Newton's Method and Generation of a Determinantal Family of Iteration Functions

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Abstract

It is well-known that Halley's method can be obtained by applying Newton's method to the function $f/\sqrt{f'}$. Gerlach [3], gives a generalization of this approach, and for each $m \geq 2$, recursively defines an iteration function $G_m(x)$ having order m. Kalantari et al. [6], and Kalantari [8] derive and characterize a determinantal family of iteration functions, called the Basic Family, $B_m(x)$, $m \geq 2$. In this paper we prove, $G_m(x) = B_m(x)$. On the one hand, this implies that $G_m(x)$ enjoys the previously derived properties of $B_m(x)$, i.e., the closed formula, its efficient computation, an expansion formula which gives its precise asymptotic constant, as well as its multipoint versions. On the other hand, this gives a new insight on the Basic Family and Newton's method.

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1 Introduction.

Let $f(x)$ be a polynomial of degree $n \geq 2$ with complex coefficients. The following theorem gives a recipe for constructing high order method for the approximation of roots of f:

Theorem 1.1. (Gerlach [3]) Set $F_1(x) = f(x)$, and for each $m \geq 2$, recursively define

$$F_m(x) = \frac{F_{m-1}(x)}{F'_{m-1}(x)^{\frac{1}{m}}}.$$

Then, the function

$$G_m(x) = x - \frac{F_{m-1}(x)}{F'_{m-1}(x)}$$

defines an iteration function whose order of convergence for simple roots is m. $\square$.

The functions $G_2(x) = x - f/f'$ and $G_3(x) = x - ff'/(f'^2 - ff''/2)$ are the well-known Newton and Halley iteration functions, respectively. For more on Halley's method see Halley [4], Traub [12], Scavo and Thoo [11], Ypma [13], and Kalantari [10]. The fact that Halley's method can be obtained by applying Newton's method to the function $f/\sqrt{f'}$ is well-known, see Bateman [1]. No closed formula for $G_m(x)$ was given previously. Indeed it not even clear that $G_m(x)$ would simplify into a rational function of x, $f(x)$, and its derivatives. Ford and Pennline [2], give a rational formulation of $G_m(x)$. More precisely, they show:

Theorem 1.2. (Ford and Pennline [2]) The iteration function $G_m(x)$ can be written as

$$G_m(x) = x - f(x)\frac{Q_m(x)}{Q_{m+1}(x)},$$

where $Q_2(x) \equiv 1$, and

$$Q_{m+1}(x) = f'(x)Q_m(x) - \frac{1}{m-1}f(x)Q'_m(x).\square$$

Ford and Pennline however do not give a closed form for $G_m(x)$. In this paper we obtain a closed formula for $G_m(x)$. We do this by proving the equivalence of the family $\{G_m(x)\}_{m=2}^{\infty}$ to a family of iteration functions, $\{B_m(x)\}_{m=2}^{\infty}$, called the Basic Family.

To define the Basic Family, let $D_0(x) \equiv 1$, and for each $m \geq 1$, let

$$D_m(x) = det \begin{pmatrix} f'(x) & \frac{f''(x)}{2!} & \dots & \frac{f^{(m-1)}(x)}{(m-1)!} & \frac{f^{(m)}(x)}{(m)!} \\ f(x) & f'(x) & \ddots & \ddots & \frac{f^{(m-1)}(x)}{(m-1)!} \\ 0 & f(x) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{f''(x)}{2!} \\ 0 & 0 & \dots & f(x) & f'(x) \end{pmatrix},$$

where det denotes the determinant. Also, for each $i = m+1, \ldots, n+m-1$, define

$$\widehat{D}_{m,i}(x) = det \begin{pmatrix} \frac{f''(x)}{2!} & \frac{f'''(x)}{3!} & \cdots & \frac{f^{(m)}(x)}{(m)!} & \frac{f^{(i)}(x)}{i!} \\ f'(x) & \frac{f''(x)}{2!} & \ddots & \frac{f^{(m-1)}(x)}{(m-1)!} & \frac{f^{(i-1)}(x)}{(i-1)!} \\ f(x) & f'(x) & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \frac{f''(x)}{2!} & \frac{f^{(i-m+2)}(x)}{(i-m+2)!} \\ 0 & 0 & \cdots & f'(x) & \frac{f^{(i-m+1)}(x)}{(i-m+1)!} \end{pmatrix}.$$

Theorem 1.3. (Kalantari et al. [6], Kalantari [8]) For each $m \geq 2$, define

$$B_m(x) \equiv x - f(x) \ \frac{D_{m-2}(x)}{D_{m-1}(x)}. \tag{1}$$

Let θ be a simple root of $f(x)$. Then,

$$B_m(x) = \theta + \sum_{i=m}^{m+n-2} (-1)^m \frac{\widehat{D}_{m-1,i}(x)}{D_{m-1}(x)} (x-\theta)^m. \tag{2}$$

In particular, there exists $r > 0$ such that given any $x_0 \in N_r(\theta) = \{x : |x - \theta| < r\}$, the fixed-point iteration

$$x_{k+1} = B_m(x_k), \quad k = 1, 2, \ldots, \tag{3}$$

is well-defined, it converges to θ having order m. Specifically,

$$\lim_{k\to\infty} \frac{(\theta - x_{k+1})}{(\theta - x_k)^m} = (-1)^{m-1} \frac{\widehat{D}_{m-1,m}(\theta)}{D_{m-1}(\theta)} = (-1)^{m-1} \frac{\widehat{D}_{m-1,m}(\theta)}{f'(\theta)^{m-1}}. \ \square \tag{4}$$

Theorem 1.3, in particular gives the order of convergence of each $B_m(x)$, and its asymptotic error constant. In [9] the theorem is used to prove the following important property of the Basic Family: given a simple root θ of $f(x)$, there exists a neighborhood of θ so that given any x_0 in this neighborhood, we have

$$\theta = \lim_{m\to\infty} B_m(x_0).$$

The expansion formula given in part (2) of Theorem 1.3 is a consequence of a nontrivial determinantal generalization of Taylor's theorem with confluent divided differences, given in Kalantari [8]. In particular, this determinantal Taylor's theorem gives a more general development of the Basic Family, and its multipoint version. Each member B_m of the Basic Family blossoms into m iteration functions, $B_m^{(1)} = B_m, \ldots, B_m^{(m)}$, where $B_m^{(k)}$ is a k-point iteration function. The order of convergence of these iteration functions is derived in [7]. A simple development of the Basic Family for square and cube roots is given in [5].

2 The Equivalence of G_m and B_m.

Let $f(x)$ be a polynomial of degree n with complex coefficients. We prove that for each $m \geq 2$, $G_m(x) = B_m(x)$, as applied to the polynomial $f(x)$. We remark that the equivalence of these two iteration functions for polynomials implies their equivalence for more general functions. This is because given an input x_0, $G_m(x_0)$ and $B_m(x_0)$ can be viewed as the value of these iteration functions as applied to the Taylor polynomial of the function at x_0. Hence, the two iteration functions are equivalent for more general functions (sufficiently differentiable) if and only if they are identical for all polynomials.

The following is a key result. For simplicity of notation throughout the rest of the paper we will suppress the variable x in $f^{(j)}(x)$, $D_j(x)$, etc.

Lemma 2.1. (Kalantari [9]) For $j < 0$, set $D_j = 0$. Then, for all $m \geq 1$ we have,

$$D_m = \sum_{i=1}^{n} (-1)^{i-1} \frac{f^{i-1} f^{(i)}}{i!} D_{m-i}.$$

Theorem 2.1. For each $m \geq 1$, we have

$$D'_m = \frac{m+1}{f}\Big(f' D_m - D_{m+1}\Big) = (m+1) \sum_{i=2}^{n} (-1)^{i} \frac{f^{i-2} f^{(i)}}{i!} D_{m+1-i}.$$

Proof. Observe that the second equality in the lemma is the immediate consequence of Lemma 2.1 as written for D_{m+1}. We prove the first equality in the lemma by induction on m. For $m = 1$, $D_1 = f'$, and $D_2 = f'^2 - ff''/2$. We have,

$$\frac{2}{f}(f' D_1 - D_2) = \frac{2}{f}(f'^2 - f'^2 + \frac{ff''}{2}) = f'' = D'_1.$$

Hence, the lemma is true for $m = 1$. Now assume that $m > 1$. From Lemma 2.1, we have

$$D_m = \sum_{i=1}^{n} (-1)^{i-1} \frac{f^{i-1} f^{(i)}}{i!} D_{m-i}.$$

Differentiating D_m, and grouping terms corresponding to all those that differentiate f^{i-1} $(i > 1)$ into one group, those that differentiate $f^{(i)}$ into a second group, and those that differentiate D_{m-i} into a third group we get,

$$D'_m = A + B + C,$$

where

$$A = \sum_{i=2}^{n} (-1)^{i-1} (i-1) \frac{f^{i-2} f' f^{(i)}}{i!} D_{m-i},$$

$$B = \sum_{i=1}^{n} (-1)^{i-1} \frac{f^{i-1} f^{(i+1)}}{i!} D_{m-i},$$

$$C = \sum_{i=1}^{n} (-1)^{i-1} \frac{f^{i-1} f^{(i)}}{i!} D'_{m-i}.$$

By the induction hypothesis, we have

$$D'_{m-i} = \frac{m+1-i}{f} \Big(f' D_{m-i} - D_{m+1-i} \Big).$$

Thus,

$$C = C_1 + C_2,$$

where

$$C_1 = \sum_{i=1}^{n} (-1)^{i-1} (m+1-i) \frac{f^{i-2} f' f^{(i)}}{i!} D_{m-i},$$

$$C_2 = -\frac{mf'}{f} D_m + \sum_{i=2}^{n} (-1)^{i} (m+1-i) \frac{f^{i-2} f^{(i)}}{i!} D_{m+1-i}.$$

Note that

$$A + C_1 = \frac{m(f')^2}{f} D_{m-1} + m \sum_{i=2}^{n} (-1)^{i-1} \frac{f^{i-2} f' f^{(i)}}{i!} D_{m-i}.$$

Since in B, $f^{(n+1)} \equiv 0$, and by changing the summation index, we get

$$B = \sum_{i=1}^{n-1} (-1)^{i-1} \frac{f^{i-1} f^{(i+1)}}{i!} D_{m-i} = \sum_{i=2}^{n} (-1)^{i-2} \frac{f^{i-2} f^{(i)}}{(i-1)!} D_{m+1-i}.$$

Thus,

$$B + C_2 = -\frac{mf'}{f} D_m + (m+1) \sum_{i=2}^{n} (-1)^{i} \frac{f^{i-2} f^{(i)}}{i!} D_{m+1-i}.$$

From Lemma 2.1 we also have

$$\frac{m(f')^2}{f} D_{m-1} - \frac{mf'}{f} D_m = f' \Big(\frac{m}{f} \Big) (f' D_{m-1} - D_m) = m \sum_{i=2}^{n} (-1)^{i} \frac{f^{i-2} f' f^{(i)}}{i!} D_{m-i}.$$

Thus,

$$D'_m = A + C_1 + B + C_2 = (m+1) \sum_{i=2}^{n} (-1)^{i} \frac{f^{i-2} f^{(i)}}{i!} D_{m+1-i}. \square$$

From Theorem 1.2 and Theorem 2.1 it follows that Q_m and D_{m-2} satisfy the same recurrence relationship. Since $Q_2 = D_0$, it follows that $Q_m = D_{m-2}$. Thus, $F_{m-1}/F'_{m-1} = f D_{m-2}/D_{m-1}$, i.e., $G_m = B_m$, for all $m \geq 2$. In what follows we give a direct proof of this using Theorem 2.1, also directly relating F_m and D_m.

Theorem 2.2. Define $\widehat{F}_1 = f$, and for each $m \geq 2$, define $\widehat{F}_m = f D_{m-1}^{-\frac{1}{m}}$. Then,

$$B_m(x) = x - \frac{\widehat{F}_{m-1}}{\widehat{F}'_{m-1}} = G_m(x).$$

Proof. From differentiation of $\widehat{F}_m$ and the identity $D'_{m-1} = \frac{m}{f}(f' D_{m-1} - D_m)$ derived in Theorem 2.1, we get

$$\widehat{F}'_m = f' D_{m-1}^{-\frac{1}{m}} - \frac{1}{m} f D_{m-1}^{-\frac{m+1}{m}} D'_{m-1} = D_{m-1}^{-\frac{m+1}{m}} D_m.$$

It follows that

$$\frac{\widehat{F}_{m-1}}{\widehat{F}'_{m-1}} = f \frac{D_{m-2}}{D_{m-1}}.$$

To prove that $G_m = B_m$ and to directly relate F_m and D_m, it suffices to show $\widehat{F}_m = F_m$. But they have the same initial condition and satisfy the same recursion since we have

$$\widehat{F}_{m-1}(\widehat{F}'_{m-1})^{-\frac{1}{m}} = f D_{m-2}^{-\frac{1}{m-1}} D_{m-2}^{\frac{1}{m-1}} D_{m-1}^{-\frac{1}{m}} = f D_{m-1}^{-\frac{1}{m}} = \widehat{F}_m. \square$$

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