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NYSTROM'S INTERPOLATION FORMULA IN THE SOLUTION OF SINGULAR INTEGRAL EQUATIONS DISCRETIZED BY THE GAUSS-JACOBI QUADRATURE

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ABSTRACT

The numerical solution of Singular Integral Equations of Cauchy-type at a discrete set of points t_i, is obtained through discretization of the original equation with the Gauss-Jacobi quadrature. The natural or Nyström's interpolation formula is used to approximate the solution of the equation for points different than t_i. Uniform convergence of the interpolation formula is shown for C^1 functions. Finally, error bounds are derived and for smooth function it is shown that Nyström's formula converges faster than Lagrange's interpolation polynomials.

Mathematical Reviews subject classification: 65R20, 45E05, 45L10

1. Introduction

In this paper we consider Nyström's interpolation formula, applied in the numerical solution of singular integral equations of the form,

$$(1.1)\quad aw(s)y(s) + \frac{b}{\pi}\int_{-1}^{1} w(t)\frac{y(t)}{t-s}\,dt + \lambda\int_{-1}^{1} w(t)K(s,t)y(t)dt = f(s),\ -1<s<1.$$

where the kernel K(s,t) and the input function f(s) are Hölder continuous and the constant coefficients a,b satisfy the relation $a^2+b^2=1$. The weight function w(t) can be found by using Nöether's index theory [12], i.e.,

$$(1.2)\qquad w(t) = (1-t)^{\alpha}\,(1+t)^{\beta}$$

where α,β satisfy the relation $\alpha+\beta=-k$; here k is the index of (1.1). Without any loss of generality we may assume that k=1, which implies that $-1<\alpha,\beta<0$. Then an additional condition of the following form is required,

$$(1.3)\qquad \frac{1}{\pi}\int_{-1}^{1} w(t)y(t)dt = N,$$

where N is a given constant, in order to ensure the uniqueness of the solution of (1.1).

A direct numerical method based on a Gaussian-type quadrature approximation of the integral parts in (1.1), (1.2) was originally proposed by Erdogan and Gupta [4] for the case a=0. Subsequently, Krenk [11] extended the previous method to the more general case of (1.1), i.e. with $a\neq 0$. These methods approximate (1.1) and (1.3) by the Gauss-Jacobi quadrature on a special set of points which results in the system of algebraic equations

$$(1.4) \qquad (A_n + \lambda C_n)\underline{y}_n^* = \underline{f}_n$$

where

$$(1.5) \qquad (A_n)_{i,j} = \frac{bw_j}{t_j - s_i}, \quad (A_n)_{n,j} = w_j$$

$$(1.6) \qquad (C_n)_{i,j} = \pi w_j K(s_i,t_j), \quad (C_n)_{n,j} = 0$$

and

$$(1.7) \qquad \underline{y}_n^* = [y_n^*(t_1),\ y_n^*(t_2),\dots,y_n^*(t_n)]^T,\ \underline{f} = [f(s_1),\dots,f(s_{n-1}),N]^T.$$

The node points t_j and the collocation point s_i are the roots of the following Jacobi polynomials

$$(1.8) \qquad P_n^{(\alpha,\beta)}(t_j) = 0,\ j = 1(1)n,\ P_{n-1}^{(-\alpha,-\beta)}(s_i) = 0, i = 1,\dots,n-1$$

and w_j are the Gauss-Jacobi quadrature weights [7].
Gerasoulis [6], and Gerasoulis and Srivastav [7] have shown that (1.4) possesses a unique solution for sufficiently large n, provided that λ is not an eigenvalue of (1.1).

The linear algebraic system (1.4) gives an approximation of the solution of (1.1), (1.3) at a discrete set of points t_j. For points different than t_j an interpolation formula must be used. Krenk [10] has proposed an interpolation formula which is based on the Lagrange polynomials $L_n(t)$ at $(t_j, y_n^*(t_j))$. The convergence of $L_n(t)$ has been considered by Ioakimidis and Theocaris [9] and Tsamasphyros and Theocaris [16]. Note that the Lagrange interpolation formula is exact for polynomials of degree less or equal to n, whereas the Gaussian quadrature approximating (1.1), (1.3) is exact for polynomials of degree less or equal to 2n and 2n-1 respectively. Consequently, the use of $L_n(t)$ as an interpolation formula generally results in a significant loss of accuracy. Until recently [7], [8], [15], Nyström's formula [1] has been completely ignored in the solution of signular integral equations. Although some equivalence results for the special case a=0 of (1.1) are given in [8], the question of convergence and computational efficiency of Nyström's interpolation formula has not been studied.

In section 2 we use the Gaussian quadrature to derive Nystrom's interpolation formula. In section 3 we show that if $f(s) \in C^1[-1,1]$ and $K(s,t) \in C^1([-1,1]\times[-1,1])$ then the Nyström's approximation converges uniformly to the solution of equation (1.1), (1.3). In addition we have found that the error is bounded above by Gaussian-quadrature errors. Moreover, from the analysis of sections 2 and 3 we can easily see that the Lagrange interpolation formula can be obtained from Nyström's formula if we replace the kernel K(s,t) and the input function f(s) with their respective Lagrange polynomials. The last observation implies that in general Nyström's formula converges faster than the Lagrange interpolation formula, especially for functions which are sufficiently smooth. Finally, in section 4, a numerical example is given, which verifies the theoretically derived error bounds.

We have used throughout Krenk's [11] notation for the Jacobi polynomials and identities. Unless otherwise specified we reserve the letter t with any subscript for the node points, and the letter s with any subscript for the collocation points, i.e., given t_m, s_j then $m=1,\ldots,n$, $j=1,\ldots,n-1$ and t_m, s_j are given in (1.9).

2. Nyström's Interpolation Formula

By using the Carleman-Vekue regularization method [12] equations (1.1) and (1.3) can be reduced into an equivalent Fredholm integral equation of the form

$$(2.1) \qquad y(t) + \lambda \int_{-1}^{1} w(x) L(t,x) y(x) dx = F(t)$$

where

$$(2.2) \qquad L(t,x) = a\frac{K(t,x)}{w(t)} - \frac{b}{\pi} \int_{-1}^{1} \frac{K(s,x)ds}{w(s)(s-t)}$$

$$(2.3) \qquad F(t) = a\frac{f(t)}{w(t)} - \frac{b}{\pi} \int_{-1}^{1} \frac{f(s)ds}{w(s)(s-t)} + |b|N$$

If $K(s,t) \in C^1([-1,1]\times[-1,1])$ and $f(s) \in C^1[-1,1]$, then we can use identities ([11], (2.1)) and $\Gamma(-\alpha)\Gamma(1+\alpha) = \pi/b$ ([7]), to rewrite (2.2) and (2.3) in a more compact form:

$$(2.4) \qquad L(t,x) = 2K(t,x)P_1^{(\alpha,\beta)}(t) - \frac{b}{\pi} \int_{-1}^{1} \frac{1}{w(s)}\, g(x,t,s)ds$$

$$(2.5) \qquad F(t) = 2f(t)P_1^{(\alpha,\beta)}(t) - \frac{b}{\pi} \int_{-1}^{1} \frac{1}{w(s)}\, h(t,s)ds + |b|N$$

where

$$(2.6)\qquad g(x,t,s) = \begin{cases} [K(s,x)-K(t,x)]/(s-t) & \text{if } s\neq t \\ \frac{\partial K}{\partial s}(s,x) & \text{if } s=t \end{cases}$$

$$(2.7)\qquad h(t,s) = \begin{cases} [f(s)-f(t)]/(s-t) & \text{if } s\neq t \\ f'(s) & \text{if } s=t \end{cases}$$

and $P_1^{(\alpha,\beta)}(t)$ is the Jacobi polynomial of degree one given by

$$(2.8)\qquad P_1^{(\alpha,\beta)}(t) = \frac{1}{2}\,[-\beta+\alpha+(\alpha+\beta+2)t] = \frac{1}{2}[2\alpha+1+t].$$

We notice that by the definitions (2.6) and (2.7) $g(x,t,s)$ and $h(t,s)$ are continuous functions. This implies that $L(t,x)$ and $F(t)$ are continuous. Therefore Nyström's theory for Fredholm integral equations [1], [2], [3], can be applied to equation (2.1). If we approximate the integral part of (2.1) using the Gauss-Jacobi quadrature formula, then we obtain the functional equation

$$(2.9)\qquad y_n(t) + \lambda\pi \sum_{r=1}^{n} w_r L(t,t_r) y_n(t_r) = F(t),$$

where w_r are the Gauss-Jacobi weights defined in [7] and t_r are the roots of the following polynomial $P_n^{(\alpha,\beta)}(t_r) = 0, r = 1,\ldots,n$.
Furthermore, we can use the Gauss-Jacobi quadrature with n-1 points, to approximate $L(t,x)$ and $F(t)$, i.e.,

$$(2.10)\qquad L(t,x) = L_n(t,x) + r_n(K;t,x)$$

$$(2.11)\qquad F(t) = F_n(t) + r_n(f;t)$$

where

$$(2.12) \qquad L_n(t,x) = -b\sum_{m=1}^{n-1} w_m^* \, g(x,t,s_m) + 2K(t,x)P_1^{(\alpha,\beta)}(t)$$

$$(2.13) \qquad F_n(t) = -b\sum_{m=1}^{n-1} w_m^* \, h(t,s_m) + 2f(t)P_1^{(\alpha,\beta)}(t) + |b|N$$

$$(2.14) \qquad r_n(K;t,x) = -\frac{b}{\pi}\left[\int_{-1}^{1} \frac{1}{w(s)} g(x,t,s)ds - \sum_{r=1}^{n-1} w_r^* \, g(x,t,s_r)\right]$$

$$(2.15) \qquad r_n(f;t) = -\frac{b}{\pi}\left[\int_{-1}^{1} \frac{1}{w(s)} h(t,s)ds - \sum_{r=1}^{n-1} w_r^* \, h(t,s_r)\right],$$

w_m^* are the Gauss-Jacobi weights corresponding to $1/w(s)$ [7], and $P_{n-1}^{(-\alpha,-\beta)}(s_m) = 0, m = 1,\ldots,n-1$.

If we substitute $L(t,x)$ with $L_n(t,x)$ and $F(t)$ with $F_n(t)$ in (2.9) a new functional equation is obtained

$$(2.16) \qquad y_n^*(t) + \lambda\pi \sum_{r=1}^{n} w_r L_n(t,t_r) y_n^*(t_r) = F_n(t).$$

By setting $t=t_i$, $i=1,\ldots,n$, where $P_n^{(\alpha,\beta)}(t_i) = 0$, (2.16) is reduced to a linear algebraic system of the form,

$$(2.17) \qquad (I + \lambda\tilde{Q}_n)\underline{y}_n^* = \underline{F}_n$$

$$(2.18) \qquad (\tilde{Q}_n)_{i,j} = \pi w_j L_n(t_i,t_j) \quad , \qquad i,j = 1,\ldots,n$$

and $\underline{y}_n^* = [y_n^*(t_1),\ldots,y_n^*(t_n)]^T$, $\underline{F}_n = [F_n(t_1),\ldots,F_n(t_n)]^T$.

The results stated in the following theorem have been shown earlier by Gerasoulis, et al in [7].

Theorem 2.1 If we assume that λ is not an eigenvalue of (1.1), then there exists an integer n_o such that the system (2.17) possesses a unique solution for all $n \geq n_o$. Moreover, the systems (2.17) and (1.4) are equivalent in the following sense: $\tilde{Q}_n = A_n^{-1} C_n$ and $\underline{F}_n = A_n^{-1} \underline{f}$.

Algebraic systems (2.14) or (1.5) give a numerical approximation to the solution of (1.1) at a discrete set of points t_i. For points different than t_i an interpolation formula must be used. Equation (2.16) is a natural interpolation formula and we can use it for all t in $[-1,1]$.

By using the identities ([7], [11] (2.3)),

$$\Gamma(-\alpha)\Gamma(1+\alpha) = \pi/b \tag{2.19}$$

$$\sum_{m=1}^{n-1} \frac{w_m^*}{s_m - t} = \frac{2\Gamma(-\alpha)\Gamma(1+\alpha)}{\pi} \left[\frac{P_n^{(\alpha,\beta)}(t) - P_{n-1}^{(-\alpha,\beta)}(t) P_1^{(\alpha,\beta)}(t)}{P_{n-1}^{(-\alpha,-\beta)}(t)} \right] \tag{2.20}$$

we can rewrite (2.12), (2.13) in an equivalent form

$$L_n(t,x) = -b \sum_{m=1}^{n-1} w_m^* \frac{K(s_m,x)}{s_m - t} + 2K(t,x) \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(-\alpha,-\beta)}(t)} \tag{2.21}$$

$$F_n(t) = -b \sum_{m=1}^{n-1} w_m^* \frac{f(s_m)}{s_m - t} + 2f(t) \frac{P_n^{(\alpha,\beta)}(t)}{P_{n-1}^{(-\alpha,-\beta)}(t)} + |b| N \tag{2.22}$$

for all $t \neq s_j$,

and

$$(2.23) \qquad L_n(s_j,x) = -b\sum_{\substack{m=1\\m\neq j}}^{n-1} w_m^* \; g(x,s_j,s_m) - bw_j^* \frac{\partial K(s_j,x)}{\partial s} + 2K(s_j,x)P_1^{(\alpha,\beta)}(s_j)$$

$$(2.24) \qquad F_n(s_j) = -b\sum_{\substack{m=1\\m\neq j}}^{n-1} w_m^* \; h(s_j,s_m) - bw_j^* f'(s_j) + 2f(s_j)P_1^{(\alpha,\beta)}(s_j)+|b|N.$$

In the remainder of this section we will show that Nyström's interpolation formula (2.16) can be further reduced into a more computationally efficient form for all $t\neq s_j$. To do that we set

$$(2.25) \qquad f_1(t) = f(t) - \lambda\pi\sum_{r=1}^{n} w_r K(t,t_r)y_n^*(t_r)$$

and rewrite (2.16) as

$$(2.26) \qquad y_n^*(t) = -b\sum_{m=1}^{n-1} w_m^* \frac{f_1(s_m)}{s_m-t} + 2f_1(t)\frac{P_n^{(\alpha,\beta)}(t)}{P_{n-1}^{(-\alpha,-\beta)}(t)} + |b|N, \text{ for } t\neq s_j.$$

From (1.4) we have that

$$(2.27) \qquad f_1(s_m) = b\sum_{r=1}^{n} w_r \frac{y_n^*(t_r)}{t_r-s_m}.$$

If we introduce (2.27) into (2.26), after interchanging the order of summation and using identities (2.19) and (2.20) we obtain

$$(2.28) \qquad y_n^*(t) = -2b\sum_{r=1}^{n} w_r \frac{y_n^*(t_r)}{t_r-t}\left[P_1^{(\alpha,\beta)}(t_r) - P_1^{(\alpha,\beta)}(t) + \frac{P_n^{(\alpha,\beta)}(t)}{P_{n-1}^{(-\alpha,-\beta)}(t)}\right] +$$

$$2f_1(t)\frac{P_n^{(\alpha,\beta)}(t)}{P_{n-1}^{(-\alpha,-\beta)}(t)} + |b|N, \text{ for } t\neq s_j.$$

Since $\left[P_1^{(\alpha,\beta)}(t_r) - P_1^{(\alpha,\beta)}(t)\right] = \frac{t_r - t}{2}$ and $\sum_{r=1}^{n} w_r y_n^*(t_r) = N$, we finally obtain

$$y_n^*(t) = 2\,\frac{P_n^{(\alpha,\beta)}(t)}{P_{n-1}^{(-\alpha,-\beta)}(t)}\left[-b\sum_{r=1}^{n} w_r \frac{y_n^*(t_r)}{t_r - t} + f_1(t)\right], \text{ for } t \neq s_j \tag{2.29}$$

It is obvious that the last equation is more computationally efficient interpolation formula than (2.16). Moreover, it is not difficult to see that (2.29) can be derived directly from (1.1) by using the appropriate Gauss-Jacobi quadrature.

3. <u>The Convergence of Nyström's Interpolation formula</u>

If we assume that $L(x,t) \in C([-1,1]\times[-1,1])$, and $F(t) \in C[-1,1]$, then we can apply Nyström's theory to the Fredholm integral equation (2.1). Let us define the maximum norm in $C[-1,1]$ by

$$||y||_\infty = \max_{-1\le x\le 1} |y(x)|, \tag{3.1}$$

and introduce the linear operators

$$\mathcal{L}y = \int_{-1}^{1} w(x)L(t,x)y(x)dx \tag{3.2}$$

$$\mathcal{L}_n y = \pi \sum_{r=1}^{n} w_r L(t,t_r)y(t_r) \tag{3.3}$$

$$\mathcal{L}_n^* y = \pi \sum_{r=1}^{n} w_r L_n(t,t_r)y(t_r). \tag{3.4}$$

The norm of a linear operator T on $C[-1,1]$ is defined by

$$||T|| = \sup_{\substack{||y||_\infty \neq 0 \\ y\in C[-1,1]}} \frac{||Ty||_\infty}{||y||_\infty} \tag{3.5}$$

We rewrite equations (2.1), (2.9), (2.10) by using the operator notation as follows

$$(3.6) \qquad (I + \lambda L)y = F$$

$$(3.7) \qquad (I + \lambda L_n)y_n = F$$

$$(3.8) \qquad (I + \lambda L_n^*)y_n^* = F_n$$

where I is the identity operator. In the remainder of this section we will show that Nyström's interpolant $y_n^*(t)$ given by (3.8) converges to the solution y(t) of (3.6). To do so we will need to assume that

A1. $\max_{-1\le x,t\le 1} |L_n(x,t) - L(x,t)| \to 0$ as $n\to\infty$

A2. $||F-F_n||_\infty \to 0$ as $n\to\infty$,

and we will derive later the conditions under which the previous assumptions are satisfied. Now we are ready to present the following theorems:

<u>Theorem 3.1</u> $||L_n-L_n^*||\to 0$ as $n\to\infty$

<u>Proof</u> The theorem follows immediately from the inequality

$$(3.9) \qquad ||L_n-L_n^*||\le \max_{-1\le x,t\le 1} |L_n(x,t)-L(x,t| \pi \sum_{r=1}^{n} w_r,$$

assumption A1 and the fact that $\sum_{r=1}^{n} w_r \le C < \infty$ (Szegö [13], p. 350). □

Theorem 3.2 For all $y \in C[-1,1]$, $||(L_n-L)y||_\infty \to 0$ as $n\to\infty$.

Proof See Gerasoulis [7], Appendix B(i), and Atkinson [1], [2]. □

Theorem 3.3 If λ is not an eigenvalue of (3.6), then $(I+\lambda L_n)^{-1}$ exists for all $n \geq N(\lambda)$, and it is uniformly bounded by a constant B, i.e., $||(I+\lambda L_n)^{-1}|| \leq B$.

Proof It follows from Theorem 3.2 (for more details see Atkinson [1] p. 98 and p. 105). □

Corollary 3.1 Under the assumptions of Theorem (3.3) $(I+\lambda L_n^*)^{-1}$ exists and it is uniformly bounded for $n \geq n_o$.

Proof From the identity

$$(3.10) \qquad (I+\lambda L_n^*)^{-1} = (I+\lambda L_n)^{-1} + [\tfrac{1}{\lambda}I-(I+\lambda L_n)^{-1}(L_n-L_n^*)]^{-1}(I+\lambda L_n)(L_n-L_n^*)(I+\lambda L_n)^{-1}$$

and Theorem (3.3) we see that $(I+\lambda L_n^*)^{-1}$ exists whenever

$$(3.11) \qquad [\tfrac{1}{\lambda} I-(I+\lambda L_n)^{-1}(L_n-L_n^*)]^{-1}$$

exists. This can be shown by using Theorems (3.1), (3.3) and the inequality

$$(3.12) \qquad ||(I+\lambda L_n)^{-1}(L_n-L_n^*)|| \leq B||L_n-L_n^*|| < \frac{1}{|\lambda|} \text{ for } n \geq n_o.$$

The uniform boundedness can be easily shown by taking norms in (3.10). □

Now we present the main theorem in this section.

Theorem 3.4 If λ is not an eigenvalue of equation (1.1), then the Nyström's interpolant $y_n^*(t)$ given in (3.8) converges uniformly to the unique solution $y(t)$ of (1.1), (1.3).

Proof

We need to show $||y-y_n^*||_\infty \to 0$ as $n\to\infty$. From (3.7) - (3.8):

$$(3.13) \qquad (I+\lambda L_n)(y_n-y_n^*) = \lambda(L_n^*-L_n)y_n^* + F-F_n.$$

which can be further reduced to

$$(3.14) \qquad ||y_n-y_n^*||_\infty \le ||(I+\lambda L_n)^{-1}||(\lambda||L_n^*-L_n|| \; ||y_n^*||_\infty+||F-F_n||_\infty).$$

Similarly, from (3.6) and (3.8);

$$(3.15) \qquad ||y-y_n||_\infty \le ||(I+\lambda L_n)^{-1}|| \; |\lambda| \; ||(L-L_n)y||_\infty$$

Finally, from the obvious inequality:

$$(3.16) \qquad ||y-y_n^*||_\infty \le ||y_n-y_n^*||_\infty+||y_n-y||_\infty$$

and (3.14)-(3.15) we readily obtain the error bound:

$$(3.17) \quad ||y-y_n^*||_\infty \le ||(I+\lambda L_n)^{-1}||\{|\lambda| \; ||(L-L_n)y||_\infty+|\lambda| \; ||(L_n^*-L_n)||\cdot||y_n^*||_\infty+||F-F_n||_\infty\}$$

which tends to zero uniformly by Theorems (3.1)-(3.3) and assumption A2. □

The following Theorem provides the conditions under which assumptions A1 and A2 are satisfied.

<u>Theorem 3.5</u> If $K(s,t)\epsilon C^1([-1,1]x[-1,1])$ and $f(s)\epsilon C^1[-1,1]$, then A1 and A2 are satisfied.

<u>Proof</u> For A1 we must show that the sequence $\{r_n(K;t,x)\}$ converges uniformly. From Szegö [13], p. 350 we know that $\{r_n(K;t,x)\}$ converges to zero pointwise in $[-1,1]x[-1,1]$. Moreover,

$$(3.18)\qquad |r_n(K;x,t)|\le(A_1+C_1)\max_{-1\le x,t,s\le 1}|g(x,t,s)|$$

$$(3.19)\qquad |r_n(K;x_1,t_1)-r_n(K;x_2,t_2)|\le(A_1+C_1)\max_{-1\le s\le 1}|g(x_1,t_1,s)-g(x_2,t_2,s)|$$

where

$$A_1 = \frac{b}{\pi}\int_{-1}^{1}\frac{ds}{w(s)}\quad\text{and}\quad C_1 = \frac{b}{\pi}\sum_{r=1}^{n-1} w_r^*.$$

Since $g(x,t,s)$ is continuous in $[-1,1]x[-1,1]x[-1,1]$, (3.18), (3.19) imply that $\{r_n(K;x,t)\}$ is a uniformly bounded equicontinuous family of functions, and from Arzela-Ascoli lemma $\{r_n(K;x,t)\}$ must converge to zero uniformly (see Atkinson [1], p. 92 for a similar argument).
We may use a similar argument to show assumption A2. □

In order to illustrate the use of (3.17), we consider smooth input functions and derive the order of convergence for the special case $a=0$, $b=1$, $\alpha=\beta=-1/2$ of equations (1.1) and (1.3).

<u>Theorem 3.6</u> If $K(s,t)\epsilon C^P([-1,1]x[-1,1])$ and $f(s)\epsilon C^P([-1,1])$, then for all n such that $2n\le p-1$,

$$(3.20)\qquad ||y-y_n^*||_\infty\le\left(\frac{B_1}{2n}+B_2\right)/[(2n-1)!2^{2n-1}],$$

where B_1,B_2 are constants, independent of n.

Proof

From (2.1), (2.4) and (2.7) it is easy to see that if $K(s,t)\epsilon C^P([-1,1]x[-1,1])$ $f(s)\epsilon C^P([-1,1]x[-1,1])$ then $L(t,x)\epsilon C^{P-1}([-1,1]x[-1,1])$ and $y(x)\epsilon C^{P-1}[-1,1]$. The Gaussian quadrature error formulae ([5], p. 75) and the assumption $2n\le p-1$ imply:

$$(3.21) \qquad ||F-F_n||_\infty \le \max_{-1\le t\le 1} \left|\frac{\partial^{(2n-2)}h(t,s)}{\partial s^{(2n-2)}}\right| / [(2n-2)!2^{2n-1}]$$

$$(3.22) \qquad ||L_n-L_n^*|| \le \max_{-1\le t\le 1} |L_n(x,t)-L(x,t)| \le \max_{-1\le t\le 1} \left|\frac{\partial^{(2n-2)}g(x,t,s)}{\partial s^{(2n-2)}}\right| / [(2n-2)!2^{2n-1}]$$

However, by using Taylor's expansion of $h(t,s)$ we can show that

$$(3.23) \qquad \max_{-1\le t\le 1} \left|\frac{\partial^{(2n-2)}h(t,s)}{\partial s^{(2n-2)}}\right| = ||f^{(2n-1)}||_\infty/(2n-1).$$

Hence, (4.4), (4.5) can be further reduced to

$$(3.24) \qquad ||F-F_n||_\infty \le ||f^{(2n-1)}||_\infty / [(2n-1)!2^{2n-1}]$$

$$(3.25) \qquad ||L_n-L_n^*|| \le \max_{-1\le s,t\le 1} \left|\frac{\partial^{(2n-1)}K(s,t)}{\partial s^{(2n-1)}}\right| / [(2n-1)!2^{2n-1}].$$

Similarly, we obtain the inequality:

$$(3.26) \qquad ||Ly-L_n y||_\infty \le \max_{-1\le x,t\le 1} \left|\frac{\partial^{(2n)}[L(t,x)y(x)]}{\partial x^{(2n)}}\right| / [(2n)!2^{2n-1}],$$

Finally, we combine (3.22)-(3.26), and Theorem (3.3) to derive (3.20), where the constants B_1 and B_2 are given below:

(3.27) $$B_1 = B \max_{-1 \le t,x \le 1} \left| \frac{\partial^{(2n)}[L(t,x)y(x)]}{\partial x^{(2n)}} \right|$$

(3.28) $$B_2 = B\left(||f^{(2n-1)}||_\infty + ||y_n^*||_\infty \max_{-1 \le s,t \le 1} \left| \frac{\partial^{(2n-1)}K(s,t)}{\delta s^{(2n-1)}} \right|\right). \quad \square$$

We note that while the constant B_2 can be evaluated, B_1 remains generally unknown. However, its contribution to the error bound becomes insignificant for sufficiently large n and p, and its evaluation is unnecessary for deriving the order of convergence. From (3.20) it is clear that Nyström's interpolant converges at least as fast as $\underline{O}(1/[(2n-1)!2^{2n-1}])$ for this special case of (1.1), (1.3).

By following the same analysis as in Theorem (3.6), we can easily derive the order of convergence for the general case of equations (1.1) and (1.3); we will have to use the Gauss-Jacobi ([5], p. 76) instead of the Gauss-Chebyshev quadrature error formulae.

Finally, the order of convergence derived above implies that Nyström's interpolation formula will converge very fast especially for smooth input functions. This observation has been verified with several numerical examples, one of which is outlined in the next section.

4. A Numerical Example

In this section we consider the integral equation (1.1), (1.3) with $\alpha=\beta=-1/2$, $a=0$, $b=1$ $K(s,t)=0$, $f(s)=\cos(s)$ and $N=1$. This equation arises in the stress analysis of a crack in an isotropic medium [15]. The solution at ± 1 is the stress intensity factor, which is an important quantity in engineering. We can easily show that $B=1$ $B_1=0$ and $B_2 \le 1$, so that (3.20) is reduced to:

(4.1) $\|y-y_n^*\|_\infty \leq 1/[(2n-1)!2^{2n-1}]$.

The error in the evaluation of y(1) is given in columns 1 and 2 of Table 1. The exact solution y(1) is easily derived by inverting (1.1), (1.3) and is $y(1)=J_0(1)+1= 1.7651....$, where $J_0(1)$ is the Bessel function of order zero. We can see from columns 1 and 2, that Nyström's interpolant converges much faster than the Lagrange interpolation formula. This is an expected result since Nyström's formula uses the exact value of K(s,t) and f(s), while the Lagrange's formula uses the Lagrange interpolatory polynomials $K_n(s,t)$, $f_n(s)$ instead. By comparing columns 2 and 3 we see that the error bound given in (4.1) agrees very well with the actual error. This result is due to the fact that the error bound (3.21) of the Gauss-Chebyshev quadrature is very sharp.

Similar results have been obtained for several different integral equations. The convergence of Nyström's formula was always faster than the Lagrange's formula even in cases of low differentiability of the input functions.

Table 1

The error in the evaluation of y(+1) of eg. (1.1), (1.3) with a=0, b=1, K(s,t)=0, f(s)=cos(s), N=1.

n	Error of Lagrange Polynomials [10]	Error of Nyström's interpolant (2.29)	Error bound (4.1)
3	$1.1\cdot10^{-1}$	$4.3\cdot10^{-5}$	$2.0\cdot10^{-4}$
4	$4.9\cdot10^{-3}$	$1.9\cdot10^{-7}$	$1.6\cdot10^{-6}$
5	$2.4\cdot10^{-3}$	$5.3\cdot10^{-10}$	$5.4\cdot10^{-9}$
6	$4.2\cdot10^{-5}$	$1.0\cdot10^{-11}$	$1.2\cdot10^{-11}$
7	$2.1\cdot10^{-5}$	$1.4\cdot10^{-15}$	$2.0\cdot10^{-14}$
8	$1.9\cdot10^{-7}$	$1.0\cdot10^{-18}$	$2.3\cdot10^{-17}$

NOTE: All computations have been performed in DEC-20 FORTRAN, with a double precision arithmetic.

References

1. K. Atkinson, A survey of Numerical Methods for the solution of Fredholm Integral Equations of the Second Kind, SIAM, Philadelphia, 1976.

2. K. Atkinson, The Numerical solution of Fredholm Integral equations of the second kind, SIAM J. Num. Anal. 4 (1967), 337.

3. K. Atkinson, Extensions of the Nyström Method for the Numerical Solution of Linear Integral Equations, U. S. Army Math. Res. Ctr. Tech. Rep. #686, Madison 1966.

4. F. Erdogan and G. D. Gupta, On the Numerical Solution of Singular Integral Equations, Quart. Appl. Math., 30 (1972), 525-534.

5. Davis and Rabinowitz, Numerical Integration, Blaisdell Publ. Company, 1967.

6. A. Gerasoulis, On the Existence of Approximate Solutions for Singular Integral Equations of Cauchy Type Discretized by Gauss-Chebyshev Quadrature Formulae, BIT, 21, (1981), 377-380.

7. A. Gerasoulis and R. P. Srivastav, On the Solvability of singular integral Equations via Gauss-Jacobi Quadrature. To appear in Computer Mathematics an Int. J., 12, No. 1 (1982).

8. N. Ioakimidis, On the Natural Interpolation Formula for Cauchy Type Singular Integral Equations of the First Kind, Computing, 26 (1981), 73-77.

9. N. Ioakimidis and P. S. Theocaris, On Convergence of Two Direct Methods for Solution of Cauchy Type Singular Integral Equations of the first kind, BIT, 20 (1980), 83-87.

10. S. Krenk, On the Use of the Interpolation Polynomial for solutions of Singular Integral Equations, Q. Applied Math., 32 (1975), 479-484.

11. S. Krenk, On Quadrature Formulas for Singular Integral Equations of the First and the Second Kind, Q. Appl. Math. (1975), pp. 225-232.

12. N. Muskhelishvili, Singular Integral Equations, P. Noordhoff, Groningen, 1953.

13. G. Szegö. Orthogonal Polynomials, Fourth Ed., Amer. Math. Soc., Providence, R. I., 1975.

14. P. S. Theocaris and N. I. Ioakimidis, Numerical Integration Methods for the Solution of Singular Integral Equations, Q. Appl. Math. 35 (1977), 173-182.

15. P. S. Theocaris and N. I. Ioakimidis, A Remark on the Numerical Solution of Singular Integral Equations and the Determination of Stress-Intensity Factors, J. Eng. Math. 18 (1979), 213-222.

16. G. Tsamasphyros and P. S. Theocaris, Equivalence and Convergence of Direct and Indirect Methods for the Numerical Solution of Singular Integral Equations, Computing 27 (1981), 71-80.