}$.

With more detailed estimates on y we can produce tighter bounds on its Fourier coefficients. In Chapter 4 such estimates are numerically derived in a rigorous fashion. One of the results from this analysis is a pair of bounding functions which provide upper and lower bounds on SOPS to (1.2) at a given parameter value. Formally, a *bounding function* is defined to be an interval valued function $[\ell(t), u(t)]$ where $\ell, u : \mathbb{R} \to \mathbb{R}$.

These functions ℓ, u are constructed in Chapter 4 using rigorous numerics, and in particular interval arithmetic. As a matter of computational convenience, these functions are defined as piecewise constant functions which change value only finitely many times (see Figure 5.2). For functions of this form, calculating a supremum over a bounded domain is reduced to finding the maximum of a finite set, and calculating an integral is reduced into a finite sum. For elementary functions such as sin or cos, interval arithmetic packages have been developed which allow us to rigorously bound their image over arbitrary domains [Rum99].

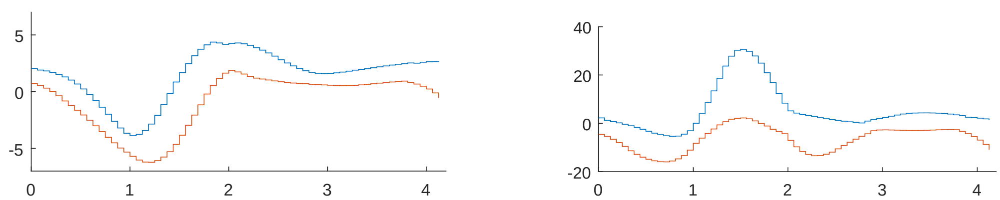
a: Bounds for y b: Bounds for y'



k	$A_{k,0}$	$B_{k,0}$
1	$[-0.103, \quad 0.181]$	$[-0.544, -0.317]$
2	$[-0.238, \quad 0.110]$	$[-0.142, \quad 0.187]$
3	$[-0.207, \quad 0.228]$	$[-0.205, \quad 0.211]$

k	$A_{k,1}$	$B_{k,1}$
1	$[-0.154, \quad 0.205]$	$[-0.673, -0.192]$
2	$[-0.215, \quad 0.031]$	$[-0.100, \quad 0.179]$
3	$[-0.094, \quad 0.109]$	$[-0.090, \quad 0.125]$

c: Bounds for y'' d: Bounds for y'''



k	$A_{k,2}$	$B_{k,2}$
1	$[-0.384, \quad 0.525]$	$[-0.848, -0.103]$
2	$[-0.205, \quad 0.037]$	$[-0.094, \quad 0.155]$
3	$[-0.051, \quad 0.077]$	$[-0.054, \quad 0.071]$

k	$A_{k,3}$	$B_{k,3}$
1	$[-0.995, \quad 1.160]$	$[-1.713, \quad 0.715]$
2	$[-0.279, \quad 0.053]$	$[-0.120, \quad 0.194]$
3	$[-0.039, \quad 0.068]$	$[-0.045, \quad 0.063]$

Figure 5.2: Depicted in the figures are functions $\ell^s, u^s : \mathbb{R} \to \mathbb{R}$ which bound a periodic function y and its derivatives $y^{(s)}$. Depicted in the tables are the values for $A_{k,s}$ and $B_{k,s}$ produced by Algorithm 5.4.1 which bound the Fourier coefficients $c_k = a_k + ib_k$ of y.

Algorithm 5.4.1 describes a method for obtaining rigorous bounds on the Fourier coefficients of a periodic function y. This algorithm applies the inner product $\langle \cdot, \cdot \rangle$ to bounds not just on the function y but on its derivatives as well. Examples of these bounds are given in Figure 5.2, where we note that by the third Fourier coefficient, the tightest estimate is given by the third derivative. We will use $y^{(s)}$ denotes the s^{th}

derivative of a function y, whereas we will use Y^s to denote a bounding function of index s, which bounds the derivative $y^{(s)}$.

We have stated Algorithm 5.4.1 so that it does not estimate the zeroth Fourier coefficient, as periodic solutions to (1.2) necessarily have a trivial zeroth Fourier coefficient. The algorithm could be modified in the obvious way to bound the zeroth Fourier coefficient of a function as well.

Algorithm 5.4.1. *Take as input projection dimension $M \in \mathbb{N}$, period bounds $[\underline{L}, \overline{L}]$, and a collection of interval-valued functions:*

$$\{Y^s(t) = [\ell^s(t), u^s(t)] : \ell^s, u^s : \mathbb{R} \to \mathbb{R}\}_{s=0}^{S}.$$

The output is an (α-parameterless) cube $X \subseteq \mathbb{R}^1 \times \Omega^S$.

1. *Define $I_\omega := [2\pi/\overline{L}, 2\pi/\underline{L}]$.*

2. *For $1 \leq k \leq M$ and $0 \leq s \leq S$ define $\delta_c, \delta_s \in \mathbb{R}_+$ so that:*

$$\delta_c \geq \sup_{\omega \in I_\omega, y^s \in Y^s} \int_{\underline{L}}^{\overline{L}} |\cos(\omega k t) y^s(t)|\, dt, \quad \delta_s \geq \sup_{\omega \in I_\omega, y^s \in Y^s} \int_{\underline{L}}^{\overline{L}} |\sin(\omega k t) y^s(t)|\, dt,$$

   and define $a^+_{k,s}, a^-_{k,s}, b^+_{k,s}, b^-_{k,s} \in \mathbb{R}_+$ so that:

$$\begin{aligned} a^+_{k,s} &\geq \quad \delta_c + \sup_{\omega \in I_\omega, y^s \in Y^s} \int_0^{\underline{L}} \cos(\omega k t) y^s(t) dt \\ a^-_{k,s} &\leq -\delta_c + \inf_{\omega \in I_\omega, y^s \in Y^s} \int_0^{\underline{L}} \cos(\omega k t) y^s(t) dt \\ b^+_{k,s} &\geq \quad \delta_s + \sup_{\omega \in I_\omega, y^s \in Y^s} \int_0^{\underline{L}} \sin(\omega k t) y^s(t) dt \\ b^-_{k,s} &\leq -\delta_s + \inf_{\omega \in I_\omega, y^s \in Y^s} \int_0^{\underline{L}} \sin(\omega k t) y^s(t) dt. \end{aligned}$$

3. *For $1 \leq k \leq M$ and $0 \leq s \leq S$ define:*

$$A'_{k,s} := \frac{1}{2\pi k^s} \left[\inf_{\omega \in I_\omega} \frac{a^-_{k,s}}{\omega^{s-1}}, \sup_{\omega \in I_\omega} \frac{a^+_{k,s}}{\omega^{s-1}} \right], \quad B'_{k,s} := \frac{1}{2\pi k^s} \left[\inf_{\omega \in I_\omega} \frac{b^-_{k,s}}{\omega^{s-1}}, \sup_{\omega \in I_\omega} \frac{b^+_{k,s}}{\omega^{s-1}} \right]. \tag{5.53}$$

Define the intervals $A_{k,s}$ and $B_{k,s}$ as follows:

$$A_{k,s} := \begin{cases} A'_{k,s} & \text{if } s \equiv 0 \pmod 4 \\ -B'_{k,s} & \text{if } s \equiv 1 \pmod 4 \\ -A'_{k,s} & \text{if } s \equiv 2 \pmod 4 \\ B'_{k,s} & \text{if } s \equiv 3 \pmod 4 \end{cases}, \quad B_{k,s} := \begin{cases} -B'_{k,s} & \text{if } s \equiv 0 \pmod 4 \\ -A'_{k,s} & \text{if } s \equiv 1 \pmod 4 \\ B'_{k,s} & \text{if } s \equiv 2 \pmod 4 \\ A'_{k,s} & \text{if } s \equiv 3 \pmod 4 \end{cases}.$$

4. *For $1 \le k \le M$ define:*

$$A_k := \bigcap_{0 \le s \le S} A_{k,s}, \qquad B_k := \bigcap_{0 \le s \le S} B_{k,s}.$$

5. *For each $1 \le k \le M$, define $\bar{a}_k := mid(A_{k,S})$, $\bar{b}_k := mid(B_{k,S})$, $\bar{c}_k = \bar{a}_k + i\bar{b}_k$, and $\bar{c}_{-k} = \bar{c}_k^*$. Define $y_M^S(t,\omega)$ as in (5.54), and define $C_0 > 0$ so that (5.55) holds.*

$$y_M^S(t,\omega) := \sum_{k=-M}^{M} \bar{c}_k (i\omega k)^S e^{i\omega k t} \tag{5.54}$$

$$C_0 \ge \sup_{\omega \in I_\omega, y^S \in Y^S} \frac{1}{2\pi\omega^{S-1}} \int_0^{\overline{L}} \left| y^S(t) - y_M^S(t,\omega) \right| dt. \tag{5.55}$$

6. *Define a cube $X := X_M \times X_\infty \subseteq \mathbb{R}^1 \times \Omega^S$ by:*

$$X_M := I_\omega \times \prod_{k=1}^{M} A_k \times B_k$$

$$X_\infty := \left\{ c_k \in \mathbb{C} : |c_k| \le C_0/k^S \right\}_{k=M+1}^{\infty}.$$

Proposition 5.4.2. *Let the cube X be the output of Algorithm 5.4.1 with input $M \in \mathbb{N}$, $[\underline{L}, \overline{L}] \subseteq \mathbb{R}$ and bounding functions $\{Y^s\}_{s=0}^{S}$. Fix a function $\hat{y}$ with period L and continuous derivatives $\hat{y}^{(s)}$ for $0 \le s \le S$. If $L \in [\underline{L}, \overline{L}]$ and $\hat{y}^{(s)}(t) \in Y^s(t)$ for all $0 \le s \le S$ and $t \in [0, \overline{L}]$, then the frequency and Fourier coefficients of $\hat{y}$ satisfy $(\omega, \{c_k\}_{k=1}^{\infty}) \in X$.*

Proof. We organize the proof into the steps of the algorithm.

1. If the period of $\hat{y}$ is $L \in [\underline{L}, \overline{L}]$ then it will have frequency $\hat{\omega} = 2\pi/L$ and $\hat{\omega} \in [2\pi/\overline{L}, 2\pi/\underline{L}]$.

2. Let us define

$$a_{k,s} := \left\langle \cos(\hat{\omega}kt), \hat{y}^{(s)}(t) \right\rangle, \qquad b_{k,s} := \left\langle \sin(\hat{\omega}kt), \hat{y}^{(s)}(t) \right\rangle.$$

We show that $a_{k,s} \in [a^-_{k,s}, a^+_{k,s}]$. Since $L \in [\underline{L}, \overline{L}]$ it follows that:

$$\begin{aligned} \left\langle \cos(\hat{\omega}kt), \hat{y}^{(s)}(t) \right\rangle &= \int_0^L \cos(\hat{\omega}kt)\hat{y}^{(s)}(t)dt \\ &= \int_0^{\underline{L}} \cos(\hat{\omega}kt)\hat{y}^{(s)}(t)dt + \int_{\underline{L}}^L \cos(\hat{\omega}kt)\hat{y}^{(s)}(t)dt. \end{aligned} \tag{5.56}$$

To estimate the rightmost summand in (5.56) we calculate:

$$\left|\int_{\underline{L}}^L \cos(\hat{\omega}kt)\hat{y}^{(s)}(t)dt\right| \le \int_{\underline{L}}^{\overline{L}} \left|\cos(\hat{\omega}kt)\hat{y}^{(s)}(t)\right| dt \le \sup_{\omega\in I_\omega, y^s\in Y^s} \int_{\underline{L}}^{\overline{L}} \left|\cos(\omega kt)y^s(t)\right| dt \le \delta_c.$$

We obtain a bound on $a_{k,s}$ by appropriately taking an infimum and a supremum in (5.56) as follows:

$$\inf_{\omega\in I_\omega, y^s\in Y^s} \int_0^{\underline{L}} \cos(\omega kt)y^s(t)dt - \delta_c \le a_{k,s} \le \sup_{\omega\in I_\omega, y^s\in Y^s} \int_0^{\underline{L}} \cos(\omega kt)y^s(t)dt + \delta_c.$$

Hence $a_{k,s} \in [a^-_{k,s}, a^+_{k,s}]$, and by analogy $b_{k,s} \in [b^-_{k,s}, b^+_{k,s}]$.

3. Let $c_k = a_k + ib_k$ denote the Fourier coefficients of $\hat{y}$. We show that $a_k \in A_{k,s}$ and $b_k \in B_{k,s}$. Firstly, we calculate the derivative $\hat{y}^{(s)}$ as follows:

$$\hat{y}^{(s)}(t) = \sum_{k\in\mathbb{Z}} c_k(i\hat{\omega}k)^s e^{i\hat{\omega}kt}.$$

We can express the Fourier coefficients of $\hat{y}$ in terms of the Fourier coefficients of its derivatives $\hat{y}^{(s)}$; below, we calculate c_k in terms of $a_{k,s}$ and $b_{k,s}$ as follows:

$$\begin{aligned} \int_0^{2\pi/\hat{\omega}} c_k(i\hat{\omega}k)^s e^{i\hat{\omega}kt} \cdot e^{-i\hat{\omega}kt}dt &= \left\langle \hat{y}^{(s)}(t), e^{i\hat{\omega}kt} \right\rangle \qquad (5.57) \\ \frac{2\pi}{\hat{\omega}} c_k(i\hat{\omega}k)^s &= \left\langle \hat{y}^{(s)}(t), \cos(\hat{\omega}kt) \right\rangle - i \left\langle \hat{y}^{(s)}(t), \sin(\hat{\omega}kt) \right\rangle \\ i^s a_k + i^{s+1} b_k &= \frac{a_{k,s} - i\, b_{k,s}}{2\pi\hat{\omega}^{s-1}k^s}. \end{aligned}$$

From the definition of $A'_{k,s}$ and $B'_{k,s}$ in (5.53) it follows that:

$$\frac{a_{k,s}}{2\pi\hat{\omega}^{s-1}k^s} \in A'_{k,s}, \qquad \frac{b_{k,s}}{2\pi\hat{\omega}^{s-1}k^s} \in B'_{k,s}.$$

By matching the real and imaginary parts, which only depend on s (mod 4), we obtain that $a_k \in A_{k,s}$ and $b_k \in B_{k,s}$.

4. Since $a_k \in A_{k,s}$ and $b_k \in B_{k,s}$ for all k and $0 \le s \le S$, it follows that:

$$a_k \in \bigcap_{0\le s\le S} A_{k,s}, \qquad\qquad b_k \in \bigcap_{0\le s\le S} B_{k,s}.$$

5. We calculate c_k for $k \ge M+1$ starting from (5.57) and using the fact that the functions $e^{i\hat{\omega}kt}$ are L^2–orthogonal:

$$\begin{aligned} c_k(i\hat{\omega}k)^S &= \frac{1}{2\pi/\hat{\omega}} \left\langle e^{i\hat{\omega}kt}, \hat{y}^{(S)}(t) \right\rangle \\ &= \frac{1}{2\pi/\hat{\omega}} \left\langle e^{i\hat{\omega}kt}, \hat{y}^{(S)}(t) - \sum_{j=-M}^{M} \bar{c}_j (i\hat{\omega}j)^S e^{i\hat{\omega}jt} \right\rangle \\ &= \frac{1}{2\pi/\hat{\omega}} \left\langle e^{i\hat{\omega}kt}, \hat{y}^{(S)}(t) - y_M^S(t,\hat{\omega}) \right\rangle. \end{aligned}$$

By taking absolute values, and the suprema over $\omega \in I_\omega$ and $y^S \in Y^S$ we obtain the following.

$$\begin{aligned} \left|c_k(i\hat{\omega}k)^S\right| &\le \frac{1}{2\pi/\hat{\omega}} \int_0^L \left|e^{-i\hat{\omega}kt}\right| \left|\hat{y}^{(S)}(t) - y_M^S(t,\hat{\omega})\right| dt \\ |c_k| k^S &\le \sup_{\omega\in I_\omega, y^S\in Y^S} \frac{1}{2\pi\omega^{S-1}} \int_0^{\overline{L}} \left|y^S(t) - y_M^S(t,\omega)\right| dt \\ &\le C_0. \end{aligned}$$

Hence $|c_k| \le C_0/k^S$ for all $k \ge M+1$.

6. In Step 1 we showed that $\hat{\omega} \in I_\omega$. In Steps 2-4 we showed that $c_k \in [X]_k$ for $1 \le k \le M$, and in Step 5 we showed that $|c_k| \le C_0/k^S$ for $k \ge M+1$.

□

Algorithm 5.4.3. *Take as input an (α-parameterless) cube $X \subseteq \mathbb{R}^1 \times \Omega^s$. The output is an ($\alpha$-parameterless) cube $X' \subseteq \mathbb{R}^1 \times \tilde{\Omega}^s$.*

1. For $[X]_1 = A_1 \times B_1$, with $A_1 = [\underline{A}_1, \overline{A}_1]$ and $B_1 = [\underline{B}_1, \overline{B}_1]$, define an interval

$\Theta \subseteq \mathbb{R}$ so that:

$$\Theta \supseteq \begin{cases} \bigcup_{a_1 \in A_1, b_1 \in B_1} \tan^{-1}(b_1/a_1) & \text{if } \underline{A}_1 > 0 \\ \bigcup_{a_1 \in A_1, b_1 \in B_1} \tan^{-1}(b_1/a_1) + \pi & \text{if } \overline{A}_1 < 0 \\ \bigcup_{a_1 \in A_1, b_1 \in B_1} -\tan^{-1}(a_1/b_1) + \frac{\pi}{2} & \text{if } \underline{B}_1 > 0 \\ \bigcup_{a_1 \in A_1, b_1 \in B_1} -\tan^{-1}(a_1/b_1) - \frac{\pi}{2} & \text{if } \overline{B}_1 < 0 \\ [-\pi, \pi] & \text{otherwise.} \end{cases}$$

2. Rotate every Fourier coefficient's phase by $-\Theta k$. That is, define:

$$A'_1 := \left[\inf_{a_1 \in A_1, b_1 \in B_1} \sqrt{a_1^2 + b_1^2}, \sup_{a_1 \in A_1, b_1 \in B_1} \sqrt{a_1^2 + b_1^2}\right], \qquad B'_1 := [0, 0],$$

and for $2 \le k \le M$ define intervals $A'_k, B'_k \subseteq \mathbb{R}$ such that:

$$A'_k \supseteq \bigcup_{\theta \in \Theta, a_k \in A_k, b_k \in B_k} \cos(\theta k) a_k + \sin(\theta k) b_k$$
$$B'_k \supseteq \bigcup_{\theta \in \Theta, a_k \in A_k, b_k \in B_k} -\sin(\theta k) a_k + \cos(\theta k) b_k.$$

3. Define a cube $X' := X'_M \times X'_\infty \subseteq \mathbb{R}^1 \times \Omega^S$ by

$$X'_M := I_\omega \times \prod_{k=1}^{M} A'_k \times B'_k$$
$$X'_\infty := \left\{c_k \in \mathbb{C} : |c_k| \le C_0/k^S\right\}_{k=M+1}^{\infty}.$$

Proposition 5.4.4. *For an input cube X, let X' denote the output of Algorithm 5.4.3. Suppose that $y : \mathbb{R} \to \mathbb{R}$ is a periodic function given as in* (5.2) *with frequency and Fourier coefficients satisfying $(\omega, \{c_k\}_{k=1}^{\infty}) \in X$. Then there exists some $\tau \in \mathbb{R}$ such that the Fourier coefficients c' of $y(t + \tau)$ satisfy $(\omega, \{c'_k\}_{k=1}^{\infty}) \in X'$. Furthermore c'_1 is a real non-negative number.*

Proof. We organize the proof into the steps of the algorithm.

1. Write the first Fourier coefficient of y as $c_1 = a_1 + ib_1$. We may write $c_1 = re^{i\theta}$ where $r = \sqrt{a_1^2 + b_1^2}$ and if $c_1 \neq 0$, then θ is unique up to an integer multiple of

2π. By the rules for arctan we can calculate:

$$\theta = \begin{cases} \tan^{-1}(b_1/a_1) & \text{if } a_1 > 0 \\ \tan^{-1}(b_1/a_1) + \pi & \text{if } a_1 < 0 \\ -\tan^{-1}(a_1/b_1) + \frac{\pi}{2} & \text{if } b_1 > 0 \\ -\tan^{-1}(a_1/b_1) - \frac{\pi}{2} & \text{if } b_1 < 0. \end{cases}$$

Since $a_1 \in A_1$ and $b_1 \in B_1$, it follows that $\theta \in \Theta$.

2. For any τ we can calculate the Fourier series of $y(t+\tau)$ as follows:

$$y(t+\tau) = \sum_{k\in\mathbb{Z}} c_k e^{i\omega k(t+\tau)} = \sum_{k\in\mathbb{Z}} c_k e^{i\omega k\tau} e^{i\omega kt}.$$

If we choose $\tau = -\theta/\omega$, then $c_1' = c_1 e^{i\omega\tau} = \sqrt{a_1^2 + b_1^2}$ is a real, non-negative number and moreover $c_1' \in [X']_1$.

3. The Fourier coefficients of $y(t+\tau)$ are given by $c_k' = e^{-ik\theta} c_k$, hence $(\omega, \{c_k'\}_{k=1}^{\infty}) \in X'$.

□

5.4.2 Bounds for Wright's Equation

The culmination of this subsection is Algorithm 5.4.7 which, for a given range of parameters, constructs a collection of cubes covering the solution space to $F_\alpha = 0$. This algorithm begins with pointwise bounds on SOPS to (1.2). To obtain these pointwise bounds we use Algorithm 4.5.1, wherein we augment Algorithm 4.2.2 with Step 7 defined in Algorithm 4.2.4. By Theorem 4.5.2, for a given range of parameters I_α the output of this algorithm is a collection of bounding functions Ξ, such that if there is a SOPS to the exponential version of Wright's equation at parameter $\alpha \in I_\alpha$, then it will be bounded by one of the bounding functions in Ξ. Recall that solutions to the exponential version of Wright's equation solve (1.1) where $f(x) = e^x - 1$, and can be transformed into the quadratic version of Wright's equation (1.2) using the change of variable $y = e^x - 1$. Each bounding function in the output of Algorithm 4.5.1 is associated with intervals Q and $\bar{Q}$, bounding the amount of time a SOPS is respectively positive and negative, from which we can bound the period of the SOPS as $L = Q + \bar{Q}$. We state a slightly reformulated version of Theorem 4.5.1 below:

Theorem 5.4.5 (See Theorem 4.5.2)**.** *Fix some $I_\alpha = [\alpha_{min}, \alpha_{max}]$ such that $\alpha_{min} \geq \frac{\pi}{2}$. Suppose that $x : \mathbb{R} \to \mathbb{R}$ is periodic with period L, and is a SOPS to* (1.1) *at parameter $\alpha \in I_\alpha$ with $f(x) = e^x - 1$. Furthermore, assume without loss of generality that $x(0) = 0$ and $x'(0) > 0$.*

If $\mathcal{L}$ and Ξ denote the output of Algorithm 4.5.1 run with input I_α, then there exists some $[\underline{L}_i, \overline{L}_i] \in \mathcal{L}$ and $[\ell_i, u_i] \in \Xi$ for which $L \in [\underline{L}_i, \overline{L}_i]$ and $x(t) \in [\ell_i(t), u_i(t)]$ for all t.

Remark 5.4.6. *The set Ξ we refer to in Chapter 5, is referred to as $\mathcal{X}$ in [Jaq18]. In both cases the variable is used to denote a collection of bounding functions output by Algorithm 4.5.1. We have made this change because in Chapter 4 the variable $\mathcal{X} \subseteq C^1(\mathbb{R}, \mathbb{R})$ is used to denote a fixed collection of functions.*

The higher derivatives of a function can be very useful in constructing bounds on its Fourier coefficients and their rate of decay. While the bounding functions constructed in Chapter 4 are not even continuous, we can use them to construct bounding functions for the derivative of SOPS to Wright's equation via a bootstrapping argument. Namely, by taking a derivative on both sides of (1.2) we obtain an equation for the second derivative of solutions to (1.2). In a similar manner, can obtain an expression for the third derivative of solutions to (1.2), both of which are presented below:

$$
\begin{aligned}
y''(t) &= -\alpha \left[y'(t-1) \left[1 + y(t) \right] + y(t-1) y'(t) \right] \\
y'''(t) &= -\alpha \left[y''(t-1)[1 + y(t)] + 2y'(t-1)y'(t) + y(t-1)y''(t) \right].
\end{aligned}
$$

Note that we can always express the derivative $y^{(s)}(t)$ in terms of $y^{(r)}(t)$ and $y^{(r)}(t-1)$ where $0 \leq r \leq s-1$. That is, we can inductively define functions $f^s : \mathbb{R}^{2s} \to \mathbb{R}$ such that for all t we have:

$$
y^{(s)}(t) = f^s \left(y(t), y(t-1), y'(t), y'(t-1), \ldots, y^{(s-1)}(t), y^{(s-1)}(t-1) \right). \tag{5.58}
$$

If we start with a bounding function for y, then by appropriately adding and multiplying the bounding functions for $y^{(r)}$, taking wider bounds whenever necessary, we can obtain bounding functions for any derivative of y (see for example Figure 5.2).

Algorithm 5.4.7 proceeds by first constructing bounding functions for y and its derivatives, and then applying Algorithm 5.4.1 to obtain a cube containing its Fourier coefficients. Then it applies Algorithm 5.4.3 to impose the phase condition that $c_1 = c_1^*$. In this manner we obtain a collection of cubes which contains all of the Fourier

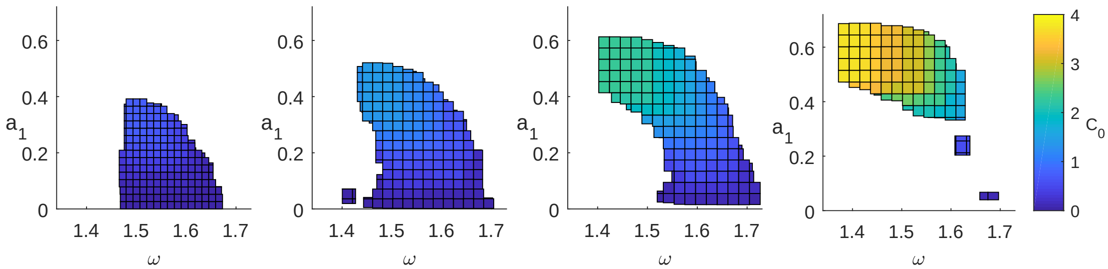


Figure 5.3: Depicted above is the output of Algorithm 5.4.7 projected onto the (ω, a_1) plane. From left to right, the input I_α was taken to be $[\frac{\pi}{2}, 1.6]$, $[1.6, 1.7]$, $[1.7, 1.8]$, and $[1.8, 1.9]$. Note that C_0 increases with α, a_1, and period length $2\pi/\omega$.

coefficients to SOPS to (1.2). We then apply Algorithm 5.3.1 to each cube, discarding it if possible. This allows us to discard between 5% and 20% of cubes (see N'_{grid} in Table 5.1).

One problem however, is that the Fourier projection of two distinct bounding functions often overlap considerably. To address this we combine overlapping cubes together. While we could combine all of our cubes into one big cube, this would not be efficient. Instead, we divide our cover along a grid in the $\omega \times a_1$ plane (see Figure 5.3).

Algorithm 5.4.7. *Fix an interval of $I_\alpha \subseteq [\alpha_{min}, \alpha_{max}]$, integers $M, S \in \mathbb{N}$ and a subdivision number $N \in \mathbb{N}$, and the computational parameters for Algorithm 4.5.1. The output is a (finite) collection of cubes $\mathcal{S} = \{X_i \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s\}$.*

1. *Let $\Xi, \mathcal{L}$ be the output of Algorithm 4.5.1 with input I_α and appropriate computational parameters.*

2. *Use the change of variables $y = e^x - 1$ to define a collection of functions:*

$$\mathcal{Y}^0 := \left\{ Y_i(t) = [e^{\ell_i(t)} - 1, e^{u_i(t)} - 1] : [\ell_i(t), u_i(t)] \in \Xi \right\}.$$

3. *Inductively define $\mathcal{Y}^s$ for $1 \leq s \leq S$ so that corresponding to each $Y_i^0 \in \mathcal{Y}^0$ there*

exists a $Y_i^s = [\underline{Y_i^s}, \overline{Y_i^s}] \in \mathcal{Y}^s$ *such that for* f^s *defined in* (5.58) *we have:*

$$\underline{Y_i^s}(t) \leq \inf_{\{y^r\}_{r=0}^{s-1} \in \{Y_i^r\}_{r=0}^{s-1}} f^s\left(y^0(t), y^0(t-1), \ldots, y^{s-1}(t), y^{s-1}(t-1)\right)$$
$$\overline{Y_i^s}(t) \geq \sup_{\{y^r\}_{r=0}^{s-1} \in \{Y_i^r\}_{r=0}^{s-1}} f^s\left(y^0(t), y^0(t-1), \ldots, y^{s-1}(t), y^{s-1}(t-1)\right).$$

4. *Define* $\mathcal{S}' := \{X_i' \subseteq \mathbb{R}^1 \times \Omega^s\}$ *to be the collective output of Algorithm 5.4.1 run with* $M \in \mathbb{N}$, *and each of the sets* $L_i \in \mathcal{L}$ *and* $\{Y_i^s\}_{s=0}^S \in \{\mathcal{Y}^s\}_{s=0}^S$ *as input.*

5. *Define* $\mathcal{S}'' := \{X_i'' \subseteq \mathbb{R}^1 \times \tilde{\Omega}^s\}$ *to be the collective output of Algorithm 5.4.3 run with each of the sets* $X_i' \in \mathcal{S}'$ *as input.*

6. *Define* $\mathcal{S}'''$ *by taking the product of* I_α *with the cubes in* $\mathcal{S}''$. *That is, define* $\mathcal{S}''' := \{I_\alpha \times X_i'' \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s : X_i'' \in \mathcal{S}''\}$.

7. *For each* $X \in \mathcal{S}'''$, *let* $\{flag, X'\}$ *denote the output of Algorithm 5.3.1 with input* X. *If* $flag = 1$, *then remove* X *from* $\mathcal{S}'''$. *Otherwise replace* X *by* X'.

8. *Subdivide the* $\omega \times a_1$ *space covered by* $\mathcal{S}'''$ *into an* $N \times N$ *grid. That is, define an index set* $B := \{1, 2, \ldots, N\} \times \{1, 2, \ldots, N\}$ *and define intervals* $I^\omega, I^{a_1} \subseteq \mathbb{R}$ *so that:*

$$I^\omega \supseteq \bigcup_{X \in \mathcal{S}'''} \pi_\omega(X), \qquad I^{a_1} \supseteq \bigcup_{X \in \mathcal{S}'''} \pi_{a_1}(X).$$

Subdivide I^ω *and* I^{a_1} *into* N *subintervals of equal width,* $\{I_i^\omega\}_{i=1}^N$ *and* $\{I_i^{a_1}\}_{i=1}^N$, *so that* $I^\omega = \bigcup_{i=1}^N I_i^\omega$ *and* $I^{a_1} = \bigcup_{i=1}^N I_i^{a_1}$.

9. *For each* $\beta = (\beta_1, \beta_2) \in B$, *take the union of cubes in* $\mathcal{S}'''$ *whose* (ω, a_1)*–projection intersects* $I_{\beta_1}^\omega \times I_{\beta_2}^{a_1}$. *That is, define:*

$$\tilde{X}_\beta := \{(\alpha, \omega, c) \in \mathbb{R}^2 \times \tilde{\Omega}^s : \omega \in I_{\beta_1}^\omega, [c]_1 \in I_{\beta_2}^{a_1}\},$$

and define X_β *to be a cube such that:*

$$X_\beta \supseteq \bigcup_{X \in \mathcal{S}'''} X \cap \tilde{X}_\beta.$$

10. *Define* $\mathcal{S} := \{X_\beta : \beta \in B\}$.

Theorem 5.4.8. *Fix an interval $I_\alpha = [\alpha_{min}, \alpha_{max}]$ such that $\alpha_{min} \geq \frac{\pi}{2}$, and let $\mathcal{S}$ denote the output of Algorithm 5.4.7. If a function y as given in* (5.2) *is a SOPS to Wright's equation at $\alpha \in I_\alpha$, then there exists a time translation so that its Fourier coefficients are in $\bigcup \mathcal{S}$.*

Proof. Every SOPS y to the quadratic version of Wright's equation given in (1.2) corresponds to a SOPS x to the exponential version of Wright's equation given in (1.1) with $f(x) = e^x - 1$. Fix a SOPS $x : \mathbb{R} \to \mathbb{R}$ to the exponential version of Wright's equation with period L. We organize the proof into the steps of the algorithm.

1. By Theorem 5.4.5 there exists an interval $L_i \in \mathcal{L}$ and a bounding function $[\ell_i, u_i] \in \Xi$ and such that $L \in L_i$ and $x(t) \in [\ell_i(t), u_i(t)]$ for all $t \in \mathbb{R}$.

2. The change of variables between the exponential and quadratic versions of Wright's equation is given by $y = e^x - 1$. Hence for the interval $L_i \in \mathcal{L}$ and the bounding function $Y_i \in \mathcal{Y}^0$, it follows that $L \in L_i$ and $y(t) \in Y_i(t)$ for all $t \in \mathbb{R}$.

3. Since $y \in Y_i^0$ it follows that its derivatives satisfy $y^{(s)} \in Y_i^s$ for all $0 \leq s \leq S$.

4. Let ω and c denote the frequency and Fourier coefficients of y respectively. If X_i' is the output of Algorithm 5.4.1 with input $M \in \mathbb{N}$, L_i and $\{Y_i^s\}_{s=0}^S$, then by Proposition 5.4.2 it follows that $(\omega, \{c_k\}_{k=1}^\infty) \in X_i'$.

5. Let X_i'' denote the output of Algorithm 5.4.3 with input X_i'. By Theorem 5.4.4, there exists a $\tau \in \mathbb{R}$ such that the Fourier coefficients c' of $y(t+\tau)$ satisfy $(\omega, \{c_k'\}_{k=1}^\infty) \in X_i''$.

6. We have shown that if y is a SOPS to (1.2) at parameter α having frequency ω, then up to a time translation $(\alpha, \omega, c) \in \bigcup \mathcal{S}'''$. By Theorem 2.1.4 the SOPS to (1.2) at parameter $\alpha \in I_\alpha$ correspond to the non-trivial zeros of F in $\bigcup \mathcal{S}'''$. (Note that F defined here in Chapter 5 is equal to $\frac{1}{\alpha} G$ for G defined in Chapter 2.) Hence, if there is a solution $F(\hat{x}) = 0$ for some $x \in \mathbb{R}^2 \times \tilde{\Omega}^s$ with $\pi_\alpha(\hat{x}) \in I_\alpha$, then $\hat{x} \in \bigcup \mathcal{S}'''$.

7. Let $\{flag, X_i^{(4)}\}$ denote the output of Algorithm 5.3.1 with input $X_i''' \in \mathcal{S}'''$. By Theorem 5.3.2 we can replace each $X''' \in \mathcal{S}'''$ with $X_i^{(4)}$, and it will still be the case that $\bigcup \mathcal{S}'''$ contains all of the solutions to $F = 0$. In particular, if $flag = 1$ then $X_i^{(4)} = \emptyset$ and we may remove X_i''' in this case.

8. If $(\alpha, \omega, c) \in \bigcup \mathcal{S}'''$ and $a_1 = [c]_1$, then by construction $\omega \in I^\omega$ and $a_1 \in I^{a_1}$. As $I^\omega \times I^{a_1} = \bigcup_{(\beta_1,\beta_2)\in B} I^\omega_{\beta_1} \times I^{a_1}_{\beta_2}$, then there is some $(\beta_1, \beta_2) \in B$ such that $(\omega, a_1) \in I^\omega_{\beta_1} \times I^{a_1}_{\beta_2}$.

9. As $\bigcup_{X\in\mathcal{S}'''} X \subseteq \bigcup_{\beta\in B} \tilde{X}_\beta$, then it follows that $\bigcup_{X\in\mathcal{S}'''} X \subseteq \bigcup_{\beta\in B} X_\beta$. That is to say $\bigcup \mathcal{S}''' \subseteq \bigcup \mathcal{S}$.

10. Hence, $\bigcup \mathcal{S}$ contains the Fourier coefficients of any possible SOPS.

$\square$

5.5 Global Algorithm

After Algorithm 5.4.7 has constructed a collection of cubes $\mathcal{S}$ covering the solution space to $F = 0$, we run a branch and prune algorithm. This algorithm iteratively inspects the elements in $X \in \mathcal{S}$ and then constructs three new lists of cubes: $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$. To summarize, first we compute the output $Prune(X) = \{flag, X'\}$ from Algorithm 5.3.1. If $flag = 1$, then there are no solutions in X, and we can remove X from $\mathcal{S}$. If $flag = 2$, then the cube is in the neighborhood of the Hopf bifurcation, and we add X' to $\mathcal{B}$. If $flag = 3$, then for all $\alpha \in \pi_\alpha(X)$ there exists a unique solution to $F_\alpha = 0$ in X', and we add X' to $\mathcal{A}$. If X' is too small, then we add it to $\mathcal{R}$. If the Krawczyk operator appears to be effective at reducing the size of the cube, then the pruning operation is performed again. Otherwise X' is subdivided along some lower dimension and the resulting pieces are added back to $\mathcal{S}$.

The most obvious difference between our algorithm and the classical algorithm is that we are working in infinite dimensions. While we store $2M + 1$ real valued coordinates in a given cube, as in [GZ07, DK13] the subdivision is only performed along a subset of these dimensions. Choosing which dimension to subdivide along can greatly affect the efficiency of a branch and bound algorithm, and there are heuristic methods for optimizing this choice [CR97]. However since we are finding all the zeros along a 1-parameter family of solutions, these branching methods are not entirely applicable. To determine which dimension to subdivide we select the dimension with the largest weighted diameter. That is, for a collection of weights $\{\lambda_i\}_{i=0}^d$ we define:

$$w(X,i) := \begin{cases} \lambda_i \cdot \operatorname{diam}(\pi_\alpha(X)) & \text{if } i = 0, \\ \lambda_i \cdot \operatorname{diam}\left([\tilde{\pi}'_M(X)]_i\right) & \text{otherwise.} \end{cases} \tag{5.59}$$

Algorithm 5.5.1 (Branch & Bound)**.** *Take as input a collection of cubes $\mathcal{S} = \{X_i \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s\}$ with $M \geq 5$ and $s > 2$, and as computational parameters: a halting criteria $\epsilon > 0$, a continue-pruning criteria $\delta \geq 0$, a maximum subdivision dimension $0 \leq d \leq 2M$ and a set of weights $\{\lambda_i\}_{i=0}^d$. The output is three lists of cubes: $\mathcal{A}, \mathcal{B}$ and $\mathcal{R}$.*

1. *If $\mathcal{S}$ is empty, terminate the algorithm.*
2. *Select an element $X \in \mathcal{S}$ and remove X from $\mathcal{S}$.*
3. *Define $\{flag, X'\} = Prune(X)$ to be the output of Algorithm 5.3.1 with input X.*
4. *If $flag = 1$, then reject X and GOTO Step 1.*
5. *If $flag = 2$, then add X' to $\mathcal{B}$ and GOTO Step 1.*
6. *If $flag = 3$, then add X' to $\mathcal{A}$ and GOTO Step 1.*
7. *If $\max_{0 \leq i \leq d} w(X', i) < \epsilon$, then add X' to $\mathcal{R}$ and GOTO Step 1.*
8. *Define $m = \lfloor d/2 \rfloor$. If $(1+\delta) < \frac{vol(\tilde{\pi}'_m(X))}{vol(\tilde{\pi}'_m(X'))}$, then define $X := X'$ and GOTO Step 3.*
9. *Subdivide X' into two pieces, X'_1 and X'_2, along a dimension which maximizes $w(X', i)$, and so that $X' = X'_1 \cup X'_2$. Add the two new cubes to $\mathcal{S}$ and GOTO Step 1.*

Theorem 5.5.2. *Let $\mathcal{S} = \{X_i \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s\}$ with $M \geq 5$ and $s > 2$. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{R}$ be the output of Algorithm 5.5.1 run with input $\mathcal{S}$ and various computational parameters.*

(i) If $F(\hat{x}) = 0$ for some $\hat{x} \in \bigcup \mathcal{S}$, then $\hat{x} \in \bigcup \mathcal{A} \cup \mathcal{B} \cup \mathcal{R}$.

(ii) For each $X \in \mathcal{A}$ and $\alpha \in \pi_\alpha(X)$, there is a unique $\hat{x} = (\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) \in X$ such that $F(\hat{x}) = 0$.

(iii) For each $X \in \mathcal{B}$, if there is a solution $\hat{x} \in X$ to $F = 0$, then $\hat{x}$ is on the principal branch.

Proof. We prove the claims of the theorem.

(i) Suppose there is some solution $\hat{x} \in X$ for some $X \in \mathcal{S}$. We show that $\hat{x} \in \bigcup \mathcal{S} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{R}$ at every step of the algorithm. If we replace X by X' as in Step 3, then $\hat{x} \in X'$ by Theorem 5.3.2. In Step 4, if $flag = 1$ then in fact $X' = \emptyset$, so X could not have contained any solutions in the first place. In Steps 5, 6 and 7, the cube X' is added to one of $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{R}$. Hence, as $\hat{x} \in X'$ then $\hat{x} \in \bigcup \mathcal{S} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{R}$. If in Step 8 we decide to prune the cube X' again, then we may repeat the argument made for Steps 3-7. In Step 9 we divide X' into two new cubes X'_1 and X'_2 for which $X' = X'_1 \cup X'_2$. Hence $\hat{x}$ will be contained in at least one of X'_1 or X'_2, and both cubes are added to $\mathcal{S}$, so we cannot lose the solution in Step 9.

Thus we have shown that $\hat{x} \in \bigcup \mathcal{S} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{R}$ at every step. Since the algorithm can only stop when $\mathcal{S} = \emptyset$, it follows that every solution $\hat{x}$ initially contained in $\bigcup \mathcal{S}$ will eventually be contained in $\bigcup \mathcal{A} \cup \mathcal{B} \cup \mathcal{R}$.

(ii) The only way a cube X' can be added to $\mathcal{A}$ is in Step 6. That is, for some cube $X \in \mathcal{S}$ the output of Algorithm 5.3.1 returned $\{3, X'\}$. Thus, it follows from Theorem 5.3.2 that for all $\alpha \in \pi_\alpha(X)$ there is a unique $\hat{x} = (\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) \in X$ such that $F(\hat{x}) = 0$.

(iii) The only way a cube X' can be added to $\mathcal{B}$ is in Step 5. That is, for some cube $X \in \mathcal{S}$ the output of Algorithm 5.3.1 returned $\{2, X'\}$. Thus, it follows from Corollary 2.3.11 that the only solutions to $F = 0$ in X' are those on the principal branch.

□

If a cube has no zeros inside of it yet there is a solution close to its boundary, then proving that the cube does not contain any solutions can be very difficult, resulting in an excessive number of subdivisions. This phenomenon is common to branch and bound algorithms and is referred to as the cluster effect [SN04]. As we wish to enumerate not just isolated solutions but a 1-parameter family of solutions, the difficulty of the cluster effect is multiplied. Furthermore, we cannot expect that the boundary of a cube will almost never contain a solution. In particular, when we subdivide a cube we may also bisect the curve of solutions, and further subdivisions will not remedy this problem (see Figure 5.4). As such, we should not expect that $\mathcal{R} \neq \emptyset$.

To address this issue we apply Algorithm 5.5.3 to the output of Algorithm 5.5.1. In Step 1 we recombine cubes in $\mathcal{R}$ which overlap in the α dimension. In Step 2 we split the cubes in $\mathcal{R}$ along the α-dimension to make them easier to prune, which we do in Step 3. Ideally by Step 4 all of the cubes have been removed from $\mathcal{R}$, having been added to either $\mathcal{A}$ or $\mathcal{B}$.

Even if $\mathcal{R} = \emptyset$ at this point, it is not immediately clear that the only solutions are on the principal branch. For two distinct cubes $X_1, X_2 \in \mathcal{A}$, if there is some α_0 such that $\alpha_0 \in \pi_\alpha(X_1)$ and $\alpha_0 \in \pi_\alpha(X_2)$, then there could very well be two distinct solutions at the parameter α_0. In fact, since we subdivide along the α–dimension it is to be expected that a cube will share an α–value with one or two other cubes. In Steps 6-9 of Algorithm 5.5.3 we check to make sure that when two cubes have α–values in common, then there is a unique solution associated to each $\alpha_0 \in \pi_\alpha(X_1) \cap \pi_\alpha(X_2)$.

Algorithm 5.5.3. *Take as input sets $\mathcal{A}, \mathcal{B}, \mathcal{R}$ produced by Algorithm 5.5.1 and a computational parameter $n \in \mathbb{N}$. The output is a pair of intervals $I_\alpha^{\mathcal{A}}$, $I_\alpha^{\mathcal{B}}$ and either success or failure.*

1. *Combine the elements in $\mathcal{R}$ whose α-components overlap in more than just a point. That is, for all $X, Y \in \mathcal{R}$, if $diam(\pi_\alpha(X) \cap \pi_\alpha(Y)) > 0$, then replace X and Y in the set $\mathcal{R}$ with a new cube Z containing $X \cup Y$.*

2. *Subdivide each $X \in \mathcal{R}$ along the α-dimension.*

3. *For all $X \in \mathcal{R}$ calculate $\{flag, X'\} = Prune^{(n)}(X)$, the output of Algorithm 5.3.1*

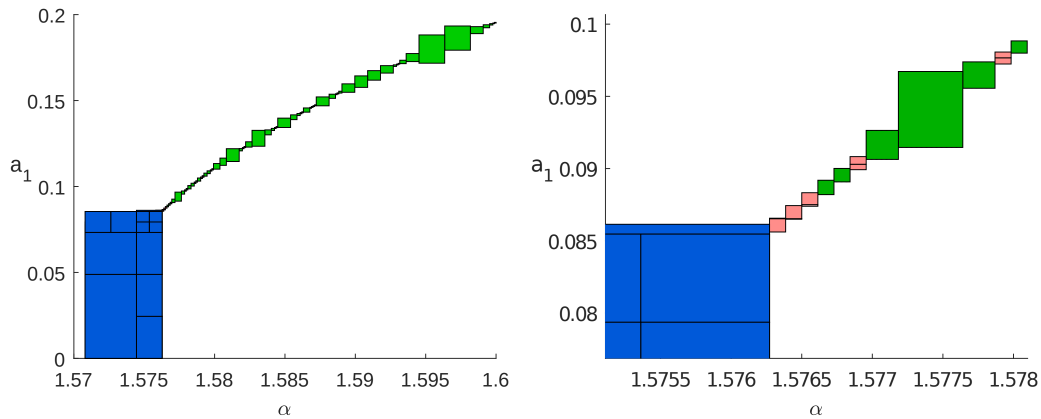


Figure 5.4: An example output of Algorithm 5.5.1. The cubes in $\mathcal{A}$ are in green, the cubes in $\mathcal{B}$ are in dark blue, and the cubes in $\mathcal{R}$ are in pink.

iterated at most n times with initial input X. If $flag = 1$, then remove X from $\mathcal{R}$. If $flag = 2$, then remove X from $\mathcal{R}$ and add X' to $\mathcal{B}$. If $flag = 3$, then remove X from $\mathcal{R}$ and add X' to $\mathcal{A}$.

4. *If $\mathcal{R} \neq \emptyset$ then return FAILURE.*

5. *Define $I_\alpha^{\mathcal{A}} = \bigcup_{X \in \mathcal{A}} \pi_\alpha(X)$ and $I_\alpha^{\mathcal{B}} = \bigcup_{X \in \mathcal{B}} \pi_\alpha(X)$.*

6. *Construct a cover $\mathcal{I}'_{\mathcal{B}}$ of the parts of cubes in $\mathcal{A}$ which intersect with $\bigcup \mathcal{B}$. That is, define $\mathcal{I}_{\mathcal{B}} = \{X \in \mathcal{A} : \pi_\alpha(X) \cap I_\alpha^{\mathcal{B}}\}$. Then define $\mathcal{I}'_{\mathcal{B}}$ by, for each $X \in \mathcal{I}_{\mathcal{B}}$, taking the α-component of X and setting it equal to $\pi_\alpha(X) \cap I_\alpha^{\mathcal{B}}$ and adding the modified cube to $\mathcal{I}'_{\mathcal{B}}$.*

7. *For all $X \in \mathcal{I}'_{\mathcal{B}}$ calculate $\{flag, X'\} = Prune^{(n)}(X)$, the output of Algorithm 5.3.1 iterated n–times with initial input X. If $flag \neq 2$ then return FAILURE.*

8. *Construct a cover $\mathcal{I}'_{\mathcal{A}}$ of the parts of cubes in $\mathcal{A}$ which intersect with another cube in $\mathcal{A}$. That is, define $\mathcal{I}_{\mathcal{A}} = \{(X, Y) \in \mathcal{A} \times \mathcal{A} : X \neq Y, \pi_\alpha(X) \cap \pi_\alpha(Y) \neq \emptyset\}$. Then define $\mathcal{I}'_{\mathcal{A}}$ by, for each $(X, Y) \in \mathcal{I}_{\mathcal{A}}$, defining a new cube Z which contains $X \cup Y$, replacing the α-component of Z by $\pi_\alpha(X) \cap \pi_\alpha(Y)$, and adding Z to $\mathcal{I}'_{\mathcal{A}}$.*

9. *For all $Z \in \mathcal{I}'_{\mathcal{A}}$ calculate $\{flag, Z'\} = Prune^{(n)}(Z)$, the output of Algorithm 5.3.1 iterated n–times with initial input Z. If $flag \neq 3$ then return FAILURE.*

10. *If the algorithm reaches this point, return SUCCESS.*

Theorem 5.5.4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{R}$ denote the output of Algorithm 5.5.1 run with input $\mathcal{S} = \{X_i \subseteq \mathbb{R}^2 \times \tilde{\Omega}^s\}$ where $M \geq 5$ and $s > 2$. Suppose having received input $\mathcal{A}, \mathcal{B}, \mathcal{R}$ and $n \in \mathbb{N}$, Algorithm 5.5.3 returns SUCCESS and intervals $I_\alpha^{\mathcal{A}}$ and $I_\alpha^{\mathcal{B}}$.*

(i) If $\alpha \in I_\alpha^{\mathcal{A}} \backslash I_\alpha^{\mathcal{B}}$, then there is a unique solution $\hat{x}_\alpha = (\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) \in \bigcup \mathcal{S}$ such that $F_\alpha(\hat{\omega}_\alpha, \hat{c}_\alpha) = 0$.

(ii) If $\alpha \in I_\alpha^{\mathcal{B}}$, then the only solutions to $F_\alpha = 0$ in $\bigcup \mathcal{S}$ are on the principal branch.

Proof. We describe the first 4 steps of the algorithm and then prove the theorem.

1. Let $\mathcal{R}$ denote the initial input to the algorithm and $\mathcal{R}'$ denote the resulting set produced by Step 1. By its construction, it follows that $\bigcup \mathcal{R} \subseteq \bigcup \mathcal{R}'$.

2. If we subdivide the cubes in $\mathcal{R}'$, then it is still true that $\bigcup \mathcal{R} \subseteq \bigcup \mathcal{R}'$.

3. As described in the proof of Theorem 5.5.2, if $flag = 1, 2, 3$ then it is appropriate to respectively, discard X, add X' to $\mathcal{B}$ and add X' to $\mathcal{A}$. Appropriate, that is, in the sense that the conclusion of Theorem 5.5.1 will hold for these modified sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$.

4. If we cannot show that every region of phase-space lies in either $\mathcal{A}$ or $\mathcal{B}$ then we are unable to prove the theorem. Otherwise, every solution to $F = 0$ in $\bigcup \mathcal{S}$ is contained in $\bigcup \mathcal{A} \cup \mathcal{B}$.

We prove claim (i). If $\alpha \in I_\alpha^{\mathcal{A}} \backslash I_\alpha^{\mathcal{B}}$ there is a solution $\hat{x}_\alpha$ to $F_\alpha = 0$ in $\bigcup \mathcal{A}$. Suppose there exists a second distinct solution $\hat{x}'_\alpha$ to $F_\alpha = 0$. Since each cube $X \in \mathcal{A}$ contains a unique solution for all $\alpha \in \pi_\alpha(X)$, there would exist distinct cubes $X, Y \in \mathcal{A}$ such that $\hat{x}_\alpha \in X$ and $\hat{x}'_\alpha \in Y$. It follows then that there exists some $Z \in \mathcal{I}'_\alpha$ such that $\hat{x}_\alpha, \hat{x}'_\alpha \in Z$. Since it is determined by Step 9 that $flag = 3$ in the output of $Prune^{(n)}(Z)$, therefore by Theorem 5.3.2 there exists a unique solution to $F = 0$ in Z. Thereby $\hat{x}_\alpha = \hat{x}'_\alpha$, and if $\alpha \in I_\alpha^{\mathcal{A}} \backslash I_\alpha^{\mathcal{B}}$, then there is a unique solution $\hat{x}_\alpha = (\alpha, \hat{\omega}_\alpha, \hat{c}_\alpha) \in \bigcup \mathcal{S}$ such that $F_\alpha(\hat{\omega}_\alpha, \hat{c}_\alpha) = 0$.

We prove claim (ii). Suppose there exists some $\hat{x}_\alpha$ such that $\alpha \in I_\alpha^{\mathcal{B}}$ and $F_\alpha(\hat{\omega}, \hat{c}) = 0$. Since the algorithm passed through Step 4, it follows that $\hat{x}_\alpha \in \bigcup \mathcal{A} \cup \mathcal{B}$. If $\hat{x}_\alpha \in \bigcup \mathcal{B}$, then $\hat{x}_\alpha$ is on the principal branch by Theorem 5.5.2. If $\hat{x}_\alpha \in \bigcup \mathcal{A}$, then there exists a cube $X \in \mathcal{I}'_\mathcal{B}$ such that $\hat{x}_\alpha \in X$. If the Algorithm 5.5.3 is successful, then when Algorithm 5.3.1 is run n–times with initial input X it will produce $flag = 2$. Hence by Theorem 5.3.2 this solution $\hat{x}_\alpha \in \bigcup \mathcal{A}$ must be on the principal branch.

□

Proof of Theorem 1.2.3. We implemented the algorithms discussed in this chapter using MATLAB version R2017b (see [Jaq] for the code). The calculations were performed on Intel Xeon E5-2670 and Intel Xeon E5-2680 processors, and used INTLAB for the interval arithmetic [Rum99]. A summary of the algorithms' runtime is given in Table 5.1.

For the intervals I_α taking the values (containing at least) $[\frac{\pi}{2}, 1.6]$, $[1.6, 1.7]$, $[1.7, 1.8]$, and $[1.8, 1.9]$, we ran Algorithm 4.5.1, augmenting Algorithm 4.2.2 with the seventh step given in Algorithm 4.2.4. In Algorithm 4.5.1 we used computational parameters $i_0 = 2$, $j_0 = 20$, $n_{Time} = 32$, $N_{Period} = 10$, $N_{Prune} = 4$, $\epsilon_1 = 0.05$ and $\epsilon_2 = 0.05$.

We then ran Algorithm 5.4.7 using computational parameters $M = 10$ and $S = 3$, and $N = 15$ producing outputs $\mathcal{S}_{I_\alpha}$ (see Figure 5.3). By Theorem 5.4.8, if y is a SOPS at parameter $\alpha \in I_\alpha$ given as in (5.2), then $(\alpha, \omega, c) \in \bigcup \mathcal{S}_{I_\alpha}$. By Theorem 2.1.4 the SOPS to (1.2) at parameters $\alpha \in I_\alpha$ are in bijective correspondence with the nontrivial zeros of F inside $\bigcup \mathcal{S}_{I_\alpha}$.

On each of the collections of cubes $\mathcal{S}_{I_\alpha}$ we ran Algorithm 5.5.1, using the following computational parameters: For the stopping criterion we used $\epsilon = 0.0001$ for $\alpha \in [\frac{\pi}{2}, 1.6]$ and $\epsilon = 0.01$ otherwise. For the continue-pruning criterion, in every case we used $\delta = 0.5$. For the maximal subdivision dimension, in each case we used $d = 6$, corresponding to the variables $\alpha, \omega, a_1 \in \mathbb{R}$ and $c_2, c_3 \in \mathbb{C}$. For the set of weights, in each case we used $\lambda_0 = 8$ (corresponding to α) and $\lambda_i = 1$ otherwise.

The output of Algorithm 5.5.1 are sets $\mathcal{A}_{I_\alpha}, \mathcal{B}_{I_\alpha}, \mathcal{R}_{I_\alpha}$. On each of these resulting outputs we ran Algorithm 5.5.3 using $n = 5$, and in each case it was successful, producing sets $I_\alpha^\mathcal{A}$ and $I_\alpha^\mathcal{B}$. When $I_\alpha = [\frac{\pi}{2}, 1.6]$ then $I_\alpha^\mathcal{B} = [\frac{\pi}{2}, \frac{\pi}{2} + 0.00550]$ and $I_\alpha^\mathcal{A} = [\frac{\pi}{2} + 0.00550, 1.6]$, and otherwise $I_\alpha^\mathcal{A} = I_\alpha$. By Theorem 5.5.2, this shows that for all $\alpha \in [\frac{\pi}{2} + 0.00550, 1.9]$ there exists a unique solution to $F_\alpha = 0$ in $\bigcup \mathcal{S}$, and if $\alpha \in [\frac{\pi}{2}, \frac{\pi}{2} + 0.00550]$ then the only solutions that exist are on the principal branch. By Theorem 2.3.7 there are no solutions at $\alpha = \frac{\pi}{2}$ on or off the principal branch, and by Theorem 2.3.8 there are no folds in the principal branch for $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 0.00683]$. Hence for all $\alpha \in (\frac{\pi}{2}, 1.9]$ there exists a unique solution to (1.2). □

I_α	N_{bf}	N'_{grid}	N_{grid}	T_{bf}	T_{grid}	T^*_{bb}	T_{verify}
$[\frac{\pi}{2}, 1.6]$	774	614	181	361.8	3.8	2.5*	1.3
$[1.6, 1.7]$	953	861	165	422.2	4.9	3.3*	1.1
$[1.7, 1.8]$	603	566	143	290.1	3.2	10.9*	0.4
$[1.8, 1.9]$	292	277	97	179.1	1.6	61.7*	0.6

Table 5.1: Computational benchmarks from the computer-assisted proof of Theorem 1.2.3. N_{bf} – the number of bounding functions output by Algorithm 4.5.1. N'_{grid} – the number of cubes in $\mathcal{S}'''$ after Step 7 in Algorithm 5.4.7. N_{grid} – the number of cubes output by Algorithm 5.4.7. T_{bf} – the run time (min.) of Algorithm 4.5.1. T_{grid} – the run time (min.) of Algorithm 5.4.7. T^*_{bb} – the run time (min.) of Algorithm 5.5.1 parallelized using 20 workers. T_{verify} – the run time (min.) of Algorithm 5.5.3.

Proof of Theorem 1.2.4. By Theorem 1.2.3 for all $\alpha \in (\frac{\pi}{2}, 1.9]$ there exists a unique solution to (1.2). By Theorem 1.2.2 and [Xie91] there exists a unique SOPS to (1.2) for $\alpha \in [1.9, 6.0]$ and $\alpha \geq 5.67$ respectively. Hence there exists a unique SOPS to (1.2) for all $\alpha > \frac{\pi}{2}$. □

Remark 5.5.5. *It is apparent from Table 5.1 that the two principal computational bottlenecks are Algorithm 4.5.1 (see column T_{bf}) and Algorithm 5.5.1 (see column T^*_{bb}). Algorithm 5.5.1 takes more time to finish as α increases, largely driven by a corresponding increase in C_0 (see Figure 5.3). On the other hand, Algorithm 4.5.1 takes less time to finish as α increases. The interval $I_\alpha = [\frac{\pi}{2}, 1.6]$, however, provides an exception to this trend. This is due to our augmentation of Algorithm 4.2.2 by the seventh step given in Algorithm 4.2.4, an improvement which is most effective for α near $\frac{\pi}{2}$.*

Chapter 6

Future Directions

Proof of Theorem 1.2.5. By [MP88] every global solution to (1.1) has a positive, integer valued lap number $V(x,t)$. For non-zero x the lap number will be an odd integer, defined by fixing the smallest possible $\sigma \geq t$ such that $x(\sigma) = 0$ and defining:

$$V(x,t) = \begin{cases} \text{the \# of zeros (counting multiplicity) of } x(s) \text{ in } (\sigma - 1, \sigma]\text{; or} \\ 1 \text{ if no } \sigma \text{ exists.} \end{cases}$$

Let us fix x_0 as a periodic solution to (1.1) with period L_0. For any $t \in \mathbb{R}$ the lap number $V(x_0,t)$ remains constant, and we can define $N := V(x_0,t)$. If $N = 1$ then x_0 must be a SOPS. If $N \geq 3$ then define the integer $n := \frac{N-1}{2}$ and $r := 1 - nL_0$. By [MP88], it follows that $2/N < L_0 < 2/(N-1)$, hence $0 < r < N^{-1}$. Defining $x_1(t) := x_0(rt)$ and $\alpha_1 = r\alpha_0$ we calculate the derivative of $x_1(t)$ as: $x_1'(t) = -\alpha_1 f(x_0(rt-1))$. We may further compute:

$$x_0(rt-1) = x_0(rt-1+nL_0) = x_0(r(t-1)) = x_1(t-1).$$

Hence it follows that $x_1'(t) = -\alpha_1 f(x_1(t-1))$. Thus we have shown that if $V(x_0) \geq 3$ then x_0 is a rescaling of a periodic solution x_1 with period length $L_1 = L_0/r > 2$. Hence x_0 is a rescaling of a SOPS.

□

6.1 Dynamical Questions

One pertinent question that remains concerns the period length of SOPS to Wright's equation.

Conjecture 6.1.1. *The period length of SOPS to* (1.2) *increases monotonically in* α *for all* $\alpha > \frac{\pi}{2}$.

By Corollary 2.3.9 the period length increases monotonically when $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 6.830\times10^{-3}]$. The rigorous numerics performed in this thesis strongly suggests this to be true when $\alpha \leq 6$, and when $\alpha \geq 3.8$ the period length L satisfies $|L - \alpha^{-1}e^{\alpha}| < 7.66\alpha^{-1}$ by [Nus82]. However Conjecture 6.1.1 is unresolved for $\alpha > \frac{\pi}{2} + 6.830 \times 10^{-3}$.

Another question, proposed in [BCKN14], is the generalized Wright's conjecture.

Conjecture 6.1.2. *For every $\alpha > 0$ the set $\overline{U(\alpha)}$, the closure of the forward extension by the semiflow of a local unstable manifold at zero, is the global attractor for* (1.2)*.*

This is known to be true for $\alpha \leq \frac{\pi}{2}$ by Theorem 2.3.7 and is unresolved for $\alpha > \frac{\pi}{2}$. Conjecture 6.1.2 can be reduced to a question about the number of rapidly oscillating periodic solutions, and moreover Conjecture 6.1.1 implies Conjecture 6.1.2. To wit, by the Poincaré-Bendixson theorem for monotone feedback systems [MPS96], the ω-limit set of any initial data to (1.2) is either 0 or a periodic orbit. The lap number organizes the attractor into Morse sets S_N by [MP88], and by [FMP89] there is always a connecting orbit from the unstable manifold of the origin to the Morse set S_N. Hence, to prove Conjecture 6.1.2, it would suffice to show that each Morse set consists of exactly one periodic orbit.

By Theorem 1.2.5 there are no isolas of periodic orbits, so multiple rapidly oscillating periodic solutions can only arise if there is a fold in one of the branches of rapidly oscillating periodic solutions. If Conjecture 6.1.1 holds, then such a fold can be ruled out using the rescaling equation in Theorem 1.2.5. In particular, if there are two SOPS at parameters $\alpha_1 < \alpha_2$ with period lengths L_1, L_2 and the equality $\alpha_0 = \alpha_1(1 + nL_1) = \alpha_2(1 + nL_2)$ holds, then there will be two distinct rapidly oscillating periodic solutions at parameter α_0. This equality cannot hold if $L_1 < L_2$ whenever $\alpha_1 < \alpha_2$. Thereby Conjecture 6.1.1 implies Conjecture 6.1.2.

There are still further questions about Wright's equation. In [MM96] the authors show a semi-conjugacy of Wright's equation, and negative feedback systems more generally, onto a family of finite dimensional ODEs. Outside the dynamics described by this semi-conjugacy, are there any other interesting dynamics in (1.2)? Furthermore, do the stable and unstable manifolds of the periodic orbits in (1.2) intersect transversely?

6.2 Computational Questions

There are many future directions for the rigorous numerics of infinite dimensional dynamical systems. Perhaps one of the most striking features of Figure 5.1 and Figure 5.4 is the non-uniform size of cubes. This seems to be a result of applying the branch and bound method to a 1-parameter family of solutions instead of a collection of isolated solutions. One approach would be to first validate a neighborhood around the branch of solutions (*á la* [Les10]) and then use a branch and bound method to ensure that there are no solutions outside of this neighborhood. In this paper, we used a collection of weights $\{\lambda\}_{i=0}^{d}$ to mitigate this problem. When using all equal weights ($\lambda_i = 1$ for all i), the vast majority of cubes output by Algorithm 5.5.1 ended up in $\mathcal{R}$. Having a better heuristic for deciding along which dimension to branch would be very useful, particularly so if it does away with the *a priori* need to select a maximal subdivision dimension d as a computational parameter.

Integral to the success of the algorithms in Chapter 5 (allowing it to finish in finite time) are the estimates derived in Chapter 4 which bound *all* of the slowly oscillating periodic solutions to Wright's equation. Since most initial conditions are attracted to the single SOPS in Wright's equation, it was sufficient for the methods in Chapter 4 to be relatively simple. Future work could be done toward bounding all periodic orbits in more general solutions. Examples of this are when there are rapidly oscillating periodic solutions of interest, or when there are multiple (unstable) solutions, or when the dimension is higher, or when one considers instead an ODE or a PDE.

Another question, explored in [LMJ17], is "what is the best Banach space to work in?" In this paper we consider the space Ω^S of Fourier coefficients with algebraic decay. In Algorithm 5.4.7, the estimates for obtaining *a priori* estimates on the Fourier coefficients of SOPS always improve in absolute terms by using larger value of S. However, the value of C_0 will increase when using a larger S. It would likely be beneficial to initially run Algorithm 5.4.7 with a large S, and then convert these bounds into a smaller

S so that C_0 will shrink as well. However, for other applications and other infinite dimensional problems, the question of what is the optimal Banach space remains.

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