ON THE COMPLEXITY OF MATRIX BALANCING*

B. Kalantari, L. Khachiyan, and A. Shokoufandeh

Department of Computer Science

Rutgers University, New Brunswick, NJ 08903

Abstract

An $n \times n$ matrix with nonnegative entries is said to be *balanced* if for each $i = 1, \ldots, n$, the sum of the entries of its i-th row is equal to the sum of the entries of its i-th column. An $n \times n$ matrix A with nonnegative entries is said to be *balancable via diagonal similarity scaling* if there exists a diagonal matrix X with positive diagonal entries such that XAX^{-1} is balanced. We give upper and lower bounds on the entries of X, and prove the necessary sensitivity analysis in the required accuracy of the minimization of an associated convex programming problem. These results are used to prove the polynomial time solvability of computing X to any prescribed accuracy.

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1. Introduction.

An $n \times n$ matrix A with nonnegative entries is said to be *balanced* if for each $i = 1, \ldots, n$, the sum of the entries of its i-th row is equal to the sum of the entries of its i-th column:

$$A\mathbf{1} = A^T\mathbf{1}, \tag{1.1}$$

where $\mathbf{1}$ is the n-vector of all ones. A is said to be *balancable via diagonal similarity scaling* (or simply balancable) if there exists a diagonal matrix X with positive diagonal entries such that XAX^{-1} is balanced, i.e.,

$$XAX^{-1}\mathbf{1} = X^{-1}A^TX\mathbf{1}. \tag{1.2}$$

The matrix balancing problem can be defined in more generality:
An $n \times n$ matrix $B = (b_{ij})$ with arbitrary real entries is said to be *balanced in the l_p-norm* $(p > 0)$ if for each $i = 1, \ldots, n$, its i-th row and column have the same l_p-norm. An invertible diagonal matrix $Y = \text{diag}(y_1, \ldots, y_n)$ *balances B in the l_p-norm* if for each $i = 1, \ldots, n$ the l_p-norm of the i-th row and column of YBY^{-1} are identical, i.e.

$$\sum_{j=1}^{n} |b_{ij}\frac{y_i}{y_j}|^p = \sum_{j=1}^{n} |b_{ji}\frac{y_j}{y_i}|^p, \quad i = 1, \ldots, n. \tag{1.3}$$

Clearly, an invertible diagonal matrix $Y = \text{diag}(y_1, \ldots, y_n)$ balances B in the l_p-norm if and only if the positive diagonal matrix $X = \text{diag}(|y_1|^p, \ldots, |y_n|^p)$ balances the nonnegative matrix $A = (|b_{ij}|^p)$ in l_1-norm. The general matrix balancing problem in l_p-norm can thus be reduced to the case of nonnegative matrix balancing via a positive diagonal matrix.

Osborne [5] considered the case of $p = 2$ and its application in pre-conditioning a given matrix B in order to increase the accuracy of the computation of its eigenvalues (B and YBY^{-1} have the same set of eigenvalues). Through an iterative process, Osborne [5] showed that if $Y^* = \text{diag}(y_1^*, \ldots, y_n^*)$ satisfies (1.3), then the vector $y^* = (y_1^*, \ldots, y_n^*)$ is the minimizer of the function

$$\phi_B(y) = \Big(\sum_{i,j=1}^{n} |b_{ij}\frac{y_i}{y_j}|^2 \Big)^{1/2},$$

where the minimization ranges over all invertible $Y = \text{diag}(y_1, \ldots, y_n)$. Conversely, the minimizer of $\phi_B(y)$, if it exists satisfies (1.3). Note that $\phi_B(y)$ is the Frobenius norm of the matrix YBY^{-1} satisfying

$$\frac{1}{\sqrt{n}}\phi_B(y) \leq \|YBY^{-1}\|_2 \leq \phi_B(y).$$

In view of the above inequality and Osborne's result, the balancing of the nonnegative matrix $A = (|b_{ij}|^2)$, also bounds the quantity

$$\nu(B) = \inf\{\|YBY^{-1}\|_2 : y \in \Re^n, \quad Y = \text{diag}(y_1, \ldots, y_n), \quad \Pi_{i=1}^n y_i \neq 0\}.$$

For a description of $\nu(B)$ as a generalized eigenvalue problem see Boyd, El Ghaoui, Feron, and Balakrishnan [1].

The problem of nonnegative matrix balancing has been treated by several researchers. Balancability has been termed as *line-sum-symmetric scaling*, see Eaves, Hoffman, Rothblum, and Schneider [2], and *balancing*, see Grad [3], Schneider and Zenios [7]. Characterization theorems on nonnegative balancable matrices has been given by Osborne [5] and Eaves et. al [2]. Other results on matrix balancing including applications and iterative algorithms are given by Osborne [5], Grad [3], and Schneider and Zenios [7].

From now on we shall consider nonnegative matrices, and we shall say a nonnegative matrix is *balanced* to mean that it is balanced in the sense of (1.1) and (1.2). In this paper we prove the polynomial time solvability of the problem of balancing a nonnegative matrix to any prescribed accuracy.

Clearly, without loss of generality we may assume that a given $n \times n$ nonnegative matrix $A = (a_{ij})$ satisfies $a_{ii} = 0$, for all $i = 1, \ldots, n$. Corresponding to such a matrix A there exists a directed graph $G_A = (V, E)$ where $V = \{1, \ldots, n\}$, and where $E = \{(i, j) : a_{ij} > 0\}$. Without loss of generality we may also assume that G_A, when viewed as an undirected graph, is connected. Otherwise, after a permutation of $V = \{1, \ldots, n\}$ the given matrix A can be replaced by $\text{diag}(A_1, \ldots, A_r)$, where each of $A_1, \ldots, A_r$ is a square matrix whose corresponding directed graph is connected. Thus A is balancable if and only if $A_1, \ldots, A_r$

are balancable. Moreover, it can be shown that for each $i = 1, \ldots, r$, A_i is balancable if and only if the corresponding graph G_{A_i} is strongly connected (Theorem 1).

Definition 1. Given a positive tolerance ε, a nonnegative $n \times n$-matrix A is said to be *balanced to the absolute error of* ε if $\|A\mathbf{1} - A^T\mathbf{1}\| \le \varepsilon$. A positive diagonal matrix X is said to *balance* A *to the absolute error* of ε if the matrix XAX^{-1} is balanced to the absolute error of ε:

$$\|XAX^{-1}\mathbf{1} - X^{-1}A^TX\mathbf{1}\| \le \varepsilon. \tag{1.4}$$

Definition 2. Given a positive tolerance ε, a nonnegative $n \times n$-matrix A is said to be *balanced to the relative error of* ε if $\|A\mathbf{1} - A^T\mathbf{1}\|/\mathbf{1}^TA\mathbf{1} \le \varepsilon$. A positive diagonal matrix X is said to *balance* A *to the relative error* of ε if the matrix XAX^{-1} is balanced to the relative error of ε:

$$\frac{\|XAX^{-1}\mathbf{1} - X^{-1}A^TX\mathbf{1}\|}{\mathbf{1}^TXAX^{-1}\mathbf{1}} \le \varepsilon. \tag{1.5}$$

Letting

$$f(x) = \sum_{i,j=1}^{n} a_{ij}\frac{x_i}{x_j}, \quad g(x) = \ln f(x),$$

it is easy to see that (1.4) and (1.5) can be written as $\|\nabla f(x)\| \le \varepsilon$, and $\|\nabla g(x)\| \le \varepsilon$, respectively. Neither $f(x)$ nor $g(x)$ are convex. However, if we apply the change of variable

$$x = (e^{w_1}, \ldots, e^{w_n})^T \in \Re^n,$$

we obtain the functions

$$F(w) = \sum_{i,j=1}^{n} a_{ij}e^{w_i - w_j}, \quad G(w) = \ln F(w)$$

both of which are convex over $\Re^n$. In particular, if $w^* = (w_1^*, \ldots, w_n^*)$ is the exact minimizer of $G(w)$, then $X(w^*) = \text{diag}(e^{w_1^*}, \ldots, e^{w_n^*})$ exactly balances A. Moreover, if $\|\nabla F(w)\| \le \varepsilon$, $X(w)$ satisfies (1.4) and if $\|\nabla G(w)\| \le \varepsilon$, $X(w)$ satisfies (1.5).

In this paper we show the polynomial-time solvability of the balancing problem. Specifically, we prove the following complexity result (Theorem 5):

Let A be an $n \times n$ nonnegative matrix, $a_{ii} = 0$, for all $i = 1, \ldots, n$. Suppose that $G_A = (V, E)$ is strongly connected. Let $a_{\min} = \min\{a_{ij} : (i,j) \in E\}$, $\sigma = \sum_{(i,j)\in E} a_{ij}$, $v = a_{\min}/\sigma$, and $m = |E|$. For any given accuracy $\varepsilon \in (0,1)$, in $O\Big(n^2 m \ln\big(\frac{n}{\varepsilon}\ln\frac{1}{v}\big)\Big)$ arithmetic operations over $O\Big(\ln\big(\frac{n}{\varepsilon v}\big)\Big)$-bit numbers, we can compute a positive diagonal matrix $X = \operatorname{diag}(e^{w_1}, \ldots, e^{w_n})$ so that XAX^{-1} is balanced to the relative error of ε, and it is balanced to the absolute error of $e\sigma\varepsilon$.

In order to obtain the above result we first state a characterization on balancable matrices, Theorem 1. In particular this theorem implies that an arbitrary nonnegative matrix is balancable if and only if its corresponding graph is the union of strongly connected graphs.

In Theorem 2 we prove

An $n \times n$ nonnegative matrix A with $a_{ii} = 0$, for all $i = 1, \ldots, n$, and G_A strongly connected, can be balanced by a diagonal matrix $X^* = \operatorname{diag}(e^{w_1^*}, \ldots, e^{w_n^*})$ such that

$$|w_i^*| \leq \frac{n-1}{2} \ln \frac{e}{(n-1)v}, \quad i = 1, \ldots, n.$$

We give an example of ill-behaved balancable matrices for which the above bound on balancing factors is sharp up to a constant factor.

To obtain the necessary bound on the accuracy of the minimization of $G(w)$ we prove in Theorem 3, and Corollary 2 that

For any given $\varepsilon \in (0,1)$, the minimization of $G(w)$ to an absolute accuracy of $\delta = \varepsilon^2/16$ gives a point $(w_1, \ldots, w_n)$ so that (1.5) is satisfied with $X(w) = \operatorname{diag}(e^{w_1}, \ldots, e^{w_n})$, i.e. if w^* is an exact minimizer of $G(w)$ in $\Re^n$, $G(w) - G(w^*) \leq \delta$ implies that $X(w)$ balances A to a relative error of ε.

Again, we give a simple example for which the above bound is optimal up to a constant factor. In Theorem 4 we prove

For any given $\varepsilon \in (0,1)$, $\|w - w^*\| \leq \varepsilon^2/16\sqrt{2}$ *implies* $G(w) - G(w^*) \leq \delta = \varepsilon^2/16$.

The above results will imply the polynomial-time solvability of the problem of balancing to any prescribed relative or absolute error, via the ellipsoid algorithm, or interior-point Newton methods, see e.g. Nesterov and Nemirovskii [4].

The remainder of the paper is organized as follows. In Section 2 we state a characterization result on balancable matrices. In Section 3 we derive our bounds on balancing matrices. In Section 4 we bound the required absolute accuracy of the minimization of $G(w)$. Finally, in Section 5 we prove the polynomial time solvability of the balancing problem.

2. Characterization of Nonnegative Balancable Matrices.

The following characterization of balancable matrices is due to Eaves, Hoffman, Rothblum, and Schneider [2]. The equivalence of conditions (ii) and (iii) is essentially due to Osborne [5]. For the sake of completeness we provide a proof of this theorem. The proof can be viewed as an alternative proof to that of [2].

Theorem 1. *Let A be an $n \times n$ nonnegative matrix, $a_{ii} = 0$, for all $i = 1, \ldots, n$, whose graph $G_A = (V, E)$ when viewed as an undirected graph is connected. The following statements are equivalent:*

(i): *G_A is strongly connected.*

(ii): *$f(x) = \sum_{(i,j)\in E} a_{ij} \frac{x_i}{x_j}$ attains its infimum over $\Omega = \{x \in \Re^n : \sum_{i=1}^n x_i = 1, \ x_i > 0, \ \ i = 1, \ldots, n\}$*

(iii): *A is balancable.*

(iv): *There exist* $n \times n$ *circuit matrices* $C_1, \ldots, C_q$ *(i.e., each* C_k *is a matrix with* 0-1 *entries whose graph* G_k *is a simple cycle through* $n_k \leq n$ *vertices), and a positive diagonal matrix* $X^* = \mathrm{diag}(x_1^*, \ldots, x_n^*)$ *such that*

$$X^* A X^{*^{-1}} = \sum_{k=1}^{q} \alpha_k C_k$$

with some positive weights $\alpha_1, \ldots, \alpha_q$.

Proof. (i)⇒(ii): Let $\bar{x}$ be a boundary point of Ω. Then $\bar{x}_i > 0$ and $\bar{x}_j = 0$ for some $i, j \in \{1, \ldots, n\}$. Since G_A is strongly connected, there exist edges $(i_1 = i, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k = j) \in E$, where $i_1, \ldots, i_k$ are distinct. Then

$$f(x) \geq \sum_{l=1}^{k-1} a_{i_l i_{l+1}} \frac{x_{i_l}}{x_{i_{l+1}}}.$$

As $x \in \Omega$ converges to $\bar{x}$, the right hand side of the above approaches $+\infty$.
(ii) ⇒ (iii): Let $x^* \in \Omega$ be a minimizer of f over Ω. Since f is homogeneous of degree zero, x^* minimizes f over $\{x : x > 0\}$. Hence $\nabla f(x^*) = 0$, which gives

$$\sum_{j=1}^{n} a_{ij} \frac{x_i^*}{x_j^*} = \sum_{j=1}^{n} a_{ji} \frac{x_j^*}{x_i^*}.$$

(iii)⇒(iv): Let $A^* = X^* A X^{*^{-1}}$. Since A^* is balanced, $A^* \in K$, where

$$K = \{Y = (y_{ij}) \geq 0 : \sum_{j=1}^{n} y_{ij} - \sum_{j=1}^{n} y_{ji} = 0, \quad i = 1, \ldots, n, \quad y_{ij} = 0 \; for \; (i,j) \notin E\}.$$

So A^* is a positive combination of generators of the cone K. But the graph corresponding to any generator of K is a simple cycle, proving (iv).
(iv)⇒(i): From (iv), G_A can be decomposed as the union of directed cycles. Given this and since G_A is connected, it is easy to argue the reachability from any vertex i to any other vertex j. □

Corollary 1. *Letting* $A_k = X^{*^{-1}} C_k X^*$, *we have*

$$A = \sum_{k=1}^{q} \alpha_k A_k,$$

where X^* *simultaneously balances each of the* A_k's.

Remark 1. Theorem 1 implies that the balancability of A can be tested in $O(|E|)$ time.

Remark 2. It can be shown that under the assumption of Theorem 1 the balancing matrix X^* is unique up to a scalar factor (see Osborne [5] and Eaves et al. [2]).

3. Bounds on Balancing Matrix.

Theorem 2. *Let A be an $n \times n$ nonnegative matrix, $a_{ii} = 0$, for all $i = 1, \ldots, n$, and G_A strongly connected. There exists a positive diagonal $X^* = \mathrm{diag}(x_1^*, \ldots, x_n^*)$ balancing A such that*

$$\left[\frac{e}{(n-1)v}\right]^{\frac{1-n}{2}} < x_i^* < \left[\frac{e}{(n-1)v}\right]^{\frac{n-1}{2}}, \quad i = 1, \ldots, n, \tag{3.1}$$

where

$$v = \frac{a_{\min}}{\sigma}, \quad a_{\min} = \min\{a_{ij} \ : (i,j) \in E\}, \quad \sigma = \sum_{(i,j)\in E} a_{ij}. \tag{3.2}$$

Proof. Suppose $X^* = \mathrm{diag}(x^*)$ balances A. From (3.2) and that x^* is a minimizer of $f(x)$ over $\{x : x > 0\}$ we have

$$a_{min} \sum_{(i,j)\in E} \frac{x_i^*}{x_j^*} \leq \sum_{(i,j)\in E} a_{ij} \frac{x_i^*}{x_j^*} = f(x_1^*, \ldots, x_n^*) \leq f(1, \ldots, 1) = \sigma. \tag{3.3}$$

Suppose without loss of generality that

$$\frac{\max\{x_1^*, \ldots, x_n^*\}}{\min\{x_1^*, \ldots, x_n^*\}} = \frac{x_1^*}{x_2^*}.$$

Since G_A is strongly connected, there exists a simple directed path from 1 to 2, say $P = (i_1, i_2), \ldots, (i_{t-1}, i_t)$, where $i_1 = 1$, $i_t = 2$, and $t \leq n$. From (3.3) and the arithmetic-geometric mean inequality we get

$$\frac{x_1^*}{x_2^*} = \prod_{(i,j)\in P} \frac{x_i^*}{x_j^*} \leq \left[\frac{1}{(t-1)} \sum_{(i,j)\in P} \frac{x_i^*}{x_j^*}\right]^{t-1} \leq \left[\frac{1}{(t-1)v}\right]^{t-1} < \left[\frac{e}{(t-1)v}\right]^{t-1} \leq \left[\frac{e}{(n-1)v}\right]^{n-1},$$

where the last inequality follows from $v \leq 1/n$. Replacing X^* by tX^* with $t = 1/\sqrt{x_1^* x_2^*}$, (3.1) follows. □

We now give an example of nonnegative balancable matrices which are ill-behaved. Consider $n = 2k+1$ and $\varepsilon \in (0,1)$. Let A be $n \times n$ nonnegative matrix with the following positive entries:

$$a_{i,i+1} = a_{2k+2-i,2k+1-i} = 1, \tag{3.4}$$

$$a_{i+1,i} = a_{2k+1-i,2k+2-i} = \varepsilon, \tag{3.5}$$

for $i = 1, \ldots, k$, and

$$a_{n1} = a_{1n} = 1. \tag{3.6}$$

Observe that the graph G_A of matrix A can be decomposed into two cycles, through its n vertices. Next, define the diagonal matrix $X^* = \mathrm{diag}(x_1^*, \ldots, x_n^*)$ as follows

$$x^*_{2k+2-i} = x^*_i = \varepsilon^{\frac{k+2-2i}{4}}, \quad i = 1, \ldots, k+1. \tag{3.7}$$

To illustrate (3.4) – (3.7), consider the case $k = 3$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \varepsilon \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X^* = \mathrm{diag}(\varepsilon^{\frac{3}{4}}, \varepsilon^{\frac{1}{4}}, \varepsilon^{-\frac{1}{4}}, \varepsilon^{-\frac{3}{4}}, \varepsilon^{-\frac{1}{4}}, \varepsilon^{\frac{1}{4}}, \varepsilon^{\frac{3}{4}}).$$

It is easy to check the positive entries of $X^*AX^{*^{-1}}$ are given by

$$a^*_{i,i-1} = a^*_{i-1,i} = \sqrt{\varepsilon}, \quad i = 2, \ldots, n,$$

$$a^*_{1n} = a^*_{n1} = 1.$$

From this it follows that

$$\sum_{j=1}^{n} a^*_{ij} = \sum_{j=1}^{n} a^*_{ji} = 2\sqrt{\varepsilon}, \quad i = 2, \ldots, n-1,$$

and

$$\sum_{j=1}^{n} a^*_{ij} = \sum_{j=1}^{n} a^*_{ji} = 1 + \sqrt{\varepsilon}, \quad i = 1, n.$$

Thus X^* balances A, and

$$\frac{\max\{x_1^*,\ldots,x_n^*\}}{\min\{x_1^*,\ldots,x_n^*\}} = \left(\frac{1}{\varepsilon}\right)^{(n-1)/4}.$$

Since X^* is unique up to scalar multiplication, the above bound holds for any balancing of A. For comparison, Theorem 1 gives

$$\frac{\max\{x_1^*,\ldots,x_n^*\}}{\min\{x_1^*,\ldots,x_n^*\}} < \left[\frac{e}{n-1}\cdot\frac{(n-1)\varepsilon+n+1}{\varepsilon}\right]^{n-1} \sim \left[\frac{e}{\varepsilon}\right]^{n-1}.$$

Remark 3. It follows from the proof of Theorem 2 that (3.1) can be strengthened as follows:

$$\left[\frac{e}{dv}\right]^{\frac{-d}{2}} < x_i^* < \left[\frac{e}{dv}\right]^{\frac{d}{2}}, \quad i=1,\ldots,n,$$

where d is the longest (directed) shortest path between a pair of vertices in G_A.

4. A Sensitivity Theorem for the Convex Program.

For a given $n\times n$ nonnegative matrix A, consider the function

$$G(w) = \ln F(w),$$

where

$$F(w) = f(e^{w_1},\ldots,e^{w_n}) = \sum_{(i,j)\in E} a_{ij}e^{w_i-w_j}.$$

Theorem 3. *Let A be an $n\times n$ nonnegative matrix, $a_{ii}=0$, for all $i=1,\ldots,n$, and G_A strongly connected. Let $w^*\in\Re^n$ be an exact minimizer of $G(w)$. Suppose that for a given $\delta>0$, we have $w\in\Re^n$ satisfying*

$$G(w) - G(w^*) \le \delta.$$

Then

$$\|\nabla G(w)\| \le 2\sqrt{e^{2\delta}-1}.$$

Before proving the theorem we state the following corollary which gives the required accuracy of the optimization of $G(w)$. From Theorem 3 and the fact that $x/2 < \ln(1+x)$ for all $x\in(0,1)$, we get

Corollary 2. *Given* $\varepsilon \in (0,1)$*, let*

$$\delta = \frac{\varepsilon^2}{16}.$$

If $w \in \Re^n$ *satisfies* $G(w) - G(w^*) \leq \delta$*, then* $\|\nabla G(w)\| \leq \varepsilon$ *(and hence the diagonal matrix* $X(w) = \mathrm{diag}(e^{w_1}, \ldots, e^{w_n})$ *balances* A *to the relative error of* ε*.)* □

Observe that the bound of Corollary 2 is optimal up to a constant factor: the identity matrix balances A to the relative error of ε where

$$A = \begin{pmatrix} 0 & \frac{1}{2} + \frac{\varepsilon}{2\sqrt{2}} \\ \frac{1}{2} - \frac{\varepsilon}{2\sqrt{2}} & 0 \end{pmatrix},$$

and

$$\delta = G(0) - G(w^*) = -\frac{1}{2}\ln(1 - \frac{\varepsilon^2}{2}) \sim \frac{\varepsilon^2}{4}.$$

In order to prove Theorem 3 we need some auxiliary lemmas.

Lemma 1. *For any given number* $a \geq 1$*, and* $x \in \Re^n$ *define*

$$\sigma(x,a) = \sqrt{\frac{\sum_{i=1}^n (x_i - a)^2}{n(n-1)}}.$$

Let

$$B(n,a) = \max\{\sigma(x,a) : \frac{1}{n}\sum_{i=1}^n x_i = a, \quad \prod_{i=1}^n x_i = 1, \quad x > 0\}.$$

The above optimization problem has an optimal solution $(x_1, \ldots, x_n)$ *such that* $x_1 = \ldots = x_{n-1} \leq x_n$.

Proof. For $n = 1, 2$ there is nothing to prove. We first prove the lemma for $n = 3$. Consider

$$\max\{\sum_{i=1}^3 (x_i - a)^2 : \frac{1}{3}\sum_{i=1}^3 x_i = a, \quad \prod_{i=1}^3 x_i = 1, \quad x_i > 0, \quad i = 1,2,3\}.$$

Since $a \geq 1$, from the relationship between arithmetic-geometric means, the feasible region is nonempty. Optimality condition gives

$$2(x_i - a) = \lambda_1 + \frac{\lambda_2}{x_i}, \quad i = 1,2,3,$$

where λ_1, and λ_2 are Lagrange multipliers. Since x_i's are solution to quadratic equation $2x(x-a) = \lambda_1 x + \lambda_2$, it follows that two of the x_i's have the same value. This implies that either there exists an optimal solution of the form $x_1 = x_2 = a - R$, $x_3 = a + 2R$, where R is a positive number satisfying

$$(a + 2R)(a - R)^2 = 1, \tag{4.1}$$

or there exists an optimal solution with $x_1 = a - 2r$, $x_2 = x_3 = a + r$, where r is a positive number satisfying

$$(a - 2r)(a + r)^2 = 1. \tag{4.2}$$

The corresponding values for $(x_1 - a)^2 + (x_2 - a)^2 + (x_3 - a)^2$ are $6R^2$, and $6r^2$. From (4.1) and (4.2) we get

$$(a + 2R)(a - R)^2 - (a - 2r)(a + r)^2 = 0. \tag{4.3}$$

Equivalently, (4.3) can be written as

$$3(r^2 - R^2) + 2(r^3 + R^3) = 0. \tag{4.4}$$

But (4.4) implies that $r < R$. Hence the proof of the lemma for $n = 3$.

Next we prove the lemma for $n > 3$. Let $x = (x_1, x_2, \ldots, x_n)$ be an optimal solution for $B(n, a)$, where $n > 3$. Since for any permutation π of the set $\{1, \ldots, n\}$, $x_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ is also an optimal solution, without loss of generality we may assume that $x_1 \leq x_2 \ldots \leq x_n$. Suppose there exists i, j and k such that $x_i < x_j \leq x_k$. Consider the 3-dimensional minimization problem that results when all the variables except the i-th, the j-th and the k-th variables stay fixed at the value of the corresponding component of x. Note that from homogeneity of the constraint set, this 3-dimensional minimization can be reduced to the problem of computing $B(3, a')$, for some $a' \geq 1$ having an optimal solution which is a scalar multiple of (x_i, x_j, x_k). But this contradicts the correctness of the lemma for $n = 3$. □

Corollary 3. *$B(n, a) = a\xi$, where $\xi = \xi(n) \in [0, 1)$ satisfies*

$$a^n (1 - \xi)^{n-1} (1 + (n - 1)\xi) = 1. \tag{4.5}$$

Proof. From Lemma 1 there exists an optimal solution satisfying

$$x_1 = \ldots = x_{n-1} = a(1-\xi),$$

for some $\xi \in [0,1)$. Since the average value of the x_i's is a, we get $x_n = a(1+(n-1)\xi)$. Also product of x_i's is 1. Hence, (4.5) holds. □.

Lemma 2. $B(n,a) \leq B(n-1,a) \ldots \leq B(2,a) = \sqrt{a^2-1}$.

Proof. The fact that $B(2,a) = \sqrt{a^2-1}$ is immediate from Corollary 2. Next we prove the monotonicity of $B(n,a)$ in n. From (4.5) we have

$$n \ln a + (n-1)\ln(1-\xi) + \ln(1+(n-1)\xi) = 0. \tag{4.6}$$

Treating n as a continuous variable, and upon implicit differentiation of (4.6) we obtain

$$\frac{d\xi}{dn} B + A = 0,$$

where

$$B = \frac{-n\xi}{(1+(n-1)\xi)(1-\xi)},$$

and

$$A = \ln(a(1-\xi)) + \frac{\xi}{(1+(n-1)\xi)}.$$

Since $B < 0$, it suffices to show that $A \leq 0$. By (4.6),

$$\ln(a(1-\xi)) = \frac{1}{n}\Big(\ln(1-\xi) - \ln(1+(n-1)\xi)\Big) = \frac{1}{n}\ln\Big(\frac{(1-\xi)}{(1+(n-1)\xi)}\Big). \tag{4.7}$$

From (4.7) we have

$$A = \frac{1}{n}\ln\Big(1 - \frac{n\xi}{(1+(n-1)\xi)}\Big) - \frac{\xi}{(1+(n-1)\xi)} = \frac{1}{n}\ln(1-nu) + u, \tag{4.8}$$

where $u = n\xi/[1+(n-1)\xi]$. Using the fact that $e^x \geq 1+x$ for all x, from (4.8) it follows that $A \leq 0$, and hence the proof of monotonicity of $B(n,a)$. □

Lemma 3. *Suppose that* $A = X^{*^{-1}} C X^*$*, where* C *is a circuit matrix with* t *equal to the size of the circuit, and* $F(w) = \sum_{(i,j)\in E} a_{ij} e^{w_i - w_j}$*. Then*

$$\frac{1}{t}\|\nabla F(w)\| \leq 2B(t, \frac{1}{t}F(w)).$$

Proof. Without loss of generality assume that the corresponding circuit in G_A is $\{(1,2),\ldots,(t-1,t)\}$. Thus

$$F(w) = e^{\Delta w_1 - \Delta w_2} + e^{\Delta w_2 - \Delta w_3} + \ldots + e^{\Delta w_t - \Delta w_1}, \tag{4.9}$$

where $w_i^* = \ln x_i^*$, and where $\Delta w_i = w_i - w_i^*$ for all $i = 1,\ldots,n$. From (4.9) it follows that

$$\nabla F(w) = (e^{\Delta w_1 - \Delta w_2} - e^{\Delta w_n - \Delta w_1}, \ldots, e^{\Delta w_t - \Delta w_1} - e^{\Delta w_{t-1} - \Delta w_t})^T. \tag{4.10}$$

Let $z = (z_1,\ldots,z_t)^T$, where

$$z_1 = e^{\Delta w_1 - \Delta w_2}, \quad z_2 = e^{\Delta w_2 - \Delta w_3}, \ldots, z_t = e^{\Delta w_t - \Delta w_1},$$

and let $z' = (z_2, z_3, \ldots, z_t, z_1)^T$. Then

$$\|\nabla F(w)\| = \|z - z'\|. \tag{4.11}$$

Let

$$a = \frac{1}{t}F(w), \quad \widehat{a} = (a, a, \ldots, a)^T \in \Re^t.$$

From (4.11) and triangle inequality we get

$$\|\nabla F(w)\| \le \|z - \widehat{a}\| + \|z' - \widehat{a}\| = 2\|z - \widehat{a}\|. \tag{4.12}$$

Now from (4.12) and Lemma 1 we obtain

$$\frac{1}{t}\|\nabla F(w)\| \le \frac{2}{t}\|z - \widehat{a}\| \le \frac{2}{t}\sqrt{\frac{t}{t-1}}\|z - \widehat{a}\| = 2\sigma(z, \widehat{a}) \le 2B(t, \frac{1}{t}F(w)). \quad \Box$$

Lemma 4. *Let β_k, $k = 1,\ldots,q$ be a set of positive numbers satisfying $\sum_{k=1}^q \beta_k = 1$. Let $\gamma \ge 1$ be any given number. For $x \in \Re^q$, let $H(x) = \sum_{k=1}^q \beta_k \sqrt{x_k^2 - 1}$. Then*

$$\max\{H(x) : \sum_{k=1}^q \beta_k x_k \le \gamma, \quad x_k \ge 1, \quad k = 1,\ldots,q\} = \sqrt{\gamma^2 - 1}.$$

Proof. Note that $H(x)$ is concave and that the feasible region is convex. Thus, to prove the lemma it suffices to check that $x = (\gamma,\ldots,\gamma) \in \Re^q$ is a constrained stationary point. But this can easily be established. $\Box$

Proof of Theorem 3. Let $A = \sum_{k=1}^{q} \alpha_k A_k$, where $A_k = X^{*^{-1}} C_k X^*$'s are the decomposition components ensured by Theorem 1 (iv). For each $k = 1, \ldots, q$, let E_k be the simple cycle in G_A corresponding to A_k, and let n_k be the size of the cycle. In particular, the positive entries of A_k are given by

$$a_{ij}^{(k)} = \alpha_k \frac{x_j^*}{x_i^*}, \quad (i,j) \in E_k.$$

Let $w^* = (\ln x_1^*, \ldots, \ln x_n^*)$, and $\Delta w_i = w_i - w_i^*$, $i = 1, \ldots, n$. For each $k = 1, \ldots, q$, define

$$F_k(w) = \sum_{(i,j) \in E_k} e^{\Delta w_i - \Delta w_j}. \tag{4.13}$$

Then we have

$$F(w) = \sum_{k=1}^{q} \alpha_k F_k(w), \tag{4.14}$$

and

$$F(w^*) = \sum_{k=1}^{q} \alpha_k n_k. \tag{4.15}$$

Suppose

$$G(w) - G(w^*) \le \delta,$$

where $G(w) = \ln F(w)$. Equivalently, suppose that

$$\frac{F(w)}{F(w^*)} \le e^{\delta}. \tag{4.16}$$

From (4.14) and (4.15) we obtain

$$\frac{F(w)}{F(w^*)} = \frac{\sum_{k=1}^{q} \alpha_k n_k \frac{F_k(w)}{n_k}}{\sum_{k=1}^{q} \alpha_k n_k}. \tag{4.17}$$

Define

$$\beta_k = \frac{\alpha_k n_k}{\sum_{k=1}^{q} \alpha_k n_k}, \quad k = 1, \ldots, k. \tag{4.18}$$

With this notation (4.16) can be written as

$$\sum_{k=1}^{q} \beta_k \frac{F_k(w)}{n_k} \le e^{\delta}. \tag{4.19}$$

Since X^* simultaneously balances each A_k, see Corollary 1, we have

$$F_k(w) \ge n_k, \quad k = 1, \ldots, q. \tag{4.20}$$

From (4.17) it follows that

$$\nabla G(w) = \frac{\nabla F(w)}{F(w)} = \frac{\sum_{k=1}^{q} \alpha_k \nabla F_k(w)}{\sum_{k=1}^{q} \alpha_k F_k(w)}. \tag{4.21}$$

By (4.18), (4.20), and (4.21)

$$\|\nabla G(w)\| \leq \frac{\sum_{k=1}^{q} \alpha_k \|\nabla F_k(w)\|}{\sum_{k=1}^{q} \alpha_k n_k} = \sum_{k=1}^{q} \beta_k \frac{\|\nabla F_k(w)\|}{n_k}.$$

Hence from Lemma 3 we obtain

$$\|\nabla G(w)\| \leq 2 \sum_{k=1}^{q} \beta_k B(n_k, \frac{1}{n_k} F_k(w)). \tag{4.22}$$

Let

$$a_k = \frac{F_k(w)}{n_k}, \quad k = 1, \ldots, q. \tag{4.23}$$

Note that from (4.20) $a_k \geq 1$. From this, (4.22), and Lemma 2 we have

$$\|\nabla G(w)\| \leq 2 \sum_{k=1}^{q} \beta_k B(2, a_k) = 2 \sum_{k=1}^{q} \beta_k \sqrt{a_k^2 - 1}. \tag{4.24}$$

Since $\sum_{k=1}^{q} \beta_k a_k \leq e^{\delta}$, and $\sum_{k=1}^{q} \beta_k = 1$, from (4.24) and Lemma 4 we conclude

$$\|\nabla G(w)\| \leq 2\sqrt{e^{2\delta} - 1}.$$

Hence the proof of Theorem 3. □

5. Polynomial Time Solvability of the Matrix Balancing Problem.

To complete the proof of polynomial time solvability we also need the following result.

Theorem 4. *Let A be an $n \times n$ nonnegative matrix, $a_{ii} = 0$, for all $i = 1, \ldots, n$, and G_A strongly connected. Let w^* be an exact minimizer of $G(w)$ in $\Re^n$. Given $\varepsilon \in (0, 1)$, define*

$$r = \frac{\varepsilon^2}{16\sqrt{2}}. \tag{5.1}$$

The condition

$$\|w - w^*\| \leq r \tag{5.2}$$

implies

$$G(w) - G(w^*) \leq \delta = \frac{\varepsilon^2}{16}. \tag{5.3}$$

Proof. From (5.2) it follows that

$$|\Delta w_i - \Delta w_j| \le \sqrt{2}r, \tag{5.4}$$

where as before we use the notation $\Delta w_i = w_i - w_i^*, \quad i = 1, \ldots, n$. Recall from the proof of Theorem 3 that

$$\frac{F(w)}{F(w^*)} = \sum_{k=1}^{q} \beta_k \frac{F_k(w)}{n_k}, \tag{5.5}$$

where $\sum_{k=1}^{q} \beta_k = 1$, see (4.17) and (4.18). Also recall that for each $k = 1, \ldots, q$ we have

$$F_k(w) = e^{\Delta w_{t_1} - \Delta w_{t_2}} + e^{\Delta w_{t_2} - \Delta w_{t_3}} + \ldots + e^{\Delta w_{t_{n_k - 1}} - \Delta w_{t_{n_k}}}, \tag{5.6}$$

where $\{t_1, \ldots, t_{n_k}\}$ is a subset of $\{1, \ldots, n\}$. Using (5.4) and (5.6) we get

$$F_k(w) \le n_k e^{\sqrt{2}r},$$

which in view of (5.5) yields

$$\frac{F(w)}{F(w^*)} \le \sum_{k=1}^{q} \beta_k e^{\sqrt{2}r} = e^{\sqrt{2}r}.$$

Substituting (5.1) into the above inequality we obtain (5.3). □

From Theorems 2, Theorem 3, and Corollary 2 it follows that in order to balance a given nonnegative $n \times n$ matrix A whose diagonal entries are zero and G_A strongly connected, to the relative error of ε, it suffices to solve the convex program

$$\min \left\{ G(w) = \ln \sum_{i,j=1}^{n} a_{ij} e^{w_i - w_j} : \|w\|_2 \le R = \frac{\sqrt{n}(n-1)}{2} \ln \frac{e}{(n-1)v} \right\},$$

with the absolute accuracy $\delta = \varepsilon^2/16$. By Theorem 4 the number of iterations of the ellipsoid method for solving the above convex program does not exceed

$$2n(n+1) \ln \frac{R}{r} = O\Big(n^2 \ln \big(\frac{n}{\varepsilon} \ln \frac{1}{v} \big) \Big).$$

These operations are to be performed over the numbers ω_i's having at most $O\Big(\ln \big(\frac{n}{\varepsilon v} \big) \Big)$ digits before and after the decimal point. From Theorem 4, Theorem 3, and Corollary 2 we conclude that if $\|w - w^*\| \le r$, then $F(w)/F(w^*) \le e^\delta$. Thus, $\|w - w^*\| \le r$ implies

$$\frac{\|\nabla F(w)\|}{e^\delta F(w^*)} \le \frac{\|\nabla F(w)\|}{F(w)} = \|\nabla G(w)\| \le \varepsilon.$$

Equivalently,

$$\|\nabla F(w)\| \leq e^{\delta} F(w^*)\varepsilon \leq eF(0)\varepsilon = e\sigma\varepsilon.$$

We thus have

Theorem 5. *Let A be an $n \times n$ nonnegative matrix, $a_{ii} = 0$, for all $i = 1, \ldots, n$. Suppose that $G_A = (V, E)$ is strongly connected. Let $a_{\min} = \min\{a_{ij} \ : (i,j) \in E\}$, $\sigma = \sum_{(i,j)\in E} a_{ij}$, $v = a_{\min}/\sigma$, and $m = |E|$. For any given accuracy $\varepsilon \in (0,1)$, in $O\Big(n^2 m \ln\big(\frac{n}{\varepsilon}\ln\frac{1}{v}\big)\Big)$ arithmetic operations over $O\Big(\ln\big(\frac{n}{\varepsilon v}\big)\Big)$-bit numbers, we can compute a positive diagonal matrix $X = \mathrm{diag}(e^{w_1}, \ldots, e^{w_n})$ so that XAX^{-1} is balanced to the relative error of ε, and it is balanced to the absolute error of $e\sigma\varepsilon$.* $\Box$

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