November, 1983

THE ENERGY FLOW EQUATION

by

R. Vichnevetsky

DCS-TR-136

Department of Computer Science
Rutgers University
New Brunswick, New Jersey

THE ENERGY FLOW EQUATION

R. Vichnevetsky
Rutgers University
New Brunswick, NJ 08903

ABSTRACT

An equation which describes the flow of energy in energy conservative semi- and full discretizations of hyperbolic equations is derived. While the form of this equation for semi-discretizations verifies known principles of group velocity and wave propagation in periodic structures, its form and strict applicability to discrete-space-discrete-time systems like those resulting from the full discretization of hyperbolic equations are new results.

THE ENERGY FLOW EQUATION

R. Vichnevetsky
Rutgers University
New Brunswick, NJ 08903

1. INTRODUCTION

Attention has been focused recently on energy and wave propagation as useful concepts for the analysis of errors and instability in the numerical approximation of hyperbolic equations. In this process energy is expressed as a quantity integrated over space at a fixed time, and energy flow as a quantity integrated over time at a fixed point in space. While energy may be expressed in Fourier space by the familiar Parseval relation, a similar expression for energy flow across a fixed point in the computational domain is not as evident from first principles and the purpose of this paper is to carry out its strict derivation.

Specifically, an energy flow equation for energy conservative discretizations of hyperbolic equations is obtained in the form of an integral in Fourier space, which invokes the concepts of energy spectral density and group velocity. While the form of this equation for semi-discretizations verifies known principles of wave propagation in periodic structures, its form and strict applicability to the discrete-space-discrete-time systems resulting from full-discretizations of hyperbolic equations are to our knowledge, new results.

In order to make contact with the use of energy and norms in the manner which is usual in numerical analysis, the relevant functions are placed in appropriate Hilbert spaces; this is in contrast with the standard formulation of wave propagation in physics and engineering, in which energy flow rates are usually finite, energy is not, and wave functions do not necessarily vanish at infinity.

2. ENERGY FLOW IN A SEMI-DISCRETE APPROXIMATION OF $U_t + cU_x = 0$

Consider as a model of hyperbolic systems, the simple advection equation

$$\frac{\partial U(x,t)}{\partial t} + c\,\frac{\partial U(x,t)}{\partial x} = 0. \qquad (1)$$

and its semi-discrete approximation on a regular division of the x axis:

$$\left.\begin{aligned} \frac{du_j}{dt} &= -c\left(\frac{u_{j+1} - u_{j-1}}{2h}\right) \\ u_j(t) &\simeq U(jh,t) \end{aligned}\right\} \qquad (2)$$

This approximation is energy conservative in the Cauchy sense. That is, for the approximation (2) on the entire x - axis, the energy of $\{u_j(t)\}$ defined as

$$\mathcal{E}(t) \equiv h \sum_{j=-\infty}^{\infty} |u_j(t)|^2 \qquad (3)$$

is a constant ($\mathcal{E}(t)$ is also $\| u_j(t)\|_2^2$, the square of the ℓ_2 norm of $\{u_j\}$; the notation $\mathcal{E}(t)$ shall however be used for simplicity). To prove that $\mathcal{E}$ is a constant, we multiply (2) by $2u_j$ and sum over all j to obtain:

$$\frac{d\mathcal{E}}{dt} = 2h\sum_j u_j \frac{du_j}{dt} = \left[c\, u_j\, u_{j+1}\right]_{-\infty}^{\infty} \qquad (4)$$

This expression vanishes for solutions $\{u_j(t)\}$ that are in ℓ_2 (i.e. for which (3) is finite): q.e.d.

Consider now the half space

$$D \equiv [0, \infty) \qquad (5)$$

on which the approximation (2) applies, with an inflow boundary in $x = 0$.
The definition of energy is now

$$\mathcal{E}(t) \equiv h \sum_{j>0} |u_j(t)|^2 \tag{6}$$

Proceeding as before we find

$$\frac{d\mathcal{E}}{dt} = 2h \sum_{j>0} u_j \frac{du_j}{dt} = c\, u_0 u_1 \tag{7}$$

Since the discrete approximation is energy conservative in D, energy can only come from the boundary. It follows that (7) is also the expression of the energy flow rate from the boundary into D, which we denote by $\phi_0(t)$:

$$\phi_0(t) = c\, u_0 u_1 \tag{8}$$

We shall now derive a computable expression for the total energy flow from $x = 0$ into D:

$$\mathcal{E}^\infty \equiv \int_{-\infty}^{\infty} \phi_0(t)\, dt = \lim_{t\to\infty} h \sum_{j>0} |u_j(t)|^2 = \int_{-\infty}^{\infty} c\, u_0(t)\, u_1(t)\, dt \tag{9}$$

in response to an arbitrary boundary condition $u_0(t)$ which is in $\mathcal{L}_2$ (meaning here that

$$\int_{-\infty}^{\infty} |u_0(t)|^2\, dt \tag{10}$$

is finite).

We note in passing that for the original equation (1), the total energy flow is easily derived analytically:

$$\mathcal{E}^{\infty}_{\text{exact}} = \lim_{t\to\infty} \int_0^{\infty} |U(x,t)|^2 dx = c\int_{-\infty}^{\infty} |U(0,t)|^2 dt \tag{11}$$

That an analytic derivation of $\mathcal{E}^{\infty}$ is not as simple in the discrete case, is due to the fact that the discrete approximation is <u>dispersive</u>, while the original equation is not.

<u>Leftgoing and Rightgoing Solutions</u>

Before proceeding with the evaluation of (9), we recall some known results, (see [5,7]) which shall be needed later on:

Let $\hat{u}(\omega)$ denote the t-Fourier transforms of $u(t)$:

$$\hat{u}(\omega) \equiv \int_{-\infty}^{\infty} u(t)\, e^{-i\omega t}\, dt \tag{12}$$

The Fourier transform of a solution $\{u_j(t)\}$ of the Cauchy problem for the semi-discretization (2) may be decomposed into two types of solutions:

$$\{u_j(t)\} = \{p_j(t)\} + \{q_j(t)\} \tag{13}$$

where the Fourier transform of $\{p_j\}$ satisfies (with $\bar{\omega} \equiv \omega h/c$):

$$\frac{\hat{p}_{j+1}}{\hat{p}_j} = -i\bar{\omega} + \sqrt{1-\bar{\omega}^2} \equiv \hat{E}_1(\omega) \tag{14}$$

and that of $\{q_j\}$ satisfies:

$$\frac{\hat{q}_{j+1}}{\hat{q}_j} = -i\bar{\omega} - \sqrt{1-\bar{\omega}^2} \equiv \hat{E}_2(\omega) \tag{15}$$

The discrete x-Fourier transform of $\{u_j^n\}$ is defined as

$$\bar{u}(\xi,t) \equiv h \sum_{j=-\infty}^{\infty} u_j\, e^{-i\xi j h} \tag{16}$$

To solutions of $\{p_j\}$ type correspond in Fourier space wave numbers

$$|\xi_1| \text{ in } [0, \pi/2h)$$

while to solutions of $\{q_j\}$ type correspond wave numbers

$$|\xi_2| \text{ in } (\pi/2h, \pi/h)$$

The relation of wave number ξ to frequency ω is the dispersion relation:

$$\omega = -\frac{c}{h}\sin(\xi h) \tag{17}$$

The group velocity

$$\mathcal{V} = -\frac{d\omega}{d\xi} = c\cos(\xi h) = \pm c\sqrt{1-\overline{\omega}^2} \tag{18}$$

is positive for solution of $\{p_j\}$ type (which are thus right-going solutions) and is negative for solutions of $\{q_j\}$ type, (which are thus leftgoing)

While the transforms (12) and (16) have infinite domains, (in t and x, respectively), they may also be applied to semi-infinite or finite domains by simply assuming that the domains are still infinite, but that $u = 0$ on the outside. This artifice makes it possible to analyse phenomena attached to a fixed point in space an to solutions defined for $t > 0$ only, with an initial condition in $t = 0$, as shall be illustrated below.

Evaluation of $\mathcal{E}^\infty$

The expression (9) of the energy flow may be rewritten by use of Plancherel's theorem in t-Fourier space as:

$$\mathcal{E}^\infty = \int_{-\infty}^{\infty} c\, \overline{\hat{u}_0}\, \hat{u}_1 \frac{d\omega}{2\pi} \tag{19}$$

where $\overline{\hat{u}_0}$ is the complex conjugate of $\hat{u}_0$

Since the solution in D is coming entirely from the left, we have, by (14):

$$\hat{u}_1(\omega) = \hat{E}_1(\omega) \cdot \hat{u}_0(\omega) \tag{20}$$

and (19) thus becomes:

$$\begin{aligned}\mathcal{E}^\infty &= \int_{-\infty}^{\infty} c\, \overline{\hat{u}_0}\, \hat{E}_1 \hat{u}_0 \frac{d\omega}{2\pi} \\ &= \int_{-\infty}^{\infty} c\, |\hat{u}_0|^2\, R_e[\hat{E}_1] \frac{d\omega}{2\pi}\end{aligned} \tag{21}$$

We note that

$$c\, R_e[\hat{E}_1(\omega)] = \begin{cases} c\sqrt{1-\left(\frac{\omega h}{c}\right)^2} = \mathcal{V}(\omega) & \text{when } |\omega| \leq c/h \\ 0 & \text{when } |\omega| > c/h \end{cases} \tag{22}$$

where $\mathcal{V}(\omega)$ is the group velocity (18) for rightgoing solutions. A consequence of (22) is that Fourier components of $\hat{u}_o(\omega)$ with a frequency $|\omega|$ beyond the cut-off frequency

$$\omega_c = \frac{c}{h} \tag{23}$$

contribute no energy to D, and we obtain the following form of the <u>energy flow equation</u> for the semi-discretization (2):

$$\boxed{\begin{aligned} \mathcal{E}^\infty \equiv \int_{-\infty}^{\infty} \phi_o(t)\, dt &= \lim_{t\to\infty} h \sum_{j>0}^{\infty} |u_j(t)|^2 \\ &= \int_{-\omega_c}^{\omega_c} |\hat{u}_o(\omega)|^2\, \mathcal{V}(\omega)\, \frac{d\omega}{2\pi} \end{aligned}} \tag{24}$$

We also note in passing that

$$c \cdot \hat{E}_1(\omega) \equiv W(\omega) \tag{25}$$

in (21) is the characteristic admittance (or inverse of the characteristic impedance) of the semi-discretization (2). The energy flow equation (24) may thus also be expressed as

$$\mathcal{E}^\infty = \int_{-\infty}^{\infty} |\hat{u}_o(\omega)|^2\, R_e[W(\omega)]\, \frac{d\omega}{2\pi} \tag{24a}$$

It is a general property of dispersive media that energy flow is determined by the real part of the characteristic admittance (see e.g. Brillouin (1946, 1960)).

3. SPECTRAL DISTRIBUTION OF THE ENERGY GENERATED BY THE BOUNDARY

As $t \to \infty$ the energy in D may be expressed with Parseval's relation in x-Fourier space as:

$$\mathcal{E}^{\infty} = \lim_{t\to\infty} \int_{-\pi/2h}^{\pi/2h} |\overline{U}(\xi, t)|^2 \frac{d\xi}{2\pi} \tag{26}$$

where $|\overline{U}(\xi,t)|^2$ is the spectral distribution of energy in D.

But, while the energy flow equation (24) gives the total energy in D, it does not tell what the spectral distribution is. This shall now be derived:

Since $d\omega = -\mathcal{V} d\xi$, we may rewrite (24) as:

$$\mathcal{E}^{\infty} = \int_{-\pi/2h}^{\pi/2h} |\hat{U}_0(\omega) \cdot \mathcal{V}|^2 \frac{d\xi}{2\pi} \tag{27}$$

(where ω and ξ are linked by the dispersion relation (17))

Whence, equating the integrand with that of (26), we find:

$$\boxed{|\hat{U}_0(\omega)\, \mathcal{V}(\omega)| = |\overline{U}(\xi, \infty)|} \tag{28}$$

which relates

$$|\overline{U}(\xi, \infty)| \quad \text{for } U \text{ on } [x > 0,\ t \to \infty]$$

to its source at the boundary

$$|\hat{U}_0(\omega)| \quad \text{for } U \text{ on } [x = 0,\ t \in (-\infty, \infty)]$$

and where $\xi \in [-\pi/2h, \pi/2h]$ and $\omega \in [-c/h, c/h]$ are linked by the dispersion relation (17)

4. ENERGY FLOW THROUGH AN ARBITRARY POINT IN COMPUTATIONAL SPACE (SEMI-DISCRETE CASE)

The energy flow equation may be applied to any mesh point which is inside of the computational domain: Consider for instance the Cauchy problem for (2) (i.e. $x \in (-\infty, \infty)$) , and an arbitrary mesh point x_A. Through any such point, there may exist a rightgoing energy flow which arrives from the left and which is expressed by

$$\mathcal{E}^{\infty}_{R,A} = \int_{-\omega_c}^{\omega_c} |\hat{p}_A(\omega)|^2 \, \mathcal{V}(\omega) \frac{d\omega}{2\pi} \qquad (29)$$

In this expression, $|\hat{p}_A(\omega)|$ may be derived from initial conditions in the half space to the left of x_A,

$$\overline{p}(\xi, 0) = h \sum_{j<A} u_j(0)\, e^{-i\xi j h} \qquad (30a)$$
$$((|\xi| < \pi/2h)$$

and relation (28) which can be rewritten as:

$$|\hat{p}_A(\omega)| = |\overline{p}(\xi, 0)/\mathcal{V}(\omega)| \qquad (30b)$$

(where ω and ξ are linked by the dispersion relation (17))
Likewise, through any mesh point x_A there may exist a leftgoing energy flow which arrives from the right and which is expressed by

$$\mathcal{E}^{\infty}_{L,A} = \int_{-\omega_c}^{\omega_c} |\hat{q}_A(\omega)|^2 \, \mathcal{V}(\omega) \frac{d\omega}{2\pi} \qquad (31)$$

In this expression, $|\hat{q}_A(\omega)|$ may be derived from initial conditions in the half space to the right of x_A

$$\overline{q}(\xi, 0) = h \sum_{j>A} u_j(0)\, e^{-i\xi j h}$$
$$(\pi/2h < |\xi| < \pi/h) \qquad (32a)$$

and relation (28) which can be rewritten as:

$$|\hat{q}_A(\omega)| = |\bar{q}(\xi, 0)/\mathcal{V}(\omega)| \qquad (32b)$$

(where ω and ξ are linked by the dispersion relation (17))

5. ENERGY FLOW IN FULLY DISCRETE APPROXIMATIONS OF $U_t + cU_x = 0$.

We now derive the energy flow equation for those full discretizations which are obtained by applying to (2) a discrete time marching method (described by $\mathbb{M}$):

$$\mathbb{M}(\mathbb{Z}) \cdot u_j^n = \mathbb{A} \cdot u_j^n \tag{33}$$

Here, $\{u_j^n\}$ is a fully discrete approximation of $U(x,t)$:

$$u_j^n \simeq U(jh, n\Delta t) \tag{34}$$

and $\mathbb{Z}\cdot$ is the usual time shift operator.

We limit ourselves to energy conservative time-marching methods: These are the methods which admit sinusoidal solutions in both space and time:

$$u_j^n = e^{i(\xi jh + \omega n\Delta t)} \tag{35}$$

Their spectral function or symbol is defined as:

$$i\mu(\omega) = \frac{\mathbb{M}(\mathbb{Z}) \cdot e^{i\omega n\Delta t}}{e^{i\omega n\Delta t}} = \mathbb{M}(e^{i\omega\Delta t}) \tag{36}$$

where energy conservation implies that $\mu(\omega)$ is a real function of ω.
Two examples are:
The leapfrog method (LF):

$$\left.\begin{aligned} &\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \mathbb{A} \cdot u_j^n \\ &\mathbb{M} = \frac{\mathbb{Z} - \mathbb{Z}^{-1}}{2\Delta t} \quad ; \mu(\omega) = \frac{\sin(\omega\Delta t)}{\Delta t} \end{aligned}\right\} \tag{37}$$

The Crank Nicolson method (CN)

$$\frac{U_j^{n+1} - U_j^{n}}{\Delta t} = \frac{1}{2}\left(\mathbb{A}\cdot U_j^{n} + \mathbb{A}\cdot U_j^{n+1}\right)$$

$$\mathbb{M} = \frac{2}{\Delta t}\left(\frac{\mathbb{Z} - 1}{\mathbb{Z} + 1}\right) \quad ; \quad \mu(\omega) = \frac{2}{\Delta t}\tan\left(\frac{\omega \Delta t}{2}\right) \tag{38}$$

The t-Fourier transform of $\{U_j^n\}$ in the fully discrete case is the sum

$$\overline{U}_j(\omega) = \Delta t \sum_{n=-\infty}^{\infty} U_j^n e^{-i\omega n \Delta t} \tag{39}$$

A solution of the Cauchy problem for energy conservative full-discretization of the form (33) may still be decomposed as in (13):

$$\{U_j^n\} = \{p_j^n\} + \{q_j^n\} \tag{40}$$

where $\{p_j^n\}$ and $\{q_j^n\}$ now have discrete Fourier transforms which satisfy the relation obtained by replacing ω with $\mu(\omega)$ in (14) - (15).

$$\frac{\overline{p}_{j+1}}{\overline{p}_j} = -i\overline{\mu} + \sqrt{1-\overline{\mu}^2} = \hat{E}_1(\mu(\omega)) \tag{41}$$

$$\frac{\overline{q}_{j+1}}{\overline{q}_j} = -i\overline{\mu} - \sqrt{1-\overline{\mu}^2} = \hat{E}_2(\mu(\omega)) \tag{42}$$

The dispersion relation remains similar to (17), also with $\mu(\omega)$ replacing ω

$$\bar{\mu}(\omega) \equiv \frac{\mu(\omega)\, h}{c} = -\sin(\xi h) \tag{43}$$

and the group velocity of the full discretizations becomes

$$\mathcal{V}_F(\omega) = -\frac{d\omega}{d\xi} = -\frac{d\omega}{d\mu}\frac{d\mu}{d\xi}$$

$$= \frac{d\omega}{d\mu}\,\mathcal{V}_s(\mu) \tag{44}$$

where $\mathcal{V}_s$ is the group velocity (18) which applies to the semi-discrete case. The cut-off frequency is now that for which $\mu = c/h$, that is:

Leapfrog (LF):

$$\omega_{c,F} = \frac{1}{\Delta t} \arcsin\left(\frac{c\Delta t}{h}\right) \tag{45}$$

and

Crank Nicolson(CN):

$$\omega_{c,F} = \frac{2}{\Delta t} \arctan\left(\frac{c\Delta t}{2h}\right) \tag{46}$$

Consider the Crank Nicolson method at first.

Upon multiplication of (37) by $h\left(U_j^n + U_j^{n+1}\right)$ and summing over all $j > 0$, we find

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}^n}{\Delta t} = c\left(\frac{U_0^n + U_0^{n+1}}{2}\right)\left(\frac{U_1^n + U_1^{n+1}}{2}\right) = \phi_0^n \tag{47}$$

We then sum over all n to obtain the energy flow equation :

$$\mathcal{E}^\infty = \Delta t \sum_{n=-\infty}^{\infty} \phi_0^n = \int_{-\infty}^{\infty} c\, \overline{U}_0(\omega)\, \overline{U}_1(\omega) \cos^2\left(\frac{\omega \Delta t}{2}\right) \frac{d\omega}{2\pi}$$

$$= \int_{-\omega_c}^{\omega_c} c\, |\overline{U}_0(\omega)|^2\, R_e[\hat{E}_1(\mu(\omega))] \cos^2\left(\frac{\omega \Delta t}{2}\right) \frac{d\omega}{2\pi}$$

$$= \int_{-\omega_c}^{\omega_c} c\, |\overline{U}_0(\omega)|^2 \sqrt{1 - \overline{\mu}(\omega)^2}\, \cos^2\left(\frac{\omega \Delta t}{2}\right) \frac{d\omega}{2\pi} \tag{48}$$

where the right hand side has been evaluated in Fourier space, and use has been made of the fact that for rightgoing solutions

$$\overline{U}_1(\omega) = \overline{U}_0(\omega) \cdot \hat{E}_1(\mu(\omega)) \tag{49}$$

The similar result with the leapfrog method is obtained by multiplying (33) by $2h\,u_j^n$ and summing over all $j>0$:

$$\frac{h\sum_j u_j^n u_j^{n+1} - h\sum_j u_j^n u_j^{n-1}}{2\Delta t} = c\,u_0^n u_1^n \qquad (50)$$

If we multiply by Δt and sum over all n we obtain, as before by evaluation in t-Fourier space, the energy flow equation:

$$\mathcal{E}^\infty = \Delta t \sum_{n=-\infty}^{\infty} \Phi_0^n = \int_{-\infty}^{\infty} c\,\overline{u}_0(\omega)\,\overline{u}_1(\omega)\,\frac{1}{\cos(\omega\Delta t)}\,\frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} c\,|\overline{u}_0(\omega)|^2\,\mathrm{Re}\left[\hat{E}_1(\mu(\omega))\right]\frac{1}{\cos(\omega\Delta t)}\,\frac{d\omega}{2\pi}$$

$$= \int_{-\omega_{c,F}}^{\omega_{c,F}} c\,|\overline{u}_0(\omega)|^2\sqrt{1-\bar{\mu}(\omega)^2}\,\frac{1}{\cos(\omega\Delta t)}\,\frac{d\omega}{2\pi} \qquad (51)$$

The apparently different expressions (48) and (51) have of course a common form: Equation (24) contains the general principle, which applies to the full discretization as well that:

> In Fourier space, energy flow equals energy density ($|\overline{u}(\omega)|^2$ times group velocity γ.

This principle is generally known to apply to propagating media described by systems of difference-differential equations such as (2) (see for instance Brillouin (1946), pg)) and could have been used directly to derive some of the preceding results (see e.g. [5] where this has been done). But there exists no proof, to our knowledge, that it continues to apply to the discrete space discrete time systems such as those obtained in the numerical approximation of hyperbolic equations. We will, by way of proof, verify that this principle does indeed apply strictly in the two fully discrete cases considered here. (a general, case independent proof will be given separately)

The expression which corresponds to (24) in the fully discrete case is the general form of the energy flow equation :

$$\boxed{\begin{aligned} \mathcal{E}^{\infty} &= \Delta t \sum_{n=-\infty}^{\infty} \phi_0^n = \lim_{n\to\infty} \sum_{j>0} |U_j^n|^2 \\ &= \int_{-\omega_{c,F}}^{\omega_{c,F}} |\overline{U}_0(\omega)|^2\, \gamma_F(\omega)\, \frac{d\omega}{2\pi} \\ &= \int_{-\omega_{c,F}}^{\omega_{c,F}} |\overline{U}_0(\omega)|^2\, \gamma_S(\omega) \left(\frac{d\omega}{d\mu}\right) \frac{d\omega}{2\pi} \end{aligned}} \tag{52}$$

and it may be verified that it contains both (48) and (51). Indeed, from (37) and (38) :

CN:

$$\frac{d\mu}{d\omega} = \cos(\omega\Delta t) \tag{53}$$

LF:

$$\frac{d\mu}{d\omega} = \frac{1}{\cos^2(\omega\Delta t/2)} \tag{54}$$

The characteristic admittance $W(\omega)$ is now

$$W(\omega) = c\,\hat{E}_1(\mu(\omega)) \cdot \frac{d\omega}{d\mu} \qquad (55)$$

It consists of two terms

$$W(\omega) = W_1(\omega)\, W_2(\omega) \qquad (56)$$

where

$$W_1(\omega) = c\,\hat{E}_1(\mu(\omega)) \qquad (57)$$

describes the propagation properties of the full discretization, and

$$W_2(\omega) = \left(\frac{d\omega}{d\mu}\right) \qquad (58)$$

describes the filtering effect of the discrete time marching.

6. SPECTRAL DISTRIBUTION OF THE ENERGY GENERATED BY THE BOUNDARY (FULL DISCRETE CASE).

By the same argument as in the semi-discrete case, we may derive the spectral distribution of energy in D as $t \to \infty$, in response to an imposed condition $\{U_0^n\}$ at the boundary:

$$\mathcal{E}^\infty = \lim_{n \to \infty} \int_{-\pi/2h}^{\pi/2h} |\hat{U}^n(\xi)|^2 \frac{d\xi}{2\pi} \tag{59}$$

Since $\omega = -\gamma_F d\xi$ we may rewrite (52) as:

$$\mathcal{E}^\infty = \lim_{n \to \infty} \int_{-\pi/2h}^{\pi/2h} |\hat{U}_0(\omega)\, \gamma_F(\omega)|^2 \frac{d\xi}{2\pi} \tag{60}$$

(where ω and ξ are linked by the dispersion relation (43)) Whence, equating the integrands of (59) and (60), we obtain

$$\boxed{|\hat{U}_0(\omega)\, \gamma_F(\omega)| = |\hat{U}^\infty(\xi)|} \tag{61}$$

which relates

$$|\hat{U}^\infty(\xi)|\,;\ \xi \in \left(-\frac{\pi}{2h}, \frac{\pi}{2h}\right) \text{ on } [x > 0,\ t \to \infty]$$

to its source at the boundary

$$|\hat{U}_0(\omega)|\,;\ \omega \in (-\omega_{c,F}, \omega_{c,F}) \text{ on } [x = 0,\ t \in (-\infty, \infty)]$$

7. ENERGY FLOW THROUGH AN ARBITRARY POINT IN COMPUTATIONAL SPACE (FULLY DISCRETE CASE)

The energy flow equation may be applied to any mesh point which is inside of the computational domain: as in Section4, consider the Cauchy problem for (2) and an arbitrary mesh point x_A .Through any such point there may exist a rightgoing energy flow which originate from the left, expressed by:

$$\mathcal{E}^{\infty}_{R,A} = \int_{-\omega_{c,F}}^{\omega_{c,F}} |\overline{p_A}(\omega)|^2 \, \mathcal{V}_F(\omega) \, \frac{d\omega}{2\pi} \tag{62}$$

In this expression, $|\overline{p_A}(\omega)|$ may be derived from initial conditions in the half space to the left of x_A .

$$|\overline{p^{\circ}}(\xi)| = h \sum_{j<A} u_j^{\circ} \, e^{-i\xi jh} \qquad (|\xi| < \pi/2h) \tag{63}$$

and relation (61) which can be rewritten as:

$$|\overline{p_A}(\omega)| = |\overline{p^{\circ}}(\xi) \;/\; \mathcal{V}_F(\omega)| \tag{64}$$

(where ω and ξ are linked by the dispersion relation (43)).

Likewise, through any mesh point x_A there may exist a leftgoing energy flow which originates from the right, expressed by

$$\mathcal{E}^{\infty}_{L,A} = \int_{-\omega_{c,F}}^{\omega_{c,F}} |\overline{q_A}(\omega)|^2 \, \mathcal{V}_F(\omega) \, \frac{d\omega}{2\pi} \tag{65}$$

In this exoression, $|\overline{q}_A(\omega)|$ may be derived from initial conditions in the half space to the right of x_A

$$\overline{q}^{\circ}(\xi) = h \sum_{j>A} u_j^{\circ} e^{-i\xi jh} \qquad \left(\frac{\pi}{2h} < |\xi| < \frac{\pi}{h}\right)$$

(66)

and relation (61) which can be rewritten as:

$$|\overline{q}_A(\omega)| = |\overline{q}^{\circ}(\xi)| / |\gamma_F^{-}(\omega)|$$

(67)

(where ω and ξ are linked by the dispersion relation (43)).

8. AN EXAMPLE

Consider the semi-infinite space $D = x \geqslant 0$ on which $U_t + cU_x = 0$ is approximated with (2) in space and the Crank Nicolson method in time,

- and a boundary condition in $x=0$ which is the rectangular pulse:

$$U_o(t) = \begin{cases} 1. & \text{for} \quad 1. < t \leqslant T+1 \\ 0 & \text{elsewhere} \end{cases} \tag{68}$$

where for convenience, T is chosen a multiple of Δt

The expression of $\mathcal{E}^{\infty}$ is easily derived: from

$$\left| \overline{U}_o(\omega) \right| = 2 \cdot \Delta t \left| \frac{\sin(\omega T/2)}{\sin(\omega \Delta t/2)} \right| \tag{69}$$

and (48), we obtain:

$$\mathcal{E}^{\infty} = \lim_{n \to \infty} h \sum_{j>0} |U_j^n|^2$$

$$= \int_0^{\omega_c} \left| 2 \cdot \Delta t \, \frac{\sin(\omega T/2)}{\sin(\omega \Delta t/2)} \right|^2 \sqrt{1 - \left(\frac{2h}{c \Delta t} \tan\left(\frac{\omega \Delta t}{2}\right) \right)^2} \cos\left(\frac{\omega \Delta t}{2}\right)^2 \frac{d\omega}{\pi}$$

(70)

where ω_c is (46). Moreover, from (47) we see that

$$\frac{d\mathcal{E}}{dt} = 0 \quad \text{for} \quad t > T + 1 + \Delta t \tag{71}$$

Thus

$$\mathcal{E}^{\infty} = \mathcal{E}(\tau) \quad \text{for any} \quad \tau > T + 1 + \Delta t \tag{72}$$

Given in Table 1 is $\mathcal{E}$ as a function of t^n, obtained by numerical integration of (2) - (38). The final value $\mathcal{E}^{\infty}$ = 8.9804277...agrees to within arithmetical accuracy with the value of the integral (70) evaluated by numerical quadrature.

(Acknowledgements are due to E.C. Pariser who programmed this example and obtained the numerical results).

TIME	ENERGY
0.25000	0.000000000
0.50000	0.000000000
0.75000	0.000000000
1.00000	0.000000000
1.25000	3.891109268E-03
1.50000	3.489979291E-02
1.75000	9.644010353E-02
2.00000	0.187572606
2.25000	0.306924281
2.50000	0.452730326
2.75000	0.622887367
3.00000	0.815016068
3.25000	1.026530756
3.50000	1.254713505
3.75000	1.496789689
4.00000	1.750003746
4.25000	2.011688243
4.50000	2.279330847
4.75000	2.550630056
5.00000	2.823542149
5.25000	3.096316654
5.50000	3.367519723
5.75000	3.636045132
6.00000	3.901113090
6.25000	4.162257587
6.50000	4.419303380
6.75000	4.672334157
7.00000	4.921653651
7.25000	5.167741744
7.50000	5.411207654
7.75000	5.652742360
8.00000	5.893072309
8.25000	6.132916279
8.50000	6.372947052
8.75000	6.613759211
9.00000	6.855844041
9.25000	7.099572120
9.50000	7.345183813
9.75000	7.592787474
10.0000	7.842364841
10.2500	8.093782774
10.5000	8.346810257
10.7500	8.601139372
11.0000	8.856408866
11.2500	8.980427765
11.5000	8.98042/765
11.7500	8.980427765
12.0000	8.980427765
12.2500	8.980427765
12.5000	8.98042/765
12.7500	8.98042/765
13.0000	8.980427765
13.2500	8.98042/765
13.5000	8.98042/765
13.7500	8.98042/765
14.0000	8.98042/765
14.2500	8.98042/765
14.5000	8.980427765
14.7500	8.980427765
15.0000	8.980427765

REFERENCES

1. Brillouin, L. (1946) "Wave Propagation in Periodic Structures". McGraw Hill.

2. Brillouin, L. (1960). "Wave Propagation and Group Velocity". Academic Press, New York.

3. Trefethen, L.N. (1982a) "Group Velocity in Finite Difference Schemes". SIAM Review 24, 113-136.

4. Trefethen, L.N. (1982b). "Wave Propagation and Stability for Finite Difference Schemes". Ph. D. Thesis, Stanford University. Department of Computer Science

5. Vichnevetsky, R. (1981)."Energy and Group Velocity in Semi-discretizations of Hyperbolic Equations" Mathematics and Computers in Simulation 23 (1981a), 333-343

6. Vichnevetsky, R. , (1983) "Group Velocity and Reflection Phenomena in Numerical Approximations of Hyperbolic Equations". Journal of the Franklin Institute, Pergamon Press .

7. Vichnevetsky, R. and J.B. Bowles. (1982) "Fourier Analysis of Numerical Approximations of Hyperbolic Equations"(Book) SIAM, Philadelphia, PA.