FUNDAMENTAL GROUPS OF OPEN MANIFOLDS OF NON-NEGATIVE RICCI CURVATURE

by

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ABSTRACT OF THE DISSERTATION

Fundamental groups of open manifolds of non-negative Ricci curvature

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We study the fundamental groups of open n-manifolds of non-negative Ricci curvature, via the method of Gromov-Hausdorff convergence. In 1968, Milnor conjectured that any open n-manifold M of non-negative Ricci curvature has a finitely generated fundamental group. In this thesis, we verify this conjecture under various geometrical conditions. We show that the Milnor conjecture holds when M has dimension 3, or when the Riemannian universal cover of M has Euclidean volume growth and the unique tangent cone at infinity, or when $\pi_1(M)$-action on the Riemannian universal cover satisfies the no small almost subgroup condition.

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Dedication

To my mom.

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Chapter 1

Introduction

In 1968, Milnor proposed the following conjecture on the fundamental groups of open manifolds with non-negative Ricci curvature [25].

Conjecture 1.0.1. *Let M be an open n-manifold of* $\mathrm{Ric} \geq 0$*, then $\pi_1(M)$ is finitely generated.*

The Milnor conjecture remains open. If this conjecture is true, then by the work of Milnor, Gromov, Kapovitch, and Wilking, $\pi_1(M)$ has polynomial growth and contains a nilpotent subgroup of index bounded by some constant $C(n)$ [25, 17, 22].

For open manifolds with non-negative sectional curvature, Toponogov's triangle comparison controls the small-scale geometry from the large one. This bounds the number of Gromov's short generators [16], and finite generation follows. Actually, any open manifold with non-negative sectional curvature has finite topology [9]. However, for non-negative Ricci curvature, the manifold may have infinite topology [31]. Unlike sectional curvature, Ricci curvature lacks a strong relation between the large-scale geometry and the small-scale one, which is the main difficulty when studying Ricci curvature.

It is natural to ask: *on what additional conditions does the Milnor conjecture hold?* In this thesis paper, we aim to achieve new results regarding this question.

The Milnor conjecture was proved under various additional assumptions in the past decades. For instance, for a manifold with Euclidean volume growth, Anderson and Li independently proved that the fundamental group is finite [2, 23]. Sormani discovered geometric properties for the shortest geodesic loop that represents the short generator [29]. With this, she showed that the Milnor conjecture holds if the manifold has small linear diameter growth or linear volume growth. Liu classified open 3-manifolds of nonnegative Ricci curvature using minimal surface theory and Perelman's work on 3-manifolds [24]. In particular, it confirms the Milnor conjecture in dimension 3.

The first result we present is a new proof of the Milnor conjecture in dimension 3, using structure results for limits spaces of manifolds with Ricci curvature bounded below [4, 5, 7, 11] and equivariant Gromov-Hausdorff convergence [14].

Theorem A. *The Milnor conjecture is true in dimension* 3.

In this thesis, we also verify the Milnor conjecture for manifolds with additional conditions on the Riemannian universal covers at infinity. Recall that for any open n-manifold (M,x) of $\mathrm{Ric}\geq 0$, and any sequence $r_i\to\infty$, passing to a subsequence if necessary, we obtain a tangent cone of M at infinity, which is the Gromov-Hausdorff limit [18] of

$$(r_i^{-1}M,x)\xrightarrow{GH}(Y,y).$$

In general, M may not have a unique tangent cone at infinity [5]. In other words, (Y,y) may depend on the scaling sequence r_i. By splitting theorem [4], Y is a metric product $\mathbb{R}^k\times Y'$, where Y' has no lines. Cheeger and Colding showed that when M has Euclidean volume growth, any tangent cone of M at infinity is a metric cone $(\mathbb{R}^k\times C(Z),(0,z))$ of dimension n [4], where $C(Z)$ has $\mathrm{diam}(Z)<\pi$ and the vertex z. However, the dimension of maximal Euclidean factor k may not be unique among all tangent cones of M at infinity [12].

Theorem B. *Let M be an open n-manifold of* $\mathrm{Ric}\geq 0$. *If there is an integer k such that any tangent cone at infinity of the Riemannian universal cover of M is a metric cone, whose maximal Euclidean factor has dimension k, then $\pi_1(M)$ is finitely generated.*

If $k=0$, then in fact $\pi_1(M)$ is finite (Proposition 4.1.5). Theorem B, in particular, confirms the Milnor conjecture for any manifold whose universal cover has Euclidean volume growth and the unique tangent cone at infinity.

Corollary 1.0.2. *Let M be an open n-manifold of* $\mathrm{Ric}\geq 0$. *If the Riemannian universal cover of M has Euclidean volume growth and the unique tangent cone at infinity, then $\pi_1(M)$ is finitely generated.*

Under the assumption in Theorem B, the Riemannian universal cover $\widetilde{M}$ of M may have different tangent cones at infinity, even with different dimensions.

Our third main result involves the following condition on $\pi_1(M,x)$-action on $\widetilde{M}$.

Definition 1.0.3. Let $\epsilon, \eta, r > 0$ and (M,x) be an n-manifold. For a closed subgroup G of $\mathrm{Isom}(M)$ acting freely on M, we say that G-action has no ϵ-small η-subgroup at $q \in M$ with scale r, if for any nontrivial symmetric subset A of G with

$$\frac{d_H(Aq, A^2q)}{\mathrm{diam}(Aq)} \leq \eta,$$

$D_{r,q}(A) \geq r\epsilon$ holds. We say that G-action has no ϵ-small η-subgroup on $B_1(x)$ with scale r, if it has no ϵ-small η-subgroup at every point in $B_1(x)$ with scale r.

We will explain the motivation behind Definition 1.0.3, and its relation to volume, later in Chapter 5. We conjecture that a volume lower bound would imply the no small almost subgroup condition.

Conjecture 1.0.4. *Given $n, v > 0$, there exist positive constants $\epsilon(n,v)$ and $\eta(n,v)$ such that if an n-manifold (M,x) satisfies*

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{vol}(B_1(x)) \geq v,$$

then any isometric free G-action on M has no ϵ-small η-subgroup on $B_{1/2}(x)$ with scale $r \in (0, 1/2]$.

We show that if $\pi_1(M,x)$-action has no ϵ-small η-subgroup on $\widetilde{M}$ with all scales, then the Milnor conjecture is true.

Theorem C. *Let (M,x) be an open n-manifold with $\mathrm{Ric}_M \geq 0$. If there are ϵ, $\eta > 0$ such that $\pi_1(M,x)$-action has no ϵ-small η-subgroup on $\widetilde{M}$ with all scales $r > 0$, then $\pi_1(M,x)$ is finitely generated.*

Actually, we can bound the number of short generators in terms of ϵ and η, which is stronger than the finite generation (see Sections 2.3, 5.1 and 6.6). Note that if Conjecture 1.0.4 is true, then Theorem 1 would verify the Milnor conjecture for manifolds whose universal covers have Euclidean volume growth.

Theorem C is a joint work with Xiaochun Rong.

We roughly illustrate our approach to the main theorems. When $\pi_1(M,x)$ is not finitely generated, for instance, $\pi_1(M)=\mathbb{Q}$ the additive group of rationals, then on M we would have infinitely many geodesic loops representing the generators with length to infinity, and on the Riemannian universal cover $(\widetilde{M},\tilde{x})$ we would have orbit points at $\tilde{x}$ coming from the generators with arbitrary far distances to $\tilde{x}$. The equivariant Gromov-Hausdorff convergence [14] provides an ideal platform to study this phenomenon. For a sequence $r_i\to\infty$, passing to a subsequence if necessary, we can consider the equivariant Gromov-Hausdorff convergence (see Sections 2.1 and 2.3 for details):

$$\begin{array}{ccc}(r_i^{-1}\widetilde{M},\tilde{x},\pi_1(M,x)) & \xrightarrow{GH} & (\widetilde{Y},\tilde{y},G)\\ \Big\downarrow \pi & & \Big\downarrow \pi \\ (r_i^{-1}M,x) & \xrightarrow{GH} & (Y,y).\end{array}$$

We expect to understand the fundamental group by studying limit spaces (Y,y) and $(\widetilde{Y},\tilde{y},G)$, where the structure theory for Ricci limit spaces can be applied. If $\pi_1(M,x)$ is not finitely generated, then we expect to see the consequence of non-finite generation at infinity. Indeed, in terms of the base manifold (M,x), we can choose a special sequence $r_i\to\infty$ so that the corresponding limit space (Y,y) does not have a pole at y [29]. In terms of the universal cover $(\widetilde{M},\tilde{x},\pi_1(M,x))$, we can choose a special sequence $r_i\to\infty$ so that $(\widetilde{Y},\tilde{y},G)$ has non-connected orbit $G\cdot\tilde{y}$ (see Theorem 2.3.6). These observations play a crucial rule in proving our main results. For example, in proving Theorem B, under certain stability condition on $\widetilde{M}$ at infinity, the key is to show that $\pi_1(M,x)$-action also has certain stability at infinity in terms of equivariant Gromov-Hausdorff convergence, then a contradiction to non-connected orbit would follow. We will provide more details on our approach later in Section 2.3.

We organize the thesis as follows. In Chapter 2, we recall Gromov-Hausdorff topology, structure theory on Ricci limit spaces, and Gromov's short generators. We use these properties frequently through the thesis. Theorems A and B are proved in Chapters 3 and 4 respectively. In Chapter 5, we explore the relation between volume and group actions, which leads to Definition 1.0.3. In Chapter 6, we prove some technical results with the no small almost group assumption, then prove Theorem C.

Chapter 2

Preliminaries

2.1 Gromov-Hausdorff topology

We recall some basic properties on Gromov-Hausdorff topology and its equivariant version. The main reference is [28].

Let X and Y be two bounded closed subsets of a metric space (Z, d). The Hausdorff distance between A and B is defined as

$$d_H(X, Y) = \inf\{\epsilon > 0 \mid B_\epsilon(X) \subseteq Y, B_\epsilon(Y) \subseteq X\},$$

where $B_\epsilon(A)$ is the open ϵ-neighborhood of A. The Hausdorff distance describes the global closeness between A and B. Moreover, $d_H(A, B) = 0$ if and only if $A = B$.

Gromov generalized this notion of distance to the set of all compact metric spaces. For two compact metric spaces X and Y, we define their Gromov-Hausdorff distance

$$d_{GH}(X, Y) = \inf_{(Z,f,g)}\{d_H(f(X), g(Y)) \mid \text{isometric embeddings } f : X \to Z,\ g : Y \to Z$$
$$\text{for some metric space } (Z, d)\}.$$

Roughly speaking, the Gromov-Hausdorff distance between X and Y is small, if these two spaces look alike. $d_{GH}(X, Y) = 0$ if and only if X is isometric to Y. Gromov-Hausdorff distance gives a weak topology on the set of all compact metric spaces. We are particularly interested in the convergence with respect to Gromov-Hausdorff topology, which is a powerful tool in studying a class of Riemannian manifolds, due to the following pre-compactness theorem by Gromov [18].

Theorem 2.1.1. *Let M_i be a sequence of complete n-manifolds of*

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{diam}(M_i) \leq D.$$

Then after passing to a subsequence, the sequence converges in the Gromov-Hausdorff topology to some limit space X.

An alternative formulation of Gromov-Hausdorff convergence uses approximation maps.

Definition 2.1.2. Let X and Y be two compact metric spaces. We say that

$$d_{GH}(X, Y) \leq \epsilon,$$

if there is a map $f : X \to Y$, called an ϵ Gromov-Hausdorff approximation, being
(1) ϵ-isometric, that is, $|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| \leq \epsilon$ for all $x_1, x_2 \in X$, and
(2) ϵ-onto, that is, $Y = B_\epsilon(f(X))$.

d_{GH} in Definition 2.1.2 may be different in values from the one defined by Hausdorff distance. However, they generate the same topology on the set of all compact metric spaces.

Lemma 2.1.3. *Let X_i be a sequence of compact metric spaces of bounded diameter. Then $X_i \xrightarrow{GH} X$ if and only if there is a sequence of ϵ_i Gromov-Hausdorff approximations $f_i : X \to Y$ for some $\epsilon_i \to 0$.*

The notion of Gromov-Hausdorff convergence can be extended to complete metric spaces of unbounded diameter.

Definition 2.1.4. Let (X, x) and (Y, y) be two pointed complete metric spaces. We say that

$$d_{GH}((X, x)(Y, y)) \leq \epsilon,$$

if there is a map $f : (X, x) \to (Y, y)$ satisfying
(1) $d(f(x), y) \leq \epsilon$,
(2) $|d(z_1, z_2) - d(f(z_1), f(z_2))| \leq \epsilon$ for all $z_1, z_2 \in B_{1/\epsilon}(x)$,
(3) $B_{1/\epsilon}(y) \subseteq B_\epsilon(f(B_{1/\epsilon}(x)))$.

Fukaya and Yamaguchi introduced equivariant Gromov-Hausdorff convergence. It takes the symmetries of spaces into account when considering Gromov-Hausdorff convergence. Let (X, x) be a pointed metric space, and let G be a closed subgroup of

$\mathrm{Isom}(X)$. In other words, we assume that G acts effectively and isometrically on X. We write the triple (X, x, G) for simplicity. We also put

$$G(R) = \{g \in G \mid d(g \cdot x, x) \leq R\}.$$

Definition 2.1.5. Let (X, x, G) and (Y, y, H) be two spaces. We say that

$$d_{GH}((X, x, G), (Y, y, H)) \leq \epsilon,$$

if there is a triple of maps (f, ψ, ϕ):

$$f : (X, x) \to (Y, y), \quad \psi : G\left(\frac{1}{\epsilon}\right) \to H, \quad \phi : H\left(\frac{1}{\epsilon}\right) \to G$$

such that:

(1) f satisfies the conditions in Definition 2.1.4,

(2) if $t \in G(\frac{1}{\epsilon})$, $z \in B_{1/\epsilon}(x)$, $t \cdot z \in B_{1/\epsilon}(y)$, then $d(f(t \cdot z), \psi(t) \cdot f(z)) \leq \epsilon$,

(3) if $s \in H(\frac{1}{\epsilon})$, $z \in B_{1/\epsilon}(x)$, $\phi_i(s) \cdot z \in B_{1/\epsilon}(y)$, then $d(f(\phi(s) \cdot z), s \cdot f(z)) \leq \epsilon$.

$d_{GH}((X, x, G), (Y, y, H)) = 0$ in the equivariant Gromov-Hausdorff topology means that there is an isometry $F : (X, x) \to (Y, y)$ and a group isomorphism $\Psi : G \to H$ such that $F(x) = y$ and $F(g \cdot z) = \Psi(g) \cdot F(z)$ for all $g \in G$ and $z \in X$.

Fukaya and Yamaguchi showed that for any Gromov-Hausdorff convergent sequence with isometric actions, we can always find a subsequence converging in the equivariant Gromov-Hausdorff topology [14].

Theorem 2.1.6. *Let*

$$(X_i, x_i) \xrightarrow{GH} (Y, y)$$

be a Gromov-Hausdorff convergent sequence of pointed metric spaces. Let G_i be a closed subgroup of $\mathrm{Isom}(X_i)$ *for each i. Then passing to a subsequence, we can obtain an equivariant Gromov-Hausdorff convergent sequence*

$$(X_i, x_i, G_i) \xrightarrow{GH} (Y, y, H),$$

where H is a closed subgroup of $\mathrm{Isom}(Y)$. *Moreover, the corresponding sequence of quotient spaces converges as well:*

$$(X_i/G_i, \bar{x}_i) \xrightarrow{GH} (Y/H, \bar{y}).$$

Equivariant Gromov-Hausdorff convergence is useful when studying fundamental groups of Riemannian manifolds with curvature bounds. Let (M_i, x_i) be a sequence of complete n-manifolds of

$$\mathrm{Ric}_{M_i} \geq -(n-1),$$

and let $(\widetilde{M}, \tilde{x}_i)$ be the corresponding sequence of Riemannian universal covers. For each i, $\pi_1(M_i, x_i)$ acts isometrically on $\widetilde{M}_i$. Passing to a subsequence if necessary, we obtain the convergence diagram below:

$$\begin{array}{ccc}
(\widetilde{M}_i, \tilde{x}_i, \pi_1(M_i, x_i)) & \xrightarrow{GH} & (\widetilde{Y}, \tilde{y}, G) \\
\big\downarrow \pi_i & & \big\downarrow \pi \\
(M_i, x_i) & \xrightarrow{GH} & (Y = \widetilde{Y}/G, y).
\end{array}$$

2.2 Structure of Ricci limit spaces

Next we recall some structure results on Ricci limit spaces, developed mainly by Cheeger, Colding, and Naber [4, 5, 6, 7, 11].

Definitions 2.2.1. Let $n, v > 0$ and $\kappa \in \mathbb{R}$. We denote $\mathcal{M}(n, \kappa)$ as the set of all metric spaces (X, x) such that (X, x) is the Gromov-Hausdorff limit of a sequence of complete n-manifolds (M_i, x_i) of

$$\mathrm{Ric}_{M_i} \geq (n-1)\kappa.$$

We denote $\mathcal{M}(n, \kappa, v)$ as the set of all metric spaces (X, x) such that (X, x) is the Gromov-Hausdorff limit of a sequence of complete n-manifolds (M_i, x_i) of

$$\mathrm{Ric}_{M_i} \geq (n-1)\kappa, \quad \mathrm{vol}(B_1(x_i)) \geq v.$$

Cheeger and Colding proved the splitting theorem for Ricci limit spaces [4], which is a generalization to splitting theorem for manifolds of non-negative Ricci curvature [8].

Theorem 2.2.2. *[4] Let (X, x) be Gromov-Hausdorff limit of a sequence of complete n-manifolds (M_i, x_i) such that*

$$\mathrm{Ric}_{M_i} \geq -(n-1)\epsilon_i$$

for some $\epsilon_i \to 0$. If X contains a line, then X splits isometrically as $\mathbb{R} \times Y$ for some length metric space Y.

Theorem 2.2.2 is crucial in studying the tangent cones of Ricci limit spaces.

Definition 2.2.3. Let $(X, x) \in \mathcal{M}(n, -1)$ be a Ricci limit space, and $y \in X$ be a point. Let $r_i \to \infty$ be a sequence. Passing to a subsequence if necessary, we obtain Gromov-Hausdorff convergence

$$(r_i X, y) \xrightarrow{GH} (C_y X, o).$$

We say that $(C_y X, o)$ is a tangent cone of X at y.

Let $(C_y X, v)$ be a tangent cone of some Ricci limit space $X \in \mathcal{M}(n, -1)$ at $y \in X$. By a standard diagonal argument, we can find a sequence $s_i \to \infty$ such that

$$(s_i M_i, x_i) \xrightarrow{GH} (C_y X, o).$$

Since the sequence $(s_i M_i, x_i)$ satisfies the curvature condition in Theorem 2.2.2, we conclude that $C_y X$ splits isometrically as $\mathbb{R}^k \times Y$, where the metric space Y does not contain any line.

In general, X may have different tangent cones at y with different dimensions. Moreover, $(C_y X, o)$ may not has a pole at o (see Definition 2.3.9). When $(X, x) \in \mathcal{M}(n, -1, v)$ is a non-collapsed Ricci limit space, we have a more detailed description on its tangent cones.

Definition 2.2.4. Let (Z, d_Z) be a compact metric space. Let $C(Z)$ be the quotient space of $X \times [0, \infty)$ by identifying $X \times \{0\}$ as a point, the vertex of $C(Z)$. We define a metric on $C(Z)$ by

$$d((x, r), (y, s)) = \begin{cases} \sqrt{r^2 + s^2 - 2rs \cos d_Z(x, y)}, & \text{if } d_Z(x, y) \leq \pi; \\ r + s, & \text{if } d_Z(x, y) \geq \pi. \end{cases}$$

We say that $(C(Z), d)$ is the metric cone over (Z, d_Z).

Theorem 2.2.5. *Let $(X, x) \in \mathcal{M}(n, -1, v)$ be a non-collapsed Ricci limit space. For any point $y \in X$ and any tangent cone of X at y, $(C_y X, o)$ is a metric cone $(C(Z), o)$ with vertex o and* $\operatorname{diam}(Z) \leq \pi$.

For a manifold with non-negative Ricci curvature, we can similarly define tangent cones at infinity by blowing down the metric.

Definition 2.2.6. Let (M, x) be an open n-manifold of $\mathrm{Ric} \geq 0$. We say that $(C_\infty M, o)$ is a tangent cone of M at infinity, if there is $r_i \to \infty$ such that

$$(r_i^{-1} M, x) \xrightarrow{GH} (C_\infty M, o)$$

Theorem 2.2.7. *[4] Let (M, x) be an open n-manifold of $\mathrm{Ric} \geq 0$. If M has Euclidean volume growth, that is,*

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r(x))}{r^n} > 0,$$

then any tangent cone of M at infinity (Y, y) is a metric cone $(C(Z), y)$ with vertex y and $\mathrm{diam}(Z) \leq \pi$.

For a Ricci limit space (X, x), we use tangent cones to define regular points of X.

Definition 2.2.8. Let $(X, x) \in \mathcal{M}(n, -1)$ be a Ricci limit space and y be a point in X. We say that y is k-regular, if any tangent cone of X at y is isometric to the Euclidean space $\mathbb{R}^k$.

We denote $\mathcal{R}^k(X)$, or $\mathcal{R}^k$, as the set of all k-regular points in X.

Colding and Naber showed that there is a unique k such that k-regular points are abundant [12].

Theorem 2.2.9. *Let $(X, x) \in \mathcal{M}(n, -1)$ be a Ricci limit space. Then there is a unique integer $0 \leq k \leq n$ such that $\mathcal{R}^k$ has full measure in X with respect to any renormalized measure.*

One may see the definition of renormalized measure in [5]. In this thesis, we only need to use the fact that $\mathcal{R}^k$ is dense in X.

We call the integer k in Theorem 2.2.9 as the dimension of X in the Colding-Naber sense. It is a open question whether the dimension in the Colding-Naber sense equals to the Hausdorff dimension. For the context below, when mentioning the dimension of some Ricci limit space, we always use the dimension in the Colding-Naber sense.

This notion of dimension is semi-continuous with respect to Gromov-Hausdorff convergence [20, 21]:

Theorem 2.2.10. *Let (X_i, x_i) be a sequence of Ricci limit spaces in $\mathcal{M}(n,-1)$. Suppose that $\dim(X_i) \le k$ and*

$$(X_i, x_i) \xrightarrow{GH} (X_\infty, x_\infty).$$

Then $\dim(X_\infty) \le k$.

Apply Theorem 2.2.10 to tangent cones of X, we obtain:

Corollary 2.2.11. *Let $(X, x) \in \mathcal{M}(n,-1)$ be a Ricci limit space with dimension k. Then for any $y \in X$ and any C_yX, a tangent cone of X at y, C_yX has dimension at most k. In particular, $\mathcal{R}^l(X)$ is empty for all $l > k$.*

For non-collapsed Ricci limit spaces, we have a better description of the set of n-regular points [6].

Theorem 2.2.12. *Let $(X, x) \in \mathcal{M}(n,-1,v)$ be a non-collapsed Ricci limit space. Then $X - \mathcal{R}^n$ has Hausdorff dimension at most $n-2$.*

The set of regular points also can be equivalently defined in an effective way. Let X be a Ricci limit space. Let $\epsilon, \delta > 0$ and let k be an integer. We define

$$\mathcal{R}^k_{\epsilon,\delta} = \{y \in X \mid d_{GH}(B_r(y), B^k_r(0)) \le \epsilon r \text{ for all } 0 < r < \delta\},$$

where $B^k_r(0)$ is the ball of radius r in $\mathbb{R}^k$. It is clear that

$$\mathcal{R}^k = \bigcap_{\epsilon>0} \bigcup_{\delta>0} \mathcal{R}^k_{\epsilon,\delta}.$$

One important property of non-collapsed Ricci limit spaces is volume convergence [7, 4].

Theorem 2.2.13. *Let (M_i, x_i) be a sequence of complete n-manifolds of*

$$\mathrm{Ric}_{M_i} \ge -(n-1), \quad \mathrm{vol}(B_1(x_i)) \ge v > 0$$

converging to (X, x). Then for all $R > 0$ and any $q_i \in M_i$ converging to $q \in X$,

$$\mathrm{vol}(B_R(q_i)) \to \mathcal{H}^n(B_R(q)),$$

where $\mathcal{H}^n$ *is the* n*-dimensional Hausdorff measure on* X*. In particular,* X *has Hausdorff dimension* n*.*

On a non-collapsed Ricci limit space, x being effectively regular is equivalent to the almost maximality of local volume [4]. For simplicity, we denote $\Psi(\delta|n)$ as a positive function depending on δ and n such that $\lim_{\delta\to 0} \Psi(\delta|n) = 0$.

Theorem 2.2.14. *Let* $\delta, v > 0$*. Let* $(X, x) \in \mathcal{M}(n, -\delta, v)$ *be a non-collapsed Ricci limit space.*
(1) If

$$d_{GH}(B_1(x), B_1^n(0)) \leq \delta,$$

then

$$\mathcal{H}^n(B_1(x)) \geq (1 - \Psi(\delta|n))\mathrm{vol}(B_1^n(0)),$$

where $\Psi(\delta|n)$ *is some positive function with* $\lim_{\delta\to 0} \Psi(\delta|n) = 0$*.*
(2) If

$$\mathcal{H}^n(B_1(x)) \geq (1 - \delta)\mathrm{vol}(B_1^n(0)),$$

then

$$d_{GH}(B_1(x), B_1^n(0)) \leq \Psi(\delta|n).$$

Ricci limit spaces of dimension 1 are well-understood due to the work by Honda and Chen [10, 19].

Theorem 2.2.15. *Suppose that* $(X, x) \in \mathcal{M}(n, -1)$ *has dimension* 1*, then* X *is a one-dimensional manifold.*

Since we use equivariant Gromov-Hausdorff convergence frequently, it is crucial to know about the isometry group of a Ricci limit space. Cheeger, Colding, and Naber showed that the isometry group is always a Lie group [5, 12].

Theorem 2.2.16. *For any* $X \in \mathcal{M}(n, -1)$*, its isometry group is a Lie group.*

2.3 Gromov's short generators

To study fundamental groups, it is natural to take the universal cover space into account. This is because $\pi_1(M, x)$ acts on the Riemannian universal cover $\widetilde{M}$ isometrically as covering transformations. Gromov introduced the notion of short generators of $\pi_1(M, x)$ [16].

Definition 2.3.1. We say that $\{\gamma_1, ..., \gamma_i, ...\} \subseteq \pi_1(M, x)$ is a set of short generators of $\pi_1(M, x)$, if

$$d(\gamma_1\tilde{x}, \tilde{x}) \leq d(\gamma\tilde{x}, \tilde{x}) \text{ for all } \gamma \in \pi_1(M, x),$$

and for each i, $\gamma_i \in \pi_1(M, x) - \langle\gamma_1, ..., \gamma_{i-1}\rangle$ satisfies

$$d(\gamma_i\tilde{x}, \tilde{x}) \leq d(\gamma\tilde{x}, \tilde{x}) \text{ for all } \gamma \in \pi_1(M, x) - \langle\gamma_1, ..., \gamma_{i-1}\rangle,$$

where $\langle\gamma_1, ..., \gamma_{i-1}\rangle$ is the subgroup generated by $\gamma_1, ..., \gamma_{i-1}$.

The basic properties of short generators below follow directly from the definition.

Proposition 2.3.2. *Let $\{\gamma_1, ..., \gamma_k\}$ be a set of short generators of Γ. Then*
(1) for any $k > l > 0$,

$$d(\gamma_k\tilde{x}, \gamma_l\tilde{x}) \geq d(\gamma_k\tilde{x}, \tilde{x});$$

(2) for any $k > 1$,

$$d(H_k \cdot \tilde{x}, (\Gamma - H_k) \cdot \tilde{x}) = d(\tilde{x}, \gamma_k\tilde{x}),$$

where $H_k = \langle\gamma_1, ..., \gamma_{k-1}\rangle$.

Proof. (1) Suppose that the contrary holds, that is, there are some $k > l > 0$ such that

$$d(\gamma_k\tilde{x}, \gamma_l\tilde{x}) < d(\gamma_k\tilde{x}, \tilde{x}).$$

Consequently,

$$d(\gamma_l^{-1}\gamma_k\tilde{x}, \tilde{x}) < d(\gamma_k\tilde{x}, \tilde{x}).$$

According to the method choosing short generators, we should choose $\gamma_l^{-1}\gamma_k$ as the k-th short generator instead of γ_k, a contradiction.

(2) Because $e \in H_k$ and $\gamma_k \in \Gamma - H_k$, it is clear that for each k,

$$d(H_k \cdot \tilde{x}, (\Gamma - H_k) \cdot \tilde{x}) \leq d(\tilde{x}, \gamma_k \tilde{x}).$$

Suppose that there are $h \in H_k$ and $g \in \Gamma - H_k$ such that

$$d(h\tilde{x}, g\tilde{x}) < d(\tilde{x}, \gamma_k \tilde{x}).$$

Then we should choose $g^{-1}h \in \Gamma - H_k$ instead of γ_k as the k-th short generator, because

$$d(g^{-1}h\tilde{x}, \tilde{x}) < d(\tilde{x}, \gamma_k \tilde{x}).$$

This completes the proof. $\square$

Using Proposition 2.3.2(1) and Toponogov's comparison theorem, Gromov showed that for any complete n-manifold with lower sectional curvature bound, the number of short generators of $\pi_1(M, x)$ can be uniformly controlled for any $x \in M$ [16].

Theorem 2.3.3. *Given n and R, there exists a constant $C(n, R)$ such that the following holds.*

Let M be a complete (compact or non-compact) n-manifold with $\sec \geq -1$, *and x be any point in M. Then the number of short generators of $\pi_1(M, x)$ with length $\leq R$ can be bounded by $C(n, R)$.*

Theorem 2.3.3 confirms the Milnor conjecture for manifolds with non-negative sectional curvature by a scaling trick. In fact, Let $C = C(n, 1)$ be the constant in Theorem 2.3.3. Suppose that $\pi_1(M, x)$ has at least $C + 1$ many short generators $\gamma_1, ..., \gamma_{C+1}, ...$. Put $L = d(\gamma_{C+1}\tilde{x}, \tilde{x})$ and consider the scaled metric $L^{-1}M$. On $L^{-1}M$, we have $\sec_{L^{-1}M} \geq 0$, but $\pi_1(L^{-1}M, x)$ contains $C + 1$ many short generators of length ≤ 1, a contradiction to Theorem 2.3.3. To sum up, we showed that $\pi_1(M, x)$ has at most C many short generators. Note that this is actually stronger than finite generation, since we can bound the number of short generators at every point.

Regarding Ricci curvature, Kapovitch and Wilking proved the following result on the number of short generators [22].

Theorem 2.3.4. *Given n and R, there exists a constant $C(n,R)$ such that the following holds.*

Let M be a complete (compact or non-compact) n-manifold with $\mathrm{Ric} \geq -(n-1)$, *and x be any point in M. Then there exists a point $y \in B_1(x)$ such that the number of short generators of $\pi_1(M,y)$ with length $\leq R$ can be bounded by $C(n,R)$.*

Unlike Theorem 2.3.3, Theorem 2.3.4 bounds the number of short generators at some unspecified point y around x. Due to this, Theorem 2.3.3 fails to confirm the Milnor conjecture.

Note that if one can show that under additional conditions, the number of short generators of $\pi_1(M,x)$ can be uniformly bounded for any x, then partial result of the Milnor conjecture would follow. This is the approach used in Chapters 5 and 6 to prove Theorem C.

The other approach than bounding the number of short generators uniformly is using a contradicting argument: assuming that $\pi_1(M,x)$ has infinitely many short generators, then we seek to find a contradiction from the consequences of non-finite generation.

For an open n-manifold M, if $\Gamma = \pi_1(M,x)$ is not finitely generated, then Γ would have infinitely many short generators. Using the properties of these generators (Proposition 2.3.2) and Gromov-Hausdorff convergence, we can see the impact of non-finite generation from the tangent cones of M or $\widetilde{M}$ at infinity.

Definition 2.3.5. Let (M,x) be an open n-manifold with $\mathrm{Ric} \geq 0$ and an isometric Γ-action. We say that $(C_\infty M, o, G)$, a pointed metric space $(C_\infty M, o)$ with isometric G-action, is an equivariant tangent cone of (M,Γ) at infinity, if there is $r_i \to \infty$ such that

$$(r_i^{-1}M, x, \Gamma) \xrightarrow{GH} (C_\infty M, o, G).$$

Lemma 2.3.6. *Let (M,x) be an open n-manifold with* $\mathrm{Ric} \geq 0$ *and $(\widetilde{M},\tilde{x})$ be its universal cover. Suppose that $\Gamma = \pi_1(M,x)$ has infinitely many short generators $\{\gamma_1,...,\gamma_i,...\}$. Then in the following equivariant tangent cone of $(\widetilde{M},\Gamma)$ at infinity*

$$(r_i^{-1}\widetilde{M}, x, \Gamma) \xrightarrow{GH} (\widetilde{Y}, \tilde{y}, G),$$

the orbit $G \cdot \tilde{y}$ *is not connected, where* $r_i = d(\gamma_i \tilde{x}, \tilde{x}) \to \infty$.

Proof. On $r_i^{-1}\widetilde{M}$, γ_i has displacement 1 at $\tilde{x}$. By Proposition 2.3.2(2), $\gamma_i \cdot \tilde{x}$ has distance 1 from the orbit $H_i \cdot \tilde{x}$, where $H_i = \langle \gamma_1, ..., \gamma_{i-1} \rangle$. From the equivariant convergence

$$(r_i^{-1}\widetilde{M}, \tilde{x}, H_i, \gamma_i) \xrightarrow{GH} (\widetilde{Y}, \tilde{y}, H_\infty, \gamma_\infty),$$

we conclude $d(\gamma_\infty \cdot \tilde{y}, H_\infty \cdot \tilde{y}) = 1$. Moreover, for any element $g \in G - H_\infty$, we can find a sequence $g_i \in \Gamma - H_i$ such that

$$(r_i^{-1}\widetilde{M}, \tilde{x}, g_i) \xrightarrow{GH} (\widetilde{Y}, \tilde{y}, g).$$

Again by Proposition 2.3.2(2), we see that $d(g \cdot \tilde{y}, H_\infty \cdot \tilde{y}) \geq 1$. We divide the orbit $G \cdot \tilde{y}$ into two non-empty subsets: $H_\infty \cdot \tilde{y}$ and $(G - H_\infty) \cdot \tilde{y}$. Since these two subsets have distance 1 between them, we conclude that the orbit $G \cdot \tilde{y}$ must be non-connected. □

Corollary 2.3.7. *Let* M *be an open* n*-manifold of* Ric ≥ 0. *Suppose that for any equivariant tangent cone of* $(\widetilde{M}, \Gamma)$ *at infinity* $(C_\infty \widetilde{M}, \tilde{o}, G)$, *the orbit* $G \cdot \tilde{o}$ *is connected, then* $\pi_1(M, x)$ *is finitely generated.*

It is unknown to the author whether there is M that fails the condition in Corollary 2.3.7. However, given a generic M, usually it is hard to check this condition. One wishes to find natural geometrical constraints that imply such a condition, then the partial result of the Milnor conjecture would follow. This idea leads to Theorem B (see Chapter 4.1 for details).

Sormani has applied a similar idea in her work [29]. For short generators of $\pi_1(M, x)$, we can view them as covering transformations on $\widetilde{M}$ as in Lemma 2.3.6, and we can also view them as geodesic loops in M at x. The properties of these geodesic loops were discovered by Sormani [29].

Theorem 2.3.8. *Let* M *be a complete* n*-manifold of* Ric ≥ 0 *and dimension* $n \geq 3$. *Let* $\{\gamma_1, \gamma_2, ...\}$ *be a set of short generators of* $\pi_1(M, x)$. *Suppose that each* γ_i *has unit speed minimal representative geodesic loops* σ_i *of* x *of length* d_i, *then for each* i

(1) the loop σ_i *is minimal on* $[0, d_i/2]$ *and is also minimal on* $[d_i/2, d_i]$*;*

(2) there is a universal constant S_n such that if $y \in \partial B_{Rd_i}(x)$ where $R \geq 1/2 + S_n$, then

$$d(y, \sigma_i(d_i/2)) \geq (R - 1/2)d_i + 2S_n d_i$$

We indicate the geometry of Theorem 2.3.8. The first half of Theorem 2.3.8 implies that the midpoint of σ_i is a cut point. In particular, a minimal geodesic of length d_i can not go through $\sigma_i(d_i/2)$. The second half is called uniform cut theorem, in the sense that there is a uniform lower bounder on the distance between any minimal geodesic of length d_i and the midpoint $\sigma_i(d_i/2)$ (see [29]). Passing this property to infinity, we see the consequence of non-finite generation at infinity.

Definition 2.3.9. Let X be a complete length metric space. We say that X has a pole at $x \in X$, if for any $y \in X$, there is a ray starting at x going through y.

Theorem 2.3.10. *[29] Let (M, x) be an open n-manifold with* $\mathrm{Ric} \geq 0$*. Suppose that Γ has infinitely many short generators $\{\gamma_1, ..., \gamma_i, ...\}$. Then in the following tangent cone of M at infinity*

$$(r_i^{-1}M, x) \xrightarrow{GH} (Y, y),$$

Y can not have a pole at y, where $r_i = d(\gamma_i \tilde{x}, \tilde{x}) \to \infty$.

We finish Chapter 2 with a handful reduction by Wilking on the Milnor conjecture [32]:

Theorem 2.3.11. *Let M be an open n-manifold with* $\mathrm{Ric}_M \geq 0$*. If $\pi_1(M)$ is not finitely generated, then it contains a non-finitely generated abelian subgroup.*

Chapter 3

Milnor conjecture in dimension 3

We prove Theorem A in this Chapter. We start with a topological result by Evans and Moser [13].

Theorem 3.0.1. *Let M be a 3-manifold. If $\pi_1(M)$ is abelian and not finitely generated, then $\pi_1(M)$ is torsion free.*

Evans and Moser [13] actually showed that $\pi_1(M)$ is a subgroup of the additive group of rationals. Being torsion free is sufficient to prove Theorem A.

Let M be an open 3-manifold with $\mathrm{Ric}_M \geq 0$. Suppose that $\Gamma = \pi_1(M,x)$ is not finitely generated, then by Theorems 2.3.11 and 3.0.1, without lose of generality, we can assume that Γ is abelian and torsion free. Let $\{\gamma_1, ..., \gamma_i, ...\}$ be an infinite set of short generators at x. Passing to a subsequence if necessary, we obtain the following equivariant Gromov-Hausdorff convergence:

$$\begin{array}{ccc} (r_i^{-1}\widetilde{M},\tilde{x},\Gamma) & \xrightarrow{GH} & (\widetilde{Y},\tilde{y},G) \\ \downarrow \pi & & \downarrow \pi \\ (r_i^{-1}M,x) & \xrightarrow{GH} & (Y=\widetilde{Y}/G,y). \end{array},$$

where $r_i = d(\gamma_i\tilde{x},\tilde{x}) \to \infty$. By Theorem 2.3.6, the orbit $G\cdot\tilde{y}$ must be non-connected. We also know that y is not a pole of Y according to Theorem 2.3.10. We prove Theorem A by eliminating all the possibilities regarding the dimension of Y and $\widetilde{Y}$ above in the Colding-Naber sense. There are three possibilities listed as below, and we rule out each of them, which finishes the proof of Theorem A:

Case 1. $\dim(\widetilde{Y}) = 3$ (Lemma 3.0.4);

Case 2. $\dim(Y) = \dim(\widetilde{Y}) = 2$ (Lemma 3.0.5);

Case 3. $\dim(Y) = 1$ (Lemma 3.0.6).

One important observation follows from Theorem 3.0.1: the limit orbit $G\cdot\tilde{y}$ can not be discrete.

Proposition 3.0.2. *Let (M,x) be an open n-manifold with $\mathrm{Ric}_M\geq 0$ and $(\widetilde{M},\tilde{x})$ be its universal cover. Suppose that $\Gamma=\pi_1(M,x)$ is torsion free, then for any $s_i\to\infty$ and any convergent sequence*

$$(s_i^{-1}\widetilde{M},\tilde{x},\Gamma)\overset{GH}{\longrightarrow}(C_\infty\widetilde{M},\tilde{o},G),$$

the orbit $G\cdot\tilde{o}$ is not discrete.

Lemma 3.0.3. *Let (M_i,x_i) be a sequence of complete n-manifolds and $(\widetilde{M}_i,\tilde{x}_i)$ be their universal covers. Suppose that the following sequence converges*

$$(\widetilde{M}_i,\tilde{x}_i,\Gamma_i)\overset{GH}{\longrightarrow}(\widetilde{X},\tilde{x},G),$$

where $\Gamma_i=\pi_1(M_i,x_i)$ is torsion free for each i. If the orbit $G\cdot\tilde{x}$ is discrete in $\widetilde{X}$, then there is an integer N such that

$$\#\Gamma_i(1)\leq N$$

for all i, where $\#\Gamma_i(1)$ is the number of elements in

$$\Gamma_i(1)=\{\gamma\in\Gamma_i\mid d(\gamma\tilde{x}_i,\tilde{x}_i)\leq 1\}.$$

Proof. We claim that if a sequence $\gamma_i\in\Gamma_i$ converges to $g\in G$ with g fixing $\tilde{x}$, then $g=e$, the identity element, and $\gamma_i=e$ for all i sufficiently large. In fact, suppose that $\gamma_i\neq e$ for some subsequence. Since γ_i is torsion free, we always have $\mathrm{diam}(\langle\gamma_i\rangle\cdot\tilde{x}_i)=\infty$. Together with $d(\gamma_i\tilde{x}_i,\tilde{x}_i)\to 0$, we see that $G\cdot\tilde{x}$ can not be discrete, a contradiction to the assumption.

Therefore, there exists i_0 large such that for all $g\in G(2)$ and any two sequences with $\gamma_i\overset{GH}{\to}g$ and $\gamma_i'\overset{GH}{\to}g$, $\gamma_i=\gamma_i'$ holds for all $i\geq i_0$. In particular, we conclude that

$$\#\Gamma_i(1)\leq\#G(2)<\infty$$

for all $i\geq i_0$. □

Proof of Proposition 3.0.2. The proof follows directly from Lemma 3.0.3. If $G \cdot \tilde{o}$ is discrete, then there is N such that $\#\Gamma(s_i) \le N$ for all i. On the other hand, $\#\Gamma(s_i) \to \infty$ because Γ is torsion free, a contradiction. □

Now we begin to rule out the possibilities listed in the beginning of this chapter.

Lemma 3.0.4. *Case 1 can not happen.*

Proof. When $\dim(\widetilde{Y}) = 3$, $\widetilde{Y}$ is a non-collapsing limit space, that is, there is $v > 0$ such that

$$\mathrm{vol}(B_1(\tilde{x}, r_i^{-1}\widetilde{M})) \ge v$$

for all i. By relative volume comparison, this implies that $\widetilde{M}$ has Euclidean volume growth

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_r(\tilde{x}))}{r^n} \ge v.$$

By Theorem 2.2.7, $\widetilde{Y}$ is a metric cone $\mathbb{R}^k \times C(Z)$ with vertex $\tilde{y} = (0, z)$, where $C(Z)$ has vertex z and $\mathrm{diam}(Z) < \pi$. We rule out all the possibilities of $k \in \{0, 1, 2, 3\}$.

If $k = 3$, then $\widetilde{Y} = \mathbb{R}^3$. Thus $\widetilde{M}$ is isometric to $\mathbb{R}^3$ [7].

If $k = 2$, then by Theorem 2.2.12, $\widetilde{Y} = \mathbb{R}^3$ holds.

If $k = 1$, then $Y = \mathbb{R} \times C(Z)$. By Proposition 4.1.2, the orbit $G \cdot \tilde{y}$ is contained in $\mathbb{R} \times \{z\}$. Applying Lemma 2.3.6, we see that $G \cdot \tilde{y}$ is not connected. Note that a non-connected orbit in $\mathbb{R}$ is either a $\mathbb{Z}$-translation orbit, or a $\mathbb{Z}_2$-reflection orbit. In particular, the orbit $G \cdot \tilde{y}$ must be discrete. This contradicts with Proposition 3.0.2.

If $k = 0$, then $Y = C(Z)$ with no lines. Again by Proposition 4.1.2, the orbit $G \cdot \tilde{y}$ must be a single point $\tilde{y}$, a contradiction to Lemma 2.3.6. □

Lemma 3.0.5. *Let (M, x) be an open n-manifold with $\mathrm{Ric}_M \ge 0$ and $(\widetilde{M}, \tilde{x})$ be its universal cover. Assume that $\Gamma = \pi_1(M, x)$ is torsion free. Then for any $s_i \to \infty$ and any convergent sequence*

$$\begin{array}{ccc}
(s_i^{-1}\widetilde{M}, \tilde{x}, \Gamma) & \xrightarrow{GH} & (C_\infty\widetilde{M}, \tilde{o}, G) \\
\downarrow \pi & & \downarrow \pi \\
(s_i^{-1}M, x) & \xrightarrow{GH} & (C_\infty M, o),
\end{array}$$

$\dim(C_\infty\widetilde{M}) = \dim(C_\infty M)$ *can not happen. In particular, Case 2 can not happen.*

Proof of Lemma 3.0.5. We claim that when $\dim(C_\infty\widetilde{M}) = \dim(C_\infty M) = k$, G must be a discrete group. If this claim holds, then the desired contradiction follows from Proposition 3.0.2.

It remains to verify the claim. Suppose that G_0 is non-trivial, then we pick $g \neq e$ in G_0. We first show that there is a k-regular point $\tilde{q} \in C_\infty\widetilde{M}$ such that $d(g\tilde{q},\tilde{q}) > 0$ and $\tilde{q}$ projects to a k-regular point $q \in C_\infty M$. In fact, let $\mathcal{R}_k(C_\infty M)$ be the set of k-regular points in $C_\infty M$. Since $\mathcal{R}_k(C_\infty M)$ is dense in $C_\infty M$, its pre-image $\pi^{-1}(\mathcal{R}_k(C_\infty M))$ is also dense in $C_\infty\widetilde{M}$. Let $\tilde{q}$ be a point in the pre-image such that $d(g\tilde{q},\tilde{q}) > 0$. Note that any tangent cone at $\tilde{q}$ splits $\mathbb{R}^k$-factor isometrically. By Corollary 2.2.11, it follows that any tangent cone at $\tilde{q}$ is isometric to $\mathbb{R}^k$. In other words, $\tilde{q}$ is k-regular.

Along a one-parameter subgroup of G_0 containing g, we can choose a sequence of elements $g_j \in G_0$ with $d(g_j\tilde{q},\tilde{q}) = 1/j \to 0$. We consider a tangent cone at $\tilde{q}$ and q respectively coming from the sequence $j \to \infty$. Passing to some subsequences if necessary, we obtain

$$\begin{array}{ccc}
(jC_\infty\widetilde{M},\tilde{q},G,g_j) & \xrightarrow{GH} & (C_{\tilde{q}}C_\infty\widetilde{M},\tilde{o}',H,h) \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
(jC_\infty M,q) & \xrightarrow{GH} & (C_qC_\infty M,o').
\end{array}$$

with $C_{\tilde{q}}C_\infty\widetilde{M}/H = C_qC_\infty M$ and $d(h\tilde{o}',\tilde{o}') = 1$. On the other hand, since both q and $\tilde{q}$ are k-regular, $C_{\tilde{q}}C_\infty\widetilde{M} = C_qC_\infty M = \mathbb{R}^k$. This is a contradiction to $H \neq \{e\}$. Hence the claim holds. $\square$

To rule out the last case $\dim(Y) = 1$, we make use Sormani's pole group theorem (Theorem 2.3.10) and Theorem 2.2.15.

Lemma 3.0.6. *Case 3 can not happen.*

Proof. By Theorem 2.2.15, Y is a topological manifold of dimension 1. Since Y is non-compact, Y is either a line $(-\infty,\infty)$ or a half line $[0,\infty)$. By Theorem 2.3.10, Y can not have a pole at y. Thus there is only one possibility left: $Y = [0,\infty)$ but y is not the endpoint $0 \in [0,\infty)$. Put $d = d_Y(0,y) > 0$. We rule out this case by a rescaling argument and Lemmas 3.0.4, 3.0.5 above. (In general, it is possible for an open manifold to have a tangent cone at infinity as $[0,\infty)$, with base point not being 0; see example 3.0.7.)

Let $\alpha(t)$ be a unit speed ray in M starting from x, and converging to the unique ray from y in $Y=[0,\infty)$ with respect to the sequence $(r_i^{-1}M,x)\xrightarrow{GH}(Y,y)$. Let $z_i\in r_i^{-1}M_i$ be a sequence of points converging to $0\in Y$, then $r_i^{-1}d_M(x,z_i)\to d$. For each i, let $c_i(t)$ be a minimal geodesic from z_i to $\alpha(dr_i)$, and q_i be a closest point to x on c_i. We re-parametrize c_i so that $c_i(0)=q_i$. With respect to the sequence $(r_i^{-1}M,x)\xrightarrow{GH}(Y,y)$, c_i sub-converges to the unique segment between 0 and $2d\in[0,\infty)$. Clearly,

$$r_i^{-1}d_M(z_i,\alpha(dr_i))\to 2d,\quad r_i^{-1}d_i\to 0,$$

where $d_i=d_M(x,c_i(0))$.

If $d_i\to\infty$, then we rescale M and $\widetilde{M}$ by $d_i^{-1}\to 0$. Passing to some subsequences if necessary, we obtain

$$\begin{array}{ccc}(d_i^{-1}\widetilde{M},\tilde{x},\Gamma) & \xrightarrow{GH} & (\widetilde{Y}',\tilde{y}',G')\\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi}\\ (d_i^{-1}M,x) & \xrightarrow{GH} & (Y',y').\end{array}$$

If $\dim(Y')=1$, then we know that $Y'=(-\infty,\infty)$ or $[0,\infty)$. On the other hand, since

$$d_i^{-1}d_M(c_i(0),z_i)\to\infty,\quad d_i^{-1}d_M(c_i(0),\alpha(dr_i))\to\infty,\quad d_i^{-1}d_M(c_i,x)=1,$$

c_i sub-converges to a line c_∞ in Y' with $d(c_\infty,y')=1$. Clearly this can not happen in $Y'=(-\infty,\infty)$ nor $[0,\infty)$. If $\dim(\widetilde{Y}')=3$, then $\widetilde{M}$ has Euclidean volume growth. Thus with the sequence r_i^{-1}, the corresponding limit spaces Y and $\widetilde{Y}$ must satisfy $\dim(Y)=1$ and $\dim(\widetilde{Y})=3$, which is already covered in Lemma 3.0.4. The only situation left is $\dim(\widetilde{Y}')=\dim(Y')=2$. By Lemma 3.0.5, this also leads to a contradiction. In conclusion, $d_i\to\infty$ can not happen.

If there is some $R>0$ such that $d_i\le R$ for all i, then on M, c_i subconverges to a line c with $c(0)\in B_{2R}(x)$. Consequently, M splits off a line isometrically [8], a contradiction to $Y=[0,\infty)$. This completes the proof. $\square$

Example 3.0.7. We construct a surface (S, x) isometrically embedded in $\mathbb{R}^3$ such that S has a tangent cone at infinity as $[0, \infty)$, but x does not correspond to 0. We first construct a subset of xy-plane by gluing intervals. Let $r_i \to \infty$ be a positive sequence with $r_{i+1}/r_i \to \infty$. Starting with a interval $I_1 = [-r_1, r_2]$, we attach a second interval $I_2 = [-r_3, r_4]$ perpendicularly to I_1 by identifying $r_2 \in I_1$ and $0 \in I_2$. Repeating this process, suppose that I_k is attached, then we attach the next interval $I_{k+1} = [-r_{2k+1}, r_{2k+2}]$ perpendicularly to I_k by identifying $r_{2k} \in I_k$ and $0 \in I_{k+1}$. In this way, we construct a subset T in the xy-plane consisting of segments. We can smooth the ϵ-neighborhood of T in $\mathbb{R}^3$ so that it has sectional curvature $\geq -C$, where $\epsilon, C > 0$. We call this surface S. Let $p \in S$ be a point closest to $0 \in I_1$ as the base point. If we rescale (S, x) by $r_{2k+1}^{-1} \to 0$, then

$$(r_{2k+1}^{-1} S, x) \xrightarrow{GH} ([-1, \infty), 0)$$

because $r_{i+1}/r_i \to \infty$. In other words, S has a tangent cone at infinity as the half line, but the base point does not correspond to the end point in this half line.

Chapter 4

Stability of Euclidean factors at infinity

4.1 Introduction

We prove Theorem B in this chapter. For convenience, we introduce the following notion for the stability assumption in Theorem B.

Definition 4.1.1. Let M be an open n-manifold with $\mathrm{Ric} \geq 0$, and let k be an integer. We say that M is *k-Euclidean at infinity*, if any tangent cone of M at infinity (Y, y) is a metric cone, whose maximal Euclidean factor has dimension k, that is, (Y, y) splits as $(\mathbb{R}^k \times C(Z), (0, z))$, where $C(Z)$ is a metric cone with $\mathrm{diam}(Z) < \pi$ and vertex z.

We point out that, Definition 4.1.1 implies a uniform control on the diameter of Z: for any tangent cone of M at infinity $(\mathbb{R}^k \times C(Z), (0, z))$, $\mathrm{diam}(Z) \leq \pi - \eta(M)$ for some positive constant $\eta(M)$ (see Lemma 4.2.1 and Remark 4.2.2).

For a metric cone $\mathbb{R}^k \times C(Z)$ with $\mathrm{diam}(Z) < \pi$, it is known that its isometry group has a splitting structure.

Proposition 4.1.2. *Let $(\mathbb{R}^k \times C(Z), (0, z)) \in \mathcal{M}(n, 0)$ be a metric cone, where $C(Z)$ has vertex z and $\mathrm{diam}(Z) < \pi$. Then for any isometry g of $C(Z)$, we have*

$$g \cdot (\mathbb{R}^k \times \{z\}) \subseteq \mathbb{R}^k \times \{z\}.$$

Proposition 4.1.3. *Let $Y = \mathbb{R}^k \times C(Z) \in \mathcal{M}(n, 0)$ be a metric cone with $\mathrm{diam}(Z) < \pi$. Then its isometry group $\mathrm{Isom}(Y)$ splits as $\mathrm{Isom}(\mathbb{R}^k) \times \mathrm{Isom}(Z)$.*

Proof. For any $F \in \mathrm{Isom}(\mathbb{R}^k \times C(Z))$, we write

$$F(v, w) = (F_1(v, w), F_2(v, w))$$

for any $v \in \mathbb{R}^k$ and $w \in C(Z)$. We claim that F_1 and F_2 are independent of w and v, respectively.

In fact, notice that for any line $\sigma(t) = (\sigma_1(t), \sigma_2(t))$ in Y, because the $C(Z)$-factor contains no lines, σ_2 must be a constant map. For any $w \in C(Z)$ and any arbitrary $v_1, v_2 \in \mathbb{R}^k$, let $\sigma_1(t)$ be the line through v_1 and v_2 in $\mathbb{R}^k$. We consider the line

$$\sigma(t) = (\sigma_1(t), w).$$

Through the isometry F, $F \circ \sigma(t)$ is another line, whose $C(Z)$-component must be constant. This shows that $F_2(v_1, w) = F_2(v_2, w)$. Also, for any $v \in \mathbb{R}^k$, the image of $\{v\} \times C(Z)$ under F is a cone orthogonal to the $\mathbb{R}^k$-factor, thus $F(\{v\} \times C(Z)) = (\{v'\} \times C(Z))$ for some $v' \in \mathbb{R}^k$, This shows that F_1 is independent of elements in $C(Z)$.

Therefore, any $F \in \mathrm{Isom}(Y)$ can be written as

$$F(v, w) = (F_1(v, w), F_2(v, w)) = (F_1(v), F_2(w))$$

with $F_1 \in \mathrm{Isom}(\mathbb{R}^k)$ and $F_2 \in \mathrm{Isom}(C(Z))$. $\square$

Due to Proposition 4.1.3, for a metric cone $Y = \mathbb{R}^k \times C(Z) \in \mathcal{M}(n, 0)$, where $\mathrm{diam}(Z) < \pi$, there is a natural projection map:

$$p : \mathrm{Isom}(Y) \to \mathrm{Isom}(\mathbb{R}^k).$$

Throughout this Chapter, we always use p to denote this projection map.

We explain our approach to Theorem B as follows. We consider all the possible equivariant tangent cones of $(\widetilde{M}, \pi_1(M, x))$ at infinity (see Definition 2.3.5). Our main discovery is that, if $\widetilde{M}$ is k-Euclidean at infinity, there is certain equivariant stability among all the equivariant tangent cones of $(\widetilde{M}, \pi_1(M, x))$ at infinity:

Theorem 4.1.4. *Let M be an open n-manifold of $\mathrm{Ric} \geq 0$. Suppose that $\pi_1(M)$ is abelian and $\widetilde{M}$ is k-Euclidean at infinity. Then there exist a closed abelian subgroup K of $O(k)$ and an integer $l \in [0, k]$ such that for any equivariant tangent cone of $(\widetilde{M}, \pi_1(M, x))$ at infinity $(\widetilde{Y}, \tilde{y}, G) = (\mathbb{R}^k \times C(Z), (0, z), G)$, the projected G-action on $\mathbb{R}^k$-factor $(\mathbb{R}^k, 0, p(G))$ satisfies that $p(G) = K \times \mathbb{R}^l$, with K fixing 0 and the subgroup $\{e\} \times \mathbb{R}^l$ acting as translations in $\mathbb{R}^k$.*

Theorem B follows directly from Theorem 4.1.4 due to Theorem 2.3.11 and Corollary 2.3.7.

We illustrate our approach to Theorem 4.1.4. Put $\Gamma = \pi_1(M, x)$. Given two equivariant tangent cones of $(\widetilde{M}, \Gamma)$ at infinity $(\widetilde{Y}_i, \tilde{y}_i, G_i)$ $(i = 1, 2)$, assume that their projected actions $(\mathbb{R}^k, 0, p(G_i))$ are different. We consider the set of all equivariant tangent cones of $(\widetilde{M}, \Gamma)$ at infinity, denoted by $\Omega(\widetilde{M}, \Gamma)$. It is known that $\Omega(\widetilde{M}, \Gamma)$ is compact and connected in the equivariant Gromov-Hausdorff topology. Consequently, for any $\epsilon > 0$, there are finitely many spaces $(W_j, w_j, H_j) \in \Omega(\widetilde{M}, \Gamma)$ $(j = 1, ..., l)$ such that $(W_1, w_1, H_1) = (\widetilde{Y}_1, \tilde{y}_1, G_1)$, $(W_l, w_l, H_l) = (\widetilde{Y}_2, \tilde{y}_2, G_2)$, and

$$d_{GH}((W_j, w_j, H_j), (W_{j+1}, w_{j+1}, H_{j+1})) \leq \epsilon$$

for all $j = 1, ..., l-1$. When $\widetilde{M}$ is k-Euclidean at infinity, $W_i = \mathbb{R}^k \times C(Z_i)$ with $\mathrm{diam}(Z_i) \leq \pi - \eta(\widetilde{M})$. Under this control, the associated chain of $\mathbb{R}^k$-factor in W_j with projected $p(H_j)$-action $\{(\mathbb{R}^k, 0, p(H_j))\}_{j=1}^{l}$ form a $\psi(\epsilon)$ chain, where $\psi(\epsilon)$ is a positive function with $\psi(\epsilon) \to 0$ as $\epsilon \to 0$.

To see a contradiction without involving the complexity in general situation, we restrict to the special case that all $p(H_j)$-actions fix 0. Then this leads to the following stability of isometric actions on the unit sphere $S^{k-1} \subseteq \mathbb{R}^k$: if (S^{k-1}, K_1) and (S^{k-1}, K_2) are sufficiently close in the equivariant Gromov-Hausdorff topology, then either K_1 and K_2 are conjugate in $O(k)$, or $\dim(K_1) \neq \dim(K_2)$ (see Proposition 4.3.3). It turns out this stability is enough for us to derive a contradiction. For instance, if $p(G_1) = \{e\}$ and $p(G_2) = \mathbb{Z}_2$, there is no ϵ-chain $\{(\mathbb{R}^k, 0, p(H_j))\}_{j=1}^{l}$, with $p(H_i)$ fixing 0, between $(\mathbb{R}^k, 0, \{e\})$ and $(\mathbb{R}^k, 0, \mathbb{Z}_2)$, given that ϵ is small (see Lemma 4.2.3). To deal with the general situation where these $p(H_i)$-action may not fix 0, we develop a key technical tool, referred as critical rescaling (see Section 4.2 for details).

To complete the introduction to chapter 4, we show that if $\widetilde{M}$ is 0-Euclidean at infinity, then $\pi_1(M)$ must be finite.

Proposition 4.1.5. *Let M be an open n-manifold of $\mathrm{Ric} \geq 0$. If $\widetilde{M}$ is 0-Euclidean at infinity, then $\pi_1(M)$ is finite.*

Proof. Suppose that $\Gamma = \pi_1(M, x)$ is an infinite group, then there are elements $\gamma_i \in$

$\pi_1(M,x)$ with $r_i := d(\gamma_i\tilde{x},\tilde{x}) \to \infty$. Passing to a subsequence if necessary, we consider an equivariant tangent cone of $(\widetilde{M},\Gamma)$ at infinity:

$$(r_i^{-1}\widetilde{M},\tilde{x},\Gamma) \xrightarrow{GH} (\widetilde{Y},\tilde{y},G).$$

By our choice of r_i, there is $g \in G$ such that $d(g\cdot\tilde{y},\tilde{y}) = 1$. On the other hand, since $\widetilde{M}$ is 0-Euclidean at infinity, $(\widetilde{Y},\tilde{y})$ is a metric cone with no lines and $\tilde{y}$ is the unique vertex. Thus the orbit $G\cdot\tilde{y}$ must be a single point $\tilde{y}$ by Proposition 4.1.2. A contradiction. $\square$

Corollary 4.1.6. *Let (M,x) be an open n-manifold of* $\mathrm{Ric} \geq 0$*. If its universal cover $(\widetilde{M},\tilde{x})$ has Euclidean volume growth and non-maximal diameter growth*

$$\limsup_{R\to\infty} \frac{\mathrm{diam}(\partial B_R(\tilde{x}))}{R} < 2,$$

then $\pi_1(M,x)$ is a finite group. Consequently, M itself has Euclidean volume growth.

Here we use extrinsic metric on $\mathrm{diam}(\partial B_R(\tilde{x}))$, so we always have

$$\frac{\mathrm{diam}(\partial B_R(\tilde{x}))}{R} \leq 2.$$

4.2 A critical rescaling argument

In this section, we develop the critical rescaling argument, a key technical tool as mentioned in the introduction, to prove a special case of Theorem 4.1.4: if there is $(\widetilde{Y},\tilde{y},G) \in \Omega(\widetilde{M},\Gamma)$ such that $p(G)$ is trivial, then for any $(\widetilde{W},\tilde{w},H) \in \Omega(\widetilde{M},\Gamma)$, $p(H)$ is also trivial (see Proposition 4.2.5). The proof of Theorem 4.1.4 is also modeled on the proofs in this section.

We first show that if M is k-Euclidean at infinity, then for any $Y = \mathbb{R}^k \times C(Z) \in \Omega(M)$, there is a uniform gap between Y and any Ricci limit space splitting off a $\mathbb{R}^{k+1}$-factor. This is indeed a direct consequence of being k-Euclidean at infinity.

Lemma 4.2.1. *Let M be an open n-manifold of* $\mathrm{Ric} \geq 0$*. If M is k-Euclidean at infinity, then there is $\epsilon(M) > 0$ such that for any $(Y,y) \in \Omega(M)$ and any Ricci limit space $\mathbb{R}^l \times X \in \mathcal{M}(n,0)$ with $l > k$, we have*

$$d_{GH}((Y,y),(\mathbb{R}^l \times X,(0,x))) \geq \epsilon(M).$$

Proof. Suppose the contrary, then we would have a sequence $(Y_i, y_i) \in \Omega(M)$ such that as $i \to \infty$,

$$d_{GH}((Y_i, y_i), (\mathbb{R}^{l_i} \times X_i), (0, x_i)) \to 0,$$

where $\mathbb{R}^{l_i} \times X_i \in \mathcal{M}(n, 0)$ and $k < l_i \leq n$. By pre-compactness, we can pass to a subsequence and have convergence

$$(\mathbb{R}^{l_i} \times X_i, (0, x_i)) \xrightarrow{GH} (\mathbb{R}^{l_\infty} \times X_\infty, (0, x_\infty))$$

with integer $l_\infty > k$. For the corresponding subsequence of (Y_i, y_i), it has the same limit. By a standard diagonal argument, $(\mathbb{R}^{l_\infty} \times X_\infty, (0, x_\infty))$ is also a tangent cone of M at infinity. This is a contradiction to the assumption that M is k-Euclidean at infinity. $\square$

Remark 4.2.2. Note that for a metric cone $C(Z) \in \mathcal{M}(n, 0)$, $C(Z)$ splits off a line if and only if $\operatorname{diam}(Z) = \pi$. From this perspective, Lemma 4.2.1 implies that if M is k-Euclidean at infinity, then there exists $\eta(M) > 0$ such that for any $\mathbb{R}^k \times C(Z) \in \Omega(M)$, Z has diameter no more than $\pi - \eta(M)$.

Next we prove a gap phenomenon between two classes of group actions on spaces in $\Omega(M)$, which is a key property needed in the critical rescaling argument.

Lemma 4.2.3. *Let M be an open n-manifold of* Ric ≥ 0. *Suppose that $\widetilde{M}$ is k-Euclidean at infinity. Then there exists a constant $\epsilon(M) > 0$ such that the following holds.*

For two spaces $(\widetilde{Y}_j, \tilde{y}_j, G_j) \in \Omega(\widetilde{M}, \Gamma)$ with $(\widetilde{Y}_j, \tilde{y}_j) = (\mathbb{R}^k \times C(Z_j), (0, z_j))$ $(j = 1, 2)$ if

(1) $p(G_1)$ is trivial, and

(2) there is $g \in G_2$ such that $p(g) \neq e$ and $d(g \cdot \tilde{y}_2, \tilde{y}_2) \leq 1$,

then

$$d_{GH}((\widetilde{Y}_1, \tilde{y}_1, G_1), (\widetilde{Y}_2, \tilde{y}_2, G_2)) \geq \epsilon(M).$$

For the Euclidean space $\mathbb{R}^k$ with isometric G-action and a non-identity element $g \in G$, if $d(g \cdot 0, 0) \leq 1$, then it is obvious that

$$d_{GH}((\mathbb{R}^k, 0, G), (\mathbb{R}^k, 0, \{e\})) \geq 1/2.$$

Let $(\widetilde{Y}_1, \tilde{y}_1, G_1)$ and $(\widetilde{Y}_2, \tilde{y}_2, G_2)$ be two spaces in $\Omega(\widetilde{M}, \Gamma)$ as in Lemma 4.2.3. Roughly speaking, Lemma 4.2.1 assures that for ϵ sufficiently small, any ϵ-approximation from $(\widetilde{Y}_1, \tilde{y}_1)$ to $(\widetilde{Y}_2, \tilde{y}_2)$ can not map $\mathbb{R}^k$-factor to non-Euclidean cone factor $C(Z_2)$. In other words, an ϵ-approximation map should map $\mathbb{R}^k$-factor to $\mathbb{R}^k$-factor. Together with the $p(G_j)$-action on $\mathbb{R}^k$-factor, we see that there should be a gap between $(\widetilde{Y}_1, \tilde{y}_1, G_1)$ and $(\widetilde{Y}_2, \tilde{y}_2, G_2)$.

Proof of Lemma 4.2.3. Suppose the contrary, then we have two sequences in $\Omega(M, \Gamma)$: $\{(\widetilde{Y}_{i1}, \tilde{y}_{i1}, G_{i1})\}$ and $\{(\widetilde{Y}_{i2}, \tilde{y}_{i2}, G_{i2})\}$ such that
(1) $p(G_{i1})$ is trivial,
(2) there is $g_i \in G_{i2}$ such that $p(g_i) \neq e$ and $d(g_i \cdot \tilde{y}_{i2}, \tilde{y}_{i2}) \leq 1$,
(3) $d_{GH}((\widetilde{Y}_{i1}, \tilde{y}_{i1}, G_{i1}), (\widetilde{Y}_{i2}, \tilde{y}_{i2}, G_{i2})) \to 0$ as $i \to \infty$.
Passing to some subsequences if necessary, the above two sequences converge to the same limit:

$$(\widetilde{Y}_{i1}, \tilde{y}_{i1}, G_{i1}) \xrightarrow{GH} (\widetilde{Y}_\infty, \tilde{y}_\infty, G_\infty),$$
$$(\widetilde{Y}_{i2}, \tilde{y}_{i2}, G_{i2}) \xrightarrow{GH} (\widetilde{Y}_\infty, \tilde{y}_\infty, G_\infty),$$

with $(\widetilde{Y}_\infty, \tilde{y}_\infty) = (\mathbb{R}^k \times C(Z_\infty), (0, z_\infty))$. By Lemma 4.2.1, $C(Z_\infty)$ does not split off any line, and thus

$$(\mathbb{R}^k \times \{z_{i1}\}, \tilde{y}_{i1}, p(G_{i1})) \xrightarrow{GH} (\mathbb{R}^k \times \{z_\infty\}, \tilde{y}_\infty, p(G_\infty)),$$
$$(\mathbb{R}^k \times \{z_{i2}\}, \tilde{y}_{i2}, p(G_{i2})) \xrightarrow{GH} (\mathbb{R}^k \times \{z_\infty\}, \tilde{y}_\infty, p(G_\infty)).$$

From the first sequence, we see that $p(G_\infty)$ is trivial because $p(G_{i1}) = \{e\}$. On the other hand, $p(G_{i2})$ contains some element β_i with $d_{\mathbb{R}^k}(\beta_i \cdot 0, 0) \leq 1$. β_i sub-converges to some element $\beta_\infty \in G_\infty$ with $d(\beta_\infty \cdot 0, 0) \leq 1$. If $\beta_\infty \neq e$, then $p(G_\infty)$ is non-trivial. If $\beta_\infty = e$, then we consider the subgroup $H_i = \langle \beta_i \rangle$. The sequence of subgroups H_i sub-converges to some non-trivial subgroup H_∞ of $p(G_\infty)$ because $D_1(H_i) \geq 1/20$, where $D_1(H_i)$ is the displacement of H_i on $B_1(0) \subseteq \mathbb{R}^k$. In either case, $p(G_\infty)$, a contradiction. □

Remark 4.2.4. The gap in Lemma 4.2.1 plays a key role in the above proof; it guarantees that symmetries on the non-Euclidean cone factor and on the Euclidean factor can not interchange. If there is no gap between the non-Euclidean cone factor $C(Z)$ and spaces splitting off lines, then Lemma 4.2.3 would fail. As an example, we construct a continuous family of metric cones (Y_t, y_t, G_t) $(-\delta \leq t \leq \delta)$ such that $Y_t = \mathbb{R}^2 \times C(Z_t)$, where $\mathrm{diam}(Z_t) \leq \pi$. As $t \to 0$,

$$d_{GH}((\mathbb{R}^2 \times \{z_{-t}\}, (0, z_{-t}), p(G_{-t})), (\mathbb{R}^2 \times z_t, (0, z_t), p(G_t))) \not\to 0.$$

For $|t| < \delta$ small, we put $Y_t = \mathbb{R}^2 \times C(S^1_t)$, where S^1_t is the round circle of diameter $\pi - |t|$. When $t = 0$, then $Y_t = \mathbb{R}^4$. Next we define G_t-action on Y_t. For $t > 0$, $G_t = S^1$ acting as rotations on the $C(S^1_t)$-factor; while for $t \leq 0$, $G_t = S^1$ acting as rotations about the origin on $\mathbb{R}^2$-factor. It is clear that (Y_t, y_t, G_t) is a continuous path in the equivariant Gromov-Hausdorff topology. However, $p(G_t)$ is trivial for $t > 0$ while $p(G_t) = S^1$ for $t < 0$; they can not be arbitrarily close as $t \to 0$.

We are ready to prove the following rudimentary version of Theorem B.

Proposition 4.2.5. *Let (M, x) be an open n-manifold of* Ric ≥ 0*, whose universal cover is k-Euclidean at infinity. If there is $(\widetilde{Y}, \tilde{y}, G) \in \Omega(\widetilde{M}, \Gamma)$ such that $p(G)$ is trivial, then for any space $(\widetilde{W}, \tilde{w}, H) \in \Omega(\widetilde{M}, \Gamma)$, $p(H)$ is also trivial.*

Proof. We argue by contradiction. Suppose that there are $r_i \to \infty$ and $s_i \to \infty$ such that

$$(r_i^{-1}\widetilde{M}, \tilde{x}, \Gamma) \xrightarrow{GH} (\widetilde{Y}_1, \tilde{y}_1, G_1),$$
$$(s_i^{-1}\widetilde{M}, \tilde{x}, \Gamma) \xrightarrow{GH} (\widetilde{Y}_2, \tilde{y}_2, G_2),$$

where $p(G_1)$ is trivial but $p(G_2)$ is not. Scaling down the sequence s_i^{-1} by a constant if necessary, we assume that there is $g \in G_2$ such that $p(g_2) \neq e$ and $d(g \cdot \tilde{y}_2, \tilde{y}_2) \leq 1$. We pass to a subsequence and assume that $t_i := s_i^{-1}/r_i^{-1} \to \infty$. This enables us to regard the above first sequence as a rescaling of the second one. Put

$$(N_i, q_i, \Gamma_i) = (s_i^{-1}\widetilde{M}, \tilde{x}, \Gamma).$$

In this way, we can rewrite these two convergent sequences as $(t_i \to \infty)$:

$$(N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}_1, \tilde{y}_1, G_1),$$

$$(t_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}_2, \tilde{y}_2, G_2).$$

We look for a contradiction in some intermediate rescaling sequence. For each i, we define a set of scales

$$\begin{aligned} L_i := \{ \ 1 \leq l \leq t_i \mid \ & d_{GH}((lN_i, q_i, \Gamma_i), (W, w, H)) \leq \epsilon/3 \\ & \text{for some space } (W, w, H) \in \Omega(\widetilde{M}, \Gamma) \\ & \text{such that } H \text{ has some element } h \\ & \text{with } p(h) \neq e \text{ and } d(h \cdot \tilde{w}, \tilde{w}) \leq 1 \}, \end{aligned}$$

where $\epsilon = \epsilon(M) > 0$ is the constant in Lemma 4.2.3. It is clear that $t_i \in L_i$ for all i large, thus L_i is non-empty. We choose $l_i \in L_i$ such that $\inf L_i \leq l_i \leq \inf L_i + 1/i$. We regard this l_i as the critical rescaling sequence.

Claim 1: $l_i \to \infty$. Suppose that l_i subconverges to $C < \infty$, then for this subsequence, we can pass to a subsequence again and obtain the convergence

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (C \cdot \widetilde{Y}_1, \tilde{y}_1, G_1).$$

Since $l_i \in L_i$, by definition of L_i and the above convergence, we conclude that

$$d_{GH}((C \cdot \widetilde{Y}_1, \tilde{y}_1, G_1), (W, w, H)) \leq \epsilon/2$$

for some space (W, w, H) such that there is $h \in H$ with $p(h) \neq e$ and $d(h \cdot \tilde{w}, \tilde{w}) \leq 1$. On the other hand, on $(C \cdot \widetilde{Y}_1, \tilde{y}_1, G_1)$, $p(G_1)$ is trivial. This is a contradiction to the choice of ϵ and Lemma 4.2.3. Hence Claim 1 is true.

Next we consider the convergence

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}', \tilde{y}', G') \in \Omega(\widetilde{M}, \Gamma).$$

We will derive a contradiction by ruling out all the possibilities of $p(G')$-action on the $\mathbb{R}^k$-factor of $\widetilde{Y}'$.

Claim 2: $p(G')$ is non-trivial. For each i, because $l_i \in L_i$, we know that

$$d_{GH}((l_i N_i, q_i, \Gamma_i), (W_i, w_i, K_i)) \leq \epsilon/3$$

for some $(W_i, w_i) \in \Omega(\widetilde{M})$ with K_i-action such that there is $k_i \in K_i$ with $p(k_i) \neq e$ and $d(k_i \cdot \tilde{w}_i, \tilde{w}_i) \leq 1$. Since $(l_i N_i, q_i, \Gamma_i)$ converges to $(\widetilde{Y}', \tilde{y}', G')$, the limit space satisfies

$$d_{GH}((\widetilde{Y}', \tilde{y}', G'), (W_i, w_i, K_i)) \leq \epsilon/2$$

for i large. By Lemma 4.2.3, $p(G')$ is nontrivial.

By Claim 2, there is some $g' \in G'$ such that $p(g') \neq e$. We put $d := d(g' \cdot \tilde{y}', \tilde{y}')$. If $d \leq 1$, we consider the scaling sequence $l_i/2$:

$$(l_i/2 \cdot N_i, q_i, \Gamma_i) \xrightarrow{GH} (1/2 \cdot \widetilde{Y}', \tilde{y}', G').$$

Note that on $(1/2 \cdot \widetilde{Y}', \tilde{y}', G')$, there is some element $g' \in G'$ with $p(g') \neq e$ and $d(g' \cdot \tilde{y}', \tilde{y}') \leq 1/2$. This shows that $l_i/2 \in L_i$ for i large, which is a contradiction our choice of l_i with $\inf L_i \leq l_i \leq \inf L_i + 1/i$. If $d > 1$, then we consider the scaling sequence $l_i/(2d)$:

$$(l_i/(2d) \cdot N_i, q_i, \Gamma_i) \xrightarrow{GH} (1/(2d) \cdot \widetilde{Y}, \tilde{y}, H').$$

and a similar contradiction would arise because $l_i/(2d) \in L_i$. In any case, we see a contradiction. This completes the proof. □

Remark 4.2.6. In the above proof when defining L_i, we include all the contradictory H-actions with $p(H) \neq \{e\}$ and a constraint on its displacement. In particular, this allows $p(H)$ to be different from $p(G_2)$. Doing so is necessary. For example, if $p(G_2) = \mathbb{Z}$ and we require $p(H) = \mathbb{Z}$ when defining L_i, then the intermediate sequence

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}', \tilde{y}', G')$$

may have a limit group with $p(G') = \mathbb{Z}_p$, where p is a large integer. This $\mathbb{Z}_p$ acts on $\mathbb{R}^k$ as rotations on a plane about a point far away from 0, so that $(\mathbb{R}^k, 0, \mathbb{Z}_p)$ and $(\mathbb{R}^k, 0, \mathbb{Z})$ are very close. In this situation, one can not derive a contradiction by dividing l_i by a constant.

4.3 Stability of isometric actions

For an isometric G-action on a Riemannian manifold M, we always assume that G is a closed subgroup of $\mathrm{Isom}(M)$. The goal of this section is Proposition 4.3.3 below, a stability result on isometric actions on any compact manifold M, which will be used in the proof of Theorem 4.1.4 with M being the unit sphere S^{k-1}.

For convenience, we introduce a definition first.

Definition 4.3.1. Let (Y_j, y_j) be a metric space with isometric Lie group G_j-action $(j = 1, 2)$. We say that (Y_1, y_1, G_1) is equivalent to (Y_2, y_2, G_2), if

$$d_{GH}((Y_1, y_1, G_1), (Y_2, y_2, G_2)) = 0;$$

or equivalently, there is an isometry $F : Y_1 \to Y_2$ with $F(y_1) = y_2$, and a Lie group isomorphism $\psi : G_1 \to G_2$ such that $F(g_1 \cdot x_1) = \psi(g_1) \cdot F(x_1)$ for any $g_1 \in G$ and $x_1 \in Y_1$.

Remark 4.3.2. For any isometry $F : Y_1 \to Y_2$, it induces an Lie group isomorphism $C_F : \mathrm{Isom}(Y_1) \to \mathrm{Isom}(Y_2)$ by conjugation, that is, $C_F(g) = F \circ g \circ F^{-1}$. It is direct to check C_F satisfies

$$C_F(g_1) \cdot F(x_1) = F(g_1 \cdot x_1)$$

for any $x_1 \in Y_1$ and any $g_1 \in \mathrm{Isom}(Y_1)$. This implies that the Lie group isomorphism ψ in Definition 4.3.1 must be the conjugation map C_F. Indeed, consider the composition $\mathrm{id} = F^{-1} \circ F$, then $C_{F^{-1}} \circ \psi$ satisfies

$$(C_{F^{-1}} \circ \psi)(g_1) \cdot x_1 = g_1 \cdot x_1$$

for all $x_1 \in Y_1$ and $g_1 \in G_1$. This shows that $\psi = (C_{F^{-1}})^{-1} = C_F$.

Proposition 4.3.3. *Let (M, G) be a compact Riemannian manifold with isometric G-action. Then there exists a constant $\epsilon > 0$, depending on (M, G), such that the following holds.*

For any isometric H-action on M, if

$$d_{GH}((M, G), (M, H)) \leq \epsilon,$$

then either (M, G) is equivalent to (M, H), or $\dim(H) < \dim(G)$.

One may compare Proposition 4.3.3 with the result below by Grove and Karcher [15].

Theorem 4.3.4. *Let M a compact Riemannian manifold. Then there exists $\epsilon(M) > 0$ such that for any two isometric G-actions*

$$\mu_1, \mu_2 : G \times M \to M$$

with $d_M(\mu_1(g,x), \mu_2(g,x)) \leq \epsilon(M)$ for all $g \in G$ and $x \in M$, these two actions are conjugate by an isometry.

We mention that the stability of group actions can be traced back to Palais [27]. He shows that any two C^1-close G-actions, as diffeomorphisms on M, can be conjugated by a diffeomorphism, where G is a compact Lie group. Grove and Karcher use the center of mass technique, and explicitly construct the conjugation map. They also interpret the C^1-closeness in terms of curvature bounds of M, when one of actions is by isometries. For our purpose, we restrict our attention to isometric actions only here.

Proposition 4.3.3 is different from Theorem 4.3.4 in the following aspects. Proposition 4.3.3 considers two isometric actions with possibly different groups. For instance, $G = S^1$ and we can take $H = \mathbb{Z}_p \subseteq G$ with large integer p. Even if one assume $G = H$, the closeness of these two actions in the equivariant Gromov-Hausdorff topology is weaker than the pointwise closeness condition in Theorem 4.3.4. For example, we know that there is a sequence of circle actions on the standard torus $T^2 = S^1 \times S^1$ converging to T^2-action:

$$(T^2, S^1_i) \xrightarrow{GH} (T^2, T^2).$$

Thus for any $\epsilon > 0$, we can find two different circle actions in the tail of this sequence such that

$$d_{GH}((T^2, S^1_j), (T^2, S^1_k)) \leq \epsilon,$$

where j, k are sufficiently large. However, these circle actions are not pointwise close. This example also illustrates that the ϵ in Proposition 4.3.3 has to depend on the G-action.

To prove Proposition 4.3.3, we recall some facts on equivariant Gromov-Hausdorff convergence [14]. Given $(M, H_i) \xrightarrow{GH} (M, G)$, one can always assume that the identity map on M gives equivariant ϵ_i-approximations for some $\epsilon_i \to 0$. We endow $\mathrm{Isom}(M)$

with a natural bi-invariant metric d from its action on M:

$$d_G(g_1, g_2) = \max_{x \in M} d_M(g_1 \cdot x, g_2 \cdot x).$$

Then H_i converges to the limit G with respect to the Hausdorff distance induced by $(\mathrm{Isom}(M), d)$.

In our proof of Proposition 4.3.3, we use the following results.

Proposition 4.3.5. *[15] Let $\mu_1, \mu_2 : H \to G$ be two homomorphisms of compact Lie group H into the Lie group G with a bi-invariant Riemannian metric. There exists $\epsilon(G) > 0$ such that if $d(\mu_1(h), \mu_2(h)) \leq \epsilon(G)$ for all $h \in H$, then the subgroups $\mu_1(H)$ and $\mu_2(H)$ are conjugate in G.*

Proposition 4.3.6. *[26] Let G be a Lie group with left-invariant Riemannian metric. Then there exists a constant $\epsilon(G) > 0$ such that if $\phi : H \to G$ is a map from a Lie group H to G such that*

$$d(\phi(h_1 h_2), \phi(h_1)\phi(h_2)) \leq \epsilon < \epsilon(G)$$

for all $h_1, h_2 \in H$, then there is a Lie group homomorphism $\bar{\phi} : H \to G$ with

$$d(\bar{\phi}(h), \phi(h)) \leq 2\epsilon$$

for all $h \in H$.

We call such a map $\phi : H \to G$ with

$$d(\phi(h_1 h_2), \phi(h_1)\phi(h_2)) \leq \epsilon$$

an ϵ-*homomorphism*. In practice, we may start with some bi-invariant distance function d on G. We can equip G with a bi-invariant Riemannian metric d_0 (we can do this because G is compact). Then there is some constant $C \geq 1$ such that $C^{-1} d_0 \leq d_G \leq C d_0$. With this observation, for a sequence of ϵ_i-homomorphisms with respect to d, it is a sequence of $C\epsilon_i$-homomorphisms with respect to d_0. Therefore, we can still apply Proposition 4.3.6 for i large.

Proof of Proposition 4.3.3. Let $\{H_i\}$ be a sequence of group actions on M such that $(M, H_i) \xrightarrow{GH} (M, G)$. We show that if $\dim(H_i) \geq \dim(G)$, then (M, H_i) is equivalent to (M, G) for all i large.

As pointed out, for $(M, H_i) \xrightarrow{GH} (M, G)$, it is equivalent to consider the Hausdorff convergence $H_i \xrightarrow{H} G$ in $(\mathrm{Isom}(M), d)$, where d is given by

$$d(g_1, g_2) = \max_{x \in M} d_M(g_1 \cdot x, g_2 \cdot x).$$

We know that there is $\epsilon_i \to 0$ such that $d_H(H_i, G) \leq \epsilon_i$. For each $h \in H_i$, we choose $\phi_i(h)$ as an element in G that is ϵ_i-close to h. This defines a map

$$\phi_i : H_i \to G.$$

It is straight-forward to check that ϕ_i is a $3\epsilon_i$-homomorphism with respect the metric $d|_G$:

$$\begin{aligned} & d(\phi_i(h_1 h_2), \phi_i(h_1)\phi_i(h_2)) \\ \leq\ & d(\phi_i(h_1 h_2), h_1 h_2) + d(h_1 h_2, h_1 \phi_i(h_2)) + d(h_1 \phi_i(h_2), \phi_i(h_1)\phi_i(h_2)) \\ \leq\ & 3\epsilon_i \end{aligned}$$

for any $h_1, h_2 \in H_i$. Apply Proposition 4.3.6, we obtain a sequence of Lie group homomorphisms:

$$\bar{\phi}_i : H_i \to G.$$

Claim: $\bar{\phi}_i$ is a Lie group isomorphism for all i large. We first show that $\bar{\phi}_i$ is injective. Suppose that $\ker(\bar{\phi}_i) \neq \{e\}$, then we have a sequence of non-trivial subgroups converging to $\{e\}$:

$$\ker(\bar{\phi}_i) \xrightarrow{H} \{e\}.$$

However, there exists $\delta > 0$ such that any non-trivial subgroup of $\mathrm{Isom}(M)$ has displacement at least δ on M. This is because $\mathrm{Isom}(M)$ is a Lie group, which can not have arbitrarily small non-trivial subgroups. Thus $\ker(\bar{\phi}_i) = \{e\}$ for all i large. Recall the assumption that $\dim(H_i) \geq \dim(G)$. Since $\bar{\phi}_i$ is injective, we must have $\dim(H_i) = \dim(G)$. Also note that the image $\bar{\phi}_i(H_i)$ is $C\epsilon_i$-dense in G, thus $\bar{\phi}_i$ must be surjective for i large.

Now H_i has two embeddings into $\mathrm{Isom}(M)$:

$$\iota_i : H_i \to \mathrm{Isom}(M), \quad \bar{\phi}_i : H_i \to G \subseteq \mathrm{Isom}(M),$$

where ι_i is the inclusion map. Note that

$$d(h, \bar{\phi}_i(h)) \leq C\epsilon_i \to 0$$

for all $h \in H_i$ and some constant C. By Proposition 4.3.5, we conclude that for i large $G = \bar{\phi}_i(H_i)$ is conjugate to H_i as subgroups in $\mathrm{Isom}(M)$. In other words, there is some isometry $g_i \in \mathrm{Isom}(M)$ such that $g_i^{-1} G g_i = H_i$. This shows that (M, H_i) and (M, G) are equivalent for i large. $\square$

4.4 Proof of equivariant stability at infinity

Without mentioning, we always assume that groups in this section are abelian. We prove Theorem 4.1.4 in two steps. First we show that for all

$$(\widetilde{Y}, \tilde{y}, G) = (\mathbb{R}^k \times C(Z), (0, z), G) \in \Omega(\widetilde{M}, \Gamma),$$

the isotropy subgroup of $p(G)$ at 0 is independent of $(\widetilde{Y}, \tilde{y}, G)$, and $(\mathbb{R}^k, 0, p(G))$ satisfies property *(P)* (see Definition 4.4.2 below). Secondly, we prove the non-compact factor in $p(G)$ is also independent of $(\widetilde{Y}, \tilde{y}, G)$. The proof of each step shares the same structure as Proposition 4.2.5: we show that there exists a gap between two certain classes of group actions, then choose a critical rescaling to derive a desired contradiction in the corresponding limit space.

Recall that once we specify a point in $\mathbb{R}^k$ as the origin 0, then every element in $\mathrm{Isom}(\mathbb{R}^k) = \mathbb{R}^k \rtimes O(k)$ can be written as (A, v), where $A \in O(k)$ fixing 0 and $v \in \mathbb{R}^k$. For convenience, we introduce a definition.

Definition 4.4.1. Let $(\mathbb{R}^k, 0, G)$ be the k dimensional Euclidean space with an isometric abelian G-action. We say that $(\mathbb{R}^k, 0, G)$ satisfies *property (P)*, if
(P) for any element $(A, v) \in G$, $(A, 0)$ is also an element of G.

Property *(P)* has the following consequence.

Lemma 4.4.2. *If* $(\mathbb{R}^k, 0, G)$ *satisfies property (P), then*

(1) any compact subgroup of G *fixes* 0*;*

(2) G *admits decomposition* $G = \mathrm{Iso}_0 G \times \mathbb{R}^l \times \mathbb{Z}^m$*, and any element in the subgroup* $\{e\} \times \mathbb{R}^l \times \mathbb{Z}^m$ *is a translation, where* $\mathrm{Iso}_0 G$ *is the isotropy subgroup of* G *at* 0*.*

Proof. (1) Let K be any compact subgroup of G. Suppose that K does not fix 0. Then there is $g = (A, v) \in K$ such that

$$0 \neq g \cdot 0 = (A, v) \cdot 0 = v.$$

By assumption, $(A, 0) \in G$. Hence $(A, v) \cdot (A^{-1}, 0) = (I, v)$ is also an element of G. Because G is abelian, $(A, 0)$ and (I, v) commutes. This implies that $A \cdot v = v$, and thus $(A, v)^k = (A^k, kv)$ for any integer k. We see that the subgroup generated by (A, v) can not be contained in any compact group, a contradiction.

(2) This follows from (1) and the structure of abelian Lie groups. □

It is clear that (2) in Lemma 4.4.2 is equivalent to property *(P)*.

Lemma 4.4.3. *Let* G *be an abelian subgroup of* $\mathrm{Isom}(\mathbb{R}^k)$*, and* x *be a point in* $\mathbb{R}^n$*. For a sequence* $r_i \to \infty$*, consider the following equivariant tangent cones of* $(\mathbb{R}^n, G)$ *at* x *and infinity:*

$$(r_i \mathbb{R}^k, x, G) \xrightarrow{GH} (\mathbb{R}^k, 0, G_x),$$

$$(r_i^{-1} \mathbb{R}^k, x, G) \xrightarrow{GH} (\mathbb{R}^k, 0, G_\infty).$$

Then both $(\mathbb{R}^k, 0, G_x)$ *and* $(\mathbb{R}^k, 0, G_\infty)$ *satisfy property (P).*

Proof. Let K be the subgroup of G fixing x. It is clear that

$$(r_i \mathbb{R}^k, x, G) \xrightarrow{GH} (\mathbb{R}^k, 0, K \times \mathbb{R}^l),$$

where l is the dimension of the orbit $G \cdot x$ and $\{e\} \times \mathbb{R}^l$ acts as translations.

Next we check that $(\mathbb{R}^k, 0, G_\infty)$ satisfies property P. Let (A, v) be an element in G_∞ with $v \neq 0$. Due to the convergence, this means there are a sequence $(A_i, t_i v_i) \in G$ with $t_i / r_i \to 1$, $A_i \to A$ and $v_i \to v$. For each fixed integer k,

$$(r_i^{-1} \mathbb{R}^k, x, (A_k, t_k v_k)) \xrightarrow{GH} (\mathbb{R}^k, 0, (A_k, 0)).$$

Thus $(A_k, 0) \in G_\infty$. Since $A_k \to A$ as $k \to \infty$ and G_∞ is closed, we conclude that $(A, 0) \in G_\infty$. Therefore, $(\mathbb{R}^k, 0, G_\infty)$ satisfies property *(P)*. $\square$

Remark 4.4.4. Let M be an open n-manifold of $\mathrm{Ric} \geq 0$. Suppose that

$$(\widetilde{Y}, \tilde{y}, G) = (\mathbb{R}^k \times C(Z), (0, z), G) \in \Omega(\widetilde{M}, \Gamma)$$

is a metric cone with isometric G-action, where $C(Z)$ has vertex z and $\mathrm{diam}(Z) < \pi$. We do not know any example so that $(\mathbb{R}^k, 0, p(G))$ does not satisfy property *(P)*. However, by Lemma 4.4.3, we can always find ones with property *(P)* in $\Omega(\widetilde{M}, \Gamma)$ by passing to the equivariant tangent cone of $(\widetilde{Y}, \tilde{y}, G)$ at $\tilde{y}$, or at infinity $(j \to \infty)$:

$$(j\widetilde{Y}, \tilde{y}, G) \xrightarrow{GH} (\widetilde{Y}, \tilde{y}, G_{\tilde{y}}),$$

$$(j^{-1}\widetilde{Y}, \tilde{y}, G) \xrightarrow{GH} (\widetilde{Y}, \tilde{y}, G_\infty).$$

Remark 4.4.5. If $(\mathbb{R}^k, 0, G)$ does not satisfy property *(P)*, then there is an element $(A, v) \in G$, but $(A, 0) \notin G$. After blowing down

$$(j^{-1}\mathbb{R}^k, 0, G, (A, v)) \xrightarrow{GH} (\mathbb{R}^k, 0, G_\infty, (A, 0)).$$

Thus $(A, 0) \in G_\infty$. Note that $\mathrm{Iso}_0 p(G)$ is preserved as a subgroup of $\mathrm{Iso}_0 p(G_\infty)$. Hence $\mathrm{Iso}_0 p(G)$ is a proper subgroup of $\mathrm{Iso}_0 p(G_\infty)$.

We restate Theorem 4.1.4 in terms of Definition 4.4.1.

Theorem 4.4.6. *Let M be an open n-manifold with abelian fundamental group and* $\mathrm{Ric} \geq 0$*, whose universal cover $\widetilde{M}$ is k-Euclidean at infinity. Then there exist a closed abelian subgroup K of $O(k)$ and an integer $l \in [0, k]$ such that for any space $(\widetilde{Y}, \tilde{y}, G) \in \Omega(\widetilde{M}, \Gamma)$, $(\mathbb{R}^k, 0, p(G))$ satisfies property (P) and $p(G) = K \times \mathbb{R}^l$.*

We first establish a gap phenomenon between two classes of actions with property *(P)* but different projected isotropy groups.

Lemma 4.4.7. *Let M be an open n-manifold of* $\mathrm{Ric} \geq 0$*, whose universal cover is k-Euclidean at infinity. Let K be an isometric action on $\mathbb{R}^k$ fixing 0. Then there exists $\epsilon > 0$, depending on M and K-action, such that the following holds.*

For any two spaces $(\widetilde{Y}_j, \tilde{y}_j, G_j) \in \Omega(\widetilde{M}, \Gamma)$ $(j = 1, 2)$ *satisfying*

(1) $(\mathbb{R}^k, 0, p(G_j))$ *satisfies property (P)* $(j = 1, 2)$*,*

(2) $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_1))$ *is equivalent to* $(\mathbb{R}^k, 0, K)$*,*

(3) $\dim(\mathrm{Iso}_0 p(G_2)) \geq \dim(K)$ *and* $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_2))$ *is not equivalent to* $(\mathbb{R}^k, 0, K)$*,*

then

$$d_{GH}((\widetilde{Y}_1, \tilde{y}_1, G_1), (\widetilde{Y}_2, \tilde{y}_2, G_2)) \geq \epsilon.$$

Proof. Suppose that there are two sequences in $\Omega(M)$: $\{(\widetilde{Y}_{ij}, \tilde{y}_{ij}, G_{ij})\}_i$ $(j = 1, 2)$ such that for all i,

(1) $(\mathbb{R}^k, 0, p(G_{ij}))$ satisfies property *(P)* $(j = 1, 2)$;

(2) $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_{i1}))$ is equivalent to $(\mathbb{R}^k, 0, K)$;

(3) $\dim(K_i) \geq \dim(K)$ and $(\mathbb{R}^k, 0, K_i)$ is not equivalent to $(\mathbb{R}^k, 0, K)$, where $K_i = \mathrm{Iso}_0 p(G_{i2})$;

(4) $d_{GH}((\widetilde{Y}_{i1}, \tilde{y}_{i1}, G_{i1}), (\widetilde{Y}_{i2}, \tilde{y}_{i2}, G_{i2})) \to 0$ as $i \to \infty$.

After passing to some subsequences, this gives convergence

$$(\widetilde{Y}_{i1}, \tilde{y}_{i1}, G_{i1}) \xrightarrow{GH} (\widetilde{Y}_\infty, \tilde{y}_\infty, G_\infty),$$

$$(\widetilde{Y}_{i2}, \tilde{y}_{i2}, G_{i2}) \xrightarrow{GH} (\widetilde{Y}_\infty, \tilde{y}_\infty, G_\infty),$$

with $(\widetilde{Y}_\infty, \tilde{y}_\infty) = (\mathbb{R}^k \times C(Z_\infty), (0, z_\infty))$, where $\mathrm{diam}(Z_\infty) < \pi$ and z_∞ is the vertex of $C(Z_\infty)$ (Lemma 4.2.1). Consequently,

$$(\mathbb{R}^k, 0, p(G_{i1})) \xrightarrow{GH} (\mathbb{R}^k, 0, p(G_\infty)),$$

$$(\mathbb{R}^k, 0, p(G_{i2})) \xrightarrow{GH} (\mathbb{R}^k, 0, p(G_\infty)).$$

Because each $(\mathbb{R}^k, 0, p(G_{ij}))$ satisfies property *(P)* for all i and j, we conclude that

$$(S^{k-1}, K) \xrightarrow{GH} (S^{k-1}, K_\infty),$$

$$(S^{k-1}, K_i) \xrightarrow{GH} (S^{k-1}, K_\infty),$$

where S^{k-1} is the unit sphere in $\mathbb{R}^k$ and $K_\infty = \mathrm{Iso}_0 p(G_\infty)$. Note that $(S^{k-1}, 0, K_\infty)$ is equivalent to $(S^{k-1}, 0, K)$. By Proposition 4.3.3, for all i sufficiently large, either $\dim(K_i) < \dim(K)$ or $(S^{k-1}, 0, K_i)$ is equivalent to $(S^{k-1}, 0, K)$. This contradicts the hypothesis (3) on K_i. $\square$

Lemma 4.4.8. *Let M be an open n-manifold of* $\mathrm{Ric} \geq 0$ *and abelian fundamental group. Suppose that $\widetilde{M}$ is k-Euclidean at infinity. Then for any space $(\widetilde{Y}, \tilde{y}, G) \in \Omega(\widetilde{M}, \Gamma)$, $p(G)$-action on $(\mathbb{R}^k, 0, G)$ satisfies property (P). Moreover,* $\mathrm{Iso}_0 p(G)$ *is independent of $(\widetilde{Y}, \tilde{y}, G)$.*

The key to prove Lemma 4.4.8 is the following lemma.

Lemma 4.4.9. *Let M be an open n-manifold of* $\mathrm{Ric} \geq 0$ *and abelian fundamental group. Suppose that $\widetilde{M}$ is k-Euclidean at infinity. Then for any two spaces $(\widetilde{Y}_j, \tilde{y}_j, G_j) \in \Omega(\widetilde{M}, \Gamma)$ with $(\mathbb{R}^k, 0, p(G_j))$ satisfying property (P)* $(j = 1, 2)$*, $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_1))$ must be equivalent to $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_2))$.*

We prove Lemma 4.4.9 by induction, in terms of the following order on the set of all compact abelian Lie groups.

Definition 4.4.10. For a compact Lie group K, we define $D(K) = (\dim K, \# K/K_0)$. For two compact Lie groups K and H, with $D(K) = (l_1, l_2)$ and $D(H) = (m_1, m_2)$, we say that $D(K) < D(H)$, if $l_1 < m_1$, or if $l_1 = m_1$ and $l_2 < m_2$. We say that $D(K) \leq D(H)$, if $D(K) = D(H)$ or $D(K) < D(H)$.

Proof of Lemma 4.4.9. We argue by contradiction. Suppose that there are two spaces $(\widetilde{Y}_j, \tilde{y}_j, G_j) \in \Omega(\widetilde{M}, \Gamma)$ such that $(\mathbb{R}^k, 0, p(G_j))$ satisfies property *(P)* $(j = 1, 2)$, and $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_1))$ is not equivalent to $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_2))$. We derive a contradiction by the critical rescaling argument and Lemma 4.4.7.

We argue this by induction on $\min\{D(K_1), D(K_2)\}$, where $K_j = \mathrm{Iso}_0 p(G_j)$ $(j = 1, 2)$. Without lose of generality, we assume that $D(K_1) \leq D(K_2)$. Assuming that the above can not happen when $D(K_1) < (m_1, m_2)$, we will derive a contradiction for $D(K_1) = (m_1, m_2)$.

Let $r_i \to \infty$ and $s_i \to \infty$ be two sequences such that

$$(r_i^{-1}\widetilde{M}, \tilde{x}, \Gamma) \xrightarrow{GH} (\widetilde{Y}_1, \tilde{y}_1, G_1),$$

$$(s_i^{-1}\widetilde{M}, \tilde{x}, \Gamma) \xrightarrow{GH} (\widetilde{Y}_2, \tilde{y}_2, G_2),$$

and $t_i := (s_i^{-1})/(r_i^{-1}) \to \infty$. Put $(N_i, q_i, \Gamma_i) = (r_i^{-1}\widetilde{M}, \tilde{x}, \Gamma)$, then we have

$$(N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}_1, \tilde{y}_1, G_1),$$

$$(t_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}_2, \tilde{y}_2, G_2).$$

We know that $(\mathbb{R}^k, 0, p(G_j))$ satisfies property *(P)* $(j = 1, 2)$, $D(K_1) = (m_1, m_2) \leq D(K_2)$, and $(\mathbb{R}^k, 0, K_1)$ is not equivalent to $(\mathbb{R}^k, 0, K_2)$.

For each i, we define a set of scales

$$\begin{aligned} L_i := \{\ 1 \leq l \leq t_i \mid\ & d_{GH}((lN_i, q_i, \Gamma_i), (W, w, H)) \leq \epsilon/3 \\ & \text{for some space } (W, w, H) \in \Omega(\widetilde{M}, \Gamma) \\ & \text{such that } (\mathbb{R}^k, 0, p(H)) \text{ satisfies property } \textit{(P)}; \\ & \text{moreover, } D(\mathrm{Iso}_0 p(H)) > (m_1, m_2), \text{ or} \\ & D(\mathrm{Iso}_0 p(H)) = (m_1, m_2) \text{ but } (\mathbb{R}^k, 0, \mathrm{Iso}_0 p(H)) \\ & \text{is not equivalent to } (\mathbb{R}^k, 0, K_1)\}. \end{aligned}$$

We choose the above $\epsilon > 0$ as follows: by Lemma 4.4.7, there is $\epsilon > 0$, depending on M and $(\mathbb{R}^k, 0, K_1)$ such that for any $(W_j, w_j, H_i) \in \Omega(\widetilde{M}, \Gamma)$ $(j = 1, 2)$ satisfying
(1) $(\mathbb{R}^k, 0, p(H_j))$ satisfies property *(P)* $(j = 1, 2)$,
(2) $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(H_1))$ is equivalent to $(\mathbb{R}^k, 0, K_1)$,
(3) $d_{GH}((W_1, w_1, H_1), (W_2, w_2, H_2)) \leq \epsilon$,
then $\dim(\mathrm{Iso}_0 p(H_2)) < \dim(K_1)$, or $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(H_2))$ is equivalent to $(\mathbb{R}^k, 0, K_1)$.

Since $t_i \in L_i$ for i large, we choose $l_i \in L_i$ with $\inf L_i \leq l_i \leq \inf L_i + 1/i$.

Claim 1: $l_i \to \infty$. Suppose that $l_i \to C < \infty$ for some subsequence, then for this subsequence,

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (C \cdot \widetilde{Y}_1, \tilde{y}_1, G_1).$$

Together with the fact that $l_i \in L_i$, we know that there is some space $(W, w, H) \in \Omega(\widetilde{M}, \Gamma)$ with the properties below:
(1) $(\mathbb{R}^k, 0, p(H))$ satisfies property *(P)*,
(2) $D(\mathrm{Iso}_0 p(H)) > (m_1, m_2)$, or $D(\mathrm{Iso}_0 p(H)) = (m_1, m_2)$ but $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(H))$ is not equivalent to $(\mathbb{R}^k, 0, K_1)$,

(3) $d_{GH}((C\cdot \widetilde{Y}_1,\tilde{y}_1,G_1),(W,w,H))\le \epsilon/2$.

Since $(\mathbb{R}^k,0,p(H))$ satisfies property *(P)*, we see that $(\mathbb{R}^k,0,\mathrm{Iso}_0 p(H))$ is equivalent to $(C\cdot\mathbb{R}^k,0,\mathrm{Iso}_0 p(H))$. By Lemma 4.4.7 the choice of ϵ, we conclude that either $\dim(\mathrm{Iso}_0 p(H))<\dim K_1$, or $(\mathbb{R}^k,0,\mathrm{Iso}_0 p(H))$ is equivalent to $(\mathbb{R}^k,0,K_1)$, which is a contradiction to the condition (2) above. We have verified Claim 1.

Passing to a subsequence if necessary, we have convergence

$$(l_iN_i,q_i,\Gamma_i)\overset{GH}{\longrightarrow}(\widetilde{Y}',\tilde{y}',G').$$

To draw a contradiction, the goal is ruling out all the possibilities of $p(G')$-action.

Claim 2: $D(K')\ge(m_1,m_2)$, where $K'=\mathrm{Iso}_0 p(G')$. If $D(K')<(m_1,m_2)$, we pass to the equivariant tangent cone of $(\widetilde{Y}',\tilde{y}',G')$ at $\tilde{y}'$. In this way, we have $(\widetilde{Y}',\tilde{y}',G'_{\tilde{y}'})$ with $(\mathbb{R}^k,0,p(G'_{\tilde{y}'}))$ satisfying property *(P)* (see Remark 4.4.4). Note that $(\mathbb{R}^k,0,\mathrm{Iso}_0 p(G'_{\tilde{y}'}))$ is equivalent to $(\mathbb{R}^k,0,K')$ and $D(K')<(m_1,m_2)$. We know that this can not happen due to the induction assumption.

Claim 3: $(\mathbb{R}^k,0,p(G'))$ satisfies property *(P)*, and $D(K')=(m_1,m_2)$. In fact, we pass to the equivariant tangent cone of $(\widetilde{Y}',\tilde{y}',G')$ at infinity:

$$(j^{-1}\widetilde{Y}',\tilde{y}',G')\overset{GH}{\longrightarrow}(\widetilde{Y}',\tilde{y}',G'_\infty).$$

For this space, $(\mathbb{R}^k,0,p(G'_\infty))$ satisfies property *(P)* (see Remark 4.4.4). Suppose that Claim 3 fails, then $D(\mathrm{Iso}_0 p(G'_\infty))>(m_1,m_2)$ (see Remark 4.4.5). We choose a large integer J such that

$$d_{GH}((J^{-1}\widetilde{Y}',\tilde{y}',G'),(\widetilde{Y}',\tilde{y}',G'_\infty))\le\epsilon/4.$$

Hence for all i large, we have

$$d_{GH}((J^{-1}l_iN_i,q_i,\Gamma_i),(\widetilde{Y}',\tilde{y}',G'_\infty))\le\epsilon/3.$$

This implies that $l_i/J\in L_i$ for all i large, which is a contradiction to our choice of l_i.

Claim 4: $(\mathbb{R}^k,0,K')$ is equivalent to $(\mathbb{R}^k,0,K_1)$. Suppose not, then we consider the sequence $l_i/2$:

$$(l_i/2\cdot N_i,q_i,\Gamma_i)\overset{GH}{\longrightarrow}(1/2\cdot\widetilde{Y}',\tilde{y}',G').$$

Since $(\mathbb{R}^k, 0, p(G'))$ satisfies property *(P)*, $(1/2 \cdot \mathbb{R}^k, 0, K')$ is equivalent to $(\mathbb{R}^k, 0, K')$, which is not equivalent to $(\mathbb{R}^k, 0, K_1)$. This means that $l_i/2 \in L_i$ for i large. A contradiction.

This leads to the ultimate contradiction: Because $l_i \in L_i$, there is some space $(W, w, H) \in \Omega(\widetilde{M}, \Gamma)$ satisfying the conditions (1)(2) in the proof of Claim 1, and

$$d_{GH}((\widetilde{Y}', \tilde{y}', G'), (W, w, H)) \leq \epsilon/2.$$

On the other hand, by Claims 3, 4, Lemma 4.4.7 and the choice of ϵ, (W, w, H) can not fulfill condition (2) (cf. proof of Claim 1).

For the remaining base case $D(K_1) = (0,0)$, note that in the above proof, the induction assumption is only used in Claim 2 to conclude $D(K') \geq (m_1, m_2)$. For the base case $(m_1, m_2) = (0,0)$, it is trivial that $D(K') \geq (0,0)$. □

Proof of Lemma 4.4.8. With Lemma 4.4.9, it is enough to show that for any space $(\widetilde{Y}, \tilde{y}, G) \in \Omega(\widetilde{M}, \Gamma)$, $(\mathbb{R}^k, 0, p(G))$ always satisfies property *(P)*. Suppose the contrary, that is, $(\mathbb{R}^k, 0, p(G))$ does not satisfy property *(P)* for some $(\widetilde{Y}, \tilde{y}, G)$ in $\Omega(\widetilde{M}, \Gamma)$. We pass to the equivariant tangent cone of $(\widetilde{Y}, \tilde{y}, G)$ at $\tilde{y}$ and at infinity respectively (see Remark 4.4.4). We obtain $(\widetilde{Y}, \tilde{y}, G_{\tilde{y}})$ and $(\widetilde{Y}, \tilde{y}, G_\infty)$. For these two spaces, $(\mathbb{R}^k, 0, p(G_{\tilde{y}}))$ and $(\mathbb{R}^k, 0, p(G_\infty))$ always satisfy property *(P)*. By Lemma 4.4.9, $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_{\tilde{y}}))$ is equivalent to $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_\infty))$.

On the other hand, because $(\mathbb{R}^k, 0, p(G))$ does not satisfy property *(P)*, $\mathrm{Iso}_0 p(G)$ is a proper subgroup of $\mathrm{Iso}_0 p(G_\infty)$ (Remark 4.4.5). Since $\mathrm{Iso}_0 p(G) = \mathrm{Iso}_0 p(G_{\tilde{y}})$, we conclude that $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_{\tilde{y}}))$ and $(\mathbb{R}^k, 0, \mathrm{Iso}_0 p(G_\infty))$ can not be equivalent, a contradiction to Lemma 4.4.9. □

Lemmas 4.4.2 and 4.4.8 imply that there exists a closed subgroup K of $O(k)$ such that for any $(\widetilde{Y}, \tilde{y}, G) \in \Omega(\widetilde{M}, \Gamma)$, $(\mathbb{R}^k, 0, p(G))$ satisfies property *(P)*, and $p(G) = K \times \mathbb{R}^l \times \mathbb{Z}^m$. To finish the proof of Theorem 4.4.6, we need to show that l is independent of $(\widetilde{Y}, \tilde{y}, G)$ and m is always 0.

We prove the following gap lemma on the non-compact factor of $p(G)$, which does not require Lemma 4.4.7.

Lemma 4.4.11. *Let M be an open n-manifold of* $\mathrm{Ric} \geq 0$*. Suppose that $\widetilde{M}$ is k-Euclidean at infinity. Then there exists $\epsilon(M) > 0$ such that the following holds.*

For any two spaces $(\widetilde{Y}_j, \tilde{y}_i, G_j) \in \Omega(\widetilde{M}, \Gamma)$ $(j = 1, 2)$ satisfying

(1) $(\mathbb{R}^k, 0, p(G_j))$ satisfies property (P) $(j = 1, 2)$,

(2) $p(G_1) = \mathrm{Iso}_0 p(G_1) \times \mathbb{R}^l$ (cf. Lemma 4.4.2),

(3) $p(G_2)$ contains $\mathbb{R}^l \times \mathbb{Z}$ as a closed subgroup; for this extra $\mathbb{Z}$ subgroup, it has generator γ with $d_{\mathbb{R}^k}(\gamma \cdot 0, 0) \leq 1$.

Then

$$d_{GH}((\widetilde{Y}_1, \tilde{y}_1, G_1), (\widetilde{Y}_2, \tilde{y}_2, G_2)) \geq \epsilon(M).$$

Proof. We argue by contradiction. Suppose that there are two sequences of spaces in $\Omega(\widetilde{M}, \Gamma)$: $\{(\widetilde{Y}_{ij}, \tilde{y}_{ij}, G_{ij})\}_i$ $(j = 1, 2)$ such that

(1) $(\mathbb{R}^k, 0, p(G_{ij}))$ satisfies property *(P)* $(j = 1, 2)$;

(2) $p(G_{i1}) = K_{i1} \times \mathbb{R}^l$, where $K_{i1} = \mathrm{Iso}_0 p(G_{i1})$;

(3) $p(G_{i2})$ contains $\mathbb{R}^l \times \mathbb{Z}$ as a closed subgroup; for this extra $\mathbb{Z}$ subgroup, it has generator γ_i with $d_{\mathbb{R}^k}(\gamma_i \cdot 0, 0) \leq 1$;

(4) $d_{GH}((\widetilde{Y}_{i1}, \tilde{y}_{i1}, G_{i1}), (\widetilde{Y}_{i2}, \tilde{y}_{i2}, G_{i2})) \to 0$ as $i \to \infty$.

This gives the convergence

$$(\widetilde{Y}_{i1}, \tilde{y}_{i1}, G_{i1}) \xrightarrow{GH} (\widetilde{Y}_\infty, \tilde{y}_\infty, G_\infty),$$

$$(\widetilde{Y}_{i2}, \tilde{y}_{i2}, G_{i2}) \xrightarrow{GH} (\widetilde{Y}_\infty, \tilde{y}_\infty, G_\infty);$$

thus

$$(\mathbb{R}^k, 0, p(G_{i1})) \xrightarrow{GH} (\mathbb{R}^k, 0, p(G_\infty)),$$
$$(\mathbb{R}^k, 0, p(G_{i2})) \xrightarrow{GH} (\mathbb{R}^k, 0, p(G_\infty)).$$

Since $p(G_{i1}) = K_{i1} \times \mathbb{R}^l$ and $(\mathbb{R}^k, 0, p(G_{i1}))$ satisfies property *(P)*, we conclude that $(\mathbb{R}^k, 0, p(G_\infty))$ also satisfies property *(P)*, and $p(G_\infty) = K_\infty \times \mathbb{R}^l$ with K_∞ fixing 0. On the other hand, by hypothesis (3), $p(G_{i2})$ contains a proper closed subgroup $H_i = K_{i2} \times \mathbb{R}^l$ with $K_{i2} = \mathrm{Iso}_0 p(G_{i2})$. Moreover, there is some element $\alpha_i \in p(G_{i2})$ outside H_i such that $d(H_i \cdot 0, \alpha_i \cdot 0) \in (1, 3)$. This yields

$$(\mathbb{R}^k, 0, H_i, \alpha_i) \xrightarrow{GH} (\mathbb{R}^k, 0, H_\infty, \alpha_\infty),$$

where α_∞ is outside H_∞ with $d(H_\infty \cdot 0, \alpha_\infty \cdot 0) \in (1,3)$. Therefore, $p(G_\infty)$ also contains $\mathbb{R}^l \times \mathbb{Z}$ as a closed subgroup, a contradiction. $\square$

Proof of Theorem 4.4.6. By Lemmas 4.4.2 and 4.4.8, for any space $(\widetilde{Y}, \tilde{y}, G) \in \Omega(\widetilde{M}, \Gamma)$, $(\mathbb{R}^k, 0, p(G))$ always satisfies property *(P)*, and $p(G) = K \times \mathbb{R}^l \times \mathbb{Z}^m$, where K is a closed subgroup of $O(k)$ independent of $(\widetilde{Y}, \tilde{y}, G)$. It remains to show that l is independent of $(\widetilde{Y}, \tilde{y}, G)$ and m is always 0.

We argue by contradiction with the critical rescaling argument and Lemma 4.4.11. By passing to the tangent cones at the base point of spaces in $\Omega(\widetilde{M}, \Gamma)$, we can choose contradictory two spaces $(\widetilde{Y}_j, \tilde{y}_j, G_j) \in \Omega(\widetilde{M}, \Gamma)$ $(j = 1, 2)$ with $p(G_1) = K \times \mathbb{R}^l$ and $p(G_2)$ containing $\mathbb{R}^l \times \mathbb{Z}$ as a closed subgroup. We rule out this by induction on l. Assuming that this can not happen for $p(G_1)$ has non-compact factor with dimension $0, .., l-1$, we prove the case $p(G_1) = K \times \mathbb{R}^l$.

Let $r_i \to \infty$ and $s_i \to \infty$ be two sequences such that

$$(r_i^{-1}\widetilde{M}, \tilde{x}, \Gamma) \xrightarrow{GH} (\widetilde{Y}_1, \tilde{y}_1, G_1),$$

$$(s_i^{-1}\widetilde{M}, \tilde{x}, \Gamma) \xrightarrow{GH} (\widetilde{Y}_2, \tilde{y}_2, G_2),$$

and $t_i := (s_i^{-1})/(r_i^{-1}) \to \infty$. Rescale s_i^{-1} down by a constant if necessary, we assume that the extra $\mathbb{Z}$ subgroup in $p(G_2)$ has generator γ with $d_{\mathbb{R}^k}(\gamma \cdot 0, 0) \leq 1$. We put $(N_i, q_i, \Gamma_i) = (r_i^{-1}\widetilde{M}, \tilde{x}, \Gamma)$, then

$$(N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}_1, \tilde{y}_1, G_1),$$

$$(t_iN_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}_2, \tilde{y}_2, G_2).$$

For each i, we define

$$\begin{aligned} L_i := \{\ 1 \leq l \leq t_i \mid\ & d_{GH}((lN_i, q_i, \Gamma_i), (W, w, H)) \leq \epsilon/3 \\ & \text{for some space } (W, w, H) \in \Omega(\widetilde{M}, \Gamma) \text{ such that} \\ & p(H) \text{ contains } \mathbb{R}^l \times \mathbb{Z} \text{ as a closed subgroup;} \\ & \text{moreover, this extra } \mathbb{Z}\text{-subgroup} \\ & \text{has generator } h \text{ with } d_{\mathbb{R}^k}(h \cdot 0, 0) \leq 1.\} \end{aligned}$$

In the above definition, we choose $\epsilon = \epsilon(M) > 0$ as the constant in Lemma 4.4.11. $t_i \in L_i$ for i large. We choose $l_i \in L_i$ with $\inf L_i \le l_i \le \inf L_i + 1/i$.

Claim 1: $l_i \to \infty$. If $l_i \to C$, then

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (C \cdot \widetilde{Y}_1, \tilde{y}_1, G_1).$$

The projection to Euclidean factor $(C \cdot \mathbb{R}^k, 0, p(G_1))$ is equivalent to $(\mathbb{R}^k, 0, p(G'))$, because the later one satisfies property *(P)* and $p(G') = K \times \mathbb{R}^l$. Since $l_i \in L_i$,

$$d_{GH}((C \cdot \widetilde{Y}_1, \tilde{y}_1, G_1), (W, w, H)) \le \epsilon/2$$

for some $(W, w, H) \in \Omega(\widetilde{M}, \Gamma)$ with $p(H)$ containing $\mathbb{R}^l \times \mathbb{Z}$ as a closed subgroup. Moreover, the extra $\mathbb{Z}$-subgroup has generator h with $d_{\mathbb{R}^k}(h \cdot 0, 0) \le 1$. This is a contradiction to our choice of ϵ and Lemma 4.4.11.

Next we consider convergence

$$(l_i N_i, q_i, \Gamma_i) \xrightarrow{GH} (\widetilde{Y}', \tilde{y}', G').$$

Claim 2: $p(G') = K \times \mathbb{R}^l$. Indeed, because $(\mathbb{R}^k, 0, p(G'))$ satisfies property *(P)*, we can write $p(G') = K \times \mathbb{R}^{l'} \times \mathbb{Z}^{m'}$. We can also assume that $l' \ge l$ due to induction assumption. If $l' > l$ or $m' \ne 0$, then $p(G')$ contains $\mathbb{R}^l \times \mathbb{Z}$ as a closed subgroup. Consequently, $l_i/d \in L_i$ for some constant $d \ge 2$, which contradicts our choice of l_i. Hence Claim 2 holds.

We derive the desired contradiction: $l_i \in L_i$ so

$$d_{GH}((\widetilde{Y}', \tilde{y}', G'), (W, w, H)) \le \epsilon/2$$

for some space $(W, w, H) \in \Omega(\widetilde{M}, \Gamma)$, where $p(H)$ contains $\mathbb{R}^l \times \mathbb{Z}$ as a closed subgroup, and the extra $\mathbb{Z}$-subgroup has generator h with $d_{\mathbb{R}^k}(h \cdot 0, 0) \le 1$. A contradiction to Lemma 4.4.11.

For the remaining base case $p(G_1) = K$ $(l = 0)$, the above proof also goes through. Indeed, note that we use the induction assumption to conclude that $l' \ge l$ in Claim 2. For $l = 0$, $l' \ge 0$ holds trivially. $\square$

Remark 4.4.12. Theorems B and 4.1.4 remains true if one replace the k-Euclidean at infinity condition by the following: there is k such that any tangent cone of $\widetilde{M}$ at infinity splits as $(\mathbb{R}^k \times X, (0, x))$, where (X, x) satisfies

(1) X has no lines,

(2) any isometry of X fixes x.

With this assumption, tangent cones of $\widetilde{M}$ at infinity may not be metric cones nor be polar spaces. Nevertheless, we still have the desired properties on $\mathrm{Isom}(\widetilde{Y})$ for any $\widetilde{Y} = \mathbb{R}^k \times X \in \Omega(\widetilde{M})$ (cf. Propositions 4.1.2 and 4.1.3):

(1) $\mathrm{Isom}(\mathbb{R}^k \times X) = \mathrm{Isom}(\mathbb{R}^k) \times \mathrm{Isom}(X)$,

(2) $g \cdot (v, x) = (p(g) \cdot v, x)$ for any $g \in \mathrm{Isom}(\widetilde{Y})$ and any $v \in \mathbb{R}^k$, where $p : \mathrm{Isom}(\widetilde{Y}) \to \mathrm{Isom}(\mathbb{R}^k)$ is the natural projection.

These properties are all that we required in our proof of Theorems B and 4.1.4.

Chapter 5

No small subgroups and no small almost subgroups

5.1 Introduction

We first introduce the motivation and main results of Chapters 5 and 6.

For a metric space (X, x) and a subset A of $\mathrm{Isom}(X)$, put

$$D_{r,x}(A) = \sup_{f \in A, q \in B_r(x)} d(fq, q).$$

If the base point x is clear, we write $D_r(A)$ for simplicity. We regard A as being ϵ-small at x with scale r, if $D_{r,x}(A) < \epsilon r$. Cheeger, Colding ,and Naber showed that for any $X \in \mathcal{M}(n, -1)$ coming from a sequence of complete manifolds, its isometry group $\mathrm{Isom}(X)$ is a Lie group, by ruling out non-trivial small subgroups of $\mathrm{Isom}(X)$ [6, 11]. More precisely, they showed that for $X \in \mathcal{M}(n, -1)$, if there is a sequence of subgroups H_i of $\mathrm{Isom}(X)$ such that $D_{R,z}(H_i) \to 0$ for all $R > 0$ and all $z \in X$, then $H_i = \{e\}$ for i large. Our first result in this Chapter is a quantitative version of no small subgroup property for non-collapsing Ricci limit spaces.

Theorem 5.1.1 (No small subgroup)**.** *Given $n, v > 0$, there exists a positive constant $\delta(n, v)$ such that for any Ricci limit space $(X, x) \in \mathcal{M}(n, -1, v)$ and any nontrivial subgroup H in $\mathrm{Isom}(X)$, $D_{r,x}(H) \geq r\delta$ holds for all $r \in (0, 1]$.*

This quantitative lower bound on displacements fails if one drops the volume lower bound assumption. As an application of Theorem 5.1.1, we derive a stability result on fundamental groups for non-collapsing manifolds with bounded diameter.

Theorem 5.1.2. *Let (M_i, x_i) be a sequence of closed n-manifolds with*

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(B_1(x_i)) \geq v > 0$$

If the following sequences converge

$$\begin{array}{ccc}(\widetilde{M}_i, \tilde{x}_i, \Gamma_i) & \xrightarrow{GH} & (\widetilde{X}, \tilde{x}, G) \\ \Big\downarrow \pi_i & & \Big\downarrow \pi \\ (M_i, x_i) & \xrightarrow{GH} & (X, x),\end{array}$$

then Γ_i is isomorphic to G for all i large.

Theorem 5.1.2 implies finiteness of fundamental groups in [1]. For more applications of Theorem 5.1.1 on the structure of fundamental groups, see Section 5.3.

Next we extend this idea of "no small subgroups" to certain subsets that are very close to being subgroups. We say that a subset A of a group is symmetric, if $e \in A$ and $A = A^{-1} = \{a^{-1} | a \in A\}$. We denote d_H as the Hausdorff distance between two subsets in some metric space. Theorem 5.1.1 implies the following result:

Proposition 5.1.3. *Given $n, v > 0$, there exist positive constants $\epsilon(n, v)$ and $\eta(n, v)$ such that the following holds.*

Let (M, x) be a complete n-manifold with

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{vol}(B_1(x)) \geq v.$$

For any free isometric G-action on (M, x) and any nontrivial symmetric subset A of G, if

$$\sup_{q \in B_1(x)} \frac{d_H(Aq, A^2q)}{\mathrm{diam}(Aq)} \leq \eta,$$

then $D_{r,x}(A) \geq r\epsilon$ for all $r \in (0, 1/2]$.

The ratio $d_H(Aq, A^2q)/\mathrm{diam}(Aq)$ describes how close a symmetric subset A is close to being a subgroup with respect to its orbit at q. Note that when this ratio equals zero, A is indeed a subgroup. If this ratio is less than η, we regard A as an η almost subgroup at q. Inspired by Proposition 5.1.3, we introduce the definition below.

Definition 5.1.4. Let $\epsilon, \eta, r > 0$ and (M, x) be an n-manifold. For a subgroup G of $\mathrm{Isom}(M)$ acting freely on M, we say that G-action has no ϵ-small η-subgroup at $q \in M$ with scale r, if for any nontrivial symmetric subset A of G with

$$\frac{d_H(Aq, A^2q)}{\mathrm{diam}(Aq)} \leq \eta,$$

$D_{r,q}(A) \geq r\epsilon$ holds. We say that G-action has no ϵ-small η-subgroup on $B_1(x)$ with scale r, if it has no ϵ-small η-subgroup at every point in $B_1(x)$ with scale r.

We conjecture that no small almost subgroup property always holds when there is a lower bound on the volume, which is also mentioned in Chapter 1.

Conjecture 5.1.5 (No small almost subgroup)**.** *Given $n, v > 0$, there exist positive constants $\epsilon(n, v)$ and $\eta(n, v)$ such that if an n-manifold (M, x) satisfies*

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{vol}(B_1(x)) \geq v,$$

then any isometric free G-action on M has no ϵ-small η-subgroup on $B_{1/2}(x)$ with scale $r \in (0, 1/2]$.

We point out that to verify Conjecture 5.1.5, it is sufficient to answer the question below:

Question 5.1.6. *Let (M_i, x_i) be a sequence of complete n-manifolds of*

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{vol}(B_1(x_i)) \geq v > 0,$$

and f_i be a sequence of isometries on M_i. Suppose that the following sequences converge ($r_i \to \infty$):

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, f_\infty),$$
$$(r_i M_i, x_i, f_i) \xrightarrow{GH} (X', x', \mathrm{id}),$$

where id *means the identity map. Is it true that $f_\infty = \mathrm{id}$ always holds?*

For more explanations related to Question 5.1.6 and Conjecture 5.1.5, see Section 5.4. We also confirm Question 5.1.6, thus Conjecture 5.1.5 holds under a stronger curvature condition $\sec_M \geq -1$ (See Corollary 5.4.8).

The main result of Chapters 5 and 6 is the following theorem on bounding the number of short generators under the small almost subgroup condition. Let $S(x)$ be a set of short generators of $\pi_1(M, x)$. For convenience, we denote $S(x, R)$ as the subset of $S(x)$ consisting of elements with length less than R, and $|S(x, R)|$ as the cardinality of this subset.

Theorem 5.1.7. *Given $n, R, \epsilon, \eta > 0$, there exists a constant $C(n, R, \epsilon, \eta)$ such that for any complete n-manifold M with $\mathrm{Ric}_M \geq -(n-1)$, if $\pi_1(M, x)$ is abelian and $\pi_1(M, x)$-action has no ϵ-small η-subgroup on $B_1(\tilde{x})$ with scale $r \in (0, 1]$, then $|S(x, R)| \leq C(n, R, \epsilon, \eta)$.*

As indicated in Section 2.3, Theorem 5.1.7 implies Theorem C. It also follows from Theorem C that, an verification on Conjecture 5.1.5 would imply that Milnor conjecture holds for manifolds whose universal covers have Euclidean volume growth.

Proof of Theorem 5.1.7 is closely related to a rescaling argument. For $r_i \to \infty$, we consider

$$\begin{array}{ccccccc} (\widetilde{M}_i, \tilde{x}_i, \Gamma_i) & \xrightarrow{GH} & (\widetilde{X}, \tilde{x}, G) & \quad & (r_i\widetilde{M}_i, \tilde{x}_i, \Gamma_i) & \xrightarrow{GH} & (\widetilde{X}', \tilde{x}', G') \\ \downarrow \pi_i & & \downarrow \pi & & \downarrow \pi_i & & \downarrow \pi' \\ (M_i, x_i) & \xrightarrow{GH} & (X, x), & & (r_iM_i, x_i) & \xrightarrow{GH} & (X', x'), \end{array} \qquad (*)$$

where $\Gamma_i = \pi_1(M_i, x_i)$. The limit group G (resp. G'), as a closed subgroup of $\mathrm{Isom}(\widetilde{X})$ (resp. $\mathrm{Isom}(\widetilde{X}')$), is a Lie group [6, 11]. Our main technical result is the following relation between G and G' when Γ_i-action has no small almost subgroup.

Theorem 5.1.8 (Dimension monotonicity of symmetries)**.** *Let (M_i, x_i) be a sequence of complete n-manifolds with abelian fundamental groups Γ_i and $\mathrm{Ric}_{M_i} \geq -(n-1)$. Consider the convergent sequence and any rescaling sequence as in $(*)$. If there are $\epsilon, \eta > 0$ such that for each i, Γ_i-action has no ϵ-small η-subgroups on $B_1(\tilde{x}_i)$ with scale $r \in (0, 1]$, then*

$$\dim(G') \leq \dim(G).$$

We prove a more detailed relation between G and G' in Chapter 6. Also note that in Theorem 5.1.8, there is no requirements on volume and $\widetilde{X}$ can be collapsed. If in $(*)$ there exists a sequence of symmetric subsets A_i with $A_i \overset{GH}{\to} \{e\}$ and

$$\frac{d_H(A_i\tilde{x}_i, A_i^2\tilde{x}_i)}{\mathrm{diam}(A_i\tilde{x}_i)} \to 0,$$

then $\dim(G') \leq \dim(G)$ may fail (see Examples 3.2).

We now indicate our approach to Theorem 5.1.8, and Theorem 5.1.7 by assuming Theorem 5.1.8.

For Theorem 5.1.8, let us first consider an easy case: $G = \mathbb{R}$ and $G' = \mathbb{R} \times S^1$. Assume that S^1-action is free at some $q \in X'$ and let $q_i \in r_iM_i$ converging to q. Let γ be the element of order 2 in S^1 and $\gamma_i \in \Gamma_i$ such that

$$(r_iM_i, x_i, \gamma_i) \xrightarrow{GH} (X', x', \gamma).$$

Put $A_i = \{e, \gamma_i^{\pm 1}\}$. Then with respect to the above sequence, $A_i \overset{GH}{\to} \langle \gamma \rangle$ and thus the scaling invariant

$$\frac{d_H(A_i q_i, A_i^2 q_i)}{\mathrm{diam}(A_i q_i)} \to 0.$$

Note that before rescaling $D_{1,x_i}(A_i) \to 0$, a contradiction to assumption that Γ_i-action has no ϵ-small η-subgroup on $B_1(\tilde{x}_i)$. Next we consider a typical situation: $G = \mathbb{R}$ and $G' = \mathbb{R}^2$. The difficulty compared with the previous case is that, there is no indication to choose a sequence of collapsed almost subgroups from $G' = \mathbb{R}^2$. Our strategy is finding a suitable intermediate rescaling sequence, from which we are able to pick up a sequence of small almost groups (See Chapter 6 for details). This is similar to the critical rescaling argument in Chapter 4.

In the proof of Theorem 5.1.7, suppose that there is a contradicting sequence

$$\begin{array}{ccc}
(\widetilde{M_i}, \tilde{x}_i, \Gamma_i) & \xrightarrow{GH} & (\widetilde{X}, \tilde{x}, G) \\
\Big\downarrow{\scriptstyle \pi_i} & & \Big\downarrow{\scriptstyle \pi} \\
(M_i, x_i) & \xrightarrow{GH} & (X, x)
\end{array}$$

satisfying the following conditions:

(1) $\mathrm{Ric}_{M_i} \geq -(n-1)$,

(2) $\pi_1(M_i, x_i)$ is abelian, whose action has no ϵ-small η-subgroup on $B_1(\tilde{x}_i)$ with scale $r \in (0, 1]$;

(3) $|S(x_i, R)| \to \infty$.

Roughly speaking, we derive a contradiction by induction on the dimension of G. Assume that $\dim(G) = 0$, or G is discrete. Recall that there is a sequence of ϵ_i-equivariant maps [14]

$$\psi_i : \Gamma_i(R) \to G(R), \quad \Gamma_i(R) = \{\gamma \in \Gamma_i \mid d(\gamma\tilde{x}_i, \tilde{x}_i) \leq R\}$$

for some $\epsilon_i \to 0$. By the discreteness of $G(R)$ and no small almost subgroup assumption, it is not difficult to check that $|S(x_i, R)|$ is uniformly bounded (see Proposition 6.2.3 for details), a contradiction to (3). Assume that there is no such contradicting sequence with $\dim(G) \leq k$, while there is one with $\dim(G) = k+1$. We shall obtain a contradiction by constructing a new contradicting sequence with $\dim(G) \leq k$. For a sequence $m_i \to \infty$, let Γ_{i,m_i} be the subgroup of Γ_i generated by the first m_i short generators at x_i. If for some $m_i \to \infty$,

$$(\widetilde{M}_i, \tilde{x}_i, \Gamma_{i,m_i}) \xrightarrow{GH} (\widetilde{X}, \tilde{x}, H).$$

and $\dim(H) \leq k$, then we are done. Without lose of generality, we assume that $\dim(H) = k+1$ for all $m_i \to \infty$. For some $m_i \to \infty$ with $|\beta_i| \to 0$, where $\beta_i = \gamma_{i,m_i+1}$ is the (m_i+1)-th short generator in Γ_i, we consider a sequence of intermediate coverings,

$$\begin{array}{ccc} (\widetilde{M}_i, \tilde{x}_i, \langle \Gamma_{i,m_i}, \beta_i \rangle) & \xrightarrow{GH} & (\widetilde{X}, \tilde{x}, K) \\ \big\downarrow \pi_i & & \big\downarrow \pi \\ (\overline{M}_i = \widetilde{M}_i/\Gamma_{i,m_i}, \bar{x}_i, \langle \overline{\beta_i} \rangle) & \xrightarrow{GH} & (\overline{X}, \bar{x}, \Lambda). \end{array}$$

Because $d(\beta_i \tilde{x}_i, \tilde{x}_i) \to 0$ and $\dim(H) = \dim(K)$, one can show that Λ is discrete and fixes $\bar{x}$. Put $r_i = \mathrm{diam}(\langle \overline{\beta_i} \rangle \cdot \bar{x}_i) \to 0$ and consider the rescaling sequences

$$\begin{array}{ccc} (r_i^{-1}\widetilde{M}_i, \tilde{x}_i, \Gamma_{i,m_i}, \langle \Gamma_{i,m_i}, \beta_i \rangle) & \xrightarrow{GH} & (\widetilde{X}', \tilde{x}', H', K') \\ \big\downarrow \pi_i & & \big\downarrow \pi \\ (r_i^{-1}\overline{M}_i, \bar{x}_i, \langle \overline{\beta_i} \rangle) & \xrightarrow{GH} & (\overline{X}', \bar{x}', \Lambda'). \end{array}$$

By Theorem 5.1.8, $\dim(K') \leq \dim(K) = k+1$. If $\dim(H') < \dim(K')$, then we reduce the dimension successfully. One can check that $(r_i^{-1}\overline{M}_i, \bar{x}_i)$ is a desired contradicting sequence. If $\dim(H') = \dim(K')$, then we will look into the isotropy subgroups of H' and K' at $\tilde{x}'$, and use an induction argument on the number of the connected components of the isotropy subgroups (See Chapter 6 for details).

5.2 Volume and no small subgroups

We start with the following characterization of identity maps on Ricci limit spaces.

Lemma 5.2.1. *Let $(X, x) \in \mathcal{M}(n, -1)$ be a Ricci limit space. If $g \in \mathrm{Isom}(X)$ has trivial action on $B_1(x)$, then $g = e$.*

Proof. The proof is a modification of the arguments of Theorem 4.5 in [6] and Theorem 1.14 in [11]. Let k be the dimension of X in the Colding-Naber sense [11] and $\mathcal{R}^k$ be the set of points of which any tangent cone is isometric to $\mathbb{R}^k$. Recall that the effective regular set $\mathcal{R}^k_{\epsilon,\delta}$ is defined as all points $y \in X$ such that

$$d_{GH}(B_r(y), B^k_r(0)) \leq \epsilon r$$

for all $0 < r < \delta$.

We recall the uniform Reifenberg property proved in [11]: almost every $y \in \mathcal{R}^k$ and almost every $z \in \mathcal{R}^k$ have the property that for any $\epsilon > 0$, there exist $\delta > 0$ and a geodesic γ_{yz} connecting y and z such that $\gamma_{yz} \subseteq \mathcal{R}^k_{\epsilon,\delta}$.

Suppose that g is not the identity element. Let H be the closure of the subgroup generated by g, then clearly $H|_{B_1(x)} = \mathrm{id}$. Since $H \neq \{e\}$, for any $\epsilon > 0$, there exist $\theta \in (0, \epsilon)$ and a k-regular point $w \in (\mathcal{R}_k)_{\epsilon,\theta}$ such that

$$\theta^{-1} D_{\theta,w}(H) \geq 1/20.$$

On the other hand, because H acts trivially on $B_1(x)$, there are $\eta > 0$ and a k-regular point $y \in B_{1/2}(x) \cap (\mathcal{R}_k)_{\epsilon,\eta}$ with

$$\eta^{-1} D_{\eta,y}(H) = 0.$$

We further assume that the points w and y chosen above satisfy the uniform Reifenberg property, that is, there are $\lambda < \min\{\theta, \eta\}$ and such that γ_{wy} lies in $\mathcal{R}^k_{\epsilon,\lambda}$.

If $\lambda^{-1} D_{\lambda,w}(H) \leq 1/20$, then by intermediate value theorem, we can find $r \in [\lambda, \eta]$ such that

$$r^{-1} D_{r,w}(H) = 1/20.$$

If $\lambda^{-1} D_{\lambda,w}(H) > 1/20$, together with

$$\lambda^{-1} D_{\lambda,y}(H) = 0 < 1/20,$$

we can find z along γ_{wy} such that

$$\lambda^{-1} D_{\lambda,z}(H) = 1/20.$$

Replace the arbitrary $\epsilon > 0$ by a sequence $\epsilon_i \to 0$. Then we can find $\tau_i \geq r_i \to 0$, $z_i \in \mathcal{R}^k_{\epsilon_i,\tau_i}$ such that

$$D_{r_i,z_i}(H) = r_i/20.$$

Consequently,

$$(r_i^{-1} B_{r_i}(z_i), z_i, H) \xrightarrow{GH} (B_1^k(0), 0, H_\infty)$$

with $D_{1,0}(H_\infty) = 1/20$. However, there is no such a subgroup H_∞ of $\mathrm{Isom}(\mathbb{R}^k)$, a contradiction. $\square$

Next we prove Theorem 5.1.1 as follows; the proof is also similar to the one of Theorem 4.5 in [6].

Proof of Theorem 5.1.1. We show that $D_{1,x}(H) \geq \delta(n,v)$. For the result $D_{r,x}(H) \geq r\delta(n,v)$, with a possibly different δ, for all $r \in (0,1]$, we can scale the metric by r^{-1}. Then by relative volume comparison on $(r^{-1}X, x)$ the unit ball has volume $\mathrm{vol}(B_1(x)) \geq C(n)v$ and thus $D_{1,x}(H) \geq \delta(n, C(n)v)$ on $(r^{-1}X, x)$ for all $r \in (0,1]$. Scale the metric back to (X,x), we have $D_{r,x}(H) \geq \delta(n, C(n)v)r$.

Suppose the contrary, then there is a sequence of spaces $(X_i, x_i) \in \mathcal{M}(n,-1,v)$ and nontrivial subgroups H_i of $\mathrm{Isom}(X_i)$ with $D_1(H_i) \to 0$. By Lemma 5.2.1, passing to a subsequence if necessary,

$$(X_i, x_i, H_i) \xrightarrow{GH} (X, x, \{e\}).$$

We will find a subsequence $i(j)$, $\epsilon_{j(i)} \to 0$, $\tau_{i(j)} \geq r_{i(j)} > 0$, $z_{i(j)} \in \mathcal{R}_{\epsilon_{i(j)},\tau_{i(j)}} \subseteq X_{i(j)}$ and $D_{r_{i(j)},z_{i(j)}}(H_{i(j)}) = \dfrac{1}{20} r_{i(j)}$. Then

$$(r_{i(j)}^{-1} B_{r_{i(j)}}(z_{i(j)}), z_{i(j)}, H_{i(j)}) \xrightarrow{GH} (B_1^n(0), 0, H_\infty)$$

with $D_1(H_\infty) = 1/20$; and the desired contradiction follows.

Fix a regular point $y \in B_1(x) \subset X$. For each $\epsilon > 0$, there is $\delta > 0$ such that $y \in \mathcal{R}_{\epsilon,\delta}$. Pick a sequence of regular points $y_i \in X_i$ converging to y. Put

$$\eta_i = d_{GH}(\delta^{-1} B_\delta(y_i), \delta^{-1} B_\delta(y)) \to 0.$$

Because $y \in \mathcal{R}_{\epsilon,\delta}$,

$$d_{GH}(\delta^{-1} B_\delta(y_i), B_1^n(0)) \leq \eta_i + \epsilon.$$

By Theorem 2.2.14, for all $0 < s \leq \delta$,

$$d_{GH}(s^{-1} B_s(y_i), B_1^n(0)) \leq \Phi(\eta_i + \epsilon, \delta | n).$$

In other words, $y_i \in \mathcal{R}_{\Phi_i,\delta}$. Also, because $H_i \to \{e\}$,

$$\delta^{-1} D_{\delta,y_i}(H_i) \to 0.$$

For each ϵ, pick $i(\epsilon)$ large such that for all $i \geq i(\epsilon)$, we have

$$\eta_i \leq \delta \text{ and } \delta^{-1} D_{\delta,y_i}(H_i) \leq \frac{1}{20}.$$

Now consider a sequence $\epsilon_j \to 0$, then $y \in \mathcal{R}_{\epsilon_j,\delta_j}$ for some $\delta_j \to 0$. There is a subsequence $i(j)$ such that

1. $\eta_{i(j)} \leq \delta_j$, thus $y_{i(j)} \in \mathcal{R}_{\Phi_{i(j)},\delta_j}$, where $\Phi_{i(j)} = \Phi(\delta_j + \epsilon_j|n)$;
2. $D_{\delta_j,y_{i(j)}}(H_{i(j)}) \leq \dfrac{1}{20}\delta_j$.

On each $X_{i(j)}$, there is $\theta_{i(j)} > 0$, $w_{i(j)} \in \mathcal{R}_{\Phi_{i(j)},\theta_{i(j)}}$ such that

$$D_{\theta_{i(j)},w_{i(j)}}(H_{i(j)}) \geq \frac{1}{20}\theta_{i(j)}.$$

The remaining proof is essentially the same as Theorem 4.5 in [CC00a]. □

For fundamental group actions on universal covers, one can prove no small subgroup property alternatively by using Dirichlet domains and volume convergence.

Corollary 5.2.2. *Let (M_i, x_i) be a sequence of complete n-manifolds and $(\widetilde{M}_i, \tilde{x}_i)$ be the sequence of their universal covers with*

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{vol}(B_1(\tilde{x}_i)) \geq v > 0.$$

If H_i is a sequence of subgroups of $\pi_1(M_i, x_i)$ with $D_1(H_i) \to 0$, then $H_i = \{e\}$ for all i sufficiently large.

Proof. Suppose the contrary, i.e. $\#H_i \geq 2$ for some subsequence. Passing to this subsequence, we may assume that $\#H_i \geq 2$ for all i. Let h_i be a sequence of nontrivial elements in H_i. Notice that by Lemma 5.2.1, $D_1(H_i) \to 0$ implies that $H_i \xrightarrow{GH} \{e\}$. Passing to some subsequences if necessary, we consider

$$\begin{array}{ccc}
(\widetilde{M}_i, \tilde{x}_i, H_i) & \xrightarrow{GH} & (\widetilde{X}, \tilde{x}, \{e\}) \\
\downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi} \\
(\overline{M}_i = \widetilde{M}_i/H_i, \bar{x}_i) & \xrightarrow{GH} & (\widetilde{X}/\{e\}, \bar{x}).
\end{array}$$

By Theorem 2.2.13,

$$\mathrm{vol}(B_1(\tilde{x}_i)) \to \mathcal{H}^n(B_1(\tilde{x}));$$
$$\mathrm{vol}(B_1(\bar{x}_i)) \to \mathcal{H}^n(B_1(\bar{x})) = \mathcal{H}^n(B_1(\tilde{x})),$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on the limit spaces.

On the other hand, there is $\epsilon_i \to 0$ such that $H_i(B_1(x_i)) \subseteq B_{1+\epsilon_i}(x_i)$. Let F_i be the Dirichlet domain centered at $\tilde{x}_i$. Then we have

$$\begin{aligned}\mathrm{vol}(B_1(\bar{x}_i)) &= \frac{1}{2}(\mathrm{vol}(B_1(\tilde{x}_i) \cap F_i) + \mathrm{vol}(h_i(B_1(\tilde{x}_i) \cap F_i))) \\ &\leq \frac{1}{2} \cdot \mathrm{vol}(B_{1+\epsilon_i}(\tilde{x}_i)) \\ &\to \frac{1}{2} \cdot \mathcal{H}^n(B_1(\tilde{x})).\end{aligned}$$

We end in a contradiction. □

5.3 Applications of no small subgroups

As one of applications of Theorem 5.1.1, we prove Theorem 5.1.2. Recall that Theorem 5.1.1 implies that if (M, x) satisfies

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{vol}(B_1(\tilde{x})) \geq v > 0,$$

then any nontrivial subgroup H of Γ has $D_{1,\tilde{x}}(H) \geq \delta(n, v)$. Under a stronger volume condition

$$\mathrm{vol}(B_1(x)) \geq v > 0,$$

we show that such a lower bound on displacement holds for any nontrivial covering transformation.

Lemma 5.3.1. *Given n and $v > 0$, there is a constant $\delta(n, v) > 0$ such that for any n-manifold (M, x) with*

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{vol}(B_1(x)) \geq v$$

and any nontrivial element $\gamma \in \pi_1(M, x)$, we have $D_{1,\tilde{x}}(\gamma) \geq \delta$.

Proof. We argue by contradiction. Suppose that we have the following convergent sequences

$$\begin{array}{ccc} (\widetilde{M}_i, \tilde{x}_i, \Gamma_i) & \xrightarrow{GH} & (\widetilde{X}, \tilde{x}, G) \\ \downarrow \pi_i & & \downarrow \pi \\ (M_i, x_i) & \xrightarrow{GH} & (X, x). \end{array}$$

with

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{vol}(B_1(x)) \geq v;$$

and a sequence of nontrivial elements $\gamma_i \in \Gamma_i$ converging to the identity map, where $\Gamma_i = \pi_1(M, x)$.

By [1], there are positive constants $L(n, v)$ and $N(n, v)$ such that for any subgroup in $\pi_1(M, x)$ generated by elements of length $\leq L$, this subgroup has order $\leq N$ (In [1], only closed manifolds with bounded diameter are considered, but its proof extends to open manifolds). Since $\gamma_i \to \mathrm{id}$, for all i large γ_i has length $\leq L$, thus has order $\leq N$. Consequently, the sequence of subgroups generated by γ_i also converges to $\{e\}$. By Theorem 5.1.1, this implies that $\langle \gamma_i \rangle$, and thus γ_i, is identity for i large. □

With Lemma 5.3.1, we prove Theorem 5.1.2, the stability of π_1 under equivariant GH convergence for non-collapsing manifolds with bounded diameter.

Proof of Theorem 5.1.2. We first notice that G is a discrete group (intuitively, otherwise M_i would be collapsed). In fact, we consider $\langle \Gamma_i(L) \rangle$, the subgroup generated by loops of length $\leq L$, where $L = L(n, v)$ is the constant mentioned in the proof of Lemma 5.3.1. We consider

$$(\widetilde{M}_i, \tilde{x}_i, \Gamma_i(L)) \xrightarrow{GH} (\widetilde{X}, \tilde{x}, H).$$

Since each $\Gamma_i(L)$ has order $\leq N$, so does H. Note that H contains G_0, thus $G_0 = \{e\}$ and G is discrete.

By [14], there exists a sequence of subgroups H_i of Γ_i such that

$$(\widetilde{M}_i, \tilde{x}_i, H_i) \xrightarrow{GH} (\widetilde{X}, \tilde{x}, G_0)$$

and Γ_i / H_i is isomorphic to G/G_0 for all i large. In our situation, $G_0 = \{e\}$ and thus $H_i \overset{GH}{\to} \{e\}$. By Theorem 5.3.1, we see that $H_i = \{e\}$ for all i large. Consequently, Γ_i is isomorphic to G for all i large. □

We also prove two structure theorems below on fundamental groups of closed manifolds with non-collapsing conditions on the universal covers.

Theorem 5.3.2. *Given $n, D, v > 0$, there exists a constant $C(n, D, v)$ such that if a complete n-manifold (M, x) with finite fundamental group satisfies*

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(B_1(\tilde{x})) \geq v > 0,$$

then $\pi_1(M)$ contains an abelian subgroup of index $\leq C(n, v)$. Moreover, this subgroup can be generated by at most n elements.

Theorem 5.3.3. *Given $n, v > 0$, there exists a constant $C(n, v)$ such that if a complete n-manifold (M, x) satisfies*

$$\mathrm{Ric}_M \geq 0, \quad \mathrm{diam}(M) = 1, \quad \mathrm{vol}(B_1(\tilde{x})) \geq v > 0,$$

then $\pi_1(M)$ contains an abelian subgroup of index $\leq C(n, v)$. Moreover, this subgroup can be generated by at most n elements.

Theorems 5.3.2 and 5.3.3 generalize Theorems D and E in [26], where the curvature conditions are on sectional curvature. Given Theorem 8 in [22] and Theorem 4.1 [6], actually their proof [26] extends to the Ricci case. Here we give an alternative approach by applying Theorem 5.1.1 and Kapovitch-Wilking's work [22]. Our new proof also gives a bound of generators of the abelian subgroup.

Theorems 5.3.2 and 5.3.3 partially verify the following conjectures respectively.

Conjecture 5.3.4. *Given n and D, there exists a constant $C(n, D)$ such that the following holds. Let M be an n-manifold with finite fundamental group and*

$$\mathrm{Ric}_M \geq -(n-1), \quad \mathrm{diam}(M) \leq D,$$

then $\pi_1(M)$ contains an abelian subgroup of index $\leq C(n, D)$. Moreover, this subgroup can be generated by at most n elements.

Conjecture 5.3.5 (Fukaya-Yamaguchi)**.** *Given n, there exists a constant $C(n)$ such that for any n-manifold with nonnegative Ricci curvature, its fundamental group contains an abelian subgroup of index $\leq C(n)$. Moreover, this subgroup can be generated by at most n elements.*

We make use of some results on nilpotent groups.

Lemma 5.3.6. *Any compact connected nilpotent Lie group is abelian, thus a torus.*

Lemma 5.3.7. *[30] Let* Γ *be a nilpotent group generated by* n *elements* $x_1, ..., x_n$*. Then every element in* $[\Gamma, \Gamma]$ *is a product of* n *commutators* $[x_1, g_1], ..., [x_n, g_n]$ *for suitable* $g_i \in G$ $(i = 1, ..., n)$.

Proof of Theorem 5.3.2. Suppose that the statement does not hold, then we have a contradicting sequence

$$\begin{array}{ccc} (\widetilde{M}_i, \tilde{x}_i, \Gamma_i) & \xrightarrow{GH} & (\widetilde{X}, \tilde{x}, G) \\ \Big\downarrow{\scriptstyle \pi_i} & & \Big\downarrow{\scriptstyle \pi} \\ (M_i, x_i) & \xrightarrow{GH} & (X, x) \end{array}$$

with finite fundamental groups and

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{diam}(M_i) = D, \quad \mathrm{vol}(B_1(\tilde{x}_i)) \geq v > 0,$$

but any abelian subgroup in $\pi_1(M_i)$ has index larger than i. By [22], Γ_i is $C(n)$-nilpotent with a cyclic chain of length $\leq n$. Thus without lose of generality, we may assume that Γ_i is nilpotent with a cyclic chain of length $\leq n$ for all i, and thus G is a nilpotent Lie group.

By Diameter Ratio Theorem [22], $\mathrm{diam}(\widetilde{M}_i)$ has an upper bound $\widetilde{D}(n, D)$. Thus the limit space $\widetilde{X}$ and its limit group G are compact. G_0, as a connected compact nilpotent Lie group, must be a torus. We call this torus T. Since G is compact, there is a sequence of subgroups H_i converging to T such that $\Gamma_i/H_i \simeq G/T$. In particular,

$$[\Gamma_i : H_i] = [G : T] < \infty.$$

We complete the proof once we show that H_i is abelian and can be generated by at most n-elements.

Since Γ_i is nilpotent with a cyclic chain of length $\leq n$, H_i can be generated by at most n-elements. To show that Γ_i is abelian, we consider the commutator subgroup $[H_i, H_i]$. We claim that $[H_i, H_i] \xrightarrow{GH} e$, then by Corollary 5.2.2, $[H_i, H_i] = e$ and thus H_i is abelian. Indeed, for any sequence γ_i in $[H_i, H_i]$, by lemma 5.3.7 it can be written as $\prod_{j=1}^{n}[x_{i,j}, h_{i,j}]$, where $\{x_{i,j}\}_{j=1}^{n}$ are generators of H_i and $h_{i,j} \in H_i$. Since the limit group

T is compact, passing to a subsequence if necessary, we may assume that $x_{i,j} \to x_j \in T$ and $h_{i,j} \to h_j \in T$. Because T is abelian, $[x_{i,j}, h_{i,j}] \to [x_j, h_j] = e$ and thus $\gamma_i \to e$. $\square$

Next we consider closed manifolds with nonnegative Ricci curvature.

Lemma 5.3.8. *Given n, there exists a constant $C(n)$ such that the following holds. Let M be a closed n-Riemannian manifold with*

$$\mathrm{Ric}_M \geq 0, \quad \mathrm{diam}(M) = 1.$$

Then $\widetilde{M}$ splits isometrically as $N \times \mathbb{R}^k$ with $\mathrm{diam}(N) \leq C(n)$.

Proof. By Cheeger-Gromoll splitting theorem [8], we know that $\widetilde{M}$ splits isometrically as $N \times \mathbb{R}^k$, where N is compact and simply connected. Suppose that we have a contradicting sequence: M_i with

$$\mathrm{Ric}_{M_i} \geq 0, \quad \mathrm{diam}(M_i) = 1,$$

but N_i, the compact factor of $\widetilde{M_i}$, has diameter $\to \infty$. By generalized Margulis Lemma [22], it is easy to see that $\Gamma_i = \pi_1(M_i, x_i)$ is $C(n)$-nilpotent. Hence without lose of generality, we may assume that Γ_i itself is nilpotent.

Put $r_i = \mathrm{diam}(N_i) \to \infty$ and consider the rescaling sequence

$$\begin{array}{ccc} (r_i^{-1} N_i \times \mathbb{R}^k, \tilde{x}_i, \Gamma_i) & \xrightarrow{GH} & (Y \times \mathbb{R}^k, \tilde{x}, G) \\ \Big\downarrow{\scriptstyle \pi_i} & & \Big\downarrow{\scriptstyle \pi} \\ (r_i^{-1} M_i, x_i) & \xrightarrow{GH} & \textit{point} \end{array}$$

where G is a nilpotent Lie group acting transitively on the limit space $Y \times \mathbb{R}^k$. Let K be the subgroup of G acting trivially on $\mathbb{R}^k$-factor. Then K acts effectively and transitively on Y. In particular, Y is a compact topological manifold homeomorphic to K/Iso. By Lemma 5.3.6, K_0 is a torus, which acts freely and effectively on Y. With these facts, it is easy to verify that Y itself is also a torus.

On the other hand, we have $r_i^{-1} N_i \xrightarrow{GH} Y$. Since each N_i is simply connected and Y is a compact manifold, Y must be simply connected. We obtained a contradiction. $\square$

Remark 5.3.9. We point out that in [26]'s proof, they also need Lemma 5.3.8, but their proof of this diameter bound has a mistake.

Proof of Theorem 5.3.3. We argue by contradiction. Suppose the contrary, then we have a contradicting sequence M_i with

$$\mathrm{Ric}_{M_i} \geq 0, \quad \mathrm{diam}(M_i) = 1, \quad \mathrm{vol}(B_1(\tilde{x}_i)) \geq v > 0,$$

but any abelian subgroup of $\pi_1(M_i)$ has index $> i$. By generalized Margulis Lemma [22], we may assume that for each i, $\pi_1(M_i)$ is nilpotent with a cyclic chain of length at most n.

By Lemma 5.3.8, $\widetilde{M_i}$ splits as $N_i \times \mathbb{R}^{k_i}$ isometrically with $\mathrm{diam}(N_i) \leq C(n)$. Since $k_i \leq n$ for all i, passing to a subsequence, we may assume $k_i = k$ for all i. Passing to a subsequence again, we obtain the following convergent sequences.

$$\begin{array}{ccc} (N_i \times \mathbb{R}^k, \tilde{x}_i) & \xrightarrow{GH} & (N \times \mathbb{R}^k, \tilde{x}) \\ \downarrow & & \downarrow \\ (M_i, x_i) & \xrightarrow{GH} & (X, x), \end{array}$$

where N is compact. From the assumption that $\mathrm{vol}(B_1(\tilde{x}_i)) \geq v > 0$, it is obvious that $\mathrm{vol}(N_i) \geq v_0 > 0$ for some v_0.

Let $p_i : \mathrm{Isom}(N_i \times \mathbb{R}^k) \to \mathrm{Isom}(\mathbb{R}^k)$ and $q_i : \mathrm{Isom}(N_i \times \mathbb{R}^k) \to \mathrm{Isom}(N_i)$ be the natural projection maps. Consider $\overline{q_i(\Gamma_i)}$ acting on N_i and the corresponding convergent sequence

$$(N_i, \overline{q_i(\Gamma_i)}) \xrightarrow{GH} (N, G).$$

N is compact and thus G is also compact. Then by a similar argument in the proof of Theorem 5.3.2, we can show that $\overline{q_i(\Gamma_i)}$, and thus $q_i(\Gamma_i)$, is C_1-abelian, where C_1 is a constant independent of i. Also, $x_i(\Gamma_i)$ acts co-compactly on $\mathbb{R}^k$, thus by Bieberbach theorem, $x_i(\Gamma_i)$ is $C_2(n)$-abelian.

Finally, we treat Γ_i as a subgroup of $q_i(\Gamma_i) \times x_i(\Gamma_i)$. It is easy to check that Γ_i contains an abelian subgroup of index $\leq C_1C_2$. Moreover, this subgroup can be generated by at most n-elements because Γ_i is nilpotent with a cyclic chain of length at most n. $\square$

5.4 No small almost subgroups and almost identity isometries

We explore the relations between volume, no small almost subgroup property and isometries that are close to identity maps in this section. We present two equivalent statements implying Conjecture 5.1.5 (See Proposition 5.4.3). We also show that Conjecture 5.1.5 holds when sectional curvature has a lower bound (See Corollary 5.4.8).

Using Theorem 5.1.1, we first prove Proposition 5.1.3.

Proof of Proposition 5.1.3. We argue by contradiction. Suppose that there is a sequence of complete n-manifolds (M_i, x_i) with

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{vol}(B_1(x_i)) \geq v;$$

and a sequence of symmetric subsets $A_i \neq \{e\}$ of Γ_i with $D_{1,x_i}(A_i) \to 0$ and

$$\sup_{q \in B_1(x_i)} \frac{d_H(A_i q, A_i^2 q)}{\mathrm{diam}(A_i q)} \to 0.$$

Let $\delta = \delta(n, v)$ be the constant in Theorem 5.1.1. For any positive integer j, we can choose $i(j)$ large with

$$D_1(A_{i(j)}) \leq \delta/2.$$

For this $i(j)$, there is $\theta(j) > 0$ such that

$$\theta(j)^{-1} D_{\theta(j)}(A_{i(j)}) \geq \delta/2.$$

Otherwise A_i would fix $\tilde{x}_i$. By intermediate value theorem, there is $r(j) \in [\theta(j), 1]$ such that

$$r(j)^{-1} D_{r(j)}(A_{i(j)}) = \delta/2.$$

For simplicity, we just call $r(j)$ as r_i and the subsequence $i(j)$ as i.

We know that

$$\sup_{q \in B_{r_i}(x_i)} \frac{d_H(A_i q, A_i^2 q)}{\mathrm{diam}(A_i q)} \to 0.$$

Then after rescaling $r_i^{-1} \to \infty$,

$$(r_i^{-1} M_i, x_i, A_i) \xrightarrow{GH} (X', x', A_\infty).$$

A_∞ satisfies $D_1(A_\infty) = \delta/2$. By Theorem 5.1.1, A_∞ is not a subgroup. So there is some point $q \in B_1(x')$ such that $A_\infty^2 q \neq A_\infty q$ (See Lemma 5.2.1).

On the other hand, for a sequence q_i converging to q,

$$r_i^{-1} d_H(A_i q_i, A_i^2 q_i) \leq \epsilon_i \cdot r_i^{-1} D_1(A_i) = \epsilon_i \delta/2 \to 0$$

for some sequence $\epsilon_i \to 0$. Thus $A_\infty q = A_\infty^2 q$, a contradiction. $\square$

Remark 5.4.1. Conjecture 5.1.5 says that there is no small almost subgroup, where almost subgroup means

$$\frac{d_H(Ax, A^2x)}{\mathrm{diam}(Ax)} \leq \eta$$

for a single point x, while in Proposition 5.1.3, we showed that for an almost subgroup in the sense of

$$\sup_{q \in B_1(x)} \frac{d_H(Aq, A^2q)}{\mathrm{diam}(Aq)} \leq \eta,$$

its displacement can not be too small.

Remark 5.4.2. Theorem 5.1.1 and Proposition 5.1.3 are evidents supporting Conjecture 5.1.5. Later, we will see that Conjecture 5.1.5 holds under a stronger curvature condition $\sec_M \geq -1$ (See Corollary 5.4.8).

Proposition 5.4.3. *Let (M_i, x_i) be a sequence of complete n-manifolds with*

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{vol}(B_1(x_i)) \geq v > 0.$$

Let G_i be a group acting isometrically and freely on M for each i. Suppose that one of the following statements holds:

(1) For any sequence $f_i \in G_i$ and $r_i \to \infty$ with

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, \mathrm{id}),$$

$$(r_i M_i, x_i, f_i) \xrightarrow{GH} (X', x', f'_\infty),$$

if f'_∞ fixes x', then $f'_\infty = \mathrm{id}$.

(2) For any sequence $f_i \in G_i$ and $r_i \to \infty$ with

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, f_\infty),$$

$$(r_iM_i, x_i, f_i) \xrightarrow{GH} (X', x', \mathrm{id}),$$

then $f_\infty = \mathrm{id}$.

Then there are $\epsilon, \eta > 0$ *such that for all* i, G_i*-action has no* ϵ*-small* η*-almost subgroup at* x_i *with scale* 1.

Moreover, (1) and (2) are equivalent.

Proof. We first show that G_i-action has no ϵ-small η-almost subgroup at x_i by assuming (1). Suppose that each G_i contains a symmetric subset A_i with $D_{1,x_i}(A_i) \to 0$ and

$$\frac{d_H(A_ix_i, A_i^2x_i)}{\mathrm{diam}(A_ix_i)} \to 0.$$

We rescale the sequence by r_i^{-1} as in the proof of Proposition 5.1.3

$$(r_i^{-1}M_i, x_i, A_i) \xrightarrow{GH} (X', x', A_\infty)$$

so that $D_1(A_\infty) = \delta/2$, where $\delta = \delta(n, v)$ is the constant in Theorem 5.1.1; and thus A_∞ is not a subgroup. At point x', A_∞-orbit satisfies

$$\begin{aligned} d_H(A_\infty x', A_\infty^2 x') &= \lim_{i\to\infty} d_H(A_ix_i, A_i^2x_i) \quad (\text{On } r_i^{-1}M_i) \\ &\leq \lim_{i\to\infty} \epsilon_i \mathrm{diam}(A_ix_i) \\ &\leq \lim_{i\to\infty} \epsilon_i\delta/2 \to 0. \end{aligned}$$

This means that there is an non-identity element $a \in A_\infty^3$ fixing x'. Therefore, we have a sequence $a_i \in A_i^3$ such that

$$(M_i, x_i, a_i) \xrightarrow{GH} (X, x, \mathrm{id});$$

$$(r_i^{-1}M_i, x_i, a_i) \xrightarrow{GH} (X', x', a).$$

On the other hand, by assumptions we have $a = \mathrm{id}$. A contradiction.

Proof of (2)⇒(1). Suppose that there exists $r_i \to \infty$ and $f_i \in G_i$ such that

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, \mathrm{id}),$$

$$(r_iM_i, x_i, f_i) \xrightarrow{GH} (X', x', f'_\infty)$$

where f'_∞ fixes x', but $f'_\infty \neq \mathrm{id}$.

Without lose of generality, we assume that f'_∞ has finite order. Actually, if f'_∞ has infinite order, then $\overline{\langle f'_\infty \rangle}$ has a circle subgroup. We take $A_i = \{e, f_i^{\pm 1}, ..., f_i^{\pm k_i}\}$ such that $k_i \to \infty$ and

$$(M_i, x_i, A_i) \xrightarrow{GH} (X, x, \{e\}).$$

After rescaling r_i, the limit of A_i contains $\overline{\langle f'_\infty \rangle}$. So there is $g_i \in A_i$ such that

$$(M_i, x_i, g_i) \xrightarrow{GH} (X, x, \mathrm{id}),$$

$$(r_i M_i, x_i, g_i) \xrightarrow{GH} (X', x', g'_\infty),$$

where g'_∞ fixes x' and has finite order.

Let $N < \infty$ be the order of f'_∞. By Theorem 5.1.1,

$$D_1(f'_\infty) \geq \delta/N.$$

By intermediate value theorem, there is a rescaling sequence $s_i \to \infty$ such that $r_i/s_i \to \infty$ and

$$(s_i M_i, x_i, f_i) \xrightarrow{GH} (X'', x'', f''_\infty),$$

with f''_∞ satisfying $D_1(f''_\infty) = \delta/(2N)$. By Theorem 5.1.1, it is clear that f''_∞ has order $\geq 2N$. Now we result in the following sequence:

$$(s_i M_i, x_i, f_i^N) \xrightarrow{GH} (X'', x'', (f''_\infty)^N \neq \mathrm{id});$$

$$(r_i M_i, x_i, f_i^N) \xrightarrow{GH} (X', x', \mathrm{id}).$$

This contradicts with the assumption.

Proof of (1)⇒(2). The proof is very similar to the one of *(2)⇒(1).* If the statement is false, then one can find a contradiction to (1) in some intermediate rescaling sequence.

□

Remark 5.4.4. According to Proposition 5.4.3, if statement (1) or (2) above holds for any sequence (M_i, x_i, f_i) with

$$\mathrm{Ric}_{M_i} \geq -(n-1), \quad \mathrm{vol}(B_1(x_i)) \geq v > 0,$$

then Conjecture 5.1.5 would follow from Bishop-Gromov relative volume comparison and Proposition 5.4.3. Note that both statements fail if when remove the lower volume bound. For (2), recall that horns can appear as Ricci limit spaces [4], so one can construct a sequence (M_i, x_i, f_i) such that (X, x, f_∞) is a horn with a rotational isometry, while (X', x', f'_∞) is a half line with identity isometry.

We can further reduce statement (2), or (1), in Proposition 5.4.3 to the following situation:

Without lose of generality, we can assume that both X, X' are Euclidean cones, and f_∞ has finite order.

In fact, if f_∞ has infinite order, we consider the following sequence of symmetric subsets $A_i = \{e, f_i^{\pm 1}, ..., f_i^{\pm k_i}\}$. We choose $k_i \to \infty$ slowly so that

$$(r_i M_i, x_i, A_i) \xrightarrow{GH} (X', x', \{e\}).$$

Since before rescaling r_i, the limit of f_i fixes x. Thus the limit of A_i contains a circle subgroup fixing x. As a result, there is $g_i \in A_i$ such that

$$(M_i, x_i, g_i) \xrightarrow{GH} (X, x, g_\infty),$$

$$(r_i M_i, x_i, g_i) \xrightarrow{GH} (X', x', \mathrm{id}),$$

where g_∞ fixes x and has finite order.

Reduction to metric cones follows directly from the lemma below and a standard rescaling argument by passing to tangent cones (See Theorem 2.2.5). More precisely, under the conditions of Proposition 5.4.3, we can find $s_i \to \infty$, $s'_i \to \infty$ with $s'_i/s_i \to \infty$ and

$$(s_i M_i, x_i) \xrightarrow{GH} (C_p X, o)$$

$$(s'_i M_i, x_i) \xrightarrow{GH} (C_{x'} X', o').$$

Lemma 5.4.5. *Let (Y, y) be an non-collapsing Ricci limit space and f be any isometry of Y fixing y. Suppose that f has finite order k, then for any $r_i \to \infty$ and any convergent subsequence*

$$(r_i Y, y, f) \xrightarrow{GH} (C_y Y, o, f_y),$$

f_y has order k.

Proof. Because f has finite order, for any $r_i \to \infty$ and any convergent subsequence, we have

$$(r_iY, y, \langle f\rangle) \xrightarrow{GH} (C_yY, o, \langle f_y\rangle).$$

Since f has order k, f_y has order at most k. Suppose that f_y has order $l < k$. This implies that

$$(r_iY, y, f^l) \xrightarrow{GH} (C_yY, o, e).$$

Together with the fact that $\langle f\rangle$ is a discrete group, we see that

$$(r_iY, y, \langle f^l\rangle) \xrightarrow{GH} (C_yY, o, \{e\}).$$

By Theorem 5.1.1, $\langle f^l\rangle = e$, contradiction. □

We show that Theorem 5.4.3(2) holds when $\sec_{M_i} \geq -1$ (volume condition is not required in this situation).

Lemma 5.4.6. *Let (M_i, x_i) be a sequence of n-manifolds with $\sec_{M_i} \geq -1$ and f_i be a sequence of isometries of M_i. Suppose that*

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, f_\infty);$$

$$(r_iM_i, x_i, f_i) \xrightarrow{GH} (X', x', \mathrm{id}).$$

Then $f_\infty = \mathrm{id}$.

For $0 < r \leq R$, we define the (r, R)-scale segment domain at x as follows.

$$S_r^R(x) = \{\gamma|_{[0,r]} \mid \gamma \text{ is a unit speed minimal geodesic from } x \text{ of length at least } R\}.$$

Note that $S_r^R(x)$ is always a subset of $B_r(x)$, but it may not be equal to $B_r(x)$. We also define the r-scale exponential map at x $(0 < r < 1)$:

$$\begin{aligned} \exp_p^r : S_r^1(x) &\to B_1(x) \\ \exp_p(v) &\mapsto \exp_p(r^{-1}v). \end{aligned}$$

Lemma 5.4.7. *If $(M_i, x_i) \overset{GH}{\to} (X, x)$ and (X, x) is a metric cone with vertex x, then $S_1^1(x_i) \overset{GH}{\to} B_1(x)$.*

Proof. For any $z \in B_1(x)$ with $z \neq x$, put $d = d(z, x)$. Let γ be the unique unit speed minimal geodesic from x to z. Extend γ to a ray starting at x and put $q := \gamma(2)$. Pick $q_i \in M_i$ with $q_i \to q$. For each i, let γ_i be a unit speed minimal geodesic from x_i to q_i. It is clear that the image of $\gamma_i|_{[0,1]}$ is in $S^1_1(x_i)$. γ_i converges to a minimal geodesic from x to q, which must be $\gamma|_{[0,2]}$. In particular, $\gamma_i(d) \to z$. $\square$

Proof of Lemma 5.4.6. As discussed above on the reduction, we may assume that both X and X' are metric cones (Note that both X and X' are Alexandrov spaces, thus their tangent cones are always metric cones [3]).

For each i, we consider the commutative diagram:

$$\begin{array}{ccc} r_i S^1_{r_i^{-1}}(x_i) & \xrightarrow{f_i} & r_i S^1_{r_i^{-1}}(f_i(x_i)) \\ \Big\downarrow r_i \exp_{x_i}^{r_i^{-1}} & & \Big\downarrow r_i \exp_{f_i(x_i)}^{r_i^{-1}} \\ B_1(x_i) & \xrightarrow{f_i} & B_1(f_i(x_i)) \end{array}$$

Let $S(x')$ be the limit of $r_i S^1_{r_i^{-1}}(x_i)$. Since after rescaling r_i, $f_i \overset{GH}{\to} e$, $S(x')$ is also the limit of $r_i S^1_{r_i^{-1}}(f_i(x_i))$. By Toponogov theorem, $r_i \exp_{x_i}^{r_i^{-1}}$ and $r_i \exp_{f_i(x_i)}^{r_i^{-1}}$ are $L(n)$-Lipschitz maps. Passing to a subsequence, these two sequences of maps converge to α and $\alpha' : S(x') \to B_1(x)$ as $i \to \infty$ respectively. By Lemma 5.4.7, α and α' are surjective. We claim that $\alpha = \alpha'$. In fact, if for some $q \in S(x')$, $\alpha(q) \neq \alpha'(q)$, then we can find minimal geodesics γ_i and γ_i' from x_i such that

$$(r_i M_i, \gamma_i(r_i^{-1} d), \gamma_i'(r_i^{-1} d)) \xrightarrow{GH} (X, q, q)$$

$$(M_i, \gamma_i(d), \gamma_i'(d)) \xrightarrow{GH} (X, \alpha(q), \alpha'(q)),$$

where $d = d(x, q)$. By Toponogov comparison theorem, we see a bifurcation of minimal geodesics at q. But we know this can not happen in X' [3].

Now we have a commutative diagram of limit spaces

$$\begin{array}{ccc} S(x') & \xrightarrow{\mathrm{id}} & S(x') \\ \Big\downarrow \alpha & & \Big\downarrow \alpha \\ B_1(x) & \xrightarrow{f_\infty} & B_1(x), \end{array}$$

where f_∞ is an isometry and α is surjective. Therefore, $f_\infty = \mathrm{id}$. $\square$

Since we have showed that Proposition 5.4.3(2) holds when $\sec_M \geq -1$, we see that the following corollary holds.

Corollary 5.4.8. *Conjecture 5.1.5 holds when* (M, x) *satisfies*

$$\sec_M \geq -1, \quad \mathrm{vol}(B_1(x)) \geq v > 0.$$

Chapter 6

Dimension monotonicity of symmetries

6.1 Introduction

We state the dimension monotonicity of symmetries, which is the main technical tool to prove Theorem C.

Theorem 6.1.1. *Let (M_i, x_i) be a sequence of complete n-manifolds with*

$$\mathrm{Ric}_{M_i} \geq -(n-1)$$

and Γ_i be a discrete abelian group acting freely and isometrically on M_i for each i. Suppose that each Γ_i-action has no ϵ-small η-subgroup on $B_1(x_i)$ with scale $r \in (0,1]$. If the following two sequences converge ($r_i \to \infty$):

$$(M_i, x_i, \Gamma_i) \xrightarrow{GH} (X, x, G),$$

$$(r_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x', G'),$$

then the following holds:
(1) $\dim(G') \leq \dim(G)$*;*
(2) If G' contains a compact subgroup K' with $K'_0 = \mathbb{T}^l$, then G contains a subgroup K fixing x with $K_0 = \mathbb{T}^l$ and K/K_0 being isomorphic to K'/K'_0.

To roughly illustrate why no small almost subgroup condition is the key criterion for dimension monotonicity of symmetries, regardless whether the manifolds are collapsed or not, we consider the following examples.

Examples 6.1.2. Let $M_i = \mathbb{R} \times (S^3, \frac{1}{i} d_0)$, where d_0 is the standard metric on S^3, and x_i be a point in M_i. S^3 admits a circle group S^1 acting freely and isometrically on S^3.

For a number $\theta \in S^1 = [0, 2\pi]/\sim$, we denote $R(\theta)$ as the corresponding isometry on S^3. We define two isometries of M_i by

$$\alpha_i(x, y) =(x + i^{-2}, R(2\pi/i)y);$$
$$\beta_i(x, y) =(x + i^{-3}, R(2\pi/i)y).$$

As $i \to \infty$, both $\langle\alpha_i\rangle$-action and $\langle\beta_i\rangle$-action converges to standard $\mathbb{R}$-translations in the limit space $\mathbb{R}$, because S^3-factor disappears in the limit. Now we rescale this sequence by $r_i = i$. Then $r_iM_i = \mathbb{R} \times (S^3, g_0)$, on which α_i and β_i acts as

$$\alpha_i(x, y) =(x + i^{-1}, R(2\pi/i)y);$$
$$\beta_i(x, y) =(x + i^{-2}, R(2\pi/i)y).$$

It is clear that

$$(r_iM_i, x_i, \langle\alpha_i\rangle, \langle\beta_i\rangle) \xrightarrow{GH} (\mathbb{R} \times S^3, x', \mathbb{R}, \mathbb{R} \times S^1).$$

The limit group of $\langle\alpha_i\rangle$ is $\mathbb{R}$ acting as

$$t \cdot (x, y) = (x + t, R(2\pi t)y), \ t \in \mathbb{R},$$

while the limit group of $\langle\beta_i\rangle$ has an extra dimension. This extra dimension comes from a sequence of collapsed almost subgroups in $\langle\beta_i\rangle$. More precisely, if we put $B_i = \{e, \beta_i^{\pm 1}, ..., \beta_i^{\pm(i-1)}\}$, then on (M_i, x_i) we have $D_{1,x_i}(B_i) \to 0$ and

$$\frac{d_H(B_ix_i, B_i^2x_i)}{\mathrm{diam}(B_ix_i)} \to 0.$$

On $(M_i, x_i, \langle\alpha_i\rangle)$, there is no such small almost subgroup. We can take the same symmetric subsets $A_i = \{e, \alpha_i^{\pm 1}, ..., \alpha_i^{\pm(i-1)}\}$. Although A_i satisfies

$$d_H(A_ix_i, A_i^2x_i) \to 0, \quad D_{1,x_i}(A_i) \to 0,$$

the ratio is away from 0 for all i

$$\frac{d_H(A_ix_i, A_i^2x_i)}{\mathrm{diam}(A_ix_i)} \geq 1/2\pi.$$

The proof of Theorem 6.1.1 is technical and involved. We have illustrated on how to rule out $G = \mathbb{R}$ with $G' = \mathbb{R} \times S^1$ in the introduction. Here we give some indications on how to rule out $G = \mathbb{R}$ with $G' = \mathbb{R}^2$. Suppose that G'-action is standard translation for simplicity. One may consider a parameter s changing the scale from 1 to r_i as $1 + s(r_i - 1)$, $s \in [0, 1]$. In this way, one may imagine that there is a path, consisting of intermediate rescaling limits, and varying from $\mathbb{R}$-action to $\mathbb{R}^2$-translation. Then we can find an intermediate rescaling sequence $s_i \to \infty$ with $r_i/s_i \to \infty$ and

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (Y, q, H),$$

where H-action is very close to $\mathbb{R}^2$-translation in the equivariant Gromov-Hausdorff topology but $H \neq \mathbb{R}^2$. If $H = \mathbb{R} \times \mathbb{Z}$, then we can apply a scaling trick to rule it out (See proof of Proposition 6.3.1(1) later for details). If $H = \mathbb{R} \times S^1$, then we result in the case that we know can not happen. The situation that needs some additional arguments is $H = \mathbb{R}$, whose action is very close to $\mathbb{R}^2$-translation. We take a closer look at such an $\mathbb{R}$-action.

Example 6.1.3. Consider $M_i = \mathbb{R} \times (S^1, i \cdot d_0)$ and $\mathbb{R}$ acting on M_i by

$$t(x, y) = (x + t/i, R(2\pi t)y), \quad t \in \mathbb{R}.$$

Then $d_{GH}((M_i, x_i, \mathbb{R}), (\mathbb{R}^2, 0, \mathbb{R}^2)) \leq 2/i$.

Note that in this particular example, $\mathbb{R}$-action on M_i has almost subgroups. For $A = [-1, 1] \subseteq \mathbb{R}$, we have

$$\frac{d_H(Ax_i, A^2 x_i)}{\mathrm{diam}(Ax_i)} \leq 1/(2\pi i^2).$$

A key observation is that such phenomenon also happens in the general case: if a $\mathbb{R}$-action is very close to some $\mathbb{R}^2$-action, then it must contain some almost subgroup (See Lemma 6.3.7). With this observation, we can rule out such intermediate rescaling sequence from the no small almost subgroup assumption.

We start with some definitions.

Definition 6.1.4. Let G be a Lie group. We say that a symmetric subset A of G is one-parameter, if A has one of the following forms:

I. $A = \{e, g^{\pm 1}, ..., g^{\pm k}\}$ for some $g \in G$ and $k \in \mathbb{Z}^+$;

II. $A = \{\exp(tv) \mid t \in [-1, 1]\}$ for some $v \in \mathfrak{g}$, the Lie algebra of G.

Definition 6.1.5. Let $\eta > 0$ and (Y, q, G) be a space. We say that G-action has no η-subgroup of one-parameter at $y \in Y$, if for any one-parameter symmetric subset $A \subset G$ with $\operatorname{diam}(Ay) > 0$, we have

$$\frac{d_H(Ay, A^2y)}{\operatorname{diam}(Ay)} \geq \eta.$$

Lemma 6.1.6. *Let $(Y, q, \mathbb{R})$ be a space. Then $\mathbb{R}$ contains a one-parameter symmetric subset A of form I with*

$$\frac{d_H(Aq, A^2q)}{\operatorname{diam}(Aq)} < \eta$$

if and only if it contains a one-parameter symmetric subset B of form II with

$$\frac{d_H(Bq, B^2q)}{\operatorname{diam}(Bq)} < \eta.$$

Proof. Suppose that $\mathbb{R}$ contains a one-parameter symmetric subset A of form I with

$$\frac{d_H(Aq, A^2q)}{\operatorname{diam}(Aq)} < \eta.$$

We write A as $\{e, g^{\pm 1}, ..., g^{\pm k}\}$. For $g^{2k} \in A^2$, there is $g^n \in A$ with

$$d(g^{2k}q, g^nq) < \eta \cdot \operatorname{diam}(Aq).$$

Case 1: $n \geq 0$.

Since $g \in \mathbb{R}$, $g = \exp(v)$ for some $v \in \mathfrak{g} = \mathbb{R}$. Consider

$$B = \{\exp(tv) \mid t \in [-k, k]\}.$$

For any $s \in [0, k]$,

$$d(\exp((2k - s)v)q, \exp((n - s)v)q) < \eta \cdot \operatorname{diam}(Aq) \leq \eta \cdot \operatorname{diam}(Bq)$$

with $\exp((n - s)v) \in B$. Thus B is a one-parameter symmetric subset of form II with

$$\frac{d_H(Bq, B^2q)}{\operatorname{diam}(Bq)} < \eta.$$

Case 2: $n < 0$.

In this case, we have $2k - n > 2k$ and

$$d(g^{2k-n}q, q) = d(g^{2k}q, g^nq) < \eta \cdot \operatorname{diam}(Aq).$$

Now $A' := \{e, g^{\pm 1}, ..., g^{\pm(2k-n-1)}\}$ satisfies

$$\frac{d_H(A'q, (A')^2 q)}{\mathrm{diam}(A'q)} < \eta$$

and the condition in Case 1. By the same method as in Case 1, we are able to construct a desired subset B.

Conversely, if we have $B = \{ \exp(tv) \mid t \in [-1, 1]\}$ with

$$\frac{d_H(Bq, B^2 q)}{\mathrm{diam}(Bq)} < \eta.$$

For each positive integer k, define $B_k = \{\exp(\pm\frac{j}{k}v) \mid j = 0, \pm 1, ..., \pm k\}$. It is clear that $B_k q$ converges to Bq in the Hausdorff sense. Thus for k sufficiently large, $A := B_k$ is a one-parameter symmetric subset of form I with the desired property. □

6.2 Convergence of actions with no small almost subgroups

Under the assumption of no small almost subgroup, we recover some properties that were proved for non-collapsing manifolds in Chapter 5 (Cf. Theorems 5.1.1, 5.1.2 and Proposition 5.4.3).

Lemma 6.2.1. *Let (M_i, x_i) be a sequence of n-manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)$ and Γ_i be groups acting freely and isometrically on M_i. Suppose that Γ_i-action has no ϵ-small η-subgroup on $B_1(x_i)$ with scale 1 for some $\epsilon, \eta > 0$. We consider the convergent sequence*

$$(M_i, x_i, \Gamma_i) \xrightarrow{GH} (X, x, G).$$

Then G has no nontrivial subgroup H of $D_{1,x}(H) \leq \epsilon/2$.

Proof. Suppose that G has a nontrivial subgroup H with $D_1(H) \leq \epsilon/2$. Without lose of generality, we will assume that H is a closed subgroup, thus compact. Let γ be an non-identity element of finite order in H and K be the finite group generated by γ. It is clear that $D_1(K) \leq \epsilon/2$. Choose $\gamma_i \in \Gamma_i$ converging to γ, and define $A_i := \{e, f_i^{\pm 1}, ..., f_i^{\pm(k-1)}\}$, where k is the order of γ. We have

$$(M_i, x_i, A_i) \xrightarrow{GH} (X, x, K).$$

By Lemma 5.2.1, there is $q \in B_1(x)$ such that the K-orbit at q is not a single point. Choose $q_i \in B_1(x_i)$ converging to q. Then for i large, A_i has displacement $D_{1,q_i}(A_i) \le \epsilon$, but

$$\frac{d_H(A_i q_i, A_i^2 q_i)}{\mathrm{diam}(A_i q_i)} \to 0$$

as $i \to \infty$. A contradiction to the condition that Γ_i-action has no ϵ-small η-subgroup on $B_1(x_i)$. $\square$

Lemma 6.2.2. *Let (M_i, x_i) be a sequence of n-manifolds with $\mathrm{Ric}_{M_i} \ge -(n-1)$ and Γ_i be groups acting freely and isometrically on M_i. Suppose that Γ_i-action has no ϵ-small η-subgroup on $B_1(x_i)$ with scale $r \in (0,1]$ for some $\epsilon, \eta > 0$. Let f_i be a sequence of elements in Γ_i. We consider the following convergent sequences ($r_i \to \infty$):*

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, f_\infty);$$

$$(r_i M_i, x_i, f_i) \xrightarrow{GH} (X', x', f'_\infty).$$

(1) If $f'_\infty = \mathrm{id}$, then $f_\infty = \mathrm{id}$;

(2) If $f_\infty = \mathrm{id}$ and f'_∞ fixes x', then $f'_\infty = \mathrm{id}$.

Proof. We first prove (2). Suppose that $f'_\infty \neq \mathrm{id}$. Let H be the closure of the subgroup generated by f'_∞ in $\mathrm{Isom}(X')$. We choose a sequence of symmetric subsets A_i as follows:
If H is a finite group of order k, we put $A_i := \{e, f_i^{\pm 1}, ..., f_i^{\pm(k-1)}\}$;
If H is an infinite group, we put $A_i := \{e, f_i^{\pm 1}, ..., f_i^{\pm k_i}\}$ with $k_i \to \infty$ slowly so that

$$(M_i, x_i, A_i) \xrightarrow{GH} (X, x, \{e\}),$$

$$(r_i M_i, x_i, A_i) \xrightarrow{GH} (X', x', H).$$

By Lemma 5.2.1, there is $q \in B_1(x')$ such that H-orbit at q is not a single point. Thus at q, we have

$$\frac{d_H(Hq, H^2 q)}{\mathrm{diam}(Hq)} = 0.$$

Pick $q_i \in B_1(x_i) \subset M_i$ with

$$(r_i M_i, q_i) \xrightarrow{GH} (X', q).$$

It is clear that for i sufficiently large, we have $D_{1,x_i}(A_i) \leq \epsilon$, but

$$\frac{d_H(A_i q_i, A_i^2 q_i)}{\mathrm{diam}(A_i q_i)} < \eta/2.$$

This contradicts with that Γ_i-action has no ϵ-small η-subgroup at $B_1(x_i)$.

(1) follows from (2) by Lemma 6.2.1 and a rescaling trick as we applied in the proof of Proposition 5.4.3. □

Lemma 6.2.3. *Let (M_i, x_i) be a sequence of n-manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)$ and Γ_i be groups acting freely and isometrically on M_i. Suppose that Γ_i-action has no ϵ-small subgroup on $B_1(x_i)$ with scale 1 for some $\epsilon, \eta > 0$. If*

$$(M_i, x_i, \Gamma_i) \overset{GH}{\longrightarrow} (X, x, G)$$

with G being discrete, then there is $N > 0$ such that

$$\#\Gamma_i(1) \leq N < \infty.$$

Furthermore, if there is $D > 0$ such that $\mathrm{diam}(M_i) \leq D$, then Γ_i is isomorphic to G for all i large.

Proof. Notice that if $\gamma_i \to e$, then $\gamma_i = e$ for all i large. Indeed, because G is a discrete group, it is easy to see that the group generated by γ_i also converges to $\{e\}$. On the other hand, every nontrivial subgroup of Γ_i has displacement at least ϵ on $B_1(x_i)$. Therefore, $\gamma_i = e$. This implies that there is a large number i_0 such that if two sequences $\gamma_i \overset{GH}{\to} g$, $\gamma_i' \overset{GH}{\to} g$ with $g \in G(2)$, then $\gamma_i = \gamma_i'$ for all $i \geq i_0$. Thus for all $i \geq i_0$,

$$\#\Gamma_i(1) \leq \#G(2) < \infty.$$

When $\mathrm{diam}(M_i) \leq D$, there is a sequence of subgroups $H_i \overset{GH}{\to} G_0 = \{e\}$ such that Γ_i/H_i is isomorphic to $G/G_0 = G$ for all i large [14]. By assumptions, we conclude that $H_i = \{e\}$ for all i large and complete the proof. □

6.3 Free actions

We deal with a special case of dimension monotonicity in this section: G-action is free at x. Our goal is the following.

Proposition 6.3.1. *Theorem 6.1.1 holds when G action is free at x, that is,*

(1) $\dim(G') \leq \dim(G)$. *Moreover,*

(2) G' *has no nontrivial compact subgroups.*

It is direct to prove (2) in Proposition 6.3.1:

Proof of Proposition 6.3.1(2). Suppose that G' has a nontrivial compact subgroup K. Without lose of generality, we may assume that K is a finite group of prime order k. Let γ be a generator of K. We choose a sequence of elements $\gamma_i \in \Gamma_i$ converging to γ, and consider the symmetric subset $A_i = \{e, \gamma_i^{\pm 1}, ..., \gamma_i^{\pm(k-1)}\}$. Clearly

$$(r_i M_i, x_i, A_i) \xrightarrow{GH} (X, x, K).$$

Before rescaling r_i, since $\mathrm{diam}(A_i x_i) \to 0$ and G-action is free at x, we conclude that $A_i \to e$. By Lemma 6.2.2, K acts freely at x'. With respect to the metric $r_i M_i$,

$$\mathrm{diam}(A_i x_i) \to \mathrm{diam}(Kx) > 0.$$

Also $d_H(A_i x_i, A_i^2 x_i) \to 0$ because K is a subgroup. This gives $\dfrac{d_H(A_i x_i, A_i^2 x_i)}{\mathrm{diam}(A_i x_i)} \to 0$. However, $D_{1,x_i}(A_i) < \epsilon$ for i large. A contradiction to the no small almost subgroup assumption. $\square$

Corollary 6.3.2. *Under the assumptions of Proposition 6.3.1, G'-action is free.*

Proof. Otherwise, G' would have an isotropy subgroup, which is compact. $\square$

Lemma 6.3.3. *Under the assumptions of Proposition 6.3.1, G'-action on X' has no η-subgroup of one-parameter.*

Proof. Suppose that there is a point $q \in X'$ and a one-parameter symmetric subset A of G' with $\mathrm{diam}(Aq) < \infty$ and $\dfrac{d_H(Aq, A^2 q)}{\mathrm{diam}(Aq)} < \eta$.

By Corollary 6.3.2, G'-action is free. Thus $\mathrm{diam}(Aq) \in (0, \infty)$. Pick a sequence of symmetric subsets $A_i \subseteq \Gamma_i$ and a sequence of points $q_i \in B_1(x_i)$ such that

$$(r_i M_i, q_i, A_i) \xrightarrow{GH} (X', q, A).$$

By a similar argument we used in Proposition 6.3.1(2), before rescaling r_i, we have $D_{1,q_i}(A_i) \to 0$ but $\dfrac{d_H(A_i q_i, A_i^2 q_i)}{\mathrm{diam}(A_i q_i)} < \eta$ for i large. A contradiction. $\square$

Lemma 6.3.4. *Let (Y, q, G) be a space and g be an element in G. Suppose that $\langle g\rangle$-action is free at q and has no η-subgroup of one-parameter at q. If $d(q, gq) \geq r$ and $d(q, g^N q) \leq R$ for some N, then*

(1) $d(q, g^j q) \geq \eta r$ for all j. In particular, $\langle g\rangle q$ is ηr-disjoint;

(2) $d(q, g^j q) \leq \eta^{-1} R$ for all $-N < j < N$;

(3) there is a constant $C = C(n, \eta, r, R)$ such that $N \leq C$.

Proof. (1) If $d(q, g^j q) < \eta r$ for some j, we consider $A = \{e, g^{\pm 1}, ..., g^{\pm j}\}$. Then $\mathrm{diam}(Aq) \geq d(q, gq) \geq r$. Thus

$$\frac{d_H(Aq, A^2 q)}{\mathrm{diam}(Aq)} < \frac{\eta r}{r} = \eta.$$

A contradiction.

(2) This time we put $A = \{e, g^{\pm 1}, ..., g^{\pm N}\}$. Then

$$\mathrm{diam}(Aq) \leq \eta^{-1} d_H(Aq, A^2 q) \leq \eta^{-1} d(q, g^N q) = \eta^{-1} R.$$

(3) This follows from (1),(2), relative volume comparison (of limit renormalized measure) and a standard packing argument. □

Remark 6.3.5. To prove Lemma 6.3.4(3) only, the assumptions above can be weakened. Instead of assuming that $\langle g\rangle$-action has no η-subgroup of one-parameter at q, we can assume the following condition:

For every nontrivial symmetric subset B of $A = \{e, g^{\pm 1}, ..., g^{\pm N}\}$, we have

$$\frac{d_H(Bq, B^2 q)}{\mathrm{diam}(Bq)} \geq \eta.$$

Under this condition, we can show that the points $\{q, g^1 q, ..., g^N q\}$ are ηr-disjoint by the similar method. The remaining proof is the same.

Remark 6.3.6. If $Y \in \mathcal{M}(n, -1)$ is a limit space of a sequence of manifolds M_i with $\mathrm{Ric}_{M_i} \geq -(n-1)\epsilon_i \to 0$, then the constant C in Lemma 6.3.4 only depends on n, η and R/r.

We prove a key lemma for Proposition 6.3.1(1), which states that there exists an equivariant Gromov-Hausdorff distance gap between $\mathbb{R}^k$-actions with no almost subgroups and any $(\mathbb{R}^k \times \mathbb{Z})$-actions.

Lemma 6.3.7. *There exists a constant $\delta(n, \eta) > 0$ such that the following holds.*

Let (Y, q, G) be a space such that $G = \mathbb{R}^k$ and G-action has no η-subgroup of one-parameter at q. Let (Y', q', G') be another space with
(C1) G' contains $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup,
(C2) this $\mathbb{Z}$ subgroup has generator whose displacement at q' is less than 1.

Then

$$d_{GH}((Y, q, G), (Y', q', G')) > \delta(n, \eta).$$

Proof. Recall that we assume that $(Y, q) \in \mathcal{M}(n, -1)$, and it is clear that $k \leq n$. We first select a basis of $\mathbb{R}^k$ as follows. Fix any element $v_1 \neq e$ in $\mathbb{R}^k$. There is $t_1 > 0$ such that $d(t_1 v_1 q, q) = 1/n$ and $d(t v_1 q, q) < 1/n$ for all $t \in (0, t_1)$. Put $e_1 = t_1 v_1$ as the first element in the basis. Consider the quotient space $(Y/\mathbb{R}e_1, \bar{q}, \mathbb{R}^{k-1})$. Select an element $\bar{e}_2 \in \mathbb{R}^{k-1}$ such that $d(\bar{e}_2 \bar{q}, \bar{q}) = 1/n$ and $d(t\bar{e}_2 \bar{q}, \bar{q}) < 1/n$ for all $t \in (0, 1)$. $\bar{e}_2$ corresponds to a coset in $\mathbb{R}^k$. In this coset, choose e_2 such that $d(e_2 q, q) = d(\bar{e}_2 \bar{q}, \bar{q})$. By our choice of e_2, it is easy to see that $d(te_2 q, q) = d(t\bar{e}_2 \bar{q}, \bar{q})$ for all $t \in (0, 1)$. Continue this process until we obtain a basis $\{e_1, ..., e_k\}$ in $\mathbb{R}^k$.

We claim that the basis we choose has the following property: For $z = \sum_{j=1}^k \alpha_j e_j$ with $|\alpha_j| \leq 1$ for all j and $|\alpha_m| = 1$ for some m, we have $d(zq, q) \geq r(n, \eta)$, where $r(n, \eta) > 0$ is a small constant. In fact, first notice that by our choice of e_m,

$$d\left((\sum_{j=1}^{m} \alpha_j e_j) q, q \right) \geq d(e_j q, q) = 1/n.$$

If $d(\alpha_{m+1} e_{m+1} q, q) < 1/2n$, then clearly

$$d\left((\sum_{j=1}^{m+1} \alpha_j e_j) q, q \right) \geq 1/2n.$$

If $d(\alpha_{m+1} e_{m+1} q, q) \geq 1/2n$, by Lemma 6.3.4,

$$|\alpha_{m+1}| \geq \frac{1}{2C(n, \eta, 1/2n, 1/n)} =: r_1(n, \eta).$$

Consequently,

$$d\left((\sum_{j=1}^{m+1} \alpha_j e_j) q, q \right) \geq r_1(n, \eta).$$

Iterate this process at most $k-m-1(<n)$ times, we result in the desired estimate $d(zq,q)\geq r(n,\eta)$.

We set $\delta=1/100$ now and will further determine it later. Let $L=\langle e_1,..,e_k\rangle$ be the lattice generated by $e_1,...,e_k$. Notice that Lq is 1-dense in the orbit Gq. Let $e'_j\in G'$ be an element δ-close to e_j $(j=1,...,k)$. Let $L':=\langle e'_1,...,e'_k\rangle$ be the subgroup of G' generated by these elements. Notice that conditions *(C1)(C2)* guarantee that there is $w'\in G'$ such that $d(w'q',q')=d(w'q',L'q')\in(8,10)$. Let $w\in G=\mathbb{R}^k$ be the element δ-close to w'. Since Lq is 1-dense in Gq, there is $v\in L$ such that $d(v,w)<1$. We write $v=\sum_{j=1}^k\beta_je_j$ $(\beta_j\in\mathbb{Z})$. Put $M:=\max_j(|\beta_j|)$ and $z=\frac{1}{M}v$. Then $z=\sum_{j=1}^k\alpha_je_j$ with $|\alpha_j|\leq 1$ for all j and $|\alpha_m|=1$ for some m. By our choice of $\{e_1,...,e_k\}$, we have $d(zq,q)\geq r(n,\eta)$. Also, $d(Mzq,z)\leq 12$. Apply Lemma 6.3.4, we conclude that $M\leq C_0(n,\eta)$. Consequently, if we set δ with $nC_0(n,\eta)\delta\leq 1/100$, then $v':=\sum_{j=1}^k\beta_je'_j$ is 1/100-close to v. This leads to a contradiction because $d(v'q',L'q')>6$. □

Remark 6.3.8. Inspecting the proof above, we see that only property (3) in Lemma 6.3.4 is applied. Hence we may replace the condition that $\mathbb{R}^k$-action has no η-subgroup of one-parameter at q by the following one:

There exists a function $C(r,R)>0$ such that for all $z\in\mathbb{R}^k$ with $d(zq,q)\geq r$ and $d(Nzq,q)\leq R$, we have $N\leq C(r,R)$.

Correspondingly, the equivariant Gromov-Hausdorff distance gap δ will depend on n and the function C.

Remark 6.3.9. Another observation on the proof of Lemma 6.3.7 is that, we find a contradiction when two orbits Gq and $G'q'$ are close. Therefore, only the properties of the orbits at the base points matter in this proof.

Remark 6.3.10. Later in Section 6.5, we will generalize Lemma 6.3.7 to the case when G is a nilpotent Lie group diffeomorphic to $\mathbb{R}^k$.

Lemma 6.3.11. *Under the assumption of Proposition 6.3.1, for any $s_j\to\infty$, passing to a subsequence if necessary, we consider a tangent cone at x:*

$$(s_jX,x,G)\xrightarrow{GH}(C_xX,v,G_x).$$

Then $G_x = \mathbb{R}^{\dim(G)}$.

Proof. We prove the case $G = \mathbb{R}^k$. For the general case, we consider pseudo-action instead and the proof is similar. We know that G_x has no nontrivial compact subgroups from Proposition 6.3.1(2). It is also clear that G_x contains $\mathbb{R}^k$. As a result, if G_x is not $\mathbb{R}^k$, it must contain $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup. To prove that $G_x = \mathbb{R}^k$, it is enough to show the following: There is $\delta_0 > 0$, which depends on (X, x, G), such that for any $s \geq 1$ and for any space (Y', q', G') with

(C1) G' contains $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup,

(C2) this $\mathbb{Z}$ subgroup has generator whose displacement at q' is less than 1,

then

$$d_{GH}((sX, x, G), (Y', q', G')) > \delta_0.$$

By Remark 6.3.8, it suffices to prove the following claim.

Claim: There exists a function $C(r, R)$ such that for any $\tau \in (0, 1]$ and any $z \in \mathbb{R}^k$ with $d(zx, x) \geq \tau r$ and $d(Nzx, x) \leq \tau R$, we have $N \leq C(r, R)$.

For $r > 0$, we define

$$A(r) = \{v \in \mathbb{R}^k \mid d(vx, x) = r,\ d(tvx, x) \leq r \text{ for all } 0 < t < 1\},$$

It is clear that $A(r)$ is compact. For $R > 0$, we define a function on $A(r)$:

$$\begin{aligned} F_{r,R} : A(r) \to \quad & \mathbb{R}^+ \\ v \ \mapsto & \sup\{t > 0 \mid d(tvx, x) = R\}. \end{aligned}$$

Since $\mathbb{R}^k$ is a closed subgroup, $F_{r,R}(v)$ exists and is finite for each $v \in A(r)$. Though $F_{r,R}$ may not be continuous in general, we can check that it is always upper semi-continuous. In fact, given $v_j \in A(r)$ with $v_j \to v$, we put $t_j = F_{r,R}(v_j)$ for simplicity. Then $d(t_j v_j x, x) = R$ and $d(tv_j x, x) > R$ for all $t > t_j$. It is clear that $\limsup_{j\to\infty} t_j < \infty$. Since $d(tvx, x) \geq R$ for all $t > \limsup_{j\to\infty} t_j$, we conclude that $\limsup_{j\to\infty} t_j \leq F_{r,R}(v)$. Let $M_{r,R} < \infty$ be the maximum of $F_{r,R}$ on $A(r)$. If we have $z \in \mathbb{R}^k$ with $d(zx, x) \geq r$ and $d(Nzx, x) \leq R$, then $N \leq M_{r,R}$. By our construction of $F_{r,R}$, we see that

$$M_{\tau r, \tau R} \leq M_{\tau_0 r, R}$$

for all $\tau \in [\tau_0, 1]$. Here $\tau_0 > 0$ is a very small number that will be determined later. This shows that claim holds for $\tau \in [\tau_0, 1]$. It remains to prove that claim also holds when $\tau \in (0, \tau_0]$ for sufficiently small τ_0.

We further define

$$\Omega(R) = \{tv \mid t \in [0,1],\ v \in \mathbb{R}^k \text{ with } d(vx, x) = R$$
$$\text{and } d(svx, x) > R \text{ for all } s > 1\}.$$

Observe that $D_1(\Omega(R)) \to 0$ as $R \to 0$. So there is τ_0 small such that

$$D_1(\Omega(R(\tau))) < \epsilon$$

for all $\tau \leq \tau_0$. By assumptions, for any symmetric subset $B \neq \{e\}$ of $\Omega(R(\tau_0))$, we have

$$\frac{d_H(Bx, B^2x)}{\mathrm{diam}(Bx)} \geq \eta.$$

By Lemma 6.3.4, Remarks 6.3.5 and 6.3.6, there is some constant $C_0(n, \eta, r, R)$ such that the claim holds for $\tau \in (0, \tau_0]$. Put $C(r, R) = \max\{C_0(n, \eta, r, R), M_{\tau_0 r, R}\}$ and we proves the claim. □

Now we prove Proposition 6.3.1(1) by induction on $\dim(G)$.

Proof of Proposition 6.3.1(1). We first show that statement holds when $\dim(G) = 0$. In this case, we claim that $G' = \{e\}$. In fact, suppose that G' has an nontrivial element g', then we pick $\gamma_i \in \Gamma_i$ converging to g'. Because G-action is free at x, before rescaling $\gamma_i \to e \in G$. By Lemma 6.2.3, $\gamma_i = e$ for i large. Hence γ_i can not converge to $g' \neq e$ after rescaling.

Assuming the statement also holds for $\dim(G) = 1, ..., k-1$, we verify the case $\dim(G) = k$.

We make the following reductions: By a standard rescaling and diagonal argument, we may assume that

$$(t_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (C_x X, v, G_x)$$

with $t_i \to \infty$ and $r_i/t_i \to 0$. Here $C_p X$ is a tangent cone at x and $G_p = \mathbb{R}^k$. By Lemma 6.3.3, G_x-action has no η-subgroup of one-parameter. Now we replace

$$(M_i, x_i, \Gamma_i) \xrightarrow{GH} (X, x, G)$$

by

$$(t_iM_i, x_i, \Gamma_i) \xrightarrow{GH} (C_xX, v, \mathbb{R}^k)$$

and continue the proof.

We know that $G'_0 = \mathbb{R}^l$ because it is abelian and it has no nontrivial compact subgroup. We show that $l \leq k$. Suppose that the contrary holds. In other words, G' contains $\mathbb{R}^{k+1}$ as a closed subgroup. Then G' would contain $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup. Scaling the sequence down by a constant, we may assume that for this $\mathbb{Z}$ subgroup, its generator has displacement at x' less than 1.

Put $\delta(n, \eta) > 0$ as the constant in Lemma 6.3.7. For each i, consider the following set of scales

$$\begin{aligned} S_i := \{\ 1 \leq s \leq r_i \mid d_{GH}((sM_i, x_i, \Gamma_i), (Y, q, H)) \leq \delta/3 \text{ for some space } (Y, q, H) \\ \text{satisfying the following conditions} \\ \textit{(C1) } H \text{ contains } \mathbb{R}^k \times \mathbb{Z} \text{ as a closed subgroup,} \\ \textit{(C2) } \text{this } \mathbb{Z} \text{ subgroup has generator whose displacement} \\ \text{at } q \text{ is less than } 1.\} \end{aligned}$$

S_i is nonempty for i sufficiently large because $r_i \in S_i$. Pick critical rescalings $s_i \in S_i$ with $\inf S_i \leq s_i \leq \inf S_i + 1/i$.

Step 1: $s_i \to \infty$.

Otherwise passing to a subsequence, $s_i \to s < \infty$. Then

$$(s_iM_i, x_i, \Gamma_i) \xrightarrow{GH} (sX, x, \mathbb{R}^k).$$

Since $s_i \in S_i$, for each i, there is (Y_i, q_i, H_i) with *(C1)(C2)* and for i large,

$$d_{GH}((Y_i, q_i, H_i), (sX, x, \mathbb{R}^k)) \leq \delta/2.$$

This would contradict Lemma 6.3.7 because $\mathbb{R}^k$-action on sX has no η-subgroup of one-parameter.

Step 2: $r_i/s_i \to \infty$.

If $r_i/s_i \leq C$ for some $C \geq 1$. Then consider

$$(\frac{r_i}{2C}M_i, x_i, \Gamma_i) \xrightarrow{GH} (\frac{1}{2C}X', x', G').$$

This would imply that $r_i/2C \in S_i$ for i large. This contradicts to $r_i/\inf(S_i) \leq C$.

Step 3: Passing to a subsequence, $(s_iM_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty)$, where H_∞ contains $\mathbb{R}^k$ as a proper closed subgroup.

By Proposition 6.3.1(2), H_∞ has no nontrivial compact subgroups and thus $(H_\infty)_0 = \mathbb{R}^m$. If $m < k$, we consider

$$(s_iM_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty)$$

and its rescaling sequence $(r_i/s_i \to \infty)$

$$(r_iM_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x', G')$$

with G' containing $\mathbb{R}^{k+1}$ $(k > m)$. This contradicts with the induction assumptions. It remains to rule out the case $H_\infty = \mathbb{R}^k$ to finish Step 3. By Lemma 6.3.3, H_∞-action has no η-subgroup of one-parameter. Together with the fact that $s_i \in S_i$, *(C2)* and Lemma 6.3.7, we can rule out this case.

Step 4: We claim that H_∞ contains $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup. If this claim holds, we draw a contradiction as follows. Let h be the generator of this $\mathbb{Z}$ subgroup. We also know that h moves q_∞. Put $l = d(hq_\infty, q_\infty) > 0$. If $l \leq 1$, then we choose $t_i = s_i/2 \to \infty$. Then

$$(t_iM_i, x_i, \Gamma_i) \xrightarrow{GH} (\frac{1}{2}Y_\infty, q_\infty, H_\infty).$$

Hence $t_i \in S_i$ for i sufficiently large. But $t_i < \inf(S_i)$, which is a contradiction. If $l > 1$, then we put $t_i = s_i/2l$ and we will result in a similar contradiction.

It remains to verify the claim that H_∞ contains $\mathbb{R}^k \times \mathbb{Z}$ as a proper closed subgroup. From Step 3, we know that H_∞ contains $\mathbb{R}^k$. If $\dim(H_\infty) > k$, since H_∞ is abelian and has no nontrivial compact subgroups, then H_∞ contains $\mathbb{R}^{k+1}$ and the claim follows. If $\dim(H_\infty) = k$, then H_∞ contains $\mathbb{R}^k \times \mathbb{Z}$ by Proposition 6.3.1(2). $\square$

Remark 6.3.12. Note that through the proof above, we also eliminate $G = \mathbb{R}^k$ while $G' = \mathbb{R}^k \times \mathbb{Z}$. The proof of Proposition 6.3.1(1) is a prototype for the proof of the general case. Here we choose a critical rescaling sequence with limit $(Y_\infty, q_\infty, H_\infty)$, then make use of Proposition 6.3.1(2), Lemmas 6.3.3 and 6.3.7 to rule out every possibility of

$(Y_\infty, q_\infty, H_\infty)$. When dealing with general G-action, we will first extend Proposition 6.3.1(2) and Lemma 6.3.3 (See Proposition 6.4.1 and Lemma 6.5.1), then apply a similar critical rescaling argument as the proof of Proposition 6.3.1(1).

6.4 Compact subgroups in G'

We look into the compact subgroups of G' and prove Theorem 6.1.1(2) in this section. By Proposition 6.3.1(2), we know that if G' has nontrivial compact subgroups, then G-action has nontrivial isotropy subgroups at x. We restate Theorem 6.1.1(2) here for convenience:

Proposition 6.4.1. *Suppose that G' contains a compact subgroup K' with $K'_0 = \mathbb{T}^l$. Then G contains a subgroup K fixing x with $K_0 = \mathbb{T}^l$ and K/K_0 being isomorphic to K'/K'_0.*

Remark 6.4.2. Actually one can show that K is isomorphic to K', but Proposition 6.4.1 is sufficient for applications. In fact, even $\#K/K_0 \geq \#K'/K'_0$ will be sufficient.

Lemma 6.4.3. *Suppose that $(M_i, x_i, f_i) \overset{GH}{\longrightarrow} (X, x, \mathrm{id})$ and $f_i \in \Gamma_i$. Let $r_i \to \infty$ be a rescaling sequence. After passing to a subsequence, we have $(r_i M_i, x_i, f_i) \overset{GH}{\longrightarrow} (X', x', f)$. If $\overline{\langle f \rangle}$ is a compact group, then $f = e$.*

Proof. Suppose $f \neq e$. Since $(M_i, x_i, f_i) \overset{GH}{\longrightarrow} (X, x, \mathrm{id})$, there is a sequence $k_i \to \infty$ slowly such that $A_i := \{e, f_i^{\pm 1}, ..., f_i^{\pm k_i}\} \overset{GH}{\to} \{e\}$. But after rescaling r_i, the limit of A_i contains a compact subgroup $\overline{\langle f \rangle}$. By the same proof of Proposition 6.3.1(2), we see a contradiction to the no small almost subgroup assumption. □

Lemma 6.4.4. *Let $\mathcal{S}$ be a circle factor in G'_0, then there is a sequence of symmetric subsets $A_i \subseteq \Gamma_i$ such that*

$$(r_i M_i, x_i, A_i) \overset{GH}{\longrightarrow} (X', x', \mathcal{S}).$$

Before rescaling we have

$$(M_i, x_i, A_i) \overset{GH}{\longrightarrow} (X, x, A_\infty)$$

with A_∞ fixing x and containing a circle group.

Proof. Select an element $\gamma' \in \mathcal{S}$ such that $\overline{\langle \gamma' \rangle} = \mathcal{S}$, and a sequence $\gamma_i \in \Gamma_i$ with

$$(r_i M_i, x_i, \gamma_i) \xrightarrow{GH} (X', x', \gamma').$$

Put $A_i := \{e, \gamma_i^{\pm 1}, ..., \gamma_i^{\pm k_i}\}$, where $k_i \to \infty$ slowly such that

$$(r_i M_i, x_i, A_i) \xrightarrow{GH} (X', x', \mathcal{S}).$$

Before rescaling r_i, let A_∞ be the limit of A_i and γ be the limit of γ_i. By no small almost subgroup assumption, A_∞ satisfies $D_1(A_\infty) \geq \epsilon$. In particular, $A_\infty \neq \{e\}$. Moreover, A_∞ fixes x because after rescaling $\mathrm{diam}(\mathcal{S}x') < \infty$. We claim that γ has infinite order. In fact, suppose that γ has finite order. Let N be the order of $\langle \gamma \rangle$, then

$$(M_i, x_i, \gamma_i^N) \xrightarrow{GH} (X, x, \mathrm{id}).$$

But after rescaling r_i, we have

$$(r_i M_i, x_i, \gamma_i^N) \xrightarrow{GH} (X', x', (\gamma')^N).$$

Since $(\gamma')^N \neq e$, by Lemma 6.4.3 we result in a contradiction.

Since γ has infinite order and $\overline{\langle \gamma \rangle}$ is contained in the isotropy subgroup at x, we know that $\overline{\langle \gamma \rangle}$ is compact and thus contains a circle subgroup S^1. It is clear that A_∞ contains this circle. □

Lemma 6.4.5. *Let $\mathbb{T}^l$ be a torus subgroup of G'. Then G also contains $\mathbb{T}^l$, whose action fixes x.*

Proof. Let $\mathcal{S}_j$ $(j = 1, ..., l)$ be the j-th circle factor in $\mathbb{T}^l$. For each j, by the proof of lemma 6.4.4, we can choose symmetric subsets $A_{i,j} \subseteq \Gamma_i$ with the following properties:
(1). $(r_i M_i, x_i, A_{i,j}) \xrightarrow{GH} (X', x', \mathcal{S}_j)$;
(2). $A_{i,j}$ is generated a single element $\gamma_{i,j}$, $A_{i,j} = \{e, \gamma_{i,j}^{\pm 1}, ..., \gamma_{i,j}^{\pm k_{i,j}}\}$;
(3). $(M_i, x_i, A_{i,j}) \xrightarrow{GH} (X, x, A_{\infty,j})$ with $A_{\infty,j}$ fixing x and containing a circle S^1.

We claim that the set $\cup_{j=1}^l A_{\infty,j}$ contains l independent circles. We argue this by induction on j. By property (3), the claim holds for $l = 1$. Assuming it holds for l, we consider the case $l+1$. By induction assumption, $\cup_{j=1}^l A_{\infty,j}$ contains l independent circles and $A_{\infty,l+1}$ contains an additional circle. Suppose that $\cup_{j=1}^{l+1} A_{\infty,j}$ does not have $l+1$ independent circles, then $A_{\infty,l+1} \subset \mathbb{T}^l$, where $\mathbb{T}^l$ is the torus generated by l independent circles in $\cup_{j=1}^{l} A_{\infty,j}$. Recall that $A_{i,j+1}$ is generated by $\gamma_{i,j+1}$ with property (2). Since $\gamma_{j+1} \in \mathbb{T}^l$, there exists a sequence $\beta_i = \prod_{j=1}^{l} \gamma_{i,j}^{p_{i,j}}$ such that $|p_{i,j}| \leq k_{i,j}$ and

$$(M_i, x_i, \beta_i) \xrightarrow{GH} (X, x, \gamma_{j+1}).$$

After rescaling r_i,

$$(r_i M_i, x_i, \beta_i) \xrightarrow{GH} (X', x', \beta').$$

By our choice of β_i, its limit β' is outside $\mathcal{S}_{l+1}$. Now consider the sequence $z_i = \beta_i^{-1}\gamma_{i,j+1}$. Before rescaling $z_i \overset{GH}{\to} e$, while after rescaling r_i,

$$z_i \overset{GH}{\to} z' = (\beta')^{-1}\gamma'_{j+1} \neq e.$$

However, $\overline{\langle z' \rangle}$ is a compact group, which is a contradiction to Lemma 6.4.3. $\square$

For finite subgroups of G', there is a similar property.

Lemma 6.4.6. *Let F' be a finite group of G', then G contains a subgroup isomorphic to F', whose action fixes x.*

Proof. Let $g'_1, ..., g'_k$ be a set of generators of F'. We present F' as

$$\langle g'_1, ..., g'_k | R_1, ..., R_l \rangle,$$

where $R_1, ..., R_l = e$ are relations among these generators. For each generator g'_j, there is sequence $\gamma_{i,j} \in \Gamma_i$ such that

$$(r_i M_i, x_i, \gamma_{i,j}) \xrightarrow{GH} (X', x', g'_j).$$

Before rescaling, passing to a subsequence if necessary, we have

$$(M_i, x_i, \gamma_{i,j}) \xrightarrow{GH} (X, x, g_j).$$

In this way, we obtain k elements $g_1, ..., g_k$ in G. Let F be the subgroup generated by these k elements. It is clear that F-action fixes x. We show that F is isomorphic to F'.

Let W be a word consisting of $g_1, ..., g_k$. Correspondingly, we have words $W_i \in \Gamma_i$ and $W' \in G'$ of the same form. Clearly,

$$(M_i, x_i, W_i) \xrightarrow{GH} (X, x, W);$$

$$(r_i M_i, x_i, W_i) \xrightarrow{GH} (X, x, W').$$

Recall that W' generates a finite group. Thus by Lemmas 6.4.3 and 6.2.2(1), $W = e$ if and only if $W' = e$. This shows that F and F' has the same presentation. $\square$

We prove Proposition 6.4.1.

Proof of Proposition 6.4.1. By Lemma 6.4.5, we know that G has a compact subgroup $\mathbb{T}^l$. We pick a set of generators $\bar{g}'_j$ $(j = 1, ..., k)$ of K'/K'_0 and present the group K'/K'_0 as

$$\langle \bar{g}'_1, ..., \bar{g}'_k | \bar{R}_1, ..., \bar{R}_l \rangle,$$

For each $j = 1, ..., k$, let $g'_j \in K'$ be an element representing $\bar{g}'_j \in K'/K'_0$. Then on K', the relations $\bar{R}_1, ..., \bar{R}_l = e$ are equivalent to $R_1, ..., R_k \in K'_0$, where $R_1, ..., R_k$ are the words of the same form as $\bar{R}_1, ..., \bar{R}_l$ (replace $\bar{g}'_j$ by g'_j). By a similar application of Lemmas 6.4.3 and 6.2.2 as we used in the proof of Lemma 6.4.6, we can find a desired subgroup K in G. $\square$

We finish this section by the following results on passing isotropy group to any tangent cone.

Lemma 6.4.7. *For $(M_i, x_i, \Gamma_i) \xrightarrow{GH} (X, x, G)$ and $s_j \to \infty$, passing to a subsequence if necessary we consider a tangent cone at x:*

$$(s_j X, x, G) \xrightarrow{GH} (C_p X, v, G_x).$$

If G is a compact group fixing x with $G_0 = \mathbb{T}^l$, then $(G_x)_0 = \mathbb{T}^l$ and

$$\#\pi_0(G_x) \leq \#\pi_0(G).$$

Proof. It is clear that G_x fixes v. We first prove the case $G = \mathbb{T}^l$. By Proposition 6.4.1, we know that G contains a subgroup isomorphic to G_x. Since $G = \mathbb{T}^l$, G_x has to be a subgroup of $\mathbb{T}^l$. We show that $(G_x)_0 = \mathbb{T}^l$, which will imply that $G_x = \mathbb{T}^l$. Suppose that $(G_x)_0 = \mathbb{T}^m$ with $m < l$. Notice that $G = \mathbb{T}^l$ contains exactly $2^l - 1$ many non-identity elements of order 2. From the sequence $\{(s_j X, x, G)\}_j$, we obtain $2^l - 1$ different sequences of elements with order 2 in G. It is clear that, passing to a subsequence if necessary, their limits are contained in $(G_x)_0$ and have order 2. On the other hand, $(G_x)_0 = \mathbb{T}^m$ has $2^m - 1$ many non-identity elements of order 2. Thus there must be two sequences $\{\alpha_{1,j}\}$, $\{\alpha_{2,j}\}$ such that

$$\alpha_{k,j} \neq e, \ \alpha_{k,j}^2 = e \ (k = 1, 2), \ \alpha_{1,j} \neq \alpha_{2,j}$$

but their limits are the same. Then $\beta_j = \alpha_{1,j}\alpha_{2,j} \neq e$ would converge to e. On the other hand, β_j has order 2; thus by the no small almost subgroup assumption, $D_1(\beta_j) \geq \epsilon > 0$. This is a contradiction.

For the general case, G may have multiple components. Apply the same argument above, we see that $(G_x)_0 = \mathbb{T}^l$; and thus the result follows from Proposition 6.4.1. $\square$

Remark 6.4.8. In Lemma 6.4.7, one can actually show that G_p is isomorphic to G. The current statement is sufficient for proving dimension monotonicity of symmetries.

Corollary 6.4.9. *For* $(M_i, x_i, \Gamma_i) \xrightarrow{GH} (X, x, G)$ *with* $G_0 = \mathbb{R}^k \times \mathbb{T}^l$, *and* $s_j \to \infty$, *passing to a subsequence if necessary, we consider a tangent cone at* x*:*

$$(s_j X, x, G) \xrightarrow{GH} (C_x X, v, G_x).$$

Then $G_x = \mathbb{R}^k \times K$, *where* K*-action fixes* v, $K_0 = \mathbb{T}^l$ *and*

$$\#\pi_0(K) \leq \#\pi_0(\mathrm{Iso}(x, G)).$$

Proof. We put K as the limit of $\mathrm{Iso}(x, G)$ with respect to the sequence

$$(s_j X, x, G) \xrightarrow{GH} (C_x X, v, G_x).$$

With Lemmas 6.3.11 and 6.4.7, it remains to check that G_x has the splitting $\mathbb{R}^k \times K$. In fact, note that $K \cap \mathbb{R}^k = e$ and $\langle \mathbb{R}^k, K \rangle = G_x$. Hence the splitting follows. $\square$

Remark 6.4.10. For a space (Y, q, H), as long as the orbit $H \cdot q$ is connected, we always have the splitting $H = \mathbb{R}^k \times \mathrm{Iso}(q, H)$.

6.5 Proof of dimension monotonicity

We complete the proof of Theorem 6.1.1 in this section. We make some reductions at first. By Lemma 6.4.9, a standard rescaling and diagonal argument, we may pass to a tangent cone of X at x and assume that $G = \mathbb{R}^k \times \mathrm{Iso}(x, G)$ (See Remark 6.4.10). We will always assume this reduction when proving Theorem 6.1.1(1).

For a space (X, x, G) with $G = \mathbb{R}^k \times \mathrm{Iso}(x, G)$, we define $\dim_R(G) = k$ and $\dim_T(G) = \dim(\mathrm{Iso}(x, G))$ as the dimension of $\mathbb{R}$-factors and torus factors in G respectively. We will prove Theorem 6.1.1 by a triple induction on $\dim_T(G)$, $\dim_R(G)$ and $\#\pi_0(G)$. Also note that the case $\dim_T(G) = 0$ with $\#G/G_0 = 1$ is proved as Proposition 6.3.1(1); and the case $\dim_R(G) = \dim_T(G) = 0$ follows from Lemma 6.2.3. When we say such G in the induction assumptions, we always mean that such limit group is possible to exist as the limit of (M_i^n, x_i, Γ_i) (for example, k is always no greater than n).

Triple induction:

- *Induction on $\#\pi_0(G)$:* Under the reductions, suppose that Theorem 6.1.1(1) holds when

  (1) $G_0 = \mathbb{R}^k \times \mathbb{T}^l$ with $\#\pi_0(G) \leq m$, or

  (2) $\dim_T(G) = l$ with $\dim_R(G) < k$, or

  (3) $\dim_T(G) < l$.

  Then it holds for $G_0 = \mathbb{R}^k \times \mathbb{T}^l$ with $\#\pi_0(G) = m + 1$.

- *Induction on* $\dim_R(G)$*:* Under the reductions, suppose that Theorem 6.1.1(1) holds when

  (1) $\dim_T(G) = l$ with $\dim_R(G) \leq k$, or

  (2) $\dim_T(G) < l$.

  Then it holds for $G = \mathbb{R}^{k+1} \times \mathbb{T}^l$.

- *Induction on* $\dim_T(G)$*:* Under the reductions, suppose that Theorem 6.1.1(1) holds for $\dim_T(G) \leq l$, then it holds for $G = \mathbb{T}^{l+1}$.

Applying these three inductions above repeatedly, we will eventually cover every possible G. More precisely, we start with the base case $\dim_R(G) = \dim_T(G) = 0$ (Lemma 6.2.3). Together with Proposition 6.3.1(1), induction on $\dim_R(G)$ and on $\#G/G_0$, we conclude that Theorem 6.1.1 holds for $G = \mathbb{R}^k \times F$, where F is a finite group fixing x. Then by induction on $\dim_T(G)$, we know it also holds for $G = S^1$. After that, apply inductions on $\dim_R(G)$ and on $\#\pi_0(G)$ again, and we cover the case $G = \mathbb{R}^k \times \mathrm{Iso}(x, G)$ with $\mathrm{Iso}(x, G)_0 = S^1$. We continue this process and finish the proof of Theorem 6.1.1(1).

All these three induction arguments are similar to the proof of Proposition 6.3.1(1): choose a critical rescaling sequence and rule out every possibility in the corresponding limit. To illustrate this strategy, we consider the case $G = \mathbb{R} \times S^1$ as an example. By Proposition 6.4.5, we know that G' has no torus of dimension > 1. Thus we need to rule out the cases like $G' = \mathbb{R}^3$ or $G' = \mathbb{R}^2 \times S^1$. In either case, G' contains $\mathbb{R}^2 \times \mathbb{Z}$ as a closed subgroup. For $\delta > 0$ small, we consider

$$\begin{aligned} S_i := \{\ 1 \leq s \leq r_i \mid\ & d_{GH}((sM_i, x_i, \Gamma_i), (Y, q, H)) \leq \delta \text{ for some space } (Y, q, H) \\ & \text{with } H\text{-action satisfying the following conditions} \\ & \textit{(C1) } H \text{ contains } \mathbb{R}^2 \times \mathbb{Z} \text{ as a closed subgroup },\\ & \textit{(C2) } \text{This } \mathbb{Z} \text{ subgroup has generator whose displacement} \\ & \quad \text{at } q \text{ is less than } 1. \end{aligned}$$

Pick $s_i \in S_i$ with $\inf(S_i) \leq s_i \leq \inf S_i + 1/i$. Assume $s_i \to \infty$ and we consider

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty).$$

Like step 4 in the proof of Proposition 6.3.1(1), if H_∞ contains $\mathbb{R}^2 \times \mathbb{Z}$ as a closed subgroup, then we will obtain a contradiction by scaling s_i down by a constant. One can also apply induction assumptions to rule out the cases like $H_\infty = \mathbb{R} \times F$ or $H_\infty = S^1$. If $H_\infty = \mathbb{R} \times S^1$ but S^1-action is free at q_∞, then we can apply the result in free case. The last case we want to eliminate is that $H_\infty = \mathbb{R} \times S^1$ with S^1-action fixing q_∞.

Here comes the distinction between general case and free case in Section 6.2: for general limit G-action, rescaling limit group H_∞-action may have η-subgroups at x'. The observation is that, if H_∞ contains a torus of the same dimension as $\dim_T(G)$ and this torus fixes x', then actions of $\mathbb{R}^k$ subgroups in G' should have no η-subgroups of one-parameter at x' (See Lemma 6.5.1 below for the precise statement). With this in hand, then together with an equivariant GH-distance gap between $(Y_\infty, q_\infty, H_\infty)$ and the spaces we used to define S_i (See Lemma 6.5.4), we can rule out the case $H_\infty = \mathbb{R}\times S^1$ when δ is sufficiently small.

Following this idea, we prove the lemma below.

Lemma 6.5.1. *Suppose that* $\mathrm{Iso}(x, G)$ *has identity component* $\mathbb{T}^l$*. Further suppose that* $\mathrm{Iso}(x', G')$ *contains a torus of dimension* l*, that is,* $G'_0 = \mathbb{R}^k \times \mathbb{T}^l$ *with* $\mathbb{T}^l$ *fixing* x' *(torus factor in* G'_0 *can not have dimension* $> l$ *by Lemma 6.4.5). Then* $\mathbb{R}^k$*-action on* X' *has no* η*-subgroup of one-parameter at* x'*.*

Remark 6.5.2. In Lemma 6.5.1, G' contains infinitely many subgroups isomorphic to $\mathbb{R}^k$, but their orbits at x' are exactly the same. Thus the condition that $\mathbb{R}^k$-action has no η-subgroup of one-parameter at x' has no ambiguity.

One may regard Lemma 6.5.1 as a generalization of Lemma 6.3.3, where $\mathrm{Iso}(x, G)$ is trivial (also compare with the proof of Lemma 6.4.5).

Lemma 6.5.3. *Let* η *be the constant in the no small subgroup group assumption and* $f_i \in \Gamma_i$*. Suppose that the following sequences converge* $(r_i \to \infty)$

$$(M_i, x_i, f_i) \xrightarrow{GH} (X, x, \mathrm{id})$$

$$(r_i M_i, x_i, f_i) \xrightarrow{GH} (X', x', f).$$

Then the following can NOT happen: for some integer k *and some point* $q \in X'$*,* $A_\infty = \{e, f^{\pm 1}, ..., f^{\pm k}\}$ *satisfies*

$$\frac{d_H(A_\infty q, A_\infty^2 q)}{\mathrm{diam}(A_\infty q)} < \eta.$$

Proof. $\mathrm{diam}(A_\infty q)$ is the denominator so $f \neq \mathrm{id}$. Put $A_i = \{e, f_i^{\pm 1}, ..., f_i^{\pm k}\}$, then

$$(M_i, x_i, A_i) \xrightarrow{GH} (X, x, \{e\})$$

and

$$(r_i M_i, x_i, A_i) \xrightarrow{GH} (X', x', A_\infty).$$

Clearly this contradicts with *(P1).* □

Proof of Lemma 6.5.1. Suppose that $\mathbb{R}^k$-action has η-subgroup of one-parameter at x'. We will show that G contains $\mathbb{T}^{l+1}$, which contradicts with the assumption.

We follow the proof of Lemma 6.4.5. For each circle factor $\mathcal{S}_j$ in G' ($j = 1, ..., k$), we can pick $A_{i,j} = \{e, \gamma_{i,j}^{\pm 1}, ..., \gamma_{i,j}^{\pm k_{i,j}}\} \subset \Gamma_i$ with properties (1)-(3) as in the proof of Lemma 6.4.5. We also know that $\{A_{\infty,j}\}_{j=1}^{l}$ contains l independent circles.

Since $\mathbb{R}^k$-action has η-subgroup of one-parameter at x', it contains a one-parameter symmetric subset $\mathcal{T}$ such that

$$\frac{d_H(\mathcal{T}x', \mathcal{T}^2 x')}{\mathrm{diam}(\mathcal{T}x')} < \eta.$$

By Lemma 6.1.6, we may assume that $\mathcal{T}$ has form II, and thus we may write $\mathcal{T}$ as $\{tg \mid t \in [-1, 1]\}$. Put $F := \pi_0(\mathrm{Iso}(x, G))$, which is a finite group. We choose a large integer m_0 such that $\frac{1}{m_0}g$ satisfies the following property: for each integer $N = 1, ..., \#F + 1$, $\mathcal{T}_N := \{e, \frac{N}{m_0}g, \frac{2N}{m_0}g, ..., \frac{k_N N}{m_0}g\}$ satisfies

$$\frac{d_H(\mathcal{T}_N x', (\mathcal{T}_N)^2 x')}{\mathrm{diam}(\mathcal{T}_N x')} < \eta,$$

where k_N is the largest integer with $k_N N \leq m_0$.

Choose $f_i \in \Gamma_i$ with

$$(r_i M_i, x_i, f_i) \xrightarrow{GH} (X', x', \frac{1}{m_0} g).$$

Let f be a limit of f_i before rescaling. By Lemma 6.5.3, the know that $f^N \neq e$ for all $N = 1, ..., \#F + 1$.

Claim : For all $N = 1, ..., \#F + 1$, f^N is outside the torus subgroup generated by l independent circles in $\cup_{j=1}^{l} A_{\infty,j}$.

Suppose that f^N is in the group generated by l independent circles in $\{A_{\infty,j}\}_{j=1}^{l}$. Then there is $\beta_i = \prod_{j=1}^{l} \gamma_{i,j}^{p_{i,j}}$ with $|p_{i,j}| \leq k_{i,j}$ such that

$$(M_i, x_i, \beta_i) \xrightarrow{GH} (X, x, f^N).$$

After rescaling r_i,

$$(r_i M_i, x_i, \beta_i) \xrightarrow{GH} (X', x', \beta')$$

with $\beta' \in \mathbb{T}^l$. We consider $z_i = \beta_i^{-1} f_i^N$. It is clear that $z_i \overset{GH}{\to} e$, while after rescaling $r_i \to \infty$, $z_i \overset{GH}{\to} z' \neq e$ because $\beta' \in \mathbb{T}^l$ and $\frac{N}{m_0} g$ is in some closed $\mathbb{R}$ subgroup. Put

$C = \{e, z^{\pm 1}, ..., z^{\pm k}\}$. Since $\mathbb{T}^l$-action fixes x', the orbit Cx' is identically the same as $\mathcal{T}_N x'$. Apply Lemma 6.5.3 and we obtain the desired contradiction.

Finally, notice that all these f^N ($N = 1, ..., \#F + 1$) lie in $\mathrm{Iso}(x, G)$, which consists of exactly $\#F$ connected components, so there must be some N such that $f^N \in G_0$. Together with the Claim we just showed, we see that $\mathrm{Iso}(x, G)$ contains $\mathbb{T}^{l+1}$, which is a contradiction. $\square$

Besides Lemma 6.5.1, another ingredient is an equivariant Gromov-Hausdorff gap like Lemma 6.3.7. Actually here we only need to modify the statement of Lemma 6.3.7, because in its proof, we only used the properties of G-orbit at q (Remark 6.3.9).

Lemma 6.5.4. *There exists a constant $\delta(n, \eta) > 0$ such that the following holds.*

Let (Y, q, G) be a space with $G = \mathbb{R}^k \times \mathrm{Iso}(q, G)$. Suppose that $\mathbb{R}^k$-action on Y has no η-subgroup of one-parameter at q. Let (Y', q', G') be another space with

(C1) G' contains $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup,

(C2) this $\mathbb{Z}$ subgroup has generator whose displacement at q' is less than 1.

Then

$$d_{GH}((Y, q, G), (Y', q', G')) > \delta(n, \eta).$$

With all these preparations, we start the triple induction described in the beginning of this section. We begin with the easiest one among these three: induction on $\dim_T(G)$. Actually for this one, we do not even need the preparations above.

Proof of Induction on $\dim_T(G)$. Under the reductions, assuming that the Theorem 6.1.1 holds when $\dim_T(G) \leq l$, we need to verify the case $G = \mathbb{T}^{l+1}$ with G fixing x. Out goal is to rule out $\dim(G') > l + 1$. We argue by contradiction, suppose that for some $r_i \to \infty$ and some convergent subsequence

$$(r_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x', G'),$$

we have $\dim(G') > l + 1$.

By Lemma 6.4.5, we know that G' can not contain a torus of dimension $> l + 1$. As a result, if $\dim(G') > l + 1$, then G' contains a closed $\mathbb{R}$ subgroup, and thus contains a closed $\mathbb{Z}$ subgroup.

For $\delta = 1/10$, we consider the following set of scales for each i,

$$S_i := \{\ 1 \le s \le r_i \mid d_{GH}((sM_i, x_i, \Gamma_i), (Y, q, H)) \le \delta/3 \text{ for some space } (Y, q, H)$$
$$\text{satisfying the following conditions}$$
$$\textit{(C1) } H \text{ contains } \mathbb{Z} \text{ as a closed subgroup,}$$
$$\textit{(C2) } \text{this } \mathbb{Z} \text{ subgroup has generator whose displacement}$$
$$\text{at } q \text{ is less than } 1.\}$$

(See Remark 6.5.5 for explanations on the definition of S_i)

Since G' contains a closed $\mathbb{R}$ subgroup, we conclude that $r_i \in S_i$ for i large. Pick $s_i \in S_i$ with $\inf(S_i) \le s_i \le \inf(S_i) + 1/i$.

We show that $s_i \to \infty$. In fact, suppose s_i subconverges to $s < \infty$, then after passing to a subsequence, we have

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (sX, x, G).$$

Since $s_i \in S_i$, each $(s_i M_i, x_i, \Gamma_i)$ is δ-close to some space (Y_i, q_i, H_i) with conditions *(C1)(C2)*. G fixes x while H_i contains some element h_i moving q_i with displacement less than 1. Furthermore, by condition *(C1)* the orbit $\langle h_i \rangle q_i$ has infinite diameter. Obviously, such (Y_i, q_i, H_i) can not be δ close to (sX, x, G). A contradiction.

As Step 2 in the proof of Proposition 6.3.1, we follow the same argument and conclude that $r_i/s_i \to \infty$.

Now consider the convergent sequence

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty).$$

and we make the following observations:

1. If $\dim(H_\infty) \le l + 1$, or $\mathrm{Iso}(q_\infty, H_\infty)$ has dimension less than $l + 1$, then we would obtain a contradiction to the induction assumptions by passing to the tangent cone at q_∞ and applying the fact that $r_i/s_i \to \infty$.
2. If $\dim(H_\infty) > l + 1$, then H_∞ contains a closed $\mathbb{R}$ subgroup and we follow the methods used in Step 4 of the proof of Proposition 6.3.1 to draw a contradiction. More precisely, we can rescale s_i down by a constant but this smaller sequence still belongs

to S_i for i large, and this leads to a contradiction to our choice of s_i.

3. If $H_\infty = \mathbb{T}^l$ fixing q_∞, then we also end in a contradiction. This is because each (s_iM_i, x_i, Γ_i) is $\delta/3$ close to some (Y_i, q_i, H_i), where H_i has some element h_i moving q_i with displacement less than 1 and $\mathrm{diam}(\langle h_i\rangle q_i) = \infty$. This can not happen for $\delta = 1/10$.

Therefore, the only situation left is that, H_∞ contains $(H_\infty)_0 = \mathbb{T}^{l+1}$ as a proper subgroup with $\mathbb{T}^{l+1}$-action fixing q_∞. By Proposition 6.4.1, H_∞ does not contain any element of finite order outside $(H_\infty)_0$. If H_∞ contain a closed $\mathbb{Z}$ subgroup, then we can rule out this case as we did in observation 2 above.

We have ruled out every possibility of $(Y_\infty, q_\infty, H_\infty)$, and this completes the proof.

$\square$

Remark 6.5.5. When defining S_i in the proof above, we only require that (Y, q, H) contains some $\mathbb{Z}$ subgroup moving q (but not too far). So logically, if $G' = \mathbb{R}$, which may happen, then such S_i is still nonempty and we can still pick s_i close to $\inf(S_i)$. However, in this case, we will not find any contradiction. Inspecting the proof above, we used the fact that G' actually has higher dimension than G to rule out every possibility of $(Y_\infty, q_\infty.H_\infty)$.

Next we prove induction on $\dim_R(G)$.

Proof of Induction on $\dim_R(G)$. Under the reductions, assuming that Theorem 6.1.1 holds when

(1) $\dim_T(G) = l$ with $\dim_R(G) \le k$, or

(2) $\dim_T(G) < l$,

we need to show that if $G = \mathbb{R}^{k+1} \times \mathbb{T}^l$ with $\mathbb{T}^l$ fixing x, then for any rescaling sequence $r_i \to \infty$ and any convergent subsequence

$$(r_iM_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x', G'),$$

we have $\dim(G') \le (k+1) + l$.

We argue by contradiction. Suppose that there is a rescaling sequence $r_i \to \infty$ such that the corresponding limit group G' has dimension $> (k+1) + l$. By Proposition

6.4.1, we know that G' has no torus factor of dimension $> l$, thus it must contain $\mathbb{R}^{k+2}$ as a closed subgroup. In particular, G' contains a closed subgroup $\mathbb{R}^{k+1} \times \mathbb{Z}$.

Let $\delta = \delta(n, \eta) > 0$ be the constant in Lemma 6.5.4. We consider

$$\begin{aligned} S_i := \{\ 1 \leq s \leq r_i \mid\ & d_{GH}((sM_i, x_i, \Gamma_i), (Y, q, H)) \leq \delta/3 \text{ for some space } (Y, q, H) \\ & \text{satisfying the following conditions} \\ & \textit{(C1) } H \text{ contains } \mathbb{R}^{k+1} \times \mathbb{Z} \text{ as a closed subgroup,} \\ & \textit{(C2) } \text{this } \mathbb{Z} \text{ subgroup has generator whose displacement} \\ & \quad \text{at } q \text{ is less than } 1.\} \end{aligned}$$

We know that $r_i \in S_i$ for i large. Pick $s_i \in S_i$ such that $\inf(S_i) \leq s_i \leq \inf(S_i) + 1/i$.

We show that $s_i \to \infty$. Suppose that s_i sub-converges to $s < \infty$, then

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (sX, x, G).$$

For i large, since $s_i \in S_i$, there is some space (Y_i, q_i, H_i) with conditions *(C1)(C2)* above and

$$d_{GH}((sX, x, G), (Y_i, q_i, H_i)) \leq \delta/2.$$

Recall that by the reductions at the beginning of this section and Lemma 6.5.1, we may assume that $\mathbb{R}^{k+1}$-action has no η-subgroup of one-parameter at x ($\mathbb{R}^{k+1} \subseteq G$). We apply Lemma 6.5.4 and obtain the desired contradiction.

Follow the same proof as Step 2 in Proposition 6.3.1, we derive that $r_i/s_i \to \infty$.

We consider

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty).$$

If $\dim(H_\infty) > (k+1)+l$, then H_∞ contains $\mathbb{R}^{k+1} \times \mathbb{Z}$; and just like step 4 in the proof of Proposition 6.3.1, we will get a contradiction by rescaling down s_i by a constant. Thus we always have $\dim(H_\infty) \leq (k+1)+l$. If $\dim(H_\infty) < (k+1)+l$, or $\dim(H_\infty) = (k+1)+l$ but $\mathrm{Iso}(q_\infty, H_\infty)$ has dimension $< l$, then we consider

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty)$$

and its rescaling sequence ($r_i/s_i \to \infty$)

$$(r_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x, G').$$

Apply the induction assumptions, we rule out such cases.

The only remaining case is $(H_\infty)_0 = \mathbb{R}^{k+1} \times \mathbb{T}^l$ with $\mathbb{T}^l$-action fixing q_∞. By Lemma 6.5.1, $\mathbb{R}^{k+1}$-action has no η-subgroup of one-parameter at q_∞. If H_∞ is connected, we apply Lemma 6.5.4 once again and end in a contradiction. If H_∞ has finitely many components, then the contradiction arises from Proposition 6.4.1. If H_∞ has infinitely many components, then again by Proposition 6.4.1, H_∞ contains $\mathbb{R}^{k+1} \times \mathbb{Z}$ as a closed subgroup, which would contradict with our choice of s_i. □

We finish the proof of Theorem 6.1.1(1) by verifying the last induction on $\#\pi_0(G)$.

Proof of Induction on $\#\pi_0(G)$. Under the reductions, assuming that Theorem 6.1.1 holds when

(1) $G_0 = \mathbb{R}^k \times \mathbb{T}^l$ with $\#G/G_0 \le m$, or

(2) $\dim_T(G) = l$ with $\dim_R(G) < k$, or

(3) $\dim_T(G) < l$.

We need to verify the case $G_0 = \mathbb{R}^k \times \mathbb{T}^l$ with $\#\pi_0(G) = m+1$. By reductions, we know that $G = \mathbb{R}^k \times \mathrm{Iso}(x, G)$.

We argue by contradiction. Suppose that for some $r_i \to \infty$,

$$(r_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x, G')$$

$\dim(G') > k+l$ happens. By Lemma 6.4.5, G' contains $\mathbb{R}^{k+1}$ as a closed subgroup, and thus it contains $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup.

Let $\delta(n,\eta) > 0$ be the constant in Lemma 6.5.4. We consider

$$\begin{aligned} S_i := \{\ 1 \le s \le r_i \mid\ & d_{GH}((sM_i, x_i, \Gamma_i), (Y, q, H)) \le \delta/3 \text{ for some space } (Y, q, H) \\ & \text{satisfying the following conditions} \\ & \textit{(C1) } H \text{ contains } \mathbb{R}^k \times \mathbb{Z} \text{ as a closed subgroup,} \\ & \textit{(C2) } \text{this } \mathbb{Z} \text{ subgroup has generator whose displacement} \\ & \quad \text{at } q \text{ is less than } 1.\} \end{aligned}$$

S_i is nonempty because $r_i \in S_i$ for i large. We pick $s_i \in S_i$ with $\inf(S_i) \le s_i \le \inf(S_i) + 1/i$.

By Lemma 6.5.4 and the same argument we applied before, we conclude that $s_i \to \infty$. By our choice of s_i, we also have $r_i/s_i \to \infty$.

We consider

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty).$$

If $\dim(H_\infty) > k + l$, then it contains $\mathbb{R}^k \times \mathbb{Z}$ as a closed subgroup, and we get a contradiction by scaling down s_i by a constant. If $\dim(H_\infty) < k+l$, or $\dim(H_\infty) = k+l$ but $\mathrm{Iso}(q_\infty, H_\infty)$ has dimension $< l$, or $\dim(H_\infty) = k + l$ with $\dim(\mathrm{Iso}(q_\infty, H_\infty)) = l$ but number of connected components of $\mathrm{Iso}(q_\infty, H_\infty)$ being less than $m + 1$, then we consider

$$(s_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (Y_\infty, q_\infty, H_\infty)$$

and its rescaling sequence ($r_i/s_i \to \infty$)

$$(r_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x, G').$$

Apply the induction assumptions and passing to the tangent cone if necessary, we rule out these cases.

The only remaining case is $(H_\infty)_0 = \mathbb{R}^k \times \mathbb{T}^l$ with $\mathbb{T}^l$ fixing q_∞ and $\mathrm{Iso}(q_\infty, H_\infty)$ having at least $m + 1$ many components. According to Proposition 6.4.1, $\mathrm{Iso}(q_\infty, H_\infty)$ has exactly $m + 1$ many components. If $\#\pi_0(H_\infty)$ is finite, then by Proposition 6.4.1 again, $\#\pi_0(H_\infty) = m + 1$ and $H_\infty = \mathbb{R}^k \times \mathrm{Iso}(q_\infty, H_\infty)$. Apply Lemmas 6.5.1 and 6.5.4 here, we result in a desired contradiction. If $\#\pi_0(H_\infty) = \infty$, then H_∞ contains a closed subgroup $\mathbb{R}^k \times \mathbb{Z}$, and we can scale down s_i by a suitable constant to rule out this case. $\square$

One can regard the tuple $(\dim_T(G), \dim_R(G), \#\pi_0(G))$ as an order. Recall that for a space (X, x, G) with $G = \mathbb{R}^k \times \mathrm{Iso}(x, G)$, we define $\dim_R(G) = k$ and $\dim_T(G) = \dim(\mathrm{Iso}(x, G))$. For a general G-action on (X, x), we introduce the following definition.

Definition 6.5.6. Let (X, x, G) be a space, we put $\overline{G}$ as the subgroup generated by G_0 and $\mathrm{Iso}(x, G)$. We define $\dim_T(\overline{G}) = \dim(\mathrm{Iso}(x, G))$ and $\dim_R(\overline{G}) = \dim(G) - \dim_T(\overline{G})$ (Compare with Corollary 6.4.9).

Definition 6.5.7. Let (Y_1, q_1, H_1) and (Y_2, q_2, H_2) be two spaces. We say that

$$(Y_1, q_1, H_1) < (Y_2, q_2, H_2),$$

if one of the following holds:

(1) $\dim_T(\overline{H}_1) < \dim_T(\overline{H}_2)$;

(2) $\dim_T(\overline{H}_1) = \dim_T(\overline{H}_2)$, $\dim_R(\overline{H}_1) < \dim_R(\overline{H}_2)$;

(3) $\dim_T(\overline{H}_1) = \dim_T(\overline{H}_2)$, $\dim_R(\overline{H}_1) = \dim_R(\overline{H}_2)$, $\#\pi_0(\overline{H}_1) < \#\pi_0(\overline{H}_2)$.

We say that $(Y_1, q_1, H_1) \sim (Y_2, q_2, H_2)$, if

$$\dim_T(\overline{H}_1) = \dim_T(\overline{H}_2), \quad \dim_R(\overline{H}_1) = \dim_R(\overline{H}_2), \quad \#\pi_0(\overline{H}_1) = \#\pi_0(\overline{H}_2).$$

Similarly, we can define $(Y_1, q_1, H_1) \lesssim (Y_2, q_2, H_2)$.

With respect to this order, the three inductions in the proof of Theorem 6.1.1(1) mean that, if Theorem 6.1.1(1) holds for all (X_1, x_1, G_1) with $(X_1, x_1, G_1) < (X, x, G)$, then it holds for (X, x, G). With this definition, we derive the following proposition from Theorem 6.1.1:

Proposition 6.5.8. *Let (M_i, x_i) be a sequence of complete n-manifolds with*

$$\mathrm{Ric}_{M_i} \geq -(n-1)$$

and Γ_i be a discrete abelian group acting freely and isometrically on M_i for each i. Suppose that each Γ_i-action has no ϵ-small η-subgroup on $B_1(x_i)$ with scale $r \in (0, 1]$. If the following two sequences converge ($r_i \to \infty$):

$$(M_i, x_i, \Gamma_i) \xrightarrow{GH} (X, x, G),$$

$$(r_i M_i, x_i, \Gamma_i) \xrightarrow{GH} (X', x', G'),$$

then $(X', x', G') \lesssim (X, x, G)$. Moreover, if $(X', x', G') \sim (X, x, G)$, then $G' = \overline{G'}$.

Proof. The first part follows from Theorem 6.1.1 (also see the reduction step at the beginning of this section). For the second part, when $(X', x', G') \sim (X, x, G)$, suppose that $\overline{G'}$ is proper in G', then G' either contains $\overline{G'} \times \mathbb{Z}$ as a closed subgroup, or contains a finite extension of $\overline{G'}$. Both cases are ruled out in the proof of Theorem 6.1.1. $\square$

Remark 6.5.9. Notice that Theorem 6.1.1 can eliminate $G = S^1$ fixing base point with $G' = \mathbb{R}^2$, while Proposition 6.5.8 can not. However, this is sufficient for the argument in next section and streamlines the proof (See proof of Theorem 5.1.7 in Section 6.6).

We prove a corollary to end this section, which will be used in Section 6.6 to bound the number of short generators.

Corollary 6.5.10. *Let (M_i, x_i) be a sequence of complete n-manifolds with*

$$\mathrm{Ric}_{M_i} \geq -(n-1).$$

Let Γ_i be a discrete abelian group acting freely and isometrically on M_i for each i and H_i be a subgroup of Γ_i. Suppose that each Γ_i-action has no ϵ-small η-subgroup on $B_1(x_i)$ with scale $r \in (0, 1]$. If the following two sequences converge ($r_i \to \infty$):

$$(M_i, x_i, \Gamma_i, H_i) \xrightarrow{GH} (X, x, G, H),$$

$$(r_i M_i, x_i, \Gamma_i, H_i) \xrightarrow{GH} (X', x', G', H')$$

with $\overline{G} = \overline{H}$ and H' being a proper subgroup of G', then $(X', x', H') < (X, x, G)$.

Proof. By Theorem 6.5.8,

$$(X', x', H') \lesssim (X', x', G') \lesssim (X, x, G);$$

$$(X', x', H') \lesssim (X, x, H) \sim (X, x, G).$$

Suppose that $(X', x', H') \sim (X, x, G)$ happens, then

$$(X', x', H') \sim (X', x', G').$$

On the other hand, by the second part of Theorem 6.5.8, we have $H' = \overline{H'}$ and $G' = \overline{G'}$. Since H' is a proper subgroup of G', $\overline{H'}$ is proper in $\overline{G'}$. In other words,

$$(X', x', H') < (X', x', G').$$

We end in a contradiction. ∎

Remark 6.5.11. Later in Section 6.6, we bound the number of short generators by induction on the order introduced in Definition 6.5.7. Notice that for any space (X, x, G), if there is a series of spaces

$$(X, x, G) > (X_1, x_1, G_1) > (X_2, x_2, G_2) > ... > (X_i, x_i, G_i) > ...,$$

then this series must stop at certain k, that is, $\overline{G_k} = \{e\}$.

6.6 Bounding number of short generators

We prove Theorems 5.1.7 and C. Recall that to prove results on Milnor conjecture, by Theorem 2.3.11, it suffices to check abelian fundamental groups.

Proof of Theorems 5.1.7. Suppose that there exists a contradicting convergent sequence of n-manifolds with $\mathrm{Ric}_{M_i} \geq -(n-1)$

$$\begin{tikzcd} (\widetilde{M_i}, \tilde{x}_i, \Gamma_i) \arrow[r, "GH"] \arrow[d, "\pi_i"] & (\widetilde{X}, \tilde{x}, G) \arrow[d, "\pi"] \\ (M_i, x_i) \arrow[r, "GH"] & (X, x) \end{tikzcd}$$

satisfying the following conditions:

(1) Γ_i can be generated by loops of length less than R,

(2) $|S(x_i)| \geq 2^i$,

(3) Γ_i is abelian and Γ_i-action has no ϵ-small η-subgroup on $B_1(\tilde{x}_i)$ with scale $r \in (0, 1]$.

To derive a contradiction, the goal is to show that $|S(x_i)| \leq N$ for some N. We rule out such contradicting sequence above by induction on the order of limit space $(\widetilde{X}, \tilde{x}, G)$ (See Remark 6.5.11).

If G is discrete, then by Lemma 6.2.3, there is N such that $\#\Gamma_i(R) \leq N$ for all i large. In particular, $|S(x_i)|$ can not diverge to infinity. A contradiction.

Assuming that the statement holds for all possible limit spaces $(\widetilde{X}_1, \tilde{x}_1, G_1)$ with

$$(\widetilde{X}_1, \tilde{x}_1, G_1) < (\widetilde{X}, \tilde{x}, G),$$

we show that it also holds for $(\widetilde{X}, \tilde{x}, G)$.

Given each $\epsilon > 0$, by basic properties of short basis and Bishop-Gromov relative volume comparison, number of short generators with length between ϵ and R is bounded by some constant $C(n,R,\epsilon)$. Thus number of short generators with length less than ϵ is larger than $2^i - C(n,R,\epsilon) \to \infty$. By a diagonal argument and passing to a subsequence, we can pick $\epsilon_i \to 0$ such that number of short generators with length less than ϵ_i is larger than 2^i. Replacing M_i by $\widetilde{M}_i/\langle \Gamma_i(\epsilon_i)\rangle$, we may assume that $\Gamma_i = \langle \Gamma_i(\epsilon_i)\rangle$.

We introduce some notations here. For an integer m, we denote $\gamma_{i,m}$ as the m-th short generator of Γ_i. For a sequence $m_i \to \infty$ below, we always assume that $m_i \leq |S(x_i)|$. We consider H_i as the subgroup in Γ_i generated by first m_i short generators and H as a limit group of H_i

$$(\widetilde{M}_i, \tilde{x}_i, H_i) \xrightarrow{GH} (\widetilde{X}, \tilde{x}, H).$$

Case 1: There is a sequence $m_i \to \infty$ such that $(\widetilde{X}, \tilde{x}, H) < (\widetilde{X}, \tilde{x}, G)$

If this happens, we replace M_i by $\widetilde{M}_i/\Gamma_{i,m_i}$ and finish the induction step.

Case 2: For any sequence $m_i \to \infty$, $(\widetilde{X}, \tilde{x}, H) \sim (\widetilde{X}, \tilde{x}, G)$.

Recall that this means $\overline{H} = \overline{G}$ (See Definition 6.5.7). We pass this to tangent cone of $\widetilde{X}$ at $\tilde{x}$ (See Corollary 6.4.9). By a standard diagonal argument, there is some $s_i \to \infty$ slowly such that $\epsilon_i s_i \to 0$ and

$$(s_i\widetilde{M}_i, \tilde{x}_i, \Gamma_i, H_i) \xrightarrow{GH} (C_{\tilde{x}}\widetilde{X}, \tilde{o}, G_{\tilde{x}}, H_{\tilde{x}}).$$

We can assume that $G_{\tilde{x}} = H_{\tilde{x}}$ here. Otherwise,

$$(C_{\tilde{x}}\widetilde{X}, \tilde{o}, H_{\tilde{x}}) < (\widetilde{X}, \tilde{x}, G)$$

and we can apply the induction assumption to rule out such a sequence. We replace M_i by $s_i M_i$ and continue the proof.

Now we have

$$(\widetilde{M}_i, \tilde{x}_i, \Gamma_i, H_i) \xrightarrow{GH} (\widetilde{X}, \tilde{x}, G, H)$$

with $G = H$. We consider intermediate coverings $\overline{M}_i = \widetilde{M}_i/H_i$ and $K_i = \Gamma_i/H_i$

$$(\overline{M}_i, \bar{x}_i, K_i) \xrightarrow{GH} (\overline{X}, \bar{x}, \{e\}).$$

Since K_i is generated by elements with length less than $\epsilon_i \to 0$, we have

$$\operatorname{diam}(K_i \cdot \bar{x}_i) \to 0.$$

Put $r_i = \mathrm{diam}(K_i \cdot \bar{x}_i)^{-1} \to \infty$. Rescaling the above sequences by r_i and passing to a subsequence, we obtain the following convergent sequences:

$$\begin{array}{ccc} (r_i\widetilde{M}_i, \tilde{x}_i, \Gamma_i, H_i) & \xrightarrow{GH} & (\widetilde{X}', \tilde{x}', G', H') \\ \downarrow & & \downarrow \\ (r_i\overline{M}_i, \bar{x}_i, K_i) & \xrightarrow{GH} & (\overline{X}', \bar{x}', \Lambda) \end{array}$$

with $\mathrm{diam}(\Lambda \cdot \bar{x}) = 1$. In particular, we conclude that H' is a proper subgroup of G'. By Corollary 6.5.10,

$$(\widetilde{X}', \tilde{x}', H') < (\widetilde{X}, \tilde{x}, G).$$

Claim : On $\overline{M}$, $\pi_1(\overline{M}_i, \bar{x}_i)$ can be generated by loops of length less than 1.

Indeed, $r_i|\gamma_{i,m_i}| \leq 1$ because

$$\begin{aligned} r_i^{-1} &= \mathrm{diam}(K_i \cdot \bar{x}_i) \\ &= \sup_{\gamma \in \Gamma_i} d(\gamma H_i \cdot \tilde{x}_i, H_i \cdot \tilde{x}_i) \\ &\geq d(\gamma_{i,m_i+1} H_i \cdot \tilde{x}_i, H_i \cdot \tilde{x}_i) \\ &= d(\gamma_{i,m_i+1} t \cdot \tilde{x}_i, \tilde{x}_i) \text{ (for some } t \in H_i) \\ &\geq d(\gamma_{i.m_i} \cdot \tilde{x}_i, \tilde{x}_i). \end{aligned}$$

The last inequality follows from the method by which we select short generators.

Now we have the following new contradicting sequence:

$$\begin{array}{ccc} (r_i\widetilde{M}_i, \tilde{x}_i, \Gamma_{i,m_i}) & \xrightarrow{GH} & (\widetilde{X}', \tilde{x}', H') \\ \downarrow & & \downarrow \\ (r_i\overline{M}_i, \bar{x}_i) & \xrightarrow{GH} & (\overline{X}', \bar{x}') \end{array}$$

with $(\widetilde{X}', \tilde{x}', H') < (X, x, G)$. Applying the induction assumption, we can rule out the existence of such a sequence and complete the proof. $\square$

Remark 6.6.1. In the proof above, if $\dim_T(\overline{H'}) = \dim_T(\overline{G})$ and $\dim_R(\overline{H'}) = \dim_R(\overline{G})$, then

$$(\widetilde{X}', \tilde{x}', H') < (\widetilde{X}, x, G)$$

means $\#\pi_0(\mathrm{Iso}(\tilde{x}', H')) < \#\pi_0(\mathrm{Iso}(\tilde{x}, G))$. Therefore, when the dimension does not decrease, we actually did an induction on the number of connected components of the isotropy subgroup, as indicated in Section 5.1.

Theorem C follows from Theorem 5.1.7 by a rescaling trick indicated in Section 2.3.

Proof of Theorem C. By Theorem 2.3.11, we can assume that $\pi_1(M,x)$ is abelian. Let $\{\gamma_1, ..., \gamma_i, ...\}$ be a set of short generators at x. We show that there are at most C many short generators, where $C = C(n,0,\epsilon,\eta,1)$ is the constant in Theorem 5.1.7. Suppose that there are at least $C+1$ many short generators. We put R as the length of γ_{C+1}. Then on $(R^{-1}\widetilde{M}, \tilde{x}, \pi_1(M,x))$, $\pi_1(M,x)$-action has ϵ-small η-almost subgroup for all scales $r \in (0,1]$, but there are $C+1$ many short generators of length ≤ 1. This is a contradiction to Theorem 5.1.7. $\square$

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