THREE ESSAYS IN THE THEORY OF PREFERENCES

By

SEYED HASSAN NOSRATABADI

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ABSTRACT OF THE DISSERTATION

Three Essays in the Theory of Preferences

by SEYED HASSAN NOSRATABADI

Dissertation Director:

Oriol Carbonell-Nicolau

This dissertation consists of three chapters. The first chapter addresses the classical questions of utility representation and maximization. It relaxes the notion of *weak upper continuity* (Campbell and Walker (1990)) to obtain a property called *partial weak upper continuity* and shows that both maximization of preferences and representation by a utility function can be achieved under this new property.

The rest of this dissertation focuses on extending revealed preference theory to accommodate behavioral anomalies observed in the experimental data. In particular, I offer a framework to expand the theory of revealed preferences to the case where a DM's choice is not completely identified with a single preferences.

In Chapter 2, I use a divide and conquer procedure in order to expand the revealed preference theory to accommodate behavioral anomalies such as *attraction effect*, *compromise effect*, and *reverse dominance effect*. These effects are induced when a third alternative is used as a "decoy" to change the relative ranking from a pair. The-

refore, they could be rationalized using the notion of *referenced preferences*: that is, when the pairwise preference $\succsim^p$ between a pair of alternatives $(x \sim^p y)$ is referenced when a third alternative r is added to the menu $x \succ^r y$. I model such behavior as a partially rational inductive divide and conquer procedure where the deviation from WARP only take place on tripletons where references operate. Keeping the rational choice axioms on sets with higher cardinalities retains the predictive power in the classical theory to the extent possible. In order to do this, I assume a DM only drops reference elements when facing more than one such elements in a set. I show that under this particular division rule the choice is characterized via simple majority rule over the collection of referenced preference.

In Chapter 3, I consider a variation of the results in Chapter 2 where in dividing a set of alternative in an inductive manner, the DM considers all possible first-diminished subsets. I show that such division rule results in a more sophisticated behavior than the simple majority rule. Here the DM uses here referenced preferences over the pairwise preference in a consecutive manner to squeeze the choice problem. In particular, if the DM has a pairwise preference $\succsim^p$ and two referenced preference $\succsim^{r_1}$ and $\succsim^{r_2}$, then she makes her choice by making consecutive short-lists of the choice problem using the preferences in a following consecutive manner:

(i) $S \xrightarrow{\succsim^p} S^p \xrightarrow{\succsim^{r_1}} S_1^p \xrightarrow{\succsim^{r_2}} S_{12}^p$

(i) $S \xrightarrow{\succsim^p} S^p \xrightarrow{\succsim^{r_2}} S_2^p \xrightarrow{\succsim^{r_1}} S_{21}^p$

This, indeed is an extension of the *rational short-list method* in Manzini and Mariotti (2007) with three preference. The distinction, however, is that the method is enogenized in this set up.

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Chapter 1

Partially Weakly Upper Continuous Preferences: Representation and Maximal Elements

The contents of this chapter has been published in Nosratabadi (2014).

1.1 Introduction

The question of existence of a representation for a preference relation has been addressed by many scholars. In order to guarantee desired implications, different notions of continuity have always been of interest. Eilenberg (1941), Debreu (1954) and Jaffray (1975) presented different classes of preferences for which a continuous representation exists. Rader (1963) and Bosi and Mehta (2002) limited the argument to upper semicontinuity. A wider class of preference relations for which a maximizable representation exists is weakly upper continuous as defined in Campbell and Walker

(1990)[1]. The purpose of this paper is to generalize the result in Campbell and Walker (1990) to a wider class of preference relations called *partially weakly upper continuous*.

Tian and Zhou (1995) argued that a necessary and sufficient condition for a transitive preference relation to have a maximal element is *transfer weak upper continuity*. However, this concept is not sufficient for representation[2]. Mathematically, representability is not a topological property but an order one[3]. This might suggest that it is possible to weaken the topological structure imposed by weak upper continuity, without losing either representabilty nor maximizability. We show that *partial weak upper continuity* does this job. The main idea behind this concept is to relax imposition of topological structure at jumps. This will give us advantages regarding changes in preferences and topologies[4].

1.2   Notations And Definitions

In all definitions below, X is a topological space, $\succsim$ is a preference relation and $\succ$ is the associated strict preference relation.

A preference relation on a set X is a binary relation. A *rational* preference relation is a preference relation which is complete and transitive. Define $L(x) = \{y \in X : x \succ y\}$, and $U(x) = \{y \in X : y \succsim x\}$. For any $x \in X$ and $A \subseteq X$ we use notation: $x \succsim (\succ)A$ to denote $x \succsim (\succ)z, \forall z \in A$. Define indifference relation $\sim$ as: $x \sim y \Leftrightarrow x \succsim y$ and $y \succsim x$. $\sim$ defines an equivalence relation on X. For each $x \in X$, let $[x]$ be the equivalence class for x and $X/\sim$ be the quotient set of $\sim$. $X/\sim$ inherits order from the parent set X. The order is defined as: $[x] \succsim [y] \Leftrightarrow x \succsim y, \forall x \in [x], \forall y \in [y]$. Abusing notation, we use $(X/\sim, \succsim)$ to show this order.

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[1]The authors refer to the mentioned property as weak lower continuity. We will be using weak upper continuity instead, since it is more consistent with other terminologies used in this paper.

[2]See Example 3.

[3]See footnote 7.

[4]See Examples 1 and 2.

Definition 1: Suppose that X is a topological space and $\succsim$ is a preference relation on X. We say $\succsim$ is *upper semicontinuous* if $U(x)$ is closed for all $x \in X$.

Definition 2: Suppose that X is a topological space and $\succsim$ is a preference relation on X. For $x, y \in X$ with $x \succ y$, we say $\succsim$ is *weakly upper continuous at the pair* (y, x) if there exists an open set containing y, $V(y)$, such that $x \succsim V(y)$.

Definition 3: Suppose that X is a topological space and $\succsim$ is a preference relation. We say $\succsim$ is *weakly upper continuous in X* if $\succsim$ is weakly upper continuous at any pair (y, x) with $x \succ y$.

Definition 4: Suppose that X is a topological space and $\succsim$ is a preference relation on X. For $[x], [y] \in X/\sim$ with $[x] \succ [y]$, we say $([y], [x])$ is a *jump* in $X/\sim$ if there does not exist $[z] \in X/\sim$ such that $[x] \succ [z] \succ [y]$.

Proposition 1. *Suppose X is a second countable topological space, and $\succsim$ is a rational preference relation on X which is weakly upper continuous. Then $(X/\sim, \succsim)$ possesses at most countably many jumps.*

Definition 5: Suppose that X is a topological space and $\succsim$ is a preference relation on X. Suppose J, the set of jumps in $X/\sim$, is countable and $C = \{(t, v) : t \in [t'], v \in [v'], ([t'], [v']) \in J)\}$. We say $\succsim$ is *partially weakly upper continuous* if $\succsim$ is weakly upper continuous at any $(y, x) \in (X \times X)\backslash C^5$.

Definition 6[6]: Suppose that X is a topological space and $\succsim$ is a preference relation

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[5]Note that for Definition 5 to hold $(X/\sim, \succsim)$ needs to have countably many jumps (even if $(X, \succsim)$ has uncountably many jumps. For the distinction see Example 1.

[6]This is Definition 4(3) in Tian and Zhou (1992) letting $K = X$.

on X. We say $\succsim$ is *transfer weakly upper continuous* on X if for any $x, y \in X$ with $x \succ y$ there exists $x' \in X$ and an open set containing y, $V(y)$, such that $x' \succsim V(y)$.

1.3 Two Main Theorems

We start this section by stating the first theorem which shows that partial weak upper continuity is sufficient for representation.

Theorem 1. *Suppose X is a second countable topological space and $\succsim$ is a rational preference relation on X. Then there exists a utility function representing $\succsim$ if $\succsim$ is partially weakly upper continuous.*

Now we look at the maximal elements for partially weakly upper continuous relations.

Theorem 2. *Suppose X is a topological space and $\succsim$ is a transitive preference relation on X. Then $\succsim$ has a maximal element on X if it is partially weakly upper continuous.*

Remark: Theorem 1 still works if $\succ$ is transitive (this is often called weak transitivity). Theorem 2 works under pseudotransitivity[7] of $\succ$.

Corollary: *Suppose X is a second countable topological space and $\succsim$ is a rational preference relation on X. Then there exists a maximizable utility function representing $\succsim$ if $\succsim$ is partially weakly upper continuous.*

In the following, we introduce some examples to show the distinction among theses different concepts of continuity.

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[7]A strict preference relation $\succ$ is pseudotransitive if $x' \succ x \succsim y \succ y' \implies x' \succ y'$ when $x \neq y$.

Example 1: Suppose $X = \mathbb{R}^+$ and suppose $\succsim^*$ is a rational preference relation on X. Let $[\bar{0}] := [0, 1]$, $[\bar{1}] := (1, 2]$ and $[\bar{2}] := (2, +\infty]$. Suppose that $\succsim^*$ satisfies the following properties:

(i) $x \sim^* y, \forall x, y \in [\bar{0}]$

(ii) $x \sim^* y, \forall x, y \in [\bar{1}]$

(iii) $x \sim^* y, \forall x, y \in [\bar{2}]$

(iv) $[\bar{1}] \succ^* [\bar{2}] \succ^* [\bar{0}]$

$\succsim^*$ has three different equivalence classes: $[\bar{0}], [\bar{1}]$ and $[\bar{2}]$. $\succsim^*$ is not weakly upper continuous (and therefore not upper semicontinuous) since $3 \succ^* 1$ but any open set containing 1 has an element $z \in [\bar{1}]$ and $z \succ^* 3$. $\succsim^*$ is partially weakly upper continuous since strict preferences happen just in jumps.

Note that $\succsim^*$ has uncountably many *jumps in X*. For every $t \in [\bar{i}]$ and $v \in [\bar{j}] : i, j \in \{0, 1, 2\}$, (t, v) is a jump in X. However, $\succsim^*$ only possesses three jumps in $X/\sim$, namely: $([\bar{0}], [\bar{1}]), ([\bar{0}], [\bar{2}])$ and $([\bar{2}], [\bar{1}])$.

Example 2: Suppose $X = \mathbb{N}$ with the normal order of natural numbers $(\geq)$. Obviously, X has countably many jumps. Let $\mathbb{N}_k = \{1, 2, 3, ..., k\}$. Now let $\mathcal{T} = \{\mathbb{N}_{3k}\}_{k=1}^\infty \cup \{\emptyset, \mathbb{N}\}$ be a topology on X. $\geq$ is not upper semicontinuous nor weakly upper continuous since $2 \geq 1$, but every open set containing 1 also contains $3 \geq 2$.

Now suppose (x, y) is not a jump. Then there exists $z' \in \mathbb{N}$ such that $x \leq z' \leq y$ and $z' = 3\bar{k}$ for some $\bar{k}$. $\mathbb{N}_{3\bar{k}}$ is an open set containing x and $y \geq z, \forall z \in \mathbb{N}_{3\bar{k}}$. So $\geq$ is weakly upper continuous on non-jump pairs, and therefore is partially weakly upper continuous. Indeed, for topologies of type $\mathcal{T} = \{\mathbb{N}_{nk}\}_{k=1}^\infty \cup \{\emptyset, \mathbb{N}\}$ for $n > 2$ the same conclusion can be reached. Therefore, there are infinitely many topologies for which $\geq$ is partially weakly upper continuous but not weakly upper continuous.

Our final example shows that partial weak upper continuity is strictly stronger than the transfer concept.

Example 3: Suppose $\succsim_L$ is lexicographic order on $X = [0,1] \times [0,1] \subset \mathbb{R}^2$. For each $q \in Q^c$ we have $(q,1) \succ_L (q,0)$. Let $C_q = \{x_q \in X : (q,1) \succsim_L x_q \succsim_L (q,0)\}$. For $q_1 > q_2$ we have $x_{q_1} \succ_L x_{q_2}, \forall x_{q_1} \in C_{q_1}, \forall x_{q_2} \in C_{q_2}$. Since there are uncountably many irrational numbers in $[0,1]$ we conclude that $\succsim_L$ is not *order separable in the sense of Debreu*[8], and therefore is not representable. However, it is transfer weakly upper continuous. To see this note that X is a compact subset of $\mathbb{R}^2$ and obviously $(1,1)$ is the maximal element of $\succsim_L$. Using Tian and Zhou (1995) we conclude $\succsim_L$ is transfer weakly upper continuous.

On the other hand, lexicographic order is not partially weakly upper continuous, since $\succsim_L$ does not possess any jumps, and $(0,1) \succ_L (0,0)$ but every open set containing $(0,0)$ has an element $z = (z_1, z_2) \in X$ such that $z_1 > 0$, and therefore $z \succ_L (0,1)$.

1.4 Conclusion

It is shown that partial weak upper continuity of a preference relation is sufficient for both representability and maximizability. This generalizes the result in Campbell and Walker (1990). Furthermore, we show transfer weak upper continuity is not sufficient for representation. This makes partial weak upper continuity more desirable, as it guarantees both representability and maximizability.

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[8]$\succsim$ is *order separable in the sense of Debreu* if there exists a countable subset $Z \subseteq X$ such that for any $x \succ y$ there exists an element $z \in Z$ such that $x \succsim z \succsim y$.
A rational preference relation is representable if and only if it is order separable in the sense of Debreu.

1.5 Appendix - Proofs

Throughout this dissertation, and in order to avoid confusion, I preserve the symbol ∎ for Halmos Q.E.D. sign for the proofs of the main statements, and, wherever applied, □ for subproofs.

Proof of Proposition 1

Proof. By Theorem 3 in Campbell and Walker (1990) we conclude that $\succsim$ is representable on X and therefore on $X/\sim$. Now using Corollary 1.6.14 in Bridges and Mehta (2013) $(X/\sim, \succsim)$ has countably many jumps.   ∎

Proof of Theorem 1

Proof. Let $\mathbb{B} = \{B_1, B_2, B_3, ...\}$ be an enumeration of the countable base. For any x, let:

$$N_1(x) = \{n : x \succsim B_n\}$$

Let $J = \{([t_1], [v_1]), ([t_2], [v_2]), ([t_3], [v_3]), ...\}$ be the countable set of jumps in $X/\sim$. For any $x \in X$ define $N_2(x)$ as follows:

$$N_2(x) = \{n : ([t_n], [v_n]) \in J, x \succsim v_n\}$$

Define the utility function, $u(.)$, as follows:

$$u(x) = \sum_{n \in N_1(x)} (1/2)^n + \sum_{n \in N_2(x)} (1/2)^n$$

To prove that $u(.)$ is a utility function we first show $x \succ y$ implies $u(x) > u(y)$. Suppose $x \succ y$. For each $B_n \in \mathbb{B}$, by transitivity of $\succsim$ we conclude $x \succ y \succsim B_n$ implies $x \succ B_n$. This implies $N_1(y) \subseteq N_1(x)$. Next, let $k \in N_2(y)$, then $([t_k], [v_k])$ is a jump and $x \succ y \succ v_k$ implies $x \succ v_k$. So $k \in N_2(x)$. So $N_2(y) \subseteq N_2(x)$.

To complete the proof note that if $([y], [x])$ is a jump then we have $N_2(y) \subset N_2(x)$ and therefore $u(x) > u(y)$. If it is not a jump then there exist $z^* : [x] \succ [z^*] \succ [y]$. Now if $([z^*], [x])$ is a jump by the same argument $u(x) > u(y)$. If $([z^*], [x])$ is not a jump then, by partial weak upper continuity of $\succsim$, there exists an open set $V(z^*)$ containing z^* for which $x \succsim V(z^*)$. Since X is second countable, $V(z^*) = \bigcup_{i \in A} B_i$ for some $A \subseteq \mathbb{N}$, so there exists $k \in \mathbb{N}$ such that $z^* \in B_k$. $z^* \succ y$ implies $z \notin B_n, \forall n \in N_1(y)$, and we conclude $k \in N_1(x) \backslash N_1(y)$ so $N_1(y) \subset N_1(x)$. This will give us the strict inclusion in all cases. We conclude:

$$u(x) = \sum_{n \in N_1(x)} (1/2)^n + \sum_{n \in N_2(x)} (1/2)^n > \sum_{n \in N_1(y)} (1/2)^n + \sum_{n \in N_2(y)} (1/2)^n = u(y)$$

Now suppose $x \succsim y$. By transitivity of $\succsim$ we have $N_1(y) \subseteq N_1(x)$, and $N_2(y) \subseteq N_2(x)$, therefore $u(x) \geq u(y)$. On the other hand, suppose $u(x) \geq u(y)$. $y \succ x$ implies $u(y) > u(x)$ and therefore can not be true. Since $\succsim$ is complete we have $x \succsim y$. This completes the proof. ∎

Proof of Theorem 2

Proof. Suppose that $\succsim$ is a transitive preference relation. Let J be the set of jumps in $X/ \sim$. We use the result in Tian and Zhou (1995). Since X is compact we only need to show $\succsim$ is transfer weakly upper continuous. To do this, suppose $x \succ y$. If there does not exist $z \in X$ such that $z \succ x$ (i.e. x is a maximal element) then we are done. If not, there exists $z^* \in X : z^* \succ x$. Since $z^* \succ x \succ y$, by transitivity we have $z^* \succ y$. So $([y], [z^*]) \notin J$. By partial weak upper continuity of $\succsim$ on X, there exists an open set $V(y)$ containing y such that $z^* \succsim V(y)$. Letting $x' = z^*$ in the Definition 6 we conclude $\succsim$ is transfer weakly upper continuous and therefore has a maximal element. ∎

Chapter 2

A Model of Referenced Preferences

2.1 Introduction

Following the seminal work of Kahneman and Tversky (1979), the relevance of "reference points" in decision making has been widely accepted. In the simplest domain of revealed preference theory, reference points could be interpreted as third alternatives that influence the relative ranking of a pair. Formally, an alternative r is said to be a reference if

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\};$$

that is, if r *helps* to break the indifference between the pair of alternatives x, y. The notion of "helping" one alternative over the other can be formalize in the theory of revealed preference in a fashion which is familiar to economists. Formally, if the choice from the doubleton $\{x, y\}$ is interpreted to reveal the pairwise preference over the two alternatives; that is,

$$x \succsim^p y \iff x \in \boldsymbol{c}\{x, y\},$$

then, we define a preference over the two alternative x, y, referenced via a third alternative r, by the following:

$$x \succsim^r y \iff x \in \boldsymbol{c}\{x, y, r\}.$$

The notation should simply read "x is at least as good as y is presence of r".

Referencing preferences via third alternatives has been widely studied in the experiments. To exhaust all the relevant experimental evidence, let us consider an example. Let a decision maker's (DM) pairwise preference $\succsim^p$ over the alternatives $\{x, y, z, r, r'\}$ be as follows: (brackets represent indifference classes):

$$[r'] \succ^{\boldsymbol{p}} [x \sim^{\boldsymbol{p}} y \sim^{\boldsymbol{p}} z] \succ^{\boldsymbol{p}} [r]$$

It is a well-established experimental fact that decoy (reference) alternatives can influence the choice from a pair.[1] The three categories in Figure 2.1, logically, exhaust all types of such "referential effects":

Figure 2.1: Third Alternatives as Reference Points

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[1]One interpretation is that the decoy sangs the relative importance of different attributes.

All three effects in Figure 2.1 are experimentally well-documented[2]. As shown in Figure 2.1, the effect of a reference point is categorized by the position in which the point lies in the pairwise comparison. In the case of *attraction effect* the reference point (r) is acting from a lower indifference class into a higher one. In this example, r "references" the pairwise preference over the alternatives $\{x, y, z\}$ so that, in presence of r, the relative ranking is given by $z \succ^r x \sim^r y$. Implementing the reference point r in choice, DM behaves inconsistent with the *weak axiom of revealed preference* (WARP). For example,

$$\boldsymbol{c}\{x, z, r\} = \{z\}, \text{ and } \boldsymbol{c}\{x, z\} = \{x, z\}, \text{ or,}$$

$$\boldsymbol{c}\{x, y, z, r\} = \{z\} \text{ and } \boldsymbol{c}\{x, y, r\} = \{x, y\}.$$

On the other hand, if the reference point lies in a higher indifference class, then the phenomenon is referred to as *reverse dominance effect*. Note that, obviously, r' (red) is assumed not to be available to the DM. A decision making scenario in which such reverse dominance comes to effect is when a favorite alternative is out of stock. Lastly, when the effect is within an indifference class, then it is called *compromise effect*. In Figure 2.1, for example, x, z together reference to y. Just like attraction effect, the last two effects also violate WARP.

The behavior of a classical DM (CDM) under WARP is fully identifiable with $\succsim^p$.[3] However, the behavior of the RDM in Figure 2.1 depends on the collection of *referenced preferences* as well; that is, DM's behavior depends on $(\succsim^p, \{\succsim^r\}_{r \in \Re})$. We refer to such a DM as the referential DM (RDM). This paper design a straightforward platform to extend the classical revealed preference theory to the case that referenced

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[2]See Huber, Payne, and Puto (1982), Knetsch and Sinden (1984), and Doyl, O'Conner, Reynolds, and Bottomly (1999) for example

[3]This is shown in Arrow (1959).

preference are also rooted in the process of decision making. Using the platform, the paper next fully identifies the behavior of the RDM under attraction and reverse dominance effect. The case of compromise effect remains an open problem.[4]

2.1.1 RDM as an Inductive Divide and Conquer Procedure

In our setup, the RDM uses a third alternative (referential level in Figure 2.2) to reference her pairwise preference. *How can the choice be identified using referenced preferences for any level beyond the referential level?* In answering this question, we use an inductive divide and conquer procedure outlined in the following:

1. **No Binary Cycle (NBC)**: Choice over doubleton is transitive.

2. **Referential Inductive Division Correspondence (d)**: d assigns to any set S with $|S| = k \geq 3$ a collection of its subsets with cardinality $k - 1$ (first-order diminished subsets). That is, $|A| = k - 1$ for all $A \in d(S)$. The blue part of this figure, for example, illustrates that the menu $\{x, y, r\}$ is divided into all of its doubleton subsets. In general, we assume that d does so for all sets with at most on reference point in them. For sets with more than one reference point we assume that elements of $d(S)$ are derived only by dropping reference points from S.

3. **Contraction Rule (α')** : For S with $|S| = k \geq 3$,

   If x is chosen in S ($x \in c(S)$), but loses to y in $A \in d(S)$ ($x \notin c(A)$ and $y \in c(A)$), then x wins over y in a menu $B \in d(S)$ ($x \in c(B)$ and $y \notin c(B)$).

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[4]However, easy to incorporate in he model, the treatment of the compromise effect is distinguished from the other two effect since the effect is comprised of two references at the same time. For example, in Figure 2.1 it can be said that x, z together help y.

α' can be thought as a variation of α in Sen (1971)'s decomposition of WARP. Note that α enforces a contraction rule over all the subsets of S. However, α' only does so on some of the subsets. Using Sen's words, α' is simply the following consistency condition:

X, Y are both from Pakistan. If X is the champion of the world, but has lost to Y in a regional contest (for example, Pakistani championship), then X should have defeated Y in another regional contest (Asian championship, for example).

4. **Expansion Rule (β')** : For S with $|S| = k \geq 3$,

    If, against any given y, x compensates a loss in $A \in \boldsymbol{d}(S)$ with a win in some $B \in \boldsymbol{d}(S)$, then x is chosen in S.

    Logically, β' is the converse statement of α', and can be interpreted as a variation of β in the sense of Sen (1971).

The use of the inductive divide and conquer representation of WARP in identification of RDM's choice is straightforward. We simply retain both contraction and expansion rules on levels beyond referential level (sets with more than three elements). Figure 2.2 shows this process in the case of attraction effect. Note that RDM still satisfies α' on the referential level. That is, attraction effect only violates β' on triplets[5]. Also, the behavior of the RDM on the triplets reveals the referenced preference (in this case there is only one reference r under which a reranking $x \succ^r y \succ^r z$ takes place). Retaining both contraction and expansion rules on higher levels will allow us to construct the choice in any arbitrary set following those rules.[6] In this example, x is the only alternative in the menu $\{x, y, z, r\}$ that satisfies both rules. Figure 2.3 on

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[5]other effect same argument

[6]if possible

the other hand shows that in the presence of multiple reference points RDM divides the choice problem by only dropping the reference points. The blue part in this figure, for example, shows that the menu $\{x, y, z, r_1, r_2\}$ is divide into the two smaller problems $\{x, y, z, r_1\}$ and $\{x, y, z, r_2\}$. Applying the contraction and expansion rules, the choice in the menu $\{x, y, z, r_1, r_2\}$ then will include all three alternatives x, y, z. That is, the choice is determined by a simple majority rule: r_1 favors both x, y and r_2 favors x. Therefore, each of the three alternatives has one vote which makes them favorable if the DM used a simple majority rule.

Figure 2.2: RDM's Choice - The Case of Single Reference

Figure 2.3: RDM's Choice with Multiple References

Indeed, it is shown that the intuition in the example in Figure 2.3 extends to any

arbitrary choice problem. That is, choice in any arbitrary set S is characterized via a simple majority rule, where each $\succsim^r$ simply votes for $\operatorname*{argmax}_{S} \succsim^r$.

2.1.2   On the Related Literature

The behavior of a RDM, in general, is "procedural". A well-known procedure in the literature, rational short-list method, is introduced in Manzini and Mariotti (2007).[7] The method assumes that the DM uses a preference to make a short-list of the alternatives available, and then finalizes her choice by applying another preference on that list. This procedure is a special case of a RDM who is only influenced by a unique referenced preference. The first ranking matches the pairwise preference and the second one is induced via a third alternative. Formally, if $\succsim^p$ and $\succsim r$ are the pairwise preference and the unique referenced preference, then chose in any set is characterized by

$$\boldsymbol{c}(S) = \operatorname*{argmax}_{\operatorname*{argmax}_{S} \succsim^p} \succsim^r .$$

Consequently, our results *endogenize* the notion of rational short-list method.

Our approach is in essence different from the literature where behavioral anomalies of interest are caused by a notion of "inattention". Masatlioglu, Nakajima, and Ozbay (2012) and Ok, Ortoleva, and Riella (2015) are predominant papers in this category where DM only pays attention to (considers) a subset of a given menu.[8] They provide proper axioms under which the behavioral anomalies can be represented by a preference and a collection of "consideration sets". In general, however, either the utility function or the consideration sets are not uniquely identified in these models. Our approach, on the other hand, fully identifies choice by the collection of referenced

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[7]Manzini and Mariotti (2007) uses the method to explain cyclical choice over doubletons. However, here we are looking at its applications to the sets with more than two elements, where the choice over doubletons satisfy the transitivity condition.

[8]In addition to attraction effect, Masatlioglu et al. (2012) is also concerned with cyclical choice and choosing pairwisely unchosen. Lleras, Masatlioglu, Nakajima, and Ozbay (2017) is concentrated on choice overload phenomenon.

preference. One interpretation is as follows: in any given set the RDM only pays attention to those elements that have the majority of votes from referenced preferences. The idea of endogenous formation of references through weakening WARP is originated in Ok et al. (2015). Following their motivation, referenced preferences are also endogenously derived in this work. Some predictions of the theory developed here, however, are different from these works. In particular, it is natural in our treatment that a RDM who admits a pairwise preference relation (complete and transitive) is not unaware of her most preferred alternatives; that is, she is never inattentive towards a dominant alternative in favor of a dominated one.[9]

The notion of multiple rationales in Kalai, Rubinstein, and Spiegler (2002) is closely connected to the reference-dependent model in Tversky and Kahneman (1991). These rationales can be thought of as reference preference revealed in my treatment. However, this work is an effort to endogenize these reference preferences as opposed to their focus which is to find a minimal number of exogenously given rationales with which the choice behavior is rationalized. Similarly in Cherepanov, Feddersen, and Sandroni (2013) the concept of rationales are exogenously given. However, referential revealed preference theory produces results which are consistent with theirs. To see this once again, we can interpret the rationales as reference preferences in my treatment. Following their motivating example, assume that a decision maker who is choosing from the alternatives $\{x, y, z\}$ has the two following rationales (reference preferences): $x \succ^{r_1} y \succ^{r_1} z$ and $z \succ^{r_2} y \succ^{r_2} x$. As my treatment predicts y can not be the choice in the menu xyz as it is not a maximum element under any of the reference preferences. This matches the prediction in Cherepanov et al. (2013) as choosing y is not rationalizable with respect to any of the rationales regardless of the structure of DM's innate preference.[10]

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[9]For a detailed discussion on this see Section 2.4.3.

[10]The concept of innate preference in Cherepanov et al. (2013) is a fixed preference which is only used if it matches a rationale, however, I distance myself from such concept and allow for refining pairwise relation via references. This difference has not relevance in terms of theoretical

2.2 Preliminaries

Let X be a finite set. X is the set of all "relevant" alternatives for the DM. Therefore, it contains not only the concrete options available to the DM, but also, for example, alternatives that she has chosen before, or phantom alternatives that are not available to choose but presented to her (e.g., items that are out of stock, or shows that are sold out). In terms of the nature of the elements, X might be alternatives available for grocery shopping, different colleges to attend, various policies to be followed by the policy maker, etc. Let 2^X be the power set of X. Also let

$$\mathfrak{X}^k := \{A \subseteq X : |A| = k\};$$

that is the set of all subsets of X with cardinality equal to k, and

$$\mathfrak{X}^{\geq k} := \{A \subseteq X : |A| \geq k\};$$

that is the set of all subsets of X with cardinality of at least k. In order to simplify the domain of the discussion on choice I only consider the sets that have at least two elements, as the choice over the empty set and the singletons are trivially interpreted. A *choice correspondence* on X is a function $\boldsymbol{c} : \mathfrak{X}^{\geq 2} \to 2^X$ such that for all $A \in \mathfrak{X}^{\geq 2}$ we have $\boldsymbol{c}(A) \subseteq A$. $\boldsymbol{c}$ is called a *non-empty valued* choice correspondence if $\boldsymbol{c}(A) \neq \emptyset$ for all $A \in \mathfrak{X}^{\geq 2}$. We make the common notational abuses:

$$\boldsymbol{c}\{x, y, z\} := \boldsymbol{c}(\{x, y, z\}) \text{ and } \boldsymbol{c}\{x, y\} := \boldsymbol{c}(\{x, y\}),$$

for all $x, y, z \in X$.

For $S \subseteq X$, unless otherwise stated, whenever used throughout this paper let $S \in \mathfrak{X}^{\geq 3}$; that is let S have at least three elements. For $x \in S$ let $S - x := S \setminus \{x\}$;

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ramifications of these two theories.

that is the set which is derived by removing x from S.

A binary relation R on X is a subset of $X \times X$. Let $\mathcal{R}$ be the asymmetric relation derived from R; that is

$$x\mathcal{R}y \iff xRy \text{ and } \neg(yRx).$$

A *cycle* of order k in R is a set $\{x_1, x_2, \ldots, x_k\}$ with $x_i \in X$ such that

$$x_1 \, \mathcal{R} \, x_2 \, \mathcal{R} \, \ldots \, \mathcal{R} \, x_k \, \mathcal{R} \, x_1.$$

R is said to be *acyclic* if it does not posses any cycle of any order. A *preference* relation on X is a binary relation which is transitive and complete. For a binary relation R on X, and $S \subseteq X$, x is called a *maximum* element of R on S if

$$xRy : \forall y \in S.$$

Let

$$\operatorname*{argmax}_{S} R := \{x \in S : x \text{ is a maximum for } R \text{ on } S\};$$

x is called a *maximal* element of R on S if there does note exist $y \in A$ such that $y\mathcal{R}x$, where $\mathcal{R}$ is the asymmetric relation derived from R.

A *cover* for $S \subseteq X$ is a family of sets, $\{A_i\}_{i=1}^n$ such that $A_i \subseteq S$ for all i and

$$S = \bigcup_{i=1}^{n} A_i.$$

For a choice correspondence $\boldsymbol{c}$ define the relation $\succsim^p$ on X by

$$x \succsim^p y \iff x \in \boldsymbol{c}\{x, y\}.$$

Let $\succ^p$ and $\sim^p$ be asymmetric and symmetric parts of $\succsim^p$. Note that $\succsim^p$ matches the notion of revealed preference in the sense of Samuelson (1938). We call $\succsim^p$ the

pairwise revealed preference throughout this paper. We next define the key notion of references.

Definition 1. (References) For a choice correspondence c and $S \subseteq X$ we say r is a *revealed reference*[11] in S if there exits two distinct elements $x, y \in S$, both different from r such that

$$c\{x, y, r\} \subset c\{x, y\}.[12]$$

Note that references are not chosen in sets where they operate; that is $r \notin c\{x, y, r\}$ in the previous definition. Let

$$\mathfrak{R}(S) = \{r \in S : r \text{ is a reference in } S\};$$

that is the set of references in S.

2.2.1   Referential Inductive Division Correspondence

Assume that a decision maker rationalizes her choice in a menu by her relative choice in submenus. Recall that, wherever operational, reference are not elements of choice themselves and only relevant by refining the pairwise relation.[13] Therefore assume that submenus are derived by removing references from the original menu, one at a time, to the extent that referential effect is preserved; This creates a family of *first-order diminished* subsets that constitutes a cover for the original set. To see this in a formal setting, let

$$R_1(S) = \{S - x : x \in \mathfrak{R}(S)\}, \text{ and, } R_2(S) = \{S - x : x \in S \setminus \mathfrak{R}(S)\}.$$

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[11]The term is originally introduced in Ok et al. (2015). For the sake of parsimony in writing, I will drop the term "revealed" for the rest of this paper.

[12]I am borrowing the definition of references from Ok et al. (2015).

[13]This is naturally derived from my treatment. See Corollary 14 in Appendix.

R_1 is the family of first-order diminished subsets of S derived from removing references, one at a time, from S. R_2, in a similar fashion, is the family derived from removing non-reference elements. Next let

$$\boldsymbol{d}(S) = \begin{cases} R_1(S) & S \text{ has at least two references} \\ R_2(S) & \text{Otherwise} \end{cases}$$

To explore the nature of $\boldsymbol{d}(S)$ and see why it is a division correspondence, let us consider the following cases:

(i) If $\mathfrak{R}(S) = \varnothing$ then there is no reference in S and therefore $\boldsymbol{d}(S) = R_2(S)$. To show that $\boldsymbol{d}(S)$ is a cover take $x \in S$. Since $|S| \geq 3$ there exists $y \in S$ different from x and $S - y \in \boldsymbol{d}(S)$. Obviously $x \in \boldsymbol{c}(S - y)$.

(ii) If $\mathfrak{R}(S) = \{r\}$ then $\boldsymbol{d}(S) = R_2(S)$ and all elements of $\boldsymbol{d}(S)$ are derived by taking an element different than r out of S; that is r is in all elements of $\boldsymbol{d}(S)$. To show that $\boldsymbol{d}(S)$ is a cover take $x \in S$. If $x = r$ then $x \in A$ for all $A \in \boldsymbol{d}(S)$ and therefore $\boldsymbol{d}(S)$ is a cover for S. So assume $x \neq r$. Since $|S| \geq 3$ it follows that there exists a non-reference element $y \in S$ and $S - y \in \mathfrak{B}(S)$. Obviously $x \in S - y$.

(iii) If $\mathfrak{R}(S) = \{r_1, r_2\}$ then $\boldsymbol{d}(S) = R_1(S) \cup \{r_1, r_2\}$. That is all elements of $\boldsymbol{d}(S)$ are of the form $S - r$ for some reference r in S. $\{r_1, r_2\}$ is added to to make sure $\boldsymbol{d}(S)$ is a division correspondence.

(iv) If $|\mathfrak{R}(S)| \geq 2$ then $\boldsymbol{d}(S) = R_1(S)$. That is all elements of $\boldsymbol{d}(S)$ are of the form $S - r$ for some reference r in S. To show that $\boldsymbol{d}(S)$ is a cover for S take $x \in S$. If $x \in \mathfrak{R}(S)$ take a reference r different from x and note that $S - r \in \boldsymbol{d}(S)$. Obviously $x \in S - r$. So assume x is not a reference in S. Take a reference r in S. It follows that $x \in S - r$.

As the argument above shows, the covers are built by removing the references from a menu, one at a time. If such procedure leads to blockage of the referential effect (that is when there is only one reference in the menu) the process is performed by removing non-referential elements. Obviously in the case of no references the cover contains all first-order diminished subsets.[14]

2.3 Choice Axioms

Axiom 1. No Binary Cycle - NBC: We say a choice correspondence $\boldsymbol{c}$ satisfies NBC on S if for all $x, y, z \in S$

$$x \succsim^p y \ \text{ and } \ y \succsim^p z \ \text{ implies } \ x \succsim^p z.$$

If $S = X$, then we simply say $\boldsymbol{c}$ satisfies NBC.

The following two definitions ease formalizing contraction and expansion rules.

Definition 2. (Beating) Let $S \subseteq X$. For $x, y \in S$ we say x *beats* y in S whenever $x \in \boldsymbol{c}(S)$ and $y \notin \boldsymbol{c}(S)$.

Definition 3. (Dominance) Let $S \subseteq X$. We say x *dominates* y *in* S relative to the division correspondence $\boldsymbol{d}$ and we write

$$x \underset{S}{\blacktriangleright}^{\boldsymbol{d}} y,$$

if there exists $\bar{A} \in \boldsymbol{d}(S)$ such that x beats y in $\bar{A}$, and there does not exist $A \in \boldsymbol{d}(S)$ such that y beats x in A.

Now we can formalized the three choice axioms outlined in the section before. For axioms 1-2 let $\boldsymbol{c}$ be a choice correspondence and $\boldsymbol{d}$ a division correspondence on $\boldsymbol{c}$.

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[14]The rationalizability motivation proposed here is particularly close to that of "divide and conquer" in Plott (1973). However, here, covers are not arbitrary and are specified via references.

Axiom 2. Contraction Rule - α': S satisfies α' relative to $\boldsymbol{d}$ if

$$x \in \boldsymbol{c}(S) \implies x \text{ is a maximal element of } \blacktriangleright_S^{\boldsymbol{d}}.$$

A choice correspondence $\boldsymbol{c}$ satisfies α' on S relative to $\boldsymbol{d}$ if all $A \in \mathcal{P}^{\geq 3}(S)$ satisfy α'. If $S = X$, then we simply say $\boldsymbol{c}$ satisfies α' relative to $\boldsymbol{d}$.

Axiom 3. Expansion Rule - β': We say S satisfies β' relative to $\boldsymbol{d}$ if

$$x \text{ is a maximal element of } \blacktriangleright_S^{\boldsymbol{d}} \implies x \in \boldsymbol{c}(S).$$

We say a choice correspondence $\boldsymbol{c}$ satisfies β' on S relative to $\boldsymbol{d}$ if all $A \in \mathcal{P}^{\geq 3}(S)$ satisfy β'. If $S = X$, then we simply say $\boldsymbol{c}$ satisfies β' relative to $\boldsymbol{d}$.

2.4   Referential Decision Maker

For the remainder of this paper assume $\boldsymbol{c}$ is a *non-empty valued* choice correspondence. Let $S \subseteq X$. For the sake of parsimony I use the following notation: $\mathcal{I}^p(S) =:$ $\underset{S}{\operatorname{argmax}} \succsim^p$. $\mathcal{I}^p(S)$ is, therefore, the set of best alternatives in S from the perspective of the pairwise revealed preference. S is called *fully indecisive* if $S \subseteq \mathcal{I}^p(S)$.

Definition 4. (Classical DM: CDM) A choice correspondence $\boldsymbol{c}$ is called a *CDM* if it satisfies WARP.

From the fundamental theorem of revealed preferences implies that the choice for CDM is completely pinned down by her transitive choice over doubletons. In the formal sense, if $\boldsymbol{c}$ is a CDM on S then

$$\boldsymbol{c}(S) = \mathcal{I}^p(S);$$

that is a CDM's choice is not dependent on *context*. Obviously such contextlessness is violated in phenomena like attraction effect. In order to capture these contextual effects I next introduce the notion of *referential* DM.

As discussed in the introduction, the main source of the behavioral anomalies of concern in this paper is the β' on tripletons where a reference operates. In order to do this let

$$\mathfrak{A} = \{A \in \mathfrak{X}^3 \text{ such that } A \text{ is fully indecisive }\}.$$

For all $S \in \mathfrak{X}^{\geq 4} \cup \mathfrak{A}$.

Definition 5. (Referential DM: RDM) Let $\boldsymbol{d}$ be an referential inductive division correspondence. A choice correspondence $\boldsymbol{c}$ is called a *RDM* on S if it satisfies NBC, α' relative to $\boldsymbol{d}$, and β' relative to $\boldsymbol{d}$ on $\mathfrak{A}$.[15,16]

2.4.1 RDM's Properties

Definition 6. A choice correspondence $\boldsymbol{c}$ is called a *pure* referential DM if it is a RDM but not a CDM.

This next proposition formalizes how the *minimal* deviation from WARP introduce via Wβ' can generate references.

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[15]Note that references are defined by refining the pairwise relation. Therefore, if $\boldsymbol{c}$ is a choice function (that in the absence of pairwise refinement), then there are no references in X and Proposition 2 implies that a RDM is a CDM; that is the classical and referential revealed preference theories coincide in the absence of indecisiveness.

[16]The notion of *path independence* in Plott (1973) is related to CDM and RDM. We say a choice correspondence $\boldsymbol{c}$ satisfies path independence if for all $S \subseteq X$ and two finite covers of S, v_1 and v_2, we have

$$c\Big(\bigcup_{v \in v_1} c(v) \Big) = c\Big(\bigcup_{v \in v_2} c(v) \Big).$$

To explore this relation first note that a CDM admits to WARP and therefore satisfies path independence. On the other hand, path independence does not imply WARP. To see this, consider the following example which is introduced in Plott (1973). Let $\boldsymbol{c}\{x, y, r\} = \{x, y\}, \boldsymbol{c}\{x, y\} = \{x, y\}, \boldsymbol{c}\{x, r\} = \{x, r\}$, and $\boldsymbol{c}\{y, r\} = \{y, r\}$. This choice structure satisfies path independence (see Plott (1973) for the proof), but it obviously violates WARP. There is no logical relation between RDM and a path independent choice structure however. To see this consider the following choice correspondence: $\boldsymbol{c}\{x, y, r\} = \{x\}, \boldsymbol{c}\{x, y\} = \{x, y\}, \boldsymbol{c}\{x, r\} = \{x\}$, and $\boldsymbol{c}\{y, r\} = \{y\}$. This is typical case of attraction effect and therefore consistent with RDM. However this choice correspondence does not satisfy path independence. To see this consider the two following covers: $v_1 = \big\{\{x, y, r\}\big\}$ and $v_2 = \big\{\{x, r\}, \{y, r\}\big\}$. Then $\boldsymbol{c}\big(\bigcup_{v \in v_1} v\big) = \boldsymbol{c}\{x, y, r\} = \{x\}$ and $\boldsymbol{c}\big(\bigcup_{v \in v_2} v\big) = \boldsymbol{c}(\boldsymbol{c}\{x, r\} \cup \boldsymbol{c}\{y, r\}) = \boldsymbol{c}\{x, y\} = \{x, y\}$. To see that path independence does not imply RDM consider, again, the example from Plott (1973). Since $x \sim^p y \sim^p r$ we conclude that $\{x, y, r\}$ is fully indecisive. If $\boldsymbol{c}$ was a RDM then we would have $\boldsymbol{c}\{x, y, r\} = \{x, y, r\}$ which is not possible.

Proposition 2. *Let* $\boldsymbol{c}$ *be a RDM. Then*

$$\mathfrak{R}(X) \neq \varnothing \iff \boldsymbol{c} \text{ is a pure RDM}$$

The proof of Proposition 2 is presented in Section 2.8.1. Note that $\boldsymbol{c}$ in this proposition has all the rationalities of a CDM except, possibly, β' on referential tripletons. Proposition 2, therefore, simply states that the emergence of references in my treatment is *equivalent* to violation of β' on referential tripletons. Such relaxation, as a result, guarantees the proportionate deviation from WARP.[17]

Note that it is obvious from the definitions that if $\boldsymbol{c}$ is a RDM (resp. CDM) then $\boldsymbol{c}$ is a RDM (resp. CDM) on all $S \subseteq X$. Next I explore some basic, yet critical, properties of a RDM. Recall from my motivational examples that references lie on indifference curves below the ones where they operate. One of the benefits of my approach via decomposition theorem is that such phenomenon is naturally derived from my setup. This next proposition, for which the proof is presented in Section 2.8.1, formalizes this observation.

Proposition 3. *Let* $\boldsymbol{c}$ *be a RDM and* $S \subseteq X$. *Also let* $r \in \mathfrak{R}(S)$ *and* $x, y \in S$ *such that*

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

Then $x \sim^p y \succ^p r$.

Having defined both notions of CDM and RDM, an important question arises: *How is the choice behavior of RDM related to that of CDM?* As discussed before CDM's choice is independent of context and summarized by $\boldsymbol{c}(S) = \mathcal{I}^p(S)$. For RDM, on the other hand, such contextlessness is violated. Nonetheless, RDM still satisfies α'. This leads to a natural conjecture: *RDM's choice must still be rationalizable by* $\succsim^p$. This is in fact true and formalized in the next proposition.

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[17]Relaxing β' on fully indecisive tripletons, in fact, captures another behavioral anomaly referred to as *compromise effect* in the literature. This effect pertains to references that act on the element from the same indifference curve. However consistent with my framework, explaining this effect is beyond the scope of this work.

Proposition 4. *Let **c** be a RDM and $S \subseteq X$. Then*

$$\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S).$$

The proofs of Proposition 4 is presented in Section 2.8.1. The intuition is quite simple. As long as DM is "capable" to reveal her rational preference in all pairs of alternatives, that is if $\succsim^p$ is a complete and transitive relation, then she is completely "aware" of all available options and the manner in which she ranks them. Therefore, there is no reason to ignore dominant alternatives in favor of dominated ones.[18]

2.4.2 Referenced Preferences

After observing the pairwise revealed preference $(\succsim^p)$ from doubletons, the next step is to observe where pairwise relation is refined as a result of adding a third alternative. This revelation of references is then followed by observation in regards to the manner references rank other elements, in particular those in a highest indifference curve, giving birth to *revealed reference relations*.[19] This notion is introduced in the following definition.

Definition 7. (Reference Relation) Let $S \subseteq X$ and $r \in \mathfrak{R}(S)$. For two distinct element $x, y \in S$, both different from r, define

$$x \succsim^r y \iff x \in \boldsymbol{c}\{x, y, r\}.$$

Also let $\succ^r$ and $\sim^r$ be the asymmetric and symmetric parts of $\succsim^r$.

This next proposition explores the nature of the reference relations.

Proposition 5. *Let **c** be a RDM, $S \subseteq X$, and $r \in \mathfrak{R}(S)$. Then*

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[18]This intuitive result, indeed, is key to the distinction between predictions of my rational approach and the existing behavioral treatments in the literature. I elaborate on this point in Section 2.4.3.

[19]As in the case of references, I drop the word "revealed" from the rest of this paper.

(i) $x \succsim^r y \implies x \succsim^p y$.

(ii) $\succsim^r$ *defines a complete binary relation on* $\mathcal{I}^p(S)$.

The proof of Proposition 5 is presented in Section 2.8.1 . Proposition 5.i speaks to the evolution of the pairwise revealed preference by adapting to a reference point r. To see this let us look at an example that shows the reverse direction may not hold. Let $S = \{x, y, r\}$ such that $\boldsymbol{c}\{x, y\} = \{x, y\}, \boldsymbol{c}\{x, r\} = \{x\}, \boldsymbol{c}\{y, r\} = \{y\}$, and $\boldsymbol{c}\{x, y, r\} = \{x\}$. This is typical case of attraction effect (or statues quo bias). $\boldsymbol{c}$ is a RDM on S, and $y \succsim^p x$, but $x \succ^r y$. Proposition 5.ii, on the other hand, guarantees one of the two essential characteristics of reference relations: *complete-ness*.[20] As I show in Section 2.5 reference relations are also *acyclic*, completing their characterization.[21]

2.4.3 Maximal References

Definition 8. We say $r \in S$ is a *maximal* reference if there exists $x, y \in \mathcal{I}^p(S)$ such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y.\}$$

We also use the following notation:

$$\mathfrak{R}_M(S) := \{r \in S \text{ such that } r \text{ is a maximal reference in } S\}.$$

Definition 9. For a maximal reference r and $x \in S$ we say r *refers* x in S (or

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[20] As discussed before for any arbitrary sets S, a RDM's behavior satisfies $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$ and, therefore, $\mathcal{I}^p(S)$ is the domain of relevance for $\succsim^r$. This intuition is in fact true; that is $\underset{S}{\text{argmax}} \succsim^r = \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^r$. For the proof of this latter statement see footnote 28 in the Appendix.

[21] $\succsim^r$ are assumed to be transitive in Tversky and Kahneman (1991). However, for the purpose of maximization the weaker notion of acyclicity of references is sufficient.

equivalently, r is a *x-maximal* reference) if

$$x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r .$$

We also use the the notation:

$$\mathfrak{R}_M^x(S) := \{r \in \mathfrak{R}_M(S) : r \text{ is a } x\text{-maximal reference}\}.$$

We call $\mathfrak{R}_M^x(S)$ the *maximal reference set* of x in S.

Note that

$$\mathfrak{R}(S) \supseteq \mathfrak{R}_M(S) = \bigcup_{x \in \mathcal{I}^p(S)} \mathfrak{R}_M^x(S).$$

The difference between the maximal and non-maximal references is illustrated in Figure 2.4. In this figure, brackets represent indifference classes (indifference curves) and the most preferred class, $\mathcal{I}^p(S)$, is the one of the left containing x_1, x_2, and x_3. Recall from the motivational examples (and also from Proposition 3) that references are from lower indifference curves and therefore all referential effects (presented by both dashed and solid arrows) are from a class on the right to one on the left. x_7 is a reference that refines the pairwise relation between x_4 and x_5 in favor of x_4, however, it does not affect the pairwise relation between the elements of the most preferred class and therefore not a maximal reference (these are shown by dashed arrows). x_6, and x_4, on the other hand, are maximal references since they affect the pairwise relation in $\mathcal{I}^p(S)$ (these are shown by solid arrows). In particular, $\{x_2\} = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{x_6}$, and $\{x_1, x_3\} = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{x_4}$. As shown in this figure, references can operate in more than one class. For example, x_6 refine the pairwise relation in both first and second class. Also note that even though x_5 is dominated from the preceptive of pairwise comparison, it is not a reference since there is no observation of refinement of the pairwise relation for this element. Finally, and obviously from my setup, elements

of $\mathcal{I}^p(S)$ do not induce referential effects. Note that the notion of referring is only defined for maximal references. In Figure 2.4, x_6 refers to x_2 and x_4 refers to x_1, and x_3.

$$\left[x_1 \sim^p x_2 \sim^p x_3\right] \succ^p \left[x_4 \sim^p x_5\right] \succ^p \left[x_6 \sim^p x_7\right]$$

$$\mathcal{I}^p(S)$$

Non-Maximal Reference(s): x_7
Maximal References: x_4, x_6

Figure 2.4: Maximal and Non-Maximal References

A natural question arises here: *why is the notion of referring only defined for maximal references?* This is purposeful, and indeed, speaks to the key difference that distinguishes the rational treatment in referential revealed preference theory from the behavioral approaches taken in the existing literature.[22] As I show in Section 2.5 (Theorem 4) the maximal references are the sole determinants of the RDM's behavior that goes beyond WARP. Indeed, recall from Proposition 4 that RDM's behavior satisfies $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$. As a result only those references that effect the relative ranking of elements of $\mathcal{I}^p(S)$ play a role in the characterization of choice.

Let me consider an example. Assume that a DM is choosing from a menu in a restaurant. The menu consists of five options: i. beef ribs with a side of lentil soup, ii. pork ribs with side that does not contain lentil, nor beans, iii. a vegetarian dish (veg1) containing both lentil and beans, iv. a vegetarian dish (veg2) that does not contain lentil, nor beans, and v. vegan dish that contains beans. Assume that price

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[22]See, for example, Masatlioglu et al. (2012), Ok et al. (2015), and Lleras et al. (2017). Masatlioglu et al. (2012), Lleras et al. (2017) only consider choice functions. Here I consider the implications of their approach on choice correspondence.

is of no concern for the DM. DM is an absolute meat lover who, even though does not mind vegetarian dishes, will not choose them over meat options. Assume the following pairwise preference:

$$[\text{beef ribs} \sim^p \text{pork ribs}] \succ^p [\text{veg1} \sim^p \text{veg 2}] \succ^p [\text{vegan}].$$

DM associates the variety of lentil and beans in the menu as a sign of chef's specialty and therefore veg1 and vegan dish acts as references. Now consider the two following scenarios:

Assume, in the first scenario, that the restaurant is out of ribs (both pork and beef). Then veg1 and veg2 are the most favored alternatives from the perspective of pairwise comparison. In this scenario, offering beans, vegan dish acts as a reference that effect the most favored class and therefore a maximal reference under which DM chooses veg1.

$$[\underbrace{\text{veg1} \sim^p \text{veg2}}_{\Im(S)}] \succ^p [\text{vegan}]$$

Scenario 1

In the second scenario, all options in the menu are available. In this case, offering lentil, veg1 also acts as a reference and the following referential effects are observed:

$$[\underbrace{\text{beef ribs} \sim^p \text{pork ribs}}_{\Im(S)}] \succ^p [\text{veg1} \sim^p \text{veg2}] \succ^p [\text{vegan}]$$

Scenario 2

Veg1 is a maximal reference (solid arrow) that favors beef ribs over pork ribs. However vegan dish is not a maximal reference anymore (dashed arrow) as it fails to effect the pairwise relation between the most preferred options. The choice of veg1

in this scenario is consistent if an arbitrary notion of "inattention" (or attraction) is employed. In words, a DM's might use the vegan dish as a reference and become inattentive towards the dominant options of meat, and therefore choose the dominated alternative veg1. However, such prediction is not consistent with the referential revealed preference theory; that is WARP deviations are only caused by maximal references and, therefore, the consistent choice in this scenario is beef ribs. In words, a RDM who admits to a complete and transitive preference relation on the entire set of alternative is not "irrational" in her inattention.[23]

2.5   Main Results

Now I can proceed to the main results of the paper. Recall form Proposition 5.ii that $\succsim^r$ is a complete relation on $\mathcal{I}^p(S)$. The first result completes the characterization of reference relations asserting that they are also acyclical.[24]

Theorem 3. *(**Acyclicity of Reference Relations**) Let c be a RDM and $r \in \mathfrak{R}(S)$. $\succsim^r$ defines an acyclic relation on $\mathcal{I}^p(S)$.*

The proof of Theorem 3 is presented in Section 2.8.1. The final theorem in this paper is a characterization of choice on an arbitrary set with arbitrary number of references.

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[23]To explore this distinction in more detail, let me employ the jargon of inattention and say veg1 attracts attention to beef ribs, and vegan dish attracts attention to veg1. Then DM chooses to be *rational* in her inattention by only using those references that affect her most preferred alternatives. The aforementioned interpretation is, in a sense, in line with the talking point of the *rational inattention* models introduced in Sims (2003). These models, however, are indistinguishable from the processing capacity constrain (Shannon entropy) under which the choice is made. This constraint is lacking in the current work, and therefore, to avoid confusion, I keep the jargon of this work akin to that of classical revealed preference theory.

[24]Reference acyclicity (RA) also appears in Ok et al. (2015). Two things are important to note here. The notion in these authors' paper refers to the manner references operates on each other and, therefore, different from the acyclic "revealed binary relation" induced here. Indeed, the two notions are not logically nested. Second, RA is an axiom in Ok et al. (2015) and a result in this paper.

Theorem 4. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Then $x \in \boldsymbol{c}(S)$ if and only if $x \in \mathcal{I}^p(S)$ and for all $t \in \mathcal{I}^p(S)$, the number of x-maximal references in S is greater than or equal to the number of t-maximal references in S.*

Proof. We need to show that

$$x \in \boldsymbol{c}(S) \iff x \in \mathcal{I}^p(S), \text{ and } |\mathfrak{R}^x_M(S)| \geq |\mathfrak{R}^t_M(S)| \text{ for all } t \in \mathcal{I}^p(S).$$

I start with following lemmas the proof of which, in order to keep the flow of the argument, are presented in Section 2.8.1.

Lemma 10. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$ that possesses at least two references. There exists $\bar{r} \in \mathfrak{R}(S)$ such that $x \in \boldsymbol{c}(S - \bar{r})$.*

Lemma 11. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$ that possesses at least two maximal references. For any reference r in S assume*

$$t^* \in \boldsymbol{c}(S - r) \iff |\mathfrak{R}^{t^*}_M(S - r)| \geq |\mathfrak{R}^t_M(S - r)|, \text{ for all } t \in \mathcal{I}^p(S - r.)$$

Also assume that $x \in \boldsymbol{c}(S)$ and $x, y \in \boldsymbol{c}(S - \bar{r})$ for a reference $\bar{r}$ in S. If $\bar{r}$ refers to y then it refers to x.

We now, start the proof of the theorem by considering two cases.

Case 1: $\mathfrak{R}_M(S) = \emptyset$.

Note that in this case all elements in $\mathcal{I}^p(S)$ posses zero maximal references. Take $x \in \boldsymbol{c}(S)$. From Proposition 4 we have $x \in \mathcal{I}^p(S)$. Also, in S, the number of x-maximal references (zero) is greater than or equal to the number of t-maximal references (zero), for all $t \in \mathcal{I}^p(S)$ which completes $\Rightarrow$ direction of the proof. Next

assume $x \in \mathcal{I}^p(S)$ (and note that it is true in case that, in S, the number of x-maximal references is greater than or equal to the number of t-maximal references, for all $t \in \mathcal{I}^p(S)$). It directly follows from Lemma 15 that $x \in \boldsymbol{c}(S)$. This completes the proof of $\Leftarrow$ and, hence, the proof of the statement for Case 1.

<u>Case 2</u>: $\mathfrak{R}_M(S) \neq \varnothing$

Assume S at least has one maximal reference. We prove the statement by induction on $|\mathfrak{R}_M(S)|$. Note that since the set of maximal references in S is non-empty we conclude the set of references in S is also non-empty.

Induction Base: Let $\mathfrak{R}_M(S) = \{r\}$.

($\Rightarrow$): Let $x \in \boldsymbol{c}(S)$. Since r is the unique maximal reference in S Lemma 18 implies $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \overset{r}{\succsim}$. First note $x \in \mathcal{I}^p(S)$. Also, r, as the single maximal reference in S, refers to x and the result follows.

($\Leftarrow$): Now assume $x \in \mathcal{I}^p(S)$ and

$$|\mathfrak{R}_M^x(S)| \geq |\mathfrak{R}_M^t(S)| \text{ for all } t \in \mathcal{I}^p(S).$$

We must show $x \in \boldsymbol{c}(S)$. Note that since r is a maximal reference in S Corollary 19 implies that $\underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \overset{r}{\succsim} \neq \varnothing$. So take $z \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \overset{r}{\succsim}$. This means r refers to z. Since the number of x-maximal references in S is greater or equal to the number of z-maximal references, and S only possesses a single maximal reference, r, we conclude r refers to x. Therefore $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \overset{r}{\succsim}$. Note that $\boldsymbol{c}$ satisfies the conditions in Lemma 18 and we conclude from $\Leftarrow$ of that theorem that $x \in \boldsymbol{c}(S)$.

Induction Hypothesis: Assume that the statement is true for $A \subseteq X$ with $|\mathfrak{R}_M(A)| = k$ and let $|\mathfrak{R}(S)| = k + 1$. Note that we have $|\mathfrak{R}(S)| \geq |\mathfrak{R}_M(S)| \geq 2$ in our induction. Therefore we have $\boldsymbol{d}(S) = R_1(S)$. That is all elements of $\boldsymbol{d}(S)$ are of the form $S - r$ for some $r \in \mathfrak{R}(S)$.

($\Rightarrow$): Let $x \in \boldsymbol{c}(S)$. Note that Lemma 24 implies $x \in \mathcal{I}^p(S)$. From Lemma 10 we conclude $x \in \boldsymbol{c}(S - \bar{r})$ for some $\bar{r} \in \mathfrak{R}(S)$.

Recall that we must show that, in S, the number of x-maximal references is greater than or equal to the number of t-maximal references, for all $t \in \mathcal{I}^p(S)$. Consider the set $S - \bar{r}$. Note that $\mathcal{I}^p(S - \bar{r}) = \mathcal{I}^p(S)$. Take $y \in \mathcal{I}^p(S) = \mathcal{I}^p(S - \bar{r}) \subseteq S - \bar{r}$. We consider two cases here:

Case 1: $y \notin \boldsymbol{c}(S - \bar{r})$.

Since $x \in \boldsymbol{c}(S - \bar{r})$ and $y \notin \boldsymbol{c}(S - \bar{r})$ then induction hypothesis implies that

$$|\mathfrak{R}_M^x(S - \bar{r})| > |\mathfrak{R}_M^y(S - \bar{r})|,$$

and therefore adding $\bar{r}$ to the set $S - \bar{r}$ does not increase the number of y-maximal references over the number of x-maximal references; that is,

$$|\mathfrak{R}_M^x(S)| \geq |\mathfrak{R}_M^y(S)|,$$

and the proof in this case is complete.

Case 2: $y \in \boldsymbol{c}(S - \bar{r})$.

Note that since $x, y \in \boldsymbol{c}(S - \bar{r})$ induction hypothesis implies

$$|\mathfrak{R}_M^x(S - \bar{r})| = |\mathfrak{R}_M^y(S - \bar{r})|.$$

Note that all the conditions of Lemma 19 are met (considering induction hypothesis) and therefore we conclude if $\bar{r}$ refers to y then it also refers to x. This means adding $\bar{r}$ to the set $S - \bar{r}$ does not increase the number of y-maximal references over the number of x-maximal references; that is,

$$|\mathfrak{R}_M^x(S)| \geq |\mathfrak{R}_M^y(S)|.$$

This complete the proof in this case and, in turn, the proof of $\Rightarrow$ of the theorem.

($\Leftarrow$): Take $x \in \mathcal{I}^p(S)$ and assume that

$$|\mathfrak{R}_M^x(S)| \geq |\mathfrak{R}_M^t(S)|, \text{ for all } t \in \mathcal{I}^p(S). \tag{2.1}$$

We must show $x \in c(S)$. First note in our induction there are at least two references in S and therefore $|S| \geq 4^{25}$ and, therefore, S satisfies β'. s a result in order to show $x \in \boldsymbol{c}(S)$ we prove that there does not exist $y \in S$ such that $y \underset{S}{\blacktriangleright} x$; that is we show x is a maximal element of $\underset{S}{\blacktriangleright}$ on S. To do this first recall that all elements of $\boldsymbol{d}(S)$ are of the form $S - r$ for some reference r is S. Therefore assume that there exist $y \in S$ and $r^* \in \mathfrak{R}(S)$ such that y beats x in $S - r^*$. In particular $y \in \boldsymbol{c}(S - r^*)$ and therefore by Proposition 4 $y \in \mathcal{I}^p(S - r^*) = \mathcal{I}^p(S)$. Note that since in our induction

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[25]To see this note that since r is a maximal reference in S there exists two distinct elements, t_1, t_2 in $\mathcal{I}^p(S)$, both different from r. Next since $|\mathfrak{R}(S)| \geq 2$ there exists $s \neq r$ in S. Since s is a reference in S it follows from Proposition 6. iv that $s \notin \mathcal{I}^p(S)$. This in turn means $t_1, t_2 \succ^p s$ and there for t_1, t_2 are different from s. Therefore we conclude t_1, t_2, r, s are four distinct elements. This means $|S| \geq 4$.

$|\mathfrak{R}_M(S)| \geq 2$ it follows that there exists a maximal reference in $S - r^*$. Therefore induction hypothesis applies to this set and we conclude

$$|\mathfrak{R}_M^y(S - r^*)| > |\mathfrak{R}_M^x(S - r^*)|.$$

First note that this along with (1) implies $|\mathfrak{R}_M^x(S)| = |\mathfrak{R}_M^y(S)|$. Second note that since dropping r^* lowers the number of maximal references of x relative to y it has to be the case that r^* refers to x and r^* does not refer to y. Since the number of maximal references that refer to x in S is equal to the number of maximal references that refer to y there must exist a maximal reference r^{**} such that r^{**} refers to y, and r^{**} does not refer to x. To complete the proof of this direction consider the set $S - r^{**}$. Since x, y have the equal number of maximal references in S dropping r^{**} from S should lower the number of y-maximal references below the number of x-maximal references; that is

$$|\mathfrak{R}_M^x(S - r^{**})| > |\mathfrak{R}_M^y(S - r^{**})|$$

.

By induction assumption this means x beats y in $S - r^{**}$. Therefore x is not dominated by y and β' implies that $x \in c(S)$. This completes this direction of the proof. ■

2.6   Discussion

2.6.1   Interpretation of the Main Results

Some implication of the final theorem are important to note. First, if there are no maximal references in the set S then the result formulated in Theorem 4 is simplified to

$$c(S) = \mathcal{I}^p(S);$$

that is, in the absence of maximal references the referential revealed preference theory coincides with the classical revealed preference theory. As we discussed in Section 2.4.3 this is a key distinction between the rational treatment in this paper and the behavioral ones in the existing literature.

Second, assume that r is the unique maximal reference in the set S. Then from Theorem 4 the choice is characterized by those elements of $\mathcal{I}^p(S)$ to which r refers. That is:

$$\boldsymbol{c}(S) = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r = \operatorname*{argmax}_{\operatorname*{argmax}_{S}\succsim^p} \succsim^r .$$

That is the choice in the set S is rationalized by, first, applying $\succsim^p$ on S, and second, applying $\succsim^r$ on the resulting set. This is exactly the *rational shortlist method (RSM)* in Manzini and Mariotti (2007) applied to choice correspondence; that is, the *methodology* of sequential rationalizability arises, endogenously, in the referential revealed preference theory.

Third, it is implied from the result in Theorem 4 that opposing references *nullify* the referential effect. To clarify, assume that a DM who is indecisive between two items x, y when confronted with the menu xy, prefers x over y when a reference r_1 is introduced into the menu, and y over x when an opposing reference r_2 is added. Therefore r_1, and r_2 are opposing maximal references that refer to x and y, respectively, and Theorem 4 predicts that DM's choice on the menu $xyzr_1r_2$ should be xy; that is, she acts as if there were no references in the menu. This intuitive prediction is supported by the results in Teppan and Felfering (2012).

2.6.2   Sen's Decomposition and RDM

In terms of presentation Sen's Dcomposition (Sen (1971)) is reminiscent to the decomposition provided in this paper. In this section I provide an example to show that the notion of RDM can not be constructed via Sen's Decomposition. In order to do

this let me introduced Sen's decomposition:

Sen's Property α: If $x \in B \subseteq A$ and $x \in \boldsymbol{c}(A)$, then $x \in \boldsymbol{c}(B)$.

Sen's Property β: If $x, y \in \boldsymbol{c}(A)$, $A \subseteq B$ and $y \in \boldsymbol{c}(B)$, then $x \in \boldsymbol{c}(B)$.

Sen (1971) proves that WARP is decomposable to properties α and β. The following example shows that a RDM need not satisfy either of these properties.

Example 1. Consider the following choice correspondence where circles represent choice.

$$\{\textcircled{x}, \textcircled{y}, r_1, r_2\}$$

$$\{\textcircled{x}, y, r_1\}, \{x, \textcircled{y}, r_2\}, \{\textcircled{x}, r_1, r_2\}, \{\textcircled{y}, r_1, r_2\}$$

$$\{\textcircled{x}, \textcircled{y}\}, \{\textcircled{x}, r_1\}, \{\textcircled{y}, r_1\}, \{\textcircled{x}, r_2\} \{\textcircled{y}, r_2\}, \{\textcircled{r_1}, r_2\}$$

This choice correspondence satisfies α' and NBC, and β' above the referential level, and therefore is a RDM. However, it violates β. The violation of β is obviously a result of referential effect where pairwise relation is getting refined via a reference. For example, $y \in \boldsymbol{c}\{x, y, \}$ and $\{x, y\} \subseteq \{x, y, r_1\}$, but $y \notin \boldsymbol{c}\{x, y, r_1\}$. It also violates α since one would expect the opposing referential effects to *cancel out* each other as results in Teppan and Felfering (2012) suggests. For example, since r_1 refines the pairwise relation in favor of x and r_2 does in favor of y the referential effect disappears when both decoys are introduced in the menu. Under α such nullification is impossible.

2.6.3   Context-Based Decision Making and Welfare Analysis

The referential revealed preference theory, as one probably expects, produces testable predictions. One major precondition is the notion of "best in one attribute" vs. "better in more attributes". To see that assume that a DM who is indecisive between the four alternatives x_1, x_2, x_3, x_4 also faces some less favorite alternatives r_1, r_2, r_3 that have features she likes. Therefore she has the following pairwise preference relation:

$$\left[x_1 \sim^p x_2 \sim^p x_3 \sim^p x_4 \right] \succ^p r_1 \succ^p r_2 \succ^p r_3$$

In particular, assume that, r_1, r_2, and r_3 present favorite color, style, and the discount offered, respectively. Obviously the prediction here is that she will choose an alternative from $\{x_1, x_2, x_3, x_4\}$. Assume from each reference point the following ranking is observed:

$$r_1 \text{ (Color)} : x_1 \succ^{r_1} x_2 \succ^{r_1} x_3 \succ^{r_1} x_4$$

$$r_1 \text{ (Style)} : x_2 \succ^{r_2} x_3 \succ^{r_2} x_4 \succ^{r_2} x_1,$$

$$r_1 \text{ (Dicsount)} : x_3 \succ^{r_3} x_2 \succ^{r_3} x_4 \succ^{r_1} x_1.$$

Therefore x_1, x_2, and x_3 all have a reference that puts them at the most preferred alternative. x_4 does note have such support from any reference. From the point of view the theory in this paper, a switch from x_4 to any alternatives in the set $\{x_1, x_2, x_3\}$ is welfare improving. However when only compared to x_1, x_4 is indeed better in both style and the offered discount, and x_1 is only favorite in color. That is in the absence of alternatives x_2, x_3, switching from x_1 to x_4, indeed, increases welfare.

The notion of revealed preferences developed in Bernheim and Rangel (2009) has some relations to the referential revealed preference in terms of welfare analysis. However, the theory in this paper is also concerned with keeping the predictive power of the classical theory to the extent possible. If I interpret the "ancillary condition" in these authors' paper as the referential effect then we can explore the relation. To make this formal, assume that the ancillary condition are captured by the referential effect introduce via a references that refines the pairwise relation between any arbitrary pair x, y. Next define a variation of Bernheim and Rangel (2009)'s *strict unambiguous preference relation*, P^*, in the following manner:

$$x P^* y \text{ if and only if for all } \{x, y, r_i\} \text{ we have } y \notin \boldsymbol{c}\{x, y, r_i\}.$$

Under such case, and for all $S \subseteq X$ with $x, y \in S$ there are no maximal reference that refer to y (since $x \succ^{r_i} y$ for all i), and therefore Theorem 4 implies for all $S \subseteq X$, with $x, y \in S$ we have

$$x P^* y \implies y \notin \boldsymbol{c}(S).$$

That is if x is strictly unambiguously chosen over y in via all references then x is strictly unambiguously chosen over y in all possible set of alternatives that contain both x, y. However such prediction is not necessarily true for the *weak unambiguous choice* relation, R'. Using the same interpretation this a variation of this revealed preference relation could be defined as:

$$x R' y \text{ if and only if for all } \{x, y, r_i\}, \ y \in \boldsymbol{c}\{x, y, r_i\} \text{ implies } x \in \boldsymbol{c}\{x, y, r_i\}.$$

Now consider the following example, where a DM has the follwoign pairwise prefe-

rence:

$$x \sim^p y \sim^p z \succ^p r_1 \succ^p r_2,$$

with the following acyclic reference relations:

$$r_1 : x \sim^{r_1} y, y \succ^{r_1} z, z \succ^{r_1} x,$$

$$r_2 : x \sim^{r_2} y, y \succ^{r_2} z, z \succ^{r_2} x.$$

It is true that $x, y \in \boldsymbol{c}\{x, y, r_i\}$ for $i \in \{1, 2\}$. However, Theorem 4 implies

$$\boldsymbol{c}\{x, y, z, r_1, r_2\} = \{y\};$$

that is we can not conclude x is "weakly unambiguously chosen over y" in all possible scenarios, if it indeed does so via any possible references. Therefore, for $S \subseteq X$ with $x, y \in S$ and $xR'y$ then:

$$y \in \boldsymbol{c}(S) \;\;\not\Longrightarrow\;\; x \in \boldsymbol{c}(S).$$

2.6.4 RDM and Other Behavioral Anomalies

In this section I discuss how the theory developed in this paper can be used in order to address other behavioral anomalies in the literature. Another behavioral anomaly which has been experimentally documented is the tendency to retain status quo. Assume that a DM, to start, is indecisive between alternatives x and y. In period 1, she is given one of these alternatives, and then makes a choice from the two in period 2. Figure 2.5 depicts the choice scenario in period 2. The main idea in status quo bias is that the initial *endowments* produce biases that refines the pairwise relation.[26] Assume that in period 1 the decision maker is given the alternative x. A risk averse

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[26]This effect and its twin, *endowment effect*, are essentially the same. For a detailed discussion on this see Tversky and Kahneman (1991).

DM who assigns higher weights to losses than gains will favor x in period 2 to retain her status quo. Let $\succsim^p$ denote the preference derived from the pairwise comparison in period 1 (the one that matches the revealed preference theory) and (a, t_i) denote the alternative a in time i. Then $(x, t_2) \sim^p (y, t_2)$, but $(x, t_2) \succ^{(x,t_1)} (y, t_2)$. Obviously both (x, t_2) and (y, t_2) dominate (x, t_1) in period 2 as the consumption in period 1 is not available anymore. The same argument works in the reverse direction when the initial endowment is y.[27] Therefore, under such interpretation, referential revealed preference theory also explain this effect. The significance of this effect has also been vastly documented in the literature. See, for example, Samuelson and Zeckhauser (1988), Tversky and Kahneman (1991), Knetsch and Sinden (1984), and Knetsch (1989).

Figure 2.5: Status-Que Bias (or Endowment Effect)

As mentioned in Footnote 17 relaxing β' on fully indecisive tripletons will explain the compromise effect. In order to be explain choosing pairwise unchosen phenomenon one could, along with β', relax α' from the referential level as well. The framework

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[27]Our approach to status quo bias is similar to that of *phantom* alternatives. A phantom alternative is an alternative which, for example because of being out of stock, is in the consideration set of the DM but can not be chosen. The significance of phantom alternatives are documented in the experimental works. See for example Doyl et al. (1999) for the case of phantom decoys.

here, because of NBC, is not consistent with the cyclical choice. However, one could think of an extension of the results for the case when weaker notions of transitivity are assumed at the pairwise level, eg. pseudotransitivity, or acyclicity.

2.7 Conclusion

In this paper, I develop "revealed preference"-type theory (in the sense of Samuelson (1938)) that is consistent with the decoy effect. The main innovation of this paper is that I search for a minimal deviation from WARP (that is one which is proportionate to the extend WARP is violated in the data). This is done using a decomposition theorem of WARP. I extract the axiomatic approach from WARP. Next I track those WARP-rationals which are the sole reason of the inability of the classical theory in explaining behavioral anomalies such as attraction effect. Removing these rationals from the classical DM, naturally, gives birth to the notion of the referential DM. I show that the rational treatment in this paper preserves the predictive power of the classical theory to the extent possible; that is, RDM's choice behavior is completely characterized by the observations on doubletons and tripletons (as compared to the classical revealed preference theory where such characterization is made with only observations on doubletons.) In addition, the methodology of sequential rationalizability in Manzini and Mariotti (2007) arises, endogenously, as an untapped potential in WARP coming to effect in the referential revealed preference theory.

2.8 Appendix

2.8.1 Proofs

Throughout this section, and in order to avoid confusion, I preserve the symbol ∎ for Halmos Q.E.D. sign for the proofs of the main statements, and, wherever applied,

□ for subproofs. Let $\mathcal{F}$ be any of the operators $\mathfrak{R}, R_1, \mathfrak{B}$, or $\mathcal{I}^p$. I also make the following notational abuse throughout the arguments in this section.

$$\mathcal{F}(\{x, y, z\}) := \mathcal{F}\{x, y, z\}$$

For $S \subseteq X$, let

$$S^- = \{A - x : x \in A\}.$$

Recall that $R_1(S)$ is the collection of of subsets of S derived from taking references out of S, one at a time, and $R_2(S)$ is the collection of all subsets of S derived from taking non-reference elements out. S^-, on the other hand, is the collection of all subsets of S derived from removing any arbitrary element. Note that if $\mathfrak{R}(S) = \emptyset$ then $d(S) = R_2(S) = S^-$.

Proof of Proposition 2

Proposition 2. *Let c be a RDM. Then*

$$\mathfrak{R}(X) \neq \emptyset \iff c \text{ is a pure RDM.}$$

Proof. ($\Rightarrow$): Take $r \in \mathfrak{R}(X)$. Then there exists $x, y \in X$ such that

$$c\{x, y, r\} \subset c\{x, y\}.$$

Therefore we conclude c does not satisfy WARP, and as a result, is not a CDM. This completes this direction of the proof.

($\Leftarrow$): Assume that c is a pure RDM. Therefore there exists a tripleton, T, which is not fully indecisive, and a $x \in T$ such that $x \in T$ and x is a maximal element of $\blacktriangleright_{T}$, but $x \notin c(T)$. Let $T = \{x, y, z\}$. If $d(T) \neq \{\{x, y\}, \{x, z\}, \{y, z\}\}$ then there should be a reference in T and there is nothing to prove. So assume $d(T) =$

$\{\{x,y\}, \{x,z\}, \{y,z\}\}$, and wlog, assume $y \in \boldsymbol{c}(T)$. Then α' implies $\boldsymbol{c}\{x,y\} = \{x,y\}$. We only need to show $z \notin \boldsymbol{c}(T)$. Assume otherwise. Then α' implies that $\boldsymbol{c}\{x,z\} = \{x,z\}$ and $\boldsymbol{c}\{y,z\} = \{y,z\}$. This mean T is fully indecisive; a contradiction. Therefore it should be that $z \notin \boldsymbol{c}(T)$ and we conclude

$$\boldsymbol{c}\{x,y,z\} \subseteq \boldsymbol{c}\{x,y\}.$$

This mean $z \in \mathfrak{R}(X)$ which completes this direction of the proof. ∎

Proof of Propositions 3 and 4

I start by the following basic observations about RDM.

Proposition 6. *Let $\boldsymbol{c}$ be a RDM. The following is true:*

(i) If S is fully indecisive then $\boldsymbol{c}(S) = S$.

(ii) $|\mathfrak{R}\{x,y,z\}| \leq 1$.

Proof. (i) We prove this by induction on $|S|$.

Induction Base: For $|S| = 3$ let $S = \{x,y,z\}$. First note that since S is fully indecisive it satisfies β'. Take $t \in \{x,y,z\}$. First note that since $\boldsymbol{d}\{x,y,z\}$ is a cover for S and the fact that $x \sim^p y \sim^p z$ we conclude that $q \not\underset{S}{\blacktriangleright} t$ for all $q \in S$. Therefore, by β', we conclude $t \in \boldsymbol{c}(S)$. Therefore $\boldsymbol{c}\{x,y,z\} = \{x,y,z\}$.

Induction Hypothesis: Now assume that the statement is true if $|S| = k$. Let $|S| = k + 1$. Take $x \in S$. Take $t \in S$. Note that for $A \in S^-$ we have A is fully indecisive and therefore by induction base $\boldsymbol{c}(A) = A$. Next note that $\boldsymbol{d}(S) \subseteq S^-$. Therefore for all $A \in \boldsymbol{d}(S)$ we have $\boldsymbol{c}(A) = A$. This implies

there does not exists $q \in S$ such that $q \underset{S}{\blacktriangleright} t$. β' implies $t \in \boldsymbol{c}(S)$. This means $\boldsymbol{c}(S) = S$.

(ii) Consider the set $\{x, y, z\}$. By definition $t \in \mathfrak{R}\{x, y, z\}$ implies $t \notin \boldsymbol{c}\{x, y, z\}$. Since $\boldsymbol{c}\{x, y, z\} \neq \emptyset$ then we conclude $|\mathfrak{R}\{x, y, z\}| < 3$. It suffice to show $|\mathfrak{R}\{x, y, z\}| \neq 2$. Assume, wlog, $\mathfrak{R}\{x, y, z\} = \{y, z\}$. First note that this means $\boldsymbol{c}\{x, y, z\} = \{x\}$. Next since y, z are references we should have

$$\{x\} = \boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, z\},$$

and

$$\{x\} = \boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, y\},$$

which, respectively, imply

$$\boldsymbol{c}\{x, z\} = \{x, z\},$$

and

$$\boldsymbol{c}\{x, y\} = \{x, y\}.$$

Next NBC implies $x \sim^p y \sim^p z$ and therefore we conclude $\{x, y, z\}$ is fully indecisive and by part (i) we conclude $\boldsymbol{c}\{x, y, z\} = \{x, y, z\}$ which is a contradiction. Therefore $|\mathfrak{R}\{x, y, z\}| \leq 1$.

∎

Proposition 3. *Let $\boldsymbol{c}$ be a RDM. Also let $r \in \mathfrak{R}(S)$ and $x, y \in S$ such that*

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

Then $x \sim^p y \succ^p r$.

Proof. Take $r \in \mathfrak{R}(S)$ and $x, y \in S$ such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

First note that $r \notin \boldsymbol{c}\{x, y, r\}$, and, since $\boldsymbol{c}\{x, y, r\} \neq \emptyset$, it has to be the case that $\boldsymbol{c}\{x, y\} = \{x, y\}$. Therefore $x \sim^p y$. Since r is also a reference in $\{x, y, r\}$ from Proposition 6.ii we conclude $\mathfrak{R}\{x, y, r\} = \{r\}$. This implies

$$\boldsymbol{d}(S) = \{\{x, r\}, \{y, r\}\}.$$

If, wlog, $r \in \boldsymbol{c}\{x, r\}$, then NBC implies $x \sim^p y \sim^p r$ and therefore $\{x, y, z\}$ is fully indecisive and by Proposition 6.i we will have $\boldsymbol{c}\{x, y, z\} = \{x, y, z\}$ which is not possible. So it has to be the case that $\boldsymbol{c}\{x, r\} = \{x\}$ and $\boldsymbol{c}\{y, r\} = \{y\}$ which imply, respectively, $x \succ^p r$, and $y \succ^p r$. ■

This next statement is a direct result of the Proposition 3 and is used in the future results of the paper.

Corollary 12. *Let $\boldsymbol{c}$ be a RDM and $r \in \mathfrak{R}(S)$. Then $r \notin \mathcal{I}^p(S)$.*

Proof. Take $r \in \mathfrak{R}(S)$. From the definition of reference we conclude that there exists two distinct elements $x, y, \in S$, both different from r such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

Now Proposition 3 implies $x \sim^p y \succ^p r$. This means $r \notin \mathcal{I}^p(S)$. ■

Proposition 4. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Then*

$$\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$$

Proof. We prove this by induction on $|S|$.

Induction Base: For $|S| = 3$ let $S = \{x, y, z\}$. Let $x \in \boldsymbol{c}\{x, y, z\}$. We need to show $x \in \mathcal{I}^p\{x, y, z\}$. If $\mathfrak{R}(S) = \emptyset$ then $\boldsymbol{d}(S) = \{x, y, z\}^-$ and α' implies that $x \in \boldsymbol{c}\{x, y\}$ and $x \in \boldsymbol{c}\{x, z\}$ which in turn implies $x \succsim^p y$ and $x \succsim^p z$. This means $x \in \mathcal{I}^p\{x, y, z\}$ and the proof is complete. So assume $\mathfrak{R}\{x, y, z\} \neq \emptyset$. Since $x \in \boldsymbol{c}\{x, y, z\}$ then the definition of reference implies $x \notin \mathfrak{R}\{x, y, z\}$. Also note that prp:properties.ii implies that there only exists one reference in $\{x, y, z\}$. Wlog, assume $\mathfrak{R}\{x, y, z\} = \{z\}$. Then Proposition 3 implies $x \sim^p y \succ^p z$ and therefore $x \in \mathcal{I}^p\{x, y, z\}$ and the proof is complete.

Induction Hypothesis: Now assume the statement is true if $|S| = k$. Let $|S| = k+1$. Assume $x \in \boldsymbol{c}(S)$ and take an element y in S different from x. We must show $x \succsim^p y$. Let

$$^{xy}\boldsymbol{d}(S) = \{A \in \boldsymbol{d}(S) : x, y \in A\}.$$

Claim: $^{xy}\boldsymbol{d}(S) \neq \emptyset$.

Proof. By contradiction assume $^{xy}\boldsymbol{d}(S) = \emptyset$. Since $\boldsymbol{d}(S)$ is a cover for S there has to be a set in it that contain x, and a set that contains y. note that it follows from the contradiction that

$$\boldsymbol{d}(S) = \{S - x, S - y\}.$$

Since $|S| > 3$ and $\boldsymbol{d}(S)$ only has two elements it has to be the case that $\boldsymbol{d}(S) = R_1(S)$ which in turn means $\mathfrak{R}(S) = \{x, y\}$.

Next note that $x \in \boldsymbol{c}(S - y)$. To see this, by contradiction assume $x \notin \boldsymbol{c}(S - y)$. Take $z \in \boldsymbol{c}(S - y)$. Since x only appears in $S - y$ this implies that z dominates x in S and therefore $x \notin \boldsymbol{c}(S)$ which is a contradiction. Therefore assume $x \in \boldsymbol{c}(S - y)$.

Note that from induction assumption we conclude that $x \in \mathcal{I}^p(S - y)$. This means $x \succsim^p t$ for all t in S different from y.

Finally since $x \in \mathfrak{R}(S)$ it follows from the definition that there exist two distinct elements $t_1, t_2 \in S$ both different from x, such that

$$\boldsymbol{c}\{t_1, t_2, x\} \subset \boldsymbol{c}\{t_1, t_2\}.$$

Now Proposition 3 implies $t_1 \sim^p t_2 \succ^p x$. Since t_1, t_2 are distinct elements, at least one of them is different from y. This contradicts our earlier observation that $x \succsim^p t$ for all t different from y. $\qquad\square$

To finish the proof let $^{xy}\boldsymbol{d}(S) \neq \varnothing$. If $A \in {}^{xy}\boldsymbol{d}(S)$ exists such that $x \in \boldsymbol{c}(A)$ then by induction assumption we conclude $x \succsim^p y$ and the proof is complete. So assume for all $A \in {}^{xy}\boldsymbol{d}(S)$ we have $x \notin \boldsymbol{c}(A)$. First note that this implies $y \notin \boldsymbol{c}(A)$ for all $A \in {}^{xy}\boldsymbol{d}(S)$. To see this note that if y is chosen on an element of $^{xy}\boldsymbol{d}(S)$, it would mean that y beats x in that set. And since x is never chosen in the elements of $^{xy}\boldsymbol{d}$ x does not beat y in any of those elements. This means $y \blacktriangleright_S x$ which is not possible because of α' and the fact that $x \in \boldsymbol{c}(S)$.

Therefore take $A_1 \in {}^{xy}\boldsymbol{d}(S)$ and $z \in \boldsymbol{c}(A_1)$ different from x, y. Note that by induction assumption this implies $z \succsim^p y$. Next this implies that z beats x in A_1. Since $x \in \boldsymbol{c}(S)$ α' implies that there exists $A_2 \in \boldsymbol{d}(S)$ such that x beats z in A_2. This from induction assumption on A_2 implies $x \succsim^p z$. Combining this latter fact with $z \succsim^p y$, from NBC it follows that $x \succsim^p y$. This completes the proof of the proposition. $\qquad\blacksquare$

I finish this section by proving the following lemma and corollary that are used in the proof of the main results.

Lemma 13. *Let $\boldsymbol{c}$ be a RDM. If $x \in \boldsymbol{c}\{x, y, z\}$ for all pair of distinct elements $y, z \in S$, both different from x, then $x \in \boldsymbol{c}(S)$.*

Proof. We prove this by induction on $|S|$. For $|S| = 3$ there is nothing to prove. Assume that the statement is true for all the sets with cardinality k. Let $|S| = k+1$ and take $x \in S$ be such that $x \in \boldsymbol{c}\{x, y, z\}$ for all two distinct elements $y, z \in S$, both different from x. Let

$$^x\boldsymbol{d}(S) = \{A \in \boldsymbol{d}(S) : x \in A\}.$$

Note that since $\boldsymbol{d}(S)$ is a cover for S we conclude $^x\boldsymbol{d}(S) \neq \emptyset$. Take $A \in \ ^x\boldsymbol{d}(S)$. Note that since

$$x \in \boldsymbol{c}\{x, y, z\},$$

for two distinct elements $y, z \in S$ and, since $A \subseteq S$, we conclude that

$$x \in \boldsymbol{c}\{x, y, z\},$$

for two distinct elements $y, z \in A$, both different from x. Finally induction assumption implies that $x \in \boldsymbol{c}(A)$. Since A was an arbitrary element of $^x\boldsymbol{d}(S)$ we conclude x is chosen in all elements of $^x\boldsymbol{d}(S)$. This means x is not dominated by any element of S. Therefore β' implies that $x \in \boldsymbol{c}(S)$. ∎

Corollary 14. *Let $\boldsymbol{c}$ be a RDM. If $r \in \mathfrak{R}(S)$ then $r \notin \boldsymbol{c}(S)$.*

Proof. Take $r \in \mathfrak{R}(S)$. Then Corollary 12 implies $r \notin \mathcal{I}^p(S)$. Next Proposition 4 implies $r \notin \boldsymbol{c}(S)$. ∎

Proof of Proposition 5

Proposition 5. *Let $\boldsymbol{c}$ be a RDM and $r \in \mathfrak{R}(S)$. Then*

(i) $x \succsim^r y \implies x \succsim^p y$.

(ii) $\succsim^r$ defines a complete binary relation on $\mathcal{I}^p(S)$.

Proof. (i) Assume $x \succsim^r y$. If follows from the definition that $x \in \boldsymbol{c}\{x, y, r\}$. Note that Corollary 14 implies that $x \notin \mathfrak{R}\{x, y, r\}$. If r is a reference in $\{x, y, r\}$

then from Proposition 3 we conclude $x \sim^p y$ which completes the proof. If y is a reference in $\{x, y, r\}$ again Proposition 3 implies that $x \succ^p y$ in which case the proof is also complete. So assume there are no references in $\{x, y, r\}$. Then $\boldsymbol{d}(S) = \{\{x, y\}, \{x, r\}, \{y, r\}\}$. Since $x \in \boldsymbol{c}\{x, y, r\}$ from α' we conclude that $x \in \boldsymbol{c}\{x, y\}$ and therefore $x \succsim^p y$.

(ii) Take $x, y \in \mathcal{I}^p(S)$. Note that since r is a reference in S it follows from Corollary 12 that $r \notin \mathcal{I}^p(S)$. Next Proposition 4 implies that $r \notin \boldsymbol{c}\{x, y, r\}$. Since $\boldsymbol{c}$ is non-empty valued we have $x \in \boldsymbol{c}\{x, y, r\}$ or $y \in \boldsymbol{c}\{x, y, r\}$, which in turn imply $x \succsim^r y$ or $y \succsim^r x$. This completes the proof.

$\blacksquare$

The Case of No Maximal Reference

Lemma 15. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. If $\mathfrak{R}_M(S) = \emptyset$ then $\boldsymbol{c}(S) = \mathcal{I}^p(S)$.*

Proof. Assume $\mathfrak{R}_M(S) = \emptyset$. We need to show $\boldsymbol{c}(S) = \mathcal{I}^p(S)$. To do this first note that Proposition 4 implies $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$. So we only need to prove $\mathcal{I}^p(S) \subseteq \boldsymbol{c}(S)$. Take $x \in \mathcal{I}^p(S)$ and consider the set $\{x, y, z\}$ for two distinct elements $y, z \in S$, both different from x.

Step 1: $\mathfrak{R}\{x, y, z\} = \emptyset$.

Proof. By contradiction assume $\mathfrak{R}\{x, y, z\} \neq \emptyset$. First note that by Proposition 6.ii it has to be the case that there is only one reference in $\{x, y, z\}$. Second by Corollary 12 and the fact that $x \in \mathcal{I}^p(S)$ we conclude $x \notin \mathfrak{R}(S)$. Wlog, assume z is the reference in $\{x, y, z\}$. Then it has to be the case that

$$\boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, y\},$$

which implies $\boldsymbol{c}\{x, y\} = \{x, y\}$, which in turn implies $y \succsim^p x$. Since $x \in \mathcal{I}^p(S)$ NBC

implies $y \in \mathcal{I}^p(S)$. This means $z \in \mathfrak{R}_M(S)$ which is not possible. This completes the proof of Step 1. $\qquad \square$

Step 2: $x \in \boldsymbol{c}\{x, y, z\}$.

Proof. From Step 1 we have $\mathfrak{R}\{x, y, z\} = \emptyset$. Therefore

$$\boldsymbol{d}\{x, y, z\} = \{\{x, y\}, \{y, z\}, \{x, z\}\}.$$

We first argue that y or z can not be the single choice in $\boldsymbol{c}\{x, y, z\}$. To do this, and wlog, assume $\boldsymbol{c}\{x, y, z\} = \{y\}$. Then α' implies $y \succsim^p x$ which in turn means $x, y \in \mathcal{I}^p(S)$. Then we conclude that

$$\boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, y\},$$

which means z is a maximal reference in S which is impossible. Second assume $\{y, z\} \subseteq \boldsymbol{c}\{x, y, z\}$. We show that $x \in \boldsymbol{c}\{x, y, z\}$. From α' we conclude $y, z \succsim^p x$ and, since $x \in \mathcal{I}^p(S)$, it follows that $x, y, z \in \mathcal{I}^p(S)$ which means $\{x, y, z\}$ is fully indecisive. Using Proposition 6.i we conclude $\boldsymbol{c}\{x, y, z\} = \{x, y, z\}$ which obviously means $x \in \boldsymbol{c}\{x, y, z\}$. Since $\boldsymbol{c}\{x, y, z\} \neq \emptyset$ the proof of this Step 2 is complete. $\qquad \square$

To finish the proof note that by Lemma 24.ii we conclude $x \in \boldsymbol{c}(S)$. This means $\mathcal{I}^p(S) \subseteq \boldsymbol{c}(S)$. Therefore $\boldsymbol{c}(S) = \mathcal{I}^p(S)$. $\qquad \blacksquare$

For the sake of avoiding repetition, I make the the following obvious observations that are used in the remaining argument in several occasions.

Let $\boldsymbol{c}$ be an RDM and $S \subseteq X$. Then

(i) By Corollary 12 for a reference $r \in \mathfrak{R}(S)$, we have $r \notin \mathcal{I}^p(S)$. Therefore dropping r from the set S does note change the maximal class of the resulting

subset; that is, $\mathcal{I}^p(S - r) = \mathcal{I}^p(S)$. Note that, using the same argument we deduce that

$$\mathcal{I}^p(S) = \mathcal{I}^p(S \setminus R),$$

for all $R \subseteq \mathfrak{R}(S)$; that is the maximal class of S is the same in all subsets of S which are derived from removing any arbitrary numbers of references from S.

(ii) Maximal references in $S-r$ and S only (potentially) differ in r. That is $\mathfrak{R}_M(S) = \mathfrak{R}_M(S-r)$ if r is not a maximal reference in S and $\mathfrak{R}_M(S) = \mathfrak{R}_M(S-r) \cup \{r\}$ if r is a maximal reference in S. Similarly such preservation happens for maximal reference sets of elements; that is for $t \in \mathcal{I}^p(S)$, we have $\mathfrak{R}^t_M(S) = \mathfrak{R}^t_M(S - r)$ if r is not a maximal reference in S and $\mathfrak{R}^t_M(S) = \mathfrak{R}^t_M(S - r) \cup \{r\}$ if r is a maximal reference in S.

Proof of Theorem 3

Lemma 16. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Also assume $\mathfrak{R}(S) = \mathfrak{R}_M(S) = \{r\}$. Then*

$$\boldsymbol{c}(S) = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r .\,[28]$$

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[28] As discussed before, the domain of relevance for $\succsim^r$ is $\mathcal{I}^p(S)$; that is, indeed, $\operatorname*{argmax}_{S} \succsim^r = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r$. To see this note that obviously $\operatorname*{argmax}_{S} \succsim^r \subseteq \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r$. To see the other inclusion take $x \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r$ and $y \in S$ such that $y \notin \mathcal{I}^p(S)$. Since $x \in \mathcal{I}^p(S)$ this means $x \succ^p y$. We must show $x \succsim^r y$. First note that by Corollary 12 we conclude that $r \notin \mathcal{I}^p(S)$ and since $x \in \mathcal{I}^p(S)$ we have $x \succ^p r$. Consider the set $\{x, y, r\}$. We must show $x \in \boldsymbol{c}\{x, y, r\}$. Note that $x \in \mathcal{I}^p(S)$ implies $x \in \mathcal{I}^p\{x, y, r\}$ and by Corollary 12 we conclude x is not a reference in $\{x, y, r\}$. Also note that since there is no pairwise indifference between either x, y or x, r we conclude y, r are also not references in $\{x, y, r\}$. As a result there is no reference in $\{x, y, r\}$ and we conclude

$$\boldsymbol{d}\{x, y, r\} = \big\{\{x, y\}, \{y, r\}, \{x, r\}\big\}.$$

Since $x \succ^p y$ and $x \succ^p r$ it follows that y, r are dominated by x in $\{x, y, r\}$ and α' implies $y, r \notin \boldsymbol{c}\{x, y, r\}$. Finally since $\boldsymbol{c}\{x, y, r\} \neq \emptyset$ we conclude $x \in \boldsymbol{c}\{x, y, r\}$. This in turn means $x \succsim^r y$. Therefore, $x \in \operatorname*{argmax}_{S} \succsim^r$; that is $\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r \subseteq \operatorname*{argmax}_{S} \succsim^r$. From this we conclude

$$\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r = \operatorname*{argmax}_{S} \succsim^r .$$

Proof. First note that since r is a reference in S it follows that there exists two distinct elements in S both different from r and therefore $|S| \geq 3$.

($\Leftarrow$): Take $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. Obviously $x \in \mathcal{I}^p(S)$. We must show $x \in \boldsymbol{c}(S)$. To do this we first make the following claim.

Claim: For two distinct elements $y, z \in S$, both different from x, we have $x \in \boldsymbol{c}\{x, y, z\}$.

Proof. Take two distinct elements $y, z \in S$, both different from x, and consider the set $\{x, y, z\}$. We consider two cases here:

Case 1: $\mathfrak{R}\{x, y, z\} = \varnothing$.

Note that $\mathfrak{R}\{x, y, z\} = \varnothing$ implies $\mathfrak{R}_M\{x, y, z\} = \varnothing$. Since $\boldsymbol{c}$ is a RDM on $\{x, y, z\}$ and there are no maximal references in $\{x, y, z\}$ Lemma 15 implies $\boldsymbol{c}\{x, y, z\} = \mathcal{I}^p\{x, y, z\}$. Since $x \in \mathcal{I}^p(S)$ we conclude $x \in \mathcal{I}^p\{x, y, z\}$ and therefore $x \in \boldsymbol{c}\{x, y, z\}$. This completes the proof of this case.

Case 2: $\mathfrak{R}\{x, y, z\} \neq \varnothing$.

First note that by Proposition 6.ii we conclude $|\mathfrak{R}\{x, y, z\}| = 1$. Second Corollary 12 and the fact that $x \in \mathcal{I}^p(S)$ imply $x \notin \mathfrak{R}\{x, y, z\}$. Wlog, assume $z \in \mathfrak{R}\{x, y, z\}$. It follows from the definition of references that

$$\boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, y\},$$

and by Proposition 3 we conclude $y \sim^p x$. Since $x \in \mathcal{I}^p(S)$ this implies $y \in \mathcal{I}^p(S)$. Therefore we conclude $z \in \mathfrak{R}_M(S)$ which means $z = r$. Since $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$ we conclude $x \succsim^r y$ which in turn implies $x \in \boldsymbol{c}\{x, y, r\}$. This completes the proof in this case and therefore the claim. $\qquad\square$

To finish the proof of $\Leftarrow$ note that Lemma 24 implies $x \in \boldsymbol{c}(S)$. This finishes the proof of $\Leftarrow$.

$(\Rightarrow)$: Take $x \in \boldsymbol{c}(S)$. We must show $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$.

First note that since r is the only reference in S we conclude $\boldsymbol{d}(S) = R_2(S)$ and that all the elements in $\boldsymbol{d}(S)$ are of the form $S - t$ for some t different from r. Also note that $r \in A$ for all $A \in \boldsymbol{d}(S)$.

We prove this by <u>strong induction</u> on $|S|$. Note that for $|S| = 3$ the statement is obvious; that is if $S = \{x, y, r\}$ then if follows from the definition of $\succsim^r$ that $x \in c(S)$ if and only if $x \succsim^r y$. We base our induction on $|S| = 4$.

Induction Base: Assume $|S| = 4$. Let $S = \{x, y, z, r\}$. Assume, wlog, that $x \in \boldsymbol{c}(S)$. We need to show $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. First note that

$$\boldsymbol{d}(S) = \big\{\{x, y, r\}, \{x, z, r\}, \{y, z, r\}\big\}.$$

Note that since r is a reference in S, Proposition 3 implies that at least two elements in the set $\{x, y, z\}$ are preferred to r under pairwise comparison. Therefore it follows from Proposition 4 that $r \notin \boldsymbol{c}\{x, y, r\}$ and $r \notin \boldsymbol{c}\{y, z, r\}$.

Next note that since $r \notin \mathcal{I}^p(S)$ we only need to show $x \succsim^r y$ and $x \succsim^r z$. To do

this, and wlog, consider the set $\{x, y, r\}$. Note that if $x \notin \boldsymbol{c}\{x, y, r\}$ then it has to be the case that $\boldsymbol{c}\{x, y, r\} = \{y\}$. Since $\{x, y, r\}$ is the only element in $\boldsymbol{d}(S)$ that contains both x, y this would imply that y dominates x in S and therefore $x \notin \boldsymbol{c}(S)$ which is not possible. So we have $x \in \boldsymbol{c}\{x, y, r\}$. This by definition implies $x \succsim^r y$. Therefore $x \in \operatorname*{argmax}_{S} \succsim^r = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r$. This completes the proof for induction base.

Induction Hypothesis: Assume the statement is true for all sets with cardinality less than or equal to k and let $|S| = k+1$. Take $x \in \boldsymbol{c}(S)$. We first make the following claim.

Claim: $\succsim^r$ does not posses any cycles of the length less than or equal to $k-1$ in $\mathcal{I}^p(S)$.

Proof. Let $t \leq k-1$. We must show that $\succsim^r$ does not possess any cycle of degree t in $\mathcal{I}^p(S)$. To do this let $A = \{x_1, x_2, \ldots, x_t\} \subseteq \mathcal{I}^p(S)$ and and assume $x_1 \succ^r x_2 \succ^r \ldots \succ^r x_t$. We show $x_1 \succsim^r x_t$. Consider the set $A \cup \{r\}$. First note that since $\boldsymbol{c}$ is a RDM on S it follows that $\boldsymbol{c}$ is also an RDM in $A \cup \{r\}$. Second, the number of elements in A, t, is less than $k-1$ and therefore we have $|A \cup \{r\}| < |S| = k+1$. Third, take an element $x_i \in A$. Note that since

$$x_i \succ^r x_{i+1} \text{ and } x_i \sim^p x_{i+1}$$

we conclude r is the unique maximal reference in $A \cup \{r\}$. The last three observations together imply that induction hypothesis is applicable to $A \cup \{r\}$.

To finish the poof of this claim first note that since r is a reference in $A \cup \{r\}$ by Corollary 14 we have $r \notin \boldsymbol{c}(A \cup \{r\})$. Since $\boldsymbol{c}(A \cup \{r\}) \neq \emptyset$ take $x_i \in \boldsymbol{c}(S)$. Induction hypothesis implies that $x_i \succsim^r x_j$ for all $j \neq i$. So it has to be the case that $x_i = x_1$.

This means $x_1 \succsim^r x_t$. So the proof of the claim is complete. $\qquad\square$

In order to proof the lemma by <u>contradiction</u> assume $y \succ^r x$, for $y \in \mathcal{I}^p(S)$. Let

$$^{xy}\boldsymbol{d}(S) = \{A \in \boldsymbol{d}(S) : x, y \in A\};$$

that is, the set of all elements of $\boldsymbol{d}(S)$ that contain both elements x, y.

Next recall that $r \in A$ for all $A \in \boldsymbol{d}(S)$ which in turn implies $r \in A$ for all $A \in {}^{xy}\boldsymbol{d}(S)$. Also since $y \succ^r x$ and $y \sim^p x$ we conclude r is a maximal reference in A for all $A \in {}^{xy}\boldsymbol{d}(S)$. To summarize for all $A \in {}^{xy}\boldsymbol{d}(S)$ we have made the following observations: i. $x, y, r \in A$, ii. r is a maximal reference in A. Note that this means induction hypothesis is applicable to all of these sets.

<u>Step 1</u>: $^{xy}\boldsymbol{d}(S) \neq \varnothing$.

Proof. First note that in our induction $|S| \geq 5$ and, therefore, there is at least an element $z \in S$ that is different from x, y, r. Since $\boldsymbol{d}(S) = R_2(S)$ we conclude $S - z$ is an element of $\boldsymbol{d}(S)$. Obviously $x, y \in \boldsymbol{c}(S - z)$. This implies $S - z \in {}^{xy}\boldsymbol{d}(S)$ and we conclude $^{xy}\boldsymbol{d}(S) \neq \varnothing$. $\qquad\square$

<u>Step 2</u>: $x, y \notin \boldsymbol{c}(A)$ for all $A \in {}^{xy}\boldsymbol{d}(S)$.

Proof. Take $A \in {}^{xy}\boldsymbol{d}(S)$. Since $y \succ^r x$ we conclude $x \notin \operatorname*{argmax}_{\mathcal{I}^p(A)} \succsim^r$ and therefore induction hypothesis implies that $x \notin \boldsymbol{c}(A)$. Note that since A was an arbitrary element of $^{xy}\boldsymbol{d}(S)$ we conclude that x does not beat y in all $A \in {}^{xy}\boldsymbol{d}(S)$.

Next we show that y is also not chosen in all elements of $^{xy}\boldsymbol{d}(S)$. To do this take $A \in {}^{xy}\boldsymbol{d}(S)$. If $y \in \boldsymbol{c}(A)$, since $x \notin \boldsymbol{c}(A)$, we conclude that y beats x in A. Since x never beats y it follows that $y \blacktriangleright_S x$ and therefore α' implies $x \notin \boldsymbol{c}(S)$ which is not

possible. Since A was arbitrary we conclude y is not chosen in all elements of $^{xy}\boldsymbol{d}(S)$ as well. $\qquad\square$

Now let $\mathcal{T} = \{z \in S : z \in \boldsymbol{c}(A) \text{ for some } A \in {}^{xy}\boldsymbol{d}(S)\}$. First note that since $^{xy}\boldsymbol{d}(S) \neq \emptyset$ and also that $\boldsymbol{c}$ is non-empty valued we conclude $\mathcal{T} \neq \emptyset$. Second by Step 2 it follows that $\{x, y\} \cap \mathcal{T} = \emptyset$. Finally since r is a (maximal) reference in all elements of $^{xy}\boldsymbol{d}(S)$ Corollary 14 implies that r is never chosen in elements of $^{xy}\boldsymbol{d}(S)$ and therefore $r \notin \mathcal{T}$. To summarize we have made the following observation: $\mathcal{T} \cap \{x, y, r\} = \emptyset$.

Step 3: $\mathcal{T} \subseteq \mathcal{I}^p(S)$.

Proof. Take $z \in \mathcal{T}$. Then there exists $A \in {}^{xy}\boldsymbol{d}(S)$ such that $z \in \boldsymbol{c}(A)$. Since $x \in A$ we conclude from Proposition 4 that $z \succsim^p x$. Next $x \in \mathcal{I}^p(S)$ implies $z \in \mathcal{I}^p(S)$. $\qquad\square$

Step 4: For all $z \in \mathcal{T}$ there exists $t \in \mathcal{I}^p(S)$ such that $t \succ^r z$.

Proof. Take $z \in \mathcal{T}$. Then there exists $A_1 \in {}^{xy}\boldsymbol{d}(S)$ such that $z \in \boldsymbol{c}(A_1)$. Note that by Step 2 we have $x \notin \boldsymbol{c}(A_1)$, and therefore z beats x in A_1. Since $x \in \boldsymbol{c}(S)$, α' implies that x is not dominated by z and therefore it has to be the case that x beats z_2 in A_2 for some $A_2 \in \boldsymbol{d}(S)$.[29]

Next, since $\boldsymbol{c}$ is a RDM in A_2, and $z \notin \boldsymbol{c}(A_2)$ induction hypothesis implies $z \notin \operatorname*{argmax}_{\mathcal{I}^p(A_2)} \succsim^r$ and it follows that there exists $t \in \mathcal{I}^p(A_2)$ such that $t \succ^r z$. To show that $t \in \mathcal{I}^p(S)$, note that it follows from the definition of reference relations that $t \in \boldsymbol{c}\{t, z, r\}$. Since $\boldsymbol{c}$ is a RDM on $\{t, z, r\}$ Lemma 24 implies $t \in \mathcal{I}^p\{t, z, r\}$. This means $t \succsim^p z$. Since $z \in \mathcal{I}^p(S)$ we conclude $t \in \mathcal{I}^p(S)$. $\qquad\square$

Step 5: For $z_1 \in \mathcal{T}$ there exists $z_2 \in \mathcal{T}$ such that $z_1 \succ^r z_2$.

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[29] In fact, $A_2 = S - y$.

Proof. Take $z_1 \in \mathcal{T}$. Consider the set $S - z_1$. It follows that $S - z_1 \in \boldsymbol{d}(S)$. Note that $z_1 \notin \{x, y, r\}$. Therefore we conclude $S - z_1 \in {}^{xy}\boldsymbol{d}(S)$. Take $z_2 \in \boldsymbol{c}(S - z_1)$.[30] Since $S - z_1 \in {}^{xy}\boldsymbol{d}(S)$ it follows that $z_2 \in \mathcal{T}$.

Next we show that $z_1 \succ^r z_2$. To do this note that induction hypothesis implies $z_2 \in \underset{\mathcal{I}^p(S-z_1)}{\text{argmax}} \succsim^r$ which means $z_2 \succsim^r t$ for all $t \in \mathcal{I}^p(S - z_1)$. Also by Step 4 there exists $t^* \in \mathcal{I}^p(S)$ such that $t^* \succ^r z_2$. Since $z_2 \succsim^r t$ for all $t \in \mathcal{I}^p(S - z_1)$, and also there is an element t^* in $\mathcal{I}^p(S)$ such that $t^* \succ^r z_2$, there is only one element that can be preferred to z_2 under r and that is the element which is not in $S - z_1$: z_1. This means $z_1 \succ^r z_2$. $\qquad\square$

To finish the proof of the lemma note that since $x, y \notin \mathcal{T}$ we have $|\mathcal{T}| = t$ for $t \leq k - 1$. Take $z_1 \in \mathcal{T}$ from Step 5 there exists $z_2 \in \mathcal{T}$ such that $z_2 \succ^r z_1$. Applying Step 5 one more time we conclude that there exists $z_3 \in \mathcal{T}$ such that $z_3 \succ^r z_2$. By repeating this argument t times we conclude

$$z_1 \succ^r z_2 \succ^r \ldots \succ^r z_t \succ^r z_{t+1}$$

Since $|\mathcal{T}| = t$ this always produces a $\succsim^r$-cycle in $\mathcal{I}^p(S)$ of degree $t \leq k - 1$. This contradicts the starting claim. So proof is complete. $\qquad\blacksquare$

We next prove the following lemma which states this intuitive statement: *non-maximal references are irrelevant to RDM's choice behavior.*

Lemma 17. *Let $\boldsymbol{c}$ be a RDM and $S \subseteq X$. Assume $\mathfrak{R}_M(S) = \{r\}$. Also Let S_r be the subset of S which is derived by removing all the non-maximal references from S. Then*

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[30]Note that since $|S| \geq 5$ there exists an element in $S - z_1$ different from x, y, r and therefore an element z_2 exists in $\mathcal{T}$.

$$\boldsymbol{c}(S) = \boldsymbol{c}(S_r)$$

Proof. First note that if r is the only reference in S; that is if $|\mathfrak{R}(S)| = 1$ then $S_r = S$ and the statement is obvious. So for the rest of this argument assume $|\mathfrak{R}(S)| \geq 2$. Note that this latter fact implies $\boldsymbol{d}(S) = R_1(S)$. Therefore elements of $\boldsymbol{d}(S)$ are derive by taking references out of the set S. We start with the following tow claims.

<u>Claim 1</u>: If $x \in \boldsymbol{c}(S)$ then there exists a non-maximal reference t such that $x \in \boldsymbol{c}(S - t)$.

Proof. Take $x \in \boldsymbol{c}(S)$. First note that $\mathcal{I}^p(S - r) = \mathcal{I}^p(S)$. Since $S - r$ does note posses any maximal reference Lemma 15 implies $\boldsymbol{c}(S - r) = \mathcal{I}^p(S - r) = \mathcal{I}^p(S)$. This means x does not beat any elements of $\mathcal{I}^p(S)$ in $S - r$. Now take a reference $t_1 \neq r$ in S and consider the set $S - t_1$. If $x \in \boldsymbol{c}(S - t_1)$ there is nothing to prove. So assume $x \notin \boldsymbol{c}(S - t_1)$ and take $y \in \boldsymbol{c}(S - t_1)$. It follows that y beats x in $S - t_1$. Since $x \in \boldsymbol{c}(S)$, α' implies that x is not dominated by y and therefore there exists a reference t_2 such that x beats y in $S - t_2$. Note that since x is beating y in $S - t_2$ we conclude $t_2 \neq r$ and therefore t_2 is a non-maximal reference. This finishes the proof of this claim. □

<u>Claim 2</u>: If $x \notin \boldsymbol{c}(S)$ then there exists a non-maximal reference $t \neq r$ in S such that $x \notin \boldsymbol{c}(S - t)$.

Proof. First note that $|S| \geq 4$.[31] Recall from the proof of Claim 1 that $\boldsymbol{c}(S - r) = \mathcal{I}^p(S)$. We consider two cases here.

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[31] To see this note that since r is a maximal reference in S there exists, at least two distinct elements, t_1, t_2 in $\mathcal{I}^p(S)$, both different from r. Next since $|\mathfrak{R}(S)| \geq 2$ there exists a reference $s \neq r$ in S. Since s is a reference in S it follows from Proposition 6. iv that $s \notin \mathcal{I}^p(S)$. This in turn means $t_1, t_2 \succ^p s$ and there for t_1, t_2 are different from s. Therefore we conclude t_1, t_2, r, s are four distinct elements. This means $|S| \geq 4$.

<u>Case 1</u>: $x \notin \boldsymbol{c}(S - r)$.

Since $\boldsymbol{c}(S - r) = \mathcal{I}^p(S)$ this means that $x \notin \mathcal{I}^p(S)$. Now consider the set $S - t$ for a reference $t \neq r$ in S. Proposition 4 implies that $\boldsymbol{c}(S - t) \subseteq \mathcal{I}^p(S - t) = \mathcal{I}^p(S)$ and therefore we conclude $x \notin \boldsymbol{c}(S - t)$. This completes the proof in this case.

<u>Case 2</u>: $x \in \boldsymbol{c}(S - r)$.

First from $|S| \geq 4$ it follows that S satisfies β'. Since $x \notin \boldsymbol{c}(S)$ we conclude that there exists $y \in S$ such that y dominates x in S. Therefore there exists a reference t in S such that $y, x \in \boldsymbol{c}(S - t)$ and y is chosen in $S - t$ and x is not chosen in $S - t$. So it only remains to prove that $t \neq r$. This follows from the fact that $x \in \boldsymbol{c}(S - r)$; that is y does not beat x in $S - r$. This completes the proof in this case. $\square$

Now we proceed to prove the Lemma.

First we show $\boldsymbol{c}(S) \subseteq \boldsymbol{c}(S_r)$. To do this take $x \in \boldsymbol{c}(S)$. By Claim 1 there exists a non-maximal reference $t_1 \neq r$ in S such that $x \in \boldsymbol{c}(S - t)$. If $S - t = S_r$ the proof is complete. If not then there must exist another reference different from r in S and therefore $|\mathfrak{R}(S - t)| \geq 2$. Applying Claim 1 one more time we conclude there exits a non-maximal reference t_2 in S such that $x \in \boldsymbol{c}((S - t_1) - t_2)$. By repeating this algorithm for a finite number of time, and after taking all non-maximal references out of S, we conclude $x \in \boldsymbol{c}(S_r)$. This means $\boldsymbol{c}(S) \subseteq \boldsymbol{c}(S_r)$.

To show $\boldsymbol{c}(S_r) \subseteq \boldsymbol{c}(S)$ we prove that $x \notin \boldsymbol{c}(S) \implies x \notin \boldsymbol{c}(S_r)$. Assume $x \notin \boldsymbol{c}(S)$. By Claim 2 there exists a non-maximal reference $t_1 \neq r$ in S such that $x \notin \boldsymbol{c}(S - t)$. If $S - t = S_r$ the proof is complete. If not then there must exist another reference different from r in S and therefore $|\mathfrak{R}(S-t)| \geq 2$. Applying Claim 2 one more time we conclude there exits a non-maximal reference t_2 in S such that $x \notin \boldsymbol{c}((S - t_1) - t_2)$. By repeating this algorithm for a finite number of time, and after taking all non-

maximal references out of S, we conclude $x \notin \boldsymbol{c}(S_r)$. This implies $\boldsymbol{c}(S_r) \subseteq \boldsymbol{c}(S)$, and therefore $\boldsymbol{c}(S) = \boldsymbol{c}(S_r)$. ∎

Lemma 18. *Let $\boldsymbol{c}$ be a RDM such that $\mathfrak{R}_M(S) = \{r\}$. Then*

$$c(S) = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r .$$

Proof. Let S_r be the subset of S which is derived by taking all the non-maximal references out of S. Note that $\mathcal{I}^p(S_r) = \mathcal{I}^p(S)$. It follows that $\boldsymbol{c}$ is a RDM on S_r and S_r only possesses one reference, r, which is maximal. Then from Lemmas 17, and 16 we conclude

$$x \in \boldsymbol{c}(S) \overset{Lem.17}{\iff} x \in \boldsymbol{c}(S_r) \overset{Lem.16}{\iff} x \in \operatorname*{argmax}_{\mathcal{I}^p(S_r)} \succsim^r = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r .$$

Therefore the proof is complete. ∎

Theorem 3. (Acyclicity of Reference Relations) *Let $\boldsymbol{c}$ be a RDM and $r \in \mathfrak{R}(S)$. $\succsim^r$ defines an acyclic relation on $\mathcal{I}^p(S)$.*

Proof. Let $t \leq |\mathcal{I}^p(S)|$. We must show that r does not possess any cycle of degree t. To do this let $A = \{x_1, x_2, \ldots, x_t\} \subseteq \mathcal{I}^p(S)$ and assume $x_1 \succ^r x_2 \succ^r \ldots \succ^r x_t$. We show $x_1 \succsim^r x_t$. Consider the set $A \cup \{r\}$. First assume that r is not a maximal reference in $A \cup \{r\}$ and consider the set $\{x_1, x_t, r\}$. Note that since $x_1 \succ^p r$ Proposition 4 implies that $r \notin \boldsymbol{c}\{x_1, x_2, r\}$. Since r is not a maximal reference in $A \cup \{r\}$ then it should be the case that $\boldsymbol{c}\{x_1, x_t, r\} = \boldsymbol{c}\{x_1, x_t\} = \{x_1, x_t\}$, which, in turn, means $x_1 \succsim^r x_t$. To finish the proof assume that r is a maximal reference in $A \cup \{r\}$. First note that $\mathcal{I}^p(A \cup \{r\}) = A$. Second, since $r \notin \boldsymbol{c}(A \cup \{r\})$ and $\boldsymbol{c}(A \cup \{r\}) \neq \emptyset$ take $x_i \in \boldsymbol{c}(A \cup \{r\})$. Then Lemma 18 implies that $x_i \in \operatorname*{argmax}_{A} \succsim^r$; that is $x_i \succsim^r x_j$. Therefore $x_i = x_1$ and as a result $x_1 \succsim^r x_t$. This completes the proof. [32] ∎

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[32] It worths noting that $\succsim^r$, indeed, defines an acyclic relation on S. To see this assume, for

This next direct corollary of Theorem 3 is useful in the proof of the final theorem.

Corollary 19. *Let c be a RDM and $S \subseteq X$. For a maximal reference r is S we have*

$$\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r \neq \emptyset.$$

Proof. From Proposition 5.ii and Theorem 3 we conclude $\succsim^r$ defines a complete and acylice relation on $\mathcal{I}^p(S)$. Therefore $\succsim^r$ attains a maximum on $\mathcal{I}^p(S)$ and we conclude $\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^r \neq \emptyset.$ ∎

Proofs of Lemmas 10, 11

Lemma 10. *Let c be a RDM and $S \subseteq X$ such that $|\mathfrak{R}(S)| \geq 2$. If $x \in c(S)$ then there exists a reference r in S such that $x \in c(S - r)$.*

Proof. Assume $|\mathfrak{R}(S)| \geq 2$. Note that all elements of $d(S)$ are of the form $S - r$ for some references in r in S. Now take $S - r_1$ in $\mathfrak{R}(S)$. If $x \in c(S - r_1)$ then there is nothing to prove. So take y different from x such that $y \in c(S - r_1)$. If follows that y beats x in $S - r_1$. Since $x \in c(S)$ and c satisfies α' we conclude there exists $r_2 \in \mathfrak{R}(S)$ such that x beats y in $S - r_2$.[33] In particular, we have $x \in c(S - r_2)$. ∎

Lemma 11. *Let c be a RDM and $S \subseteq X$ that possesses at least two maximal references. For any reference r in S assume*

$$t^* \in c(S - r) \iff |\mathfrak{R}_M^{t^*}(S - r)| \geq |\mathfrak{R}_M^t(S - r)|, \text{ for all } t \in \mathcal{I}^p(S - r.)$$

$x_i \in S$, $x_1 \succ^r x_2 \succ^r \ldots \succ^r x_n$. Let $A = \{x_1, x_2, \ldots, x_n\} \cup \{r\}$. Note that $x_1 \in \operatorname*{argmax}_A \succsim^r$ we conclude that $x_1 \in c(A)$ which in turn, using Proposition 4 implies $x_1 \in \mathcal{I}^p(A)$. If $x_n \in \mathcal{I}^p(A)$ then Proposition 5.i implies that $x_{n-1} \in \mathcal{I}^p(A)$ which in turn, using the same proposition, implies $x_i \in \mathcal{I}^p(A)$ for all $i \in \{1, 2, \ldots, n\}$. Therefore from Theorem 3 we conclude $x_1 \succsim^r x_n$ and the proof is complete. If, otherwise, $x_n \notin \mathcal{I}^p(S)$ then we have $x_1 \succ^p x_n$ and Proposition 5.i implies that $x_n \succsim^r x_1$ can not hold and therefore $x_n \notin c\{x_1, x_n, r\}$. Since, $r \notin c\{x_1, x_n, r\}$ we conclude $x_1 \in c\{x_1, x_n, r\}$ and therefore $x_1 \succsim^r x_n$.

[33]Note that such r_2 exists since $|\mathfrak{R}(S)| \geq 2$.

Also assume that $x \in \boldsymbol{c}(S)$ and $x, y \in \boldsymbol{c}(S - \bar{r})$ for a reference $\bar{r}$ in S. If $\bar{r}$ refers to y then $\bar{r}$ refers to x.

Proof. To start we partition the set $\mathcal{I}^p(S)$ into the following sets:

$$\mathcal{T}_0 = \{t \in \mathcal{I}^p(S) : t \notin \boldsymbol{c}(S - \bar{r})\},$$

$$\mathcal{T}_1 = \{t \in \boldsymbol{c}(S - \bar{r}) : \bar{r} \text{ does not refer to } t\},$$

$$\mathcal{T}_2 = \{t \in \boldsymbol{c}(S - \bar{r}) : \bar{r} \text{ refers to } t\}.$$

Note that by Corollary 14 $r \notin \mathcal{I}^p(S)$ and, as a result, $\{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2\}$ partitions the set $\mathcal{I}^p(S)$.

Assume $\bar{r}$ refers to y. Note that by definition $\bar{r}$ is a maximal reference. Also note that since $y \in \boldsymbol{c}(S - \bar{r})$ it follows that $y \in \mathcal{T}_2$ and therefore $\mathcal{T}_2 \neq \emptyset$. By <u>contradiction</u> assume that $\bar{r}$ does not refer to x. Note that this implies $x \in \mathcal{T}_1$.

<u>Step 1:</u>

$$|\mathfrak{R}^z_M(S)| > |\mathfrak{R}^x_M(S)|, \tag{2.1}$$

for all $z \in \mathcal{T}_2$.

Proof. Take an arbitrary element $z \in \mathcal{T}_2$. We have x, z are chosen in $S - \bar{r}$ and therefore

$$|\mathfrak{R}^x_M(S - \bar{r})| = |\mathfrak{R}^z_M(S - \bar{r})|.$$

Since $\bar{r}$ refers to z and not to x, adding it to set the set $S - \bar{r}$ increases the number of z-maximal references over x-maximal references; that is

$$|\mathfrak{R}_M^z(S)| > |\mathfrak{R}_M^x(S)|.$$

$\square$

Step 2: x does note beat z in A, for all $A \in \boldsymbol{d}(S)$, and all $z \in \mathcal{T}_2$.

Proof. First note that since $|\mathfrak{R}(S)| \geq |\mathfrak{R}_M(S)| \geq 2$ then all $A \in \boldsymbol{d}(S)$ are of the form $S - r$ for some reference r in S. So take an arbitrary element $A = S - r \in \boldsymbol{d}(S)$. Also take an arbitrary element $z \in \mathcal{T}_2$. From (1), and in S, the number of z-maximal references is more than the number of $x-$maximal references, and therefore dropping r will in the worse case equalize these two numbers (that worse case happens when r refers to z and not x). Therefore

$$|\mathfrak{R}_M^z(S - r)| \geq |\mathfrak{R}_M^x(S - r)| \tag{2.2}$$

To complete the proof of this step we show that $x \in \boldsymbol{c}(S - r)$ implies $z \in \boldsymbol{c}(S - r)$ and therefore x does not beat z in $S - r$. In order to do this assume $x \in \boldsymbol{c}(S - r)$. Then it follows that

$$|\mathfrak{R}_M^x(S - r)| \geq |\mathfrak{R}_M^t(S - r)|, \tag{2.3}$$

for all $t \in \mathcal{I}^p(S - r)$. From (2) and (3) we conclude

$$|\mathfrak{R}_M^z(S - r)| \geq |\mathfrak{R}_M^t(S - r)|,$$

for all $t \in \mathcal{I}^p(S - r)$. This implies $z \in \boldsymbol{c}(S - r)$. Therefore we conclude x does not beat z in $S - r$. $\square$

Step 3: There exists a x-maximal reference in S.

Proof. Since there are at least two maximal references in S we conclude that $S - \bar{r}$ has at least one maximal reference. Take $r \in \mathfrak{R}_M(S - \bar{r})$, By Corollary [19] we conclude $\operatorname*{argmax}_{\mathcal{I}^p(S-\bar{r})} \succsim^r \neq \varnothing$ and therefore r refers to an element of $\mathcal{I}^p(S - \bar{r})$; that is each maximal references in the set $S - \bar{r}$ refers, at least, to one element of $\mathcal{I}^p(S - \bar{r})$. Since $x \in \boldsymbol{c}(S - \bar{r})$ it follows that, in $S - \bar{r}$, the number of x-maximal references is greater than or equal to the number of t-maximal references, for all $t \in \mathcal{I}^p(S - \bar{r})$. Since there exists at least one maximal references in $S - \bar{r}$ if x has no maximal reference in $S - r$ this last assertion would be violated. Therefore there exists a x-maximal reference in $S - r$. $\square$

As a result of Step 3 let r_x be a (maximal) reference in S that refers to x.

Step 4: $x \notin \boldsymbol{c}(S - r_x)$.

Proof. Take $z \in \mathcal{T}_2$. Recall from (1) that z has more maximal references in S than x. Since r_x refers to x dropping r_x from S results in

$$|\mathfrak{R}_M^z(S - r_x)| > |\mathfrak{R}_M^x(S - r_x)|.$$

It follows that $x \notin \boldsymbol{c}(S - r_x)$. $\square$

Step 5:
$$|\mathfrak{R}_M^z(S - r_x)| \geq |\mathfrak{R}_M^t(S - r_x)|,$$

for all $z \in \mathcal{T}_2$, and all $t \in \mathcal{T}_0$.

Proof. Take an arbitrary element $t \in \mathcal{T}_0$ and an arbitrary element $z \in \mathcal{T}_2$. First note that since $\bar{r}$ refers to z (and r_x might or might not refer to z) then dropping $\bar{r}$ from S does not increase the number of maximal references of z relative to dropping r_x. Therefore we have

$$|\mathfrak{R}_M^z(S - r_x)| \geq |\mathfrak{R}_M^z(S - \bar{r})|.$$

Next recall that we have i. z is chosen in $S - \bar{r}$ and ii. t is not chosen in $S - \bar{r}$. Therefore we conclude

$$|\mathfrak{R}^z_M(S - \bar{r})| > |\mathfrak{R}^t_M(S - \bar{r})| \text{ (or equivalently } |\mathfrak{R}^z_M(S - \bar{r})| \geq |\mathfrak{R}^t_M(S - \bar{r})| + 1).$$

Finally consider the two sets $S - r_x$, and $S - \bar{r}$. Since these sets are derived by dropping only of reference from the set S it is true that the difference between the number of t-maximal references in these two sets is less then or equal to 1; that is

$$|\mathfrak{R}^t_M(S - \bar{r})| + 1 \geq |\mathfrak{R}^t_M(S - r_x)|.$$

Combining these assertion in order we conclude

$$|\mathfrak{R}^z_M(S - r_x)| \geq |\mathfrak{R}^z_M(S - \bar{r})| \geq |\mathfrak{R}^t_M(S - \bar{r})| + 1 \geq |\mathfrak{R}^t_M(S - r_x)|.$$

This complete the proof of this step. $\qquad\qquad\square$

Step 6: for $z \in \mathcal{T}_2$ and $t \in \mathcal{T}_1$ we have:

$$|\mathfrak{R}^z_M(S - r_x)| \geq |\mathfrak{R}^t_M(S - r_x)|$$

Proof. Take an arbitrary element $z \in \mathcal{T}_2$ and an arbitrary element $t \in \mathcal{T}_1$. Note that since $\bar{r}$ refers to z (and r_x might or might not refer to z) then dropping $\bar{r}$ from S does not increase the number of maximal references of z relative to dropping r_x. Therefore we have

$$|\mathfrak{R}^z_M(S - r_x)| \geq |\mathfrak{R}^z_M(S - \bar{r})|.$$

Next recall z, t are chosen in $S - \bar{r}$. Therefore it follows that

$$|\mathfrak{R}_M^z(S - \bar{r})| = |\mathfrak{R}_M^t(S - \bar{r})|.$$

Combining these two assertions then we conclude

$$|\mathfrak{R}_M^z(S - r_x)| \geq |\mathfrak{R}_M^z(S - \bar{r})| = |\mathfrak{R}_M^t(S - r_x)|.$$

This complete the proof of this claim. $\qquad\square$

Step 7: There exists $z^* \in \mathcal{T}_2$ such that $z^* \in \boldsymbol{c}(S - r_x)$.

Proof. Take $z \in \boldsymbol{c}(S - r_x)$. First note that this implies Then

$$|\mathfrak{R}_M^z(S - r_x)| \geq |\mathfrak{R}_M^t(S - r_x)|$$

for all $t \in \mathcal{I}^p(S - r_x)$. Second Proposition 4 implies that $\boldsymbol{c}(S - r_x) \subseteq \mathcal{I}^p(S - r_x) = \mathcal{I}^p(S)$ and we conclude $z \in \mathcal{I}^p(S)$. Recall that $\{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2\}$ partitions the set $\mathcal{I}^p(S)$. If $z \in \mathcal{T}_2$ there is nothing to prove. So assume z in either $\mathcal{T}_0$ or $\mathcal{T}_1$; that is $z \in \mathcal{T}_0 \cup \mathcal{T}_1$. Next take $z^* \in \mathcal{T}_2$. By Steps $5, 6$ we conclude

$$|\mathfrak{R}_M^{z^*}(S - r_x)| \geq |\mathfrak{R}_M^z(S - r_x)|,$$

Combining the last two equations we conclude

$$|\mathfrak{R}_M^{z^*}(S - r_x)| \geq |\mathfrak{R}_M^t(S - r_x)|,$$

for all $t \in \mathcal{I}^p(S - r_x)$. It follows that $z^* \in \boldsymbol{c}(S - r_x)$. This completes the proof of this step. $\qquad\square$

To finish the proof of the Lemma note that by Step 7 there exists $z^* \in \mathcal{T}_2$ such

that $z^* \in \boldsymbol{c}(S - r_x)$. Also from Step 4 we have $x \notin \boldsymbol{c}(S - r_x)$. This means z^* beats x in $S - r_x$. Next from Step 2 we have x does not beat z^* in all $A \in \boldsymbol{d}(S)$. This implies $z^* \underset{S}{\blacktriangleright} x$. This means x is not a maximal element of $\underset{S}{\blacktriangleright}$, which in turn by α', implies $x \notin \boldsymbol{c}(S)$. A contradiction. ∎

2.8.2 RDM: Majority Rule Vs. Condorcet Criterion

The notion of RDM can be conveniently perceived as a voting situation where candidate are represented by the elements of the most preferred class, $\mathcal{I}^p(S)$, and voters are references whose rankings of the candidates are given with the associated reference preference. In the view of the result in Theorem 4, then, RDM is consistent with the majority rule; that is: *the candidate who has the majority of the votes is elected.* RDM, however, is not consistent with the Condorcet criterion. To see this let $S = \{x_1, x_2, x_3, x_4, r_1, r_2, r_3\}$, where r_i is a reference (voters), and $\{x_1, x_2, x_3, x_4\} = \mathcal{I}^p(S)$ (candidates). Assume the following preference for each voter:

$$r_1 : x_1 \succ^{r_1} x_2 \succ^{r_1} x_3 \succ^{r_1} x_4$$

$$r_2 : x_2 \succ^{r_2} x_3 \succ^{r_2} x_4 \succ^{r_2} x_1$$

$$r_3 : x_3 \succ^{r_3} x_2 \succ^{r_3} x_4 \succ^{r_3} x_1$$

From Theorem 4 it follows that $\boldsymbol{c}(S) = \{x_1, x_2, x_3\}$. However, x_1 does not satisfy the Condorcet criterion. In particular, if restricted to only x_1, x_4, the latter has more votes than the former.

Chapter 3

Rational Filters

3.1 Introduction

Reference-dependent choice model have received quiet an attention in the literature as they help explaining behavioral anomalies. One explanation for such effects is through relaxing *independence of irrelevant alternatives*. That is a third alternative, a reference point, can filter the naive pairwise preference between two alternative. This is indeed the essence of the rational shortlist method introduced in Manzini and Mariotti (2007), where a DM uses preference relations consecutively to rationalize her choice. To clarify, consider the following scenario where a DM is choosing from Brands A-E with the following pairwise preference:

$$\left[\text{A} \sim^p \text{B} \sim^p \text{C}\right] \succ^p \text{D} \sim^p \text{E}$$

Next assume from the respective of pairwisely dominated Brand D the following rational filtering is observed:

$$A \succ^D B \succ^D C.$$

and that Brand E induces the following rational filtering on the most favorite alternatives:

$$B \sim^E C \succ^E A.$$

The choice under classical rationality assumption, induced by *the weak axiom of revealed preferences*, should be consistent with the pairwise choice and therefore consists of all elements A, B, C. However, under presence of the rational filter D choice will be filtered to only A. On the other hand, and in the presence of E the choice should be refined to B, C. What would be the "joint" effect of filters D, E? In particular, if the filters work sequentially then the choice of C is not rationalizable, since after surviving the pairwise comparison, and E filtering, C is ranked lower than B under D.

In this chapter, I show that the rational short-list method can be extended using the divide and conquer representation of WARP developed in the first chapter. In particular, we assume that the DM uses the following division correspondence:

Comprehensive Inductive Division Correspondence (d): d assigns to any set S with $|S| = k \geq 3$ the collection of all its subsets with cardinality $k - 1$ (first-order diminished subsets).

Figure 3.1 illustrate the natrue of a comprehensive division correspondence.

As seen in this the menu $\{x, y, z, r_1, r_2\}$ is divided into all of its doubleton subsets. An important observation is that the DM who used the comprehensive division correspondence, does not simply use the majority rule in making her choice. She refines her choice by applying the referenced preference in a consecutive manner. This is illustrate in Figure 3.2. She first makes a short-list $\{x, y, z\}$ using the pairwise preference $\succsim^p$. Then using the two referenced preference in a consecutive manner she eliminates y. This is because, even though equality attractive when the reference
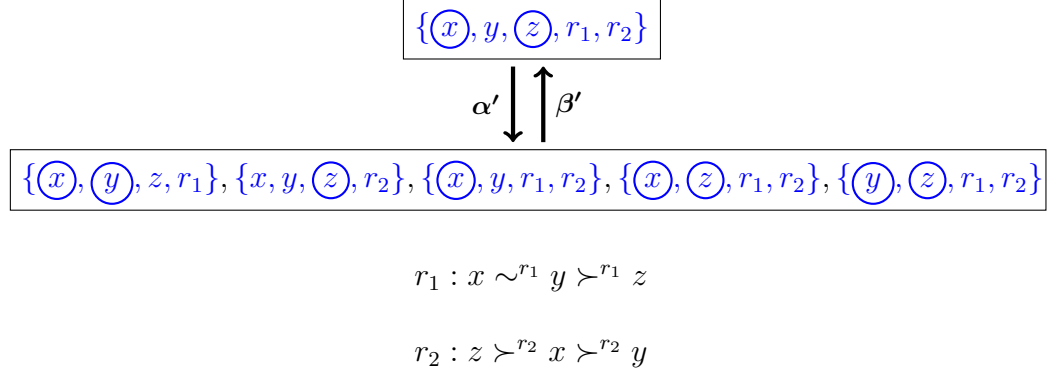
$$\{\textcircled{x}, y, \textcircled{z}, r_1, r_2\}$$

$$\alpha' \downarrow \uparrow \beta'$$

$$\{\textcircled{x}, \textcircled{y}, z, r_1\}, \{x, y, \textcircled{z}, r_2\}, \{\textcircled{x}, y, r_1, r_2\}, \{\textcircled{x}, \textcircled{z}, r_1, r_2\}, \{\textcircled{y}, \textcircled{z}, r_1, r_2\}$$

$$r_1 : x \sim^{r_1} y \succ^{r_1} z$$

$$r_2 : z \succ^{r_2} x \succ^{r_2} y$$

Figure 3.1: RDM's Choice with Multiple References

point r_1 is used, y is less attractive than x when the reference point r_2 is further used.

$$\{x, y, z, r_1, r_2\} \xrightarrow{\succsim^p} \{x, y, z\} \begin{array}{c} \xrightarrow{\succsim^{r_1}} \{x, y\} \xrightarrow{\succsim^{r_2}} \{x\} \\ \\ \xrightarrow{\succsim^{r_2}} \{z\} \xrightarrow{\succsim^{r_1}} \{z\} \end{array} \longrightarrow \text{Choice: } \{x, z\}$$
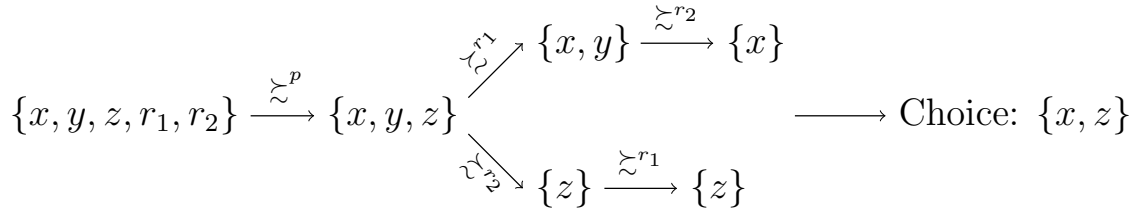
Figure 3.2: Rational Filtering

Rational filtering, therefore, endogenizes the notion of rational short-list method in Manzini and Mariotti (2007). It also extends it to the case with three preferences. It is important to note that the case with more than three preferences remains ans open problem.

3.1.1 Related Literature

The approaches that have been taken can are divided in two categories. rational, and behavioral. In the latter category the models developed in Masatlioglu et al. (2012) and Ok et al. (2015) capture the concept using the behavioral notion of "inattention". In the former category the results in Cherepanov et al. (2013) provides explanation where rationals (reference preference) are formed, respectively, exogenously and endogenously.

3.2 Preliminaries

Let X be a finite set. X is the set of all "relevant" alternatives for the DM. Therefore, it contains not only the concrete options available to the DM, but also, for example, alternatives that she has chosen before, or phantom alternatives that are not available to choose but presented to her (e.g., items that are out of stock, or shows that are sold out). In terms of the nature of the elements, X might be alternatives available for grocery shopping, different colleges to attend, various policies to be followed by the policy maker, etc. Let 2^X be the power set of X. Also let

$$\mathfrak{X}^k := \{A \subseteq X : |A| = k\};$$

that is the set of all subsets of X with cardinality equal to k, and

$$\mathfrak{X}^{\geq k} := \{A \subseteq X : |A| \geq k\};$$

that is the set of all subsets of X with cardinality of at least k. In order to simplify the domain of the discussion on choice I only consider the sets that have at least two elements, as the choice over the empty set and the singletons are trivially interpreted. Therefore let $\mathfrak{X} := \mathfrak{X}^{\geq 2}$. A *choice correspondence* on X is a function $\boldsymbol{c} : \mathfrak{X} \to 2^X$ such that for all $A \in \mathfrak{X}$ we have $\boldsymbol{c}(A) \subseteq A$. $\boldsymbol{c}$ is called a *non-empty valued* choice correspondence if $\boldsymbol{c}(A) \neq \emptyset$ for all $A \subseteq X$. We make the common notational abuses:

$$\boldsymbol{c}\{x, y, z\} := \boldsymbol{c}(\{x, y, z\}) \text{ and } \boldsymbol{c}\{x, y\} := \boldsymbol{c}(\{x, y\}),$$

for all $x, y, z \in X$.

Let $S \subseteq X$. Unless otherwise stated, whenever used throughout this paper let

$S \in \mathfrak{X}^{\geq 3}$; that is let S have at least three elements. Similar to X, for S let

$$\mathfrak{S}^k := \{A \subseteq S : |A| = k\} \quad \text{and} \quad \mathfrak{S}^{\geq k} := \{A \subseteq S : |A| \geq k\}.$$

For $x \in S$ let $S - x := S \setminus \{x\}$; that is the set which is derived by removing x from S.

A binary relation R on X is a subset of $X \times X$. Let $\mathcal{R}$ be the asymmetric relation derived from R; that is

$$x\mathcal{R}y \iff xRy \text{ and } \neg(yRx).$$

A *cycle* of order k in R is a set $\{x_1, x_2, \ldots, x_k\}$ with $x_i \in X$ such that

$$x_1 \mathcal{R} x_2 \mathcal{R} \ldots \mathcal{R} x_k \mathcal{R} x_1.$$

R is said to be *acyclic* if it does not posses any cycle of any order. A *preference* relation on X is a binary relation which is transitive and complete. For a binary relation R on X, and $S \subseteq X$, x is called a *maximum* element of R on S if

$$xRy : \forall y \in S.$$

Let

$$\operatorname*{argmax}_{S} R := \{x \in S : x \text{ is a maximum for } R \text{ on } S\};$$

x is called a *maximal* element of R on S if there does note exist $y \in A$ such that $y\mathcal{R}x$, where $\mathcal{R}$ is the asymmetric relation derived from R.

A *cover* for $S \subseteq X$ is a family of sets, $\{A_i\}_{i=1}^n$ such that $A_i \subseteq S$ for all i and

$$S = \bigcup_{i=1}^n A_i.$$

For a choice correspondence $\boldsymbol{c}$ define the relation $\succsim^p$ on X by

$$x \succsim^p y \iff x \in \boldsymbol{c}\{x,y\}.$$

Let $\succ^p$ and $\sim^p$ be asymmetric and symmetric parts of $\succsim^p$. Note that $\succsim^p$ matches the notion of revealed preference in the sense of Samuelson (1938). We call $\succsim^p$ the *pairwise revealed preference* throughout this paper. We next define the key notion of references.

In order to extend the model in Chapter 2 to the case where reference have correlated effect, it turns out, an expansion of the consideration sets with which a DM makes here choice does the job. To do this let us introduce the a formal definition of a comprehensive division correspondence.

Definition 20. A correspondence $\boldsymbol{d} : \mathcal{P}^{\geq 3}(X) \to \mathcal{P}^{\geq 2}(X)$ is called a *comprehensive inductive division correspondence* if

$$\boldsymbol{d}(S) = \{S \setminus \{x\} : x \in S\},$$

for all $S \in \mathcal{P}^{\geq 3}(X)$.

Note that here a DM considers her respective choice in *all* of the first-order diminished sets.

3.3 Filtering Decision Maker

Recall the axiom of weak expansion rule β'' from Chapter 2.

Definition 21. (Filtering DM: FDM) Let $\boldsymbol{d}$ be an comprehensive inductive division correspondence. A choice correspondence $\boldsymbol{c}$ is called a *RDM* on S if it satisfies NBC, α', and β''.

The basic properties of a RDM follows directly from those of RDM in Chapter 2. In particular, references are indeed elements of a lower indifference class and change the relative importance of two more preferable alternatives.[1]. Also the decision maker still chooses an alternative from the most preferable indifference class. That is $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$ for all $S \subseteq X$.[2]

3.4 Reference Transitivity

Definition 22. (References) For a choice correspondence $\boldsymbol{c}$ and $S \subseteq X$ we say r is a *reference* in S if there exits two distinct elements $x, y \in S$, both different from r such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y\}.$$

We say r is a *maximal* reference if there exists $x, y \in \mathcal{I}^p(S)$ such that

$$\boldsymbol{c}\{x, y, r\} \subset \boldsymbol{c}\{x, y.\}.$$

Definition 23. (Reference Relation) Let $S \subseteq X$ and $r \in \mathfrak{R}(S)$. For two distinct element $x, y \in S$, both different from r, define

$$x \succsim^r y \iff x \in \boldsymbol{c}\{x, y, r\}.$$

Also let $\succ^r$ and $\sim^r$ be the asymmetric and symmetric parts of $\succsim^r$.

As shown in Chapter 2 if $\boldsymbol{c}$ is non-empty valued then $\succsim^r$ defines a complete and acyclic binary relation on $\mathcal{I}^p(S)$.[3]. In order to make the choice of a DM more structured I strengthen the latter result by enforcing transitivity of reference as an axiom.

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[1]See Proposition 3 in Chapter 2.
[2]See Proposition 4 in Chapter 2.
[3]See Proposition 5 and Theorem 3 in Chapter 2.

Axiom 4. (Reference Transitivity - RT): $\succsim^r$ defines a transitive binary relation on $\mathcal{I}^p(S)$.

3.5 Main Results

As discussed in Chapter 2, the referential effects for a RDM are completely separable. In particular, the from Theorem **??** it follows that a choice in a set is characterized by the element(s) that have the "most number" of references. That is the effect of references from the respective of the DM are completely separable. Let me consider a example here. Suppose that the menu in a restaurant included the item $\{x, y, z, r_1, r_2\}$. DM's preferences on the menu is defined by

$$x \sim^p y \sim^p z \succ^p r_1 \sim^p r_2.$$

Next assume that r_1, r_2 are maximal references that produce the following reference preference relation over the most favorable alternatives; that is $\mathcal{I}^p(S) = \{x, y, z\}$:

$$r_1 : x \succ^{r_1} y \succ^{r_1} z,$$

$$r_2 : y \sim^{r_2} z \succ^{r_2} x.$$

That is from the perspective of r_1 DM likes x the best, and from the perspective of r_2 does y, z. If DM completely separates the referential effects then the choice in the menu would consist of $\{x, y, z\} = \mathcal{I}^p(S)$. That is DM's acts as her behavior was satisfying WARP. On the other hand it is viable to think that a DM considers the *join* effects of the references. That is after adopting to a reference point r_1 she further "filters" her choice by adopting to r_2 and vice versa.[4]. In this example under r_1 the

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[4]This indeed is generalizing the notion of sequential rationality introduced in Manzini and Mariotti (2007)

unique most preferable alternative is x and applying r_2 does not change the choice. However, y, z are most preference under r_2 but $y \succ^{r_1} z$. If referential effects are joint then the choice of z is not desirable from the DM.

To make this formal, let $S \subseteq X$ be a set that has at most two maximal reference. Let $M_1(S) = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1}$. For a reference r_1, define

$$r_{12}(S) = \underset{M_1(S)}{\operatorname{argmax}} \succsim^{r_2},$$

and respectively,

$$r_{21}(S) = \underset{M_2}{\operatorname{argmax}} \succsim^{r_1}.$$

Also define

$$\mathfrak{r}(S) = r_{12}(S) \cup r_{21}(S).$$

Note that if there are no maximal references in S then $\mathfrak{r}(S) = \mathcal{I}^p(S)$. Also Let $\underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1 r_2} = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1} \cap \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_2}$. The next theorem formalized the notion of joint referential effect.

Theorem 5. *Let $S \subseteq X$ have at most two maximal references. Then*

$$\boldsymbol{c}(S) = \mathfrak{r}(S).$$

The results in Theorem 5 shows the endogenous formation of the notion of *rational shortlist model* introduced in Manzini and Mariotti (2007). The two preferences are exogenously given in their work. From the point of view of joint referential effects introduce in this paper, a rational referential DM has a "naıve" preference which is free of referential effects. Such preference however get updated adopting to reference points one after the other. The order does not matter here and the results are consisted of both directions of the reference influence.

3.6 Conclusion

This chapter uses the inductive divide and conquer procedure developed in Chapter 2 in order to capture the behavior of a DM who uses referenced preferences in a more sophisticated manner than a simple majority rule. It is shown that a comprehensive division correspondence yields a sequentially rationalizable behavior in the sense of Manzini and Mariotti (2007). Therefore, endogenizing the notion, it also extends it to the case with three preference (a pairwise preference and two referenced preferences.) The case with more than three preferences remains an open problem.

3.7 Appendix - Proofs

Proof of Theorem 5

The Case of No Maximal Reference

Lemma 24. *Let c be a RDM. If $x \in c\{x, y, z\}$ for all pair of distinct elements $y, z \in S$, both different from x, then $x \in c(S)$.*

Proof. We prove this by induction on $|S|$. For $|S| = 3$ there is nothing to prove. Assume that the statement is true for all the sets with cardinality k. Let $|S| = k + 1$ and take $x \in S$ be such that $x \in c\{x, y, z\}$ for all two distinct elements $y, z \in S$, both different from x. Let

$$^{x}S^{-} = \{A \in S^{-} : x \in A\}.$$

Obviously $^{x}S^{-} \neq \emptyset$. Take $A \in {}^{x}S^{-}$. Note that since

$$x \in c\{x, y, z\},$$

for two distinct elements $y, z \in S$ and, since $A \subseteq S$, we conclude that

$$x \in \boldsymbol{c}\{x, y, z\},$$

for two distinct elements $y, z \in A$, both different from x. Finally induction assumption implies that $x \in \boldsymbol{c}(A)$. Since A was an arbitrary element of $^xS^-$ we conclude x is chosen in all elements of $^xS^-$. This means x is not dominated by any element of S. Therefore β' implies that $x \in \boldsymbol{c}(S)$. ∎

Lemma 25. *If* $\mathfrak{R}_M(S) = \varnothing$ *then* $\boldsymbol{c}(S) = \mathcal{I}^p(S)$.

Proof. Assume $\mathfrak{R}_M(S) = \varnothing$. We need to show $\boldsymbol{c}(S) = \mathcal{I}^p(S)$. First note that in the proof of Lemma ??? ($\Rightarrow$) we only used α'. Since an RDM satisfies α' we conclude that from Lemma ??? ($\Rightarrow$) that $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$. So we only need to prove $\mathcal{I}^p(S) \subseteq \boldsymbol{c}(S)$. Take $x \in \mathcal{I}^p(S)$ and consider the set $\{x, y, z\}$ for two distinct elements $y, z \in S$, both different from x.

    <u>Step 1</u>: $\mathfrak{R}\{x, y, z\} = \varnothing$.

Proof. By contradiction assume $\mathfrak{R}\{x, y, z\} \neq \varnothing$. First note that by Proposition 6.ii it has to be the case that there is only one reference in $\{x, y, z\}$. Second by Corollary 12 and the fact that $x \in \mathcal{I}^p(S)$ we conclude $x \notin \mathfrak{R}(S)$. Wlog, assume z is the reference in $\{x, y, z\}$. Then it has to be the case that

$$\boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, y\},$$

which implies $\boldsymbol{c}\{x, y\} = \{x, y\}$, which in turn implies $y \succsim^p x$. Since $x \in \mathcal{I}^p(S)$ NBC implies $y \in \mathcal{I}^p(S)$. This means $z \in \mathfrak{R}_M(S)$ which is not possible. This completes the proof of Step 1. □

    <u>Step 2</u>: $x \in \boldsymbol{c}\{x, y, z\}$.

Proof. From Step 1 we have $\mathfrak{R}\{x, y, z\} = \emptyset$. Therefore

$$\boldsymbol{d}\{x, y, z\} = \big\{\{x, y\}, \{y, z\}, \{x, z\}\big\}.$$

We first argue that y or z can not be the single choice in $\boldsymbol{c}\{x, y, z\}$. To do this, and wlog, assume $\boldsymbol{c}\{x, y, z\} = \{y\}$. Then α' implies $y \succsim^p x$ which in turn means $x, y \in \mathcal{I}^p(S)$. Then we conclude that

$$\boldsymbol{c}\{x, y, z\} \subset \boldsymbol{c}\{x, y\},$$

which means z is a maximal reference in S which is impossible. Second assume $\{y, z\} \subseteq \boldsymbol{c}\{x, y, z\}$. We show that $x \in \boldsymbol{c}\{x, y, z\}$. From α' we conclude $y, z \succsim^p x$ and, since $x \in \mathcal{I}^p(S)$, it follows that $x, y, z \in \mathcal{I}^p(S)$ which means $\{x, y, z\}$ is fully indecisive. Using Proposition 6.i we conclude $\boldsymbol{c}\{x, y, z\} = \{x, y, z\}$ which obviously means $x \in \boldsymbol{c}\{x, y, z\}$. Since $\boldsymbol{c}\{x, y, z\} \neq \emptyset$ the proof of this Step 2 is complete. $\square$

To finish the proof note that by Lemma 24.ii we conclude $x \in \boldsymbol{c}(S)$. This means $\mathcal{I}^p(S) \subseteq \boldsymbol{c}(S)$. Therefore $\boldsymbol{c}(S) = \mathcal{I}^p(S)$. $\blacksquare$

The Case of Single Maximal Reference

Lemma 26. *If* $\mathfrak{R}_M(S) = \{r\}$ *then* $\boldsymbol{c}(S) = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$.

Proof. We prove this by induction on $|S|$. For $k = 3$ assume $S = \{x_1, x_2, r\}$. Obviously $x \in \boldsymbol{c}\{x_1, x_2, r\}$ if and only if $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. Now assume for $|S| = k$ the statement is true and let $|S| = k + 1$. First note that $\boldsymbol{c}(S) \subseteq \mathcal{I}^p(S)$. Take $x \in \boldsymbol{c}(S)$. First note that by Theorem 25 we conclude that $\boldsymbol{c}(S - r) = \mathcal{I}^p(S)$. Assume there exists $y \in S$ and $A \in S^-$ such that y beats x in A. Therefore $r \in A$ and by induction assumption $y \succ^r x$. Note that $y \in \mathcal{I}^p(S)$. This means y beats x in all elements of S^- except $S - r$. Since $y \in \boldsymbol{c}(S - r)$ then we conclude x does not beat y in any elements

of S^- and therefore $y \underset{S}{\blacktriangleright} x$. This contradicts the assumption that $x \in \boldsymbol{c}(S)$ and $\boldsymbol{c}$ satisfies α'. This establishes that $\boldsymbol{c}(S) \subseteq \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$.

To show the other inclusion take $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r$. Note that by induction assumption and since $x \in \underset{\mathcal{I}^p(S-)}{\operatorname{argmax}} \succsim^r$ for all $t \neq r$ we conclude $x \in \boldsymbol{c}(S - r)$. Also since $x \in \mathcal{I}^p(S)$ we conclude from Theorem 25 that $x \in \boldsymbol{c}(S - r)$. Now β' implies $x \in \boldsymbol{c}(S)$. This establishes that $\underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^r \subseteq \boldsymbol{c}(S)$. Therefore the proof is complete. ∎

The Case of Double Maximal References

Lemma 27. (WARP Lemma) *Let $\succsim$ be a preference relation on $S \subseteq X$. For $A \subseteq B \subseteq S$ we have*

$$\underset{B}{\operatorname{argmax}} \succsim \cap A = \underset{A}{\operatorname{argmax}} \succsim,$$

whenever $\underset{B}{\operatorname{argmax}} \succsim \cap A \neq \emptyset$.

Proof. Take $x \in \underset{B}{\operatorname{argmax}} \succsim \cap A$. Obviously $x \in A$. Also $x \succsim y$ for all $y \in B \supseteq A$ and therefore $x \in \underset{A}{\operatorname{argmax}} \succsim$. To see the other inclusion take $x \in \underset{A}{\operatorname{argmax}} \succsim$. Since $\underset{B}{\operatorname{argmax}} \succsim \cap A \neq \emptyset$ also take $y \in \underset{B}{\operatorname{argmax}} \succsim \cap A$. Since $y \in A$ and $x \in \underset{A}{\operatorname{argmax}} \succsim$ we conclude $x \succsim y$. Since $y \in \underset{B}{\operatorname{argmax}} \succsim$ it follows that $y \succsim t$ for all $t \in B$. Therefore transitivity of $\succsim$ implies $x \succsim t$ for all $t \in B$, which in turn means $x \in \underset{B}{\operatorname{argmax}} \succsim$. Finally $x \in A$ and we conclude $x \in \underset{B}{\operatorname{argmax}} \succsim \cap A$. ∎

Lemma 28. *The following are true:*

   (i) $x \in \mathfrak{r}(S)$ *then* $x \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{rt}$ *for some* $t \in \{1, 2\}$.

  (ii) *For* $x \in \mathfrak{r}(S)$ *if* $y \succ^{r_1} x$ *then* $x \succ^{r_2} y$.

 (iii) *For* $x, y \in r_{12}(S)$ *we have* $x \sim^{r_i} y$ *for all* $i \in \{1, 2\}$.

Proof. (i) Wlog, assume that $x \in r_{12}(S)$. Then

$$x \in \underset{M_1(S)}{\operatorname{argmax}} \succsim^{r_2}$$

this means $x \in M_1(S) = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1}$

(ii) Since $y \succ^{r_1} x$ then we conclude that $x \notin M_1(S)$ and therefore $x \notin r_{12}(S)$.

This means $x \in r_{21}(S) = \underset{M_2(S)}{\operatorname{argmax}} \succsim^{r_1}$. Since $y \succ^{r_1} x$ then then we conclude

$y \notin M_2(S)$, which by transitivity of $\succsim^{r_2}$ and the fact that $x \in M_2(S)$ implies

$x \succ^{r_2} y$.

(iii)

$$x, y \in r_{12}(S) = \underset{M_1(S)}{\operatorname{argmax}} \succsim^{r_2}$$

This, first means $x, y \in M_1(S)$ and therefore $x \sim^{r_1} y$, second, $x \sim^{r_2} y$.

∎

Lemma 29. *The followings are true:*

i. If $x_i \in \mathfrak{r}(S)$ then $x_i \in \mathfrak{r}(S - x_t)$ for $t \neq i$.

ii. $\underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1 r_2} \neq \emptyset$ if and only if $\mathfrak{r}(S) = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1 r_2}$

Proof. i . Take $x_i \in \mathfrak{r}(S)$ and $t \neq i$. Wlog assume $x_i \in r_{12}(S)$, so $x_i \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1}$

. First note that by WA Lemma we conclude

$$x_i \in \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1} \cap \mathcal{I}^p(S - x_t) = \underset{\mathcal{I}^p(S-x_t)}{\operatorname{argmax}} \succsim^{r_1} .$$

Let $N(S) = \underset{\mathcal{I}^p(S)}{\operatorname{argmax}} \succsim^{r_1}$ and $N(S - x_t) = \underset{\mathcal{I}^p(S-x_t)}{\operatorname{argmax}} \succsim^{r_1}$. Note that $x_i \in$

$N(S - x_t) \cap N(S)$. This latter fact means $N(S - x_t) \subseteq N(S)$. To see that

note take $x_j \in N(S - x_t)$. This means $x_j \succsim^{r_1} x$. Since $x_i \in N(S)$ and $x \in S$ from

transitivity of $\succsim^{r_1}$ it follows that $x_j \in N(S)$. Next using WA Lemma one more time we conclude

$$x_i \in \operatorname*{argmax}_{N_1(S)} \succsim^{r_2} \cap N_1(S - x_t) = \operatorname*{argmax}_{N_1(S-x_t)} \succsim^{r_2} = r_{12}(S - x_t) \subseteq \mathfrak{r}(S - x_t)$$

ii. ($\Leftarrow$): Since $\mathfrak{r}(S) \neq \varnothing$ then this is obvious.

($\Rightarrow$): Now assume $\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2} \neq \varnothing$ and take $x_i \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$. Then we have $x_i \in M_1(S)$ and since $M_1(S) \subseteq \mathcal{I}^p(S)$ by WA Lemma

$$x_i \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_2} \cap M_1(S) = \operatorname*{argmax}_{M_1(S)} \succsim^{r_2} = r_{12}(S) \subseteq \mathfrak{r}(S).$$

This establishes that $\operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2} \subseteq \mathfrak{r}(S)$. Now take $x_i \in \mathfrak{r}(S)$. By contradiction assume $x_i \notin \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$. Since this latter set is non-empty take $x_j \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$. Wlog, it follows that $x_j \succ^{r_1} x_i$. This means $x_i \notin M_1(S)$ which in turn means $x_i \notin r_{12}(S)$. Also since $x_j \in M_2(S)$ and $x_j \succ^{r_1} x_i$ it follows that $x \notin r_{21}(S)$ and therefore $x_i \notin \mathfrak{r}(S)$. This is a contradiction. This establishes that $x_i \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$ and therefore $\mathfrak{r}(S) = \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1 r_2}$

■

Theorem ??. *Let $S \subseteq X$ have at most two maximal references. Then*

$$\boldsymbol{c}(S) = \mathfrak{r}(S)$$

Proof. Let $S = \{x_1, x_2, \ldots, x_n, r_1, r_2\}$. We prove the result by induction on n.

Induction Base: ($\Rightarrow$) For $k = 2$ assume $S = \{x_1, x_2, r_1, r_2\}$ and assume wlog $x_1 \in \boldsymbol{c}(S)$. If $x_1 \succsim^{r_i} x_2$ for all $i \in \{1, 2\}$ then $x \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_i}$ for all $i \in \{1, 2\}$ and therefore $x \in \mathfrak{r}(S)$. So assume there exists $i \in \{1, 2\}$ such that $x_2 \succ^{r_i} x_1$. Then α' implies $x_1 \succ^{r_j} x_2$ for $j \neq i$ which implies $x_1 \in r_{ji}(S) \subseteq r_i r_j(S)$.

($\Longleftarrow$) Now assume $x_1 \in \mathfrak{r}(S)$. We show that x_2 does not dominate x_1 in S (that is we show that $x_2 \underset{S}{\blacktriangleright} x_1$ is not true). For this assume, wlog, $x_2 \succ^{r_1} x_1$, then $x_1 \notin r_{12}(S)$ which implies $x_1 \in r_{21}(S)$. This means $x_1 \succ^{r_2} x_2$. Therefore β' implies $x_1 \in \boldsymbol{c}(S)$.

Induction Assumption: Assume that for S with $n-1$ non-maximal reference elements the statement is true and assume S has n non-maximal reference elements.

($\Longrightarrow$): Let $x_i \in \boldsymbol{c}(S)$.

<u>Claim</u>: $x_i \in \text{argmax} \succsim^{r_i}$ for some $i \in \{1, 2\}$.

Proof. By contradiction assume that $x_i \notin \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^{r_t}$ for all $t \in \{1, 2\}$. Take $x_j \in \mathfrak{r}(S)$. Then Lemma 29.ii and for all $t \notin \{i, j\}$ we conclude from induction assumption that

$$x_i \in \boldsymbol{c}(S - x_t).$$

Next note that since $x_j \in \mathfrak{r}(S)$ then Lemma 28.ii, and wlog, $x_j \in \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^{r_1}$. Since $x_i \notin \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^{r_1}$ Then Lemma 26 implies that $\boldsymbol{c}(S - r_2) \cap \{x_i, x_j\} = \{x_j\}$. This means that x_j beats x_i in $S - r_2$. By the argument before we also conclude that x_i does not beat x_j in all $A \in S^-$ and therefore $x_j \underset{S}{\blacktriangleright} x_i$. This contradicts $x_i \in \boldsymbol{c}(S)$ and the fact that $\boldsymbol{c}$ satisfies α'. $\qquad\square$

To complete the proof we consider two cases:

<u>Case I</u>: If $x_i \in \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^{r_1 r_2}$ by Lemma 29.ii then $x_i \in \mathfrak{r}(S)$ and the proof is complete.

<u>Case II</u>: Wlog, assume that $x_i \in \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^{r_1}$ and $x_i \notin \underset{\mathcal{I}^p(S)}{\text{argmax}} \succsim^{r_2}$. Note that

the latter, by Lemma 26 implies that $x_i \notin \boldsymbol{c}(S - r_1)$. Obviously $x_i \notin r_{21}(S)$. By contradiction assume $x_i \notin r_{12}(S)$. Take $x_j \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1}$ such that $x_j \in r_{12}(S)$. Note that since $r_{12}(S) \neq \emptyset$ such x_j exists. Next note that it has to be the case that $x_j \succ^{r_2} x_i$. Now consider the set $S - x_t$ for $t \notin \{i, j\}$. By Lemma 29.i $x_j \in \mathfrak{r}(S - x_t)$. Also $x_i \notin \mathfrak{r}(S - x_t)$ which by induction hypothesis implies $x_i \notin \boldsymbol{c}(S - x_t)$. This means x_j beats x_i in $S - x_t$. Since $x_j, x_i \in \boldsymbol{c}(S - r_2)$ and $x_i \notin \boldsymbol{c}(S - r_1)$ it follows that x_i does not beat x_j in any elements of S^- which in turn means $x_j \mathrel{\underset{S}{\blacktriangleright}} x_i$. This contradicts the fact that $x_i \in \boldsymbol{c}(S)$ and $\boldsymbol{c}$ satisfies α'.

($\Leftarrow$): Now assume that $x_i \in \mathfrak{r}(S)$. First note that by Lemma 29.i $x_i \in \mathfrak{r}(S - x_t)$ and by induction assumption $x_i \in \boldsymbol{c}(S - x_t)$ for all $t \neq i$. Wlog, assume $x_i \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1}$, which implies $x_i \in \boldsymbol{c}(S - r_2)$. If $x_i \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_2}$ then $x_i \in \boldsymbol{c}(S - r_1)$ and β' implies $x_i \in \boldsymbol{c}(S)$. So take x_j such that $x_j \succ^{r_2} x_i$. Therefore Lemma 26 implies that x_j beats x_i in $S - r_1$. Since $x_i \in \mathfrak{r}(S)$ this implies $x_i \in \operatorname*{argmax}_{\mathcal{I}^p(S)} \succsim^{r_1}$. Next by Lemma 28.iii $x_i \succ^{r_1} x_j$ and therefore x_i beats x_j in $S - r_2$. Now from β' it follows that $x_i \in \boldsymbol{c}(S)$. $\blacksquare$

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