REPRESENTATION THEORY AND COHOMOLOGY THEORY OF MEROMORPHIC OPEN STRING VERTEX ALGEBRAS

by

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ABSTRACT OF THE DISSERTATION

Representation theory and Cohomology theory of Meromorphic Open String Vertex Algebras

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In this dissertation we systematically study the meromorphic open-string vertex algebra, its representation theory, and its the cohomology theory. Meromorphic open-string vertex algebra (MOSVA hereafter) is a natural noncommutative generalization of vertex algebra. It is the algebraic structure of vertex operators satisfying associativity, but not necessarily commutativity. We review the axiomatic system of MOSVA and its left modules given by Huang and give the definition of right modules and bimodules. We prove that the rationality of iterates follows from the axioms. We introduce a pole-order condition which is used to simplify the axiomatic system and give a formulation by series with formal variables. We introduce the skew-symmetry operator, define the opposite MOSVA analogous to the opposite algebra of an associative algebra, and study the relation between modules for a MOSVA and modules for the opposite MOSVA. We consider the Möbius structure on MOSVA and its modules, and prove that the contragredient of a module with Möbius structure is also a module. We compute an example of MOSVA that is constructed from the two-dimensional sphere. We use rational function taking values in the algebraic completion to develop cohomology theory of MOSVA and its bimodules. We prove that the first cohomology of a MOSVA is isomorphic to the set of outer derivations. We prove also that if a MOSVA has vanishing first cohomology for every bimodule, then the its left modules of finite length and satisfying a composability condition is completely reducible.

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Dedication

To my wife, Na He, to my parents, Ding Qi and Zhe Zhao, and to my teachers, Wu-Xing Cai, Wan-Lin Chen, Chuan-Long Huang and Zhao-Xin Huang.

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Chapter 1

Introduction

Vertex (operator) algebras are algebraic structures formed by (meromorphic) vertex operators. In mathematics, they arose naturally in the study of representations of infinite-dimensional Lie algebras and the Monster group (see [FLM] and [B]). In physics, they arose in the study of two-dimensional conformal field theory (2d CFT hereafter, see [BPZ] and [MS]). One of the most important properties of the vertex operators for a vertex (operator) algebra is the commutativity, which plays an important role in the study of these algebras and their representation theory. Mathematically, the commutativity, especially the equivalent commutator formula, makes it possible to use the Lie-theoretic methods to study vertex (operator) algebras and modules. Many results are proved based on the commutativity. In physical terms, vertex operators for a vertex (operator) algebra or a module correspond to fields of a special kind: meromorphic fields. The commutativity of vertex operators is closely related to the locality of meromorphic fields in two-dimensional conformal field theory. This commutativity is one of the most important reasons for the success of the mathematical construction of 2d CFT using the vertex (operator) algebras, its modules and the intertwining operators among the modules.

However, if we want to use vertex-algebraic methods to study quantum field theories in general, the commutativity might not hold even for meromorphic fields. One important class of quantum field theories is the nonlinear σ-model with the target manifold being a Riemannian manifold. If we want to realize certain differential operators on the manifold as components of some vertex operators, then these vertex operators cannot be commutative.

On the other hand, vertex (operator) algebras also have associativity, which is even more fundamental. In physical terms, associativity of vertex operators can be viewed as a strong form of the operator product expansion (OPE hereafter) of meromorphic fields. And the OPE of fields is expected to hold for all quantum field theories. This is one of the motivations for studying algebraic structures of suitable vertex operators that have associativity but not necessarily commutativity. In 2003, Huang and Kong introduced and constructed open-string vertex algebras in [HK]. In 2012, Huang introduced the notion of meromorphic open-string vertex algebras in [H3], a special case of open-string vertex algebra for which the correlation functions are rational functions.

Our motivation of studying meromorphic open-string vertex algebras (MOSVAs hereafter) are the following: first, just as vertex (operators) algebras can be viewed as analogues of commutative associative algebras, MOSVAs can be viewed as analogues of associative algebras that are not necessarily commutative. In particular, all vertex (operator) algebras are MOSVAs. So all the results for MOSVA also hold for vertex (operator) algebras. Since all correlation functions are rational functions, it is easier to deal with issues related to convergence and analytic extensions for MOSVAs than general open-string vertex algebras.

In 2012, Huang also constructed an example of MOSVA using parallel sections of tensor products of tangent bundles on any fixed Riemannian manifold (see [H4]). More importantly, Huang constructed modules generated by eigenfunctions of Laplacian operator. In physics, the eigenfunctions correspond to quantum states of a particle, which can be viewed as a degenerated form of a string. Elements of the MOSVA modules generated by eigenfunctions can be viewed as suitable string-theoretic excitations of the particle states. It is Huang's idea that the MOSVAs constructed from Riemannian manifolds, together with modules generated by Laplacian eigenfunctions and the still-to-be-defined intertwining operators among these modules may lead to a mathematical construction of the quantum two-dimensional nonlinear σ-model. Huang also hopes that this will shed lights on the four-dimensional Yang-Mills theory, which, though much more difficult, is indeed analogous to the two-dimensional nonlinear σ-model whose target manifold is a Lie group.

Another motivation for studying MOSVA is brought by the progress of developing cohomological methods. In the representation theory of various algebras, one of the main tools is the cohomological method. The powerful tool of homological algebra often provides a unified treatment of many results in representation theory. Such a unified treatment not only gives solutions to open problems, but also provides a conceptual understanding of the results. Here we shall particular mention the following results in associative algebras. Let A be an associative algebra. For an A-bimodule M, we use $\hat{H}^n(A,M)$ to denote the n-th Hochschild cohomology of A with coefficients in M. When A is commutative and M is a module (viewed as A-bimodule with the left and right A-module structures to be both the one from the original A-module structure), we use $H^n(A,M)$ to denote the the n-th Harrison cohomology of A with coefficients in M.

1. The first Hochschild cohomology $\hat{H}^1(A,M)$ is isomorphic to the quotient of the space of derivations from A to M by the space of inner derivations from A to M. When A is commutative, the first Harrison cohomology $H^1(A,M)$ is isomorphic to the space of derivations from A to M.

2. The second Hochschild cohomology $\hat{H}^2(A,A)$ is in one-to-one correspondence with the set of first-order deformations of A. When A is commutative, the second Harrison cohomology $H^2(A,A)$ is in one-to-one correspondence with the set of the first-order deformations of A.

3. All the left A-modules are completely reducible if and only if for every A-bimodules B and every $n\in\mathbb{Z}_+$, the Hochschild cohomology $\hat{H}^n(A,B)=0$.

In [H1], Huang introduced the cohomology of a grading-restricted vertex algebra. As vertex algebras can be viewed as an analogue to commutative associative algebras, the cohomology introduced in [H1] can be viewed as an analogue of Harrison cohomology. In [H2], using the cohomology established in [H1], Huang established the analogues of the results (1) and (2) for a grading-restricted vertex algebra V and grading-restricted V-modules. To define this cohomology, Huang introduced a larger complex in [H1] such that the complex for the grading-restricted vertex algebra is a subcomplex, just as the Harrison complex is a subcomplex of the Hochschild complex for the commutative associative algebra. In particular, the larger complex can be viewed as the analogue of the Hochschild complex. But this complex was defined in [H1] only for a grading-restricted vertex algebra.

In the dissertation, we give the definition of this larger complex for meromorphic open-string vertex algebras V and V-bimodules W that are not necessarily grading-restricted but satisfy the pole-order condition. Using the cohomology of this larger complex, we can establish the analogues of results (1), (2) and (3). The results (1) and (3) will be presented in this dissertation. The result (2) will be presented in the future paper [Q4]. Since a vertex algebra is also a special kind of MOSVA, all these results above also applies to vertex algebra.

The dissertation is organized as follows:

Chapter 2 focus on the study of the MOSVA. We recall the definitions of a MOSVA V in [H3] and discuss the following topics for a MOSVA V: the $\overline{V}$-valued map interpretation of vertex operators; the rationality of products of any numbers of vertex operators implies the rationality of iterates of any numbers of vertex operators; the pole-order condition, together with rationality of products and iterates of two vertex operators and other axioms, implies the rationality of any numbers of vertex operators; the formal variable formulation of MOSVA with the pole-order condition; the opposite MOSVA V^{op} of a MOSVA V. Many of the results in this chapter relies on the technique of analytic continuation of functions with several complex variables. We also gave an exposition section to these lemmas.

Chapter 3 focus on the study of modules for a MOSVA V. We recall the definition of left V-modules in [H3] and define right V-modules and V-bimodules. Aside from the discussion of the topics in Chapter 2 under the context of left V-modules, right V-modules and V-bimodules, we also discuss the following topics: the relation between V-modules and V^{op}-modules; compatibility condition of a V-bimodule W in terms of the left vertex operator Y_W^L and the skew-symmetry operator $Y_W^{s(R)}$ of the right vertex operator Y_W^R; Möbius structure on MOSVAs and modules; Contragredient of a Möbius V-module is also a Möbius V-module.

Chapter 4 computes an example of MOSVA constructed from parallel sections of the tensors of tangent bundle of the 2-dimensional sphere. Some exposition is given on the geometric backgrounds. Due to the limitation of time, we have not discussed the result for general n-dimensional spheres. Nor have we discussed the modules generated by the Laplacian eigenfunctions. These important topics will have to wait for future work.

Chapter 5 establishes the cohomology theory for MOSVAs and bimodules. For a MOSVA V and a V-bimodule $W = \coprod_{n\in\mathbb{C}} W_{[n]}$, we discuss $\overline{W}$-valued rational functions, where $\overline{W} = \prod_{n\in\mathbb{C}} W_{[n]}$ is the algebraic completion of W. We also study series of $\overline{W}$-valued rational functions and prove that the associativity (of Y_W^L and of $Y_W^{s(R)}$) and commutativity (of Y_W^L and $Y_W^{s(R)}$) hold when acting on $\overline{W}$-valued rational functions satisfying certain convergence conditions. Then we use the linear maps from $V^{\otimes n}$ to the space of $\overline{W}$-valued rational function that satisfy $\mathbf{d}$-conjugation properties, D-derivative properties and composable condition are used to construct the cochain complex. The coboundary operators for the cochain complex is defined using the $\overline{W}$-valued rational functions that the relevant series converge to. which is the key (as observed by Huang) for the cohomology theory to work as the defining series have disjoint regions of convergence.

Chapter 6 applies the cohomology theory to give a cohomological criterion of reductivity for left modules for MOSVAs. For a MOSVA V and a V-bimodule M that are lower-bounded (not necessarily grading-restricted) and satisfy the pole-order condition, let $\hat{H}^1_\infty(V, M)$ be the first cohomology of V with the coefficients in M. For a left V-module W, a left V-submodule W_2 of W and a graded subspace W_1 of W such that as a graded vector space, $W = W_1 \oplus W_2$, let π_{W_1} and π_{W_2} be the projections from W to W_1 and W_2, respectively. For a left V-module W and a left V-submodule W_2, we say that the pair (W, W_2) satisfies the composability condition if there exists a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ and such that for $k, l \in \mathbb{N}$, $w_2' \in W_2$, $w_1 \in W_1$, $v_1, \dots, v_{k+l}, v \in V$, the series

$$\langle w_2', Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)\pi_{W_2} Y_W(v, z)\pi_{W_1} Y_W(v_{k+1}, z_{k+1}) \cdots \pi_{W_1} Y_W(v_{k+l}, z_{k+l})w_1\rangle$$

is absolutely convergent the region $|z_1| > \cdots > |z_k| > |z| > \cdots > |z_{k+l}| > 0$ to a suitable rational function. We say that a left V-module W satisfies the composability condition if for every proper nonzero left V-submodule W_2 of W, the pair (W, W_2) satisfies the composability condition. We prove in this paper that if $\hat{H}^1_\infty(V, M) = 0$ for every $\mathbb{Z}$-graded V-bimodule M, then every left V-module of finite-length satisfying the composability condition is completely reducible. Since the first cohomology of V with coefficients in W is the quotient of the space of derivations from V to M by the space of inner derivations, the condition $\hat{H}^1_\infty(V, M) = 0$ in our main theorem above can also be formulated as the condition that every derivation from V to M is inner.

Chapter 2

Meromorphic open string vertex algebras

2.1 Basic Definitions

We first recall the notion of meromorphic open-string vertex algebra given in [H3].

2.1.1 The axiomatic definition

Definition 2.1.1. A *meromorphic open-string vertex algebra* (hereafter MOSVA) is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ (graded by *weights*) equipped with a *vertex operator map*

$$\begin{aligned} Y_V : V \otimes V &\to V[[x, x^{-1}]] \\ u \otimes v &\mapsto Y_V(u, x)v \end{aligned}$$

and a *vacuum* $\mathbf{1} \in V$, satisfying the following axioms:

1. Axioms for the grading:

   (a) *Lower bound condition*: When n is sufficiently negative, $V_{(n)} = 0$.

   (b) $\mathbf{d}$*-bracket formula*: Let $\mathbf{d}_V : V \to V$ be defined by $\mathbf{d}_V v = nv$ for $v \in V_{(n)}$. Then for every $v \in V$

   $$[\mathbf{d}_V, Y_V(v, x)] = x\frac{d}{dx}Y_V(v, x) + Y_V(\mathbf{d}_V v, x).$$

2. Axioms for the vacuum:

   (a) *Identity property*: Let 1_V be the identity operator on V. Then $Y_V(\mathbf{1}, x) = 1_V$.

   (b) *Creation property*: For $u \in V$, $Y_V(u, x)\mathbf{1} \in V[[x]]$ and $\lim_{z\to 0} Y_V(u, x)\mathbf{1} = u$.

3. *D-derivative and D-bracket properties*: Let $D_V : V \to V$ be the operator given by

$$D_V v = \lim_{x\to 0} \frac{d}{dx} Y_V(v, x)\mathbf{1}$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx} Y_V(v, x) = Y_V(D_V v, x) = [D_V, Y_V(v, x)].$$

4. *Rationality*: Let $V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$ be the graded dual of V. For $u_1, \cdots, u_n, v \in V, v' \in V'$, the series

$$\langle v', Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \cdots, z_n$, with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i \neq j \leq n$. For $u_1, u_2, v \in V$ and $v' \in V'$, the series

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2, v \in V$ and $v' \in V'$, we have

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle = \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

Such a meromorphic open-string vertex algebra is denoted by $(V, Y_V, \mathbf{1})$ or simply by V.

Definition 2.1.2. A meomorphic open-string vertex algebra V is said to be *grading-restricted* if $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$.

Throughout this thesis, all meromorphic open-string vertex algebras are assumed to be grading-restricted.

Remark 2.1.3. If in addition, V satisfies commutativity, namely, for every $u_1, u_2, v \in V, v' \in V'$

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$$

converges absolutely when $|z_1| > |z_2| > 0$ to the same rational function that

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

converges to when $|z_2| > |z_1 - z_2| > 0$, [FHL] shows that in this case the Jacobi identity for vertex algebras holds and V is a vertex algebra with lower bounded $\mathbb{Z}$-grading. So MOSVA can be treated as a noncommutative generalization to the vertex algebra. We have to redevelop a lot of basic results for MOSVA, since in vertex operator algebras these results are proved using commutativity.

2.1.2 Some immediate consequences

Axioms 1, 2 and 3 make it possible to carry over some facts of vertex algebras to MOSVA:

Proposition 2.1.4. *Let V be a MOSVA. Then*

1. *For $u \in V$, $Y_V(u, x)$ can be regarded as a formal series in $End(V)[[x, x^{-1}]]$*

$$Y_V(u, x) = \sum_{n\in\mathbb{Z}} (Y_V)_n(u)x^{-n-1}$$

   where $(Y_V)_n(u) : V \to V$ is a linear map for every $n \in \mathbb{Z}$. If u is homogeneous, then $(Y_V)_n(u)$ is a map of weight $\mathrm{wt}\ u - n - 1$.

2. *For fixed $u, v \in V$, $Y_V(u, x)v$ is lower truncated, i.e, there are at most finitely many negative powers of x.*

3. *For $u \in V$,*

$$Y_V(u, x)\mathbf{1} = e^{xD_V}u$$

4. *Formal Taylor theorem: for $u \in V$,*

$$Y_V(u, x + y) = Y_V(e^{yD_V}u, x) = e^{yD_V}Y_V(u, x)e^{-yD_V},$$

   in $End(V)[[x, x^{-1}, y]]$.

Remark 2.1.5. In the statement of the formal Taylor's theorem, the series $Y(u, x+y)$ should not be regarded as a series with one single variable $x + y$. Rather, it should be regarded as a series with two variables x and y, *expanded* from the series with the single variable $x + y$ with positive powers of y. Details are discussed in [LL]

Proof.

1. Follows from the linearity of $Y_V(u, x)v$ in both u and v, and the $\mathbf{d}$-bracket formula.

2. When u, v are homogeneous, the coefficient $(Y_V)_n(u)v$ of x^{-n-1} in $Y_V(u, x)v$ is also homogeneous of weight $m = \text{wt } u + \text{wt } v - n - 1$. As n gets sufficient large, m becomes sufficiently negative and by the lower bound condition, $V_{(m)} = 0$. So $(Y_V)_n(u)v = 0$ when n gets sufficiently large, hence the series $\sum\limits_{n\in\mathbb{Z}} (Y_V)_n(u)vx^{-n-1}$ is lower truncated.

3. Use the D-derivative property and induction, it is easy to show for $n = 0, 1, ...,$
$$D_V^n v = \lim_{x\to 0} \frac{d^n}{dx^n} Y(v, x)\mathbf{1}$$
So $Y_V(v, x)\mathbf{1}$, as a power series, has $D_V^n v/n!$ as the coefficient of x^n. Hence
$$Y_V(v, x)\mathbf{1} = \sum_{n=0}^{\infty} \left(\frac{1}{n!} D_V^n v\right) x^n = e^{xD_V} v$$

4. The first equality follows from the D_V-derivative formula. The second equality follows from the exponentiation of the D_V-bracket formula.

$\square$

2.1.3 On the product and iterate of two vertex operators

Note that Axiom 4 and 5 are formulated using complex functions. To understand these axioms correctly, let's consider the example the rationality of the product of two vertex operators. Let $u_1, u_2, v \in V$ and consider the formal series

$$\begin{aligned}
Y_V(u_1, x_1)Y_V(u_2, x_2)v &= Y_V(u_1, x_1) \left(\sum_{m\in\mathbb{Z}} (Y_V)_m(u_2)vx_2^{-m-1} \right) \\
&= \sum_{m\in\mathbb{Z}} Y_V(u_1, x_1) \left((Y_V)_m(u_2)v\right) x_2^{-m-1} \\
&= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} (Y_V)_n(u_1)((Y_V)_m(u_2)v)x_1^{-n-1}x_2^{-m-1}
\end{aligned}$$

This is a formal series in $V[[x_1, x_2, x_1^{-1}, x_2^{-1}]]$. Note that there are only finitely many negative powers of x_2, since $Y_V(u_2, x_2)v$ is lower truncated. However, in general there are infinitely many negative and positive powers of x_1, since although $Y_V(u_1, x_1)((Y_V)_m(u_2)v)$ is lower truncated for each $m \in \mathbb{Z}$, the lower bound of powers might not be uniform with respect to m.

Pairing the formal series with $v' \in V'$

$$\langle v', Y_V(u_1, x_1)Y_V(u_2, x_2)v\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle v', (Y_V)_n(u_1)((Y_V)_m(u_2)v)\rangle x_1^{-n-1}x_2^{-m-1}$$

we get a formal series in $\mathbb{C}[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$. Since the weight of $(Y_V)_n(u_1)((Y_V)_m(u_2)v)$ will be larger than wt v' when n becomes sufficient negative, in this series there are at most finitely many positive powers of x_1.

After substituting x_1, x_2 by two complex numbers z_1, z_2, we will get a series of complex numbers:

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle v', (Y_V)_n(u_1)((Y_V)_m(u_2)v)\rangle z_1^{-n-1}z_2^{-m-1}.$$

The first part of rationality says that this series of complex numbers converges absolutely when $|z_1| > |z_2| > 0$, and the limit is a rational function in z_1, z_2, with possible poles only at $z_1 = 0, z_2 = 0, z_1 = z_2$. If we denote the rational function by

$$\frac{f(z_1, z_2)}{z_1^{p_1}z_2^{p_2}(z_1 - z_2)^{p_{12}}},$$

where $f(z_1, z_2)$ is a polynomial function in z_1, z_2, then the series of complex numbers is precisely the series expansion of the rational function in the region $|z_1| > |z_2| > 0$, i.e., each $(z_1 - z_2)^{-1}$ factor is expanded as $z_1^{-1}\sum\limits_{k=0}^{\infty}(z_2/z_1)^k$.

To interpret the above in terms of formal variables, we let

$$\iota_{12} : V[x_1, x_1^{-1}, x_2, x_2^{-1}, (x_1 - x_2)^{-1}] \to V[[x_1, x_2, x_1^{-1}, x_2^{-1}]]$$

be the map that expands $(x_1 - x_2)^{-1}$ by the positive powers of x_2. Then the rationaly above amounts to say that for every $v' \in V', u_1, u_2, v \in V$, the formal series $\langle v', Y_V(u_1, x_1)Y_V(u_2, x_2)v\rangle$ can be obtained by applying ι_{12} to the rational function, i.e.

$$\langle v', Y_V(u_1, x_1)Y_V(u_2, x_2)v\rangle = \iota_{12}\left(\frac{f(x_1, x_2)}{x_1^{p_1}x_2^{p_2}(x_1 - x_2)^{p_{12}}}\right)$$

Since the series $Y_V(u_2, x_2)v$ is lower-truncated, the power p_2 of x_2 in the denominator is bounded above by a constant that depends only on u_2 and v.

Also we consider the formal series

$$\begin{aligned} Y_V(Y_V(u_1, x_0)u_2, x_2)v &= Y_V(\sum_{m\in\mathbb{Z}}(Y_V)_m(u_1)u_2x_0^{-m-1}, x_2)v \\ &= \sum_{m\in\mathbb{Z}} Y_V((Y_V)_m(u_1)u_2, x_2)vx_0^{-m-1} \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(Y_V)_n((Y_V)_m(u_1)u_2)vx_2^{-n-1}x_0^{-m-1} \end{aligned}$$

in $V[[x_0, x_2, x_0^{-1}, x_2^{-1}]]$. Similarly, there are finitely many negative powers of x_0, since $Y_V(u_1, x_0)u_2$ is lower truncated. However there might be infinitely many negative powers of x_2.

Pairing the formal series with with $v' \in V'$

$$\langle v', Y_V(Y_V(u_1, x_0)u_2, x_2)v\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle v', (Y_V)_n((Y_V)_m(u_1)u_2)v\rangle x_2^{-n-1}x_0^{-m-1}$$

we get a formal series in $\mathbb{C}[[x_0, x_2, x_0^{-1}, x_2^{-1}]]$. Since the weight of $(Y_V)_n((Y_V)_m(u_1)u_2)v$ will be larger than wt v' when n becomes sufficient negative, in this series there are at most finitely many positive powers of x_2.

After substituting $x_0 = z_1 - z_2, x_2 = z_2$, we will get a series of complex numbers

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle v', Y_V((Y_V)_m(u_1)u_2)_nv\rangle z_2^{-n-1}(z_1 - z_2)^{-m-1}.$$

The second part of rationality states that the series converges absolutely when $|z_2| > |z_1 - z_2| > 0$. Together with associativity, we know that the sum is equal to the same rational function which $\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$ converges to. In other words, the series $\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$ is the series expansion of the same function but in the different region region $|z_2| > |z_1 - z_2| > 0$, i.e., each z_1^{-1} factor is expanded as $(z_2 + z_1 - z_2)^{-1} = z_2^{-1}\sum_{k=0}^{\infty}[(z_1 - z_2)/z_2]^k$. As a consequence, in the series $\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle$, there are only finitely positive powers of z_2 and finitely many negative powers of $(z_1 - z_2)$.

To interpret the above in terms of the formal variables, let

$$\iota_{20} : \mathbb{C}[x_0, x_2, x_0^{-1}, x_2^{-1}, (x_0 + x_2)^{-1}] \to \mathbb{C}[[x_0, x_2, x_0^{-1}, x_2^{-1}]]$$

be the map that expands $(x_0+x_2)^{-1}$ by the positive power of x_0. Since the rational function is the same, the formal series $\langle v', Y_V(Y_V(u_1,x_0)u_2,x_2)v\rangle$ can be obtained by applying ι_{20} to the rational function after substituting $x_1 = x_0+x_2$, i.e.

$$\langle v', Y_V(Y_V(u_1,x_0)u_2,x_2)v\rangle = \iota_{20}\left(\frac{f(x_0+x_2,x_2)}{(x_0+x_2)^{p_1}x_2^{p_2}x_0^{p_{12}}}\right)$$

In particular, since the series $Y_V(u_1,x_0)u_2$ is lower-truncated, the power of p_{12} of x_0 in this series is bounded above by a constant that depends only on u_1 and u_2.

As a consequence of the discussion above, we have the following weak associativity in terms of correlation functions.

Proposition 2.1.6. *Let V be a MOSVA. Let ι_{12} and ι_{20} be defined as above. Then for every $v' \in V, u_1, u_2, v \in V$,*

$$\iota_{12}^{-1}\langle v', Y_V(u_1,x_1)Y_V(u_2,x_2)v\rangle = \iota_{20}^{-1}\langle v', Y_V(Y_V(u_1,x_0)u_2,x_2)v\rangle|_{x_1=x_0+x_2}$$

Remark 2.1.7. Here we shall experience the first difference to usual VOA. In case the commutativity is also present, then a similar argument shows that p_1 is also controlled above by u_1 and v. Then because p_1, p_2, p_{12} are independent of the choice of v', letting v' vary in V' we will be able to see that the formal series $Y(u_1,x_1)Y(u_2,x_2)v$, after multiplying suitable powers of x_1, x_2 and (x_1-x_2), is a power series in $V[[x_1,x_2]]$. However, we don't have commutativity for MOSVA. In the most general sense p_1 can be dependent to the choice of v'. So the best we can say is, for the integers p_2 and p_{12}, the formal series

$$x_2^{p_2}(x_1-x_2)^{p_{12}}Y_V(u_1,x_1)Y_V(u_2,x_2)v \in V[[x_1,x_1^{-1},x_2]]$$

In general, there might not exists an integer p_1 such that

$$x_1^{p_1}x_2^{p_2}(x_1-x_2)^{p_{12}}Y_V(u_1,x_1)Y_V(u_2,x_2)v \in V[[x_1,x_2]]$$

unless we know something about the pole $z_1=0$ of the rational function determined by $\langle v', Y_V(u_1,z_1)Y_V(u_2,z_2)v\rangle$, as described in the following proposition.

Proposition 2.1.8. *Let $(V, Y_V, \mathbf{1})$ be a MOSVA. Assume that for every $u_1, u_2, v \in V$, there exists a positive integer p_1 such that for every $v' \in V'$, the order of the pole $z_1 = 0$*

of the rational function determined by $\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$ *is bounded above by* p_1*. Then the following associativity holds: for* $u_1, u_2, v \in V$ *and the integer* p_1 *above,*

$$(x_0 + x_2)^{p_1} Y_V(u_1, x_0 + x_2)Y_V(u_2, x_2)v = (x_0 + x_2)^{p_1} Y_V(Y_V(u_1, x_0)u_2, x_2)v$$

where both sides are understood as Laurent series in $V[[x_0, x_2]][x_0^{-1}, x_2^{-1}]$

Proof. One sees easily that for every $v' \in V'$, there exists $p_1, p_2, p_{12} \in \mathbb{Z}_+$ such that

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} \langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$$

converges to a polynomial function. Thus with the formal variables x_1, x_2, the formal series

$$x_1^{p_1} x_2^{p_2} (x_1 - x_2)^{p_{12}} \langle v', Y_V(u_1, x_1)Y_V(u_2, x_2)v\rangle$$

has no negative powers of x_1, x_2. Thus as a formal series with coefficients in W,

$$x_1^{p_1} x_2^{p_2} (x_1 - x_2)^{p_{12}} Y_V(u_1, x_1)Y_V(u_2, x_2)v$$

has no negative powers of x_1, x_2 and thus sits in $V[[x_1, x_2]]$. Replace x_1 by $x_0 + x_2$ and divide the resulting power series by $x_0^{p_{12}} x_2^{p_2}$, we see that

$$(x_0 + x_2)^{p_1} Y_V(u_1, x_0 + x_2)Y_V(u_2, x_2)v \in V[[x_0, x_2]][x_0^{-1}, x_2^{-1}]$$

Similarly, we see that

$$(x_0 + x_2)^{p_1} Y_V(Y_V(u_1, x_0)u_2, x_2)v \in V[[x_0, x_2]][x_0^{-1}, x_2^{-1}]$$

The conclusion then follows from Proposition 2.1.6. $\square$

Remark 2.1.9. So to recover the commonly-known weak associativity for MOSVAs, extra conditions on the correlation functions has to be assumed.

2.1.4 On the product of any number of vertex operators

The rationality of the product of n vertex operators can be understood in a similar fashion: For fixed $u_1, u_2, ..., u_n, v \in V$, the formal series

$$\begin{aligned} &Y_V(u_1, x_1)Y_V(u_2, x_2) \cdots Y_V(u_n, x_n)v \\ &\quad = \sum_{k_n \in \mathbb{Z}} \cdots \sum_{k_1 \in \mathbb{Z}} (Y_V)_{k_1}(u_1)((Y_V)_{k_2}(u_2)(\cdots((Y_V)_{k_n}(u_n)v)\cdots))x_1^{-k_1-1}x_2^{-k_2-1}\cdots x_n^{-k_n-1} \end{aligned}$$

is in $V[[x_1,...,x_n,x_1^{-1},...,x_n^{-1}]]$. There are finitely many negative powers of x_n but there are infinitely many positive and negative powers of all other variables. However the "peeling off" trick (See for example [LL], Section 3.10) works here: if we look at the coefficient of fixed power of x_n, then this coefficient is a series in $V[[x_1,...,x_{n-1},x_1^{-1},...,x_{n-1}^{-1}]]$, with only finitely many negative powers of x_{n-1}. Similarly if we look at the coefficient of fixed powers of x_{n-1} and x_n, then it will be a series in $V[[x_1,...,x_{n-2},x_1^{-1},...,x_{n-2}^{-1}]]$ with finitely many negative powers of x_{n-2}. Similar story is true consecutively for $x_{n-3},...,x_1$.

Evaluating $x_1 = z_1,...,x_n = z_n$ and pair it with $v' \in V'$, we get a series of complex numbers

$$\begin{aligned}&\langle v', Y_V(u_1,z_1)Y_V(u_2,z_2)\cdots Y_V(u_n,z_n)v\rangle\\ &\quad= \sum_{k_n\in\mathbb{Z}}\cdots\sum_{k_1\in\mathbb{Z}}\langle v',(Y_V)_{k_1}(u_1)((Y_V)_{k_2}(u_2)(\cdots((Y_V)_{k_n}(u_n)v)\cdots))\rangle z_1^{-k_1-1}z_2^{-k_2-1}\cdots z_n^{-k_n-1}.\end{aligned}$$

The rationality states that the series converges absolutely when $|z_1| > |z_2| > \cdots > |z_n| > 0$ to a rational function with possible poles only at $z_i = 0, 1 \le i \le n$ and $z_i = z_j, 1 \le i \ne j \le n$. Equivalently, the series of complex numbers is precisely the series expansion of the rational function in the region $|z_1| > |z_2| > \cdots > |z_n| > 0$, i.e., for any $1 \le i < j \le n$, every $(z_i - z_j)^{-1}$ factor in the rational function is expanded as $z_i^{-1}\sum_{k=0}^{\infty}(z_j/z_i)^k$. In particular, we know that there are at most finitely many negative powers of z_n and finitely many positive powers of z_1.

To interpret the above in terms of formal variables, let S be the multiplicative set in the polynomial ring $\mathbb{C}[x_1,...,x_n]$ generated by $x_i, i = 1,...,n$, and $(x_i - x_j)^{-1}, 1 \le i \ne j \le n$. Consider the localization $\mathbb{C}[x_1,...,x_n]_S$ of the polynomial ring with S. Let

$$\iota_{12..n} : \mathbb{C}[x_1,...x_n]_S \to \mathbb{C}[[x_1,...,x_n,x_1^{-1},...,x_n^{-1}]]$$

be the map that expands $(x_i - x_j)^{-1}$ for each $1 \le i < j \le n$ as the series with positive powers in x_j. Then for every $v' \in V', u_1,...,u_n, v \in V$, we have

$$\langle v', Y_V(u_1,x_1)\cdots Y_V(u_n,x_n)v\rangle = \iota_{12...n}\left(\frac{f(x_1,...,x_n)}{\prod_{i=1}^n x_i^{p_i}\prod_{1\le i<j\le n}(x_i-x_j)^{p_{ij}}}\right)$$

for some polynomial $f(x_1,...,x_n) \in \mathbb{C}[x_1,...,x_n]$ and some integers $p_i, i = 1,...,n, p_{ij}, 1 \le i < j \le n$.

Remark 2.1.10. We should mention that the rationality of the product of n vertex operators turns out to be a very subtle issue. It is proved in [FHL], Section 3.5 that the rationality involving any number of vertex operators holds automatically for a lower bounded $\mathbb{Z}$-graded vertex algebra, for which all the axioms are formulated using only two vertex operators. However for MOSVA, commutativity does not hold in general. So the argument in [FHL] fails for such algebras. To make sense of the product of any number of vertex operators, it is necessary to assume Axiom 4 for every number $n \geq 2$.

Definition 2.1.11. A MOSVA is said to satisfy the *pole-order condition*, if for every $v' \in V', u_1, u_2, v \in V$, the order of the pole $z_1 = 0$ of the rational function $\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$ converges to is bounded above by a constant that depends on u_1 and v.

The condition here is stronger than that the version used in Proposition 2.1.8. All the vertex algebras satisfy this condition because of commutativity. With the pole-order condition, we only need to assume Axiom 4 for $n = 2$. The rationality of the product of $n > 2$ vertex operators is a consequence.

Proposition 2.1.12. *Let* $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$, $Y_V : V \otimes V \to V[[x, x^{-1}]]$ *and* $\mathbf{1} \in V_{(0)}$ *satisfy Axiom 1, 2, 3, 5, the Axiom 4 with only* $n = 2$*, and the pole-order condition. Then Axiom 4 holds for every* $n > 2$*. Moreover, for the rational function determined by* $\langle v', Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v\rangle$*, the order of poles* $z_i = 0$ *is bounded above by a constant that depends only on* u_i *and* v *for* $i = 1, ..., n$*; the order of poles* $z_i = z_j$ *is bounded above by a constant that depends only on* u_i *and* u_j *for* $1 \leq i < j \leq n$.

Proof. We first prove the rationality of the product of three vertex operators. Without loss of generality, let $v' \in V', u_1, u_2, u_3, v \in V$ be homogeneous elements. We first prove that for some positive integers p_1, p_2, p_{12},

$$(x_1 - x_2)^{p_{12}}(x_1 + x_3)^{p_1}(x_2 + x_3)^{p_2}\langle v', Y_V(u_1, x_1 + x_3)Y_V(u_2, x_2 + x_3)Y_V(u_3, x_3)v\rangle,$$

as a series in $\mathbb{C}[[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]]$ where all negative powers of $(x_1 + x_3)$ and $(x_2 + x_3)$ are expanded in positive powers of x_3, is both upper and lower-truncated. In other words, it is indeed a Laurent polynomial in $\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]$.

We start by peeling off the variable x_3. First note the power of x_3 is lower-truncated. To see it is also upper-truncated, we use the weak associativity to see that

$$
\begin{aligned}
&(x_1 - x_2)^{p_{12}}(x_1 + x_3)^{p_1}(x_2 + x_3)^{p_2}\langle v', Y_V(u_1, x_1 + x_3)Y_V(u_2, x_2 + x_3)Y_V(u_3, x_3)v\rangle\\
=&(x_1 - x_2)^{p_{12}}(x_1 + x_3)^{p_1}(x_2 + x_3)^{p_2}\langle v', Y_V(u_1, x_1 + x_3)Y_V(Y_V(u_2, x_2)u_3, x_3)v\rangle\\
=&(x_1 - x_2)^{p_{12}}(x_1 + x_3)^{p_1}(x_2 + x_3)^{p_2}\langle v', Y_V(Y_V(u_1, x_1)Y_V(u_2, x_2)u_3, x_3)v\rangle
\end{aligned}
$$

Note that the second equality is guaranteed by the pole-order condition: since p_1 depends only on u_1 and v, we don't need to worry about the infinitely many terms given by $Y_V(u_2, x_3)u_3$. If p_1 is only independent of the choice of v' and depends on u_2 as in Proposition 2.1.8, then the equality may not hold.

We claim that the series on the right-hand-side is upper-truncated in x_3. This can be seen by writing the series as

$$
\langle v', Y_V(Y_V(u_1, u_1)Y_V(u_2, x_2)u_3, x_3) = \sum_{m,n}\langle v', Y_V(u_{mn}, x_3)v\rangle x_1^{-m-1}x_2^{-n-1}
$$

Note that the lowest weight of the components of u_{mn} is nonnegative. So the coefficient of x_3^{-p-1} in $Y_V(u_{mn}, x_3)v$ is nonzero only when $\min_{m,n\in\mathbb{Z}} \text{wt } u_{mn} + \text{wt } v - p - 1 = \text{wt } v'$. Thus

$$
-p - 1 = \text{wt } v' - \text{wt } v - \min_{m,n\in\mathbb{Z}} \text{wt } u_{mn} \le \text{wt } v' - \text{wt } v
$$

Hence the power of x_3 is upper-truncated.

Now we compute the coefficient of x_3^{-m-1} for each $m \in \mathbb{Z}$.

$$
\begin{aligned}
&(x_1 - x_2)^{p_{12}}(x_1 + x_3)^{p_1}(x_2 + x_3)^{p_2}\langle v', e^{x_3 D_V}Y_V(u_1, x_1)Y_V(u_2, x_2)e^{-x_3 D_V}Y_V(u_3, x_3)v\rangle\\
=&(x_1 - x_2)^{p_{12}}\sum_{k_1=0}^{p_1}\binom{p_1}{k_1}x_1^{p_1-k_1}x_3^{k_1}\sum_{k_2=0}^{p_2}\binom{p_2}{k_2}x_2^{p_2-k_2}x_3^{k_2}\cdot\\
&\qquad\langle v', \sum_{i=1}^{\infty}\frac{1}{i!}x_3^i D_V^i Y_V(u_1, x_1)Y_V(u_2, x_2)\sum_{j=1}^{\infty}\frac{1}{j!}(-x_3)^j D_V^j\sum_{m\in\mathbb{Z}}(Y_V)_m(u_3)vx_3^{-m-1}\rangle\\
=&\sum_{m\in\mathbb{Z}}\sum_{k_1=0}^{p_1}\sum_{k_2=0}^{p_2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^j}{i!j!}(x_1 - x_2)^{p_{12}}\binom{p_1}{k_1}x_1^{p_1-k_1}\binom{p_2}{k_2}x_2^{p_2-k_2}\cdot\\
&\qquad\langle v', D_V^i Y_V(u_1, x_1)Y_V(u_2, x_2)D_V^j(Y_V)_m(u_3)v\rangle x_3^{-m+k_1+k_2+i+j-1}\\
=&\sum_{m\in\mathbb{Z}}\sum_{k_1=0}^{p_1}\sum_{k_2=0}^{p_2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^j}{i!j!}(x_1 - x_2)^{p_{12}}\binom{p_1}{k_1}x_1^{p_1-k_1}\binom{p_2}{k_2}x_2^{p_2-k_2}\cdot
\end{aligned}
$$

$$\langle v', D_V^i Y_V(u_1,x_1)Y_V(u_2,x_2)D_V^j(Y_V)_{m+k_1+k_2+i+j}(u_3)v\rangle x_3^{-m-1}$$

So the coefficient of x_3^{-m-1} is

$$\sum_{k_1=0}^{p_1}\sum_{k_2=0}^{p_2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^j}{i!j!}(x_1-x_2)^{p_{12}}\binom{p_1}{k_1}x_1^{p_1-k_1}\binom{p_2}{k_2}x_2^{p_2-k_2}\cdot$$
$$\langle v', D_V^i Y_V(u_1,x_1)Y_V(u_2,x_2)D_V^j(Y_V)_{m+k_1+k_2+i+j}(u_3)v\rangle$$

This is actually a finite sum, since for each fixed m, $(Y_V)_{m+k_1+k_2+i+j}(u_3)v = 0$ when $\mathrm{wt}\ u_3 + \mathrm{wt}\ v - (m+k_1+k_2+i+j) < 0$. So both i and j has an upper bound. And for each fixed m, p_1, p_2, i, j, we know that

$$(x_1-x_2)^{p_{12}}\langle v', D_V^i Y_V(u_1,x_1)Y_V(u_2,x_2)D_V^j(Y_V)_{m+k_1+k_2+i+j}(u_3)v\rangle$$

is a Laurent polynomial. Therefore, we proved that for each $m\in\mathbb{Z}$, the coefficient of x_3^{-m-1} is a finite sum of Laurent polynomials. So the power of x_1 and x_2 is both upper- and lower-truncated for each m where m ranges in a finite set. So the powers of x_1 and x_2 in the series are also upper- and lower-truncated.

Since all the powers of x_1, x_2, x_3 are lower-truncated, we can find positive integers p_{13}, p_{23}, p_3 such that

$$(x_1+x_3)^{p_1}(x_2+x_3)^{p_2}x_3^{p_3}x_1^{p_{13}}x_2^{p_{23}}(x_1-x_2)^{p_{12}}\langle v', Y_V(u_1,x_1)Y_V(u_2,x_2)Y_V(u_3,x_3)v\rangle$$

is a polynomial in $\mathbb{C}[x_1,x_2,x_3]$. So the transformation $x_1\mapsto x_1-x_3, x_2\mapsto x_2-x_3, x_3\mapsto x_3$, makes sense and leads to the conclusion that

$$x_1^{p_1}x_2^{p_2}x_3^{p_3}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}(x_2-x_3)^{p_{23}}\langle v', Y_V(u_1,x_1)Y_V(u_2,x_2)Y_V(u_3,x_3)v\rangle \quad (2.1)$$

is a polynomial in $\mathbb{C}[x_1,x_2,x_3]$.

It remains to prove that all the powers $p_1, p_2, p_3, p_{12}, p_{23}, p_{13}$ are bounded above by constants that depend only on the corresponding elements. We start by noting that p_3 is bounded above by a constant that depends only on u_3 and v, due to the lower truncation of $Y_V(u_3,x_3)v$. If we rewrite Formula (2.1) as a sum

$$x_1^{p_1}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}\sum_{n\in\mathbb{Z}}x_2^{p_2}x_3^{p_3}(x_2-x_3)^{p_{23}}\langle v', (Y_V)_n(u_1)Y_V(u_2,x_2)Y_V(u_3,x_3)v\rangle x_1^{-n-1},$$

and apply the pole-order condition to each summand, then we see that p_2 is bounded above by a constant that depends only on u_2 and v. The associativity tells that Formula (2.1) equals

$$x_1^{p_1}x_2^{p_2}x_3^{p_3}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}(x_2-x_3)^{p_{23}}\langle v', Y_V(u_1,x_1)Y_V(Y_V(u_2,x_2-x_3)u_3,x_3)v\rangle.$$

Then p_{23} is bounded above by a constant that depends only on u_2 and u_3. Similarly, the associativity tells that Formula (2.1) equals

$$x_1^{p_1}x_2^{p_2}x_3^{p_3}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}(x_2-x_3)^{p_{23}}\langle v', Y_V(Y_V(u_1,x_1-x_2)u_2,x_2)Y_V(u_3,x_3)v\rangle.$$

Then p_{12} is bounded above by a constant that depends only on u_1 and u_2. Also, the associativity tells that Formula (2.1) equals

$$x_1^{p_1}x_2^{p_2}x_3^{p_3}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}(x_2-x_3)^{p_{23}}\langle v', Y_V(Y_V(u_1,x_1-x_2)Y_V(u_2,x_2-x_3)u_3,x_3)v\rangle,$$

which can be rewritten as

$$x_1^{p_1}x_2^{p_2}x_3^{p_3}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}(x_2-x_3)^{p_{23}}\langle v', \sum_{n\in\mathbb{Z}}(Y_V)_n(Y_V(u_1,x_1-x_3)Y_V(u_2,x_2-x_3)u_3)vx_3^{-n-1}\rangle,$$

Apply the pole-order condition to each summand, we see that p_{13} is bounded above by a constant that depends only on u_1 and u_3. Finally, the associativity tells that Formula (2.1) is equal to

$$x_1^{p_1}x_2^{p_2}x_3^{p_3}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}(x_2-x_3)^{p_{23}}\langle v', Y_V(u_1,x_1)Y_V(Y_V(u_2,x_2-x_3)u_3,x_3)v\rangle,$$

which can be rewritten as

$$x_1^{p_1}x_2^{p_2}x_3^{p_3}(x_1-x_2)^{p_{12}}(x_1-x_3)^{p_{13}}(x_2-x_3)^{p_{23}}\sum_{n\in\mathbb{Z}}\langle v', Y_V(u_1,x_1)Y_V((Y_V)_n(u_2)u_3,x_3)v(x_2-x_3)^{-n-1}\rangle$$

Apply the pole-order condition to each summand, to see that p_1 is bounded above by a constant that depends only on u_1 and v.

The general case can be done by induction. For brevity, we only give a sketch without the details:

1. The formal series
$$\prod_{i=1}^{n-1}(x_i+x_n)^{p_i}\prod_{1\le r<s\le n-1}(x_r-x_s)^{p_{rs}}\langle v', Y_V(u_1,x_1+x_n)\cdots Y_V(u_{n-1},x_{n-1}+x_n)Y_V(u_n,x_n)v\rangle$$

in $\mathbb{C}[[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]]$ has lower-truncated powers of x_n. Use weak associativity to see it is equal to

$$\prod_{i=1}^{n-1}(x_i + x_n)^{p_i}\langle v', Y_V(Y_V(u_1, x_1)\cdots Y_V(u_{n-1}, x_{n-1}u_n, x_n)v\rangle$$

and thus the power of x_n is upper-truncated.

2. Compute the coefficient of x_n^{-m-1}, which looks like

$$\sum_{m\in\mathbb{Z}}\sum_{k_1,...,k_{n-1}=0}^{p_1,...,p_{n-1}}\sum_{i,j=0}^{\infty}\prod_{i=1}^{n-1}\binom{p_i}{k_i}x_i^{p_i-k_i}\frac{(-1)^j}{i!j!}\prod_{1\le r<s\le n-1}(x_r - x_s)^{p_{rs}}$$
$$\langle v', D_V^i Y(u_1, x_1)Y(u_2, x_2)\cdots Y(u_{n-1}, x_{n-1})D_V^j(Y_V)_{m+i+j+k_1+\cdots+k_{n-1}}(u_n)v\rangle(-1)^j$$

argue it is a finite sum of Laurent polynomials in $x_1, ..., x_{n-1}$ which is seen by the induction hypothesis.

3. So the series we are considering is a Laurent polynomial. Find all the integers such that

$$x_n^{p_n}\prod_{i=1}^{n-1}x_i^{p_{in}}(x_i + x_n)^{p_i}\prod_{1\le r<s\le n-1}(x_r - x_s)^{p_{rs}}\cdot$$
$$\langle v', Y_V(u_1, x_1 + x_n)\cdots Y_V(u_{n-1}, x_{n-1} + x_n)Y_V(u_n, x_n)v\rangle$$

is a polynomial in $\mathbb{C}[x_1, ..., x_n]$. Then perform the transformation $x_i \mapsto x_i - x_n, x_n \mapsto x_n$ to see the rationality.

4. Repeatedly use associativity and the pole-order condition obtained in the previous steps to show the dependence of the upper bounds of the order of poles.

□

Proposition 2.1.13. *Let* $V = \coprod_{n\in\mathbb{Z}} V_{[n]}$, $Y_V : V \otimes V \to v[[x, x^{-1}]]$ *satisfy axioms for the grading, D-derivative property, D-commutator formula, and the following weak associativity with pole order condition: for every* $u_1, u_2, v \in V$, *there exists an integer* p_1 *that depends only on* u_1 *and* v, *such that*

$$(x_0 + x_2)^{p_1}Y_V(Y_V(u_1, x_0)u_2, x_2)v = (x_0 + x_2)^{p_1}Y_V(u_1, x_0 + x_2)Y_V(u_2, x_2)v$$

as formal series in $V[[x_0, x_0^{-1}, x_2, x_2^{-1}]]$, *then* $(V, Y_V, \mathbf{1})$ *forms a MOSVA.*

Remark 2.1.14. From this proof one observes that the pole-order condition is crucial for the formal variable approach. Although this condition holds for all the existing MOSVAs, with the absence of commutativity this is still not a natural assumption. That is why for the rest of the thesis, we will still develop the theory of MOSVA without this condition.

2.2 $\overline{V}$-valued map interpretation

Let

$$\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$$

be the algebraic completion of the graded space $\coprod_{n\in\mathbb{Z}} V_{(n)}$. Let

$$\widehat{V} = \prod_{n\in\mathbb{Z}} V_{(n)}^{**}$$

be the full dual space of $V' = \coprod_{n\in\mathbb{Z}} V_{(n)}^*$. In general, $\overline{V}$ is a subspace of $\widehat{V}$. When V is grading-restricted, we have $\overline{V} = \widehat{V}$.

We shall interpret the vertex operators as $\overline{V}$-valued maps for grading-restricted meromorphic open-string vertex algebras (which is of the most interest and shall be our main focus). The modification for non-grading-restricted MOSVAs will be discussed in Remarks 2.2.10, 2.2.14, 2.2.18, 2.2.23, 2.3.12, 2.3.15 and 2.4.9.

2.2.1 One single vertex operator

Since each vertex operator $Y_V(u,x), u \in V$ admits a series expansion

$$Y_V(u,x) = \sum_{n\in\mathbb{Z}} (Y_V)_n(u)x^{-n-1}$$

When u is homogeneous, each $(Y_V)_n(u) : V \to V$ of weight wt $(u) - n - 1$. Replacing x by a nonzero complex number z and apply $Y_V(u,z)$ to a homogeneous $v \in V$. Then each $(Y_V)_n(u)vz^{-n-1}$ is homogeneous of weight wt (u) + wt $(v) - n - 1$, hence the infinite sum

$$\sum_{n\in\mathbb{Z}} (Y_V)_n(u)vz^{-n-1}$$

gives an element in $\prod_{n\in\mathbb{Z}} V_{(n)} = \overline{V}$. This also holds for any $u, v \in V$ since they are finite sums of homogeneous elements. So we conclude the following:

Summary 2.2.1. *For a given nonzero $z \in \mathbb{C}$, the vertex operator map give rise to the following map*

$$Y_V(\cdot, z)\cdot : V \otimes V \to \overline{V}$$

Remark 2.2.2. Note that here $Y_V(u, z)v$ is regarded as one single element in $\overline{V}$, instead of a series of elements in V.

Remark 2.2.3. This interpretation works no matter whether V is grading-restricted or not.

2.2.2 Product of two vertex operators

Note that for fixed nonzero $z_1 \in \mathbb{C}$ and $u_1 \in V$, the map

$$Y_V(u_1, z_1)\cdot : V \to \overline{V}$$

accepts only inputs of V. To apply the vertex operator $Y_V(u_1, z_1)$ to $Y_V(u_2, z_2)v$, the following steps should be carried out:

1. For each $k \in \mathbb{Z}$, apply the projection operator $\pi_k : \prod\limits_{n\in\mathbb{Z}} V_{(n)} \to V_{(k)}$ to the $\overline{V}$-element, so as to get an element $\pi_k Y_V(u_2, z_2)v \in V$.

2. Apply the vertex operator $Y_V(u_1, z_1)$ to each $\pi_k Y_V(u_2, z_2)$, to get

$$Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2) \in \prod_{n\in\mathbb{Z}} V_{(n)} = \overline{V}$$

3. Sum up all $k \in \mathbb{Z}$ to get the following infinite series

$$\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v$$

   of elements in $\overline{V}$.

To make sense of the infinite series, we need to define a topology on $\overline{V}$.

2.2.3 A note on topology

Here we recall some definitions and notions in topology.

Definition 2.2.4. Let X be a complex vector space. A seminorm on V is a map $p : X \to [0, \infty)$ satisfying

1. $p(x + y) \leq p(x) + p(y), \forall x, y \in X$;

2. $p(\lambda x) = |\lambda| p(x), \forall x \in X, \lambda \in \mathbb{C}$.

Definition 2.2.5. Let X be a vector space over $\mathbb{C}$, $\mathcal{P}$ be a set of seminorms. Define a topology on X by

1. A basic neighborhood of 0 is a set N of the form

$$N = \{x \in V : p(x) < \epsilon_p \text{ for finitely many } p \in \mathcal{P}, \epsilon_p > 0\}$$

   Note that N is convex, containing 0. Also note that the intersection of two basic neighborhoods is also a basic neighborhood.

2. A neighborhood of 0 is a set that contains a basic neighborhood.

3. A set U is a neighborhood of p if $U \supseteq p + N$ for some neighborhood of 0.

4. U is open if U is a neighborhood for every $x \in U$, i.e., $\forall p \in V, \exists \epsilon_1, ..., \epsilon_n \in \mathbb{R}_{>0}, \exists p_1, ..., p_n \in \mathcal{P}, \{q : p_1(q - x) < \epsilon_1, ..., p_n(q - x) < \epsilon_n\} \subseteq U$

Theorem 2.2.6. *With the topology defined above, X is a locally convex topological vector space.*

Proof. See [R]. $\square$

Definition 2.2.7. Let $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ be a graded vector space over $\mathbb{C}$. Let $V' = \coprod_{n \in \mathbb{Z}} V^*_{(n)}$ be the graded dual and let $\widehat{V} = \prod_{n \in \mathbb{Z}} V_{(n)}$ be full dual space of V'. The following set of seminorms

$$\{p_{v'} : \widehat{V} \to \mathbb{C}, p_{v'}(v) \to \langle v', v \rangle\}$$

defines a locally convex topological vector space structure on $\widehat{V}$.

Definition 2.2.8. Let $\{v_\lambda : \lambda \in \Lambda\}$ be an indexed family of elements in $\widehat{V}$. We say the series

$$\sum_{\lambda\in\Lambda} v_\lambda$$

converges absolutely in $\widehat{V}$ if for every $v' \in V'$, the complex series

$$\sum_{\lambda\in\Lambda} \langle v', v_\lambda\rangle$$

converges absolutely. In this case, the sum of the series is a well-defined element in $\widehat{V}$.

Remark 2.2.9. It suffices to check the definition for homogeneous $v' \in V'$, i.e. for each $l \in \mathbb{Z}$ and each $v' \in V^*_{(l)}$.

Remark 2.2.10. When V is grading-restricted, as $\overline{V} = \widehat{V}$, so in this case, the sum of an absolutely convergent infinite series do fall in $\overline{V}$. In general, $\overline{V}$ is only a linear subspace of $\widehat{V}$ and might not necessarily be closed. So the sum of an absolutely convergent infinite series in $\overline{V}$ does not necessarily fall in $\overline{V}$.

Now we investigate the expression $Y_V(u_1, z_1)Y_V(u_2, z_2)v$. The rationality states that for fixed $z_1, z_2 \in \mathbb{C}$ such that $|z_1| > |z_2| > 0$, the double series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle v', (Y_V)_n(u_1)((Y_V)_m(u_2)v)\rangle z_1^{-n-1} z_2^{-m-1}.$$

converges absolutely to a complex number, for every $v' \in V$. Thus at the very least, the sum of the double series $Y_V(u_1, z_1)Y_V(u_2, z_2)v$ is well-defined in $\overline{V}$.

Moreover, note that when v', u_1, u_2, v are homogeneous, the coefficient $\langle v', (Y_V)_n(u_1)((Y_V)_m(u_2)v)\rangle$ is nonzero only when

$$m + n = \text{wt }(u_1) + \text{wt }(u_2) + \text{wt }(v) - \text{wt }(v') - 2$$

So in this case,

$$\begin{aligned}&\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle \\ &= \sum_{m+n=\text{wt}(u_1)+\text{wt}(u_2)+\text{wt}(v)-\text{wt}(v')-2} \langle v', (Y_V)_n(u_1)((Y_V)_m(u_2)v)\rangle z_1^{-n-1} z_2^{-m-1}.\end{aligned}$$

And rationality states that the series converges absolutely when $|z_1| > |z_2| > 0$. In particular, any rearrangement of the series on the right-hand-side converges to the same value.

Proposition 2.2.11. *For any $u_1, u_2, v \in V$ and any complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the single series*

$$\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v$$

of elements in $\overline{V}$ is absolutely convergent, i.e.,

$$\sum_{k\in\mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v\rangle$$

is absolutely convergent for any $v' \in V'$. Moreover, the sum of the series is equal to the sum of

$$Y_V(u_1, z_1)Y_V(u_2, z_2)v$$

Proof. We first deal with homogeneous $u_1, u_2, v \in V$. Since

$$Y_V(u_2, z_2)v = \sum_{m\in\mathbb{Z}} (Y_V)_m(u_2)vz_2^{-m-1}$$

For $k \in \mathbb{Z}$, we apply π_k:

$$\pi_k Y_V(u_2, z_2)v = (Y_V)_{m(k)}(u_2)vz_2^{-m(k)-1}$$

to get an element in $V_{(k)}$, where $m(k) = \text{wt }(u_2) + \text{wt }(v) - k - 1$. So

$$Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v = \sum_{n\in\mathbb{Z}} (Y_V)_n(u_1)\left((Y_V)_{m(k)}(u_2)v\right) z_2^{-m(k)-1} z_1^{-n-1}$$

is an element in $\overline{V}$, with each component in $V_{(l)}$ being

$$\pi_l Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v = (Y_V)_{n(l)}(u_1)\left((Y_V)_{m(k)}(u_2)v\right) z_2^{-m(k)-1} z_1^{-n(l)-1},$$

where $n(l) = \text{wt }(u_1) + \text{wt }(u_2) + \text{wt }(v) - l - m(k) - 2$. Hence the infinite sum $\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v$ of elements in $\overline{V}$ converges absolutely to an element in $\overline{V}$ if for each $l \in \mathbb{Z}$ and each $v' \in V_{(l)}^*$, the infinite sum

$$\sum_{k\in\mathbb{Z}} \langle v', (Y_V)_{n(l)}(u_1)\left((Y_V)_{m(k)}(u_2)v\right)\rangle z_2^{-m(k)-1} z_1^{-n(l)-1}$$

converges absolutely. When $|z_1| > |z_2| > 0$, this is true because the series is a rearrangement of the following absolutely convergent series

$$\sum_{m+n=\text{wt}(u_1)+\text{wt}(u_2)+\text{wt}(v)-l-2} \langle v', (Y_V)_n(u_1)((Y_V)_m(u_2)v)\rangle z_1^{-n-1} z_2^{-m-1}.$$

So when $u_1, u_2, v \in V$ are homogeneous, we proved that the series $\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v$ converges absolutely to an element in $\overline{V}$ when $|z_1| > |z_2| > 0$, where the element is given by $Y_V(u_1, z_1)Y_V(u_2, z_2)v$.

For general nonhomogeneous u_1, u_2, v, write

$$u_1 = \sum_{j_1 \text{ finite}} u_1^{(p^1_{j_1})}, u_2 = \sum_{j_2 \text{ finite}} u_2^{(p^2_{j_2})}, v = \sum_{m \text{ finite}} v^{(q_m)}$$

We already know from above that for each fixed j_1, j_2, m and each fixed $l \in \mathbb{Z}$ and $v' \in V^*_{(l)}$,

$$\langle v', Y_V(u_1^{(p_{j_1})}, z_1)Y_V(u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle = \sum_{k\in\mathbb{Z}} \langle v', Y_V(u_1^{(p_{j_1})}, z_1)\pi_k Y_V(u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle,$$

it follows that

$$\begin{aligned}
\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle &= \sum_{j_1,j_2,m \text{ finite}} \langle v', Y_V(u_1^{(p_{j_1})}, z_1)Y_V(u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle \\
&= \sum_{j_1,j_2,m \text{ finite}} \sum_{k\in\mathbb{Z}} \langle v', Y_V(u_1^{(p_{j_1})}, z_1)\pi_k Y_V(u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle \\
&= \sum_{k\in\mathbb{Z}} \sum_{j_1,j_2,m \text{ finite}} \langle v', Y_V(u_1^{(p_{j_1})}, z_1)\pi_k Y_V(u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle \\
&= \sum_{k\in\mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v\rangle,
\end{aligned}$$

the third equality of which is justified because a finite sum of absolutely convergent series is still absolutely convergent, and for absolutely convergent series the order of summation can be rearranged. So we proved that the sum $\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v$ absolutely converges to an element in $\overline{V}$ when $|z_1| > |z_2| > 0$, where the element is given by $Y_V(u_1, z_1)Y_V(u_2, z_2)v$. $\square$

Summary 2.2.12. *For fixed z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the product of two vertex operators gives rise to the following map*

$$Y_V(\cdot, z_1)Y_V(\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

which is equal to the map

$$\sum_{k\in\mathbb{Z}} Y_V(\cdot, z_1)\pi_k Y_V(\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

Remark 2.2.13. We emphasize that the sum $\sum_{k\in\mathbb{Z}} Y_V(u_1, z_2)\pi_k Y_V(u_2, z_2)v$ should be regarded as a single series in $\overline{V}$, while $Y_V(u_1, z_1)Y_V(u_2, z_2)v = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(Y_V)_m(u_1)((Y_v)_n(u_2)v)z_1^{-m-1}z_2^{-n-1}$ are regarded as a double series in V. The proof above amounts to conclude the the absolute convergence of the former series from the absolute convergence of the latter series, which is guaranteed by the rationality. When $u_1, u_2, v \in V$ are homogeneous, the double series $Y_V(u_1, z_1)Y_V(u_2, z_2)v$ reduces to a single sum and indeed coincides with the series $\sum_{k\in\mathbb{Z}} Y_V(u_1, z_2)\pi_k Y_V(u_2, z_2)v$. When $u_1, u_2, v \in V$ are not homogeneous, these two series no longer coincide and should not be recognized as identical to each other. It is the sums of these series that are identical, not the series themselves.

Remark 2.2.14. When V is not grading-restricted:

1. The statement of Summary 2.2.12 do not hold when V is not grading-restricted. In regarding to Remark 2.2.10, unless we know $\overline{V}$ is a closed linear subspace of $\widehat{V}$, we can only conclude that $Y_V(u_1, z_1)Y_V(u_2, z_2)v$ and $\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v$ are elements in $\widehat{V}$.

2. However, the conclusions in Proposition 2.2.11 do hold, as essentially we are realizing the single complex series $\sum_{k\in\mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)v\rangle$ as a rearrangement of the absolutely convergent double complex series $\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v\rangle$ and use this realization to prove that the single series is absolutely convergent. Although the sum falls outside $\overline{V}$, the series still converges absolutely.

2.2.4 Product of any number of vertex operators

The above discussion generalizes to the product of any number of vertex operators. Instead of getting into too much technical details, we sketch the steps here:

1. Rationality of the product of n vertex operators states that for each $k \in \mathbb{Z}$ and each $v' \in V_{(k)}^*$, when $|z_1| > |z_2| > \cdots > |z_n| > 0$, the multi-series

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)\cdots Y_V(u_n, z_n)v\rangle$$

absolutely converges. Hence $Y_V(u_1, z_1)Y_V(u_2, z_2)\cdots Y_V(u_n, z_n)v$ is well-defined in $\overline{V}$.

2. When all $u_1, ..., u_n$ and v are homogeneous, we have

$$\begin{aligned}&\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)\cdots Y_V(u_n, z_n)v\rangle \\ &= \sum_{\substack{m_1+\cdots+m_n=\mathrm{wt}(u_1)+\cdots+\mathrm{wt}(u_n)-k-n \\ m_1,...,m_n\in\mathbb{Z}}} \langle v', (Y_V)_{m_1}(u_1)(\cdots((Y_V)_{m_n}(u_n)v)\cdots)z_1^{-m_1-1}\cdots z_n^{-m_n-1}\rangle\end{aligned}$$

Rationality states that this series is absolutely convergent when $|z_1| > \cdots > |z_n| > 0$.

3. We argue that when $|z_1| > |z_2| > \cdots > |z_n|$, the series

$$\sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_{k_2}\cdots Y_V(u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v$$

of elements of $\overline{V}$ converges absolutely to an element in $\overline{V}$ identical to that $Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v$ converges to. This is done by first arguing that for each $l \in \mathbb{Z}$, each $v' \in V^*_{(l)}$ and each homogeneous $u_1, u_2, ..., u_n, v \in V$,

$$\sum_{k_1,k_2,...,k_{n-1}\in\mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_{k_2}\cdots\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle$$

is a rearrangement of the multi-series

$$\sum_{\substack{m_1+\cdots+m_n=\mathrm{wt}(u_1)+\cdots+\mathrm{wt}(u_n)-k-n \\ m_1,...,m_n\in\mathbb{Z}}} \langle v', (Y_V)_{m_1}(u_1)(\cdots((Y_V)_{m_n}(u_n)v)\cdots)\rangle z_1^{-m_1-1}\cdots z_n^{-m_n-1}.$$

Once we set up the equality for homogeneous elements, using the fact that a finite sum of absolutely convergent series is absolutely convergent and thus the order of summation can be rearranged, we generalize the equality to nonhomogeneous $u_1, u_2, ..., u_n, v$ by a finite sum argument. Technical details are skipped here.

Summary 2.2.15. *For any $u_1, ..., u_n, v \in V$ and any $z_1, ..., z_n \in \mathbb{C}$ satisfying $|z_1| > |z_2| > \cdots > |z_n| > 0$, the series*

$$\sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_2\cdots Y_V(u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v$$

of elements in $\overline{V}$ converges absolutely, The sum is equal to the $\overline{V}$-element given by

$$Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v$$

For fixed $z_1, z_2, ..., z_n \in \mathbb{C}$ satisfying $|z_1| > \cdots > |z_n| > 0$, the product of any number of vertex operators gives rise to a map

$$Y_V(\cdot, z_1)Y_V(\cdot, z_2) \cdots Y_V(\cdot, z_n)\cdot : V^{\otimes n} \otimes V \to \overline{V}$$

and is equal to the sum

$$\sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_V(\cdot, z_1)\pi_{k_1}Y_V(\cdot, z_2)\pi_{k_2} \cdots Y_V(\cdot, z_{n-1})\pi_{k_{n-1}}Y_V(\cdot, z_n)\cdot : V^{\otimes n} \otimes V \to \overline{V}$$

Remark 2.2.16. So we also know that when $|z_1| > |z_2| > \cdots > |z_n| > 0$,

$$\sum_{k_1,...,k_{n-1}\in\mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_{k_2} \cdots Y_V(u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle$$

converges absolutely to a rational function with the only possible poles at $z_i = 0, i = 1, 2, ..., n; z_i = z_j, i, j = 1, 2, ..., n$. As we will see, this makes it easy to discuss of the region of convergence.

Remark 2.2.17. Just as in Remark 2.2.13, We emphasize that for general $u_1, ..., u_n, v \in V$, the multiseries

$$\sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_{k_2} \cdots Y_V(u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v$$

is a $(n-1)$-multiseries of elements in $\overline{V}$ and should not be recognized as the same series as

$$Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v = \sum_{m_1,...,m_n\in\mathbb{Z}} (Y_V)_{m_1}(u_1)(\cdots((Y_V)_{m_n}(u_n)v)\cdots)z_1^{-m_1-1} \cdots z_n^{-m_n-1},$$

which is a n-multiseries of elements in V, though their sums are equal when $|z_1| > \cdots > |z_n| > 0$.

Remark 2.2.18. When V is not grading-restricted:

1. As shown in Remark 2.2.14, $Y_V(u_2, z_2)Y_V(u_3, z_3)v$ does not necessarily sit in $\overline{V}$. The π_k in the series $\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)Y_V(u_3, z_3)v$ has to extend to $\widehat{V}$, i.e., $\pi_k : \widehat{V} \to V_{(k)}^{**}$.

2. As $\pi_k Y_V(u_2, z_2) Y_V(u_3, z_3)v$ does not fall in V, we would interpret the action of $Y_V(u_1, z_1)$ on the $V_{(k)}^{**}$-element simply as a rearrangement of summation. More precisely,

$$Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2) Y_V(u_3, z_3)$$
$$= \sum_{\substack{\text{wt } u_2 + \text{wt } u_3 - n - p - 2 = k \\ n, p \in \mathbb{Z}}} \left((Y_V)_n(u_1)(Y_V)_m(u_2)(Y_V)_p(u_3) v z_1^{-n-1} z_2^{-m-1} z_3^{-m-1} \right)$$

So although $\pi_k Y_V(u_2, z_2) Y_V(u_3, z_3)$ falls in the space $V_{(k)}^{**}$ much larger than $V_{(k)}$, $Y_V(u_1, z_1)$ still "act" on it in the sense above. Such an action is well-defined because of the rationality of products of three vertex operators.

3. The interpretation extends to the product of n vertex operators in a similar way, i.e.

$$Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2) \cdots Y_V(u_n, z_n) v$$
$$= \sum_{\substack{\text{wt } u_2 + \cdots + \text{wt } u_n - m_2 - \cdots - m_n - n + 1 = k \\ m_2, \ldots, m_n \in \mathbb{Z}}} \left((Y_V)_{m_1}(u_1) \cdots (Y_V)_{m_n}(u_n) v \right) z_1^{-m_1-1} \cdots z_n^{-m_n-1}$$

4. The conclusion of Summary 2.2.15 has to be modified, as everything now sits in $\widehat{V}$. Since all the $Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v$ spans only a very small subspace in $\overline{V}$, we don't think it necessary to give an explicit formulation.

5. The conclusion of Remark 2.2.16 still hold, as we did prove the absolute convergence of the series $\sum_{k \in \mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_{k_1} Y_V(u_2, z_2) \cdots \pi_{k_{n-1}} Y_V(u_n, z_n) v \rangle$.

2.2.5 Iterate of two vertex operators

The $\overline{V}$-description to vertex operators directly applies to the iterate of two vertex operators. Let's fix a nonzero $z_2 \in \mathbb{C}$ and $v \in V$. Then the map

$$Y_V(\cdot, z_2)v : V \to \overline{V}$$

accepts only inputs in V. Likewise, for each $k \in \mathbb{Z}$, we apply the projection operator π_k to get an element $\pi_k Y_V(u_1, z_1 - z_2)v$ in V. Then we apply the vertex operator to get $Y_V(\pi_k(Y_V(u_1, z_1 - z_2)u_2), z_2)v \in \overline{V}$. Finally we sum up to get the infinite series

$$\sum_{k \in \mathbb{Z}} Y_V(\pi_k(Y_V(u_1, z_1 - z_2)u_2), z_2)v$$

of elements in $\overline{V}$. The following proposition makes sense of the infinite series for some choices of z_1, z_2:

Proposition 2.2.19. *For any $u_1, u_2, v \in V$ and any complex numbers z_1, z_2 satisfying $|z_2| > |z_1 - z_2| > 0$, the single series*

$$\sum_{k\in\mathbb{Z}} Y_V(\pi_k(Y_V(u_1, z_1 - z_2)u_2), z_2)v$$

of elements in $\overline{V}$ converges absolutely, i.e., the complex series

$$\sum_{k\in\mathbb{Z}} \langle v', Y_V(\pi_k(Y_V(u_1, z_1 - z_2)u_2), z_2)v\rangle$$

converges absolutely. Its sum is equal to the $\overline{V}$ element given by

$$Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v$$

Proof. We first verify this for homogeneous $u_1, u_2, v \in V$. Since

$$Y_V(u_1, z_1 - z_2)u_2 = \sum_{m\in\mathbb{Z}} (Y_V)_m(u_1)u_2(z_1 - z_2)^{-m-1}$$

For $k \in \mathbb{Z}$, we apply π_k:

$$\pi_k Y_V(u_1, z_1 - z_2)u_2 = (Y_V)_{m(k)}(u_1)u_2(z_1 - z_2)^{-m(k)-1}$$

where $m(k) = \text{wt } (u_1) + \text{wt } (u_2) - k - 1$. So

$$Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v = \sum_{n\in\mathbb{Z}} (Y_V)_n((Y_V)_{m(k)}(u_1)u_2)z_2^{-n-1}(z_1 - z_2)^{-m(k)-1}$$

gives an element in $\overline{V}$, with the projection in $V_{(l)}$ being

$$\pi_l Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v = (Y_V)_{n(l)}((Y_V)_{m(k)}(u_1)u_2)z_2^{-n(l)-1}(z_1 - z_2)^{-m(k)-1}$$

where $n(l) = \text{wt } u_1 + \text{wt } u_2 + \text{wt } v - m(k) - l - 2$. The summation $\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v$ makes sense in $\overline{V}$ if for every $l \in \mathbb{Z}$ and every $v' \in V_{(l)}^*$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \langle v', Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle \\
&\quad = \sum_{k\in\mathbb{Z}} \langle v', (Y_V)_{n(l)}((Y_V)_{m(k)}(u_1)u_2)\rangle z_2^{-n(l)-1}(z_1 - z_2)^{-m(k)-1}
\end{aligned}$$

converges. Note that the the second part of the rationality (Axiom 4) states that when $|z_2| > |z_1 - z_2| > 0$, the double series

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\langle v', (Y_V)_m\left((Y_V)_n(u_1)u_2\right)v\rangle(z_1-z_2)^{-n-1}z_2^{-m-1}$$

converges absolutely. So $Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v$ is an element in $\overline{V}$. Moreover, since u_1, u_2, v are homogeneous, $\langle v', (Y_V)_m\left((Y_V)_n(u_1)u_2\right)v\rangle$ is zero except when wt (u_1) + wt (u_2) + wt $(v) - n - m - 2 = l$. So the series

$$\sum_{\substack{m+n=\mathrm{wt}(u_1)+\mathrm{wt}(u_2)+\mathrm{wt}(v)-l-2\\ m,n\in\mathbb{Z}}}\langle v', (Y_V)_m\left((Y_V)_n(u_1)u_2\right)v\rangle(z_1 - z_2)^{-n-1}z_2^{-m-1}$$

converges absolutely. With a rearrangement we will recover the series we want

$$\sum_{k\in\mathbb{Z}}\langle v', (Y_V)_{n(l)}((Y_V)_{m(k)}(u_1)u_2)\rangle z_2^{-n(l)-1}(z_1 - z_2)^{-m(k)-1},$$

So we proved that the sum $\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v$ absolutely converges to an element in $\overline{V}$ when $|z_2| > |z_1 - z_2| > 0$, where the element is given by $Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)$.

For general nonhomogeneous u_1, u_2, v, write

$$u_1 = \sum_{j_1 \text{ finite}} u_1^{(p^1_{j_1})}, u_2 = \sum_{j \text{ finite}} u_2^{(p^2_{j_2})}, v = \sum_{m \text{ finite}} v^{(q_m)}$$

We already know from above that for each fixed j_1, j_2, m and each fixed $l \in \mathbb{Z}$ and $v' \in V^*_{(l)}$,

$$\langle v', Y_V(Y_V(u_1^{(p_{j_1})}, z_1 - z_2)u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle = \sum_{k\in\mathbb{Z}}\langle v', Y_V(\pi_k Y_V(u_1^{(p_{j_1})}, z_1 - z_2), z_2)v^{(q_m)}\rangle,$$

it follows that

$$\begin{aligned}
\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle &= \sum_{j,m,n \text{ finite}}\langle v', Y_V(Y_V(u_1^{(p_{j_1})}, z_1 - z_2)u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle\\
&= \sum_{j,m,n \text{ finite}}\sum_{k\in\mathbb{Z}}\langle v', Y_V(\pi_k Y_V(u_1^{(p_{j_1})}, z_1 - z_2)u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle\\
&= \sum_{k\in\mathbb{Z}}\sum_{j,m,n \text{ finite}}\langle v', Y_V(\pi_k Y_V(u_1^{(p_{j_1})}, z_1 - z_2)u_2^{(p_{j_2})}, z_2)v^{(q_m)}\rangle\\
&= \sum_{k\in\mathbb{Z}}\langle v', Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v\rangle
\end{aligned}$$

the third equality of which is justified because a finite sum of absolutely convergent series is still absolutely convergent, and for absolutely convergent series the order of summation can be rearranged. So we proved that the sum $\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v$ absolutely converges to an element in $\overline{V}$ when $|z_2| > |z_1 - z_2| > 0$, where the element is given by $Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v$. □

Summary 2.2.20. *For fixed z_1, z_2 satisfying $|z_2| > |z_1 - z_2| > 0$, the iterate of two vertex operators gives rise to a map*

$$Y_V(Y_V(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

which is equal to the sum

$$\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

Remark 2.2.21. Just as in Remark 2.2.13, We emphasize that for general $u_1, u_2, v \in V$, the series

$$\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)v$$

is a single series of elements in $\overline{V}$ and should not be recognized as the same series as

$$Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} (Y_V)_n((Y_V)_m(u_1)v)(z_1 - z_2)^{-m-1} z_2^{-n-1}$$

which is a double series of elements in V. It is their sums that are equal when $|z_2| > |z_1 - z_2| > 0$, not the series themselves.

Taking the associativity in Axiom 5 into account, we have the following:

Summary 2.2.22. *For fixed z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, the following maps*

$$Y_V(\cdot, z_1)Y_V(\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

$$\sum_{k\in\mathbb{Z}} Y_V(\cdot, z_1)\pi_k Y_V(\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

$$Y_V(Y_V(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

$$\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes V \to \overline{V}$$

are equal.

Remark 2.2.23. When V is not grading-restricted, all the maps in Summary 2.2.22 are $\widehat{V}$-valued.

2.3 Rationality of the iterate of n vertex operators

We shall use the $\overline{V}$-valued maps interpretation to prove the rationality of the iteration of any number of vertex operators, i.e., the series

$$\langle v', Y_V(Y_V(\cdots Y_V(u_1, z_1 - z_2)u_2, \cdots, z_{n-1} - z_n)u_n, z_n)v\rangle$$

converges absolutely to the same rational function

$$\langle v', Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v\rangle$$

for certain choices of $(z_1, ..., z_n) \in \mathbb{C}^n$. As this is an analytic statement, some facts in complex analysis will be needed.

2.3.1 A note on complex analysis

Definition 2.3.1. A *multicircular domain* $E \subseteq \mathbb{C}^n$ (centered at the origin) is an open subset such that

$$(z_1, ..., z_n) \in E \text{ implies } (z_1 e^{i\theta_1}, ..., z_n e^{i\theta_n}) \in E$$

for every $\theta_1, ..., \theta_n \in \mathbb{R}$. The *trace* of a multicircular domain $E \subset \mathbb{C}$ is given by

$$\mathrm{Tr}E = \{(|z_1|, ..., |z_n|) \in \mathbb{R}_+^n : (z_1, ..., z_n) \in E\}$$

We need the following results in several complex variable functions (see for example [KW]

Lemma 2.3.2. *A multicircular domain $E \subset \mathbb{C}^n$ is connected if and only if* $\mathrm{Tr}E \subset \mathbb{R}_+^n$ *is connected.*

Proof. Since the map $\phi : \mathbb{C}^n \to \mathbb{R}_+^n : (z_1, ..., z_n) \to (|z_1|, ..., |z_n|)$ is continuous, thus if E is connected then $\mathrm{Tr}E$ is also connected. Conversely, if $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$, then both E_1 and E_2 are multicircular, and $\phi(E) = \phi(E_1) \cup \phi(E_2)$. We claim that $\phi(E_1) \cap \phi(E_2) = \emptyset$. Suppose this is not the case, then there exists two points $(z_1, ..., z_n) \in E_1$ and $(w_1, ..., w_n) \in E_2$ such that $|z_1| = |w_1|, ..., |z_n| = |w_n|$. Hence there exists $\theta_1, ..., \theta_n \in \mathbb{R}$ such that $z_1 = e^{i\theta_1} w_1, ..., z_n = e^{i\theta_n} w_n$. But since E_2 is also multicircular, this is to say that $(z_1, ..., z_n)$ is also in E_2, $\square$

Theorem 2.3.3. *Let E be a connected multicircular domain. Let f be a holomorphic function on E. Then there is a unique n-variable Laurent series with center 0 and constant coefficients which converges to $f(z_1, ..., z_n)$ at every point of E for some total ordering of its terms. It is the series*

$$\sum_{\alpha_1 \in \mathbb{Z}, ..., \alpha_n \in \mathbb{Z}} c_{\alpha_1 \ldots \alpha_n} z_1^{-\alpha_1 - 1} \cdots z_n^{-\alpha_n - 1}$$

whose coefficients are given by the formula

$$c_{\alpha_1 \ldots \alpha_n} = \frac{1}{(2\pi i)^n} \int \cdots \int_{T(0,r)} f(z) z_1^{\alpha_1} \cdots z_n^{\alpha_n} dz_1 \cdots dz_n$$

for any $r = (r_1, ..., r_n) > 0$ in the trace of E. The series will actually be absolutely convergent on E and it will converge uniformly to f on any compact subset of E.

Proof. See Theorem 1.5.4, Theorem 2.7.1 and the discussion in Section 2.8 of [KW]. $\square$

Remark 2.3.4. If the Laurent series is lower-truncated in z_n, let $-M_n$ be a lower bound of the powers of z_n, then one can recover the coefficient of z_n from the derivatives of $z_n^{M_n} f(z_1, ..., z_n)$. More precisely, we have

$$\sum_{\alpha_1, ..., \alpha_{n-1} \in \mathbb{Z}} c_{\alpha_1 \ldots \alpha_n} z_1^{\alpha_1} \cdots z_{n-1}^{\alpha_{n-1}} = \frac{1}{(\alpha_n + M_n)!} \lim_{z_n = 0} \left(\frac{\partial}{\partial z_n} \right)^{\alpha_n + M_n} (z_n^{M_n} f(z_1, ..., z_n))$$

Lemma 2.3.5. *Let f be a rational function in z_1, z_2. Let T be a connected multicircular domain on which the lowest power of z_2 in the Laurent series expansion of $f(z_1, z_2)$ is the same as the order of pole $z_2 = 0$. Let S be a nonempty open subset of T and S' be the image of S via the projection $(z_1, z_2) \mapsto z_1$. Assume that for any fixed $k_2 \in \mathbb{Z}$, the series*

$$\sum_{k_1 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1}$$

converges absolutely for every $z_1 \in S'$*, and*

$$\sum_{k_2 \in \mathbb{Z}} \left(\sum_{k_1 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1} \right) z_2^{k_2}, \tag{2.2}$$

viewed as a series whose terms are $\left(\sum_{k_1 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1}\right) z_2^{k_2}$*, is lower-truncated in* z_2 *and converges absolutely to* $f(z_1, z_2)$ *whenever* $(z_1, z_2) \in S$*. Then the corresponding multi-series*

$$\sum_{k_1, k_2 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1} z_2^{k_2} \tag{2.3}$$

converges absolutely to $f(z_1, z_2)$ *whenever* $|z_1| > |z_2| > 0$*.*

Proof. Fix $z_1 \in \mathbb{C}$ and let $z_2 \in \mathbb{C}$ such that $(z_1, z_2) \in S$. Since S is open and the series

$$\sum_{k_2 \in \mathbb{Z}} \left(\sum_{k_1 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1} \right) w_2^{k_2},$$

is lower-truncated in w_2, one can find a real number $r \geq |z_2| > 0$ (depending on z_1) such that the series converges absolutely in the region $\{w_2 \in \mathbb{C} : 0 < |w_2| < r\}$ to $f(z_1, w_2)$. By assumption, the power of w_2 is lower-truncated. Let M_2 be the lowest power of w_2. Then

$$w_2^{M_2} \sum_{k_2 \in \mathbb{Z}} \left(\sum_{k_1 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1} \right) w_2^{k_2},$$

is a power series with variable w_2, which converges uniformly in the region $\{w_2 \in \mathbb{C} : |w_2| < r\}$ to $w_2^{M_1} f(z_1, w_2)$. In particular, $w_2^{M_1} f(z_1, w_2)$ is defined when $w_2 = 0$. From the limit ratio test and the fact that $\lim_{n \to \infty} n/(n+1) = 1 < 2$, one sees that the derivative of the series

$$\sum_{k_2 \in \mathbb{Z}} (k_2 + M_2) \left(\sum_{k_1 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1} \right) w_2^{k_2 + M_2 - 1}$$

converges uniformly in the region $\{w_2 \in \mathbb{C} : |w_2| < r/2\}$. Thus we can perform term-by-term partial differentiation, then evaluate $w_2 = 0$, to conclude that for each $k_2 \in \mathbb{Z}$

$$\sum_{k_1 \in \mathbb{Z}} a_{k_1 k_2} z_1^{k_1} = \lim_{w_2 = 0} \left(\frac{\partial}{\partial w_2} \right)^{k_2 + M_2} (w_2^{M_2} f(z_1, w_2))$$

As the left-hand-side is an absolutely convergent series, the right-hand-side is a holomorphic function in z_1 that is defined on an open set in $\mathbb{C}$ (the image of S via the projection $(z_1, z_2) \mapsto z_1$, which is open). Thus one can find an annulus where the left-hand-side converges absolutely. In particular, if we use $g_{k_2}(z_1)$ to denote the holomorphic function defined by the right-hand-side, then $g_{k_2}(z_1)$ is defined on an annulus. With Theorem 2.3.3, we know that $\sum\limits_{k_1\in\mathbb{Z}} a_{k_1k_2} z_1^{k_1}$ is precisely the Laurent series expansion of $g_{k_2}(z_1)$, with

$$a_{k_1k_2} = \int_\gamma z_1^{-k_1-1} g_{k_2}(z_1) dz_1$$

for some circle γ.

Now we consider the Laurent series expansion of $f(z_1, z_2)$. By Theorem 2.3.3, this function can be expanded uniquely as a Laurent series in z_1, z_2 on T. By assumption, the lowest power of z_2 in this double series is bounded below by the order of pole $z_2 = 0$. As pointed out above, the function $z_2^{M_2} f(z_1, z_2)$ is defined when $z_2 = 0$, so the order of pole is bounded below by $-M_2$. Thus for each $k_2 \in \mathbb{Z}$, the coefficient of $z_2^{k_2}$ in this series expansion is precisely

$$\lim_{z_2=0} \left(\frac{\partial}{\partial z_2}\right)^{k_2+M_2} (z_2^{M_2} f(z_1, z_2))$$

that coincides with $g_{k_2}(z_1)$. Moreover, from the way of expansion, one easily sees that $g_{k_2}(z_1)$ is a polynomial function in z_1 (with possibly finitely negative powers of z_1). Thus $g_{k_2}(z_1)$ is defined on γ and one can perform the integration on γ. Therefore, the coefficient of $z_1^{k_1} z_2^{k_2}$ in the Laurent series expansion of $f(z_1, z_2)$ is precisely

$$\int_\gamma z_1^{-k_1-1} g_{k_2}(z_1) dz_1 = \int_\gamma z_1^{-k_1-1} \lim_{z_2=0} \left(\frac{\partial}{\partial z_2}\right)^{k_2+M_2} (z_2^{M_2} f(z_1, z_2)) dz_1$$

which coincides with $a_{k_1k_2}$.

So we proved that the double series

$$\sum_{k_1,k_2\in\mathbb{Z}} a_{k_1k_2} z_1^{k_1} z_2^{k_2}$$

is precisely the Laurent series expansion of the function $f(z_1, z_2)$ in the T. In particular, this double series converges absolutely for every $(z_1, z_2) \in T$.

□

To generalize the above lemma, we need the following lemma:

Lemma 2.3.6 (Hartogs' Theorem)**.** *Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$. Suppose that for each $i = 1, ..., n$ and each fixed $z_1, ..., z_{i-1}, z_{i+1}, ..., z_n$, the function*

$$\zeta \mapsto f(z_1, ..., z_{i-1}, \zeta, z_{i+1}, ..., z_n)$$

is holomorphic, in the classical one-variable sense, on the set

$$U(z_1, ..., z_{i-1}, z_{i+1}, ..., z_n) := \{\zeta \in \mathbb{C} : (z_1, ..., z_{i-1}, \zeta, z_{i+1}, ..., z_n) \in U\}$$

Then f is holomorphic on U.

Proof. See [K], Theorem 1.2.5. □

Lemma 2.3.7. *Let n be a positive integer. Let f be a rational function in $z_1, ..., z_n$. Let T be a connected multicircular domain on which the lowest power of z_n in the Laurent series expansion of $f(z_1, ..., z_n)$ is the same as the negative of the order of pole $z_n = 0$. Let S be a nonempty open subset of T and S' be the image of S via the projection $(z_1, ..., z_n) \mapsto (z_1, ..., z_{n-1})$. Assume that for each fixed $k_n \in \mathbb{Z}$, the series*

$$\sum_{k_1, k_2, ..., k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 ... k_{n-1} k_n} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}}$$

converges absolutely for every $(z_1, z_2, ..., z_{n-1}) \in S'$, and

$$\sum_{k_n \in \mathbb{Z}} \left(\sum_{k_1, k_2, ..., k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 ... k_{n-1} k_n} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} \right) z_n^{k_n}, \tag{2.4}$$

viewed as a series whose terms are $\left(\sum_{k_1, k_2, ..., k_{n-1} \in \mathbb{Z}} a_{k_1 k_2 ... k_{n-1} k_n} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} \right) z_n^{k_n}$, is lower-truncated in z_n and converges to $f(z_1, ..., z_n)$ for every $(z_1, z_2, ..., z_{n-1}, z_n) \in S$. Then the corresponding Laurent series

$$\sum_{k_1, k_2, ..., k_{n-1}, k_n \in \mathbb{Z}} a_{k_1 k_2 ... k_n} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} z_n^{k_n}, \tag{2.5}$$

converges absolutely to $f(z_1, ..., z_n)$ for every $(z_1, ..., z_n) \in T$

Proof. Fix $z_1, ..., z_{n-1} \in \mathbb{C}$ and let $z_n \in \mathbb{C}$ such that $(z_1, ..., z_n) \in S$. Since the series

$$\sum_{k_n \in \mathbb{Z}} \left(\sum_{k_1, ..., k_{n-1} \in \mathbb{Z}} a_{k_1 ... k_n} z_1^{k_1} \cdots z_{n-1}^{k_{n-1}} \right) w_n^{k_n},$$

is lower-truncated in w_n and converges when $w_n = z_n$ and S is open, one can find a real number $r \geq |z_n| > 0$ (depending on $z_1, ..., z_{n-1}$) such that the series converges absolutely in the region $\{w_n \in \mathbb{C} : 0 < |w_n| < r\}$ to $f(z_1, ..., z_{n-1}, w_n)$. Let $-M_n$ be the lowest power of w_n. Then

$$w_n^{M_n} \sum_{k_n \in \mathbb{Z}} \left(\sum_{k_1, ..., k_{n-1} \in \mathbb{Z}} a_{k_1 ... k_n} z_1^{k_1} \cdots z_{n-1}^{k_{n-1}} \right) w_n^{k_n},$$

is a power series with variable w_n, which converges uniformly in the region $\{w_n \in \mathbb{C} : |w_n| < r\}$ to $w_n^{M_1} f(z_1, ..., z_{n-1}, w_n)$. In particular, $w_n^{M_1} f(z_1, ..., z_{n-1}, w_n)$ is defined when $w_n = 0$. From the limit ratio test and the fact that $\lim_{n \to \infty} n/(n+1) = 1 < 2$, one sees that the derivative of the series

$$\sum_{k_n \in \mathbb{Z}} (k_n + M_n) \left(\sum_{k_1, ..., k_{n-1} \in \mathbb{Z}} a_{k_1 ... k_n} z_1^{k_1} \cdots z_{n-1}^{k_{n-1}} \right) w_2^{k_2 + M_2 - 1}$$

converges uniformly in the region $\{w_n \in \mathbb{C} : |w_n| < r/2\}$. Thus we can perform term-by-term partial differentiation, then evaluate $w_n = 0$, to conclude that for each $k_n \in \mathbb{Z}$

$$\sum_{k_1, ..., k_{n-1} \in \mathbb{Z}} a_{k_1 ... k_n} z_1^{k_1} \cdots z_{n-1}^{k_{n-1}} = \lim_{w_n = 0} \left(\frac{\partial}{\partial w_n} \right)^{k_n + M_n} (w_n^{M_2} f(z_1, ..., z_{n-1}, w_n))$$

We denote the right-hand-side function as $g_{k_n}(z_1, ..., z_{n-1})$ and prove that it is holomorphic on S'. Here we will use Hartogs' theorem. Fix $i = 1, ..., n-1$ and $z_1, ..., z_{i-1}, z_{i+1}, ..., z_{n-1}$. Then for every $\zeta \in S'(z_1, ..., z_{i-1}, z_{i+1}, ..., z_{n-1})$, since the left-hand-side series

$$\sum_{k_1, ..., k_n \in \mathbb{Z}} a_{k_1 ... k_n} z_1^{k_1} \cdots z_{i-1}^{k_{i-1}} \zeta^{k_i} z_{i+1}^{k_{i+1}} \cdots z_{n-1}^{k_{n-1}}$$

is an absolutely convergent multi-Laurent series in $z_1, ..., \zeta, ..., z_{n-1}$, in particular, we can arrange it as an absolutely convergent single Laurent series in ζ:

$$\sum_{k_i \in \mathbb{Z}} \left(\sum_{k_1, ..., k_{i-1}, k_{i+1}, ..., k_n \in \mathbb{Z}} a_{k_1 ... k_n} z_1^{k_1} \cdots z_{i-1}^{k_{i-1}} z_{i+1}^{k_{i+1}} \cdots z_{n-1}^{k_{n-1}} \right) \zeta^{k_i}$$

where the parenthesis sum is now treated as the coefficient of the single Laurent series. Then from the absolute convergence, the limit function of this single Laurent series is holomorphic at every $\zeta \in S'(z_1, ..., z_{i-1}, z_{i+1}, ..., z_{n-1})$. In other words, the right-hand-side function $g_{k_n}(z_1, ..., z_{i-1}, \zeta, z_{i+1}, ..., z_n)$ is holomorphic in the classical one-variable sense on $s'(z_1, ..., z_{i-1}, z_{i+1}, ..., z_{n-1})$. So from Hartog's theorem, we conclude that $g_{k_n}(z_1, ..., z_{n-1})$ is holomorphic on S'.

Since in the multicircular domain $\{(e^{i\theta_1}z_1, ..., e^{i\theta_{n-1}}z_{n-1}) : (z_1, ..., z_{n-1}) \in S', \theta_1, ..., \theta_{n-1} \in [0, 2\pi)\}$, the left-hand-side series converges absolutely, $g_{k_n}(z_1, ..., z_{n-1})$ is defined on this multicircular domain. With Theorem 2.3.3, we know that $\sum\limits_{k_1,...,k_{n-1}\in\mathbb{Z}} a_{k_1...k_n} z_1^{k_1} \cdots z_{n-1}^{k_{n-1}}$ is precisely the Laurent series expansion of $g_{k_n}(z_1, ..., z_n)$, with

$$a_{k_1...k_n} = \int \cdots \int_\gamma z_1^{-k_1-1} \cdots z_{n-1}^{-k_{n-1}-1} g_{k_n}(z_1, ..., z_{n-1}) dz_1 \cdots dz_{n-1}$$

where $\gamma = \{(r_1e^{i\theta}, \cdots r_{n-1}e^{i\theta_{n-1}}) : \theta_1, ..., \theta_{n-1} \in [0, 2\pi)\}$ with any $(r_1, ..., r_n) \in \mathrm{Tr}\ S'$.

Now we consider the Laurent series expansion of $f(z_1, ..., z_n)$. By Theorem 2.3.3, this function can be expanded uniquely as a Laurent series in $z_1, ..., z_n$ on T. By assumption, the lowest power of z_n in this series is bounded below by the order of pole $z_n = 0$. As pointed out above, the function $z_n^{M_2} f(z_1, ..., z_n)$ is defined when $z_n = 0$, so the order of pole is bounded below by $-M_2$. Thus for each $k_n \in \mathbb{Z}$, the coefficient of $z_n^{k_2}$ in this series expansion is precisely

$$\lim_{z_n=0} \left(\frac{\partial}{\partial z_n}\right)^{k_n+M_n} (z_n^{M_n} f(z_1, ..., z_n))$$

that coincides with $g_{k_n}(z_1, ..., z_n)$. Thus we can perform the integration on the multicircle γ, to see the coefficient of $z_1^{k_1} \cdots z_n^{k_n}$ in the Laurent series expansion of $f(z_1, ..., z_n)$ is precisely

$$\begin{aligned}
&\int \cdots \int_\gamma z_1^{-k_1-1} \cdots z_n^{-k_{n-1}-1} g_{k_2}(z_1, ..., z_{n-1}) dz_1 \cdots dz_{n-1} \\
&= \int \cdots \int_\gamma z_1^{-k_1-1} \cdots z_{n-1}^{k_{n-1}} \lim_{z_n=0} \left(\frac{\partial}{\partial z_n}\right)^{k_n+M_n} (z_n^{M_n} f(z_1, ..., z_n)) dz_1 \cdots dz_{n-1}
\end{aligned}$$

which coincides with $a_{k_1...k_n}$.

So we proved that the multiseries

$$\sum_{k_1,...,k_n\in\mathbb{Z}} a_{k_1...k_n} z_1^{k_1} \cdots z_n^{k_n}$$

is precisely the Laurent series expansion of the function $f(z_1, ..., z_n)$ in the region T. In particular, this multiseries converges absolutely in T. $\square$

The following generalization of Lemma 2.3.7 can be proved similarly and will be frequently quoted in future papers.

Lemma 2.3.8. *Let n be a positive integer. Let f be a rational function in $z_1, ..., z_n$. Let T be a connected multicircular domain on which the lowest power of z_n in the Laurent series expansion of $f(z_1, ..., z_n)$ is the same as the order of pole $z_n = 0$. Let S be a nonempty open subset of T and S' be the image of S via the projection $(z_1, ..., z_n) \mapsto (z_1, ..., z_{n-1})$. Assume that for each fixed $k_{p+1}, ..., k_n \in \mathbb{Z}$, the series*

$$\sum_{k_1,k_2,...,k_p\in\mathbb{Z}} a_{k_1k_2...k_{n-1}k_n} z_1^{k_1} z_2^{k_2} \cdots z_p^{k_p}$$

converges absolutely for every $(z_1, z_2, ..., z_p) \in S'$, and

$$\sum_{k_n\in\mathbb{Z}} \left(\sum_{k_1,k_2,...,k_p\in\mathbb{Z}} a_{k_1k_2...k_{n-1}k_n} z_1^{k_1} z_2^{k_2} \cdots z_p^{k_p} \right) z_{p+1}^{k_{p+1}} \cdots z_n^{k_n},$$

viewed as a series whose terms are $\left(\sum_{k_1,k_2,...,k_p\in\mathbb{Z}} a_{k_1k_2...k_{n-1}k_n} z_1^{k_1} z_2^{k_2} \cdots z_p^{k_p} \right) z_{p+1}^{k_{p+1}} \cdots z_n^{k_n}$, *satisfies the following:*

1. *The series is lower truncated in z_n. Moreover, for every $i = p+2, ..., n-1$, every fixed $k_{i+1}, ..., k_n \in \mathbb{Z}$, the series is lower-truncated in z_i.*

2. *The series converges absolutely to $f(z_1, ..., z_n)$ for every $(z_1, z_2, ..., z_{n-1}, z_n) \in S$.*

Then the corresponding Laurent series

$$\sum_{k_1,k_2,...,k_{n-1},k_n\in\mathbb{Z}} a_{k_1k_2...k_n} z_1^{k_1} z_2^{k_2} \cdots z_{n-1}^{k_{n-1}} z_n^{k_n},$$

converges absolutely to $f(z_1, ..., z_n)$ for every $(z_1, ..., z_n) \in T$

Proof. For convenience, we only give a sketch of the steps here:

1. Fix $k_{p+1}, ..., k_n \in \mathbb{Z}$. We first conclude that there exists positive integers $M_n, ..., M_{p+1}$, such that

$$\sum_{k_1,...,k_p\in\mathbb{Z}} a_{k_1...k_n} z_1^{k_1} \cdots z_p^{k_p} = \lim_{(w_{p+1},...,w_n)\to(0,...,0)} \left(\frac{\partial}{\partial w_{p+1}}\right)^{k_{p+1}+M_{p+1}} \cdots \left(\frac{\partial}{\partial w_n}\right)^{k_n+M_n} (w_{p+1}^{M_{p+1}} \cdots w_n^{M_n} f(z_1, ..., z_p, w_{p+1}, ..., w_n)).$$

2. Denote by $g_{k_{p+1}\dots k_n}(z_1,...,z_p)$ the function that maps $(z_1,...,z_p)$ to the right-hand-side. Then we use Hartogs' theorem to conclude that the $g_{k_{p+1}\dots k_n}(z_1,...,z_p)$ is holomorphic on S'.

3. One sees that the left-hand-side series is precisely the Laurent series expansion of $g_{k_{p+1}\dots k_n}(z_1,...,z_p)$ on the multicircular domain$\{(e^{i\theta_1}z_1,...,e^{i\theta_p}z_p):(z_1,...,z_p)\in S',\theta_1,...,\theta_p\in[0,2\pi)\}$. In particular,
$$a_{k_1\dots k_n}=\int\cdots\int_\gamma z_1^{-k_1-1}\cdots z_p^{-k_p-1}g_{k_{p+1}\dots k_n}(z_1,...,z_p)dz_1\cdots dz_p$$
where $\gamma=\{(r_1e^{i\theta},\cdots r_{n-1}e^{i\theta_p}):\theta_1,...,\theta_p\in[0,2\pi)\}$ for any $(r_1,...,r_n)\in$Tr S'.

4. Consider the Laurent series expansion of $f(z_1,...,z_n)$ on T. Argue that the series coefficient of $z_1^{k_1}\cdots z_n^{k_n}$ in the Laurent series expansion of $f(z_1,...,z_n)$ is precisely
$$\int\cdots\int_\gamma z_1^{-k_1-1}\cdots z_p^{-k_p-1}g_{k_{p+1}\dots k_n}(z_1,...,z_p)dz_1\cdots dz_p$$
which coincides with $a_{k_1\dots k_n}$. Thus the series
$$\sum_{k_n\in\mathbb{Z}}\left(\sum_{k_1,k_2,\dots,k_p\in\mathbb{Z}}a_{k_1k_2\dots k_{n-1}k_n}z_1^{k_1}z_2^{k_2}\cdots z_p^{k_p}\right)z_{p+1}^{k_{p+1}}\cdots z_n^{k_n}$$
converges absolutely for every $(z_1,...,z_n)\in T$.

□

For iterated series that are "locally" upper truncated, we also have a similar result.

Lemma 2.3.9. *Let n be a positive integer. Let f be a rational function in $z_1,...,z_n$. Let T be a connected multicircular domain on which the highest power of z_n in the Laurent series expansion of $f(z_1,...,z_n)$ is the same as the negative of the order of pole $z_n=\infty$. Let S be a nonempty open subset of T and S' be the image of S via the projection $(z_1,...,z_n)\mapsto(z_1,...,z_{n-1})$. Assume that for each fixed $k_n\in\mathbb{Z}$, the series*
$$\sum_{k_1,k_2,\dots,k_{n-1}\in\mathbb{Z}}a_{k_1k_2\dots k_{n-1}k_n}z_1^{k_1}z_2^{k_2}\cdots z_{n-1}^{k_{n-1}}$$
converges absolutely for every $(z_1,z_2,...,z_{n-1})\in S'$, and
$$\sum_{k_n\in\mathbb{Z}}\left(\sum_{k_1,k_2,\dots,k_{n-1}\in\mathbb{Z}}a_{k_1k_2\dots k_{n-1}k_n}z_1^{k_1}z_2^{k_2}\cdots z_{n-1}^{k_{n-1}}\right)z_n^{k_n},$$

viewed as a series whose terms are $\left(\sum_{k_1,k_2,\ldots,k_{n-1}\in\mathbb{Z}} a_{k_1k_2\ldots k_{n-1}k_n} z_1^{k_1} z_2^{k_2}\cdots z_{n-1}^{k_{n-1}}\right) z_n^{k_n}$, *is upper-truncated in* z_n *and converges to* $f(z_1,...,z_n)$ *for every* $(z_1,z_2,...,z_{n-1},z_n)\in S$. *Then the corresponding Laurent series*

$$\sum_{k_1,k_2,\ldots,k_{n-1},k_n\in\mathbb{Z}} a_{k_1k_2\ldots k_n} z_1^{k_1} z_2^{k_2}\cdots z_{n-1}^{k_{n-1}} z_n^{k_n},$$

converges absolutely to $f(z_1,...,z_n)$ *for every* $(z_1,...,z_n)\in T$

Proof. It suffices to perform the transformation $z_n \mapsto 1/z_n$ and apply the Lemma 2.3.7 □

2.3.2 Iterate of three vertex operators

With the above preparation, we can start to deal with vertex operators. To make it easier, we first investigate the iterate of three vertex operators, i.e.

$$\langle v', Y_V(Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle.$$

To show that this series converges absolutely and to find the region of convergence, we need the following intermediate proposition:

Proposition 2.3.10. *For any* $u_1,u_2,u_3,v\in V, v'\in V'$, *fixed* $z_1,z_2,z_3\in\mathbb{C}$ *satisfying* $|z_2|>|z_1-z_2-z_3|, |z_2|>|z_1-z_2|>0, |z_2|>|z_3|>0$, *the series*

$$\sum_{k\in\mathbb{Z}}\langle v', Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)\pi_l Y_V(u_3, z_3)v\rangle$$

converges absolutely to the rational function that is determined by

$$\langle v', Y_V(u_1,z_1)Y_V(u_2,z_2)Y_V(u_3,z_3)v\rangle$$

Proof. From Summary 2.2.15 and Remark 2.2.16,

$$\langle v', Y_V(u_1,z_1)Y_V(u_2,z_2)Y_V(u_3,z_3)v\rangle = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle v', Y_V(u_1,z_1)\pi_k Y_V(u_2,z_2)\pi_l Y_V(u_3,z_3)v\rangle$$

gives a rational function on $\{(z_1,z_2,z_3)\in\mathbb{C}^3 : |z_1|>|z_2|>|z_3|>0\}$ that has the only possible poles at $z_1=0, z_2=0, z_3=0, z_1=z_2, z_2=z_3, z_1=z_3$. Denote this rational function by $f(z_1,z_2,z_3)$. Then

$$f(z_1,z_2,z_3) = \frac{g(z_1,z_2,z_3)}{z_1^{p_1}z_2^{p_2}z_3^{p_3}(z_1-z_2)^{p_{12}}(z_2-z_3)^{p_{23}}(z_1-z_3)^{p_{13}}}$$

for some integers $p_1, p_2, p_3, p_{12}, p_{23}, p_{13} \geq 0$ and some polynomial $g(z_1, z_2, z_3)$.

Now we fix l and consider the series

$$\sum_{k\in\mathbb{Z}} Y_V(u_1, z_1)\pi_k Y_V(u_2, z_2)(\pi_l Y_V(u_3, z_3)v)$$

where $\pi_l Y_V(u_3, z_3)v$ is an element in V. As part of an absolutely convergent double series, it is also absolutely convergent. From Summary 2.2.22, it is equal to

$$\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$ for each fixed l. We sum up all $l \in \mathbb{Z}$ to see that

$$\sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v\right)$$

viewed as a series whose terms are $\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v$, converges absolutely and the sum is equal to $Y_V(u_1, z_1)Y_V(u_2, z_2)Y_V(u_3, z_3)v$ when $z_1, z_2, z_3 \in \mathbb{C}$ satisfy $|z_1| > |z_2| > |z_3| > 0, |z_2| > |z_1 - z_2| > 0$. In other words,

$$\sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \langle v', Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v\rangle\right)$$

viewed as a complex series whose terms are $\sum_{k\in\mathbb{Z}} \langle v', Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v\rangle$, converges to $f(z_1, z_2, z_3)$ when $|z_1| > |z_2| > |z_3| > 0, |z_2| > |z_1 - z_2| > 0$.

To see that the double series

$$\sum_{l\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle v', Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_2| > |z_3| + |z_1 - z_2|, |z_3| > 0, |z_1 - z_2| > 0$, we need to apply Lemma 2.3.7 with the following parameter transformation $\zeta_1 = z_1 - z_2, \zeta_2 = z_2, \zeta_3 = z_3$. Let

$$T = \{(\zeta_1, \zeta_2, \zeta_3) : |\zeta_2| > |\zeta_3| + |\zeta_1|, |\zeta_1| > |\zeta_3| > 0\}\}$$

With Lemma 2.3.2, we see that T is a connected multicircular domain. Moreover, T is a subset of $\{(\zeta_1, \zeta_2, \zeta_3) : |\zeta_i| > |\zeta_3|, i = 1, 2\}$. Now we express the function $f(z_1, z_2, z_3)$in terms of the variables $\zeta_1, \zeta_2, \zeta_3$ as

$$f(\zeta_1 + \zeta_2, \zeta_2, \zeta_3) = \frac{g(\zeta_1 + \zeta_2, \zeta_2, \zeta_3)}{(\zeta_1 + \zeta_2)^{p_1}\zeta_2^{p_2}\zeta_3^{p_3}\zeta_1^{p_{12}}(\zeta_2 - \zeta_3)^{p_{23}}(\zeta_1 + \zeta_2 - \zeta_3)^{p_{13}}},$$

admits an expansion as Laurent series in $\zeta_1, \zeta_2, \zeta_3$ by the following steps

1. Expand the negative powers of $\zeta_1+\zeta_2$ as a power series ζ_1. The resulted series converges when $|\zeta_2|>|\zeta_1|$.

2. Expand the negative powers of $\zeta_2-\zeta_3$ as a power series of ζ_3. The resulted series converges when $|\zeta_2|>|\zeta_3|$.

3. Expand the negative powers of $\zeta_1+\zeta_2-\zeta_3$ as power series of $\zeta_1-\zeta_3$, then further expand all the positive power of $\zeta_1-\zeta_3$ as polynomials. The resulted series converges in $|\zeta_2|>|\zeta_1-\zeta_3|$.

Obviously elements if $(\zeta_1,\zeta_2,\zeta_3)\in T$, then all the above conditions are satisfied (note that $|\zeta_2|>|\zeta_3|+|\zeta_1|$ implies that $|\zeta_2|>|\zeta_1-\zeta_3|$ by triangle inequality). Thus $f(\zeta_1+\zeta_2,\zeta_2,\zeta_3)$ is expanded as an absolutely convergent Laurent series in T. From Theorem 2.3.3, the Laurent series is unique. Note that the lowest power of ζ_3 in this Laurent is $-p_3$.

Set

$$S=\{(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3:|\zeta_1+\zeta_2|>|\zeta_2|>|\zeta_3|>0,|\zeta_2|>|\zeta_1|>0\}\cap T$$

Obviously, S is a nonempty open subset of T. We know that the series

$$\sum_{k\in\mathbb{Z}}\langle w',Y_W^L(\pi_k^V Y_V(u_1,\zeta_1)u_2,\zeta_2)\pi_l^W Y_W^L(u_3,\zeta_3)w\rangle$$

is absolutely convergent whenever $(\zeta_1,\zeta_2,\zeta_3)\in S$, and the series

$$\sum_{l\in\mathbb{C}}\left(\sum_{k\in\mathbb{Z}}\langle w',Y_W^L(\pi_k^V Y_V(u_1,\zeta_1)u_2,\zeta_2)\pi_l^W Y_W^L(u_3,\zeta_3)w\rangle\right),$$

viewed as a series whose terms are $\sum_{k\in\mathbb{Z}}\langle w',Y_W^L(\pi_k^V Y_V(u_1,\zeta_1)u_2,\zeta_2)\pi_l^W Y_W^L(u_3,\zeta_3)w\rangle$, is lower-truncated in z_3 and absolutely convergent to $f(\zeta_1+\zeta_2,\zeta_2,\zeta_3)$ whenever $(\zeta_1,\zeta_2,\zeta_3)\in S$. Thus Lemma 2.3.7 implies that the series

$$\sum_{l\in\mathbb{C}}\sum_{k\in\mathbb{Z}}\langle w',Y_W^L(\pi_k^V Y_V(u_1,\zeta_1)u_2,\zeta_2)\pi_l^W Y_W^L(u_3,\zeta_3)w\rangle$$

converges absolutely when $(\zeta_1,\zeta_2,\zeta_3)\in T$.

Finally, since the expansion of the rational function is done in the region

$$\{(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3:|\zeta_2|>|\zeta_1-\zeta_3|,|\zeta_2|>|\zeta_1|>0,|\zeta_2|>|\zeta_3|>0\},$$

the Laurent series also converges absolutely in this region. That is to say, in terms of variables z_1, z_2, z_3, the series

$$\sum_{l\in\mathbb{C}}\sum_{k\in\mathbb{Z}}\langle w', Y_W^L(\pi_k^V Y_V(u_1, z_1-z_2)u_2, z_2)\pi_l^W Y_W^L(u_3, z_3)w\rangle$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_2| > |z_1 - z_2 - z_3|, |z_2| > |z_1 - z_2| > 0, |z_2| > |z_3| > 0$. □

Proposition 2.3.11. *For any $u_1, u_2, u_3, v \in V, v' \in V'$, fixed $z_1, z_2, z_3 \in \mathbb{C}$ satisfying $|z_3| > |z_1 - z_3|, |z_2 - z_3| > |z_1 - z_2| > 0$, the series*

$$\langle v', Y_V(Y_V(Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3, z_3)v\rangle$$

converges absolutely to the rational function determined by

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)Y_V(u_3, z_3)v\rangle$$

Proof. We proceed similarly based on the result above: in the double series

$$\sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v$$

we fix k and consider the series

$$\sum_{l\in\mathbb{Z}} Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2)\pi_l Y_V(u_3, z_3)v$$

where $\pi_k Y_V(u_1, z_1-z_2)u_2$ is an element in V. As part of an absolutely convergent double series, this series is also absolutely convergent. From Summary 2.2.22, it is equal to

$$\sum_{l\in\mathbb{Z}} Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3, z_3)v$$

when $|z_2| > |z_3| > |z_2 - z_3| > 0$. We sum up all $k \in \mathbb{Z}$. From the proof of the previous proposition,

$$\sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}} Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3, z_3)v\right),$$

viewed as a series whose terms are $\sum\limits_{l\in\mathbb{Z}} Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3, z_3)v$, converges absolutely and the sum is equal to $Y_V(u_1, z_1)Y_V(u_2, z_2)Y_V(u_3, z_3)v$ when

$z_1, z_2, z_3 \in \mathbb{C}$ satisfy $|z_2| > |z_2 - z_3| > 0, |z_2| > |z_3| + |z_1 - z_2|, |z_1 - z_2| > 0, |z_3| > 0$. In other words,

$$\sum_{k\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} \langle v', Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle \right),$$

viewed as a series whose terms are $\sum_{l\in\mathbb{Z}} \langle v', Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle$, converges absolutely to $f(z_1, z_2, z_3)$ when $|z_2| > |z_2 - z_3|, |z_2| > |z_3| + |z_1 - z_2|, |z_1 - z_2| > 0, |z_3| > 0$. Moreover, one sees that the power of $(z_1 - z_2)$ in this series is lower-truncated.

We claim that the double series

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \langle v', Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle,$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_3| > |z_1 - z_2| + |z_2 - z_3|, |z_2 - z_3| > |z_1 - z_2| > 0$.

To apply Lemma 2.3.7, we perform the transformation $\zeta_1 = z_1 - z_2, \zeta_2 = z_2 - z_3, \zeta_3 = z_3$. Set

$$T = \{(\zeta_1, \zeta_2, \zeta_3) : |\zeta_3| > |\zeta_1| + |\zeta_2|, |\zeta_2| > |\zeta_1| > 0\}$$

With Lemma 2.3.2, we see that T is a connected multicircular domain Moreover, T is a subset of $\{(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 : |\zeta_i| > |\zeta_1|, i = 2, 3\}$. We express the function $f(z_1, z_2, z_3)$ in terms of the variables $\zeta_1, \zeta_2, \zeta_3$ as

$$f(\zeta_1 + \zeta_2 + \zeta_3, \zeta_2 + \zeta_3, \zeta_3) = \frac{g(\zeta_1 + \zeta_2 + \zeta_3, \zeta_2 + \zeta_3, \zeta_3)}{(\zeta_1 + \zeta_2 + \zeta_3)^{p_1}(\zeta_2 + \zeta_3)^{p_2}\zeta_3^{p_3}\zeta_1^{p_{12}}\zeta_2^{p_{23}}(\zeta_1 + \zeta_2)^{p^{13}}},$$

which admits a Laurent series expansion in the following steps:

1. Expand negative powers of $\zeta_1 + \zeta_2 + \zeta_3$ as power series of $\zeta_1 + \zeta_2$, then further expand the positive powers of $\zeta_1 + \zeta_2$ as polynomials in ζ_1 and ζ_2. This series converges absolutely when $|\zeta_3| > |\zeta_1 + \zeta_2|$

2. Expand negative powers of $\zeta_2 + \zeta_3$ as power series of ζ_2. This series converges absolutely when $|\zeta_3| > |\zeta_2|$

3. Expand negative powers of $\zeta_1 + \zeta_2$ as power series of ζ_1. This series converges absolutely when $|\zeta_2| > |\zeta_1|$

Obviously if $(\zeta_1, \zeta_2, \zeta_3) \in E$, then all the above conditions are satisfied (Note that $|\zeta_3| > |\zeta_1| + |\zeta_2|$ implies that $|\zeta_3| > |\zeta_1 + \zeta_2|$ by triangle inequality). Thus $f(\zeta_1+\zeta_2, \zeta_2+\zeta_3, \zeta_3)$ is expressed as an absolutely convergent Laurent series. From Theorem 2.3.3, the Laurent series is unique.

Set

$$S = \{(\zeta_1, \zeta_2, \zeta_3) : |\zeta_2| > |\zeta_3| + |\zeta_1|, |\zeta_1| > 0, |\zeta_3| > 0\} \cap T.$$

So S is a nonempty open subset of T. We know that the series

$$\sum_{l\in\mathbb{Z}} \langle w', Y_W^L(\pi_l^V Y_V(\pi_k^V Y_V(u_1, \zeta_1)u_2, \zeta_2)u_3, \zeta_3)w\rangle$$

converges absolutely when $(\zeta_1, \zeta_2, \zeta_3) \in S$, and the series

$$\sum_{k\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} \langle w', Y_W^L(\pi_l^V Y_V(\pi_k^V Y_V(u_1, \zeta_1)u_2, \zeta_2)u_3, \zeta_3)w\rangle \right),$$

viewed as a series whose terms are $\sum_{l\in\mathbb{Z}} \langle w', Y_W^L(\pi_l^V Y_V(\pi_k^V Y_V(u_1, \zeta_1)u_2, \zeta_2)u_3, \zeta_3)w\rangle$, converges absolutely to $f(\zeta_1 + \zeta_2, \zeta_2 + \zeta_3, \zeta_3)$ when $(\zeta_1, \zeta_2, \zeta_3) \in S$. Thus Lemma 2.3.7 implies that the series

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \langle w', Y_W^L(\pi_l^V Y_V(\pi_k^V Y_V(u_1, \zeta_1)u_2, \zeta_2)u_3, \zeta_3)w\rangle,$$

converges absolutely when $(\zeta_1, \zeta_2, \zeta_3) \in T$.

Finally, as the expansion is done in the region

$$\{(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 : |\zeta_3| > |\zeta_1 + \zeta_2| > 0, |\zeta_3| > |\zeta_2| > |\zeta_1| > 0\}$$

the series also converges absolutely in this region. That is to say, in terms of variables z_1, z_2, z_3, the series

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \langle w', Y_W^L(\pi_l^V Y_V(\pi_k^V Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)w\rangle$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_3| > |z_1 - z_3|, |z_3| > |z_2 - z_3| > |z_1 - z_2| > 0$.

Now we claim that the triple series

$$\begin{aligned}
&\langle v', Y_V(Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle \\
&\quad = \sum_{m,n,p\in\mathbb{Z}} \langle v', (Y_V)_p((Y_V)_n((Y_V)_m(u_1)u_2)u_3)v\rangle (z_1 - z_2)^{-m-1}(z_2 - z_3)^{-n-1} z_3^{-p-1}
\end{aligned}$$

of elements in V converges absolutely when $|z_3| > |z_1-z_2|+|z_2-z_3|, |z_2-z_3| > |z_1-z_2| > 0$. We start by the special case when $v' \in V, u_1, u_2, u_3, v \in V$ are homogeneous, when the triple series degenerates to a double series

$$\sum_{\substack{m+n+p=\mathrm{wt}(u_1)+\mathrm{wt}(u_2)+\mathrm{wt}(u_3)-\mathrm{wt}(v')-3\\ m_1,\dots,m_n\in\mathbb{Z}}} \langle v', (Y_V)_p((Y_V)_n((Y_V)_m(u_1)u_2)u_3)v\rangle (z_1-z_2)^{-m-1}(z_2-z_3)^{-n-1}z_3^{-p-1}.$$

Note that in this case,

$$\pi_k Y_V(u_1, z_1-z_2)u_2 = (Y_V)_{n(k)}(u_1)u_2(z_1-z_2)^{-n(k)-1}$$

where $n(k) = \mathrm{wt}\ (u_1) + \mathrm{wt}\ (u_2) - k - 1$, and

$$\pi_l Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3 = (Y_V)_{m(l)}((Y_V)_{n(k)}(u_1)u_2)u_3(z_1-z_2)^{-n(k)}(z_2-z_3)^{-m(l)-1}$$

where $m(l) = k - l - 1 + \mathrm{wt}\ (u_3)$. So

$$\begin{aligned} &Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3, z_3)v \\ &\quad= \sum_p (Y_V)_p((Y_V)_{m(l)}((Y_V)_{n(k)}(u_1)u_2)u_3)v(z_1-z_2)^{-n(k)}(z_2-z_3)^{-m(l)-1}z_3^{-p-1} \end{aligned}$$

and finally

$$\begin{aligned} &\langle v', Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3, z_3)v\rangle \\ &\quad= \langle v', (Y_V)_p((Y_V)_{m(l)}((Y_V)_{n(k)}(u_1)u_2)u_3)v\rangle (z_1-z_2)^{-n(k)}(z_2-z_3)^{-m(l)-1}z_3^{-p-1} \end{aligned}$$

where $p = l - \mathrm{wt}\ v' - 1 + \mathrm{wt}\ v$. Since the double series

$$\begin{aligned} &\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \langle v', Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3, z_3)v\rangle \\ &\quad= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \langle v', (Y_V)_p((Y_V)_{m(l)}((Y_V)_{n(k)}(u_1)u_2)u_3)v\rangle (z_1-z_2)^{-n(k)}(z_2-z_3)^{-m(l)-1}z_3^{-p-1} \end{aligned}$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_3| > |z_1-z_2|+|z_2-z_3|, |z_2-z_3| > |z_1-z_2| > 0$, so does the double series

$$\sum_{\substack{m+n+p=\mathrm{wt}(u_1)+\mathrm{wt}(u_2)+\mathrm{wt}(u_3)-\mathrm{wt}(v')-3\\ m_1,\dots,m_n\in\mathbb{Z}}} \langle v', (Y_V)_p((Y_V)_n((Y_V)_m(u_1)u_2)u_3)v\rangle (z_1-z_2)^{-m-1}(z_2-z_3)^{-n-1}z_3^{-p-1}$$

as it is a rearrangement.

For nonhomogeneous $v' \in V', u_1, u_2, u_3, v \in V$, we similarly write

$$v' = \sum_{i \text{ finite}} (v')^{(i)}, u_1 = \sum_{j_1 \text{ finite}} u_1^{(p^1_{j_1})}, u_2 = \sum_{j_2 \text{ finite}} u_2^{(p^2_{j_2})}, u_3 = \sum_{j_3 \text{ finite}} u_3^{(p^3_{j_3})}, v = \sum_{m \text{ finite}} v^{(q_m)}.$$

It follows that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle v', Y_V(\pi_l Y_V(\pi_k Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle \\
=&\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\sum_{i,j_1,j_2,j_3,m \text{ finite}} \langle (v')^{(i)}, Y_V(\pi_l Y_V(\pi_k Y_V(u_1^{(p^1_{j_1})}, z_1 - z_2)u_2^{(p^2_{j_2})}, z_2 - z_3)u_3^{(p^3_{j_3})}, z_3)v^{(q_m)}\rangle \\
=&\sum_{i,j_1,j_2,j_3,m \text{ finite}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \langle (v')^{(i)}, Y_V(\pi_l Y_V(\pi_k Y_V(u_1^{(p^1_{j_1})}, z_1 - z_2)u_2^{(p^2_{j_2})}, z_2 - z_3)u_3^{(p^3_{j_3})}, z_3)v^{(q_m)}\rangle \\
=&\sum_{i,j_1,j_2,j_3,m \text{ finite}} \langle (v')^{(i_1)}, Y_V(Y_V(Y_V(u_1^{(p^1_{j_1})}, z_1 - z_2)u_2^{(p^2_{j_2})}, z_2 - z_3)u_3^{(p^3_{j_3})}, z_3)v^{(q_m)}\rangle \\
=&\langle v', Y_V(Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle.
\end{aligned}$$

the third equality of which is justified because a finite sum of absolutely convergent series is still absolutely convergent, and for absolutely convergent series the order of summation can be rearranged. So we proved that the triple series $\langle v', Y_V(Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)v\rangle$ converges absolutely to $f(z_1, z_2, z_3)$ when $|z_3| > |z_1 - z_2| + |z_2 - z_3|, |z_2 - z_3| > |z_1 - z_2| > 0$, where $f(z_1, z_2, z_3)$ is the same rational function that $\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)Y_V(u_3, z_3)v\rangle$ converges to. $\square$

Remark 2.3.12. Although we are guided by the $\overline{V}$-valued map interpretation laid down in the previous sections, in the proof we only used the absolute convergence of the corresponding complex series. So all the proofs and discussion here extends to the case when V is not grading-restricted.

2.3.3 Iterate of any number of vertex operators

With induction one can prove:

Proposition 2.3.13. *For $u_1, u_2, ..., u_n, v \in V, v' \in V'$, the series*

$$\langle v', Y_V(Y_V(\cdots Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3 \cdots, z_{n-1} - z_n)u_n, z_n)v\rangle$$

converges absolutely in the region

$$\left\{(z_1, ..., z_n) \in \mathbb{C}^n : \begin{array}{l} |z_n| > |z_i - z_n| > 0, i = 1, ..., n; \\ |z_i - z_{i+1}| > |z_j - z_i| > 0, 1 \le j < i \le n-1 \end{array}\right\}$$

to the rational function that

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)\cdots Y_V(u_n, z_n)v\rangle$$

converges to.

Proof. Here we only give a brief sketch without going into the details. Assume the conclusion is true for the product of $n-1$ vertex operators.

1. Use Summary 2.2.15 to write

   $$\begin{aligned}&Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v\\ &\quad= \sum_{k_1,\ldots,k_{n-1}\in\mathbb{Z}} Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_{k_2}\cdots Y_V(u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\end{aligned}$$

   and hence

   $$\begin{aligned}&\langle v', Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v\rangle\\ &\quad= \sum_{k_1,\ldots,k_{n-1}\in\mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_{k_2}\cdots Y_V(u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle\end{aligned}$$

   when $|z_1| > |z_2| > \cdots > |z_n| > 0$.

2. For each fixed k_{n-1}, we use the induction hypothesis to see that

   $$\begin{aligned}&\sum_{k_1,\ldots,k_{n-2}\in\mathbb{Z}} \langle v', Y_V(u_1, z_1)\pi_{k_1}Y_V(u_2, z_2)\pi_{k_2}\cdots Y_V(u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle\\ &= \sum_{k_1,\ldots,k_{n-2}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_1}Y_V(\cdots\pi_{k_{n-2}}Y_V(u_1, z_1-z_2)u_2,\cdots)u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle\end{aligned}$$

   when $|z_1| > |z_2| > \cdots > |z_n| > 0, |z_{n-1}| > |z_i - z_{n-1}| > 0, i = 1, ..., n-2, |z_i - z_{i+1}| > |z_j - z_i| > 0, 1 \le j < i \le n-2$. In particular, the right hand side, as an $(n-1)$-multiseries in $z_1 - z_2, z_2 - z_3, ..., z_{n-2} - z_{n-1}, z_{n-1}$, converges absolutely.

3. Summing up all k_{n-1}'s to see that

   $$\sum_{k_{n-1}\in\mathbb{Z}}\sum_{k_1,\ldots,k_{n-2}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_1}Y_V(\cdots\pi_{k_{n-2}}Y_V(u_1, z_1-z_2)u_2,\cdots)u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle$$

   viewed as a single complex series whose terms are

   $$\sum_{k_1,\ldots,k_{n-2}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_1}Y_V(\cdots\pi_{k_{n-2}}Y_V(u_1, z_1-z_2)u_2,\cdots)u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle$$

converges to the rational function that $\langle v', Y_V(u_1, z_1)\cdots Y_V(u_{n-1}, z_{n-1})Y_V(u_n, z_n)v\rangle$ converges to, when $|z_1| > |z_2| > \cdots > |z_n| > 0, |z_{n-1}| > |z_i - z_{n-1}| > 0, i = 1, ..., n-2, |z_i - z_{i+1}| > |z_j - z_i| > 0, 1 \leq j < i \leq n-2$. Note that the power of z_n is lower-truncated.

4. With the help of a parameter transformation, we apply Lemma 2.3.7 to see that the series

$$\sum_{k_1,...,k_{n-2},k_{n-1}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_1}Y_V(\cdots\pi_{k_{n-2}}Y_V(u_1, z_1 - z_2)u_2, \cdots)u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v\rangle$$

converges absolutely when

$$|z_{n-1}| > |z_n| + |z_1 - z_2| + \cdots + |z_{n-2} - z_{n-1}|, |z_n| > 0, |z_{n-2} - z_{n-1}| > \cdots > |z_1 - z_2| > 0$$

to the rational function that $\langle v', Y_V(u_1, z_1)\cdots Y_V(u_{n-1}, z_{n-1})v\rangle$ converges to. From the way of expansion, one can further enlarge the region of convergence and obtain a generalization of Proposition 2.3.10, i.e., the series

$$\langle v', Y_V(Y_V(Y_V(\cdots Y_V(u_1, z_1 - z_2)u_2, \cdots)u_{n-2}, z_{n-2} - z_{n-1})u_{n-1}, z_{n-1})Y_V(u_n, z_n)v\rangle$$

converges absolutely in the region

$$\left\{ (z_1, ..., z_n) \in \mathbb{C}^n : \begin{array}{c} |z_n| > 0, |z_{n-1}| > |z_i - z_{n-1}| > 0, i = 1, ..., n-2 \\ |z_{n-1}| > |z_i - z_{n-1} - z_n| > 0, i = 1, ..., n-2 \\ |z_{j-1} - z_j| > |z_i - z_{j-1}| > 0, 1 \leq i \leq n-2, i+2 \leq j \leq n \end{array} \right\} \tag{2.6}$$

to the rational function that

$$\langle v', Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v\rangle$$

converges to.

5. For each fixed $k_1 \in \mathbb{Z}$, we use the associativity to see that

$$\begin{aligned} &\sum_{k_2,...,k_{n-1}\in\mathbb{Z}} Y_V(\pi_{k_1}Y_V(\cdots\pi_{k_{n-2}}Y_V(u_1, z_1 - z_2)u_2, \cdots)u_{n-1}, z_{n-1})\pi_{k_{n-1}}Y_V(u_n, z_n)v \\ = &\sum_{k_2,...,k_{n-1}\in\mathbb{Z}} Y_V(\pi_{k_{n-1}}Y_V(\pi_{k_1}Y_V(\cdots\pi_{k_{n-2}}Y_V(u_1, z_1 - z_2)u_2, \cdots)u_{n-1}, z_{n-1})u_n, z_n)v \end{aligned}$$

when $|z_{n-1}| > |z_n| + |z_{n-2} - z_{n-1}| + \cdots + |z_1 - z_2|, |z_{n-2} - z_{n-1}| > \cdots > |z_1 - z_2| > 0, |z_n| > |z_{n-1} - z_n|$. In particular, we know that

$$\sum_{k_2,\dots,k_{n-1}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_{n-1}} Y_V(\pi_{k_1} Y_V(\cdots \pi_{k_{n-2}} Y_V(u_1, z_1-z_2)u_2, \cdots)u_{n-1}, z_{n-1})u_n, z_n)v\rangle,$$

as part of an absolutely convergent n-multiseries in $z_1 - z_2, z_2 - z_3, ..., z_{n-2} - z_{n-1}, z_{n-1} - z_n, z_n$, converges absolutely.

6. Summing up all k_1's to see that

$$\sum_{k_1\in\mathbb{Z}} \left(\sum_{k_2,\dots,k_{n-1}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_{n-1}} Y_V(\pi_{k_1} Y_V(\cdots \pi_{k_{n-2}} Y_V(u_1, z_1 - z_2)u_2, \cdots)u_{n-1}, z_{n-1})u_n, z_n)v\rangle \right),$$

viewed as a single complex series whose terms are

$$\left(\sum_{k_2,\dots,k_{n-1}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_{n-1}} Y_V(\pi_{k_1} Y_V(\cdots \pi_{k_{n-2}} Y_V(u_1, z_1 - z_2)u_2, \cdots)u_{n-1}, z_{n-1})u_n, z_n)v\rangle \right),$$

converges to the rational function that $\langle v', Y_V(u_1, z_1) \cdots Y_V(u_{n-1}, z_{n-1})Y_V(u_n, z_n)v\rangle$ converges to, when $|z_{n-1}| > |z_n| + |z_{n-2} - z_{n-1}| + \cdots + |z_1 - z_2|, |z_{n-2} - z_{n-1}| > \cdots > |z_1 - z_2| > 0, |z_n| > |z_{n-1} - z_n|$.

7. With the help of a parameter transformation, we apply Lemma 2.3.7 to see that the series

$$\sum_{k_1,k_2,\dots,k_{n-1}\in\mathbb{Z}} \langle v', Y_V(\pi_{k_{n-1}} Y_V(\pi_{k_1} Y_V(\cdots \pi_{k_{n-2}} Y_V(u_1, z_1-z_2)u_2, \cdots)u_{n-1}, z_{n-1})u_n, z_n)v\rangle,$$

converges absolutely when

$$|z_n| > |z_{n-1}-z_n|+|z_{n-2}-z_{n-1}|+\cdots+|z_1-z_2|, |z_{n-1}-z_n| > |z_{n-2}-z_{n-1}| > \cdots > |z_1-z_2| > 0$$

to the rational function that $\langle v', Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v\rangle$ converges to. From the way of expansion, one can further enlarge the region of convergence, thus proving that the series

$$\langle v', Y_V(Y_V(\cdots Y_V(u_1, z_1 - z_2)u_2, \cdots)u_{n-1}, z_{n-1})u_n, z_n)v\rangle$$

converges absolutely in the region

$$\left\{ (z_1, ..., z_n) \in \mathbb{C}^n : \begin{array}{l} |z_n| > |z_i - z_n| > 0, i = 1, ..., n; \\ |z_i - z_{i+1}| > |z_j - z_i| > 0, 1 \le j < i \le n-1 \end{array} \right\}$$

to the rational function that

$$\langle v', Y_V(u_1, z_1)\cdots Y_V(u_n, z_n)v\rangle$$

converges to.

$\square$

Because of the rationality, it is easy to obtain the following analogue of Summary 2.2.15:

Summary 2.3.14. *For any $u_1, ..., u_n \in V$ and any $z_1, ..., z_n$ satisfying $|z_n| > |z_i - z_n| > 0, i = 1, ..., n; |z_i - z_{i+1}| > |z_j - z_i| > 0, 1 \leq j < i \leq n-1$,*

$$\begin{aligned}&Y_V(Y_V(\cdots Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3 \cdots, z_{n-1} - z_n)u_n, z_n)v\\ &= \sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_V(\pi_{k_1} Y_V(\cdots Y_V(\pi_{k_{n-1}} Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3 \cdots, z_{n-1} - z_n)u_n, z_n)v\end{aligned}$$

For fixed $z_1, z_2, ..., z_n \in \mathbb{C}$ satisfying $|z_n| > |z_i - z_n| > 0, i = 1, ..., n; |z_i - z_{i+1}| > |z_j - z_i| > 0, 1 \leq j < i \leq n-1$, the iterate of any number of vertex operators gives rise to the following map

$$Y_V(Y_V(\cdots Y_V(Y_V(\cdot, z_1 - z_2)\cdot, z_2 - z_3)\cdots, z_{n-1} - z_n)\cdot, z_n)\cdot : V^{\otimes n} \otimes V \to \overline{V}$$

If in addition, $|z_1| > |z_2| > \cdots > |z_n|$, then the map coincides with

$$Y_V(\cdot, z_1)Y_V(\cdot, z_2)\cdots Y_V(\cdot, z_n)\cdot : V^{\otimes n} \otimes V \to \overline{V}$$

Remark 2.3.15. When V is not grading-restricted:

1. just as Remark 2.3.12 mentioned, the proof of Proposition 2.3.13 is also valid. So the rationality of iterate of n vertex operators still holds.

2. In the Summary 2.3.14, all the maps involved are actually $\widehat{V}$-valued. However as we have explained in Remark 2.2.18, an explicit formulation is not necessary.

2.4 Opposite MOSVA

In this section, for a given MOSVA, we will introduce its opposite MOSVA using the skew-symmetry opposite vertex operator. This can be viewed as the analogue of the opposite associative algebra of a given associative algebra. Analogously, we prove that a right module for the MOSVA is the same as the left module for the opposite MOSVA, and a left module for the MOSVA is the same as the right module for the opposite MOSVA. The rationality of iterates we proved in the previous section will be used in the proofs of these theorems.

2.4.1 The opposite vertex operator

Definition 2.4.1. Let $(V, Y_V, \mathbf{1})$ be a MOSVA. The opposite vertex operator map Y_V^s of Y_V is defined as follows

$$\begin{aligned} Y_V^s: \quad & V \otimes V \to V[[x, x^{-1}]] \\ & u \otimes v \mapsto e^{xD_V} Y_V(v, -x)u \end{aligned}$$

where $e^{xD_V} Y_V(v, -x)u$ is understood as a single series that is obtained by multiplying two formal series e^{xD_V} and $Y_V(v, -x)u$.

One sees easily that the series defining the skew-symmetry opposite vertex operator is lower truncated. Moreover, for any nonzero complex number z that is substituting x, the resulted complex series gives a well-defined element in $\overline{V}$.

2.4.2 Rationality and Associativity

Proposition 2.4.2. *For $v' \in V', u_1, u_2, v \in V$, the double complex series*

$$\langle v', Y_V^s(u_1, z_1) Y_V^s(u_2, z_2) v \rangle$$

converges absolutely when $|z_1| > |z_2| > 0$ to a rational function with the only possible poles $z_1 = 0, z_2 = 0, z_1 = z_2$.

Proof. The proof will be divided in three steps.

1. From the conclusion of Theorem 2.3.13 that the series

$$\langle v', Y_V(Y_V(v, -z_2)u_2, -z_1 + \zeta_2)u_1\rangle$$

converges absolutely when $|-z_1 + \zeta_2| > |z_2| > 0$ to a rational function that has the only possible poles at $z_2 = 0, z_1 = \zeta_2, z_1 + z_2 = \zeta_2$, with the D-conjugation property (See Part (4) of Proposition 2.1.4) and Lemma 2.3.7, we can prove that the series

$$\langle v', Y_V(e^{\zeta_2 D_V} Y_V(v, -z_2)u_2, -z_1)u_1\rangle$$

converges absolutely when $|z_1| > |z_2 - \zeta_2|, |z_1| > |\zeta_2|, |z_2| > 0$ to a rational function that has the only possible poles at $z_1 + z_2 = \zeta_2, z_2 = 0, z_1 = \zeta_2$. The argument is very similar to that in the proof of Theorem 2.3.13 and is omitted here.

2. Since $\zeta_2 = z_2$ is contained in the region of the convergence, we then evaluate $\zeta_2 = z_2$ to see that the series

$$\langle v', Y_V(e^{z_2 D_V} Y_V(v, -z_2)u_2, -z_1)u_1\rangle$$

converges absolutely when $|z_1| > |z_2| > 0$ to the rational function determined by

$$\langle v', Y_V(Y_V(v, -z_2)u_2, -z_1 + z_2)u_1\rangle$$

that has the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$.

3. Now we argue that for every $v' \in V', u_1, u_2, v \in V$, the series

$$\langle v', e^{z_1 D_V} Y_V(e^{z_2 D_V} Y_V(v, -z_2)u_2, -z_1)u_1\rangle \tag{2.7}$$

converges absolutely in the same region S. We first note that the adjoint $D_V' : V^* \to V^*$ of D_V, defined by

$$\langle D_V' v', v\rangle = \langle v', D_V v\rangle, v' \in V', v \in V,$$

restrics to a homogeneous linear operator on V' of weight -1. Thus for every $z \in \mathbb{C}^\times$, the action of $e^{zD_V'}$ on $v' \in V'$ is a finite sum of elements of V'. So the series (2.7) is the same as

$$\langle e^{z_1 D_V'} v', Y_V(e^{z_2 D_V} Y_V(v, -z_2)u_2, -z_1)u_1\rangle$$

which is a finite sum of series that converges absolutely to rational functions with the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$. Thus the sum also converges absolutely in the same region to a rational function of the same type.

$\square$

Remark 2.4.3. The argument we have used in dealing with e^{zD} operator can be generalized to products and iterates of any numbers of vertex operators. One should also note that we don't need V to be grading-restricted. The same result also holds for left modules, right modules and bimodules for MOSVAs. For brevity, in the future we will not repeat the argument, but refer to this remark when we need the e^{zD} operator.

Proposition 2.4.4. *For $v' \in V', u_1, ..., u_n, v \in V$, the complex n-multiseries*

$$\langle v', Y_V^s(u_1, z_1) \cdots Y_V^s(u_n, z_n)v\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n; z_i = z_j, 1 \leq i < j \leq n$.

Proof. Likewise, the proof is divided into three steps. For brevity, we only state the conclusions of each step.

1. With the conclusion of Theorem 2.3.13, the D-conjugation property (See Part (4) of Proposition 2.1.4) and Lemma 2.3.8 we can prove that the series

   $$\langle v', Y_V(e^{\zeta_2 D_V} \cdots Y_V(e^{\zeta_{n-1}D_V} Y_V(e^{\zeta_n D_V} Y_V(v, -z_n)u_n, -z_{n-1})u_{n-1}, -z_{n-2}) \cdots, -z_1)u_1\rangle$$

   converges absolutely when

   $$|z_k| > |\zeta_{k+1} + (-z_{k+1} + \zeta_{k+2}) + \cdots + (-z_{n-1} + \zeta_n) - z_n|, k = 1, ..., n-1,$$
   $$|z_k| > |\zeta_{k+1} + (-z_{k+1} + \zeta_{k+2}) + \cdots + (-z_i + \zeta_{i+1})|, k = 1, ..., n-1, i = k, ..., n-1.$$

   to the rational function determined by

   $$\langle v', Y_V(\cdots Y_V(Y_V(Y_V(v, -z_n)u_n, -z_{n-1}+\zeta_n)u_{n-1}, -z_{n-2}+\zeta_{n-1}) \cdots, -z_1+\zeta_2)u_1\rangle$$

   that has the only possible poles at

   $$-z_n + (z_{n-1} + \zeta_n) + \cdots + (-z_k + \zeta_{k+1}) = 0, k = 1, ..., n-1;$$
   $$(-z_i + \zeta_{i+1}) + \cdots + (-z_k + \zeta_{k+1}) = 0, k = 1, ..., n-1, i = k, ..., n-1.$$

2. Since $\zeta_i = z_i, i = 2, ..., n$ is contained in the region of the convergence, we then evaluate $\zeta_i = z_i$ for every $i = 2, ..., n$, to see that the series

$$\langle v', Y_V(e^{z_2 D_V} \cdots Y_V(e^{z_{n-1} D_V} Y_V(e^{z_n D_V} Y_V(v, -z_n)u_n, -z_{n-1})u_{n-1}, -z_{n-2}) \cdots, -z_1)u_1\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to the rational function determined by

$$\langle v', Y_V(\cdots Y_V(Y_V(Y_V(v, -z_n)u_n, -z_{n-1} + z_n)u_{n-1}, -z_{n-2} + z_{n-1}) \cdots, -z_1 + z_2)u_1\rangle$$

that has the only possible poles at $z_i = 0, i = 1, ..., n; z_i = z_j, 1 \leq i < j \leq n$.

3. Finally we use Remark 2.4.3 to conclude that the series

$$\langle v', Y_V^s(u_1, z_1) \cdots Y_V^s(u_n, z_n)v\rangle,$$

which is precisely

$$\langle v', e^{z_1 D_V} Y_V(e^{z_2 D_V} \cdots Y_V(e^{z_{n-1} D_V} Y_V(e^{z_n D_V} Y_V(v, -z_n)u_n, -z_{n-1})u_{n-1}, -z_{n-2}) \cdots, -z_1)u_1\rangle,$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function that has the same types of poles.

$\square$

Proposition 2.4.5. *For $v' \in V', u_1, u_2, v \in V$, the complex double series*

$$\langle v', Y_V^o(Y_V^o(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$

Proof. 1. With the rationality of products the D-conjugation property and Lemma 2.3.7, we can prove that the series

$$\langle v', e^{\zeta D_V} Y_V(v, -z_2)e^{-\zeta D_V} Y_V(u_2, -z_1 + z_2)u_1\rangle$$

converges absolutely when $|z_2| > |\zeta|, |z_2| > |z_1 - z_2 + \zeta|, |z_1 - z_2| > 0$ to the rational function determined by

$$\langle v', Y_V(v, -z_2 + \zeta)Y_V(u_2, -z_1 + z_2)u_1\rangle$$

that has the only possible poles at $z_2 = \zeta, z_1 = z_2, z_1 - z_2 + \zeta = z_2$

2. Since $\zeta = -z_1 + z_2$ is contained in the region of the convergence, we then evaluate $\zeta = -z_1 + z_2$ to see that the series

$$\langle v', e^{(-z_1+z_2)D_V} Y_V(v, -z_2) e^{(z_1-z_2)D_V} Y_V(u_2, -z_1 + z_2)u_1\rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to the rational function determined by

$$\langle v', Y_V(v, -z_1) Y_V(u_2, -z_1 + z_2)u_1\rangle$$

that has the only possible poles at $z_1 = 0, z_1 = z_2, z_2 = 0$.

3. Finally we use Remark 2.4.3 to conclude that the series

$$\langle v', Y_V^s(Y_V^s(u_1, z_1 - z_2)u_2, z_2)v\rangle,$$

which is precisely

$$\langle v', e^{z_2 D_V} Y_V(v, -z_2) e^{(z_1-z_2)D_V} Y_V(u_2, -z_1 + z_2)u_1\rangle,$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function that has the same types of poles.

□

2.4.3 $(V, Y_V^s, \mathbf{1})$ forms a MOSVA

Proposition 2.4.6. *Given a MOSVA $(V, Y_V, \mathbf{1})$, with the opposite vertex operator map*

$$\begin{aligned} Y_V^s : \quad & V \otimes V \to V[[x, x^{-1}]] \\ & u \otimes v \mapsto e^{xD_V} Y_V(v, -x)u \end{aligned}$$

$(V, Y_V^s, \mathbf{1})$ is also a MOSVA.

Proof. 1. The lower bound condition is trivial. We verify the $\mathbf{d}_V$-bracket formula: for every $u \in V$

$$[\mathbf{d}_V, Y_V^s(u, x)] = x\frac{d}{dx} Y_V^s(u, x) + Y_V^s(\mathbf{d}_V u, x)$$

Without loss of generality, let u, v be homogeneous element

$[\mathbf{d}_V, Y_V^s(u, x)]v = \mathbf{d}_V Y_V^s(u, x)v - Y_V^s(u, x)\mathbf{d}_V v$

$$
\begin{aligned}
&= \mathbf{d}_V e^{xD_V} Y_V(v,-x)u - e^{xD_V} Y_V(\mathbf{d}_V v,-x)u \\
&= \sum_{m,n} \mathbf{d}_V \left(\frac{1}{m!} D_V^m (Y_V)_n(v)u \right) x^m (-x)^{-n-1} \\
&\qquad - \sum_{m,n} \frac{1}{m!} D_V^m (\mathrm{wt}\ v)(Y_V)_n(v) u x^m (-x)^{-n-1} \\
&= \sum_{m,n} (\mathrm{wt}\ u - n - 1 + \mathrm{wt}\ v + m - \mathrm{wt}\ v) \frac{1}{m!} D_V^m (Y_V)_n(v) u x^m (-x)^{-n-1} \\
&= \sum_{m,n} (\mathrm{wt}\ u - n - 1 + m) \frac{1}{m!} (D_V^m (Y_V)_n(v)u) x^m (-x)^{-n-1} \\
&= \sum_{m,n} (\mathrm{wt}\ u) \frac{1}{m!} (D_V^m (Y_V)_n(v)u) x^m (-x)^{-n-1} \\
&\qquad + \sum_{m,n} (-n + m - 1) \frac{1}{m!} (D_V^m (Y_V)_n(v)u) x^m (-x)^{-n-1} \\
&= (\mathrm{wt}\ u) e^{xD_V} Y_V(v,-x)u + x \frac{d}{dx} \left(e^{xD_V} Y_V(v,-x)u \right) \\
&= Y_V^s(\mathbf{d}_V u, x)v + x \frac{d}{dx} Y_V^s(u,x)v
\end{aligned}
$$

2. Since for $v \in V$,

$$Y_V^s(\mathbf{1}, x)v = e^{xD_V} Y_V(v,-x)\mathbf{1} = e^{xD_V} e^{-xD_V} v = v,$$

the identity property follows. Since for $u \in V$,

$$Y_V^s(u,x)\mathbf{1} = e^{zD_V} Y_V(\mathbf{1}, -x)u = e^{zD_V} u,$$

the creation property follows.

3. It follows directly from $Y_V^s(u,x)\mathbf{1} = e^{zD_V} u$ that

$$Du = \lim_{x \to 0} \frac{d}{dx} Y_V^s(u,x)\mathbf{1}$$

We prove the D-derivative formula as follows:

$$
\begin{aligned}
Y_V^s(D_V u, x)v &= e^{xD_V} Y_V(v,-x) D_V u = e^{xD_V} D_V Y_V(v,-x)u + e^{xD_V} [D_V, Y_V(v,-x)]u \\
&= e^{xD_V} D_V Y_V(v,-x)u + e^{xD_V} \frac{d}{d(-x)} Y_V(v,-x)u \\
&= \frac{d}{dx} e^{xD_V} Y_V(v,-x)u = \frac{d}{dx} Y_V^s(u,x)v
\end{aligned}
$$

Then the D_V-bracket formula follows from

$$\begin{aligned}[D_V, Y_V^s(u,x)]v &= D_V Y_V^s(u,x)v - Y_V^s(u,x)D_V v \\ &= D_V e^{xD_V} Y(v,-x)u - e^{xD_V} Y(D_V v, -x)u \\ &= e^{xD_V} D_V Y_V(v,-x)u + e^{xD_V} \frac{d}{d(-x)} Y_V(v,-x)u = Y^o(D_V u, x)V\end{aligned}$$

4. This has been done in Proposition 2.4.4 and Proposition 2.4.5

5. Fix $u_1, u_2, v \in V$ and $v' \in V'$. Let $S_1 = \{(z_1, z_2) : |z_1| > |z_2| > 0\}$ and $S_2 = \{(z_1, z_2) : |z_2| > |z_1 - z_2| > 0\}$. A careful analysis of the proof to Proposition 2.4.4 shows that, the series

$$\langle v', Y_V^o(u_1, z_1)Y_V^o(u_2, z_2)v\rangle$$

converges absolutely in S_1 to the same rational function as that

$$\langle v', e^{z_1 D_V} Y_V(Y_V(v, -z_2)u - 2, -z_1 + z_2)u_1\rangle$$

converges to (in the region $|z_1 - z_2| > |z_2| > 0$). Also the proof to Proposition 2.4.5 shows that when $|z_2| > |z_1 - z_2| > 0$, the series

$$\langle v', Y_V^o(Y_V^o(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

converges absolutely in S_2 to the same rational function as that

$$\langle v', e^{z_1 D_V} Y_V(v, -z_1)Y_V(u_2, -z_1 + z_2)u_1\rangle$$

converges to (in the region $|z_1| > |z_1 - z_2| > 0$). From the associativity of Y_V, we know that these rational functions are identical. In other words, $\langle v', Y_V^o(u_1, z_1)Y_V^o(u_2, z_2)v\rangle$ and $\langle v', Y_V^o(Y_V^o(u_1, z_1 - z_2)u_2, z_2)v\rangle$ converges absolutely to the same rational function respectively in the region S_1 and S_2. So in $S_1 \cap S_2$ their sums are equal.

$\square$

2.4.4 Other Remarks

Remark 2.4.7. Given a MOSVA $(V, Y_V, \mathbf{1})$, from the fact that

$$(Y^o)^o(u,x)v = e^{xD_V} Y_V^s(v,-x)u = e^{xD_V} e^{-xD_V} Y(u,x)v = Y(u,x)v$$

we have $(V^{op})^{op} = V$.

Remark 2.4.8. For a vertex algebra with a lower-bounded grading, we know that $Y_V = Y_V^s$ because this is precisely the skew-symmetry identity. Conversely, if a MOSVA V satisfies $Y_V = Y_V^s$, i.e. for $v' \in V', u_1, u_2, v \in V$ and any $x \neq 0$,

$$\langle v', Y_V(u,z)v\rangle = \langle v', e^{zD_V} Y_V(v,-z)u\rangle$$

then V is a vertex algebra with a lower-bounded grading, since associativity and skew-symmetry identity imply the Jacobi identity (see [H5] Proposition 2.2 and [LL], Section 3.6.)

Remark 2.4.9. The discussion here works also when V is not grading-restricted.

Chapter 3

Modules of meromorphic open string vertex algebras

Throughout the whole chapter, all MOSVAs are assumed to be grading-restricted.

3.1 Left V-modules

The notion of left V-module for a meromorphic open-string vertex algebra was introduced in [H3]. Here we recall the definition.

3.1.1 The axiomatic definition

Definition 3.1.1. Let $(V, Y_V, \mathbf{1})$ be a meromorphic open-string vertex algebra. A *left V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{m\in\mathbb{C}} W_{[m]}$ (graded by *weights*), equipped with a *vertex operator map*

$$\begin{array}{rcl} Y_W^L : V \otimes W & \to & W[[x, x^{-1}]] \\ u \otimes w & \mapsto & Y_W^L(u, x)v, \end{array}$$

an operator $\mathbf{d}_W$ of weight 0 and an operator D_W of weight 1, satisfying the following axioms:

1. Axioms for the grading:

   (a) *Lower bound condition*: When $\mathrm{Re}(m)$ is sufficiently negative, $W_{[m]} = 0$.

   (b) $\mathbf{d}$-*grading condition*: for every $w \in W_{[m]}$, $\mathbf{d}_W w = mw$.

   (c) $\mathbf{d}$-*bracket property*: For $u \in V$,

   $$[\mathbf{d}_W, Y_W^L(u, x)] = Y_W^L(\mathbf{d}_V u, x) + x\frac{d}{dx}Y_W^L(u, x).$$

2. The *identity property*: $Y_W^L(\mathbf{1}, x) = 1_W$.

3. The *D-derivative property* and the *D-commutator formula*: For $u \in V$,

$$\begin{aligned} \frac{d}{dx} Y_W^L(u,x) &= Y_W^L(D_V u, x) \\ &= [D_W, Y_W^L(u,x)]. \end{aligned}$$

4. *Rationality*: For $u_1, \dots, u_n \in V, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n) v \rangle$$

   converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \dots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \dots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2 \in V, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2) v \rangle$$

   converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2 \in V, w \in W$, $w' \in W'$,

$$\langle w', Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) v \rangle = \langle w', Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2) v \rangle$$

   when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A left V-module is said to be *grading-restricted* if $\dim W_{[m]} < \infty$ for every $m \in \mathbb{C}$.

We denote the left V-module just defined by $(W, Y_W^L, \mathbf{d}_W, D_W)$ or simply W when there is no confusion.

3.1.2 Some immediate consequences

Similarly, the following proposition holds

Proposition 3.1.2. *Let V be a MOSVA and W be a left V-module. then*

1. *For $u \in V$, $Y_W^L(u,x)$ can be regarded as a formal series in End$(W)[[x, x^{-1}]]$*

$$Y_W^L(u,x) = \sum_{n \in \mathbb{Z}} (Y_W^L)_n(u) x^{-n-1}$$

   where $(Y_W^L)_n(u) : W \to W$ is a linear map for every $n \in \mathbb{Z}$. If u is homogeneous, then $(Y_W^L)_n(u)$ is a map of weight $\mathrm{wt}\ u - n - 1$.

2. *For fixed $u \in V, w \in W$, $Y_W^L(u,x)w$ is lower truncated, i.e, there are at most finitely many negative powers of x.*

3. *Formal Taylor theorem: for $u \in V$,*

$$Y_W^L(u, x+y) = Y_V(e^{yD_V}u, x) = e^{yD_W}Y_W^L(u,x)e^{-yD_W},$$

in $End(W)[[x, y, x^{-1}]]$.

Proof. Similar to the argument of Proposition 2.1.4. For the second statement, note that W is lower truncated in the sense that $W_{[m]} < 0$ when Re$m << 0$. □

Remark 3.1.3. If we let

$$\frac{f(z_1, ..., z_n)}{\prod_{i=1}^{n} z_i^{p_i} \prod_{1\le i<j\le n} (z_i - z_j)^{p_{ij}}}$$

be the rational function determined by the series

$$\langle w', Y_W^L(u_1, z_1)\cdots Y_W^L(u_n, z_n)w\rangle,$$

then for homogeneous $u_1, ..., u_n \in V, w \in W, w' \in W'$, we can explicitly compute the total degree of the homogeneous polynomial $f(z_1, ..., z_n)$ in terms of the weights and p_i, p_{ij}'s. We start by expanding the series as

$$\sum_{k_1,...,k_n} \langle w', (Y_W^L)_{k_1}(u_1)\cdots(Y_W^L)_{k_n}(u_n)w\rangle z_1^{-k_1-1}\cdots z_n^{-k_n-1}$$

then the coefficients are nonzero only when

$$\text{wt } w' = \text{wt } u_1 - k_1 - 1 + \cdots + \text{wt } u_n - k_n - 1 + \text{wt } w$$

In particular,

$$\text{Re wt } w' = \text{wt } u_1 - k_1 - 1 + \cdots + \text{wt } u_n - k_n - 1 + \text{Re wt } w$$

Thus

$$\begin{aligned}\deg f &= \sum_{i=1}^{n} p_i + \sum_{1\le i<j\le n} p_{ij} + (-k_1 - 1 - k_2 - 1 - \cdots - k_n - 1)\\ &= \sum_{i=1}^{n} p_i + \sum_{1\le i<j\le n} p_{ij} + \text{Re wt } w' - \sum_{i=1}^{n} \text{wt } u_i - \text{Re wt } w\end{aligned}$$

In particular, when there are only two vertex operators, the total degree of the homogeneous polynomial in the numerator is just

$$p_1 + p_2 + p_{12} + \text{Re wt } w' - \text{wt } u_1 - \text{wt } u_2 - \text{Re wt } w$$

3.1.3 $\widehat{W}$-valued map interpretation

In Chapter 2, we interpreted vertex operators as $\overline{V}$-valued maps, under the assumption that V is grading-restricted. Since we always work with grading-restricted MOSVA, all the results concerning non-grading-restricted MOSVAs were given in Remarks. This is no long the case when we talk about modules: modules $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ that are not grading-restricted ($\dim W_{[n]}$ need not be finite) arise naturally in our studies. In this scenario, the full dual $\widehat{W} = \prod_{n\in\mathbb{C}} W_{[n]}^{**}$ of the graded dual $W' = \coprod_{n\in\mathbb{C}} W_{[n]}^{*}$ no longer coincides with the algebraic completion $\overline{W} = \prod_{n\in\mathbb{C}} W_{[n]}$ of W. As a result, we need to modify all the previous summaries and interpret vertex operators as $\widehat{W}$-valued maps.

For $m \in \mathbb{C}$, let $\pi_m^W : \widehat{W} \to W_{[m]}^{**}$ be the projection operator. This projection operator can be restricted to $\overline{W}$ to give the projection $\overline{W} \to W_{[m]}$, which we also denote by π_m^W.

For one single operator, a similar discussion to Summary 2.2.1 with Proposition 3.1.2 will lead us to the following summary:

Summary 3.1.4. *For $u \in V, w \in W$ and any nonzero complex number z, the summation*

$$Y_W^L(u, z)w = \sum_{n\in\mathbb{Z}} Y_W^L(u)_n w z^{-n-1}$$

gives an element in $\overline{W}$. For a given nonzero $z \in \mathbb{C}$, the vertex operator map give rise to the following map

$$Y_W^L(\cdot, z)\cdot : V \otimes W \to \overline{W} \subset \widehat{W}$$

Since the $Y_W^L(u, z)w \in \overline{W}$, there is not much trouble in understanding the product of two vertex operators $Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)w$: for each $r \in \mathbb{C}$ we apply the operator $\pi_m^W : \overline{W} \to W_{[m]}$ to the $\overline{W}$ element $Y_W^L(u_2, z_2)w$, then act $Y_W^L(u_1, z_1)$ to the W element $\pi_m^W Y_W^L(u_1, z_1)w$. The proof of Proposition 2.2.11 applies similarly here: we know that the series of $\overline{W}$-elements

$$\sum_{r\in\mathbb{C}} Y_W^L(u_1, z_1)\pi_m^W Y_W^L(u_2, z_2)w$$

converges absolutely in the sense of Definition 2.2.8 and it should coincide with the element $Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)w$. However since $\overline{W}$ is in general not a closed subspace of $\widehat{W}$, we can only say that the sum of the series is an element in $\widehat{W}$ and it does not necessarily fall in $\overline{W}$. So the summary must be modified as follows

Summary 3.1.5. *For any $u_1, u_2 \in V$, $w \in W$ and any complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the single series*

$$\sum_{r\in\mathbb{C}} Y_W^L(u_1, z_1)\pi_m^W Y_W^L(u_2, z_2)w$$

of elements in $\overline{W}$ converges absolutely, i.e., for any $w' \in W'$,

$$\sum_{r\in\mathbb{C}} \langle w', Y_W^L(u_1, z_1)\pi_m^W Y_W^L(u_2, z_2)w\rangle$$

converges absolutely. Moreover, the sum of the series is equal to the sum of the double series

$$Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)w$$

For fixed z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the product of two vertex operators gives rise to the following map

$$Y_W^L(\cdot, z_1)Y_W^L(\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$

which is equal to the map

$$\sum_{r\in\mathbb{C}} Y_W^L(\cdot, z_1)\pi_m^W Y_W^L(\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$

Remark 3.1.6. Although the summation is over $\mathbb{C}$, it is easy to see that only countably many indexes will be involved, as every element in W must be a *finite* sum of homogeneous elements, and vertex operators acting on homogeneous elements $w \in W_{[m]}$ only gives a series of elements with weights in the equivalent class $m + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$. Also note that when paired to $w' \in W'$, the complex series is still of integral power. No fractional powers should arise here.

When we want to consider the product of three vertex operators, say $Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w$, problems arise, as $Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w \in \widehat{W}$, thus $\pi_m^W Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w$ is no longer an element in $W_{[m]}$, but in a larger space $W_{[m]}^{**}$, and we don't have any definition of $Y_W^L(u_1, z_1)$ acting $W_{[m]}^{**}$.

One way to resolve the problem is to understand each term $Y_W^L(u_1, z_1)\pi_m^W Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w$ simply as part of the triple series $Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w$. More precisely, for homogeneous $u_2, u_3 \in V$ and $w \in W$, as

$$\pi_m^W Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w = \sum_{\substack{\mathrm{wt}\, u_2 + \mathrm{wt}\, u_3 + \mathrm{wt}\, w - n_2 - n_3 - 2 = m \\ n_2, n_3 \in \mathbb{Z}}} (Y_W^L)_{n_2}(u_2)(Y_W^L)_{n_3}(u_3) w z_2^{-n_2-1} z_3^{-n_3-1}$$

we naturally have

$$\begin{aligned}
& Y_W^L(u_1, z_1)\pi_m^W Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w \\
=& Y_W^L(u_1, z_1)\left(\sum_{\substack{\mathrm{wt}\, u_2 + \mathrm{wt}\, u_3 + \mathrm{wt}\, w - n_2 - n_3 - 2 = m \\ n_2, n_3 \in \mathbb{Z}}} (Y_W^L)_{n_2}(u_2)(Y_W^L)_{n_3}(u_3) w z_2^{-n_2-1} z_3^{-n_3-1}\right) \\
=& \sum_{n_1 \in \mathbb{Z}} (Y_W^L)_{n_1}(u_1)\left(\sum_{\substack{\mathrm{wt}\, u_2 + \mathrm{wt}\, u_3 + \mathrm{wt}\, w - n_2 - n_3 - 2 = m \\ n_2, n_3 \in \mathbb{Z}}} (Y_W^L)_{n_2}(u_2)(Y_W^L)_{n_3}(u_3) w z_2^{-n_2-1} z_3^{-n_3-1}\right) z_1^{-n_1-1} \\
=& \sum_{\substack{\mathrm{wt}\, u_2 + \mathrm{wt}\, u_3 + \mathrm{wt}\, w - n_2 - n_3 - 2 = m \\ n_1, n_2, n_3 \in \mathbb{Z}}} (Y_W^L)_{n_1}(u_1)(Y_W^L)_{n_2}(u_2)(Y_W^L)_{n_3}(u_3) w z_1^{-n_1-1} z_2^{-n_2-1} z_3^{-n_3-1}
\end{aligned}$$

If we treat the element in the parenthesis as an element of $W_{[m]}^{**}$, then the sum gives an element in $\widehat{W}$. So summing up all $r \in \mathbb{C}$ will yield a series in $\widehat{W}$. However, after pairing it with w', we see that the resulted complex series $\sum_{m \in \mathbb{C}} \langle w', Y_W^L(u_1, z_1)\pi_m^W Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w\rangle$ is just a rearrangement of the absolutely convergent triple series $Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w$. For nonhomogeneous $u_2, u_3 \in V$ and $w \in W$, we use the same argument as in Proposition 2.2.11 to write the corresponding series as a finite sum of absolutely convergent series.

With the above argument in mind, one can modify the arguments in Summary 2.2.15 similarly, to get

Summary 3.1.7. *For any $u_1, ..., u_n \in V$, $w \in W$ and any $z_1, ..., z_n \in \mathbb{C}$ satisfying*

$|z_1| > |z_2| > \cdots > |z_n| > 0$, the series

$$\sum_{m_1,\ldots,m_{n-1}\in\mathbb{Z}} Y_W^L(u_1,z_1)\pi_{m_1}^W Y_W^L(u_2,z_2)\pi_{m_2}^W \cdots Y_W^L(u_{n-1},z_{n-1})\pi_{m_{n-1}}^W Y_W^L(u_n,z_n)w$$

of elements in $\widehat{W}$ converges absolutely, The sum is equal to the $\widehat{W}$ element given by

$$Y_W^L(u_1,z_1)\cdots Y_W^L(u_n,z_n)w$$

For fixed $z_1, z_2, ..., z_n \in \mathbb{C}$ satisfying $|z_1| > \cdots > |z_n| > 0$, the product of any number of vertex operators gives rise to a map

$$Y_W^L(\cdot,z_1)Y_W^L(\cdot,z_2)\cdots Y_W^L(\cdot,z_n)\cdot : V^{\otimes n}\otimes W \to \widehat{W}$$

and is equal to the sum

$$\sum_{m_1,\ldots,m_{n-1}\in\mathbb{Z}} Y_W^L(\cdot,z_1)\pi_{m_1}^W Y_W^L(\cdot,z_2)\pi_{m_2}^W \cdots Y_W^L(\cdot,z_{n-1})\pi_{m_{n-1}}^W Y_W^L(\cdot,z_n)\cdot : V^{\otimes n}\otimes W \to \widehat{W}$$

Remark 3.1.8. We put all $\pi_{m_i}^W$'s on for completeness. In practice it is absolutely fine to omit any number of them.

Remark 3.1.9. Another way to resolve this issue is to extend the vertex operator actions to $\widehat{W}$ using the double adjoint process. Let $L : W \to W$ be a homogeneous linear operator, then L can be extended to $\widehat{W} \to \widehat{W}$ by the double adjoint process: first define the adjoint L' on L by

$$\langle L'w', w\rangle = \langle w', Lw\rangle$$

One checks that L' is also a homogeneous operator on W'. In particular, for every $w' \in W'$, $L'w' \in W'$. Thus the image of L' on W' still falls in W' (if L is not homogeneous, then we only know that $L'W' \subseteq W^*$ and not necessarily in W'). Hence $L' : W' \to W'$ is an operator. We then define the extension $L : \widehat{W} \to \widehat{W}$ by

$$\langle w', L\widehat{w}\rangle = \langle L'w', \widehat{w}\rangle$$

In particular, the operators $\mathbf{d}_W$, a^{d_W} $(a \in \mathbb{C}^\times)$, D_W, and $(Y_W^L)_n(u), (Y_W^R)_n(u)$ for $n \in \mathbb{Z}$ obtained from the vertex operators with homogeneous $u \in V$ can be extended to operators on $\widehat{W}$. For convenience, we will not add an arc on these extended operators, but simply use the same notation.

Now let $L : W \to W$ be a finite sum of homogeneous linear operators on L, by extending each summands we see that L also admits an extension to $\widehat{W}$. In particular, the components of vertex operators $(Y_W^L)_n(u)$ and $(Y_W^R)_n(u)$ admits an extension for every $u \in V$ and $n \in \mathbb{Z}$.

Thus, for each fixed $z \in \mathbb{C}^\times$, the actions of the vertex operator $Y_W^L(u,z)\cdot$ and $Y_W^R(\cdot,z)u$ on an element $\widehat{w} \in \widehat{W}$ amounts be giving infinite sums of elements $\widehat{W}$. To make sense of $Y_W^L(u,z)\widehat{w}$ and $Y_W^R(\widehat{w},z)u$, both $\widehat{w}$ and z has to be chosen carefully so that these infinite sums converge absolutely. For example, if $\widehat{w}$ is chosen as $Y_W^L(u_1,z_1)\cdots Y_W^L(u_n,z_n)$ for $|z_1| > \cdots > |z_n| > 0$ and z is chosen such that $|z| > |z_1|$, then $Y_W^L(u,z)\widehat{w}$ converges absolutely and thus is a well-defined element in $\widehat{W}$.

Similar to the principle above, the e^{zD_W} operator on W is extended to an operator on $\widehat{W}$, provided that z and $\widehat{w}$ are carefully chosen to make sure the series $\sum_{i=0}^{\infty} 1/i!z^i D_W^i \widehat{w}$ converges absolutely.

Remark 3.1.10. From the extension process, one can easily check that the equality of two vertex operator actions on W extends to $\widehat{W}$, provided the actions are well-defined on $\widehat{W}$. For example, let W be a left V-module, for $u_1, u_2 \in V$, if $\widehat{w} \in \widehat{W}$ and $z_1, z_2 \in \mathbb{C}$ are chosen such that both $Y_W^L(u_1,z_1)Y_W^L(u_2,z_2)\widehat{w}$ and $Y_W^L(Y_V(u_1,z_1-z_2)u_2,z_2)\widehat{w}$ converges absolutely, then the sums of these series in $\widehat{W}$ are equal.

We remind the reader that it is crucial to check if the actions are well-defined, i.e., the corresponding series in $\widehat{W}$ converges absolutely.

3.1.4 Rationality of iterates

For left V-modules, it is relatively easier to make the modifications on the iterates. As the MOSVA is always assumed to be grading-restricted, the interpretation

$$Y_W^L(Y_V(u_1,z_1-z_2)u_2,z_2)w = \sum_{k\in\mathbb{Z}} Y_W^L(\pi_k Y_V(u_1,z_1-z_2)u_2,z_2)w$$

and

$$Y_W^L(Y_V(\cdots Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3 \cdots, z_{n-1} - z_n)u_n, z_n)w$$
$$= \sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_W^L(\pi_{k_{n-1}} Y_V(\cdots Y_V(\pi_{k_1} Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3 \cdots, z_{n-1} - z_n)u_n, z_n)w$$

in Chapter 2 still applies here, for any $u_1, ..., u_n \in V, w \in W$. Of course, the resulted element may still fall outside of $\overline{W}$. But the following conclusions similarly hold.

Summary 3.1.11. *For fixed z_1, z_2 satisfying $|z_2| > |z_1 - z_2| > 0$, the iterate of two vertex operators gives rise to a map*

$$Y_W^L(Y_V(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$

which is equal to the sum

$$\sum_{k\in\mathbb{Z}} Y_W^L(\pi_k Y_V(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$

Summary 3.1.12. *For fixed z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, the following maps*

$$Y_W^L(\cdot, z_1)Y_W^L(\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$
$$\sum_{m\in\mathbb{C}} Y_W^L(\cdot, z_1)\pi_m^W Y_W^L(\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$
$$Y_W^L(Y_W^L(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$
$$\sum_{m\in\mathbb{C}} Y_W^L(\pi_m^W Y_W^L(\cdot, z_1 - z_2)\cdot, z_2)\cdot : V \otimes V \otimes W \to \widehat{W}$$

are equal.

For the iterate of n vertex operators, the module version is formulated similarly as in Chapter 2:

Proposition 3.1.13. *For $u_1, u_2, ..., u_n \in V, w \in W, w' \in W'$, the series*

$$\langle w', Y_W^L(Y_V(\cdots Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3 \cdots, z_{n-1} - z_n)u_n, z_n)w\rangle$$

converges absolutely in the region

$$\left\{ (z_1, ..., z_n) \in \mathbb{C}^n : \begin{array}{l} |z_n| > |z_{n-1} - z_n| + |z_{n-2} - z_{n-1}| + \cdots + |z_1 - z_2|, \\ |z_i - z_{i+1}| > \displaystyle\sum_{j=1}^{i-1} |z_j - z_{j+1}|, i = 3, ..., n-1 \\ |z_2 - z_3| > |z_1 - z_2| > 0 \end{array} \right\}$$

to the rational function that

$$\langle w', Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) \cdots Y_W^L(u_n, z_n) w\rangle$$

converges to.

Proof. The formulations in Section 2.2 and 2.3 and the proof of Proposition 2.3.13 carry over to the modules. Alternatively, one can formulate a proof using the extended operators. Here we only show this approach in detail for 3 vertex operators. The argument of induction is similar to that in Proposition 2.3.13.

We first prove the following intermediate conclusion: for any $u_1, u_2, u_3 \in V, w \in W, w' \in w'$, fixed $z_1, z_2, z_3 \in \mathbb{C}$ satisfying $|z_2| > |z_1 - z_2 - z_3|, |z_2| > |z_1 - z_2| > 0, |z_2| > |z_3| > 0$, the series

$$\langle w', Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3) Y_W^L(u_3, z_3) w\rangle$$

converges absolutely to the rational function that

$$\langle w', Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) Y_W^L(u_3, z_3) w\rangle$$

converges to.

From the rationality of products, we know that

$$\begin{aligned}
&\langle w', Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) Y_W^L(u_3, z_3) w\rangle \\
&\quad = \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \langle w', (Y_W^L)_{k_1}(u_1)(Y_W^L)_{k_2}(u_2)(Y_W^L)_{k_3}(u_3) w\rangle z_1^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1}
\end{aligned}$$

converges absolutely in the region $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| > |z_2| > |z_3| > 0\}$ to a rational function that has the only possible poles at $z_1 = 0, z_2 = 0, z_3 = 0, z_1 = z_2, z_2 = z_3, z_1 = z_3$. Denote this rational function by $f(z_1, z_2, z_3)$. Then

$$f(z_1, z_2, z_3) = \frac{g(z_1, z_2, z_3)}{z_1^{p_1} z_2^{p_2} z_3^{p_3} (z_1 - z_2)^{p_{12}} (z_2 - z_3)^{p_{23}} (z_1 - z_3)^{p_{13}}} \tag{3.1}$$

for some integers $p_1, p_2, p_3, p_{12}, p_{23}, p_{13} \geq 0$ and some polynomial $g(z_1, z_2, z_3)$.

Now for each fixed $k_3 \in \mathbb{Z}$, we consider the series

$$\langle w', Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) (Y_W^L)_{k_3}(u_3) w\rangle$$

$$= \sum_{k_1,k_2\in\mathbb{Z}} \langle w', (Y_W^L)_{k_1}(u_1)(Y_W^L)_{k_2}(u_2)(Y_W^L)_{k_3}(u_3)w\rangle z_1^{-k_1-1}z_2^{-k_2-1}.$$

As part of an absolutely convergent series, it is also absolutely convergent. From associativity, its sum is equal to

$$\langle w', Y_W^L(Y_V(u_1, z_1-z_2)u_2, z_2)(Y_W^L)_{k_3}(u_3)w\rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$ for each fixed l. We multiply it with $z_3^{-k_3-1}$ and sum up all $k_3 \in \mathbb{Z}$ to see that

$$\begin{aligned}&\sum_{k_3\in\mathbb{Z}} \langle w', Y_W^L(Y_V(u_1, z_1-z_2)u_2, z_2)(Y_W^L)_{k_3}(u_3)w\rangle z_3^{-k_3-1}\\ &\quad= \sum_{k_3\in\mathbb{Z}} \left(\sum_{k_1,k_2\in\mathbb{Z}} \langle w', (Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle (z_1-z_2)^{-k_1-1}z_2^{-k_2-1}\right) z_3^{-k_3-1}\end{aligned}$$

viewed as a single complex series whose terms are

$$\left(\sum_{k_1,k_2\in\mathbb{Z}} \langle w', (Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle (z_1-z_2)^{-k_1-1}z_2^{-k_2-1}\right) z_3^{-k_3-1},$$

converges to $f(z_1, z_2, z_3)$ when $|z_1| > |z_2| > |z_3| > 0, |z_2| > |z_1 - z_2| > 0$. Moreover, one checks easily that the power of z_3 is lower-truncated.

We now use Lemma 2.3.7 to elaborately show that the series

$$\begin{aligned}&\langle w', Y_W^L(Y_V(u_1, z_1-z_2)u_2, z_2)Y_W^L(u_3, z_3)w\rangle\\ &\quad= \sum_{k_1,k_2,k_3\in\mathbb{Z}} \langle w', (Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle (z_1-z_2)^{-k_1-1}z_2^{-k_2-1}z_3^{-k_3-1}\end{aligned}$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_2| > |z_1 - z_2 - z_3|, |z_2| > |z_1 - z_2| > 0, |z_2| > |z_3| > 0$. First we set $\zeta_1 = z_1 - z_2, \zeta_2 = z_2, \zeta_3 = z_3$. Let

$$T = \{(\zeta_1, \zeta_2, \zeta_3) : |\zeta_2| > |\zeta_3| + |\zeta_1|, |\zeta_1| > 0, |\zeta_3| > 0\}$$

With Lemma 2.3.2, we see that T is a connected multicircular domain. Now we express the function $f(z_1, z_2, z_3)$ in terms of the variables $\zeta_1, \zeta_2, \zeta_3$ as

$$f(\zeta_1+\zeta_2, \zeta_2, \zeta_3) = \frac{g(\zeta_1+\zeta_2, \zeta_2, \zeta_3)}{(\zeta_1+\zeta_2)^{p_1}\zeta_2^{p_2}\zeta_3^{p_3}\zeta_1^{p_{12}}(\zeta_2-\zeta_3)^{p_{23}}(\zeta_1+\zeta_2-\zeta_3)^{p_{13}}},$$

which admits a Laurent series expansion in $\zeta_1, \zeta_2, \zeta_3$ by the following steps:

1. Expand the negative powers of $\zeta_1+\zeta_2$ as a power series in ζ_1. The resulted series converges when $|\zeta_2|>|\zeta_1|$.

2. Expand the negative powers of $\zeta_2-\zeta_3$ as a power series in ζ_3. The resulted series converges when $|\zeta_2|>|\zeta_3|$.

3. Expand the negative powers of $\zeta_1+\zeta_2-\zeta_3$ as power series in $\zeta_1-\zeta_3$, then further expand all the positive power of $\zeta_1-\zeta_3$ as polynomials. The resulted series converges in $|\zeta_2|>|\zeta_1-\zeta_3|$.

Obviously if $(\zeta_1,\zeta_2,\zeta_3)\in T$, then all the above conditions are satisfied (note that $|\zeta_2|>|\zeta_3|+|\zeta_1|$ implies that $|\zeta_2|>|\zeta_1-\zeta_3|$ by triangle inequality). Thus $f(\zeta_1+\zeta_2,\zeta_2,\zeta_3)$ is expanded as an absolutely convergent Laurent series in T. From Theorem 2.3.3, the Laurent series is unique. Note that the lowest power of ζ_3 in this Laurent is $-p_3$.

Set

$$S=\{(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3:|\zeta_1+\zeta_2|>|\zeta_2|>|\zeta_3|>0,|\zeta_2|>|\zeta_1|>0\}\cap T$$

Obviously, S is a nonempty open subset of T. We know that the series

$$\sum_{k_1,k_2\in\mathbb{Z}}\langle w',(Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle\zeta_1^{-k_1-1}\zeta_2^{-k_2-1}\zeta_3^{-k_3-1}$$

is absolutely convergent whenever $(\zeta_1,\zeta_2,\zeta_3)\in S$, and the series

$$\sum_{k_3\in\mathbb{Z}}\left(\sum_{k_1,k_2\in\mathbb{Z}}\langle w',(Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle\zeta_1^{-k_1-1}\zeta_2^{-k_2-1}\right)\zeta_3^{-k_3-1},$$

viewed as a series whose terms are $\sum_{k\in\mathbb{Z}}\langle w',Y_W^L(\pi_k^V Y_V(u_1,\zeta_1)u_2,\zeta_2)\pi_l^W Y_W^L(u_3,\zeta_3)w\rangle$, is lower-truncated in ζ_3 and absolutely convergent to $f(\zeta_1+\zeta_2,\zeta_2,\zeta_3)$ whenever $(\zeta_1,\zeta_2,\zeta_3)\in S$. Thus Lemma 2.3.7 implies that the series

$$\sum_{k_3\in\mathbb{Z}}\sum_{k_1,k_2\in\mathbb{Z}}\langle w',(Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle\zeta_1^{-k_1-1}\zeta_2^{-k_2-1}\zeta_3^{-k_3-1}$$

converges absolutely when $(\zeta_1,\zeta_2,\zeta_3)\in T$. Finally, since the expansion of the rational function is given in the region

$$\{(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3:|\zeta_2|>|\zeta_1-\zeta_3|,|\zeta_2|>|\zeta_1|>0,|\zeta_2|>|\zeta_3|>0\},$$

the Laurent series also converges absolutely in this region. That is to say, in terms of variables z_1, z_2, z_3, the series

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}} \langle w', (Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle (z_1-z_2)^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1}$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_2| > |z_1 - z_2 - z_3|, |z_2| > |z_1 - z_2| > 0, |z_2| > |z_3| > 0$.

Now we prove that for any $u_1, u_2, u_3 \in V, w \in W, w' \in w'$, fixed $z_1, z_2, z_3 \in \mathbb{C}$ satisfying $|z_3| > |z_1 - z_3|, |z_3| > |z_2 - z_3| > |z_1 - z_2| > 0$, the series

$$\langle w', Y_W^L(Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2 - z_3)u_3, z_3)w\rangle$$

converges absolutely to the rational function that

$$\langle w', Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)Y_W^L(u_3, z_3)w\rangle$$

converges to.

The process is similar: in the series

$$\begin{aligned}&\langle w', Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2)Y_W^L(u_3, z_3)w\rangle \\ &\quad = \sum_{k_1,k_2,k_3\in\mathbb{Z}} \langle w', (Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle (z_1-z_2)^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1}\end{aligned}$$

we fix k_1 and consider the series

$$\begin{aligned}&\langle w', Y_W^L((Y_V)_{k_1}(u_1)u_2, z_2)Y_W^L(u_3, z_3)w\rangle \\ &\quad = \sum_{k_2,k_3\in\mathbb{C}} \langle w', (Y_W^L)_{k_2}((Y_V)_{k_1}(u_1)u_2)(Y_W^L)_{k_3}(u_3)w\rangle z_2^{-k_2-1} z_3^{-k_3-1}.\end{aligned}$$

As part of an absolutely convergent series, this series is also absolutely convergent. From associativity, its sum is equal to

$$\begin{aligned}&\langle w', Y_W^L(Y_V((Y_V)_{k_1}(u_1)u_2, z_2 - z_3)u_3, z_3)w\rangle \\ &\quad = \sum_{k_2,k_3\in\mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle (z_2 - z_3)^{-k_2-1} z_3^{-k_3-1}.\end{aligned}$$

when $|z_2| > |z_3| > |z_2 - z_3| > 0$. We multiply it with $(z_1 - z_2)^{-k_1-1}$ sum up all $k_1 \in \mathbb{Z}$. With the conclusion of the previous proposition, we see that

$$\sum_{k_1\in\mathbb{Z}} \langle w', Y_W^L(Y_V((Y_V)_{k_1}(u_1)u_2, z_2 - z_3)u_3, z_3)w\rangle (z_1 - z_2)^{-k_1-1}$$

$$= \sum_{k_1\in\mathbb{Z}} \left(\sum_{k_2,k_3\in\mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle (z_2-z_3)^{-k_2-1} z_3^{-k_3-1} \right) (z_1-z_2)^{-k_1-1}.$$

viewed as a single complex series whose terms are

$$\left(\sum_{k_2,k_3\in\mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle (z_2-z_3)^{-k_2-1} z_3^{-k_3-1} \right) (z_1-z_2)^{-k_1-1},$$

converges absolutely to $f(z_1,z_2,z_3)$ when $|z_2|>|z_2-z_3|, |z_2|>|z_3|+|z_1-z_2|, |z_1-z_2|>0, |z_3|>0$, for the same $f(z_1,z_2,z_3)$ as that in Formula (3.1). Moreover, one sees that the power of (z_1-z_2) in this series is lower-truncated.

We similarly use Lemma 2.3.7 to elaborately show that the series

$$\begin{aligned}&\langle w', Y_W^L(Y_V(Y_V(u_1,z_1-z_2)u_2,z_2-z_3)u_3,z_3)w\rangle \\ &\quad = \sum_{k_1,k_2,k_3\in\mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle (z_1-z_2)^{-k_1-1}(z_2-z_3)^{-k_2-1} z_3^{-k_3-1}.\end{aligned}$$

converges absolutely to $f(z_1,z_2,z_3)$ when $|z_3|>|z_1-z_3|, |z_2-z_3|>|z_1-z_2|>0$.

First we perform the transformation $\zeta_1=z_1-z_2, \zeta_2=z_2-z_3, \zeta_3=z_3$. Set

$$T=\{(\zeta_1,\zeta_2,\zeta_3): |\zeta_3|>|\zeta_1|+|\zeta_2|, |\zeta_2|>|\zeta_1|>0\}$$

With Lemma 2.3.2, we see that T is a connected multicircular domain. Moreover, T is a subset of $\{(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3: |\zeta_i|>|\zeta_1|, i=2,3\}$. We express the function $f(z_1,z_2,z_3)$ in terms of the variables ζ_1,ζ_2,ζ_3 as

$$f(\zeta_1+\zeta_2+\zeta_3,\zeta_2+\zeta_3,\zeta_3) = \frac{g(\zeta_1+\zeta_2+\zeta_3,\zeta_2+\zeta_3,\zeta_3)}{(\zeta_1+\zeta_2+\zeta_3)^{p_1}(\zeta_2+\zeta_3)^{p_2}\zeta_3^{p_3}\zeta_1^{p_{12}}\zeta_2^{p_{23}}(\zeta_1+\zeta_2)^{p^{13}}},$$

which admits a Laurent series expansion in the following steps:

1. Expand negative powers of $\zeta_1+\zeta_2+\zeta_3$ as power series in $\zeta_1+\zeta_2$, then further expand the positive powers of $\zeta_1+\zeta_2$ as polynomials in ζ_1 and ζ_2. This series converges absolutely when $|\zeta_3|>|\zeta_1+\zeta_2|$

2. Expand negative powers of $\zeta_2+\zeta_3$ as power series in ζ_2. This series converges absolutely when $|\zeta_3|>|\zeta_2|$

3. Expand negative powers of $\zeta_1+\zeta_2$ as power series in ζ_1. This series converges absolutely when $|\zeta_2|>|\zeta_1|$

Obviously if $(\zeta_1, \zeta_2, \zeta_3) \in T$, then all the above conditions are satisfied (Note that $|\zeta_3| > |\zeta_1| + |\zeta_2|$ implies that $|\zeta_3| > |\zeta_1 + \zeta_2|$ by triangle inequality). Thus $f(\zeta_1+\zeta_2, \zeta_2+\zeta_3, \zeta_3)$ is expressed as an absolutely convergent Laurent series. From Theorem 2.3.3, the Laurent series is unique.

Set

$$S = \{(\zeta_1, \zeta_2, \zeta_3) : |\zeta_2| > |\zeta_3| + |\zeta_1|, |\zeta_1| > 0, |\zeta_3| > 0\} \cap T.$$

So S is a nonempty open subset of T. We know that the series

$$\sum_{k_2, k_3 \in \mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle \zeta_2^{-k_2-1}\zeta_3^{-k_3-1}\zeta_1^{-k_1-1}.$$

converges absolutely when $(\zeta_1, \zeta_2, \zeta_3) \in S$, and the series

$$\sum_{k_1 \in \mathbb{Z}} \left(\sum_{k_2, k_3 \in \mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle \zeta_2^{-k_2-1}\zeta_3^{-k_3-1} \right) \zeta_1^{-k_1-1},$$

viewed as a series whose terms are

$$\sum_{k_2, k_3 \in \mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle \zeta_2^{-k_2-1}\zeta_3^{-k_3-1}\zeta_1^{-k_1-1},$$

converges absolutely to $f(\zeta_1 + \zeta_2, \zeta_2 + \zeta_3, \zeta_3)$ when $(\zeta_1, \zeta_2, \zeta_3) \in S$. Thus Lemma 2.3.7 implies that the triple series

$$\sum_{k_1 \in \mathbb{Z}} \sum_{k_2, k_3 \in \mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle \zeta_2^{-k_2-1}\zeta_3^{-k_3-1}\zeta_1^{-k_1-1},$$

converges absolutely when $(\zeta_1, \zeta_2, \zeta_3) \in T$. Finally, as the expansion is done in the region

$$\{(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 : |\zeta_3| > |\zeta_1 + \zeta_2| > 0, |\zeta_3| > |\zeta_2| > |\zeta_1| > 0\}$$

the series also converges absolutely in this region. That is to say, in terms of variables z_1, z_2, z_3, the series

$$\sum_{k_1, k_2, k_3 \in \mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_2}((Y_V)_{k_1}(u_1)u_2)u_3)w\rangle (z_2 - z_3)^{-k_2-1} z_3^{-k_3-1}(z_1 - z_2)^{-k_1-1}$$

converges absolutely to $f(z_1, z_2, z_3)$ when $|z_3| > |z_1 - z_3|, |z_3| > |z_2 - z_3| > |z_1 - z_2| > 0$. $\square$

Remark 3.1.14. One can also use the approach with the projection operator after having extended the vertex operators to $\widehat{W}$. The convergence of corresponding series still holds because they are part of absolutely convergent series.

Summary 3.1.15. *For any $u_1, ..., u_n \in V, w \in W$ and any $z_1, ..., z_n$ satisfying $|z_n| > |z_{n-1}-z_n|+|z_{n-2}-z_{n-1}|+\cdots+|z_1-z_2|, |z_{n-1}-z_n| > |z_{n-2}-z_{n-1}| > \cdots > |z_1-z_2| > 0$,*

$$\begin{aligned} &Y_W^L(Y_W^L(\cdots Y_W^L(Y_W^L(u_1, z_1-z_2)u_2, z_2-z_3)u_3\cdots, z_{n-1}-z_n)u_n, z_n)w \\ &\quad = \sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_W^L(\pi_{k_{n-1}}Y_V(\cdots Y_V(\pi_{k_1}Y_V(u_1, z_1-z_2)u_2, z_2-z_3)u_3\cdots, z_{n-1}-z_n)u_n, z_n)w \end{aligned}$$

For fixed $z_1, z_2, ..., z_n \in \mathbb{C}$ satisfying $|z_n| > |z_{n-1}-z_n| + |z_{n-2}-z_{n-1}| + \cdots + |z_1 - z_2|, |z_{n-1}-z_n| > |z_{n-2}-z_{n-1}| > \cdots > |z_1-z_2| > 0$, the iteration of any number of vertex operators gives rise to the following map

$$Y_W^L(Y_V(\cdots Y_V(Y_V(\cdot, z_1-z_2)\cdot, z_2-z_3)\cdots, z_{n-1}-z_n)\cdot, z_n)\cdot : V^{\otimes n}\otimes W \to \widehat{W}$$

If in addition, $|z_1| > |z_2| > \cdots > |z_n|$, then the map coincides with

$$Y_W^L(\cdot, z_1)Y_W^L(\cdot, z_2)\cdots Y_W^L(\cdot, z_n)\cdot : V^{\otimes n}\otimes W \to \widehat{W}$$

Remark 3.1.16. Because of the rationality of iterates, we know that for fixed $z_1, ..., z_n \in \mathbb{C}$ such that $|z_1| > \cdots > |z_n| > |z_{n-1}-z_n|+\cdots+|z_1-z_2|, |z_{n-1}-z_n| > \cdots > |z_1-z_2| > 0$, the vector subspace spanned by $\{Y_W^L(u_1, z_1)\cdots Y_W^L(u_n, z_n)w : u_1, ..., u_n \in V, w \in W\}$ in $\widehat{W}$ is isomorphic to that spanned by $\{Y_W^L(Y_V(\cdots(Y_V(u_1, z_1-z_2)u_2, z_2)\cdots, z_{n-1} - z_n)u_{n-1}, z_n)w : u_1, ..., u_n \in V, w \in W\}$. Taking account the change of parameters, the subspace is at most of $\dim W$ times the continuum. This in general is much smaller than the full $\widehat{W}$. So in general, the maps we mention above take values in a much smaller subspace of $\widehat{W}$ than $\widehat{W}$ itself.

Remark 3.1.17. Similarly we can prove the rationality of products and iterates of any number of vertex operators.

3.1.5 Pole-order condition and formal variable formulation

Similar to the discussion in Chapter 2, we have the pole-order condition for modules.

Definition 3.1.18. Let V be a MOSVA. Let $W = \coprod_{m\in\mathbb{C}} W_{[m]}, Y_W^L : V\otimes W \to W[[x,x^{-1}]]$ satisfy axioms for gradings, rationality of products and iterates of two vertex operators and associativity in Definition 3.1.1 . Y_W^L is said to satisfy the *pole-order condition*, if for every $w' \in W', u_1, u_2 \in V, w \in W$, the order of the pole $z_1 = 0$ of the rational function that$\langle w', Y_W^L(u_1,z_1)Y_W^L(u_2,z_2)w\rangle$ converges to is bounded above by an integer that depends only on u_1 and w.

Remark 3.1.19. With the same notations and assumptions in Definition 3.1.18, we see that for every $u_1, u_2 \in V, w \in W$, p_1 appearing in the weak associativity

$$(x_0+x_2)^{p_1}Y_W^L(u_1,x_0+x_2)Y_W^L(u_2,x_2)w = (x_0+x_2)^{p_1}Y_W^L(Y_V(u_1,x_0)u_2,x_2)w$$

can be chosen as an integer that depends only on u_1 and w. Conversely, if W and Y_W^L satisfy axioms for gradings, weak associativity with the choice of p_1 depending only on u_1 and w, then one can prove that Y_W^L satisfies the rationality of products and iterates for two vertex operators, associativity and the pole-order condition.

Remark 3.1.20. Note that this condition holds automatically when the commutativity holds. Therefore for vertex algebras, we don't need any extra condition to have a formal variable formulation.

Proposition 3.1.21. *Let V be a MOSVA. Let $W = \coprod_{n\in\mathbb{C}} W_{[n]}$, $Y_W^L : V\otimes W \to W[[x,x^{-1}]]$ satisfy the axioms for the grading, the D-derivative and D-commutator properties, rationality of products and iterates of two vertex operators, associativity, and the pole-order condition in Definition 3.1.18. Then rationality of products holds for any numbers of vertex operators. More precisely, for every $u_1,...,u_n \in V, w' \in W', w\in W$, the series*

$$\langle w', Y_W^L(u_1,z_1)\cdots Y_W^L(u_n,z_n)w\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1,...,n$ and $z_i = z_j$. Moreover, for each $i = 1,...,n$, the order of the pole $z_i = 0$ is bounded above by an integer that depends only on u_i and w; for each i,j with $1 \le i < j \le n$, the order of the pole $z_i = z_j$ is bounded above by an integer that depends only on u_i and u_j.

Proof. The proof is similar to that in Proposition 2.1.12. We shall not repeat it here. □

In regards of Remark 3.1.19, we have the following theorem:

Theorem 3.1.22. *Let V be a MOSVA, Let $W = \coprod_{n\in\mathbb{C}} V_{[n]}$, $Y_W^L : V \otimes W \to W[[x, x^{-1}]]$, $\mathbf{d}_W : W \to W$ of weight 0, and $D_W : W \to W$ of weight 1 satisfy axioms for the grading, D-derivative property, D-commutator formula, and the following weak associativity with pole-order condition: for every $u_1, u_2 \in V$, $w \in W$, there exists an integer p_1 that depends only on u_1 and w, such that*

$$(x_0 + x_2)^{p_1} Y_W^L(Y_V(u_1, x_0)u_2, x_2)w = (x_0 + x_2)^{p_1} Y_W^L(u_1, x_0 + x_2) Y_W^L(u_2, x_2)w$$

as formal series in $W[[x_0, x_0^{-1}, x_2, x_2^{-1}]]$, then $(W, Y_W^L, \mathbf{d}_W, D_W)$ forms a left V-module, with Y_W^L satisfying the pole-order condition.

Not only is the pole-order condition crucial for the formal variable approach, it also provides the following surprising result.

Proposition 3.1.23. *For every $u_1, ..., u_n \in V, w \in W$ and $z_1, ..., z_n \in \mathbb{C}$ satisfying $|z_1| > \cdots > |z_n| > 0$, the sum of the series*

$$Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n)w$$

takes value in $\overline{W}$.

Proof. Consider the easier case when $n = 2$. Thus, we want to argue that the projection of the series

$$Y_W^L(u_1, z_1) Y_W^L(u_2, z_2)w$$

onto any fixed homogeneous subspace of W is a finite sum. In other words, for every fixed $r \in \mathbb{Z}$, we want to have

$$\sum_{m\in\mathbb{Z}} (u_1)(Y_W^L)_{r-m}(u_2)_m w z_1^{-r+m-1} z_2^{-m-1}$$

to be a finite sum. Obviously, m is upper-truncated. So it suffices to prove that m is lower-truncated.

From the pole condition, we know that there exists p_1, p_2, p_{12}, such that

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) w = \sum_{i_1, i_2 = 0}^{\infty} b_{i_1 i_2} z_1^{i_1} z_2^{i_2} \in W[[z_1, z_2]]$$

Multiplying both sides with the series

$$z_1^{-p_1} z_2^{-p_2} \iota_{12}(z_1 - z_2)^{-p_{12}} = \sum_{j=0}^{\infty} \binom{-p_{12}}{j} z_1^{-p_1 - p_{12} - j} z_2^{j - p_2}$$

to see that

$$\begin{aligned} Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) w &= \sum_{i_1, i_2 = 0}^{\infty} b_{i_1 i_2} z_1^{i_1} z_2^{i_2} \sum_{k=0}^{\infty} \binom{-p_{12}}{k} z_1^{-p_1 - p_{12} - k} z_2^{-p_2 + k} \\ &= \sum_{i_1, i_2 = 0}^{\infty} \sum_{k=0}^{\infty} b_{i_1 i_2} \binom{-p_{12}}{k} z_1^{i_1 - p_1 - p_{12} - k} z_2^{i_2 - p_2 + k} \\ &= \sum_{i_1 = 0}^{\infty} \sum_{i_2' = -p_2}^{\infty} \sum_{k=0}^{\infty} b_{i_1, i_2' + p_2 - k} \binom{-p_{12}}{k} z_1^{i_1 - p_1 - p_{12} - k} z_2^{i_2'} \\ &= \sum_{i_1' \in \mathbb{Z}} \sum_{i_2' = -p_2}^{\infty} \sum_{k=0}^{\infty} b_{i_1' + p_1 + p_{12} + k, i_2' + p_2 - k} \binom{-p_{12}}{k} z_1^{i_1'} z_2^{i_2'} \end{aligned}$$

Thus we have the equality

$$(u_1)_{-i_1' - 1} (u_2)_{-i_2' - 1} w = \sum_{k=0}^{\infty} b_{i_1' + p_1 + p_{12} + k, i_2' + p_2 - k} \binom{-p_{12}}{k}$$

Or,

$$(u_1)_{m_1} (u_2)_{m_2} w = \sum_{k=0}^{\infty} b_{-m_1 - 1 + p_1 + p_{12} + k, -m_2 - 1 + p_2 - k} \binom{-p_{12}}{k}$$

Thus

$$\sum_{m \in \mathbb{Z}} (u_1)_{r-n} (u_2)_n w = \sum_{n \in \mathbb{Z}} \sum_{k=0}^{\infty} b_{-r + n - 1 + p_{12} + k, -n - 1 - k} \binom{-p_{12}}{k}$$

Note that the sum of the indices is $-k - 2 + p_{12}$. It is clear that there are at most finitely many (i_1, i_2) satisfies $i_1 + i_2 = -k - 2 + p_{12}$ and $b_{i_1 i_2} \neq 0$. Thus the sum is indeed finite. Thus the left-hand-side is also a finite sum.

In general, the pole-order condition yields integers p_{ij} depending only on u_i and u_j, such that

$$\prod_{i=1}^{n} z_i^{p_i} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n) w = \sum_{i_1, \ldots, i_n = 0}^{\infty} b_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n}.$$

is in $W[[z_1, ..., z_n]]$. Thus

$$
\begin{aligned}
&Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n)w \\
&\quad = \prod_{i=1}^{n} z_i^{-p_i} \prod_{1 \le i < j \le n} \iota_{ij}(z_i - z_j)^{-p_{ij}} \sum_{i_1,\ldots,i_n=0}^{\infty} b_{i_1 \ldots i_n} z_1^{i_1} \cdots z_n^{i_n} \\
&\quad = \prod_{1 \le i < j \le n} \sum_{k_{ij}=0}^{\infty} \binom{-p_{ij}}{k_{ij}} z_i^{-p_{ij}-k_{ij}} z_j^{k_{ij}} \sum_{i_1,\ldots,i_n=0}^{\infty} b_{i_1 \ldots i_n} z_1^{i_1-p_1} \cdots z_n^{i_n-p_n} \\
&\quad = \prod_{1 \le i < j \le n-1} \sum_{k_{ij}=0}^{\infty} \binom{-p_{ij}}{k_{ij}} \sum_{k_{1n},\ldots,k_{n-1n}=0}^{\infty} \prod_{1 \le i < n} \binom{-p_{in}}{k_{in}} z_i^{-p_{in}-k_{in}} z_n^{k_{in}} \sum_{i_1,\ldots,i_n=0}^{\infty} b_{i_1 \ldots i_n} z_1^{i_1-p_1} \cdots z_n^{i_n-p_n} \\
&\quad = \prod_{1 \le i < j \le n-1} \sum_{k_{ij}=0}^{\infty} \binom{-p_{ij}}{k_{ij}} \sum_{k_{1n},\ldots,k_{n-1n}=0}^{\infty} \prod_{1 \le i < n} \binom{-p_{in}}{k_{in}} z_1^{-p_{1n}-k_{1n}} \cdots z_{n-1}^{-p_{n-1n}-k_{n-1n}} z_n^{k_{1n}+\cdots+k_{n-1n}} \\
&\qquad\quad \sum_{i_1,\ldots,i_n=0}^{\infty} b_{i_1 \ldots i_n} z_1^{i_1-p_1} \cdots z_n^{i_n-p_n} \\
&\quad = \prod_{1 \le i < j \le n-1} \sum_{k_{ij}=0}^{\infty} \binom{-p_{ij}}{k_{ij}} \sum_{k_{1n},\ldots,k_{n-1n}=0}^{\infty} \prod_{1 \le i < n} \binom{-p_{in}}{k_{in}} \\
&\qquad\quad \sum_{i_1,\ldots,i_n=0}^{\infty} b_{i_1 \ldots i_n} z_1^{i_1-p_1-p_{1n}-k_{1n}} \cdots z_{n-1}^{i_{n-1}-p_{n-1}-p_{n-1,n}-k_{n-1,n}} z_n^{i_n-p_n+\sum_{i=1}^{n-1} k_{in}} \\
&\quad = \prod_{1 \le i < j \le n-1} \sum_{k_{ij}=0}^{\infty} \binom{-p_{ij}}{k_{ij}} \sum_{k_{1n},\ldots,k_{n-1n}=0}^{\infty} \prod_{1 \le i < n} \binom{-p_{in}}{k_{in}} \\
&\qquad\quad \sum_{i_1,\ldots,i_n=0}^{\infty} b_{i_1 \ldots i_n} z_1^{i_1-p_1-p_{1n}-k_{1n}} \cdots z_{n-1}^{i_{n-1}-p_{n-1}-p_{n-1,n}-k_{n-1,n}} z_n^{i_n-p_n+\sum_{i=1}^{n-1} k_{in}} \\
&\quad = \sum_{i_1,\ldots,i_n=0}^{\infty} \sum_{k_{ij} \ge 0, 1 \le i < j \le n} \prod_{1 \le i < j \le n} \binom{-p_{ij}}{k_{ij}} b_{i_1 \ldots i_n} z_1^{i_1-p_1-\sum_{j>1}(p_{1j}+k_{1j})} z_2^{i_2-p_2-\sum_{j>2}(p_{2j}+k_{2j})+k_{12}} \\
&\qquad\quad \cdots z_l^{i_l-p_l-\sum_{j>l}(p_{lj}+k_{lj})+\sum_{j<l} k_{jl}} \cdots z_{n-1}^{i_{n-1}-p_{n-1}-p_{n-1,n}-k_{n-1,n}+\sum_{j=1}^{n-2} k_{j,n-1}} z_n^{i_n-p_n+\sum_{i=1}^{n-1} k_{in}}
\end{aligned}
$$

If we set

$$i'_1 = i_1 - p_1 - \sum_{j>1}(p_{1j} + k_{1j})$$

$$i'_2 = i_2 - p_2 - \sum_{j>2}(p_{2j} + k_{2j}) + k_{12}$$

$\ldots\ldots$

$$i'_l = i_l - p_l - \sum_{j>l}(p_{lj} + k_{lj}) + \sum_{j<l} k_{jl}$$

$$\cdots\cdots$$

$$i'_{n-1} = i_{n-1} - p_{n-1} - p_{n-1,n} - k_{n-1,n} + \sum_{j<n-1} k_{j,n-1}$$

$$i'_n = i_n - p_n + \sum_{i=1}^{n-1} k_{in}$$

Then the series can be written as

$$\sum_{i'_1,\ldots,i'_n\in\mathbb{Z}} \sum_{\substack{k_{ij}\geq 0, 1\leq i<j\leq n\\ \text{plus other inequalities}}} \prod_{1\leq i<j\leq n} \binom{-p_{ij}}{k_{ij}}$$

$$b_{i'_1+p_1+\sum_{j>1}(p_{1j}+k_{1j}),\ldots,i'_l+p_l+\sum_{j>l}(p_{lj}+k_{lj})-\sum_{j<l}k_{jl},\ldots,i'_n+p_n-\sum_{j<n}k_{in}} z_1^{i'_1}\cdots z_n^{i'_n}$$

For fixed $i'_1, ..., i'_n$, since i_n is nonnegative, for each fixed i'_n we have $\sum_{i=1}^{n-1} k_{i_n} \leq i'_n + p_n$. In particular, all $k_{1n}, ..., k_{n-1,n}$ are bounded above. Since i_{n-1} is nonnegative, we have $\sum_{j<n-1} k_{j,n-1} \leq i'_{n-1} + p_{n-1} + p_{n-1,n} + k_{n-1,n}$ where $k_{n-1,n}$ is bounded above. Thus all $k_{j,n-1}$'s are bounded above. Repeating the argument to see that all k_{ij}'s are bounded above. Thus the summation involving the k_{ij}'s are all finite.

We thus have

$$(u_1)_{-i'_1-1}\cdots(u_n)_{-i'_n-1}w = \sum_{k_{ij}\text{ finite}} \prod_{1\leq i<j\leq n} \binom{-p_{ij}}{k_{ij}}$$

$$b_{i'_1+p_1+\sum_{j>1}(p_{1j}+k_{1j}),\ldots,i'_l+p_l+\sum_{j>l}(p_{lj}+k_{lj})-\sum_{j<l}k_{jl},\ldots,i'_n+p_n-\sum_{j<n}k_{in}}$$

In other words,

$$(u_1)_{m_1}\cdots(u_n)_{m_n}w = \sum_{k_{ij}\text{ finite}} \prod_{1\leq i<j\leq n} \binom{-p_{ij}}{k_{ij}}$$

$$b_{-m_1-1+p_1+\sum_{j>1}(p_{1j}+k_{1j}),\ldots,-m_l-1+p_l+\sum_{j>l}(p_{lj}+k_{lj})-\sum_{j<l}k_{jl},\ldots,-m_n-1+p_n-\sum_{j<n}k_{in}}$$

To show that for each fixed $r \in \mathbb{Z}$,

$$\sum_{m_1+\cdots+m_n=r} (u_1)_{m_1}\cdots(u_n)_{m_n} w z_1^{-m_1-1}\cdots z_n^{-m_n-1}$$

$$= \sum_{m_1+\cdots+m_n=r} \sum_{k_{ij}\text{ finite}} \prod_{1\leq i<j\leq n} \binom{-p_{ij}}{k_{ij}} z_1^{-m_1-1}\cdots z_n^{-m_n-1}$$

$$b_{-m_1-1+p_1+\sum_{j>1}(p_{1j}+k_{1j}),\ldots,-m_l-1+p_l+\sum_{j>l}(p_{lj}+k_{lj})-\sum_{j<l}k_{jl},\ldots,-m_n-1+p_n-\sum_{j<n}k_{in}}$$

is a finite sum, it suffices to notice that that the sum of the indices of b is indeed a fixed constant

$$-r-n+\sum_{i=1}^{n} p_i + \sum_{1\le i<j\le n} p_{ij}$$

It is clear that there are at most finitely many $(i_1, ..., i_n)$ such that the sum of all i_j's is equal to the constant, and $b_{i_1...i_n} \neq 0$. Thus the sum above must be finite. □

Remark 3.1.24. This result is observed by Huang. His argument uses $\overline{W}$-valued rational functions, which is more conceptual than what was shown above.

3.2 Right V-modules

We now introduce the notion of right V-modules. This notion and the elementary properties have been known to Huang.

3.2.1 Basic definitions

Definition 3.2.1. Let $(V, Y_V, \mathbf{1})$ be a meromorphic open-string vertex algebra. A *right V-module* is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ (graded by *weights*), equipped with a *vertex operator map*

$$\begin{aligned} Y_W^R : W\otimes V &\to W[[x,x^{-1}]] \\ w\otimes u &\mapsto Y_W^R(w,x)u, \end{aligned}$$

an operator $\mathbf{d}_W$ and an operator D_W of weight 1, satisfying the following axioms:

1. Axioms for the grading:

   (a) *Lower bound condition*: When $\mathrm{Re}(n)$ is sufficiently negative, $W_{[n]} = 0$.

   (b) $\mathbf{d}$-*grading condition*: for every $w \in W_{[n]}, \mathbf{d}_W w = nw$.

   (c) $\mathbf{d}$-*bracket property*: For $w \in W$,

   $$\mathbf{d}_W Y_W^R(w,x) - Y_W^R(w,x)\mathbf{d}_V = Y_W^R(\mathbf{d}_W w, x) + x\frac{d}{dx}Y_W^R(w,x).$$

2. The *Creation property*: For $w \in W$, $Y_W^R(w,x)\mathbf{1} \in W[[x]]$ and $\lim_{x\to 0} Y_W^R(w,x)\mathbf{1} = w$.

3. The *D-derivative property* and the *D-commutator formula*: For $u \in V$,

$$\begin{aligned} \frac{d}{dx} Y_W^R(w, x) &= Y_W^R(D_W w, x) \\ &= D_W Y_W^R(w, x) - Y_W^R(w, x) D_V. \end{aligned}$$

4. *Rationality*: For $u_1, \dots, u_n \in V, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W^R(w, z_1) Y_V(u_1, z_2) \cdots Y_V(u_{n-1}, z_n) u_n \rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \dots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \dots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2 \in V, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W^R(Y_W^R(w, z_1 - z_2) u_1, z_2) u_2 \rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2 \in V, w \in W$, $w' \in W'$,

$$\langle w', Y_W^R(w, z_1) Y_V(u_1, z_2) u_2 \rangle = \langle w', Y_W^R(Y_W^R(w, z_1 - z_2) u_1, z_2) u_2 \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A right V-module is said to be *grading-restricted* if $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$.

When there is no confusion, we also denote the right V-module just defined by $(W, Y_W^R, \mathbf{d}_W, D_W)$ or simply W.

Remark 3.2.2. The right module is defined with the following philosophy: all the properties of intertwining operators of type $\binom{W}{WV}$ that make sense hold. With such a formulation, all the issues on convergence can be analyzed similarly as the usual vertex operators.

3.2.2 Some immediate consequences

Proposition 3.2.3. *Let V be a MOSVA and W be a right V-module. then*

1. *For $u \in V$, $Y_W^R(\cdot, x)u$ can be regarded as a formal series in End$(W)[[x, x^{-1}]]$*

$$Y_W^R(\cdot, x)u = \sum_{n\in\mathbb{Z}} (Y_W^R)_n(u) x^{-n-1}$$

where $(Y_W^R)_n(u) : W \to W$ is a linear map for every $n \in \mathbb{Z}$. If u is homogeneous, then $(Y_W^R)_n(u)$ is a map of weight $\mathrm{wt}\ u - n - 1$.

2. *For fixed $u \in V$ and $w \in W$, $Y_W^R(w, x)u$ is lower truncated, i.e, there are at most finitely many negative powers of x.*

3. *For $w \in W$,*

$$Y_W^R(w, x)\mathbf{1} = e^{xD_W} w$$

4. *Formal Taylor theorem: for $w \in V$,*

$$Y_W^R(w, x + y) = Y_W^R(e^{yD_W} w, x) = e^{yD_W} Y_W^R(w, x) e^{-yD_V},$$

in End$(W)[[x, y, x^{-1}]]$.

Proof. The arguments for (1), (2) and (4) are similar to those for Proposition 2.1.4. To see (3), one first note that

$$D_W w = \lim_{x\to 0} Y_W^R(D_W w, x)\mathbf{1} = \lim_{x\to 0} \frac{d}{dx} Y_W^R(w, x)$$

(the first equality follows from the creation property, the second from D-derivative property), then apply the arguments in Proposition 2.1.4. □

3.2.3 $\widehat{W}$-valued map interpretation

Just like what we did for left V-modules, similar results work for right modules: for one vertex operator, we have

Summary 3.2.4. *For $u \in V, w \in W$ and any nonzero complex number z, the summation*

$$Y_W^R(w, z)u = \sum_{n\in\mathbb{Z}} Y_W^R(u)_n w z^{-n-1}$$

gives an element in $\overline{W}$. For a given nonzero $z \in \mathbb{C}$, the vertex operator map give rise to the following map

$$Y_W^R(\cdot, z)\cdot : W \otimes V \to \overline{W} \subset \widehat{W}$$

For the product of two vertex operators, we have

Summary 3.2.5. *For any $u_1, u_2 \in V$, $w \in W$ and any complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the single series*

$$\sum_{k\in\mathbb{Z}} Y_W^R(w, z_1)\pi_k Y_V(u_1, z_2)u_2$$

of elements in $\overline{W}$ converges absolutely, i.e., for any $w' \in W'$,

$$\sum_{k\in\mathbb{Z}} \langle w', Y_W^R(w, z_1)\pi_k Y_V(u_1, z_2)u_2\rangle$$

converges absolutely. Moreover, the sum of the series is equal to the sum of the double series

$$Y_W^R(w, z_1)Y_V(u_1, z_2)u_2$$

For fixed z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the product of two vertex operators gives rise to the following map

$$Y_W^R(\cdot, z_1)Y_W^R(\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

which is equal to the map

$$\sum_{k\in\mathbb{Z}} Y_W^R(\cdot, z_1)\pi_k Y_W^R(\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

For the product of three vertex operators, although we don't know if $Y_W^R(w, z_1)Y_V(u_1, z_1)u_2$ sits in $\overline{W}$, since $Y_V(u_1, z_2)Y_V(u_2, z_3)u_3$ is in $\overline{V}$, the expression

$$Y_W^R(w, z_1)\pi_k Y_V(u_1, z_1)Y_V(u_2, z_2)u_3$$

can be understood just as in Chapter 2. So no modifications is needed to give the following:

Summary 3.2.6. *For any $u_1, ..., u_n \in V$, $w \in W$ and any $z_1, ..., z_n \in \mathbb{C}$ satisfying $|z_1| > |z_2| > \cdots > |z_n| > 0$, the series*

$$\sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_W^R(w, z_1)\pi_{k_1} Y_V(u_1, z_2)\pi_2 \cdots Y_V(u_{n-1}, z_{n-2})\pi_{k_{n-1}} Y_V(u_{n-1}, z_n)u_n$$

of elements in $\widehat{W}$ *converges absolutely, The sum is equal to the* $\widehat{W}$ *element given by*

$$Y_W^R(w, z_1)Y_V(u_1, z_2)\cdots Y_V(u_{n-1}, z_n)u_n$$

For fixed $z_1, z_2, ..., z_n \in \mathbb{C}$ *satisfying* $|z_1| > \cdots > |z_n| > 0$*, the product of any number of vertex operators gives rise to a map*

$$Y_W^R(\cdot, z_1)Y_V(\cdot, z_2)\cdots Y_V(\cdot, z_n)\cdot : W \otimes V^{\otimes n} \to \widehat{W}$$

and is equal to the sum

$$\sum_{k_1,...,k_{n-1}\in\mathbb{Z}} Y_W^R(\cdot, z_1)\pi_{k_1}Y_V(\cdot, z_2)\pi_{k_2}\cdots Y_V(\cdot, z_{n-1})\pi_{k_{n-1}}Y_V(\cdot, z_n)\cdot : W \otimes V^{\otimes n} \to \widehat{W}$$

Remark 3.2.7. We put all π_{k_i}'s on for completeness. In practice it is absolutely fine to omit any number of them.

3.2.4 Rationality of iterates

For the iterate of two vertex operators, since $Y_W^R(w, z_1 - z_2)u_1 \in \overline{W}$, we still have the following interpretation:

Summary 3.2.8. *For any* $u_1, u_2 \in V, w \in W$ *and any complex numbers* z_1, z_2 *satisfying* $|z_2| > |z_1 - z_2| > 0$*, the single series*

$$\sum_{m\in\mathbb{C}} Y_W^R(\pi_m^W Y_W^R(w, z_1 - z_2)u_1, z_2)u_2$$

of elements in $\overline{W}$ *converges absolutely, i.e., the complex series*

$$\sum_{m\in\mathbb{C}} \langle w', Y_W^R(\pi_m^W Y_W^R(w, z_1 - z_2)u_1, z_2)u_2\rangle$$

converges absolutely. The sum is equal to the $\widehat{W}$ *element given by*

$$Y_W^R(Y_W^R(w, z_1 - z_2)u_1, z_2)u_2$$

For fixed z_1, z_2 *satisfying* $|z_2| > |z_1 - z_2| > 0$*, the iterate of two vertex operators gives rise to a map*

$$Y_W^R(Y_W^R(\cdot, z_1 - z_2)\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

which is equal to the sum

$$\sum_{m\in\mathbb{C}} Y_W^R(\pi_m^W Y_W^R(\cdot, z_1 - z_2)\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

Summary 3.2.9. *For fixed z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, the following maps*

$$Y_W^R(\cdot, z_1)Y_V(\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

$$\sum_{k\in\mathbb{Z}} Y_W^R(\cdot, z_1)\pi_k Y_V(\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

$$Y_W^R(Y_W^R(\cdot, z_1 - z_2)\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

$$\sum_{m\in\mathbb{C}} Y_W^R(\pi_m^W Y_W^R(\cdot, z_1 - z_2)\cdot, z_2)\cdot : W \otimes V \otimes V \to \widehat{W}$$

are equal.

For the iterate of more than three vertex operators, problems arise, as we don't know how to define $Y_W^R(\cdot, z_n)u_n$ on the $\widehat{W}$-element $Y_W^R(\cdots Y_W^R(Y_W^R(w, z_1 - z_2)u_1, z_2 - z_3)\cdots, z_{n-1} - z_n)u_{n-1}$. We have to prove the convergence of the iterate first.

Proposition 3.2.10. *For $u_1, u_2, ..., u_n \in V, w \in W, w' \in W'$, the series*

$$\langle w', Y_W^R(Y_W^R(\cdots Y_W^R(Y_W^R(w, z_1 - z_2)u_1, z_2 - z_3)u_2 \cdots, z_{n-1} - z_n)u_{n-1}, z_n)u_n\rangle$$

converges absolutely in the region

$$\left\{ (z_1, ..., z_n) \in \mathbb{C}^n : \begin{array}{l} |z_n| > |z_{n-1} - z_n| + |z_{n-2} - z_{n-1}| + \cdots + |z_1 - z_2|, \\ |z_i - z_{i+1}| > \sum_{j=1}^{i-1} |z_j - z_{j+1}|, i = 3, ..., n-1 \\ |z_2 - z_3| > |z_1 - z_2| > 0 \end{array} \right\}$$

to the rational function that

$$\langle w', Y_W^R(w, z_1)Y_V(u_1, z_2)\cdots Y_V(u_{n-1}, z_n)u_n\rangle$$

converges to.

Proof. The steps are similar to the proof of Proposition 2.3.13. We give only a sketch here: Assume the conclusion is true for the iterate of $n-1$ vertex operators:

1. Use Summary 3.2.6 to write

$$Y_W^R(w, z_1)Y_V(u_1, z_2)\cdots Y_V(u_{n-1}, z_n)u_n$$

$$= \sum_{k_1,\dots,k_{n-1}\in\mathbb{Z}} Y_W^R(w,z_1)\pi_{k_1}Y_V(u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n$$

and hence

$$\langle w', Y_W^R(w,z_1)Y_V(u_1,z_2)\cdots Y_V(u_{n-1},z_n)u_n\rangle$$
$$= \sum_{k_1,\dots,k_{n-1}\in\mathbb{Z}} \langle w', Y_W^R(w,z_1)\pi_{k_1}Y_V(u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n\rangle$$

when $|z_1|>|z_2|>\cdots>|z_n|>0$.

2. For each fixed k_2, we use the associativity to see that

$$\sum_{k_1,k_3,\dots,k_{n-1}\in\mathbb{Z}} \langle w', Y_W^R(w,z_1)\pi_{k_1}Y_V(u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n\rangle$$
$$= \sum_{r_1\in\mathbb{C}}\sum_{k_3,\dots,k_{n-1}\in\mathbb{Z}} \langle w', Y_W^R(\pi_{r_1}^W Y_W^R(w,z_1-z_2)u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n\rangle$$

when $|z_1|>|z_2|>\cdots>|z_n|>0, |z_2|>|z_1-z_2|>0$. In particular, the right hand side, as part of an absolutely convergent n-multiseries in $z_1-z_2, z_2, ..., z_n$, converges absolutely.

3. Summing up all k_2's to see that

$$\sum_{k_2\in\mathbb{Z}}\left(\sum_{r_1\in\mathbb{C}}\sum_{k_3,\dots,k_{n-1}\in\mathbb{Z}} \langle w', Y_W^R(\pi_{r_1}^W Y_W^R(w,z_1-z_2)u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n\rangle\right)$$

viewed as a single complex series whose terms are

$$\left(\sum_{r_1\in\mathbb{C}}\sum_{k_3,\dots,k_{n-1}\in\mathbb{Z}} \langle w', Y_W^R(\pi_{r_1}^W Y_W^R(w,z_1-z_2)u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n\rangle\right),$$

converges to the rational function that $\langle w', Y_W^R(w,z_1)Y_V(u_1,z_2)\cdots Y_V(u_{n-1},z_n)u_n\rangle$ converges to, when $|z_1|>|z_2|>\cdots>|z_n|>0, |z_2|>|z_1-z_2|>0$.

4. With the help of a parameter transformation, we apply Lemma 2.3.7 to see that the series

$$\sum_{k_2\in\mathbb{Z}}\sum_{r_1\in\mathbb{C}}\sum_{k_3,\dots,k_{n-1}\in\mathbb{Z}} \langle w', Y_W^R(\pi_{r_1}^W Y_W^R(w,z_1-z_2)u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n\rangle$$

converges absolutely to the rational function that $\langle w', Y_W^R(w,z_1)Y_V(u_1,z_2)\cdots Y_V(u_{n-1},z_n)u_n\rangle$ converges to, when $|z_2|>|z_1-z_2|+|z_3|, |z_1-z_2|>0, |z_3|>|z_4|>\cdots>|z_n|>0$. In other words, the series

$$\langle w', Y_W^R(Y_W^R(w,z_1-z_2)u_1,z_2)Y_V(u_2,z_3)\cdots Y_V(u_{n-1},z_n)u_n\rangle$$

converges absolutely in the region

$$\{(z_1,...,z_n)\in\mathbb{C}^n : |z_2|>|z_1-z_2|+|z_3|, |z_1-z_2|>0, |z_3|>|z_4|>\cdots>|z_n|>0\}$$

to the rational function that

$$\langle w', Y_W^R(w,z_1)Y_V(u_1,z_2)\cdots Y_V(u_{n-1},z_n)u_n\rangle$$

converges to.

5. For each fixed $r\in\mathbb{C}$, we use the the induction hypothesis to see that

$$\sum_{k_2,\dots,k_{n-1}\in\mathbb{Z}}\langle w', Y_W^R(\pi_{r_1}^W Y_W^R(w,z_1-z_2)u_1,z_2)\pi_{k_2}Y_V(u_2,z_3)\cdots\pi_{k_{n-1}}Y_V(u_{n-1},z_n)u_n\rangle$$
$$=\sum_{r_2,\dots,r_{n-1}\in\mathbb{C}}\langle w', Y_W^R(\pi_{r_2}Y_W^R(\cdots\pi_{r_{n-1}}Y_W^R(\pi_{r_1}Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2,\cdots)u_{n-1},z_n)u_n\rangle$$

when $|z_2|>|z_1-z_2|+|z_3|, |z_1-z_2|>0, |z_3|>|z_4|>\cdots>|z_n|>|z_2-z_3|+\cdots+|z_n-z_{n-1}|, |z_{n-1}-z_n|>\cdots>|z_2-z_3|>0$. In particular, the right hand side, as an $(n-1)$-multiseries in $z_2-z_3,...,z_{n-2}-z_{n-1},z_{n-1}-z_n,z_n$, converges absolutely.

6. Summing up all r_1's to see that

$$\sum_{r_1\in\mathbb{C}}\left(\sum_{r_2,\dots,r_{n-1}\in\mathbb{C}}\langle w', Y_W^R(\pi_{r_2}Y_W^R(\cdots\pi_{r_{n-1}}Y_W^R(\pi_{r_1}Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2,\cdots)u_{n-1},z_n)u_n\rangle\right),$$

viewed as a single complex series whose terms are

$$\left(\sum_{r_2,\dots,r_{n-1}\in\mathbb{C}}\langle w', Y_W^R(\pi_{r_2}Y_W^R(\cdots\pi_{r_{n-1}}Y_W^R(\pi_{r_1}Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2,\cdots)u_{n-1},z_n)u_n\rangle\right),$$

converges to the rational function that $\langle w', Y_W^R(w,z_1)Y_V(u_1,z_2)\cdots Y_V(u_{n-1},z_n)u_n\rangle$ converges to, when $|z_2|>|z_1-z_2|+|z_3|, |z_1-z_2|>0, |z_3|>|z_4|>\cdots>|z_n|>|z_2-z_3|+\cdots+|z_n-z_{n-1}|, |z_{n-1}-z_n|>\cdots>|z_2-z_3|>0$.

7. With the help of a parameter transformation, we apply Lemma 2.3.7 to see that the series

$$\sum_{r_1,r_2,\dots,r_{n-1}\in\mathbb{C}} \langle w', Y_W^R(\pi_{r_2}Y_W^R(\cdots\pi_{r_{n-1}}Y_W^R(\pi_{r_1}Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2,\cdots)u_{n-1},z_n)u_n\rangle,$$

converges absolutely to the rational function that $\langle w', Y_W^R(w,z_1)Y_V(u_1,z_2)\cdots Y_V(u_{n-1},z_n)u_n\rangle$ converges to, when $|z_n| > |z_{n-1}-z_n| + |z_{n-2}-z_{n-1}| + \cdots + |z_1-z_2|, |z_{n-1}-z_n| > |z_{n-2}-z_{n-1}| > \cdots > |z_1-z_2| > 0$.

□

Because of this proposition, we can now understand the action of $Y_W^R(\cdot,z_3)u_3$ on $\pi_m^W Y_W^R(Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2 \in W_{[m]}^{**}$ as the part of the triple series $Y_W^R(Y_W^R(Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2,z_3)u_3$. More precisely, for homogeneous $u_1,u_2\in V$ and $w\in W$, as

$$\begin{aligned}
&\pi_m^W Y_W^R(Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2\\
&\quad= \sum_{\substack{\mathrm{wt}\,u_1+\mathrm{wt}\,u_2+\mathrm{wt}\,w-n_2-n_3-2=m\\ n_2,n_3\in\mathbb{Z}}} (Y_W^R)_{n_2}(u_2)(Y_W^R)_{n_1}(u_1)w(z_1-z_2)^{-n_1-1}(z_2-z_3)^{-n_2-1}
\end{aligned}$$

we naturally have

$$\begin{aligned}
&Y_W^R(\pi_m^W Y_W^R(Y_W^R(w,z_1-z_2)u_1,z_2-z_3)u_2,z_3)u_3\\
=&Y_W^R\left(\sum_{\substack{\mathrm{wt}\,u_1+\mathrm{wt}\,u_2+\mathrm{wt}\,w-n_2-n_3-2=m\\ n_2,n_3\in\mathbb{Z}}} (Y_W^R)_{n_2}(u_2)(Y_W^R)_{n_1}(u_1)w(z_1-z_2)^{-n_1-1}(z_2-z_3)^{-n_2-1}, z_3\right)u_3\\
=&\sum_{n_3\in\mathbb{Z}}(Y_W^R)_{n_3}(u_3)\cdot\\
&\qquad\left(\sum_{\substack{\mathrm{wt}\,u_1+\mathrm{wt}\,u_2+\mathrm{wt}\,w-n_2-n_3-2=m\\ n_2,n_3\in\mathbb{Z}}} (Y_W^R)_{n_2}(u_2)(Y_W^R)_{n_1}(u_1)w(z_1-z_2)^{-n_1-1}(z_2-z_3)^{-n_2-1}\right)z_3^{-n_3-1}\\
=&\sum_{\substack{\mathrm{wt}\,u_1+\mathrm{wt}\,u_2+\mathrm{wt}\,w-n_2-n_3-2=m\\ n_1,n_2,n_3\in\mathbb{Z}}} \left((Y_W^R)_{n_1}(u_1)(Y_W^R)_{n_2}(u_2)(Y_W^R)_{n_1}(u_1)w(z_1-z_2)^{-n_1-1}(z_2-z_3)^{-n_2-1}\right)z_3^{-n_3-1}
\end{aligned}$$

If we treat the element in the parenthesis as an element of $W_{[m]}^{**}$, then the sum gives an element in $\widehat{W}$. So summing up all $m\in\mathbb{C}$ will yield a series in $\widehat{W}$. However, after pairing it with w', we see that the resulted complex series $\sum_{r\in\mathbb{C}}\langle w', Y_W^R(u_1,z_1)\pi_m^W Y_W^R(u_2,z_2)Y_W^R(u_3,z_3)w\rangle$

is just a rearrangement of the absolutely convergent triple series $\langle w', Y_W^R(u_1, z_1)Y_W^R(u_2, z_2)Y_W^R(u_3, z_3)w\rangle$. For nonhomogeneous $u_2, u_3 \in V$ and $w \in W$, we use the same argument as in Proposition 2.2.11 to write the corresponding series as a finite sum of absolutely convergent series.

Remark 3.2.11. Alternatively, we can also extend the operator $Y_W^R(\cdot, z)u$ using the double adjoint process in the same way as in Remark 3.1.9. The proof of Proposition 3.2.10 can also be rewritten without using the projection operators. The details are similar to those in Proposition 3.1.13.

Similarly, we have the following summary

Summary 3.2.12. *For any* $u_1, ..., u_n \in V, w \in W$ *and any* $z_1, ..., z_n$ *satisfying* $|z_n| > |z_{n-1}-z_n|+|z_{n-2}-z_{n-1}|+\cdots+|z_1-z_2|, |z_{n-1}-z_n| > |z_{n-2}-z_{n-1}| > \cdots > |z_1-z_2| > 0$,

$$\begin{aligned}
&Y_W^R(Y_W^R(\cdots Y_W^R(Y_W^R(w, z_1-z_2)u_1, z_2-z_3)u_2\cdots, z_{n-1}-z_n)u_{n-1}, z_n)u_n \\
&\quad = \sum_{m_1,...,m_{n-1}\in\mathbb{Z}} Y_W^R(\pi_{m_{n-1}}Y_W^R(\cdots Y_W^R(\pi_{m_1}Y_W^R(w, z_1-z_2)u_1, z_2-z_3)u_2\cdots, z_{n-1}-z_n)u_{n-1}, z_n)u_n
\end{aligned}$$

For fixed $z_1, z_2, ..., z_n \in \mathbb{C}$ *satisfying* $|z_n| > |z_{n-1}-z_n| + |z_{n-2}-z_{n-1}| + \cdots + |z_1 - z_2|, |z_{n-1}-z_n| > |z_{n-2}-z_{n-1}| > \cdots > |z_1-z_2| > 0$, *the iteration of any number of vertex operators gives rise to the following map*

$$Y_W^R(Y_W^R(\cdots Y_W^R(Y_W^R(\cdot, z_1-z_2)\cdot, z_2-z_3)\cdots, z_{n-1}-z_n)\cdot, z_n)\cdot : W\otimes V^{\otimes n} \to \widehat{W}$$

If in addition, $|z_1| > |z_2| > \cdots > |z_n|$, *then the map coincides with*

$$Y_W^R(\cdot, z_1)Y_V(\cdot, z_2)\cdots Y_V(\cdot, z_n)\cdot : W\otimes V^{\otimes n} \to \widehat{W}$$

3.2.5 Pole-order condition and formal variable formulation

We similarly have the following pole-order condition for right V-modules. All the proofs are similar to those proved for the left modules. We shall not repeat them here.

Definition 3.2.13. Let V be a MOSVA. Let $W = \coprod_{m\in\mathbb{C}} W_{[m]}, Y_W^R : W\otimes V \to W[[x, x^{-1}]]$ satisfy axioms for gradings, rationality of products and iterates of two vertex operators and associativity in Definition 3.1.1 . Y_W^L is said to satisfy the *pole-order condition*, if for every $w' \in W', u_1, u_2 \in V, w \in W$, the order of the pole $z_1 = 0$ of the rational function that$\langle w', Y_W^R(w, z_1)Y_W^R(u_1, z_2)u_2\rangle$ converges to is bounded above by an integer that depends only on w and u_2.

Remark 3.2.14. With the same notations and assumptions in Definition 3.2.13, we see that for every $u_1, u_2 \in V, w \in W$, p_1 appearing in the weak associativity

$$(x_0 + x_2)^{p_1} Y_W^R(w, x_0 + x_2)Y_W^L(u_1, x_2)u_2 = (x_0 + x_2)^{p_1} Y_W^R(Y_W^R(w, x_0)u_1, x_2)u_2$$

can be chosen as an integer that depends only on w and u_2. Conversely, if W and Y_W^L satisfy axioms for gradings, weak associativity with the choice of p_1 depending only on w and u_2, then one can prove that Y_W^R satisfies the rationality of products and iterates for two vertex operators, associativity and the pole-order condition.

Proposition 3.2.15. *Let V be a MOSVA. Let $W = \coprod_{n\in\mathbb{C}} W_{[n]}$, $Y_W^R : W \otimes V \to W[[x, x^{-1}]]$ satisfy the axioms for the grading, the D-derivative and D-commutator properties, rationality of products and iterates of two vertex operators, associativity, and the pole-order condition in Definition 3.2.13. Then rationality of products holds for any numbers of vertex operators. More precisely, for every $u_1, ..., u_n \in V, w' \in W', w \in W$, the series*

$$\langle w', Y_W^R(w, z_1)Y_V(u_1, z_2) \cdots Y_V(u_{n-1}, z_n)u_n\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j$. Moreover, the order of the pole $z_1 = 0$ is bounded above by an integer that depends only on w and u_n; for each $i = 2, ..., n$, the order of the pole $z_i = 0$ is bounded above by an integer that depends only on u_{i-1} and u_n; for each $i = 2, ..., n$, the order of the pole $z_1 = z_i$ is bounded above by an integer that depends only on w and u_{i-1}; for each i, j with $2 \leq i < j \leq n$, the order of the pole $z_i = z_j$ is bounded above by an integer that depends only on u_{i-1} and u_{j-1}.

In regards of Remark 3.1.19, we have the following theorem:

Theorem 3.2.16. *Let V be a MOSVA, Let $W = \coprod_{n\in\mathbb{C}} V_{[n]}$, $Y_W^R : W \otimes V \to W[[x, x^{-1}]]$, $\mathbf{d}_W : W \to W$ of weight 0, and $D_W : W \to W$ of weight 1 satisfy axioms for the grading, D-derivative property, D-commutator formula, and the following weak associativity with* *pole-order condition: for every $u_1, u_2 \in V$, $w \in W$, there exists an integer p_1 that depends only on w and u_2, such that*

$$(x_0 + x_2)^{p_1} Y_W^R(Y_W^R(w, x_0)u_1, x_2)u_2 = (x_0 + x_2)^{p_1} Y_W^R(w, x_0 + x_2) Y_V(u_1, x_2)u_2$$

as formal series in $W[[x_0, x_0^{-1}, x_2, x_2^{-1}]]$, then $(W, Y_W^R, \mathbf{d}_W, D_W)$ forms a right V-module, with Y_W^R satisfying the pole-order condition.

Proposition 3.2.17. *For every $u_1, ..., u_n \in V, w \in W$ and $z_1, ..., z_n \in \mathbb{C}$ satisfying $|z_1| > \cdots > |z_n| > 0$, the sum of the series*

$$Y_W^R(w, z_1) Y_V(u_1, z_2) \cdots Y_V(u_{n-1}, z_n)u_n$$

takes value in $\overline{W}$.

3.2.6 V-modules and V^{op}-modules

If V is a vertex algebra, then a left V-module automatically makes a right V-module. More generally, we have the following proposition:

Proposition 3.2.18. *Given a right V-module $(W, Y_W^R, \mathbf{d}_W, D_W)$, we define the vertex operator map*

$$\begin{aligned} Y_W^{s(R)} :& V \otimes W \to W \\ & v \otimes w \mapsto e^{xD_W} Y_W^R(w, -x)v \end{aligned}$$

Then $(W, Y_W^{s(R)}, \mathbf{d}_W, D_W)$ is a left V^{op}-module.
Conversely, given a left V^{op}-module $(W, Y_W^{s(R)}, \mathbf{d}_W, D_W)$, we define the vertex operator map

$$\begin{aligned} Y_W^R :& W \otimes V \to W \\ & w \otimes v \mapsto e^{xD_W} Y_W^{s(R)}(v, -x)w \end{aligned}$$

then $(W, Y_W^R, \mathbf{d}_W, D_W)$ is a right V-module.

Proof. Let $(W, Y_W^R, \mathbf{d}_W, D_W)$ be a right V-module. We verify all the axioms of the left V^{op}-module.

1. The grading of W obviously satisfy the lower bound condition and the $\mathbf{d}$-grading condition. The proof of the $\mathbf{d}$-commutator formula is similar to that in the proof of Proposition 2.4.6.

2. The identity property follows from Proposition 3.2.3

$$Y_W^{s(R)}(\mathbf{1},x)w = e^{xD_W}Y_W^R(w,-x)\mathbf{1} = e^{xD_W}e^{-xD_W}w = w$$

3. We first prove the D-derivative property

$$\begin{aligned}\frac{d}{dx}Y_W^{s(R)}(v,x)w &= \frac{d}{dx}\left(e^{xD_W}Y_W^R(w,-x)v\right) = D_We^{xD_W}Y_W^R(w,-x)v + e^{xD_W}\frac{d}{dx}\left(Y_W^R(w,-x)v\right)\\ &= e^{xD_W}D_WY_W^R(w,-x)v + e^{xD_W}\frac{d}{dx}\left(Y_W^R(w,-x)v\right)\\ &= e^{xD_W}[D_W,Y_W^R(w,-x)]v + e^{xD_W}Y_W^R(w,-x)D_Vv + e^{xD_W}\frac{d}{dx}\left(Y_W^R(w,-x)v\right)\\ &= e^{xD_W}\frac{d}{d(-x)}Y_W^R(w,-x)v + e^{xD_W}Y_W^R(w,-x)D_Vv + e^{xD_W}\frac{d}{dx}\left(Y_W^R(w,-x)v\right)\\ &= e^{xD_W}Y_W^R(w,-x)D_Vv = Y_W^{s(R)}(D_Vv,x)w\end{aligned}$$

The D-commutator formula follows

$$\begin{aligned}[D_W,Y_W^{s(R)}(v,x)]w &= D_We^{xD_W}Y_W^R(w,-x)v - e^{xD_W}Y_W^R(D_Ww,-x)v\\ &= e^{xD_W}D_WY_W^R(w,-x)v + e^{xD_W}\frac{d}{dx}\left(Y_W^R(w,-x)v\right)\\ &= \frac{d}{dx}\left(e^{xD_W}Y_W^R(w,-x)v\right) = \frac{d}{dx}Y_W^{s(R)}(v,x)w\end{aligned}$$

4. It suffices to replace Y_V by Y_W^R and Y_V^s by $Y_W^{s(R)}$ in the arguments of Proposition 2.4.4 and Proposition 2.4.5.

5. It suffices to replace Y_V by Y_W^R and Y_V^s by $Y_W^{s(R)}$ in the arguments of Part (5) of Proposition 2.4.6.

The converse can be proved similarly. We omit the details here. □

Similarly, one can prove the following theorem:

Proposition 3.2.19. *Given a left V-module $(W, Y_W^L, \mathbf{d}_W, D_W)$, we define the vertex operator map*

$$\begin{aligned}Y_W^{s(L)}:&W\otimes V\to W\\ &w\otimes v\mapsto e^{xD_W}Y_W^L(v,-x)w\end{aligned}$$

Then $(W, Y_W^{s(L)}, \mathbf{d}_W, D_W)$ *is a left* V^{op}*-module.*

Conversely, given a right V^{op}*-module* $(W, Y_W^{s(L)}, \mathbf{d}_W, D_W)$*, we define the vertex operator map*

$$\begin{aligned} Y_W^L : & V \otimes W \to W \\ & v \otimes w \mapsto e^{xD_W} Y_W^{s(L)}(w, -x)v \end{aligned}$$

then $(W, Y_W^L, \mathbf{d}_W, D_W)$ *is a left* V*-module.*

3.3 V-bimodules

In this section we define V-bimodules. Many results in the previous sections can be generalized to V-bimodules. We will list these results without giving any explicit arguments. Then we will discuss some convergence results that will be used in the later Chapters.

3.3.1 The definition and the summaries

Definition 3.3.1. Let $(V, Y_V, \mathbf{1})$ be a meromorphic open-string vertex algebra. A V-*bimodule* is a vector space equipped with a left V-module structure and right V-module structure such that these two strutcure are compatible. More precisely, a V*-bimodule* is a $\mathbb{C}$-graded vector space

$$W = \coprod_{n \in \mathbb{C}} W_{[n]}$$

equipped with a *left vertex operator map*

$$\begin{aligned} Y_W^L : V \otimes W &\to W[[x, x^{-1}]] \\ u \otimes w &\mapsto Y_W^L(u, x)v, \end{aligned}$$

a *right vertex operator map*

$$\begin{aligned} Y_W^R : W \otimes V &\to W[[x, x^{-1}]] \\ w \otimes u &\mapsto Y_W^R(w, x)u, \end{aligned}$$

and linear operators d_W, D_W on W satisfying the following conditions.

1. $(W, Y_W^L, \mathbf{d}_W, D_W)$ is a left V-module.

2. $(W, Y_W^R, \mathbf{d}_W, D_W)$ is a right V-module.

3. *Compatibility*:

   (a) *Rationality of left and right vertex operator maps*: For $u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V$, $w \in W$, the series

   $$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n) Y_W^R(w, z_{n+1}) Y_V(u_{n+1}, z_{n+2}) \cdots Y_V(u_{n+m-1}, z_{n+m}) u_{n+m} \rangle$$

   converges absolutely in the region $|z_1| > |z_2| > \cdots > |z_n| > |z_{n+1}| > \cdots > |z_{n+m}| > 0$ to a rational function in $z_1, ..., z_n, z_{n+1}, ..., z_{n+m}$.

   (b) *Associativity for left and right vertex operator maps*: For $u, v \in V$, $w \in W$ and $w' \in W'$, the series

   $$\langle w', Y_W^L(u, z_1) Y_W^R(w, z_2) v \rangle$$

   $$\langle w', Y_W^R(Y_W^L(u, z_1 - z_2) w, z_2) v \rangle$$

   converges absolutely in the region $|z_1| > |z_2| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

The V-bimodule just defined is denoted by $(W, Y_W^L, Y_W^R, \mathbf{d}_W, D_W)$ or simply by W.

Remark 3.3.2. It is possible to generalize the definition to allow the left and right module structure on W to wield different $\mathbf{d}$ and D operators. Since we don't have any essential examples and certain subtlety also arise when modules are not grading-restricted, we choose not to discuss it here.

Remark 3.3.3. If V is a vertex algebra and W is a V-module (a vertex algebra module), then W can be regarded as a bimodule of the MOSVA V. Just as a module of a commutative associative algebra A can be viewed as a A-bimodule when A is viewed as an associative algebra. However, not all V-bimodules come in that way. In general, on the same space W one may have two different V-module action that are compatible, so as to make W a V-bimodule.

3.3.2 $\widehat{W}$-valued map interpretation and rationality of iterates

Likewise, we have the following summaries

Summary 3.3.4. *For any $u_1, u_2 \in V$, $w \in W$ and any complex numbers z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the single series*

$$\sum_{r\in\mathbb{C}} Y_W^L(u_1, z_1)\pi_r^W Y_W^R(w, z_2)u_2$$

of elements in $\overline{W}$ converges absolutely, i.e., for any $w' \in W'$,

$$\sum_{r\in\mathbb{C}} \langle w', Y_W^L(u_1, z_1)\pi_r^W Y_W^R(w, z_2)u_2\rangle$$

converges absolutely. Moreover, the sum of the series is equal to the sum of the double series

$$Y_W^L(u_1, z_1)Y_W^R(w, z_2)u_2$$

For fixed z_1, z_2 satisfying $|z_1| > |z_2| > 0$, the product of two vertex operators gives rise to the following map

$$Y_W^L(\cdot, z_1)Y_W^R(\cdot, z_2)\cdot : V \otimes W \otimes V \to \widehat{W}$$

which is equal to the map

$$\sum_{r\in\mathbb{C}} Y_W^L(\cdot, z_1)\pi_r^W Y_W^R(\cdot, z_2)\cdot : V \otimes W \otimes V \to \widehat{W}$$

Summary 3.3.5. *For any $u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V$, $w \in W$ and any $z_1, ..., z_{n+m} \in \mathbb{C}$ satisfying $|z_1| > |z_2| > \cdots > |z_{n+m}| > 0$, the series*

$$\sum_{r_1,...,r_n\in\mathbb{C}} \sum_{k_{n+1},...k_{n+m-1}\in\mathbb{Z}} Y_W^L(u_1, z_1)\pi_{r_1}^W Y_W^L(u_2, z_2)\cdots \pi_{r_{n-1}}^W Y_W^L(u_n, z_n)\pi_{r_n}^W Y_W^R(w, z_{n+1})$$

$$\pi_{k_{n+1}} Y_V(u_{n+1}, z_{n+2})\cdots \pi_{k_{n+m-1}} Y_V(u_{n+m-1}, z_{n+m})u_{n+m}$$

of elements in $\widehat{W}$ converges absolutely, The sum is equal to the $\widehat{W}$ element given by

$$Y_W^L(u_1, z_1)\cdots Y_W^L(u_n, z_n)Y_W^R(w, z_{n+1})Y_V(u_{n+1}, z_{n+2})\cdots Y_V(u_{n+m-1}, z_{n+m})u_{n+m}$$

For fixed $z_1, z_2, ..., z_n, z_{n+1}, ..., z_{n+m} \in \mathbb{C}$ satisfying $|z_1| > \cdots > |z_{n+m}| > 0$, the product of any number of vertex operators gives rise to a map

$$Y_W^L(\cdot, z_1)\cdots Y_W^L(\cdot, z_n)Y_W^R(\cdot, z_{n+1})Y_V(\cdot, z_{n+2})\cdots Y_V(\cdot, z_{n+m})\cdot : V^{\otimes n} \otimes W \otimes V^{\otimes m} \to \widehat{W}$$

and is equal to the sum

$$\sum_{r_1,\dots,r_n\in\mathbb{C}}\sum_{k_{n+1},\dots,k_{n+m-1}\in\mathbb{Z}} Y_W^L(\cdot,z_1)\pi_{r_1}^W\cdots Y_W^L(\cdot,z_n)\pi_{r_n}^W Y_W^R(\cdot,z_{n+1})$$

$$\pi_{k_{n+1}}Y_V(\cdot,z_2)\pi_{k_{n+2}}\cdots Y_V(\cdot,z_{n-1})\pi_{k_{n+m-1}}Y_V(\cdot,z_n)\cdot : W\otimes V^{\otimes n}\to \widehat{W}$$

Summary 3.3.6. *For fixed z_1, z_2 satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, the following maps*

$$Y_W^L(\cdot,z_1)Y_W^R(\cdot,z_2)\cdot : V\otimes W\otimes V\to\widehat{W}$$

$$\sum_{r\in\mathbb{C}} Y_W^L(\cdot,z_1)\pi_r^W Y_W^R(\cdot,z_2)\cdot : V\otimes W\otimes V\to\widehat{W}$$

$$Y_W^R(Y_W^L(\cdot,z_1-z_2)\cdot,z_2)\cdot : V\otimes W\otimes V\to\widehat{W}$$

$$\sum_{r\in\mathbb{C}} Y_W^R(\pi_r^W Y_W^L(\cdot,z_1-z_2)\cdot,z_2)\cdot : V\otimes W\otimes V\to\widehat{W}$$

are equal.

Also, the rationality of iterates holds:

Proposition 3.3.7. *For $u_1,\dots,u_n,u_{n+1},\dots u_{n+m}\in V, w\in W, w'\in W'$, the series*

$$\langle w', Y_W^R(\cdots Y_W^R(Y_W^L(Y_V(\cdots(Y_V(u_1,z_1-z_2)\cdots u_n,z_n-z_{n+1})w,z_{n+1}-z_{n+2})\cdots u_{n+m-1},z_{n+m})u_{n+m}\rangle$$

converges absolutely in the region

$$\left\{(z_1,\dots,z_n)\in\mathbb{C}^n : \begin{array}{c} |z_n| > |z_{n-1}-z_n| + |z_{n-2}-z_{n-1}| + \cdots + |z_1 - z_2|, \\ |z_i - z_{i+1}| > \displaystyle\sum_{j=1}^{i-1}|z_j - z_{j+1}|, i = 3,\dots,n-1 \\ |z_2 - z_3| > |z_1 - z_2| > 0 \end{array}\right\}$$

to the same rational function that

$$\langle w', Y_W^L(u_1,z_1)\cdots Y_W^L(u_n,z_n)Y_W^R(w,z_{n+1})Y_V(u_{n+1},z_{n+2})\cdots Y_V(u_{n+m-1},z_{n+m})u_{n+m}\rangle$$

converges to.

Summary 3.3.8. *For any $u_1,\dots,u_{n+m}\in V, w\in W$ and any $z_1,\dots,z_{n+m}$ satisfying $|z_{n+m}| > |z_{n+m-1} - z_{n+m}| + |z_{n+m-2} - z_{n+m-1}| + \cdots + |z_1 - z_2|, |z_{n+m-1} - z_{n+m}| >$*

$|z_{n+m-2} - z_{n+m-1}| > \cdots > |z_1 - z_2| > 0,$

$$Y_W^R(\cdots Y_W^R(Y_W^L(Y_V(\cdots(Y_V(u_1, z_1 - z_2)\cdots u_n, z_n - z_{n+1})w, z_{n+1} - z_{n+2})\cdots u_{n+m-1}, z_{n+m})u_{n+m}$$
$$= \sum_{r_1,...,r_n\in\mathbb{C}} \sum_{k_{n+1},...k_{n+m-1}\in\mathbb{Z}} Y_W^L(u_1, z_1)\pi_{r_1}^W Y_W^L(u_2, z_2)\cdots \pi_{r_{n-1}}^W Y_W^L(u_n, z_n)\pi_{r_n}^W Y_W^R(w, z_{n+1})$$
$$\pi_{k_{n+1}} Y_V(u_{n+1}, z_{n+2})\cdots \pi_{k_{n+m-1}} Y_V(u_{n+m-1}, z_{n+m})u_{n+m}$$

For fixed $z_1, z_2, ..., z_n \in \mathbb{C}$ satisfying $|z_{n+m}| > |z_{n+m-1} - z_{n+m}| + |z_{n+m-2} - z_{n+m-1}| + \cdots + |z_1 - z_2|, |z_{n+m-1} - z_{n+m}| > |z_{n+m-2} - z_{n+m-1}| > \cdots > |z_1 - z_2| > 0$, the iteration of any number of vertex operators gives rise to the following map

$$Y_W^R(\cdots Y_W^R(Y_W^L(Y_V(\cdots(Y_V(\cdot, z_1 - z_2)\cdots)\cdot, z_n - z_{n+1})\cdot, z_{n+1} - z_{n+2})\cdots)\cdot, z_{n+m})\cdot : V^{\otimes n}\otimes W\otimes V^{\otimes m} \to \widehat{W}$$

If in addition, $|z_1| > |z_2| > \cdots > |z_{n+m}|$, then the map coincides with

$$Y_W^L(\cdot, z_1)\cdots Y_W^L(\cdot, z_n)Y_W^R(\cdot, z_{n+1})Y_V(\cdot, z_{n+2})\cdots Y_V(\cdot, z_n)\cdot : V^{\otimes n} \otimes W \otimes V^{\otimes m} \to \widehat{W}$$

3.3.3 The pole-order condition and formal variable formulation

For V-bimodules, we can define the following pole-order condition:

Definition 3.3.9. Let V be a MOSVA with Y_V satisfies the pole-order condition in Definition 2.1.11. Let $W = \coprod_{n\in\mathbb{C}} W_{[n]}$, $Y_W^L : V \otimes W \to W[[x, x^{-1}]], Y_W^R : W \otimes V \to W[[x, x^{-1}]], \mathbf{d}_W : W \to W$ satisfy the following

1. The axioms for grading in Definition 3.1.1 hold for $(W, Y_W^L, \mathbf{d}_W)$. The axioms for grading in Definition 3.2.1 hold for $(W, Y_W^R, \mathbf{d}_W)$.

2. The rationality of products and iterates of two vertex operators, and the associativity in Definition 3.1.1 hold for Y_W^L. The rationality of products and iterates of two vertex operators, and the associativity in Definition 3.2.1 hold for Y_W^R.

3. The compatibility condition in Definition 3.3.1 hold for two vertex operators.

We say the pair (Y_W^L, Y_W^R) satisfies the *pole-order condition* if

1. Y_W^L satisfies the pole-order condition in Definition 3.1.18. Y_W^R satisfies the pole-order condition in Definition 3.2.13.

2. For every $u_1, u_2 \in V$, there exists $C > 0$, such that for every $w' \in W', w \in W$, the pole $z_1 = 0$ of the rational functions determined by

$$\langle w', Y_W^L(u_1, z_1) Y_W^R(w, z_2) u_2 \rangle$$

has order less than C.

Proposition 3.3.10. *Let V be a MOSVA. Let $W = \coprod_{n\in\mathbb{C}} W_{[n]}$, $Y_W^L : V \otimes W \to W[[x, x^{-1}]], Y_W^R : W \otimes V \to W[[x, x^{-1}]], \mathbf{d}_W : W \to W, D_W : W \to W$ satisfy the following*

1. *The axioms for grading in Definition 3.1.1 hold for $(W, Y_W^L, \mathbf{d}_W)$. The axioms for grading in Definition 3.2.1 hold for $(W, Y_W^R, \mathbf{d}_W)$.*

2. *The rationality of products and iterates of two vertex operators in Definition 3.1.1 hold for Y_W^L. The rationality of products and iterates of two vertex operators in Definition 3.2.1 hold for Y_W^R.*

3. *The D-derivative and D-commutator properties in Definition 3.1.1 hold for Y_W^L. The D-derivative and D-commutator properties in Definition 3.2.1 hold for Y_W^R.*

4. *The compatibility condition in Definition 3.3.1 holds for two vertex operators.*

Then the compatibility condition holds for any numbers of vertex operators. More precisely, for every $u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V, w' \in W', w \in W$, the series

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n) Y_W^R(w, z_{n+1}) Y_V(u_{n+1}, z_{n+2}) \cdots Y_V(u_{n+m-1}, z_{n+m}) u_{n+m} \rangle$$

converges absolutely when $|z_1| > \cdots > |z_{n+m}| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n+m$ and $z_i = z_j, 1 \le i < j \le n+m$. Moreover,

- *For each $i = 1, ..., n$ the order of the pole $z_i = 0$ is bounded above by an integer that depends only on u_i and u_{n+m}.*

- *The order of the pole $z_{n+1} = 0$ is bounded above by an integer that depends only on w and u_{n+m}.*

- *For each $i = n+2, ..., n+m$, the order of the pole $z_i = 0$ is bounded above by an integer that depends only on u_{i-1} and u_{n+m}.*
- *For each i, j with $1 \leq i < j \leq n$, the order of the pole $z_i = z_j$ is bounded above by an integer that depends only on u_i and u_j*
- *For each $i = 1, ..., n$, the order of the pole $z_i = z_{n+1}$ is bounded above by an integer that depends only on u_i and w.*
- *For each $i = n+2, ..., n+m$, the order of the pole $z_{n+1} = z_i$ is bounded above by an integer that depends only on w and u_{i-1}.*
- *For each i, j with $n+2 \leq i < j \leq n+m$, the order of the pole $z_i = z_j$ is bounded above by an integer that depends only on u_{i-1} and u_{j-1}.*

Theorem 3.3.11. *Let V be a MOSVA, Let $W = \coprod_{n\in\mathbb{C}} V_{[n]}$, $Y_W^L : V \otimes W \to W[[x, x^{-1}]], Y_W^R : W \otimes V \to W[[x, x^{-1}]], \mathbf{d}_W : W \to W$ of weight 0, and $D_W : W \to W$ of weight 1 satisfy axioms for the grading, D-derivative property, D-commutator formula, and the following weak associativities with pole-order condition:*

1. *For every $u_1, u_2 \in V$, $w \in W$, there exists an integer p_1 that depends only on w and u_2, such that*
$$(x_0 + x_2)^{p_1} Y_W^L(Y_V(u_1, x_0)u_2, x_2)w = (x_0 + x_2)^{p_1} Y_W^L(u_1, x_0 + x_2)Y_V(u_2, x_2)w$$
as formal series in $W[[x_0, x_0^{-1}, x_2, x_2^{-1}]]$,

2. *For every $u_1, u_2 \in V$, $w \in W$, there exists an integer p_1 that depends only on w and u_2, such that*
$$(x_0 + x_2)^{p_1} Y_W^R(Y_W^R(w, x_0)u_1, x_2)u_2 = (x_0 + x_2)^{p_1} Y_W^R(w, x_0 + x_2)Y_V(u_1, x_2)u_2$$
as formal series in $W[[x_0, x_0^{-1}, x_2, x_2^{-1}]]$,

3. *For every $u_1, u_2 \in V$, $w \in W$, there exists an integer p_1 that depends only on u_1 and u_2, such that*
$$(x_0 + x_2)^{p_1} Y_W^R(Y_W^L(u_1, x_0)w, x_2)u_2 = (x_0 + x_2)^{p_1} Y_W^L(u_1, x_0 + x_2)Y_W^R(w, x_2)u_2$$
as formal series in $W[[x_0, x_0^{-1}, x_2, x_2^{-1}]]$,

Then $(W, Y_W^L, Y_W^R, \mathbf{d}_W, D_W)$ *forms* V*-bimodule, with* Y_W^L, Y_W^R *and the pair* (Y_W^L, Y_W^R) *satisfying the corresponding pole-order conditions.*

Proposition 3.3.12. *For every* $u_1, ..., u_n, u_{n+1}, ..., u_{n+m} \in V, w \in W$ *and* $z_1, ..., z_n \in \mathbb{C}$ *satisfying* $|z_1| > \cdots > |z_n| > 0$*, the sum of the series*

$$Y_W^L(u_1, z_1) \cdots Y_W^L(u_n, z_n) Y_W^R(w, z_{n+1}) \cdots Y_W^R(u_{n+m-1}, z_{n+m}) u_{n+m}$$

takes value in $\overline{W}$*.*

3.3.4 In terms of the opposite MOSVAs

Recall that in the previous sections, we proved that for a MOSVA $(V, Y_V, \mathbf{1})$, the space V with the following vertex operator

$$Y_V^s : V \otimes V \to V[[x, x^{-1}]]$$
$$Y_V^s(u, x)v = e^{xD_V} Y_V(v, -x)u$$

and the vacuum $\mathbf{1} \in V$ also forms a MOSVA, called the opposite MOSVA of V and denoted V^{op}. We also proved that a right V-module $(W, Y_W^R, \mathbf{d}_W, D_W)$ is equivalent to a left V^{op}-module $(W, Y_W^{s(R)}, \mathbf{d}_W, D_W)$, where $Y_W^{s(R)}$ is defined by

$$Y_W^{s(R)}(v, x)w = e^{xD_W} Y_W^R(w, -x)v.$$

In Chapter 5, we will use the $Y_W^{s(R)}$ operator extensively. For convenience, we list some properties here.

Proposition 3.3.13. *Let* V *be a MOSVA and* W *be a right* V*-module. Then*

1. *For* $u \in V$*,* $Y_W^{s(R)}(u, x)$ *can be regarded as a formal series in* End$(W)[[x, x^{-1}]]$

   $$Y_W^{s(R)}(u, x) = \sum_{n \in \mathbb{Z}} (Y_W^{s(R)})_n(u) x^{-n-1}$$

   where $(Y_W^{s(R)})_n(u) : V \to V$ *is a linear map for every* $n \in \mathbb{Z}$*. If* u *is homogeneous, then* $(Y_W^{s(R)})_n(u)$ *is a map of weight* wt $u - n - 1$*.*

2. *For fixed* $u, v \in V$*,* $Y_V(u, x)v$ *is lower truncated, i.e, there are at most finitely many negative powers of* x*.*

3. *D-conjugation property: for $u \in V$,*

$$Y_W^{s(R)}(u, x+y) = Y_W^{s(R)}(e^{yD_V}u, x) = e^{yD_W}Y_W^{s(R)}(u, x)e^{-yD_W},$$

in $End(V)[[x, x^{-1}, y]]$.

4. $\mathbf{d}$*-conjugation property: for $u \in V$,*

$$e^{y\mathbf{d}_V}Y_W^{s(R)}(u, x)e^{-y\mathbf{d}_V} = Y_W^{s(R)}(e^{y\mathbf{d}_V}u, xy)$$

in $End(V)[[x, x^{-1}, y, y^{-1}]]$.

Theorem 3.3.14. *Let W be a V-bimodule. Then the compatibility condition can be formulated in terms of Y_W^L and $Y_W^{s(R)}$ as follows*

1. *For every $n \in \mathbb{Z}_+$, $l = 1, ..., n$, $u_1, ..., u_n \in V$, $w \in W$, $w' \in W'$,*

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l)Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w\rangle$$

converges absolutely to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i < j \leq n$.

2. *For every $u_1, u_2 \in V, w \in W, w' \in W'$,*

$$\langle w', Y_W^L(u_1, z_1)Y_W^{s(R)}(u_2, z_2)w\rangle$$

$$\langle w', Y_W^{s(R)}(u_2, z_2)Y_W^L(u_1, z_1)w\rangle$$

converges absolutely to a common rational function respectively in the region $|z_1| > |z_2| > 0$ and $|z_2| > |z_1| > 0$.

Proof. We only give a sketch here. From the compatibility condition of Y_W^L and Y_W^R,

$$\langle w', Y_W^L(u_1, z_1-z_{l+1}) \cdots Y_W^L(u_l, z_l-z_{l+1})Y_W^R(w, -z_{l+1})Y_V(u_n, -z_{l+1}+z_n) \cdots Y_V(u_{l+2}, -z_{l+1}+z_{l+2})u_{l+1}\rangle$$

converges absolutely when

$$|z_1 - z_{l+1}| > \cdots > |z_l - z_{l+1}| > |z_{l+1}| > |z_{l+1} - z_n| > \cdots > |z_{l+1} - z_{l+2}| > 0$$

to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i < j \leq n$. Then one uses Lemma 2.3.7 to argue that

$$\langle w', Y_W^L(u_1, z_1-z_{l+1}) \cdots Y_W^L(u_l, z_l-z_{l+1}) Y_W^R(\cdots Y_W^R(Y_W^R(w, -z_n)u_n, -z_{n-1}+z_n)u_{n-1}, \cdots, -z_{l+1}+z_{l+2})u_{l+1}\rangle$$

converges absolutely when

$$|z_1 - z_{l+1}| > \cdots > |z_l - z_{l+1}| > |z_{l+1} - z_{l+2}| + \cdots + |z_{n-1} - z_n| + |z_n|;$$
$$|z_i - z_{i+1}| > |z_{i+1} - z_{i+2}| + \cdots + |z_{n-1} - z_n| + |z_n| > 0, i = 1, ..., n-1.$$

to the same rational function. If we further expand the negative powers of $z_i - z_{l+1}$ as a power series in z_{l+1} for $i = 1, ..., l$, and further expand the negative powers of $-z_i + z_{i+1}$ as a power series z_{i+1} for $i = l+1, ..., n-1$, the resulting series in $z_1, ..., z_n$ is precisely

$$\langle w', e^{-z_{l+1}D_W} Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) e^{z_{l+1}D_W} Y_W^R(\cdots e^{z_n D_W} Y_W^R(w, -z_n)u_n, \cdots, -z_{l+1})u_{l+1}\rangle$$

One uses Lemma 2.3.7 that this series converges absolutely when

$$|z_1| > \cdots > |z_n| > 0$$

to the same rational function. Thus we proved that the series

$$\langle w', e^{-z_{l+1}D_W} Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$. The conclusion of (1) then follows from Remark 2.4.3, which allows us to apply another $e^{z_{l+1}D_W}$ to the front and keep the convergence (though the rational function might change).

For (2), note that

$$\langle w', e^{z_2 D_W} Y_W^L(u_1, z_1 - z_2) Y_W^R(w, -z_2)u_2\rangle = \langle w', e^{z_2 D_W} Y_W^R(Y_W^L(u_1, z_1)w, -z_2)u_2$$

when $|z_1 - z_2| > |z_2| > |z_1| > 0$. Both sides converge to the same rational function. If the negative powers of $z_1 - z_2$ in the series on the left-hand-side are expanded as a power series in z_2, then the resulting series is precisely $\langle w', Y_W^L(u_1, z_1) Y_W^{s(R)}(u_2, z_2)w\rangle$ and converges absolutely in the region $|z_1| > |z_2| > 0, |z_1 - z_2| > 0$ to the same rational function. We then use Lemma 2.3.7 to see that $\langle w', Y_W^L(u_1, z_1) Y_W^{s(R)}(u_2, z_2)w\rangle$ converges absolute when $|z_1| > |z_2| > 0$ to the same rational function as the right-hand-side, while the right-hand-side is precisely $\langle w', Y_W^s(R)(u_2, z_2) Y_W^L(u_1, z_1) w\rangle$. Thus the conclusion is proved. $\square$

Remark 3.3.15. The associativity relation of Y_W^L and Y_W^R translates to the commutativity relation of Y_W^L and the skew-symmetry operator $Y_W^{s(R)}$.

Remark 3.3.16. The pole-order condition can also be expressed in terms of $Y_W^{s(R)}$. More precisely, if V is a MOSVA and W is a V-bimodule with all vertex operators satisfying the corresponding pole-order condition, then

1. For every $u_1 \in V, w \in W$, there exists $C > 0$ such that for every $w' \in W', u_2 \in V$, the pole $z_1 = 0$ of the rational function determined by
$$\langle w', Y_W^{s(R)}(u_1, z_1) Y_W^{s(R)}(u_2, z_2) w\rangle$$
has order less than C. In fact, C can be chosen to be the same upper bound of the order pole $z_1 = 0$ for $\langle w', Y_W^R(w, z_1) Y_V(u_2, z_2) u_1\rangle$.

2. For every $u_1, u_2 \in V$, there exists $C > 0$ such that for every $w' \in W', w \in W$, the pole $z_1 = z_2$ of the rational function determined by
$$\langle w', Y_W^L(u_1, z_1) Y_W^{s(R)}(u_2, z_2) w\rangle$$
has order less than C. In fact, C can be chosen to be the same upper bound of the order pole $z_1 = 0$ for $\langle w', Y_W^L(u_1, z_1) Y_W^R(w, z_2) u_2\rangle$.

Remark 3.3.17. Similarly, one can prove that for the rational function determined by
$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w\rangle$$
the order of the pole $z_i = 0$ is bounded above by a constant that depends only on u_i and w, $i = 1, ..., n$; and the order of the pole $z_i = z_j$ is bounded above by a constant that depends only on u_i and u_j, $1 \leq i < j \leq n$.

3.4 Möbius Structure and Contragredient Modules

In this section we define Möbius structure on MOSVAs and the left (right, bi-) modules for such MOSVAs. With this structure, we prove that the graded dual of a grading-restricted left module for a MOSVA forms a Möbius right module for the MOSVA. For Möbius left modules that are not grading-restricted, we prove the same result under a pole-order condition stronger than that in Definition 2.1.11. The results in this section generalize the theory of contragredient modules for Möbius vertex algebras developed in [FHL] and [HLZ].

3.4.1 Basic definitions

Definition 3.4.1. A *Möbius MOSVA* is a MOSVA $(V, Y_V, \mathbf{1})$ with a representation ρ_V of the Lie algebra $\mathfrak{sl}(2)$ on V, given by

$$L_V(0) = \rho_V(L_0) = \mathbf{d}_V, L_V(-1) = \rho_V(L_1) = D_V, L_V(1) = \rho_V(L_1)$$

where $\{L_{-1}, L_0, L_1\}$ is a basis of $\mathfrak{sl}(2)$ with Lie commutators

$$[L_0, L_{-1}] = L_{-1}, [L_0, L_1] = -L_1, \text{ and } [L_{-1}, L_1] = -2L_0,$$

and the following conditions hold for every $u \in V$:

$$[L_V(1), Y_V(u, x)] = Y(L_V(1)u, x) + 2xY(L_V(0)u, x) + x^2 Y(L_V(-1)u, x)$$

We will use the notation $(V, Y_V, \mathbf{1}, \rho_V)$ to denote a Möbius MOSVA. When there is no confusion, we will simply use the notation V.

Remark 3.4.2. Since $\mathbf{d}_V = L_V(0)$ and $[L_V(0), L_V(1)] = -L_V(1)$, we know that $L_V(1)$ is actually a linear operator of weight -1. Since the grading on V is lower-bounded, the operator is actually locally nilpotent, i.e., for every $v \in V$, there exists $m \in \mathbb{Z}_+$ such that $L_V(1)^m v = 0$. Moreover, with the identity property and creation property, we can see that

$$L_V(j)\mathbf{1} = 0, j = 0, \pm 1$$

Proposition 3.4.3. *Let $(V, Y_V, \mathbf{1}, \rho_V)$ be a Möbius MOSVA. Then the opposite MOSVA $(V, Y_V^s, \mathbf{1}, \rho_V)$ is also a Möbius MOSVA.*

Proof. It suffices to check the commutator formula

$$[L_V(1), Y_V^s(u,x)] = Y_V^s(L_V(1)u,x) + 2xY_V^s(L_V(0)u,x) + x^2Y_V^s(L_V(-1)u,x).$$

We first compute the left-hand-side:

$$\begin{aligned}[L_V(1), Y_V^s(u,x)]v &= L_V(1)Y_V^s(u,x)v - Y_V^s(u,x)L_V(1)v \\ &= L_V(1)e^{xL_V(-1)}Y_V(v,-x)u - e^{xL_V(-1)}Y_V(L_V(1)v,-x)u\end{aligned}$$

In order to interchange $L_V(1)$ and $e^{xL_V(-1)}$ that appear in the first term, we note that for every $n \in \mathbb{N}$,

$$L_V(1)L_V(-1)^n = L_V(-1)^nL_V(1) + L_V(-1)^{n-1}2nL_V(0) + n(n-1)L_V(-1)^{n-1},$$

which can be easily proved by induction. Then a straightforward computation shows that

$$L_V(1)e^{xL_V(-1)} = e^{xL_V(-1)}L_V(1) + 2xe^{xL_V(-1)}L_V(0) + x^2e^{xL_V(-1)}L_V(-1).$$

So the left-hand-side is

$$\begin{aligned}&e^{xL_V(-1)}L_V(1)Y_V(v,-x)u + 2xe^{xL_V(-1)}L_V(0)Y_V(v,-x)u \\ &\quad + x^2e^{xL_V(-1)}L_V(-1)Y_V(v,-x)u - e^{xL_V(-1)}Y(L_V(1)v,-x)u\end{aligned}$$

Then we use the commutator relation between $L_V(j), j = 0, \pm 1$ and $Y_V(v,-x)$ to deal with the first three terms. The first term is equal to

$$\begin{aligned}&e^{xL_V(-1)}Y_V(v,-x)L_V(1)u + e^{xL_V(-1)}Y_V(L_V(1)v,-x)u \\ &\quad - 2xe^{xL_V(-1)}Y_V(L_V(0)v,-x)u + x^2e^{xL_V(-1)}Y_V(L_V(-1)v,-x)u\end{aligned}$$

The second term is equal to

$$2xe^{xL_V(-1)}Y_V(v,-x)L_V(0)u + 2xe^{xL_V(-1)}Y_V(L_V(0)v,-x)u - 2x^2e^{xL_V(-1)}Y_V(L_V(-1)v,-x)u$$

The third term is equal to

$$x^2e^{xL_V(-1)}Y_V(v,-x)L_V(-1)u + x^2e^{xL_V(-1)}Y_V(L_V(-1)v,-x)u$$

The summation of the above three formulas, together with the fourth term, would then simplify to the right-hand-side. $\square$

Definition 3.4.4. Let $(V, Y_V, \mathbf{1}, \rho_V)$ be a Möbius MOSVA. A *Möbius left V-module W* is a left V-module $(W, Y_W^L, \mathbf{d}_W, D_W)$ with a representation ρ_W of the Lie algebra $\mathfrak{sl}(2)$ on W, such that

$$L_W(0) = \rho_W(L_0), L_W(-1) = \rho_W(L_{-1}) = D_W, L_W(1) = \rho_W(L_1),$$

and for every $u \in V$,

$$[L_W(0), Y_W^L(u, x)] = Y_W^L(L_V(0)u, x) + xY_W^L(L_V(-1)u, x)$$

$$[L_W(1), Y_W^L(u, x)] = Y_W^L(L_V(1)u, x) + 2xY_W^L(L_V(0)u, x) + x^2Y_W^L(L_V(-1)u, x),$$

and for every $n \in \mathbb{C}, w \in W_{[n]}$, there exists $m \in \mathbb{N}$ such that $(L_W(0) - n)^m w = 0$.

We will use the notation (W, Y_W^L, ρ_W) to denote Möbius left V-modules. The operator $\mathbf{d}_W$ can be defined as the semisimple part of $L_W(0)$, and the operator D_W is just $L_W(-1)$. So the representation ρ_W has all the information of these two operators and thus we don't need to include them in the notation. When there is no confusion, we will simply use W.

Remark 3.4.5. In [HLZ], modules in which $L_W(0)$ is not semisimple are called generalized modules. In the MOSVA setting, we don't use this terminology because we are not requiring the operator $\mathbf{d}_W$ to be coincide with $L_W(0)$. Indeed, given $L_W(0)$ satisfying the commutator formulas, one can define $\mathbf{d}_W$ as the semisimple of $L_W(0)$. By similar arguments as those in [HLZ], we have

$$[\mathbf{d}_W, (Y_W^L)_n(v)] = [L_W(0), (Y_W^L)_n(v)] \text{ for all } v \in V \text{ and } n \in Z;$$

$$[\mathbf{d}_W, L_W(j)] = [L_W(0), L_W(j)] \text{ for } j = 0, \pm 1.$$

Thus a Möbius left V-module is still a left V-module and should not be entitled with the word "generalized".

Remark 3.4.6. In accordance with convention, when we discuss MOSVA and modules with Möbius structure, we will refer $\mathbf{d}$-commutator formula as $L(0)$-commutator formula, D-derivative property and D-commutator formula as $L(-1)$-commutator formula.

Definition 3.4.7. Let $(V, Y_V, \mathbf{1}, \rho_V)$ be a Möbius MOSVA. A *Möbius right V-module* W is a right V-module $(W, Y_W^R, \mathbf{d}_W, D_W)$ with a representation ρ_W of the Lie algebra $\mathfrak{sl}(2)$ on W, such that

$$L_W(0) = \rho_W(L_0), L_W(-1) = \rho_W(L_{-1}) = D_W, L_W(1) = \rho_W(L_1),$$

and for every $w \in W$,

$$[L_W(0), Y_W^R(u,x)] = Y_W^L(L_W(0)w, x) + xY_W^L(L_W(-1)w, x)$$

$$L_W(1)Y_W^R(w,x) - Y_W^R(w,x)L_V(1) = Y_W^L(L_W(1)w,x) + 2xY_W^L(L_W(0)w,x) + x^2Y_W^L(L_W(-1)w,x),$$

and for every $n \in \mathbb{C}, w \in W_{[n]}$, there exists $m \in \mathbb{N}$ such that $(L_W(0) - n)^m w = 0$

Remark 3.4.8. With similar arguments as Proposition 3.4.3, one can prove the Möbius version of Theorem 3.2.18. In particular, (W, Y_W^R, ρ_W) is a Möbius right V-module if and only if $(W, Y_W^{s(R)}, \rho_W)$ is a Möbius left V^{op}-module, where $Y_W^{s(R)}$ and Y_W^R are skew-symmetry opposite vertex operators to each other. This will be used in the proof of Theorem 3.4.14 and 3.4.17.

Definition 3.4.9. Let $(V, Y_V, \mathbf{1}, \rho_V)$ be a Möbius MOSVA. A *Möbius V-bimodule W* is a V-bimodule $(W, Y_W^L, Y_W^R, \mathbf{d}_W, D_W)$ with a representation ρ_W of the Lie algebra $\mathfrak{sl}(2)$ on W, such that (W, Y_W^L, ρ_W) forms a Möbius left V-module, and (W, Y_W^R, ρ_W) forms a Möbius right V-module.

3.4.2 The opposite vertex operator

Definition 3.4.10. Let $(V, Y_V, \mathbf{1}, \rho_V)$ be a Möbius MOSVA and (W, Y_W^L, ρ_W) be a Möbius left V-module. We define the *opposite vertex operator* on W associated to $u \in V$ by

$$Y_W^o(u,x) = Y_W^L(e^{xL(1)}(-x^{-2})^{L(0)}u, x^{-1}).$$

For homogeneous $u \in V$, we have

$$\begin{aligned} Y_W^o(u,x) &= \sum_{n\in\mathbb{Z}} (Y_W^o)_n(u)x^{-n-1} \\ &= \sum_{n\in\mathbb{Z}} \left((-1)^{\mathrm{wt}\ u} \sum_{m=0}^{\infty} \frac{1}{m!} (Y_W^L)_{-n-m-2+2\mathrm{wt}\ u}(L(1)^m v) \right) x^{-n-1} \end{aligned}$$

Note that since $L(1)$ is locally nilpotent, the summation about variable m is actually finite. Thus each component $(Y_W^o)_n(u)$ is well-defined. Also, the order of summation can be switched at our convenience.

Remark 3.4.11. The opposite vertex operator we are defining here should not be confused with the skew-symmetry operator we introduced in the previous section.

Proposition 3.4.12. *For every $u_1, ..., u_n \in V, w \in W, w' \in W'$, the series*

$$\langle w', Y_W^o(u_n, z_n) \cdots Y_W^o(u_1, z_1)w\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i < j \leq n$.

Proof. It suffices to consider the case when $u_1, ..., u_n \in V$ are homogeneous. In this case,

$$\begin{aligned} &\langle w', Y_W^o(u_n, z_n) \cdots Y_W^o(u_1, z_1)w\rangle \\ = &\sum_{m_1,...,m_n \text{ finite}} (-1)^{\mathrm{wt}\ u_1 + \cdots + \mathrm{wt}\ u_n} z_1^{-2\mathrm{wt}\ u_1} \cdots z_n^{-2\mathrm{wt}\ u_n} \langle w', Y_W^L(L(1)^{m_n}u)_n, z_n^{-1})w \cdots Y_W^L(L(1)^{m_1}u_1, z_1^{-1})\rangle. \end{aligned}$$

By the rationality of Y_W^L, for fixed $m_1, ..., m_n$, $\langle w', Y_W^L(L(1)^{m_n}u)_n, z_n^{-1})w \cdots Y_W^L(L(1)^{m_1}u_1, z_1^{-1})\rangle$ converges absolutely when $|z_n^{-1}| > \cdots > |z_1|^{-1} > 0$ to a rational function of the form

$$\frac{f(z_1^{-1}, ..., z_n^{-1})}{\prod_{i=1}^n z_i^{-p_i} \prod_{1\leq i<j\leq n} (z_i^{-1} - z_j^{-1})^{p_{ij}}} = \frac{f(z_1^{-1}, ..., z_n^{-1}) \prod_{i=1}^n z_i^{p_i + \sum_{j=i+1}^n p_{ij}}}{\prod_{1\leq i<j\leq n} (z_j - z_i)^{p_{ij}}}$$

As the polyonomial $f(z_1^{-1}, ..., z_n^{-n})$ provides negative powers of $z_i, i = 1, ..., n$, this fraction is a rational function with possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i < j \leq n$. Then $\langle w', Y_W^o(u_n, z_n) \cdots Y_W^o(u_1, z_1)w\rangle$, as a finite sum of absolutely convergent series, also converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i < j \leq n$. $\square$

Proposition 3.4.13. *For every $u_1, u_2 \in V, w \in W, w' \in W'$. the series*

$$\langle w', Y_W^o(Y_V(u_2, z_2 - z_1)u_1, z_1)w\rangle$$

converges absolutely when $|z_1| > |z_2 - z_1| > 0$ *to a rational function with the only possible poles at* $z_1 = 0, z_2 = 0, z_1 = z_2$. *Moreover,*

$$\langle w', Y_W^o(u_2, z_2) Y_W^o(u_1, z_1) w\rangle = \langle w', Y_W^o(Y_V(u_2, z_2 - z_1)u_1, z_1)w\rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$

Proof. We will use Formula (5.2.35) in [FHL]: for every $u \in V$, we have

$$e^{xL(1)}(-x^{-2})^{L(0)} Y_V(u, x_0) = Y_V\left(e^{(x+x_0)L(1)}(-(x+x_0)^{-2})^{L(0)}u, -\frac{x_0}{(x+x_0)x}\right) e^{xL(1)}(-x^{-2})^{L(0)}$$

as formal series in $(\mathrm{End}\ V)[[x, x^{-1}, x_0, x_0^{-1}]]$ where all the negative powers of $x + x_0$ are expanded as power series in x_0. The proof of the formula can be found in [FHL], Section 5.2. The idea is to use the $L(0)$-commutator formula and $L(1)$-commutator formula to obtain $L(0)$-conjugation formula and $L(1)$-conjugation formula. No other property was needed. So the proof carries over to MOSVAs and their modules.

To apply this formula, we first study the formal series

$$\langle w', Y_W^L\left(e^{x_1 L(1)}(-x_1^{-2})^{L(0)} Y_V(u_2, x_0)u_1, x_1^{-1}\right) w\rangle$$

in $\mathbb{C}[[x_0, x_0^{-1}, x_1, x_1^{-1}]]$. By the formula above, the formal series is equal to

$$\left\langle w', Y_W^L\left(Y_V\left(e^{(x_1+x_0)L(1)}(-(x_1+x_0)^{-2})^{L(0)}u, -\frac{x_0}{(x_1+x_0)x_1}\right) e^{x_1L(1)}(-x_1^{-2})^{L(0)}u_1, x_1^{-1}\right) w\right\rangle \tag{3.2}$$

in $\mathbb{C}[[x_0, x_0^{-1}, x_1, x_1^{-1}]]$, with all the negative powers of $x_1 + x_0$ expanded as power series in x_0. Moreover, it is easy to see that this series has at most finitely many negative powers of x_0 and at most finitely many positive powers of x_1.

In order to substitute x_0 and x_1 by complex numbers z_0 and z_1, we first note from the rationality of iterates of two vertex operators, for complex numbers $z_0, z_1, \zeta_0, \zeta_1$ with $|\zeta_1| > |z_0\zeta_1/((z_1+\zeta_0))| > 0$, i.e., $|z_1 + \zeta_0| > |z_0| > 0, |\zeta_1| > 0$, the complex series

$$\begin{aligned}
&\left\langle w', Y_W^L\left(Y_V\left(e^{(z_1+\zeta_0)L(1)}(-(z_1+\zeta_0)^{-2})^{L(0)}u, -\frac{z_0\zeta_1}{(z_1+\zeta_0)}\right) e^{z_1L(1)}(-z_1^{-2})^{L(0)}u_1, \zeta_1\right) w\right\rangle \\
&\qquad = \sum_{i \text{ finite}} \sum_{m,n} a_{mni}(z_1+\zeta_0)^i \left(-\frac{z_0\zeta_1}{(z_1+\zeta_0)}\right)^{-m-1} (\zeta_1)^{-n-1}
\end{aligned}$$

(with variables $-z_0\zeta_1/((z_1+\zeta_0))$ and ζ_1) converges absolutely to a rational function with the only possible poles at $z_0 = 0, z_1 = 0, \zeta_1 = 0, z_1 + \zeta_0 = 0, z_1 + \zeta_0 = z_0$ (note the operators $e^{(z_1+\zeta_0)L(1)}$ and $e^{z_1L(1)}$ acts as polynomials, and $(-(z_1+\zeta_0)^{-2})^{-L(0)}$ and $(-z_1^{-2})^{L(0)}$ acts by a scalar multiplication on homogeneous elements). Note that in this expansion, the power of $(-z_0\zeta_1/(z_1+\zeta_0))$ is lower-truncated, i.e., m is bounded above. In particular, the power of z_0 is lowert-truncated. Moreover, for each fixed m, the power of ζ_1 is lower-truncated, i.e., n is also bounded above for each fixed m.

Now, we further expand the negative powers of $z_1+\zeta_0$ as power series in ζ_0, i.e.,

$$\begin{aligned}&\sum_{i \text{ finite}}\sum_{m,n} a_{mni}(z_1+\zeta_0)^i\left(-\frac{z_0\zeta_1}{(z_1+\zeta_0)}\right)^{-m-1}\zeta_1^{-n-1}\\ &\quad=\sum_{i \text{ finite}}\sum_{m,n} a_{mni}(-1)^{m+1}z_0^{-m-1}\zeta_1^{-m-n-2}\left(\sum_{k=0}^{\infty}\binom{m+1+i}{k}z_1^{m+1+i-k}\zeta_0^k\right).\end{aligned}$$

The resulting iterated series on the right-hand-side converges absolutely to the rational function when $|z_1+\zeta_0|>|z_0|>0, |z_1|>|\zeta_0|, |\zeta_1|>0$. We check that all the conditions of Lemma 2.3.8 is satisfied. Thus the complex series corresponding to the iterated series on the right-hand-side is precisely the Laurent series expansion of the rational function when $|z_1|>|\zeta_0|, |z_1|>|\zeta_0-z_0|, |\zeta_1|>0, |z_0|>0$. In particular, the complex series converges absolutely when $|z_1|>|\zeta_0|, |z_1|>|\zeta_0-z_0|, |\zeta_1|>0, |z_0|>0$. Now we substitute $\zeta_0=z_0, \zeta_1=z_1^{-1}$ to see that the complex series

$$\left\langle w', Y_W^L\left(Y_V\left(e^{(z_1+z_0)L(1)}(-(z_1+z_0)^{-2})^{L(0)}u, -\frac{z_0z_1^{-1}}{(z_1+z_0)}\right)e^{z_1L(1)}(-z_1^{-2})^{L(0)}u_1, z_1^{-1}\right)w\right\rangle$$

converges absolutely when $|z_1|>|z_0|>0$ to a rational function with the only possible poles at $z_0=0, z_1=0, z_1+z_0=0$. And this series is precisely the complex series obtained from substituting $x_0=z_0$ and $x_1=z_1$ in the formal series (3.2).

We then perform the transformation $z_0\mapsto z_2-z_1$ to see that the complex series

$$\begin{aligned}&\left\langle w', Y_W^L\left(Y_V\left(e^{z_2L(1)}(-z_2^{-2})^{L(0)}u, -\frac{z_2-z_1}{z_2z_1}\right)e^{z_1L(1)}(-z_1^{-2})^{L(0)}u_1, z_1^{-1}\right)w\right\rangle\\ =&\langle w', Y_W^L(Y_V(e^{z_2L(1)}(-z_2^{-2})^{L(0)}u, -z_1^{-1}+z_2^{-1})e^{z_1L(1)}(-z_1^{-2})^{L(0)}u_1, z_1^{-1})w\rangle\end{aligned}$$

which is equal to

$$\langle w', Y_W^o(Y_V(u_2, z_2-z_1)u_1, z_1)w\rangle,$$

converges absolutely when $|z_1|>|z_1-z_2|>0$ to a rational function with the only possible poles at $z_1=0, z_2=0, z_1=z_2$.

Now we use the definition of Y_W^o to rewrite the left-hand-side as

$$\langle w', Y_W^L(e^{z_2 L(1)}(-z_2^{-2})^{L(0)}u_2, z_2^{-1})Y_W^L(e^{z_1 L(1)}(-z_1^{-2})^{L(0)}u_1, z_1^{-1})w\rangle$$

This series converges absolutely when $|z_1^{-1}| > |z_2^{-1}| > 0$ to a rational function with the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$. Moreover, by associativity, when $|z_2^{-1}| > |z_1^{-1}| > |z_1^{-1} - z_2^{-1}| > 0$, i.e., $|z_1| > |z_2| > |z_1 - z_2| > 0$, it is equal to

$$\langle w', Y_W^L(Y_V(e^{z_2 L(1)}(-z_2^{-2})^{L(0)}u_2, z_2^{-1} - z_1^{-1})e^{z_1 L(1)}(-z_1^{-2})^{L(0)}u_1, z_1^{-1})w\rangle$$

Thus left-hand-side is equal to right-hand-side when $|z_1| > |z_2| > |z_1 - z_2| > 0$. $\square$

3.4.3 Contragredient of a Möbius left V-module

We first discuss the results for grading-restricted Möbius left V-modules. Then we deal with the non-grading-restricted case with a stronger pole-order condition.

Theorem 3.4.14. *Let $(V, Y_V, \mathbf{1}, \rho_V)$ be a Möbius MOSVA and (W, Y_W^L, ρ_W) be a grading-restricted Möbius left V-module. On the graded dual $W' = \coprod_{n\in\mathbb{C}} W_{[n]}^*$, we define a vertex operator action of V by*

$$\langle Y_W'(u,z)w', w\rangle = \langle w', Y_W^o(u,z)w\rangle = \langle w', Y_W^L(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})w\rangle,$$

and an $\mathfrak{sl}(2)$-action ρ_W' by $\rho_W'(L_j) = L'(j)$ for $j = 0, \pm 1$, where

$$\langle L_W'(j)w', w\rangle = \langle w', L_W(-j)w\rangle.$$

Then (W', Y_W', ρ_W') forms a Möbius right V-module.

Proof. The commutator formulas for $L_W'(0), L_W'(-1)$ and $L_W'(1)$ follows from the computations in [HLZ], Lemma 2.22. The argument there carries over to MOSVAs and requires some work. For brevity we will not include them here but redirect the reader to [HLZ], Page 59 to 61. From Remark 3.4.8, it suffice to verify that $(W', Y_W', \mathbf{d}_W', L_W'(-1))$ forms a left V^{op}-module.

1. The lower bound condition obviously hold. The **d**-grading condition and **d**-commutator formula follow from the discussions in Remark 3.4.5. In particular, from the **d**-commutator formula and the lower bound condition, one sees that the series $Y_W'(u,z)w'$ is lower truncated.

2. To see the identity property, note that $L_V(1)\mathbf{1} = 0$ and $L_V(0)\mathbf{1} = 0$, thus $e^{xL(1)}(-x^{-2})^{L(0)}\mathbf{1} = \mathbf{1}$. So $Y_W^o(\mathbf{1}, x) = Y_W^L(\mathbf{1}, x^{-1}) = 1_W$. Then follows $Y_W'(\mathbf{1}, x)w' = w'$.

3. The $L(-1)$-derivative property is verified in [FHL]. See [FHL], Page 47 and 48.

4. Since W is grading restricted, $(W')' = W$. Thus for the rationality of products, it suffices to verify that for every $w' \in W', w \in W, u_1, ..., u_n \in V$, the series

$$\langle Y_W'(u_1, z_1) \cdots Y_W'(u_n, z_n)w', w\rangle = \langle w', Y_W^o(u_n, z_n) \cdots Y_W^o(u_1, z_1)w\rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n$ and $z_i = z_j, 1 \leq i < j \leq n$. This was shown in Proposition 3.4.12. For the rationality of iterates, it suffices to show that for every $w' \in W', w \in W, u_1, u_2 \in V$, the series

$$\langle Y_W'(Y_V^s(u_1, z_1 - z_2)u_2, z_2)w', w\rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$. We first use the definition of Y_V^s, then use $L(-1)$-conjugation property to see that

$$\begin{aligned}
&\langle Y_W'(Y_V^s(u_1, z_1 - z_2)u_2, z_2)w', w\rangle \\
=&\langle Y_W'(e^{(z_1-z_2)L(-1)}Y_V(u_2, z_2 - z_1)u_1, z_2)w', w\rangle \\
=&\langle Y_W'(Y_V(u_2, z_2 - z_1)u_1, z_1)w', w\rangle
\end{aligned}$$

Note that from Remark 2.1.5, this series is still in variables z_2 and $z_1 - z_2$, where z_1 should be regarded as the sum $z_2 + z_1 - z_2$ and thus negative powers of z_1 should be expanded as power series in $(z_1 - z_2)$. Then we use the definition of Y_W' to see that this series is equal to

$$\langle w', Y_W^o(Y_V(u_2, z_2 - z_1)u_1, z_1)w\rangle.$$

And the proof of Proposition 3.4.13 shows that this series converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to the same rational function as $\langle w', Y_W^o(u_2, z_2)Y_W^o(u_1, z_1)w\rangle$

5. The associativity follows from the discussion above and Proposition 3.4.13.

□

Definition 3.4.15. The module (W', Y'_W, ρ'_W) is referred as the contragredient module of (W, Y_W^L, ρ_W). In case there is no confusion, we just use W' to denote it. From the results in Section 5 and Proposition 3.4.3 , one easily sees that W' also a Möbius right V-module.

Remark 3.4.16. When W is not grading restricted, one has to verify the rationality with w taking value in the much larger space $(W')'$. So the above proof does not work. To construct the contragredient module for non-grading-restricted modules, an additional condition has to be assumed.

Theorem 3.4.17. *Let $(V, Y_V, \mathbf{1}, \rho_V)$ be a Möbius MOSVA and (W, Y_W^L, ρ_W) be a Möbius left V-module. If the vertex operator Y_W^L satisfies the strong pole-order condition, that there exists a real number C, such that for every homogeneous $u_1, u_2 \in V, w' \in W'$ and $w \in W$, the order of the pole $z_1 = 0$ in the rational function given by*

$$\langle w', Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) w\rangle$$

is bounded above by $\text{wt } u_1 + Re \text{ wt } w + C$*, then with Y'_W and ρ'_W are defined in the same way as the above theorem, (W', Y'_W, ρ'_W) forms a Möbius left V^{op}-module.*

Proof. It suffices to deal with the rationality and associativity axioms. The idea is to use the formal variable approach. With the strong pole-order condition, we proceed to verify the weak associativity with the pole-order condition in Theorem 3.1.22 based on the results of Proposition 3.4.12 and 3.4.13. Then the conclusion follows from the theorem.

Let $w' \in W', u_1, u_2 \in V, w \in W$ be homogeneous. We rewrite the series

$$\langle Y'_W(u_1, z_1) Y'_W(u_2, z_2) w', w\rangle = \langle w', Y_W^o(u_2, z_2) Y_W^o(u_1, z_1) w\rangle$$

as

$$\sum_{m_1=0}^{\text{finite}}\sum_{m_2=0}^{\text{finite}}(-1)^{\text{wt }u_2+\text{wt }u_1}\frac{1}{m_1!}\frac{1}{m_2!}z_2^{m_2-2\text{wt }u_2}z_1^{m_1-2\text{wt }u_1}\cdot \tag{3.3}$$
$$\langle w', Y_W^L(L(1)^{m_2}u_2, z_2^{-1})Y_W^L(L(1)^{m_1}u_1, z_1^{-1})w\rangle$$

We shall use the computations in Remark 3.1.3 to give an explicit upper bound of the order of the pole $z_1 = 0$ for the rational function given by each term in the sum.

For each fixed m_1, m_2, the rational function determine by $\langle w', Y_W^L(L(1)^{m_2}u_2, z_2^{-1})Y_W^L(L(1)^{m_1}u_1, z_1^{-1})w\rangle$ is of the form

$$\frac{f(z_2^{-1}, z_1^{-1})}{z_2^{-p_1}z_1^{-p_2}(z_2^{-1}-z_1^{-1})^{p_{12}}} = \frac{f(z_2^{-1}, z_1^{-1})}{z_2^{-p_1-p_{12}}z_1^{-p_2-p_{12}}(z_1-z_2)^{p_{12}}}$$

Note that Y_W^L satisfies the strong pole-order condition, thus

$$p_1 \leq \text{wt }(L(1)^{m_2}u_2) + \text{Re wt }w + C$$

Let d be the degree of f as a polynomial in z_2^{-1}, z_1^{-1}. From Remark 3.1.3,

$$d = p_1 + p_2 + p_{12} + \text{wt }w' - \text{wt }(L(1)^{m_2}u_2) - \text{wt }(L(1)^{m_1}u_1) - \text{wt }w$$

Note that though wt w' and wt w might be complex numbers, their difference is supposed to be an integer. In particular, we know that

$$\text{wt }w' - \text{wt }w = \text{Re wt }w' - \text{Re wt }w$$

If we write

$$f(x_1, x_2) = \sum_{k=0}^{d} a_k x_1^k x_2^{d-k},$$

then

$$f(z_2^{-1}, z_1^{-1}) = \sum_{k=0}^{d} a_k z_2^{-k} z_1^{-d+k}$$

where the lowest possible power of z_1 is $-d$. Therefore, the order of pole of the rational function that each term in (3.3) converges to is bounded above by

$$\begin{aligned} d - p_2 - p_{12} - m_1 + 2\text{wt }u_1 &= p_1 + \text{wt }w' - \text{wt }(L(1)^{m_2}u_2) - \text{wt }(L(1)^{m_1}u_1) - \text{wt }w - m_1 + 2\text{wt }u_1 \\ &= p_1 + \text{Re wt }w' - \text{wt }(L(1)^{m_2}u_2) - \text{Re wt }w + \text{wt }u_1 \end{aligned}$$

$$\leq \text{Re wt } w' + \text{wt } u_1 + C.$$

This upper bound is independent of m_1, m_2. Thus we have proved that the order of the pole $z_1 = 0$ of the rational function given by

$$\langle Y'_W(u_1, z_1) Y'_W(u_2, z_2) w', w\rangle = \langle w', Y^o_W(u_2, z_2) Y^o_W(u_1, z_1) w\rangle$$

is controlled above by the real number that depends only u_1 and w_1. So with the assumption here, the vertex operator Y'_W satisfies the pole-order condition as in Definition 2.1.11.

Now with the conclusion of Proposition 3.4.12 and 3.4.13, we know that one can choose $q_1 = \text{wt } w' + \text{wt } u_1 + C$ depending only on u_1 and w', q_2 depending only on u_2 and w', q_{12} depending only on u_1 and u_2, such that

$$(z_0+z_2)^{q_1} z_2^{q_2} z_0^{q_{12}} \langle Y'_W(u_1, z_0+z_2) Y'_W(u_2, z_2) w', w\rangle = (z_0+z_2)^{q_1} z_2^{q_2} z_0^{q_{12}} \langle Y'_W(Y^s_V(u_1, z_0) u_2, z_2) w', w\rangle$$

converges absolutely to a polynomial function. Thus as formal series with coefficients in W',

$$(x_0+x_2)^{q_1} x_2^{q_2} x_0^{q_{12}} \langle Y'_W(u_1, x_0+x_2) Y'_W(u_2, x_2) w' = (x_0+x_2)^{q_1} x_2^{q_2} x_0^{q_{12}} \langle Y'_W(Y^s_V(u_1, x_0) u_2, x_2) w' \in W'[[x_0, x_2]].$$

has no negative powers of x_0 and x_2. Thus they all live in $W'[[x_0, x_2]]$. The weak associativity relation is then seen by dividing both sides by $x_2^{q_2}$ and $x_0^{q_{12}}$. Moreover, the choice of q_1 depends only on u_1 and w'. Thus, as a consequence of Theorem 3.1.22, the rationality and associativity axioms hold. $\square$

Remark 3.4.18. This strong pole-order condition is natural because it is satisfied by all the Möbius left modules for vertex algebras.

Chapter 4

MOSVA constructed from 2d unit sphere

In this chapter we study the MOSVA constructed on the 2-dimensional sphere. Throughout this chapter, S^2 denotes the unit sphere in the three-dimensional Euclidean space. TS^2 denotes the tangent bundle of S^2. We will use the parallel sections of the tensor bundles $(TS^2)^{\otimes k}, k = 0, 1, ...$ to construct the MOSVA.

4.1 Basic Geometry Facts

Most of the results in this section can be found in [KN], and [P] and [T]. To be self-contained, we will still give some brief arguments regarding these facts.

Let E be a vector bundle over a connected Riemannian manifold M with a connection ∇. Fix a point $p \in M$ and a piecewise smooth path $\gamma : [0, 1] \to M$ based at p. A *smooth section along* γ is a smooth map $X : [0, 1] \to E$ with $X(t) \in E_{\gamma(t)}$. A smooth section X along γ is *parallel* if $\nabla_{\dot{\gamma}} X = 0$. For every vector $X_0 \in E_p$, from the existence and uniqueness of ODE, there exists a parallel smooth section $X : [0, 1] \to E$ with $X(0) = X_0$. This gives rise to the notion of *parallel translation*, which is a family of linear maps $P_\gamma(t) : E_p \to E_{\gamma(t)}$ that maps X_0 to $X(t)$. In case γ is a loop with $\gamma(0) = \gamma(1) = p$, then $P_\gamma(1) : E_p \to E_p$ is an automorphism of the fiber E_p. The subgroup of $GL(E_p)$ generated by all such $P_\gamma(1)$'s is the *holonomy group* of E at p. Since M is connected, the holonomy group of E at different p are isomorphic to each other. We should use the notation $\mathrm{Hol}(E)$ to denote it.

Theorem 4.1.1. *The holonomy group on TS^2 is precisely $SO(2, \mathbb{R})$.*

Proof. Fix p, q, r on the sphere. Let $\gamma_1, \gamma_2, \gamma_3$ be geodesics connecting pq, qr and rp. Let $\alpha_p, \alpha_q, \alpha_r$ be the angles formed by the tangent vectors of spherical triangle pqr. Let

$v \in T_pM$ be a unit vector. One sees easily that that composition of parallel transport along γ_1, γ_2 and γ_3 will end up with a unit vector $w \in T_pM$, such that the angle of w and v is $\alpha_1 + \alpha_2 + \alpha_3$, which by Gauss-Bonnet theorem, is precisely $\pi + \text{Area}(pqr)$. As p, q, r varies, the angle varies from $(\pi, 2\pi]$. Wrapping around the loop again to see that the angle varies from $(2\pi, 4\pi]$ that covers all the rotations in $SO(2, \mathbb{R})$. $\square$

Let $X : M \to E$ be a smooth section. X is *parallel* if for every $Y \in T$, $\nabla_Y X = 0$. Equivalently, for every $p \in M$ and every piecewise smooth path γ based at p, $X_{\gamma(t)} = P_{\gamma(t)} X_p$. The space of parallel sections of E is denoted by $\Pi(E)$.

Proposition 4.1.2. *Let E be a vector bundle over a Riemannian manifold M with a connection. Fix a point p on M. The space of parallel sections $\Pi(E)$ of E is isomorphic to the fixed point subspace $E_p^{Hol(E)}$ of E_p under the action of the holonomy group $Hol(E)$.*

Proof. Given a parallel section, its restriction at p is obviously a vector fixed by all parallel transports along piecewise smooth loops. Conversely, given a vector that is invariant under the holonomy group action, the parallel transport along the path connecting p and any other point q on M yields a parallel vector field. Easy to see that these two operations are inverse to each other. $\square$

Proposition 4.1.3. *Let $E_1, E_2, \ldots$ be a sequence of vector bundles on M. Let $E = \bigoplus_{i=1}^{\infty} E_i$. Then the parallel sections of E is the direct sum of the parallel sections of E_i, i.e., $\Pi(E) = \bigoplus_{i=1}^{\infty} \Pi(E_i)$.*

Proof. Obviously $\bigoplus_{i=1}^{\infty} \Pi(E_i) \subset \Pi(E)$. We show the inverse inclusion here. Let X be a parallel section of E. Fix any $p \in M$ and piecewise smooth path γ based on p. Consider $X_p = \sum_{i \text{ finite}} (X_i)_p$, which is a finite sum of components in $(E_1)_p, (E_2)_p, \ldots$ The parallel transport $P_{\gamma(1)}^E$ applied on X amounts to the sum of the action of $P_{\gamma(1)}^{E_i}$ on X_i. Since it is a direct sum, we necessarily have $P_{\gamma(1)}^{E_i}(X_i)_p = (X_i)_p$. Since γ is arbitrarily chosen, we see that $(X_i)_p \in (E_i)_p^{\text{Hol}(E)}$. That is to say, X_p is a finite sum of elements in $(E_i)_p^{\text{Hol}(E)}$. Thus X is a finite sum of parallel sections in E_i. So we proved that $\Pi(E) \subset \bigoplus_{i=1}^{\infty} \Pi(E_i)$. $\square$

4.2 Complexified parallel tensor fields on S^2

We start with the following elementary problem: fix an arbitrary positive integer k, we proceed to determine all the parallel k-tensor fields on the tangent bundle $(TS^2)^{\otimes k}$.

Proposition 4.2.1. *The holonomy group of* $\otimes^k TS^2$ *is* $SO(2,\mathbb{R})/\{\pm 1\}$ *when* k *is even;* $SO(2,\mathbb{R})$ *when* k *is odd. In any case,* $SO(2,\mathbb{R})$ *acts on the vector tensor space* $\otimes^k T_pS^2$*:* $v_1, ..., v_k \in T_pS^2, g \in SO(2,\mathbb{R})$,

$$g(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$$

Proof. For simplicity we just proceed with $k = 2$. For higher k's the proof easily generalizes. Let $\gamma : [0,1] \to S^2$ be a path based on p and denote by $T_{\gamma(t)} : T_pS^2 \to T_{\gamma(t)}S^2$ the parallel transportation along γ over the tangent bundle, and $T^2_{\gamma(t)} : (TS^2 \otimes TS^2)_p \to (TS^2 \otimes TS^2)_{\gamma(t)}$ the parallel transportation along γ over the tensor bundle $TS^2 \otimes TS^2$. We claim that for every $v_1, v_2 \in T_pS^2$,

$$T^2_{\gamma(t)}(v_1 \otimes v_2) = T_{\gamma(t)}v_1 \otimes T_{\gamma(t)}v_2.$$

As the equality stands for $t = 0$, it suffices to verify that the derivatives of both sides are equal. By the definition of parallel transportations,

$$\begin{aligned} \frac{d}{dt}\left[T^2_{\gamma(t)}(v_1 \otimes v_2)\right] &= \nabla_{\dot\gamma} T^2_{\gamma(t)}(v_1 \otimes v_2) = 0 \\ \frac{d}{dt}\left[T_{\gamma(t)}v_1 \otimes T_{\gamma(t)}v_2\right] &= \nabla_{\dot\gamma}(T_{\gamma(t)}v_1 \otimes T_{\gamma(t)}v_2) \\ &= \nabla_{\dot\gamma}T_{\gamma(t)}v_1 \otimes T_{\gamma(t)}v_2 + T_{\gamma(t)}v_1 \otimes \nabla_{\dot\gamma}T_{\gamma(t)}v_2 \\ &= 0 \end{aligned}$$

So the claim is proved.

As every element in the holonomy group of $TS^2 \otimes TS^2$ is of the form $T^2_{\gamma(1)}$, the claim we proved implies that the map $Hol_p(TS^2) \to Hol_p(TS^2 \otimes TS^2) : T_{\gamma(1)} \mapsto (T_{\gamma(1)}, T_{\gamma(1)})$ is surjective. All it remains is to study the kernel of the map. Fixing γ and a basis of T_pS^2, so that $T_{\gamma(1)}$ admits a matrix representation

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then the matrix representation of $(T_{\gamma(1)}, T_{\gamma(1)})$ on $T_pS^2 \otimes T_pS^2$ is simply the Kronecker product

$$\begin{bmatrix} a_{11}A & a_{12}A \\ a_{21}A & a_{22}A \end{bmatrix}$$

Now if $T_{\gamma(1)}$ lies in the kernel, then the above 4-by-4 matrix must be the identity. To achieve this, it is necessary that $a_{12} = a_{21} = 0, a_{11}^2 = a_{22}^2 = a_{11}a_{22} = 1 \Rightarrow A = id$ or $A = -id$. $\square$

Corollary 4.2.2. *For the complexified tensor bundle $TS^2 \otimes_R \mathbb{C}$ where $\mathbb{C}$ is regarded as a trivial bundle with two-dimension fiber over S^2, we have $Hol((TS^2 \otimes_{\mathbb{R}} \mathbb{C})^{\otimes k}) = Hol((TS^2)^{\otimes k})$.*

Proof. It suffices to notice that $\nabla_{\dot\gamma}(v(t) \otimes a(t)) = \nabla_{\dot\gamma} v(t) \otimes a(t)$. All the computation above generalizes easily. $\square$

Proposition 4.2.3. *For odd k, $\Pi((TS^2)^{\otimes k}) = 0$.*

Proof. It suffices to notice that $-id$ is an element in the holonomy group. $\square$

Proposition 4.2.4. *For even k, the parallel k-tensor fields are described as follows: Pick a generic*

$$g(\theta) := \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

in $SO(2,\mathbb{R})$. Let $v_1 \in \mathbb{C}^2$ be an eigenvector of $g(\theta)$ with eigenvalue $e^{i\theta}$ over the complexified tangent space $(T_pS^2)^{\mathbb{C}}$ and v_2 be its complex conjugate. Then the invariant subspace of $\otimes^k T_pS^2$ (aka, parallel k-tensor fields) are spanned by the real parts and the imaginary parts of the k-tensors $v_{i_1} \otimes \cdots \otimes v_{i_k}$ satisfying $\#\{j : i_j = 1\} = \#\{j : i_j = 2\}$.

Proof. To see the second conclusion, let $I = (i_1, ..., i_k)$ be a sequence of length k with $i_j \in \{1,2\}$ for each $j = 1, ..., k$. Set $v_I = v_{i_1} \otimes \cdots \otimes v_{i_k}$ and $N(I) = \#\{j : i_j = 1\} - \#\{j : i_j = 2\}$. It's easy to see that as I ranges through all sequences, all vectors v_I's form a basis of complexified $(\otimes^k T_pS^2)^{\mathbb{C}}$ and each v_I is an eigenvector of $g(\theta)$ with eigenvalue $e^{iN(I)\theta}$. So the complexified invariant space is spanned by those v_I's with $N(I) = 0$. Since v_1 and v_2 are conjugate to each other, the complexified invariant space is closed under conjugation. Therefore the real invariant space in T_pS^2 is spanned by real and imaginary parts of v_I (with $N(I)=0$). $\square$

4.3 Parallel Sections of $T(\widehat{TM}_-)$ and the MOSVA

Recall that

$$\widehat{TM}_- = TM \otimes_{\mathbb{R}} (M \times t^{-1}\mathbb{C}[t^{-1}])$$

and

$$T(\widehat{TM}_-) = \mathbb{C} \oplus \widehat{TM}_- \oplus \widehat{TM}_- \otimes_{\mathbb{C}} \widehat{TM}_- \oplus \cdots$$

We consider the following grading structure on $T(\widehat{TM}_-)$, that

$$T(\widehat{TM}_-) = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{m=0}^{n} \bigoplus_{k_1,\dots,k_m \in \mathbb{Z}_+}^{k_1+\cdots k_m = n} (TM \otimes_{\mathbb{R}} (M \times \mathbb{C}t^{-k_1})) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (TM \otimes_{\mathbb{R}} (M \times \mathbb{C}t^{-k_m})) \right)$$

Thus, for any open subset U of M,

$$\Pi T(\widehat{TM}_-) = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{m=0}^{n} \bigoplus_{k_1,\dots,k_m \in \mathbb{Z}_+}^{k_1+\cdots k_m = n} \Pi[(TM \otimes_{\mathbb{R}} (M \times \mathbb{C}t^{-k_1})) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} (TM \otimes_{\mathbb{R}} (M \times \mathbb{C}t^{-k_m}))] \right)$$

which is isomorphic to

$$T(\widehat{T_pM}_-)^{\mathrm{Hol}(U)} = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{m=0}^{n} \bigoplus_{k_1,\dots,k_m \in \mathbb{Z}_+}^{k_1+\cdots k_m = n} [(T_pM^{\mathbb{C}})^{\otimes m}]^{\mathrm{Hol}(U)} \right)$$

In case $M = S^2$, it is isomorphic to

$$\bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{n} \bigoplus_{k_1,\dots,k_m \in \mathbb{Z}_+}^{k_1+\cdots k_m = n} span \left\{ \{(v_{i_1} \otimes t^{-k_1}) \otimes \cdots \otimes (v_{i_m} \otimes t^{-k_m}) : i_j \in \{1,2\}, \#\{j : i_j = 1\} = \#\{j : i_j = 2\} \right\}$$

In other words, our MOSVA V is a graded vector space $V = \oplus_{n=0}^{\infty} V_n$, with each V_n spanned by the elements

$$(v_{i_1} \otimes t^{-k_1}) \otimes \cdots \otimes (v_{i_m} \otimes t^{-k_m}) : 0 \le m \le n, k_1, \dots, k_m \ge 0, k_1 + \cdots + k_m = n,$$
$$i_1, \dots, i_m \in \{1,2\}, \#\{j : i_j = 1\} = \#\{j : i_j = 2\}$$

The vertex operator is defined the same way as in [H3]. We recall the definition here. Let $\mathfrak{h} = T_pM$, which is a finite-dimensional Euclidean space over $\mathbb{R}$. Let

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$$

be the ambient vector space of the Heisenberg algebra. Note that $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_- \oplus \hat{\mathfrak{h}}_0 \oplus \hat{\mathfrak{h}}_+$. Let $N(\hat{\mathfrak{h}})$ be the quotient of the tensor algebra $T(\hat{h})$ of $\hat{h}$ modulo the two-sided ideal I generated by

$$(a \otimes t^m) \otimes (b \otimes t^n) - (b \otimes t^n) \otimes (a \otimes t^m) - m(a,b)\delta_{m+n,0}\mathbf{k},$$
$$(a \otimes t^k) \otimes (b \otimes t^0) - (b \otimes t^0) \otimes (a \otimes t^k),$$
$$(a \otimes t^k) \otimes \mathbf{k} - \mathbf{k} \otimes (a \otimes t^k)$$

for $a, b \in \mathfrak{h}$, $m \in \mathbb{Z}_+$, $n \in -\mathbb{Z}_+$, $k \in \mathbb{Z}$. Note that in the quotient, there are no relations between $X \otimes t^m$ and $Y \otimes t^n$ for $m, n \in Z_+$ and for $m, n \in \mathbb{Z}_-$. Also note that $N(\hat{\mathfrak{h}}) \simeq T(\hat{\mathfrak{h}}_-) \otimes T(\hat{\mathfrak{h}}_0) \otimes T(\hat{\mathfrak{h}}_+) \otimes T(\mathbb{C}\mathbf{k})$ as vector spaces.

Let $\mathbb{C} = \mathbb{C}\mathbf{1}$ be a one-dimensional vector space on which $\mathfrak{h}$ acts by 0. Define the action of $\mathbf{k}$ by 1 and $\hat{\mathfrak{h}}_+$ by 0. One can prove that the induced module $N(\hat{\mathfrak{h}}) \otimes_{N(\hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_0)} \mathbb{C}$ is isomorphic to $T(\hat{\mathfrak{h}}_-)$ as a vector space. We regard $T(\hat{\mathfrak{h}}_-)$ now as an $N(\hat{\mathfrak{h}})$-module and denote the action of $h \otimes t^k$ by $h(k)$. Then $T(\hat{\mathfrak{h}}_-)$ is spanned by $h_1(-k_1) \cdots h_m(-k_m)\mathbf{1}$ for $m \in \mathbb{N}, h_1, ..., h_m \in \mathfrak{h}, k_1, ..., k_m \in \mathbb{Z}_+$.

Huang proved the following theorem in [H3]

Theorem 4.3.1 (Huang, 2012)**.** *The left $N(\hat{\mathfrak{h}})$-module $T(\hat{\mathfrak{h}}_-)$ forms a grading-restricted MOSVA with the following vertex operator action:*

$$\begin{aligned} &Y(h_1(-k_1) \cdots h_m(-k_m)1, x) \\ &\quad = \,\mathop{\circ}\limits_{\circ}\, \frac{1}{(k_1-1)!} \frac{d^{k_1-1}}{dx^{k_1-1}} h_1(x) \cdots \frac{1}{(k_1-1)!} \frac{d^{k_m-1}}{dx^{k_m-1}} h_k(x) \,\mathop{\circ}\limits_{\circ}\end{aligned}$$

where $h_i(x) = \sum_{n \in \mathbb{Z}} h_i(n) x^{-n-1}$

With the knowledge of the parallel sections of the bundle $T(\widehat{TM}_-)$, from the conclusions of [H4], we have the following theorem:

Theorem 4.3.2. *The subspace $V = \bigoplus_{n \geq 0} V_n \subset N(\widehat{h}_-)$ with*

$$V_n = span \left\{ v_{i_1}(-k_1) \cdots v_{i_m}(-k_m)\mathbf{1} : \begin{array}{c} 0 \leq m \leq n, k_1, ...k_m \geq 0, \sum_{j=1}^{m} k_j = n \\ , i_1, ..., i_k \in \{1, 2\}, \#\{j : i_j = 1\} = \#\{j : i_j = 2\} \end{array} \right\},$$

together with the following vertex operator

$$Y(v_{i_1}(-k_1)\cdots v_{i_m}(-k_m)1,x) = {}^{\circ}_{\circ}\frac{1}{(k_1-1)!}\frac{d^{k_1-1}}{dx^{k_1-1}}v_{i_1}(x)\cdots\frac{1}{(k_1-1)!}\frac{d^{k_m-1}}{dx^{k_m-1}}v_{i_k}(x){}^{\circ}_{\circ}$$

and $\mathbf{1} \in V_0$ *forms a grading-restricted MOSVA.*

Chapter 5

Cohomology theory of MOSVA

In this chapter we develop the cohomology theory of bimodules of a meromorphic open string vertex algebras. The theory is generalized from that in [H1].

5.1 Classical Theory

5.1.1 Hochschild cochain complex of an associative algebra

Recall that for an associative algebra A over $\mathbb{C}$ and an A-bimodule M, the set of linear maps from $A^{\otimes n}$ to M, namely $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M)$, is defined to be the n-th Hochschild cochain complex with coefficients in M, for each natural number n. When $n = 0$, $A^{\otimes 0}$ is identified with the base field $\mathbb{C}$ and thus the zero-th Hochschild cochain complex is canonically isomorphic to M. When A is commutative and the the right action of A on M is identical to the left action, the set $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M)$ is referred as Harrison cochain complex.

For each natural number n, the coboundary map

$$\delta^n : \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M) \to \mathrm{Hom}_{\mathbb{C}}(A^{\otimes (n+1)}, M)$$

is defined by the following formula

$$\begin{aligned}(\delta^n f)(a_0 \otimes \cdots \otimes a_n)\\ = a_0 \cdot f(a_1 \otimes \cdots \otimes a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1}) \cdot a_n\end{aligned}$$

To see that the sequence $(\mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M), \delta^n)$ form a cochain complex, we need to verify that for each $n \in \mathbb{Z}_+$,

$$\delta^n \circ \delta^{n-1} = 0.$$

Let $f \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n})$. We compute as follows:

$$
\begin{aligned}
&\delta^{n-1}(\delta^n f)(a_1 \otimes \cdots \otimes a_{n+2})\\
=&a_1 \delta^n f(a_2 \otimes \cdots \otimes a_{n+2}) + \sum_{i=1}^{n+1} (-1)^i \delta^n f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2})\\
&+ (-1)^{n+2} \delta^n f(a_1 \otimes \cdots \otimes a_{n+1}) a_{n+2}\\
=&a_1(a_2 f(a_3 \otimes \cdots \otimes a_{n+2}) && (1)\\
&+ \sum_{i=2}^{n+1} (-1)^{i+1} a_1 f(a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}) && (2)\\
&+ (-1)^{n+3} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) a_{n+2}) && (3)\\
&+ (-1) a_1 a_2 f(a_3 \otimes \cdots \otimes a_{n+2}) && (4)\\
&+ \sum_{i=2}^{n+1} (-1)^i a_1 f(a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}) && (5)\\
&+ \sum_{i=3}^{n+1} (-1)^i \sum_{j=1}^{i-2} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}) && (6)\\
&+ \sum_{i=2}^{n+1} (-1)^i (-1)^{i-1} f(a_1 \otimes \cdots \otimes a_{i-1} a_i a_{i+1} \otimes \cdots \otimes a_{n+2}) && (7)\\
&+ \sum_{i=1}^{n} (-1)^i (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} a_{i+2} \otimes \cdots \otimes a_{n+2}) && (8)\\
&+ \sum_{i=1}^{n-1} (-1)^i \sum_{j=i+2}^{n+1} (-1)^{j-1} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+2}) && (9)\\
&+ \sum_{i=1}^{n} (-1)^i (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) a_{n+2} && (10)\\
&+ (-1)^{n+1} (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} a_{n+2} && (11)\\
&+ (-1)^{n+2} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) a_{n+2} && (12)\\
&+ \sum_{j=1}^{n} (-1)^{n+2} (-1)^j f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) a_{n+2} && (13)\\
&+ (-1)^{n+2} (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} a_{n+2}, && (14)
\end{aligned}
$$

where (1), (2) and (3) comes from the first term of $\delta^{n-1}(\delta^n f)$ term, (12), (13), (14) comes from the last $\delta^{n-1}(\delta^n f)$ term, and (5) to (11) comes from the middle terms $\delta^{n-1}(\delta^n f)$. Among these middle terms, (4) and (5) come from the the first term $\delta^n f$ respectively when $i = 1$ and $i > 1$. (11) and (12) come from the last term of $\delta^n f$

respectively when $i < n+1$ and $i = n+1$. (6) through (9) come from the middle term of $\delta^n f$, where (6) and (9) deals with the case when $j < i-1$ and $j > i$, and (7) and (8) deals the the case $j = i-1$ and $j = i$.

It is easy to see that the paired sums $(1)+(4), (2)+(5), (3)+(12), (6)+(9), (7)+(8), (10)+(13), (11)+(14)$ are all zero. Thus $\delta^n(\delta^{n-1}f) = 0$ for any $f \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M)$, and therefore $(\mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M), \delta^n)$ form a cochain complex.

5.1.2 Approached by the language of operad

For the convenience of generalization to MOSVA, we will rewrite the above prove using the language of operads.

Let $M_1, ..., M_n, M, N$ be vector spaces. Let $\alpha : M_1 \otimes \cdots \otimes M_n \to M$ be a linear map. For a fixed integer i between 1 and n, let $\beta : N \to M_i$ be a linear map. We define $\alpha \circ_i \beta$ to be the map obtained by composing α with β at the i-th spot. More precisely, $\alpha \circ_i \beta$ is a linear map from $M_1 \otimes \cdots \otimes M_{i-1} \otimes N \otimes M_{i+1} \otimes \cdots \otimes M_n$ to M, such that

$$(\alpha \circ_i \beta)(m_1 \otimes \cdots \otimes m_{k-1} \otimes x \otimes m_{i+1} \otimes \cdots \otimes m_n) = \alpha(m_1 \otimes \cdots \otimes m_{i-1} \otimes \beta(x) \otimes m_{i+1} \otimes \cdots \otimes m_n)$$

for any $m_1 \in M_1, ..., m_{i-1} \in M_{i-1}, x \in N, m_{i+1} \in M_{i+1}, ...m_n \in M_n$,

Now for the associative algebra A, let $E_A : A \otimes A \to A$ be the multiplication map, i.e. $E_A(a_1 \otimes a_2) = a_1 a_2$ for $a_1, a_2 \in A$. For the A-bimodule M, let $E_M^l : A \otimes M \to M$ and $E_M^r : M \otimes A \to M$ be the map given by the left and right action of A on M, i.e, $E_M^l(a \otimes m) = am, E_M^r(m \otimes a) = ma$ for $a \in A, m \in M$. Let $f \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, M)$, i.e., f is a linear map from $A^{\otimes} n$ to M. Then the δ map can be written as

$$\delta f = E_M^l \circ_2 f + \sum_{i=1}^{n} (-1)^i f \circ_i E_A + (-1)^{n+1} E_M^r \circ_1 f$$

Using this notation, we compute $\delta^2 f$ as follows:

$$\begin{aligned}
\delta^2 f =& E_M^l \circ_2 \delta f + \sum_{i=1}^{n+1} (-1)^i \delta f \circ_i E_A + (-1)^{n+2} E_M^r \circ_1 \delta f \\
=& E_M^l \circ_2 (E_M^l \circ_2 f) + \sum_{j=1}^{n} (-1)^j E_M^l \circ_2 (f \circ_j E_A) + (-1)^{n+1} E_M^l \circ_2 (E_M^r \circ_1 f) \\
&+ \sum_{i=1}^{n+1} (-1)^i \left((E_M^l \circ_2 f) \circ_i E_A + \sum_{j=1}^{n} (-1)^j (f \circ_j E_A) \circ_i E_A + (-1)^{n+1} (E_M^r \circ_1 f) \circ_i E_A \right)
\end{aligned}$$

$$+(-1)^{n+2}\left(E^r_M\circ_1(E^L_M\circ_2 f)+\sum_{j=1}^{n}(-1)^j E^r_M\circ_1(f\circ_j E_A)+(-1)^{n+1}E^r_M\circ_1(E^r_M\circ_1 f)\right)$$

We rearrange the terms and indexes, to write the above as

$$\delta^2 f = E^l_M\circ_2(E^l_M\circ_2 f)+\sum_{i=1}^{n}(-1)^i E^l_M\circ_2(f\circ_i E_A)+\sum_{i=1}^{n+1}(-1)^i(E^l_M\circ_2 f)\circ_i E_A \tag{1}$$

$$+(-1)^{n+1}E^l_M\circ_2(E^r_M\circ_1 f)+(-1)^{n+2}E^r_M\circ_1(E^l_M\circ_2 f) \tag{2}$$

$$+\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^i(-1)^j(f\circ_j E_A)\circ_i E_A \tag{3}$$

$$+\sum_{i=1}^{n+1}(-1)^{n+1+i}(E^r_M\circ_1 f)\circ_i E_A+\sum_{i=1}^{n}(-1)^{n+2+i}E^r_M\circ_1(f\circ_i E_A)-E^r_M\circ_1(E^r_M\circ_1 f) \tag{4}$$

We argue that all (1), (2), (3), (4) are zero.

For (1), note that

$$E^l_M\circ_2(E^l_M\circ_2 f)=(E^l_M\circ_2 f)\circ_1 E_A$$

so the sum of the first term and the $i=1$ term in the third summation cancel out. Also note that

$$E^l_M\circ_2(f\circ_i E_A)=(E^l_M\circ_2)\circ_{i+1}E_A$$

So the second sum and the third sum without $i=1$ differs by an index shift and a (-1) factor. That way they cancels out.

For (2), note that

$$E^l_M\circ_2(E^r_M\circ_1 f)=E^r_M\circ_1(E^l_M\circ_2 f)$$

So they cancel out.

For (3), note that if $j\le i-1$, then

$$(f\circ_j E_A)\circ_i E_A=(f\circ_{i-1}E_A)\circ_j E_A$$

and if $j\ge i$, then

$$(f\circ_j E_A)\circ_i E_A=(f\circ_i E_A)\circ_{j+1}E_A.$$

This hints that we should write (3) into two parts

$$\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(-1)^{i+j}(f\circ_j E_A)\circ_i E_A+\sum_{i=1}^{n}\sum_{j=i}^{n}(-1)^{i+j}(f\circ_j E_A)\circ_i E_A$$

Here the first sum starts from $i = 2$ because when $i = 1$, the inner sum does not exist. Similarly the second sum ends at $i = n$ because when $i = n + 1$, the inner sum does not exist. We compute the first sum as follows

$$\begin{aligned}
&\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(-1)^{i+j}(f \circ_j E_A) \circ_i E_A \\
=&\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(-1)^{i+j}(f \circ_{i-1} E_A) \circ_j E_A && \text{use the identity above} \\
=&\sum_{j=1}^{n}\sum_{i=j+1}^{n+1}(-1)^{i+j}(f \circ_{i-1} E_A) \circ_j E_A && \text{change the order of summation} \\
=&\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}(-1)^{i+j}(f \circ_{j-1} E_A) \circ_i E_A && \text{interchange } i \text{ and } j \\
=&\sum_{i=1}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}(f \circ_j E_A) \circ_i E_A && \text{shift the index } j
\end{aligned}$$

So the first sum is precisely the negative of the second sum. Thus the two sums add up to be zero.

For (4), Note that

$$(E_M^r \circ_1 f) \circ_{n+1} E_A = E_M^r \circ_1 (E_M^r \circ_1 f)$$

so the $(n + 1)$-th in the first sum cancels out with the third term. Also note that

$$(E_M^r \circ_1 f) \circ_i E_A = E_M^r \circ_1 (f \circ_i E_A)$$

The rest of the first sum cancels out with the second sum.

So we managed to prove $\delta^2 = 0$ with the language of operads. As we will see, it will be easier to generalize this argument to MOSVA.

Elements in $\text{Ker}\delta^n$ are called n-th cocycles. Elements in $\text{Im}\delta^{n-1}$ are called n-th coboundaries. The quotient of $\text{Ker}\delta^n$ modulo $\text{Im}\delta^{n-1}$ is called the n-th Hochschild cohomology group.

To construct an analogue of MOSVAs, the main challenge is to figure out the appropriate analogue of chain complexes, and take care of the parameter appropriately when we perform the related operations. Most of the hard work has been done by Huang in [H1], where he constructed the analogue of Harrison cohomology for vertex algebras. Here we develop the analogue of Hochschild cohomology of MOSVA.

5.2 $\overline{W}$-valued rational functions

Throughout this chapter, V is a MOSVA; $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ is a V-bimodule that is not necessarily grading-restricted; $W' = \coprod_{n\in\mathbb{C}} W_{[n]}^*$ is the graded dual of W. We shall assume that all the pole-order conditions hold for V and W.

5.2.1 Definition and basic properties

For $n \in \mathbb{Z}_+$, the configuration spaces is following region in $\mathbb{C}^n$

$$F_n\mathbb{C} = \{(z_1, ..., z_n) \in \mathbb{C}^n : z_i \neq z_j, i \neq j\}$$

We use $\overline{W}$ to denote the algebraic completion $\prod_{n\in\mathbb{C}} W_{[n]}$ of W. Note that the dual $(W')^*$ of W' does not coincide with $\overline{W}$. Also note that any homogeneous linear map $L : W \to W$ extends to a map $\overline{W} \to \overline{W}$ by the formal linearity

$$L(\overline{w}) = L\left(\sum_{k\in\mathbb{C}} \pi_k \overline{w}\right) = \sum_{k\in\mathbb{C}} L(\pi_k \overline{w})$$

where π_k is the projection of W onto $W_{[k]}$. More generally, any linear map $L : W \to W$ that is a finite linear combination of homogeneous linear maps can be extended to $L : \overline{W} \to \overline{W}$. For convenience, we will not introduce new notations to distinguish the extended map from the original map.

Definition 5.2.1. For $n \in \mathbb{Z}_+$, we consider the configuration space

$$F_n\mathbb{C} = \{(z_1, ..., z_n) \in \mathbb{C}^n : z_i \neq z_j, i \neq j\}$$

A *$\overline{W}$-valued rational function in $z_1, ..., z_n$ with the only possible poles at $z_i = z_j, i \neq j$* is a map

$$f : F_n\mathbb{C} \to \overline{W}$$

$$(z_1, ..., z_n) \mapsto f(z_1, ..., z_n)$$

such that

1. For any $w' \in W'$,

$$\langle w', f(z_1, ..., z_n)\rangle$$

is a rational function in $z_1, ..., z_n$ with the only possible poles at $z_i = z_j, i \neq j$.

2. There exists integers $p_{ij}, 1 \leq i < j \leq n$ and a formal series $g(x_1, ..., x_n) \in W[[x_1, ..., x_n]]$, such that for every $w' \in W'$ and $(z_1, ..., z_n) \in F_n\mathbb{C}$,

$$\prod_{1\leq i<j\leq n} (z_i - z_j)^{p_{ij}} \langle w', f(z_1, ..., z_n)\rangle = \langle w', g(z_1, ..., z_n)\rangle$$

as a polynomial function.

For simplicity, we will simply call such maps $\overline{W}$*-valued rational function* when there is no confusion. The space of all such functions will be denoted by $\widetilde{W}_{z_1,...,z_n}$.

Remark 5.2.2. From the second condition, we know that the order of poles of the rational function $\langle w', f(z_1, ..., z_n)\rangle$ is independent of the choice of w'. So for every $w' \in W'$,

$$(z_1, ..., z_n) \mapsto \prod_{1\leq i<j\leq n} (z_i - z_j)^{p_{ij}} \langle w', f(z_1, ..., z_n)\rangle$$

is a holomorphic (in fact, polynomial) function on $\mathbb{C}^n$, which can be expanded as a multiple power series

$$\sum_{i_1,...,i_n=0}^{\infty} a_{i_1...i_n}(w') z_1^{i_1} \cdots z_n^{i_n}$$

For each $i_1, ..., i_n \in \mathbb{N}$, $w' \mapsto a_{i_1...i_n}(w')$ is an element in $(W')^*$. The second condition further specifies that there exists $b_{i_1...i_n} \in W$, such that $a_{i_1...i_n}(w') = \langle w', b_{i_1...i_n}\rangle$. Thus, $\prod_{1\leq i<j\leq n}(z_i - z_j)^{p_{ij}} f(z_1, ..., z_n)$ can be expanded as

$$\sum_{i_1,...,i_n=0}^{\infty} b_{i_1...i_n} z_1^{i_1} \cdots z_n^{i_n} \in W[[z_1, ..., z_n]]$$

and therefore, $f(z_1, ..., z_n)$ can be expanded as

$$\frac{\sum\limits_{i_1,...,i_n=0}^{\infty} b_{i_1...i_n} z_1^{i_1} \cdots z_n^{i_n}}{\prod\limits_{1\leq i<j\leq n} (z_i - z_j)^{p_{ij}}} \in W[[z_1, ..., z_n]][(z_1 - z_2)^{-1}, ..., (z_{n-1} - z_n)^{-1}]$$

For $1 \leq i < j \leq n$, one can further expand the negative powers of $z_i - z_j$ as a power series in z_j and multiply them out. It is clear that in the resulting series

$$\sum_{k_1,...,k_n\in\mathbb{Z}} f_{k_1...k_n} z_1^{k_1} \cdots z_n^{k_n} \tag{5.1}$$

each coefficient $f_{k_1...k_n}$ is a finite sum of various $b_{i_1...i_n}$'s. Thus $f_{k_1...k_n} \in W$. So (5.1) is a series in $W[[z_1, z_1^{-1}, ..., z_n, z_n^{-1}]]$ that converges absolutely to $f(z_1, ..., z_n)$ in the region

$$\{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1| > \cdots > |z_n|\}$$

We will also consider the expansion of $f(z_1, ..., z_n)$ in other regions. In all the regions that arise in our applications, all the coefficients of the corresponding series sit in W.

In an earlier draft of the paper, f takes value in the larger space $(W')^*$. The following observation by Huang says that we can use $\overline{W}$ instead of $(W')^*$:

Proposition 5.2.3. *Let $f : F_n\mathbb{C} \to (W')^*$ satisfying the two conditions in Definition 5.2.1. In addition, assume that there exists a complex number C, such that for every $i_1, ..., i_n \in \mathbb{N}$, the coefficient $b_{i_1,...,i_n}$ of the power $x_1^{i_1} \cdots x_n^{i_n}$ in the series $g(x_1, ..., x_n)$ are homogeneous, with*

$$\text{wt } b_{i_1...i_n} - i_1 - \cdots - i_n = C$$

Then f takes value in $\overline{W}$.

Proof. We write

$$g(z_1, ..., z_n) = \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_n=k} b_{i_1,...,i_n} z_1^{i_1} \cdots z_n^{i_n}$$

Note that for every $k \in \mathbb{N}$,

$$\sum_{i_1+\cdots+i_n=k} b_{i_1,...,i_n} z_1^{i_1} \cdots z_n^{i_n}$$

is a finite sum of homogeneous elements in W of weight $k + C$. As k varies, the weight of these elements varies. Thus for fixed $(z_1, ..., z_n) \in F_n\mathbb{C}$, $g(z_1, ..., z_n)$ is an infinite sum of homogeneous elements in W, thus an element in $\overline{W}$. As $f(z_1, ..., z_n)$ is simply a quotient of $g(z_1, ..., z_n)$ and products of $(z_i - z_j)$, the same holds for $f(z_1, ..., z_n)$. $\square$

Remark 5.2.4. The arguments above can be easily modified to give a much simpler proof to Proposition 3.1.23.

Remark 5.2.5. It is possible to develop a cohomology theory with $(W')^*$-valued rational functions that do not satisfy the second condition. We choose not to do that because we do not need to consider such general MOSVAs and modules.

Proposition 5.2.6. *Let $n \in \mathbb{Z}_+$ and take $l = 0, ..., n$. Let $u_1, ..., u_n \in V$ and $w \in W$ satisfying the condition that $\forall u \in V, Y_W^L(u, x)w \in W[[x]]$, $Y_W^{s(R)}(u, x)w \in W[[x]]$. Then for every $w' \in W'$, the series*

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w\rangle \tag{5.2}$$

converges absolutely when $|z_1| > \cdots > |z_n|$ to a rational function with the only possible poles at $z_i = z_j, 1 \leq i < j \leq n$.

Proof. From rationality we know that the series (5.2) converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function with the only possible poles at $z_i = 0, i = 1, ..., n, z_i = z_j, 1 \leq i < j \leq n$. From the assumption, we see that the lowest power of z_n is nonnegative. Therefore, z_n is allowed to take zero. So $z_n = 0$ s not a pole.

Assume that $z_{n-1} = 0$ is a pole. By associativity, the series

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(Y_V^s(u_{n-1}, z_{n-1} - z_n)u_n, z_n)w\rangle,$$

is obtained by the expanding some rational function in some certain region, during which one of the steps expands the negative powers of $z_{n-1} = z_n + (z_{n-1} - z_n)$ as a series with positive powers $z_{n-1} - z_n$. So there should be infinitely many negative powers of z_n. However, since $Y_W^{s(R)}(u, z_n)w$ has no negative powers of z_n, in particular, for $u = (Y_V)_k(u_{n-1})u_n$ with any $k \in \mathbb{Z}$. Thus this series has no negative powers of z_n. So it is impossible for $z_{n-1} = 0$ to be a pole.

Similarly, assume $z_{n-2} = 0$ is a pole, we use the associativity again to see that

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(Y_V^s(u_{n-2}, z_{n-2} - z_n)Y_V^s(u_{n-1}, z_{n-1} - z_n)u_n, z_n)w\rangle,$$

is obtained by the expanding some rational function in some certain region, during which one of the steps expands the negative powers of $z_{n-2} = z_n + (z_{n-2} - z_n)$ as a series with positive powers $z_{n-2} - z_n$. So there should be infinitely many negative powers of z_n, which is not possible.

Similarly one can argue that $z_{n-3} = 0$ is not a pole, ..., $z_{l+1} = 0$ is not a pole. To see that z_l is not a pole, we use the commutativity of Y_W^L and $Y_W^{s(R)}$ to move all the

Y_W^L to the right. The resulting series

$$\langle w', Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) w \rangle$$

converges absolutely when $|z_{l+1}| > \cdots > |z_n| > |z_1| > \cdots > |z_l| > 0$ to the same rational function as (5.2) does. As $Y_W^L(u, z_l)w$ has no negative powers of z_l for every u, we then see that z_l is allowed to take zero and thus $z_l = 0$ is not a pole. Then we apply associativity of Y_W^L and argue similarly that $z_{l-1} = 0$ is not a pole, ..., z_1 is not a pole. □

Notation 5.2.7. We denote the rational function that the series

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w \rangle$$

converges to by

$$R\left(\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w \rangle \right).$$

By the previous proposition, it is of the form

$$\frac{h(z_1, ..., z_n)}{\prod\limits_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}}}$$

for some polynomial $h(z_1, ..., z_n)$ and some integers $p_{ij}, 1 \leq i < j \leq n$. The polynomial depends on the choice of $w' \in W', w \in W, u_1, ..., u_n \in V$. But since V and W satisfies the pole-order condition, for each $1 \leq i < j \leq n$, the integer p_{ij} depends only on u_i and u_j. It is important that $R\left(\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w \rangle \right)$ is defined whenever $z_i \neq z_j, 1 \leq i < j \leq n$. The inequality $|z_1| > \cdots > |z_n|$ is not necessary.

Notation 5.2.8. For every $(z_1, ..., z_n) \in F_n\mathbb{C}$, the linear functional

$$w' \mapsto R\left(\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w \rangle \right)$$

determines an element in $(W')^*$ that will be denoted by

$$E\left(Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w \right).$$

As will be seen soon, this element is indeed in $\overline{W}$. It is important that this element of $\overline{W}$ is defined whenever $z_i \neq z_j, 1 \leq i < j \leq n$. The inequality $|z_1| > \cdots > |z_n|$ is not necessary.

Remark 5.2.9. The E-notation was introduced by Huang in [H1]. Instead of dealing with the series, we are dealing with the holomorphic function obtained by the analytic extension of the sum of the series. With the E-notation, the commutativity of Y_W^L and $Y_W^{s(R)}$ can now be expressedas

$$E(Y_W^L(u_1, z_1)Y_W^{s(R)}(u_2, z_2)w) = E(Y_W^{s(R)}(u_2, z_2)Y_W^L(u_1, z_1)w)$$

Notice that the series in the left-hand-side only makes sense in $|z_1| > |z_2| > 0$, and the series in the right-hand-side only makes sense in $|z_2| > |z_1| > 0$. So we will not be able to find $z_1, z_2 \in \mathbb{C}$ such that $Y_W^L(u_1, z_1)Y_W^{s(R)}(u_2, z_2)w$ and $Y_W^{s(R)}(u_2, z_2)Y_W^L(u_1, z_1)w$ are equal as elements in $\overline{W}$. However, as they both converge to a common rational function that determines an element in $\overline{W}$ defined for every $(z_1, z_2) \in F_2\mathbb{C}$.

Example 5.2.10. Let V be a MOSVA and W be a V-bimodule. Assume that both V and W satisfies the pole-order condition. Fix $n \in \mathbb{Z}_+$ and $l \in \mathbb{N}$ such that $0 \leq l \leq n$. For every $w \in W$ such that for every $u \in V$, $Y_W^L(u, x)w \in W[[x]], Y_W^{s(R)}(u, x)w \in W[[x]]$, and for every $u_1, ..., u_n \in V$, the map from $F_n\mathbb{C}$ to $\overline{W}$ defined by

$$(z_1, ..., z_n) \mapsto E\left(Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l)Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w\right) \quad (5.3)$$

is a $\overline{W}$-valued rational function in $z_1, ..., z_n$ with the only possible poles at $z_i = z_j$, $1 \leq i < j \leq n$.

Proof. The first condition is seen from the discussions above. The second condition follows from Theorem 3.3.10. For homogeneous $u_1, ..., u_n \in V, w \in W$, one can computes that the power series

$$\prod_{1\leq i<j\leq n} (z_i - z_j)^{p_{ij}} Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l)Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w$$

satisfies the the conditions in Proposition 5.2.3 (C can be chosen as $\sum\limits_{i=1}^{n}$ wt $u_i +$ wt $w - \sum\limits_{1\leq i<j\leq n} p_{ij}$.) Thus the rational function in question takes value in $\overline{W}$. $\square$

Notation 5.2.11. We will use the notation

$$E_W^{(l,n-l)}(u_1 \otimes \cdots \otimes u_n; w)$$

to denote the rational function (5.3), with $n \in \mathbb{Z}_+, l \in \mathbb{N}$, $u_1, ..., u_n \in V$ and $w \in W$ chosen the same way as in the previous example. In particular, $E_W^{(l,n-l)}(u_1 \otimes \cdots \otimes u_n; w) \in \widetilde{W}_{z_1 \ldots z_n}$. It is also clear that when $W = V$ and $w = \mathbf{1}$, $E_V^{(l,n-l)}(u_1, ..., u_n; \mathbf{1})$ is the same for every $l = 0, ..., n$. In this case we will use the notation

$$E_V^{(n)}(u_1, ..., u_n)$$

without explicitly mentioning l and $\mathbf{1}$.

5.2.2 Series of $\overline{W}$-valued rational functions

In this paper we will be frequently dealing with series of $\overline{W}$-valued rational function. Here we illustrate some examples. Let $(z_1, ..., z_n) \in F_n\mathbb{C}$. Let $u_1, ..., u_n \in V$ and $w \in W$ such that $Y_W^L(u, x)w \in W[[x]]$ and $Y_W^R(w, x)u \in W[[x]]$ for every $u \in V$. Let $v \in V$ and x be a formal variable. Note that the components $(Y_W^L)_n(v)$ of the vertex operator $Y_W^L(v, x)$ are sums of homogeneous linear operators on W that extends naturally to $\overline{W}$. In particular, they acts on

$$\begin{aligned}\overline{w} &= (E_W^{(l,n-l)}(u_1 \otimes \cdots \otimes u_n; w))(z_1, ..., z_n) \\ &= E(Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w)\end{aligned}$$

Thus the vertex operator $Y_W^L(u, x)$ acting on $\overline{w}$ is the following *single* series of elements in $\overline{W}$:

$$\begin{aligned}Y_W^L(v, x)\overline{w} &= \sum_{n\in\mathbb{Z}} (Y_W^L)_n(u)\overline{w}x^{-n-1} \\ &= \sum_{n\in\mathbb{Z}} (Y_W^L)_n(u) E(Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w)x^{-n-1}\end{aligned}$$

If we pair the above with $w' \in W'$, then the coefficient of x^{-n-1} in $\langle w', Y_W^L(v, x)\overline{w}\rangle$ is just

$$\langle w', (Y_W^L)_n(v) E(Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n)w)\rangle$$

which is a rational function in $z_1, ..., z_n$ with the only possible poles at $z_i = z_j, 1 \leq i < j \leq n$. Moreover, if n is sufficiently negative, the coefficient is zero. Thus the series $\langle w', Y_W^L(v, x)\overline{w}\rangle$ has at most finitely many positive powers.

Proposition 5.2.12. *Let $u_1, ..., u_n \in V, w \in W$ be chosen as above. Then the single series*

$$Y_W^L(v, z)E(Y_W^L(u_1, z_1)\cdots Y_W^L(u_l, z_l)Y_W^{s(R)}(u_{l+1}, z_{l+1})\cdots Y_W^{s(R)}(u_n, z_n)w)$$

converges absolutely when

$$|z| > |z_i|, i = 1, ..., n$$

to the $\overline{W}$-valued rational function

$$E(Y_W^L(v, z)Y_W^L(u_1, z_1)\cdots Y_W^L(u_l, z_l)Y_W^{s(R)}(u_{l+1}, z_{l+1})\cdots Y_W^{s(R)}(u_n, z_n)w)$$

Proof. For every $w' \in W'$, we know that the series

$$\langle w', Y_W^L(v, z)Y_W^L(u_1, z_1)\cdots Y_W^L(u_l, z_l)Y_W^{s(R)}(u_{l+1}, z_{l+1})\cdots Y_W^{s(R)}(u_n, z_n)w)\rangle$$

converges absolutely when $|z| > |z_1| > \cdots > |z_n|$ to the rational function with the only possible poles at $z = z_i, i = 1, ..., n, z_i = z_j, 1 \leq i < j \leq n$. For each fixed $n \in \mathbb{Z}$, the coefficient of z^{-n-1} is precisely the sum of the series

$$\langle w', (Y_W^L)_n(v)Y_W^L(u_1, z_1)\cdots Y_W^L(u_l, z_l)Y_W^{s(R)}(u_{l+1}, z_{l+1})\cdots Y_W^{s(R)}(u_n, z_n)w)\rangle$$

in $z_1, ..., z_n$, which is the same as the coefficient of $Y_W^L(v, z)\overline{w}$. From the upper truncation of z, we know that the series is obtained by expanding the negative powers of $z - z_i$ as a power series in z_i. Thus it converges absolutely whenever $|z| > |z_i|, i = 1, ..., n$. □

Proposition 5.2.13. *Let $u_1, ..., u_n \in V, w \in W$ be chosen as above. Let $m \in \mathbb{Z}_+, v_1, ..., v_m \in V$. Then for each $p = 0, ..., m$, the series*

$$Y_W^L(v_1, z_1)\cdots Y_W^L(v_p, z_p)Y_W^{s(R)}(v_{p+1}, z_{p+1})\cdots Y_W^{s(R)}(v_m, z_m)\cdot$$
$$E(Y_W^L(u_1, z_{m+1})\cdots Y_W^L(u_l, z_{m+l})Y_W^{s(R)}(u_{l+1}, z_{m+l+1})\cdots Y_W^{s(R)}(u_n, z_{m+n})w)$$

converges absolutely when

$$|z_1| > \cdots > |z_m| > |z_i|, i = m+1, ..., m+n.$$

to the $\overline{W}$-valued rational function

$$E(Y_W^L(v_1, z_1)\cdots Y_W^L(v_p, z_p)Y_W^{s(R)}(v_{p+1}, z_{p+1})\cdots Y_W^{s(R)}(v_m, z_m)\cdot$$
$$Y_W^L(u_1, z_{m+1})\cdots Y_W^L(u_l, z_{m+l})Y_W^{s(R)}(u_{l+1}, z_{m+l+1})\cdots Y_W^{s(R)}(u_n, z_{m+n})w)$$

Proof. It suffices to notice that for each $w' \in W'$, the series

$$\langle w', Y_W^L(v_1, z_1) \cdots Y_W^L(v_p, z_p) Y_W^{s(R)}(v_{p+1}, z_{p+1}) \cdots Y_W^{s(R)}(v_m, z_m) \cdot \\ E(Y_W^L(u_1, z_{m+1}) \cdots Y_W^L(u_l, z_{m+l}) Y_W^{s(R)}(u_{l+1}, z_{m+l+1}) \cdots Y_W^{s(R)}(u_n, z_{m+n}) w) \rangle$$

coincides with the expansion of the rational function

$$R(\langle w', Y_W^L(v_1, z_1) \cdots Y_W^L(v_p, z_p) Y_W^{s(R)}(v_{p+1}, z_{p+1}) \cdots Y_W^{s(R)}(v_m, z_m) \cdot \\ Y_W^L(u_1, z_{m+1}) \cdots Y_W^L(u_l, z_{m+l}) Y_W^{s(R)}(u_{l+1}, z_{m+l+1}) \cdots Y_W^{s(R)}(u_n, z_{m+n}) w) \rangle$$

in the region $\{(z_1, ..., z_{m+n}) : |z_1| > \cdots > |z_m| > |z_i|, i = m+1, ..., m+n\}$. □

Remark 5.2.14. In terms of the E-notation, we have

$$E(Y_W^L(v, z) E(Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w)) \\ = E(Y_W^L(v, z) Y_W^L(u_1, z_1) \cdots Y_W^L(u_l, z_l) Y_W^{s(R)}(u_{l+1}, z_{l+1}) \cdots Y_W^{s(R)}(u_n, z_n) w))$$

and

$$E(Y_W^L(v_1, z_1) \cdots Y_W^L(v_p, z_p) Y_W^{s(R)}(v_{p+1}, z_{p+1}) \cdots Y_W^{s(R)}(v_m, z_m) \cdot \\ E(Y_W^L(u_1, z_{m+1}) \cdots Y_W^L(u_l, z_{m+l}) Y_W^{s(R)}(u_{l+1}, z_{m+l+1}) \cdots Y_W^{s(R)}(u_n, z_{m+n}) w)) \\ = E(Y_W^L(v_1, z_1) \cdots Y_W^L(v_p, z_p) Y_W^{s(R)}(v_{p+1}, z_{p+1}) \cdots Y_W^{s(R)}(v_m, z_m) \cdot \\ Y_W^L(u_1, z_{m+1}) \cdots Y_W^L(u_l, z_{m+l}) Y_W^{s(R)}(u_{l+1}, z_{m+l+1}) \cdots Y_W^{s(R)}(u_n, z_{m+n}) w)$$

Here is another type of series of $\overline{W}$-valued rational functions that will be considered. Let $u_1, ..., u_{n+1} \in V, w \in W$ such that $Y_W^L(u, x)w \in W[[x]]$ and $Y_W^R(w, x)u \in W[[x]]$. Let $(\zeta, z_3, ..., z_n) \in F_n\mathbb{C}$.

$$E_W^{(l,n-l)}(Y_V(u_1, z_1 - \zeta) Y_V(u_2, z_2 - \zeta)\mathbf{1} \otimes u_3 \otimes \cdots \otimes u_{n+1}; w)(\zeta, z_3, ..., z_{n+1})$$

which expands as

$$\sum_{k_1, k_2 \in \mathbb{Z}} E_W^{(l,n-l)}((Y_V)_{k_1}(u_1)(Y_V)_{k_2}(u_2)\mathbf{1} \otimes u_3 \otimes \cdots \otimes u_{n+1}; w)(\zeta, z_3, ..., z_{n+1})(z_1 - \zeta)^{-k_1 - 1}(z_2 - \zeta)^{-k_2 - 1}$$

For each $k_1, k_2 \in \mathbb{Z}$, the coefficients of $(z_1 - \zeta)^{-k_1-1}(z_2 - \zeta)^{-k_2-1}$ is a $\overline{W}$-valued rational function in $\zeta, z_3, ..., z_{n+1}$.

Proposition 5.2.15. *Let $u_1, ..., u_{n+1} \in V$ and $w \in W$ be chosen as above. Then the series*

$$E_W^{(l,n-l)}(Y_V(u_1, z_1 - \zeta)Y_V(u_2, z_2 - \zeta)\mathbf{1} \otimes u_3 \otimes \cdots \otimes u_{n+1}; w)(\zeta, z_3, ..., z_{n+1}) \qquad (5.4)$$

converges absolutely when

$$|z_3 - \zeta| > |z_1 - \zeta| > |z_2 - \zeta|$$

to the $\overline{W}$-valued rational function

$$E(Y_W^L(u_1, z_1) \cdots Y_W^L(u_{l+1}, z_{l+1})Y_W^{s(R)}(u_{l+2}, z_{l+2}) \cdots Y_W^{s(R)}(u_{n+1}, z_{n+1})w)$$

Proof. For every $w' \in W'$, we know that the series

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_{l+1}, z_{l+1})Y_W^{s(R)}(u_{l+2}, z_{l+2}) \cdots Y_W^{s(R)}(u_{n+1}, z_{n+1})w\rangle$$

converges absolutely when $|z_1| > \cdots > |z_{n+1}|$ to a rational function with the only possible poles at $z_i = z_j, 1 \leq i < j \leq n+1$. By associativity and Lemma *.*.* in [Q1], we know that the series

$$\langle w', Y_W^L(Y_V(u_1, z_1-\zeta)Y_V(u_2, z_2-\zeta)\mathbf{1}, \zeta)Y_W^L(u_3, z_3) \cdots Y_W^L(u_{l+1}, z_{l+1})Y_W^{s(R)}(u_{l+2}, z_{l+2}) \cdots Y_W^{s(R)}(u_{n+1}, z_{n+1})u$$

with variables $z_1 - \zeta, z_2 - \zeta, \zeta, z_3, ..., z_n$ that expands as

$$\sum_{k_1,...,k_{n+2}\in\mathbb{Z}} \langle w', (Y_W^L)_{k_3}((Y_V)_{k_1}(u_1)(Y_V)_{k_2}(u_2)\mathbf{1})(Y_W^L)_{k_4}(u_3) \cdots (Y_W^L)_{k_{l+2}}(u_{l+1})$$
$$\cdot (Y_W^{s(R)})_{k_{l+3}}(u_{l+2}) \cdots (Y_W^{s(R)})_{k_{n+2}}(u_{n+1})w\rangle(z_1 - \zeta)^{-k_1-1}(z_2 - \zeta)^{-k_2-1}\zeta^{-k_3-1}z_3^{-k_4-1} \cdots z_{n+1}^{-k_{n+}}$$

is obtained from the following expansion of the rational function:

1. Expand the negative powers of $z_1 - z_2 = z_1 - \zeta - (z_2 - \zeta)$ as a power series of $z_2 - \zeta$.

2. For $s = 1, 2$ and $j = 3, ..., n$, expand the negative powers of $z_s - z_j = \zeta + (z_s - \zeta + z_j)$ as a power series of $z_s - \zeta + z_j$, then further expand the positive powers of $z_s - \zeta + z_j$ as polynomials of $z_s - \zeta$ and z_j. Note that this expansion is the same as first expand the negative powers of $z_s - z_j = (z_j - \zeta) + (z_s - \zeta)$ as power series of $(z_s - \zeta)$, then further expand all the negative powers of $z_j - \zeta$ as power series of z_j.

3. For $3 \leq i < j \leq n$, expand the negative powers of $z_i - z_j$ as power series of z_j.

Thus the series converges absolutely when

$$|z_1 - \zeta| > |z_2 - \zeta|, |\zeta| > |z_3| > \cdots > |z_n|,$$
$$|z_j - \zeta| > |z_1 - \zeta|, j = 3, ..., n.$$

The result then follow by noticing that the coefficients of the series (5.4), paired with w', are precisely the partial sums of the above series with respect to $k_3, ..., k_{n+2}$. In particular, the series (5.4) is obtained from the following expansions of the rational function

$$R(\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_{l+1}, z_{l+1}) Y_W^{s(R)}(u_{l+2}, z_{l+2}) \cdots Y_W^{s(R)}(u_{n+1}, z_{n+1}) w\rangle)$$

1. Expand the negative powers of $z_1 - z_2 = z_1 - \zeta - (z_2 - \zeta)$ as a power series of $z_2 - \zeta$.

2. For $s = 1, 2$ and $j = 3, ..., n$, expand the negative powers of $z_s - z_j = (z_j - \zeta) + (z_s - \zeta)$ as power series of $(z_s - \zeta)$.

Thus the series (5.4) converges absolutely when $|z_3 - \zeta| > |z_1 - \zeta| > |z_2 - \zeta|$. $\square$

Proposition 5.2.16. *Let $m, n \in \mathbb{Z}_+$. Let $\alpha_1, ..., \alpha_n$ be chosen such that $\alpha_1 + \cdots + \alpha_n = m + n$. Then the series*

$$E_W^{(l,n-l)}(Y_V(u_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(u_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1} \otimes \cdots \otimes Y_V(u_1^{(n)}, z_1^{(n)} - \zeta_n) \cdots Y_V(u_{\alpha_n}^{(n)}, z_{\alpha_n}^{(n)} - \zeta_n)\mathbf{1})(\zeta_1, ..., \zeta_n)$$

converges absolutely when

$$|\zeta_i - \zeta_j| > |z_s^{(i)} - \zeta_i| + |z_t^{(j)} - \zeta_j|, 1 \leq i < j \leq n, s = 1, ..., \alpha_i, t = 1, ..., \alpha_j$$
$$|z_s^{(i)} - \zeta_i| > |z_t^{(i)} - \zeta_i|, i = 1, ..., n, 1 \leq s < t \leq \alpha_i.$$

to the $\overline{W}$-valued rational function

$$E(Y_W^L(u_1^{(1)}, z_1^{(1)}) \cdots Y_W^L(u_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)}) \cdots Y_W^L(u_1^{(l)}, z_1^{(l)}) \cdots Y_W^L(u_{\alpha_l}^{(l)}, z_{\alpha_l}^{(l)})$$
$$\cdot Y_W^{s(R)}(u_1^{(l+1)}, z_1^{(l+1)}) \cdots Y_W^{s(R)}(u_{\alpha_{l+1}}^{(l+1)}, z_{\alpha_{l+1}}^{(l+1)}) \cdots Y_W^{s(R)}(u_1^{(n)}, z_1^{(n)}) \cdots Y_W^{s(R)}(u_{\alpha_n}^{(n)}, z_{\alpha_n}^{(n)}) w)$$

Proof. It suffices argue similarly that the series, paired with any $w' \in W'$, is obtained from the expansion of the corresponding rational function in the region. $\square$

We end this section by a proposition dealing with the mixture of the above two types of series of $\overline{W}$-valued rational functions

Proposition 5.2.17. *Fix $m, n \in \mathbb{Z}_+$. Let $\alpha_0, \alpha_1, \cdots, \alpha_n$ be chosen such that $\alpha_0 + \alpha_1 + \cdots + \alpha_n = m + n$. Then for every $l_0 = 0, ..., \alpha_0$, the the series*

$$Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)})$$
$$\cdot E_W^{(l,n-l)}(Y_V(u_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(u_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1} \otimes \cdots \otimes Y_V(u_1^{(n)}, z_1^{(n)} - \zeta_n) \cdots Y_V(u_{\alpha_n}^{(n)}, z_{\alpha_n}^{(n)} - \zeta_n)\mathbf{1})(\zeta_1, ..$$

converges absolutely when

$$|z_1^{(0)}| > \cdots > |z_{\alpha_0}^{(0)}| > |\zeta_i| + |z_t^{(i)} - \zeta_i|, i = 1, ..., n, t = 1, ..., \alpha_i$$
$$|z_1^{(i)} - \zeta_i| > \cdots > |z_{\alpha_i}^{(i)} - \zeta_i|, i = 1, ..., n$$
$$|\zeta_i - \zeta_j| > |z_s^{(i)} - \zeta_i| + |z_t^{(j)} - \zeta_j|, 1 \leq i < j \leq n, s = 1, ..., \alpha_i, t = 1, ..., \alpha_j.$$

to the $\overline{W}$-valued rational function

$$E(Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)})$$
$$\cdot Y_W^L(u_1^{(1)}, z_1^{(1)}) \cdots Y_W^L(u_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)}) \cdots Y_W^L(u_1^{(l)}, z_1^{(l)}) \cdots Y_W^L(u_{\alpha_l}^{(l)}, z_{\alpha_l}^{(l)})$$
$$\cdot Y_W^{s(R)}(u_1^{(l+1)}, z_1^{(l+1)}) \cdots Y_W^{s(R)}(u_{\alpha_{l+1}}^{(l+1)}, z_{\alpha_{l+1}}^{(l+1)}) \cdots Y_W^{s(R)}(u_1^{(n)}, z_1^{(n)}) \cdots Y_W^{s(R)}(u_{\alpha_n}^{(n)}, z_{\alpha_n}^{(n)})w)$$

Proof. It suffices to notice that the series in question, paired with any $w' \in W'$, is obtained from the following expansions of the corresponding rational function:

1. For $s = 1, ..., \alpha_0, i = 0, ..., n, t = 1, ..., \alpha_i$, expand the negative powers $z_s^{(0)} - z_t^{(i)}$ as power series of $z_t^{(i)}$. When $i \geq 1$, one further expands the positive powers of $z_t^{(i)} = \zeta_i + (z_t^{(i)} - \zeta_i)$ as polynomials of ζ_i and $z_t^{(i)}$.

2. For $i = 1, ..., n, 1 \leq s < t \leq \alpha_i$, expand the negative powers of $z_s^{(i)} - z_t^{(i)} = z_s^{(i)} - \zeta_i - (z_t^{(i)} - \zeta_i)$ as power series of $(z_t^{(i)} - \zeta_i)$.

3. For $1 \leq i < j \leq n, s = 1, ..., \alpha_i, t = 1, ..., \alpha_j$, expand the negative powers of $z_s^{(i)} - z_t^{(j)} = (\zeta_i - \zeta_j) + (z_s^{(i)} - \zeta_i - z_t^{(j)} + \zeta_j)$ as power series of $(z_s^{(i)} - \zeta_i - z_t^{(j)} + \zeta_j)$,

then further expand the positive powers of $(z_s^{(i)} - \zeta_i - z_t^{(j)} + \zeta_j)$ as polynomials of $(z_s^{(i)} - \zeta_i)$ and $(z_t^{(j)} - \zeta_j)$.

Thus the series in question converges absolutely when

$$|z_1^{(0)}| > \cdots > |z_{\alpha_0}^{(0)}| > |\zeta_i| + |z_t^{(i)} - \zeta_i|, i = 1, ..., n, t = 1, ..., \alpha_i$$
$$|z_1^{(i)} - \zeta_i| > \cdots > |z_{\alpha_i}^{(i)} - \zeta_i|, i = 1, ..., n$$
$$|\zeta_i - \zeta_j| > |z_s^{(i)} - \zeta_i| + |z_t^{(j)} - \zeta_j|, 1 \leq i < j \leq n, s = 1, ..., \alpha_i, t = 1, ..., \alpha_j.$$

□

5.2.3 Associativity and commutativity extended to $\overline{W}$-valued rational functions

In this subsection we consider the vertex operator action on more general $\overline{W}$-valued rational functions. Let $(z_1, ..., z_n) \in F_n\mathbb{C}$. Let $v \in V$ and x be a formal variable. Let f be a $\overline{W}$-valued rational function. Then $(Y_W^L)_n(v)$ acts on the $\overline{W}$-element

$$\overline{w} = f(z_1, ..., z_n)$$

Thus the vertex operator $Y_W^L(u, x)$ acting on $\overline{w}$ is the following *single* series of elements in $\overline{W}$:

$$\begin{aligned} Y_W^L(v, x)\overline{w} &= \sum_{n\in\mathbb{Z}} (Y_W^L)_n(u)\overline{w}x^{-n-1} \\ &= \sum_{n\in\mathbb{Z}} (Y_W^L)_n(u)f(z_1, ..., z_n)x^{-n-1} \end{aligned}$$

If we pair the above with $w' \in W'$, then the coefficient of x^{-n-1} in $\langle w', Y_W^L(v, x)\overline{w}\rangle$ is just

$$\langle w', (Y_W^L)_n(u)f(z_1, ..., z_n))\rangle$$

which is rational function in $z_1, ..., z_n$ with the only possible poles at $z_i = z_j, 1 \leq i < j \leq n$. Moreover, if n is sufficiently negative, the coefficient is zero. Thus the series $\langle w', Y_W^L(v, x)\overline{w}\rangle$ has at most finitely many positive powers.

Similarly, for $v_1, v_2 \in V$ and formal variables x_1, x_2, the series

$$Y_W^L(v_1, x_1)Y_W^L(v_2, x_2)\overline{w}$$

and

$$Y_W^L(Y_V(v_1, x_0)v_2, x_2)\overline{w}$$

are understood as *double* series of elements in $\overline{W}$

$$\begin{aligned}
Y_W^L(v_1, x_1)Y_W^L(v_2, x_2)\overline{w} &= \sum_{k_1, k_2 \in \mathbb{Z}} (Y_W^L)_{k_1}(v_1)(Y_W^L)_{k_2}(v_2)\overline{w}x_1^{-k_1-1}x_2^{-k_2-1} \\
&= \sum_{k_1, k_2 \in \mathbb{Z}} (Y_W^L)_{k_1}(v_1)(Y_W^L)_{k_2}(v_2)f(z_1, ..., z_n)x_1^{-k_1-1}x_2^{-k_2-1} \\
Y_W^L(Y_V(v_1, x_0)v_2, x_2)\overline{w} &= \sum_{k_1, k_2 \in \mathbb{Z}} (Y_W^L)_{k_1}((Y_V)_{k_1}(v_1)v_2)\overline{w}x_0^{-k_1-1}x_2^{-k_2-1} \\
&= \sum_{k_1, k_2 \in \mathbb{Z}} (Y_W^L)_{k_1}((Y_V)_{k_1}(v_1)v_2)f(z_1, ..., z_n)x_0^{-k_1-1}x_2^{-k_2-1}
\end{aligned}$$

In general, we don't know if these two series converge. But if $\overline{w}$ is chosen appropriately and one of them converges absolutely under certain conditions, then the other also converges absolutely. More precisely,

Proposition 5.2.18. *Let $v_1, v_2 \in V$. Let $f \in \widetilde{W}_{z_3,...,z_{n+2}}$ such that for every $(z_1, ..., z_{n+2}) \in F_{n+2}\mathbb{C}$ with $|z_1| > |z_2| > |z_i|, i = 3, ..., n+2$, the series*

$$Y_W^L(v_1, z_1)Y_W^L(v_2, z_2)f(z_3, ..., z_n)$$

converges absolutely to a $\overline{W}$-valued rational function. Then for every $(z_1, ..., z_{n+2}) \in F_{n+2}\mathbb{C}$ such that $|z_2| > |z_1 - z_2| + |z_i|, i = 3, ..., n+2$, the series

$$Y_W^L(Y_V(v_1, z_1 - z_2)v_2, z_2)f(z_3, ..., z_n)$$

also converges absolutely to the same $\overline{W}$-valued rational function.

Proof. By Definition 5.2.1 and Remark 5.2.2, we know that $f(z_3, ..., z_n)$ can be expanded in the region

$$\{(z_3, ..., z_{n+2}) \in \mathbb{C}^n : |z_3| > \cdots > |z_n|\}$$

as an absolutely convergent series

$$f(z_3, ..., z_{n+2}) = \sum_{k_3, ..., k_{n+2} \in \mathbb{Z}} f_{k_3 ..., k_{n+2}} z_3^{k_3} \cdots z_{n+2}^{k_{n+2}}$$

in $W[[z_3, z_3^{-1}, ..., z_{n+2}, z_{n+2}^{-1}]]$. This expansion is obtained by expanding each negative power of $z_i - z_j$ as a power series in z_j, for $3 \le i < j \le n+2$. Thus the series is lower-truncated in z_{n+2}. The coefficient of each fixed power of z_{n+2}, as a series in $z_3, ..., z_{n+1}$, is lower-truncated in z_{n+1}. In general, for each $i = 3, ..., n+1$ and each $k_{i+1}, ..., k_{n+2} \in \mathbb{Z}$, the coefficient of $z_{i+1}^{k_{i+1}} \cdots z_{n+2}^{k_{n+2}}$, as a series in $z_3, ..., z_i$, is lower-truncated in z_i. If we pick $M_i \in \mathbb{Z}$ such that the lowest power of z_i is $-M_i$, then we can recover the coefficient of the series from the following formula

$$f_{k_3 \dots k_{n+2}} = \lim_{z_3=0} \cdots \lim_{z_{n+2}=0} \left(\frac{\partial}{\partial z_3}\right)^{k_3+M_3} \cdots \left(\frac{\partial}{\partial z_{n+2}}\right)^{k_{n+2}+M_{n+2}} (z_3^{M_3} \cdots z_{n+2}^{M_{n+2}} f(z_3, ..., z_n))$$

Now by assumption, the series

$$\begin{aligned} & Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) f(z_3, ..., z_{n+2}) \\ = & \sum_{k_1, k_2 \in \mathbb{Z}} (Y_W^L)_{k_1}(u_1) (Y_W^L)_{k_2}(u_2) f(z_3, ..., z_{n+2}) z_1^{-k_1-1} z_2^{-k_2-1} \end{aligned}$$

converges absolutely when $|z_1| > |z_2| > |z_i|, i = 3, ..., n+2$ to a $\overline{W}$-valued rational function. This means the following iterated series

$$\begin{aligned} & Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) \left(\sum_{k_3, ..., k_{n+2} \in \mathbb{Z}} f_{k_3 ..., k_{n+2}} z_3^{k_3} \cdots z_{n+2}^{k_{n+2}} \right) \\ = & \sum_{k_1, k_2 \in \mathbb{Z}} \left(\sum_{k_3, ..., k_{n+2} \in \mathbb{Z}} (Y_W^L)_{k_1}(u_1) (Y_W^L)_{k_2}(u_2) f_{k_3 ..., k_{n+2}} z_3^{k_3} \cdots z_{n+2}^{k_{n+2}} \right) z_1^{-k_1-1} z_2^{-k_2-1}, \end{aligned}$$

viewed as a double series in z_1, z_2 whose coefficients are

$$\sum_{k_3, ..., k_{n+2} \in \mathbb{Z}} (Y_W^L)_{k_1}(u_1) (Y_W^L)_{k_2}(u_2) f_{k_3 ..., k_{n+2}} z_3^{k_3} \cdots z_{n+2}^{k_{n+2}},$$

converges absolutely when $|z_1| > \cdots > |z_{n+2}|$ to the same $\overline{W}$-valued rational function. Moreover, the power of z_2 is lower-truncated. And for each fixed power of z_2, the power of z_1 in coefficient series is also lower-truncated. Thus by Lemma *.*.* in [Q1], the series

$$\sum_{k_1, ..., k_{n+2} \in \mathbb{Z}} (Y_W^L)_{k_1}(u_1) (Y_W^L)_{k_2}(u_2) f_{k_3 ..., k_{n+2}} z_1^{-k_1-1} z_2^{-k_2-1} z_3^{k_3} \cdots z_{n+2}^{k_{n+2}}$$

is precisely the expansion of the $\overline{W}$-valued rational function

$$E(Y_W^L(u_1, z_1) Y_W^L(u_2, z_2) f(z_3, ..., z_{n+2})) \tag{5.5}$$

in the region

$$\{(z_1, ..., z_{n+2}) \in \mathbb{C}^{n+2} : |z_1| > \cdots > |z_{n+2}|\}$$

In particular, the series converges absolutely in this region.

By associativity, for fixed $k_3, ..., k_{n+2}$, when $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have

$$\begin{aligned} Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)f_{k_3...,k_{n+2}} &= Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2)f_{k_3...,k_{n+2}} \\ &= \sum_{k_1,k_2\in\mathbb{Z}} (Y_W^L)_{k_1}((Y_V)_{k_2}(u_1)u_2)(z_1 - z_2)^{-k_1-1}z_2^{-k_2-1} \end{aligned}$$

Thus the series

$$\begin{aligned} &Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2) \sum_{k_3,...,k_{n+2}\in\mathbb{Z}} f_{k_3...,k_{n+2}} z_3^{k_3} \cdots z_{n+2}^{k_{n+2}} \\ &= \sum_{k_3,...,k_{n+2}\in\mathbb{Z}} \left(\sum_{k_1,k_2\in\mathbb{Z}} (Y_W^L)_{k_1}((Y_V)_{k_2}(u_1)u_2)f_{k_3...,k_{n+2}}(z_1 - z_2)^{-k_1-1}z_2^{-k_2-1} \right) z_3^{k_3} \cdots z_{n+2}^{k_{n+2}} \end{aligned}$$

viewed as a series in $z_3, ..., z_{n+2}$ whose coefficients are

$$\sum_{k_1,k_2\in\mathbb{Z}} (Y_W^L)_{k_1}((Y_V)_{k_2}(u_1)u_2)f_{k_3...,k_{n+2}}(z_1 - z_2)^{-k_1-1}z_2^{-k_2-1},$$

converges absolutely when

$$|z_1| > |z_2| > \cdots > |z_{n+2}|, |z_2| > |z_1 - z_2| > 0.$$

Moreover, for every $i = n + 2, ..., 3$ and every $k_{i+1}, ..., k_{n+2}$, the coefficient series of $z_{i+1}^{k_{i+1}} \cdots z_{n+2}^{k_{n+2}}$ is lower-truncated in z_i. One then sees from the Lemma *.*.* in [Q1] that the series

$$\begin{aligned} &Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2) \sum_{k_3,...,k_{n+2}\in\mathbb{Z}} f_{k_3...,k_{n+2}} z_3^{k_3} \cdots z_{n+2}^{k_{n+2}} \\ &= \sum_{k_1,k_2,k_3,...,k_{n+2}\in\mathbb{Z}} (Y_W^L)_{k_1}((Y_V)_{k_2}(u_1)u_2)f_{k_3...,k_{n+2}}(z_1 - z_2)^{-k_1-1}z_2^{-k_2-1}z_3^{k_3} \cdots z_{n+2}^{k_{n+2}} \end{aligned}$$

is the expansion of the $\overline{W}$-valued rational function (5.5) in the region

$$\{(z_1, ..., z_{n+2}) \in \mathbb{C}^{n+2} : |z_2| > |z_1 - z_2| + |z_3|, |z_1 - z_2| > 0, |z_3| > \cdots > |z_{n+2}|\}$$

In particular, the series converges absolutely in the region. We then sum up all $k_3, ..., k_{n+1}$, to see that the double series

$$Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2)f(z_3, ..., z_{n+2}) = \sum_{k_1,k_2\in\mathbb{Z}} (Y_W^L)_{k_1}((Y_V)_{k_2}(u_1)u_2)f(z_3, ..., z_{n+2})$$

of elements in $\overline{W}$ is precisely the expansion of the $\overline{W}$-rational function (5.5) in the region

$$\{(z_1, ..., z_{n+2}) : |z_2| > |z_1 - z_2| + |z_i|, i = 3, ..., n+2\}$$

In particular, the double series converges absolutely in the region. $\square$

Corollary 5.2.19. *For $u_1, u_2 \in V$ and $f \in \widetilde{W}_{z_3,...,z_{n+2}}$ chosen as above, we have*

$$Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)f(z_3, ..., z_{n+2}) = Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2)f(z_3, ..., z_{n+2})$$

for every $(z_1, ..., z_{n+2}) \in F_{n+2}\mathbb{C}$ such that $|z_1| > |z_2| > |z_1 - z_2| + |z_i|, i = 3, ..., n+2$. Moreover, we have

$$E(Y_W^L(u_1, z_1)Y_W^L(u_2, z_2)f(z_3, ..., z_{n+2})) = E(Y_W^L(Y_V(u_1, z_1 - z_2)u_2, z_2)f(z_3, ..., z_{n+2}))$$

where both sides are regarded as $\overline{W}$-valued rational functions in $\widetilde{W}_{z_1,...,z_{n+2}}$.

One can generalize the above conclusions to the product of any numbers of Y_W^L and $Y_W^{s(R)}$ vertex operators. For convenience, we list the conclusions we will need in this paper in the following theorem.

Theorem 5.2.20. *1. Let $u_1, u_2 \in V$, $f \in \widetilde{W}_{z_3,...,z_{n+2}}$ such that*

$$Y_W^L(u_1, z_1)Y_W^{s(R)}(u_2, z_2)f(z_3, ..., z_{n+2})$$

converges absolutely to a $\overline{W}$-valued rational function for every $(z_1, ..., z_{n+2}) \in F_{n+2}\mathbb{C}$ such that $|z_1| > |z_2| > |z_i|, i = 3, ..., n+2$. Then the series

$$Y_W^{s(R)}(u_2, z_2)Y_W^L(u_1, z_1)f(z_3, ..., z_{n+2})$$

also converges absolutely to the same $\overline{W}$-valued rational function for every $(z_1, ..., z_{n+2}) \in F_{n+2}\mathbb{C}$ such that $|z_2| > |z_1| > |z_i|, i = 3, ..., n+2$. Moreover, we have

$$E(Y_W^L(u_1, z_1)Y_W^{s(R)}(u_2, z_2)f(z_3, ..., z_{n+2})) = E(Y_W^{s(R)}(u_2, z_2)Y_W^L(u_1, z_1)f(z_3, ..., z_{n+2}))$$

as elements in $\widetilde{W}_{z_1,...,z_{n+2}}$.

2. Let $u_1, ..., u_m \in V$, $f \in \widetilde{W}_{z_{m+1},...,z_{m+n}}$ *such that*

$$Y_W^L(u_1, z_1) \cdots Y_W^L(u_m, z_m) f(z_{m+1}, ..., z_{m+n})$$

converges absolutely to a $\overline{W}$*-valued rational function for every* $(z_1, ..., z_{m+n}) \in F_{m+n}\mathbb{C}$ *such that* $|z_1| > \cdots > |z_m| > |z_i|, i = m+1, ..., m+n$. *Then the series*

$$Y_W^L(Y_V(u_1, z_1 - \zeta) \cdots Y_V(u_m, z_m - \zeta)\mathbf{1}, \zeta) f(z_{m+1}, ..., z_{m+n})$$

also converges absolutely to the same $\overline{W}$*-valued rational function whenever* $(z_1, ..., z_{m+n}) \in F_{m+n}\mathbb{C}, |\zeta| > |z_1 - \zeta| + |z_i|, i = m+1, ..., m+n, |z_1 - \zeta| > \cdots > |z_m - \zeta|$. *Moreover, we have*

$$\begin{aligned} &E(Y_W^L(u_1, z_m) \cdots Y_W^L(u_m, z_m) f(z_{m+1}, ..., z_{m+n})) \\ &\quad = E(Y_W^L(Y_V(u_1, z_1 - \zeta) \cdots Y_V(u_m, z_m - \zeta)\mathbf{1}, \zeta) f(z_{m+1}, ..., z_{m+n})) \end{aligned}$$

as elements in $\widetilde{W}_{z_1,...,z_{m+n}}$.

3. Let $u_1, ..., u_m \in V$, $f \in \widetilde{W}_{z_{m+1},...,z_{m+n}}$ *such that*

$$Y_W^{s(R)}(u_m, z_m) \cdots Y_W^{s(R)}(u_1, z_1) f(z_{m+1}, ..., z_{m+n})$$

converges absolutely to a $\overline{W}$*-valued rational function for every* $(z_1, ..., z_{m+n}) \in F_{m+n}\mathbb{C}$ *such that* $|z_m| > \cdots > |z_1| > |z_i|, i = m+1, ..., m+n$. *Then the series*

$$Y_W^{s(R)}(Y_V(u_1, z_1 - \zeta) \cdots Y_V(u_m, z_m - \zeta)\mathbf{1}, \zeta) f(z_{m+1}, ..., z_{m+n})$$

also converges absolutely to the same $\overline{W}$*-valued rational function whenever* $(z_1, ..., z_{m+n}) \in F_{m+n}\mathbb{C}, |\zeta| > |z_1 - \zeta| + |z_i|, i = m+1, ..., m+n, |z_1 - \zeta| > \cdots > |z_m - \zeta|$. *Moreover, we have*

$$\begin{aligned} &E(Y_W^{s(R)}(u_m, z_m) \cdots Y_W^{s(R)}(u_1, z_1) f(z_{m+1}, ..., z_{m+n})) \\ &\quad = E(Y_W^{s(R)}(Y_V(u_1, z_1 - \zeta) \cdots Y_V(u_m, z_m - \zeta)\mathbf{1}, \zeta) f(z_{m+1}, ..., z_{m+n})) \end{aligned}$$

as elements in $\widetilde{W}_{z_1,...,z_{m+n}}$.

5.3 The cochain complex and the cohomology group

Let n be a fixed positive integer. We will define cochain complexes from linear maps $V^{\otimes n} \to \widetilde{W}_{z_1,...,z_n}$ satisfying some natural properties.

5.3.1 Linear maps $V^{\otimes n} \to \widetilde{W}_{z_1 \dots z_n}$ satisfying D-derivative and $\mathbf{d}$-conjugation properties

Definition 5.3.1. A linear map $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ is said to have the D-derivative property if

1. For $i = 1, ..., n$, $v_1, ..., v_n \in V, w' \in W'$,

$$\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes D_V v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle = \frac{\partial}{\partial z_i} \langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle$$

2. For $v_1, ..., v_n \in V, w' \in W'$,

$$\langle w', D_W(\Phi(v_1 \otimes \cdots v_n))(z_1, ..., z_n)\rangle = \left(\frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n}\right) \langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle$$

Definition 5.3.2. A linear map $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ is said to have the $\mathbf{d}$-conjugation property if for $v_1, ..., v_n \in V, w' \in W', (z_1, ..., z_n) \in F_n\mathbb{C}$ and $z \in \mathbb{C}^\times$ so that $(zz_1, ..., zz_n) \in F_n\mathbb{C}$,

$$\langle w', z^{\mathbf{d}_W} (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle = \langle w', (\Phi(z^{\mathbf{d}_V} v_1 \otimes \cdots z^{\mathbf{d}_V} v_n))(zz_1, ..., zz_n)\rangle$$

Proposition 5.3.3. *Let $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\dots,z_n}$ be a linear map satisfying the D-derivative property.*

1. *For $v_1, ..., v_n \in V, w' \in W', (z_1, ..., z_n) \in F_n\mathbb{C}, z \in \mathbb{C}$ and $1 \le i \le n$ such that $(z_1, ..., z_{i-1}, z_i + z, z_{i+1}, ..., z_n) \in F_n\mathbb{C}$, the power series expansion of*

$$\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_{i-1}, z_i + z, z_{i+1}, ..., z_n)\rangle$$

   in positive powers of z is equal to the power series

$$\langle w', (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes e^{zD_V} v_i \otimes v_{i+1} \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle$$

   in z, which converges absolutely when $|z| < \min_{1 \le i < j \le n} |z_i - z_j|$.

2. *For $v_1, ..., v_n \in V, w' \in W', (z_1, ..., z_n) \in F_n\mathbb{C}, z \in \mathbb{C}$ so that $(z_1 + z, ..., z_n + z) \in F_n\mathbb{C}$, the power series expansion*

$$\langle w', (\Phi(v_1 \otimes \cdots \otimes v_n))(z_1 + z, ..., z_n + z)\rangle$$

in positive powers of z is equal to the power series

$$\langle w', e^{zD_W}(\Phi(v_1 \otimes \cdots \otimes v_n))(z_1, ..., z_n)\rangle.$$

which converges absolutely when $|z| < \min_{1\leq i\leq n} |z_i|$

Proof. The argument of D-conjugation property carries over. □

Definition 5.3.4. For every $n \in \mathbb{N}$, we define $\hat{C}_0^n(V, W)$ to be the set of all linear maps from $V^{\otimes n} \to \widetilde{W}_{z_1,...,z_n}$ that satisfies D-derivative property and $\mathbf{d}$-conjugation property.

Example 5.3.5. Let $l = 0, 1, ..., n$. For every $w \in W$ such that for every $u \in V$, $Y_W^L(u, x)w \in W[[x]], Y_W^{s(R)}(u, x)w \in W[[x]]$, one checks easily that the map

$$u_1 \otimes \cdots \otimes u_n \mapsto E_W^{(l,n-l)}(u_1 \otimes \cdots u_n; w)$$

is a linear map $V^{\otimes n} \to \widetilde{W}_{z_1 ... z_n}$ that has the D-derivative property and $\mathbf{d}$-conjugation property.

Notation 5.3.6. We will use the notation $E_{W,w}^{(l,n-l)}$ to denote the map in the previous example.

Let $\Phi \in \hat{C}_0^n(V, W)$, $u^{(1)}, ..., u^{(n)} \in V$. Consider the following series of $\overline{W}$-valued rational functions:

$$\Phi(Y_V(u^{(1)}, z^{(1)} - \zeta_1)\mathbf{1} \otimes Y_V(u^{(2)}, z^{(2)} - \zeta_2)\mathbf{1} \otimes \cdots \otimes Y_V(u^{(n)}, z^{(n)} - \zeta_n)\mathbf{1})(\zeta_1, ..., \zeta_n) \tag{5.6}$$

which is a series in variables $z^{(i)} - \zeta_i, i = 1, ..., n$ with

$$\sum_{k_1,...,k_n\in\mathbb{Z}} \Phi((Y_V)_{k_1}(u^{(1)})\mathbf{1}\otimes\cdots\otimes(Y_V)_{k_n}(u^{(n)})\mathbf{1})(\zeta_1, ..., \zeta_n)(z^{(1)}-\zeta_1)^{-k_1-1} \cdots (z^{(n)}-\zeta_n)^{-k_n-1}$$

For each $k_1, ..., k_n \in \mathbb{Z}$, the coefficient of $(z^{(1)} - \zeta_1)^{-k_1-1} \cdots (z^{(n)} - \zeta_n)^{-k_n-1}$ is a $\overline{W}$-valued rational function with variables $\zeta_1, ..., \zeta_n$. If paired with $w' \in W'$, then for the complex series

$$\sum_{k_1,...,k_n\in\mathbb{Z}} \langle w', \Phi((Y_V)_{k_1}(u^{(1)})\mathbf{1}\otimes\cdots\otimes(Y_V)_{k_n}(u^{(n)})\mathbf{1})(\zeta_1, ..., \zeta_n)\rangle(z^{(1)}-\zeta_1)^{-k_1-1} \cdots (z^{(n)}-\zeta_n)^{-k_n-1}$$

the coefficient of $(z^{(1)} - \zeta_1)^{-k_1-1} \cdots (z^{(n)} - \zeta_n)^{-k_n-1}$ is a rational function with possible poles at $\zeta_i = \zeta_j$ for $1 \leq i < j \leq n$.

Proposition 5.3.7. *The series (5.6) converges absolutely when*

$$|\zeta_i - \zeta_j| > |z^{(i)} - \zeta_i| + |z^{(j)} - \zeta_j|$$

to $\Phi(u_1 \otimes \cdots \otimes u_n)(z^{(1)}, ..., z^{(n)})$.

Proof. From the creation property, we know that the series is the same as

$$\langle w', \Phi(e^{(z^{(1)}-\zeta_1)D_V} u_1 \otimes \cdots \otimes e^{(z^{(n)}-\zeta_n)D_V} u_n)(\zeta_1, ..., \zeta_n)\rangle$$

We repeatedly use Proposition 5.3.3 to see that the series converges absolutely to the rational function

$$R(\langle w', \Phi(u_1 \otimes \cdots \otimes u_n)(\zeta_1 + z^{(1)} - \zeta_1, ..., \zeta_n + z^{(n)} - \zeta_n)\rangle)$$

when $|z^{(s)} - \zeta_s| < |\zeta_i - \zeta_j|, s = 1, ..., n, s \leq i < j \leq n; |z^{(s)} - \zeta_s| < |z^{(t)} - \zeta_j|, s = 2, ..., n, 1 \leq t < s \leq j \leq n$. Note that the rational function is the same as

$$R(\langle w', \Phi(u_1 \otimes \cdots \otimes u_n)(z_1, ..., z_n)\rangle)$$

that has the only possible poles at $z^{(i)} = z^{(j)}, 1 \leq i < j \leq n$ and does not depend on $\zeta_1, ..., \zeta_n$. We then apply Lemma *.*.* in [Q1] to see that the series (5.6) coincides with the expansion of the rational function by expanding the negative powers of $z^{(i)} - z^{(j)} = \zeta_i - \zeta_j + (z^{(i)} - \zeta_i - z^{(j)} + \zeta_j), 1 \leq i < j \leq n$. Thus the series (5.6) converges absolutely when

$$|\zeta_i - \zeta_j| > |z^{(i)} - \zeta_i| + |z^{(j)} - \zeta_j|, 1 \leq i < j \leq n$$

□

5.3.2 Linear maps $V^{\otimes n} \to \widetilde{W}_{z_1 \ldots z_n}$ composable with vertex operators

Definition 5.3.8. Let $\Phi : V^{\otimes n} \to \widetilde{W}_{z_1,\ldots,z_n}$ be a linear map. Let $m \in \mathbb{Z}_+$. Φ is said to be composable with m vertex operators if for every $\alpha_0, \alpha_1, ..., \alpha_n \in \mathbb{Z}_+$ such that $\alpha_0 + \cdots + \alpha_n = m + n$, every $l_0 = 0, ..., \alpha_0$, and every $v_1^{(0)}, ..., v_{\alpha_0}^{(0)}, ..., v_1^{(n)}, ..., v_{\alpha_n}^{(n)} \in V$, the series of $\overline{W}$-valued rational functions

$$
\begin{aligned}
&Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)})\\
&\qquad \cdot \Phi(Y_V(v_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1}\\
&\qquad \otimes \cdots\\
&\qquad \otimes Y_V(v_1^{(n)}, z_1^{(n)} - \zeta_n) \cdots Y_V(v_{\alpha_n}^{(n)}, z_{\alpha_n}^{(n)} - \zeta_n)\mathbf{1})(\zeta_1, ..., \zeta_n)\\
&= \sum_{\substack{k_1^{(0)}, ..., k_{\alpha_0}^{(0)},\\ ..., k_1^{(n)}, ..., k_{\alpha_n}^{(n)} \in \mathbb{Z}}} (Y_W^L)_{k_1^{(0)}}(u_1^{(0)}) \cdots (Y_W^L)_{k_{l_0}^{(0)}}(u_{l_0}^{(0)}) (Y_W^{s(R)})_{k_{l_0+1}^{(0)}}(u_{l_0+1}^{(0)}) \cdots (Y_W^{s(R)})_{k_{\alpha_0}^{(0)}}(u_{\alpha_0}^{(0)})\\
&\qquad\qquad \cdot \Phi((Y_V)_{k_1^{(1)}}(u_1^{(1)}) \cdots (Y_V)_{k_{\alpha_1}^{(1)}}(u_{\alpha_1}^{(1)})\mathbf{1}\\
&\qquad\qquad \otimes (Y_V)_{k_1^{(2)}}(u_1^{(2)}) \cdots (Y_V)_{k_{\alpha_2}^{(2)}}(u_{\alpha_2}^{(2)})\mathbf{1}\\
&\qquad\qquad \otimes \cdots\\
&\qquad\qquad \otimes (Y_V)_{k_1^{(n)}}(u_1^{(n)}) \cdots (Y_V)_{k_{\alpha_n}^{(n)}}(u_{\alpha_n}^{(n)})\mathbf{1})(\zeta_1, ..., \zeta_n)\rangle\\
&\qquad\qquad \prod_{i=1}^{\alpha_0} (z_i^{(0)})^{-k_i^{(0)}-1} \prod_{i=1}^{n} \prod_{j=1}^{\alpha_i} (z_j^{(i)} - \zeta_i)^{-k_j^{(i)}-1}
\end{aligned}
$$

converges absolutely when

$$
\begin{aligned}
&|z_1^{(0)}| > \cdots > |z_{\alpha_0}^{(0)}| > |\zeta_i| + |z_t^{(i)} - \zeta_i|, i = 1, ..., n, t = 1, ..., \alpha_i\\
&|z_1^{(i)} - \zeta_i| > \cdots > |z_{l_i}^{(i)} - \zeta_i|, i = 1, ..., n;\\
&|z_s^{(i)} - \zeta_i - z_t^{(j)} + \zeta_j| < |\zeta_i - \zeta_j|, 1 \leq i < j \leq n, 1 \leq s \leq \alpha_i, 1 \leq t \leq \alpha_j.
\end{aligned}
$$

and the sum can be analytically extended to a rational function in $z_1^{(1)}, ..., z_{\alpha_1}^{(1)}, ..., z_1^{(n)}, ..., z_{\alpha_n}^{(n)}$ that is independent of $\zeta_1, ..., \zeta_n$ and has the only possible poles at $z_s^{(i)} = z_t^{(j)}$, for $1 \leq i < j \leq n, s = 1, ..., \alpha_i, t = 1, ..., \alpha_j$. We require in addition that for each i, j, s, t, the order of the pole $z_s^{(i)} = z_t^{(j)}$ is bounded above by a constant that depends only on $u_s^{(i)}$ and $u_t^{(j)}$.

Definition 5.3.9. We denote by $\hat{C}_m^n(V, W)$ the set of linear maps $V^{\otimes n} \to \widetilde{W}_{z_1, ..., z_n}$ in $\hat{C}_0^n(V, W)$ and are composable with m vertex operators. It is easy to see that

$$\hat{C}_0^n(V, W) \supseteq \hat{C}_1^n(V, W) \supseteq \hat{C}_2^n(V, W) \supseteq \cdots$$

We denote by $\hat{C}_\infty^n(V, W)$ the intersection of all $\hat{C}_m^n(V, W)$ for $m = 0, 1, 2, ...$

Example 5.3.10. Fix $n \in \mathbb{Z}_+$ and $l = 0, ..., n$. For $w \in W$ satisfying $\forall u \in V, Y_W^L(u, x)w \in W[[x]]$ and $Y_W^R(w, x)u \in W[[x]]$, the map $E_{W;w}^{(l,n-l)}$ is an element in $\hat{C}_\infty^n(V, W)$.

Proof. This was proved in Proposition 5.2.17. □

Remark 5.3.11. If V is a grading-restricted vertex algebra and W is a grading-restricted V-module, then W can be viewed as a V-bimodule when we regard V as a MOSVA. One can check easily that Definition 5.3.8 is equivalent to Definition 3.5 in [H1]. The sets $\hat{C}_m^n(V, W)$ we are defining here is precisely the same as the $\widehat{C}_m^n(V, W)$ in [H1].

Remark 5.3.12. Let $\Phi \in \hat{C}_1^n(V, W)$. Then Definition 5.3.8 implies the following

1. For every $\beta = 1, ..., n$, $v^{(1)}, ..., v_1^{(\beta)}, v_2^{(\beta)}, ..., v^{(n)} \in V$, the series

$$\Phi(Y_V(v^{(1)}, z^{(1)}-\zeta_1)\mathbf{1}\otimes\cdots\otimes Y_V(v_1^{(\beta)}, z_1^{(\beta)}-\zeta_\beta)Y_V(v_2^{(\beta)}, z_2^{(\beta)}-\zeta_\beta)\mathbf{1}\otimes\cdots\otimes Y_V(v^{(n)}, z^{(n)}-\zeta_n)\mathbf{1})(\zeta_1, ..., \zeta_n)$$

which expands as

$$\sum_{\substack{k^{(1)},...,k_1^{(\beta)},k_2^{(\beta)}\in\mathbb{Z}\\ k^{(\beta+1)},...,k^{(n)}\in\mathbb{Z}}} \Phi((Y_V)_{k^{(1)}}(v^{(1)})\mathbf{1} \otimes \cdots \otimes (Y_V)_{k_1^{(\beta)}}(v_1^{(\beta)})(Y_V)_{k_2^{(\beta)}}\mathbf{1} \otimes \cdots \otimes (Y_V)_{k^{(n)}}(v^{(n)})\mathbf{1})(\zeta_1, ..., \zeta_n)$$
$$\cdot(z^{(1)} - \zeta_1)^{-k^{(1)}-1} \cdots (z_1^{(\beta)} - \zeta_\beta)^{-k_1^{(\beta)}-1}(z_2^{(\beta)} - \zeta_\beta)^{-k_2^{(\beta)}-1} \cdots (z^{(n)} - \zeta_n)^{-k^{(n)}-1}$$

converges absolutely when

$$|z_1^{(\beta)} - \zeta_\beta| > |z_2^{(\beta)} - \zeta_\beta|, |\zeta_\beta - \zeta_i| > |z_s^{(\beta)} - \zeta_\beta| + |z^{(i)} - \zeta_i|, s = 1, 2, 1 \le i \le n, i \ne \beta.$$
$$|\zeta_i - \zeta_j| > |z^{(i)} - \zeta_i| + |z^{(j)} - \zeta_j|, 1 \le i < j \le n, i, j \ne \beta,$$

to a rational function that depends only on $z^{(1)}, ..., z_1^{(i)}, z_2^{(i)}, ..., z^{(n)}$, with the only possible poles at $z_1^{(\beta)} = z_2^{(\beta)}; z_s^{(\beta)} = z^{(i)}, s = 1, 2, i = 1, ..., \beta - 1, \beta + 1, ..., n; z^{(i)} = z^{(j)}, 1 \le i < j \le n, i, j \ne \beta$. From arguments similar to those in Proposition 5.3.7, this is equivalent to say that the following series

$$\Phi(v^{(1)} \otimes \cdots \otimes Y_V(v_1^{(\beta)}, z_1^{(\beta)} - \zeta_\beta)Y_V(v_2^{(\beta)}, z_2^{(\beta)} - \zeta_\beta)\mathbf{1} \otimes \cdots \otimes v^{(n)})(z^{(1)}, ..., \zeta_\beta, ..., z^{(n)})$$
$$= \sum_{k_1^{(\beta)},k_2^{(\beta)}\in\mathbb{Z}} \Phi(v^{(1)} \otimes \cdots \otimes (Y_V)_{k_1^{(\beta)}}(v_1^{(\beta)})(Y_V)_{k_2^{(\beta)}}\mathbf{1} \otimes \cdots \otimes v^{(n)})(z^{(1)}, ..., \zeta_\beta, ..., z^{(n)})$$
$$(z_1^{(\beta)} - \zeta_\beta)^{-k_1^{(\beta)}-1}(z_2^{(\beta)} - \zeta_\beta)^{-k_2^{(\beta)}-1}$$

of $\overline{W}$-valued rational functions converges absolutely when

$$|z_1^{(\beta)}-\zeta_\beta|>|z_2^{(\beta)}-\zeta_\beta|, |z^{(i)}-z_s^{(\beta)}|>|z_s^{\beta}-\zeta_\beta|, i=1,...,\beta-1,\beta+1,...,n, s=1,2.$$

2. For $u_1,...,u_{n+1}\in V$, the series

$$\begin{aligned}&Y_W^L(u_1,z_1)\Phi(u_2\otimes\cdots\otimes u_{n+1})(z_2,...,z_{n+1})\\ =&\sum_{k\in\mathbb{Z}}(Y_W^L)_k(u_1)\Phi(u_2\otimes\cdots\otimes u_{n+1})(z_2,...,z_{n+1})z_1^{-k-1}\end{aligned}$$

of $\overline{W}$-elements converges absolutely when $|z_1|>|z_i|>0, i=2,...,n+1$ to an $\overline{W}$-valued rational function (here the operator $(Y_W^L)_k(u_1)$ is extended to $\overline{W}\to\overline{W}$.)

3. For $u_1,...,u_{n+1}\in V$, the series

$$\begin{aligned}&Y_W^{s(R)}(u_{n+1},z_{n+1})\Phi(u_1\otimes\cdots\otimes u_n)(z_1,...,z_n)\\ =&\sum_{k\in\mathbb{Z}}(Y_W^{s(R)})_k(u_{n+1})\Phi(u_1\otimes\cdots\otimes u_n)(z_1,...,z_n)z_{n+1}^{-k-1}\end{aligned}$$

of $\overline{W}$-elements converges absolutely when $|z_{n+1}|>|z_i|>0, i=1,...,n$ to an $\overline{W}$-valued rational function (here the operator $(Y_W^{s(R)})_k(u_{n+1})$ is extended to $\overline{W}\to\overline{W}$.)

Definition 5.3.13. Let $\Phi\in\hat{C}_m^n(V,W)$.

1. For every $i=1,...,n$, we define the map

$$\Phi\circ_i E_V^{(2)}: V^{\otimes(n+1)}\to\widetilde{W}_{z_1,...,z_{n+1}}$$

by setting

$$(\Phi\circ_i E_V^{(2)})(v_1\otimes\cdots\otimes v_{n+1})$$

to be the $\overline{W}$-valued rational function

$$E(\Phi(v_1\otimes\cdots\otimes Y_V(v_i,z_i-\zeta)Y_V(v_{i+1},z_{i+1}-\zeta)\mathbf{1}\otimes v_{i+2}\otimes\cdots\otimes v_{n+1})(z_1,...,z_{i-1},\zeta,z_{i+2},...,z_{n+1}))$$

2. We define the map

$$E_W^{(1,0)}\circ_2\Phi: V^{\otimes(n+1)}\to\widetilde{W}_{z_1,...,z_{n+1}}$$

by setting

$$(E_W^{(1,0)} \circ_2 \Phi)(v_1 \otimes \cdots \otimes v_{n+1})$$

to be the $\overline{W}$-valued rational function

$$E(Y_W^L(v_1, z_1)\Phi(v_2 \otimes \cdots \otimes v_{n+1})(z_2, ..., z_{n+1}))$$

3. We define the map

$$E_W^{(0,1)} \circ_2 \Phi : V^{\otimes(n+1)} \to \widetilde{W}_{z_1,...,z_{n+1}}$$

by setting

$$(E_W^{(0,1)} \circ_2 \Phi)(v_1 \otimes \cdots \otimes v_{n+1})$$

to be the $\overline{W}$-valued rational function

$$E(Y_W^{s(R)}(v_{n+1}, z_{n+1})\Phi(v_1 \otimes \cdots \otimes v_n)(z_1, ..., z_n))$$

Proposition 5.3.14. *The maps $\Phi \circ_i E_V^{(2)}, E_W^{(1,0)} \circ_2 \Phi$ and $E_W^{(0,1)} \circ_2 \Phi$ are elements of $\hat{C}_{m-1}^{n+1}(V, W)$*

Proof. Let $\alpha_0, ..., \alpha_{n+1} \in \mathbb{N}$ such that $\alpha_0 + \cdots + \alpha_{n+1} = m + n$. Let $l_0 = 0, ..., \alpha_0$. Take $v_s^{(j)} \in V, j = 0, 1, , ..., n+1, s = 1, ..., \alpha_j$.

1. For the first conclusion, we first note that the associativity implies that

$$\begin{aligned}
&Y_V(v_1^{(i)}, z_1^{(i)} - \zeta) \cdots Y_V(v_{\alpha_i}^{(i)}, z_{\alpha_i}^{(i)} - \zeta) Y_V(v_1^{(i+1)}, z_1^{(i+1)} - \zeta) \cdots Y_V(v_{\alpha_{i+1}}^{(i+1)}, z_{\alpha_{i+1}}^{(i+1)} - \zeta)\mathbf{1} \\
=&Y_V(Y_V(v_1^{(i)}, z_1^{(i)} - \zeta_i) \cdots Y_V(v_{\alpha_i}^{(i)}, z_{\alpha_i}^{(i)} - \zeta_i)\mathbf{1}, \zeta_i - \zeta) \\
&\quad \cdot Y_V(Y_V(v_1^{(i+1)}, z_1^{(i+1)} - \zeta_{i+1}) \cdots Y_V(v_{\alpha_{i+1}}^{(i+1)}, z_{\alpha_{i+1}}^{(i+1)} - \zeta_{i+1})\mathbf{1}, \zeta_{i+1} - \zeta)\mathbf{1}
\end{aligned}$$

when

$$\begin{aligned}
&|z_1^{(i)} - \zeta| > \cdots > |z_{\alpha_i}^{(i)} - \zeta| > |z_1^{(i+1)} - \zeta| > \cdots > |z_{\alpha_{i+1}}^{(i+1)} - \zeta| \\
&|z_s^{(i)} - \zeta_i| > |z_t^{(i)} - \zeta_i|, 1 \leq s < t \leq \alpha_i, |z_s^{(i+1)} - \zeta_{i+1}| > |z_t^{(i+1)} - \zeta_{i+1}|, 1 \leq s < t \leq \alpha_{i+1} \\
&|\zeta_i - \zeta| > |\zeta_{i+1} - \zeta| + |z_s^{(i)} - \zeta_i| + |z_t^{(i+1)} - \zeta_{i+1}|, s = 1, ..., \alpha_i, t = 1, ..., \alpha_{i+1}.
\end{aligned}$$

Then we can prove that the series

$$Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)})$$
$$\cdot(\Phi \circ_i E_V^{(2)})(Y_V(v_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1}$$
$$\otimes \cdots \otimes Y_V(v_1^{(n+1)}, z_1^{(n+1)} - \zeta_{n+1}) \cdots Y_V(v_{\alpha_{n+1}}^{(1)}, z_{\alpha_{n+1}}^{(n+1)} - \zeta_{n+1})\mathbf{1})(\zeta_1, ..., \zeta_{n+1})$$
$$= \sum_{\substack{k_1^{(0)},...,k_{\alpha_0}^{(0)} \\ ...k_1^{(n+1)},...,k_{\alpha_{n+1}}^{(n+1)} \in \mathbb{Z}}} (Y_W^L)_{k_1^{(0)}}(u_1) \cdots (Y_W^L)_{k_{l_0}^{(0)}}(u_{l_0})(Y_W^{s(R)})_{k_{l_0+1}}(u_{l_0+1}) \cdots (Y_W^{s(R)})_{k_{\alpha_0}}(u_{\alpha_0})$$
$$\cdot(\Phi \circ_i E_V^{(2)})((Y_V)_{k_1^{(1)}}(v_1^{(1)}) \cdots (Y_V)_{k_{\alpha_1}^{(1)}}(v_{\alpha_1}^{(1)})\mathbf{1}$$
$$\otimes \cdots \otimes (Y_V)_{k_1^{(n+1)}}(v_1^{(n+1)}) \cdots (Y_V)_{k_{\alpha_{n+1}}^{(n+1)}}(v_{\alpha_{n+1}}^{(n+1)})\mathbf{1})(\zeta_1, ..., \zeta_{n+1})$$
$$\cdot \prod_{j=1}^{\alpha_0} z_j^{(0)} \prod_{j=1}^{n} (z_1^{(j)} - \zeta_j)^{-k_1^{(j)}-1} \cdots (z_{\alpha_j}^{(j)} - \zeta_j)^{-k_{\alpha_j}^{(j)}-1}$$

is the expansion of the $\overline{W}$-rational function

$$E(Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)})$$
$$\cdot\Phi(Y_V(v_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1}$$
$$\otimes \cdots$$
$$\otimes Y_V(v_1^{(i)}, z_1^{(i)} - \zeta) \cdots Y_V(v_{\alpha_i}^{(i)}, z_{\alpha_i}^{(i)} - \zeta) Y_V(v_1^{(i+1)}, z_1^{(i+1)} - \zeta) \cdots Y_V(v_{\alpha_{i+1}}^{(i+1)}, z_{\alpha_{i+1}}^{(i+1)} - \zeta)\mathbf{1}$$
$$\otimes \cdots$$
$$\otimes Y_V(v_1^{(n+1)}, z_1^{(n+1)} - \zeta_{n+1}) \cdots Y_V(v_{\alpha_{n+1}}^{(n+1)}, z_{\alpha_{n+1}}^{(n+1)} - \zeta_{n+1})\mathbf{1})(\zeta_1, ..., \zeta, \zeta_{i+2}, ..., \zeta_{n+1}))$$

in the region

$$|z_1^{(0)}| > \cdots > |z_{\alpha_0}^{(0)}| > |\zeta_i| + |z_t^{(i)} - \zeta_i|, i = 1, ..., n+1, t = 1, ..., \alpha_i$$
$$|z_1^{(i)} - \zeta_i| > \cdots > |z_{l_i}^{(i)} - \zeta_i|, i = 1, ..., n+1;$$
$$|z_s^{(i)} - \zeta_i - z_t^{(j)} + \zeta_j| < |\zeta_i - \zeta_j|, 1 \leq i < j \leq n+1, 1 \leq s \leq \alpha_i, 1 \leq t \leq \alpha_j.$$

2. For the second conclusion, we first note that from Theorem 5.2.20 Part (2),

$$Y_W^L(v_1^{(1)}, z_1^{(1)}) \cdots Y_W^L(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)})\overline{w} = Y_W^L(Y_V(v_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1}, \zeta_1)\overline{w}$$

where

$$\overline{w} = E(\Phi(Y_V(v_1^{(2)}, z_1^{(2)} - \zeta_2) \cdots Y_V(v_{\alpha_2}^{(2)}, z_{\alpha_2}^{(2)} - \zeta_2)\mathbf{1}$$

$$\otimes \cdots$$

$$\otimes Y_V(v_1^{(n+1)}, z_1^{(n+1)} - \zeta_{n+1}) \cdots Y_V(v_{\alpha_{n+1}}^{(n+1)}, z_{\alpha_{n+1}}^{(n+1)} - \zeta_{n+1})\mathbf{1})(\zeta_2, ..., \zeta_{n+1})) \in \overline{W}$$

and $z_1^{(1)}, ..., z_{\alpha_{n+1}}^{(n+1)} \in \mathbb{C}$ such that

$$|z_1^{(1)}| > \cdots > |z_{\alpha_1}^{(1)}| > |z_j^{(i)}|, i = 2, ..., n+1, j = 1, ..., \alpha_i;$$

$$|\zeta_1| > |z_1^{(1)} - \zeta_1| + |z_j^{(i)}|, i = 2, ..., n+1, j = 1, ..., \alpha_i;$$

$$|z_1^{(1)} - \zeta| > \cdots > |z_{\alpha_1}^{(1)} - \zeta_1|;$$

$$z_s^{(i)} \neq z_t^{(j)}, 1 \leq i \leq j \leq n+1, s = 1, ..., \alpha_i, t = 1, ..., \alpha_j, s \neq t \text{ when } i = j;$$

Then with the commutativity of Y_W^L and $Y_W^{s(R)}$ operators, we can prove that the series

$$\begin{aligned}
&Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)}) \\
&\quad \cdot (E_W^{(1,0)} \circ_2 \Phi)(Y_V(v_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1} \\
&\qquad\qquad \otimes \cdots \otimes Y_V(v_1^{(n+1)}, z_1^{(n+1)} - \zeta_{n+1}) \cdots Y_V(v_{\alpha_{n+1}}^{(1)}, z_{\alpha_{n+1}}^{(n+1)} - \zeta_{n+1})\mathbf{1})(\zeta_1, ..., \zeta_{n+1}) \\
&= \sum_{\substack{k_1^{(0)}, ..., k_{\alpha_0}^{(0)} \\ ... k_1^{(n+1)}, ..., k_{\alpha_{n+1}}^{(n+1)} \in \mathbb{Z}}} (Y_W^L)_{k_1^{(0)}}(u_1) \cdots (Y_W^L)_{k_{l_0}^{(0)}}(u_{l_0}) (Y_W^{s(R)})_{k_{l_0+1}}(u_{l_0+1}) \cdots (Y_W^{s(R)})_{k_{\alpha_0}}(u_{\alpha_0}) \\
&\qquad\qquad \cdot (E_W^{(1,0)} \circ_2 \Phi)((Y_V)_{k_1^{(1)}}(v_1^{(1)}) \cdots (Y_V)_{k_{\alpha_1}^{(1)}}(v_{\alpha_1}^{(1)})\mathbf{1} \\
&\qquad\qquad\qquad \otimes \cdots \otimes (Y_V)_{k_1^{(n+1)}}(v_1^{(n+1)}) \cdots (Y_V)_{k_{\alpha_{n+1}}^{(n+1)}}(v_{\alpha_{n+1}}^{(n+1)})\mathbf{1})(\zeta_1, ..., \zeta_{n+1}) \\
&\qquad\qquad \cdot \prod_{j=1}^{\alpha_0} z_j^{(0)} \prod_{j=1}^{n} (z_1^{(j)} - \zeta_j)^{-k_1^{(j)}-1} \cdots (z_{\alpha_j}^{(j)} - \zeta_j)^{-k_{\alpha_j}^{(j)}-1}
\end{aligned}$$

is the expansion of the $\overline{W}$-rational function

$$\begin{aligned}
&E(Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) \cdot Y_W^L(v_1^{(1)}, z_1^{(1)}) \cdots Y_W^L(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)}) \\
&\qquad \cdot Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)}) \\
&\qquad \cdot \Phi(Y_V(v_1^{(2)}, z_1^{(2)} - \zeta_2) \cdots Y_V(v_{\alpha_2}^{(2)}, z_{\alpha_2}^{(1)} - \zeta_2)\mathbf{1} \\
&\qquad\quad \otimes \cdots \\
&\qquad\quad \otimes Y_V(v_1^{(n+1)}, z_1^{(n+1)} - \zeta_{n+1}) \cdots Y_V(v_{\alpha_{n+1}}^{(n+1)}, z_{\alpha_{n+1}}^{(n+1)} - \zeta_{n+1})\mathbf{1})(\zeta_2, ..., \zeta_{n+1}))
\end{aligned}$$

in the region

$$|z_1^{(0)}| > \cdots > |z_{\alpha_0}^{(0)}| > |\zeta_i| + |z_t^{(i)} - \zeta_i|, i = 1, ..., n+1, t = 1, ..., \alpha_i$$

$$|z_1^{(i)}-\zeta_i|>\cdots>|z_{l_i}^{(i)}-\zeta_i|, i=1,...,n+1;$$

$$|z_s^{(i)}-\zeta_i-z_t^{(j)}+\zeta_j|<|\zeta_i-\zeta_j|, 1\le i<j\le n+1, 1\le s\le \alpha_i, 1\le t\le \alpha_j.$$

3. For the third conclusion, we first note that from Theorem 5.2.20 Part (2),

$$\begin{aligned}&Y_W^{s(R)}(v_{\alpha_{n+1}}^{(n+1)},z_{\alpha_{n+1}}^{(n+1)})\cdots Y_W^{s(R)}(v_1^{(n+1)},z_1^{(n+1)})\overline{w}\\&\quad=Y_W^{s(R)}(Y_V(v_1^{(n+1)},z_1^{(n+1)}-\zeta_{n+1})\cdots Y_V(v_{\alpha_{n+1}}^{(n+1)},z_{\alpha_{n+1}}^{(n+1)}-\zeta_{n+1})\mathbf{1},\zeta_{n+1})\overline{w}\end{aligned}$$

where

$$\begin{aligned}\overline{w}=E(\Phi(&Y_V(v_1^{(1)},z_1^{(1)}-\zeta_1)\cdots Y_V(v_{\alpha_1}^{(1)},z_{\alpha_1}^{(1)}-\zeta_1)\mathbf{1}\\&\otimes\cdots\\&\otimes Y_V(v_1^{(n)},z_1^{(n)}-\zeta_n)\cdots Y_V(v_{\alpha_n}^{(n)},z_{\alpha_n}^{(n)}-\zeta_n)\mathbf{1})(\zeta_1,...,\zeta_n))\in\overline{W}\end{aligned}$$

and $z_1^{(1)},...,z_{\alpha_{n+1}}^{(n+1)}\in\mathbb{C}$ such that

$$|z_{\alpha_{n+1}}^{(n+1)}|>\cdots>|z_1^{(n+1)}|>|z_s^{(i)}|, i=1,...,n, s=1,...,\alpha_i;$$

$$|\zeta_{n+1}|>|z_1^{(n+1)}-\zeta_{n+1}|+|z_s^{(i)}|, i=1,...,n, s=1,...,\alpha_i;$$

$$|z_1^{(n+1)}-\zeta_{n+1}|>\cdots>|z_{\alpha_{n+1}}^{(n+1)}-\zeta_{n+1}|;$$

$$z_s^{(i)}\ne z_t^{(j)}, 1\le i\le j\le n+1, s=1,...,\alpha_i, t=1,...,\alpha_j, s\ne t \text{ when } i=j$$

for every $\overline{w}\in\overline{W}$ and $z_1^{(n+1)},...,z_{\alpha_{n+1}}^{(n+1)}\in\mathbb{C}$ such that the left-hand-side converges. Then with the commutativity of Y_W^L and $Y_W^{s(R)}$ operators, we can prove that the series

$$\begin{aligned}&Y_W^L(u_1^{(0)},z_1^{(0)})\cdots Y_W^L(u_{l_0}^{(0)},z_{l_0}^{(0)})Y_W^{s(R)}(u_{l_0+1}^{(0)},z_{l_0+1}^{(0)})\cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)},z_{\alpha_0}^{(0)})\\&\quad\cdot(E_W^{(0,1)}\circ_2\Phi)(Y_V(v_1^{(1)},z_1^{(1)}-\zeta_1)\cdots Y_V(v_{\alpha_1}^{(1)},z_{\alpha_1}^{(1)}-\zeta_1)\mathbf{1}\\&\qquad\qquad\otimes\cdots\otimes Y_V(v_1^{(n+1)},z_1^{(n+1)}-\zeta_{n+1})\cdots Y_V(v_{\alpha_{n+1}}^{(1)},z_{\alpha_{n+1}}^{(n+1)}-\zeta_{n+1})\mathbf{1})(\zeta_1,...,\zeta_{n+1})\\&=\sum_{\substack{k_1^{(0)},...,k_{\alpha_0}^{(0)}\\ ...k_1^{(n+1)},...,k_{\alpha_{n+1}}^{(n+1)}\in\mathbb{Z}}}(Y_W^L)_{k_1^{(0)}}(u_1)\cdots(Y_W^L)_{k_{l_0}^{(0)}}(u_{l_0})(Y_W^{s(R)})_{k_{l_0+1}}(u_{l_0+1})\cdots(Y_W^{s(R)})_{k_{\alpha_0}}(u_{\alpha_0})\\&\qquad\cdot(E_W^{(0,1)}\circ_2\Phi)((Y_V)_{k_1^{(1)}}(v_1^{(1)})\cdots(Y_V)_{k_{\alpha_1}^{(1)}}(v_{\alpha_1}^{(1)})\mathbf{1}\\&\qquad\qquad\otimes\cdots\otimes(Y_V)_{k_1^{(n+1)}}(v_1^{(n+1)})\cdots(Y_V)_{k_{\alpha_{n+1}}^{(n+1)}}(v_{\alpha_{n+1}}^{(n+1)})\mathbf{1})(\zeta_1,...,\zeta_{n+1})\\&\qquad\cdot\prod_{j=1}^{\alpha_0}z_j^{(0)}\prod_{j=1}^{n}(z_1^{(j)}-\zeta_j)^{-k_1^{(j)}-1}\cdots(z_{\alpha_j}^{(j)}-\zeta_j)^{-k_{\alpha_j}^{(j)}-1}\end{aligned}$$

is the expansion of the $\overline{W}$-rational function

$$E(Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_{l_0}^{(0)}, z_{l_0}^{(0)}) \cdot Y_W^L(v_1^{(1)}, z_1^{(1)}) \cdots Y_W^{s(R)}(u_{l_0+1}^{(0)}, z_{l_0+1}^{(0)})$$
$$\cdot Y_W^{s(R)}(v_1^{(n+1)}, z_1^{(n+1)}) \cdots Y_W^{s(R)}(v_{\alpha_{n+1}}^{(n+1)}, z_{\alpha_{n+1}}^{(n+1)})$$
$$\cdot \Phi(Y_V(v_1^{(1)}, z_1^{(1)} - \zeta_1) \cdots Y_V(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)} - \zeta_1)\mathbf{1}$$
$$\otimes \cdots$$
$$\otimes Y_V(v_1^{(n)}, z_1^{(n)} - \zeta_n) \cdots Y_V(v_{\alpha_n}^{(n)}, z_{\alpha_n}^{(n)} - \zeta_n)\mathbf{1})(\zeta_1, ..., \zeta_n))$$

in the region

$$|z_1^{(0)}| > \cdots > |z_{\alpha_0}^{(0)}| > |\zeta_i| + |z_t^{(i)} - \zeta_i|, i = 1, ..., n+1, t = 1, ..., \alpha_i$$
$$|z_1^{(i)} - \zeta_i| > \cdots > |z_{l_i}^{(i)} - \zeta_i|, i = 1, ..., n+1;$$
$$|z_s^{(i)} - \zeta_i - z_t^{(j)} + \zeta_j| < |\zeta_i - \zeta_j|, 1 \le i < j \le n+1, 1 \le s \le \alpha_i, 1 \le t \le \alpha_j.$$

□

5.3.3 The coboundary operators and the cochain complex

For $m, n \in \mathbb{Z}_+$, we define the coboundary operator as follows

$$\hat{\delta}_m^n : \hat{C}_m^n(V, W) \to \hat{C}_{m-1}^{n+1}(V, W)$$

by

$$\hat{\delta}_m^n \Phi = E_W^{(1,0)} \circ_2 \Phi + \sum_{i=1}^n (-1)^i \Phi \circ_i E_V^{(2)} + (-1)^{n+1} E_W^{(0,1)} \circ_2 \Phi$$

More explicitly, $\hat{\delta}_m^n \Phi$ is a map from $V^{\otimes(n+1)}$ to $\widetilde{W}_{z_1,...,z_{n+1}}$ satisfying

$$((\hat{\delta}_m^n \Phi)(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, ..., z_{n+1})$$
$$= E(Y_W^L(v_1, z_1)(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, ..., z_{n+1}))$$
$$- E((\Phi(Y_V(v_1, z_1 - \zeta_1)Y_V(v_2, z_2 - \zeta_1)\mathbf{1} \otimes v_3 \otimes \cdots \otimes v_{n+1}))(\zeta_1, z_3, ..., z_{n+1}))$$
$$+ E((\Phi(v_1 \otimes Y_V(v_2, z_2 - \zeta_2)Y_V(v_3, z_3 - \zeta_2)\mathbf{1} \otimes v_4 \otimes \cdots \otimes v_{n+1})(z_1, \zeta_2, z_4, ..., z_{n+1}))$$
$$- \cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ (-1)^i E\left(\begin{array}{l} (\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - \zeta_i)Y_V(v_{i+1}, z_{i+1} - \zeta_i)\mathbf{1} \otimes v_{i+2} \otimes \cdots \otimes v_{n+1})) \\ \quad (z_1, ..., z_{i-1}, \zeta_i, z_{i+2}, ..., z_{n+1}) \end{array} \right)$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(-1)^n E(\Phi(v_1 \otimes \cdots \otimes v_{n-1} \otimes Y_V(v_n, z_n - \zeta_n) Y_V(z_{n+1} - \zeta_n)\mathbf{1}))(z_1, ..., z_{n-1}, \zeta_n))$$

$$+(-1)^{n+1} E(Y_W^{s(R)}(u_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots v_n))(z_1, ..., z_n))$$

One can also write

$$\begin{aligned}
&((\hat{\delta}_m^n(\Phi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, ..., z_{n+1})\\
=&E(Y_W^L(v_1, z_1)(\Phi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, ..., z_{n+1}))\\
&+\sum_{i=1}^{n}(-1)^i E\left(\begin{array}{l}(\Phi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i - \zeta_i) Y_V(v_{i+1}, z_{i+1} - \zeta_i)\mathbf{1} \otimes v_{i+2} \otimes \cdots \otimes v_{n+1}))\\ \quad (z_1, ..., z_{i-1}, \zeta_i, z_{i+2}, ..., z_{n+1})\end{array}\right)\\
&+(-1)^{n+1} E(Y_W^{s(R)}(u_{n+1}, z_{n+1})(\Phi(v_1 \otimes \cdots v_n))(z_1, ..., z_n))
\end{aligned}$$

provided that $i = 1$ and $i = n$ term in the sum is well-understood.

When $n = 1$, we have

$$\begin{aligned}
&(\hat{\delta}_m^1(\Phi)(v_1 \otimes v_2))(z_1, z_2)\\
&= E(Y_W^L(v_1, z_1)(\Phi(v_2))(z_2)) - E(\Phi(Y_V(u_1, z_1 - \zeta) Y_V(u_2, z_2 - \zeta)\mathbf{1}))(\zeta) + E(Y_W^o(v_2, z_2)(\Phi(v_1))(z_1))
\end{aligned}$$

Remark 5.3.15. It is crucial that in all the explicit summations above, we are not adding series, but adding the analytic extensions of the sums of these series, which are $\overline{W}$-valued rational functions, aka., $\overline{W}$-elements that depends on $z_1, ..., z_{n+1}$. Those series refuse to be added up directly because the region of convergence of the first series and that of the last series do not intersect.

Theorem 5.3.16. *For every $m \in \mathbb{Z}_+, n \in \mathbb{N}$, $\hat{\delta}_m^n(\hat{C}_m^n(V, W)) \subseteq \hat{C}_{m-1}^{n+1}(V, W)$.*

Proof. This follows from Proposition 5.3.14. □

Theorem 5.3.17. *For $m, n \in \mathbb{Z}_+$, $\hat{\delta}_{m-1}^{n+1} \circ \hat{\delta}_m^n = 0$*

Proof. Let $\Phi \in \hat{C}_m^n(V, W)$. We compute as follows:

$$\begin{aligned}
&\hat{\delta}_{m-1}^{n+1}(\hat{\delta}_m^n \Phi)\\
=&E_W^{(1,0)} \circ_2 \hat{\delta}_m^n \Phi + \sum_{i=1}^{n+1}(-1)^i (\hat{\delta}_m^n \Phi) \circ_i E_V^{(2)} + E_W^{(0,1)} \circ_0 \hat{\delta}_m^n \Phi
\end{aligned}$$

$$
\begin{aligned}
=&E_W^{(1,0)}\circ_2(E_W^{(1,0)}\circ_2\Phi+\sum_{j=1}^{n}\Phi\circ_j E_V^{(2)}+E_W^{(0,1)}\circ_0\Phi)\\
&+\sum_{i=1}^{n+1}(-1)^i\left((E_W^{(1,0)}\circ_2\Phi)\circ_i E_V^{(2)}+\sum_{j=1}^{n}(-1)^j(\Phi\circ_j E_V^{(2)})\circ_i E_V^{(2)}+(-1)^{n+1}(E_W^{(0,1)}\circ_1\Phi)\circ_i E_V^{(2)}\right)\\
&+(-1)^{n+2}\left(E_W^{(0,1)}\circ_1(E_W^{(1,0)}\circ_2\Phi)+\sum_{j=1}^{n}(-1)^jE_W^{(0,1)}\circ_1(\Phi\circ_j E_V^{(2)})+(-1)^{n+1}E_W^{(0,1)}\circ_1(E_W^{(0,1)}\circ_1\Phi)\right)
\end{aligned}
$$

We rearrange the terms and indexes to write $\hat{\delta}_{m-1}^{n+1}(\hat{\delta}_m^n\Phi)$ as

$$
E_W^{(1,0)}\circ_2(E_W^{(1,0)}\circ_2\Phi)+\sum_{i=1}^{n}(-1)^iE_W^{(1,0)}\circ_2(\Phi\circ_i E_V^{(2)})+\sum_{i=1}^{n+1}(-1)^i(E_W^{(1,0)}\circ_2\Phi)\circ_i E_V^{(2)} \tag{I}
$$

$$
+(-1)^{n+1}E_W^{(1,0)}\circ_2(E_W^{(0,1)}\circ_1\Phi)+(-1)^{n+2}E_W^{(0,1)}\circ_1(E_W^{(1,0)}\circ_2\Phi) \tag{II}
$$

$$
+\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^i(-1)^j(\Phi\circ_j E_V^{(2)})\circ_i E_V^{(2)} \tag{III}
$$

$$
+\sum_{i=1}^{n+1}(-1)^{n+1+i}(E_W^{(0,1)}\circ_1\Phi)\circ_i E_V^{(2)}+\sum_{i=1}^{n}(-1)^{n+2+i}E_W^{(0,1)}\circ_1(\Phi\circ_i E_V^{(2)})-E_W^{(0,1)}\circ_1(E_W^{(0,1)}\circ_1\Phi) \tag{IV}
$$

We argue that (I), (II), (III) and (IV) are all zero.

For (I), we need the following lemma

Lemma 5.3.18. $E_W^{(1,0)}\circ_2(E_W^{(1,0)}\circ_2\Phi)=(E_W^{(1,0)}\circ_2\Phi)\circ_1 E_V^2$

Proof. For any $v_1,...,v_{n+2}\in V,(z_1,...,z_{n+2})\in F_n\mathbb{C}$, we have

$$
\begin{aligned}
&[E_W^{(1,0)}\circ_2(E_W^{(1,0)}\circ_2\Phi)(v_1\otimes\cdots\otimes v_{n+2})](z_1,...,z_{n+2})\\
=&[E_W^{(1,0)}(v_1;[(E_W^{(1,0)}\circ_2\Phi)(v_2\otimes\cdots\otimes v_{n+2})](z_2,...,z_{n+2}))](z_1)\\
=&E(Y_W^L(v_1,z_1)[E_W^{(1,0)}(v_2;[\Phi(v_3\otimes\cdots\otimes v_{n+2})](z_3,...,z_{n+2}))](z_2)\\
=&E(Y_W^L(v_1,z_1)Y_W^L(v_2,z_2)[\Phi(v_3\otimes\cdots\otimes v_{n+2})](z_3,...,z_{n+2}),
\end{aligned}
$$

and

$$
\begin{aligned}
&[(E_W^{(1,0)}\circ_2\Phi)\circ_1 E_V^{(2)}](v_1\otimes\cdots\otimes v_{n+2})(z_1,...,z_{n+2})\\
=&[(E_W^{(1,0)}\circ_2\Phi)(Y_V(v_1,z_1-\zeta)Y_V(v_2,z_2-\zeta)\mathbf{1}\otimes v_3\otimes\cdots\otimes v_{n+2})](\zeta,z_3,...,z_{n+2})\\
=&[E_W^{(1,0)}(Y_V(v_1,z_1-\zeta)Y_V(v_2,z_2-\zeta)\mathbf{1};[\Phi(v_3\otimes\cdots\otimes v_{n+2})](z_3,...,z_{n+2}))](\zeta)\\
=&E(Y_W^L(Y_V(v_1,z_1-\zeta)Y_V(v_2,z_2-\zeta)\mathbf{1},\zeta)[\Phi(v_3\otimes\cdots\otimes v_{n+2})](z_3,...,z_{n+2}))
\end{aligned}
$$

It follows from Theorem 5.2.20 Part (2) and the identity property of vacuum that these rational functions are equal. □

We also need the following lemma

Lemma 5.3.19. $E_W^{(1,0)} \circ_2 (\Phi \circ_i E_V^{(2)}) = (E_W^{(1,0)} \circ_2 \Phi) \circ_{i+1} E_V^{(2)}$

Proof. For any $v_1, ..., v_{n+2} \in V, (z_1, ..., z_{n+2}) \in F_n\mathbb{C}$, we have

$$
\begin{aligned}
&[E_W^{(1,0)} \circ_2 (\Phi \circ_i E_V^{(2)})(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[E_W^{(1,0)}(v_1; [(\Phi \circ_i E_V^{(2)})(v_2 \otimes \cdots \otimes v_{n+2})](z_2, ..., z_{n+2}))](z_1)\\
=&[E_W^{(1,0)}(v_1; [\Phi(v_2 \otimes \cdots \otimes [E_V^{(2)}(v_{i+1}, v_{i+2})](z_{i+1} - \zeta, z_{i+2} - \zeta) \otimes \cdots \otimes v_{n+2})](z_2, ..., \zeta, ..., z_{n+2}))](z_1)\\
=&E(Y_W(v_1, z_1)[\Phi(v_2 \otimes \cdots \otimes Y_V(v_{i+1}, z_{i+1} - \zeta)Y_V(v_{i+2}, z_{i+2} - \zeta)\mathbf{1} \otimes \cdots \otimes v_{n+2})](z_2, ..., \zeta, ..., z_{n+2})
\end{aligned}
$$

and

$$
\begin{aligned}
&[(E_W^{(1,0)} \circ_2 \Phi) \circ_{i+1} E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[(E_W^{(1,0)} \circ_2 \Phi)(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_{i+1}, v_{i+2})](z_{i+1} - \zeta, z_{i+2} - \zeta) \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., z_{n+2})\\
=&[E_W^{(1,0)}(v_1; [\Phi(v_2 \otimes \cdots \otimes [E_V^{(2)}(v_{i+1}, v_{i+2})](z_{i+1} - \zeta, z_{i+2} - \zeta) \otimes \cdots \otimes v_{n+2})](z_2, ..., \zeta, ..., z_{n+2}))](z_1)\\
=&E(Y_W(v_1, z_1)[\Phi(v_2 \otimes \cdots \otimes Y_V(v_{i+1}, z_{i+1} - \zeta)Y_V(v_{i+2}, z_{i+2} - \zeta)\mathbf{1} \otimes \cdots \otimes v_{n+2})](z_2, ..., \zeta, ..., z_{n+2})
\end{aligned}
$$

So they are equal. □

So the second sum and the third sum without $i = 1$ differs by an index shift and a (-1) factor. That way they cancels out.

For (II), we need the following lemma:

Lemma 5.3.20.

$$E_W^{(1,0)} \circ_2 (E_W^{(0,1)} \circ_1 \Phi) = E_W^{(0,1)} \circ_1 (E_W^{(1,0)} \circ_2 f)$$

Proof. For any $v_1, ..., v_{n+2} \in V, (z_1, ..., z_{n+2}) \in F_n\mathbb{C}$, we have

$$
\begin{aligned}
&[E_W^{(1,0)} \circ_2 (E_W^{(0,1)} \circ_1 \Phi)(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[E_W^{(1,0)}(v_1; [(E_W^{(0,1)} \circ_1 \Phi)(v_2 \otimes \cdots \otimes v_{n+2})](z_2, ..., z_{n+2})](z_1)\\
=&[E_W^{(1,0)}(v_1; [E_W^{(0,1)}([\Phi(v_2 \otimes \cdots \otimes v_{n+1})](z_2, ..., z_{n+1}); v_{n+2})(z_{n+2})](z_1)
\end{aligned}
$$

$$=E(Y_W^L(v_1,z_1)Y_W^{s(R)}(v_{n+2},z_{n+2})[\Phi(v_2\otimes\cdots\otimes v_{n+1})](z_2,...,z_{n+1}),$$

and

$$\begin{aligned}&[E_W^{(0,1)}\circ_1(E_W^{(1,0)}\circ_2\Phi)(v_1\otimes\cdots\otimes v_{n+2})](z_1,...,z_{n+2})\\ =&[E_W^{(0,1)}([(E_W^{(1,0)}\circ_2\Phi)(v_1\otimes\cdots\otimes v_{n+1})](z_1,...,z_{n+1});v_{n+2})](z_{n+2})\\ =&[E_W^{(0,1)}([E_W^{(1,0)}(v_1;[\Phi(v_2\otimes\cdots\otimes v_{n+1})](z_2,...,z_{n+1}))(z_1);v_{n+2})(z_{n+2})\\ =&E(Y_W^{s(R)}(v_{n+2},z_{n+2})Y_W^L(v_1,z_1)[\Phi(v_2\otimes\cdots\otimes v_{n+1})](z_2,...,z_{n+1}),\end{aligned}$$

It follows from Theorem 5.2.20 Part (1) that these rational functions are equal. □

So the two terms in (II) add up to zero.

For (III), We need the following lemmas

Lemma 5.3.21. *If $j\le i-1$, then*

$$(\Phi\circ_j E_V^{(2)})\circ_i E_V^{(2)}=(\Phi\circ_{i-1}E_V^{(2)})\circ_j E_V^{(2)}.$$

Proof. Consider the case when $j<i-1$. Then for any $v_1,...,v_{n+2}\in V,(z_1,...,z_{n+2})\in F_n\mathbb{C}$, we have

$$\begin{aligned}&[(\Phi\circ_j E_V^{(2)})\circ_i E_V^{(2)}(v_1\otimes\cdots\otimes v_{n+2})](z_1,...,z_{n+2})\\ =&[\Phi\circ_j E_V^{(2)}(v_1\otimes\cdots\otimes[E_V^{(2)}(v_i,v_{i+1})](z_i-\zeta,z_{i+1}-\zeta)\otimes\cdots\otimes v_{n+2})](z_1,...,\zeta,...,z_{n+2})\\ =&[\Phi(v_1\otimes\cdots\otimes[E_V^{(2)}(v_j,v_{j+1})](z_j-\eta,z_{j+1}-\eta)\\ &\qquad\otimes\cdots\otimes[E_V^{(2)}(v_i,v_{i+1})](z_i-\zeta,z_{i+1}-\zeta)\otimes\cdots\otimes v_{n+2})](z_1,...,\eta,...,\zeta,...,z_{n+2})\\ =&E([\Phi(v_1\otimes\cdots\otimes Y_V(v_j,z_j-\eta)Y_V(v_{j+1},z_{j+1}-\eta)\mathbf{1}\\ &\qquad\otimes\cdots\otimes Y_V(v_i,z_i-\zeta)Y_V(v_{i+1},z_{i+1}-\zeta)\mathbf{1}\otimes\cdots\otimes v_{n+2})](z_1,...,\eta,...,\zeta,...,z_{n+2}))\end{aligned}$$

and

$$\begin{aligned}&[(\Phi\circ_{i-1}E_V^{(2)})\circ_j E_V^{(2)}(v_1\otimes\cdots\otimes v_{n+2})](z_1,...,z_{n+2})\\ =&[\Phi\circ_{i-1}E_V^{(2)}(v_1\otimes\cdots\otimes[E_V^{(2)}(v_j,v_{j+1})](z_j-\zeta,z_{j+1}-\zeta)\otimes\cdots\otimes v_{n+2})](z_1,...,\zeta,...,z_{n+2})\\ =&[\Phi(v_1\otimes\cdots\otimes[E_V^{(2)}(v_j,v_{j+1})](z_j-\zeta,z_{j+1}-\zeta)\\ &\qquad\otimes\cdots\otimes[E_V^{(2)}(v_i,v_{i+1})](z_i-\eta,z_{i+1}-\eta)\otimes\cdots\otimes v_{n+2})](z_1,...,\zeta,...,\eta,...,z_{n+2})\end{aligned}$$

$$=E([\Phi(v_1\otimes\cdots\otimes Y_V(v_j,z_j-\zeta)Y_V(v_{j+1},z_{j+1}-\zeta)\mathbf{1}$$
$$\otimes\cdots\otimes Y_V(v_i,z_i-\eta)Y_V(v_{i+1},z_{i+1}-\eta)\mathbf{1}\otimes\cdots\otimes v_{n+2})](z_1,...,\zeta,...,\eta,...,z_{n+2}))$$

Since the resulting $\overline{W}$-valued rational functions are independent of the choice of ζ and η, they are equal.

Now consider the case when $j=i-1$. Then for any $v_1,...,v_{n+2}\in V,(z_1,...,z_{n+2})\in F_n\mathbb{C}$, we compute the left-hand-side as follows:

$$[(\Phi\circ_{i-1}E_V^{(2)})\circ_i E_V^{(2)}(v_1\otimes\cdots\otimes v_{n+2})](z_1,...,z_{n+2})$$
$$=[\Phi\circ_{i-1}E_V^{(2)}(v_1\otimes\cdots\otimes[E_V^{(2)}(v_i,v_{i+1})](z_i-\zeta,z_{i+1}-\zeta)\otimes\cdots\otimes v_{n+2})](z_1,...,\zeta,...,z_{n+2})$$
$$=[\Phi(v_1\otimes\cdots\otimes[E_V^{(2)}(v_{i-1},[E_V^{(2)}(v_i,v_{i+1})](z_i-\zeta,z_{i+1}-\zeta))](z_{i-1}-\eta,\zeta-\eta)$$
$$\otimes\cdots\otimes v_{n+2})](z_1,...,z_{i-2},\eta,z_{i+2}...,z_{n+2})$$
$$=E([\Phi(v_1\otimes\cdots\otimes Y_V(v_{i-1},z_{i-1}-\eta)Y_V(Y_V(v_i,z_i-\zeta)Y_V(v_{i+1},z_{i+1}-\zeta)\mathbf{1},\zeta-\eta)\mathbf{1}$$
$$\otimes v_{i+2}\otimes\cdots\otimes v_{n+2})](z_1,...,z_{i-2},\eta,z_{i+2}...,z_{n+2})),$$
$$=E([\Phi(v_1\otimes\cdots\otimes Y_V(v_{i-1},z_{i-1}-\eta)Y_V(v_i,z_i-\eta)Y_V(v_{i+1},z_{i+1}-\eta)Y_V(\mathbf{1},\zeta-\eta)\mathbf{1}$$
$$\otimes v_{i+2}\otimes\cdots\otimes v_{n+2})](z_1,...,z_{i-2},\eta,z_{i+2}...,z_{n+2})),$$
$$=E([\Phi(v_1\otimes\cdots\otimes Y_V(v_{i-1},z_{i-1}-\eta)Y_V(v_i,z_i-\eta)Y_V(v_{i+1},z_{i+1}-\eta)\mathbf{1}\otimes\cdots\otimes v_{n+2})]$$
$$(z_1,...,z_{i-2},\eta,z_{i+2}...,z_{n+2}))$$

where the fourth equality follows from the associativity in V, the fifth equality follows from the identity property of the vacuum. Also by Definition 5.3.8, the resulting rational function is independent of η.

Now we compute the right-hand-side as follows:

$$[(\Phi\circ_{i-1}E_V^{(2)})\circ_{i-1}E_V^{(2)}(v_1\otimes\cdots\otimes v_{n+2})](z_1,...,z_{n+2})$$
$$=[\Phi\circ_{i-1}E_V^{(2)}(v_1\otimes\cdots\otimes[E_V^{(2)}(v_{i-1},v_i)](z_{i-1}-\zeta,z_i-\zeta)\otimes\cdots\otimes v_{n+2})](z_1,...,z_{i-2},\zeta,z_{i+1},...,z_{n+2})$$
$$=[\Phi(v_1\otimes\cdots\otimes[E_V^{(2)}([E_V^{(2)}(v_{i-1},v_i)](z_{i-1}-\zeta,z_i-\zeta),v_{i+1})](\zeta-\eta,z_{i+1}-\eta)$$
$$\otimes\cdots\otimes v_{n+2})](z_1,...,z_{i-2},\eta,z_{i+2},...,z_{n+2})$$
$$=E([\Phi(v_1\otimes\cdots\otimes Y_V(Y_V(v_{i-1},z_{i-1}-\zeta)Y_V(v_i,z_i-\zeta)\mathbf{1},\zeta-\eta)Y_V(v_{i+1},z_{i+1}-\eta)\mathbf{1}$$
$$\otimes\cdots\otimes v_{n+2})](z_1,...,z_{i-2},\eta,z_{i+2},...,z_{n+2}))$$

$$=E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_{i-1}, z_{i-1} - \eta) Y_V(v_i, z_i - \eta) Y_V(\mathbf{1}, \zeta - \eta) Y_V(v_{i+1}, z_{i+1} - \eta)\mathbf{1}$$
$$\otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-2}, \eta, z_{i+2}, ..., z_{n+2}))$$
$$=E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_{i-1}, z_{i-1} - \eta) Y_V(v_i, z_i - \eta) Y_V(v_{i+1}, z_{i+1} - \eta)\mathbf{1} \otimes \cdots \otimes v_{n+2})]$$
$$(z_1, ..., z_{i-2}, \eta, z_{i+2}, ..., z_{n+2}))$$

where the fourth equality follows from the associativity extended to $\widehat{V}$-valued rational functions (see Theorem 5.2.20 Part (2)), the fifth equality follows from the identity property of the vacuum. Also by Definition 5.3.8, the resulting rational function is independent of η. So the left-hand-side and the right-hand-side are equal.

□

Lemma 5.3.22. *If $j \geq i$, then*

$$(\Phi \circ_j E_V^{(2)}) \circ_i E_V^{(2)} = (\Phi \circ_i E_V^{(2)}) \circ_{j+1} E_V^{(2)}.$$

Proof. Consider the case when $j > i$. Then for any $v_1, ..., v_{n+2} \in V, (z_1, ..., z_{n+2}) \in F_n\mathbb{C}$, we have

$$[(\Phi \circ_j E_V^{(2)}) \circ_i E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})$$
$$=[\Phi \circ_j E_V^{(2)}(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i, v_{i+1})](z_i - \zeta, z_{i+1} - \zeta) \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., z_{n+2})$$
$$=[\Phi(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i, v_{i+1})](z_i - \zeta, z_{i+1} - \zeta)$$
$$\otimes \cdots \otimes [E_V^{(2)}(v_j, v_{j+1})](z_j - \eta, z_{j+1} - \eta) \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., \eta, ..., z_{n+2})$$
$$=E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \zeta) Y_V(v_{i+1}, z_{i+1} - \zeta)\mathbf{1}$$
$$\otimes \cdots \otimes Y_V(v_j, z_j - \eta) Y_V(v_{j+1}, z_{j+1} - \eta)\mathbf{1} \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., \eta, ..., z_{n+2}))$$

and

$$[(\Phi \circ_{i-1} E_V^{(2)}) \circ_j E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})$$
$$=[\Phi \circ_{i-1} E_V^{(2)}(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_j, v_{j+1})](z_j - \zeta, z_{j+1} - \zeta) \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., z_{n+2})$$
$$=[\Phi(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i, v_{i+1})](z_i - \zeta, z_{i+1} - \zeta)$$
$$\otimes \cdots \otimes [E_V^{(2)}(v_j, v_{j+1})](z_j - \eta, z_{j+1} - \eta) \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., \eta, ..., z_{n+2})$$
$$=E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \zeta) Y_V(v_{i+1}, z_{i+1} - \zeta)\mathbf{1}$$

$$\otimes \cdots \otimes Y_V(v_j, z_j - \eta) Y_V(v_{j+1}, z_{j+1} - \eta)\mathbf{1} \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., \eta, ..., z_{n+2}))$$

They are equal because the resulting $\overline{W}$-valued rational functions are independent of ζ and η.

Now consider the case when $j = i$. Then for any $v_1, ..., v_{n+2} \in V, (z_1, ..., z_{n+2}) \in F_n\mathbb{C}$, we compute the left-hand-side as follows:

$$\begin{aligned}
&[(\Phi \circ_i E_V^{(2)}) \circ_i E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[\Phi \circ_i E_V^{(2)}(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i, v_{i+1})](z_i - \zeta, z_{i+1} - \zeta) \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-1}, \zeta, z_{i+2}, ..., z_{n+2})\\
=&[\Phi(v_1 \otimes \cdots \otimes [E_V^{(2)}([E_V^{(2)}(v_i, v_{i+1})](z_i - \zeta, z_{i+1} - \zeta), v_{i+2})](\zeta - \eta, z_{i+2} - \eta)\\
&\qquad \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-1}, \eta, z_{i+3}, ..., z_{n+2})\\
=&E([\Phi(v_1 \otimes \cdots \otimes Y_V(Y_V(v_i, z_i - \zeta) Y_V(v_{i+1}, z_{i+1} - \zeta)\mathbf{1}, \zeta - \eta) Y_V(v_{i+2}, z_{i+2} - \eta)\mathbf{1}\\
&\qquad \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-1}, \eta, z_{i+3}, ..., z_{n+2}))\\
=&E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \eta) Y_V(v_{i+1}, z_{i+1} - \eta) Y_V(\mathbf{1}, \zeta - \eta) Y_V(v_{i+2}, z_{i+2} - \eta)\mathbf{1}\\
&\qquad \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-1}, \eta, z_{i+3}, ..., z_{n+2}))\\
=&E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \eta) Y_V(v_{i+1}, z_{i+1} - \eta) Y_V(v_{i+2}, z_{i+2} - \eta)\mathbf{1} \otimes \cdots \otimes v_{n+2})]\\
&\quad (z_1, ..., z_{i-1}, \eta, z_{i+3}, ..., z_{n+2}))
\end{aligned}$$

where the fourth equality follows from the associativity extended to $\widehat{V}$-valued rational functions (see Theorem 5.2.20 Part (2)), the fifth equality follows from the identity property of the vacuum. Also by Definition 5.3.8, the resulting rational function is independent of η.

Now we compute the right-hand-side as follows

$$\begin{aligned}
&[(\Phi \circ_i E_V^{(2)}) \circ_{i+1} E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[\Phi \circ_{i-1} E_V^{(2)}(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_{i+1}, v_{i+2})](z_{i+1} - \zeta, z_{i+2} - \zeta) \otimes \cdots \otimes v_{n+2})](z_1, ..., \zeta, ..., z_{n+2})\\
=&[\Phi(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i, [E_V^{(2)}(v_{i+1}, v_{i+2})](z_{i+1} - \zeta, z_{i+2} - \zeta))](z_i - \eta, \zeta - \eta)\\
&\qquad \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-1}, \eta, z_{i+3} ..., z_{n+2})\\
=&E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \eta) Y_V(Y_V(v_{i+1}, z_{i+1} - \zeta) Y_V(z_{i+2} - \zeta)\mathbf{1}, \zeta - \eta)\mathbf{1}\\
&\qquad \otimes v_{i+3} \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-1}, \eta, z_{i+3} ..., z_{n+2})),
\end{aligned}$$

$$=E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \eta) Y_V(v_{i+1}, z_{i+1} - \eta) Y_V(z_{i+2} - \eta) Y_V(\mathbf{1}, \zeta - \eta)\mathbf{1}$$

$$\otimes v_{i+3} \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{i-1}, \eta, z_{i+3} ..., z_{n+2})),$$

$$=E([\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \eta) Y_V(v_{i+1}, z_{i+1} - \eta) Y_V(z_{i+2} - \eta)\mathbf{1} \otimes v_{i+3} \otimes \cdots \otimes v_{n+2})]$$

$$(z_1, ..., z_{i-1}, \eta, z_{i+3} ..., z_{n+2})).$$

where the fourth equality follows from the associativity in V, the fifth equality follows from the identity property of the vacuum. The resulting $\overline{W}$-valued rational function is independent of η. So the left-hand-side and the right-hand-side are equal. □

Once we proved these two lemmas, we write (III) as

$$\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(-1)^{i+j}(\Phi \circ_j E_V^{(2)}) \circ_i E_V^{(2)} + \sum_{i=1}^{n}\sum_{j=i}^{n}(-1)^{i+j}(\Phi \circ_j E_V^{(2)}) \circ_i E_V^{(2)}$$

Here the first sum starts from $i = 2$ because when $i = 1$, the inner sum does not exist. Similarly the second sum ends at $i = n$ because when $i = n + 1$, the inner sum does not exist. The first sum is computed as follows

$$\begin{aligned}
&\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(-1)^{i+j}(\Phi \circ_j E_V^{(2)}) \circ_i E_V^{(2)} && \\
=&\sum_{i=2}^{n+1}\sum_{j=1}^{i-1}(-1)^{i+j}(\Phi \circ_{i-1} E_V^{(2)}) \circ_j E_V^{(2)} && \text{use the identity above}\\
=&\sum_{j=1}^{n}\sum_{i=j+1}^{n+1}(-1)^{i+j}(\Phi \circ_{i-1} E_V^{(2)}) \circ_j E_V^{(2)} && \text{change the order of summation}\\
=&\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}(-1)^{i+j}(\Phi \circ_{j-1} E_V^{(2)}) \circ_i E_V^{(2)} && \text{interchange } i \text{ and } j\\
=&\sum_{i=1}^{n}\sum_{j=i}^{n}(-1)^{i+j+1}(\Phi \circ_j E_V^{(2)}) \circ_i E_V^{(2)} && \text{shift the index } j
\end{aligned}$$

So the first sum is precisely the negative of the second sum. Thus the two sums add up to be zero.

For (IV), we need the following lemma

Lemma 5.3.23.

$$(E_W^{(0,1)} \circ_1 \Phi) \circ_{n+1} E_V^{(2)} = E_W^{(0,1)} \circ_1 (E_W^{(0,1)} \circ_1 \Phi)$$

Proof. For any $v_1, ..., v_{n+2} \in V, (z_1, ..., z_{n+2}) \in F_n\mathbb{C}$, we compute the left-hand-side as follows:

$$
\begin{aligned}
&[(E_W^{(0,1)} \circ_1 \Phi) \circ_{n+1} E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[E_W^{(0,1)} \circ_1 \Phi(v_1 \otimes \cdots \otimes v_n \otimes [E_V^{(2)}(v_{n+1}, v_{n+2})](z_{n+1} - \zeta, z_{n+2} - \zeta))](z_1, ..., z_n, \zeta)\\
=&[E_W^{(0,1)}([\Phi(v_1 \otimes \cdots \otimes v_n)](z_1, ..., z_n); [E_V^{(2)}(v_{n+1}, v_{n+2})](z_{n+1} - \zeta, z_{n+2} - \zeta))](\zeta)\\
=&E(Y_W^{s(R)}(Y_V(v_{n+1}, z_{n+1} - \zeta)Y_V(v_{n+2}, z_{n+2} - \zeta)\mathbf{1}, \zeta)[\Phi(v_1, ..., v_n)](z_1, ..., z_n))\\
=&E(Y_W^{s(R)}(v_{n+2}, z_{n+2})Y_W^{s(R)}(v_{n+1}, z_{n+1})Y_W^{s(R)}(\mathbf{1}, \zeta)[\Phi(v_1, ..., v_n)](z_1, ..., z_n))\\
=&E(Y_W^{s(R)}(v_{n+2}, z_{n+2})Y_W^{s(R)}(v_{n+1}, z_{n+1})[\Phi(v_1, ..., v_n)](z_1, ..., z_n)),
\end{aligned}
$$

where the fourth equality follows from the associativity of $Y_W^{s(R)}$ extended to $\overline{W}$-valued rational functions (see Theorem 5.2.20 Part (3)). The fifth equality follows from the identity property of vacuum.

Now we compute the right-hand-side as follows:

$$
\begin{aligned}
&[E_W^{(0,1)} \circ_1 (E_W^{(0,1)} \circ_1 \Phi)(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[E_W^{(0,1)}([E_W^{(0,1)} \circ_1 \Phi(v_1 \otimes \cdots \otimes v_{n+1})](z_1, ..., z_{n+1});, v_{n+2})](z_{n+2})\\
=&[E_W^{(0,1)}([E_W^{(0,1)}([\Phi(v_1 \otimes \cdots \otimes v_{n+1})](z_1, ..., z_n);, v_{n+1})](z_{n+1});, v_{n+2})](z_{n+2})\\
=&E(Y_W^{s(R)}(v_{n+2}, z_{n+2})Y_W^{s(R)}(v_{n+1}, z_{n+1})[\Phi(v_1 \otimes \cdots \otimes v_n)](z_1, ...z_n))
\end{aligned}
$$

So it is equal to the left-hand-side. □

So the $(n + 1)$-th term in the first sum cancels out with the third term.

We also need the following lemma

Lemma 5.3.24. $(E_W^{(0,1)} \circ_1 \Phi) \circ_i E_V^{(2)} = E_W^{(0,1)} \circ_1 (\Phi \circ_i E_V^{(2)})$

Proof. For $v_1, ..., v_{n+2} \in V, (z_1, ..., z_{n+2}) \in F_n\mathbb{C}$,

$$
\begin{aligned}
&[(E_W^{(0,1)} \circ_1 \Phi) \circ_i E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2})\\
=&[E_W^{(0,1)} \circ_1 \Phi(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i \otimes v_{i+1})](z_i - \zeta, z_{i+1} - \zeta) \otimes \cdots \otimes v_{n+2}](z_1, ..., \zeta, ..., z_{n+2})\\
=&[E_W^{(0,1)}]([\Phi(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i \otimes v_{i+1})](z_i - \zeta, z_{i+1} - \zeta) \otimes \cdots \otimes v_{n+1})](z_1, ..., \zeta, ..., z_{n+1}); v_{n+2})(z_{n+2})\\
=&E(Y_W^{s(R)}(v_{n+2}, z_{n+2})[\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \zeta)Y_V(v_{i+1}, z_{i+1} - \zeta)\mathbf{1} \otimes \cdots \otimes v_{n+1})](z_1, ..., \zeta, ..., z_{n+1}))
\end{aligned}
$$

and

$$
\begin{aligned}
&[E_W^{(0,1)} \circ_1 (\Phi \circ_i E_V^{(2)})(v_1 \otimes \cdots \otimes v_{n+2})](z_1, ..., z_{n+2}) \\
=&[E_W^{(0,1)}([\Phi \circ_i E_V^{(2)}(v_1 \otimes \cdots \otimes v_{n+1})](z_1, ..., z_{n+1}); v_{n+2})](z_{n+2}) \\
=&[E_W^{(0,1)}]([\Phi(v_1 \otimes \cdots \otimes [E_V^{(2)}(v_i \otimes v_{i+1})](z_i - \zeta, z_{i+1} - \zeta) \otimes \cdots \otimes v_{n+1})](z_1, ..., \zeta, ..., z_{n+1}); v_{n+2})(z_{n+2}) \\
=&E(Y_W^{s(R)}(v_{n+2}, z_{n+2})[\Phi(v_1 \otimes \cdots \otimes Y_V(v_i, z_i - \zeta)Y_V(v_{i+1}, z_{i+1} - \zeta)\mathbf{1} \otimes \cdots \otimes v_{n+1})](z_1, ..., \zeta, ..., z_{n+1}))
\end{aligned}
$$

So they are equal. □

Therefore, the rest of the first sum cancels out with the second sum.

□

Remark 5.3.25. We remind the readers again that all the equalities in the lemmas above are in the space of $\overline{W}$-valued rational functions. The only requirements on the parameters $z_1, ..., z_{n+1}$ is that they are mutually distinct to each other.

We have given the definitions of $\hat{C}_m^n(V, W)$ and $\hat{\delta}_m^n$ for all integers $m \geq 1, n \geq 1$. Here we discuss the case $n = 0$.

Definition 5.3.26. We define $\hat{C}^0(V, W)$ to be the set of *vaccum-like* vectors $w \in W$, i.e., $w \in W_{(0)}$ and $D_W w = 0$

Proposition 5.3.27. *Let $w \in W$ be a vaccum-like vector. Then for every $v \in V, Y_W^L(v, x)w \in W[[x]], Y_W^{s(R)}(v, x)w \in W[[x]]$.*

Proof. Fix $v \in V$. From the D-commutator formula, we have

$$\frac{d}{dx} Y_W^L(v, x)w = D_W Y_W^L(v, x).$$

Thus for the series $e^{-xD_W} Y_W^L(v, x)w$, we have

$$\frac{d}{dx}\left(e^{-xD_W} Y_W^L(v, x)w\right) = -e^{-xD_W} D_W Y_W^L(v, x)w + e^{-xD_W} D_W Y_W^L(v, x) = 0,$$

which then shows $e^{-xD_W} Y_W^L(v, x)w$ has only constant term. If we denote this constant term by $v_{-1}w$, then $Y_W^L(v, x)w = e^{xD_W} v_{-1}w \in W[[x]]$. One similarly proves the conclusion for $Y_W^{s(R)}$. □

Definition 5.3.28. We define $\hat{\delta}^0 : \hat{C}^0(V,W) \to \mathrm{Hom}(V, \widetilde{W_z})$ by the following: for $w \in \hat{C}^0(V,W)$

$$((\hat{\delta}^0(w))(v))(z) = E(Y_W^L(v,z)w - Y_W^{s(R)}(v,z)w)$$

Proposition 5.3.29. *For every $w \in \hat{C}^0(V,W)$, $\hat{\delta}^0(w) \in C^1_\infty(V,W)$, and $\hat{\delta}^1(\hat{\delta}^0(w)) = 0$*

Proof. It is easy to check that $\hat{\delta}^0(w)$ satisfies the **d**-conjugation property and D-derivative property. From the arguments in Example 5.2.10, $\hat{\delta}^0(w)$ is composable with any numbers of vertex operators. The last conclusion follows from a computation that is essentially the same as those in Theorem 5.3.17. $\square$

Thus we proved the following theorem:

Theorem 5.3.30. *For any $m \in \mathbb{Z}_+$, the following sequence*

$$\hat{C}^0(V,W) \xrightarrow{\hat{\delta}^0} \hat{C}^1_m(V,W) \xrightarrow{\hat{\delta}^1_m} \hat{C}^2_{m-1}(V,W) \xrightarrow{\hat{\delta}^2_m} \hat{C}^3_{m-2}(V,W) \to \cdots \to \hat{C}^{m+1}_0(V,W)$$

forms a cochain complex. The following sequence

$$\hat{C}^0(V,W) \xrightarrow{\hat{\delta}^0} \hat{C}^1_\infty(V,W) \xrightarrow{\hat{\delta}^1} \hat{C}^2_\infty(V,W) \xrightarrow{\hat{\delta}^2_\infty} \hat{C}^3_\infty(V,W) \to \cdots \hat{C}^m_\infty(V,W) \cdots$$

forms a cochain complex.

5.3.4 Cohomology groups

Definition 5.3.31. For every $n \in \mathbb{N}$, the *n-th cohomology group* is defined as

$$\hat{H}^n_\infty(V,W) = \ker \hat{\delta}^n_\infty / \mathrm{im} \hat{\delta}^{n-1}_\infty$$

Remark 5.3.32. We can similarly define the cohomology groups $\hat{H}^n_m(V,W)$ with $\hat{C}^n_m(V,W)$.

5.3.5 Derivations and the first cohomology

In this section we will study the first cohomology $\hat{H}^1_\infty(V,W)$ and prove isomorphic to the vector space spanned by the outer derivations.

Definition 5.3.33. A linear map $f : V \to W$ is a *derivation* if f is of weight 0, and

$$f(Y_V(u_1, x)u_2) = Y_W^L(u_1, x)f(u_2) + Y_W^R(f(u_1), x)u_2 \in W[[x, x^{-1}]]$$

The space consisting of derivations will be denoted by $\mathrm{Der}(V, W)$.

Lemma 5.3.34. *Let $f : V \to W$ be a derivation. Then*

1. $f(\mathbf{1}) = 0$.

2. *For $v \in V$, $f(e^{zD_V}v) = e^{zD_W}f(v)$. Thus $f(Y_V(v, z)\mathbf{1}) = Y_W^R(f(v), z)\mathbf{1}$.*

3. *The map $\Phi_f : V \to \widetilde{W}_z$ defined by*

$$(\Phi_f(v))(z) = f(e^{zD_V}v)$$

is in $\hat{C}^1_\infty(V, W)$

Proof. 1. We compute as follows:

$$f(\mathbf{1}) = f(Y_V(\mathbf{1}, z)\mathbf{1}) = Y_W^L(\mathbf{1}, z)f(\mathbf{1}) + Y_W^R(f(\mathbf{1}), z)\mathbf{1} = f(\mathbf{1}) + e^{zD_W}f(\mathbf{1}).$$

So $e^{zD_W}f(\mathbf{1}) = 0$ for every $z \in \mathbb{C}$. In particular, $f(\mathbf{1}) = 0$.

2. We compute as follows:

$$f(e^{zD_V}v) = f(Y_V(v, z)\mathbf{1}) = Y_W^L(v, z)f(\mathbf{1}) + Y_W^R(f(v), z)\mathbf{1} = e^{zD_W}f(v).$$

3. We first verify that Φ_f satisfies the $\mathbf{d}$-conjugation property and D-derivative property. We compute as follows: fix $a \in \mathbb{C}$,

$$\begin{aligned} a^{d_W}(\Phi_f(v))(z) &= a^{d_W}Y_W^R(f(v), z)\mathbf{1} = Y_W^R(a^{d_W}f(v), a^{-1}z)a^{d_W}\mathbf{1} = a^{wt(f(v))}Y_W^R(f(v), a^{-1}z)\mathbf{1} \\ &= a^{wt(v)}(\Phi_f(v))(a^{-1}z) = (\Phi_f(a^{d_W}v))(a^{-1}z). \end{aligned}$$

For any $z_0 \in \mathbb{C}$,

$$\begin{aligned} (\Phi_f(e^{z_0D_V}v))(z) &= f(e^{z_0D_V}e^{zD_V}v) = f(e^{(z_0+z)D_V}v) = (\Phi_f(v))(z_0 + z); \\ e^{z_0D_W}(\Phi_f(v))(z) &= e^{z_0D_W}e^{zD_W}f(v) = e^{(z_0+z)D_W}f(v) = (\Phi_f(v))(z + z_0). \end{aligned}$$

The conclusion that Φ_f is composable with any numbers of vertex operators essentially follows from following, that for every $m \in \mathbb{Z}_+$, every $\alpha_0 \in \mathbb{N}, \alpha_1 \in \mathbb{Z}_+$ such that $\alpha_0 + \alpha_1 = m+1$, every $l = 0, ..., \alpha_0$, every $i = 1, ..., \alpha_1$, the series

$$\begin{aligned}&Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_l^{(0)}, z_l^{(0)}) Y_W^{s(R)}(u_{l+1}^{(0)}, z_{l+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)})\\ &\quad \cdot Y_W^L(v_1^{(1)}, z_1^{(1)}) \cdots Y_W^L(v_{i-1}^{(1)}, z_{i-1}^{(1)}) Y_W^R(f(v_i), z_i^{(1)}) Y_V(v_{i+1}^{(1)}, z_{i+1}^{(1)}) \cdots Y_V(v_{\alpha_1}^{(1)}, z_{\alpha_1}^{(1)}) \mathbf{1}\end{aligned}$$

is the expansion of the $\overline{W}$-valued rational function

$$\begin{aligned}&E(Y_W^L(u_1^{(0)}, z_1^{(0)}) \cdots Y_W^L(u_l^{(0)}, z_l^{(0)}) Y_W^L(v_1^{(1)}, z_1^{(1)}) \cdots Y_W^L(v_{i-1}^{(1)}, z_{i-1}^{(1)})\\ &\quad \cdot Y_W^{s(R)}(u_{l+1}^{(0)}, z_{l+1}^{(0)}) \cdots Y_W^{s(R)}(u_{\alpha_0}^{(0)}, z_{\alpha_0}^{(0)}) Y_W^{s(R)}(v_n^{(1)}, z_n^{(1)}) \cdots Y_W^{s(R)}(v_{i+1}^{(1)}, z_{i+1}^{(1)}) e^{z_i^{(1)} D_W} f(v_i^{(1)}))\end{aligned}$$

in the region $|z_1^{(0)}| > \cdots > |z_{\alpha_0}^{(0)}| > |z_1^{(1)}| > \cdots > |z_{\alpha_1}^{(1)}|$. For brevity, we shall not elaborate the technical details.

□

Theorem 5.3.35. *As vector spaces, Der(V, W) is isomorphic to* $\ker \hat{\delta}_m^1 \subseteq \hat{C}_m^1(V, W)$ *for every* $m \geq 1$.

Proof. Given $f \in \mathrm{Der}(V, W)$, we prove that the map $\Phi_f : V \to \widetilde{W}_z$ defined by $(\Phi_f(v))(z) = e^{zD_W} f(v)$ is an element in $\ker \hat{\delta}_\infty^1$. Fix $u_1, u_2 \in V$. Then for $|z_1| > |z_2|, |\zeta| > |z_1 - \zeta| > |z_2 - \zeta|$,

$$(\Phi_f(Y_V(u_1, z_1 - \zeta) Y_V(u_2, z_2 - \zeta)\mathbf{1}))(\zeta) = f(e^{\zeta D_V} Y_V(u_1, z_1 - \zeta) Y_V(u_2, z_2 - \zeta)\mathbf{1})$$

Since $e^{\zeta D_V} Y_V(u_1, z_1 - \zeta) Y_V(u_2, z_2 - \zeta)\mathbf{1} = Y_V(u_1, z_1) e^{z_2 D_V} u_2$ for every $z_1, z_2, \zeta \in \mathbb{C}$ such that $|z_1| > |z_2|$, we have

$$E[(\Phi_f(Y_V(u_1, z_1 - \zeta) Y_V(u_2, z_2 - \zeta)\mathbf{1}))(\zeta)] = E[f(Y_V(u_1, z_1) e^{z_2 D_V} u_2)]$$

Now let $z_1, z_2 \in \mathbb{C}$ such that $|z_1| > |z_2|$, we have

$$\begin{aligned}f(Y_V(u_1, z_1) e^{z_2 D_V} u_2) &= \sum_{i=0}^{\infty} \frac{1}{i!} z_2^i f(Y_V(u_1, z_1) D_V^i u_2)\\ &= \sum_{i=0}^{\infty} \frac{1}{i!} z_2^i (Y_W^L(u_1, z_1) f(D_V^i u_2) + Y_W^R(f(u_1), z_1) D_V^i u_2)\\ &= Y_W^L(u_1, z_1) f(e^{z_2} D_V u_2) + Y_W^R(f(u_1), z_1) e^{z_2 D_V} u_2\end{aligned}$$

$$= Y_W^L(u_1, z_1)(\Phi_f(u_2))(z_2) + e^{z_1 D_W} Y_W^{s(R)}(e^{z_2 D_V} u_2, -z_1) f(u_1)$$

From the D-conjugation property, the second term on the right-hand-side converges absolutely when $|z_1| > |z_2|$ to the $\overline{W}$-valued function $E(e^{z_1 D_W} Y_W^{s(R)}(u_2, -z_1+z_2) f(u_1))$, which is equal to $E(Y_W^s(R)(u_2, z_2) e^{z_1 D_W} f(u_1)) = E(Y_W^{s(R)}(u_2, z_2)(\Phi_f(u_1))(z_1)$. Thus we have

$$E((\Phi_f(Y_V(u_1, z_1-\zeta) Y_V(u_2, z_2-\zeta)\mathbf{1}))(\zeta)) = E(Y_W^L(u_1, z_1)(\Phi_f(u_2))(z_2)) + E(Y_W^{s(R)}(u_2, z_2)(\Phi_f(u_1))(z_1)$$

and hence $\Phi_f \in \ker \hat{\delta}^1_\infty$.

Conversely, given $\Phi \in \ker \hat{\delta}^1_m \subseteq \hat{C}^1_m(V, W)$, we note first that for every $v \in V$, $(\Phi(v))(z)$ can be expanded as a series in $W[[z, z^{-1}]]$. Also, since $z = 0$ is not a pole of the $\overline{W}$-valued rational function $(\Phi(v))(z)$, the series expansion of $(\Phi(v))(z)$ has no negative powers. Thus one can evaluate $z = 0$ to get a map $(\Phi(\cdot))(0) : V \to W$. Denote this map by $f : V \to W$. By D-derivative property, $(\Phi(v))(z) = e^{z D_W} f(v) = f(e^{z D_V} v)$. Thus by $\Phi \in \ker \hat{\delta}^1_m$,

$$E(f(e^{\zeta D_V} Y_V(u_1, z_1-\zeta) Y_V(u_2, z_2-\zeta)\mathbf{1})) = E(Y_W^L(u_1, z_1) e^{z_2 D_W} f(u_2) + E(Y_W^{s(R)}(u_2, z_2) e^{z_1 D_W} f(u_1)).$$

From the discussions above, this can be simplified as

$$E(f(Y_V(u_1, z_1) e^{z_2 D_V} u_2)) = E(Y_W^L(u_1, z_1) e^{z_2 D_W} f(u_2)) + E(Y_W^R(f(u_1), z_1) e^{z_2 D_W} u_2)$$

Evaluate $z_2 = 0$ to see that

$$E(f(Y_V(u_1, z_1) u_2)) = E(Y_W^L(u_1, z_1) f(u_2)) + E(Y_W^R(f(u_1), z_1) u_2)$$

So the rational function defined by the complex series $f(Y_V(u_1, z_1)u_2), Y_W^L(u_1, z_1) f(u_2)$ and $Y_W^R(f(u_1), z_1)u_2)$ satisfies the above equation. Hence one can find a $r > 0$, such that for every $z_1 \in \mathbb{C}, 0 < |z| < r$,

$$f(Y_V(u_1, z_1)u_2) = Y_W^L(u_1, z_1) f(u_2) + Y_W^R(f(u_1), z_1)u_2.$$

Therefore,

$$f(Y_V(u_1, x)u_2) = Y_W^L(u_1, x) f(u_2) + Y_W^R(f(u_1), x)u_2.$$

as formal series in $W[[x, x^{-1}]]$. $\square$

Remark 5.3.36. In particular, we showed that for every $m \in \mathbb{Z}_+$, $\ker \hat{\delta}^1_m$ are isomorphic to $\mathrm{Der}(V, W)$. So for maps $\Phi : V \to \widetilde{W}_z$ that are cocycles, being composable with one vertex operator is equivalent to being composable with any number of vertex operators. Whether or not this holds for cocycles in $\hat{C}^2_m(V, W)$ is still a problem that needs further investigation.

Definition 5.3.37. A linear map $f : V \to W$ is an *inner derivation* if there exists $w \in C^0(V, W)$ such that for every $v \in V$,

$$f(v) = Y_W^L(v, 0)w - Y_W^{s(R)}(v, 0)w$$

Proposition 5.3.38. *Let $f : V \to W$ be an inner derivation. Then $\Phi_f : V \to \widetilde{W}_z$ is in the image of $\hat{\delta}^0$. Conversely, for any Φ in the image of $\hat{\delta}^0$, $(\Phi(\cdot))(0)$ is an inner derivation. Thus the space of inner derivations and $\mathrm{im}\hat{\delta}^0(V, W)$ are isomorphic as vector spaces.*

Proof. The first conclusion follows from the following computation

$$(\Phi_f(v))(z) = f(e^{zD_V}v) = Y_W^L(e^{zD_V}v, 0)w - Y_W^{s(R)}(e^{zD_V}v, 0)w) = Y_W^L(v, z)w - Y_W^{s(R)}(v, z)w = \hat{\delta}^0(w).$$

To see the second conclusion, we note that there exists $w \in \hat{C}^0(V, W)$, such that

$$(\Phi(v))(z) = Y_W^L(v, z)w - Y_W^{s(R)}(v, z)w$$

Since Φ is in $\ker \hat{\delta}^1$, $(\Phi(\cdot))(0) : V \to W$ is a derivation. The conclusion is then seen from evaluating $z = 0$ in the above equality. □

Theorem 5.3.39. *For every $m \in \mathbb{Z}_+$,*

$$\hat{H}^1_m(V, W) \simeq \{\textit{Derivation } V \to W\}/\{\textit{Inner derivation } V \to W\}$$

Chapter 6

Reductivity theorem

6.1 The classical theory

Let A be an associative algebra over a field k and M be a finite dimensional left A-module. Regarding the complete reducibility of M, we have the following criterion:

Theorem 6.1.1. *If for every A-bimodule B, the first Hochschlid cohomology $HH^1(A, B) = 0$, then every A-submodule M_2 of M is complemented, i.e., there exists a left A-submodule M_1 such that $M = M_1 \oplus M_2$.*

We shall give a rough sketch of the proof, as it helps to understand the more technical case for MOSVAs.

Proof. Let M_2 be an A-submodule of M and let N_1 be a subspace of M such that $M = N_1 \oplus M_2$. Let $\pi_1 : M \to N_1$ and $\pi_2 : M \to M_2$ be the corresponding projection operators. Then for every $a, b \in A, m \in M$, since

$$(ab)x = a(bx)$$

we have

$$\pi_1(ab)m = (\pi_1 a \circ \pi_1 b)m$$

$$\pi_2(ab)m = (\pi_2 a \circ \pi_1 b)m + (\pi_2 a \circ \pi_2 b)m$$

Now let $H = Hom_k(N_1, M_2)$. We define a A-bimodule structure on H by defining the left and right action of $a \in A$ on $f \in H$

$$(L(a)f)(x) = af(x), (fa)(x) = f(\pi_1 ax)$$

(since $(f(ab))(x) = f(\pi_1(ab)x) = f(\pi_1 a \circ \pi_1 bx) = (fa)(\pi_1 bx) = ((fa)b)(x)$ for every $a, b \in A$, we have a right action). Consider the map $\Delta : A \to H$ defined by

$$\Delta(a) = \pi_2 a : x \mapsto \pi_2 ax$$

We check that Δ is a $(1, H)$-cocycle.

$$a\Delta(b) - \Delta(ab) + \Delta(a)b : x \mapsto a\pi_2 bx - \pi_2 abx + \pi_2 a\pi_1 bx = 0$$

Since $HH^1(A, H) = 0$, there exists a map $f : N_1 \to M_2$ such that

$$\Delta(a) = af - fa$$

Define $F : N_1 \to M$ by $F(x) = x - f(x)$ and let M_1 be the image of F. We claim that M_2 is a A-submodule: if $m \in M_2$, then there exists $x \in N_1$ such that $m = x - f(x)$. Then

$$\begin{aligned} am &= ax - af(x) = ax - (\Delta(a))(x) + (fa)(x) \\ &= ax - \pi_2 ax + f(\pi_1 ax) \\ &= ax - \pi_2 ax + f(ax - \pi_2(ax)) \\ &= F(ax - \pi_2 ax) \in \mathrm{Im} F \end{aligned}$$

Since M is finite-dimensional, the map f is nilpotent and hence $F(x)$ is invertible. This shows that N_1 and M_1 are isomorphic as vector spaces. Hence $M = M_1 \oplus M_2$ $\square$

To generalize the above arguments to MOSVAs, the following questions must be answered:

1. Does the MOSVA-analogue of $\pi_1 a$ and $\pi_2 a$ make sense?

2. What is the MOSVA-analogue of $\mathrm{Hom}_k(N_1, M_1)$?

3. How to guarantee the isomorphism of N_1 and M_1 for the MOSVA-analogue.

The answers to the above questions are not trivial. One has to deal with the technical issues brought by the related convergence.

6.2 The space $H(W_1, W_2)$ and $H^N(W_1, W_2)$

In this section we give the definition of the spaces analogous to the space of linear functions of two left V-modules. We start by defining the space of rational functions which take values in the algebraic completion of a module and possess possible poles at zero. Then we will choose linear maps from one module to the space of rational functions to construct the spaces.

6.2.1 The space $\widehat{W}_\zeta$ and $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta))$

Definition 6.2.1. A $\overline{W}$-valued rational function in ζ with the only possible pole at ζ is a map

$$f : F_1\mathbb{C} = \mathbb{C}^\times \to \overline{W}$$
$$\zeta \mapsto f(\zeta)$$

satisfying the following condition: there exists an integer k and a series $g(x) \in W[[x]]$, for every $w' \in W'$

$$\zeta^k \langle w', f(\zeta)\rangle = \langle w', g(\zeta)\rangle$$

is a polynomial function.

In other words, $f(\zeta)$ is obtained from evaluating $x = \zeta$ of a lower-truncated formal series $\sum_{n\in\mathbb{Z}} f_n x^{-n-1}$ with all coefficients $f_n \in W$. We know that $f_n = 0$ when n is sufficiently large. For any fixed $w' \in W'$, $\langle w', f_n\rangle = 0$ when n is sufficiently negative.

Remark 6.2.2. For each $u \in V$, the vertex operator $Y^L_{W_2}(u, z)$ can act on it in the usual sense:

$$Y^L_{W_2}(u, z)f(\zeta) = \sum_{k\in\mathbb{Z}} Y^L_{W_2}(u, z)f_k\zeta^{-k-1}$$

We can also interpret the vertex operator action of u as

$$Y^L_{W_2}(u, z)f(\zeta) = \sum_{m\in\mathbb{C}} Y^L_{W_2}(u, z)\pi^W_m(f(\zeta))$$

Notation 6.2.3. We denote by $(\widehat{W})_\zeta$ the space of $\overline{W}$-valued rational functions in ζ satisfying Definition 6.2.1. By an abuse of terminology, we will still call elements of $(\widehat{W})_\zeta$ as $\widehat{W}$-valued rational function.

Definition 6.2.4. Let W_1, W_2 be two grading-restricted left V-modules. Let $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$ be the space of $\mathbb{C}$-linear maps from W_1 to $(\widehat{W_2})_\zeta$. Let a be a nonzero real number. Define $a^{\mathbf{d}_H}$ operator on $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$ by

$$((a^{\mathbf{d}_H}\phi)(w_1))(\zeta) = a^{\mathbf{d}_{W_2}}(\phi(a^{-\mathbf{d}_{W_1}}w_1)(a^{-1}\zeta)$$

It is easy to see that $a^{\mathbf{d}_H}$ is a linear operator on $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$. By taking the derivative with respect to the variable a, one recovers the $\mathbf{d}_H$ operator on $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$. We will call eigenvectors of $\mathbf{d}_H$ as $\mathbf{d}_H$-homoegenous maps. The corresponding eigenvalues will be referred as weights.

Remark 6.2.5. For $\phi \in \mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$ homogenous of weight m, and for $w_1 \in W$ homogeneous, we know that

$$a^m(\phi(w_1))(\zeta) = ((a^{\mathbf{d}_H}\phi)(w_1))(\zeta) = a^{\mathbf{d}_{W_2}}(\phi(a^{\mathbf{d}_{W_1}}w_1))(a^{-1}\zeta)$$

Expanded in series, the coefficients of ζ^{-k-1} on both sides are

$$a^m\phi(w_1)_k = a^{\mathbf{d}_{W_2}}\phi(w_1)_k a^{-\mathrm{wt}\ w_1+k+1}$$

Thus as an element in W_2, $\phi(w_1)_k$ is also homogeneous of weight $\mathrm{wt}\ w_1 + m - k - 1$, for each integer k.

Definition 6.2.6. For $\phi \in \mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$, we define the operator D_H by

$$((D_H\phi)(w_1))(\zeta) = \frac{\partial}{\partial\zeta}(\phi(w_1))(\zeta)$$

Proposition 6.2.7. *If* $\phi \in \mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$ *is homogeneous of weight* $\mathrm{wt}\ \phi$*, then* $D_H\phi$ *is also homogeneous of weight* $\mathrm{wt}\ \phi + 1$*.*

Proof. First we compute the components of $D_H\phi$:

$$[(D_H\phi)(w_1)](\zeta) = \sum_{n\in\mathbb{Z}}(-n-1)\phi_n(w_1)\zeta^{-n-2}$$

Since ϕ is of weight $\mathrm{wt}\ \phi$,

$$[(a^{\mathbf{d}_H}\phi)(w_1)](\zeta) = a^{\mathbf{d}_{W_2}}(\phi(a^{-\mathbf{d}_{W_1}}w_1))(a^{-1}\zeta)$$

$$= \sum_{n\in\mathbb{Z}} a^{-\mathrm{wt}\ w_1+n+1} a^{\mathbf{d}_{W_2}} \phi_n(w_1)\zeta^{-n-1}$$

$$= \sum_{n\in\mathbb{Z}} a^{\mathrm{wt}\ \phi} \phi_n(w_1)\zeta^{-n-1}$$

So we have $a^{\mathbf{d}_{W_2}}\phi_n(w_1) = a^{\mathrm{wt}\ \phi+\mathrm{wt}\ w_1-n-1}\phi_n(w_1)$. In particular, $\phi_n(w_1)$ is also homogeneous of weight $\mathrm{wt}\ \phi + \mathrm{wt}\ w_1 - n - 1$. Therefore,

$$\begin{aligned}
[(a^{\mathbf{d}_H} D_H\phi)w_1](\zeta) &= a^{\mathbf{d}_{W_2}}[(D_H\phi)(a^{-\mathbf{d}_{W_1}}w_1)](a^{-1}\zeta) \\
&= \sum_{n\in\mathbb{Z}} a^{-\mathrm{wt}\ w_1} a^{\mathbf{d}_{W_2}}\phi_n(w_1) a^{n+2}(-n-1)\zeta^{-n-2} \\
&= \sum_{n\in\mathbb{Z}} a^{-\mathrm{wt}\ w_1+\mathrm{wt}\ \phi+\mathrm{wt}\ w_1-n-1}\phi_n(w_1)a^{n+2}(-n-1)\zeta^{-n-2} \\
&= \sum_{n\in\mathbb{Z}} a^{\mathrm{wt}\ \phi+1}\phi_n(w_1)(-n-1)\zeta^{-n-2} \\
&= a^{\mathrm{wt}\ \phi+1}[(D_H\phi)w_1](\zeta)
\end{aligned}$$

$\square$

Remark 6.2.8. The space $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$ is too large for our use. So we proceed to find an appropriate subspace.

6.2.2 Composable condition

Definition 6.2.9. Let $m \in \mathbb{Z}_+$. A linear map $\phi : W_1 \to (\widehat{W_2})_\zeta$ is composable with m vertex operators if for every $l = 0, ..., m, u_1, ..., u_m \in V, w_1 \in W_1, w_2' \in W_2'$, the series

$$\langle w_2', Y_{W_2}^L(u_1, z_1)\cdots Y_{W_2}^L(u_l, z_l)(\phi(Y_{W_1}^L(u_{l+1}, z_{l+1})\cdots Y_{W_1}^L(u_m, z_m)w_1))(\zeta)\rangle$$

converges absolutely in the region

$$|z_1| > \cdots > |z_l| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$$

to a rational function with poles at

$$z_i = 0, i = 1, ..., m$$

$$z_i - z_j = 0, 1 \le i < j \le m$$

$$z_i - \zeta = 0, i = 1, ..., m$$

$$\zeta = 0.$$

Moreover, there exists integers $r \in \mathbb{N}$ depending only on the choice of ϕ and w_1, $q_i \in \mathbb{N}$ for $i = 1, ..., m$ depending only on the choice of u_i and w_1, $p_i \in \mathbb{N}$ for $i = 1, ..., m$ depending only on the choice of u_i and ϕ, $p_{ij} \in \mathbb{N}$ for $1 \leq i < j \leq m$ depending only on the choice of u_i and u_j, and $g(z_1, ..., z_m, \zeta) \in W[[z_1, ..., z_m, \zeta]]$, such that for every $w_2' \in W_2'$,

$$\zeta^r \prod_{i=1}^m z_i^{q_i} \prod_{i=1}^m (z_i - \zeta)^{p_i} \prod_{1 \leq i < j \leq m} (z_i - z_j)^{p_{ij}}$$
$$\cdot R\left(\langle w_2', Y_{W_2}^L(u_1, z_1) \cdots Y_{W_2}^L(u_l, z_l)(\phi(Y_{W_1}^L(u_{l+1}, z_{l+1}) \cdots Y_{W_1}^L(u_m, z_m)w_1))(\zeta)\rangle\right)$$

is a polynomial and is equal to $\langle w_2', g(z_1, ..., z_m, \zeta)\rangle$. Moreover, for $i = 1, ..., m$, p_i depends only on u_i and ϕ; for $1 \leq i < j \leq m$, p_{ij} depends only on u_i and u_j.

Remark 6.2.10. The second part of the composable condition holds if and only if all the following conditions on the rational function hold:

1. The order of the pole $\zeta = 0$ are bounded above by an integer r depending only on the choice of ϕ and w_1.

2. For $i = 1, ..., m$, there exists $q_i \in \mathbb{N}$ depending only on the choice of u_i and w_1, such that the order of pole $z_i = 0$ is bounded above by p_i.

3. For $i = 1, ..., m$, there exists $p_i \in \mathbb{N}$ depending only on the choice of u_i and ϕ, such that the order of pole $z_i = \zeta$ is bounded above by p_i.

4. For each $i, j = 1, ..., m, i \neq j$, there exists $p_{ij} \in \mathbb{N}$ depending only on the choice of u_i and u_j, such that the order of pole $z_i = z_j$ is bounded above by p_{ij}.

Remark 6.2.11. With the same argument as in Remark 5.2.2,

$$Y_{W_2}^L(u_1, z_1) \cdots Y_{W_2}^L(u_l, z_l)(\phi(Y_{W_1}^L(u_{l+1}, z_{l+1}) \cdots Y_{W_1}^L(u_m, z_m)w_1))(\zeta)$$

is a series in $W[[z_1, z_1^{-1}, ..., z_m, z_m^{-1}, \zeta, \zeta^{-1}]]$ that converges absolutely to a $\overline{W}$-valued rational function with only possible poles specified as above.

Remark 6.2.12. One could also say that the following formal series

$$y^r \prod_{i=1}^{m} x_i^{q_i} \prod_{i=1}^{m} (x_i - y)^{p_i} \prod_{1 \le i < j \le m} (x_i - x_j)^{p_{ij}}$$
$$\cdot Y_{W_2}^L(u_1, x_1) \cdots Y_{W_2}^L(u_l, x_l)(\phi(Y_{W_1}^L(u_{l+1}, x_{l+1}) \cdots Y_{W_1}^L(u_m, x_m)w_1))(y)$$

has no negative powers and thus is in $W[[x_1, ..., x_m, y]]$.

Proposition 6.2.13. *Let* $\phi : W_1 \to (\widehat{W_2})_\zeta$ *be a map that is composable with* m *vertex operators, then* $D_H\phi : W_1 \to (\widehat{W_2})_\zeta$ *is also composable with* m *vertex operators, then* $D_H\phi$

Proof. It suffices to notice that the series

$$\langle w_2', Y_{W_2}^L(u_1, z_1) \cdots Y_{W_2}^L(u_l, z_l)(D_H\phi(Y_{W_1}^L(u_{l+1}, z_{l+1}) \cdots Y_{W_1}^L(u_m, z_m)w_1))(\zeta)\rangle$$

is simply the partial derivative of ζ of the absolutely convergent series

$$\langle w_2', Y_{W_2}^L(u_1, z_1) \cdots Y_{W_2}^L(u_l, z_l)(\phi(Y_{W_1}^L(u_{l+1}, z_{l+1}) \cdots Y_{W_1}^L(u_m, z_m)w_1))(\zeta)\rangle.$$

The conclusion then follows from complex analysis. $\square$

Remark 6.2.14. We will be mainly using the composable condition formulated with a different set of parameters: for every $l = 0, ..., m, u_1, ..., u_m \in V, w_1 \in W_1, w_2' \in W_2'$, the series

$$\langle w_2', Y_{W_2}^L(u_1, z_1+\zeta) \cdots Y_{W_2}^L(u_l, z_l+\zeta)(\phi(Y_{W_1}^L(u_{l+1}, z_{l+2}+\zeta) \cdots Y_{W_1}^L(u_{m-1}, z_m+\zeta)Y_{W_1}^L(u_m, \zeta)w_1))(\zeta)\rangle$$

converges absolutely in the region

$$|z_1 + \zeta| > \cdots > |z_m + \zeta| > |\zeta| > 0$$

to a rational function with poles at

$$z_i + \zeta = 0, i = 1, ..., m$$
$$z_i - z_j = 0, 1 \le i < j \le m$$
$$z_i = 0, i = 1, ..., m$$
$$\zeta = 0.$$

Moreover, there exists integers $r \in \mathbb{N}$ depending only on the choice of ϕ and w_1, $q_i \in \mathbb{N}$ for $i = 1, ..., m$ depending only on the choice of u_i and w_1, $p_i \in \mathbb{N}$ for $i = 1, ..., m$ depending only on the choice of u_i and ϕ, $p_{ij} \in \mathbb{N}$ for $1 \leq i < j \leq m$ depending only on the choice of u_i and u_j, and $g(z_1, ..., z_m, \zeta) \in W[[z_1, ..., z_m, \zeta]]$, such that for every $w_2' \in W_2'$,

$$\zeta^r \prod_{i=1}^{m}(z_i + \zeta)^{q_i} \prod_{i=1}^{m} z_i^{p_i} \prod_{1 \leq i < j \leq m} (z_i - z_j)^{p_{ij}}$$
$$\cdot R\big(\langle w_2', Y_{W_2}^L(u_1, z_1 + \zeta) \cdots Y_{W_2}^L(u_l, z_l + \zeta)$$
$$\cdot(\phi(Y_{W_1}^L(u_{l+1}, z_{l+2} + \zeta) \cdots Y_{W_1}^L(u_{m-1}, z_m + \zeta) Y_{W_1}^L(u_m, \zeta) w_1))(z_{l+1} + \zeta)\rangle\big)$$

is a polynomial and is equal to $\langle w_2', g(z_1, ..., z_m, \zeta)\rangle$.

6.2.3 N-weight-degree condition

Definition 6.2.15. Let $m \in \mathbb{Z}_+$ and $\phi : W_1 \to (\widehat{W_2})_\zeta$ be a homogeneous linear map that is composable with m vertex operators. For $N \in \mathbb{Z}$, ϕ is said to satisfy the N-weight-degree condition if for every homogeneous $u_1, ..., u_m \in V$, the Laurent series expansion of the rational function

$$R\big(\langle w_2', Y_{W_2}^L(u_1, z_1) \cdots Y_{W_2}^L(u_l, z_l)$$
$$\cdot (\phi(Y_{W_1}^L(u_{l+1}, z_{l+1}) \cdots Y_{W_1}^L(u_m, z_m) w_1))(\zeta)\rangle\big) \tag{6.1}$$

in the region $|z_m| > |z_1 - z_m| > \cdots > |z_l - z_m| > |\zeta - z_m| > |z_{l+1} - z_m| > \cdots > |z_{m-1} - z_m| > 0$, as a Laurent series in $z_1 - z_m, ..., z_l - z_m, \zeta - z_m, z_{l+1} - z_m, ..., z_{m-1} - z_m$ with coefficients in $\mathbb{C}[[z_m, z_m^{-1}]]$, has total degree at least as large as $N - \text{wt } u_1 - \cdots - \text{wt } u_m - \text{wt } \phi$.

Proposition 6.2.16. *Let $\phi : W_1 \to (\widehat{W_2})_\zeta$ be a homogeneous linear map that satisfies the N-weight-degree condition. Then $D_H\phi : W_1 \to (\widehat{W_2})_\zeta$ also satisfies the condition.*

Proof. It suffices to notice that the total degree of each monomial in expansion of the rational function

$$R\big(\langle w_2', Y_{W_2}^L(u_1, z_1) \cdots Y_{W_2}^L(u_l, z_l)$$

$$\cdot\,(\phi(Y^L_{W_1}(u_{l+1},z_{l+1})\cdots Y^L_{W_1}(u_m,z_m)w_1))(\zeta)\rangle\big)$$

in the region $|z_m| > |z_1 - z_m| > \cdots > |z_l - z_m| > |\zeta - z_m| > |z_{l+1} - z_m| > \cdots > |z_{m-1} - z_m| > 0$ as Laurent series in $z_1 - z_m, ..., z_l - z_m, \zeta - z_m, z_{l+1} - z_m, ..., z_{m-1} - z_m$ with coefficients in $\mathbb{C}[[z_m, z_m^{-1}]]$, is one less than that for ϕ. Thus the lowest total degree is at least as large as

$$N - (\sum_{i=1}^{m} \text{wt } u_i + \text{wt } \phi) - 1 = N - (\sum_{i=1}^{m} \text{wt } u_i + \text{wt } D_H\phi).$$

Hence $D_H\phi$ also satisfies the N-weight-degree condition. □

Remark 6.2.17. We will be mainly using the condition formulated with a different set of parameters: for every homogeneous $u_1, ..., u_m \in V$, the Laurent series expansion of the rational function

$$R\big(\langle w_2', Y^L_{W_2}(u_1, z_1+\zeta)\cdots Y^L_{W_2}(u_l, z_l+\zeta)$$
$$\cdot\,(\phi(Y^L_{W_1}(u_{l+1}, z_{l+2}+\zeta)\cdots Y^L_{W_1}(u_{m-1}, z_m+\zeta)Y^L_{W_1}(u_m,\zeta)w_1))(z_{l+1}+\zeta)\rangle\big) \tag{6.2}$$

in the region $|\zeta| > |z_1| > \cdots > |z_m| > 0$, viewed as a Laurent series in $z_1, ..., z_m$ with coefficients in $\mathbb{C}[[\zeta, \zeta^{-1}]]$, has total degree at least as large as $N - \text{wt } u_1 - \cdots - \text{wt } u_m - \text{wt } \phi$.

Remark 6.2.18. We note that the order of $z_1, ..., z_m$ in the definition given in Remark 6.2.17 can be switched. From the composability, we know that the rational function (6.2) has poles at $\zeta = 0$, $z_i = 0, z_i + \zeta = 0$ for $i = 1, ..., m$, and $z_i = z_j$ for $1 \le i < j \le m$. The expansion in $|\zeta| > |z_1| > \cdots > |z_m|$ amounts to expand all the negative powers of $z_i + \zeta$ as a power series in z_i for $i = 1, ..., m$, and expand all the negative powers of $z_i - z_j$ as a powers series in z_j for $1 \le i < j \le m$. It is easy to see that the lower bound of the total degree is not changed if we expand the negative powers of $z_i - z_j$ instead as a power series in z_i for some $i \neq j$.

Example 6.2.19. Let V be a MOSVA satisfying the pole-order condition. Let W be a left V-module. Then for $W_1 = W, W_2 = W$ and for a fixed $v \in V$, we consider the map

$$\begin{aligned}\phi_v :& W \to (\widehat{W})_\zeta \\ & w \mapsto Y_W^L(v,\zeta)w\end{aligned}$$

First we note that this map is homogeneous of weight wt v.

$$\begin{aligned}a^{\mathbf{d}_{W_2}}(\phi_v(w_1))(\zeta) &= a^{\mathbf{d}_{W_2}} Y_W^L(v,\zeta)w_1 \\ &= Y_W^L(a^{\mathbf{d}_V} v, a\zeta)a^{\mathbf{d}_{W_1}} w_1 \\ &= a^{\mathrm{wt}\ v}(\phi_v(a^{\mathbf{d}_{W_1}} w_1))(a\zeta)\end{aligned}$$

where the second equality follows from the $\mathbf{d}$-conjugation property of Y_W^L.

Then we verify that this map is composable with any number of vertex operators. The convergence part follows directly from the rationality of products: for every $m \in \mathbb{Z}_+$, $w' \in W', u_1, ..., u_{m+1} \in V, w \in W$

$$\langle w', Y_W^L(u_1, z_1) \cdots Y_W^L(u_{m+1}, z_{m+1})w\rangle$$

converges absolutely to a rational function of the form

$$\frac{f(z_1, ..., z_m, z_{m+1})}{\prod_{i=1}^{m+1} z_i^{p_i} \prod_{1\le i<j\le m+1} (z_i - z_j)^{p_{ij}}}$$

For any fixed l between 1 and $m+1$, replace u_l by v, z_l by ζ and u_j by u_{j-1}, z_j by z_{j-1} for every $j = l+1, ..., m+1$, to see the that ϕ_v converges to the rational function in the specified area. The second part of composable condition follows from the pole-order condition.

Let N be a lower bound of the weights of V. We show that ϕ satisfies the N-weight-degree condition. Note that the series

$$\begin{aligned}&\langle w', Y_W^L(Y_V(u_1, z_1) \cdots\cdots Y_V(u_l, z_l) \\ &\qquad \cdot Y_V(v, z_{l+1})Y_V(u_{l+1}, z_{l+1}) \cdots Y_V(u_m, z_m)u_{m+1}, \zeta)w\rangle \\ = &\sum_{n_1,...,n_m\in\mathbb{Z}} \langle w', Y_W^L((Y_V)_{n_1}(u_1) \cdots (Y_V)_{n_l}(u_l) \\ &\qquad \cdot (Y_V)_{n_{l+1}}(v)(Y_V)_{n_{l+2}}(u_{l+1}) \cdots (Y_V)_{n_m}(u_{m-1})u_m, \zeta)w\rangle z_1^{-n_1-1} \cdots z_m^{-n_m-1}\end{aligned}$$

is the expansion of the rational function

$$R\left(\langle w_2', Y_W^L(u_1, z_1+\zeta)\cdots Y_W^L(u_l, z_l+\zeta)\right.$$
$$\left.\cdot\, Y_W^L(v, z_{l+1}+\zeta)Y_W^L(u_{l+1}, z_{l+2}+\zeta)\cdots Y_W^L(u_{m-1}, z_m+\zeta)Y_W^L(u_m,\zeta)w_1\rangle\right)$$

which is precisely

$$R\left(\langle w', Y_W^L(u_1, z_1+\zeta)\cdots Y_W^L(u_l, z_l+\zeta)\right.$$
$$\left.\cdot\,(\phi_v(Y_W^L(u_{l+1}, z_{l+2}+\zeta)\cdots Y_W^L(u_{m-1}, z_m+\zeta)Y_W^L(u_m,\zeta)w))(z_{l+1}+\zeta)\rangle\right)$$

in the region

$$|\zeta| > |z_1| > \cdots > |z_m| > 0$$

Since for every $n_1, ..., n_m \in \mathbb{Z}, \text{wt } (Y_V)_{n_1}(u_1)\cdots(Y_V)_{n_m}(u_m)u_{m+1} \geq N$, the indices $n_1, ..., n_m$ satisfies the following inequality

$$\text{wt } u_1 + \cdots + \text{wt } u_m + \text{wt } u_{m+1} - n_1 - \cdots - n_m - m \geq N,$$

In other words, as a Laurent series with coefficients in $\mathbb{C}[[z_{m+1}, z_{m+1}^{-1}]]$ in variables $z_1, ..., z_m$, the total degree has to be at least as large as $N - (\text{wt } u_1 + \cdots + \text{wt } u_m + \text{wt } u_{m+1})$.

Remark 6.2.20. We remind the reader that every grading-restricted vertex algebra with nonnegative grading satisfies N-weight-degree condition, where N can be any lower bound of the weights of the vertex algebra. So the story in this chapter works for any such vertex algebras.

Definition 6.2.21. Let $H(W_1, W_2)$ be the subspace of $\text{Hom}(W_1, (\widehat{W_2})_\zeta)$ spanned by all the homogeneous maps ϕ that are composable with any number of vertex operators. For $N \in \mathbb{Z}$, let $H^N(W_1, W_2)$ be the subspace of $H(W_1, W_2)$ spanned by the maps satisfying the N-weight-degree condition.

Notation 6.2.22. For simplicity, we will use the notations H and H^N respectively for the space $H(W_1, W_2)$ and $H^N(W_1, W_2)$. We will use $H_{[n]}$ and $H_{[n]}^N$ respectively to denote the weight n subspace of H and H^N. A generic element in H or H^N will be denoted by ϕ.

6.3 V-bimodule structure on $H^N(W_1, W_2)$

In this section, we endow the space $H^N(W_1, W_2)$ a V-bimodule structure. In general, this bimodule is not necessarily grading-restricted.

6.3.1 The left V-module structure on H^N

Definition 6.3.1. We define the following left action of V on H^N: for $u \in V, \phi \in H^N$

$$[(Y_H^L(u,z)\phi)(w_1)](\zeta) = \iota_{\zeta z} E(Y_{W_2}^L(u, z+\zeta)(\phi(w_1))(\zeta))$$

The definition is understood as follows:

1. Since ϕ is composable with any number of vertex operators, in particular, for any $w_2' \in W_2'$, the complex series

$$\langle w_2', Y_{W_2}^L(u, z+\zeta)(\phi(w_1))(\zeta)\rangle$$

converges absolutely in the region

$$|z+\zeta| > |\zeta| > 0$$

to a rational function with poles at $z+\zeta = 0, \zeta = 0, z = 0$. Because of the N-weight-degree condition, the order of the pole $z = 0$ is controlled above by $\mathrm{wt}\ u + \mathrm{wt}\ \phi - N$.

2. Expand the negative powers of $(z+\zeta)$ as a power series of z. Then we get a complex series

$$\sum_{n\in\mathbb{Z}} c_n(w_2' \otimes u \otimes \phi \otimes w_1; \zeta) z^{-n-1}$$

Note that because of the N-weight-degree condition, the lowest power of z is at least as large as $-\mathrm{wt}\ u - \mathrm{wt}\ \phi + N$. The coefficient of each power of z is then a rational function with poles at $\zeta = 0$.

3. For each fixed $\zeta \neq 0$, the linear functional $w_2' \mapsto c_n(w_2' \otimes u \otimes \phi \otimes w_1; \zeta)$ would then be defining an element in $\widehat{W_2}$. From the definition of composability, note that the series $\sum_{n\in\mathbb{Z}} c_n(\cdot \otimes u \otimes \phi \otimes w_1; \zeta) z^{-n-1}$ of $\widehat{W}$-elements is obtained from expanding

a localized formal series in $W[[z,\zeta]][z^{-1},\zeta^{-1},(z+\zeta)^{-1}]$. Thus the coefficients $c_n(\cdot\otimes u\otimes\phi\otimes w_1;\zeta)$ is a lower-truncated series in $W_2((\zeta))$. One then checks easily that for every $n\in\mathbb{Z}$ and every fixed $u\in V, w_1\in W_1$, $c_n(\cdot\otimes u\otimes\phi\otimes w_1;\zeta)\in(\widehat{W_2})\zeta$. Therefore, for each $n\in\mathbb{Z}$, the map

$$w_1\mapsto c_n(\cdot\otimes u\otimes\phi\otimes w_1;\zeta)$$

is an element of $\mathrm{Hom}(W_1,(\widehat{W_2})_\zeta)$. If we denote this map by $(Y_H^L)_n(u)\phi$, then we have

$$Y_H^L(u,z)\phi=\sum_{n\in\mathbb{Z}}(Y_H^L)_n(u)\phi z^{-n-1}$$

as a series with coefficients in $\mathrm{Hom}(W_1,(\widehat{W_2})_\zeta)$.

4. In terms of coefficients, we have the following identity

$$\begin{aligned}\sum_{n_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}}&\langle w_2',(Y_{W_2}^L)_{n_1}(u)\phi_{n_2}(w_1)\rangle(z+\zeta)^{-n_1-1}\zeta^{-n_2-1}\\&=\sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}}\langle w_2',[(Y_H^L)_{n_1}(u)\phi]_{n_2}(w_1)\rangle\zeta^{-n_2-1}z^{-n_1-1}\end{aligned}$$

when $|z+\zeta|>|\zeta|>|z|>0$. Since both sides are Laurent series expansion of a rational function, the order of summation can be switched. Thus we can also write

$$\sum_{n\in\mathbb{Z}}\langle w_2',(Y_{W_2}^L)_n(u)(\phi(w_1))(\zeta)\rangle(z+\zeta)^{-n-1}=\sum_{n\in\mathbb{Z}}\langle w_2',[((Y_H^L)_n(u)\phi)(w_1)](\zeta)\rangle z^{-n-1}$$

Because of the second part of the composability, we can drop $w_2'\in W_2'$, to see that when $|z+\zeta|>|\zeta|>|z|>0$,

$$\sum_{n\in\mathbb{Z}}(Y_{W_2}^L)_n(u)(\phi(w_1))(\zeta)(z+\zeta)^{-n-1}=\sum_{n\in\mathbb{Z}}[((Y_H^L)_n(u)\phi)(w_1)](\zeta)z^{-n-1}\qquad(6.3)$$

Proposition 6.3.2. *Let $\phi:W_1\to(\widehat{W_2})_\zeta$ be a homogeneous map composable with any number of vertex operators. Then for every $n\in\mathbb{Z},u\in V$,*

$$\mathrm{wt}\ (Y_H^L)_n(u)\phi=\mathrm{wt}\ u-n-1+\mathrm{wt}\ \phi.$$

Proof. Let $|z+\zeta|>|\zeta|>|z|>0$. Since

$$\langle w_2',a^{\mathbf{d}_{W_2}}Y_{W_2}^L(u,z+\zeta)(\phi(w_1))(\zeta)\rangle=\langle w_2',Y_{W_2}^L(a^{\mathbf{d}_V}u,az+a\zeta)a^{\mathbf{d}_{W_2}}(\phi(w_1))(\zeta)\rangle$$

$$= \langle w_2', Y_{W_2}^L(a^{\mathbf{d}_V}u, az + a\zeta)a^{\text{wt }\phi}(\phi(a^{\mathbf{d}_{W_1}}w_1))(a\zeta)\rangle$$

we thus have

$$\langle w_2', a^{\mathbf{d}_{W_2}}[(Y_H^L(u,z)\phi)(w_1)](\zeta)\rangle = \langle w_2', [(Y_H^L(a^{\mathbf{d}_V}u, az)\phi)(a^{\mathbf{d}_{W_1}}w_1)](a\zeta)\rangle$$

Expand $Y_H^L(u,z)\phi$ as the sum of $(Y_H^L)_n(u)\phi$ and use the definition of the $\mathbf{d}_H$, we see that

$$a^{\mathbf{d}_H}(Y_H^L)_n(u)\phi = a^{\text{wt }u + \text{wt }\phi - n - 1}$$

□

Remark 6.3.3. As a consequence of the N-weight-degree condition, we know that $-n-1 \geq -\text{wt } u - \text{wt } \phi + N$, and thus

$$\text{wt } (Y_H^L)_n(u)\phi = \text{wt } (u) - n - 1 + \text{wt } \phi \geq N.$$

Proposition 6.3.4. *Let $\phi : W_1 \to (\widehat{W_2})_\zeta$ be a homogeneous map composable with any number of vertex operators. Then for every $n \in \mathbb{Z}, u \in V$, $(Y_H^L)_n(u)\phi$ is also composable with any number of vertex operators and satisfies the N-weight-degree condition.*

Proof. Fix any $m \in \mathbb{Z}_+$. Since ϕ is composable with, in particular, $m+1$ vertex operators, for every $l = 0, ..., m, w_2' \in W_2', u_1, ..., u_m \in V$ and $w_1 \in W_1$, the following complex series

$$\langle w_2', Y_{W_2}^L(u_1, z_1) \cdots Y_{W_2}^L(u_l, z_l) Y_{W_2}^L(u, z+\zeta)(\phi(Y_{W_1}^L(u_{l+1}, z_{l+1}) \cdots Y_{W_1}^L(u_m, z_m)w_1))(\zeta)\rangle$$

converges absolutely when

$$|z_1| > \cdots > |z_l| > |z+\zeta| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$$

to a rational function that is of the form

$$\frac{f(z_1, ..., z_m, z, \zeta)}{(z+\zeta)^{r_1} z^p \zeta^{r_2} \prod_{i=1}^m z_i^{q_i}(z_i - \zeta)^{p_i'}(z_i - z - \zeta)^{p_i''} \prod_{1 \leq i < j \leq m} (z_i - z_j)^{p_{ij}'''}}$$

with

1. p depends only on u and ϕ;

2. p'_i depending only to u_i and ϕ, p''_i depending only on u_i and u for $i = 1, ..., m$;

3. p'''_{ij} depending only on u_i and u_j for $1 \leq i < j \leq m$;

4. r_1 depends only on u and w_1;

5. r_2 depends only on ϕ and w_1;

6. q_i depends only on u_i and ϕ for $i = 1, ..., m$.

In particular, the series, rewritten as,

$$\sum_{n_1,\ldots,n_m\in\mathbb{Z}, n\in\mathbb{Z}} \langle w'_2, (Y^L_{W_2})_{n_1}(u_1)\cdots(Y^L_{W_2})_{n_l}(u_l)(Y^L_{W_2})_n(u)(\phi((Y^L_{W_1})_{n_{l+1}}(u_{l+1})\cdots(Y^L_{W_1})_{n_m}(u_m)w_1))(\zeta)\rangle \cdot$$
$$z_1^{-n_1-1}\cdots z_l^{-n_l-1}(z+\zeta)^{-n-1} z_{l+1}^{-n_{l+1}-1}\cdots z_m^{-n_m-1}$$

converges absolutely to the same rational function in the smaller region

$$|z_1| > \cdots > |z_l| > |\zeta| + |z|, |\zeta| - |z| > |z_{l+1}| > \cdots > |z_m| > 0, |z+\zeta| > |\zeta|, |z| > 0$$

Notice $|\zeta| > |z|$, thus $|z+\zeta| > |\zeta| > |z| > 0$. By Eqn. (6.3), we know that the following iterated series

$$\sum_{n_1,\ldots,n_m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \langle w'_2, (Y^L_{W_2})_{n_1}(u_1)\cdots(Y^L_{W_2})_{n_l}(u_l)[((Y^L_H)_n(u)\phi)((Y^L_{W_1})_{n_{l+1}}(u_{l+1})\cdots(Y^L_{W_1})_{n_m}(u_m)w_1)](\zeta)\rangle \cdot \right.$$
$$\left. z^{-n-1}\right) z_1^{-n_1-1}\cdots z_l^{-n_l-1} z_{l+1}^{-n_{l+1}-1}\cdots z_m^{-n_m-1}$$

which also converges to the same rational function. We note that the series coincides with the expansion of the rational function in the region

$$|z_1| > \cdots > |z_l| > |\zeta| + |z|, |\zeta| - |z| > |z_{l+1}| > \cdots > |z_m| > 0, |z| > 0$$

where the negative powers of

- $z + \zeta$ are expanded as a power series of z;
- $z_i - \zeta$ are expanded as a power series of ζ when $i \leq l$;
- $z_i - \zeta$ are expanded as a power series of z_i when $i > l$;

- $z_i - z - \zeta$ are expanded first as a power series of $(z+\zeta)$ when $i \le l$, then further expand the positive powers of $z+\zeta$ as polynomials in z, ζ;
- $z_i - z - \zeta$ are first expanded as a power series of $(z+z_i)$ when $i > l$, then further expand the positive powers of $z+z_i$ as polynomials in z, z_i;
- $z_i - z_j$ are expanded as a power series of z_j when $i < j$.

In particular, the order of summation can be switched. Thus we know that the following series

$$\sum_{n\in\mathbb{Z}} \langle w_2', Y_{W_2}^L(u_1,z_1)\cdots Y_{W_2}^L(u_l,z_l)[((Y_H^L)_n(u)\phi)(Y_{W_1}^L(u_{l+1},z_{l+1})\cdots Y_{W_1}^L(u_m,z_m)w_1)](\zeta)\rangle z^{-n-1}$$

converges absolutely to the same rational function when

$$|z_1| > \cdots > |z_l| > |\zeta| + |z|, |\zeta| - |z| > |z_{l+1}| > \cdots > |z_m| > 0.$$

Now we rewrite the rational function as

$$\frac{\sum_{k_1=0}^{K} f_{k_1}(z_1,...,z_m,\zeta) z^{k_1}}{[1+z/\zeta]^{r_1} z^p \zeta^{r_1+r_2} \prod_{i=1}^m z_i^{q_i}(z_i-\zeta)^{p_i'+p_i''}[1-z/(z_i-\zeta)]^{p_i''} \prod_{1\le i<j\le m}(z_i-z_j)^{p_{ij}'''}}$$

and expand the bracketed denominators in the as the power series of the second term. In other words, the expansion is done in the region

$$|\zeta| > |z| > 0, |z_i - \zeta| > |z|$$

Organize the series according to the power of z, we will obtain the following:

$$\sum_{-n-1\ge -p}\left(\sum_{k_1=0}^{K}\sum_{\substack{k_{31},...,k_{3m}\ge 0\\ k_{31}+\cdots+k_{3m}\le -n-1+p-k_1}} \frac{a_{k_2k_{31}...k_{3m}} f_{k_1}(z_1,...,z_m,\zeta)}{\zeta^{r_1+r_2+k_2}\prod_{i=1}^m z_i^{q_i}(z_i-\zeta)^{p_i'+p_i''+k_{3i}}\prod_{1\le i<j\le m}(z_i-z_j)^{p_{ij}'''}}\right) z^{-n-1}$$

where $k_2 = -n-1+p-k_1-\sum_{i=1}^m k_{3i}$ is a nonnegative integer. Thus for each fixed $n \in \mathbb{Z}$, the series

$$\langle w_2', Y_{W_2}^L(u_1,z_1)\cdots Y_{W_2}^L(u_l,z_l)[((Y_H^L)_n(u)\phi)(Y_{W_1}^L(u_{l+1},z_{l+1})\cdots Y_{W_1}^L(u_m,z_m)w_1)](\zeta)\rangle$$

converges absolutely when

$$|z_1| > \cdots > |z_l| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$$

to the rational function

$$\sum_{k_1=0}^{K} \sum_{\substack{k_{31},\ldots,k_{3m}\geq 0 \\ k_{31}+\cdots+k_{3m}\leq -n-1+p-k_1}} \frac{a_{k_2k_{31}\ldots k_{3m}} f_{k_1}(z_1,\ldots,z_m,\zeta)}{\zeta^{r_1+r_2+k_2}\prod_{i=1}^{m} z_i^{q_i}(z_i-\zeta)^{p_i'+p_i''+k_{3i}}\prod_{1\leq i<j\leq m}(z_i-z_j)^{p_{ij}'''}}$$

with the only possible poles at $\zeta = 0$, $z_i = 0, z_i = \zeta$ for $i = 1,\ldots,m$, and $z_i - z_j = 0$ for $1 \leq i < j \leq m$. This proves the first part of composability.

To see the second part of composability, it suffices to verify the conditions specified in Remark 6.2.10 in the rational function above.

1. For each $i = 1,\ldots,m$, q_i depends on u_i and w_1.
2. For each $i = 1,\ldots,m$,

   $$p_i' + p_i'' + k_{3i} \leq p_i' + p_i'' - n - 1 + p - k_1 \leq p_i' + p_i'' - n - 1 + p$$

   which depends only on u_i, u, n and ϕ. In particular, it is independent of the choice of $u_j, j \neq i$, w_1 and w_2'.
3. For each $1 \leq i < j \leq m$, p_{ij}''' depends only on the choice of u_i and u_j.
4. Since $k_2 \leq -n-1-p$, we know $r_1 + r_2 + k_2 \leq r_1 + r_2 - n - 1 - p$ which depends only on u, n, ϕ and w_1. In particular, it is independent of $u_i (i = 1,\ldots,m)$ and w_2'.

Now we check the N-weight-degree condition for each $(Y_H^L)_n(u)\phi$. Sincd $\phi \in H^N$, the expansion of the rational function

$$R\left(\langle w_2', Y_{W_2}(u_1, z_1+\zeta)\cdots Y_{W_2}(u_l, z_l+\zeta)\right.$$
$$\left. Y_{W_2}(u, z+\zeta)(\phi(Y_{W_1}(u_{l+1}, z_{l+2}+\zeta)\cdots Y_{W_1}(u_m,\zeta)w_1)(z_{l+1}+\zeta)\rangle\right)$$

in the region $|\zeta| > |z_1| > \cdots > |z_l| > |z| > |z_{l+1}| > \cdots > |z_m| > 0$ as a Laurent series in $z_1,\ldots,z_m,z$ with coefficients in $\mathbb{C}[[\zeta,\zeta^{-1}]]$ has lowest total weight at least as large as $N - (\mathrm{wt}\ u_1 + \cdots + \mathrm{wt}\ u_m + \mathrm{wt}\ u + \mathrm{wt}\ \phi)$. From Remark 6.2.18, the same condition holds for the expansion in the region

$$|\zeta| > |z_1| > \cdots > |z_l| > |z_{l+1}| > \cdots > |z_m| > |z| > 0$$

From the analysis above, we see that this expansion coincides with the following series

$$\sum_{n\in\mathbb{Z}}\langle w_2', Y_{W_2}(u_1, z_1+\zeta)\cdots Y_{W_2}(u_l, z_l+\zeta)$$
$$[((Y_H^L)_n(u)\phi)(Y_{W_1}(u_{l+1}, z_{l+2}+\zeta)\cdots Y_{W_1}(u_m,\zeta)w_1)](z_{l+1}+\zeta)\rangle z^{-n-1}$$

with the coefficients of each z^{-n-1} further expanded in the region $|\zeta|>|z_1|>\cdots>|z_m|>0$. For each monomial in this expansion, the total degree of $z_1,...,z_m,z$ is nothing but the total degree of $z_1,...,z_m$ plus $-n-1$, which is at least as large as $N-(\text{wt }u_1+\cdots+\text{wt }u_m+\text{wt }u+\phi)$. Thus, the total degree of $z_1,...,z_m$ in the expansion of the rational function

$$R\left(\langle w_2', Y_{W_2}(u_1, z_1+\zeta)\cdots Y_{W_2}(u_l, z_l+\zeta)\right.$$
$$\left.[((Y_H^L)_n(u)\phi)(Y_{W_1}(u_{l+1}, z_{l+2}+\zeta)\cdots Y_{W_1}(u_m,\zeta)w_1)](z_{l+1}+\zeta)\rangle\right)$$

in the region $|\zeta|>|z_1|>\cdots>|z_m|>0$ is at least as large as

$$N-(\sum_{i=1}^m \text{wt }u_i+\text{wt }u+\text{wt }\phi)+n+1 = N-(\sum_{i=1}^m\text{wt }u_i+\text{wt }(Y_H^L)_n(u)\phi)$$

Thus for each $n\in\mathbb{Z}$, $(Y_H^L)_n(\phi)$ also satisfies the N-weight-degree condition.

□

So we have proved that for every $\phi\in H^N$, the series $Y_H^L(u,x)=\sum_{n\in\mathbb{Z}}(Y_H^L)_n(u)\phi x^{-n-1}$ is actually a series in $H^N[[x,x^{-1}]]$, i.e., the map

$$Y_H^L: V\otimes H^N\to H^N[[x,x^{-1}]]$$

gives an action of V on H^N.

Theorem 6.3.5. *$(H^N, Y_H^L, \mathbf{d}_H, D_H)$ forms a left V-module.*

Proof. We know that H^N is graded by the eigenvalues of $\mathbf{d}_H$ operator, equipped with a vertex operator map $Y_H^L: V\otimes H\to H[[x,x^{-1}]]$, an operator $\mathbf{d}_H$ of weight 0 and an operator D_H of weight 1. Now we verify all the axioms.

1. The lower bound condition and the $\mathbf{d}$-grading condition is obviously satisfied. The $\mathbf{d}$-bracket property easily follows from the weight formula proved above.

2. The identity property follows from that of $Y_{W_2}^L$.

3. The D-derivative property follows from the computation below:

$$\begin{aligned}\left[\left(\frac{d}{dz}Y_H^L(u,z)\phi\right)(w_1)\right](\zeta) &= \frac{d}{dz}\left[(Y_H^L(u,z)\phi)(w_1)\right](\zeta)\\ &= \frac{d}{dz}[\iota_{\zeta z}E(Y_{W_2}^L(u,z+\zeta)(\phi(w_1))(\zeta))]\\ &= \iota_{wz}\left[E\left(\frac{d}{dz}Y_{W_2}^L(u,z+\zeta)(\phi(w_1))(\zeta)\right)\right]\\ &= \iota_{wz}\left[E\left(\frac{d}{d(z+\zeta)}Y_{W_2}^L(u,z+\zeta)(\phi(w_1))(\zeta)\frac{d(z+\zeta)}{dz}\right)\right]\\ &= \iota_{wz}E\left(Y_{W_2}^L(D_Vu,z+\zeta)(\phi(w_1))(\zeta)\right)\\ &= [(Y_H^L(D_Vu,z+\zeta)\phi)(w_1)](\zeta).\end{aligned}$$

The D-bracket formula follows from the computation below:

$$\begin{aligned}&[(D_HY_H^L(u,z)\phi)(w_1)](\zeta)\\ =&\frac{d}{d\zeta}\iota_{wz}[E(Y_{W_2}^L(u,z+\zeta)(\phi(w_1))(\zeta))]\\ =&\iota_{wz}\left[E\left(\frac{d}{d\zeta}Y_{W_2}^L(u,z+\zeta)(\phi(w_1))(\zeta)\right)\right]\\ =&\iota_{wz}\left[E\left(\frac{d}{d(z+\zeta)}Y_{W_2}^L(u,z+\zeta)\frac{d(z+\zeta)}{d\zeta}\right)(\phi(w_1))(\zeta)+Y_{W_2}^L(u,z+\zeta)\left(\frac{d}{d\zeta}(\phi(w_1))(\zeta)\right)\right]\\ =&\iota_{wz}[E(Y_{W_2}^L(D_Vu,z+\zeta)(\phi(w_1))(\zeta))+Y_{W_2}^L(u,z+\zeta)(D_H\phi(w_1))(\zeta)]\\ =&[(Y_H^L(D_Vu,z)\phi)(w_1)](\zeta)+[(Y_H^L(u,z)(D_H\phi))(w_1)](\zeta)\end{aligned}$$

4. We prove the rationality of products of two vertex operators elaborately The rationality of products of any numbers of vertex operators will be seen immediately from Proposition 3.1.21 and the composable condition once associativity is proved.

   Since ϕ is composable with any number of vertex operators, in particular, when $m=2, l=2$, for any $u_1,u_2\in V, w_1\in W_1, w_2'\in W_2'$, we know that

$$\begin{aligned}&\langle w_2', Y_{W_2}^L(u_1,z_1+\zeta)Y_{W_2}^L(u_2,z_2+\zeta)(\phi(w_1))(\zeta)\rangle\\ =&\sum_{n_1\in\mathbb{Z}}\left(\sum_{n_2\in\mathbb{Z}}\langle w_2',(Y_{W_2}^L)_{n_1}(u_1)(Y_{W_2}^L)_{n_2}(u_2)(\phi(w_1))(\zeta)\rangle(z_2+\zeta)^{-n_2-1}\right)(z_1+\zeta)^{-n_1-1}\end{aligned} \tag{6.4}$$

converges absolutely in the region

$$S = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |z_1 + \zeta| > |z_2 + \zeta| > |\zeta| > 0\}$$

to a rational function of the form

$$\frac{f(z_1, z_2, \zeta)}{(z_1+\zeta)^{q_1}(z_2+\zeta)^{q_2}\zeta^r z_1^{p_1} z_2^{p_2}(z_1-z_2)^{p'_{12}}} \tag{6.5}$$

Note that $(\phi(w_1))(\zeta)$ is indeed also a series. We are not expanding it for simplicity. Also, from the second part of the composable condition, we can take $r, q_1, q_2, p_1, p_2, p'_{12}$ to be sufficiently large, so that they are independent of the choice of w'_2.

The identity (6.3) shows that when $|z_2 + \zeta| > |\zeta| > |z_2| > 0$,

$$\sum_{n_2\in\mathbb{Z}} (Y_{W_2}^L)_n(u_2)(\phi(w_1))(\zeta)(z_2+\zeta)^{-n-1} = \sum_{n_2\in\mathbb{Z}} [((Y_H^L)_{n_2}(u_2)\phi)(w_1)](\zeta) z_2^{-n-1}$$

Thus the right hand side of the identity (6.4) can be rewritten as

$$\begin{aligned}
&\sum_{n_1\in\mathbb{Z}} \left(\sum_{n_2\in\mathbb{Z}} \langle w'_2, (Y_{W_2}^L)_{n_1}[((Y_H^L)_{n_2}(u_2)\phi)(w_1)](\zeta)\rangle z_2^{-n_2-1} \right) (z_1+\zeta)^{-n_1-1} \\
=&\langle w'_2, Y_{W_2}^L(u_1, z_1+\zeta)[(Y_H^L(u_2, z_2)\phi)(w_1)](\zeta)\rangle
\end{aligned} \tag{6.6}$$

Thus the series on the right-hand-side of Eqn. (6.6), as a series in $z_1 + \zeta, z_2, \zeta$, converges to the rational function (6.5) in the region

$$S^{\cap} = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |z_1 + \zeta| > |z_2 + \zeta| > |\zeta| > |z_2| > 0\}$$

Note that the rational function (6.5) can be expanded as an absolutely convergent Laurent series in the region

$$S_1 = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |z_1 + \zeta| > |\zeta| + |z_2|, |\zeta| > |z_2| > 0\}$$

by expanding the negative powers of $z_1 = z_1 + \zeta - \zeta$ as a power series of ζ, the negative powers of $z_2 + \zeta$ as a power series of z_2, and the negative powers of $z_1 - z_2 = z_1 + \zeta - (\zeta + z_2)$ as a power series of $\zeta + z_2$, then further expand the positive powers of $\zeta + z_2$ as polynomials of ζ and z_2. Note that in this expansion, the power of $z_1+\zeta$ is upper-truncated by the order of pole $z_1+\zeta=\infty$. So if we set $S_1'=S_1\cap S^\cap$, then since $S_1'\neq\emptyset$, we can apply Lemma 2.3.9 to see that the series in Eqn. (6.6) converges absolutely in S_1. In particular, one can switch the order of summation. To sum up, the series

$$\langle w_2', Y_{W_2}^L(u_1, z_1+\zeta)[(Y_H^L(u_2,z_2)\phi)(w_1)](\zeta)$$

converges absolutely to the rational function (6.5) in the region S_1 and can be written as

$$\begin{aligned}
&\sum_{n_2\in\mathbb{Z}}\left(\sum_{n_1\in\mathbb{Z}}\langle w_2', (Y_{W_2}^L)_{n_1}[((Y_H^L)_{n_2}(u_2))\phi(w_1)](\zeta)\rangle(z_1+\zeta)^{-n_1-1}\right)z_2^{-n_2-1}\\
=&\sum_{n_2\in\mathbb{Z}}Y_{W_2}^L(u_1,z_1+\zeta)[((Y_H^L)_n(u_2)\phi)(w_1)](\zeta)z_2^{-n_2-1} \qquad (6.7)
\end{aligned}$$

Now for each fixed $n_2\in\mathbb{Z}$, we use the Eqn. (6.3) again, to see that when $|z_1+\zeta|>|\zeta|>|z_1|>0$, the right-hand-side of (6.7) equals to

$$\begin{aligned}
&\sum_{n_2\in\mathbb{Z}}[(Y_H^L(u_1,z_1)(Y_H^L)_n(u_2)\phi)(w_1)](\zeta)z_2^{-n_2-1} \qquad (6.8)\\
=&\sum_{n_2\in\mathbb{Z}}\left(\sum_{n_1\in\mathbb{Z}}\langle w_2', [((Y_H^L)_{n_1}(u_1)(Y_H^L)_{n_2}(u_2)\phi)(w_1)](\zeta)\rangle z_1^{-n_1-1}\right)z_2^{-n_2-1}\\
=&\langle w_2', [(Y_H^L(u_1,z_1)Y_H^L(u_2,z_2)\phi)(w_1)](\zeta)\rangle \qquad (6.9)
\end{aligned}$$

Thus the right-hand-side of Eqn. (6.8) converges absolutely to the rational function (6.5) in the region

$$\begin{aligned}
S_1^\cap &= \{(z_1,z_2,\zeta)\in S_1 : |z_1+\zeta|>|\zeta|>|z_1|>0\}\\
&= \{(z_1,z_2,\zeta)\in\mathbb{C}^3 : |z_1+\zeta|>|\zeta|+|z_2|, |\zeta|>|z_1|>0, |\zeta|>|z_2|>0\}
\end{aligned}$$

Note that the rational function (6.5) can be expanded as an absolutely convergent Laurent series in the region

$$S_2=\{(z_1,z_2,\zeta)\in\mathbb{C}^3, |\zeta|>|z_1|>|z_2|>0\}$$

by expanding the negative powers of $z_1+\zeta$ as a power series of z_1, the negative powers of $z_2+\zeta$ as a power series of z_2, the negative powers of z_1-z_2 as a power series of z_2. Moreover, the lowest power of z_2 is bounded below by the order of pole of $z_2 = 0$. So if we set $S_2' = S_2 \cap S_1^{\cap}$, then since $S_2' \neq \emptyset$, we can apply Lemma 2.3.7 to see that the double series

$$\begin{aligned}&\langle w_2', [(Y_H^L(u_1, z_1)Y_H^L(u_2, z_2)\phi)(w_1)](\zeta)\rangle\\ =&\sum_{n_1,n_2\in\mathbb{Z}}\langle w_2', [((Y_H^L)_{n_1}(u_1)(Y_H^L)_{n_2}(u_2)\phi)(w_1)](\zeta)\rangle \end{aligned} \tag{6.10}$$

converges absolutely to the rational function 6.5 in the region S_2.

Therefore, we know that

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}\langle w_2', [(Y_H^L(u_1, z_1)Y_H^L(u_2, z_2)\phi)(w_1)](\zeta)\rangle$$

is precisely

$$\frac{f(z_1,z_2,\zeta)}{\zeta^r}\iota_{\zeta z_1}\left(\frac{1}{(z_1+\zeta)^{q_1}}\right)\iota_{\zeta z_2}\left(\frac{1}{(z_2+\zeta)^{q_2}}\right),$$

which has no negative powers of z_1, z_2. For each fixed $n_1, n_2 \in \mathbb{Z}$, the coefficient of $z_1^{-n_1-1}z_2^{-n_2-1}$ is a Laurent polynomial in ζ. We denote this coefficient by $g_{n_1n_2}(\zeta)$ and claim that the $\overline{W_2}$-valued rational function determined by

$$w_2' \mapsto g_{n_1n_2}(\zeta)$$

is an element in $(\widehat{W_2})_\zeta$.

From the second part of the composable condition, the orders of poles of the rational function (6.5) are bounded above by constants that are independent of the choice of w_2'. One sees that the series

$$Y_{W_2}^L(u_1, z_1+\zeta)Y_{W_2}^L(u_2, z_2+\zeta)(\phi(w_1))(\zeta)$$

multiplied with the denominator of (6.5) is actually a power series with coefficients in W_2. The above procedure shows that the series $[(Y_H^L(u_1, z_1)Y_H^L(u_2, z_2)\phi)(w_1)](\zeta)$ converges to the same $\overline{W_2}$-valued rational function. Therefore, after multiplied with the denominators of (6.5), we should get the same power series. Thus, the series

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}[(Y_H^L(u_1, z_1)Y_H^L(u_2, z_2)\phi)(w_1)](\zeta)$$

is the expansion of the quotient of this power series and powers of $(z_1+\zeta), (z_2+\zeta)$ and ζ. The denominators are expanded as power series of z_1 and z_2. The coefficient of each $z_1^{-n_1-1}z_2^{-n_2-1}$ is precisely the $\overline{W_2}$-valued rational function determined by $w_2' \mapsto g_{n_1n_2}(\zeta)$. Thus this function is in $(\widehat{W_2})_\zeta$.

Hence, the map

$$w_1 \mapsto \big(w_2' \mapsto g_{n_1n_2}(\zeta)\big)$$

gives an element in $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$. Thus we see that

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}Y_H^L(u_1,z_1)Y_H^L(u_2,z_2)\phi$$

is indeed a power series of with coefficients in $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$. As pointed out by Proposition 6.3.4, these coefficients are necessarily in H. Thus what we get is a power series in $H[[z_1,z_2]]$.

Note that if we take homogeneous $u_1, u_2 \in V, \phi \in H$, then the coefficient of each $z_1^{-n_1-1}z_2^{-n_2-1}$ in the series

$$Y_H^L(u_1,z_1)Y_H^L(u_2,z_2)\phi = \sum_{n_1,n_2\in\mathbb{Z}} (Y_H^L)_{n_1}(u_1)(Y_H^L)_{n_2}(u_2)\phi z_1^{-n_1-1}z_2^{-n_2-1},$$

has weight

$$\mathrm{wt}\ u_1 + \mathrm{wt}\ u_2 + \mathrm{wt}\ \phi - n_1 - n_2 - 2$$

If we pair the series with some $\phi' \in (H^N)'$, then the coefficient of $z_1^{-n_1-1}z_2^{-n_2-1}$ is zero unless

$$\mathrm{wt}\ \phi' = \mathrm{wt}\ u_1 + \mathrm{wt}\ u_2 + \mathrm{wt}\ \phi - n_1 - n_2 - 2$$

Thus $-n_1-n_2-2$ would equal to a fixed number. Since the power of z_2 is bounded below, we thus know that the power of z_1 is bounded above. After multiplying $z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}$, we know that the power of z_1 is also bounded below. And thus the power of z_2 is bounded above. Therefore we proved that

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}\langle\phi', Y_H^L(u_1,z_1)Y_H^L(u_2,z_2)\phi\rangle$$

is a polynomial in $\mathbb{C}[z_1,z_2]$. It is easy to see that the total degree of the polynomial is precisely

$$\mathrm{wt}\ \phi' - \mathrm{wt}\ u_1 - \mathrm{wt}\ u_2 - \mathrm{wt}\ \phi + p_1 + p_2 + p_{12}'.$$

For nonhomogeneous $u_1, u_2 \in V, \phi \in H^N$, the conclusion also holds: a finite sum of the homogeneous polynomials is still a polynomial. Therefore we proved that for every $u_1, u_2 \in V, \phi \in H$,

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}} \langle \phi', Y_H^L(u_1, z_1) Y_H^L(u_2, z_2)\phi \rangle \in \mathbb{C}[z_1, z_2] \tag{6.11}$$

Since the complex series

$$\langle \phi', Y_H^L(u_1, z_1) Y_H^L(u_2, z_2)\phi \rangle \tag{6.12}$$

the power of z_2 is lower-truncated and the power of z_1 is upper-truncated, so if we denote the polynomial given in (6.11) by $h(z_1, z_2)$, then (6.12) must coincide with

$$\iota_{z_1 z_2}\left(\frac{h(z_1, z_2)}{z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}}}\right)$$

Thus we proved that for every $\phi' \in H', u_1, u_2 \in V, \phi \in H^N$

$$\langle \phi', Y_H^L(u_1, z_1) Y_H^L(u_2, z_2)\phi \rangle$$

converges absolutely when

$$|z_1| > |z_2| > 0$$

to a rational function with the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$.

5. We prove the rationality of the iterate of two vertex operators. As many steps are similar, here we only give a sketch. First notice that for every $w'_2 \in W'_2, u_1, u_2 \in V, \phi \in H^N, w_1 \in W_1$, from associativity of Y_W and Lemma 2.3.7, the series

$$\langle w'_2, Y_{W_2}^L(Y_V(u_1, z_1 - z_2)u_2, z_2 + \zeta)(\phi(w_1))(\zeta) \rangle$$

converges absolutely when

$$|z_2 + \zeta| > |z_1 - z_2| + |\zeta| > 0$$

to the rational function

$$R(\langle w'_2, Y_{W_2}^L(u_1, z_1 + \zeta) Y_{W_2}^L(u_2, z_2 + \zeta)(\phi(w_1))(\zeta) \rangle)$$

which is of the form

$$\frac{f(z_1, z_2, \zeta)}{z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}} (z_1 + \zeta)^{q_1} (z_2 + \zeta)^{q_2} \zeta^r}.$$

We use identity (6.3) and Lemma 2.3.7 to see that

$$\langle w'_2, [(Y_H^L(Y_V(u_1, z_1 - z_2)u_2, z_2)\phi)(w_1)](\zeta)\rangle$$

also converges to the rational function in the region

$$|\zeta| > |z_1 - z_2| + |z_2|, |z_2| > |z_1 - z_2| > 0$$

Multiplying both sides by powers of z_1, z_2 and $z_1 - z_2$, one sees that

$$\begin{aligned} & z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} \langle w'_2, [(Y_H^L(Y_V(u_1, z_1 - z_2)u_2, z_2)\phi)(w_1)](\zeta)\rangle \\ = & \frac{f(z_1, z_2, \zeta)}{\zeta^r} \iota_{\zeta z_1} \left(\frac{1}{(z_1 + \zeta)^{q_1}}\right) \iota_{\zeta z_2} \left(\frac{1}{(z_2 + \zeta)^{q_2}}\right) \end{aligned}$$

has no negative powers of z_1, z_2.

We denote the coefficient of each $z_1^{-n_1-1} z_2^{-n_2-1}$ by $g_{n_1 n_2}(\zeta)$ and claim that the $\overline{W_2}$-valued rational function

$$w'_2 \mapsto g_{n_1 n_2}(\zeta)$$

is an element in $(\widehat{W_2})_\zeta$.

Since the orders of poles of the rational function (6.5) is bounded above by constants that are independent of the choice of w'_2, one sees that the series

$$Y_{W_2}^L(Y_V(u_1, z_1 - z_2)u_2, z_2 + \zeta)(\phi(w_1))(\zeta)$$

multiplied with the denominator of (6.5) is actually a power series with coefficients in W_2. The above procedure shows that the series $[(Y_H^L(Y_V(u_1, z_1 - z_2)u_2, z_2)\phi)(w_1)](\zeta)$ converges to the same $\overline{W_2}$-valued rational function. Therefore, after multiplied with the denominators of (6.5), we should get the same power series. Thus, the series

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}} [(Y_H^L(Y_V(u_1, z_1 - z_2)u_2, z_2)\phi)(w_1)](\zeta)$$

is the expansion of the quotient of this power series and powers of $(z_1+\zeta), (z_2+\zeta)$ and ζ. The denominators are expanded as power series of z_1 and z_2. The coefficient of each $z_1^{-n_1-1}z_2^{-n_2-1}$ is precisely the $\overline{W_2}$-valued rational function determined by $w_2' \mapsto g_{n_1n_2}(\zeta)$. Thus this $\overline{W_2}$-valued rational function is in $(\widehat{W_2})_\zeta$.

Thus we know that the series

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}}Y_H^L(Y_V(u_1,z_1-z_2)u_2,z_2)\phi$$

is a power series in $H[[z_1,z_2]]$. If we take $u_1,u_2\in V, \phi\in H$ to be homogeneous elements and pair this power series to some homogeneous $\phi'\in H$, then we will get a homogeneous polynomial of degree $\mathrm{wt}\ \phi' - \mathrm{wt}\ u_1 - \mathrm{wt}\ u_2 - \mathrm{wt}\ \phi + p_1+p_2+p_{12}$. Thus for general nonhomogeneous $u_1,u_2\in V, \phi\in H, \phi'\in (H^N)'$, we have

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}}\langle \phi', Y_H^L(Y_V(u_1,z_1-z_2)u_2,z_2)\phi\rangle \in \mathbb{C}[z_1,z_2]$$

Dividing the polynomial by $z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}$ and expand the negative powers of $z_1=z_2+(z_1-z_2)$, we see that the resulting series coincides with $\langle \phi', Y_H^L(Y_V(u_1,z_1-z_2)u_2,z_2)\phi\rangle$. Thus, the series

$$\langle \phi', Y_H^L(Y_V(u_1,z_1-z_2)u_2,z_2)\phi\rangle$$

converges absolutely to a rational function that has poles at $z_1=0, z_2=0$ and $z_1=z_2$.

6. To see the associativity, it suffices to notice that

$$\langle w_2', Y_{W_2}^L(u_1,z_1+\zeta)Y_{W_2}^L(u_2,z_2+\zeta)w_1\rangle$$

converges absolutely when

$$|z_1+\zeta|>|z_2+\zeta|>|\zeta|>0$$

to the same rational function that

$$\langle w_2', Y_{W_2}^L(Y_V(u_1,z_1-z_2)u_2,z_2+\zeta)(\phi(w_1))(\zeta)\rangle$$

converges to. Thus in the region

$$|z_1+\zeta|>|z_2+\zeta|>|\zeta|>|z_1|>|z_2|>0,$$

these two series are equal. Therefore,

$$z_1^{p_1} z_2^{p_2} (z_1-z_2)^{p_{12}} Y_H^L(u_1, z_1) Y_H^L(u_2, z_2)\phi = z_1^{p_1} z_2^{p_2} (z_1-z_2)^{p_{12}} Y_H^L(Y_V(u_1, z_1-z_2)u_2, z_2)\phi$$

Hence the associativity is proved.

7. From the second part of composable condition, p_1 depends only on the choice of u_1 and ϕ, p_2 depends only on the choice of u_2 and ϕ, p'_{12} depends only on the choice of u_1 and u_2. Thus the pole-order condition is satisfied. The rationality of products of any numbers of vertex operators then follows from Proposition 3.1.21.

□

6.3.2 The right V-module structure on H^N

Definition 6.3.6. We define the following right action of V on H^N: for $u \in V, \phi \in H^N$

$$[(Y_H^R(\phi, z)u)(w_1)](\zeta) = \iota_{\zeta z} E(\phi(Y_{W_1}^L(u, \zeta)w_1)(z+\zeta))$$

The definition is understood as follows:

1. Since ϕ is composable with any number of vertex operators, in particular, for any $w'_2 \in W'_2$, the complex series

$$\langle w'_2, \phi(Y_{W_1}^L(u, \zeta)w_1)(z+\zeta)\rangle$$

converges absolutely in the region

$$|z+\zeta| > |\zeta| > 0$$

to a rational function with poles at $z+\zeta = 0, \zeta = 0, z = 0$. Moreover, the order of the pole $z = 0$ is controlled above by wt u + wt ϕ.

2. Expand the negative powers of $(z+\zeta)$ as a power series of z. Then we get a complex series

$$\sum_{n\in\mathbb{Z}} c_n(w'_2 \otimes u \otimes \phi \otimes w_1; \zeta) z^{-n-1}$$

Note that because of the N-weight-degree condition, the lowest power of z is at least as large as $-$wt $u-$wt ϕ. The coefficient of each power of z is then a rational function with poles at $\zeta = 0$.

3. For each fixed $\zeta \neq 0$, the linear functional $w_2' \mapsto c_n(w_2' \otimes u \otimes \phi \otimes w_1; \zeta)$ would then be defining an element in $\widehat{W_2}$. From the definition of composability, note that the series $\sum_{n\in\mathbb{Z}} c_n(\cdot \otimes u \otimes \phi \otimes w_1; \zeta) z^{-n-1}$ of $\widehat{W}$-elements is obtained from expanding a localized formal series in $W[[z,\zeta]][z^{-1}, \zeta^{-1}, (z+\zeta)^{-1}]$. Thus the coefficients $c_n(\cdot \otimes u \otimes \phi \otimes w_1; \zeta)$ is a lower-truncated series in $W_2((\zeta))$. One then checks easily that for every $n \in \mathbb{Z}$ and every fixed $u \in V, w_1 \in W_1$, $c_n(\cdot \otimes u \otimes \phi \otimes w_1; \zeta) \in (\widehat{W_2})\zeta$. Therefore, for each $n \in \mathbb{Z}$, the map

$$w_1 \mapsto c_n(\cdot \otimes u \otimes \phi \otimes w_1; \zeta)$$

is an element of $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$. If we denote this map by $(Y_H^R)_n(\phi)u$, then we have

$$Y_H^R(\phi, z)u = \sum_{n\in\mathbb{Z}} (Y_H^R)_n(\phi) u z^{-n-1}$$

as a series with coefficients in $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$.

4. In terms of coefficients, we have the following identity

$$\begin{aligned}&\sum_{n_2\in\mathbb{Z}}\sum_{n_1\in\mathbb{Z}} \langle w_2', \phi_{n_2}((Y_{W_1}^L)_{n_1} w_1)\rangle (z+\zeta)^{-n_1-1}\zeta^{-n_2-1}\\ &\qquad = \sum_{n_1\in\mathbb{Z}}\sum_{n_2\in\mathbb{Z}} \langle w_2', [(Y_H^R)_{n_1}(u)\phi]_{n_2}(w_1)\rangle \zeta^{-n_2-1} z^{-n_1-1}\end{aligned}$$

when $|z+\zeta| > |\zeta| > |z|$. Since both sides are Laurent series expansion of a rational function, the order of summation can be switched. Thus we can also write

$$\sum_{n\in\mathbb{Z}} \langle w_2', [\phi((Y_{W_1}^L)_n(u)w_1)](z+\zeta)\rangle \zeta^{-n-1} = \sum_{n\in\mathbb{Z}} \langle w_2', [((Y_H^R)_n(u)\phi)(w_1)](\zeta)\rangle z^{-n-1} \tag{6.13}$$

Proposition 6.3.7. *Let $\phi : W_1 \to (\widehat{W_2})_\zeta$ be a homogeneous map composable with any number of vertex operators. Then for every $n \in \mathbb{Z}, u \in V$,*

$$\mathrm{wt}\ (Y_H^R)_n(\phi)u = \mathrm{wt}\ u - n - 1 + \mathrm{wt}\ \phi.$$

Proof. Let $|z+\zeta| > |\zeta| > |z| > 0$. Since

$$\langle w_2', a^{\mathbf{d}_{W_2}} \phi(Y_{W_1}^L(u,\zeta)w_1)(z+\zeta)\rangle = \langle w_2', a^{\mathrm{wt}\ \phi}(\phi(a^{\mathbf{d}_{W_1}} Y_{W_1}^L(u,\zeta)w_1))(az+a\zeta)\rangle$$

$$= \langle w_2', a^{\mathrm{wt}\ \phi}(\phi(Y_{W_1}^L(a^{\mathbf{d}_V}u, a\zeta)a^{\mathbf{d}_{W_1}}w_1))(az+a\zeta)\rangle$$

we thus have

$$\langle w_2', a^{\mathbf{d}_{W_2}}[(Y_H^R(\phi,z)u)(w_1)](\zeta)\rangle = \langle w_2', [(Y_H^R(a^{\mathrm{wt}\ \phi}\phi, az)a^{\mathbf{d}_V}u)(a^{\mathbf{d}_{W_1}}w_1)](a\zeta)\rangle$$

Expand $Y_H^R(\phi,z)u$ as the sum of $(Y_H^R)_n(\phi)u$ and use the definition of the $\mathbf{d}_H$, we see that

$$a^{\mathbf{d}_H}(Y_H^R)_n(\phi)u = a^{\mathrm{wt}\ u+\mathrm{wt}\ \phi-n-1}$$

□

Remark 6.3.8. As a consequence of the pole condition in the definition of composability, we know that $-n-1 \geq -\mathrm{wt}\ u - \mathrm{wt}\ \phi$, and thus

$$\mathrm{wt}\ (Y_H^R)_n(\phi)u = \mathrm{wt}\ u - n - 1 + \mathrm{wt}\ \phi \geq 0.$$

Proposition 6.3.9. *Let $\phi : W_1 \to (\widehat{W_2})_\zeta$ be a homogeneous map composable with any number of vertex operators. Then for every $n \in \mathbb{Z}, u \in V$, $(Y_H^R)_n(\phi)u$ is also composable with any number of vertex operators.*

Proof. Fix any $m \in \mathbb{Z}_+$. Since ϕ is composable with, in particular, $m+1$ vertex operators, for every $l = 0, ..., m, w_2' \in W_2', u_1, ..., u_m \in V$ and $w_1 \in W_1$, the following complex series

$$\langle w_2', Y_{W_2}^L(u_1,z_1)\cdots Y_{W_2}^L(u_l,z_l)(\phi(Y_{W_1}^L(u,\zeta)Y_{W_1}^L(u_{l+1},z_{l+1})\cdots Y_{W_1}^L(u_m,z_m)w_1))(z+\zeta)\rangle$$

converges absolutely when

$$|z_1| > \cdots > |z_l| > |z+\zeta| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$$

to a rational function of the form

$$\frac{f(z_1,...,z_m,z,\zeta)}{(z+\zeta)^{r_1}z^p\zeta^{r_2}\prod_{i=1}^m z_i^{q_i}(z_i-z-\zeta)^{p_i'}(z_i-\zeta)^{p_i''}\prod_{1\leq i<j\leq m}(z_i-z_j)^{p_{ij}'''}}$$

with

1. p depends only on u and ϕ;

2. p'_i depending only to u_i and ϕ, p''_i depending only on u_i and u for $i = 1, ..., m$;

3. p'''_{ij} depending only on u_i and u_j for $1 \leq i < j \leq m$;

4. r_1 depends only on ϕ and w_1;

5. r_2 depends only on u and w_1;

6. q_i depends only on u_i and ϕ for $i = 1, ..., m$.

In particular, the series, rewritten as,

$$\sum_{n_1,...,n_m\in\mathbb{Z},n\in\mathbb{Z}} \langle w_2', (Y_{W_2}^L)_{n_1}(u_1)\cdots(Y_{W_2}^L)_{n_l}(u_l)[\phi((Y_{W_1}^L)_n(u)(Y_{W_1}^L)_{n_{l+1}}(u_{l+1})\cdots(Y_{W_1}^L)_{n_m}(u_m)w_1)](z+\zeta)\rangle\cdot$$
$$z_1^{-n_1-1}\cdots z_l^{-n_l-1}\zeta^{-n-1}z_{l+1}^{-n_{l+1}-1}\cdots z_m^{-n_m-1}$$

converges absolutely to the same rational function in the smaller region

$$|z_1| > \cdots > |z_l| > |\zeta| + |z|, |\zeta| - |z| > |z_{l+1}| > \cdots > |z_m| > 0, |z+\zeta| > |\zeta|, |z| > 0.$$

As we have $|z+\zeta| > |z| > 0$, by the identity (6.13), the series is equal to

$$\sum_{n_1,...,n_m\in\mathbb{Z}}\Bigg(\sum_{n\in\mathbb{Z}}\langle w_2', (Y_{W_2}^L)_{n_1}(u_1)\cdots(Y_{W_2}^L)_{n_l}(u_l)[((Y_H^R)_n(\phi)u)((Y_{W_1}^L)_{n_{l+1}}(u_{l+1})\cdots(Y_{W_1}^L)_{n_m}(u_m)w_1)](\zeta)\rangle\cdot$$
$$z^{-n-1}\Bigg)z_1^{-n_1-1}\cdots z_l^{-n_l-1}z_{l+1}^{-n_{l+1}-1}\cdots z_m^{-n_m-1}$$

which also converges to the same rational function. We note that the series coincides with the expansion of the rational function in the region

$$|z_1| > \cdots > |z_l| > |\zeta| + |z|, |\zeta| - |z| > |z_{l+1}| > \cdots > |z_m| > 0,$$

where the negative powers of

- $z+\zeta$ are expanded as a power series of z;
- $z_i - \zeta$ are expanded as a power series of ζ when $i \leq l$;
- $z_i - \zeta$ are expanded as a power series of z_i when $i > l$;
- $z_i - z - \zeta$ are expanded first as a power series of $(z+\zeta)$ when $i \leq l$, then further expand the positive powers of $z+\zeta$ as polynomials in z, ζ;

- $z_i - z - \zeta$ are first expanded as a power series of $(z + z_i)$ when $i > l$, then further expand the positive powers of $z + z_i$ as polynomials in z, z_i;

- $z_i - z_j$ are expanded as a power series of z_j when $i < j$.

In particular, the order of summation can be switched. Thus we know that the following series

$$\sum_{n\in\mathbb{Z}}\langle w_2', Y_{W_2}^L(u_1, z_1)\cdots Y_{W_2}^L(u_l, z_l)[((Y_H^R)_n(\phi)u)(Y_{W_1}^L(u_{l+1}, z_{l+1})\cdots Y_{W_1}^L(u_m, z_m)w_1)](\zeta)\rangle z^{-n-1}$$

converges to the same rational function when

$$|z_1| > \cdots > |z_l| > |\zeta| + |z|, |\zeta| - |z| > |z_{l+1}| > \cdots > |z_m| > 0,$$

Now we rewrite the rational function as

$$\frac{\sum_{k_1=0}^K f_{k_1}(z_1, ..., z_m, \zeta)z^{k_1}}{[1 + z/\zeta]^{r_1} z^p \zeta^{r_1+r_2} \prod_{i=1}^m z_i^{q_i}(z_i - \zeta)^{p_i'+p_i''}[1 - z/(z_i - \zeta)]^{p_i''} \prod_{1\leq i<j\leq m}(z_i - z_j)^{p_{ij}'''}}$$

and expand the bracketed denominators in the as the power series of the second term. In other words, the expansion is done in the region

$$|\zeta| > |z| > 0, |z_i - \zeta| > |z|$$

Organize the series according to the power of z, we will obtain the following:

$$\sum_{-n-1\geq -p}\left(\sum_{k_1=0}^K \sum_{\substack{k_{31},...,k_{3m}\geq 0\\ k_{31}+\cdots+k_{3m}\leq -n-1+p-k_1}} \frac{a_{k_2k_{31}...k_{3m}}f_{k_1}(z_1, ..., z_m, \zeta)}{z^p\zeta^{r_1+r_2+k_2}\prod_{i=1}^m z_i^{q_i}(z_i - \zeta)^{p_i'+p_i''+k_{3i}}\prod_{1\leq i<j\leq m}(z_i - z_j)^{p_{ij}'''}}\right) z^{-n-1}$$

where $k_2 = -n - 1 + p - k_1 - \sum_{i=1}^m k_{3i}$ is a nonnegative integer. Thus for each fixed $n \in \mathbb{Z}$, the series

$$\langle w_2', Y_{W_2}^L(u_1, z_1)\cdots Y_{W_2}^L(u_l, z_l)[((Y_H^R)_n(\phi)u)(Y_{W_1}^L(u_{l+1}, z_{l+1})\cdots Y_{W_1}^L(u_m, z_m)w_1)](\zeta)\rangle$$

converges absolutely when

$$|z_1| > \cdots > |z_l| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$$

to the rational function

$$\sum_{k_1=0}^{K} \sum_{\substack{k_{31},\dots,k_{3m}\geq 0\\ k_{31}+\cdots+k_{3m}\leq -n-1+p-k_1}} \frac{a_{k_2k_{31}\dots k_{3m}} f_{k_1}(z_1,\dots,z_m,\zeta)}{z^p \zeta^{r_1+r_2+k_2} \prod_{i=1}^{m} z_i^{q_i}(z_i-\zeta)^{p'_i+p''_i+k_{3i}} \prod_{1\leq i<j\leq m} (z_i-z_j)^{p'''_{ij}}}$$

with the only possible poles at $\zeta = 0$, $z_i = 0, z_i = \zeta$ for $i = 1, \dots, m$, and $z_i - z_j = 0$ for $1 \leq i < j \leq m$. This proves the first part of composability.

To see the second part of composability, it suffices to verify the conditions specified in Remark 6.2.10 in the rational function above.

1. For each $i = 1, \dots, m$, q_i depends only on u_i and ϕ.

2. For each $i = 1, \dots, m$,

$$p'_i + p''_i + k_{3i} \leq p'_i + p''_i + -n - 1 + p - k_1 \leq p'_i + p''_i - n - 1 + p$$

   which depends only on u_i, u, n and ϕ. In particular, it is independent of the choice of $u_j, j \neq i$, w_1 and w'_2.

3. For each $1 \leq i < j \leq m$, p'''_{ij} depends only on the choice of u_i and u_j.

4. Since $k_2 \leq -n-1-p$, we know $r_1+r_2+k_2 \leq r_1+r_2-n-1-p$ which depends only on u, ϕ, n and w_1. In particular, it is independent of the choice of $u_i (i = 1, \dots, m)$ and w'_2.

Now we check the N-weight-degree condition for each $(Y_H^R)_n(\phi)u$. Sincd $\phi \in H^N$, the expansion of the rational function

$$R\left(\langle w'_2, Y_{W_2}(u_1, z_1+\zeta)\cdots Y_{W_2}(u_l, z_l+\zeta)\right.$$
$$\left.(\phi(Y_{W_1}(u, z_{l+1}+\zeta)Y_{W_1}(u_{l+1}, z_{l+2}+\zeta)\cdots Y_{W_1}(u_m,\zeta)w_1)(z+\zeta))\rangle\right)$$

in the region $|\zeta| > |z_1| > \cdots > |z_l| > |z| > |z_{l+1}| > \cdots > |z_m| > 0$ as a Laurent series in $z_1, \dots, z_m, z$ with coefficients in $\mathbb{C}[[\zeta, \zeta^{-1}]]$ has lowest total weight at least as large as $N - (\text{wt } u_1 + \cdots + \text{wt } u_m + \text{wt } u + \text{wt } \phi)$. From Remark 6.2.18, the same condition holds for the expansion in the region

$$|\zeta| > |z_1| > \cdots > |z_l| > |z_{l+1}| > \cdots > |z_m| > |z| > 0$$

From the analysis above, we see that this expansion coincides with the following series

$$\sum_{n\in\mathbb{Z}}\langle w_2',Y_{W_2}(u_1,z_1+\zeta)\cdots Y_{W_2}(u_l,z_l+\zeta)$$
$$[((Y_H^R)_n(\phi)u)(Y_{W_1}(u_{l+1},z_{l+2}+\zeta)\cdots Y_{W_1}(u_m,\zeta)w_1)](z_{l+1}+\zeta)\rangle z^{-n-1}$$

with the coefficients of each z^{-n-1} further expanded in the region $|\zeta|>|z_1|>\cdots>|z_m|>0$. For each monomial in this expansion, the total degree of $z_1,...,z_m,z$ is nothing but the total degree of $z_1,...,z_m$ plus $-n-1$, which is at least as large as $N-(\text{wt } u_1+\cdots+\text{wt } u_m+\text{wt } u+\phi)$. Thus, the total degree of $z_1,...,z_m$ in the expansion of the rational function

$$R\big(\langle w_2', Y_{W_2}(u_1,z_1+\zeta)\cdots Y_{W_2}(u_l,z_l+\zeta)$$
$$[((Y_H^R)_n(\phi)u)(Y_{W_1}(u_{l+1},z_{l+2}+\zeta)\cdots Y_{W_1}(u_m,\zeta)w_1)](z_{l+1}+\zeta)\rangle\big)$$

in the region $|\zeta|>|z_1|>\cdots>|z_m|>0$ is at least as large as

$$N-(\sum_{i=1}^m \text{wt } u_i+\text{wt } u+\text{wt } \phi)+n+1=N-(\sum_{i=1}^m \text{wt } u_i+\text{wt } (Y_H^R)_n(\phi)u)$$

Thus for each $n\in\mathbb{Z}$, $(Y_H^L)_n(\phi)$ also satisfies the N-weight-degree condition.

□

So we have proved that for every $\phi\in H^N$, the series $Y_H^R(\phi,x)u=\sum_{n\in\mathbb{Z}}(Y_H^R)_n(\phi)ux^{-n-1}$ is actually a series in $H^N[[x,x^{-1}]]$, i.e., the map

$$Y_H^R: H^N\otimes V\to H^N[[x,x^{-1}]]$$

gives an action of V on H^N.

Theorem 6.3.10. $(H^N,Y_H^R,\mathbf{d}_H,D_H)$ *forms a right V-module.*

Proof. We know that H is graded by the eigenvalues of $\mathbf{d}_H$ operator, equipped with a vertex operator map $Y_H^R: V\otimes H\to H[[x,x^{-1}]]$, an operator $\mathbf{d}_H$ of weight 0 and an operator D_H of weight 1. Now we verify all the axioms.

1. The lower bound condition and the $\mathbf{d}$-grading condition is obviously satisfied. The $\mathbf{d}$-bracket property easily follows from the weight formula proved above.

2. The creation property follows from that the following computations:

$$\begin{aligned}[(Y_H^R(\phi,z)\mathbf{1})(w_1)](\zeta) &= [\phi(Y_{W_1}^L(\mathbf{1},\zeta)w_1)](z+\zeta)\\ &= (\phi(w_1))(z+\zeta)\\ &= [(e^{zD_H}\phi)(w_1)](\zeta)\end{aligned}$$

Thus $Y_H^R(\phi,z)\mathbf{1} = e^{zD_H}\phi$. In particular, $Y_H^R(\phi,z)\mathbf{1} \in H[[z,z^{-1}]]$ and $\lim_{z\to 0} Y_H^R(\phi,z)\mathbf{1} = \phi$.

3. The D-derivative property follows from the computation below:

$$\begin{aligned}\left[\frac{d}{dz}\left(Y_H^R(\phi,z)u\right)\right](w_1,\zeta) &= \frac{d}{dz}\left[\left(Y_H^R(\phi,z)u\right)(w_1,\zeta)\right]\\ &= \frac{d}{dz}\left[\iota_{\zeta z}E\left(\phi(Y_{W_1}(u,\zeta)w_1,z+\zeta)\right)\right]\\ &= \left[\iota_{\zeta z}E\left(\frac{d}{dz}\phi(Y_{W_1}(u,\zeta)w_1,z+\zeta)\right)\right]\\ &= \iota_{\zeta z}E\left[\frac{d}{d(z+\zeta)}\phi(Y_{W_1}(u,\zeta)w_1,z+\zeta)\frac{d(z+\zeta)}{dz}\right]\\ &= \iota_{\zeta z}E\left[(D_H\phi)(Y_{W_1}(u,\zeta)w_1,z+\zeta)\right]\\ &= [Y_H^R(D_H\phi,z)u](w_1,\zeta)\end{aligned}$$

The D-bracket formula follows from the computation below:

$$\begin{aligned}&[D_HY_H^R(\phi,z)u](w_1,\zeta)\\ &= \frac{d}{d\zeta}[Y_H^R(\phi,z)u](w_1,\zeta)\\ &= \frac{d}{d\zeta}\iota_{\zeta z}E\left[\phi(Y_{W_1}(u,\zeta)w_1,z+\zeta)\right]\\ &= \iota_{\zeta z}E\left[\frac{d}{d\zeta}\left(\phi(Y_{W_1}(u,\zeta)w_1,z+\zeta)\right)\right]\\ &= \iota_{\zeta z}E\left[\frac{d}{d(z+\zeta)}\phi(Y_{W_1}(u,\zeta)w_1,z+\zeta)\frac{d(z+\zeta)}{d\zeta}\right] + \iota_{\zeta z}E\left[\phi\left(\frac{d}{d\zeta}Y_{W_1}(u,\zeta)w_1,z+\zeta\right)\right]\\ &= \iota_{\zeta z}E\left[(D_H\phi)(Y_{W_1}(u,\zeta)w_1,z+\zeta)\right] + \iota_{\zeta z}E\left[\phi(Y_{W_1}(D_Vu,\zeta)w_1,z+\zeta)\right]\\ &= [Y_H^R(D_H\phi,z)u](w_1,\zeta) + [Y_H^R(\phi,z)D_Vu](w_1,\zeta)\end{aligned}$$

4. We prove the rationality of products of two vertex operators elaborately The rationality of products of any numbers of vertex operators will be seen immediately from Proposition 3.2.15 and the composable condition once associativity is proved.

Since ϕ is composable with any number of vertex operators, in particular, when $m = 2, l = 0$, for any $u_1, u_2 \in V, w_1 \in W_1, w_2' \in W_2'$, we know that

$$\langle w_2', [\phi(Y_{W_1}^L(u_1, z_2 + \zeta)Y_{W_1}^L(u_2, \zeta)w_1)](z_1 + \zeta)\rangle \tag{6.14}$$

converges absolutely in the region

$$S = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |z_1 + \zeta| > |z_2 + \zeta| > |\zeta| > 0\}$$

to a rational function of the form

$$\frac{f(z_1, z_2, \zeta)}{(z_1 + \zeta)^{q_1}(z_2 + \zeta)^{q_2}\zeta^r z_1^{p_1} z_2^{p_2}(z_1 - z_2)^{p_{12}'}} \tag{6.15}$$

Note that $\phi(\cdot, z_1 + \zeta)$ is indeed also a series. We are not expanding it for simplicity. The associativity of $Y_{W_1}^L$ shows that

$$Y_{W_1}^L(u_1, z_2 + \zeta)Y_{W_1}^L(u_2, \zeta)w_1 = Y_{W_1}^L(Y_V(u_1, z_2)u_2, \zeta)w_1$$

when

$$|z_2 + \zeta| > |z_2| > |\zeta| > 0.$$

Therefore (6.14) equals to

$$\begin{aligned} &\langle w_2', [\phi(Y_{W_1}^L(Y_V(u_1, z_2)u_2, \zeta)w_1)](z_1 + \zeta) \\ &= \sum_{n_1 \in \mathbb{Z}} \left(\sum_{n_2 \in \mathbb{Z}} \langle w_2', [\phi((Y_{W_1}^L)_{n_1}((Y_V)_{n_2}(u_1)u_2)w_1)](z_1 + \zeta)\rangle z_2^{-n_2 - 1} \right) \zeta^{-n_1 - 1}, \end{aligned} \tag{6.16}$$

which is a series in $z_1 + \zeta, z_2, \zeta$ converging absolutely to the rational function (6.15) in the multicircular region

$$S^{\cap} = \{(z_1, z_2, \zeta) : |z_1 + \zeta| > |z_2 + \zeta| > |z_2| > |\zeta| > 0\}$$

Note that the rational function (6.15) can be expanded as an absolutely convergent Laurent series in the region

$$S_1 = \{(z_1, z_2, \zeta) : |z_1 + \zeta| > |z_2| + |\zeta|, |z_2| > 0, |\zeta| > 0\}$$

by expanding the negative powers of $z_2+\zeta$ as power series of ζ, $z_1 = z_1+\zeta-\zeta$ as power series of ζ, $z_1-z_2 = z_1+\zeta-(\zeta+z_2)$ first as power series of $z_2+\zeta$, then expand the positive powers of $(z_2+\zeta)$ as polynomials of z_2 and ζ. Note that in this expansion, the power of ζ is upper truncated by the order of pole $\zeta=\infty$. So if we set $S_1' = S_1 \cap S^{\cap}$, then since $S_1' \neq \emptyset$, we can apply Lemma 2.3.9 to see that the series (6.14) converges absolutely to the rational function (6.15) in the region S_1. In particular, one can switch the order of summation. To sum up, the series

$$\langle w_2', (\phi(Y_{W_1}^L(Y_V(u_1,z_2)u_2,\zeta)w_1))(z_1+\zeta)\rangle$$

converges absolutely to the rational function (6.15) in the region S_1 and can be written as

$$\sum_{n_2\in\mathbb{Z}}\left(\sum_{n_1\in\mathbb{Z}}\langle w_2', (\phi((Y_{W_1}^L)_{n_1}((Y_V)_{n_2}(u_1)u_2)w_1)(z_1+\zeta)\rangle\zeta^{-n_1-1}\right)z_2^{-n_2-1} \tag{6.17}$$

converges absolutely to (6.15) in S_1

The identity (6.13) shows that when $|z_1+\zeta| > |\zeta| > |z_1| > 0$, for every fixed $n_2 \in \mathbb{Z}$,

$$\begin{aligned}&\sum_{n_1\in\mathbb{Z}}\langle w_2', (\phi((Y_{W_1}^L)_{n_1}((Y_V)_{n_2}(u_1)u_2)w_1))(z_1+\zeta)\rangle\zeta^{-n_1-1}\\ &\quad=\sum_{n_1\in\mathbb{Z}}\langle w_2', [((Y_H^R)_{n_1}((Y_V)_{n_2}(u_1)u_2)\phi)(w_1)](\zeta)\rangle z_1^{-n_1-1}\end{aligned}$$

Thus the series (6.17) can be rewritten as

$$\begin{aligned}&\sum_{n_2\in\mathbb{Z}}\left(\sum_{n_1\in\mathbb{Z}}\langle w_2', [\phi((Y_{W_1}^L)_{n_1}((Y_V)_{n_2}(u_1)u_2)w_1)](z_1+\zeta)\rangle z_1^{-n_1-1}\right)z_2^{-n_2-1}\\ =&\langle w_2', [(Y_H^R(\phi,z_1)Y_V(u_1,z_2)u_2)(w_1)](\zeta)\rangle \end{aligned}\tag{6.18}$$

in the region

$$S_1^{\cap} = \{(z_1,z_2,\zeta) : |z_1+\zeta| > |z_2|+|\zeta|, |z_1+\zeta| > |\zeta| > |z_1| > 0, |z_2| > 0\}$$

Thus the series (6.18), as a series in z_1, z_2, ζ, converges to the rational function (6.15) in the region $S_1^{\cap}$.

Note that the rational function (6.15) can be expanded as an absolutely convergent Laurent series in the region

$$S_2 = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |\zeta| > |z_1| > |z_2| > 0\}$$

by expanding the negative powers of $z_1 + \zeta$ as power series of z_1, $z_2 + \zeta$ as power series of z_2, and $z_1 - z_2$ as power series of z_2. Note that the power of z_2 is lower truncated. So if we set $S_2' = S_2 \cap S_1^{\cap}$, then since $S_2' \neq \emptyset$, we can apply Lemma 2.3.7 to see that the series on the right hand side of (6.18) converges absolutely in S_2.

Therefore, we know that

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}} \langle w'_2, [(Y_H^R(\phi, z_1) Y_V(u_1, z_2) u_2)(w_1)](\zeta) \rangle$$

is precisely

$$\frac{f(z_1, z_2, \zeta)}{\zeta^r} \iota_{\zeta z_1} \left(\frac{1}{(z_1 + \zeta)^{q_1}} \right) \iota_{\zeta z_2} \left(\frac{1}{(z_2 + \zeta)^{q_2}} \right),$$

which has no negative powers of z_1, z_2. For each fixed $n_1, n_2 \in \mathbb{Z}$, the coefficient of $z_1^{-n_1-1} z_2^{-n_2-1}$ is a Laurent polynomial in ζ. We denote this coefficient by $g_{n_1 n_2}(\zeta)$ and claim that the $\overline{W_2}$-valued rational function determined by

$$w'_2 \mapsto g_{n_1 n_2}(\zeta)$$

is an element in $(\widehat{W_2})_\zeta$.

From the second part of the composable condition, orders of poles of the rational function (6.15) is bounded above by constants that are independent of the choice of w'_2, one sees that the series

$$(\phi(Y_{W_1}^L(u_1, z_2 + \zeta) Y_{W_1}^L(u_2, \zeta) w_1))(z_1 + \zeta)$$

multiplied with the denominator of (6.15) is actually a power series with coefficients in W_2. The above procedure shows that the series $[(Y_H^R(\phi, z_1) Y_V(u_1, z_2) u_2)(w_1)](\zeta)$ converges to the same $\overline{W_2}$-valued rational function. Therefore, after multiplied with the denominators of (6.15), we should get the same power series. Thus, the series

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}} [(Y_H^R(\phi, z_1) Y_V(u_1, z_2) u_2)(w_1)](\zeta)$$

is the expansion of the quotient of this power series and powers of $(z_1+\zeta), (z_2+\zeta)$ and ζ. The denominators are expanded as power series of z_1 and z_2. The coefficient of each $z_1^{-n_1-1}z_2^{-n_2-1}$ is precisely the $\overline{W_2}$-valued rational function determined by $w_2' \mapsto g_{n_1n_2}(\zeta)$. Thus this function is in $(\widehat{W_2})_\zeta$.

Hence, the map

$$w_1 \mapsto \big(w_2' \mapsto g_{n_1n_2}(\zeta)\big)$$

gives an element in $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$. Thus we see that

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}Y_H^R(\phi, z_1)Y_V(u_1, z_2)u_2$$

is indeed a power series of with coefficients in $\mathrm{Hom}(W_1, (\widehat{W_2})_\zeta)$. As pointed out by Proposition 6.3.9, these coefficients are necessarily in H. Thus what we get is a power series in $H[[z_1, z_2]]$. Note that if we take homogeneous $u_1, u_2 \in V, \phi \in H$, then the coefficient of each $z_1^{-n_1-1}z_2^{-n_2-1}$ in the series

$$Y_H^R(\phi, z_1)Y_V(u_1, z_2)u_2 = \sum_{n_1, n_2\in\mathbb{Z}} (Y_H^R)_{n_1}((Y_V)_{n_2}(u_1)u_2)\phi z_1^{-n_1-1}z_2^{-n_2-1},$$

has weight

$$\mathrm{wt}\ u_1 + \mathrm{wt}\ u_2 + \mathrm{wt}\ \phi - n_1 - n_2 - 2$$

If we pair the series with some $\phi' \in (H^N)'$, then the coefficient of $z_1^{-n_1-1}z_2^{-n_2-1}$ is zero unless

$$\mathrm{wt}\ \phi' = \mathrm{wt}\ u_1 + \mathrm{wt}\ u_2 + \mathrm{wt}\ \phi - n_1 - n_2 - 2$$

Thus $-n_1-n_2-2$ would equal to a fixed number. Since the power of z_2 is bounded below, we thus know that the power of z_1 is bounded above. After multiplying $z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}$, we know that the power of z_1 is also bounded below. And thus the power of z_2 is bounded above. Therefore we proved that

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}'}\langle \phi', Y_H^R(\phi, z_1)Y_V(u_1, z_2)u_2\rangle$$

is a polynomial in $\mathbb{C}[z_1, z_2]$. It is easy to see that the total degree of the polynomial is precisely

$$\mathrm{wt}\ \phi' - \mathrm{wt}\ u_1 - \mathrm{wt}\ u_2 - \mathrm{wt}\ \phi + p_1 + p_2 + p_{12}'.$$

For nonhomogeneous $u_1, u_2 \in V, \phi \in H^N$, the conclusion also holds: a finite sum of the homogeneous polynomials is still a polynomial. Therefore we proved that for every $u_1, u_2 \in V, \phi \in H^N, \phi' \in (H^N)'$,

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}} \langle \phi', Y_H^R(\phi, z_1) Y_V(u_1, z_2) u_2 \rangle \in \mathbb{C}[z_1, z_2] \tag{6.19}$$

Since the complex series

$$\langle \phi', Y_H^R(\phi, z_1) Y_V(u_1, z_2) u_2 \rangle$$

the power of z_2 is lower-truncated and the power of z_1 is upper-truncated, so if we denote the polynomial given in (6.19) by $h(z_1, z_2)$, then (6.12) must coincide with

$$\iota_{z_1 z_2} \left(\frac{h(z_1, z_2)}{z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}}} \right)$$

Thus we proved that for every $\phi' \in H', u_1, u_2 \in V, \phi \in H^N$

$$\langle \phi', Y_H^R(\phi, z_1) Y_V(u_1, z_2) u_2 \rangle$$

converges absolutely when

$$|z_1| > |z_2| > 0$$

to a rational function with the only possible poles at $z_1 = 0, z_2 = 0, z_1 = z_2$.

5. We prove the rationality of the iterate of two vertex operators. First notice that for every $w'_2 \in W'_2, u_1, u_2 \in V, \phi \in H^N, w_1 \in W_1$, the series

$$\langle w'_2, [\phi(Y_{W_1}^L(u_1, z_2 + \zeta) Y_{W_1}^L(u_2, \zeta) w_1)](z_1 + \zeta) \rangle$$

converges absolutely when

$$S = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |z_1 + \zeta| > |z_2 + \zeta| > |\zeta| > 0\}$$

to the rational function (6.15).

We use identity (6.13) to see

$$\langle w'_2, [Y_H^R(\phi, z_1 - z_2) u_1](Y_{W_1}(u_2, \zeta) w_1, z_2 + \zeta) \rangle \tag{6.20}$$

converges absolutely to the rational function (6.15) in the region

$$S^{\cap} = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |z_1 + \zeta| > |z_2 + \zeta| > |\zeta| > |z_1 - z_2| > 0\}$$

Note that this series is lower-truncated in ζ. Also note that the rational function (6.15) can be expanded as an absolutely convergent Laurent series in $z_1 - z_2, z_2 + \zeta, \zeta$ in the region

$$S_1 = \{(z_1, z_2, \zeta) : |z_2 + \zeta| > |z_1 - z_2| + |\zeta|, |z_1 - z_2| > 0, |\zeta| > 0\}$$

by expanding negative powers of $z_1 + \zeta = z_2 + \zeta + (z_1 - z_2)$ as power series of $(z_1 - z_2)$, $z_1 = z_2 + \zeta + (z_1 - z_2 - \zeta)$ as a power series of $z_1 - z_2 - \zeta$, then further expand the positive powers of $z_1 - z_2 - \zeta$ as polynomials of $z_1 - z_2, \zeta$, and $z_2 = z_2 + \zeta - \zeta$ as power series of ζ. Since $S_1 \cap S^{\cap} \neq \emptyset$, we use Lemma 2.3.7 to see that the series (6.20) converges absolutely to the rational function (6.15) in the region S_1.

We use identity (6.13) again to see that

$$\langle w_2', [(Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2)(w_1)](\zeta)\rangle$$

also converges absolutely to the same rational function in the region

$$S_1^{\cap} = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |z_2 + \zeta| > |z_1 - z_2| + |\zeta|, |\zeta| > |z_2| > 0\}$$

Note that this series is lower-truncated in $z_1 - z_2$. Also note that the rational function (6.15) can be expanded as an absolutely convergent Laurent series in $z_1 - z_2, z_2, \zeta$ in the region

$$S_2 = \{(z_1, z_2, \zeta) \in \mathbb{C}^3 : |\zeta| > |z_1 - z_2| + |z_2|, |z_2| > |z_1 - z_2| > 0\}$$

by expanding negative powers of $z_1 = z_2 + (z_1 - z_2)$ as a power series in $(z_1 - z_2)$, $z_2 + \zeta$ as power series in z_2, $z_1 + \zeta$ first as power series in $z_1 = z_1 - z_2 + z_2$, then further expand the positive powers of z_1 as polynomials in $z_1 - z_2$ and z_2. Since $S_1^{\cap} \cap S_2 \neq \emptyset$, we then apply Lemma 2.3.7 to see that the series

$$\langle w_2', [(Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2)(w_1)](\zeta)\rangle$$

converges absolutely to the rational function (6.15).

Multiplying both sides by powers of z_1, z_2 and $z_1 - z_2$, one sees that

$$\begin{aligned}&z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} \langle w_2', [(Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2)(w_1)](\zeta)\rangle \\ =&\frac{f(z_1, z_2, \zeta)}{\zeta^r} \iota_{\zeta, z_1 - z_2 + z_2} \left(\frac{1}{(z_1+\zeta)^{q_1}}\right) \iota_{\zeta z_2} \left(\frac{1}{(z_2+\zeta)^{q_2}}\right)\end{aligned}$$

has no negative powers of z_1, z_2.

We denote the coefficient of each $z_1^{-n_1-1} z_2^{-n_2-1}$ by $g_{n_1 n_2}(\zeta)$ and claim that the $\overline{W_2}$-valued rational function

$$w_2' \mapsto g_{n_1 n_2}(\zeta)$$

is an element in $(\widehat{W_2})_\zeta$.

Since the orders of poles of the rational function (6.15) is bounded above by constants that are independent of the choice of w_2', one sees that the series

$$[\phi(Y_{W_1}^L(u_1, z_2+\zeta) Y_{W_1}^L(u_2, \zeta) w_1)](z_1 + \zeta)$$

multiplied with the denominator of (6.15) is actually a power series with coefficients in W_2. The above procedure shows that the series $[(Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2)(w_1)](\zeta)$ converges to the same $\overline{W_2}$-valued rational function. Therefore, after multiplied with the denominators of (6.15), we should get the same power series. Thus, the series

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}'} [(Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2)(w_1)](\zeta)$$

is the expansion of the quotient of this power series and powers of $(z_1+\zeta), (z_2+\zeta)$ and ζ. The denominators are expanded as power series of z_1 and z_2. The coefficient of each $z_1^{-n_1-1} z_2^{-n_2-1}$ is precisely the $\overline{W_2}$-valued rational function determined by $w_2' \mapsto g_{n_1 n_2}(\zeta)$. Thus this $\overline{W_2}$-valued rational function is in $(\widehat{W_2})_\zeta$.

Thus we know that the series

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2$$

is a power series in $H[[z_1, z_2]]$. If we take $u_1, u_2 \in V, \phi \in H$ to be homogeneous elements and pair this power series to some homogeneous $\phi' \in H$, then we will get a homogeneous polynomial of degree wt $\phi' -$ wt $u_1 -$ wt $u_2 -$ wt $\phi + p_1 + p_2 + p_{12}$. Thus for general nonhomogeneous $u_1, u_2 \in V, \phi \in H, \phi' \in (H^N)'$, we have

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} \langle \phi', Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2\phi\rangle \in \mathbb{C}[z_1, z_2]$$

Dividing the polynomial by $z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p'_{12}}$ and expand the negative powers of $z_1 = z_2 + (z_1 - z_2)$, we see that the resulting series coincides with $\langle \phi', Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2\rangle$. Thus, the series

$$\langle \phi', Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2\rangle$$

converges absolutely to a rational function that has poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$.

6. To see the associativity, it suffices to notice that

$$\langle w_2', [\phi(Y_{W_1}^L(u_1, z_2 + \zeta)Y_{W_1}^L(u_2, \zeta)w_1)](z_1 + \zeta)\rangle$$

converges absolutely when

$$|z_1 + \zeta| > |z_2 + \zeta| > |\zeta| > 0$$

to the same rational function that

$$\langle w_2', [\phi(Y_{W_1}^L(Y_V(u_1, z_2)u_2, \zeta)w_1)](z_1 + \zeta)\rangle$$

converges to. Thus in the region

$$|z_1 + \zeta| > |z_2 + \zeta| > |\zeta| > |z_1| > |z_2| > 0,$$

these two series are equal. Therefore,

$$z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} Y_H^R(Y_H^R(\phi, z_1 - z_2)u_1, z_2)u_2 = z_1^{p_1} z_2^{p_2} (z_1 - z_2)^{p_{12}} Y_H^R(\phi, z_1)Y_V(u_1, z_2)u_2$$

Hence the associativity is proved.

7. From the second part of composable condition, p_1 depends only on the choice of u_1 and ϕ, p_2 depends only on the choice of u_2 and ϕ, p'_{12} depends only on the choice of u_1 and u_2. Thus the pole-order condition is satisfied. The rationality of products of any numbers of vertex operators then follows from Proposition 3.2.15.

□

6.3.3 Compatibility of the left and right V-module structure

In order to conclude that H is a V-bimodule, what remains to be shown is the following compatibility condition

Proposition 6.3.11. *For every $m \in \mathbb{Z}_+$, $l = 0, ..., m$, $u_1, ..., u_m \in V, \phi \in H, \phi' \in (H^N)'$,*

$$\langle \phi', Y_H^L(u_1, z_1) \cdots Y_H^L(u_l, z_l) Y_H^R(\phi, z_{l+1}) Y_V(u_{l+1}, z_{l+2}) \cdots Y_V(u_{m-1}, z_m) u_m \rangle$$

converges absolutely when

$$|z_1| > \cdots > |z_m| > 0$$

to a rational function with the only possible poles at $z_i = 0, i = 1, ..., m; z_i = z_j, 1 \leq i < j \leq m$.

Proof. The idea of the proof is similar. Here we only give a brief sketch:

Fix $m \in \mathbb{Z}_+, u_1, ..., u_m \in V, \phi \in H^N$. It suffices to discuss only the case when $l = 1, ..., m-1$. Take also $w_2' \in W_2', w_1 \in W_1$. Since ϕ is composable with any number of vertex operators, we know that

$$\langle w_2', Y_{W_2}^L(u_1, z_1+\zeta) \cdots Y_{W_2}^L(u_l, z_l+\zeta)[\phi(Y_{W_1}^L(u_{l+1}, z_{l+2}+\zeta) \cdots Y_{W_1}^L(u_{m-1}, z_m+\zeta) Y_{W_1}^L(u_m, \zeta) w_1)](z_{l+1}+\zeta) \rangle$$

converges absolutely when

$$|z_1 + \zeta| > \cdots > |z_m + \zeta| > |\zeta| > 0$$

to a rational function of the form

$$\frac{f(z_1, ..., z_m, \zeta)}{\zeta^r \prod_{i=1}^m z_i^{p_i} (z_i + \zeta)^{q_i} \prod_{1 \leq i < j \leq m} (z_i - z_j)^{p'_{ij}}} \tag{6.21}$$

We repeatedly use associativity to see that the following series

$$\langle w_2', Y_{W_2}^L(u_1, z_1+\zeta) \cdots Y_{W_2}^L(u_l, z_l+\zeta)[\phi(Y_{W_1}^L(Y_V(u_{l+1}, z_{l+2}) \cdots Y_V(u_{m-1}, z_m) u_m, \zeta) w_1)](z_{l+1}+\zeta) \rangle$$

is the expansion of the rational function (6.21) in the region

$$\{(z_1, ..., z_m, \zeta) \in \mathbb{C}^{m+1} : |z_1+\zeta| > \cdots > |z_{l+1}+\zeta| > |\zeta|+|z_{l+2}|, |\zeta| > |z_{l+2}| > \cdots > |z_m|\}$$

with the negative powers of

- $z_i = z_i + \zeta - \zeta$ expanded as power series of ζ, for $i = 1, ..., l+1$;
- $z_i + \zeta$ expanded as power series of z_i, for $i = l+2, ..., m$;
- $z_i - z_j = z_i + \zeta - (z_j + \zeta)$ expanded as power series of $z_j + \zeta$, for $1 \leq i < j \leq l+1$;
- $z_i - z_j$ expanded as power series of z_j, for $l+2 \leq i < j \leq m$;
- $z_i - z_j = z_i + \zeta - (z_j + \zeta)$ expanded as power series of $z_j + \zeta$ first, then further expand the positive powers of $z_j + \zeta$ as polynomials of z_j and ζ, for $1 \leq i \leq l+1, l+2 \leq j \leq m$.

We repeatedly use the identities (6.3) and (6.13), to see that the series

$$\langle w_2', [(Y_H^L(u_1, z_1) \cdots Y_H^L(u_l, z_l) Y_H^R(\phi, z_{l+1}) Y_V(u_{l+1}, z_{l+2}) \cdots Y_V(u_{m-1}, z_m) u_m)(w_1)](\zeta)\rangle$$

is the expansion the rational function (6.21) in the region

$$\{(z_1, ..., z_m, \zeta) \in \mathbb{C}^{m+1} : |\zeta| > |z_1| > \cdots > |z_m| > 0\}$$

with the negative powers of

1. $z_i + \zeta$ expanded as power series of z_i, for $1 \leq i \leq m$;
2. $z_i - z_j$ expanded as power series of z_j, for $1 \leq i < j \leq m$.

Now we multiply the series with the powers of z_i's and $(z_i - z_j)$'s, to see that

$$\begin{aligned}
&\prod_{i=1}^m z_i^{p_i} \prod_{1\leq i<j\leq m} (z_i - z_j)^{p'_{ij}} \\
&\langle w_2', [(Y_H^L(u_1, z_1) \cdots Y_H^L(u_l, z_l) Y_H^R(\phi, z_{l+1}) Y_V(u_{l+1}, z_{l+2}) \cdots Y_V(u_{m-1}, z_m) u_m)(w_1)](\zeta)\rangle \\
&\qquad = \frac{f(z_1, ..., z_m, \zeta)}{\zeta^r} \prod_{i=1}^m \iota_{\zeta z_i} \frac{1}{(z_i + \zeta)^{q_i}}
\end{aligned}$$

where for each $i = 1, ..., m$ the $\iota_{\zeta z_i}$ operator expands the negative powers of $\zeta + z_i$ as a power series of z_i. Thus the left-hand-side has no negative powers of $z_1, ..., z_m$. We similarly verify that for every fixed $n_1, ..., n_m \in \mathbb{Z}$, the $\widehat{W}$-valued rational function

$$w_2' \mapsto g_{n_1 \ldots n_m}(\zeta)$$

is an element in $(\widehat{W_2})_\zeta$, where $g_{n_1 \ldots n_k}$ is the coefficient of $z_1^{-n_1-1} \cdots z_m^{-n_m-1}$ in the power series above. Thus we know that the series

$$\prod_{i=1}^{m} z_i^{p_i} \prod_{1\leq i<j\leq m} (z_i - z_j)^{p'_{ij}} Y_H^L(u_1, z_1) \cdots Y_H^L(u_l, z_l) Y_H^R(\phi, z_{l+1}) Y_V(u_{l+1}, z_{l+2}) \cdots Y_V(u_{m-1}, z_m) u_m$$

is also a power series in $H[[z_1, ..., z_m]]$. If we take $u_1, ..., u_m \in V$ and $\phi \in H$ to be homogeneous elements and pair this power series to some homogeneous $\phi' \in (H^N)'$, then we will get a homogeneous polynomial of degree wt ϕ' − wt $(u_1) - \cdots - wt(u_m)$ − wt $\phi + \sum_{i=1}^m p_i + \sum_{1\leq i<j\leq m} p'_{ij}$. So for general nonhomogeneous $u_1, ..., u_n \in V, \phi \in H^N$, we have

$$\prod_{i=1}^{m} z_i^{p_i} \prod_{1\leq i<j\leq m} (z_i - z_j)^{p_{ij}} \langle \phi', Y_H^L(u_1, z_1) \cdots Y_H^L(u_l, z_l) Y_H^R(\phi, z_{l+1}) Y_V(u_{l+1}, z_{l+2}) \cdots Y_V(u_{m-1}, z_m) u_m \rangle$$
$$\in \mathbb{C}[z_1, ..., z_m]$$

Dividing both sides by the powers of z_i's and $(z_i - z_j)$'s and expand the rational function in the region $|z_1| > \cdots > |z_m| > 0$, we see that

$$\langle \phi', Y_H^L(u_1, z_1) \cdots Y_H^L(u_l, z_l) Y_H^R(\phi, z_{l+1}) Y_V(u_{l+1}, z_{l+2}) \cdots Y_V(u_{m-1}, z_m) u_m \rangle$$

converges absolutely in the region

$$|z_1| > \cdots > |z_n| > 0$$

to a rational function with poles at $z_i = 0, i = 1, ..., m$ and $z_i = z_j, 1 \leq i < j \leq m$. Thus the rationality of the product of any number of vertex operators is proved.

$\square$

Proposition 6.3.12. *For every* $u_1, u_2 \in V, \phi \in H, \phi' \in (H^N)'$,

$$\langle \phi', Y_H^L(u_1, z_1) Y_H^R(\phi, z_2) u_2 = \langle \phi', Y_H^R(Y_H^L(u_1, z_1 - z_2)\phi, z_2) u_2$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$

Proof. Since ϕ is composable to any number of vertex operators, in particular, for every $w_2' \in W_2', w_1 \in W_1, u_1, u_2 \in V$,

$$\langle w_2', Y_{W_2}(u_1, z_1 + \zeta)(\phi(Y_{W_1}(u_2, \zeta) w_1))(z_2 + \zeta)$$

converges absolutely to a rational function with the only possible poles at $z_1+\zeta=0, z_2+\zeta=0, \zeta=0, z_1=0, z_2=0, z_1=z_2$. From the preceding proposition, we see that

$$\langle w_2', [(Y_H^L(u_1,z_1)Y_H^R(\phi,z_2)u_2)(w_1)](\zeta)\rangle$$

is precisely the expansion of the rational function in the region $|\zeta|>|z_1|>|z_2|>0$.

Now we first use Identity (6.3) to see that

$$\langle w_2', Y_{W_2}(u_1,z_1+\zeta)(\phi(Y_{W_1}(u_2,\zeta)w_1))(z_2+\zeta) = \langle w_2', [(Y_H^L(u_1,z_1-z_2)\phi)(Y_{W_1}(u_2,\zeta)w_1)](z_2+\zeta)\rangle$$

when $|z_1+\zeta|>|z_2+\zeta|>|\zeta|>|z_1-z_2|>0$. One uses Lemma 2.3.7 to see that the right-hand-side coincides with the expansion of the rational function in the region $\{(z_1,z_2,\zeta) : |z_2+\zeta|>|\zeta|+|z_1-z_2|, |\zeta|>0, |z_2|>0\}$. Then we use Identity (6.13) to see that

$$\langle w_2', [(Y_H^L(u_1,z_1-z_2)\phi)(Y_{W_1}(u_2,\zeta)w_1)](z_2+\zeta)\rangle = \langle w_2', [(Y_H^R(Y_H^L(u_1,z_1-z_2)\phi, z_2)u_2)(w_1)](\zeta)\rangle$$

One uses Lemma 2.3.7 again to see that the right-hand-side coincides with the expansion of the the rational function in the region $\{(z_1,z_2,\zeta) : |\zeta|>|z_2|>|z_1-z_2|>0\}$.

Therefore, when $|\zeta|>|z_1|>|z_2|>|z_1-z_2|>0$, we have

$$[(Y_H^L(u_1,z_1)Y_H^R(\phi,z_2)u_2)(w_1)](\zeta) = [(Y_H^L(Y_H^R(\phi,z_1-z_2)u_1,z_2)u_2)(w_1)](\zeta)$$

as $\overline{W_2}$-valued rational functions. From the second part of the composable condition, one sees that

$$(Y_H^L(u_1,z_1)Y_H^R(\phi,z_2)u_2)(w_1) = (Y_H^L(Y_H^R(\phi,z_1-z_2)u_1,z_2)u_2)(w_1)$$

defines the same $\overline{W_2}$-valued rational function, and there exists $p_1, p_2, p_{12}\in\mathbb{N}$, such that

$$z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}}(Y_H^L(u_1,z_1)Y_H^R(\phi,z_2)u_2) = z_1^{p_1}z_2^{p_2}(z_1-z_2)^{p_{12}}(Y_H^L(Y_H^R(\phi,z_1-z_2)u_1,z_2)u_2)$$

as series in $H^N[[z_1,z_2]]$. Thus for every fixed $\phi'\in(H^N)'$,

$$\langle \phi', Y_H^L(u_1,z_1)Y_H^R(\phi,z_2)u_2\rangle = \langle \phi', Y_H^L(Y_H^R(\phi,z_1-z_2)u_1,z_2)u_2\rangle$$

when $|z_1|>|z_2|>|z_1-z_2|>0$. $\square$

Combining all the results in this section, we have proved the following

Theorem 6.3.13. *For every $N\in\mathbb{Z}$, $(H^N, Y_H^L, Y_H^R, \mathbf{d}_H, D_H)$ forms a V-bimodule.*

6.4 The reductive theorem

In this section, given a left V-module W and a left V-submodule W_2 of W and assuming a composability condition, we construct a 1-cocycle in $\hat{C}^1_\infty(V, H^{(N,F(V))})$ where N is a lower bound of the weights of the elements of V, $F(V)$ is the image of a suitable linear map F from V to H^N, H^N is the V-bimodule constructed in the preceding section and $H^{(N,F(V))}$ is the V-subbimodule of H^N generated by $F(V)$. For brevity we shall denote $H^{(N,F(V))}$ simply by $H^{F(V)}$.

6.4.1 Assumption of Composability and the bimodule $H^{F(V)}$

Let W be a left V-module and W_2 a V-submodule of W. Let W_1 be a graded subspace of W such that as a graded vector space, we have

$$W = W_1 \oplus W_2.$$

Then we can also embed W_1' and W_2' into W' and we have

$$W' = W_1' \oplus W_2'.$$

Let $\pi_{W_1} : W \to W_1$ and $\pi_{W_2} : W \to W_2$ be the projections given by this graded space decomposition of of W. For simplicity, we shall use the same notations π_{W_1} and π_{W_2} to denote their natural extensions to operators on $\overline{W}_1$ and $\overline{W}_2$, respectively. By definition, we have $\pi_{W_1}+\pi_{W_2} = 1_W$, $\pi_{W_1}\circ\pi_{W_1} = \pi_{W_1}$, $\pi_{W_2}\circ\pi_{W_2} = \pi_{W_2}$, $\pi_{W_1}\circ\pi_{W_2} = \pi_{W_2}\circ\pi_{W_1} = 0$.

Since W_2 is a submodule of W, we have $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_2}) = Y_{W_2}$, $D_{W_2} = \pi_{W_2} \circ D_W \circ \pi_{W_2}$ and $\mathbf{d}_{W_2} = \pi_{W_2} \circ \mathbf{d}_W \circ \pi_{W_2}$. We also have $\pi_{W_1} \circ Y_W^L \circ (1_V \otimes \pi_{W_2}) = 0$.

Let $Y_{W_1} = \pi_{W_1} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$, $D_{W_1} = \pi_{W_1} \circ D_W \circ \pi_{W_1}$ and $\mathbf{d}_{W_1} = \pi_{W_1} \circ \mathbf{d}_W \circ \pi_{W_1}$ which is equal to the operator giving the grading on W_1. As we have done above, we use the same notations D_{W_1} and $\mathbf{d}_{W_1}$ to denote their natural extensions to $\overline{W_1}$. We also use the same convention for notations for extensions of operators on W and W_2.

Proposition 6.4.1. *The graded vector space W_1 equipped with the vertex operator map Y_{W_1} and the operator D_{W_1} is a left V-module.*

Proof. The axioms for the grading and the identity property are obvious.

For $v \in V$,

$$\begin{aligned}
\frac{d}{dz}Y_{W_1}(v,z) &= \pi_{W_1}\left(\frac{d}{dz}Y_W^L(v,z)\right)\pi_{W_1}\\
&= \pi_{W_1}Y_W^L(D_V v,z))\pi_{W_1}\\
&= Y_{W_1}(D_V v,z). \qquad (6.22)
\end{aligned}$$

Also using (6.22), we have

$$\begin{aligned}
\frac{d}{dz}Y_{W_1}(v,z) &= \pi_{W_1}Y_W^L(D_V v,z)\pi_{W_1}\\
&= \pi_{W_1}D_W Y_W^L(v,z)\pi_{W_1} - \pi_{W_1}Y_W^L(v,z)D_W\pi_{W_1}\\
&= \pi_{W_1}D_W\pi_{W_1}Y_W^L(v,z)\pi_{W_1} + \pi_{W_1}D_W\pi_{W_2}Y_W^L(v,z)\pi_{W_1}\\
&\quad - \pi_{W_1}Y_W^L(v,z)\pi_{W_1}D_W\pi_{W_1} - \pi_{W_1}Y_W^L(v,z)\pi_{W_2}D_W\pi_{W_1}\\
&= D_{W_1}Y_{W_1}(v,z) + \pi_{W_1}D_W\pi_{W_2}Y_W^L(v,z)\pi_{W_1}\\
&\quad - Y_{W_1}(v,z)D_{W_1} - \pi_{W_1}Y_W^L(v,z)\pi_{W_2}D_W\pi_{W_1}. \qquad (6.23)
\end{aligned}$$

Since W_2 is a V-submodule of W, $\pi_{W_1}D_W\pi_{W_2} = \pi_{W_1}D_{W_2}\pi_{W_2} = 0$. Again since W_2 is a V-submodule of W, $\pi_{W_1}Y_W^L(v,z)\pi_{W_2} = \pi_{W_1}Y_{W_2}(v,z)\pi_{W_2} = 0$.

For $v, v_1, \dots, v_k \in V$, $w_1' \in W_1'$ and $w_1 \in W_1$, using the properties of π_{W_1}, π_{W_2}, Y_{W_1}, Y_{W_2} given above, we have

$$\langle w_1', Y_{W_1}(v_1,z_1)\cdots Y_{W_1}(v_k,z_k)w_1\rangle = \langle w_1', Y_W^L(v_1,z_1)\cdots Y_W^L(v_k,z_k)w_1\rangle. \qquad (6.24)$$

Since the right-hand side of (6.24) is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to a rational function in $z_1, \dots, z_k$ with the only possible poles $z_i = 0$ for $i = 1, \dots, k$ and $z_i = z_j$ for $1 \le i < j \le k$, so is the left-hand side. This proves the rationality.

For $v, v_1, v_2 \in V$, $w_1' \in W_1'$ and $w_1 \in W_1$, using the properties of π_{W_1}, π_{W_2}, Y_{W_1}, Y_{W_2} again and the associativity for Y_W^L, we obtain

$$\begin{aligned}
&\langle w_1', Y_{W_1}(v_1,z_1)Y_{W_1}(v_2,z_2)w_1\rangle\\
&\quad = \langle w_1', Y_W^L(v_1,z_1)Y_W^L(v_2,z_2)w_1\rangle\\
&\quad = \langle w_1', Y_W^L(Y_V(v_1,z_1-z_2)v_2,z_2)w_1\rangle
\end{aligned}$$

$$= \langle w_1', Y_{W_1}(Y_V(v_1, z_1 - z_2)v_2, z_2)w_1 \rangle, \tag{6.25}$$

proving the associativity for Y_{W_1}. $\square$

Remark 6.4.2. Note that although W_1 is a graded subspace of W, (W_1, Y_{W_1}) is not a submodule of W since the vertex operator Y_{W_1} is not the restriction of the vertex operator Y_W^L to $V \otimes W_1$.

We now have two left V-modules W_1 and W_2. From Theorem 6.3.13, for $N \in \mathbb{Z}$, we have a V-bimodule $H^N \subset \mathrm{Hom}(W_1, \widehat{(W_2)}_z)$.

We need the following assumption (called *composability condition*) on the map $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$:

Assumption 6.4.3 (Composability condition)**.** For every $m \in \mathbb{Z}_+$, $l = 0, ..., m$, $v, v_1, \ldots, v_m \in V$, $w_2' \in W_2'$ and $w_1 \in W_1$,

$$\langle w_2', Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_l, z_l)\pi_{W_2} Y_W^L(v, \zeta)\pi_{W_1} Y_{W_1}(v_{l+1}, z_{l+1}) \cdots Y_{W_1}(v_m, z_m)w_1 \rangle \tag{6.26}$$

is absolutely convergent in the region $|z_1| > \cdots |z_l| > |\zeta| > |z_{l+1}| > \cdots > |z_m| > 0$ to a rational function in $z_1, \ldots, z_m, \zeta$ with the only possible poles $z_i = 0$ for $i = 1, \ldots, m$, $\zeta = 0$, $z_i = z_j$ for $1 \leq i < j \leq m$ and $z_i = \zeta$ for $i = 1, \ldots, m$. Moreover, the orders of the poles $\zeta = 0$, $z_i = 0$ for $i = 1, ..., m$, $z_i = \zeta$ for $i = 1, \ldots, m$ and $z_i = z_j$ for $i, j = 1, \ldots, m$, $i \neq j$ are bounded above by nonnegative integers depending only on the pairs $(v, w_1), (v_i, w_1), (v_i, v)$ and (v_i, v_j), respectively, and there exists $N \in \mathbb{Z}$ such that when (6.26) is expanded as a Laurent series in the region $|z_m| > |z_1 - z_m| > \cdots > |z_l - z_m| > |\zeta - z_m| > |z_{l+1} - z_m| > \cdots > |z_{m-1} - z_m| > 0$ as a Laurent series in $z_i - z_m$ for $i = 1, \ldots, m-1$ and $\zeta - z_m$ with Laurent polynomials in z_m as coefficients, the total degree of each monomial in $z_i - z_m$ for $i = 1, \ldots, m-1$ and $\zeta - z_m$ in the expansion is larger than or equal to $N - \sum_{i=1}^m \mathrm{wt}\ v_i + \mathrm{wt}\ v$.

We now assume that $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$ satisfies the composability condition. For $v \in V$, let $F(v) \in \mathrm{Hom}(W_1, \widehat{(W_2)}_\zeta)$ be given by

$$((F(v))(w_1))(\zeta) = \pi_{W_2} Y_W^L(v, \zeta)\pi_{W_1} w_1$$

for $w_1 \in W_1$. Thus we obtain a linear map $F : V \to \mathrm{Hom}(W_1, \widehat{(W_2)}_\zeta)$.

Proposition 6.4.4. *For $N \in \mathbb{Z}$ such that* wt $v \geq N$ *for any homogeneous $v \in V$, the image of F is in fact in H^N and is thus a map from V to H^N. Moreover, F preserve the gradings.*

Proof. Let $v \in V$ be homogeneous. For $a \in \mathbb{C}^\times$ and $w_1 \in W_1$,

$$\begin{aligned} a^{\mathbf{d}_{W_2}}((F(v))(w_1))(\zeta) &= a^{\mathbf{d}_{W_2}} \pi_{W_2} Y_W^L(v, z)\pi_{W_1} w_1 \\ &= \pi_{W_2} Y_W^L(a^{\mathbf{d}_V} v, a\zeta)\pi_{W_1} a^{\mathbf{d}_{W_1}} w_1 \\ &= a^{\mathrm{wt}\ v}((F(v))(a^{\mathbf{d}_{W_1}} w_1))(a\zeta), \end{aligned}$$

proving the $\mathbf{d}$-conjugation property of $F(v)$ and wt $F(v) =$ wt v.

The composable condition and the N-weight-degree condition obviously hold for $F(v)$ under the Assumption 6.4.3. $\square$

6.4.2 Constructing a 1-cocycle in $\hat{C}^1_\infty(V, H^{F(V)})$

Proposition 6.4.4 says in particular that $F(V)$ is independent of such lower bound N of the weights of V. Thus we shall denote $H^{(N,F(V))}$ simply by $H^{F(V)}$. Now we construct a 1-cochain $\Psi \in \hat{C}^1_\infty(V, H^{F(V)})$. Since

$$\hat{C}^1_\infty(V, H^{F(V)}) \subset \mathrm{Hom}(V, \widetilde{(H^{F(V)})}_z),$$

For $v \in V$, $w_1 \in W_1$ and $w_2' \in W_2'$,

$$\begin{aligned} \langle w_2', ((e^{zD_H} F(v))(w_1))(\zeta)\rangle &= \langle w_2', ((F(v))(w_1))(\zeta + z)\rangle \\ &= \langle w_2', \pi_{W_2} Y_W^L(v, \zeta + z)w_1\rangle \end{aligned}$$

in the region $|\zeta| > |z|$. Let $E(e^{zD_H} F(v)) \in \widetilde{(H^{F(V)})}_z$ be defined by

$$\langle w_2', ((E(e^{zD_H} F(v)))(w_1))(\zeta)\rangle = \langle w_2', \pi_{W_2} Y_W^L(v, \zeta + z)w_1\rangle$$

$v \in V$, $w_1 \in W_1$ and $w_2' \in W_2'$ in the region $\zeta + z \neq 0$.

We define

$$(\Psi(v))(z) = E(e^{zD_H} F(v)).$$

More explicitly, for $v \in V$, $w_1 \in W_1$ and $w_2' \in W_2'$,

$$\langle w_2', (((\Psi(v))(z))(w_1))(\zeta)\rangle = \langle w_2', \pi_{W_2} Y_W^L(v, \zeta + z)w_1\rangle. \tag{6.27}$$

in the region $\zeta + z \neq 0$. In the region $|\zeta| > |z|$, the series $e^{zD_H}F(v)$ is convergent absolutely to $\Psi(v)$. We shall also use $e^{zD_H}F(v)$ to denote $(\Psi(v))(z)$ in the region $|\zeta| > |z|$. By definition, $\Psi(v) \in (\widetilde{H^{F(V)}})_z$ and thus $\Psi \in \mathrm{Hom}(V, (\widetilde{H^{F(V)}})_z)$.

Theorem 6.4.5. $\Psi \in \ker \hat{\delta}^1_\infty \subset \hat{C}^1_\infty(V, H^{F(V)})$.

Proof. We first prove that Ψ satisfies the D-derivative property and the $\mathbf{d}$-conjugation property and is composable with any number of vertex operators.

For $v \in V$, $w_1 \in W_1$, in the region $|\zeta| > |z|$,

$$\begin{aligned}
&\left(\left(\frac{\partial}{\partial z}(\Psi(v))(z)\right)(w_1)\right)(\zeta)\\
&\quad = \left(\left(\frac{\partial}{\partial z}e^{zD_H}F(v)\right)(w_1)\right)(\zeta)\\
&\quad = ((e^{zD_H}D_H F(v))(w_1))(\zeta)\\
&\quad = e^{z\frac{\partial}{\partial \zeta}}\frac{\partial}{\partial \zeta}((F(v))(w_1))(\zeta)\\
&\quad = e^{z\frac{\partial}{\partial \zeta}}\frac{\partial}{\partial \zeta}\pi_{W_2}Y_W^L(v,\zeta)\pi_{W_1}w_1\\
&\quad = e^{z\frac{\partial}{\partial \zeta}}\pi_{W_2}Y_W^L(D_V v,\zeta)\pi_{W_1}w_1\\
&\quad = e^{z\frac{\partial}{\partial \zeta}}((F(D_V v))(w_1))(\zeta)\\
&\quad = (((e^{zD_H}F(D_V v)))(w_1))(\zeta)\\
&\quad = (((\Psi(D_V v))(z))(w_1))(\zeta).
\end{aligned}$$

Thus we obtain

$$\frac{\partial}{\partial z}(\Psi(v))(z) = (\Psi(D_V v))(z).$$

Also

$$\begin{aligned}
D_H(\Psi(v))(z) &= D_H E(e^{zD_H}F(v))\\
&= \frac{\partial}{\partial z}E(e^{zD_H}F(v))\\
&= \frac{\partial}{\partial z}(\Psi(v))(z)
\end{aligned}$$

for $v \in V$. Thus we also obtain

$$\frac{\partial}{\partial z}(\Psi(v))(z) = D_H(\Psi(v))(z).$$

So Ψ satisfies the D-derivative property.

For $a \in \mathbb{C}^\times$ and $v \in V$,

$$\begin{aligned} a^{\mathbf{d}_H}(\Psi(v))(z) &= a^{\mathbf{d}_H} E(e^{zD_H} F(v)) \\ &= E(e^{za^{\mathbf{d}_H} D_H a^{-\mathbf{d}_H}} a^{\mathbf{d}_H} F(v)) \\ &= E(e^{azD_H} F(a^{\mathbf{d}_V} v)) \\ &= (\Psi(a^{\mathbf{d}_V} v))(az). \end{aligned}$$

This proves the **d**-conjugation property. Thus $\Psi \in \hat{C}_0^1(V, H^{F(V)})$.

Now we prove that Ψ is composable with one vertex operator, i.e., for every $u, v \in V$, $h' \in H'$, the series

$$\begin{aligned} &\langle h', Y_H^L(u, z_1)(\Psi(v))(z_2)\rangle, \\ &\langle h', Y_H^{s(R)}(u, z_2)(\Psi(v))(z_1)\rangle, \\ &\langle h', [(\Psi(Y_V(u, z_1 - \eta)Y_V(v, z_2 - \eta)\mathbf{1}))(w_1)](\eta)\rangle \end{aligned}$$

converge absolutely respectively in $|z_1| > |z_2|, |z_2| > |z_1|$ and $|z_1 - \eta| > |z_2 - \eta|$ to rational functions with the only possible poles at $z_1 = z_2$, and order of the pole $z_1 = z_2$ is bounded above by a constant that depends only on u and v.

From the assumption of composability of $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$, we know that for every $w_2' \in W_2', w_1 \in W_1$, the series

$$\langle w_2', Y_{W_2}(u, z_1 + \zeta)\pi_{W_2} Y_W^L(v, \zeta + z_2)\pi_{W_1} w_1\rangle = \langle w_2', Y_{W_2}(u, z_1 + \zeta)[(F(v))(w_1)](\zeta + z_2)\rangle$$

converges absolutely to a rational function when $|z_1 + \zeta| > |\zeta + z_2| > 0$ with the only possible poles at $z_1 + \zeta = 0, z_2 + \zeta = 0, z_1 - z_2 = 0$. Use the identity (6.27) and Lemma 2.3.9, we see that the series

$$\langle w_2', Y_{W_2}(u, z_1 + \zeta)[(e^{z_2 D_H} F(v))(w_1)](\zeta)\rangle$$

is the expansion of the rational function in the region $|z_1+\zeta| > |z_2|+|\zeta|, |\zeta| > |z_2|$. In particular, it converges absolutely. Use the identity (6.3) and Lemma 2.3.7, we see that the series

$$\langle w_2', [Y_H^L(u,z_1)(e^{z_2 D_H}F(v))(w_1)](\zeta)\rangle$$

is the expansion of the rational function in the region $|\zeta| > |z_1| > |z_2|$. In other words, the series

$$\sum_{n_1\in\mathbb{Z}}\sum_{n_2\leq -1}\langle w_2', [((Y_H^L)_{n_1}(u)D_H^{-n_2-1}F(v))(w_1)](\zeta)\rangle z_1^{-n_1-1}z_2^{-n_2-1}$$

converges absolutely in this region to a rational function with the only possible poles at $z_1+\zeta=0, z_2+\zeta=0, z_1-z_2=0$. Moreover, one can find integers p_1 depending only on the choice of u and w_1, p_2 depending only on the choice of v and w_1, and p_{12} depending only on the choice of u and v, and a polynomial $f(z_1,z_2,\zeta)$, such that

$$\begin{aligned}(z_1-z_2)^{p_{12}}\sum_{n_1\in\mathbb{Z}}\sum_{n_2\leq -1}\frac{1}{(-n_2-1)!}\langle w_2', [((Y_H^L)_{n_1}(u)D_H^{-n_2-1}F(v))(w_1)](\zeta)\rangle z_1^{-n_1-1}z_2^{-n_2-1}\\ = \iota_{\zeta z_1}(z_1+\zeta)^{-p_1}\iota_{\zeta z_2}(z_2+\zeta)^{-p_2}f(z_1,z_2,\zeta)\end{aligned}$$

which have no negative powers of z_1, z_2. One can then see that the series

$$(z_1-z_2)^{p_{12}}\sum_{n_1\in\mathbb{Z}}\sum_{n_2\leq -1}\frac{1}{(-n_2-1)!}((Y_H^L)_{n_1}(u)D_H^{-n_2-1}F(v))z_1^{-n_1-1}z_2^{-n_2-1},$$

as a formal series in z_1, z_2 with coefficients in H, has no negative power. From the **d**-conjugation property of Y_H^L and $F(v)$, one can see that after pairing with $h'\in H'$, the series

$$\begin{aligned}&(z_1-z_2)^{p_{12}}\sum_{n_1\in\mathbb{Z}}\sum_{n_2\leq -1}\frac{1}{(-n_2-1)!}\langle h', ((Y_H^L)_{n_1}(u)D_H^{-n_2-1}F(v))\rangle z_1^{-n_1-1}z_2^{-n_2-1}\\ &= (z_1-z_2)^{p_{12}}\langle h', Y_H^L(u,z_1)e^{z_2 D_H}F(v)\rangle = (z_1-z_2)^{p_{12}}\langle h', Y_H^L(u,z_1)(\Psi(v))(z_2)\rangle\end{aligned}$$

is a polynomial in $\mathbb{C}[z_1,z_2]$. Thus the first series is the expansion of the quotient of the polynomial by $(z_1-z_2)^{p_{12}}$ in the region $|z_1|>|z_2|$. In particular, the first series converges absolutely.

For the second series, we first note that for $\phi\in H^{F(V)}, w_1\in W_1$, we can compute to see that

$$[(Y_H^{s(R)}(u,z)\phi)(w_1)](\zeta) = \iota_{\zeta z}E[\phi(Y_{W_1}(u,z+\zeta)w_1)](\zeta)$$

Similarly as the discussions after Definition of Y_H^L and Y_H^R, when $|\zeta| > |z+\zeta| > |z| > 0$

$$[(Y_H^{s(R)}(u,z)\phi)(w_1)](\zeta) = [\phi(Y_{W_1}(u,z+\zeta)w_1)](\zeta) \tag{6.28}$$

From the assumption of composability of $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$, for every $w_2' \in W_2', w_1 \in W_1$, the series

$$\langle w_2', \pi_{W_2} Y_W^L(v, \zeta + z_1)\pi_{W_1} Y_{W_1}(u, z_2+\zeta)w_1\rangle = \langle w_2', [(F(v))(Y_{W_1}(u,z_2+\zeta)w_1)](\zeta+z_1)\rangle$$

converges absolutely when $|\zeta + z_1| > |z_2 + \zeta| > 0$ to a rational function with the only possible poles at $z_1 + \zeta = 0, z_2 + \zeta = 0, z_1 - z_2 = 0$. Using Identity (6.27) and Lemma 2.3.7, we can show that the series

$$\langle w_2', [(e^{z_1 D_H} F(v))(Y_{W_1}(u, z_2+\zeta)w_1)](\zeta)\rangle$$

is the expansion of the rational function in the region $|\zeta| > |z_2 + \zeta| + |z_1|, |z_2 + \zeta| > 0$. In particular, it converges absolutely. Using the Identity (6.28) and Lemma 2.3.9, we can show that the series

$$\langle w_2', [(Y_H^{s(R)}(u,z_2)e^{z_1 D_H} F(v))(w_1)](\zeta)\rangle$$

is the expansion of the rational function in the region $|\zeta| > |z_2| > |z_1|$. In other words, the series

$$\sum_{n_2\in\mathbb{Z}}\sum_{n_1\leq -1} \frac{1}{(-n_1-1)!}\langle w_2', [((Y_H^{s(R)})_{n_2}(u)D_H^{-n_1-1}F(v))(w_1)](\zeta)\rangle$$

converges absolutely in this region to a rational function with the only possible poles at $z_1 + \zeta = 0, z_2 + \zeta = 0, z_1 - z_2 = 0$. Moreover, one can find integers p_1 depending only on the choice of v and w_1, p_2 depending only on the choice of u and w_1, and p_{12} depending only on the choice of u and v, and a polynomial $f(z_1, z_2, \zeta)$, such that

$$\begin{aligned}(z_1-z_2)^{p_{12}}\sum_{n_2\in\mathbb{Z}}\sum_{n_1\leq -1} \frac{1}{(-n_1-1)!}&\langle w_2', [((Y_H^{s(R)})_{n_2}(u)D_H^{-n_1-1}F(v))(w_1)](\zeta)\rangle\\ &= \iota_{\zeta z_1}(z_1+\zeta)^{-p_1}\iota_{\zeta z_2}(z_2+\zeta)^{-p_2}f(z_1,z_2,\zeta)\end{aligned}$$

which has no negative powers of z_1, z_2. One can then see the series

$$(z_1-z_2)^{p_{12}}\sum_{n_2\in\mathbb{Z}}\sum_{n_1\leq -1} \frac{1}{(-n_1-1)!}(Y_H^{s(R)})_{n_2}(u)D_H^{-n_1-1}F(v)$$

as a formal series in z_1, z_2 with coefficients in H, has no negative power. From the $\mathbf{d}$-conjugation property of Y_H^L and $F(v)$, one can see that after pairing with $h' \in H'$, the series

$$(z_1 - z_2)^{p_{12}} \sum_{n_2 \in \mathbb{Z}} \sum_{n_1 \leq -1} \langle h', \frac{1}{(-n_1-1)!} (Y_H^{s(R)})_{n_2}(u) D_H^{-n_1-1} F(v) \rangle$$
$$= (z_1 - z_2)^{p_{12}} \langle h', Y_H^{s(R)}(u, z_2) e^{z_1 D_H} F(v) \rangle = (z_1 - z_2)^{p_{12}} \langle h', Y_H^{s(R)}(u, z_2)(\Psi(v))(z_1) \rangle$$

is a polynomial in $\mathbb{C}[z_1, z_2]$. Thus the second series is the expansion of the quotient of the polynomial by $(z_1 - z_2)^{p_{12}}$ in the region $|z_2| > |z_1|$. In particular, the second series converges absolutely.

For the third series, note that from the associativity of Y_W^L, for every $w_2' \in W_2', w_1 \in W_1$, the series

$$\langle w_2', \pi_{W_2} Y_W^L(Y_V(u_1, z_1 - \eta) Y_V(u_2, z_2 - \eta)\mathbf{1}, \zeta + \eta) \pi_{W_1} w_1 \rangle$$

converges absolutely when $|\zeta + \eta| > |z_1 - \eta| > |z_2 - \eta|$ to a rational function that is independent of η and ζ and has the only possible poles at $z_1 + \zeta = 0, z_2 + \zeta = 0, z_1 - z_2 = 0$. Using Identity (6.27) and Lemma 2.3.7, we can show that the series

$$\langle w_2', [(e^{\eta D_H} F(Y_V(u, z_1 - \eta) Y_V(v, z_2 - \eta)\mathbf{1}))(w_1)](\zeta) \rangle$$

is the expansion of the rational function in the region $|\zeta| > |\eta| + |z_1 - \eta|, |z_1 - \eta| > |z_2 - \eta|$. Moreover, one can find integers p_1 depending only on the choice of u and w_1, p_2 depending only on the choice of v and w_1, and p_{12} depending only on the choice of u and v, and a polynomial $f(z_1, z_2, \zeta)$, such that

$$(z_1 - z_2)^{p_{12}} \langle w_2', [(e^{\eta D_H} F(Y_V(u, z_1 - \eta) Y_V(v, z_2 - \eta)\mathbf{1}))(w_1)](\zeta) \rangle$$

is the expansion of $f(z_1, z_2, \zeta)/(z_1 + \zeta)^{p_1}(z_2 + \zeta)^{p_2}$ obtained by expanding the negative powers of $z_i + \zeta$ first as a power series of z_i, then further expand the positive powers of $z_i = z_i - \eta + \eta$ as polynomials in $z_i - \eta$ and η, for $i = 1, 2$. In particular, there does not exist negative powers of $z_1 - \eta$ and $z_2 - \eta$. One can see the series

$$(z_1 - z_2)^{p_{12}} e^{\eta D_H} F(Y_V(u, z_1 - \eta) Y_V(v, z_2 - \eta)\mathbf{1}),$$

as a formal series in $z_1-\eta, z_2-\eta$ and η, does not have negative powers. From the $\mathbf{d}$-conjugation property of $F(v)$ and Y_V, one can see that after pairing with $h' \in H'$, the series

$$(z_1-z_2)^{p_{12}}\langle h', e^{\eta D_H}F(Y_V(u, z_1-\eta)Y_V(v, z_2-\eta)\mathbf{1})\rangle = (z_1-z_2)^{p_{12}}\langle h', (\Psi(Y_V(u, z_1-\eta)Y_V(v, z_2-\eta)\mathbf{1}))(\eta),$$

is a polynomial in $\mathbb{C}[z_1, z_2]$ that does not depend on η. Thus the third series is the expansion of the quotient of this polynomial by $(z_1-z_2)^{p_{12}}$ in the region $|z_1-\eta| > |z_2-\eta|$. In particular, the third series converges absolutely in this region.

Now we prove $\hat{\delta}^1_1\Psi = 0$. Let $v_1, v_2 \in V$, $w_1 \in W_1$ and $w_2' \in W_2'$. Using the properties of π_{W_1}, π_{W_2}, Y_{W_1}, Y_{W_2}, we obtain

$$\begin{aligned}\langle w_2', Y_W^L(v_1, z_1)Y_W^L(v_2, z_2)w_1\rangle \\ = \langle w_2', \pi_{W_2}Y_W^L(v_1, z_1)\pi_{W_1}Y_{W_1}(v_2, z_2)w_1\rangle + \langle w_2', Y_{W_2}(v_1, z_1)\pi_{W_2}Y_W^L(v_2, z_2)\pi_{W_1}w_1\rangle\end{aligned} \tag{6.29}$$

and

$$\langle w_2', Y_W^L(Y_V(v_1, z_1-z_2)v_2, z_2)w_1\rangle = \langle w_2', \pi_{W_2}Y_W^L(Y_V(v_1, z_1-z_2)v_2, z_2)\pi_{W_1}w_1\rangle. \tag{6.30}$$

By the rationality for Y_W^L, the left-hand sides of (6.29) and (6.30) are absolutely convergent in the region $|z_1| > |z_2| > 0$ and $|z_2| > |z_1-z_2| > 0$, respectively, to a common rational function in z_1, z_2 with the only possible poles $z_1, z_2 = 0$ and $z_1 = z_2$. In particular, the right-hand side of (6.30) is absolutely convergent in the region $|z_2| > |z_1-z_2| > 0$ to the same rational function that the the left-hand sides of (6.30) converges to. On the other hand, by Assumption 6.4.3, both terms in the right-hand side of (6.29) are also absolutely convergent in the region $|z_1| > |z_2| > 0$ to a rational function in z_1, z_2 with the only possible poles $z_1, z_2 = 0$ and $z_1 = z_2$. By the associativity of Y_W^L, (6.29) and (6.30) , we obtain

$$\begin{aligned}\langle w_2', \pi_{W_2}Y_W^L(v_1, z_1)\pi_{W_1}Y_{W_1}(v_2, z_2)w_1\rangle + \langle w_2', Y_{W_2}(v_1, z_1)\pi_{W_2}Y_W^L(v_2, z_2)\pi_{W_1}w_1\rangle \\ = \langle w_2', \pi_{W_2}Y_W^L(Y_V(v_1, z_1-z_2)v_2, z_2)\pi_{W_1}w_1\rangle\end{aligned}$$

in the region $|z_1| > |z_2| > |z_1-z_2| > 0$ or, equivalently,

$$R(\langle w_2', \pi_{W_2}Y_W^L(v_1, z_1)\pi_{W_1}Y_{W_1}(v_2, z_2)w_1\rangle) + R(\langle w_2', Y_{W_2}(v_1, z_1)\pi_{W_2}Y_W^L(v_2, z_2)\pi_{W_1}w_1\rangle)$$

$$= R(\langle w_2', \pi_{W_2} Y_W^L(Y_V(v_1, z_1 - z_2)v_2, z_2)\pi_{W_1} w_1\rangle). \tag{6.31}$$

Replacing z_1 and z_2 in (6.31) by $z_1 + \zeta$ and $z_2 + \zeta$, we obtain

$$\begin{aligned}
&R(\langle w_2', \pi_{W_2} Y_W^L(v_1, z_1 + \zeta)\pi_{W_1} Y_{W_1}(v_2, z_2 + \zeta)w_1\rangle) \\
&\quad + R(\langle w_2', Y_{W_2}(v_1, z_1 + \zeta)\pi_{W_2} Y_W^L(v_2, z_2 + \zeta)\pi_{W_1} w_1\rangle) \\
&= R(\langle w_2', \pi_{W_2} Y_W^L(Y_V(v_1, z_1 - z_2)v_2, z_2 + \zeta)\pi_{W_1} w_1\rangle).
\end{aligned} \tag{6.32}$$

By (6.27) and the definition of Y_H^R, the first term of the left-hand side of (6.32) is equal to

$$\begin{aligned}
&R(\langle w_2', ((\Psi(v_1))(z_1))(Y_{W_1}(v_2, z_2 + \zeta)w_1))(\zeta)\rangle) \\
&\quad = R(\langle w_2', (Y_H^R((\Psi(v_1))(z_1), -z_2)v_2)(w_1))(z_2 + \zeta)\rangle) \\
&\quad = R(\langle w_2', (e^{z_2 D_H} Y_H^R((\Psi(v_1))(z_1), -z_2)v_2)(w_1))(\zeta)\rangle).
\end{aligned} \tag{6.33}$$

By (6.27) and the definition of Y_H^L, the second term of the left-hand side of (6.31) is equal to

$$\begin{aligned}
&R(\langle w_2', Y_{W_2}(v_1, z_1 + \zeta)(((\Psi(v_2))(z_2))(w_1))(\zeta)\rangle) \\
&\quad = R(\langle w_2', (Y_H^L(v_1, z_1)((\Psi(v_2))(z_2))(w_1))(\zeta)\rangle).
\end{aligned} \tag{6.34}$$

By (6.27), the right-hand side of (6.31) is equal to

$$R(\langle w_2', (((\Psi(Y_V(v_1, z_1 - z_2)v_2))(z_2))(w_1))(\zeta)\rangle). \tag{6.35}$$

Using (6.33)–(6.35) and the definition of $\hat{\delta}_1^1$, we see that (6.31) becomes

$$R(\langle w_2', (((\hat{\delta}_1^1 \Psi)(v_1 \otimes v_2))(z_1, z_2))(w_1)(\zeta)\rangle) = 0$$

for $v_1, v_2 \in V$, $w_1 \in W_1$ and $w_2' \in W_2'$. Thus we obtain $\hat{\delta}_1^1 \Psi = 0$. Thus $\Psi \in \ker \hat{\delta}_1^1$ we see from Theorem 5.3.35 and Remark 5.3.36 that $\Psi \in \ker \hat{\delta}_\infty^1$. $\square$

6.4.3 The main theorem and the proof

In this section, we formulate and prove our main result on complete reducibility of modules of finite length for a meromorphic open-strong vertex algebra V.

Let W be a left V-module. Assume that W is not irreducible. Then there exists a proper nonzero left V-submodule W_2 of W. We say that the pair (W, W_2) *satisfies the composability condition* if there exists a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ as a graded vector space such that $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 6.4.3. If for every proper nonzero left V-submodule W_2 of W, the pair (W, W_2) satisfies the composability condition, we say that W *satisfies the composability condition.*

Proposition 6.4.6. *Let W be a completely reducbile left V-module. Then W satisfies the composability condition.*

Proof. Let W_2 be a left V-submodule of W. Since W is completely reducible, there is a left V-submodule W_1 of W such that W as a left V-module is the direct sum of the left V-modules W_1 and W_2. Then π_{W_1} and π_{W_2} are module maps. Thus $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 6.4.3. $\square$

Now let W be a left V-module which is not irreducible and W_2 a proper nonzero left V-submodule W_2 of W. Assuming that the pair (W, W_2) satisfies the composability condition. Then there exists a graded subspace W_1 of W such that as a graded vector space, W is the direct sum of W_1 and W_2 and $\pi_{W_2} \circ Y_W^L \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 6.4.3. By Theorem 6.3.13, Proposition 6.4.4 and Theorem 6.4.5, there exist a left V-module structure on W_1, a V-bimodule $H^N \subset \mathrm{Hom}(W_1, \widehat{(W_2)}_{z_1})$ for a lower bound N of V, a grading preserving linear map $F : V \to H^N$ and $\Psi \in \ker \hat{\delta}^1_\infty \subset \hat{C}^1_\infty(V, H^{F(V)})$, where $H^{F(V)}$ is the V-subbimodule of H^N generated by $F(V)$.

Theorem 6.4.7. *Let W, W_1, W_2 and $H^{F(V)}$ be as above. If $\hat{H}^1_\infty(V, H^{F(V)}) = 0$, then there exists another left V-submodule $\widetilde{W}_1$ of W such that W is the direct sum of the left V-submodules $\widetilde{W}_1$ and W_2.*

Proof. Since $\hat{H}^1_\infty(V, H^{F(V)}) = 0$, Ψ must be a coboundary. By definition, there exists a 0-cochain $\Phi \in \hat{C}^0_\infty(V, H)$ such that $\Psi = \hat{\delta}^0_\infty \Phi$. Note that such a 0-cochain Φ is an element of $H^{F(V)}_{[0]}$ such that $D_H \Phi = 0$ and, in particular, $((\hat{\delta}^0_\infty \Phi)(v))(z)$ is an $\overline{H^{F(V)}}$-valued holomorphic function on $\mathbb{C}$. Thus the equality $\Psi = \hat{\delta}^0_\infty \Phi$ gives

$$(\Psi(v))(z_2) = Y_H^L(v, z_2)\Phi - e^{z_2 D_H} Y_H^R(\Phi, -z_2)v. \tag{6.36}$$

Applying both sides of (6.36) to $w_1 \in W_1$, evaluating at z_1, pairing with $w' \in W'$ and using (6.27), we obtain in the region $|z_1| > |z_2| > 0$,

$$\begin{aligned}
&\langle w', \pi_{W_2} Y_W^L(v, z_1 + z_2) w_1 \rangle \\
&\quad = \langle w', ((Y_H^L(v, z_2)\Phi)(w_1))(z_1)\rangle - \langle w', ((e^{z_2 D_H} Y_H^R(\Phi, -z_2)v)(w_1))(z_1)\rangle. \qquad (6.37)
\end{aligned}$$

We now define a linear map $\eta : W_1 \to W$ by

$$\eta(w_1) = w_1 - \pi_{\mathrm{wt}\ w_1}(\Phi(w_1))(1)$$

for homogeneous $w_1 \in W_1$. Let $\widetilde{W_1} = \eta(W_1)$. We show that $\widetilde{W_1}$ is in fact a left V-submodule of W.

We first consider $w_1 - (\Phi(w_1))(z_1) \in \overline{W}$. Applying $Y_W^L(v, z_1 + z_2)$ to this element and pairing the result with $w' \in W'$, taking the rational function that the resulting series converges to, using (6.37) and then using (6.3) and (6.13), we obtain

$$\begin{aligned}
&R(\langle w', Y_W^L(v, z_1 + z_2)(w_1 - (\Phi(w_1))(z_1))\rangle) \\
&\quad = R(\langle w', (\pi_{W_1} Y_W^L(v, z_1 + z_2) w_1 + \pi_{W_2} Y_W^L(v, z_1 + z_2) w_1 - Y_{W_2}(v, z_1 + z_2)(\Phi(w_1))(z_1))\rangle) \\
&\quad = R(\langle w', (Y_{W_1}(v, z_1 + z_2) w_1 + ((Y_H^L(v, z_2)\Phi)(w_1))(z_1) \\
&\qquad\qquad - ((e^{z_2 D_H} Y_H^R(\Phi, -z_2)v)(w_1))(z_1) - Y_{W_2}(v, z_1 + z_2)(\Phi(w_1))(z_1))\rangle) \\
&\quad = R(\langle w', (Y_{W_1}(v, z_1 + z_2) w_1 - (e^{z_2 D_H} Y_H^R(\Phi, -z_2)v)(w_1)(z_1))\rangle) \\
&\quad = R(\langle w', (Y_{W_1}(v, z_1 + z_2) w_1 - (Y_H^R(\Phi, -z_2)v)(w_1)(z_1 + z_2))\rangle) \\
&\quad = R(\langle w', (Y_{W_1}(v, z_1 + z_2) w_1 - (\Phi(Y_{W_1}(v, z_1 + z_2) w_1))(z_1))\rangle). \qquad (6.38)
\end{aligned}$$

Since $D_H \Phi = 0$, $Y_H^R(\Phi, -z_2)v$ is independent of z_2 and hence $z_2 = 0$ is not a pole of the rational function (6.38). Replacing $z_1 + z_2$ by z in the two sides of (6.38), we obtain

$$R(\langle w', Y_W^L(v, z)(w_1 - (\Phi(w_1))(z_1))\rangle) = R(\langle w', (Y_{W_1}(v, z) w_1 - (\Phi(Y_{W_1}(v, z) w_1))(z_1))\rangle). \qquad (6.39)$$

Since $z_2 = 0$ is not a pole of the rational function (6.38), $z = z_1$ is not a pole of (6.39). Thus the only poles of the rational function (6.39) are $z = 0$ and $z_1 = 0$. So (6.39) must be a Laurent polynomial in z and z_1. Since $\langle w', (Y_W^L(v, z)(w_1 - \Phi(w_1)(z_1)))\rangle$ and $\langle w', (Y_{W_1}(v, z) w_1 - (\Phi(Y_{W_1}(v, z) w_1))(z_1))\rangle$ are Laurent series in z and z_1 and are convergent absolutely to a Laurent polynomial in z and z_1, these series must be Laurent polynomials in z and z_1 themselves and are equal to the Laurent polynomial (6.39). Thus we obtain

$$\langle w', Y_W^L(v,z)(w_1 - (\Phi(w_1))(z_1)))\rangle = \langle w', (Y_{W_1}(v,z)w_1 - (\Phi(Y_{W_1}(v,z)w_1))(z_1))\rangle \quad (6.40)$$

in the region $z, z_1 \neq 0$. Subtracting the Laurent polynomial $\langle w', Y_{W_1}(v,z)w_1\rangle$ from both sides of (6.40), using

$$Y_W^L(v,z)w_1 = Y_{W_1}(v,z)w_1 + \pi_{W_2}Y_W^L(v,z)w_1$$

and multiplying the resulting equality by -1, we obtain

$$-\langle w', \pi_{W_2}Y_W^L(v,z)w_1\rangle + \langle w', Y_W^L(v,z)(\Phi(w_1))(z_1)\rangle = \langle w', (\Phi(Y_{W_1}(v,z)w_1))(z_1)\rangle \quad (6.41)$$

in the region $z, z_1 \neq 0$. Moving the first term in the left-hand side of (6.41) to the tight-hand side and then taking the coefficients of z^{-n-1} of both sides of (6.41), we obtain

$$\langle w', (Y_W^L)_n(v)(\Phi(w_1))(z_1)\rangle = \langle w', (\Phi((Y_{W_1})_n(v)w_1))(z_1)\rangle + \langle w', \pi_{W_2}(Y_W^L)_n(v)w_1\rangle \quad (6.42)$$

Since w' is arbitrary, we obtain

$$(Y_W^L)_n(v)(\Phi(w_1))(z_1) = (\Phi((Y_{W_1})_n(v)w_1))(z_1) + \pi_{W_2}(Y_W^L)_n(v)w_1. \quad (6.43)$$

Applying the projection $\pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}$ to both sides of (6.43) and taking $z_1 = 1$, we have

$$\begin{aligned}
&\pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}(Y_W^L)_n(v)(\Phi(w_1))(1)\\
&\quad = \pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}(\Phi((Y_{W_1})_n(v)w_1))(1) + \pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}\pi_{W_2}(Y_W^L)_n(v)w_1. \quad (6.44)
\end{aligned}$$

Using (6.44), we have

$$\begin{aligned}
&(Y_W^L)_n(v)\eta(w_1)\\
&\quad = (Y_W^L)_n(v)(w_1 - \pi_{\mathrm{wt}\ w_1}(\Phi(w_1))(1))\\
&\quad = (Y_W^L)_n(v)w_1 - (Y_W^L)_n(v)\pi_{\mathrm{wt}\ w_1}(\Phi(w_1))(1)
\end{aligned}$$

$$= (Y_W^L)_n(v)w_1 - \pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}(Y_W^L)_n(v)(\Phi(w_1))(1)$$
$$= (Y_W^L)_n(v)w_1 - \pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}(\Phi((Y_{W_1})_n(v)w_1))(1) - \pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}\pi_{W_2}(Y_W^L)_n(v)w_1$$
$$= (Y_{W_1})_n(v)w_1 - \pi_{\mathrm{wt}\ v-n-1+\mathrm{wt}\ w_1}(\Phi((Y_{W_1})_n(v)w_1))(1)$$
$$= \eta((Y_{W_1})_n(v)w_1). \tag{6.45}$$

The formula (6.45) means in particular that the space $\widetilde{W_1}$ is invariant under the action of Y_W^L. Thus $\widetilde{W_1}$ is a submodule of W. Moreover, the sum of $\widetilde{W_1}$ and W_2 is clearly W and the intersection of $\widetilde{W_1}$ and W_2 is clearly 0. So W is equal to the direct sum of $\widetilde{W_1}$ and W_2, proving the theorem. □

We shall need the following result:

Proposition 6.4.8. *Let W be a left V-module satisfying the composability condition. Then every left V-submodule of W also satisfies the composability condition.*

Proof. Let W_0 be a left V-submodule of W. Then any proper nonzero left V-submodule W_2 of W_0 is also a proper nonzero left V-submodule of W. Then there is a graded subspace W_3 of W such that $W = W_3 \oplus W_2$ as a graded vector space and $\pi_{W_2}^W \circ Y_W^L \circ (1_V \otimes \pi_{W_3}^W)$ satisfies Assumption 6.4.3 with W_1 in Assumption 6.4.3 replaced by W_3, where $\pi_{W_2}^W$ and $\pi_{W_3}^W$ are projections from W to W_2 and W_3, respectively. Let $W_1 = W_3 \cap W_0$. Then $W_0 = W_1 \oplus W_2$ as a graded vector space. Let $\pi_{W_1}^{W_0}$ and $\pi_{W_2}^{W_0}$ be the projections from W_0 to W_1 and W_2, respectively. Then $\pi_{W_1}^{W_0} = \pi_{W_3}^W\big|_{W_0}$ and $\pi_{W_2}^{W_0} = \pi_{W_2}^W\big|_{W_0}$. So we have

$$\pi_{W_2}^{W_0} \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^{W_0}) = \pi_{W_2}^W \circ Y_W^L \circ (1_V \otimes \pi_{W_3}^W)\big|_{V\otimes W_0}.$$

Since $\pi_{W_2}^W \circ Y_W^L \circ (1_V \otimes \pi_{W_3}^W)$ satisfies Assumption 6.4.3, $\pi_{W_2}^W \circ Y_W^L \circ (1_V \otimes \pi_{W_3}^W)\big|_{V\otimes W_0}$ satisfies Assumption 6.4.3 with W in Assumption 6.4.3 replaced by W_0. Thus $\pi_{W_2}^{W_0} \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^{W_0})$ satisfies Assumption 6.4.3 with W replaced by W_0. □

Definition 6.4.9. A left V-module W is said to be of *finite length* if there is a finite sequence $W = U_1 \supset \cdots \supset U_{n+1} = 0$ of left V-modules such that U_i/U_{i+1} for $i = 1, \ldots, n$ are irreducible left V-modules. The finite sequence is called a *composition series of W*

and the positive integer n is independent of the composition series and is called the *length of* W.

Remark 6.4.10. Note that as a graded vector space, W is a direct sum of the underlying graded vector spaces of U_i/U_{i+1} for $i = 1, \dots, n$. If U_i/U_{i+1} for $i = 1, \dots, n$ are grading restricted, then W must also be grading restricted.

From Theorem 6.4.7 and Propostion 6.4.8, we obtain immediately the following main result of this paper:

Theorem 6.4.11. *Let V be a meromorphic open-string vertex algebra. If $\hat{H}^1_\infty(V, M) = 0$ for every $\mathbb{Z}$-graded V-bimodule M, then every left V-module of finite length satisfying the composability condition is completely reducible. Assume in addition that the following condition also holds: For every left V-module W satisfying the composability condition and every nonzero proper left V-submodule W_2 of W, there exists a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ as a graded vector space, $\pi_{W_2} \circ Y^L_W \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 6.4.3 and the submodule $H^{F(V)}$ of H^N is grading restricted for the grading preserving linear map $F : V \to H^N$ given by Proposition 6.4.4. Then the conclusion still holds if $\hat{H}^1_\infty(V, M) = 0$ only for every grading-restricted $\mathbb{Z}$-graded V-bimodule M.*

Proof. Let W be a left V-module of finite length satisfying the composability condition. If it is not irreducible, then there is a nonzero proper left V-submodule W_2 of V. Since W satisfies the composability condition, there is a graded subspace W_1 of W such that $W = W_1 \oplus W_2$ as a vector space and $\pi_{W_2} \circ Y^L_W \circ (1_V \otimes \pi_{W_1})$ satisfies Assumption 6.4.3. By Theorem 6.4.7, there is a left V-submodule $\widetilde{W}_1$ of W such that $W = \widetilde{W}_1 \oplus W_2$ as a left V-modules. Since W is of finite length, both $\widetilde{W}_1$ and W_2 are of lengths less than or equal to the length of W. Since W_2 is nonzero and proper in W, $\widetilde{W}_1$ is also nonzero and proper in W.

We use induction on the length of W. In the case that the length of W is 1, W is irreducible and thus is completely reducible. Assuming that when the length of W is less than n, it is completely reducible. When the length of W is n, the length of $\widetilde{W}_1$ and

W_2 are less than n. So they are completely reducible and thus W is also completely reducible.

In the case that the additional condition also holds, since $H^{F(V)}$ is grading restricted, $\hat{H}^1_\infty(V, M) = 0$ for every grading-restricted $\mathbb{Z}$-graded V-bimodule M implies in particular $\hat{H}^1_\infty(V, H^{F(V)}) = 0$. Then the conclusion of Theorem 6.4.7 still holds. Thus W is completely reducible. $\square$

Remark 6.4.12. By Theorems 5.3.39 and 6.4.11, we can replace $\hat{H}^1_\infty(V, M) = 0$ by the statement that every derivation from V to M is an inner derivation.

We also have the following conjecture:

Conjecture 6.4.13. *Let V be a meromorphic open-string vertex algebra. If every left V-module of finite length is completely reducible, then $\hat{H}^1_\infty(V, M) = 0$ for every V-bimodule M.*

Note that because of Proposition 6.4.6, we do not need to require that left V-modules satisfy the composability condition in this conjecture.

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