DIRAC COHOMOLOGY FOR HOPF–HECKE ALGEBRAS

by

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ABSTRACT OF THE DISSERTATION

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In this dissertation, a generalized version of Dirac cohomology is developed.

It is shown that Dirac operators can be defined and their cohomology can be studied for a general class of algebras, which we call Hopf–Hecke algebras. A result relating the Dirac cohomology with central characters is established for a subclass of algebras, which we call Barbasch–Sahi algebras. This result simultaneously generalizes known results on such a relation for real reductive Lie groups and for various kinds of Hecke algebras, which all go back to a conjecture of David Vogan [Vog97].

A variety of algebraic concepts and techniques is combined to create the general framework for Dirac cohomology, including central simple (super)algebras, Hopf algebras, smash products, PBW deformations, and Koszul algebras.

Classification results on the studied classes of algebras are obtained, and infinitesimal Cherednik algebras of the general linear group are studied as novel examples for algebras with a Dirac cohomology theory.

Preface

Dirac operators have played an important role in several areas of mathematics and mathematical physics, starting with the work of Paul Dirac [Dir28a, Dir28b]. While as differential operators, Dirac operators have been crucial in the Atiyah–Singer index theorem [AS63], algebraic Dirac operators have been important tools in representation theory.

This thesis is motivated by recent developments in two different fields of algebra and representation theory which it combines in order to obtain a generalized theory of Dirac cohomology: versions of Dirac cohomology for various algebraic objects and parallel results on the connection between this cohomology theory and central characters on the one hand [HP02, Ciu16] and the theory of PBW deformations of smash products on the other hand, which unify many relevant algebraic structures, like Drinfeld Hecke algebras [Dri86], symplectic reflection algebras and rational Cherednik algebras [EG02], and infinitesimal Hecke and Cherednik algebras [EGG05]. Additionally, central simple (super)algebras prove a useful ingredient for the presented theory.

The generalization of Dirac cohomology to a wider class of algebras was suggested by D. Barbasch and S. Sahi, which is why those algebras for which a generalized Dirac cohomology and a result on its connection to central characters is proved in this thesis are called Barbasch–Sahi algebras.

Some results of this thesis have already been published: The pin cover constructions of Section 4.1 – Section 4.3 and a version of Dirac cohomology and Vogan's conjecture as in Chapter 6, but largely without the use of superalgebraic concepts or central simple superalgebras are contained in the accepted article [Fla16]. Most of the classification results in Chapter 5 and the applications to infinitesimal Cherednik algebras (Chapter 7) are contained in the preprint [FS16].

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I would like to thank my wife Isabel for being wonderful.

Oh, and thanks, coffee. :)

Dedication

For my family and friends.

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Chapter 1

Introduction

The Dirac operator was introduced by Paul Dirac in 1928 [Dir28a, Dir28b] to formulate a quantum mechanical wave equation for electrons which is consistent with special relativity. The new operator serves as a "square root" of the d'Alembert operator, and just as a square root of -1 can only be found in a suitable field extension of the real numbers, the operator Dirac constructed is an element in an extended algebra of operators which act on wave functions with multiple components. The theory described by the Dirac equation was the first theory combining quantum mechanics and special relativity, it predicted the existence of antimatter years before its experimental confirmation and gave a theoretical justification for wave functions with multiple components, which had previously been used by Wolfgang Pauli to explain the spin of elementary particles as manifest, for instance, in the Stern–Gerlach experiment.

Dirac operators have subsequently played an important role in many ares of physics and mathematics, in particular, in the Atiyah–Singer index theorem [AS63]. Algebraic Dirac operators were first used by R. Parthasarathy [Par72] to realize the discrete series of a non-compact semisimple Lie group. In a certain sense, they can be viewed as "square roots" of Casimir operators and they have proven important tools to study representations of real reductive groups [Vog81], in particular, unitary representations, where they were used to investigate the classification of unitary highest weight representations [EHW83, Jak83] and of unitary representations with non-zero cohomology [VZ84].

By the work of Harish-Chandra, the study of irreducible unitary representations of a real reductive Lie group G boils down to the study of a class of unitarizable $(\mathfrak{g}, K)$-modules, where $\mathfrak{g}$ is the complexified Lie algebra of G and K is a maximal compact subgroup. These unitarizable $(\mathfrak{g}, K)$-modules are special cases of admissible $(\mathfrak{g}, K)$-modules, which were classified by Langlands in 1973 [Lan89, BW00], so the problem reduces to identifying those admissible $(\mathfrak{g}, K)$-modules which are, in fact, unitarizable. For any $(\mathfrak{g}, K)$-module M and a suitable spin representation S, the Dirac operator D acts on the tensor product module $M \otimes S$ and the kernel $\ker D$ of this action is a $\tilde{K}$-module, where $\tilde{K}$ is a certain double cover of K. For unitarizable $(\mathfrak{g}, K)$-modules M, the action of D is semisimple and Parthasarathy proved [Par72] an inequality relating the action of the Casimir operator of G on M with the action of the Casimir operator of $\tilde{K}$ on $\ker D$ which can be used as a criterion for unitarizability.

For non-unitary admissible modules M, the action of D might not be semisimple. However, David Vogan suggested [Vog97] considering the cohomology of the action of D on $M \otimes S$, the Dirac cohomology

$$H^D(M) = \ker D / (\ker D \cap \operatorname{im} D) \ .$$

He conjectured that this cohomology, if non-zero, should determine the infinitesimal character of M. Since for unitary representations, the Dirac cohomology is just the kernel of the Dirac operator and the infinitesimal character, in particular, governs the action of the Casimir element, this idea extends Parthasarathy's theory. Vogan's conjecture was proved by Huang and Pandžić [HP02].

Barbasch, Ciubotaru, and Trapa established a p-adic analog of Vogan's conjecture [BCT12], where the role of $(\mathfrak{g}, K)$-modules is played by modules of graded affine Hecke algebras. The result was extended to more general types of Hecke algebras, including rational Cherednik algebras and symplectic reflection algebras by Ciubotaru [Ciu16].

The starting point of this thesis is the observation that the different versions of Dirac cohomology and Vogan's conjecture can be treated uniformly using the theory of Hopf algebras, smash products, PBW deformations, and superalgebras. From this perspective, $(\mathfrak{g}, K)$-modules and modules of Hecke algebras are special cases of modules of a smash product algebra constructed from a Hopf algebra and a Koszul algebra, or a PBW deformation of such a smash product. If the Hopf algebra is a group algebra, these deformations include graded affine Hecke algebras, symplectic reflection algebras, and rational Cherednik algebras. If the Hopf algebra is the universal enveloping algebra of a Lie algebra, the original setting is recovered, but also novel examples like infinitesimal Cherednik algebras arise.

We explain how Dirac operators and Dirac cohomology can be defined in our general setting and how this yields an analog of Vogan's conjecture relating the Dirac cohomology with central characters (Theorem 6.4.5) for a certain subclass of deformations which includes the mentioned known special cases. An important step in the proof of the generalized Vogan's conjecture is to find a suitable concept of the pin cover of a cocommutative Hopf algebra. We construct this pin cover object explicitly (Section 4.3), but also explain how it can be derived abstractly from the theory of Hopf algebra actions on central simple superalgebras (Section 4.4).

PBW deformations of smash products of a Hopf algebra and a Koszul algebra have been an active field of research in its own right. Various special cases haven been studied by Drinfeld [Dri86], Braverman–Gaitsgory [BG96], Etingof–Ginzburg [EG02], Ram–Shepler [RS03], Etingof–Gan–Ginzburg [EGG05], Khare [Kha17], and the most general situation relevant for our purposes by Walton and Witherspoon [WW14]. In all these cases, it is known that PBW deformations can be characterized by an equivariance condition and an identity which generalizes the classical Jacobi identity for Lie algebras.

If the Hopf algebra factor of the deformed smash product is a group algebra, then an explicit classification of the deformations was given by Drinfeld [Dri86] (see also Ram–Shepler [RS03]). In analogy to this, we obtain partial results on the classification of the deformations for which our theory of Dirac cohomology applies. For instance, we prove that these deformations have to be supported on a certain subspace of bireflections in the Hopf algebra (Proposition 5.1.7). We also construct a family of explicit examples of PBW deformations parameterized by a space of Hopf algebra invariants (Section 5.2), and we prove that the family contains all PBW deformations under additional assumptions, like in the case of a group algebra. Our results are first steps towards a complete classification of the discussed PBW deformations, which would contain the classification of infinitesimal Hecke algebras and is an interesting open problem.

We study infinitesimal Cherednik algebras as a novel class of algebras with a Dirac cohomology and with an analog of Vogan's conjecture. Infinitesimal Cherednik algebras are a special case of infinitesimal Hecke algebras which were defined by Etingof–Gan–Ginzburg [EGG05] as infinitesimal analogs of Hecke algebras, where the role of the finite group is played by a Lie algebra. By definition, they are PBW deformations and some infinitesimal Hecke algebras, including all infinitesimal Cherednik algebras, satisfy the orthogonality condition which is required for our definition of a generalized Dirac operator. We derive a formula for the square of this Dirac operator for the infinitesimal Cherednik algebras associated with the general linear group (and its Lie algebra), and we show that they are, in fact, Barbasch–Sahi algebras (Proposition 7.2.14), that is, a version of Vogan's conjecture is available for them. We demonstrate that for finite-dimensional representations, the Dirac cohomology is non-zero, so by Vogan's conjecture (Corollary 7.3.12), it determines the central character, but we even show directly how it determines the precise module.

Organization of this thesis. The first two chapters following this introduction are dedicated to the necessary preliminaries. In Chapter 2, we recall definitions and basic properties of superalgebras, Clifford algebras and their substructures, central simple (super)algebras, Hopf (super)algebras, smash products, and PBW deformations thereof, before we review Dirac operators and different known versions of Dirac cohomology, for $(\mathfrak{g}, K)$-modules just as for Hecke algebras, in Chapter 3. We explain explicit pin cover constructions for groups, Lie algebras, and pointed cocommutative Hopf algebras and we interpret these using the theory of coalgebra measurings and the Skolem–Noether theorem for central simple (super)algebras in Chapter 4. In Chapter 5, we derive results on the classification of PBW deformations of certain smash products. Both the concept of pin covers and that of PBW deformations of smash products are incorporated into a generalized framework for Dirac operators and Dirac cohomology in Chapter 6, where also a version of Vogan's conjecture is proved. Finally, in Chapter 7, we study infinitesimal Cherednik algebras as novel examples of algebras with a Dirac cohomology theory. The thesis is concluded by a summary of our results and an outlook containing a selection of open questions (Chapter 8).

Chapter 2

Preliminaries: Algebraic structures

$\mathbb{F}$ will generally be a field of arbitrary characteristic, although we will frequently assume $\operatorname{char}\mathbb{F} \neq 2$ and we will want to specialize $\mathbb{F}$ or its characteristic even further for various special statements. For an $\mathbb{F}$-vector space V, we will denote the *tensor algebra*, the *symmetric algebra*, and the *exterior algebra* by the symbols $T(V)$, $S(V)$, and $\Lambda(V)$, and a typical basis element by $v_1 \dots v_m$, $v_1 \dots v_m$, and $v_1 \wedge \dots \wedge v_m$, respectively, for $m \geq 1$, $v_1, \dots, v_m \in V$.

2.1 Superalgebras

Let us use the words *superspace* and *superalgebra* for a $\mathbb{Z}_2$-graded vector space over $\mathbb{F}$ or a $\mathbb{Z}_2$-graded $\mathbb{F}$-algebra, respectively. If a is a $\mathbb{Z}_2$-homogeneous element in a superspace, we denote its degree by $|a| \in \{0, 1\}$.

As superspaces can be regarded as ordinary vector spaces, we have a tensor product for them, and the tensor product of two superspaces has a natural $\mathbb{Z}_2$-grading, where tensor products of homogeneous elements are homogeneous, their degree being the sum of the degrees of the two factors. The superspaces form a tensor category with the base field viewed as a one-dimensional even superspace as the unit object. What makes superspaces (or superalgebras) interesting is the fact that this tensor category has a non-trivial *braiding*. For superspaces V and W, we define the map

$$c_{V,W} \colon V \otimes W \to W \otimes V \ , \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \tag{2.1.1}$$

for all homogeneous $v \in V, w \in W$. Note that $c^2 = \mathrm{id}$ on any pair of superspaces, which means that the braiding is *symmetric*. Let us also observe that this braiding becomes trivial in characteristic 2, so the category of superspaces is just the category of ordinary vector spaces in this case.

The braiding affects many notions in the category of super vector spaces, for instance, it determines the algebra structure of tensor products of algebras in this category (see also Section 2.4.1): If A, A' are superalgebras with multiplication maps $\mu\colon A\otimes A\to A$ and $\mu'\colon A'\otimes A'\to A'$, then $A\otimes A'$ is the superalgebra whose underlying superspace is the tensor product space and whose multiplication is defined for homogeneous elements (and hence, for arbitrary elements) by

$$\begin{aligned}(a_1\otimes a_1')(a_2\otimes a_2') &= (\mu\otimes\mu')\circ(1\otimes c_{A'\otimes A}\otimes 1)(a_1\otimes a_1'\otimes a_2\otimes a_2')\\ &= (-1)^{|a_1'||a_2|}a_1a_2\otimes a_1'a_2' \quad \text{for all } a_1,a_2\in A, a_1',a_2'\in A' \ .\end{aligned}$$

We also use the braiding to define left actions from right actions. Let A be a superalgebra, let M be a right A-module M, that is, an A-module in the category of superspaces, and let A^{op} be the opposite algebra, that is, the superspace A with multiplication

$$A^{\mathrm{op}}\otimes A^{\mathrm{op}}\simeq A\otimes A\xrightarrow{c_{A,A}}A\otimes A\to A\simeq A^{\mathrm{op}}\ ,$$

where the second arrow is the multiplication in A. Then M is a left A^{op}-module through the map

$$A^{\mathrm{op}}\otimes M\xrightarrow{c_{A,M}}M\otimes A\to M\ ,$$

where the second arrow is the right action. In particular, any superalgebra A is a left $A\otimes A^{\mathrm{op}}$-module through the map

$$(A\otimes A^{\mathrm{op}})\otimes A\xrightarrow{\mathrm{id}_A\otimes c_{A,A}}A\otimes A\otimes A\to A\ ,$$

where the second arrow is the multiplication in A.

Now, just as for ordinary algebras, any invertible element $a\in A$ comes with a conjugation automorphism ρ_a of A defined by

$$\rho_a(b)=(a\otimes(a^{-1})^{\mathrm{op}})\cdot b \quad \text{for all } b\in A\ . \tag{2.1.2}$$

Lemma 2.1.1. $A^\times\to\mathrm{Aut}(A)$, $a\mapsto\rho_a$, *is a well-defined group homomorphism.*

Also the *supercommutator* of elements a, b in a superalgebra A can be defined using this action, namely as

$$[a,b]=(a\otimes 1^{\mathrm{op}}-1\otimes a^{\mathrm{op}})\cdot b\ . \tag{2.1.3}$$

If a, b are homogeneous with respect to the $\mathbb{Z}_2$-grading, then

$$[a, b] = ab - (-1)^{|a||b|} ba \ . \tag{2.1.4}$$

The supercommutator satisfies *super skew-symmetry* and the *super Jacobi identity*, which are $\mathbb{Z}_2$-graded analogs to their classical counterparts: for homogeneous $a, b, c \in A$,

$$[a, b] = -(-1)^{|a||b|}[b, a] \ , \tag{2.1.5}$$

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||a|}[c, [a, b]] = 0 \ . \tag{2.1.6}$$

A linear endomorphism of A is called *even* if it preserves the $\mathbb{Z}_2$-grading, or *odd* if it reverses the $\mathbb{Z}_2$-grading. The space of all linear endomorphisms $\mathrm{End}(A)$ is a direct sum of the subspaces consisting of even or odd endomorphisms, and $\mathrm{End}(A)$ is a superalgebra with these spaces as $\mathbb{Z}_2$-homogeneous subspaces.

An even endomorphism δ is called an *even derivation* if

$$\delta(ab) = \delta(a)b + a\delta(b) \quad \text{for all } a, b \in A \ , \tag{2.1.7}$$

and an odd endomorphism δ is called an *odd derivation* if

$$\delta(ab) = \delta(a)b + (-1)^{|a|} a\delta(b) \quad \text{for all homogeneous } a, b \in A \ . \tag{2.1.8}$$

The spaces of even and odd derivations together form a $\mathbb{Z}_2$-graded subspace $\mathrm{Der}(A)$ in $\mathrm{End}(A)$ which is closed under the supercommutator in $\mathrm{End}(A)$.[1]

Now any a in A defines an endomorphism $\delta_a = [a, \cdot]$.

Lemma 2.1.2. $A \to \mathrm{Der}(A), a \mapsto \delta_a$, *is a well-defined linear map which preserves the $\mathbb{Z}_2$-grading and the supercommutator.*

Proof. If $a, b \in A$ are $\mathbb{Z}_2$-homogeneous, then $\delta_a(b)$ is a linear combination of products of a and b, hence, $\delta_a(b)$ is homogeneous of degree $|a| + |b|$. Thus, δ_a is an even or odd endomorphism of A if a is even or odd, respectively.

[1] We want to avoid the subtleties of the definition of a Lie superalgebra here, since it will not be necessary for our purposes.

For all homogeneous $a, b, c \in A$,

$$\begin{aligned}[a, bc] &= abc - (-1)^{|a||bc|}bca \\ &= abc - (-1)^{|a||b|}bac + (-1)^{|a||b|}bac - (-1)^{|a||b|}(-1)^{|a||c|}bca \\ &= \delta_a(b)c + (-1)^{|a||b|}b\delta_a(c) \ ,\end{aligned}$$

so δ_a is an even or odd derivation, depending on the homogeneous degree of a.

Also, for all homogeneous $a, b, c \in A$, using super skew-symmetry and the super Jacobi identity we get

$$\begin{aligned}[\delta_a, \delta_b](c) = [a, [b, c]] - (-1)^{|a||b|}[b, [a, c]] &= -(-1)^{|b||c|}[a, [c, b]] - (-1)^{|a||b|}[b, [a, c]] \\ &= (-1)^{|b||c|}(-1)^{|a||b|}(-(-1)^{|a||b|}[a, [c, b]] - (-1)^{|b||c|}[b, [a, c]]) \\ &= (-1)^{|b||c|}(-1)^{|a||b|}(-1)^{|c||a|}[c, [b, a]] = [[a, b], c] = \delta_{[a,b]}(c) \ .\end{aligned}$$

□

2.1.1 Ideals of superalgebras

A (left/right/two-sided) ideal of a superalgebra is *graded*, if it is a graded subspace, that is, if it is spanned by homogeneous elements. For any superalgebra, we define the *graded Jacobson radical* as the intersection of all maximal (proper) graded left ideals. Any superalgebra is, in particular, an ordinary ungraded algebra, or alternatively, a trivially graded superalgebra, so we could also consider the Jacobson radical of this ungraded algebra, or equivalently, this trivially graded superalgebra. However, if the characteristic is not 2, this turns out to be the same object by a result of Bergman [Ber75] (see also [CM84, CM87]).

Proposition 2.1.3 ([Ber75, Thm. 1, Prop. 14])**.** *For any algebra over a field of characteristic not* 2*, the graded and the ungraded Jacobson radical coincide.*

A superalgebra is *simple*, if it has no non-zero proper two-sided graded ideals.

Lemma 2.1.4 ([Rac98, Thm. 3, Prop. 4])**.** *Let A be a finite-dimensional simple superalgebra. Then there is exactly one finite-dimensional irreducible A-module up to isomorphism and parity change.*

Proof. Let I be a minimal non-zero graded left ideal of A. Then I is a finite-dimensional A-module. Any submodule is a left ideal, hence, I is irreducible.

Let M be any irreducible A-module. The annihilator of M is a graded two-sided ideal properly contained in A, since it does not contain $1 \in A$. Hence, M is faithful, and there is a homogeneous $m \in M$ such that $Im \neq 0$. Consider $f \colon I \to M, x \mapsto xm$. The map is an A-module map whose kernel is the annihilator of m in I, a left-ideal in I, but not I. Hence by minimality of I, f is injective. The image of f is a non-zero submodule of M, so as M is irreducible, this must be M. Hence, f is an A-module isomorphism between I and M.

As m is homogeneous, f is an even or odd isomorphism. This completes the proof. □

A superalgebra is *semisimple* if it is the direct sum of simple superalgebras. The connection between the Jacobson radical and semisimple superalgebras is a close analog of its classical counterpart. In particular, the Jacobson radical is a graded two-sided ideal and we have the following results.

Lemma 2.1.5. *Assume A is a superalgebra which is the direct sum of minimal (non-zero) graded left ideals. Then A is semisimple.*

Proof. Let S be the set of minimal graded left ideals whose direct sum is A and let T be the set $\{IA \mid I \in S\}$. The elements of T are graded two-sided ideals in A and their sum is A. We claim that these graded two-sided ideals are, in fact, minimal: Assume K is a non-zero graded two-sided ideal in IA for $I \in S$. Then K contains a minimal graded left ideal, say $J \in S$, that is, there is a non-zero homogeneous $a \in A$ such that $0 \neq Ia \subset J$, because otherwise, K would be contained in the direct sum of the minimal graded left ideals $S \setminus \{J\}$. Hence, due to minimality of I and J and by Schur's lemma, the right-multiplication with a is an isomorphism of I and J as graded left A-modules. Let $f \colon J \to I$ be the inverse A-module isomorphism. Let b be the component of $1 \in A$ in J, then $J = Jb$ and

$$Jf(b) = f(Jb) = f(J) = I \ .$$

As K is a two-sided ideal containing J, this means, K contains I, so it contains IA. Hence, IA is a minimal graded two-sided ideal.

Being minimal graded two-sided ideals, the elements of T intersect trivially, so A is a direct sum of minimal graded two-sided ideals

$$A = \oplus_{K \in T} K \ .$$

Consider a single $K \in T$. The component of $1 \in A$ in K is an identity element in K making it a graded subalgebra, any graded two-sided ideal of which is a graded two-sided ideal in A, because $KL \subset K \cap L = 0$ for all $K \neq L \in T$. Hence, K is a simple superalgebra, as desired. □

Lemma 2.1.6. *Let A be a finite-dimensional superalgebra with Jacobson radical J. Then A/J is a semisimple superalgebra and if a homogeneous element $x \in A$ has an invertible image in the quotient algebra A/J, then x is invertible in A.*

Proof. Let $\overline{K}$ be any graded left-ideal in A/J. Then due to finite-dimensionality, there is a minimal non-zero graded left ideal $\overline{I}$ in $\overline{K}$. The preimage of $\overline{I}$ under the canonical projection is a graded left ideal I in A which contains J properly. Hence, there is a maximal graded left ideal M of A such that $M \cap I \subsetneq I$. But M contains J and I is minimal in A/J, so $\overline{M}$ and $\overline{I}$ intersect trivially in A/J. On the other hand, $I + M$ is a graded left-ideal in A which contains M properly, so it has to be A. Hence $\overline{K} = \overline{I} \oplus \overline{(M \cap K)}$.

Set $\overline{K}_0 = A/J$, and define the minimal graded left ideals $(\overline{I}_i)_{i \geq 1}$ in A/J and the graded left ideals $(\overline{K}_i)_{i \geq 1}$ in A/J recursively such that $\overline{K}_i = \overline{I}_{i+1} \oplus \overline{K}_{i+1}$ until $\overline{K}_{i+1} = 0$. Since A/J is finite-dimensional, this process will terminate, so A/J is the direct sum of minimal graded left ideals

$$A/J = \overline{I}_1 \oplus \cdots \oplus \overline{I}_n \ ,$$

which proves the first part together with the previous lemma.

For the second part, consider an even element $a \in J$. Assume $A(1 + a) \neq A$, then $A(1+a)$ is contained in a maximal graded left ideal I of A. I contains J, so in particular,

a, but also $1+a$. But then I contains 1, which is absurd. Hence, $A(1+a) = A$, and $1+a$ has a left inverse.

Consider an element x as in the assertion. Then there is an element $y \in A$ such that $yx = 1 + a$ for some $a \in J$. As x is homogeneous and as J is graded, we may assume y to be homogeneous and a to be even. But then $1+a$ has a left-inverse in A, so x has a left-inverse in A. This means the right-multiplication with x is an injective endomorphism of the finite-dimensional vector space A. Hence, this endomorphism is surjective, and x also has a right-inverse. That is, x is a unit in A, as desired. □

Corollary 2.1.7. *Let A be a finite-dimensional superalgebra over a field of characteristic not 2 with Jacobson radical J. If an element of A has an invertible image in the quotient superalgebra A/J, then it is invertible in A.*

Proof. Consider A as a trivially graded superalgebra, then any element of A corresponds to a homogeneous element, which is invertible, if it is invertible modulo the Jacobson radical of this trivially graded algebra. But in characteristic not 2, this is the same Jacobson radical by Proposition 2.1.3. □

2.2 Clifford algebras

In this section, let us assume that the characteristic of $\mathbb{F}$ is not 2, and let V be a $\mathbb{F}$-vector space with a symmetric bilinear form $\langle\cdot,\cdot\rangle\colon V\otimes V \to k$. We will recall results on Clifford algebras and their substructures mostly following Meinrenken's book [Mei13].

Definition 2.2.1. The *Clifford algebra* $C = C(V, \langle\cdot,\cdot\rangle)$ is the $\mathbb{F}$-algebra generated by elements $v \in V$ subject to the relations

$$v^2 = \langle v, v\rangle \quad \text{for all } v \in V\ . \tag{2.2.1}$$

An immediate consequence is the relation

$$v_1v_2 + v_2v_1 = 2\langle v_1, v_2\rangle \quad \text{for all } v_1, v_2 \in V \tag{2.2.2}$$

if we consider $(v_1+v_2)^2$, and since we are not in characteristic 2, the two sets of relations are in fact equivalent.

We can reformulate this definition and view C as the quotient algebra of the tensor algebra $T(V)$ (over $\mathbb{F}$) by the ideal I generated by $\{v^2 - \langle v, v\rangle\}_{v\in V}$. If $v_1, \ldots, v_m$ are elements in V for $m \geq 1$, we will denote the congruence class of $v_1 \ldots v_m$ (i.e., $v_1 \otimes \ldots \otimes v_m$) in $T(V)$ in the quotient algebra C by the same expression $v_1 \ldots v_m$, but it should become clear from the context which algebra we are working in.

Now $T(V)$ is a filtered and even $\mathbb{Z}$-graded algebra, where elements from V are assigned degree 1. More precisely, the graded slices of $T(V)$ are

$$T^0(V) = \mathbb{F}\ , \quad T^1(V) = V\ , \quad T^2(V) = V \otimes V\ ,$$

and so on, and the filtered subspaces are $\mathbb{F}$, $\mathbb{F}\oplus V$, $\mathbb{F}\oplus V\oplus V\otimes V$, and so on. Consequently, the quotient algebra C is a filtered algebra, where the filtered subspaces are just the images of the filtered subspaces of $T(V)$ under the quotient map. Let us denote them by $C_0, C_1, C_2, \ldots$. As I is generally not a homogeneous ideal with respect to the $\mathbb{Z}$-grading of $T(V)$, C does not become a $\mathbb{Z}$-graded algebra in this way. However, let us observe that the $\mathbb{Z}$-grading induces a $\mathbb{Z}_2$-grading on $T(V)$, and that I is homogeneous with respect to this grading. Hence, C is a superalgebra, and we denote the graded subspaces by C^{even} and C^{odd}. C^{even} is the image of $\mathbb{F}\oplus T^2(V)\oplus T^4(V)\ldots$ and C^{odd} is the image of $V \oplus T^3(V) \oplus T^5(V)\ldots$ under the quotient map.

To make the structure of C even more transparent, it can be instructional to compare it with the exterior algebra $\Lambda(V)$, which is the special case of a Clifford algebra where the bilinear form $\langle\cdot,\cdot\rangle$ is just 0. The exterior algebra is hence the quotient of $T(V)$ by the ideal generated by $\{v^2\}_{v\in V}$, or equivalently, by the ideal generated by $\{v_1v_2 - v_2v_1\}_{v_1,v_2\in V}$. Conventionally, we express elements in $\Lambda(V)$ using the wedge symbol: if $v_1, \ldots, v_m$ are elements in V for $m \geq 1$, then the congruence class of $v_1 \ldots v_m \in T(V)$ in the quotient algebra $\Lambda(V)$ is written as $v_1 \wedge \cdots \wedge v_m$. Since in this special case the defining ideal of $\Lambda(V)$ in $T(V)$ is homogeneous with respect to the $\mathbb{Z}$-grading, the exterior algebra is a $\mathbb{Z}$-graded algebra.

Let us pick an orthogonal basis $(e_i)_i$ of V with respect to $\langle\cdot,\cdot\rangle$, that is, a basis satisfying $\langle e_i, e_j\rangle = 0$ for $i \neq j$.[2] We define

$$f\colon T(V) \to \Lambda(V)\ , \quad e_{i_1}\dots e_{i_m} \mapsto e_{i_1}\wedge\cdots\wedge e_{i_m}\ . \tag{2.2.3}$$

If $\operatorname{char}\mathbb{F} = 0$, we also define the *quantization map* (*antisymmetrization map*)

$$q\colon T(V) \to C(V)\ , \quad v_1\dots v_m \mapsto \tfrac{1}{m!}\sum_{\sigma\in S_m}(-1)^\sigma v_{\sigma(1)}\dots v_{\sigma(m)} \quad \text{for all } v_1,\dots,v_m \in V\ , \tag{2.2.4}$$

where $(-1)^\sigma$ denotes the sign of the permutation σ.

Proposition 2.2.2 ([Mei13, 2.2.5])**.** *The map f induces a vector space isomorphism from C to $\Lambda(V)$ which is independent of the choice of the orthogonal basis $(e_i)_i$. If* $\operatorname{char}\mathbb{F} = 0$*, then q induces an isomorphism from $\Lambda(V)$ to C.*

Even in positive characteristic $\operatorname{char}\mathbb{F} = p$, q is well-defined on the p-th filtered subspace of $T(V)$, and induces an inverse isomorphism between the p-th filtered subspaces of $\Lambda(V)$ and C. In particular, we can always identify C_0 with $\mathbb{F}$ and C_1 with $\mathbb{F}\oplus V$.

Let us specialize $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$ for some $n \geq 0$. Then there is a unique non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on V up to a change of basis, defining a unique Clifford algebra $C(V, \langle\cdot,\cdot\rangle)$ up to isomorphism, which we denote by $C(n)$. We denote the matrix rings over the complex numbers by $M_m(\mathbb{C})$ for $m \geq 1$. For any $m \geq 0$, we can identify $M_{2^m}(\mathbb{C})$ with $\operatorname{End}(\Lambda(\mathbb{C}^m))$, since $\dim\Lambda(\mathbb{C}^m)) = 2^m$, and as $\Lambda(\mathbb{C}^m)$ is a superalgebra if we assign elements from $\mathbb{C}^m$ odd degree, $\operatorname{End}(\Lambda(\mathbb{C}^m))$ is a superalgebra (recall that End of a superspace is the superalgebra of all linear endomorphisms, not just the graded ones, while the graded endomorphisms form the homogeneous subspaces).

Proposition 2.2.3 ([Mei13, Prop. 2.4])**.** *For all $m \geq 0$,*

$$C(2m) \cong M_{2^m}(\mathbb{C}) \quad \textit{and} \quad C(2m+1) \cong C(2m)\otimes C(1) \cong M_{2^m}(\mathbb{C})\oplus M_{2^m}(\mathbb{C})$$

as complex algebras, while

$$C(2m) \cong \operatorname{End}(\Lambda(\mathbb{C}^m)) \quad \textit{and} \quad C(2m+1) \cong C(2m)\otimes C(1)$$

as complex superalgebras, where $C(1)$ is the superalgebra $\mathbb{C}[u]/(u^2-1)$ with an even generator u.

---

[2] For instance, we can start with any basis and orthogonalize it using the Gram–Schmidt algorithm.

This result on the structure of complex Clifford algebras immediately implies:

Corollary 2.2.4. *As ungraded algebras, $C(2m)$ has a unique irreducible module up to isomorphism (which has dimension 2^m) and $C(2m+1)$ has exactly two irreducible modules up to isomorphism (each of which has dimension 2^m, as well).*

Corollary 2.2.5. *The center of $C(2m)$ as an ungraded algebra and the supercenter of $C(2m)$ or $C(2m+1)$ as superalgebras are one-dimensional. The center of $C(2m+1)$ is two-dimensional.*

Proof. The (super)center of a full endomorphism algebra of a (super)space is one-dimensional, which proves the assertions on $C(2m)$. It also implies that the (super)center of $C(2m+1)$ is the (super)center of $C(1)$. $C(1) = \mathbb{C}[u]/(u^2-1)$ is a commutative algebra, hence, the center of $C(2m+1)$ is two-dimensional. However, the homogeneous generator u cannot be in the supercenter, as $[u,u] = 2u^2 = 2 \neq 0$. Hence, the supercenter of $C(2m+1)$ is one-dimensional. □

2.2.1 Pin group

What follows can also be found in [Mei13, 3.1.1, 3.1.2].

As in any ring, we can consider the group of units $C^\times$ of the algebra C. The units in the zeroth filtered subspace C_0 can be identified with the set $\mathbb{F}^\times$. An element $v \in V \subset C_1$ is a unit in C if and only if $v^2 = \langle v, v\rangle$ is not 0.

Let us define the *non-isotropic vectors* in V with respect to $\langle\cdot,\cdot\rangle$,

$$V^\times = \{v \in V \mid \langle v, v\rangle \neq 0\}\ , \tag{2.2.5}$$

and let $\Gamma = \Gamma(V, \langle\cdot,\cdot\rangle)$, the *Clifford group*, be the subgroup of $C^\times$ generated by non-zero scalars and non-isotropic vectors. As a set,

$$\Gamma = \Gamma(V, \langle\cdot,\cdot\rangle) = \{rv_1 \dots v_m \in C \mid r \in \mathbb{F}^\times, m \geq 0, v_1, \dots, v_m \in V^\times\}\ . \tag{2.2.6}$$

As a subgroup of the Clifford group, we define the *pin group* as

$$\mathsf{Pin} = \mathsf{Pin}(V, \langle\cdot,\cdot\rangle) = \{rv_1 \dots v_m \in \Gamma(V, \langle\cdot,\cdot\rangle) \mid r^2\langle v_1, v_1\rangle \dots \langle v_m, v_m\rangle = 1\}\ . \tag{2.2.7}$$

In particular, if $\mathbb{F}$ contains a square root of $\langle v,v\rangle$ for all $v \in V$, for instance, if $\mathbb{F}$ is algebraically closed or if $\mathbb{F} = \mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is an inner product, then we can normalize every vector in V with respect to $\langle\cdot,\cdot\rangle$, and for any $v \in V$ with $\langle v,v\rangle = 1$, we obtain an element $v \in C^\times$ which is its own inverse. In this situation, the pin group equals $\{\pm 1\}$ if $V = 0$, or otherwise

$$\mathsf{Pin} = \{v_1 \dots v_m \in C \mid v_1, \dots, v_m \in V, \langle v_1, v_1\rangle = \cdots = \langle v_m, v_m\rangle = 1\} \ . \tag{2.2.8}$$

Now any element of Pin comes with a conjugation automorphism of the superalgebra C (see Section 2.1). We want to make this more explicit. For any vector $u \in V^\times$, we define the *reflection* τ_u to be the linear transformation of V given by

$$\tau_u(v) = v - 2u\langle u,v\rangle/\langle u,u\rangle \quad \text{for all } v \in V \ . \tag{2.2.9}$$

Then τ_v^2 is the identity transformation, in particular, τ_u is an invertible linear transformation of V.

Lemma 2.2.6. *Consider any vector $u \in V^\times$ as an invertible element in C. Then the conjugation automorphism ρ_u leaves the subspace $V \subset C$ invariant. More precisely, ρ_u is the unique automorphism ρ of C satisfying $\rho(v) = \tau_u(v)$ for all $v \in V$.*

Proof. The inverse element of u in C is given by $u/\langle u,u\rangle$. Now pick any $v \in V$. As u, v are odd elements in the superalgebra C,

$$\rho_u(v) = \langle u,u\rangle^{-1}(u \otimes u^{\mathrm{op}}) \cdot v = -uvu/\langle u,u\rangle = v - 2u\langle v,u\rangle/\langle u,u\rangle = \tau_u(v) \ .$$

Also, V generates the Clifford algebra C, so any automorphism is determined by its action on V. □

We observe that this yields a group homomorphism from Pin to $\mathsf{GL}(V)$. To describe its image, we define $\mathsf{O} = \mathsf{O}(V, \langle\cdot,\cdot\rangle)$ as the group of invertible linear transformations $\mathsf{GL}(V)$ formed by those transformations $A \in \mathsf{GL}(V)$ which preserve $\langle\cdot,\cdot\rangle$:

$$\mathsf{O} = \mathsf{O}(V, \langle\cdot,\cdot\rangle) = \{A \in \mathsf{GL}(V) \mid \langle Av_1, Av_2\rangle = \langle v_1, v_2\rangle \,\forall v_1, v_2 \in V\} \ . \tag{2.2.10}$$

We call this the *orthogonal group*.

For any $u \in V^\times$, the reflection τ_u is an element in O, since for $v, w \in V$,

$$\langle \tau_u(v), w \rangle = \langle v, w \rangle - 2\langle u, v \rangle \langle u, w \rangle / \langle u, u \rangle = \langle v, \tau_u(w) \rangle \ .$$

As $\tau_u^2 = \mathrm{id}$, this proves $\langle \tau_u(v), \tau_u(w) \rangle = \langle v, w \rangle$.

But even more is true:

Theorem 2.2.7 (Cartan-Dieudonné, [Mei13, Thm. 1.1])**.** *If $\langle \cdot, \cdot \rangle$ is non-degenerate, then the group O is generated by reflections.*

Remark 2.2.8 (Historical remark)*.* The more precise fact that every orthogonal transformation of an n-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ is a product of at most n reflections was proved by Élie Cartan in 1937 [Car38a, Car38b]. This result was generalized to arbitrary fields of characteristic not 2 by Jean Dieudonné [Die48].

Lemma 2.2.9. *If $\langle \cdot, \cdot \rangle$ is non-degenerate and $\mathbb{F}$ contains a square root of $\langle v, v \rangle$ for all $v \in V$, then the conjugation action of Pin on $V \subset C$ defines a surjective group homomorphism $\phi\colon \mathsf{Pin} \to \mathsf{O}$ with kernel $\{\pm 1\} \subset C$. More explicitly, ϕ is determined by $\phi(u) = \tau_u$ for all $u \in V$ with $\langle u, u \rangle = 1$.*

Proof. The previous lemma immediately implies that the conjugation action leaves V invariant, and that this action defines a group homomorphism from Pin to O on a set of generators by $u \mapsto \tau_u$ for all $u \in V$ with $\langle u, u \rangle = 1$. Since $\mathbb{F}$ contains a square-root of $\langle v, v \rangle$ for any $v \in V$, we can normalize any vector in V. So by the Cartan–Dieudonné theorem, the image of ϕ contains a set of generators of O, so ϕ is surjective.

An element c in Pin is mapped to the identity in O if and only if

$$(c \otimes (c^{-1})^{\mathrm{op}}) \cdot v = v \quad \text{for all } v \in V \ ,$$

that is, c supercommutes with each such element v. As such elements generate the superalgebra C, this means c lies in the supercenter, so by Corollary 2.2.5, $c \in \mathbb{F}$. Since c lies in Pin, $c^2 = 1$, so $c = \pm 1$, and both of these elements are indeed mapped to the identity transformation by ϕ. $\square$

2.2.2 Pin Lie algebra

Although we use some slightly different notations, most of what follows can be found in [Mei13, 2.2.10].

Recall the definition of the quantization map q (Equation (2.2.4)). Let us define the subspace

$$\mathfrak{pin} = \mathfrak{pin}(V, \langle\cdot,\cdot\rangle) = q(V \wedge V) \subset C_2 \cap C^{\text{even}} \subset C \ , \tag{2.2.11}$$

that is, the space spanned by elements of C of the form

$$q(v \wedge w) = \tfrac{1}{2}(vw - wv) = vw - \langle v, w\rangle \quad \text{for all } v, w \in V \ . \tag{2.2.12}$$

Let us consider linear transformations of V of the form

$$A_{v,w} \colon V \to V \ , \quad A_{v,w}(w) = -2\langle v, w\rangle w + 2\langle w, w\rangle v \ , \tag{2.2.13}$$

for vectors $v, w \in V$.

Lemma 2.2.10. *For any $v, w \in V$, the derivation $\delta_{q(v\wedge w)}$ of the superalgebra C leaves the subspace $V \subset C$ invariant. More precisely, it is the unique even derivation δ of C satisfying $\delta(v') = A_{v,w}(v')$ for all $v' \in V$.*

Proof. For any $v' \in V$,

$$d_{q(v\wedge w)}(v') = [vw, v'] = vwv' - v'vw = -2\langle v, v'\rangle w + 2\langle w, v'\rangle v = A_{v,w}(v') \ .$$

Since $q(V \wedge V)$ lies in the even part of C, $\delta_{q(V\wedge V)}$ is an even derivation and as such uniquely determined by the images of elements of V, which generate the algebra C. □

This implies that:

Lemma 2.2.11. $\mathfrak{pin}$ *is a Lie algebra with the commutator (which equals the supercommutator for elements in* $\mathfrak{pin}$*).*

Proof. Consider $v, w, v', w' \in V$ and let $A = A_{v,w}$. Then

$$\begin{aligned}[q(v \wedge w), q(v' \wedge w')] &= \delta_{q(v\wedge w)}(\tfrac{1}{2}v'w' - \tfrac{1}{2}v'w') \\ &= \tfrac{1}{2}(A(v')w' + v'A(w') - A(w')v' - w'A(v')) \\ &= q((A(v') \wedge w' + v' \wedge A(w'))\end{aligned}$$

The commutator equals the supercommutator, since $\mathfrak{pin}$ is a purely even subspace of C. $\square$

In order to describe this Lie algebra more accurately, we define the Lie algebra of skew-adjoint operators $\mathfrak{so}$ on V with respect to $\langle\cdot,\cdot\rangle$,

$$\mathfrak{so} = \mathfrak{so}(V, \langle\cdot,\cdot\rangle) = \{A \in \mathrm{End}(V) \mid \langle Av, w\rangle = -\langle v, Aw\rangle\ \forall v, w \in V\}\ , \tag{2.2.14}$$

a Lie subalgebra of $\mathfrak{gl}(V)$. We can verify that for all $v, w, v', w' \in V$,

$$\langle A_{v,w}(v'), w'\rangle = -\langle v, v'\rangle\langle w, w'\rangle + \langle w, v'\rangle\langle v, w'\rangle = -\langle v', A_{v,w}(w')\rangle\ ,$$

hence, $A_{v,w} \in \mathfrak{so}$.

As we have seen in 2.1.2, the derivation action preserves the (super)commutator, so we obtain an action of the Lie algebra $\mathfrak{pin}$ on V which, in fact, yields a Lie algebra map $\mathfrak{pin} \to \mathfrak{so}$. But even more is true:

Proposition 2.2.12. *The derivation action of $\mathfrak{pin}$ on C restricted to V defines a Lie algebra homomorphism $\phi\colon \mathfrak{pin} \to \mathfrak{so}$ and $\phi(q(v \wedge w)) = A_{v,w}$. If $\langle\cdot,\cdot\rangle$ is non-degenerate, then $\phi\colon \mathfrak{pin} \to \mathfrak{so}$ is an isomorphism.*

Proof. It only remains to prove that ϕ is an isomorphism if $\langle\cdot,\cdot\rangle$ is non-degenerate. But in this case, we can choose a pair of dual bases $(v_i)_i$ and $(w_i)_i$ of V with respect to $\langle\cdot,\cdot\rangle$, and we can check directly that

$$\mathfrak{so} \to \mathfrak{pin}\ , \quad A \mapsto q(\sum_i A(v_i) \wedge w_i)\ ,$$

is an inverse map for ϕ. $\square$

Remark 2.2.13. So both Pin and $\mathfrak{pin}$ act on C leaving V invariant, the first by conjugation (in the sense of superalgebras) and the second by taking the commutator. Considering the restricted actions on V, we obtain a group homomorphism $\phi\colon \mathsf{Pin} \to \mathsf{O}$ and a Lie algebra morphism $\phi\colon \mathfrak{pin} \to \mathfrak{so}$. If $\langle\cdot,\cdot\rangle$ is non-degenerate and $\mathbb{F}$ contains a square root of $\langle v, v\rangle$ for all $v \in V$, then the first is surjective with kernel $\{\pm 1\}$ and the second is an isomorphism.

To complete the picture , have the following result:

Lemma 2.2.14. Pin *acts on* $\mathfrak{pin}$ *by the conjugate action, and this action is compatible with the maps* ϕ *in the sense that*

$$\phi((g\otimes(g^{-1})^{\mathrm{op}})\cdot x)v' = \phi(g)(\phi(x)(\phi(g)^{-1}v')) \quad \textit{for all } g \in \mathsf{Pin}, x \in \mathfrak{pin}, v' \in V \ . \quad (2.2.15)$$

Proof. Let us consider $u, v, w \in V$ such that $\langle u, u\rangle = 1$. If we view u as an element in Pin, then

$$\begin{aligned}(u \otimes (u^{-1})^{\mathrm{op}}) \cdot q(v \wedge w) &= (u \otimes u^{\mathrm{op}}) \cdot (2vw - \langle v, w\rangle) \\ &= 2\tau_u(v)\tau_u(w) - \langle \tau_u(v), \tau_u(w)\rangle = q(\tau_u(v) \wedge \tau_u(w)) \ .\end{aligned}$$

As elements u as the one considered generate Pin and as elements $q(v \wedge w)$ as the one considered span $\mathfrak{pin}$, this shows that Pin acts on $\mathfrak{pin}$.

Applying ϕ to the the result of our previous computation and acting on a vector $v' \in V$ yields

$$\begin{aligned}A_{\tau_u(v),\tau_u(w)}(v') &= -2\langle \tau_u(v), v'\rangle\tau_u(w) + 2\langle \tau_u(w), v'\rangle\tau_u(v) \\ &= \tau_u(-2\langle v, \tau_u^{-1}(v')\rangle w + 2\langle w, \tau_u^{-1}(v')\rangle v) \\ &= \phi(u)(\phi(q(u \wedge w))(\phi(u)^{-1}v')) \ ,\end{aligned}$$

as desired, and as above, this proves the desired compatibility for all of Pin and $\mathfrak{pin}$. □

2.3 Central simple (super)algebras and the Skolem–Noether theorem

All algebras in this section will be over an arbitrary field $\mathbb{F}$. Central simple superalgebras were first studied by Wall [Wal64], see also Deligne's article [Del99]. We will follow Varadarajan's book [Var04].

Definition 2.3.1. An (ungraded) algebra A is *central simple* if A is finite-dimensional, the center of A is $\mathbb{F}$, and A has no non-zero proper two-sided ideals.

Central simple algebras have various characterizations.

Proposition 2.3.2 ([Var04, Prop. 6.2.1])**.** *Let A be a finite-dimensional $\mathbb{F}$-algebra. The following are equivalent:*

- *A is a central simple algebra*

- $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A), a \otimes b^{\mathrm{op}} \mapsto (x \mapsto (a \otimes b^{\mathrm{op}}) \cdot x)$, *is an isomorphism*

- *A is semisimple and its center is $\mathbb{F}$*

Example 2.3.3. For $n \geq 1$, the complex Clifford algebra $C(n)$ is central simple if and only if n is even by Proposition 2.2.3 and Corollary 2.2.5.

An important result for central simple algebras is the following.

Theorem 2.3.4 (Skolem–Noether)**.** *Let A be a simple algebra, B a central simple algebra, and $f, g \colon A \to B$ algebra maps. Then f, g are conjugate, that is, there is an invertible element $b \in B$ such that $f(a) = bg(a)b^{-1}$ for all $a \in A$.*

An analogous notion exists also for superalgebras and this is what will be most important for us.

Definition 2.3.5. A superalgebra A is *central simple* if A is finite-dimensional, the supercenter of A is $\mathbb{F}$, and A has no non-zero proper two-sided graded ideals.

Recall from Section 2.1 that for a superalgebra A, $\mathrm{End}(A)$ denotes the space of all linear endomorphisms of A (not just the graded ones). Also recall that we have an action of $A \otimes A^{\mathrm{op}}$ on A for any superalgebra A, where the action of A^{op} is the right multiplication in A after applying the braiding in the category of superspaces.

Proposition 2.3.6 ([Var04, Thm. 6.2.5])**.** *Let A be a finite-dimensional $\mathbb{F}$-superalgebra. The following are equivalent:*

- *A is a central simple superalgebra*

- $A \otimes A^{\mathrm{op}} \to \mathrm{End}(A), a \otimes b^{\mathrm{op}} \mapsto (x \mapsto (a \otimes b^{\mathrm{op}}) \cdot x)$, *is an isomorphism*

- *A is a semisimple superalgebra and its supercenter is $\mathbb{F}$*

- *A is semisimple as an ungraded algebra and its supercenter is $\mathbb{F}$*

Remark 2.3.7. Yet another characterization of central simple (super)algebras is as matrix (super)algebras over (super) division algebras, see [Var04, 6.2]. We will not use this characterization here.

Example 2.3.8. For all $n \geq 1$, the complex Clifford algebra $C(n)$ is a central simple superalgebra by Proposition 2.2.3 and Corollary 2.2.5. In fact, the Clifford algebra over any field of characteristic not 2 with respect to a non-degenerate symmetric bilinear form is a central simple superalgebra by [Wal64, Thm. 4].

The following generalization of the classical Skolem–Noether theorem to superalgebras is almost covered by [Jab10, Thm. 3.2] or [CVOZ97, Cor. 3.6]. We will prove it in Lemma 4.4.10, because we will make use of it for our purposes.

Theorem 2.3.9 (Skolem–Noether)**.** *Let A be a simple superalgebra, B a central simple superalgebra, and $f, g\colon A \to B$ graded algebra maps. Then f, g are conjugate, that is, there is an invertible element $b \in B$ such that $f(a) = (b \otimes (b^{-1})^{\mathrm{op}}) \cdot g(a)$ for all $b \in B$.*

2.4 Hopf algebras

We will recall the definitions and basic properties of Hopf algebras. A good modern reference for the theory is Montgomery's book [Mon93].

Let us fix a base field $\mathbb{F}$ for all vector spaces and tensor products in this section. We can observe that a vector space A with maps $\eta\colon \mathbb{F} \to A$ and $\mu\colon A \otimes A \to A$ is a (unital associative) $\mathbb{F}$-algebra if and only if the following diagrams are commutative:

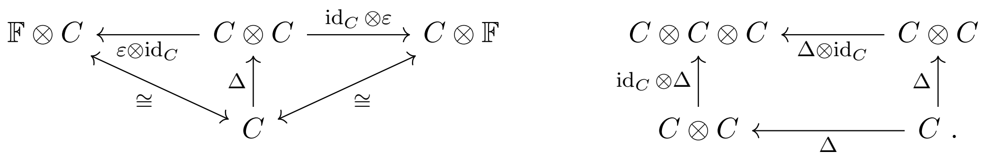
$$\begin{array}{ccccc} \mathbb{F} \otimes A & \xrightarrow{\eta \otimes \mathrm{id}_A} & A \otimes A & \xleftarrow{\mathrm{id}_A \otimes \eta} & A \otimes \mathbb{F} \\ & {\scriptstyle \cong} \searrow\!\!\nwarrow & \big\downarrow \mu & \nearrow\!\!\swarrow {\scriptstyle \cong} & \\ & & A & & \end{array} \qquad \begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathrm{id}_A} & A \otimes A \\ \big\downarrow \mathrm{id}_A \otimes \mu & & \big\downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array} .$$

More precisely, the image of 1 in $\mathbb{F}$ under the map η corresponds to the multiplicative identity element in A and the second diagram corresponds to the associativity of the multiplication map μ. Dualizing these diagrams defines the notation "coalgebra"[3]: A vector space C with maps $\varepsilon\colon C \to \mathbb{F}$ and $\Delta\colon C \otimes C \to C$ is a *coalgebra* if the following

---

[3] A mathematical folklore quotation, widely attributed to Paul Erdős, but originally due to Alfréd Rényi is: "A mathematician is a machine for turning coffee into theorems." It has the less known variation "A comathematician is a machine for turning theorems into ffee," which was communicated to the author by Fei Qi.

diagrams are commutative:



The commutativity of the second diagram is called *coassociativity*, the maps ε and Δ are called *counit* and *coproduct* (or *comultiplication*), respectively. Note that these diagrams allow the definition of algebras and coalgebras not just for vector spaces as base objects, which is the case we are most interested in, but in any monoidal category. It is convenient to use *Sweedler's notation* for images of the coproduct map: for $c \in C$, we can represent $\Delta(C) \in C \otimes C$ as a finite sum $\sum_i c_{1,i} \otimes c_{2,i}$ with suitable elements $(c_{1,i})_i$ and $(c_{2,i})_i$ from C, but we will often omit the summation sign and the index i and write

$$\Delta(c) = c_{(1)} \otimes c_{(2)} \ ,$$

instead, where a finite sum and an additional index is implied. Now the coassociativity axiom means that for any element c of a coalgebra and $n \geq 1$, there is a unique n-fold coproduct, which we denote by

$$\Delta^n(c) = c_{(1)} \otimes \ldots \otimes c_{(n+1)} \quad \text{in } C^{\otimes(n+1)} \ .$$

Using Sweedler's notation, the commutativity of the left diagram translates to the compatibility condition

$$\varepsilon(c_{(1)})c_{(2)} = c = c_{(1)}\varepsilon(c_{(2)}) \quad \text{for all } c \in C \ . \tag{2.4.1}$$

Although coalgebras are formally very similar ("dual") to algebras, they are distinguished from them by an implicit finiteness condition: any element of a coalgebra, and hence, any finite-dimensional subspace, is contained in a finite-dimensional subcoalgebra.

For vector spaces V, W let us define the map

$$\tau_{V,W} \colon V \otimes W \to W \otimes V \ , \quad v \otimes w \mapsto w \otimes v \quad \text{for all } v \in V, w \in V \ . \tag{2.4.2}$$

These maps can be regarded as the braiding in the tensor category of vector spaces. A coalgebra is called *cocommutative* if $\tau_{C,C} \circ \Delta = \Delta$.

If (C, ε, Δ) is a coalgebra and (A, η, μ) is an algebra, then the space of linear functions $\mathrm{Hom}(C, A)$ can be equipped with an algebra structure whose identity element is $(\eta \circ \varepsilon) \in \mathrm{Hom}(C, A)$ and whose product operation is defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta \quad \text{for all } f, g \in \mathrm{Hom}(C, A) \ . \tag{2.4.3}$$

Using elements,

$$(f \star g)(c) = f(c_{(1)})g(c_{(2)}) \quad \text{for all } f, g \in \mathrm{Hom}(C, A), c \in C \ . \tag{2.4.4}$$

This $\star$-product of $\mathrm{Hom}(C, A)$ is called *convolution product*. In particular, for any coalgebra C, its dual space $C^* = \mathrm{Hom}(C, \mathbb{F})$ is an algebra.

Analogs for many concepts from the theory of algebras exist for coalgebras. A coalgebra is called *simple* if it does not contain any non-zero proper subcoalgebra. With this definition, a coalgebra is simple if and only if its dual space is a simple algebra. A coalgebra is called *pointed*, if any simple subcoalgebra is one-dimensional. Over an algebraically closed field, any simple cocommutative coalgebra is one-dimensional (because the dual spaces of simple coalgebras are field extensions of the base field), so all cocommutative coalgebras are pointed.

For any algebra A, the tensor product space $A \otimes A$ can be turned into an algebra in a natural way using the twist map $\tau_{A,A}$ and similarly, the tensor products of a coalgebra with itself is a coalgebra in a natural way. Let us consider a vector space H which is an algebra with structure maps η, μ as above and at the same time a coalgebra with structure maps ε, Δ as above. Then η, μ are coalgebra maps if and only if ε, Δ are algebra maps, and if this is the case, we call H a *bialgebra*. For a bialgebra, $\mathrm{Hom}(H, H)$ is an algebra with the convolution product, and the identity map id_H is a distinguished element in it (though not the multiplicative identity, which is $\eta \circ \varepsilon$). H is a *Hopf algebra* if id_H is an invertible element in this algebra, i.e., if there is an endomorphism $S \in \mathrm{Hom}(H, H)$ such that

$$S \star \mathrm{id}_H = \eta \circ \varepsilon = \mathrm{id}_H \star S \tag{2.4.5}$$

or, equivalently, if

$$S(h_{(1)})h_{(2)} = \varepsilon(h) = h_{(1)}S(h_{(2)}) \quad \text{for all } h \in H \ . \tag{2.4.6}$$

The map S is called *antipode*, it is unique if it exists, and it is an anti-algebra and anti-coalgebra map, that is,

$$S \circ \mu = \mu \circ (S \otimes S) \circ \tau_{H,H} \quad \text{and} \quad \Delta \circ S = \tau_{H,H} \circ (S \otimes S) \circ \Delta \ . \tag{2.4.7}$$

For a commutative or cocommutative Hopf algebra, $S^2 = \mathrm{id}_H$.

An element c of a coalgebra is called *group-like* if $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$. An element h of a bialgebra is called *primitive* if $\Delta(h) = h \otimes 1 + 1 \otimes h$ and $\varepsilon(h) = 0$. If h is a group-like element in a Hopf algebra, then h is invertible with respect to the multiplication in H and $S(h) = h^{-1}$, so the set of group-like elements of a Hopf algebra is a subgroup of its multiplicative group $H^\times$, which we denote by $G(H)$. On the other hand, if h is primitive, then $S(h) = -h$ and the set of primitive elements is a Lie subalgebra of H with the commutator in H, which we denote by $P(H)$. In fact, the subsets $G(H)$ and $P(H)$ generate Hopf subalgebras, as follows.

Example 2.4.1. Let G be a group with identity element e. Then the vector space $\mathbb{F}[G]$ with basis $\{g\}_{g \in G}$ is a Hopf algebra with all structure maps defined as linear maps by letting

$$\eta(1) = e \ , \quad \mu(g \otimes h) = gh \ , \quad \varepsilon(g) = 1 \ , \quad \Delta(g) = g \otimes g \ , \quad S(g) = g^{-1}$$

for all $g, h \in G$. Note, in particular, that all elements from G are defined to be group-likes. This Hopf algebra is called *group algebra*. Any Hopf algebra H has a distinguished Hopf subalgebra $\mathbb{F}[G(H)]$.

Example 2.4.2. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. Then the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra if all elements from $\mathfrak{g}$ are defined to be primitives

$$\varepsilon(x) = 0 \ , \quad \Delta(x) = x \otimes 1 + 1 \otimes x \ , \quad S(x) = -x$$

for all $x \in \mathfrak{g}$. Any Hopf algebra has a distinguished Hopf subalgebra $\mathcal{U}(P(H))$.

Both group algebras and universal enveloping algebras are examples of cocommutative Hopf algebras and for both algebras, $S^2 = \mathrm{id}_H$.

Example 2.4.3. Let G be a finite group with identity element e, then the dual space $\mathbb{F}[G]^*$ of the group algebra is an algebra with the convolution product, and it is a Hopf algebra with the additional structure maps defined by

$$\varepsilon(f) = f(e)\ , \quad \Delta(f)(g \otimes g') = f(gg')\ , \quad S(f)(g) = f(g^{-1})$$

for all $f \in \mathbb{F}[G]^*, g, g' \in G$. It is a commutative algebra, but cocommutative as a coalgebra if and only if G is abelian.

Example 2.4.4 (Taft Hopf algebra). For $n \geq 2$, let q be a primitive n-th root of unity in $\mathbb{F}$, and let H be the $\mathbb{F}$-algebra generated by two elements g and x subject to the relations

$$g^n = 1\ , \quad x^n = 0\ , \quad gxg^{-1} = qx\ .$$

This is a Hopf algebra with the additional structure maps defined by

$$\varepsilon(g) = 1\ , \quad \Delta(g) = g \otimes g\ , \quad S(g) = g^{n-1}\ ,$$
$$\varepsilon(x) = 0\ , \quad \Delta(x) = g \otimes x + x \otimes 1\ , \quad S(x) = -g^{n-1}x\ .$$

The Taft Hopf algebra is neither commutative nor cocommutative.

2.4.1 Super Hopf algebras

We have mentioned that algebras and coalgebras can be defined in any monoidal category. In order find good analogs for bialgebras, Hopf algebras, and cocommutative coalgebras, we should require the monoidal category to be braided, i.e., there should be a braiding map $c_{V,W} \colon V \otimes W \to W \otimes V$ satisfying certain compatibility axioms. Having such a braiding, we can explain the algebra structure on the tensor products of two algebras, or the coalgebra structure on the tensor product of two coalgebras. For the category of vector spaces, the braiding is just τ. As we have seen earlier, for the category of super vector spaces, a braiding is defined by

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$$

for $\mathbb{Z}_2$-homogeneous elements v, w in super vector spaces V, W (see Section 2.1).

To make this more concrete, a *supercoalgebra* is a coalgebra C whose underlying vector space is a superspace, say with even subspace C_0 and odd subspace C_1, such that for $i = 0, 1$,

$$\Delta(C_i) \subset \bigoplus_{k+l=i} C_k \otimes C_l \ .$$

A *superbialgebra* is superalgebra which is also a supercoalgebra such that its coalgebra structure maps ε, Δ are algebra maps, or equivalently, such that its algebra structure maps η, μ are coalgebra maps. Note that a superbialgebra H is not necessarily an ungraded bialgebra if we forget the $\mathbb{Z}_2$-grading, because, for instance, the algebra structure of $H \otimes H$ depends on the braiding in the respective category, so the notion of the coproduct being an algebra maps depend on this braiding, too. However, we can identify purely even superbialgebras with ungraded bialgebras, because for purely even spaces, the braidings agree.

Finally, a *Hopf superalgebra* is a superbialgebra with an antipode, i.e., a $\star$-inverse of the identity map. Just as for bialgebras, Hopf superalgebras do not become ungraded Hopf algebras if we forget the grading, but purely even Hopf superalgebras can be identified with ungraded Hopf algebras.

A superalgebra A with multiplication map μ is called *supercommutative*, if $\mu \circ c_{A,A} = \mu$, and a coalgebra C with coproduct Δ is called *supercocommutative* if $c_{C,C} \circ \Delta = \Delta$.

2.5 Hopf algebra actions and smash products

Group algebras and universal enveloping algebras have their origins in representation theory, and in fact, the notion of a Hopf algebra has a more representation theoretic interpretation, as well: if H is a Hopf algebra with modules V, W, then the base field $\mathbb{F}$, the dual space V^*, and the tensor product $V \otimes W$ all have a natural structure as H-modules defined by

$$h \cdot 1 = \varepsilon(h) \ , \quad (h \cdot f)(v) = f(S(h) \cdot v) \ , \quad h \cdot (v \otimes w) = (h_{(1)} \cdot v) \otimes (h_2 \cdot w) \qquad (2.5.1)$$

for all $h \in H, f \in V^*, v, w \in V$. This makes the category of H-modules a tensor category itself.

Example 2.5.1. As an important consequence, we record that for any bialgebra H and an H-module V, the tensor algebra is an H-module by

$$h \cdot 1 = \varepsilon(h) \quad \text{and} \quad h \cdot (v_1 \dots v_n) = (h_{(1)} \cdot v_1) \dots (h_{(n)} \cdot v_n)$$

for all $h \in H, v_1, \dots, v_n \in V$. If H is cocommutative, then the sets of elements of the form $vw \pm wv$ in $T(V)$ are stable under this action of H for both choices of the sign $\pm$. Hence in this case, the symmetric and the exterior algebra of V are H-modules, as well.

Many representation theoretic notations carry over to Hopf algebras. Let H be a Hopf algebra. An element v in an H-module V is called *H-invariant* if

$$h \cdot v = \varepsilon(h)v \quad \text{for all } h \in H \ . \tag{2.5.2}$$

The space of H-invariants is denoted by V^H. A map f between two H-modules V, W is *H-equivariant* if

$$f(h \cdot v) = h \cdot f(v) \quad \text{for all } h \in H, v \in V \ . \tag{2.5.3}$$

Identifying $\mathrm{Hom}(V, W)$ with $V^* \otimes W$, a map $f \in \mathrm{Hom}(V, W)$ is H-equivariant if and only if it is an H-invariant element in $V^* \otimes W$.

We consider more examples of Hopf algebra modules:

Example 2.5.2. Let $\mathfrak{g}$ be a Lie algebra and let H be a cocommutative bialgebra acting on $\mathfrak{g}$ such that the Lie bracket as a map $\mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ is H-equivariant. Then $\mathcal{U}(\mathfrak{g})$ is an H-module with the H-module structure it inherits from $T(\mathfrak{g})$.

Example 2.5.3. If G is a group, then the representations of $\mathbb{F}[G]$ are exactly the $\mathbb{F}$-linear representations of G. If $\mathfrak{g}$ is a Lie algebra, then the representations of $\mathcal{U}(\mathfrak{g})$ are exactly the representations of $\mathfrak{g}$.

Example 2.5.4. There is another important Hopf algebra action. Let K be an algebra with a subalgebra H which is a Hopf algebra. Then H acts on K according to the formula

$$h \cdot k = h_{(1)} k S(h_{(2)}) \quad \text{for all } h \in H, k \in K \ . \tag{2.5.4}$$

This is called the *adjoint action*. If $g, x \in H$ are a group-like and a primitive element in H, respectively, then for all $k \in K$,

$$g \cdot k = gkg^{-1} \quad \text{and} \quad x \cdot k = [x, k] \ .$$

Example 2.5.5. In partiular, we can take $K = H$ to obtain the adjoint action of H on itself. Then for any group-like element g in H and any primitive element x in H, $g \cdot x = gxg^{-1}$ is a primitive element in H, because the coproduct is an algebra map. That is, for any Hopf algebra H, the group $G(H)$ acts on the space $P(H)$, and what is more, the commutator in H is a $\mathbb{F}[G(H)]$-equivariant Lie bracket for $P(H)$. Hence, $\mathcal{U}(P(H))$ is an $\mathbb{F}[G]$-module algebra for any Hopf algebra H.

Almost all of these examples of bialgebra actions have in common that the module in each case is itself an algebra. Let H be a bialgebra and let A be an algebra which is an H-module. A is called *H-module algebra* if

$$h \cdot 1 = \varepsilon(1) \quad \text{and} \quad h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b) \quad \text{for all } h \in H, a, b \in A \; .$$

All of the above mentioned bialgebra actions on algebras make the latter indeed H-module algebras, for the respective bialgebra H: $T(V)$, $S(V)$ and $\Lambda(V)$ for a cocommutative bialgebra acting on a module V, $\mathcal{U}(\mathfrak{g})$ for a cocommutative bialgebra acting on $\mathfrak{g}$ as described above, or an algebra which is a module for a contained Hopf algebra via the adjoint action.

Let H be a Hopf algebra with an H-module algebra A. Then we define the *smash product* (or *semidirect product*) $A \rtimes H$ as the algebra generated by H and A with the additional relation

$$h_{(1)} a S(h_{(2)}) = h \cdot a \quad \text{for all } h \in H, a \in A \; , \tag{2.5.5}$$

that is, we identify the adjoint action in the algebra generated by H and A with the given action. More explicitly, $A \rtimes H$ is the tensor product $A \otimes H$ as a vector space with a multiplication operation given by

$$(a \otimes h)(a' \otimes h') = a(h_{(1)} \cdot a') \otimes h_{(2)} h' \quad \text{for all } a, a' \in A, h, h' \in H \; . \tag{2.5.6}$$

In particular, we can construct the smash products $T(V) \rtimes H$ for any Hopf algebra H with a module V, or the smash products $S(V) \rtimes H$ and $\Lambda(V) \rtimes H$ for a cocommutative Hopf algebra with a module V.

Furthermore, for any Hopf algebra H, we can construct the smash product $\mathcal{U}(P(H)) \rtimes \mathbb{F}[G(H)]$, where the action of $G(H)$ on $\mathcal{U}(P(H))$ comes from the adjoint action of

$G(H)$ on $P(H)$ in H as described above. In fact, such smash products form an important class of Hopf algebras as described by the following result, which is usually called Cartier–Kostant theorem, Cartier–Kostant–Milner–Moore theorem, or Cartier–Gabriel–Kostant–Milner–Moore theorem.

Theorem 2.5.6 ([Mon93, Cor. 5.6.4, (3) and Thm. 5.6.5])**.** *If H is a pointed cocommutative Hopf algebra over a field of characteristic 0, for instance, if H is any cocommutative Hopf algebra over an algebraically closed field of characteristic 0, then it is isomorphic to $\mathcal{U}(P(H)) \rtimes \mathbb{F}[G(H)]$.*

2.5.1 Orthogonal modules

Let $\langle\cdot,\cdot\rangle\colon V \otimes V \to \mathbb{F}$ be a bilinear form on an H-module V.

Lemma 2.5.7. *We consider the following conditions:*

1. $\langle h_{(1)} \cdot v, h_{(2)} \cdot w\rangle = \varepsilon(h)\langle v, w\rangle$ *for all* $h \in H, v, w \in V$.
2. $\langle h \cdot v, w\rangle = \langle v, S(h) \cdot w\rangle$ *for all* $h \in H, v, w \in V$.
3. *Group-like elements of H act as orthogonal linear operators on V and primitive elements of H act as skew-adjoint linear operators on V with respect to $\langle\cdot,\cdot\rangle$.*

Then $(1) \Leftrightarrow (2) \Rightarrow (3)$ *and if H is generated by group-likes and primitives (e.g., H is cocommutative over an algebraically closed field of characteristic 0), then even* $(1) \Leftrightarrow (2) \Leftrightarrow (3)$.

Proof. Let h be an element in H and let v, w be elements in V. If we assume (1) holds, then

$$\langle h \cdot v, w\rangle = \langle h_{(1)} \cdot v, h_{(2)} \cdot (S(h_{(3)}) \cdot w)\rangle = \varepsilon(h_{(1)})\langle v, S(h_{(2)}) \cdot w\rangle = \langle v, S(h) \cdot w\rangle \ ,$$

so (2) holds. If we assume that (2) holds, then

$$\langle h_{(1)} \cdot v, h_{(2)} \cdot w\rangle = \langle v, S(h_{(1)}) \cdot (h_{(2)} \cdot w)\rangle = \varepsilon(h)\langle v, w\rangle \ ,$$

which proves (1). So (1) and (2) are equivalent.

(3) follows from (2), because $S(h) = h^{-1}$ for a group-like element $h \in H$ and $S(h) = -h$ for a primitive element $h \in H$, respectively.

(2) follows from (3) if H is generated as algebra by group-likes and primitives, because S is an anti-algebra map. □

Definition 2.5.8. In the situation of the lemma, the bilinear form $\langle \cdot, \cdot \rangle$ on V is called *H-invariant* if it satisfies any of the equivalent conditions (1), (2) considered in the lemma.

Definition 2.5.9. An H-module V of H is called *orthogonal* if V is finite-dimensional and V has an H-invariant non-degenerate symmetric bilinear form.

Lemma 2.5.10. *Let V be an orthogonal H-module. Then $V \cong V^*$.*

Proof. If $\langle \cdot, \cdot \rangle$ is an H-invariant non-degenerate symmetric bilinear form on V, then $v \mapsto \langle v, \cdot \rangle$ is an H-module isomorphism. □

Lemma 2.5.11. *Let H be a cocommutative Hopf algebra and let V be an orthogonal H-module with the bilinear form $\langle \cdot, \cdot \rangle$. Then the action of H on $T(V)$ induces an action of H on $C(V, \langle \cdot, \cdot \rangle)$.*

Proof. The defining ideal of the Clifford algebra is stable under the action of H on $T(V)$, since for all $h \in H, v, w \in V$,

$$h \cdot (vw - wv - 2\langle v, w \rangle) = (h_{(1)} \cdot v)(h_{(2)} \cdot w) - (h_{(2)} \cdot w)(h_{(1)} \cdot v) - 2\langle h_{(1)} \cdot v, h_{(2)} \cdot w \rangle \ .$$

□

2.6 PBW deformations of smash products

The theory of PBW deformations is an active field of interest in its own right, whose starting point is the Poincaré-Birkhoff-Witt (PBW) theorem for Lie algebras. If $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\cdot, \cdot]$ over any field $\mathbb{F}$, then we can pick an ordered $\mathbb{F}$-basis $(x_i)_i$ of $\mathfrak{g}$. We can construct the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ as the quotient algebra of the tensor algebra $T(\mathfrak{g})$ by the ideal I generated by element of the form

$(xy - yx - [x,y])$ for $x, y \in \mathfrak{g}$, i.e., the commutator is identified with the Lie bracket. Now the PBW theorem states that $\mathcal{U}(\mathfrak{g})$ has a basis consisting of ordered products of basis elements, meaning a basis consisting of equivalence classes of products in $T(\mathfrak{g})$ of the form $x_{i_1} \cdots x_{i_n}$ for $n \geq 0$ and $i_1 \leq \cdots \leq i_n$.

We can rephrase the PBW theorem as follows. We recall that $T(\mathfrak{g})$ is naturally $\mathbb{Z}$-graded, where elements in $\mathfrak{g}$ are assigned degree 1. If the Lie bracket is zero, then the ideal I is graded with respect to this $\mathbb{Z}$-grading and the $\mathbb{Z}$-graded quotient algebra is just the symmetric algebra $S(\mathfrak{g})$. However, if the Lie bracket is non-zero, then the ideal is not graded and $\mathcal{U}(\mathfrak{g})$ becomes a filtered algebra. So in general, $\mathcal{U}(\mathfrak{g})$ is a filtered algebra and the quotient map $T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ induces a surjective algebra map from $S(\mathfrak{g})$ to the associated graded algebra $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))$ of $\mathcal{U}(\mathfrak{g})$. The PBW theorem for Lie algebras now states that this map is, in fact, an isomorphism.

The classical PBW theorem has several generalizations of which the most general version relevant for our purposes concerns deformations of smash products between a Hopf algebra and a Koszul algebra. Let H be a Hopf algebra and let B be an H-Koszul algebra. By that we mean more explicitly that there is an H-module V, whose tensor algebra $T(V)$ thus is an H-module algebra, and an H-submodule $R \subset V \otimes V$, the "relations", such that the $\mathbb{Z}$-graded quotient H-module algebra $B = T(V)/(R)$ is Koszul. We define $A_0 = B \rtimes H$ and we want to study deformations of this smash product which are given by a deformation map $\kappa\colon R \to H \oplus (V \otimes H) \subset T(V) \rtimes H$ explicitly as

$$A_\kappa = (T(V) \rtimes H)/I_\kappa \ , \tag{2.6.1}$$

where I_κ is the ideal in $T(V) \rtimes H$ generated by elements of the form $(r - \kappa(r))$ for $r \in R$.

In this situation, again, $T(V)$ is a $\mathbb{Z}$-graded algebra and if $\kappa = 0$, then $I_\kappa = (R)$ is homogeneous and $A_\kappa = A_0$ is $\mathbb{Z}$-graded. In general, I_κ is not homogeneous, but A_κ is filtered and the quotient map $T(V) \rtimes H \to A_\kappa$ induces a surjective algebra map $A_0 \to \mathrm{gr}(A_\kappa)$.

Definition 2.6.1. We say that A_κ is a *PBW deformation* of A_0 if this map is an isomorphism.

To see how this generalizes the situation of the classical PBW theorem, we can take

$H = \mathbb{F}$, the base field, $V = \mathfrak{g}$, R the span of the elements $xy - yx$ for $x, y \in \mathfrak{g}$ and $\kappa\colon R \to H \oplus (V \otimes H)$ to be the Lie bracket of $\mathfrak{g}$.

In our general situation we have the following characterization of PBW deformations: Let us write κ as $\kappa^C + \kappa^L$ for maps $\kappa^C\colon R \to H$ and $\kappa^L\colon H \to V \otimes H$. Then

Theorem 2.6.2 ([WW14, Thm. 3.1])**.** *A_κ is a PBW deformation of A_0 if and only if κ is H-equivariant and*

- $\operatorname{im}(\kappa^L \otimes \mathrm{id} - \mathrm{id} \otimes \kappa^L) \subset R$
- $\kappa^L \circ (\kappa^L \otimes \mathrm{id} - \mathrm{id} \otimes \kappa^L) = -(\kappa^C \otimes \mathrm{id} - \mathrm{id} \otimes \kappa^C)$
- $\kappa^C \circ (\kappa^L \otimes \mathrm{id} - \mathrm{id} \otimes \kappa^L) = 0$

as maps on $(R \otimes V) \cap (V \otimes R)$ in $V \otimes V \otimes V$.

In particular, in the classical situation, for a Lie algebra $\mathfrak{g}$, we see that $\kappa^C = 0$ and $\kappa = \kappa^L$ yields a PBW deformation if and only if it satisfies the classical Jacobi identity

$$\kappa \circ (\kappa \otimes \mathrm{id}) = 0 \quad \text{on } (R \otimes V) \cap (V \otimes R)\ . \tag{2.6.2}$$

For our purposes, another special case will be of special relevance. Assume H is a cocommutative Hopf algebra with a module V and R is the space spanned by elements of the form $vw - wv$ for $v, w \in V$. As long as the characteristic of our base field $\mathbb{F}$ is not 2, we can identify R with $V \wedge V$. Then $T(V)/(R) \cong S(V)$, the symmetric algebra, and $A_0 = S(V) \rtimes H$. Let us also assume $\kappa^L = 0$, so deformation maps will be of the form $\kappa\colon V \wedge V \to H$. Then A_κ is a PBW deformation if and only if κ is H-equivariant, that is,

$$\kappa((h_{(1)} \cdot v) \wedge (h_{(2)} \cdot w)) = h \cdot \kappa(v \wedge w) \quad \text{for all } h \in H, v, w \in V \tag{2.6.3}$$

and it satisfies the Jacobi identity

$$[\kappa(u \wedge v), w] + [\kappa(v \wedge w), u] + [\kappa(w \wedge u), v] \quad \text{for all } u, v, w \in V \tag{2.6.4}$$

in $A_0 = S(V) \rtimes H$.

As we have seen in Section 2.4, the most important cocommutative Hopf algebras are group algebras, universal enveloping algebras of Lie algebras, and smash products of the former two. PBW deformations of the form $S(V) \rtimes H$ where H is the universal enveloping algebra of a Lie algebra contain deformations of universal enveloping algebras of a Lie algebra relative to the universal enveloping algebra of a Lie subalgebra and, more generally, continuous Hecke algebras as defined by Etingof–Gan–Ginzburg [EGG05], see Chapter 7.

If the Hopf algebra H is the group algebra of a finite group G over $\mathbb{C}$, and V is a finite-dimensional faithful G-module over $\mathbb{C}$, then the deformation maps $\kappa\colon V \wedge V \to \mathbb{C}[G]$ can be written as $\kappa = \sum_{g \in G} \kappa_g g$ with maps $\kappa_g\colon V \wedge V \to \mathbb{C}$ for all $g \in G$ and there is a characterization of those maps κ which yield a PBW deformation A_κ of $A_0 = S(V) \rtimes \mathbb{C}[G]$. For every $g \in G$, we have an operator $(g-1)$ acting on the module V with image $\operatorname{im}(g-1)$. Any $h \in G$ which commutes with G acts on $\operatorname{im}(g-1)$. Let S be the set of elements $g \in G$ for which $\dim \operatorname{im}(g-1) = 2$ and $\det(h|_{\operatorname{im}(g-1)}) = 1$ for all $h \in G$ which commute with g.

Theorem 2.6.3 ([Dri86],[RS03, Thm. 1.9])**.** *The maps κ yielding PBW deformations A_κ are exactly the maps of the form*

$$\kappa(v \wedge w) = \theta_1(v \wedge w) + \sum_{g \in S} \theta_g(v \wedge w) g \tag{2.6.5}$$

with $\theta_1 \in ((V \wedge V)^*)^G$ and $\theta_g \in (V \wedge V)^*$ for all $g \in S$ such that the kernel of θ_g contains the kernel of $(g-1)$ and the family θ_g is G-invariant.

This determines the family $(\theta_g)_{g \in S}$ up to a scalar for each conjugacy class in S, so the space of maps κ which yield the PBW property for A_κ has dimension $|S| + \dim((V \wedge V)^*)^G)$.

Chapter 3

Preliminaries: Dirac cohomology

As we have already mentioned, our work is motivated by two mathematical theories, the theory of algebraic Dirac operators and their applications to representation theory on the one hand and the theory of PBW deformations of smash products on the other hand. Our main results will connect the two theories leading to a generalized theory of Dirac operators. Having discussed PBW deformations in the previous chapter, we will now recall the original motivation for the Dirac operator in physics and various forms of Dirac cohomology.

3.1 Dirac operator in physics

The Dirac operator was introduced by British physicist Paul Dirac in 1928 [Dir28a, Dir28b] who sought to derive an equation for the electron (and more generally, so called fermions, a group of similar physical particles) which is compatible with both quantum mechanics and special relativity. According to quantum mechanics, the state of a particle should be described by a wave function $\phi = \phi(x, y, z, t)$, where x, y, z, t are three spatial coordinates and one time coordinate, respectively. If the particle has a mass m, then its momentum should be described by the vector field $p = p(x, y, z, t)$ of first-order derivatives

$$p = \nabla\phi = (\partial_x\phi, \partial_y\phi, \partial_z\phi) \ ,$$

and its energy should be given by the energy operator defined by

$$E\phi = |p|^2/2m = -\frac{1}{2m}\nabla^2\phi \ ,$$

where we use physical units such that the Planck constant $\hbar$ and the speed of light c both have the value 1 ("natural units"). Quantum mechanic postulates that the dynamics of the massive particle should be governed by the *Schrödinger equation*

$$i\partial_t\phi = E\phi = -\frac{1}{2m}\nabla^2\phi \ , \tag{3.1.1}$$

where i is the complex root of -1, ∂_t is the (first-order) derivative in t and $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is a second-order differential operator in the spatial coordinates. The Schrödinger equation is not compatible with special relativity, as it treats space and time coordinates on an unequal footing, which becomes clear when comparing the corresponding differential operators.

According to the theory of relativity, the quantities energy and momentum should be related by the equation

$$E^2 = m^2 + |p|^2 \tag{3.1.2}$$

(which recovers the famous equation $E = mc^2$ for $p = 0$, since $c = 1$). In an attempt to combine quantum mechanics with the theory of relativity, we could replace E and p in the relativistic energy equation by their quantum mechanical counterparts. We obtain the *Klein–Gordon equation*

$$(\nabla^2 - \partial_t^2)\phi = m^2\phi \ . \tag{3.1.3}$$

The equation visibly incorporates the relativistic concept of a unified space-time in that it is a second-order differential equation in each of the coordinates. However, being second-order in t, the equation only determines the state of the particle up to a choice of initial values for both ϕ and $\partial_t\phi$, hence, it does not control the dynamics of the particle, as desired.

Dirac's idea to solve this dilemma was to find a "square-root" of the second-order differential operator on the left-hand side of the Klein–Gordon equation, i.e., a linear differential operator D such that

$$D^2 = \nabla^2 - \partial_t^2 \ . \tag{3.1.4}$$

With such an operator at our disposal, we could replace the Klein–Gordon equation by the linear differential equation

$$D\phi = m\phi \ , \tag{3.1.5}$$

the celebrated *Dirac equation*.

Now if such an operator D exists, we can write it using coefficients c_x, c_y, c_z, c_t for the respective linear partial differential operators, but the condition

$$(c_x\partial_x + c_y\partial_y + c_z\partial_z + c_t\partial_t)^2 = D^2 = \nabla^2 - \partial_t^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2$$

implies $c_x^2 = c_y^2 = c_z^2 = (ic_t)^2 = 1$ and $c_\alpha c_\beta + c_\beta c_\alpha = 0$ for indices $\alpha, \beta \in \{x, y, z, t\}$ with $\alpha \neq \beta$, which is clearly impossible if the coefficients c_x, c_y, c_z, c_t are ordinary scalars. Dirac realized that the relations are indeed satisfied if the coefficients are chosen to be certain complex 4×4-matrices, so-called *Dirac matrices*. For instance, a set of Dirac matrices for our version of the problem is given by

$$c_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad c_y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad c_z = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad c_t = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}.$$

Mathematically, the relations defining the coefficients c_x, c_y, c_z, c_t are just the defining relation of the complex Clifford algebra associated with a four-dimensional vector space (as discussed in Section 2.2). We saw that this Clifford algebra has dimension $2^4 = 16$, it is, in fact, isomorphic to the algebra of complex 4×4-matrices.

An important physical consequence of Dirac's discovery was the implication that passing to 4×4-matrices we have to assume a four-dimensional vector-valued wave function. Choosing a suitable basis, two of the four components recover the two-component wave function Pauli had introduced "ad hoc" to explain the spin of elementary particles, as observed, for instance, in the Stern–Gerlach experiment. The experiment showed that a beam of Silver atoms splits up into two parts in a strong inhomogeneous magnetic field, and it was inferred from this observation, that elementary particles possess a quantized intrinsic angular momentum, the so called spin. Dirac's idea yielded a theoretical justification for Pauli's two-component theory of the spin, which could be seen as the low-energy limit or non-relativistic limit. The two additional components featured in Dirac's four-component wave function could be interpreted using the concept of antimatter, which makes its first appearance in Dirac's theory, before being observed only years later.

3.2 Dirac cohomology for $(\mathfrak{g}, K)$-modules

We recall the theory of Dirac cohomology for $(\mathfrak{g}, K)$-modules which was developed by Huang and Pandžić [HP02] following ideas of David Vogan [Vog97].

Let G be a connected real semisimple Lie group with finite center, let $\mathfrak{g}_0$ and $\mathfrak{g}$ be its real and its complexified Lie algebra, respectively, and let K be a maximal compact subgroup with real and complexified Lie algebra $\mathfrak{k}_0$ and $\mathfrak{k}$. Harish-Chandra showed that any irreducible unitary representation of G is determined by its *Harish-Chandra module*, the submodule of all K-finite smooth vectors, and that the Harish-Chandra module of an irreducible unitary representation is a $(\mathfrak{g}, K)$*-module*, that is, it is a $\mathfrak{g}$-module and a K-module V such that all vectors are K-finite, the $\mathfrak{g}$- and the K-action determine the same $\mathfrak{k}$-action, and the $\mathfrak{g}$-action is K-equivariant:

$$k(x(k^{-1}v))) = \mathrm{ad}_k(x)v \quad \text{for all } v \in V, k \in K, x \in \mathfrak{g} \ , \tag{3.2.1}$$

where ad is the adjoint action of K on $\mathfrak{g}$ (the definition in this form is due to James Lepowsky, see [Wal88, 3.3.1]).

Since the Lie algebra $\mathfrak{g}_0$ is semisimple, its *Killing form* B_0 is non-degenerate (in fact, the statements are equivalent), and we have a *Cartan decomposition* $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where B is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$. Complexifying these spaces and the Killing form yields the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for $\mathfrak{g}$ and the non-degenerate form B on $\mathfrak{g}$.

Remark 3.2.1. In this situation, we can observe that

- $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, so $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$,
- $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, so $\mathfrak{p}$ is a $\mathfrak{k}$-module,
- $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, so the Lie bracket defines a map $\kappa\colon \mathfrak{p} \wedge \mathfrak{p} \to \mathfrak{k}$.

Moreover, the Killing form restricts to a non-degenerate $\mathfrak{k}$-invariant symmetric bilinear form on $\mathfrak{p}$, so $\mathfrak{p}$ is an orthogonal $\mathfrak{k}$-module, and the map κ is $\mathfrak{k}$-equivariant and satisfies the Jacobi identity

$$[\kappa(x \wedge y), z] + [\kappa(y \wedge z), x] + [\kappa(z \wedge x), y] = 0 \ , \tag{3.2.2}$$

so we are precisely in the situation described by Section 2.6 and, in particular, by Equation (2.6.2).

An important tool for the study of representations of a semisimple Lie algebra $\mathfrak{g}$ is its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which allows us to employ the theory and techniques of associative algebras. Representations of $\mathfrak{g}$ can be identified with representations of $\mathcal{U}(\mathfrak{g})$ and over $\mathbb{C}$, the eigenspaces of any central element will be submodules, hence by Schur's lemma, all elements of the center act as scalars on any finite-dimensional irreducible module. We obtain an algebra map $Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$, the *central character* of the representation, and in fact, a finite-dimensional irreducible $\mathfrak{g}$-module is uniquely determined by its central character. The idea of central characters can be extended to irreducible $(\mathfrak{g}, K)$-modules, and in fact, there are only finitely many $(\mathfrak{g}, K)$-modules for any given central character.

For any semisimple Lie algebra $\mathfrak{g}$, we have a distinguished element in the center of $\mathcal{U}(\mathfrak{g})$. Let us pick bases $(W_k)_k$ and $(Z_i)_i$ of $\mathfrak{k}$ and $\mathfrak{p}$ such that $B(W_k, W_l) = -\delta_{k,l}$ and $B(Z_i, Z_j) = \delta_{i,j}$ for all i, j, k, l. Then the *Casimir element* is defined as

$$\Omega = \sum_i W_i^2 + \sum_k Z_k^2 \quad \text{in } \mathcal{U}(\mathfrak{g}) \ . \tag{3.2.3}$$

It can be shown that the definition does not depend on the choice of $(W_k)_k$ and $(Z_i)_i$ and that the Casimir element is central. Being central, it acts as a scalar on finite-dimensional irreducible $\mathfrak{g}$-modules. The Dirac operator D, which we will introduce next, can be regarded as a "square root" of the Casimir element in a certain sense. Having a central element in $\mathcal{U}(\mathfrak{g})$ which squares to Ω would allow a finer study of representations, since two irreducible modules on which Ω acts as the same scalar λ might become distinguishable from each other if the scalars by which D acts on these modules correspond to different roots of λ. However, there is a priori no such element in $\mathcal{U}(\mathfrak{g})$.

Very similar to Paul Dirac's idea described in Section 3.1, we can again resort to the Clifford algebra. More precisely, let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with the restriction of the Killing form, which is the algebra generated by $(Z_i)_i$ and the relations

$$Z_i^2 = 1 \quad \text{and} \quad Z_i Z_j = -Z_j Z_i \quad \text{for all } i \neq j \ .$$

Then the *Dirac operator* is defined as

$$D = \sum Z_i \otimes Z_i \quad \text{in } \mathcal{U}(\mathfrak{k}) \otimes C(\mathfrak{p}) \ . \tag{3.2.4}$$

To make more precise in which sense the Dirac operator "squares to Ω", we will need a Lie algebra map $\gamma\colon \mathfrak{k} \to C(\mathfrak{p})$ which exists, because $\mathfrak{p}$ is an orthogonal $\mathfrak{k}$-module, and which gives rise to a diagonal map $\Delta_C\colon \mathfrak{k} \to \mathfrak{k} \otimes C(\mathfrak{p})$ defined by $\Delta_C(x) = x \otimes 1 + 1 \otimes \gamma(x)$ for $x \in \mathfrak{k}$.

Let $\Omega_{\mathfrak{g}}, \Omega_{\mathfrak{k}}$ be the Casimir elements of $\mathfrak{g}$ and $\mathfrak{k}$, respectively, and let $\rho_{\mathfrak{g}}, \rho_{\mathfrak{k}}$ be the respective Weyl vectors, i.e., half the sum of the positive roots.

Lemma 3.2.2 ([Par72])**.** *Then*

$$D^2 = -\Omega_{\mathfrak{g}} + \Delta_C(\Omega_{\mathfrak{k}}) + C \tag{3.2.5}$$

where $C = -|\rho_{\mathfrak{g}}|^2 + |\rho_{\mathfrak{k}}|^2$.

In particular, Δ^2 differs from $-\Omega_{\mathfrak{g}}$ only by a scalar and by the image of a central element in $\mathcal{U}(\mathfrak{k})$ under the diagonal map Δ_C.

However, just as in Paul Dirac's physical problem, the Dirac operator does not act directly on the objects in question, namely $(\mathfrak{g}, K)$-modules, anymore. Instead, we have to fix an irreducible $C(\mathfrak{p})$-module S, so D will act on $M \otimes S$ for any $\mathfrak{g}$-module M. Now the *Dirac cohomology* is defined as

$$H^D(M) = \ker D/(\ker D \cap \operatorname{im} D) \ . \tag{3.2.6}$$

In certain situations, for instance, for unitary modules or for finite-dimensional modules, D acts semisimply, which implies $H^D(M) = \ker D = \ker D^2$.

In the general situation, the following result on the connection between Dirac cohomology and central characters is established, where we identify $\tilde{K}$-types with their highest weights and K-weights with central characters of $\mathfrak{g}$ using the Harish-Chandra homomorphism.

Theorem 3.2.3 ([HP02, Thm. 2.3])**.** *Let* M *be an irreducible* $(\mathfrak{g}, K)$*-module with* $H^D(M) \neq 0$*. Then the central character of* M *is given by* $\gamma + \rho_{\mathfrak{k}}$ *for any* $\tilde{K}$*-type* γ *which occurs in* $H^D(M)$*.*

Corollary 3.2.4. *Let M be an irreducible unitary $(\mathfrak{g}, K)$-module with* $\ker D \neq 0$. *Then the central character of M is given by $\gamma + \rho_{\mathfrak{k}}$ for any $\tilde{K}$-type γ which occurs in* $\ker D$.

3.3 Dirac cohomology for Hecke algebras

Motivated by the analogy between the representation theory of real reductive Lie groups and p-adic reductive Lie groups, Barbasch, Ciubotaru and Trapa established a version of Dirac cohomology for *graded affine Hecke algebras* [BCT12]. Ram and Shepler [RS03] had previously shown that graded affine Hecke algebras are special cases of a family of algebras which was studied by Drinfeld [Dri86], and which are often called *Drinfeld Hecke algebras*. Ciubotaru then realized that the program of Dirac cohomology could be carried out for this more general family of algebras [Ciu16], which also includes symplectic reflection algebras and rational Cherednik algebras.

Let us explain what Drinfeld Hecke algebras are. We consider a finite group G with a finite-dimensional faithful module V over $\mathbb{C}$. Then also the group algebra $\mathbb{C}[G]$ acts on V. Assume that, additionally, we have a linear map $\kappa\colon V \wedge V \to \mathbb{C}[G]$, then we can define $\mathcal{H}_\kappa$ to be the algebra generated by $\mathbb{C}[G]$ and V with the relations

$$gvg^{-1} = g \cdot v \quad \text{and} \quad vw - wv = \kappa(v \wedge w) \quad \text{for all } g \in G, v, w \in V \ . \tag{3.3.1}$$

$\mathcal{H}_\kappa$ becomes a filtered algebra if we assign elements from V the degree 1. Let us denote the algebra we obtain in this fashion for $\kappa = 0$ by $\mathcal{H}_0$. It is isomorphic to the smash product $S(V) \rtimes \mathbb{C}[G]$.

Definition 3.3.1. The algebra $\mathcal{H}_\kappa$ is called *Drinfeld Hecke algebra* if it is a PBW deformation of $\mathcal{H}_0$, that is, if its associated graded algebra is isomorphic to $\mathcal{H}_0$.

In order to establish his version of Dirac cohomology, Ciubotaru assumed that V carries a non-degenerate G-invariant symmetric bilinear form.

Remark 3.3.2. Let us observe and record that in this situation $\mathbb{C}[G]$ is an algebra with an orthogonal module V (by assumption). Also, it is known ([RS03, Thm. 1.5]; see Equation (2.6.3) and Equation (2.6.4)) that the PBW requirement is met if and only if the map κ is G-equivariant, that is,

$$\kappa((g\cdot v)\wedge(g\cdot w)) = g\kappa(v\wedge w)g^{-1} \quad \text{for all } g\in G, v,w\in V \quad \text{in } \mathcal{H}_0\ , \tag{3.3.2}$$

and it satisfies the following Jacobi identity

$$[\kappa(u\wedge v),w]+[\kappa(u\wedge v),w]+[\kappa(u\wedge v),w]=0 \quad \text{for all } u,v,w\in V \quad \text{in } \mathcal{H}_0\ . \tag{3.3.3}$$

Similar as for $(\mathfrak{g},K)$-modules as described in the previous section, Ciubotaru defined a Dirac operator D and Dirac cohomology as follows. Choosing a pair of dual bases of $(v_i)_i$, $(v^i)_i$ of V with respect to the assumed non-degenerate G-invariant symmetric bilinear form, we can write the *Dirac operator* as

$$D=\sum_i v_i\otimes v^i \quad \text{in } \mathcal{H}_\kappa\otimes C(V)\ ,$$

where $C(V)$ is the Clifford algebra of V and the mentioned bilinear form. The definition turns out to be independent of the choice of dual bases.

If we also fix an irreducible $C(V)$-module S, then D acts on the tensor product $M\otimes S$ for any $\mathcal{H}_\kappa$-module M. The *Dirac cohomology* of such a module M is defined as

$$H^D(M)=\ker D/(\ker D\cap \operatorname{im} D)\ .$$

It can be shown that a certain covering group $\tilde{G}$ of G acts on $H^D(M)$, $\tilde{G}$ is called the *pin cover* of G.

As an analog of Lemma 3.2.2 in Section 3.2, a formula for D^2 is derived [Ciu16, Thm. 2.7]. Let $\pi_1\colon \mathbb{C}[G]\to\mathbb{C}$ be the projection onto the span of the identity element in G (the 1-element in the algebra $\mathbb{C}[G]$) along the span of all other group elements, and let κ_1 be $\pi_1\circ\kappa$. Then the following analog of Theorem 3.2.3 is obtained:

Theorem 3.3.3 ([Ciu16, Thm. 3.14, Rem. 3.15])**.** *If $\kappa_1=0$, assume M is an $\mathcal{H}_\kappa$-module with a central character and with $H^D(M)\neq 0$. Then $H^D(M)$ determines the central character.*

More precisely, there is an algebra map from the center of $\mathcal{H}_\kappa$ to the center of $\mathbb{C}[\tilde{G}]$, and the scalar by which a central element in $\mathcal{H}_\kappa$ acts on M is just the scalar by which its image acts on an arbitrary irreducible $\tilde{G}$-submodule of $H^D(M)$.

The result applies, for instance, to rational Cherednik algebras with the parameter t equal to 0.

Chapter 4

Pin covers

4.1 Pin cover of a group

Let G be a group and let V be a G-module over the field $\mathbb{F}$. A bilinear form $\langle\cdot,\cdot\rangle$ on V is called *G-invariant* if

$$\langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \quad \text{for all } v_1, v_2 \in V, g \in G\ . \tag{4.1.1}$$

The definition can be stated without the use of elements, since it is equivalent to requiring that the adjoint operator of the action of some $g \in G$ be given by the action of g^{-1}, or also, to requiring that $\langle\cdot,\cdot\rangle$ be a G-module map, where we use the natural G-actions, the diagonal one for the tensor product and the trivial one for the field $\mathbb{F}$. The module V is called *orthogonal* if it carries a non-degenerate G-invariant symmetric bilinear form.

Assume now that $\langle v, v\rangle$ has a square-root in $\mathbb{F}$ for all $v \in V$. This is the case, for instance, if $\mathbb{F}$ is algebraically closed, or also if $\mathbb{F} = \mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is positive-definite. Recall from Section 2.2.1 that in this case, $\mathsf{Pin} = \mathsf{Pin}(V, \langle\cdot,\cdot\rangle)$ is the subgroup generated by unit vectors in $C^\times$, that $\mathsf{O} = \mathsf{O}(V, \langle\cdot,\cdot\rangle)$ is defined as the group of general linear transformations A of V which preserve $\langle\cdot,\cdot\rangle$ in the sense that

$$\langle Av, Aw\rangle = \langle v, w\rangle \quad \text{for all } v, w \in V\ ,$$

and that we have a surjective group homomorphism $\phi\colon \mathsf{Pin} \to \mathsf{O}$ which can be described with the conjugation action of $\mathsf{Pin} \subset C^\times$ on C (in the superalgebraic sense), which leaves the subspace $V \subset C$ invariant, or more explicitly by $\phi(u) = \tau_u$ for any element $u \in U$ with $\langle u, u\rangle = 1$ (see Lemma 2.2.9).

Now the action of G on V can be regarded as a group homomorphism $G \to \mathsf{O}$, and the fact that the action of G corresponds to operators in O could be taken as another definition of "orthogonal module".

Definition 4.1.1. Let the *pin cover* of G with respect to $(V, \langle\cdot,\cdot\rangle)$ be the triple $(\tilde{G}, \pi, \gamma)$ defined by the following commutative diagram (pullback) of groups:

$$\begin{array}{ccccc} \tilde{G} & \overset{\gamma}{\dashrightarrow} & \mathsf{Pin} & \hookrightarrow & C^\times \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \phi} & & \\ G & \longrightarrow & \mathsf{O} & & \end{array} \tag{4.1.2}$$

By construction, $\tilde{G}$ is a group, π and γ are group homomorphisms and π is surjective, because ϕ is so. Furthermore, if we denote the action of G on V by a dot, as in $g \cdot v$, then by commutativity of the above diagram and by Lemma 2.2.9, for all $\tilde{g} \in \tilde{G}, v \in V$,

$$\pi(\tilde{g}) \cdot v = (\gamma(\tilde{g}) \otimes \gamma(\tilde{g}^{-1})^{\mathrm{op}}) \cdot v \tag{4.1.3}$$

as elements in C.

4.2 Pin cover of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and let V be a $\mathfrak{g}$-module. A bilinear form $\langle\cdot,\cdot\rangle$ on V is called $\mathfrak{g}$-invariant if

$$\langle xv_1, v_2\rangle = -\langle v_1, xv_2\rangle \quad \text{for all } x \in \mathfrak{g}, v_1, v_2 \in V \ . \tag{4.2.1}$$

This is equivalent to the adjoint of the action of x being given by the action of $-x$, or to $\langle\cdot,\cdot\rangle$ being a $\mathfrak{g}$-module map, where $\mathfrak{g}$-acts on the tensor product diagonally and trivially on $\mathbb{F}$ (but note the different definitions of diagonal and trivial actions compared to the group case). V is called *orthogonal* if it has a non-degenerate $\mathfrak{g}$-invariant symmetric bilinear form.

Recall from Section 2.2.2 that $\mathfrak{so} = \mathfrak{so}(V, \langle\cdot,\cdot\rangle)$ was defined as the space of linear transformations A of V satisfying

$$\langle Av_1, v_2\rangle = -\langle v_1, Av_2\rangle \quad \text{for all } v_1, v_2 \in V \ ,$$

which, in fact, is a Lie subalgebra of $\mathfrak{gl}(V)$. Also recall that the pin Lie algebra $\mathfrak{pin}$ was defined as $q(V \wedge V)$ in the Clifford algebra C, where q is the quantization map, and that we defined a Lie algebra isomorphism $\phi\colon \mathfrak{pin} \to \mathfrak{so}$ which was given by the action of $\mathfrak{pin}$ on C by taking the supercommutator, which leaves $V \subset C$ invariant, or more explicitly by $\phi(q(v \wedge w)) = A_{v,w}$ (Proposition 2.2.12).

Now V is orthogonal if and only if the image of the action of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ lies in $\mathfrak{so}$.

Definition 4.2.1. Let the *pin cover* of $\mathfrak{g}$ with respect to $(V, \langle\cdot,\cdot\rangle)$ be the triple $(\mathfrak{g}, \mathrm{id}_{\mathfrak{g}}, \gamma)$, where γ is defined by the following commutative diagram of Lie algebras:

$$\begin{array}{ccccc} \mathfrak{g} & \overset{\gamma}{\dashrightarrow} & \mathfrak{pin} & \hookrightarrow & C \\ \downarrow{\scriptstyle \mathrm{id}_{\mathfrak{g}}} & & \downarrow{\scriptstyle \phi} & & \\ \mathfrak{g} & \longrightarrow & \mathfrak{so} & & \end{array} \tag{4.2.2}$$

By construction, γ is a Lie algebra homomorphism and if we denote the action of $\mathfrak{g}$ on V by a dot, then commutativity of the diagram and Proposition 2.2.12 imply, for all $x \in \mathfrak{g}, v \in V$,

$$x \cdot v = [\gamma(x), v] \tag{4.2.3}$$

as elements in C.

4.3 Pin cover of a pointed cocommutative Hopf algebra

Let H be a pointed cocommutative Hopf algebra over a field $\mathbb{F}$ of characteristic 0, and let V be an orthogonal H-module (Definition 2.5.9), that is, V has an H-invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. Let us also assume that $\mathbb{F}$ contains a square root of $\langle v, v\rangle$ for all $v \in V$, i.e., that we can normalize any non-zero vector in V with respect to $\langle\cdot,\cdot\rangle$. For instance, this is the case if $\mathbb{F} = \mathbb{C}$ or if $\mathbb{F} = \mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is positive-definite.

Let us recall (Theorem 2.5.6) that any pointed cocommutative Hopf algebra H over a field of characteristic 0 is of the form $H = \mathcal{U}(\mathfrak{g}) \rtimes \mathbb{F}[G]$ for the group of group-likes G, the Lie algebra $\mathfrak{g}$ of primitives and an action of G on $\mathfrak{g}$. By Lemma 2.5.7, V is an orthogonal module for G and $\mathfrak{g}$, so by Sections 4.1 and 4.2 we have pin covers $(\tilde{G}, \pi_G, \gamma_G)$ and $(\mathfrak{g}, \mathrm{id}_g, \gamma_{\mathfrak{g}})$ at our disposal. Our aim is now to construct a pin cover for H from these.

Let us define $\tilde{H} = \mathcal{U}(\mathfrak{g}) \rtimes \mathbb{F}[\tilde{G}]$, by construction another pointed cocommutative Hopf algebra, where the action of an element $\tilde{g} \in \tilde{G}$ on $\mathfrak{g}$ shall be given by the action of $\pi_G(\tilde{g}) \in G$. That is, $\tilde{H}$ is the algebra generated by $\mathcal{U}(\mathfrak{g})$ and $\mathbb{F}[\tilde{G}]$ with the additional relations

$$\tilde{g}x\tilde{g}^{-1} = \pi_G(\tilde{g}) \cdot x \quad \text{for all } \tilde{g} \in \tilde{G}, x \in \mathfrak{g} \ .$$

Let us further define the algebra map $\pi\colon \tilde{H} \to H$ on generators by $\pi(\tilde{g}) = \pi_G(\tilde{g})$ and $\pi(x) = x$ for all $\tilde{g} \in \tilde{G}, x \in \mathfrak{g}$.

Lemma 4.3.1. *π is a well-defined surjective Hopf algebra map.*

Proof. π is a well-defined algebra map, since for $\tilde{g} \in \tilde{G}, x \in \mathfrak{g}$, the following holds in H:

$$\pi(\tilde{g})\pi(x)\pi(\tilde{g}^{-1}) = \pi_G(\tilde{g})x\pi_G(\tilde{g})^{-1} = \pi_G(\tilde{g}) \cdot x = \pi(\pi_G(\tilde{g}) \cdot x) \ .$$

Thus, we can verify on generators that it is a Hopf algebra map. Consider $\tilde{g} \in \tilde{G}$, then

$$\varepsilon(\pi(\tilde{g})) = 1 = \pi(\varepsilon(\tilde{g})) \ , \quad \Delta(\pi(\tilde{g})) = \pi(\tilde{g}) \otimes \pi(\tilde{g}) = (\pi \otimes \pi)\Delta(\tilde{g})$$
$$\text{and} \quad S(\pi(\tilde{g})) = \pi(\tilde{g})^{-1} = \pi(S(\tilde{g})) \ .$$

The corresponding checks for $x \in \mathfrak{g}$ are trivial, because $\pi|_{\mathfrak{g}} = \mathrm{id}_{\mathfrak{g}}$. Hence indeed, π is a Hopf algebra map.

And π is surjective, since its image contains G and $\mathfrak{g}$, which generate H is an algebra. □

Similarly, we define an algebra $\gamma\colon \tilde{G} \to C$ on generators by $\gamma(\tilde{g}) = \gamma_G(\tilde{g})$ and $\gamma(x) = \gamma_{\mathfrak{g}}(x)$ for all $\tilde{g} \in \tilde{G}, x \in \mathfrak{g}$.

Lemma 4.3.2. *γ is a well-defined algebra map.*

Proof. For $\tilde{g} \in \tilde{G}, x \in \mathfrak{g}$, let us use the shortcut $p = \gamma_G(\tilde{g}) \in \mathsf{Pin}$. Then

$$pvp^{-1} = (-1)^p \pi_G(\tilde{g}) \cdot v$$

holds in C for any $v \in V$ and similarly for $\tilde{g}^{-1}, p^{-1}$. Thus for any $v \in V$, we can compute in C:

$$\begin{aligned}[\gamma(\tilde{g})\gamma(x)\gamma(\tilde{g}^{-1}), v] &= [p\gamma_{\mathfrak{g}}(x)p^{-1}, v] = p[\gamma_{\mathfrak{g}}(x), p^{-1}vp]p^{-1} \\ &= (-1)^p \pi_G(\tilde{g}) \cdot (x \cdot ((-1)^{p^{-1}} \pi_G(\tilde{g})^{-1} \cdot v)) \\ &= \pi_G(\tilde{g}) \cdot (x \cdot (\pi_G(\tilde{g})^{-1} \cdot v)) \dots ,\end{aligned}$$

but this last expression is the same as the element $\pi_G(\tilde{g})x\pi_G(\tilde{g})^{-1}$ in H acting on V. By the smash product relation in H, this is just

$$\dots = (\pi_G(\tilde{g}) \cdot x) \cdot v = (\tilde{g} \cdot x) \cdot v = [\gamma(\tilde{g} \cdot x), v] \ .$$

We recall from Lemma 2.2.14, that the pin Lie algebra is a pin group submodule of C under the adjoint action. Hence, $\gamma(\tilde{g})\gamma(x)\gamma(\tilde{g}^{-1})$ is a bivector just as $\gamma(\tilde{g} \cdot x)$, and we have just shown that their images under the isomorphism $\mathfrak{pin} \to \mathfrak{so}, b \mapsto [b, \cdot]$, coincide. Hence, the elements coincide, and γ is a well-defined algebra map, as desired. □

Definition 4.3.3. We define the pin cover of H with respect to $(V, \langle \cdot, \cdot \rangle)$ to be the triple $(\tilde{H}, \pi, \gamma)$.

Now $\tilde{H}$ is a pointed cocommutative Hopf algebra, $\pi \colon \tilde{H} \to H$ is a Hopf algebra epimorphism by Lemma 4.3.1, $\gamma \colon \tilde{H} \to C$ is an algebra map.

We want to find analogs for the compatibility conditions of π and γ as in Equations (4.1.3) and (4.2.3). To this end, it will be useful to consider C as a superalgebra as before. Let us record the following observation:

Lemma 4.3.4. *If $\tilde{g} \in \tilde{H}$ is a group-like element, then $(-1)^{\gamma(\tilde{g})}$ is just the ordinary determinant of the action of $\pi(\tilde{g})$ on V, and if $x \in \tilde{H}$ is a primitive element, then $(-1)^{\gamma(x)} = 1$.*

Proof. For a primitive element $\tilde{g}$, its action on V will be the same as the action of $\gamma(\tilde{g})$, so what we want to show is that for any p in the pin group Pin, $(-1)^p$ equals the determinant of p acting on V. We can write $p = v_1 \dots v_m$ with $m \geq 0$ and $v_i \in V$ such that $\langle v, v \rangle = 1$. Now each of the v_i acts on V as a reflection with determinant -1, so the determinant of p is $(-1)^m$, which is also its degree in the superalgebra C.

For a primitive element x, $\gamma(x)$ will be in $\mathfrak{pin}$, which is in the even part of C, so $(-1)^{\gamma(x)} = 1$. $\square$

Now let us recall (Section 2.1) that for any superalgebra A, the superalgebra $A \otimes A^{\mathrm{op}}$ acts on A from the left, where we use the superspace braiding c to turn the right action of A on itself into a left action of A^{op} on A.

Proposition 4.3.5. *For all $\tilde{h} \in \tilde{H}, v \in V$,*

$$\pi(\tilde{h}) \cdot v = (\gamma(\tilde{h}_{(1)}) \otimes \gamma(S(\tilde{h}_2))^{\mathrm{op}}) \cdot v \tag{4.3.1}$$

as elements in C.

Proof. The map

$$\tilde{H} \to C \otimes C^{\mathrm{op}} \ , \quad \tilde{h} \mapsto \gamma(\tilde{h}_{(1)}) \otimes \gamma(S(\tilde{h}_{(2)}))^{\mathrm{op}} \ ,$$

is an algebra map, because Δ and γ are algebra maps and S is an anti-algebra map. Hence we can verify the asserted equation on a set of algebra generators of H.

For $\tilde{g} \in \tilde{G}$ and $v \in V$,

$$(\gamma(\tilde{g}_{(1)}) \otimes \gamma(S(\tilde{g}_{(2)}))^{\mathrm{op}}) \cdot v = (\gamma(\tilde{g}) \otimes \gamma(\tilde{g}^{-1})^{\mathrm{op}}) \cdot v = \pi(\tilde{g}) \cdot v \quad \text{in } C$$

by Equation (4.1.3).

For $x \in \mathfrak{g}$, $v \in V$,

$$(\gamma(x_{(1)}) \otimes \gamma(S(x_{(2)}))^{\mathrm{op}}) \cdot v = (\gamma(x) \otimes 1 - 1 \otimes \gamma(x)^{\mathrm{op}}) \cdot v = [\gamma(x), v] = x \cdot v \quad \text{in } C$$

by Equation (4.2.3). This proves the desired formula. $\square$

4.4 Coalgebra measurings of central simple superalgebras

In this section, we work over an arbitrary field $\mathbb{F}$. Let C be a coalgebra and A an algebra over $\mathbb{F}$.

Definition 4.4.1. We say that C *measures* A if we have a map $C \otimes A \to A$ sending $c \otimes a$ to $c \cdot a$ such that

$$c \cdot 1 = \varepsilon(c) \quad \text{and} \quad c \cdot (ab) = (c_{(1)} \cdot a)(c_{(2)} \cdot b) \quad \text{for all } c \in C, a, b \in A \ . \tag{4.4.1}$$

In other words, a measuring is characterized by the associated map

$$A \to \mathrm{Hom}(C, A)\ , \quad a \mapsto (c \mapsto c \cdot a) \tag{4.4.2}$$

being an algebra map, where $\mathrm{Hom}(C, A)$ is an algebra with the convolution and with $\varepsilon(\cdot)1$ as the identity element (see Section 2.4).

Example 4.4.2. There is a trivial measuring $c \cdot a = \varepsilon(c)a$ for all $c \in C, a \in A$ for any coalgebra C and any algebra A.

Example 4.4.3. More generally, if C is a coalgebra, A is an algebra and $u \in \mathrm{Hom}(C, A)$ is $\star$-invertible with inverse u^-, then the map

$$C \otimes A \to A\ , \quad c \otimes a \mapsto u(c_{(1)})au^-(c_{(2)}) \quad \text{for all } c \in C, a \in A\ ,$$

is a measuring.

Example 4.4.4. If H is a bialgebra and V is an H-module algebra, then the action of H is a measuring.

Definition 4.4.5. We say that the measuring is *inner*, if it is like Example 4.4.3, that is, if there is a $\star$-invertible $u \in \mathrm{Hom}(C, A)$ such that

$$c \cdot a = u(c_{(1)})au^-(c_{(2)}) \quad \text{for all } c \in C, a \in A\ .$$

In other words, a measuring is inner, if the map in Equation (4.4.2) is conjugate to the trivial measuring corresponding to the map

$$A \mapsto \mathrm{Hom}(C, A)\ , \quad a \mapsto (c \mapsto \varepsilon(c)a) \tag{4.4.3}$$

via an invertible element in $\mathrm{Hom}(C, A)$.

Example 4.4.6. The adjoint action of a Hopf algebra on itself is an inner measuring with $u = S$, the antipode of H.

In certain situations, all measurings are inner. The following result can be viewed as a Skolem–Noether theorem for coalgebra measurings.

Theorem 4.4.7 ([Mas90a, Thm. 3.1(c)],[Kop91, Thm. 1.1])**.** *Let C be a coalgebra measuring a central simple algebra A. Then the measuring is inner.*

Remark 4.4.8. This result generalized the well-known facts that in a central simple algebra, any automorphism or any derivation is inner, where the automorphism is regarded as the measuring of a group-like element in a one-dimensional coalgebra and the derivation as the measuring of a primitive element in the universal enveloping algebra of a one-dimensional Lie algebra.

We want to prove that the theorem can be extended to the category of superspaces. In what follows, all superspaces (algebras, coalgebras, ...) will be over a field of characteristic not 2 (so the braiding is non-trivial).

We start with the Skolem–Noether theorem for superalgebras, versions of which follow from [Jab10, Thm. 3.2] or [CVOZ97, Cor. 3.6].

Definition 4.4.9. We say that two maps f, g to a superalgebra A are *conjugate* if there is an invertible $a \in A$ such that $g = (a \otimes (a^{-1})^{\mathrm{op}}) \cdot f$.

Lemma 4.4.10 (Skolem–Noether for superalgebras)**.** *Let A be a finite-dimensional simple superalgebra, B a central simple superalgebra, and f, g graded algebra maps from A to B. Then f, g are conjugate by a homogeneous element in $B^{\times}$.*

Proof. Since A is a finite-dimensional simple superalgebra and B is a central simple superalgebra, $E = A \otimes B^{\mathrm{op}}$ is a finite-dimensional simple superalgebra. The graded algebra maps f, g can be extended to graded algebra maps $f \otimes \mathrm{id}_{B^{\mathrm{op}}}, g \otimes \mathrm{id}_{B^{\mathrm{op}}}$ from E to $B \otimes B^{\mathrm{op}}$ and the latter algebra acts on B, so the simple algebra E acts on B in two ways.

By Lemma 2.1.4, B decomposes as a direct sum of copies of the simple E-module which is unique up to isomorphism and parity change. Hence there is a graded E-module automorphism s of B such that

$$s((f(a) \otimes b^{\mathrm{op}}) \cdot x) = (g(a) \otimes b^{\mathrm{op}}) \cdot s(x) \quad \text{for all } a \in A, b, x \in B \ .$$

This equation implies

$$s(f(a)) = s((f(a) \otimes 1^{\mathrm{op}}) \cdot 1) = (g(a) \otimes 1^{\mathrm{op}}) \cdot s(1) = g(a)s(1) \ ,$$
$$s(f(a)) = s((1 \otimes f(a)^{\mathrm{op}}) \cdot 1) = (1 \otimes f(a)^{\mathrm{op}}) \cdot s(1) = (-1)^{|s(1)||f(a)|} s(1) f(a) \ ,$$

for all $a \in A$. As both s and s^{-1} are B^{op}-module maps,

$$1 = s(s^{-1}(1)) = s(1 \otimes (s^{-1}(1))^{\mathrm{op}}) \cdot 1) = (1 \otimes s^{-1}(1)^{\mathrm{op}}) \cdot s(1) = \pm s(1)s^{-1}(1) \ ,$$
$$1 = s^{-1}(s(1)) = (1 \otimes s(1)^{\mathrm{op}}) \cdot s^{-1}(1) = \pm s^{-1}(1)s(1) \ ,$$

where the sign is the same in both equations, namely $+$ if s is an even automorphism of B or $-$ if s is odd.

Hence, $s(1)$ is invertible in B and, as $|s(1)^{-1}| = |s(1)|$, we get

$$g(a) = (-1)^{|s(1)||f(a)|} s(1)f(a)s(1)^{-1} = (s(1) \otimes (s(1)^{-1})^{\mathrm{op}}) \cdot f(a) \ ,$$

as desired. □

The Skolem–Noether theorem for algebras has a counterpart in the theory of coalgebras, namely the fact that certain coalgebra measurings of central simple algebras are inner. This was first proven by Koppinen [Kop91] and independently, by Masuoka [Mas90a]. A review of these ideas can also be found in Montgomery's book [Mon93, Ch. 6].

We follow the ideas of Milinski [Mil93] who derived the coalgebra results from the classical algebra result, and transfer them to superalgebras and supercoalgebras, respectively.

Lemma 4.4.11. *Let A be a central simple superalgebra, B a finite-dimensional superalgebra, and f, g graded algebra maps from A to B. Then f, g are conjugate.*

Proof. First assume B is simple. Let Z be the supercenter of B. If there was a non-zero odd element $z \in Z$, then $z^2 = 0$ and z would generate a non-trivial two-sided graded ideal in B, which contradicts simplicity of B. Hence Z is purely even and commutative, and again by simplicity of B, it is a field.

Now B is a central simple superalgebra over Z, $Z \otimes A$ is a simple superalgebra over Z and we have two graded algebra maps from $Z \otimes A$ to B sending $z \otimes a$ in $Z \otimes A$ to $zf(a)$ and $zg(a)$, respectively.

Hence, by the previous lemma, these maps are conjugate, i.e., there is an invertible homogeneous $b \in B$ such that

$$zf(a) = (b \otimes (b^{-1})^{\mathrm{op}}) \cdot (zg(a)) \quad \text{for all } z \in Z, a \in A \ ,$$

in particular, for $z = 1$ we obtain that f and g are conjugate, as desired. This completes the proof for the case where B is simple.

In general, let J be the Jacobson radical of the superalgebra B (as discussed in Section 2.1). Then by Lemma 2.1.6, B/J is a direct sum of simple superalgebras, hence, the canonical maps $\overline{f}, \overline{g} \colon A \to B/J$ are conjugate, say, by an invertible element $\overline{b} \in B/J$.

Consider B as an $A \otimes A^{\mathrm{op}}$-module via the action

$$(a \otimes (a')^{\mathrm{op}}) \cdot b = (f(a) \otimes g(a')^{\mathrm{op}}) \cdot b \quad \text{for all } a \in A, (a')^{\mathrm{op}} \in A^{\mathrm{op}}, b \in B$$

and similarly B/J using $\overline{f}$ and $\overline{g}$. Then the canonical quotient map $B \to B/J$ is an $A \otimes A^{\mathrm{op}}$-module map. Now $\overline{f}, \overline{g}$ being conjugate via $\overline{b}$ means

$$(\overline{f}(a) \otimes 1) \cdot \overline{b} = (1 \otimes \overline{g}(a)^{\mathrm{op}}) \cdot \overline{b} \quad \text{for all } a \in A \ .$$

As A is a central simple superalgebra, $A \otimes A^{\mathrm{op}}$ is a simple superalgebra, so there is an $A \otimes A^{\mathrm{op}}$-module splitting of the canonical quotient map $B \to B/J$. Hence, there is an element $b \in B$ which is mapped to $\overline{b}$ under this quotient map and which satisfies

$$(f(a) \otimes 1) \cdot b = (1 \otimes g(a)^{\mathrm{op}}) \cdot b \ .$$

b is invertible by Corollary 2.1.7, since it is invertible modulo the Jacobson radical. Hence, f, g are conjugate via b. □

Definition 4.4.12 (measuring, inner measuring for supercoalgebras)**.** Let C be a supercoalgebra and let A be a superalgebra. A map $C \otimes A \to A$, sending $c \otimes a$ to $c \cdot a$ is a *measuring* if the map

$$A \to \mathrm{Hom}(C, A) \ , \quad a \mapsto (c \mapsto c \cdot a) \tag{4.4.4}$$

is a superalgebra map. A measuring defined in this way is *inner* if it is conjugate to the trivial measuring

$$a \mapsto (c \mapsto \varepsilon(c)a) \tag{4.4.5}$$

via an invertible element $u \in \mathrm{Hom}(C, A)$, that means, if

$$c \cdot a = (u(c_{(1)}) \otimes u^{-}(c_{(2)})^{\mathrm{op}}) \cdot a \tag{4.4.6}$$

where we use the action of $A \otimes A^{\mathrm{op}}$ on A for a superalgebra A defined in Section 2.1.

Proposition 4.4.13. *Let C be a finite-dimensional supercoalgebra which measures a central simple superalgebra A. Then the measuring is inner.*

Proof. We have two graded algebra maps $f, g\colon A \to \mathrm{Hom}(C, A) = C^* \otimes A$ which are defined by $f(a)(c) = c \cdot a$ and $g(a)(c) = \varepsilon(c)a$, respectively.

C^* is a simple superalgebra, so $C^* \otimes A$ is a simple superalgebra. Hence, f, g are conjugate by an invertible element $u \in \mathrm{Hom}(C, A)$ according to the lemmas. This means

$$f(a) = (u \otimes (u^{-})^{\mathrm{op}}) \cdot g(a) \ ,$$

i.e.,

$$c \cdot a = f(a)(c) = (u(c_{(1)}) \otimes (u^{-}(c_{(3)}))^{\mathrm{op}}) \cdot \varepsilon(c_{(2)})a = (u(c_{(1)}) \otimes u^{-}(c_{(2)})^{\mathrm{op}}) \cdot a$$

for all $a \in A, c \in C$, as desired. □

To extend this result to arbitrary supercoalgebras, we use the existence of a largest inner sub-supercoalgebra. For ungraded coalgebras, the corresponding result was established by Masuoka [Mas90b]. Schneider [Sch94] found a simpler proof, whose arguments were generalized to H-comodule coalgebras by Ulm [Ulm03], where H is a bialgebra over a field. For $H = \mathbb{F}\mathbb{Z}_2$, H-comodule coalgebras are just $\mathbb{Z}_2$-graded coalgebras, i.e., supercoalgebras.

Theorem 4.4.14. *Let C be any supercoalgebra which measures a central simple superalgebra A. Then the measuring is inner.*

Proof. By [Ulm03, Thm. 4.1.2], there is a largest inner $\mathbb{Z}_2$-graded subcoalgebra C' of C, i.e., a largest graded subcoalgebra restricted to which the measuring is inner. Just as in the ungraded situation [Mon93, Thm. 5.1.1], any homogeneous element of a super-coalgebra is contained in a finite-dimensional graded subcoalgebra. Hence, any graded coalgebra is the union of its finite-dimensional graded subcoalgebras, so by Proposition 4.4.13, $C' = C$. □

Corollary 4.4.15. *Let H be a Hopf superalgebra and let A be central simple superalgebra such that A is an H-module algebra. Then there is a $\star$-invertible graded map $u \in$* $\mathrm{Hom}(H, A)$ *such that*

$$h \cdot a = (u(h_{(1)}) \otimes u^{-}(h_{(2)})^{\mathrm{op}}) \cdot a \quad \textit{for all } h \in H, a \in A \ .$$

Proof. The conditions for A to be an H-module algebra imply that the action of H is a measuring. □

Note that the corollary does not make any statement about the compatibility of the algebra structure of H with the map u. In order to make such a statement, we need the following

Definition 4.4.16. Let H be a Hopf algebra (i.e., a purely even Hopf superalgebra). A map $\sigma \in \mathrm{Hom}(H \otimes H, \mathbb{F})$ is a (normalized) *Hopf 2-cocycle* if σ is $\star$-invertible and for all $x, y, z \in H$,

$$\sigma(\sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}, z) = \sigma(x, \sigma(y_{(1)}, z_{(1)})y_{(2)}z_{(2)}) \quad \text{and} \quad \sigma(x, 1) = \varepsilon(x) = \sigma(1, x) \ .$$

The significance of this definition for us is the following result by Doi and Takeuchi [DT86].

Lemma 4.4.17 ([Mon93, Lem. 7.1.2])**.** *Let H be a Hopf algebra with multiplication map $\mu\colon H \otimes H \to H$ and with a Hopf 2-cocycle $\sigma\colon H \otimes H \to \mathbb{F}$. Then $\mu_\sigma = \sigma \star \mu$ defines an associate multiplication for the vector space H.*

The new Hopf algebra is called the *crossed product* H_σ.

Even though the map u realizing the action of a Hopf algebra H on a central simple superalgebra as an inner measuring might not be an algebra map, it is an algebra map up to a Hopf 2-cocycle in important special case, in the following sense:

Proposition 4.4.18. *In the situation of Corollary 4.4.15 assume H is a cocommutative (purely even) Hopf algebra. Then there is a Hopf 2-cocycle $\sigma\colon H\otimes H\to\mathbb{F}$ such that the map*

$$\Delta_u=(\mathrm{id}_H\otimes u)\circ\Delta\colon H\to H\otimes A\ ,\quad h\mapsto h_{(1)}\otimes u(h_{(2)})\ ,$$

is an algebra isomorphism from H_σ to $\Delta_u(H)\subset H\otimes A$.

Proof. We define $\sigma\colon H\otimes H\to A$ by

$$\sigma(x,y)=u^-(x_{(1)}y_{(2)})u(x_{(2)})u(y_{(2)})\quad\text{for all } x,y\in H\ .$$

Fix a homogeneous element $a\in A$ and let ω be the involution $b\mapsto(-1)^{|a||b|}b$ of A. Now we can use the fact that u and u^- are mutual $\star$-inverses and that u realizes the (algebra) action of H on A as an inner measuring to show that for all $x,y\in H$, in the algebra A we have

$$\begin{aligned}au^-(xy)&=\omega(u^-(x_{(1)}y_{(1)})u(x_{(2)}y_{(2)})\omega(a)\omega(u^-(x_{(3)}y_{(3)}))\\&=\omega(u^-(x_{(1)}y_{(1)})((u(x_{(2)}y_{(2)})\otimes u^-(x_{(3)}y_{(3)})^{\mathrm{op}})\cdot\omega(a)))\\&=\omega(u^-(x_{(1)}y_{(1)})((u(x_{(2)})\otimes u^-(x_{(3)})^{\mathrm{op}})\cdot((u(y_{(2)})\otimes u^-(y_{(3)})^{\mathrm{op}})\cdot\omega(a))))\\&=\omega(u^-(x_{(1)}y_{(1)})u(x_{(2)})u(y_{(2)})\omega(a)\omega(u^-(y_{(3)}))\omega(u^-(x_{(3)})))\\&=\omega(u^-(x_{(1)}y_{(1)})u(x_{(2)})u(y_{(2)}))au^-(y_{(3)})u^-(x_{(3)})\\&=((1\otimes a^{\mathrm{op}})\cdot(u^-(x_{(1)}y_{(1)})u(x_{(2)})u(y_{(2)})))u^-(y_{(3)})u^-(x_{(3)})\ ,\end{aligned}$$

hence,

$$au^-(x_{(1)}y_{(1)})u(x_{(2)})u(y_{(2)})=(1\otimes a^{\mathrm{op}})\cdot(u^-(x_{(1)}y_{(1)})u(x_{(2)})u(y_{(2)}))\ ,$$

so $\sigma(x,y)$ lies in the supercenter of A, so it lies in $\mathbb{F}$.

σ is a Hopf 2-cocycle, since for all $x, y, z \in H$, $\sigma(x,1) = \sigma(1,x) = x \cdot 1 = \varepsilon(x)$,

$$\begin{aligned}
\sigma(\sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}, z) &= u^-(x_{(1)}y_{(1)})u(x_{(2)})u(y_{(2)})u^-(x_{(3)}y_{(3)}z_{(1)})u(x_{(4)}y_{(4)})u(z_{(2)}) \\
&= u^-(x_{(1)}y_{(1)}z_{(1)})u(x_{(2)})u(y_{(2)})u(z_{(2)}) \ , \quad \text{and} \\
\sigma(x, \sigma(y_{(1)}, z_{(1)})y_{(2)}z_{(2)}) &= u^-(y_{(1)}z_{(1)})u(y_{(2)})u(z_{(2)})u^-(x_{(1)}y_{(3)}z_{(3)})u(x_{(2)})u(y_{(4)}z_{(4)}) \\
&= u^-(x_{(1)}y_{(1)}z_{(1)})u(x_{(2)})u(y_{(2)})u(z_{(2)}) \ ,
\end{aligned}$$

where we can move scalars freely within each expression and relabel the indices in Sweedler's notation as we like, since H is assumed to be cocommutative. Hence, H_σ is an associative algebra

Finally, since u is $\star$-invertible, Δ_u is injective with the inverse map

$$\Delta_u^{-1} \colon \Delta_u(H) \to H \otimes 1 \subset H \otimes C \ , \quad h \otimes c \mapsto h_{(1)} \otimes u^-(h_{(2)})c \ .$$

Now for any $x, y \in H$,

$$\begin{aligned}
\Delta_u^{-1}(\Delta_u(x)\Delta_u(y)) &= \Delta_u^{-1}(x_{(1)}y_{(1)} \otimes u(x_{(2)})u(y_{(2)})) \\
&= x_{(1)}y_{(1)} \otimes u^-(x_{(2)}y_{(2)})u(x_{(3)})u(y_{(3)}) = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} \otimes 1 \ ,
\end{aligned}$$

which is the product in H_σ. □

Remark 4.4.19. For instance, if H is a cocommutative purely even Hopf superalgebra over $\mathbb{C}$ (or $\mathbb{R}$) with an orthogonal purely odd module V with non-degenerate H-invariant (positive definite) symmetric bilinear form, then H acts on the Clifford algebra $C(V)$ by Lemma 2.5.11, which is a central simple superalgebra (see Example 2.3.3 and Example 2.3.8), so by the previous results, this action is inner via an invertible $\gamma \in \mathrm{Hom}(H, C(V))$, and there is an algebra monomorphism

$$\Delta_C \colon H_\sigma \to H \otimes C(V) \ , \quad h \mapsto h_{(1)} \otimes \gamma(h_{(2)}) \ ,$$

for the Hopf 2-cocycle $\sigma(x,y) = \gamma(x_{(1)})\gamma(y_{(1)})\gamma^-(x_{(2)}y_{(2)})$.

Finally, let us observe:

Lemma 4.4.20. *In the situation of Proposition 4.4.18,* $\mathrm{im}\,\Delta_u$ *is a graded subspace of* $H \otimes A$.

Proof. Any cocommutative coalgebra is the direct sum of its irreducible components, that is, maximal subcoalgebras each containing a unique simple subcoalgebra, because the dual space of a cocommutative coalgebra is a commutative algebra which is the direct sum of local subalgebras (see [Mon93, Thm. 5.6.3]).

Consider a fixed irreducible component C, then the proofs of Lemma 4.4.11 and Proposition 4.4.13 show that the map u realizing the measuring as an inner measuring is actually a graded map: $\mathrm{Hom}(C, A) = C^* \otimes A$ is a commutative algebra with a unique proper non-zero graded ideal, which is the Jacobson radical. Hence, the proof of Lemma 4.4.11 shows that the algebra maps

$$f, g\colon A \to \mathrm{Hom}(C, A)\ , \quad f(a)(c) = c \cdot a\ , \quad g(a)(c) = \varepsilon(c)a\ ,$$

are conjugate via a homogeneous element modulo the Jacobson radical. Hence f, g themselves are conjugate via an invertible homogeneous element, which is the graded map realizing the measuring of A by C as an inner measuring.

Hence the image of $\Delta_u = (\mathrm{id}_H \otimes u) \circ \Delta$ restricted to any irreducible component is a graded subspace of $H \otimes A$, which implies the assertion. □

Remark 4.4.21*.* Let us compare this with our observations regarding the explicit pin cover construction: For a pointed cocommutative Hopf algebra H over a field of characteristic 0, the irreducible components are just the subcoalgebras $(H^1 g)_{g \in G}$, where H^1 is the sub Hopf algebra generated by all primitive elements of H. In Section 4.3, we constructed a map γ realizing the action of H on $C(V)$ for an orthogonal module V as an inner action. In Lemma 4.3.4 we observed that $\gamma(g) \in C(V)$ is homogeneous of degree $\det(g|_V)$ (where we identify $\mathbb{Z}_2 \cong \{\pm 1\}$), so the image of γ restricted to $H^1 g$, and hence, the image of $(\mathrm{id}_H \otimes \gamma) \circ H$ restricted to $H^1 g$ is homogeneous in $H \otimes C$, the homogeneous degree being $\det(g|_V)$, for each $g \in G(H)$.

Chapter 5

Hopf–Hecke algebras

In this chapter, we fix a cocommutative Hopf algebra H over the field $\mathbb{F}$ of characteristic not 2 and a finite-dimensional H-module V. We denote the antipode of H by S and to make some formulae more readable, we will sometimes omit the parentheses when applying the linear operator S to an element of H, as in "Sh".

As H is cocommutative, it not only acts on the tensor algebra $T(V)$, but also on the symmetric algebra $S(V)$ and the exterior algebra $\Lambda(V)$. Even though the antipode and the symmetric algebra are denoted using the same symbol S, the meaning of this symbol should always be clear from the context.

Let $\kappa\colon V \wedge V \to H$ be a linear map. We define $A = A_\kappa$ as the algebra generated by H and V subject to the relations

$$h_{(1)} v S(h_{(2)}) = h \cdot v \ , \quad vw - wv = \kappa(v \wedge w) \quad \text{for all } h \in H, v, w \in V \ , \tag{5.0.1}$$

that is,

$$A = A_\kappa = (T(V) \rtimes H)/I_\kappa \ , \tag{5.0.2}$$

where I_κ is the ideal in $T(V) \rtimes H$ generated by elements of the form $vw - wv - \kappa(v \wedge w)$ for $v, w \in V$.

As explained in Section 2.6, A_κ is, in general, a filtered algebra with elements $v \in V$ in degree 1. If $\kappa = 0$, then A_κ is even a $\mathbb{Z}$-graded algebra, namely the smash product $S(V) \rtimes H$ which we denote by A_0. The quotient map $T(V) \rtimes H \to A_\kappa$ descends to a surjective algebra map $A_0 \to \operatorname{gr}(A_\kappa)$, where $\operatorname{gr}(A_\kappa)$ is the associated graded algebra of the filtered algebra A_κ.

Definition 5.0.1. A_κ is a *PBW deformation* (or *flat deformation*) of A_0 if this map is an isomorphism.

Definition 5.0.2. A_κ is a *Hopf–Hecke algebra* if H is a Hopf algebra, V is an orthogonal H-module, and A_κ is a PBW deformation of A_0.

5.1 Hopf–Hecke algebras as PBW deformations

An interesting problem is the question, which maps κ yield PBW deformations for a given choice of H and V. It is known that this can be answered in terms of two properties of the map κ.

Definition 5.1.1. For a map $\kappa\colon V \wedge V \to H$, we say:

- κ has the *PBW property* if A_κ is a PBW deformation of A_0.
- κ is *H-equivariant* if κ is an H-module map, that is, if

$$\kappa(h_{(1)} \cdot v \wedge h_{(2)} \cdot w) = \kappa(h \cdot (v \wedge w)) = h \cdot \kappa(v \wedge w) = h_{(1)}\kappa(v \wedge w)S(h_{(2)}) \quad (5.1.1)$$

  for all $h \in H, v, w \in V$ (where the first and last identity are just the definition of the action of H on $\Lambda^2(V)$ and H, respectively).

- κ has the *Jacobi property* if

$$[\kappa(u,v),w] + [\kappa(v,w),u] + [\kappa(w,u),v] = 0 \quad \text{in } T(V) \rtimes H \quad \text{for all } u,v,w \in V\ . \quad (5.1.2)$$

Maps κ with the PBW property have been studied for several special cases of H starting with the work of Drinfeld [Dri86], Braverman and Gaitsgory [BG96], Etinof and Ginzburg [EG02], see also the survey article by Shepler–Witherspoon [SW15]. The following version required for our general setting is covered by results of Walton [WW14, Thm. 3.1] or Khare [Kha17, Thm. 2.5], see also Etingof–Gan–Ginzburg [EGG05, Thm. 2.4].

Theorem 5.1.2. *κ has the PBW property if and only if κ is H-equivariant and has the Jacobi property.*

We set off to describe maps with the Jacobi property more explicitly. For instance, in the case where H is a group algebra, we have seen in Theorem 2.6.3, that the image of

κ has to be in the span of group elements satisfying a certain rank condition. To obtain a similar result in our more general situation, let us define a filtration on H depending on V. For $h \in H, v \in V$ we define a linear action

$$h \triangleright v = h \cdot v - \varepsilon(h)v \ . \tag{5.1.3}$$

Definition 5.1.3. For $i \geq 0$, we define the subspace $K_i \subset H$ as

$$K_i = \{h \in H \mid (h_{(1)} \triangleright v_1) \wedge \cdots \wedge (h_{(i+1)} \triangleright v_{i+1}) \otimes h_{(i+2)} = 0 \quad \text{for all } v_1, \ldots, v_{i+1} \in V\} \ . \tag{5.1.4}$$

Remark 5.1.4. For example, K_0 is the space of all $h \in H$ for which

$$h_{(1)} \triangleright v \otimes h_{(2)} = 0$$

for all $v \in V$. Hence, if H' is the space of $h \in H$ which act on V as $\varepsilon(h)$, then $K_0 = \Delta^{-1}(H' \otimes H) = \Delta^{-1}(H' \otimes H')$, since H is cocommutative. In particular, applying $\mathrm{id} \otimes \varepsilon$, we see that $K_0 \subset H'$.

Lemma 5.1.5. *$(K_i)_{i \geq 0}$ is a finite algebra filtration of H.*

Proof. We first check $K_i \subset K_{i+1}$. Consider h in K_i, then

$$\begin{aligned}(h_{(1)} \triangleright v_1) \wedge \cdots \wedge (h_{(i+2)} \triangleright v_{i+2}) \otimes h_{(i+3)} \\ = (h_{(1)} \triangleright v_1) \wedge \cdots \wedge (h_{(i+1)} \triangleright v_{i+1}) \wedge ((h_{(i+2)})_{(1)} \triangleright v_{i+2}) \otimes (h_{(i+2)})_{(2)} = 0\end{aligned}$$

for all $v_1, \ldots, v_{i+2} \in V$, so indeed, h lies in K_{i+1}.

Also, $\Lambda^{\dim V + 1}(V) = 0$, so $K_{\dim V} = H$, and $(K_i)_i$ is a finite filtration of subspaces.

To see that this is an algebra filtration, we observe that for $a, b \in H$ and all $v \in V$,

$$(ab) \triangleright v = (ab - \varepsilon(a)\varepsilon(b)) \cdot v = a \cdot (b \triangleright v) + \varepsilon(b)(a \triangleright v) = P(a, b, v) + Q(a, b, v)$$

using the shorthands $P(a, b, v) = a \cdot (b \triangleright v)$ and $Q(a, b, v) = \varepsilon(b)(a \triangleright v)$.

Then for all $i, j \geq 0$, $m = i + j$, $a \in K_i, b \in K_j$, and $v_1, \ldots, v_{m+1} \in V$:

$$
\begin{aligned}
&((ab)_{(1)} \triangleright v_1) \wedge \cdots \wedge ((ab)_{(m+1)} \triangleright v_{m+1}) \otimes (ab)_{(m+2)} \\
&= ((a_{(1)}b_{(1)}) \triangleright v_1) \wedge \cdots \wedge ((a_{(m+1)}b_{(m+1)}) \triangleright v_{m+1}) \otimes (a_{(m+2)}b_{(m+2)}) \\
&= (P(a_{(1)}, b_{(1)}, v_1) + Q(a_{(1)}, b_{(1)}, v_1)) \wedge \ldots \\
&\cdots \wedge (P(a_{(m+1)}, b_{(m+1)}, v_{m+1}) + Q(a_{(m+1)}, b_{(m+1)}, v_{m+1})) \otimes (a_{(m+2)}b_{(m+2)})
\end{aligned}
$$

Once we expand the $(m+1)$-fold wedge product, we obtain a sum of wedge products with factors of the form $P(\ldots)$ or $Q(\ldots)$. Since there are $m+1$ of these factors, each summand will have at least $(i+1)$ factors of the form $P(\ldots)$ or at least $(j+1)$ factors of the form $Q(\ldots)$. Hence we can swap wedge factors for the cost of a minus sign and relabel $v_1, \ldots, v_{m+1}$ using cocommutativity of H such that each summand of the expansion will contain a term of the form

$$
\begin{aligned}
&P(a_{(1)}, b_{(1)}, v_1) \wedge \cdots \wedge P(a_{(i+1)}, b_{(i+1)}, v_{(i+1)}) \\
\text{or} \quad &Q(a_{(1)}, b_{(1)}, v_1) \wedge \cdots \wedge Q(a_{(j+1)}, b_{(j+1)}, v_{(j+1)}) \ .
\end{aligned}
$$

Both terms vanish, since $a \in K_i$ and $b \in K_j$. Hence, $ab \in K_m$ and the filtration is an algebra filtration, as desired. □

However, $(K_i)_i$ is more than an algebra filtration of H.

Lemma 5.1.6. *K_i is a subcoalgebra of H and a submodule of H under the adjoint action for each $i \geq 0$.*

Proof. To see that K_i is a subcoalgebra, we consider $h \in K_i$ and we write $\Delta(h) = \sum_k r^k \otimes h^k$ for linearly independent $(h^k)_k$ in H and suitable elements $(r^k)_k$ in H. For a given index j, let p_j be a projection of H onto the span of h^j along the span of all h^k for $k \neq j$. Then

$$
\begin{aligned}
&(r^j_{(1)} \triangleright v_1) \wedge \cdots \wedge (r^j_{(i+1)} \triangleright v_{i+1}) \otimes r^j_{(i+2)} \otimes h^j \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes p_j) \left(\sum_k (r^k_{(1)} \triangleright v_1) \wedge \cdots \wedge (r^k_{(i+1)} \triangleright v_{i+1}) \otimes r^k_{(i+2)} \otimes h^k \right) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes p_j) \left(\sum_k (r^k_{(1)} \triangleright v_1) \wedge \cdots \wedge (r^k_{(i+1)} \triangleright v_{i+1}) \otimes h^k_{(1)} \otimes h^k_{(2)} \right) = 0 \ ,
\end{aligned}
$$

for all $v_1, \dots, v_{i+1} \in V$, where we have used the coassociativity of the coproduct and our assumption that $h \in K_i$. So $r^j \in K_i$, and as H is cocommutative, K_i is a subcoalgebra.

To see that K_i is a submodule of H under the adjoint action, we first note that

$$(k \cdot h) \triangleright v = (k_{(1)} h S k_{(2)}) \triangleright v = k_{(1)} \cdot (h \triangleright (S k_{(2)} \cdot v)) \quad \text{for all } h, k \in H, v \in V \ .$$

Now let us assume $h \in K_i$. Then

$$\begin{aligned}
&((k \cdot h)_{(1)} \triangleright v_1) \wedge \cdots \wedge ((k \cdot h)_{(i+1)} \triangleright v_{i+1}) \otimes (k \cdot h)_{(i+2)} \\
&= (k_{(1)} \cdot (h_{(1)} \triangleright (S k_{(2)} \cdot v_1)) \wedge \dots \\
&\qquad \cdots \wedge (k_{(2i+1)} \cdot (h_{(i+1)} \triangleright (S k_{(2i+2)} \cdot v_{i+1})) \otimes k_{(2i+3)} \cdot h_{(i+2)} \\
&= k_{(1)} \cdot ((h_{(1)} \triangleright \cdot) \wedge \dots (h_{(i+1)} \triangleright \cdot))(S k_{(2)} \cdot (v_1 \wedge \cdots \wedge v_{i+1})) \otimes k_{(3)} \cdot h_{i+2} \\
&= 0
\end{aligned}$$

for all $v_1, \dots, v_{i+1} \in V$, where we have used the action of H on $\Lambda^{i+1}(V)$. Thus indeed, $k \cdot h \in K_i$. □

As in the proof of [EGG05, Prop 2.8] we define the notation

$$(v_1, \dots, v_k | x, y) = (\kappa(x,y)_{(1)} \triangleright v_1) \wedge \cdots \wedge (\kappa(x,y)_{(k)} \triangleright v_k) \otimes \kappa(x,y)_{(k+1)} \in \Lambda^k V \otimes H \quad (5.1.5)$$

for all $v_1, \dots, v_k, x, y \in V$.

We use the filtration $(K_i)_i$ to obtain a counterpart to [EGG05, Prop. 2.8] on the "support" of maps κ with the Jacobi identity:

Proposition 5.1.7. *Assume* $\kappa \colon V \wedge V \to H$ *has the Jacobi property and* $\operatorname{char} \mathbb{F} \notin \{2, 3\}$. *Then* $\operatorname{im} \kappa \subset K_2$.

Proof. This is a word-for-word translation of [EGG05, Prop. 2.8] and the associated lemmas (Lem. 2.10, Lem. 2.11) to our situation:

Note that in A_κ,

$$[h, v] = (h_{(1)} \cdot v) h_{(2)} - \varepsilon(h_{(1)}) v h_{(2)} = (h_{(1)} \triangleright v) h_{(2)} \ ,$$

so using our new notation, the Jacobi identity reads

$$(v|x, y) + (x|y, v) + (y|v, x) = 0 \quad \text{for all } v, x, y \in V \ . \quad (5.1.6)$$

From this we get

$$(z,u|x,y) = -(z,x|y,u) - (z,y|u,x)\ ,$$

$$(z,u|x,y) = -(u,z|x,y) = (u,x|y,z) + (u,y|z,x)\ ,$$

but now taking the average on both sides and applying Equation (5.1.6) we obtain

$$(z,u|x,y) = (x,y|z,u) \quad \text{for all } z,u,x,y \in V\ . \tag{5.1.7}$$

In turn, this implies

$$(z,u,v|x,y) = (z,x,y|u,v)\ ,$$

$$(u,v,z|x,y) = (u,x,y|v,z)\ ,$$

$$(v,z,u|x,y) = (v,x,y|z,u)\ ,$$

but the right-hand sides are the same, so we can again take averages on both sides and apply Equation (5.1.6) to obtain

$$(z,u,v|x,y) = 0 \quad \text{for all } z,u,v,x,y \in V\ . \tag{5.1.8}$$

Hence if $h = \kappa(x,y) \in H$ for elements $x,y \in V$, then

$$(h_{(1)} \triangleright z) \wedge (h_{(2)} \triangleright u) \wedge (h_{(3)} \triangleright v) \otimes h_{(4)} = 0$$

for all $z,u,v \in V$, so $h \in K_2$.

Note that in order to average in this proof we implicitly used our assumption that the characteristic of the base field is not 2 or 3. □

Remark 5.1.8. Assume $H = \mathbb{C}[G]$ for a finite group G. Then a basis element $g \in G$ of $\mathbb{C}[G]$ is in K_2 if and only if $\mathrm{rnk}(g|_V - 1) \leq 2$, and a general element $\sum_g \alpha_g g$ is in K_2 if $\alpha_g = 0$ for all g for which $\mathrm{rnk}(g|_V - 1) > 2$. Hence, Proposition 5.1.7 is compatible with and, in fact, already recovers a large part of the known classification of Drinfeld Hecke algebras as explained in Theorem 2.6.3.

Remark 5.1.9. We compare this with [Kha17, Prop. 4.3] which is formulated over a field k of arbitrary characteristic, for a cocommutative bialgebra and with an additional deformation parameter λ. In case we specialize to a Hopf algebra and to $\lambda = 0$, the cited proposition yields the following result: Assume h', h'' are non-zero elements of H, $U \subset H$ is a vector space complement of kh'', $\operatorname{im}\kappa \subset h' \otimes h'' + H \otimes U$, but $\operatorname{im}\kappa \not\subset H \otimes U$. Then the operator $h'|_V - \varepsilon(h')$ has rank at most 2. In this context, let us note that our definition of the Jacobi property and our proof of Proposition 5.1.7 works even if H is a bialgebra.

We can reformulate the recent findings:

Definition 5.1.10. For each $i \geq 0$, we define the linear map

$$T_i \colon H/K_i \to \operatorname{End}(\wedge^{i+1} V) \otimes H, \qquad h \mapsto (h_{(1)} \triangleright \cdot) \wedge \cdots \wedge (h_{(i+1)} \triangleright \cdot) \otimes h_{(i+2)} \ .$$

By the definition of K_i, the map is well-defined and injective, so there is an inverse map $T_i^{-1} \colon \operatorname{im} T_i \to H/K_i$.

For a linear map $\kappa \colon V \wedge V \to H$, let us use the notation $[\kappa]_i$ for its image in $\operatorname{Hom}(V \wedge V, H/K_i)$ for any $i \geq 0$.

Corollary 5.1.11. *If $\kappa \colon V \wedge V \to H$ has the Jacobi property, then for all $x, y \in V$,*

$$[\kappa]_0(x, y) = T_0^{-1}(T_0 \circ [\kappa]_0(x, \cdot)(y) + T_0 \circ [\kappa]_0(\cdot, y)(x)) \ ,$$

$$[\kappa]_1(x, y) = T_1^{-1}(T_1 \circ [\kappa]_1(\cdot, \cdot)(x, y)) \ ,$$

and $[\kappa]_i = 0$ for all $i \geq 2$.

Proof. The three statements are reformulations of the Jacobi identity Equation (5.1.6), Equation (5.1.7) and Equation (5.1.8), respectively. □

5.2 A family of examples

Extending the class of examples we obtain from transferring the discussion in [EGG05, Sec. 2.3] to our setting, we have the following family of examples of maps with the Jacobi property and PBW property, respectively:

Definition 5.2.1. Consider elements $\tau \in (V \wedge V)^* \otimes K_0$,

$$\sigma = \sum_m \sigma_m \otimes h^m \in (V \wedge V)^* \otimes K_1 \ , \qquad \theta = \sum_i \theta_i \otimes k^i \in (V \wedge V)^* \otimes K_2 \ , \qquad (5.2.1)$$

which is to say, linear maps from $V \wedge V$ to K_0, K_1 and K_2, respectively.

Using those we define new linear maps from $V \wedge V$ to H: $\kappa_\tau(x,y) = \tau(x,y)$,

$$\kappa_\sigma(x,y) = \sum_m \sigma_m(h^m_{(1)} \triangleright x, y)h^m_{(2)} + \sigma_m(x, h^m_{(1)} \triangleright y)h^m_{(2)} \ , \tag{5.2.2}$$

$$\kappa_\theta(x,y) = \sum_i \theta_i(k^i_{(1)} \triangleright x, k^i_{(2)} \triangleright y)k^i_{(3)} \tag{5.2.3}$$

for all $x, y \in V$, and

$$\kappa = \kappa_\tau + \kappa_\sigma + \kappa_\theta \ . \tag{5.2.4}$$

Remark 5.2.2. κ_σ and κ_θ actually only depend on $[\sigma]$ and $[\theta]$ in $\mathrm{Hom}(V \wedge V, K_1/K_0)$ and $\mathrm{Hom}(V \wedge V, K_2/K_1)$, respectively. This is, because if $h \in K_0$ and $k \in K_1$, then

$$h_{(1)} \triangleright x \otimes h_{(2)} = h_{(1)} \triangleright y \otimes h_{(2)} = 0 \quad \text{and} \quad (k_{(1)} \triangleright x) \wedge (k_{(2)} \triangleright y) \otimes k_{(3)} = 0$$

for all $x, y \in V$.

Lemma 5.2.3. *Each of κ_τ, κ_σ or κ_θ as in the definition is H-equivariant if and only if the corresponding map τ, σ or θ is H-equivariant, respectively. In particular, κ is H-equivariant if τ, σ and θ are H-equivariant.*

Proof. For κ_τ, the assertion is tautological. For $\kappa_\sigma, \kappa_\theta$ let us first note that for any $h, k \in H$ and any $x \in V$,

$$k \triangleright (Sh \cdot x) = Sh_{(1)} \cdot ((h_{(2)} k Sh_{(3)}) \triangleright x) = Sh_{(1)} \cdot ((h_{(2)} \cdot k) \triangleright x)$$

using the adjoint action in H. Now a linear map from $V \wedge V$ to H is H-equivariant if the corresponding element in $(V \wedge V)^* \otimes H$ is H-invariant. So we can verify for any $h \in H, x, y \in V$:

$$\begin{aligned}
(h \cdot \kappa_\theta)(x,y) &= \sum_i \theta_i(k^i_{(1)} \triangleright (Sh_{(1)} \cdot x), k^i_{(2)} \triangleright (Sh_{(2)} \cdot y))h_{(3)} \cdot k^i_{(3)} \\
&= \sum_i \theta_i(Sh_{(1)} \cdot ((h_{(2)} \cdot k^i_{(1)}) \triangleright x), Sh_{(3)} \cdot ((h_{(4)} \cdot k^i_{(2)}) \triangleright y))h_{(5)} \cdot k^i_{(3)} \\
&= \sum_i (h_{(1)} \cdot \theta_i)((h_{(2)} \cdot k^i)_{(1)} \triangleright x, (h_{(2)} \cdot k^i)_{(2)} \triangleright y)(h_{(2)} \cdot k^i)_{(3)} \\
&= \kappa_{h \cdot \theta}(x,y) \ ,
\end{aligned}$$

and analogously for κ_σ. ∎

Remark 5.2.4. One way of obtaining H-equivariant τ, σ, θ is by choosing H-invariant elements in $(V \wedge V)^*$ and H-invariant (that is, H-central) elements in K_0, K_1, and K_2. The map κ generated according to Equation (5.2.4) will be H-equivariant and will have the Jacobi property, so $A_{H,V,\kappa}$ will be a PBW deformation. If additionally, V is an orthogonal H-module, $A_{H,V,\kappa}$ will be a Hopf–Hecke algebra.

Proposition 5.2.5. *Let κ be as in Equation* (5.2.4)*. Then it has the Jacobi property.*

In particular, if additionally, τ, σ, θ are H-equivariant, then $A = A_{H,V,\kappa}$ has the PBW property.

Proof. As in [EGG05, Thm. 2.13]: By Theorem 5.1.2, the PBW property is equivalent to the Jacobi identity if κ is H-equivariant.

To verify the Jacobi property, we consider elements $x, y, z \in V$. Recall that the Jacobi identity reads

$$0 = (\kappa(x,y)_{(1)} \triangleright z)\kappa(x,y)_{(2)} + (\kappa(y,z)_{(1)} \triangleright x)\kappa(y,z)_{(2)} + (\kappa(z,x)_{(1)} \triangleright y)\kappa(z,x)_{(2)} \ .$$

Now for all $h \in K_0$ and all $v \in V$,

$$0 = (h_{(1)} \triangleright v) \otimes h_{(2)} \ ,$$

which verifies the Jacobi identity for κ_τ.

Also, for every index m and all $x, y, z \in V$,

$$0 = (h^m_{(1)} \triangleright x) \wedge (h^m_{(2)} \triangleright y) \wedge z \otimes h^m_{(3)} \ ,$$

because $h^m \in K_1$, so applying σ_m we get

$$\begin{aligned} 0 &= \sigma_m(h^m_{(1)} \triangleright x, h^m_{(2)} \triangleright y)z \otimes h^m_{(3)} + \sigma_m(h^m_{(1)} \triangleright y, z)(h^m_{(2)} \triangleright x) \otimes h^m_{(3)} \\ &\qquad + \sigma_m(z, h^m_{(1)} \triangleright x)(h^m_{(2)} \triangleright y) \otimes h^m_{(3)} \\ &= \sigma_m(h^m_{(1)} \triangleright y, z)(h^m_{(2)} \triangleright x) \otimes h^m_{(3)} + \sigma_m(z, h^m_{(1)} \triangleright x)(h^m_{(2)} \triangleright y) \otimes h^m_{(3)} \ , \end{aligned}$$

again, because $h^m \in K_1$.

Thus,

$$0 = \sigma_m(h^m_{(1)} \triangleright x, y)(h^m_{(2)} \triangleright z)h^m_{(3)} + \sigma_m(x, h^m_{(1)} \triangleright y)(h^m_{(2)} \triangleright z)h^m_{(3)}$$
$$+ \sigma_m(h^m_{(1)} \triangleright z, x)(h^m_{(2)} \triangleright y)h^m_{(3)} + \sigma_m(z, h^m_{(1)} \triangleright x)(h^m_{(2)} \triangleright y)h^m_{(3)}$$
$$+ \sigma_m(h^m_{(1)} \triangleright y, z)(h^m_{(2)} \triangleright x)h^m_{(3)} + \sigma_m(y, h^m_{(1)} \triangleright z)(h^m_{(2)} \triangleright x)h^m_{(3)} \ ,$$

which verifies the Jacobi identity for κ_σ.

Finally for every index i and all $x, y, z \in V$,

$$0 = (k^i_{(1)} \triangleright x) \wedge (k^i_{(2)} \triangleright y) \wedge (k^i_{(3)} \triangleright z) \otimes k^i_{(4)} \ ,$$

because $k^i \in K_2$, so

$$0 = (\theta_i(k^i_{(1)} \triangleright x, k^i_{(2)} \triangleright y)(k^i_{(3)} \triangleright z) + \theta_i(k^i_{(1)} \triangleright z, k^i_{(2)} \triangleright x)(k^i_{(3)} \triangleright y)$$
$$+ \theta_i(k^i_{(1)} \triangleright y, k^i_{(2)} \triangleright z)(k^i_{(3)} \triangleright x))k^i_{(4)} \ ,$$

which verifies the Jacobi identity for κ_θ. □

Corollary 5.2.6. *In the situation of the proposition, if H is a Hopf algebra and V is an orthogonal H-module, then $A = A_\kappa$ is a Hopf–Hecke algebra.*

Definition 5.2.7. We call a PBW deformation $A = A_\kappa$ with a deformation map κ as in Equation (5.2.4) a *standard PBW deformation* or, if additionally, V is an orthogonal H-module, a *standard Hopf–Hecke algebra*.

Next, we will investigate conditions under which PBW deformations or Hopf–Hecke algebras are standard.

5.3 The pointed case

In the following, we consider the case of a pointed cocommutative Hopf algebra H over $\mathbb{F}$, a field of characteristic 0. We recall that all cocommutative Hopf algebras are pointed if $\mathbb{F}$ is algebraically closed.

Recall that by the structure theorem for cocommutative pointed Hopf algebras over a field of characteristic 0, $H = H^1 \rtimes \mathbb{F}[G(H)]$, where H^1 is the universal enveloping algebra of the Lie algebra of primitive elements in H and $\mathbb{F}[G(H)]$ is the group algebra of the group of group-likes $G(H)$ in H. For each group-like element $g \in G(H)$, $H^1 g$ is a subcoalgebra of H and $H = \bigoplus_{g\in G(H)} H^1 g$ as coalgebras.

We continue to assume that V is a finite-dimensional H-module.

Definition 5.3.1. Let κ be a linear map $V \wedge V \to H$. Then we write

$$\kappa = \sum_{g\in G(H)} \kappa_g \tag{5.3.1}$$

with component maps $\kappa_g \colon V \wedge V \to H^1 g$ for all $g \in G(H)$.

Lemma 5.3.2. *A linear map $\kappa \colon V \wedge V \to H$ has the Jacobi property if and only if κ_g has the Jacobi property for all $g \in G(H)$.*

Proof. For every $g \in G(H)$, let $p_g \colon H \to H^1 g$ be the linear projection along $\bigoplus_{g'\neq g} H^1 g'$. Then we can apply $\mathrm{id}_V \otimes p_g$ to the Jacobi identity in $V \otimes H$ to obtain the Jacobi identity for κ_g. □

Definition 5.3.3. Let C be a coalgebra. A filtration $(C_k)_{k\geq 0}$ of C as vector space is called a *coalgebra filtration* if

$$\Delta(C_k) \subset \sum_{0\leq i\leq k} C_i \otimes C_{k-i} \quad \text{for all } k \geq 0\ . \tag{5.3.2}$$

Let C_0 be the *coradical* of C, i.e. the sum of all simple subcoalgebras of C. The *coradical filtration* of C is defined inductively by $C_{k+1} = \Delta^{-1}(C_0 \otimes C + C \otimes C_k)$.

We recall well-known facts from the theory of coalgebras: The coradical filtration is a coalgebra filtration such that $C = \bigcup_{k\geq 0} C_k$ for every coalgebra C. If C is a pointed coalgebra, for instance, any cocommutative coalgebra over $\mathbb{C}$, then $C_0 = \bigoplus_{g\in G(C)} \mathbb{F}g$ for the set of group-like elements $G(C)$ in C.

We record a useful lemma.

Lemma 5.3.4. *We consider an element $g \in G(H)$ with $\mathrm{rnk}(g|_V - 1) = 1$ and we assume that $G(H)$ is finite or that V is orthogonal (i.e., there is a non-degenerate H-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on V). Then g acts diagonalizably on V.*

Proof. We can write $g|_V - 1 = f(\cdot)v$ with suitable non-zero $f \in V^*, v \in V$. It is enough to show that $f(v) \neq 0$, since then, v is an eigenvector of $g|_V$ with eigenvalue different than 1 and $\ker(g|_V - 1)$ is spanned by $(\dim V - 1)$ eigenvectors of $g|_V$ with eigenvalue 1.

In the first case, we assume $f(v) = 0$. Then for all $k \geq 1$,

$$g|_V^k = (1 + f(\cdot)v)^k = 1 + kf(\cdot)v \ ,$$

so $g|_V$ cannot have finite order, which is a contradiction. Hence $f(v) \neq 0$.

In the second case, assume $f(v) = 0$ and consider $x \in \ker f$. Then for all $y \in V$,

$$\langle x, y \rangle = \langle gx, gy \rangle = \langle x, y + f(y)v \rangle \Rightarrow 0 = f(y)\langle x, v \rangle \ ,$$

so $\langle x, v \rangle = 0$, since $f \neq 0$. In particular, $\langle v, v \rangle = 0$. Also, for all $x \in V$,

$$\langle x, x \rangle = \langle gx, gx \rangle = \langle x + f(x)v, x + f(x)v \rangle \Rightarrow 0 = 2f(x)\langle x, v \rangle \ ,$$

so $\langle x, v \rangle = 0$ for all $x \notin \ker f$. Together, these statements imply that $\langle x, v \rangle = 0$ for all $x \in V$, which is a contradiction. Hence $f(v) \neq 0$. □

We have the following information on the group-like elements g which can contribute to deformation maps κ with the Jacobi property, and their corresponding contributions κ_g (see [RS03, Sec. 1], [EGG05, Sec. 2.3]):

Proposition 5.3.5. *Let $\kappa \colon V \wedge V \to H$ be a linear map with the Jacobi property. Then the following holds for every $g \in G(H)$, where $(g-1)$ denotes the corresponding operator on V:*

- $\kappa_g = 0$ *if* $\mathrm{rnk}(g-1) \notin \{0, 1, 2\}$.
- *If* $\mathrm{rnk}(g-1) = 1$*, then* $\kappa_g(x, y) = 0$ *for all* $x, y \in V$ *satisfying*

$$((g-1) \cdot x) \otimes y - ((g-1) \cdot y) \otimes x = 0 \ .$$

- *If* $\mathrm{rnk}(g-1) = 1$ *and* g *acts diagonalizably on* V *(for instance, if* $G(H)$ *is finite or* V *is an orthogonal* H*-module, see Lemma 5.3.4), then* $\kappa_g(x, y) = 0$ *for all* $x, y \in V$ *satisfying*

$$((g-1) \cdot x) \wedge y + x \wedge ((g-1) \cdot y) = 0 \ .$$

- *If* $\mathrm{rnk}(g-1) = 2$, *then* $\kappa_g(x, y) = 0$ *for all* $x, y \in V$ *satisfying*

$$((g-1) \cdot x) \wedge ((g-1) \cdot y) = 0 \ .$$

Proof. We fix $g \in G(H)$. Then by Lemma 5.3.2, κ_g has the Jacobi property, so it is enough to consider the case $\kappa = \kappa_g$.

It is a basic statement on coalgebras that every finite-dimensional subspace is contained in a finite-dimensional subcoalgebra. Let C be such a finite-dimensional subcoalgebra of $H^1 g$ (which is a subcoalgebra of H) containing $(\mathrm{im}\,\kappa_g)$. Let $(C_k)_{k \geq 0}$ be the coradical filtration of C and let k be minimal such that $\mathrm{im}\,\kappa \subset C_k$. Note that $C_0 = \mathbb{F}g$ now, because g is the unique group-like element in C.

Then we can write $\kappa_g = \sum_i \theta_i h^i$ with suitable non-zero $(\theta_i)_i$ in $(V \wedge V)^*$ and linearly independent $(h^i)_i$ in C_k. Let J be the set of indices j such that $h^j \in C_k \setminus C_{k-1}$ (where we set $C_{-1} = 0$). Since k was chosen minimally, $J \neq \emptyset$. For every $j \in J$, let p_j be a projection of C_k onto $\mathbb{F}h^j$ along C_{k-1} and along h^i for all $i \neq j$. Then

$$(\mathrm{id} \otimes p_j) \circ \Delta(h^i) = \delta_{ij} g \otimes h^j$$

for all i.

Thus if we apply $(\mathrm{id} \otimes p_j)$ to Equation (5.1.8), this yields

$$0 = (g-1) \cdot z \wedge (g-1) \cdot u \wedge (g-1) \cdot v \otimes \theta_j(x, y) h^j$$

for all $z, u, v, x, y \in V$, so the operator $(g-1)$ has rank at most 2.

If we apply $(\mathrm{id} \otimes p_j)$ to the Jacobi identity $0 = (x|y, z) + (y|z, x) + (z|x, y)$ in $V \otimes H$ for any $x, y, z \in V$, we obtain

$$0 = (((g-1) \cdot x)\theta_j(y, z) + ((g-1) \cdot y)\theta_j(z, x) + ((g-1) \cdot z)\theta_j(x, y)) \otimes h^j \ .$$

Let us assume that $(g-1)$ has rank 1, and let us pick $f \in V^*$ and $z \in V$ such that $f((g-1) \cdot z) = 1$. Then the last equation implies

$$\theta_j(x, y) = f((g-1) \cdot z)\theta_j(x, y) = -(f \otimes \theta_j(\cdot, z))(((g-1) \cdot x) \otimes y - ((g-1) \cdot y) \otimes x) \ ,$$

so that $\theta_j(x, y) = 0$ if $((g-1) \cdot x) \otimes y - ((g-1) \cdot y) \otimes x = 0$.

If additionally, g acts diagonalizably, then we can pick f, z such that $(g-1)\cdot v = f((g-1)\cdot v)z$ for all $v \in V$, so

$$\theta_j(x,y) = -\theta_j(y,(g-1)\cdot x) + \theta_j(x,(g-1)\cdot y) = \theta_j((g-1)\cdot x \wedge y + x \wedge (g-1)\cdot y) \ ,$$

which confirms that $\theta_j(x,y) = 0$ if $(g-1)\cdot x \wedge y + x \wedge (g-1)\cdot y = 0$.

Let us assume that $(g-1)$ has rank 2. We apply $(\mathrm{id} \otimes p_j)$ to Equation (5.1.7) to obtain

$$(g-1)\cdot z \wedge (g-1)\cdot u \otimes \theta_j(x,y) = (g-1)\cdot x \wedge (g-1)\cdot y \otimes \theta_j(z,u)$$

for all $z, u, x, y \in V$. Since $(g-1)$ has rank 2, we can pick z, u such that $(g-1)\cdot z \wedge (g-1)\cdot u$ is non-zero. So $\theta_j(x,y)$ has to be zero if $(g-1)\cdot x \wedge (g-1)y = 0$.

Hence, θ_j has to vanish on the subspaces as stated for every $j \in J$. Thus, if we define

$$\kappa'(x,y) = \sum_{i \notin J} \theta_i(x,y)h^i \ ,$$

then $\kappa(x,y)$ equals $\kappa'(x,y)$ on these subspaces, but $\operatorname{im}\kappa' \subset C_{k-1}$. We repeat the argument inductively replacing κ by κ' each time until $\operatorname{im}\kappa' \subset C_{-1} = 0$. □

To compare this with the classical situation of H being the group algebra of a finite group, we note:

Corollary 5.3.6. *Let $\kappa\colon V \wedge V \to H$ be an H-equivariant $\mathbb{F}$-linear map with the Jacobi property, and fix $g \in G(H)$ such that $\operatorname{rnk}(g-1) = 1$ and g acts diagonalizably on V (which is true, for instance, if $G(H)$ is finite or V is an orthogonal H-module, see Lemma 5.3.4). Let r be the non-zero eigenvalue of $(g-1)$. Then*

$$\operatorname{im}\kappa_g \subset \{x \in H^1 g \mid gxg^{-1} = (r+1)x\} \ .$$

In particular, if H is the group algebra of a finite-group, then $\kappa_g = 0$ for all g with $\operatorname{rnk}(g-1) = 1$.

Proof. Let $v \in V$ be an eigenvector of $(g-1)$ with eigenvalue $r \in \mathbb{F} \setminus \{0\}$ such that $V = \mathbb{F}v \oplus \ker(g-1)$. Now $\kappa_g(x,y) = 0$ for all $x, y \in \ker(g-1)$ and $\kappa_g(x,y) = 0$ for all

$x, y \in \mathbb{F}v$, because in both cases,

$$(g-1) \cdot x \wedge y + x \wedge (g-1) \cdot y = 0 \ .$$

Assume $x = v$ and $y \in \ker(g-1)$. Then due to H-equivariance,

$$g\kappa_g(x,y)g^{-1} = \kappa_g(g \cdot x, g \cdot y) = (r+1)\kappa_g(x,y) \ ,$$

so indeed $\operatorname{im}\kappa_g$ lies in the subspace of $H^1 g$ on which g acts by $(r+1)$.

If H is the group algebra of a finite group, then $H^1 g = \mathbb{F}g$, so any g acts trivially on $H^1 g$, but $r + 1 \neq 1$. □

Definition 5.3.7. For every $p \geq 0$ and a linear map $\kappa \colon V \wedge V \to H$, we define

$$\kappa_{(p)} = \sum_{g \in G(H), \operatorname{rnk}(g-1)=p, \operatorname{im}\kappa_g \subset K_p} \kappa_g \ .$$

We observe that if κ has the Jacobi property, by Proposition 5.1.7 and Proposition 5.3.5, $\kappa_{(p)} = 0$ for $p > 2$ and the condition $\operatorname{im}\kappa_g \subset K_2$ in the definition of $\kappa_{(2)}$ is vacuous. We also note that if κ has the Jacobi property, then $\kappa_{(p)}$ has the Jacobi property for every $p \geq 0$ by Lemma 5.3.2, since $\kappa_{(p)}$ is a sum of κ_g's.

Lemma 5.3.8. *For every $\kappa \colon V \wedge V \to H$ with the Jacobi property, $\kappa_{(0)}$ is of the form of Equation* (5.2.4).

Proof. This is immediate from the definition of $\kappa_{(0)}$. □

Proposition 5.3.9. *Assume $G(H)$ is finite or V is an orthogonal H-module. Then for every $\kappa \colon V \wedge V \to H$ with the Jacobi property, $\kappa_{(1)}$ is of the form*

$$\kappa_{(1)}(x,y) = \sum_m \sigma_m(h^m_{(1)} \triangleright x, y)h^m_{(2)} + \sigma_m(x, h^m_{(1)} \triangleright y)h^m_{(2)}$$

with h^m in K_1 and $\sigma_m \in (V \wedge V)^*$ for every m. In particular, it is of the form of Equation (5.2.4).

Proof. By Lemma 5.3.2, it is enough to show the assertion for $\kappa = \kappa_g$ for a fixed $g \in G(H)$ with $\operatorname{rnk}(g-1) = 1$ and such that $\kappa_g \subset K_1$.

We can write $\kappa = \sum_i \sigma_i h^i$ with linearly independent h^i in $H^1 g \cap K_1$ and suitable σ_i in $(V \wedge V)^*$. Let J be the set of indices j such that h^j lies in a maximal degree d of the coradical filtration. Since $\operatorname{rnk}(g-1) = 1$, by Proposition 5.3.5 we know that

$$\sigma_j(x,y) = \tilde{\sigma}_j((g-1)\cdot x \wedge y + x \wedge (g-1)\cdot y)$$

for some $\tilde{\sigma}_j$ in $(V \wedge V)^*$. We define

$$\kappa'(x,y) = \sum_{j\in J} \tilde{\sigma}_j(h^j_{(1)} \triangleright x, y)h^j_{(2)} + \tilde{\sigma}_j(x, h^j_{(1)} \triangleright y)h^j_{(2)} \ ,$$

then by Proposition 5.2.5, κ' has the Jacobi property, so $\kappa'' = \kappa - \kappa'$ has the Jacobi property, but the image of κ'' lies in degree $\leq d-1$ of the coradical filtration, because the highest degree terms of κ and κ' cancel. We can replace κ by κ'' and proceed inductively until the image of κ'' lies in degree -1, so $\kappa'' = 0$. □

Finally, for all $g \in G(H)$ with $\operatorname{rnk}(g-1) = 2$, let us fix $\theta_g \in (V \wedge V)^*$ which do not vanish on the one-dimensional spaces $(g-1)V \wedge (g-1)V$. Now the restriction of any skew-symmetric bilinear form on V to $(g-1)V \wedge (g-1)V$ is just a scalar multiple of the restriction of θ_g.

Proposition 5.3.10. *For every $\kappa\colon V \wedge V \to H$ with the Jacobi property, $\kappa_{(2)}$ is of the form*

$$\kappa_{(2)}(x,y) = \sum_{g\in G(H),\operatorname{rnk}(g-1)=2} \theta_g(h^g_{(1)} \triangleright x, h^g_{(2)} \triangleright y)h^g_{(3)}$$

with h^g in $H^1 g \cap K_2$ for every g. In particular, it is of the form Equation (5.2.4).

Proof. By Lemma 5.3.2, it is enough to show this for $\kappa = \kappa_g$ for a fixed $g \in G(H)$ with $\operatorname{rnk}(g-1) = 2$.

We can write $\kappa = \sum_i \theta_i k^i$ with linearly independent k^i in $H^1 g \cap K_2$ and suitable θ_i in $(V \wedge V)^*$. Let J be the set of indices j such that k^j lies in a maximal degree d of the coradical filtration. Since $\operatorname{rnk}(g-1) = 2$, by Proposition 5.3.5 we know that

$$\theta_j(x,y) = \tilde{\theta}_j((g-1)\cdot x, (g-1)\cdot y) = r_j\theta_g((g-1)\cdot x, (g-1)\cdot y)$$

for some $\tilde{\theta}_j$ in $(V \wedge V)^*$ and for some $r_j \in \mathbb{F}$. We define $h^j = r_j k^j$ and

$$\kappa'(x,y) = \sum_{j\in J} \theta_g(h^j_{(1)} \triangleright x, h^j_{(2)} \triangleright y)h^j_{(3)} \ ,$$

then by Proposition 5.2.5, κ' has the Jacobi property, so $\kappa'' = \kappa - \kappa'$ has the Jacobi property, but the image of κ'' lies in degree $\leq d-1$ of the coradical filtration, because the highest degree terms of κ and κ' cancel. We can replace κ by κ'' and proceed inductively until the image of κ'' lies in degree -1, so $\kappa'' = 0$. This way we see that

$$\kappa(x, y) = \sum_p \theta_g(h^p_{(1)} \triangleright x, h^p_{(2)} \triangleright y)h^p_{(3)}$$

for some $(h^p)_p$ in $H^1 g \cap K_2$, but now we can define $h^g = \sum_p h^p$ and the assertion follows. $\square$

Definition 5.3.11. Let us denote the class of Hopf–Hecke algebras $A_{H,V,\kappa}$ by $\boldsymbol{H}$ and the class of standard Hopf–Hecke algebras by $\boldsymbol{S}$ (see Definition 5.2.7). For every PBW deformation $A = A_{H,V,\kappa}$ (even if V is not an orthogonal module) we define $\mathsf{hs}(\kappa) = \kappa_{(0)} + \kappa_{(1)} + \kappa_{(2)}$ and $\mathsf{hs}(A_{H,V,\kappa}) = A_{H,V,\mathsf{hs}(\kappa)}$. We can view hs as a map $\boldsymbol{H} \to \boldsymbol{S}$.

Remark 5.3.12. To summarize Proposition 5.2.5, Lemma 5.3.8, Proposition 5.3.9 and Proposition 5.3.10, for every κ with the Jacobi property, $\mathsf{hs}(\kappa) = \kappa_{(0)} + \kappa_{(1)} + \kappa_{(2)}$ is of the form Equation (5.2.4) and has the Jacobi property. In other words, for every PBW deformation $A = A_{H,V,\kappa}$, the deformation $\mathsf{hs}(A) = A_{H,V,\kappa_{(0)}+\kappa_{(1)}+\kappa_{(2)}}$ is standard. It might be an interesting question which maps κ have the Jacobi property other than the ones of the form Equation (5.2.4), that is, which PBW deformations $A = A_{H,V,\kappa}$ are not standard PBW deformations.

Note that by Lemma 5.3.2 and Proposition 5.3.10, it is enough to consider the case $\kappa = \kappa_g$ for a fixed group-like g with $\operatorname{rnk}(g-1) \in \{0, 1\}$, and by the results in Lemma 5.3.8, and Proposition 5.3.9, an example with orthogonal V extending our partial characterization would necessarily satisfy $\operatorname{im} \kappa_g \not\subset K_{\operatorname{rnk}(g-1)}$.

If H is the group algebra of a finite group, there can be no such maps, because by Corollary 5.3.6, $\kappa_g = 0$ for all $g \in G(H)$ with $\operatorname{rnk}(g-1) = 1$ and for all $g \in G(H)$ with $\operatorname{rnk}(g-1) = 0$, $\operatorname{im} \kappa_g \subset H^1 g = kg \subset K_0$ automatically, so $\kappa = \kappa_{(0)} + \kappa_{(2)}$. In particular, all PBW deformations are standard in this case.

Assume $H = \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, V is an H-module and $A = A_{H,V,\kappa}$ is a PBW deformation for some deformation map

$\kappa\colon V \wedge V \to H$. Then $G(H) = 1$, so $\kappa_{(1)} = \kappa_{(2)} = 0$ and $\kappa_{(0)}$ is a map of the form $V \wedge V \to K_0 \subset H$. However, we will see in Section 7.2 that there are standard deformation maps κ yielding PBW deformations (even Hopf–Hecke algebras) which may not be of this form (for a special choice of the module V). Hence hs is not surjective onto $\boldsymbol{S}$.

Chapter 6

Generalized Dirac cohomology and Vogan's conjecture

In this chapter, let $\mathbb{F}$ be a field of characteristic 0. We fix a Hopf–Hecke algebra $A = A(H, V, \langle\cdot,\cdot\rangle, \kappa)$, that is,

- H is a cocommutative Hopf algebra over $\mathbb{F}$,
- V is a finite-dimensional H-module,
- $\langle\cdot,\cdot\rangle$ is an H-invariant non-degenerate symmetric bilinear form on V (Definition 2.5.8, Definition 2.5.9),
- $\kappa\colon V \wedge V \to H$ is a map with the PBW property, that is, κ is H-equivariant (in particular, $\mathbb{F}$-linear) and has the Jacobi property (Definition 5.1.1),

and A is the algebra generated by elements from H and V subject to the relations

$$h_{(1)} v S(h_{(2)}) = h \cdot v \quad \text{and} \quad vw - wv - \kappa(v \wedge w) \quad \text{for all } h \in H, v, w \in V \ ,$$

or in other words, A is the quotient $(T(V) \rtimes H)/I_\kappa$, where I_κ is the ideal generated by elements of $T(V) \rtimes H$ of the form $vw - wv - \kappa(v \wedge w)$ for $v, w \in V$. Since κ has the PBW property, A is a PBW deformation of $A_0 = S(V) \rtimes H$ (see Chapter 5).

We also fix the Clifford algebra $C = C(V, \langle\cdot,\cdot\rangle)$.

6.1 Dirac operator and Dirac cohomology

Since A has the PBW property, there is, in particular, an injective map ι_A sending $v \in V$ to the corresponding element in A, and since C is isomorphic to $\Lambda(V)$ as filtered vector spaces, there is also an injective map ι_C sending $v \in V$ to the corresponding element in C:

$$\iota_A\colon V \to A \ , \quad \iota_C\colon V \to C \ .$$

Now A and C are H-modules, where the action on A is just the adjoint action $h \mapsto h_{(1)} \cdot S(h_{(2)})$ and the action on C is induced from the action on $T(V)$. By the smash product relation in A, ι_A is H-equivariant, and ι_C is H-equivariant, since it is induced from the natural embedding of V into $T(V)$.

We will often omit the symbols ι_A and ι_C when discussing elements in $A \otimes C$. Let us pick a pair of dual bases $(v_i)^i, (v^i)_i$ of V with respect to $\langle \cdot, \cdot \rangle$.

Definition 6.1.1. We define the *Casimir element*

$$\Omega = \sum_i \iota_A(v_i)\iota_A(v^i) = \sum_i v_i v^i \qquad \text{in } A \tag{6.1.1}$$

and the *Dirac operator*

$$D = \sum_i \iota_A(v_i) \otimes \iota_C(v^i) = \sum_i v_i \otimes v^i \qquad \text{in } A \otimes C \tag{6.1.2}$$

of the Hopf-Hecke algebra $A = A(H, V, \langle \cdot, \cdot \rangle, \kappa)$.

As immediate consequences of these definitions, we have:

Lemma 6.1.2. *The definitions of Ω and D do not depend on the choice of dual bases. Ω is an H-invariant element in A and D is an H-invariant element in $A \otimes C$.*

Proof. Let us define a map $\phi\colon V \to V^*$, $v \mapsto \langle v, \cdot \rangle$. As discussed earlier (see Lemma 2.5.10), the fact that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate and H-invariant implies that ϕ is injective and H-equivariant (there is even an equivalence of the corresponding statements). So, as V is finite-dimensional, ϕ is an H-module isomorphism.

We can view the identity map $\mathrm{id}_V \in \mathrm{End}(V)$ as an element in $V^* \otimes V$. This element can be written as $\sum \langle v_i, \cdot \rangle \otimes v^i$ in $V^* \otimes V$ for any pair of dual bases $(v_i)^i, (v^i)_i$. Since id_V is H-equivariant, the element is H-invariant. Ω and D are images of this element under suitable tensor products and compositions of the H-equivariant maps $\phi^{-1}, \iota_A, \iota_C$, so they are H-invariant, as well. $\square$

Recall from Section 2.2 the definition of the quantization map q, in particular, that $q(v \wedge w) = \frac{1}{2}(vw - wv)$ in C for all $v, w \in V$. Fixing a choice of dual bases $(v_i)^i, (v^i)_i$, we also have the following relation between Ω and D:

Lemma 6.1.3. *For Ω and D as just defined,*

$$D^2 = \Omega \otimes 1 + \sum_{i<j} \kappa(v_i \wedge v_j) \otimes q(v^i \wedge v^j) \ . \tag{6.1.3}$$

in $A \otimes C$.

Proof. This is a straight-forward computation using the relations in C:

$$\begin{aligned} D^2 &= \sum_{i,j} v_i v_j \otimes v^i v^j = \sum_{i,j} v_i v_j \otimes (\tfrac{1}{2}(v^i v^j - v^j v^i) + \langle v^i, v^j \rangle) \\ &= \sum_{i,j} v_i v_j \langle v^i, v^j \rangle \otimes 1 + \sum_{i,j} v_i v_j \otimes q(v^i \wedge v^j) \\ &= \sum_i v_i v^i \otimes 1 + \sum_{i<j} (v_i v_j - v_j v_i) \otimes q(v^i \wedge v^j) \\ &= \Omega \otimes 1 + \sum_{i<j} \kappa(v_i \wedge v_j) \otimes q(v^i \wedge v^j) \ . \end{aligned}$$

□

We are ready for the definition of Dirac cohomology. Let us fix an irreducible C-module S. Then for any A-module M, $D \in A \otimes C$ acts on $M \otimes S$.

Definition 6.1.4. We define the *Dirac cohomology* as

$$H^D(M) = H^D(M; S) = \ker D / (\operatorname{im} D \cap \ker D) \ . \tag{6.1.4}$$

As an important special case, we record:

Lemma 6.1.5. *If D acts semisimply on $M \otimes S$, then $H^D(M) = \ker D = \ker D^2$.*

Proof. If $M \otimes S$ has a basis consisting of eigenvectors of D, then $\operatorname{im} D$ and $\ker D$ are the span of all eigenvectors with non-zero and zero eigenvalues, respectively. Hence, $(\operatorname{im} D \cap \ker D) = 0$, so $H^D(M) = \ker D$. Also, the same basis is a basis of eigenvectors for D^2 and the eigenvectors for the eigenvalue 0 are the same. Hence, $H^D(M) = \ker D = \ker D^2$. □

6.2 Dirac operator and pin cover

Let us recall that C is a superalgebra with a $\mathbb{Z}_2$-grading inherited from the $\mathbb{Z}$-grading of the tensor algebra, which C is a quotient of, so the equivalence classes of elements $v \in V$ have odd degree. Let us consider A as a purely even superalgebra, then $A \otimes C$ is a superalgebra whose grading is inherited from C, in particular, all elements in $H = H \otimes 1$ are even.

Since H is a cocommutative Hopf algebra acting orthogonally on V, the action of H on the tensor algebra descends to an action on the Clifford algebra C (see Lemma 2.5.11) and in fact, this action is inner, that is, there is a $\star$-invertible γ in $\operatorname{Hom}(H, C)$ such that

$$h \cdot c = (\gamma(h_{(1)}) \otimes \gamma^{-}(h_{(2)})^{\mathrm{op}}) \cdot c \quad \text{for all } h \in H, c \in C \ , \tag{6.2.1}$$

where the action on the right-hands side is the action of $C \otimes C^{\mathrm{op}}$ on C.

Definition 6.2.1. We define the map $\Delta_C \in \operatorname{Hom}(H, A \otimes C)$ to be the composition $(\mathrm{id} \otimes \gamma) \circ \Delta$.

The following results on Δ_C are immediate:

Lemma 6.2.2. *Then Δ_C is $\star$-invertible with $\star$-inverse $\Delta_C^{-} = (S \otimes \gamma^{-}) \circ \Delta$.*

Proof. We verify that

$$\begin{aligned} \Delta_C \star \Delta_C^{-} &= \mu_{H \otimes C} \circ (\mathrm{id}_H \otimes \gamma \otimes S \otimes \gamma^{-}) \circ \Delta^3 \\ &= (\mu_H \otimes \mu_C) \circ (\mathrm{id}_H \otimes S \otimes \gamma \otimes \gamma^{-}) \circ \Delta^3 = \varepsilon \ , \end{aligned}$$

because id_H and S are $\star$-inverses just as γ and γ^{-}. A similar computation shows that $\Delta_C^{-} \star \Delta_C = \varepsilon$. Note that the braiding in the category of superspaces is trivial for the tensor product $A \otimes C$, since A is purely even. □

Lemma 6.2.3. *There is a Hopf 2-cocyle $\sigma \colon H \otimes H \to \mathbb{F}$ such that Δ_C is an algebra isomorphism from H_σ to $\operatorname{im} \Delta_C$.*

Proof. This follows directly from Proposition 4.4.18. □

Now H acts on A via the adjoint action $h \mapsto h_{(1)} \cdot S(h_{(2)})$ and also on C through its action on $T(V)$. Hence H acts on $A \otimes C$. The way we have defined Δ_C implies that this is the inner action we get from Δ_C:

Lemma 6.2.4. *We have*

$$h \cdot x = (\Delta_C(h_{(1)}) \otimes \Delta_C^-(h_{(2)})^{\mathrm{op}}) \cdot x \quad \text{for all } h \in H, x \in A \otimes C \ . \tag{6.2.2}$$

Proof. The right-hand side is just $((h_{(1)} \otimes S(h_{(2)})^{\mathrm{op}})\cdot) \otimes ((\gamma(h_{(3)}) \otimes \gamma^-(h_{(4)})^{\mathrm{op}})\cdot)$ acting on $A \otimes C$. □

In the superalgebra $A \otimes C$, we have a supercommutator $[\cdot,\cdot]$, which specialized to the ordinary commutator if at least one of the elements commuted is homogeneous of even degree.

Lemma 6.2.5. *If $x \in A \otimes C$ is an H-invariant element, then*

$$[\Delta_C(h), x] = 0 \quad \text{in } A \otimes C \quad \text{for all } h \in H \ .$$

Proof. We have

$$\begin{aligned} [\Delta_C(h), x] &= \Delta_C(h)x - (1 \otimes \Delta_C(h)^{\mathrm{op}}) \cdot x \\ &= (1 \otimes \Delta_C(h_{(1)})^{\mathrm{op}})(\Delta_C(h_{(2)}) \otimes \Delta_C^-(h_{(3)})^{\mathrm{op}}) \cdot x - (1 \otimes \Delta_C(h)^{\mathrm{op}}) \cdot x \\ &= (1 \otimes \Delta_C(h_{(1)})^{\mathrm{op}}) \cdot (\varepsilon(h_{(2)})x) - (1 \otimes \Delta_C(h)^{\mathrm{op}}) \cdot x = 0 \ , \end{aligned}$$

where we use Equation (6.2.2) and the fact that x is H-invariant, so $h \cdot x = \varepsilon(h)x$ for all $h \in H$. □

Regarding $A \otimes C$ as a superalgebra, let us recall from Lemma 2.1.2 that there is a linear map $\delta\colon A \otimes C \to \mathrm{Der}(A \otimes C)$ sending any element $x \in A \otimes C$ to $\delta_x = [x, \cdot]$ which preserves graded degrees and supercommutators.

Definition 6.2.6. We define $d = \delta_D \in \mathrm{Der}(A \otimes C)$.

Definition 6.2.7. We say that D meets the *Parthasarathy condition* if

$$D^2 \in Z(A) \otimes 1 + \operatorname{im} \Delta_C \subset A \otimes C \ ,$$

and we say that $A = A(H, V, \langle \cdot, \cdot \rangle, \kappa)$ is a *Barbasch–Sahi algebra* if D satisfies the Parthasarathy condition.

Note that since C is a central simple superalgebra and A is purely even, $Z(A) \otimes 1$ equals the supercenter of the superalgebra $A \otimes C$, where $Z(A)$ is both the center and the supercenter of A.

Lemma 6.2.8. *The following can be said about d:*

- *d is an odd derivation,*
- *$d^2 = [D^2, \cdot]$, which is also the ordinary commutator in $A \otimes C$,*
- *$d \circ \Delta_C = 0$,*
- *d preserves the subspace of H-invariants $(A \otimes C)^H$,*
- *If A is a Barbasch–Sahi algebra, i.e., if D satisfies the Parthasarathy condition, then $d|_{(A \otimes C)^H} = 0$.*

Proof. By Lemma 2.1.2, d is an odd derivation, because D is an odd element of the superalgebra $A \otimes C$, and δ is preserves the supercommutator, so

$$\delta_D^2 = \tfrac{1}{2}[\delta_D, \delta_D] = \tfrac{1}{2}\delta_{[D,D]} = \delta_{D^2} \ .$$

D^2 is an even element, so $[D^2, \cdot]$ is also the ordinary commutator.

As D is H-invariant, Lemma 6.2.5 imply that $d \circ \Delta_C = 0$.

To see that d preserves the H-invariants in $A \otimes C$, consider $h \in H, x \in (A \otimes C)^H$, then

$$h \cdot [D, x] = [h_{(1)} \cdot D, h_{(2)} \cdot x] = \varepsilon(h)[D, x] \ ,$$

as desired, where the first identity holds, since $A \otimes C$ is an H-module algebra.

Now if D satisfies the Parthasarathy condition, then $D^2 \in Z(A) \otimes 1 + \operatorname{im} \Delta_C$, but for all $h \in H$, $x \in (A \otimes C)^H$,

$$[\Delta_C(h), x] = 0$$

by Lemma 6.2.5, so D^2 (super)commutes with elements from $(A \otimes C)^H$. □

6.3 The undeformed situation

In the following, let us assume that A is not only a PBW deformation of $A_0 = S(V) \rtimes H$, but that actually $\kappa = 0$ and $A = A_0 = S(V) \rtimes H$. In this situation, we have an even more concise picture of the cohomology of the differential d.

We can identify C with $\Lambda(V)$ as superspaces using the quantization map (see Section 2.2), so we can identify $S(V) \otimes C$ with the Koszul complex of $S(V)$, and we have a differential d' on this space.

Lemma 6.3.1. *If $A = A_0$, then d preserves the subspace $S(V) \otimes C$ of $A \otimes C$ and $d|_{S(V)\otimes C}$ agrees with $2d'$.*

Proof. From Lemma 6.2.8, we know that d is an odd derivation, just as d'. We also know that elements of the form $v \otimes 1$ or $1 \otimes v$ for $v \in V$ generate both $S(V) \otimes C$ and $S(V) \otimes \Lambda(V)$. Since also the quantization map identifies these kinds of elements in the two algebras, it is enough to verify the statement for them. Now

$$d(v \otimes 1) = [D, v \otimes 1] = \sum_i (v_i v - v v_i) \otimes v^i = 0 \quad \text{and}$$

$$d(1 \otimes v) = [D, 1 \otimes v] = \sum_i v_i \otimes (v^i v + v v^i) = 2 \sum_i v_i \langle v, v^i \rangle \otimes 1 = 2v \otimes 1 \ ,$$

so d preserves the subspace $S(V) \otimes C$ of $A \otimes C$ and agrees with $2d'$. $\square$

To make the connection between d and d' even more transparent, let us define two maps,

$$f\colon A \otimes C = (S(V) \rtimes H) \otimes C \to H \otimes S(V) \otimes C,$$

$$f(s \otimes h \otimes c) = h_{(1)} \otimes s \otimes c\gamma^{-}(h_{(2)}) \ ,$$

$$g\colon H \otimes S(V) \otimes C \to A \otimes C = (S(V) \rtimes H) \otimes C,$$

$$f(h \otimes s \otimes c) = s \otimes h_{(1)} \otimes c\gamma(h_{(2)}) \ .$$

,

Proposition 6.3.2. *Then f, g are mutual inverse maps and the following diagram commutes:*

$$\begin{array}{ccc} A_0 = (S(V) \rtimes H) \otimes C & \xrightarrow{f} & H \otimes S(V) \otimes C \\ \big\downarrow d & & \big\downarrow \mathrm{id} \otimes 2d' \\ A_0 = (S(V) \rtimes H) \otimes C & \xleftarrow[g]{} & H \otimes S(V) \otimes C \end{array} .$$

Proof. To see that f, g are mutual inverses, we observe that for all $h \in H, s \in S(V), c \in C$,

$$s \otimes h_{(1)} \otimes c\gamma^{-}(h_{(2)})\gamma(h_{(3)}) = s \otimes h \otimes c = s \otimes h_{(1)} \otimes c\gamma(h_{(2)})\gamma^{-}(h_{(3)}) \ ,$$

because γ, γ^{-} are mutual $\star$-inverses.

We can now verify that the diagram commutes. For h, s, c as before,

$$\begin{aligned} g \circ (1 \otimes 2d') \circ f(s \otimes h \otimes c) &= h_{(1)} \otimes (2d'(s \otimes c\gamma^{-}(h_{(2)}))(1 \otimes \gamma(h_{(3)}))) \\ &= d(1 \otimes s \otimes c\gamma^{-}(h_{(1)}))\Delta_C(h_2) = d((1 \otimes s \otimes c\gamma^{-}(h_{(1)}))\Delta_C(h_2)) \\ &= d(h_{(1)} \otimes c\gamma^{-}(h_{(1)})\gamma(h_{(3)})) = d(h \otimes s \otimes c) \ , \end{aligned}$$

where we have used that d is an odd derivation which vanishes on the image of $\Delta_C = (\mathrm{id} \otimes \gamma) \circ \Delta$. □

Proposition 6.3.3. *If $A = A_0$, then we have the following relation of subspaces in $A \otimes C$:*

$$\ker d = \operatorname{im} \Delta_C \oplus \operatorname{im} d \ . \tag{6.3.1}$$

Proof. Since f, g are isomorphisms and the above diagram commutes, the kernel of d is the image of the kernel of $(\mathrm{id} \otimes 2d')$ under g, but d' is the differential in the Koszul complex, so $\ker d' = \mathbb{F} \oplus \operatorname{im} d'$. Hence the kernel of $(\mathrm{id} \otimes 2d')$ is $H \otimes 1 \otimes 1 + H \otimes \operatorname{im} d'$. Now g maps the first space to $\operatorname{im} \Delta_C$. Consider $h \otimes d'(y)$ for $h \in H, y \in S(V) \otimes C$, then

$$g(h \otimes d'(y)) = d'(y)(1 \otimes h_{(1)} \otimes \gamma(h_{(2)})) = d(y)\Delta_C(h) = d(y\Delta_C(h)) \ ,$$

again, since d is an odd derivative which vanishes on the image of Δ_C. Hence g maps the kernel of $(1 \otimes 2d')$ to $\operatorname{im} \Delta_C + \operatorname{im} d$, so this space has to be the kernel of d.

Finally, the sum is direct, because the image of Δ_C lies in degree 0 of the $\mathbb{Z}$-grading of $S(V)$, whereas the image of d lies in all higher degrees of this grading. □

Now let d^H be the restriction of d to the subspace $(A \otimes C)^H$. We have seen in Lemma 6.2.8, that d preserves this subspace, so d^H is an endomorphism of $(A \otimes C)^H$.

Corollary 6.3.4. *If $A = A_0$, then we have the following relation of subspaces in $(A \otimes C)^H$:*

$$\ker d^H = \operatorname{im} \Delta_C^H \oplus \operatorname{im} d^H \ . \tag{6.3.2}$$

Proof. We take H-invariants of Equation (6.3.1) and we note that d is H-equivariant, because D is H-invariant. ☐

6.4 Proof of Vogan's conjecture

We return to the general situation where $A = A_\kappa$ is a PBW deformation of A_0. As seen in Lemma 6.2.8, the odd derivation d can be restricted to an odd derivation d^H of the subspace $(A \otimes C)^H$, such that $(d^H)^2 = 0$ and d^H vanishes on the subspace $(\operatorname{im} \Delta_C)^H$. We will show now that the statement of Corollary 6.3.4 holds also in the deformed situation.

For the proof, we will heavily rely on the filtration of the algebra A. We recall that $A = A_\kappa$ is the quotient of $T(V) \rtimes H$ by the ideal I_κ. The tensor algebra $T(V)$ is $\mathbb{Z}$-graded, where elements in V have degree 1 and elements in H have degree 0, and the ideal I_κ is, in general, not homogeneous, so the quotient A is filtered. Let us denote the filtration by $0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset A$. Since we require A to be a PBW deformation of A_0, elements in V have filtered degree 1 in A, that is, they are in $F_1 \setminus F_0$.

The endomorphism $d = [D, \cdot]$ is filtered of degree one, because D is in degree 1 of our filtration. We can pass to associated graded objects. The associated graded object of A is the graded algebra $\operatorname{gr}(A) = \bigoplus_{i \geq 0} F_i / F_{i-1}$ and we have projections $\pi_i \colon F_i \to F_i / F_{i-1}$ for all $i \geq 0$. By our assumption, $\operatorname{gr}(A)$ is the algebra A_0 and $\bigoplus_{i \geq 0} \pi_i$ is an isomorphism. Note that π is H-equivariant (i.e., an H-module isomorphism), because the actions on A and A_0 both are derived from the action on $T(V) \rtimes H$. The associated graded object of the endomorphism d is the endomorphism $\operatorname{gr}(d)$ defined by

$$\operatorname{gr}(d)(\pi_i(x)) = \pi_{i+1}(d(x)) = \pi_{i+1}([D, x]) \quad \text{for all } i \geq 0, x \in F_i \ ,$$

but the last expression is just the commutator $[D, x]$ computed in $\mathrm{gr}(A)$, so this is just the differential d in the undeformed situation. Let us denote this differential $\mathrm{gr}(d)$ by d_0 and its restriction to the H-invariants by d_0^H.

We can now use the results of Section 6.3 on d_0 to infer the desired results on d.

Lemma 6.4.1. *The intersection of the spaces* $\operatorname{im} d^H$ *and* F_0 *is* 0*.*

Proof. Consider an element x in the intersection, we want to show $x = 0$. The filtration $(F_i)_i$ yields a filtration $(F_i^H)_i$ of $(A \otimes C)^H$. Let $i \geq -1$ be minimal such that $x \in d^H(F_i^H)$ and let us pick $y \in F_i^H$ such that $x = d^H(y)$. Let us assume $i \geq 0$. Now y has filtered degree i, that is, it cannot lie in F_{i-1}, due to minimality of i. Then

$$d_0^H(\pi_i(y)) = \pi_{i+1}(d(y)) = \pi_{i+1}(x) = 0 \ ,$$

since x lies in F_0. Hence, $\pi_i(y)$ lies in the kernel of d_0 and Corollary 6.3.4 applies. According to this, $\pi_i(y)$ lies in $(\operatorname{im} \Delta_C)^H$ if $i = 0$ or in $\operatorname{im} d_0^H$ if $i > 0$. Thus, y lies in $(\operatorname{im} \Delta_C)^H + F_{i-1}^H$ if $i = 0$ or in $\operatorname{im} d^H + F_{i-1}$ if $i > 0$. But d^H annihilates the subspaces $(\operatorname{im} \Delta_C)^H$ and $\operatorname{im} d^H$, so $x = d(y)$ lies in $d^H(F_{i-1})$, which contradicts the minimality of i. Hence $i = -1$ and $x = 0$, as desired. □

Proposition 6.4.2. *We have the following relation of subspaces of* $(A \otimes C)^H$*:*

$$\ker d^H = (\operatorname{im} \Delta_C)^H \oplus \operatorname{im} d^H \ .$$

Proof. By Lemma 6.2.8, the two spaces on the right-hand side are contained in the left-hand side, and by the previous lemma, the sum on the right-hand side is direct, because $\operatorname{im} \Delta_C$ lies in F_0. It remains to show that $\ker d^H$ lies in $P = (\operatorname{im} \Delta_C)^H \oplus \operatorname{im} d^H$, and we prove this by induction on the filtration.

For the base case, we note that $\ker d^H \cap F_{-1} = \{0\} \subset P$. Let us assume $\ker d^H \cap F_{i-1} \subset P$ for some $i \geq 0$, and let us consider $x \in \ker d^H \cap (F_i \setminus F_{i-1})$. Then for the graded map d_0 we get

$$d_0^H(\pi_i(x)) = \pi_{i+1}(d(x)) = 0 \ ,$$

hence, Corollary 6.3.4 applies, so $\pi(x)$ lies in $(\operatorname{im} \Delta_C)^H + \operatorname{im} d_0^H$, that is, x lies in $(\operatorname{im} \Delta_C)^H + \operatorname{im} d^H + F_{i-1}$, hence, in P. □

The last result implies what is sometimes called "algebraic Vogan's conjecture" ([HP02, Ciu16]):

Corollary 6.4.3. *For any $z \in Z(A)$, there is an odd element $y \in (A \otimes C)^H$ and a uniquely determined element $\zeta(z) \in Z(H_\sigma)$ such that*

$$z \otimes 1 = \Delta_C \circ \zeta(z) + d^H(y) \quad \text{in } A \otimes C \ , \tag{6.4.1}$$

and $\zeta\colon Z(A) \to Z(H_\sigma)$ is an algebra map.

Proof. $z \otimes 1$ is an even element in $(A \otimes C)^H$ which (super)commutes with D. Hence Proposition 6.4.2 applies, and $z \otimes 1$ can be written as $a + d^H(y)$ with unambiguous elements $a \in (\operatorname{im} \Delta_C)^H$ and $d^H(y)$.

As d is a supercommutator with an odd element, it is an odd endomorphism, and its image is a graded subspace, just as the image of Δ_C (by Lemma 4.4.20). Since the action of H is graded, the respective invariant spaces are graded subspaces, as well. Hence, we can consider the homogeneous components and assume a is even and y is odd.

The fact that a is H-invariant and in $\operatorname{im} \Delta_C$ implies with Lemma 6.2.5 that it is in the supercenter of $\operatorname{im} \Delta_C$, but being even, this means it is also in the center. Now Δ_C is an algebra isomorphism between H_σ and $\operatorname{im} \Delta_C$ (Lemma 6.2.3), so $a = \Delta_C(h)$ for an unambiguous $h \in Z(H_\sigma)$, which we take to be $\zeta(z)$.

Let us define the shorthand $\zeta' = \Delta_C \circ \zeta\colon Z(A) \to \operatorname{im} \Delta_C \subset H \otimes C$. Consider $z_1, z_2 \in Z(A)$ and $y_1, y_2 \in (A \otimes C)^H$ such that $z_i \otimes 1 = \zeta'(z_i) + d^H(y_i)$ for $i = 1, 2$. Let ω be the linear involution of $A \otimes C$ which multiplies odd elements by -1 while fixing even elements. Then

$$\begin{aligned} z_1 z_2 \otimes 1 &= (\zeta'(z_1) + d^H(y_1))(\zeta'(z_2) + d^H(y_2)) \\ &= \zeta'(z_1)\zeta'(z_2) + \zeta'(z_1) d^H(y_1) + d^H(y_1)\zeta'(z_2) + d^H(y_1) d^H(y_2) \\ &= \zeta'(z_1)\zeta'(z_2) + d^H(\omega(\zeta'(z_1)) y_1 + y_1 \zeta'(z_2) + y_1 d^H(y_2)) \ , \end{aligned}$$

because d is an odd derivation which vanishes on $\operatorname{im} \Delta_C$ and on $\operatorname{im} d$. By Lemma 6.2.3, $\operatorname{im} \Delta_C$ and hence also $(\operatorname{im} \Delta_C)^H$ are closed under multiplication, thus $\zeta'(z_1)\zeta'(z_2)$ is an element in $(\operatorname{im}\Delta_C)^H$, so it has to be the unique $\zeta'(z_1z_2)$ satisfying Equation (6.4.1) for $z = z_1z_2$. For $z = 1$, $1 \otimes 1$ is the unique $\zeta(1)$ satisfying Equation (6.4.1). Hence, ζ is an algebra map. □

To make the connection with Dirac cohomology, let us recall that this was defined for any A-module M to be

$$H^D(M) = H^D(M; S) = \ker D/(\ker D \cap \operatorname{im} D)$$

for D acting on $M \otimes S$, where S is a fixed chosen irreducible C-module.

Now for any A-module M, $M \otimes S$ is an H_σ-module via Δ_C.

Lemma 6.4.4. *This induces an action of H_σ on $H^D(M)$.*

Proof. For any $h \in H$, the supercommutator $[D, \Delta_C(h)]$ vanishes by Lemma 6.2.5. Hence, the action of $\Delta_C(h)$ on $M \otimes S$ preserves the kernel and the image of D. □

Theorem 6.4.5 (Vogan's conjecture)**.** *Let M be an A-module with central character χ and with $H^D(M) \neq 0$. Then the central character of any non-zero H_σ-submodule (U, σ) of $H^D(M)$ determines the central character χ according to the formula*

$$\chi = \sigma \circ \zeta \ ,$$

where $\zeta\colon Z(A) \to Z(H_\sigma)$ is the algebra map from Corollary 6.4.3.

Proof. Consider $z \in Z(A)$, then by Corollary 6.4.3,

$$z \otimes 1 - \Delta_C \circ \zeta(z) = d^H(y) = [D, y]$$

for some $y \in (A \otimes C)^H$. Let u be a non-zero element of U, then

$$(\chi(z) - \sigma \circ \zeta(z))u = (z \otimes 1 - \Delta_C \circ \zeta(z)) \cdot u = [D, y] \cdot u = 0 \ ,$$

because D annihilates $H^D(M)$. Hence $\chi(z) = \sigma \circ \zeta(z)$, as desired. □

Remark 6.4.6. If $\mathfrak{g}$ is a complex semisimple Lie algebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then we can take $H = \mathcal{U}(\mathfrak{k})$ and $V = \mathfrak{p}$. In this case, the map γ realizing the action of H on $C(V)$ is already an algebra map (see Section 2.2.2), the Hopf 2-cocycle is trivial and ζ is an algebra map from $Z(\mathcal{U}(\mathfrak{g}))$ to $Z(\mathcal{U}(\mathfrak{k}))$, which is essentially the situation studied by Huang–Pandžić ([HP02], see Section 3.2).

On the other hand, if G is a finite group with a faithful orthogonal module V over $\mathbb{C}$, then the pin cover construction for groups (Section 2.2.1) yields a group homomorphism $\gamma\colon \tilde{G} \to \mathsf{Pin}$, where $\tilde{G}$ is the pin cover of G, or in terms of Hopf algebras, an algebra map γ from $\tilde{H} = \mathbb{C}[\tilde{G}]$ to $C(V)$. $\tilde{H}$ acts on $M \otimes S$ via $\Delta_C = (\pi \otimes \gamma) \circ \Delta$, where $\pi\colon \tilde{H} \to H$ is the projection coming from the pin cover construction. Hence, also $H' = \tilde{H}/(\ker \Delta_C)$ acts, and a computation shows that this quotient, in fact, corresponds to a Hopf 2-cocyle deformation H_σ of $H = \mathbb{C}[G]$. This is the situation studied by Ciubotaru ([Ciu16], see Section 3.3).

Chapter 7

Infinitesimal Cherednik algebras

7.1 Motivation

We fix a cocommutative Hopf algebra H over $\mathbb{C}$ and a completely reducible H-module V. We recall that a bilinear form on an H-module is called H-invariant, if it corresponds to an H-invariant element in the the dual space of the second tensor power of the module, or equivalently, if the image of any element of the Hopf algebra under the antipode map acts as the adjoint operator on the module with respect to the form in question (see Definition 2.5.8).

Proposition 7.1.1. *If V admits both a symmetric and a skew-symmetric non-degenerate H-invariant bilinear form, then V is of the form $V \cong W \oplus W^*$ for an H-module W.*

Proof. Since V is completely reducible, we can decompose V as a direct sum of simple submodules, and we can group these simple submodules such that

$$V = \bigoplus_{i=1}^{k} V_i^{a_i} \oplus \bigoplus_{j=1}^{m} W_j^{b_j} \oplus (W_j^*)^{c_j}$$

with positive integers $(a_i)_i, (b_j)_j, (c_j)_j$ and self-dual modules $(V_i)_i$ and such that $(V_i)_i$, $(W_j)_j$, $(W_j^*)_j$ are all pairwise non-isomorphic simple H-modules. As V admits a non-degenerate H-invariant bilinear form, it is self-dual, so $b_j = c_j$ for each j. Hence, it is enough to show that a_i is even for each i.

Consider two simple submodules V' and V'' of V and let α be a non-degenerate H-invariant bilinear form on V. Then $v \mapsto \alpha(\cdot, v)$ is an H-linear map from V' to $(V'')^*$, but since V' and V'' are simple, the map has to be an isomorphism or 0. Hence the restriction of α to $V_i^{a_i}$ has to be non-degenerate for each i. This means that $V_i^{a_i}$ admits both a symmetric and a skew-symmetric non-degenerate H-invariant bilinear form for each i.

We consider a fixed index i. Since V_i is self-dual, there is an H-linear isomorphism $V_i \to V_i^*$ or, equivalently, a non-degenerate H-invariant bilinear form α on V_i. We can view α as the sum of a symmetric and a skew-symmetric bilinear form, and since α is H-invariant, both summands have to be H-invariant, as well. Since V_i is simple, the space of H-linear endomorphisms, equivalently, H-invariant bilinear forms is one-dimensional. Hence α has to be symmetric (case a) or skew-symmetric (case b).

We write $V_i^{a_i} = V_i \otimes \mathbb{C}^{a_i}$ and we pick a basis $(e_p)_{1\leq p\leq a_i}$ of $\mathbb{C}^{a_i}$. Let β be a non-degenerate H-invariant skew-symmetric (case a) or symmetric (case b) bilinear form on $V_i^{a_i}$. Now for every $1 \leq p, q \leq a_i$, the map $(v, v') \mapsto \beta(v \otimes e_p, v' \otimes e_q)$ is an H-invariant bilinear form on V_i, so it has to be a multiple of α. Hence $\beta(v \otimes e_p, v' \otimes e_q) = \gamma(e_p, e_q)\alpha(v, v')$ for scalars $(\gamma(e_p, e_q))_{p,q}$, which defines a bilinear form γ on $\mathbb{C}^{a_i}$. For β to be skew-symmetric (case a) or symmetric (case b), γ has to be skew-symmetric. Now if a_i is odd, γ cannot be non-degenerate, so there is a vector $e \in \mathbb{C}^n$ such that $\gamma(e, e') = 0$ for all $e' \in \mathbb{C}^n$, and consequently, $\beta(v \otimes e, v' \otimes e') = 0$ for all $v, v' \in V_i$ and $e' \in \mathbb{C}^n$. This is a contradiction, since β was assumed to be non-degenerate. Hence a_i has to be even, which was to be shown. □

Proposition 7.1.2. *The finite-dimensional H-modules V which admit both a symmetric and a skew-symmetric non-degenerate H-invariant bilinear form are exactly the H-modules of the form $V \cong W \oplus W^*$ for finite-dimensional H-modules W.*

Proof. It remains to show that modules of the form $W \oplus W^*$ admit both forms as required. Let $(\cdot, \cdot)\colon W \otimes W^* \to \mathbb{C}$ be the natural pairing. By definition of the contragredient action of H on W^*, the pairing is H-invariant. We define the forms α, β by

$$\alpha(y + x, y' + x') = (y, x') + (y', x) \ , \qquad \beta(y + x, y' + x') = (y, x') - (y', x)$$

for all $y, y' \in W, x, x' \in W^*$. Then since $(\cdot, \cdot)$ is H-invariant, α and β are H-invariant. By definition, they are non-degenerate, bilinear and also symmetric and skew-symmetric, respectively. □

Remark 7.1.3. One might want to look for Hopf–Hecke algebras constructed from completely reducible orthogonal H-modules V with a non-degenerate H-invariant skew-symmetric bilinear form. Then Proposition 7.1.2 tells us that these modules are exactly the ones of the form $W \oplus W^*$.

Now if we take H to be the universal enveloping algebra of the Lie algebra of a reductive algebraic group, a class of such Hopf–Hecke algebras called *infinitesimal Cherednik algebras* is defined in [EGG05].

Remark 7.1.4. The infinitesimal Hecke algebras of Sp_{2n} with the standard module $V = \mathbb{C}^{2n}$ classified in [EGG05, Sec. 4.1.2] and studied in [Kha05, TK10, DT13, LT14] are not Hopf–Hecke algebras, since the module does not have a non-degenerate invariant symmetric bilinear form; this follows from the above discussion, for instance, because we saw that a simple module cannot have a symmetric and a skew-symmetric non-degenerate invariant form at the same time.

We will study the Dirac cohomology of the infinitesimal Cherednik algebras for the group GL_n.

7.2 Infinitesimal Cherednik algebras of GL_n as Hopf–Hecke algebras

For a fixed $n \geq 1$ and with $\mathbb{F} = \mathbb{C}$, we consider the general linear group $\mathsf{GL}_n(\mathbb{C})$, its Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and its universal enveloping algebra $H = \mathcal{U}(\mathfrak{gl}_n)$. We consider the standard Lie algebra (and hence H-)module $\mathfrak{h} = \mathbb{C}^n$. We define the H-module $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h}^*$ is the usual contragredient module, and we denote the pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$ by $(\cdot, \cdot)$.

The following definitions are from [EGG05]:

Definition 7.2.1 (r_m)**.** For all $m \geq 0$, $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$, let $r_m(x, y)$ be the coefficient of τ^m in the expansion of the polynomial function with a formal variable τ

$$A \mapsto (x, (1 - \tau A)^{-1} \cdot y) \det(1 - \tau A)^{-1} \tag{7.2.1}$$

in $S(\mathfrak{gl}_n^*)$ viewed as an element in $S(\mathfrak{gl}_n^*) \simeq S(\mathfrak{gl}_n) \simeq \mathcal{U}(\mathfrak{gl}_n)$, where the first identification is via the trace pairing $\mathfrak{gl}_n \otimes \mathfrak{gl}_n \to \mathbb{C}, (A, B) \mapsto \mathrm{Tr}(AB)$ and the second identification is via the symmetrization map.

Let $\xi(z) = \sum_{m\geq 0} \xi_m z^m$ be a polynomial. We define a map $\kappa = \kappa_\xi \colon V \wedge V \to H$ by $\kappa(x, x') = \kappa(y, y') = 0$ and

$$\kappa(y, x) = \sum_{m\geq 0} \xi_m r_m(x, y) \tag{7.2.2}$$

for all $x, x' \in \mathfrak{h}^*$ and all $y, y' \in \mathfrak{h}$. Let I_κ be the ideal of $T(V) \rtimes H$ generated by elements of the form $vw - wv - \kappa(v, w)$ for $v, w \in V$. The algebra

$$\mathcal{H}_\xi = (T(V) \rtimes H)/I_\kappa$$

is called *infinitesimal Cherednik algebra*.

There is an alternative definition of κ in terms of ξ as explained in [EGG05, Sec. 4.2] (see also [DT13, Sec. 3.1]):

Definition 7.2.2. Let $\tilde{\xi}$ be the polynomial

$$\tilde{\xi}(z) = \frac{1}{2\pi^n}\partial^n(z^n\xi(z)) = \sum_{m\geq 0} \frac{1}{2\pi^n}\frac{(m+n)!}{m!}\xi_m z^m \ . \tag{7.2.3}$$

We define the notations $\langle v, w\rangle_H = v^T\overline{w}$, which is a Hermitian inner product on $\mathfrak{h}$, and $|v| = (\sum_i |v_i|^2)^{1/2}$ for all $v \in \mathfrak{h}$, the Euclidean norm.

For every non-zero $v \in \mathfrak{h}$, let $v \otimes \overline{v}$ denote the rank-one endomorphism $v\langle\cdot, v\rangle_H$ of $\mathfrak{h}$ viewed as an element in $\mathfrak{gl}_n$, so $\tilde{\xi}(v \otimes \overline{v})$ can be viewed as an element in $S(\mathfrak{gl}_n)$ or $\mathcal{U}(\mathfrak{gl}_n)$ (using the symmetrization map).

Lemma 7.2.3 ([EGG05, Sec. 4.2])**.** *With the definitions as above,*

$$\kappa(y, x) = \int_{|v|=1} (x, (v \otimes \overline{v}) \cdot y)\tilde{\xi}(v \otimes \overline{v})\, dv \ , \tag{7.2.4}$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$.

Proof. We recall results from [EGG05, Sec. 4.2]: Let $F_m \in S(\mathfrak{gl}_n^*)$ be defined by

$$F_m(A) = \int_{|v|=1} \langle A \cdot v, v\rangle_H^{m+1}\, dv$$

for all $A \in \mathfrak{gl}_n$. According to the computations in [EGG05, Sec. 4.2], $F_m(A)$ equals the coefficient of τ^{m+1} in

$$2\pi^n \frac{(m+1)!}{(m+n)!}\det(1 - \tau A)^{-1} \ .$$

As explained in [EGG05, Sec. 4.2], under the identification $S(\mathfrak{gl}_n) \simeq S(\mathfrak{gl}_n^*)$,

$$\int_{|v|=1} (x, (v \otimes \overline{v}) \cdot y)(v \otimes \overline{v})^m \, dv = \int_{|v|=1} (x, (v \otimes \overline{v}) \cdot y)\langle A \cdot v, v\rangle_H^m \, dv$$
$$= \frac{1}{m+1} dF_m|_A(y \otimes x) = 2\pi^n \frac{m!}{(m+n)!} r_m$$

where $A \in \mathfrak{gl}_n$ symbolizes the argument of a polynomial function in $S(\mathfrak{gl}_n^*)$, and where r_m is the coefficient of τ^m in $(x, (1-\tau A)^{-1} \cdot y)\det(1-\tau A)^{-1}$ as in Definition 7.2.1.

Now if we write $\tilde{\xi}(z) = \sum_{m\geq 0} \tilde{\xi}_m z^m$, then by definition, $\tilde{\xi}_m = \frac{1}{2\pi^n}\frac{(m+n)!}{m!}\xi_m$ for all $m \geq 0$, so

$$\int_{|v|=1} (x, (v \otimes \overline{v}) \cdot y)\tilde{\xi}(v \otimes \overline{v}) \, dv = \sum_{m\geq 0} 2\pi^n \frac{m!}{(m+n)!} \tilde{\xi}_m r_m(x, y) = \sum_{m\geq 0} \xi_m r_m(x, y) \ .$$

□

Remark 7.2.4. In fact, [EGG05, Thm. 4.2] says that $(T(V) \rtimes H)/I_\kappa$ has the PBW property if and only if κ is of the described form (for some polynomial ξ). Since we will see that V is an orthogonal $\mathcal{U}(\mathfrak{gl}_n)$-module, the Hopf–Hecke algebras (H, V, κ) with $H = \mathcal{U}(\mathfrak{gl}_n)$ and $V = \mathfrak{h} \oplus \mathfrak{h}^*$ are exactly the infinitesimal Cherednik algebras.

We also note that the presentation of infinitesimal Cherednik algebras is in “reverse order” here: In [EGG05], infinitesimal Cherednik algebras for a reductive algebraic group G with a module $\mathfrak{h}$ are parametrized by G-invariant distributions on the closed subscheme of “complex reflections” $\Phi \subset G$ defined by $\wedge^2(1 - g|_\mathfrak{h}) = 0$ which are supported at 1. It is shown that for $G = \mathsf{GL}_n$, those distributions are parametrized by polynomials in one variable. The relation between polynomials and resulting deformations is computed to be Equation (7.2.4). After evaluating the integral, the equivalent formulation Equation (7.2.2) is given.

The center of these algebras has been shown to be a polynomial algebra in n variables in [Tik10]. Their representation theory has been studied and, in particular, their finite-dimensional irreducible modules have been classified in [DT13]. Universal infinitesimal Cherednik algebras, which are the analogs of infinitesimal Cherednik algebras with $\xi_1, \ldots, \xi_n$ viewed as formal parameters have been identified with W-algebras of the same type and a 1-block nilpotent element in [LT14].

We want to see that $\mathcal{H}_\xi$ is a Hopf–Hecke algebra in our notation and we want to find a description of D^2.

Definition 7.2.5. Let $(\cdot,\cdot)\colon \mathfrak{h}^* \otimes \mathfrak{h} \to \mathbb{C}$ be the natural pairing, which is $\mathfrak{gl}_n$-invariant. We define a form $\langle\cdot,\cdot\rangle$ on V by

$$\langle x+y, x'+y'\rangle = (x,y') + (x',y) \tag{7.2.5}$$

for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

We pick dual bases $(x_i)_i$, $(y_i)_i$ of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, and we define

$$(v_k)_k = (x_1,\ldots,x_n,y_1,\ldots,y_n), \qquad (v^k)_k = (y_1,\ldots,y_n,x_1,\ldots,x_n)\ . \tag{7.2.6}$$

Lemma 7.2.6. *In the situation as in the definition, $\langle\cdot,\cdot\rangle$ is a symmetric $\mathfrak{gl}_n$-invariant bilinear form on V, i.e., V is an orthogonal H-module and $\mathcal{H}_\xi$ is a Hopf–Hecke algebra, and $(v_k)_k$, $(v^k)_k$ is a pair of dual bases for V with respect to $\langle\cdot,\cdot\rangle$.*

Proof. $\langle\cdot,\cdot\rangle$ makes V an orthogonal H-module with the described pair of dual bases, because the natural pairing $(\cdot,\cdot)$ is $\mathfrak{gl}_n$-invariant, as we have seen in Proposition 7.1.2 already.

By construction, $\mathcal{H}_\xi$ has the PBW property, so it is a Hopf–Hecke algebra. □

For $1 \le i,j \le n$, let E_{ij} be the elementary matrix as an element in $\mathfrak{gl}_n$. Recall (Lemma 2.5.11) that the action of a cocommutative Hopf algebra on an orthogonal module comes with an action of the Clifford algebra which can be realized as an inner measuring by a suitable map γ from the Hopf algebra to the Clifford algebra.

Proposition 7.2.7. *The map $\gamma\colon H = \mathcal{U}(\mathfrak{gl}_n) \to C$ realizing the action of H on C as an inner measuring is an algebra map given by $\gamma(E_{ij}) = \frac{1}{2}q(y_i \wedge x_j)$ for all $1 \le i,j \le n$.*

Proof. We are in the situation of Section 2.2.2 and Section 4.2, where the action of $\mathfrak{gl}_n$ on the orthogonal module V corresponds to a map $\mathfrak{gl}_n \to \mathfrak{so}(V)$. Now the action of $\mathfrak{so}(V)$ on V is realized as an inner action by mapping $A_{v,w}$ to $q(v \wedge w)$. The action of E_{ij} on V is just

$$\langle x_j,\cdot\rangle y_i - \langle y_i,\cdot\rangle x_j = \tfrac{1}{2}A_{y_i,x_j}\ ,$$

which is mapped to

$$\tfrac{1}{2}q(y_i \wedge x_j) = \tfrac{1}{4}(y_i x_j - x_j y_i) \ .$$

The Lie algebra map defined like this is promoted to an an algebra map from H to C. □

We recall the definitions of the Casimir element $\Omega = \sum_k v_k v^k$ in $A = \mathcal{H}_\xi$ and of the Dirac element $D = \sum_k v_k \otimes v^k$ in $A \otimes C$ (Definition 6.1.1) for any pair of dual bases $(v_k)_k$ and $(v^k)_k$, so, in particular, for the choice made in Definition 7.2.5.

Lemma 7.2.8. *Let $D \in A \otimes C$ be the Dirac element for $A = \mathcal{H}_\xi$. Then*

$$D^2 = \Omega \otimes 1 - 2 \int_{|v|=1} \tilde{\xi}(v \otimes \overline{v}) \otimes \gamma(v \otimes \overline{v})\, dv \ , \tag{7.2.7}$$

where $v \otimes \overline{v}$ stands for a rank-one matrix in $\mathfrak{gl}_n$ for any non-zero $v \in \mathfrak{h}$.

Proof. We invoke Equation (6.1.3) to obtain

$$\begin{aligned} D^2 &= \Omega \otimes 1 + \sum_{k<l} \kappa(v_k, v_l) \otimes q(v^k \wedge v^l) = \Omega \otimes 1 + \sum_{i,j} \kappa(y_j, x_i) \otimes q(x_j \wedge y_i) \\ &= \Omega \otimes 1 - 2 \sum_{i,j} \kappa(y_j, x_i) \otimes \gamma(E_{ij}) \ , \end{aligned}$$

where $E_{ij} = y_i \otimes x_j$ as above is an element in $\mathfrak{gl}_n$ for all i, j. Using the integral formula Equation (7.2.4) for κ, we obtain

$$\begin{aligned} \sum_{i,j} \kappa(y_j, x_i) \otimes \gamma(E_{ij}) &= \sum_{i,j} \int_{|v|=1} \tilde{\xi}(v \otimes \overline{v}) \otimes (x_i, (v \otimes \overline{v}) y_j) \gamma(E_{ij})\, dv \\ &= \int_{|v|=1} \tilde{\xi}(v \otimes \overline{v}) \otimes \gamma(v \otimes \overline{v})\, dv \ , \end{aligned}$$

as desired. □

In the following, we want to find an even more explicit expression for D^2 in terms of polynomials derived from $\tilde{\xi}$, which will allow us to prove that D satisfies the Parthasarathy condition and hence, $\mathcal{H}_\xi$ is a Barbasch–Sahi algebra. We need some auxiliary lemmas.

In the following, polynomials without further specifications are univariate and with complex coefficients.

Definition 7.2.9. For any $\varepsilon \in \mathbb{C}$, we define ∇_ε, a difference operator on univariate complex polynomials, by

$$\nabla_\varepsilon f(z) = f(z+\varepsilon) - f(z+\varepsilon-1)\ . \tag{7.2.8}$$

For $k \geq 0$, let $B_k = B_k(z)$ be the k-th *Bernoulli polynomial* defined by the generating series

$$\sum_{k\geq 0} B_k(z)\frac{t^k}{k!} = \frac{te^{tz}}{e^t-1} \tag{7.2.9}$$

([AS92, Eq. 23.1.1]).

We recall that B_k satisfies

$$\nabla_1 B_k(z) = B_k(z+1) - B_k(z) = kz^{k-1} \tag{7.2.10}$$

for every $k \geq 0$ ([AS92, Eq. 23.1.6]).

Lemma 7.2.10. *Let $p = p(z)$ be a polynomial and $\varepsilon \in \mathbb{C}$. Then there is a polynomial $f = f(z)$ satisfying $\nabla_\varepsilon f(z) = p(z)$ and f is characterized by this relation uniquely up to the constant term.*

Proof. To construct f, we write $p(z) = \sum_{i\geq 0} p_i z^i$ and we let $B_n(z)$ be the n-th Bernoulli polynomial.

Then

$$\nabla_\varepsilon f(z) = p(z) \quad \Leftrightarrow \quad \nabla_1 f(z) = p(z+1-\varepsilon) = \sum_{i\geq 0} \frac{p_i}{i+1}(i+1)(z+1-\varepsilon)^i\ ,$$

hence, $f(z) = \sum_{i\geq 0} \frac{p_i}{i+1} B_{i+1}(z+1-\varepsilon) + f_0$ satisfies this recurrence relation for any scalar f_0.

For uniqueness, let f_2 be another polynomial satisfying the same recurrence relation. Then $f_d = f - f_2$ is a polynomial satisfying $\nabla_\varepsilon f_d(z) = f_d(z+\varepsilon) - f_d(z+\varepsilon-1) = 0$. Hence f_d attains the same value at, say, all integer numbers, so it has to be a constant polynomial. □

Lemma 7.2.11. *For a fixed polynomial $p = p(z)$, let $f = f(z)$ be a polynomial satisfying $\nabla_{1/2} f(z) = p(z)$. Then*

$$p(z)\omega = f(z+\omega) + \tfrac{1}{2}p(z) - f(z+\tfrac{1}{2}) \qquad \text{in } \mathbb{C}[z,\omega] \mod (\omega^2 - \tfrac{1}{4})\cdot \tag{7.2.11}$$

Proof. We claim that for every polynomial p, there are polynomials f, q such that

$$p(z)\omega = f(z+\omega) + q(z) \qquad \text{in } \mathbb{C}[z,\omega] \mod (\omega^2 - \tfrac{1}{4}) \ . \tag{7.2.12}$$

First, we note that it is enough to show this for polynomials p of the form $p(z) = (k+1)z^k$, because those form a basis. Consider $k = 0$. Then $p(z)\omega = \omega = (z+\omega) - z$, which verifies the claim. Assume the claim is true for all non-negative integers $0 \leq k < K$ for some $K \geq 1$, and hence for all polynomials p of degree at most $K-1$. We consider $p(z) = (K+1)z^K$, $f(z) = z^{K+1}$, and $q(z) = -z^{K+1}$, then

$$p(z)\omega = f(z+\omega) + q(z) + p'(z)\omega + q'(z)$$

for polynomials p', q' (because $\omega^2 \equiv \frac{1}{4}$), and $\deg p' \leq K-1$. This proves the claim by induction.

We assume now f, q are as in Equation (7.2.12). Then we can substitute $\omega = \pm\frac{1}{2}$ to get

$$q(z) = \pm\tfrac{1}{2}p(z) - f(z \pm \tfrac{1}{2}) \ .$$

However, the two choices of substitution should yield the same result, so

$$\tfrac{1}{2}p(z) - f(z+\tfrac{1}{2}) = -\tfrac{1}{2}p(z) - f(z-\tfrac{1}{2}) \qquad \Leftrightarrow \qquad f(z+\tfrac{1}{2}) - f(z-\tfrac{1}{2}) = p(z)$$

and using the choice $\omega = \frac{1}{2}$ to obtain the above expression of q in terms of p, f, we have

$$p(z)\omega = f(z+\omega) + \tfrac{1}{2}p(z) - f(z+\tfrac{1}{2}) \ ,$$

as desired. □

Lemma 7.2.12. *Let v be a vector in $\mathfrak{h}$ with $|v| = 1$, and let $v \otimes \overline{v}$ be the corresponding rank-one matrix in $\mathfrak{gl}_n$. Then $\gamma(v \otimes \overline{v})^2 = \frac{1}{4}$ in $C(V)$.*

Proof. We write $v = \sum_i a_i y_i$ with coefficients $(a_i)_i$ in $\mathbb{C}$, then by linearity of γ,

$$\gamma(v \otimes \overline{v}) = \sum_{i,j} a_i \overline{a}_j \gamma(E_{ij}) = \tfrac{1}{2}\sum_{i,j} a_i \overline{a}_j q(y_i \wedge x_j) = \tfrac{1}{2} q(v \wedge v^*)$$

for $v^* = \sum_i \overline{a}_i x_i \in \mathfrak{h}^*$, where we used the value of $\gamma(E_{ij})$ as discussed in Proposition 7.2.7.

v and v^* can be regarded as elements in V or in $C(V)$ and in $C(V)$,

$$v^2 = \langle v, v\rangle = 0\ , \quad (v^*)^2 = \langle v^*, v^*\rangle = 0\ , \quad \text{and} \quad vv^* + v^*v = 2\langle v, v^*\rangle = 2\ ,$$

because $\langle \mathfrak{h}, \mathfrak{h}\rangle = 0$, $\langle \mathfrak{h}^*, \mathfrak{h}^*\rangle = 0$ and $\langle y_i, x_j\rangle = \delta_{ij}$. Hence,

$$\begin{aligned}\gamma(v\otimes \overline{v})^2 &= \tfrac{1}{16}(vv^*vv^* + v^*vv^*v - v(v^*)^2v - v^*v^2v^*)\\ &= \tfrac{1}{16}(v(2-vv^*)v^* + v^*(2-v^*v)v) = \tfrac{1}{8}(vv^* + v^*v) = \tfrac{1}{4}\ ,\end{aligned}$$

as desired. □

We are ready to give a refined formula for D^2.

Definition 7.2.13. Let $f_\xi = f_\xi(z)$ be the polynomial defined up to a constant by

$$\nabla_0 f_\xi(z) = f_\xi(z) - f_\xi(z-1) = \tilde{\xi}(z) = \tfrac{1}{2\pi^n}\partial^n(z^n\xi(z))$$

(the first and the last equality being the definitions of ∇_0 and $\tilde{\xi}$, respectively). Furthermore, we define $\alpha, \beta \in \mathcal{U}(\mathfrak{gl}_n)$ by

$$\alpha = \int_{|v|=1} -\tilde{\xi}(v\otimes\overline{v}) + 2f_\xi(v\otimes\overline{v})\,dv\ , \qquad \beta = \int_{|v|=1} 2f_\xi(v\otimes\overline{v} - \tfrac{1}{2})\,dv\ ,$$

and $C' = \int_{|v|=1} f_\xi(v\otimes\overline{v})\,dv$.

Proposition 7.2.14. *Let $f = f_\xi$, α, β as in the definition. Then we have the following formula for D^2:*

$$D^2 = (\Omega + \alpha)\otimes 1 - \Delta_C(\beta)\ . \tag{7.2.13}$$

Furthermore, $(\Omega+\alpha) = 2(\sum_i x_iy_i + C')$ and this element is central in A, and β, C' are central in H. In particular, D satisfies the Parthasarathy condition (Definition 6.2.7) and $\mathcal{H}_\xi$ is a Barbasch–Sahi algebra.

Proof. We fix $v \in \mathfrak{h}$ with $|v| = 1$ and define elements $z = (v\otimes\overline{v})\otimes 1$, $\omega = 1\otimes\gamma(v\otimes\overline{v})$ in $A\otimes C$. Then $z+\omega = \Delta_C(v\otimes\overline{v})$ and $\omega^2 = \frac{1}{4}$ by Lemma 7.2.12. We observe that $\nabla_{1/2}f(z-\frac{1}{2}) = \nabla_0 f(z) = \tilde{\xi}(z)$ by the definition of $f = f_\xi$. So we can apply Lemma 7.2.11 to obtain

$$\tilde{\xi}(v\otimes\overline{v})\otimes\gamma(v\otimes\overline{v}) = f(\Delta_C(v\otimes\overline{v}) - \tfrac{1}{2}) + (\tfrac{1}{2}\tilde{\xi}(v\otimes\overline{v}) - f(v\otimes\overline{v}))\otimes 1\ ,$$

which implies the new formula for D^2 with Equation (7.2.7).

We define the shorthand $M_v = v \otimes \overline{v} \in \mathfrak{gl}_n$ for any $v \in \mathfrak{h}$. To obtain the alternative formula for $(\Omega + \alpha)$ we note that

$$\begin{aligned}\Omega &= \sum_i x_i y_i + y_i x_i = \sum_i 2x_i y_i + [y_i, x_i] = \sum_i 2x_i y_i + \int_{|v|=1} (x_i, M_v \cdot y_i)\tilde{\xi}(M_v)\, dv \\ &= 2\sum_i x_i y_i + \int_{|v|=1} \tilde{\xi}(M_v)\, dv \ ,\end{aligned}$$

where we use that $\sum_i (x_i, (v \otimes \overline{v}) \cdot y_i) = \sum_i |v_i|^2 = 1$.

We want to show now that $\Omega + \alpha$ is central in A. We will prove that $\Omega + \alpha$ commutes with a set of algebra generators of $A \otimes C$, so we consider elements of $\mathfrak{h}, \mathfrak{h}^*$ and $\mathfrak{gl}_n$.

Let us first fix $y, v \in \mathfrak{h}$ such that $|v| = 1$ and $M = M_v$. We regard M as an element in a universal enveloping algebra, so M^k denotes a tensor power of M for all $k \geq 0$. If $\mu\colon \mathfrak{gl}_n^{\otimes k} \to \mathfrak{gl}_n$ is the matrix multiplication, then we have $\mu(M^k) = M$ for all $k \geq 1$, so we can compute in $A = T(V) \rtimes \mathcal{U}(\mathfrak{gl}_n)$:

$$\begin{aligned}[M^k, y] &= M^k y - y M^k = ((M^k)_{(1)} \cdot y)(M^k)_{(2)} - y M^k \\ &= \sum_{i=0}^{k} \binom{k}{i} (\mu(M^{k-i}) \cdot y) M^i - y M^k \\ &= \sum_{i=0}^{k-1} \binom{k}{i} (M \cdot y) M^i = (M \cdot y)((M+1)^k - M^k) \ ,\end{aligned}$$

because M is a primitive element, so the coproduct of M^k is just $\sum_{i=0}^{k} \binom{k}{i} M^{k-i} \otimes M^i$. Hence, for any polynomial $g = g(z)$,

$$[g(M), y] = (M \cdot y)\nabla_1 g(M) \ .$$

In particular,

$$\left[\int_{|v|=1} f(M_v)\, dv, y\right] = \int_{|v|=1} (M_v \cdot y)\nabla_1 f(M_v)\, dv = \int_{|v|=1} (M_v \cdot y)\tilde{\xi}(M_v + 1)\, dv \ .$$

On the other hand,

$$M(M \cdot v) = (M_{(1)} \cdot (M \cdot v))M_{(2)} = M \cdot (M \cdot v) + (M \cdot v)M = (M \cdot v)(M+1) \ ,$$

because $M \cdot (M \cdot v) = M \cdot v$. Hence, for any polynomial $g = g(z)$,

$$g(M)(M \cdot v) = (M \cdot v) g(M+1) \ .$$

In particular,

$$\left[\sum_i x_i y_i, y\right] = \sum_i [x_i, y] y_i = -\sum_i \int_{|v|=1} (x_i, M_v \cdot y)\tilde{\xi}(M_v) y_i \, dv$$
$$= -\int_{|v|=1} \tilde{\xi}(M_v)(M_v \cdot y)\, dv = -\int_{|v|=1} (M_v \cdot y)\tilde{\xi}(M_v + 1)\, dv \ .$$

So indeed, $\Omega + \alpha$ commutes with any $y \in \mathfrak{h}$. An exactly parallel argument shows that $\Omega + \alpha$ commutes with any $x \in \mathfrak{h}^*$. (Alternatively, this follows from the existence of an anti-involution of $\mathcal{H}_\xi$ sending $y_i \leftrightarrow x_i$ and $E_{ij} \leftrightarrow E_{ji}$ as described in [DT13, Sec. 2].)

Furthermore, we have seen already that Ω commutes with elements from $\mathfrak{gl}_n$, so it remains to show that α, β and C' are central in $\mathcal{U}(\mathfrak{gl}_n)$, too. Let q be any polynomial and consider the element $h_q = \int_{|v|=1} q(M_v)\, dv$ in $\mathcal{U}(\mathfrak{gl}_n)$. We note that h_q is invariant under the adjoint action of $U(\mathfrak{h}) \subset \mathsf{GL}(\mathfrak{h})$, the unitary group of $\mathfrak{h}$ with $\langle \cdot, \cdot \rangle_H$, because $QM_vQ^* = M_{Qv}$ for all $Q \in U(\mathfrak{h}), v \in \mathfrak{h}$ and the integral is invariant under the transformation $v \mapsto Qv$. Now $\mathfrak{gl}_n$ is just the complexified Lie algebra of $U(\mathfrak{h})$, so the center of $\mathcal{U}(\mathfrak{gl}_n)$ is just the space of $U(\mathfrak{h})$-invariants in $\mathcal{U}(\mathfrak{gl}_n)$. Hence h_q is central in $\mathcal{U}(\mathfrak{gl}_n)$, and in particular, α and β are central in $H = \mathcal{U}(\mathfrak{gl}_n)$.

The various centrality statements proven together imply that D satisfies the Parthasarathy condition. □

Remark 7.2.15. We compare this with the results in [DT13]: The polynomial f_ξ corresponds to the polynomial called "$2\pi^n f$" there, the central element $(\Omega + \alpha)$ is just "$2t_1'$" in the notation of the reference, where t_1' is the Casimir element studied there, and C' is what is denoted by C' there, as well.

7.3 Dirac cohomology for infinitesimal Cherednik algebras of GL_n

Having seen that $\mathcal{H}_\xi$ is what we call a Barbasch–Sahi algebra, we can explore the Dirac cohomology of its modules. Here we will focus on the finite-dimensional irreducible modules, which have been studied in [DT13].

Let us recall the definition of the *complete symmetric homogeneous polynomial* h_k

of degree k: it is the polynomial in n variables defined by

$$h_k(z_1, \dots, z_n) = \sum_{l_1+\cdots+l_n=k, l_i\geq 0} z_1^{l_1} \dots z_n^{l_n} \ .$$

Definition 7.3.1. Let $w = \sum_{k\geq 0} w_k z^k$ be a polynomial of degree $(\deg \xi + 1)$ satisfying

$$2\pi^n f_\xi(z) = (2\sinh(\partial/2))^{n-1} z^{n-1} w(z) \ .$$

(We observe that $2\sinh(\partial/2) = (e^{\partial/2} - e^{-\partial/2})$ is just the operator $\nabla_{1/2}$ which sends a polynomial $p(z)$ to $p(z+\frac{1}{2}) - p(z-\frac{1}{2})$.) We define a polynomial in n variables $P(\mu) = \sum_{k\geq 0} w_k h_k(\mu+\rho)$, where ρ is the Weyl vector of $\mathfrak{gl}_n$.

We denote by $C'(\lambda)$ the scalar by which C' acts on an irreducible highest weight $\mathfrak{gl}_n$-module with highest weight λ and we cite the following result from [DT13]:

Lemma 7.3.2 ([DT13, Thm. 3.2])**.** *w exists and is uniquely defined up to a constant, and $C'(\lambda) = P(\lambda)$.*

We compute the relations between w, $\tilde{\xi}$, and ξ:

Lemma 7.3.3. *w is the polynomial uniquely defined up to a constant by*

$$(2\sinh(\partial/2))^n z^{n-1} w(z) = 2\pi^n \tilde{\xi}(z+\tfrac{1}{2}) \ .$$

Equivalently, w is the polynomial uniquely defined up to a constant by

$$e^{-\partial/2}(2\sinh(\partial/2))^n z^{n-1} w(z) = \partial^n(z^n \xi(z)) \ .$$

Proof. We verify that by definition of w and f_ξ,

$$\begin{aligned}(2\sinh(\partial/2))^n z^{n-1} w(z) &= 2\pi^n(2\sinh(\partial/2)) f_\xi(z) = 2\pi^n(f_\xi(z+\tfrac{1}{2}) - f_\xi(z-\tfrac{1}{2})) \\ &= 2\pi^n \tilde{\xi}(z+\tfrac{1}{2}) \ .\end{aligned}$$

Now the polynomial $\tilde{w}$ satisfying

$$(2\sinh(\partial/2))^{n-1} z^{n-1} \tilde{w}(z) = \tilde{\xi}(z+\tfrac{1}{2})$$

is uniquely defined, so w is uniquely defined up to a constant by Lemma 7.2.10.

We can apply the bijective translation operator $e^{-\partial/2}$ on both sides and use the definition of $\tilde{\xi}$ to obtain the second assertion. ☐

From here we can go on to compute the action of D^2 and finally to study the the Dirac cohomology for all finite-dimensional irreducible $\mathcal{H}_\xi$-modules. These were classified in [DT13] and we now recall the classification.

In the following, we identify $(a_1, \dots, a_n) \in \mathbb{C}^n$ with the weight $a_1 E_{11}^* + \cdots + a_n E_{nn}^*$ of $\mathfrak{gl}_n$. For every dominant integral $\mathfrak{gl}_n$-weight $\lambda = (\lambda_1, \dots, \lambda_n)$, that is, $(\lambda_i - \lambda_{i+1})$ is a non-negative integer for all $1 \leq i < n$, let V_λ be the finite-dimensional irreducible highest weight $\mathfrak{gl}_n$-module with highest weight λ.

Definition 7.3.4. We define the set

$$\tilde{\Lambda} = \{\lambda \text{ dominant integral } \mathfrak{gl}_n\text{-weight} \mid \exists \nu_n \in \mathbb{Z}_{\geq 0} : P(\lambda) = P(\lambda - (0, \dots, 0, \nu_n + 1))\}$$

and for any $\lambda \in \tilde{\Lambda}$, we define $\nu = \nu(\xi, \lambda) \in \mathbb{Z}_{\geq 0}^n$ by letting ν_i be the minimal non-negative integer such that $\lambda' = \lambda - (0, \dots, 0, \nu_i + 1, 0, \dots, 0)$ is either not a dominant weight or $P(\lambda) = P(\lambda')$, for each $1 \leq i \leq n$.

Proposition 7.3.5. *[DT13, Thm. 4.1] For any $\lambda \in \tilde{\Lambda}$, there exists a unique irreducible finite-dimensional $\mathcal{H}_\xi$-module $L(\lambda)$. More precisely,*

$$L(\lambda) = \bigoplus_{0 \leq \nu' \leq \nu} V_{\lambda - \nu'}$$

as a $\mathfrak{gl}_n$-module, where $\nu = \nu(\xi, \lambda)$ is as defined above and $\nu' \in \mathbb{Z}_{\geq 0}^n$ runs over all tuples satisfying $0 \leq \nu_i' \leq \nu_i$ for all $1 \leq i \leq n$.

The irreducible finite-dimensional $\mathcal{H}_\xi$-modules are exactly the modules $L(\lambda)$ for $\lambda \in \tilde{\Lambda}$.

Since $\dim V$ is even, the Clifford algebra $C(V)$ has a unique finite-dimensional irreducible module S (see Section 2.2), which is a $\mathfrak{gl}_n$-module via the algebra map $\gamma \colon \mathcal{U}(\mathfrak{gl}_n) \to C(V)$. Let n_μ be the multiplicity of V_μ in $M \otimes S$ for every dominant integral $\mathfrak{gl}_n$-weight μ.

Proposition 7.3.6. *Consider $M = L(\lambda)$ for a $\lambda \in \tilde{\Lambda}$ as in Proposition 7.3.5. Then the kernel of D^2 acting on $M \otimes S$ is $\bigoplus_\mu n_\mu V_\mu$, where the sum ranges over all those μ satisfying*

$$P(\lambda) = P(\mu - (\tfrac{1}{2}, \dots, \tfrac{1}{2}))$$

(where n_μ is the full multiplicity in $M \otimes S$, as just defined).

Proof. First, we recall that $D^2 = (\Omega + \alpha) - \Delta_C(\beta)$ according to Proposition 7.2.14.

Let $v_\lambda \in V_\lambda \subset M$ be a highest weight vector. From Proposition 7.2.14 we also know that $\Omega + \alpha = 2(\sum_i x_i y_i + C')$, so $\Omega + \alpha$ acts on v_λ, and hence on M, by the scalar

$$2C'(\lambda) = 2P(\lambda) \ .$$

We want to find the scalar by which β acts on $V_\mu \subset M \otimes S$ via Δ_C, i.e., via the diagonal tensor product action, where the second tensor factor is a $\mathfrak{gl}_n$-module via γ. Recall that

$$\beta = 2 \int_{|v|=1} f_\xi(v \otimes \overline{v} - \tfrac{1}{2})\, dv \ .$$

Let $\mathfrak{t}$ be the diagonal matrices in $\mathfrak{gl}_n$. We use the twisted Harish-Chandra map $Z(\mathcal{U}(\mathfrak{gl}_n)) \to S(\mathfrak{t})$ to see that this scalar is

$$\begin{aligned} 2 \int_{|v|=1} & f_\xi(\langle \mu + \rho, (|v_1|^2, \dots, |v_n|^2) \rangle - \tfrac{1}{2})\, dv \\ &= 2 \int_{|v|=1} f_\xi(\langle \mu + \rho - (\tfrac{1}{2}, \dots, \tfrac{1}{2})), (|v_1|^2, \dots, |v_n|^2) \rangle)\, dv \\ &= 2C'(\mu - (\tfrac{1}{2}, \dots, \tfrac{1}{2})) = 2P(\mu - (\tfrac{1}{2}, \dots, \tfrac{1}{2})) \ . \end{aligned}$$

This yields the desired characterization of the highest weight submodules contained in $\ker D^2$. □

We have the following information on the structure of S as a $\mathfrak{gl}_n$-module via γ (see [Kos99, Prop. 3.17]):

Lemma 7.3.7. *The weights of S are exactly the weights $(s_1, \dots, s_n)$ in $\{\pm\frac{1}{2}\}^n$, and all weight spaces are one-dimensional. Hence $S \cong \Lambda(\mathfrak{h}) \otimes (-\frac{1}{2}\operatorname{Tr})$ as $\mathfrak{gl}_n$-modules.*

Proof. We can take S to be the left ideal generated by $u = y_1 \dots y_n$ in $C(V)$, which is irreducible (this is explained, for instance, in [Kos99, Sec. 3]).

Hence, a basis of S is given by the elements $x_1^{e_1} \dots x_n^{e_n} u$ for exponents $e_1, \dots, e_n \in \{0, 1\}$. We can compute directly

$$\gamma(E_{ii}) x_i = \tfrac{1}{4}(y_i x_i - x_i y_i) x_i = -\tfrac{1}{4} x_i y_i x_i = -\tfrac{1}{2} x_i \ ,$$

$$\gamma(E_{ii}) y_i = \tfrac{1}{4}(y_i x_i - x_i y_i) y_i = \tfrac{1}{4} y_i x_i y_i = \tfrac{1}{2} y_i \ ,$$

and $\gamma(E_{ii})$ commutes with x_j or y_j in $C(V)$ for all $j \neq i$, so

$$\gamma(E_{ii}) x_1^{e_1} \dots x_n^{e_n} u = \tfrac{1}{2}(-1)^{e_i} x_1^{e_1} \dots x_n^{e_n} u$$

for all $1 \leq i \leq n$. □

We can use this result to obtain the structure of $L(\lambda) \otimes S$:

Proposition 7.3.8. *For any $\lambda \in \tilde{\Lambda}$, $L(\lambda) \otimes S$ decomposes as*

$$\bigoplus_{0 \leq \nu' \leq \nu} \bigoplus \{V_\mu \mid \mu \text{ dominant integral weight}, \mu_i - (\lambda_i - \nu'_i) \in \{\pm\tfrac{1}{2}\}\ \forall 1 \leq i \leq n\} \,. \quad (7.3.1)$$

In particular, the irreducible modules V_μ occurring are those with dominant integral weight μ satisfying

$$\mu_i \in \{\lambda_i + \tfrac{1}{2}, \lambda_i - \tfrac{1}{2}, \dots, \lambda_i - \nu_i - \tfrac{1}{2}\}$$

for all $1 \leq i \leq n$.

Proof. Let λ' be a highest $\mathfrak{gl}_n$-weight and $V_{\lambda'}$ the corresponding irreducible highest weight $\mathfrak{gl}_n$-module. Then by the Pieri rule, $V_{\lambda'} \otimes \Lambda(\mathfrak{h})$ decomposes as

$$\bigoplus \{V_\mu \mid \mu \text{ dominant integral weight}, \mu_i - \lambda'_i \in \{0, 1\}\ \forall 1 \leq i \leq n\} \,.$$

Now since $L(\lambda) = \bigoplus_{0 \leq \nu' \leq \nu} V_{\lambda - \nu'}$ and $S = \Lambda(\mathfrak{h}) \otimes (-\frac{1}{2}\operatorname{Tr})$, $L(\lambda) \otimes S$ decomposes as asserted, and listing the weights occurring just gives the desired characterization. □

This allows us the following conclusions on the irreducible highest weight $\mathfrak{gl}_n$-submodules of $\ker D^2$:

Corollary 7.3.9. *The kernel of D^2 acting on $L(\lambda) \otimes S$ is the sum of those irreducible submodules V_μ appearing in Equation* (7.3.1) *for which $P(\lambda) = P(\mu - (\frac{1}{2}, \dots, \frac{1}{2}))$.*

We define distinguished weights $\lambda^0 := \lambda + (\frac{1}{2}, \dots, \frac{1}{2})$,

$$\lambda'^i := \lambda - (0, \dots, 0, \nu_i + 1, 0, \dots, 0) \,, \quad \text{and} \quad \lambda^i := \lambda + (\tfrac{1}{2}, \dots, \tfrac{1}{2}, -\nu_i - \tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}) \,,$$

for $1 \leq i \leq n$.

Corollary 7.3.10. *V_{λ^0} and V_{λ^n} appear with multiplicity one in the kernel of D^2 acting on $L(\lambda) \otimes S$.*

For $1 \le i < n$, if λ'^i is a dominant weight, then V_{λ^i} appears with multiplicity one in $\ker D^2$.

Proof. Examining Equation (7.3.1), we see that for each $0 \le i \le n$, there is only one possible weight ν' in the outer sum such that $\mu = \lambda^i$ can be obtained in the inner sum, so all λ^i appear with multiplicity one in $L(\lambda) \otimes S$.

Now

$$P(\lambda) = P(\lambda^0 - (\tfrac{1}{2}, \ldots, \tfrac{1}{2})) = P(\lambda^n - (\tfrac{1}{2}, \ldots, \tfrac{1}{2})) \ ,$$

hence, V_{λ^0} and V_{λ^n} lie in $\ker D^2$.

Also, if λ'^i is dominant, then by Proposition 7.3.5,

$$P(\lambda) = P(\lambda'^i) = P(\lambda^i - (\tfrac{1}{2}, \ldots, \tfrac{1}{2})) \ ,$$

so V_{λ^i} lies in $\ker D^2$ in this case, as well. □

Proposition 7.3.11. *Let M be any H_ξ-module. Then all $\mathfrak{gl}_n$-weights appearing in $\ker D^2 \subset M \otimes S$ with odd multiplicity appear in the Dirac cohomology $H^D(M)$.*

If $M = L(\lambda)$, then the multiplicity-free $\mathfrak{gl}_n$-weights of $\ker D^2$ described in Corollary 7.3.10, and, in particular, the highest $\mathfrak{gl}_n$-weight of $M \otimes S$, $\lambda^0 = \lambda + (\frac{1}{2}, \ldots, \frac{1}{2})$, appears in the Dirac cohomology.

Proof. We have already observed that D commutes with elements of the form $\Delta_C(h) \in A \otimes C$ for $h \in H = \mathcal{U}(\mathfrak{gl}_n)$ (Lemma 6.2.5). Hence, the action of D on $M \otimes S$ is a $\mathfrak{gl}_n$-module map. Now if $N \subset \ker D^2 \subset M \otimes S$ is an irreducible $\mathfrak{gl}_n$-submodule with multiplicity space N' of odd dimension, then D acts on N' such that D^2 acts as 0. Hence the action of D on N' has a Jacobi block of size 1 for the eigenvalue 0, which corresponds to a copy of N annihilated by D and not in the image of D. This copy of N appears in $H^D(M) = \ker D/(\ker D \cap \operatorname{im} D)$.

In particular, this applies to the weights described in Corollary 7.3.10. □

Corollary 7.3.12. *Any finite-dimensional irreducible representation of the infinitesimal Cherednik algebra H_ξ is uniquely determined by its Dirac cohomology.*

Proof. The highest $\mathfrak{gl}_n$-weight occurring in the Dirac cohomology determines the highest $\mathfrak{gl}_n$-weight occurring in the finite-dimensional irreducible representation, which determines the representation. □

Let us consider examples for $n = 1$ and $n = 2$ (see the examples in [DT13, Sec. 4]).

Example 7.3.13. For $n = 1$, λ is a complex number and ν is a non-negative integer (minimal) such that $P(\lambda) = P(\lambda - \nu - 1)$. Then $L(\lambda) = V_\lambda \oplus \cdots \oplus V_{\lambda-\nu}$. Hence,

$$L(\lambda) \otimes S = V_{\lambda+\frac{1}{2}} \oplus 2V_{\lambda-\frac{1}{2}} \oplus \cdots \oplus 2V_{\lambda-\nu+\frac{1}{2}} \oplus V_{\lambda-\nu-\frac{1}{2}} \ .$$

Now the only weights μ occurring in $L(\lambda) \otimes S$ such that $P(\lambda) = P(\mu - \frac{1}{2})$ are obviously $\lambda^0 = \lambda + \frac{1}{2}$ and $\lambda^n = \lambda^1 = \lambda - \nu - \frac{1}{2}$. So the kernel of D^2 and by Proposition 7.3.11 the Dirac cohomology is just $V_{\lambda^0} \oplus V_{\lambda^1}$.

Example 7.3.14. For $n = 2$, we identify weights with points in the two-dimensional plane and we consider the $\mathfrak{gl}_n$-weight $\lambda = (3, 0) - \rho$ together with a polynomial P satisfying

$$P(3, 0) = P(3, -3) \text{ and } P(3, -k) \neq P(3, 0) \neq P(3 - k, 0) \text{ for } k = 1, 2 \ .$$

Then $\nu_1 = 3$, because $-\rho = (3, 0) - (3, 0) - \rho$ is not a dominant $\mathfrak{gl}_n$-weight and $\nu_2 = 3$, because $P((3, 0) - (0, 3))$ agrees with $P(3, 0)$ by assumption. Hence the irreducible $\mathfrak{gl}_n$-submodules occurring in $M(\lambda)$ form a 3×3-grid and their highest weights are those μ satisfying $\lambda \geq \mu \geq \lambda - (3, 3)$. Each of these irreducible $\mathfrak{gl}_n$-submodules yields 4 irreducible $\mathfrak{gl}_n$-submodules when tensored with the spin module S, for an irreducible $\mathfrak{gl}_n$-module with highest weight μ they have highest $\mathfrak{gl}_n$-weight $\mu + (\pm\frac{1}{2}, \pm\frac{1}{2})$ (see Figure 7.1).

Now we specialize $P(\mu) = -27h_1(\mu) - 2h_2(\mu) + 3h_3(\mu)$ with the completely symmetric polynomials in two variables h_1, h_2, h_3. We can check that P satisfies the conditions mentioned so far and also

$$P(3, 0) = P(3, -3) = P(2, -1) = P(0, -3) \ .$$

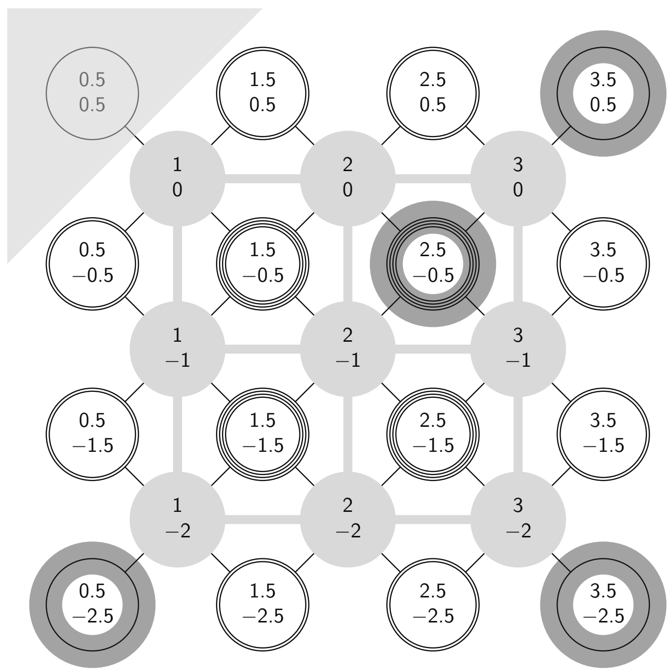


Figure 7.1: Weights of a finite-dimensional module M (filled gray circles), of the tensor product $M \otimes S$ (white circles with dark outlines indicating multiplicities), and of the kernel of D^2 (four highlighted circles, with multiplicities). The shadowed region in the top-left corner indicates non-dominant weights. As explained in Example 7.3.14, the three multiplicity-free weights in $\ker D^2$ appear in the Dirac cohomology.

Hence, the kernel of D^2 is the sum of the irreducible $\mathfrak{gl}_n$-submodules of highest weight $(3.5, 0.5) - \rho$, $(3.5, -2.5) - \rho$, $(0.5, -2.5) - \rho$ and $(2.5, -0.5) - \rho$ with full multiplicity 1, 1, 1, and 4, respectively.

By Proposition 7.3.11, the three multiplicity-free weights occur also in the Dirac cohomology.

Chapter 8

Summary and Outlook

In this thesis we explain how Dirac operators can be defined in a generalized setting for PBW deformations of smash products with an orthogonality condition. We construct pin covers of cocommutative Hopf algebras extending the known pin cover constructions for groups or Lie algebras, and we interpret these constructions using (superalgebraic versions of) the Skolem–Noether theorem, inner coalgebra measurings, and Hopf 2-cocycles. The classification of the considered class of PBW deformations is explored and partial results are presented, including the fact that they are supported on a subspace of reflections in the Hopf algebra with respect to a module. We also construct a family of examples of such PBW deformations and we show that they recover known classification results in special cases. A theory of Dirac cohomology is developed in our general framework and a version of Vogan's conjecture, which relates the central character of a module with its Dirac cohomology, is proved. It recovers versions of Vogan's conjecture for various known special cases. The theory is then applied to a novel special case of PBW deformations, namely, infinitesimal Cherednik algebras of the general linear group. We show that they, indeed, fall into the class of algebras which is covered by our version of Vogan's conjecture, and we prove that the Dirac cohomology does not vanish for and, in fact, determines any finite-dimensional irreducible module.

The discussed results raise many new questions, for instance, regarding the scope of the presented theory or regarding extensions and further generalizations. We conclude with a series of open questions and the partial answers we can give, including based on ongoing research.

8.0.1

Which infinitesimal Hecke algebras are Barbasch–Sahi algebras? Since all these algebras are PBW deformations of smash products in our sense, they are Hopf–Hecke algebras if and only if the involved Hopf algebra module is orthogonal. Are all these Hopf–Hecke algebras Barbasch–Sahi algebras, i.e., does their Dirac operator satisfy the Parthasarathy condition Definition 6.2.7? In Section 7.2, we have answered this questions in the affirmative for the general linear group and the module which is the direct sum of the standard module and its contragredient. In ongoing research, we also confirm that the infinitesimal Hecke algebra of the orthogonal group is a Barbasch–Sahi algebra. A family of infinitesimal Hecke algebras arises from symmetric spaces, but our computations indicate that for Hermitian symmetric spaces, all these infinitesimal Hecke algebras are essentially either universal enveloping algebras of Lie algebras or infinitesimal Cherendik algebras of the general linear group, which we studied. On the other hand, rational Cherednik algebras with the parameter $t \neq 0$ are Hopf–Hecke algebras which are not Barbasch–Sahi algebras, but their center consists of scalars only (see [Ciu16, Prop. 4.9 (3), Rem. 4.10]).

8.0.2

Which examples of Barbasch–Sahi algebras exist for which the role of the cocommutative Hopf algebra is not played by a group algebra or a universal enveloping algebra? For instance, if G is a finite group acting on a Lie algebra $\mathfrak{g}$, which Barbasch–Sahi algebras exists for the cocommutative smash product Hopf algebra $H = \mathcal{U}(\mathfrak{g}) \rtimes \mathbb{F}[G]$?

8.0.3

What is the classification of Barbasch–Sahi algebras, Hopf–Hecke algebras or the underlying PBW deformations? While the PBW deformations have been classified in special cases, for instance, for finite groups with faithful modules over $\mathbb{C}$ (see Theorem 2.6.3), other special cases like the classification of infinitesimal Hecke algebras are open problems.

8.0.4

Can the theory be completely carried out in the superalgebraic setting? In Section 4.4 we already explain how the pin cover can be generalized to this situation, and in ongoing research we explore how amenable the proof of Vogan's conjecture is to such a generalization. Interesting examples in this setting are deformations of universal enveloping algebras of Lie superalgebras (Dirac cohomology for Lie superalgebras was studied in [HP05]).

Can we even pass to the module category of a general quasitriangular Hopf algebra? These categories correspond to rigid braided monoidal tensor categories with a fiber functor via a Tannaka–Krein duality, and for a special choice of the quasitriangular Hopf algebra, they recover the category of superspaces. Or similarly, can we pass to Yetter–Drinfeld categories? These are a related general class of braided monoidal categories, and the superspaces form a subcategory of the Yetter–Drinfeld categorie of the Hopf algebra $\mathbb{F}[\mathbb{Z}_2]$. Do generalizations exist even to monoidal categories which do not "live over" the vector spaces (i.e., which do not have a fiber functor), like Deligne's category $\underline{\mathrm{Rep}}(S_t)$?

8.0.5

Let H be any Hopf algebra and let B be a Koszul algebra which is an H-module algebra. Can the theory be extended to PBW deformations of $B \rtimes H$, which we discussed briefly in Section 2.6? Such PBW deformations have been studied by Walton and Witherspoon in [WW14, WW18], they include the braided Chrednik algebras constructed by Bazlov and Berenstein [BB09a, BB09b]. Does a version of Vogan's conjecture hold for them?

8.0.6

Can the theory be extended to include Kostant's cubic Dirac operator [Kos99, Kos03] or Drinfeld orbifold algebras as defined by Shepler and Witherspoon [SW12]? This extension corresponds to deformation maps κ with an image not only in the Hopf algebra H, but in $H \oplus V$ or $H \oplus (V \otimes H)$, and in the Lie algebra situation, a Dirac operator with the desired properties is obtained by adding a “cubic” term.

8.0.7

Which new representation theoretic results can be obtained using Dirac operators and Dirac cohomology? For instance, which classical results can be transferred to new algebraic structures via our generalized framework? Classically, Dirac operators have proven particularly important for the study of unitary representations, so are they similarly useful for studying suitably defined unitary representations in non-classical contexts? For infinitesimal Cherednik algebras of GL_n, we have seen that the Dirac cohomology determines any finite-dimensional module and in ongoing work, we compute their Dirac cohomology completely. Can our generalized Dirac cohomology, for instance in this situation, be connected to other cohomology theories, generalizing known connections to nilpotent Lie algebra cohomology (see, for instance, [HPR06, Hua15])?

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