DECOMPOSITION OF PRINCIPAL SERIES REPRESENTATIONS AND CLEBSCH-GORDAN COEFFICIENTS

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ABSTRACT OF THE DISSERTATION

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In this thesis, following a similar procedure developed by Buttcane and Miller in [BM17] for $SL(3,\mathbb{R})$, the $(\mathfrak{g}, K)$-module structures of the minimal principal series of real reductive Lie groups $SU(2,1)$ and $Sp(4,\mathbb{R})$ are described explicitly by realizing the representations in the space of K-finite functions on $U(2)$. Moreover, by combining combinatorial techniques and contour integrations, this thesis introduces a method of calculating intertwining operators on the principal series. Upon restriction to each K-type, the matrix entries of intertwining operators are represented by Γ-functions and Laurent series coefficients of hypergeometric series. The calculation of the $(\mathfrak{g}, K)$-module structure of principal series can be generalized to real reductive Lie groups whose maximal compact subgroup is a product of $SU(2)$'s and $U(1)$'s.

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Dedication

To the little boy growing up by the sea, with a long-cherished dream of exploring science.

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Chapter 1

Introduction

1.1 Automorphic Forms and $(\mathfrak{g}, K)$-Modules

The study of automorphic forms serves as a central topic in representation theory and number theory. The *modular forms* and *Maaß forms* on the upper half plane

$$\mathbb{H} = \{x + \mathrm{i}y | x, y \in \mathbb{R}, y > 0\} \subset \mathbb{C}$$

under the action of an arithmetic subgroup Γ of $SL(2, \mathbb{Z})$ acting on $\mathbb{H}$ by fractional linear transform $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z \mapsto \frac{az+b}{cz+d}$ are two classical objects in this area, connecting the study of algebraic curves, representation theory and number theory. They are defined as certain eigenfunctions of the weight $k \in \mathbb{Z}^{\geq 0}$ Laplace operator $\Delta_k = -y^2(\partial_x^2 + \partial_y^2) + \mathrm{i}ky\partial_x$, and invariant under the action of Γ in the sense

$$f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) := \left(\frac{cz+d}{|cz+d|}\right)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z). \tag{1.1}$$

The study of automorphic forms in general passes the function from the upper half plane $\mathbb{H}$ to the real reductive Lie group $G = SL(2, \mathbb{R})$. G has a maximal compact subgroup $K = SO(2, \mathbb{R}) = \{\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) | \theta \in \mathbb{R}\}$, a real split Cartan subgroup $A = \{\left(\begin{smallmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{smallmatrix}\right) | y > 0\}$ and a nilpotent subgroup $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) | x \in \mathbb{R}\}$. These subgroups give rise to an *Iwasawa decomposition*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} 1 & \frac{ac+bd}{c^2+d^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{c^2+d^2} & 0 \\ 0 & \sqrt{c^2+d^2} \end{pmatrix} \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & -\frac{c}{\sqrt{c^2+d^2}} \\ \frac{c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix}$$

of any element of G. The Iwasawa decomposition $G = NAK$ parametrizes any element in G with the coordinates

$$(x, y, \theta) = \left(\frac{ac+bd}{c^2+d^2}, 1/(c^2+d^2), \arctan(-c/d)\right).$$

The upper half plane $\mathbb{H}$ is thus isomorphic to the hermitian symmetric space G/K, on which the fractional linear action by N is the translation along x-axis, and the action by A is the positive scalar multiple of a point.

Thus we can consider the spaces of weight k automorphic functions

$$C^\infty(\Gamma\backslash G,k)=\left\{f:G\to\mathbb{C}\text{ smooth}|f(\gamma g\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right))=e^{\mathrm{i}k\theta}f(g)\text{ for any }\gamma\in\Gamma\right\}$$

on which the group G acts by the right regular action $\pi(g)f(x)=f(xg)$. Any weight k Maaß form f defines a function $F\in C^\infty(\Gamma\backslash G,k)$ via the formula $F(g)=(f|_k g)(\mathrm{i})$. Conversely, any function $F\in C^\infty(\Gamma\backslash G,k)$ defines a function $f(x+\mathrm{i}y)=F\left(\begin{smallmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{smallmatrix}\right)$ on the upper half plane $\mathbb{H}$, which satisfies the same invariance condition under Γ as (1.1). This correspondence between weight k automorphic forms on $\mathbb{H}$ and the Γ-left invariant functions on G on which K acts on the right as a character $e^{\mathrm{i}k\theta}$ motivates the study of irreducible representations of G and their *K-types*. The concept to study is the *$(\mathfrak{g},K)$-module* or *Harish-Chandra module* introduced by Harish-Chandra and James Lepowsky in [Lep73].

1.2 Bargmann's Classification of $SL(2,\mathbb{R})$ and $GL(2,\mathbb{R})$ Irreducible $(\mathfrak{g},K)$-Modules

We define the *principal series* to be the set of K-finite smooth functions

$$I(\chi_{\delta,\lambda})=\left\{f:G\to\mathbb{C}|f\left((-1)^\epsilon\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\begin{pmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{pmatrix}g\right)=y^{\frac{\lambda+1}{2}}(-1)^{\epsilon\delta}f(g)\right\}$$

where G acts by translation on the right. Since the group G has an Iwasawa decomposition $G=NAK$, the value of f is determined by its restriction to K. We can expand $f\in I(\chi_{\delta,\lambda})$ into finite Fourier series

$$I(\chi_{\delta,\lambda})=\bigoplus_{k\equiv\delta\bmod 2}\mathbb{C}e^{\mathrm{i}k\theta}.$$

The irreducible K-representations $\mathbb{C}e^{\mathrm{i}k\theta}$ contained in the $(\mathfrak{g},K)$-module of $I(\chi_{\epsilon,\lambda})$ are called *K-types*. A representation of G is called *admissible* if all K-types occur with finite multiplicities. According to Bargmann [Bar47], the irreducible $(\mathfrak{g},K)$-modules of admissible representations of $SL(2,\mathbb{R})$ are classified by the following theorem:

Theorem 1.1 *[Bar47][Kna79][Mui09] The $(\mathfrak{g}, K)$-modules of irreducible admissible representations of $SL(2,\mathbb{R})$ can be realized as subrepresentations or quotient representations of the principal series $I(\chi_{\delta,\lambda})$ as follows:*

1. *If $\lambda + 1 \equiv \delta \bmod 2\mathbb{Z}$,*

   (a) *If $\lambda > 0$, $I(\chi_{\delta,\lambda})$ has two irreducible subrepresentations $D^{\pm}_{\lambda}$ called the discrete series representations. The quotient $W_\lambda = I(\chi_{\delta,\lambda})/(D^{+}_{\lambda} \oplus D^{-}_{\lambda})$ has finite dimension λ.*

   (b) *If $\lambda < 0$, $I(\chi_{\delta,\lambda})$ has a finite dimensional subrepresentation $W_{-\lambda}$ of dimension $-\lambda$. The quotient $I(\chi_{\delta,\lambda})/W_{-\lambda} \cong D^{+}_{-\lambda} \oplus D^{-}_{-\lambda}$ splits into a direct sum of two discrete series representations.*

   (c) *If $\delta = 1$ and $\lambda = 0$, then the principal series decomposes into two limits of discrete series, and $I(\chi_{-1,0}) = D^{+}_{0} \oplus D^{-}_{0}$.*

2. *In all other cases, $I(\chi_{\delta,\lambda})$ is irreducible, and $I(\chi_{\delta,\lambda})$ is isomorphic to $I(\chi_{\delta,-\lambda})$.*

For an arbitrary $\lambda \in \mathbb{Z}$, the (limit of) discrete series representations $D^{\pm}_{|\lambda|}$ have a decomposition into K-types

$$D^{\pm}_{|\lambda|} = \bigoplus_{\substack{k \geq |\lambda|+1 \\ k \equiv \delta \bmod 2}} \mathbb{C} e^{\pm \mathrm{i} k\theta}.$$

The *raising and lowering operators* $U_+ = \frac{1}{2}\left(\begin{smallmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{smallmatrix}\right)$ and $U_- = \frac{1}{2}\left(\begin{smallmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & -1 \end{smallmatrix}\right)$, act on the K-types by the formula

$$U_{\pm} e^{\mathrm{i}k\theta} = \frac{\lambda \pm k + 1}{2} e^{\mathrm{i}(k \pm 2)\theta}.$$

For example, the decomposition of the principal series $I(\chi_{0,1})$ can be displayed in the following diagram:

$$\begin{tikzcd}
e^{-\mathrm{i}k\theta} \arrow[r, bend left, "U_+"] & \cdots \arrow[l, bend left, "U_-"] \arrow[r, bend left, "U_+"] & e^{-2\mathrm{i}\theta} \arrow[l, bend left, "U_-"] & 1 \arrow[l, bend left, "U_-"] \arrow[r, bend left, "U_+"] & e^{2\mathrm{i}\theta} \arrow[r, bend left, "U_+"] & \cdots \arrow[l, bend left, "U_-"] \arrow[r, bend left, "U_+"] & e^{\mathrm{i}k\theta} \arrow[l, bend left, "U_-"]
\end{tikzcd}$$

In the $GL(2,\mathbb{R})$ case, consider $\delta_1, \delta_2 \in \{\pm 1\}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, and define the character $\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2}$ on the Cartan subgroup

$$H = \left\{ \left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{smallmatrix}\right) \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right) | \epsilon_1, \epsilon_2 \in \{\pm 1\}, a_1, a_2 \in \mathbb{R}_{>0} \right\}$$

by sending the elements $\left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)$ to $\epsilon_1^{\delta_1}\epsilon_2^{\delta_2}a_1^{\lambda_1}a_2^{\lambda_2}$. We define the principal series $I(\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2})$ for $GL(2,\mathbb{R})$ to be the induced representation of $\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2}$ from the Borel subgroup $B = \left\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) | ad \neq 0\right\}$ to G:

$$I(\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2}) = \{f : G \longrightarrow \mathbb{C} | f(\left(\begin{smallmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & b \\ 0 & a_2 \end{smallmatrix}\right) g) = \epsilon_1^{\delta_1}\epsilon_2^{\delta_2}a_1^{\lambda_1+\frac{1}{2}}a_2^{\lambda_2-\frac{1}{2}} f(g)\} \quad (1.2)$$

The description of the principal series and the classification of $GL(2,\mathbb{R})$ representations can be summarized as in Theorem 2.4 of [Mui09] as follows:

Theorem 1.2 *The $(\mathfrak{g}, K)$-modules of irreducible admissible representations for the group $GL(2,\mathbb{R})$ can be realized as sub- or quotient modules of the principal series $I(\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2})$. If we define*

$$s = \tfrac{\lambda_1-\lambda_2+1}{2}, \quad \mu = \tfrac{\lambda_1+\lambda_2}{2}, \quad \delta = \delta_1 + \delta_2$$

1. *If $s \notin \{\frac{k}{2} | k \in \mathbb{Z}, k \equiv \delta \bmod 2\mathbb{Z}\}$, then $I(\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2})$ is irreducible. Moreover, if we interchange the two induction parameters, $I(\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2}) \cong I(\chi_{\delta_2,\lambda_2} \times \chi_{\delta_1,\lambda_1})$.*

2. *If $\lambda_1 > \lambda_2$ and $\lambda_1 - \lambda_2 + 1 \equiv \delta \bmod 2\mathbb{Z}$, then if we set $k = \lambda_1 - \lambda_2 + 1$, the character $\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2}$ takes the form $(\chi_{\delta_0,\mu} \cdot \chi_{k,\frac{k-1}{2}}) \times (\chi_{\delta_0,\mu} \cdot \chi_{0,-\frac{k-1}{2}})$ where $\chi_{\delta,\lambda}$ is the character on $\mathbb{R}^\times$ as defined above, sending each $a \in \mathbb{R}^\times$ to $\operatorname{sgn}(a)^\delta |a|^\lambda$, and $\delta_0 \in \{\pm 1\}$, then there exists a composition series for the principal series $I(\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2})$:*

$$D_k^{\chi_{\delta_0,\mu}} \hookrightarrow I\left((\chi_{\delta_0,\mu} \cdot \chi_{k,\frac{k-1}{2}}) \times (\chi_{\delta_0,\mu} \cdot \chi_{0,-\frac{k-1}{2}})\right) \to W_k^{\chi_{\delta_0,\mu}}$$

   where $D_k^{\chi_{\delta_0,\mu}}$ is a discrete series representations, and $W_k^{\chi_{\delta_0,\mu}}$ is a finite dimensional dimension. The superscript $\chi_{\delta_0,\mu}$ indicates that the center of $GL(2,\mathbb{R})$ acts by a character $\chi_{\delta_0,\mu}$.

3. *If $\lambda_1 < \lambda_2$ and $\lambda_1 - \lambda_2 + 1 \equiv \delta \bmod 2\mathbb{Z}$, then if we set $k = \lambda_2 - \lambda_1 + 1$, the character $\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2}$ takes the form $(\chi_{\delta_0,\mu} \cdot \chi_{k,-\frac{k-1}{2}}) \times (\chi_{\delta_0,\mu} \cdot \chi_{0,\frac{k-1}{2}})$, and the principal series representation $I\left(\chi_{\delta_1,\lambda_1} \times \chi_{\delta_2,\lambda_2}\right)$ has a composition series:*

$$W_k^{\chi_{\delta_0,\mu}} \hookrightarrow I\left((\chi_{\delta_0,\mu} \cdot \chi_{k,\frac{k-1}{2}}) \times (\chi_{\delta_0,\mu} \cdot \chi_{0,-\frac{k-1}{2}})\right) \to D_k^{\chi_{\delta_0,\mu}}.$$

   The notations for specific representations are the same as in $\lambda_1 > \lambda_2$ case.

Denote by $\sigma_l = \mathrm{Ind}_{SO(2)}^{O(2)}\mathbb{C}e^{\mathrm{i}l\theta}$ *the index 2 induction of a character from* $SO(2)$ *to* $K = O(2)$. *Since the element* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in O(2)$, *and the conjugation of* $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ *by this element sends* $e^{\mathrm{i}l\theta}$ *to* $e^{-\mathrm{i}l\theta}$. *It is easy to see that* $\sigma_l \cong \sigma_{-l}$. *The* K*-type decomposition of the two irreducible* $(\mathfrak{g}, K)$*-modules* $D_k^{\chi_{\delta_0,\mu}}$ *and* $W_k^{\chi_{\delta_0,\mu}}$ *can be described as follows:*

1. *The module* $D_k^{\chi_{\delta_0,\mu}}$ *has a restriction to the maximal compact subgroup* $K = O(2)$

$$D_k^{\chi_{\delta_0,\mu}}|_{O(2)} = \bigoplus_{\substack{l \equiv k \bmod 2 \\ l \geq k}} \sigma_l.$$

2. *The module* $W_k^{\chi_{\delta_0,\mu}}$ *has restriction to the maximal compact subgroup*

$$W_k^{\chi_{\delta_0,\mu}}|_{O(2)} = \begin{cases} \bigoplus_{\substack{l \equiv k \bmod 2 \\ 1 \leq l \leq k-2}} \sigma_l & k \equiv 1 \bmod 2 \\ \chi_{\delta_0,\mu}|_{O(2)} \oplus \bigoplus_{\substack{l \equiv k \bmod 2 \\ 2 \leq l \leq k-2}} \sigma_l & k \equiv 0 \bmod 2 \end{cases}.$$

1.3 An Introduction to Results in this Thesis

In this thesis, we mainly deal with the groups $SU(2,1)$ and $Sp(4,\mathbb{R})$. The group $SU(2,1)$ is of real rank one, and the decomposition of principal series for $SU(2,1)$ has been studied in [BS80], [Joh76] and [JW77] . In this thesis, I discuss the results for $SU(2,1)$ and $Sp(4,\mathbb{R})$ by passing to the representation theory of their maximal compact subgroups.

When the minimal principal series $I(\delta,\lambda)$ of the group $SU(2,1)$ is not irreducible, we have classified 6 families of irreducible sub or quotient $(\mathfrak{g}, K)$-modules of $I(\delta,\lambda)$ depending on the induction parameters (δ,λ). In [Kra76],[BSK80] and [V+79], an algorithm to calculate the composition series and the classification of irreducible $(\mathfrak{g}, K)$-modules for real rank 1 groups like $SU(n,1)$ and $Sp(n,1)$ has already been developed. In this thesis, I will utilize the Wigner D-functions to perform the calculation explicitly and write down the K-types of the irreducible $(\mathfrak{g}, K)$-modules of $SU(2,1)$ are parametrized by a pair of half integers (j,n) satisfying appropriate parity conditions. These irreducible subquotients are

1. Holomorphic/antiholomorphic discrete series: $V_{\mathrm{disc}\pm}(\delta,\lambda)$,

2. Quaternionic discrete series $V_{\mathbb{H}}(\delta, \lambda)$,

3. Finite dimensional representations $V_{\text{fin}}(\delta, \lambda)$,

4. Two other irreducible $(\mathfrak{g}, K)$-modules $Q_{\pm}(\delta, \lambda)$.

Moreover, we can also compute the intertwining operators for the minimal principal series. This thesis has developed a computational technique based on combinatorics to calculate the intertwining integrals for $SU(2,1)$ and $Sp(4,\mathbb{R})$. They will be discussed in Chapter 5 and 6. For the long intertwining operator of the minimal principal series of $SU(2,1)$, we have another proof of the well-known result from [JW77]:

Theorem 1.3 *The long intertwining operator $A(w_0, \delta, \lambda)$ acts on each $W^{(j,n)}_{m_1,m_2}$ as a scalar $[A(w,\delta,\lambda)]_{m_1}$, with a closed form formula:*

$$[A(w,\delta,\lambda)]_{m_1} = \frac{\pi^2 2^{-\lambda-1}\Gamma(\lambda)}{\Gamma\left(1-\frac{\lambda-\delta}{2}\right)\Gamma\left(1-\frac{\lambda+\delta}{2}\right)} \frac{\Gamma\left(j+m_1-\frac{\lambda+\delta}{2}+1\right)\Gamma\left(j-m_1-\frac{\lambda-\delta}{2}+1\right)}{\Gamma\left(j+m_1+\frac{\lambda-\delta}{2}+1\right)\Gamma\left(j-m_1+\frac{\lambda+\delta}{2}+1\right)} \tag{1.3}$$

We have also shown the following new result for long intertwining operator for the minimal principal series of $Sp(4,\mathbb{R})$:

Theorem 1.4 *If the induction parameter δ of $Sp(4,\mathbb{R})$ principal series satisfies $\delta \in \{(0,0),(1,1)\}$ and $n-m_1 \equiv \delta_1$ and $n+m_2 \equiv \delta_2$ mod 2 for $i \in \{1,2\}$, the matrix entries $[A(\lambda)]^{j,n}_{m_1,m_2}$ for the long intertwining operator is the constant Laurent series coefficient* *of*

$$
\begin{aligned}
&[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \\
&\frac{(-1)^n((2j)!)^2}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{j+n-\epsilon^{j,n}_\delta}{2})}\left(\frac{\lambda_1+1}{2}\right)^{(-\frac{j+n-\epsilon^{j,n}_\delta}{2})}\left(\frac{\lambda_1-\lambda_2+1}{2}\right)^{(j)}\left(\frac{\lambda_1+\lambda_2+1}{2}\right)^{(j)}}\times \\
&\frac{1}{c^j_{m_1}c^j_{m_2}}\frac{\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(\frac{j+m_1-\epsilon^{j,n}_\delta}{2})}\left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{-j-m_2+\epsilon^{j,n}_\delta}{2})}}{\left(\frac{\lambda_2+1}{2}\right)^{\frac{m_2-n}{2}}\left(\frac{\lambda_2+1}{2}\right)^{-\frac{m_2-n}{2}}}\frac{\Gamma\left(\frac{1-\epsilon^{j,n}_\delta+j-m_1}{2}\right)}{\Gamma\left(\frac{1-\epsilon^{j,n}_\delta+j-m_2}{2}\right)}\times \\
&(1-t_1)^{\frac{-1+\epsilon^{j,n}_\delta-j+m_1}{2}}(1-t_2)^{\frac{-1-\epsilon^{j,n}_\delta+j-m_2}{2}}t_2^{\epsilon^{j,n}_\delta-2j}t_1^{-\epsilon^{j,n}_\delta}\times \\
&{}_2F_1\left(\begin{matrix}-j+m_1,\frac{\lambda_1-\lambda_2-2j-1}{2}\\-2j\end{matrix};t_1\right){}_2F_1\left(\begin{matrix}-j-m_2,\frac{\lambda_1+\lambda_2-2j-1}{2}\\-2j\end{matrix};t_2\right)\times \\
&{}_5F_4\left(\begin{matrix}-j+\epsilon^{j,n}_\delta,1,\frac{-j+m_2+1+\epsilon^{j,n}_\delta}{2},\frac{-j-n-\lambda_1+1+\epsilon^{j,n}_\delta}{2},\frac{-j-m_2+\lambda_1+\lambda_2+\epsilon^{j,n}_\delta}{2}\\-j+\epsilon^{j,n}_\delta,\frac{-j+m_1+1+\epsilon^{j,n}_\delta}{2},\frac{-j-n+\lambda_1+1+\epsilon^{j,n}_\delta}{2},\frac{-j-m_1-\lambda_1+\lambda_2+\epsilon^{j,n}_\delta}{2}+1\end{matrix};\frac{t_2^2(1-t_1)}{t_1^2(1-t_2)}\right).
\end{aligned}
\tag{1.4}
$$

where $\epsilon^{j,n}_\delta = \begin{cases} 0 & j-n\equiv\delta_i \mod 2 \\ 1 & j-n\not\equiv\delta_i \mod 2 \end{cases}.$

Chapter 2

Basic Notions

2.1 Notations

The notations used in the thesis, unless otherwise specified, are defined in this section as follows. See the book [Kna13] as a general reference.

2.1.1 Real Forms and Cartan Decomposition

The study of representations of real reductive Lie groups requires an interplay between the complex Lie group and its various real forms. The notations are specified in detail as follows

1. A real reductive Lie group G with Lie algebra $\mathfrak{g}$;
2. An anti-holomorphic involution σ on a complex Lie group $G_{\mathbb{C}}$, such that $G = G_{\mathbb{C}}^{\sigma}$, with $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ the Lie algebra of $G_{\mathbb{C}}$;
3. A Cartan involution θ on $\mathfrak{g}$, which defines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}, \mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ;
4. A maximal compact subgroup $K \subset G$ with Lie algebra $\mathfrak{k}$;
5. The complexification $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}$ and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p} \otimes \mathbb{C}$ of the spaces $\mathfrak{k}, \mathfrak{p} \subset \mathfrak{g}$, respectively;
6. A split real form $\mathfrak{g}_{\text{split}}$ of $\mathfrak{g}_{\mathbb{C}}$, and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\text{split}}$. We require that $\mathfrak{h}$ be θ-stable. Denote the complexification of $\mathfrak{h}$ as $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$;
7. Choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$, with the complexification $\mathfrak{t}_{\mathbb{C}}$;

2.1.2 Root Space Decomposition for the Complex Group

We now specify the notation for the root space decomposition of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$. For expositional convenience in this thesis, we assume rank K = rank G, so that the real reductive Lie group G has a compact Cartan subgroup $T \subset K \subset G$. In this case the complexified Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ is conjugate with $\mathfrak{h}_{\mathbb{C}}$.The notations for the Cartan subgroups for G and $G_{\mathbb{C}}$, together with their root systems, are specified as follows:

1. Let $\mathfrak{h}^*_{\mathbb{C}}, \mathfrak{t}^*_{\mathbb{C}}$ be the duals of the Cartan subalgebras $\mathfrak{h}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}$, respectively.

2. Denote by $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}), \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \subset \mathfrak{h}^*_{\mathbb{C}}$ the root system.

3. The Killing form on $\mathfrak{g}$ is $B(\cdot, \cdot)$. We define an element $H_\alpha \in \mathfrak{h}$ as the element such that $\alpha(H) = B(H, H_\alpha)$ for any $H \in \mathfrak{h}$.

4. The dual Cartan subalgebras $\mathfrak{h}^*_{\mathbb{C}}$ and $\mathfrak{t}^*_{\mathbb{C}}$ are equipped with an inner product $\langle \alpha, \beta \rangle = B(H_\alpha, H_\beta)$.

5. Choose a set of simple roots $\mathcal{S} = \{\alpha_1, \dots, \alpha_r\}$ where r is the rank of $\mathfrak{g}_{\mathbb{C}}$. Denote the set of positive and negative roots as $\Delta^{\pm}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$,$\Delta^{\pm}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ with respect to $\mathcal{S}$.

6. We call the set of all elements $\beta \in \mathfrak{h}^*$ satisfying $\langle \beta, \alpha_i \rangle > 0$ for all simple roots α_i the positive Weyl chamber.

7. Denote the simple reflection in the Weyl group W which correspond to a root α as w_α. Choose a representative s_α of w_α in the normalizer $N_{G_{\mathbb{C}}}(\mathfrak{h}_{\mathbb{C}})$ of the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$.

8. Let $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ be the coroots, and let $c_{\alpha\beta} = \langle \alpha, \check{\beta} \rangle$ be the Cartan integers.

9. Let $\{\varpi_1, ..., \varpi_r\} \subset \mathfrak{h}^*_{\mathbb{C}}$ be the set of fundamental weights, satisfying the property that $\langle \varpi_i, \check{\alpha}_j \rangle = \delta_{ij}$.

10. The weight lattice $X(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \subset \mathfrak{h}^*_{\mathbb{C}}$ is the lattice $\mathbb{Z}\varpi_1 + \ldots + \mathbb{Z}\varpi_r$ generated by the fundamental weights.

11. $\mathfrak{g}_{\mathbb{C}}$ has a root space decomposition $\mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ on which the restriction of $\mathrm{ad}_{\mathfrak{h}_{\mathbb{C}}}$ acts by the character α.

12. Let $\rho_{\mathbb{C}} = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \alpha$ denote the half sum of all positive roots

13. Let $\rho_K = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})} \alpha$ denote the half sum of all positive roots in $\Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

2.1.3 Chevalley Basis

The *Chevalley basis* serves as a basis normalized in a universal way for $\mathfrak{g}_{\mathbb{C}}$. There exists a Chevalley basis of $\mathfrak{g}_{\mathbb{C}}$ consisting of:

1. $H_{\alpha_1}, \ldots, H_{\alpha_r}$ for each simple root,

2. $X_\alpha \in \mathfrak{g}_\alpha$ for each root α,

Let α_i be a simple root and α be an arbitrary root, the Chevalley basis satisfies the following commutation relations:

1. $[H_{\alpha_i}, H_{\alpha_j}] = 0$, $[H_{\alpha_i}, X_\alpha] = (\alpha, \alpha_i)X_\alpha$, $[X_{\alpha_i}, X_{-\alpha_i}] = H_{\alpha_i}$

2. $[X_\alpha, X_\beta] = \epsilon_{\alpha\beta}(p_{\alpha\beta} + 1)X_{\alpha+\beta}$ if $\alpha + \beta$ is a root, and $-p_{\alpha\beta}\alpha + \beta, \ldots, q_{\alpha\beta}\alpha + \beta$ a maximal string of roots, and $\epsilon_{\alpha\beta} = \pm 1$

3. $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta$ is not a root.

On the level of complex matrix groups, for each simple root α, there exists an embedding

$$\phi_\alpha : \mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}H_\alpha \oplus \mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha} \hookrightarrow \mathfrak{g}_{\mathbb{C}}. \tag{2.1}$$

Let Φ_α be the Lie group homomorphism

$$\Phi_\alpha : SL(2, \mathbb{C}) \longrightarrow G_{\mathbb{C}}$$

whose differential $\mathsf{d}\Phi_\alpha = \phi_\alpha$. We define the following one parameter elements which generate $G_{\mathbb{C}}$:

1. $\chi_\alpha(t) = \Phi_\alpha(e^{tX_\alpha})$, $\chi_{-\alpha}(t) = \Phi_\alpha(e^{tX_{-\alpha}})$

2. $\tilde{w}_\alpha(t) = \chi_\alpha(t)\chi_{-\alpha}(-t^{-1})\chi_\alpha(t)$ with $\tilde{w}_\alpha = \tilde{w}_\alpha(1)$

3. $h_\alpha(t) = \tilde{w}_\alpha^{-1}\tilde{w}_\alpha(t)$

These generators satisfy the following relations:

1. $h_\alpha(uv) = h_\alpha(u)h_\alpha(v)$
2. $\tilde{w}_\alpha h_\alpha(t)\tilde{w}_\alpha^{-1} = h_\alpha(t^{-1})$
3. $\chi_\alpha(u+v) = \chi_\alpha(u)\chi_\alpha(v)$
4. $\tilde{w}_\alpha(t)\chi_\alpha(u)\tilde{w}_\alpha(t)^{-1} = \chi_{-\alpha}(-t^2u)$

In general, the simple reflection $\tilde{w}_\alpha$ acts on each one parameter subgroup $\chi_\beta(t)$ by adjoint action:

$$\tilde{w}_\alpha\chi_\beta(t)\tilde{w}_\alpha^{-1} = \chi_{w_\alpha\beta}(\eta_{\alpha\beta}t)$$

where $\eta_{\alpha\beta} = \pm 1$ depending on the roots α and β, they are called the *structure constants* of the group $G_\mathbb{C}$.

2.1.4 Universal Enveloping Algebra $U(\mathfrak{g})$

We denote by $Z(\mathfrak{g}_\mathbb{C})$ the center of the *universal enveloping algebra* $U(\mathfrak{g}_\mathbb{C})$ of the complex Lie algebra $\mathfrak{g}_\mathbb{C}$. The quadratic *Casimir element* of $U(\mathfrak{g})$ is the image of the identity homomorphism on $\mathfrak{g}_\mathbb{C}$ via the composition of the following homomorphisms :

$$\mathrm{End}_\mathbb{C}\mathfrak{g}_\mathbb{C} \quad \stackrel{\phi}{\to} \quad \mathfrak{g}_\mathbb{C}\otimes\mathfrak{g}_\mathbb{C}^* \quad \stackrel{\varphi}{\to} \quad \mathfrak{g}_\mathbb{C}\otimes\mathfrak{g}_\mathbb{C} \quad \to \quad U^{\leq 2}(\mathfrak{g}_\mathbb{C}) \subset U(\mathfrak{g}_\mathbb{C})$$

In the diagram above, the first homomorphism is the natural identification between matrices and tensor product of the vector space with its dual space, and the second map acts on the second factor of the tensor product as the dual map defined by the Killing form. In fact, if we choose

1. A basis $\{X_i\}$ of $\mathfrak{g}$;
2. A dual basis $\{\widetilde{X}_i\}$, satisfying $\langle X_i, \widetilde{X}_j\rangle = \delta_{ij}$,

then the quadratic *Casimir element* $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ is can be expressed using the dual basis:

$$\Omega = \sum_{i,j}\mathrm{Tr}(\mathrm{Ad}X_i \circ \mathrm{Ad}X_j)\widetilde{X}_i\widetilde{X}_j$$

In general, a generating set of $Z(\mathfrak{g})$ consists of elements of the form:

$$z_{\pi,n} = \sum_{i_1,\ldots,i_n} \mathrm{Tr}\pi(X_{i_1}\ldots X_{i_n})\widetilde{X}_{i_1}\ldots\widetilde{X}_{i_n}$$

for each irreducible representation π of $U(\mathfrak{g})$.

Having chosen a positive direction in the root system, we define the following two ideals of $U(\mathfrak{g}_{\mathbb{C}})$:

$$\mathcal{N}^+ = \textstyle\sum_{\alpha\in\Delta^+} U(\mathfrak{g}_{\mathbb{C}})X_\alpha, \quad \mathcal{N}^- = \textstyle\sum_{\alpha\in\Delta^-} U(\mathfrak{g}_{\mathbb{C}})X_{-\alpha}$$

Applying the Poincaré-Birkhoff-Witt theorem, there is a direct sum decomposition of $U(\mathfrak{g}_{\mathbb{C}})$ as

$$U(\mathfrak{g}_{\mathbb{C}}) \cong U(\mathfrak{h}_{\mathbb{C}}) \oplus (\mathcal{N}^+ + \mathcal{N}^-)$$

Define γ' as the projection of $Z(\mathfrak{g}_{\mathbb{C}})$ into $U(\mathfrak{h}_{\mathbb{C}})$, and $\tau : U(\mathfrak{h}_{\mathbb{C}}) \longrightarrow U(\mathfrak{h}_{\mathbb{C}})$ the linear map which acts on the generators of $U(\mathfrak{h}_{\mathbb{C}})$ in $\mathfrak{h}_{\mathbb{C}}$ by:

$$\tau(H) = H - \rho(H)1$$

The *Harish-Chandra homomorphism* $\gamma = \tau \circ \gamma'$ maps the center $Z(\mathfrak{g}_{\mathbb{C}})$ of $U(\mathfrak{g}_{\mathbb{C}})$ into $U(\mathfrak{h}_{\mathbb{C}})$. Moreover, if we denote $U(\mathfrak{h}_{\mathbb{C}})^W$ as the subalgebra of Weyl group invariants in $U(\mathfrak{h}_{\mathbb{C}})$, then γ is an algebra isomorphism independent of the choice of the positive root system.

2.2 The Real Semisimple Lie Algebras

2.2.1 Real Tori

Every real algebraic torus is isomorphic to a product of copies of $\mathbb{R}^\times$, $\mathbb{S}^1$ and $\mathbb{C}^*$[Cas06]. Therefore, for any real form G of a complex algebraic group $G_{\mathbb{C}}$ and a Cartan subgroup $H \subset G$, there is a factorization $H = H_1H_2\ldots H_k$ in which each factor H_i is isomorphic to either $\mathbb{R}^\times$, $\mathbb{S}^1$ or $\mathbb{C}^*$. The corresponding complex Cartan subgroup $H_{\mathbb{C}} \subset G_{\mathbb{C}}$ thus has a factorization $(H_1)_{\mathbb{C}}(H_2)_{\mathbb{C}}\ldots(H_k)_{\mathbb{C}}$, where each $(H_i)_{\mathbb{C}}$ is the complexification of H_i in $G_{\mathbb{C}}$. For each factor H_i of the real Cartan subgroup, we can consider the values of any algebraic character $\chi : H_{\mathbb{C}} \longrightarrow \mathbb{C}^*$ and its restriction $\chi|_{H_i}$ of χ to each factor H_i:

1. If $H_i \cong \mathbb{R}^{\times}$, then $(H_i)_{\mathbb{C}} \cong \mathbb{C}^*$ and $\chi|_{H_i}$ takes real values.

2. If $H_i \cong \mathbb{S}^1$, then $(H_i)_{\mathbb{C}} \cong \mathbb{C}^*$ and $\chi|_{H_i}$ takes values in the unit circle.

3. If $H_i \cong \mathbb{C}^*$, then $(H_i)_{\mathbb{C}} \cong \mathbb{C}^* \times \mathbb{C}^*$ and $\chi|_{H_i}$ takes complex values. Moreover, there exists another algebraic character χ' which satisfies: $\chi'(t) = \chi(\bar{t})$ for any $t \in H_i$.

2.2.2 Real and Imaginary Roots

We define the explicit conjugations between the aforementioned Lie algebras $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$. The theory is based on the correspondence between:

1. The antiholomorphic involution σ of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$;

2. The Cartan involution θ of a real semisimple Lie group $\mathfrak{g}$;

3. A maximally noncompact Cartan subalgebra of a real semisimple Lie group $\mathfrak{g}$.

The action by the Cartan involution θ on $\mathfrak{g}$ can be extended naturally to $\mathfrak{g}_{\mathbb{C}}$. Consider a θ-stable Cartan subalgebra $\mathfrak{s} \subset \mathfrak{g}$ and its complexification $\mathfrak{s}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$. The Cartan subalgebra $\mathfrak{s}$ has the decomposition $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{a}$, where $\mathfrak{s}_0 = \mathfrak{k} \cap \mathfrak{s}, \mathfrak{a} = \mathfrak{p} \cap \mathfrak{s}$. The root system of the complexified Lie algebra $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}})$ can be decomposed into the union of the following subsets [Kna13]:

1. Real roots Δ_r with $\theta\alpha = -\alpha$. In this case, α takes real values on $\mathfrak{s}$, and vanishes on $\mathfrak{a}$.

2. Imaginary roots Δ_i with $\theta\alpha = \alpha$. In this case, α takes imaginary values on $\mathfrak{s} = \mathfrak{k} \cap \mathfrak{s}$, and vanishes on $\mathfrak{s}_0 = \mathfrak{k} \cap \mathfrak{s}$. An imaginary root is called *compact* or *noncompact* depending on whether $\mathfrak{g}_\alpha \subset \mathfrak{k}$ or $\mathfrak{g}_\alpha \subset \mathfrak{p}$. We can hence decompose Δ_i further into the union of compact roots Δ_c and noncompact roots Δ_{nc}.

3. Complex roots Δ_c with $\theta\alpha \neq \pm\alpha$. In this case, α takes complex values on $\mathfrak{s}$.

Remark 2.1 *[Mat79] If α is a complex root, then the set $\{\alpha, \theta\alpha\}$ serves as the simple roots of a root system of type $A_1 \times A_1$ or A_2. In fact, if we assume that each element* *in the string $\alpha + k\theta\alpha$ formed by α and $\theta\alpha$ is a root, and let the angle between α and $\theta\alpha$ be ϕ, the number:*

$$2\frac{\langle \alpha + k\theta\alpha, \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2 + 2k\cos\phi$$

must be an integer, and $\cos\phi$ must be either $\pm\frac{1}{2}$ or 0. Therefore,

1. *When $\cos\phi = \pm\frac{1}{2}$, $\{\pm\alpha, \pm\theta\alpha, \pm\alpha \pm \theta\alpha\}$ form a root system of type A_2,*

2. *When $\cos\phi = 0$, $\{\pm\alpha, \pm\theta\alpha\}$ form a root system of type $A_1 \times A_1$.*

We can now define the *Cayley transforms*, which serve as the explicit conjugations between different Cartan subalgebras of $\mathfrak{g}_{\mathbb{C}}$.

Example 2.1 *Consider the Cartan subalgebras*

$$\mathfrak{h}_{\mathbb{C}} = \mathbb{C}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \mathfrak{t}_{\mathbb{C}} = \mathbb{C}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$, the Cartan involution which defines the maximal compact subalgebra $\mathfrak{so}(2, \mathbb{R}) \subset \mathfrak{g}$ is $\theta(X) = -X^t$.

1. *The real Cartan subalgebra $\mathfrak{h} = \mathbb{R}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is a split Cartan subalgebra in $\mathfrak{g}$. Let $\alpha \in \mathfrak{h}^*$ be the root satisfying $\alpha\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = 2$, which takes real values on $\mathfrak{h}$, and satisfies $\theta\alpha = -\alpha$. The root vectors with respect to $\mathfrak{h}_{\mathbb{C}}$ are:*

   $$E_\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), E_{-\alpha} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

   Define the Cayley transform

   $$\mathsf{d}_\alpha = \mathrm{Ad}(\exp\frac{\pi \mathsf{i}}{4}(-E_\alpha - E_{-\alpha})) = \mathrm{Ad}\left(\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -\mathsf{i} \\ -\mathsf{i} & 1 \end{smallmatrix}\right)\right)$$

   The Cayley transform d_α carries $\mathfrak{h}_{\mathbb{C}}$ to $\mathfrak{t}_{\mathbb{C}}$:

   $$\mathsf{d}_\alpha\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \mathsf{i}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

2. *The real Cartan subalgebra $\mathfrak{t} = \mathbb{R}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the compact Cartan subalgebra in $\mathfrak{g}$. The imaginary root α sends the element $\left(\begin{smallmatrix} 0 & \mathsf{i} \\ -\mathsf{i} & 0 \end{smallmatrix}\right)$ to 2, and satisfies $\theta\alpha = \alpha$. The root vectors with respect to $\mathfrak{t}_{\mathbb{C}}$ are:*

   $$e_\alpha = \frac{1}{2}\left(\begin{smallmatrix} 1 & -\mathsf{i} \\ -\mathsf{i} & -1 \end{smallmatrix}\right), \bar{e}_\alpha = \frac{1}{2}\left(\begin{smallmatrix} 1 & \mathsf{i} \\ \mathsf{i} & -1 \end{smallmatrix}\right)$$

so the root α is noncompact. We define the Cayley transform

$$\mathsf{c}_\alpha = \mathrm{Ad}(\exp\frac{\pi}{4}(\bar{e}_\alpha - e_\alpha)) = \mathrm{Ad}\left(\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{smallmatrix}\right)\right)$$

The Cayley transform c_α carries $\mathfrak{t}_\mathbb{C}$ to $\mathfrak{h}_\mathbb{C}$:

$$\mathsf{c}_\alpha \mathrm{i}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

We can see that in this case, c_α and d_α are inverse to each other.

For a general complex Lie algebra $\mathfrak{g}_\mathbb{C}$ with real form $\mathfrak{g}$, let $\mathfrak{s} \subset \mathfrak{g}$ be a real Cartan subalgebra. For any root $\alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{s}_\mathbb{C})$, there is an embedding

$$\phi_\alpha : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathfrak{g}_\mathbb{C}$$

of $\mathfrak{sl}(2, \mathbb{C})$ into the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$. If α is a noncompact imaginary root or a real root, we can use this embedding to define the *Cayley transform* or the *inverse Cayley transform* as follows:

1. If α is imaginary noncompact, let $e_\alpha, \bar{e}_\alpha$ be nonzero root vectors in $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ respectively. We can then normalize e_α such that $[e_\alpha, \bar{e}_\alpha] = \frac{2}{|\alpha|^2} H_\alpha$. In this case, the images of the generators of $\mathfrak{sl}(2, \mathbb{C})$ are:

$$\phi_\alpha \left(\begin{smallmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{smallmatrix}\right) = \frac{2}{|\alpha|^2} H_\alpha$$
$$\phi_\alpha \begin{pmatrix} \frac{1}{2} & -\frac{\mathrm{i}}{2} \\ -\frac{\mathrm{i}}{2} & -\frac{1}{2} \end{pmatrix} = e_\alpha$$
$$\phi_\alpha \begin{pmatrix} \frac{1}{2} & \frac{\mathrm{i}}{2} \\ \frac{\mathrm{i}}{2} & -\frac{1}{2} \end{pmatrix} = \bar{e}_\alpha$$

   The Cayley transform is defined as

$$\mathsf{c}_\alpha = \mathrm{Ad}\left(\phi_\alpha\left(\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{smallmatrix}\right)\right)\right) = \mathrm{Ad}\left(\exp\frac{\pi}{4}(\bar{e}_\alpha - e_\alpha)\right).$$

2. If α is real, let $E_\alpha, E_{-\alpha} = \theta E_\alpha$ be nonzero root vectors in $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ respectively. We can then normalize E_α such that $[E_\alpha, E_{-\alpha}] = \frac{2}{|\alpha|^2} H_\alpha$. In this case, the image of the generators of $\mathfrak{sl}(2, \mathbb{C})$ is:

$$\phi_\alpha \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = \frac{2}{|\alpha|^2} H_\alpha$$
$$\phi_\alpha \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = E_\alpha$$
$$\phi_\alpha \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = E_{-\alpha}$$

The inverse Cayley transform is defined as

$$\mathsf{d}_\alpha = \mathrm{Ad}\left(\phi_\alpha\left(\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -\mathsf{i} \\ -\mathsf{i} & 1 \end{smallmatrix}\right)\right)\right) = \mathrm{Ad}\left(\exp\frac{\pi\mathsf{i}}{4}(-E_{-\alpha} - E_\alpha)\right).$$

For an arbitrary Cartan subalgebra $\mathfrak{l}$ of the real Lie algebra $\mathfrak{g}$, denote the compact dimension and noncompact dimension of $\mathfrak{l}$ as $d_c = \dim \mathfrak{l} \cap \mathfrak{k}, d_{nc} = \dim \mathfrak{l} \cap \mathfrak{p}$. Then:

1. The Cayley transform c_α transforms a Cartan subalgebra with compact and noncompact dimension (d_c, d_{nc}) to a Cartan subalgebra with compact and noncompact dimension $(d_c - 1, d_{nc} + 1)$;

2. The Cayley transform d_α transforms a Cartan subalgebra with compact and noncompact dimension (d_c, d_{nc}) to a Cartan subalgebra with compact and noncompact dimension $(d_c + 1, d_{nc} - 1)$

Remark 2.2 *It is worth mentioning the embeddings of $\mathfrak{sl}(2, \mathbb{C})$ into $\mathfrak{g}_\mathbb{C}$ for a compact imaginary root. We illustrate this in the following example:*

1. *Consider a real Lie algebra $\mathfrak{g} = \mathfrak{su}(2, 1)$ and its complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(3, \mathbb{C})$. The Cartan involution corresponding to this real form is $\theta : X \mapsto -\bar{X}^t$. Take the real Cartan subalgebra:*

$$\mathfrak{t} = \mathbb{R}\mathsf{i}H_{\alpha_1} \oplus \mathbb{R}\mathsf{i}H_{\alpha_2}$$

where $H_{\alpha_1} = \mathrm{diag}(1, -1, 0)$ and $H_{\alpha_2} = \mathrm{diag}(0, 1, -1)$. The root α_1 sends $\mathsf{i}H_{\alpha_1}$ to $2\mathsf{i}$ and $\mathsf{i}H_{\alpha_2}$ to $-\mathsf{i}$, and satisfies $\theta\alpha = \alpha$. The root vectors with respect to $\mathfrak{t}_\mathbb{C}$ are:

$$e_\alpha = \left(\begin{smallmatrix} 0 & \mathsf{i} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right), e_{-\alpha} = \left(\begin{smallmatrix} 0 & 0 & 0 \\ \mathsf{i} & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$$

and $\frac{e_\alpha \pm e_{-\alpha}}{2} \in \mathfrak{k}_\mathbb{C}$. Therefore, α is a compact imaginary root, and the embedding $\phi_\alpha : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathfrak{sl}(3, \mathbb{C})$ corresponds to an embedding $\Phi_\alpha : SU(2) \longrightarrow SU(2, 1)$.

2.2.3 Compact Roots and Noncompact Roots

Recall that we are interested in the real reductive groups G such that their maximal compact subgroup K has the same rank as G. The roots $\Delta(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ can thus be identified as a subset of $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. In this situation, since all roots are imaginary, we will simply refer to the set of imaginary compact roots and imaginary noncompact roots introduced in 2.2.2 as the *compact roots* $\Delta_c(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and the *noncompact roots* $\Delta_{nc}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, respectively. The Lie algebra $\mathfrak{k}_\mathbb{C}$ will then have a decomposition:

$$\mathfrak{k}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\gamma \in \Delta_c(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \mathfrak{g}_\gamma$$

Similarly, the maximal compact subgroup K acts on $\mathfrak{p}_\mathbb{C}$ via the restriction of Ad_G to K, so $\mathfrak{p}_\mathbb{C}$ can be decomposed into weight spaces:

$$\mathfrak{p}_\mathbb{C} = \bigoplus_{\gamma \in \Delta_{nc}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \mathfrak{g}_\gamma$$

Furthermore, define the subspaces $\mathfrak{p}_\mathbb{C}^+$ and $\mathfrak{p}_\mathbb{C}^-$ as:

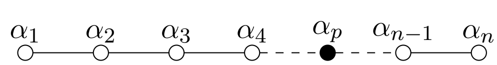
$$\mathfrak{p}_\mathbb{C}^+ = \bigoplus_{\gamma \in \Delta_{nc}^+(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \mathfrak{g}_\gamma$$

$$\mathfrak{p}_\mathbb{C}^- = \bigoplus_{\gamma \in \Delta_{nc}^-(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})} \mathfrak{g}_\gamma$$

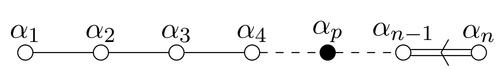
which decomposes $\mathfrak{p}_\mathbb{C}$ as the direct sum $\mathfrak{p}_\mathbb{C}^+ \oplus \mathfrak{p}_\mathbb{C}^-$. The decomposition $\Delta = \Delta_c \cup \Delta_{nc}$ can be recorded in the *Vogan diagram*, which is a decoration of the Dynkin diagram in which the *noncompact* roots are colored black.

Example 2.2 *1. If G_0 is a compact real form of $G_\mathbb{C}$, then none of the dots in the Vogan diagram is colored black.*

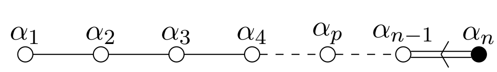
2. Assume $p + q = n + 1$, the real form $\mathfrak{su}(p, q)$ of $\mathfrak{sl}(n + 1, \mathbb{C})$ corresponds to the Vogan diagram whose p-th dot from the left colored black.



3. Assume $p + q = n$, the real form $\mathfrak{sp}(p, q)$ of $\mathfrak{sp}(2n, \mathbb{C})$ corresponds to the Vogan diagram with p-th dot from the left colored black.



4. The real form $\mathfrak{sp}(2n, \mathbb{R})$ corresponds to the Vogan diagram with the last dot in the Dynkin diagram of C_n colored black.



2.2.4 Minimal Parabolic Subgroup and Restricted Roots

For a real reductive Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, consider the following data:

1. Let $\mathfrak{a}$ be a maximal abelian Lie subalgebra in $\mathfrak{p}$;

2. Let $A_0 = \exp \mathfrak{a}$ be the connected analytic subgroup of G with Lie algebra $\mathfrak{a}$;

3. Let $M \subset K$ be the centralizer of $\mathfrak{a}$ in K;

4. Following the notations from Section VI.4 of [Kna13], if we restrict the adjoint action $\mathrm{ad}_{\mathfrak{g}}$ to $\mathfrak{a}$ there is a direct sum decomposition of $\mathfrak{g}$ into restricted root subspaces
$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha,$$
where the set of *restricted roots* $\Sigma(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^*$ is the set of nonzero simultaneous eigenvalues of the adjoint action of elements in $\mathfrak{a}$ on $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ is the simultaneous eigenspace corresponding to a restricted root $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a})$. We use the notation Σ^+ and Σ^- for the set of positive and negative restricted roots, respectively. Moreover, for a reduced restricted root α, let $\mathfrak{v}_\alpha$ be the direct sum of all the $\mathfrak{g}_{m\alpha}$'s where m is a positive integer, and for any reduced root α, let
$$\rho^{(\alpha)} = \frac{1}{2} \sum_{\substack{\nu \in \Sigma^+(\mathfrak{g},\mathfrak{a}) \\ \nu = m\alpha \\ m \text{ positive integer}}} \nu.$$

5. Take a Cartan subalgebra $\mathfrak{s}_0 \subset \mathfrak{m}$, define the real Cartan subalgebra $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{a}$ of $\mathfrak{g}$. This Cartan subalgebra is a *maximally noncompact* Cartan subalgebra of $\mathfrak{g}$.

6. The nilpotent subalgebra $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha$ defines an analytic subgroup $N^+ \subset G$.

7. The half sum of positive restricted roots $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+(\mathfrak{g},\mathfrak{a})} \alpha$;

8. The subalgebra $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ is called a *minimal parabolic* subalgebra of G. The subgroup $Q = MA_0N^+$ is called a *minimal parabolic subgroup* of G;

9. There is an *Iwasawa* decomposition for G as the product of subgroups $G = KMA_0N^+$, with a corresponding decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ on the Lie algebra;

Remark 2.3 *It is important to note that for the minimal parabolic subalgebra* $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$

$$\mathfrak{m}_{\mathbb{C}} = \mathfrak{s}_{0\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_{\alpha}.$$

The Cartan subalgebra $\mathfrak{s}_{\mathbb{C}} = \mathfrak{s}_{0\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$ *is a maximally noncompact Cartan subalgebra of* $\mathfrak{g}_{\mathbb{C}}$*, and all the imaginary roots are noncompact. Moreover, for any real roots* α*,* $\alpha|_{(\mathfrak{s}_0)_{\mathbb{C}}} = 0$*, and for any imaginary roots* α*,* $\alpha|_{\mathfrak{a}_{\mathbb{C}}}$*=0.*

The reductive group M can be factored as a commuting product $M = FM_0$, where F is a finite subgroup of M in which every nonidentity element has order 2, and M_0 is a connected compact reductive group. In fact, for any real root α of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}})$, consider the homomorphism $\phi_\alpha : \mathfrak{sl}(2, \mathbb{R}) \longrightarrow \mathfrak{g}$, which exponentiates to a homomorphism $\Phi_\alpha : \mathrm{SL}(2, \mathbb{C}) \longrightarrow G_{\mathbb{C}}$. Letting γ_α be the image of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then

$$\gamma_\alpha = \exp \frac{2\pi i H_\alpha}{|\alpha|^2} = \exp \pi(X_\alpha - \theta X_\alpha)$$

The discrete subgroup $F \subset M$ is generated by γ_α for all real roots α [Kna13].

Chapter 3

Representation of Compact Groups

As was specified in the first chapter, let $K \subset G$ be a maximal compact subgroup of G. Recall that we assume rank K = rank $G = r$ and there is a compact Cartan subalgebra $\mathfrak{t}$ for both $\mathfrak{k}$ and $\mathfrak{g}$. As before, we denote the complexification of the Lie algebra $\mathfrak{k}$ as $\mathfrak{k}_{\mathbb{C}}$ and the root system as $\Delta(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, with a choice of simple roots $\mathcal{S} = \{\alpha_1, \ldots, \alpha_r\}$. We introduce the symbol $\hat{K}$ as the set of all irreducible representations of K up to equivalence. By indexing an irreducible representation (τ, V_τ) by its highest weight μ, we can realize the set $\hat{K}$ as a subset of the weight lattice

$$X(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\mu \in \mathfrak{t}_{\mathbb{C}}^* | \langle \mu, \check{\alpha}_i \rangle \in \mathbb{Z} \text{ for all } \alpha_i \in \mathcal{S}\}.$$

In the rest of this thesis, we will focus on the real reductive groups G whose maximal compact subgroup K is a product of copies of $U(2)$'s and $SU(2)$'s. For that sake, in this chapter, we will discuss the representation theory of $U(2)$ and $SU(2)$ in detail. In this whole chapter, we let $K = U(2)$ and $\mathfrak{k} = \mathfrak{u}(2)$.

3.1 Structure of $SU(2)$ and $U(2)$

The *Pauli matrices* are the generators of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$:

$$\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_2 = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right), \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

They satisfy the commutation relations

$$[\sigma_0, \sigma_i] = 0 \tag{3.1}$$

$$[\sigma_i, \sigma_j] = \sum_{k=1}^{3} 2\mathrm{i}\epsilon_{ijk}\sigma_k \text{ if } i, j \neq 0, \tag{3.2}$$

where the *Levi-Civita symbol* ϵ_{ijk} takes value 1 if (ijk) is an even permutation of (123), -1 if (ijk) is an odd permutation of (123), and 0 if two or more elements in $\{i,j,k\}$ are equal. We multiply each Pauli matrix by $\mathsf{i}/2$, and let $\gamma_i = \frac{\mathsf{i}}{2}\sigma_i$, then γ_i are generators of the real Lie algebra $\mathfrak{u}(2) \subset \mathfrak{sl}(2,\mathbb{C})$:

$$\gamma_0 = \begin{pmatrix} \frac{\mathsf{i}}{2} & 0 \\ 0 & \frac{\mathsf{i}}{2} \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & \frac{\mathsf{i}}{2} \\ \frac{\mathsf{i}}{2} & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \frac{\mathsf{i}}{2} & 0 \\ 0 & -\frac{\mathsf{i}}{2} \end{pmatrix}.$$

The Pauli matrices γ_0 and γ_3 generate a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{u}(2)$, and the complexified Lie algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{u}(2) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{gl}(2,\mathbb{C})$ has the following decomposition

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}^{-} \oplus \mathfrak{k}_{\mathbb{C}}^{+} \oplus \mathfrak{t}_{\mathbb{C}},$$

where the positive and negative root spaces are generated by $\gamma_1 \mp \mathsf{i}\gamma_2$

$$\mathfrak{k}_{\mathbb{C}}^{-} = \mathbb{C}(\gamma_1 + \mathsf{i}\gamma_2), \mathfrak{k}_{\mathbb{C}}^{+} = \mathbb{C}(\gamma_1 - \mathsf{i}\gamma_2).$$

The commutator subalgebra $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$ is semisimple and is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, and $\mathfrak{k}_{\mathbb{C}} = \mathbb{C}\gamma_0 \oplus [\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$. The Casimir element Ω_K in the universal enveloping algebra of $U([\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}])$ of the commutator subalgebra $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$ takes the form:

$$\Omega_K = -2(\gamma_1^2 + \gamma_2^2 + \gamma_3^2).$$

The compact group $SU(2)$ is defined by

$$SU(2) = \{\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} | \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1\} \cong \mathbb{S}^3.$$

We consider the quaternions

$$\mathbb{H} = \{q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} | q_i \in \mathbb{R}, \mathsf{i}^2 = \mathsf{j}^2 = \mathsf{k}^2 = \mathsf{ijk} = -1\},$$

then there exists a group isomorphism between $SU(2)$ and the unit quaternions $\mathbb{H}^*$:

$$\begin{array}{ccc} SU(2) & \longrightarrow & \mathbb{H}^* = \{q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} | q_i \in \mathbb{R} \text{ and } \sum q_i^2 = 1\} \\ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} & \mapsto & \mathrm{Re}\alpha + (\mathrm{Im}\alpha)\mathsf{i} - (\mathrm{Re}\beta)\mathsf{j} + (\mathrm{Im}\beta)\mathsf{k} \end{array}.$$

To obtain a rotational coordinate for $SU(2)$, we need to introduce the two variable arctan function with range $(-\pi, \pi]$:

$$\arctan(x, y) = \mathrm{Arg}(x + \mathsf{i}y)$$

where Arg is the principal value of the argument function, taking value in the range $(-\pi, \pi]$. The multiplication by $\mathbb{S}^1 \cong \{e^{-\phi\gamma_3} | -\pi < \psi \leq \pi\}$ on the right defines a *Hopf fibration* of the group $SU(2)$:

$$\begin{array}{ccccccc} \mathbb{S}^1 & \longrightarrow & SU(2) & \longrightarrow & \mathbb{CP}^1 & \cong & \mathbb{S}^2 \\ & & \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} & & [\alpha : \beta] & & z = \beta/\alpha \end{array},$$

where we take

$$\phi = \arctan(-\mathrm{Im}(\alpha)\mathrm{Im}(\beta) + \mathrm{Re}(\alpha)\mathrm{Re}(\beta), -\mathrm{Im}(\beta)\mathrm{Re}(\alpha) - \mathrm{Im}(\alpha)\mathrm{Re}(\beta)). \tag{3.3}$$

We can use the zyz Euler angles (ψ, θ, ϕ) to parametrize a generic element of $SU(2)$. The ranges of these angles are

$$\phi \in (-\pi, \pi], \quad \theta \in [0, \pi], \quad \psi \in (-\pi, 3\pi].$$

If we choose the branch for the arccos function such that its value lies in the range $[0, \pi]$, and let the other two angles

$$\theta = \arccos(1 - 2|\beta|^2), \tag{3.4}$$

$$\psi = \arctan\left(\beta\bar{\alpha} + \alpha\bar{\beta}, \mathrm{Re}(2\alpha\mathrm{Im}(\beta) - 2\beta\mathrm{Im}(\alpha))\right) + \pi(1 - \epsilon(\alpha, \beta)), \tag{3.5}$$

where

$$\epsilon(\alpha, \beta) = \exp\left(-\frac{1}{2}\mathsf{i}\left(\mathrm{Arg}(\bar{\alpha}\beta) - 2\mathrm{Arg}(\bar{\alpha}) + \mathrm{Arg}\left(\overline{\alpha\beta}\right)\right)\right),$$

then $z = \beta/\alpha = e^{\mathsf{i}\psi} \tan\frac{\theta}{2}$, and the matrix

$$\mathcal{U}(\psi, \theta, \phi) = e^{-\psi\gamma_3} e^{-\theta\gamma_2} e^{-\phi\gamma_3} = \begin{pmatrix} e^{-\frac{\mathsf{i}}{2}(\phi+\psi)} \cos\frac{\theta}{2} & -e^{\frac{\mathsf{i}}{2}(\phi-\psi)} \sin\frac{\theta}{2} \\ e^{\frac{\mathsf{i}}{2}(-\phi+\psi)} \sin\frac{\theta}{2} & e^{\frac{\mathsf{i}}{2}(\phi+\psi)} \cos\frac{\theta}{2} \end{pmatrix}, \tag{3.6}$$

with ψ, θ, ϕ given by the formulas above, parametrizes a generic element $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ of the group $SU(2)$ with the Euler angles (ψ, θ, ϕ). If we write $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ as a unit quaternion, then the matrix with entries in α and β corresponds to the unit quaternion $q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$, where

$$q_0 = \cos\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\psi+\phi}{2}\right), \quad q_1 = -\cos\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{\psi+\phi}{2}\right),$$
$$q_2 = -\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\phi-\psi}{2}\right), \quad q_3 = \sin\left(\tfrac{\theta}{2}\right)\sin\left(\tfrac{-\phi+\psi}{2}\right).$$

The center of the Lie group $U(2)$ is isomorphic to $U(1)$, which can be parametrized by the exponentiation $e^{-\zeta\gamma_0}$ of the matrix γ_0. If we multiply $e^{-\zeta\gamma_0}$ to the matrix $\mathcal{U}(\psi,\theta,\phi)$, we obtain a generic element of the group $U(2)$:

$$e^{-\zeta\gamma_0}\mathcal{U}(\psi,\theta,\phi) = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}(-\zeta-\phi-\psi)}\cos\frac{\theta}{2} & -e^{\frac{\mathrm{i}}{2}(-\zeta+\phi-\psi)}\sin\frac{\theta}{2} \\ e^{\frac{\mathrm{i}}{2}(-\zeta-\phi+\psi)}\sin\frac{\theta}{2} & e^{\frac{\mathrm{i}}{2}(-\zeta+\phi+\psi)}\cos\frac{\theta}{2} \end{pmatrix}. \tag{3.7}$$

3.2 Realization of Irreducible Representations

The standard representation (π, W) of $U(2)$ can be realized in the space of linear polynomials in two variables. To be precise, let z be the column vector of two variables $(z_1, z_2)^t$, with π acting on any linear polynomial $f(z)$ via:

$$\pi(g)f(z) = f(g^{-1}z).$$

For $k \in \mathbb{Z}$, denote by $\det^k$ the one dimensional representation on which $g \in U(2)$ acts by scalar multiplication with $(\det g)^k$. Letting $j \in \frac{1}{2}\mathbb{N}$ and $n \in \frac{1}{2}\mathbb{Z}$ with $j+n \in \mathbb{Z}$, an arbitrary irreducible representation $\pi_{j,n}$ of $U(2)$ can be realized on the space $\mathrm{Sym}^{2j}W \otimes \det^{(j+n)}$, which is isomorphic to $\mathrm{Sym}^{2j}W$ as a vector space. $U(2)$ acts by right regular action on any degree $2j$ homogeneous polynomial $f \in \mathrm{Sym}^{2j}(W)\otimes\det^{(j+n)}$ in 2 variables z_1, z_2:

$$\pi_{j,n}(g)f(z) = (\det g)^{j+n}f(g^{-1}z).$$

Let $m \in \frac{1}{2}\mathbb{Z}$ such that $-j \le m \le j$ and $j \pm m$ are integers, the weight basis $\{v^j_m\}_{-j\le m\le j}$ for $\mathrm{Sym}^{2j}W \otimes \det^{(j+n)}$ is defined as:

$$v^j_m = \frac{z_1^{j-m}z_2^{j+m}}{\sqrt{(j-m)!(j+m)!}}.$$

The Lie algebra $\mathfrak{u}(2)$ acts on the weight basis $\{v^j_m\}_{-j\le m\le j}$ of $\mathrm{Sym}^{2j}W \otimes \det^{(j+n)}$ as linear operators:

$$\gamma_0 v^j_m = \mathrm{i}n v^j_m \tag{3.8}$$

$$(\gamma_1 \pm \mathrm{i}\gamma_2)v^j_m = -\mathrm{i}\sqrt{(j\mp m)(j\pm m+1)}\,v^j_{m\pm1} \tag{3.9}$$

$$\gamma_3 v^j_m = \mathrm{i}m v^j_m. \tag{3.10}$$

We define a hermitian inner product $\langle v^j_{m_1}, v^j_{m_2}\rangle = \delta_{m_1,m_2}$ on $\mathrm{Sym}^{2j}W \otimes \det^{(j+n)}$, and we require the inner product to be linear in the first argument, and conjugate linear in the second argument. We notice that γ_0 and γ_3 acts on the weight vectors v^j_m of $\mathrm{Sym}^{2j}W \otimes \det^{(j+n)}$ by multiplication of a purely imaginary number $\mathrm{i}m$. Moreover, from (3.9), we can see that the matrices of γ_1 and γ_2 actions are unitary under this hermitian inner product:

$$\langle \gamma_1 v^j_{m_1}, v^j_{m_2} \rangle = \frac{1}{2\mathrm{i}} \left(\sqrt{(j-m_1)(j+m_1+1)}\delta_{m_1+1,m_2} + \sqrt{(j-m_2)(j+m_2+1)}\delta_{m_1,m_2+1} \right) \tag{3.11}$$

$$\langle \gamma_2 v^j_{m_1}, v^j_{m_2} \rangle = \frac{1}{2} \left(-\sqrt{(j-m_1)(j+m_1+1)}\delta_{m_1+1,m_2} + \sqrt{(j-m_2)(j+m_2+1)}\delta_{m_1,m_2+1} \right). \tag{3.12}$$

Therefore, the action of the Lie algebra elements $\gamma_i \in \mathfrak{u}(2)$ on $\mathrm{Sym}^{2j}W \otimes \det^{(j+n)}$ is unitary. Using this inner product, we can define the *Wigner D-functions* $W^{(j,n)}_{m_1,m_2}(\zeta,\psi,\theta,\phi)$ as the matrix coefficients of the irreducible representation $\pi_{j,n}$:

$$\begin{aligned} W^{(j,n)}_{m_1,m_2}(\zeta,\psi,\theta,\phi) &= \langle v^j_{m_1}, e^{-\gamma_0\zeta}\mathcal{U}(\psi,\theta,\phi) v^j_{m_2} \rangle \\ &= c^j_{m_1} c^j_{m_2} e^{\mathrm{i}n\zeta} e^{\mathrm{i}(m_1\psi + m_2\phi)} d^{(j,n)}_{m_1,m_2}(\theta), \end{aligned} \tag{3.13}$$

where $c^j_m = \sqrt{(j+m)!(j-m)!}$ is a normalization factor, and the function $d^{(j,n)}_{m_1,m_2}(\theta)$ is given by the trigonometric polynomial

$$\begin{aligned} d^{(j,n)}_{m_1,m_2}(\theta) = \sum_{p=\max(0,m_1-m_2)}^{\min(j-m_2,j+m_1)} & \frac{(-1)^{m_2-m_1+p}}{(j+m_1-p)!p!(m_2-m_1+p)!(j-m_2-p)!} \\ & \sin^{m_2-m_1+2p}\left(\frac{\theta}{2}\right) \cos^{2j+m_1-m_2-2p}\left(\frac{\theta}{2}\right). \end{aligned} \tag{3.14}$$

The Wigner D-function $W^{(j,n)}_{m_1,m_2}$ satisfies the following properties:

1. **Jacobi polynomials and** $d^{(j,n)}_{m_1,m_2}(\theta)$

   The sum $d^{(j,n)}_{m_1,m_2}(\theta)$ has an expression in terms of the hypergeometric function

${}_2F_1$:

$$d^{(j,n)}_{m_1,m_2}(\theta) =$$
$$\begin{cases} \frac{\sin^{m_1-m_2}\left(\frac{\theta}{2}\right)\cos^{2j-m_1+m_2}\left(\frac{\theta}{2}\right)}{(j-m_1)!(m_1-m_2)!(j+m_2)!}\,{}_2F_1\left(\begin{matrix}-j+m_1,-j-m_2\\1+m_1-m_2\end{matrix};-\tan^2\left(\frac{\theta}{2}\right)\right) & m_1 > m_2 \\ \frac{(-1)^{-m_1+m_2}\sin^{-m_1+m_2}\left(\frac{\theta}{2}\right)\cos^{2j+m_1-m_2}\left(\frac{\theta}{2}\right)}{(j+m_1)!(-m_1+m_2)!(j-m_2)!}\,{}_2F_1\left(\begin{matrix}-j-m_1,-j+m_2\\1-m_1+m_2\end{matrix};-\tan^2\left(\frac{\theta}{2}\right)\right) & m_1 \le m_2 \end{cases} \tag{3.15}$$

For $n \geq 0$ and for $\alpha, \beta \in \mathbb{R}$, the *Jacobi polynomials* $P_n^{(\alpha,\beta)}(z)$ are a class of orthogonal polynomials defined in [AS67] as

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+\beta+n+1)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}\left(\frac{z-1}{2}\right)^m. \tag{3.16}$$

In [Res08], the Jacobi polynomial is also defined as

$$P_n^{\alpha,\beta}(x) = \binom{n+\alpha}{n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left(\begin{matrix}-n,-n-\beta,\alpha+1\end{matrix};\frac{x-1}{x+1}\right). \tag{3.17}$$

These two definitions of Jacobi polynomials are equivalent. By the definition of Jacobi polynomials and the transformation rules of hypergeometric functions, the function $d^{(j,n)}_{m_1,m_2}(\theta)$ can also be expressed in terms of Jacobi polynomials:

$$d^{(j,n)}_{m_1,m_2}(\theta) = \frac{\left(\sin\frac{\theta}{2}\right)^{m_1-m_2}\left(\cos\frac{\theta}{2}\right)^{m_1+m_2}}{(j+m_2)!(j-m_2)!}P_{j-m_1}^{(m_1-m_2,m_1+m_2)}(\cos\theta). \tag{3.18}$$

2. **Multiplicativity** If an element k of $U(2)$ is expressed in terms of Euler angles as $k = e^{-\gamma_0\zeta}\mathcal{U}(\psi,\theta,\phi)$, we can replace the notation $W^{(j,n)}_{m_1,m_2}(\zeta,\psi,\theta,\phi)$ by $W^{(j,n)}_{m_1,m_2}(k)$. For any $k_1, k_2 \in K$, since $W^{(j,n)}_{m_1,m_2}$ are the matrix coefficients of the representation $\pi_{j,n}$, we can use the multiplicative property of matrix coefficients to write $W^{(j,n)}_{m_1,m_2}(k_1k_2)$ as a sum:

$$\begin{aligned} W^{(j,n)}_{m_1,m_2}(k_1k_2) &= \langle v^j_{m_1}, k_1k_2v^j_{m_2}\rangle \\ &= \sum_{\substack{-j\le m_3\le j\\ j+m_3\in\mathbb{Z}}}\langle v^j_{m_1}, k_1v^j_{m_3}\rangle\langle v^j_{m_3}, k_2v^j_{m_2}\rangle \\ &= \sum_{\substack{-j\le m_3\le j\\ j+m_3\in\mathbb{Z}}} W^{(j,n)}_{m_1,m_3}(k_1)W^{(j,n)}_{m_3,m_2}(k_2). \end{aligned}$$

3. **Inverse Matrix**

Recall that the basis $\{v^j_m\}$ and the inner product on $\pi_{j,n}$ are chosen so that the action of $U(2)$ is unitary. Therefore, the Wigner D-functions satisfies the unitarity property:

$$\sum_{\substack{-j\leq m_3\leq j\\ j+m_3\in\mathbb{Z}}} W^{(j,n)}_{m_1,m_3}(k)\overline{W^{(j,n)}_{m_2,m_3}}(k)=\delta_{m_1,m_2}. \tag{3.19}$$

This relation is equivalent to:

$$\overline{W^{(j,n)}_{m_2,m_1}(k)}=W^{(j,n)}_{m_1,m_2}(k^{-1}).$$

To see such transformation rule directly, we can switch m_1 and m_2 in the expression (3.15) of $d^{(j,n)}_{m_1,m_2}(\theta)$ in terms of hypergeometric functions, and obtain the formula

$$d^{(j,n)}_{m_1,m_2}(\theta)=(-1)^{m_2-m_1}d^{(j,n)}_{m_2,m_1}(\theta).$$

It follows from the previous formula and the definition of Wigner D-functions that

$$(-1)^{m_2-m_1}W^{(j,-n)}_{-m_1,-m_2}(k)=W^{(j,n)}_{m_2,m_1}(k^{-1}). \tag{3.20}$$

which is equivalent to the formula (3.19) above.

4. **Differential Equations**

The right and left regular action $r(k)$ and $l(k)$ by K on any function $f\in C^\infty(K)$ are given by

$$\text{Right action: } (r(k)f)(g)=f(gk)$$
$$\text{Left action: } (l(k)f)(g)=f(k^{-1}g).$$

The corresponding action of the Lie algebra $\mathfrak{u}(2)$ as differential operators are denoted by $\mathsf{d}l$ and $\mathsf{d}r$, and we can extend the action to $\mathfrak{sl}(2,\mathbb{C})=\mathfrak{u}(2)\otimes\mathbb{C}$ by making $\mathsf{d}l$ linear and $\mathsf{d}r$ linear under multiplication by scalars:

$$\mathsf{d}l(\alpha X)=\alpha\mathsf{d}l(X),\quad \mathsf{d}r(\alpha X)=\alpha\mathsf{d}r(X).$$

The differential operators can be written down explicitly using the Euler angle coordinates. Then by direct calculation, these differential operators are

$$\begin{aligned}
&\mathsf{d}r(\gamma_0) = -\tfrac{\partial}{\partial\zeta} \quad \mathsf{d}l(\gamma_0) = \tfrac{\partial}{\partial\zeta}\\
&\mathsf{d}r(\gamma_3) = -\tfrac{\partial}{\partial\phi} \quad \mathsf{d}l(\gamma_3) = \tfrac{\partial}{\partial\psi}\\
&\mathsf{d}r(\gamma_1 \pm \mathsf{i}\gamma_2) = e^{\mp\mathsf{i}\phi}(-\cot\theta\frac{\partial}{\partial\phi} \mp \mathsf{i}\frac{\partial}{\partial\theta} + \csc\theta\frac{\partial}{\partial\psi})\\
&\mathsf{d}l(\gamma_1 \pm \mathsf{i}\gamma_2) = e^{\pm\mathsf{i}\psi}(\csc\theta\frac{\partial}{\partial\phi} \pm \mathsf{i}\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\psi}).
\end{aligned}$$

Comparing with the action of γ_i on the weight basis v^j_m in (3.8)-(3.10), the action of the differential operators $\mathsf{d}r(\gamma_i)$ and $\mathsf{d}l(\gamma_i)$ on the Wigner D-functions are

$$\begin{aligned}
\mathsf{d}r(\gamma_0)W^{(j,n)}_{m_1,m_2} &= -\mathsf{i}nW^{(j,n)}_{m_1,m_2} \qquad \mathsf{d}l(\gamma_0)W^{(j,n)}_{m_1,m_2} = \mathsf{i}nW^{(j,n)}_{m_1,m_2}\\
\mathsf{d}r(\gamma_3)W^{(j,n)}_{m_1,m_2} &= -\mathsf{i}m_2W^{(j,n)}_{m_1,m_2} \quad \mathsf{d}l(\gamma_3)W^{(j,n)}_{m_1,m_2} = \mathsf{i}m_1W^{(j,n)}_{m_1,m_2}
\end{aligned} \tag{3.21}$$

$$\mathsf{d}r(\gamma_1 \pm \mathsf{i}\gamma_2)W^{(j,n)}_{m_1,m_2} = \mathsf{i}\sqrt{(j \pm m_2)(j \mp m_2 + 1)}W^{(j,n)}_{m_1,m_2\mp1} \tag{3.22}$$

$$\mathsf{d}l(\gamma_1 \pm \mathsf{i}\gamma_2)W^{(j,n)}_{m_1,m_2} = -\mathsf{i}\sqrt{(j \mp m_1)(j \pm m_1 + 1)}W^{(j,n)}_{m_1\pm1,m_2}. \tag{3.23}$$

5. **Basis for** $L^2(K)$

   By the Peter-Weyl Theorem for $K = U(2)$, as the matrix coefficients for the finite dimensional representations of $U(2)$, the Wigner D-functions $W^{(j,n)}_{m_1,m_2}$ provides a Hilbert space basis for $L^2(K)$:

$$L^2(K) = \widehat{\bigoplus_{\substack{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}\\ n\in\frac{1}{2}\mathbb{Z}, j+n\in\mathbb{Z}\\ m_1,m_2\in\{-j,-j+1,\dots,j\}}}} \mathbb{C}W^{(j,n)}_{m_1,m_2}. \tag{3.24}$$

3.3 Tensor products and Clebsch-Gordan coefficients

For any two irreducible representations $V^{j_1} \cong \mathrm{Sym}^{2j_1}W$ and $V^{j_2} \cong \mathrm{Sym}^{2j_2}W$ of $SU(2)$, we choose the weight basis $\{v^{j_1}_{m_1}\}$ and $\{v^{j_2}_{m_2}\}$ of the two spaces, properly normalized such that γ_i acts as in (3.8)-(3.10). The tensor product $V^{j_1} \otimes V^{j_2}$ has two sets of basis: the *pure tensors* $v^{j_1}_{m_1} \otimes v^{j_2}_{m_2}$, and the weight basis of the irreducible constituents V^J of the tensor product $V^{j_1} \otimes V^{j_2}$. In fact, for the compact group $SU(2)$, each irreducible constituent in the decomposition

$$V^{j_1} \otimes V^{j_1} = \bigoplus_J (V^J)^{\oplus m_J}$$

has multiplicity $m_J = 0$ or 1, with $m_J = 1$ if and only if J satisfies the following two properties:

1. $|j_1 - j_2| \leq J \leq j_1 + j_2$
2. $J - |j_1 - j_2| \in \mathbb{Z}$.

To be more precise about the relationship between the pure tensor basis and the weight basis, we can expand the pure tensor basis into a linear combination of the weight basis:

$$v^{j_1}_{m_1} \otimes v^{j_2}_{m_2} \equiv \sum_{\substack{|j_1-j_2|\leq J\leq j_1+j_2 \\ J-|j_1-j_2|\in\mathbb{Z}}} \left({}^{J,M}_{j_1,m_1,j_2,m_2} \right) v^J_M.$$

The coefficient $\left({}^{J,M}_{j_1,m_1,j_2,m_2} \right)$ in the expression above is called the *Clebsch-Gordan coefficient*. It is zero except when $M = m_1 + m_2$. Moreover, the Clebsch-Gordan coefficients can also be used to write the product of Wigner D-functions as a linear combinations of Wigner D-functions. If we introduce the inner product on the tensor product $V^{j_1} \otimes V^{j_2}$ such that

$$\langle v^{j_1}_{m_{12}} \otimes v^{j_2}_{m_{22}}, v^{j_1}_{m_{11}} \otimes v^{j_2}_{m_{21}} \rangle = \langle v^{j_1}_{m_{12}}, v^{j_1}_{m_{11}} \rangle \langle v^{j_2}_{m_{22}}, v^{j_2}_{m_{21}} \rangle,$$

the product $W^{(j_1,n_1)}_{m_{11},m_{12}} W^{(j_2,n_2)}_{m_{21},m_{22}}$ of Wigner D-functions is thus a matrix coefficient of the representation $V^{j_1} \otimes V^{j_2}$:

$$\begin{aligned} W^{(j_1,n_1)}_{m_{11},m_{12}} W^{(j_2,n_2)}_{m_{21},m_{22}} &= \langle v^{j_1}_{m_{12}}, k v^{j_1}_{m_{11}} \rangle \langle v^{j_2}_{m_{22}}, k v^{j_2}_{m_{21}} \rangle \\ &= \langle v^{j_1}_{m_{12}} \otimes v^{j_2}_{m_{22}}, k(v^{j_1}_{m_{11}} \otimes v^{j_2}_{m_{21}}) \rangle. \end{aligned}$$

Combining this with the expansion of pure tensors into linear combinations of weight basis:

$$\begin{aligned} v^{j_1}_{m_{11}} \otimes v^{j_2}_{m_{21}} &= \sum_{J_1} \left({}^{J_1,M_1}_{j_1,m_{11},j_2,m_{21}} \right) v^{J_1}_{M_1} \\ v^{j_1}_{m_{21}} \otimes v^{j_2}_{m_{22}} &= \sum_{J_2} \left({}^{J_2,M_2}_{j_1,m_{12},j_2,m_{22}} \right) v^{J_2}_{M_2}, \end{aligned}$$

and since the matrix coefficients of nonisomorphic irreducible representations are orthogonal to each other, the product of Wigner D-functions $W^{(j_1,n_1)}_{m_{11},m_{12}} W^{(j_2,n_2)}_{m_{21},m_{22}}$ can be written as

$$W^{(j_1,n_1)}_{m_{11},m_{12}} W^{(j_2,n_2)}_{m_{21},m_{22}} = \sum_{\substack{|j_1-j_2|\le J\le j_1+j_2 \\ J-|j_1-j_2|\in\mathbb{Z} \\ M_1=m_{11}+m_{21} \\ M_2=m_{12}+m_{22}}} \binom{J,M_1}{j_1,m_{11},j_2,m_{21}} \binom{J,M_2}{j_1,m_{12},j_2,m_{22}} W^{(J,n_1+n_2)}_{M_1,M_2}. \quad (3.25)$$

The Clebsch-Gordan coefficient $\binom{J,M}{j_1,m_1,j_2,m_2}$ can be related to the *Wigner 3j-symbols* $\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}$ in the following way:

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} = \frac{(-1)^{j_2-j_1-M}}{\sqrt{2J+1}} \binom{J,M}{j_1,m_1,j_2,m_2}.$$

The Wigner $3j$-symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \neq 0$ is nonzero if and only if the following conditions are satisfied:

1. $m_i = -j_i, -j_i+1, \ldots, j_i-1, j_i$;
2. $m_1 + m_2 + m_3 = 0$;
3. $|j_1 - j_2| \le j_3 \le j_1 + j_2$;
4. $j_1 + j_2 + j_3 \in \mathbb{Z}$.

There is also a recursion relation of the Wigner $3j$-symbols, written in a symmetric manner as

$$\sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1\pm 1 & m_2 & m_3 \end{pmatrix} + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2\pm 1 & m_3 \end{pmatrix}$$
$$+ \sqrt{(j_3 \mp m_3)(j_3 \pm m_3 + 1)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\pm 1 \end{pmatrix} = 0.$$

The Clebsch-Gordan coefficients $\binom{j+j_0,m_1+m_2}{j,m_1,\frac{1}{2},m_2}$ and $\binom{j+j_0,m_1+m_2}{j,m_1,1,m_2}$ are listed in Table 3.1 and Table 3.2.

$\binom{j+j_0,m_1+m_2}{j,m_1,\frac{1}{2},m_2}$	$m_2 = -\frac{1}{2}$	$m_2 = +\frac{1}{2}$
$j_0 = -\frac{1}{2}$	$\sqrt{\frac{j+m_1}{2j+1}}$	$-\sqrt{\frac{j-m_1}{2j+1}}$
$j_0 = \frac{1}{2}$	$\sqrt{\frac{j-m_1+1}{2j+1}}$	$\sqrt{\frac{j+m_1+1}{2j+1}}$

Table 3.1: Table for Clebsch-Gordan coefficients of $V^j \otimes V^{\frac{1}{2}}$

$\left(\begin{smallmatrix} j+j_0,m_1+m_2 \\ j,m_1,1,m_2 \end{smallmatrix}\right)$	$m_2=-1$	$m_2=0$	$m_2=1$
$j_0=-1$	$\sqrt{\frac{(j+m_1)(j+m_1-1)}{2j(2j+1)}}$	$-\sqrt{\frac{(j-m_1)(j+m_1)}{j(2j+1)}}$	$\sqrt{\frac{(j-m_1)(j-m_1-1)}{2j(2j+1)}}$
$j_0=0$	$\sqrt{\frac{(j+m_1)(j-m_1+1)}{2j(j+1)}}$	$\frac{m_1}{\sqrt{j(j+1)}}$	$-\sqrt{\frac{(j-m_1)(j+m_1+1)}{2j(j+1)}}$
$j_0=1$	$\sqrt{\frac{(j-m_1+1)(j-m_1+2)}{(2j+2)(2j+1)}}$	$\sqrt{\frac{(j-m_1+1)(j+m_1+1)}{(j+1)(2j+1)}}$	$\sqrt{\frac{(j+m_1+1)(j+m_1+2)}{(2j+2)(2j+1)}}$

Table 3.2: Table for Clebsch-Gordan coefficients of $V^j \otimes V^1$

Chapter 4

Principal Series

In this section, we will define the *minimal principal series* representations of a real reductive Lie group G. The Cartan involution θ on G and $\mathfrak{g}$ has been introduced in Section 2.2.2. The notion of minimal parabolic subgroup and the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ and the restricted Weyl group $W(\mathfrak{g}, \mathfrak{a})$ has been introduced in Section 2.2.4.

4.1 Cartan Subgroups and Characters

Let $\mathfrak{s} \subset \mathfrak{g}$ be the Lie algebra of a maximally noncompact Cartan subgroup $S \subset G$. The Cartan subalgebra $\mathfrak{s}$ has a Cartan decomposition $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{a}$, such that $\mathfrak{s}_0$ and $\mathfrak{a}$ are eigenspaces of θ:

$$\theta|_{\mathfrak{s}_0} = 1, \quad \theta|_{\mathfrak{a}} = -1\cdot$$

The two conditions above are equivalent to

$$\mathfrak{s}_0 = \ker(1-\theta) \cap \mathfrak{s}, \quad \mathfrak{a} = \ker(1+\theta) \cap \mathfrak{s}.$$

For a real reductive Lie group G, the analytic subgroup $A_0 \subset G$ with Lie algebra $\mathfrak{a}$ is a connected real split torus isomorphic to a product of real tori $\mathbb{R}^{\times}$. The minimal parabolic subgroup $P \subset G$ has a Langlands decomposition $P = MA_0N$, where $M = Z_K(A_0)$ is a reductive Lie group whose Lie algebra $\mathfrak{m}$ has a Cartan subalgebra $\mathfrak{s}_0$. The maximally noncompact Cartan subgroup S containing A_0 can be decomposed into a product $S = S_0 \times A_0$, where $S_0 \subset K$ is a Cartan subgroup of M with Lie algebra $\mathfrak{s}_0$. Given such a choice of the Cartan subgroup S and minimal parabolic subgroup P, we introduce the following induction data:

1. Let δ be a character of the Cartan subalgebra $\mathfrak{s}_0 \subset \mathfrak{m}$, and denote by V_δ a representation of M with highest weight δ,

2. Let λ be a complex valued character of $A_0 \cong (\mathbb{R}^\times)^r$,

3. Let ρ be the half sum of positive restricted roots $\Sigma^+(\mathfrak{g}, \mathfrak{a})$.

We need a more explicit parametrization of the induction parameters δ and λ. It is apparent from construction that the Cartan subalgebra $\mathfrak{s}_{\mathbb{C}}$ is fixed by the Cartan involution θ. Therefore, the Cartan involution θ acts on an element γ in the weight lattice $X(\mathfrak{g}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}}) = \{\sum a_i \varpi_i | a_i \in \mathbb{Z}\}$ by

$$\theta\gamma = \gamma \circ \theta^{-1}.$$

For the sake of simplicity, we denote the weight lattice $X(\mathfrak{g}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}}) = \{\sum a_i \varpi_i | a_i \in \mathbb{Z}\}$ by X, and denote by X^θ and $X^{-\theta}$ the subgroups of elements of X fixed by θ and $-\theta$, respectively. Then the compact and split parts $\mathfrak{s}_0$ and $\mathfrak{a}$ of the Cartan subalgebra $\mathfrak{s}$ are isomorphic to

$$\mathfrak{s}_0 \cong \mathrm{Hom}_{\mathbb{Z}}(X^\theta, \mathrm{i}\mathbb{R}), \quad \mathfrak{a} \cong \mathrm{Hom}_{\mathbb{Z}}(X^{-\theta}, \mathbb{R}).$$

Therefore, as in Proposition 4.3 of [AvLTV12], a character δ on $\mathfrak{s}_0$ can be identified with an element in the quotient lattice $X/(1-\theta)X$, and a character λ on $\mathfrak{a}$ can be considered as an element in $X^{-\theta} \otimes_{\mathbb{Z}} \mathbb{C}$. By Proposition 4.3 in [AvLTV12], the set $\hat{S}$ of characters of S is the direct product

$$\hat{S} \cong (X/(1-\theta)X) \times \left(X^{-\theta} \otimes_{\mathbb{Z}} \mathbb{C}\right).$$

Example 4.1 *We use the following two examples to explicitly demonstrate how the notations introduced above can be applied to parametrize the characters on the real torus S explicitly.*

1. *$SU(n,1)$*

   The real form $SU(n,1)$ is defined by the antiholomorphic involution

   $$\sigma : g \mapsto J_{n,1}(\bar{g}^t)^{-1} J_{n,1}^{-1}$$

   where $J_{n,1}$ is the diagonal matrix with diagonal entries $(1,1,1,\ldots,1,-1)$. The Cartan involution θ is given by $\theta : g \mapsto (\bar{g}^t)^{-1}$. We shall take the Cartan subalgebra of $SU(n,1)$ to be

   $$\mathfrak{h} = \left\{ \begin{pmatrix} \mathrm{i}a & 0 & 0 & 0 & b \\ 0 & \mathrm{i}t_2 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}t_n & 0 \\ b & 0 & 0 & 0 & \mathrm{i}a \end{pmatrix} | a, b, t_i \in \mathbb{R} \right\},$$

and we set the roots to act on the elements $H_{a,b,t_2,\ldots,t_n} = \begin{pmatrix} \mathrm{i}a & 0 & 0 & 0 & b \\ 0 & \mathrm{i}t_2 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i}t_n & 0 \\ b & 0 & 0 & 0 & \mathrm{i}a \end{pmatrix}$ *by*

$$\alpha_1(H_{a,b,t_2,\ldots,t_n}) = \mathrm{i}(a - \mathrm{i}b) - \mathrm{i}t_2$$

$$\alpha_n(H_{a,b,t_2,\ldots,t_n}) = \mathrm{i}t_n - \mathrm{i}(a + \mathrm{i}b)$$

$$\alpha_i(H_{a,b,t_2,\ldots,t_n}) = \mathrm{i}(t_i - t_{i+1}) \textit{ for all } i \in \{2, \ldots, n-1\}.$$

Under this specification, the Cartan involution θ *acts on the roots by*

$$\theta(\alpha_1) = \alpha_1 - \beta$$

$$\theta(\alpha_n) = \alpha_n - \beta$$

$$\textit{and } \theta(\alpha_i) = \alpha_i \textit{ for all other } i \in \{2, \ldots, n-1\},$$

where $\beta = \alpha_1 + \ldots + \alpha_n$ *is the highest root. The matrix of* θ *under the simple root basis is* $\begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & -1 \\ \ldots & 0 & \ldots & 0 & \ldots \\ \ldots & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$, *and the matrix of* θ *under the fundamental weight basis is given by* $\begin{pmatrix} 0 & -1 & \ldots & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \ldots & 0 & 0 & 1 & 0 \\ -1 & -1 & \ldots & -1 & 0 \end{pmatrix}$. *Therefore, the sublattices* X^θ *and* $X^{-\theta}$ *of the weight lattice* X *are given in terms of their generators as*

$$X^\theta = \mathbb{Z}(\varpi_1 - \varpi_2) \oplus \ldots \oplus \mathbb{Z}(\varpi_1 - \varpi_n)$$

$$X^{-\theta} = \mathbb{Z}(\varpi_1 + \varpi_n).$$

Since $(1-\theta)X = \mathbb{Z}(\varpi_1 + \varpi_n)$, *the character* δ *can be considered as an element of the abelian group*

$$X/(1-\theta)X \cong \mathbb{Z}(\varpi_1) \oplus \ldots \oplus \mathbb{Z}(\varpi_{n-1})$$

and the continuous character λ *is an element of the line* $\mathbb{C}(\varpi_1 + \varpi_n)$. *We can therefore represent the two induction parameters* δ *and* λ *by two vectors:*

$$\delta \sim \sum_{i=1}^{n-1} \delta_i \varpi_i \longleftrightarrow (\delta_1, \ldots, \delta_{n-1}) \qquad \delta_1, \ldots, \delta_{n-1} \in \mathbb{Z};$$

$$\lambda \sim \lambda(\varpi_1 + \varpi_n) \longleftrightarrow (\lambda) \qquad \lambda \in \mathbb{C}.$$

If $\lambda \in \mathbb{Z}$, *then there is an integral character* $\chi = \sum_{i=1}^{n-1} \delta_i \varpi_i + \lambda(\varpi_1 + \varpi_n) \in X$ *such that* δ *is the image of* χ *in the quotient* $X/(1-\theta)X$ *and* $\lambda = \left(\frac{1-\theta}{2}\right)\chi$.

2. $Sp(2n,\mathbb{R})$

Consider the Cartan involution $\theta(g) = (g^t)^{-1}$ on $Sp(2n,\mathbb{R})$, which acts on the split Cartan subalgebra

$$\mathfrak{h}_{\mathbb{R}} = \bigoplus_{i=1}^{n} \mathbb{C}(E_{i,i} - E_{n+i,n+i})$$

by sending all elements to their negatives. In this case, $\theta = -\mathrm{id}$ on X. Therefore,

$$X/(1-\theta)X = \bigoplus_{i=1}^{n} \mathbb{Z}/2\mathbb{Z}\varpi_i$$

and $X^{-\theta} = X$. The discrete character δ and the continuous character λ can thus be determined by two vectors

$$\delta = \sum_{i=1}^{n} \delta_i \varpi_i \longleftrightarrow (\delta_1, \ldots, \delta_n) \qquad\qquad \delta_i \in \mathbb{Z}/2\mathbb{Z}$$
$$\lambda = \sum_{i=1}^{n} \lambda_i \varpi_i \longleftrightarrow (\lambda_1, \ldots, \lambda_n) \qquad\qquad \lambda \in \mathbb{C}.$$

If all of the λ_i's are integers satisfying $\lambda_i \equiv \delta_i \bmod 2$, then there exists an integral character $\chi = \sum_{i=1}^{n} \lambda_i \varpi_i$ such that δ is the image of χ in the quotient $X/(1-\theta)X$, and $\lambda = \left(\frac{1-\theta}{2}\right)\chi$.

The exponential map from $\mathfrak{a}$ to the split torus A_0 is a bijection. Every element $a \in A_0$ can be written as $a = \exp H_a$ for some $H_a \in A_0$. For any $\nu \in X$, we introduce the notation $a^\nu = \exp \nu(H_a)$. We define the *principal series* representation $I_P(\delta,\lambda)$ induced from the minimal parabolic subgroup $P \subset G$ as the following vector space of functions on G:

$$I_P(\delta,\lambda) = \{f : G \longrightarrow V_\delta | f(kman) = a^{-\lambda-\rho}\delta(m)^{-1} f(k)\} \tag{4.1}$$

The action $\pi_P(\delta,\lambda)$ of G on $f \in I_P(\delta,\lambda)$ is given by the *left* regular representation

$$(\pi_P(\delta,\lambda)(g)f)(h) = f(g^{-1}h).$$

Note that for expositional reasons, the principal series in this chapter is defined in an opposite way to Section 1.2, where $SL(2,\mathbb{R})$ was set up to act on the principal series by the *right* regular representation. If there is no ambiguity in the choice of the parabolic subgroup P, we can also use the notation $I(\delta,\lambda)$, omitting the parabolic subgroup P in the subscript. Also, if M is abelian, then in fact the induction parameter (δ,λ) defines a character $\chi_{\delta,\lambda} : MA_0 \longrightarrow \mathbb{C}^*$ given by

$$(\chi_{\delta,\lambda+\rho})(ma) = \delta(m)a^{\lambda+\rho}.$$

In this case, if no other ambiguity arises, we will use the notation $I(\chi_{\delta,\lambda})$ or $I(\chi)$ for the principal series induced from a character $\chi_{\delta,\lambda}$ on MA_0:

$$I_P(\chi_{\delta,\lambda}) = \{f : G \longrightarrow \mathbb{C} | f(gman) = (\chi_{\delta,\lambda+\rho})^{-1}(ma)f(g)\}.$$

If we denote by $\mathbb{C}_{\chi_{\delta,\lambda+\rho}}$ the one dimensional representation on which P acts by the character $\chi_{\delta,\lambda+\rho}$ of the Levi subgroup MA_0, the space $I_P(\chi_{\delta,\lambda})$ can be considered as the space of global sections of the line bundle $\mathcal{L}_{\chi_{\delta,\lambda}} = G \times_P \mathbb{C}_{\chi_{\delta,\lambda+\rho}}$.

If (π, V) is a (possibly infinite) dimensional representation of G, the *Harish-Chandra module* of V is its subspace consisting of the vectors $v \in V$ satisfying the following two properties:

1. The map $\phi_v : g \mapsto \pi(g)v$ is smooth;
2. The vector v is K-*finite*, i.e. the subspace generated by the orbit $\pi(K)v$ is finite dimensional.

The Harish-Chandra module of V is a $(\mathfrak{g}, K)$-module. The $\mathfrak{g}$-action comes from the derivative of the G-action

$$\mathsf{d}\pi(X)v = \frac{\mathsf{d}}{\mathsf{d}t}|_{t=0}\pi(e^{tX})v,$$

and the K-action is inherited from the G-module structure. In this thesis we will mainly discuss the *Harish-Chandra module* of the principal series of G, and we will use the same notation $I_P(\delta,\lambda)$ and $I_P(\chi_{\delta,\lambda})$ for the Harish-Chandra module of a principal series representation.

Now we restrict our study to the real reductive Lie groups G with maximal compact subgroup $K = U(2)$. Since the Lie group G has an Iwasawa decomposition

$G = KMAN$, the value of any function $f \in I_P(\delta, \lambda)$ is completely determined by its restriction to K. Therefore, as a vector space, the Harish-Chandra module $I_P(\delta, \lambda)$ is isomorphic to the following space of smooth functions on K:

$$C_\delta(K) := C^\infty(K) \otimes_M V_\delta = \{f : K \longrightarrow V_\delta | f \text{ is } K\text{-finite and } f(km) = \delta(m)^{-1} f(k)\}.$$

Compare with the Hilbert space decomposition in (3.24), using the fact that any function in $C_\delta(K)$ is K-finite, we can embed the space $C_\delta(K)$ into the space of *algebraic* direct sums of Wigner D-functions:

$$C_\delta(K) \subset \bigoplus_{\substack{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}, n \in \frac{1}{2}\mathbb{Z} \\ m_1, m_2 \in \{-j, -j+1, \ldots, j\}}} \mathbb{C} W^{(j,n)}_{m_1,m_2}.$$

consisting of finite linear combinations of Wigner D-functions

$$f(k) = \sum_{(j,n)} \sum_{m_1,m_2} a^{(j,n)}_{m_1,m_2} W^{(j,n)}_{m_1,m_2}(k)$$

satisfying the parity condition $f(km) = \delta(m)^{-1} f(k)$.

4.2 Intertwining Operators

We fix a real Cartan subalgebra $\mathfrak{a}$, and consider the following two different minimal parabolic subgroups with the same Levi subgroup:

$$P = MA_0N, \quad P^w = MA_0N^w$$

where w is an element of the restricted Weyl group $W(\mathfrak{g}, \mathfrak{a})$, and the unipotent radical of P^w is $N^w = wNw^{-1}$. Recalling that each unipotent radical N corresponds to a choice of positive restrict roots $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, we will use the notation $\bar{N}$ for the unipotent radical corresponding to the negative restricted roots $\Sigma^-(\mathfrak{g}, \mathfrak{a})$. From a fixed induction parameter (δ, λ), we can define two minimal principal series $I_P(\delta, \lambda)$ and $I_{P^w}(\delta, \lambda)$. Letting f be a K-finite vector in the principal series representation $I_P(\delta, \lambda)$, there exists a formal intertwining operator $A(P|P^w, \delta, \lambda)$ such that its action on $f \in I_P(\delta, \lambda)$ is given by

$$A(P|P^w, \delta, \lambda) f(g) = \int_{\bar{N} \cap N^w} f(g\bar{n}) \mathsf{d}\bar{n}.$$

The continuation and convergence of the intertwining operator defined by this formal integral will be discussed in the rest of this chapter.

4.2.1 The Intertwining Property

The integral operator $A(P|P^w,\delta,\lambda)$ formally satisfies the intertwining property

$$A(P|P^w,\delta,\lambda)\pi_P(\delta,\lambda)(g)=\pi_{P^w}(\delta,\lambda)(g)A(P|P^w,\delta,\lambda). \tag{4.2}$$

Denote $\rho_{\mathfrak{n}}$ as the half sum of restricted positive roots corresponding to N; thus the half sum of restricted positive roots corresponding to N^w is $w\rho_{\mathfrak{n}}$. As described in [Kna16], the right regular action by elements in M, A_0, N^w on the function $A(P|P^w,\delta,\lambda)f$ satisfies the following properties:

1. Right action by $m\in M$:

$$\left(A(P|P^w,\delta,\lambda)f\right)f(gm)=\int_{\bar{N}\cap N^w}f(gm\bar{n})\mathsf{d}\bar{n}=\delta(m)^{-1}\int_{\bar{N}\cap N^w}f(g\mathrm{Ad}(m)\bar{n})\mathsf{d}\bar{n}.$$

The measure $\mathsf{d}\bar{n}$ is given by the absolute value of the volume form in $\bigwedge^{\dim\mathfrak{n}}\bar{\mathfrak{n}}^*$, where $\bar{\mathfrak{n}}^*$ is the dual vector space of $\bar{\mathfrak{n}}$. For any vector X in the restricted root space $\mathfrak{g}_\alpha$, since m centralizes A_0, we have

$$\mathrm{Ad}(a)\mathrm{Ad}(m)X=\mathrm{Ad}(am)X=\mathrm{Ad}(m)\mathrm{Ad}(a)X=a^\alpha\mathrm{Ad}(m)X.$$

Therefore each restricted root space $\mathfrak{g}_\alpha$ is fixed by the adjoint action of M. Since the only possible image of a group homomorphism from the compact group M to $GL(1,\mathbb{R})\cong\mathbb{R}^\times$ is in $\{\pm1\}$, M acts on the top exterior power $\bigwedge^{\dim\mathfrak{n}}\bar{\mathfrak{n}}^*$ by a real character $\sigma:M\longrightarrow\{\pm1\}$, and the action of M on the measure $\mathsf{d}\bar{n}$ is

$$\mathsf{d}(\mathrm{Ad}(m)^{-1}\bar{n})=|\sigma(m)^{-1}|\mathsf{d}\bar{n}=\mathsf{d}\bar{n}.$$

Therefore, the original integral becomes

$$\begin{aligned}\left(A(P|P^w,\delta,\lambda)f\right)f(gm)&=\delta(m)^{-1}\int_{\bar{N}\cap N^w}f(g\bar{n})\mathsf{d}\bar{n}\\&=\delta(m)^{-1}\left(A(P|P^w,\delta,\lambda)f\right).\end{aligned}$$

2. Right action by $a \in A_0$:

$$\begin{aligned}(A(P|P^w,\delta,\lambda)f)(ga) &= \int_{\bar{N}\cap N^w} f(ga\bar{n})\mathsf{d}\bar{n} = a^{-(\lambda+\rho_{\mathfrak{n}})}\int_{\bar{N}\cap N^w} f(g\mathrm{Ad}(a)\bar{n})\mathsf{d}\bar{n}\\ &= a^{-(\lambda+\rho_{\mathfrak{n}})}(\det \mathrm{Ad}(a)|_{\bar{\mathfrak{n}}\cap\mathfrak{n}^w})^{-1}\int_{\bar{N}\cap N^w} f(g\bar{n})\mathsf{d}\bar{n}\\ &= a^{-(\lambda+w\rho_{\mathfrak{n}})}\int_{\bar{N}\cap N^w} f(g\bar{n})\mathsf{d}\bar{n}\\ &= a^{-(\lambda+w\rho_{\mathfrak{n}})}\left(A(P|P^w,\delta,\lambda)f\right).\end{aligned}$$

3. Right action by $n' \in N^w$:

$$\begin{aligned}(A(P|P^w,\delta,\lambda)f)(gn') &= \int_{\bar{N}\cap N^w} f(gn'\bar{n})\mathsf{d}\bar{n} = \int_{N^w/(N\cap N^w)} f(gn'\bar{n})\mathsf{d}\bar{n}\\ &= (A(P|P^w,\delta,\lambda)f)(g).\end{aligned}$$

Therefore, if the integral converges, the formal integral $A(P|P^w,\delta,\lambda)f$ is a genuine integral intertwining operator $A(P|P^w,\delta,\lambda) : I_P(\chi_{\delta,\lambda}) \longrightarrow I_{P^w}(\chi_{\delta,\lambda})$.

We can slightly change the definition of the intertwining operator so that the parabolic subgroup is fixed, and the Weyl group action changes the induction parameter (δ,λ) to $(w\delta,w\lambda)$. For $f \in I_P(\chi_{\delta,\lambda})$, we define the operator

$$A_P(w,\delta,\lambda)f(g) = \int_{\bar{N}\cap N^w} f(gw\bar{n})\mathsf{d}\bar{n}.$$

The operator $A_P(w,\delta,\lambda)$ is an intertwining operator from the minimal principal series $I_P(\delta,\lambda)$ to $I_P(w\delta,w\lambda)$, which satisfies the property

$$A_P(w,\delta,\lambda)\pi_P(\delta,\lambda)(g) = \pi_P(w\delta,w\lambda)(g)A_P(w,\delta,\lambda). \tag{4.3}$$

Since the operator $A_P(w,\delta,\lambda)$ maps between two principal series induced from the same parabolic subgroup P, and if there is no further ambiguity in the choice of P, we can drop the subscript and simply denote the intertwining operator by $A(w,\delta,\lambda)$. Also, if the induction parameter is a character $\chi_{\delta,\lambda}$ on MA_0, then we also denote the intertwining operator by $A(w,\chi_{\delta,\lambda})$.

4.2.2 Meromorphic Continuation

We assume that the induction parameter for the principal series is a character $\chi_{\delta,\lambda}$. We are interested in the long intertwining operators $A(P|\bar{P}, \chi_{\delta,\lambda})$ or $A(w_0, \chi_{\delta,\lambda})$, corresponding to the longest element of the restricted Weyl group $w_0 \in W(\mathfrak{g}, \mathfrak{a})$. For general δ and λ, the integral is singular. However, according to [VW90], for any $w \in W(\mathfrak{g}, \mathfrak{a})$, there exists a number $c_\delta > 0$ such that for any $f \in I_P(\chi_{\delta,\lambda})$ the map $\lambda \mapsto A(w, \chi_{\delta,\lambda})f$ is holomorphic in the interior of the region $\{\lambda_i \in \mathfrak{h}^*_{\mathbb{C}} | \mathrm{Re}\langle\lambda, \alpha\rangle \geq c_\delta, \text{ where } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})\}$. Moreover, according to the following theorem from [VW90], the long intertwining operator $A(w_0, \chi_{\delta,\lambda})$ depends meromorphically on $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$:

Theorem 4.1 *There exist polynomial maps $b_\delta : \mathfrak{a}^*_{\mathbb{C}} \longrightarrow \mathbb{C}$ and $D_\delta : \mathfrak{a}^*_{\mathbb{C}} \longrightarrow U(\mathfrak{g}_{\mathbb{C}})^K$, such that for $f \in I_P(\chi_{\delta,\lambda})$, if λ satisfies $\mathrm{Re}\langle\lambda, \alpha_i\rangle \geq c_\delta$ for $\alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$, we have*

$$b_\delta(\lambda)A(P|\bar{P}, \chi_{\delta,\lambda})f = A(P|\bar{P}, \chi_{\delta,\lambda+4\rho})\pi_P(\chi_{\delta,\lambda+4\rho})(D_\delta)f.$$

The theorem above ensures that there exists a meromorphic continuation of the intertwining operator $A(P|P^w, \delta, \lambda)$ to general λ, and the poles of the map $\lambda \mapsto A(P|P^w, \delta, \lambda)$ occur in the set

$$\{\lambda \in \mathfrak{a}^* | \langle\check{\alpha}, \lambda\rangle \in \mathbb{Z} \text{ for } \alpha \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \text{ with } w\alpha \in \Sigma^-(\mathfrak{g}, \mathfrak{a})\}$$

The following example for real rank 1 groups is cited from [KS80]:

Example 4.2 *If G has real rank 1, denote the reduced positive restricted root as α. Then there are two positive integers p and q counting the number of α and 2α's in the positive restricted root system, respectively, and $\rho = \frac{1}{2}(p + 2q)\alpha$. Then any complex induction parameter λ can be expressed as $\lambda = z\rho$ for some $z \in \mathbb{C}$. Pick any $f \in I(\chi_{\delta,z\rho})$, then:*

1. *If $\mathrm{Re}(z) > 0$, $A(w, \chi_{\delta,z\rho})$ is convergent. The map $z \mapsto A(w, \chi_{\delta,z\rho})$ has possible poles at $z = -\frac{1}{p+2q}\mathbb{Z}_{\geq 0}$*

2. *There exists a meromorphic function $\eta_{\delta,\alpha}(z)$ such that $A(w^{-1}, \chi_{w\delta,-z\rho})A(w, \chi_{\delta,z\rho}) = \eta_{\delta,\alpha}(z)I$.*

4.2.3 Factorization of Intertwining Operators

In order to compute the intertwining operators, we write a Weyl group element $w = s_{i_1} \dots s_{i_l}$ as a product of simple reflections, and assume that

$$P \longrightarrow P^{s_{i_1}} \longrightarrow \dots \longrightarrow P^{s_{i_1} \dots s_{i_{l-1}}} \longrightarrow P^w$$

is a chain of parabolic subgroups corresponding to this factorization. For a restricted root α_k, if we denote by $\mathfrak{v}_{\alpha_k}$ the direct sum of all restricted root spaces $\mathfrak{g}_{m\alpha_k}$ for positive integers m, the nilpotent radicals of the chain of parabolic subgroups above satisfies

$$\theta\mathfrak{n}^{s_{i_1} \dots s_{i_{k-1}}} \cap \mathfrak{n}^{s_{i_1} \dots s_{i_k}} = \theta\mathfrak{v}_{\alpha_k}.$$

We can write the intertwining operator $A(P|\bar{P}, \chi_{\delta,\lambda})$ as a composition of the lower-rank intertwining operators:

$$A(P|\bar{P}, \chi_{\delta,\lambda}) = A(P^{s_{i_1} \dots s_{i_{l-1}}}|P^w, \chi_{\delta,\lambda}) \dots A(P|P^{s_{i_1}}, \chi_{\delta,\lambda})$$

which gives rise to a factorization of the corresponding operator $A(w, \chi_{\delta,\lambda})$ on $I_P(\chi_{\delta,\lambda})$ given in [KS80] and [SV80], which is referred to as the *Langlands' Lemma* [Sha10]:

$$A(w, \chi_{\delta,\lambda}) = A(w, s_{i_1} \dots s_{i_{l-1}}\chi_{\delta,\lambda}) \dots A(s_{i_1}s_{i_2}, s_{i_1}\chi_{\delta,\lambda})A(s_{i_1}, \chi_{\delta,\lambda}). \tag{4.4}$$

4.2.4 Normalization

The material in this section is quoted from [KS80], where it is discussed in detail. There is a scalar valued meromorphic function $\eta(P|\bar{P}, \chi_{\delta,\lambda})$ on λ such that:

$$A(w_0, \chi_{\delta,\lambda})A(w_0^{-1}, w_0\chi_{\delta,\lambda}) = \eta(w_0, \chi_{\delta,\lambda})I. \tag{4.5}$$

The meromorphic function $\eta(w_0, \chi_{\delta,\lambda})$ factor into rank-one factors:

$$\eta(w_0, \chi_{\delta,\lambda}) = \prod_{\substack{\alpha \text{ reduced} \\ \langle\alpha_i,\alpha\rangle>0 \text{ for positive } \alpha_i}} \eta_{\delta,\alpha}\left(\frac{\langle\lambda, \rho^{(\alpha)}\rangle}{\langle\rho^{(\alpha)}, \rho^{(\alpha)}\rangle}\right).$$

Moreover, these $\eta_{\delta,\alpha}(z)$'s factor into meromorphic functions $\gamma_{\delta,\alpha}(z)$:

$$\eta_{\delta,\alpha}(z) = \gamma_{\delta,\alpha}(z)\overline{\gamma_{\delta,\alpha}(-\bar{z})}. \tag{4.6}$$

In terms of the long intertwining operator $A(w_0, w_0\chi_{\delta,\lambda})$ corresponding to w_0; it is related to the adjoint of the intertwining operator corresponding to w_0:

$$A(w_0, w_0\chi_{\delta,\lambda})^* = A(w_0, \chi_{\delta,-\bar{\lambda}}).$$

We can thus normalize the intertwining operator $A(w_0, \chi_{\delta,\lambda})$ by dividing it by the meromorphic factor:

$$\gamma(w_0, \chi_{\delta,\lambda}) = \prod_{\substack{\alpha \text{ reduced} \\ \langle\alpha_i,\alpha\rangle>0 \text{ for positive } \alpha_i}} \gamma_{\delta,\alpha}\left(\frac{\langle\lambda, \rho^{(\alpha)}\rangle}{\langle\rho^{(\alpha)}, \rho^{(\alpha)}\rangle}\right).$$

We denote the new normalized intertwining operator by

$$A'(w_0, \chi_{\delta,\lambda}) = \gamma(w_0, \chi_{\delta,\lambda})^{-1}A(w_0, \chi_{\delta,\lambda}).$$

4.2.5 Relation to Unitary Representations

Now we assume the induction parameter (δ, λ) satisfies $\mathrm{Re}\langle\lambda, \alpha_i\rangle > 0$ for all $\alpha_i \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$. There is a unique irreducible quotient $J(\chi_{\delta,\lambda})$ (see [BCP08]) of $I(\chi_{\delta,\lambda})$ by the kernel of the normalized intertwining operator $A'(w_0, \chi_{\delta,\lambda})$. The irreducible quotient $J(\chi_{\delta,\lambda})$ is called the *Langlands quotient* of the principal series $I(\chi_{\delta,\lambda})$. By a well-known result in [BCP08] and [KZ76], the Langlands quotient admits a hermitian form if and only if the induction parameter (δ, λ) satisfies the following symmetry with respect to the longest element w_0 of the restricted Weyl group:

$$w_0\delta \cong \delta, \quad w_0\lambda = -\bar{\lambda}.$$

Under this condition, we will use the normalized long intertwining operator $A'(w_0, \chi_{\delta,\lambda})$ to construct the hermitian form. Consider the representations (δ, V_δ) and $(w_0\delta, V_{w_0\delta})$ on M, we choose an isomorphism $\tau : V_{w_0\delta} \cong V_\delta$. Then this isomorphism τ between the induction parameters induces an isomorphism between the principal series $I(\delta, \lambda)$ and $I(w_0\delta, w_0\lambda)$, which is also denoted by τ. There exists a hermitian form on the Langlands quotient $J(\chi_{\delta,\lambda})$

$$\langle\cdot,\cdot\rangle : J(\chi_{\delta,\lambda}) \times J(\chi_{\delta,\lambda}) \longrightarrow \mathbb{C}$$

related to the normalized intertwining operator in the following way:

$$\langle v_1, v_2\rangle = \langle v_1, \tau \circ A'(w_0, \chi_{\delta,\lambda})v_2\rangle.$$

Therefore, the normalized intertwining operator $A'(w_0, \chi_{\delta,\lambda})$ induces a hermitian form on the Langlands quotient $J(\chi_{\delta,\lambda})$.

4.2.6 Calculation of Intertwining Operators

Consider a reductive Lie group G whose maximal compact subgroup is $K = U(2)$, if an Iwasawa decomposition for $\bar{n} \in \bar{N}$ is written as $\bar{n} = k(\bar{n})a(\bar{n})n(\bar{n})$, where $\bar{n} \in N, a(\bar{n}) \in A, k(\bar{n}) \in K$, then since any $f \in I(\delta, \lambda)$ satisfies the transformation property (4.1), we have

$$f(k\bar{n}) = f(kk(\bar{n})a(\bar{n})n(\bar{n})) = a(\bar{n})^{-(\lambda+\rho)}f(kk(\bar{n})).$$

Since there is an embedding of the Harish-Chandra module of $I(\delta, \lambda)$ into the space $C_\delta(K)$, we can express $f(k)$ as a linear combination of Wigner D-functions:

$$f(k) = \sum_{(j,n)} \sum_{(m_1,m_2)} a^{(j,n)}_{m_1,m_2} W^{(j,n)}_{m_1,m_2}(k).$$

Therefore, it is sufficient to compute the matrix coefficients of the intertwining operator on the basis $W^{(j,n)}_{m_1,m_2}$ of the space $C_\delta(K)$:

$$\begin{aligned} A(P|P^w, \delta, \lambda)W^{(j,n)}_{m_1,m_2} &= \int_{\bar{N}\cap N^w} W^{(j,n)}_{m_1,m_2}(k\bar{n})\mathsf{d}\bar{n} \\ &= \int_{\bar{N}\cap N^w} a(\bar{n})^{-(\lambda+\rho)} W^{(j,n)}_{m_1,m_2}(kk(\bar{n}))\mathsf{d}\bar{n} \\ &= \sum_{m_3} \left(\int_{\bar{N}\cap N^w} a(\bar{n})^{-(\lambda+\rho)} W^{(j,n)}_{m_3,m_2}(k(\bar{n}))\mathsf{d}\bar{n}\right) W^{(j,n)}_{m_1,m_3}(k), \end{aligned}$$

and the matrix coefficients of the intertwining operator $A(P|P^w, \delta, \lambda)$ are given by the formula

$$\langle W^{(j,n)}_{m_1,m_3}, A(P|P^w, \delta, \lambda)W^{(j,n)}_{m_1,m_2}\rangle = \int_{\bar{N}\cap N^w} a(\bar{n})^{-(\lambda+\rho)} W^{(j,n)}_{m_3,m_2}(k(\bar{n}))\mathsf{d}\bar{n}. \quad (4.7)$$

We will use this formula to compute the intertwining operators explicitly for the group $SU(2,1)$ and $Sp(4,\mathbb{R})$ in the next two sections.

Chapter 5

Toy Case: $SU(2,1)$

5.1 The Group $SU(2,1)$

The group $SU(2,1)$ is a real form of $SL(3,\mathbb{C})$. It is the fixed point of the anti-holomorphic involution $\sigma : g \mapsto J(\bar{g}^t)^{-1}J^{-1}$ in $SL(3,\mathbb{C})$:

$$SU(2,1) = \{g \in SL(3,\mathbb{C}) | \bar{g}^t J g = J\}$$

where J is the matrix

$$J = \mathrm{diag}(1,1,-1).$$

The real Lie algebra $\mathfrak{g}$ of $G = SU(2,1)$ is

$$\mathfrak{g} = \mathfrak{su}(2,1) = \{X \in \mathfrak{sl}(3,\mathbb{C}) | \bar{X}^t J + JX = 0\}.$$

The structure theory and representation theory of $SU(2,1)$ has been discussed in [Sha10].

5.1.1 The structure of $SU(2,1)$

First we consider the complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ and choose the following data

1. A Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ generated by $H_{\alpha_1} = E_{11} - E_{22}$ and $H_{\alpha_2} = E_{22} - E_{33}$;

2. The fundamental weights ϖ_1, ϖ_2 in $\mathfrak{h}_\mathbb{C}^*$ as dual basis for $H_{\alpha_1}, H_{\alpha_2}$, satisfying $\langle \varpi_i, H_{\alpha_j} \rangle = \delta_{ij}$ with $i, j \in \{1,2\}$;

3. The simple roots $\alpha_1 = 2\varpi_1 - \varpi_2, \alpha_2 = -\varpi_1 + 2\varpi_2$;

4. The set of positive roots $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$;

5. $\rho_\mathbb{C} = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})} \alpha = \alpha_1 + \alpha_2$;

6. The generators for the positive root spaces $X_{\alpha_1} = \left(\begin{smallmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{smallmatrix}\right)$, $X_{\alpha_2} = \left(\begin{smallmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{smallmatrix}\right)$, $X_{\alpha_1+\alpha_2} = [X_{\alpha_1}, X_{\alpha_2}] = \left(\begin{smallmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{smallmatrix}\right)$

7. Choose the generators of the negative root spaces $X_{-\alpha} = X^t_\alpha$.

The maximal compact subgroup K is defined as the set of fixed points of the Cartan involution $\theta : g \mapsto -\bar{g}^t$ on G:

$$K = G^\theta = (U(2) \times U(1))/U(1) = \{\left(\begin{smallmatrix} A & 0\\ 0 & \det(A)^{-1} \end{smallmatrix}\right) |A \in U(2)\} \cong U(2).$$

The induced isomorphism from the Lie algebra $\mathfrak{u}(2)$ to $\mathfrak{k}$ sends the Pauli matrices γ_i to the following elements in $\mathfrak{k}$:

$$U_0 = \frac{1}{2}\left(\begin{smallmatrix} \mathsf{i}&0&0\\ 0&\mathsf{i}&0\\ 0&0&-2\mathsf{i} \end{smallmatrix}\right), U_1 = \frac{1}{2}\left(\begin{smallmatrix} 0&\mathsf{i}&0\\ \mathsf{i}&0&0\\ 0&0&0 \end{smallmatrix}\right), U_2 = \frac{1}{2}\left(\begin{smallmatrix} 0&1&0\\ -1&0&0\\ 0&0&0 \end{smallmatrix}\right), U_3 = \frac{1}{2}\left(\begin{smallmatrix} \mathsf{i}&0&0\\ 0&-\mathsf{i}&0\\ 0&0&0 \end{smallmatrix}\right).$$

Under this isomorphism, the generators of the compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{su}(2,1)$ are $\mathsf{i}H_{\alpha_1} = 2U_3$ and $\mathsf{i}H_{\alpha_2} = U_0 - U_3$. The complexification of the compact Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ is thus exactly the same as the Cartan subalgebra $\mathfrak{h}_\mathbb{C}$ of $\mathfrak{sl}(3,\mathbb{C})$ defined above.

The -1 eigenspace $\mathfrak{p}$ of θ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ can be described explicitly as the following space of 3×3 matrices with complex entries:

$$\mathfrak{p} = \left\{ \left(\begin{smallmatrix} 0&0&z_1\\ 0&0&z_2\\ \bar{z}_1&\bar{z}_2&0 \end{smallmatrix}\right) |z_1, z_2 \in \mathbb{C} \right\}.$$

The matrix $\left(\begin{smallmatrix} 0&0&z_1\\ 0&0&z_2\\ \bar{z}_1&\bar{z}_2&0 \end{smallmatrix}\right)$ transforms under the adjoint action by an element $e^{aU_0+bU_3}$ in the Cartan subgroup $T \subset K$,

$$\mathrm{Ad}(e^{aU_0+bU_3}) \left(\begin{smallmatrix} 0&0&z_1\\ 0&0&z_2\\ \bar{z}_1&\bar{z}_2&0 \end{smallmatrix}\right) = \begin{pmatrix} 0 & 0 & e^{\mathsf{i}\frac{3a+b}{2}} z_1\\ 0 & 0 & e^{\mathsf{i}\frac{3a-b}{2}} z_2\\ e^{-\mathsf{i}\frac{3a+b}{2}}\bar{z}_1 & e^{-\mathsf{i}\frac{3a-b}{2}}\bar{z}_2 & 0 \end{pmatrix}. \tag{5.1}$$

Therefore, under the adjoint action of $K \cong U(2)$, we can consider the complexified space $\mathfrak{p}_\mathbb{C} = \mathfrak{p} \otimes \mathbb{C}$ as a 4-dimensional representation of K. Under the pairing between $\mathfrak{t}_\mathbb{C}$ and $\mathfrak{t}^*_\mathbb{C}$ introduced in Chapter 2, the value of any character $\chi = \chi_1\varpi_1 + \chi_2\varpi_2 \in \mathfrak{t}^*_\mathbb{C}$ on $\mathfrak{t}$ is given by

$$\chi(aU_0 + bU_3) = \mathsf{i}\frac{a+b}{2}\chi_1 + \mathsf{i}a\chi_2.$$

From this formula, the two simple roots $\alpha_1, \alpha_2 \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ act on $aU_0 + bU_3$ by

$$\alpha_1(aU_0 + bU_3) = \mathsf{i}b, \tag{5.2}$$

$$\alpha_2(aU_0 + bU_3) = \mathsf{i}\frac{3a - b}{2} \tag{5.3}$$

and the action of the highest root is

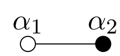
$$(\alpha_1 + \alpha_2)(aU_0 + bU_3) = \mathsf{i}\frac{3a + b}{2}. \tag{5.4}$$

5.1.2 The Cartan Subgroups of $SU(2,1)$

There are two conjugacy classes of Cartan subgroups of $SU(2,1)$: the compact Cartan subgroup isomorphic to $U(1) \times U(1)$, and the maximally noncompact Cartan subgroup isomorphic to $U(1) \times \mathbb{R}^\times$.

The maximally compact Cartan subgroup

In the root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$, the Vogan diagram of $SU(2,1)$



specifies an imaginary compact simple root α_1 and an imaginary noncompact simple root α_2. Since the root vectors of $\alpha_1 + \alpha_2$ can always be written as the commutators of root vectors of α_1 and α_2, the root $\alpha_1 + \alpha_2$ is a noncompact root. We have thus obtained the set of positive compact roots Δ_c^+ and noncompact roots Δ_{nc}^+:

$$\Delta_c^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) = \{\alpha_1\}$$

$$\Delta_{nc}^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}) = \{\alpha_2, \alpha_1 + \alpha_2\}.$$

Comparing with the discussion on $U(2)$-representations in Chapter 3, where a weight $m \in \frac{1}{2}\mathbb{Z}$ of $\mathfrak{p}_\mathbb{C}$ is equal to the half integer $-\mathsf{i}\alpha(U_3)$, and the central $U(1)$-character n is equal to $-\mathsf{i}\alpha(U_0)$. Based (5.2)-(5.4), any noncompact imaginary root

$$\alpha \in \{\pm\alpha_2, \pm\alpha_1 \pm \alpha_2\}$$

correspond to a pair of half integers denoted by $(m_\alpha, n_\alpha) = (-\mathrm{i}\alpha(U_3), -\mathrm{i}\alpha(U_0))$. Therefore, we have established the correspondence between noncompact imaginary roots with pairs of half integers (m_α, n_α):

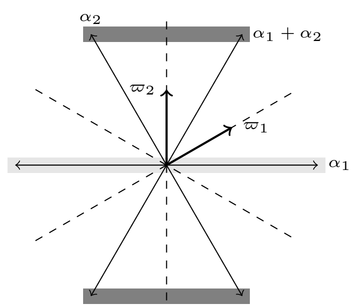
$$\pm\alpha_2 \longleftrightarrow \pm\left(-\frac{1}{2}, \frac{3}{2}\right) \tag{5.5}$$

$$\pm\alpha_1 \pm \alpha_2 \longleftrightarrow \pm\left(\frac{1}{2}, \frac{3}{2}\right). \tag{5.6}$$

We will use the description (m_α, n_α) and the corresponding noncompact roots α to refer to the weights of $\mathfrak{p}_\mathbb{C}$ interchangeably. The representation $\mathfrak{p}_\mathbb{C}$ of K can be decomposed into a direct sum of two 2 dimensional irreducible representations $V^{\frac{1}{2},\frac{3}{2}} \oplus V^{\frac{1}{2},-\frac{3}{2}}$ of $U(2)$, where the highest weights are labeled as upper indices, with weights

$$V^{\frac{1}{2},\frac{3}{2}} : \left(-\frac{1}{2}, \frac{3}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right)$$
$$V^{\frac{1}{2},-\frac{3}{2}} : \left(-\frac{1}{2}, -\frac{3}{2}\right), \left(\frac{1}{2}, -\frac{3}{2}\right).$$

The representation $V^{\frac{1}{2},\frac{3}{2}}$ corresponds to the coordinates z_1 and $\bar{z}_1$ in the space of matrices given in (5.1), and $V^{\frac{1}{2},-\frac{3}{2}}$ corresponds to the coordinates z_2 and $\bar{z}_2$ in (5.1). If we use gray for noncompact roots and light gray for compact roots, the direct sum decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus V^{\frac{1}{2},\frac{3}{2}} \oplus V^{\frac{1}{2},-\frac{3}{2}}$ can be displayed in the root system of $\mathfrak{sl}(3, \mathbb{C})$:



There is a weight basis $\{v_\alpha\}_{\alpha\in\Delta_{nc}}$ of $\mathfrak{p}_\mathbb{C}$, such v_α has weight α when considered as a vector in $\mathfrak{g}_\mathbb{C}$ under the adjoint action. By associating these weight vectors to a pair of half integers (m_α, n_α) as above, we can also label each weight vector by writing the pair (m_α, n_α) as lower indices. Therefore the two irreducible $U(2)$ subrepresentations of $\mathfrak{p}_\mathbb{C}$ have basis:

$$V^{\frac{1}{2},\frac{3}{2}} = \operatorname{span}\{v_{\alpha_2}, v_{\alpha_1+\alpha_2}\}$$
$$V^{\frac{1}{2},-\frac{3}{2}} = \operatorname{span}\{v_{-\alpha_2}, v_{-\alpha_1-\alpha_2}\}$$

where the vectors v_α's are given by

$$\begin{aligned} v_{\alpha_2} &= v_{-\frac{1}{2},\frac{3}{2}} = -X_{\alpha_2} & v_{\alpha_1+\alpha_2} &= v_{\frac{1}{2},\frac{3}{2}} = X_{\alpha_1+\alpha_2} \\ v_{-\alpha_2} &= v_{\frac{1}{2},-\frac{3}{2}} = X_{-\alpha_2} & v_{-\alpha_1-\alpha_2} &= v_{-\frac{1}{2},-\frac{3}{2}} = X_{-\alpha_1-\alpha_2}. \end{aligned}$$

The maximally noncompact Cartan subalgebra and the restricted roots

We take the noncompact imaginary root $\alpha_1+\alpha_2 \in \Delta_{nc}$ and define the transform

$$\mathsf{p}_{\alpha_1+\alpha_2} = \mathrm{Ad}\exp(\frac{\pi}{4}(\sigma(v_{\alpha_1+\alpha_2}) - v_{\alpha_1+\alpha_2})).$$

The transform $\mathsf{p}_{\alpha_1+\alpha_2}$ is motivated by the Cayley transform in Section 2.2.2 of Chapter 2. The action by $\mathsf{p}_{\alpha_1+\alpha_2}$ sends the complexified maximally compact Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ to the maximally noncompact Cartan subalgebra $\mathfrak{s}_\mathbb{C} = \mathfrak{m}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C}$, where $\mathfrak{a}$ is the subspace $\mathbb{C}(X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2})$ of $\mathfrak{p}_\mathbb{C}$, and $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The analytic subgroup $M \subset G$ with Lie algebra $\mathfrak{m}$ is

$$M = \{e^{-tU_0}e^{3tU_3} = \begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{-2it} & 0 \\ 0 & 0 & e^{it} \end{pmatrix} | t \in \mathbb{R}\}$$

The action of the Cayley transform $\mathsf{q}_{\alpha_1+\alpha_2} = \mathsf{p}^{-1}_{\alpha_1+\alpha_2}$ sends $\mathfrak{a}_\mathbb{C}$ to a subspace of $\mathfrak{t}_\mathbb{C}$ generated by

$$\mathsf{q}_{\alpha_1+\alpha_2}(X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2}) = H_{\alpha_1} + H_{\alpha_2}$$

Since $\alpha_i(H_{\alpha_1} + H_{\alpha_2}) = 2$ for $i = 1, 2$, the two simple roots α_1, α_2 have the same restriction to the line $\mathsf{q}_{\alpha_1+\alpha_2}\mathfrak{a}_\mathbb{C} \subset \mathfrak{t}_\mathbb{C}$. We denote this restriction by α_0. The set of positive restricted roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$ consists of a character $\alpha_0 = \alpha_1|_{\mathsf{q}_{\alpha_1+\alpha_2}\mathfrak{a}} = \alpha_2|_{\mathsf{q}_{\alpha_1+\alpha_2}\mathfrak{a}}$ with multiplicity 2, and $2\alpha_0 = (\alpha_1+\alpha_2)|_{\mathsf{q}_{\alpha_1+\alpha_2}\mathfrak{a}}$ with multiplicity 1. The half sum of positive restricted roots is:

$$\rho_0 = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{a})} \alpha = 2\alpha_0$$

The restricted Weyl group $W(\mathfrak{g}, \mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is an order 2 group generated by the single element

$$w_0 = \exp(2\pi U_3) = \mathrm{diag}(-1, -1, 1)$$

The adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ decomposes the real Lie algebra into restricted root spaces $\mathfrak{g}_{\pm\alpha_0}$ and $\mathfrak{g}_{\pm 2\alpha_0}$:

$$\mathfrak{g}_{\alpha_0} = \{\mathsf{p}_{\alpha_1+\alpha_2}(zX_{\alpha_1} + \bar{z}X_{\alpha_2}) = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 0 & z & 0 \\ -\bar{z} & 0 & \bar{z} \\ 0 & z & 0 \end{smallmatrix}\right) | z \in \mathbb{C}\}$$
$$\mathfrak{g}_{2\alpha_0} = \{\mathsf{p}_{\alpha_1+\alpha_2}(\mathrm{i}wX_{\alpha_1+\alpha_2}) = -\frac{1}{2} \left(\begin{smallmatrix} \mathrm{i}w & 0 & -\mathrm{i}w \\ 0 & 0 & 0 \\ \mathrm{i}w & 0 & -\mathrm{i}w \end{smallmatrix}\right) | w \in \mathbb{R}\}$$
$$\mathfrak{g}_{-\alpha_0} = \{\mathsf{p}_{\alpha_1+\alpha_2}(\bar{z}X_{-\alpha_1} + zX_{-\alpha_2}) = -\frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 0 & z & 0 \\ -\bar{z} & 0 & -\bar{z} \\ 0 & -z & 0 \end{smallmatrix}\right) | z \in \mathbb{C}\}$$
$$\mathfrak{g}_{-2\alpha_0} = \{\mathsf{p}_{\alpha_1+\alpha_2}(\mathrm{i}wX_{-\alpha_1-\alpha_2}) = \frac{1}{2} \left(\begin{smallmatrix} -\mathrm{i}w & 0 & -\mathrm{i}w \\ 0 & 0 & 0 \\ \mathrm{i}w & 0 & \mathrm{i}w \end{smallmatrix}\right) | w \in \mathbb{R}\}$$

A general element in positive restricted root space $\mathfrak{n}^+ = \mathfrak{g}_{\alpha_0} \oplus \mathfrak{g}_{2\alpha_0}$ can be denoted by

$$n_{z,w} = \left(\begin{smallmatrix} \mathrm{i}w & z & -\mathrm{i}w \\ -\bar{z} & 0 & \bar{z} \\ \mathrm{i}w & z & -\mathrm{i}w \end{smallmatrix}\right), z \in \mathbb{C}, w \in \mathbb{R}$$

Using the restricted root spaces and the $n_{z,w}$'s, we can compute the Iwasawa decomposition (as defined in Section 2.2.4) of the v_α's in the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$:

$$v_{\alpha_1+\alpha_2} = -\frac{\mathrm{i}}{2}(U_0 + U_3) + \frac{1}{2}(X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2}) + \frac{\mathrm{i}}{2}n_{0,1} \tag{5.7}$$
$$v_{-\alpha_1-\alpha_2} = \frac{\mathrm{i}}{2}(U_0 + U_3) + \frac{1}{2}(X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2}) - \frac{\mathrm{i}}{2}n_{0,1} \tag{5.8}$$
$$v_{\alpha_2} = \mathrm{i}(U_1 - \mathrm{i}U_2) - \frac{1}{2}n_{1,0} - \frac{\mathrm{i}}{2}n_{\mathrm{i},0} \tag{5.9}$$
$$v_{-\alpha_2} = \mathrm{i}(U_1 + \mathrm{i}U_2) + \frac{1}{2}n_{1,0} - \frac{\mathrm{i}}{2}n_{\mathrm{i},0} \tag{5.10}$$

Moreover, the adjoint action of the raising-lowering operators $U_1 \pm \mathrm{i}U_2$ on the v_α's satisfies:

$$\mathrm{ad}(U_1 \pm \mathrm{i}U_2)v_\alpha = -\mathrm{i}v_{\alpha_1\pm\alpha}$$

if $\alpha_1 \pm \alpha$ is a root.

5.2 The Principal Series Representations

Consider the minimal parabolic subgroup $P = MA_0N$, where A_0 is the analytic subgroup of G with Lie algebra $\mathfrak{a} = \mathbb{R}(X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2})$. We introduce the following parameters:

1. A character $\mathfrak{a} \longrightarrow \mathbb{C}$ sending $X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2}$ to a complex number λ;

2. A character $M \longrightarrow \mathbb{C}$ sending $e^{-t(U_0-3U_3)}$ to $e^{\mathrm{i}\delta t}$, where δ is an integer.

Applying the Cayley transform on $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a}$, $\mathsf{q}_{\alpha_1+\alpha_2}^{-1}(\mathfrak{s})$ is the real Lie algebra $\mathfrak{h} = \mathbb{R}\mathrm{i}(H_{\alpha_1} - H_{\alpha_2}) \oplus \mathbb{R}(H_{\alpha_1} + H_{\alpha_2})$, on which the induction parameter (δ, λ) defines a complex valued character $\chi_{\delta,\lambda} : \mathfrak{h} \longrightarrow \mathbb{C}$ such that

$$\chi_{\delta,\lambda}(H_{\alpha_1} - H_{\alpha_2}) = \delta, \chi_{\delta,\lambda}(H_{\alpha_1} + H_{\alpha_2}) = \lambda$$

and therefore the action of $\chi_{\delta,\lambda}$ on H_{α_i}'s gives:

$$\chi_{\delta,\lambda}(H_{\alpha_1}) = \tfrac{\lambda+\delta}{2}, \quad \chi_{\delta,\lambda}(H_{\alpha_2}) = \tfrac{\lambda-\delta}{2}.$$

The character $\chi_{\delta,\lambda}$ lives in the weight lattice if and only if $\lambda \pm \delta \in 2\mathbb{Z}$.

There are two Casimir elements Ω_2 and Ω_3 of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, having degree 2 and 3 respectively. Following the same notation in Section 2.1.4, denote $\{X_i\}$ to be an indexed basis of $\mathfrak{g}_{\mathbb{C}}$ and $\{\tilde{X}_i\}$ the indexed dual basis under the Killing form. If we take π_{std} to be the standard representation of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$ on a 3 dimensional complex vector space, the quadratic Casimir element can be computed using the formula:

$$\begin{aligned}
\Omega_2 &= \sum_{i,j} \mathrm{Tr}(\pi_{\mathrm{std}}(X_i)\pi_{\mathrm{std}}(X_j))\tilde{X}_i\tilde{X}_j \\
&= \frac{1}{54}(H_{\alpha_1}^2 + H_{\alpha_1}H_{\alpha_2} + H_{\alpha_2}^2) + \frac{1}{36}\sum_{\alpha\in\Delta^+}\{X_\alpha, X_{-\alpha}\} \\
&= \frac{1}{9}(H_{\alpha_1}^2 + H_{\alpha_1}H_{\alpha_2} + H_{\alpha_2}^2 + 3(H_{\alpha_1} + H_{\alpha_2})) + \frac{1}{18}\sum_{\alpha\in\Delta^+} X_{-\alpha}X_\alpha
\end{aligned}$$

where for $X, Y \in U(\mathfrak{g}_{\mathbb{C}})$, we denote by $\{X, Y\} = XY + YX \in U(\mathfrak{g}_{\mathbb{C}})$. Under the Harish-Chandra isomorphism $\gamma' : Z(\mathfrak{g}_{\mathbb{C}}) \longrightarrow S(\mathfrak{h})^W$, the image of Ω_2 is

$$\gamma'(\Omega_2) = \frac{1}{9}(H_{\alpha_1}^2 + H_{\alpha_1}H_{\alpha_2} + H_{\alpha_2}^2 - 3)$$

The cubic Casimir element Ω_3 is:

$$\begin{aligned}\Omega_3 &= \sum_{i,j,k} \mathrm{Tr}(\pi_{\mathrm{std}}(X_i)\pi_{\mathrm{std}}(X_j)\pi_{\mathrm{std}}(X_k))\tilde{X}_i\tilde{X}_j\tilde{X}_k \\ &= \frac{1}{1944}(3 - H_{\alpha_1} + H_{\alpha_2})(6 + 2H_{\alpha_1} + H_{\alpha_2})(H_{\alpha_1} + 2H_{\alpha_2}) \\ &+ \frac{1}{216}(6X_{-\alpha_1-\alpha_2}X_{\alpha_1+\alpha_2} + 6X_{-\alpha_2}X_{\alpha_2} - X_{-\alpha_1}(H_1 + 2H_2)X_{\alpha_1} \\ &+ X_{-\alpha_2}(2H_1 + H_2)X_{\alpha_2} - X_{-\alpha_1-\alpha_2}(H_1 - H_2)X_{\alpha_1+\alpha_2} \\ &- 3X_{-\alpha_2}X_{-\alpha_1}X_{\alpha_1+\alpha_2} - 3X_{-\alpha_1-\alpha_2}X_{\alpha_1}X_{\alpha_2})\end{aligned}$$

The image of Ω_3 under the Harish-Chandra homomorphism is:

$$\gamma'(\Omega_3) = -\frac{1}{2^3 3^5}(H_{\alpha_1} + 2H_{\alpha_2} - 3)(2H_{\alpha_1} + H_{\alpha_2} + 3)(H_{\alpha_1} - H_{\alpha_2} - 3) \tag{5.11}$$

After applying the character $\chi_{\delta,\lambda}$ on $S(\mathfrak{h}_{\mathbb{C}})^W$, we see that

$$\chi_{\delta,\lambda}(\gamma'(\Omega_2)) = \frac{1}{36}(3(\lambda^2 - 4) + \delta^2) \tag{5.12}$$

$$\chi_{\delta,\lambda}(\gamma'(\Omega_3)) = \frac{1}{2^5 3^5}(\delta - 3)(\delta - 3(\lambda - 2))(\delta + 3(\lambda - 2)) \tag{5.13}$$

We can induce the character $\chi_{\delta,\lambda}$ from the Levi subgroup $L = MA_0$ of the minimal parabolic subgroup $P = MA_0N$ to get the minimal principal series representation:

$$I(\chi_{\delta,\lambda}) = \{f : G \to \mathbb{C} | f(ge^{-t(U_0-3U_3)}e^{s(X_{\alpha_1+\alpha_2}+X_{-\alpha_1-\alpha_2})}n) = e^{-\mathrm{i}\delta t-(\lambda+2)s}f(g)\}$$

The value of the functions in the principal series is determined by their restriction to the maximal compact subgroup $K = U(2)$, and the Lie algebra $\mathfrak{g}$ acts as differential operators on the left. The functions on K can be expanded into Fourier series with respect to the basis $W^{(j,n)}_{m_1,m_2}$. We can apply the Iwasawa decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ in (5.7)-(5.10) and the differential operators $\mathsf{d}l(\gamma_i)$ and $\mathsf{d}r(\gamma_i)$ on K introduced in Section 3.2 to write down the action of $\mathfrak{g}$ on $C^\infty(K)$. Moreover, we can apply the product formula (3.25) to express the product of Wigner D-functions as linear combinations of Wigner D-functions. We will use the machinery developed before to prove the following proposition describing the $\mathfrak{g}$-action on $I(\chi_{\delta,\lambda})$ explicitly in the next two sections:

Proposition 5.1 *Let v_α be the weight vectors of $\mathfrak{p}_{\mathbb{C}}$ as an $U(2)$ representation, such that $\alpha \in \Delta_{nc}$ as listed in (5.5)-(5.6). If $\alpha \in \Delta_{nc}^{\pm}$, the action of the weight vectors v_α in $\mathfrak{p}_{\mathbb{C}}$ satisfies:*

$$\mathsf{d}l(v_\alpha)W^{(j,n)}_{m_1,m_2} = \frac{1}{2\sqrt{2j+1}} \sum_{j_0 \in \{\pm\frac{1}{2}\}} \begin{pmatrix} j+j_0, m_1+m_\alpha \\ J, m_1, \frac{1}{2}, m_\alpha \end{pmatrix} q_{j_0,\pm}\kappa_{j_0,\pm}(j,n,m_1;\lambda) W^{(j+j_0, n\pm\frac{3}{2})}_{m_1+m_\alpha, m_2\pm\frac{1}{2}}, \tag{5.14}$$

with the coefficients as shown in the following tables:

$q_{j_0,\pm}$	-	+
$j_0 = -\frac{1}{2}$	$\sqrt{j-m_2}$	$\sqrt{j+m_2}$
$j_0 = \frac{1}{2}$	$\sqrt{j+m_2+1}$	$\sqrt{j-m_2+1}$

$\kappa_{j_0,\pm}$	-	+
$j_0 = -\frac{1}{2}$	$-2j-m_2+n+\lambda$	$2j-m_2+n-\lambda$
$j_0 = \frac{1}{2}$	$2j-m_2+n+\lambda+2$	$2j+m_2-n+\lambda+2$

5.3 Embedding of Principal Series in $C^\infty(K)$

The $(\mathfrak{g}, K)$ module of $I(\chi_{\delta,\lambda})$ can be embedded into the space $L^2(K)$ as the subspace consisting of K-finite functions f satisfying:

$$f(ke^{-t(U_0-3U_3)}) = e^{-\mathrm{i}\delta t} f(k), \text{ for all } t \in \mathbb{R}.$$

This requires that f is a finite linear combination of Wigner D-functions $W^{(j,n)}_{m_1,m_2}$ satisfying the condition

$$e^{\mathrm{i}(n-3m_2)t} = e^{-\mathrm{i}\delta t}, \text{ for all } t \in \mathbb{R}$$

which is equivalent to the condition:

$$-n + 3m_2 = \delta \tag{5.15}$$

Therefore as a vector space, the $(\mathfrak{g}, K)$ module of the principal series $I(\chi_{\delta,\lambda})$ can be embedded into $C_\delta(K) \subset C^\infty(K)$ as an algebraic direct sum:

$$I(\chi_{\delta,\lambda}) \subset C_\delta(K) := \bigoplus_{-n+3m_2=\delta} \mathbb{C}W^{(j,n)}_{m_1,m_2} \subset C^\infty(K).$$

Since m_2 always satisfies $-j \leq m_2 \leq j$, n can only take half integer values in the interval $-3j-\delta \leq n \leq 3j-\delta$. Denote the set of the pairs (j,n) satisfying this condition as

$$\mathtt{KTypes}(\delta) = \{(j,n) \in \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} : -3j - \delta \leq n \leq 3j - \delta\}. \tag{5.16}$$

The set $\mathtt{KTypes}(\delta)$ parametrizes all the K-isotypic components of the principal series $I(\chi_{\delta,\lambda})$:

$$I(\chi_{\delta,\lambda}) = \bigoplus_{(j,n)\in\mathtt{KTypes}(\delta)} \tau^{(j,n)},$$

where $\tau^{(j,n)}$ is a direct sum of copies of irreducible representations of $U(2)$ with highest weight (j,n). The K-isotypic subspaces $\tau^{(j,n)}$ can be decomposed into the direct sum:

$$\tau^{(j,n)} = \bigoplus_{\substack{m_1\in\{-j,-j+1,\ldots,j\}\\ m_2\in\mathtt{M}(j,n,\delta)}} \mathbb{C}W^{(j,n)}_{m_1,m_2} \tag{5.17}$$

where the set $\mathtt{M}(j,n,\delta)$ is defined as:

$$\mathtt{M}(j,n,\delta) = \{m_2 \in \{-j,-j+1,\ldots,j\} : m_2 = \frac{n+\delta}{3}\}. \tag{5.18}$$

Since for each $(j,n) \in \mathtt{KTypes}(\delta)$ we have $|\mathtt{M}(j,n,\delta)| = 1$ or 0, each K-type of the $SU(2,1)$ principal series $I(\chi_{\delta,\lambda})$ has multiplicity at most 1. The K-types of $I(\chi_{\delta,\lambda})$, with their multiplicities taken into account, can be displayed on the cone in a subset of $(\frac{1}{2}\mathbb{Z})^3$ with coordinates (j,n,m_2) such that $(j,n) \in \mathtt{KTypes}(\delta)$ and $m_2 \in \mathtt{M}(j,n,\delta)$.

From this embedding of $I(\chi_{\delta,\lambda})$ into $C^\infty(K)$, the action by the Lie algebra elements of $\mathfrak{g}_\mathbb{C}$ can be realized as differential operators on K. For any $f \in C^\infty_\delta(K)$, we can extend the domain of f to a vector in $I(\chi_{\delta,\lambda})$ by applying the Iwasawa decomposition $G = KMA_0N$ and the transformation rule of the principal series. More precisely, the actions $\mathsf{d}l(X)$ and $\mathsf{d}r(X)$ of $X \in \mathfrak{g}$ on $f \in C^\infty_\delta(K) \subset I(\chi_{\delta,\lambda})$ are given by

$$(\mathsf{d}l(X)f)(k) = \tfrac{\mathsf{d}}{\mathsf{d}t}|_{t=0} f(e^{-tX}k), \quad (\mathsf{d}r(X)f)(k) = \tfrac{\mathsf{d}}{\mathsf{d}t}|_{t=0} f(ke^{tX})$$

We recall that in Section 3.2, the extension of $\mathsf{d}r$ to $\mathfrak{k}_\mathbb{C} = \mathfrak{k} \otimes \mathbb{C}$ is linear. We can follow the same rule and extend $\mathsf{d}l$ and $\mathsf{d}r$ further to $\mathfrak{g}_\mathbb{C}$ by setting $\mathsf{d}l(zX) = z\mathsf{d}l(X)$ and

$\mathsf{d}r(zX) = z\mathsf{d}r(X)$ for any $z \in \mathbb{C}$. Under such extension, the right action $\mathsf{d}r(n_{z,w})$ by elements $n_{z,w} \in \mathfrak{n}^+$ in the nilpotent radical sends any $f \in I(\chi_{\delta,\lambda})$ to 0. The element $X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2} \in \mathfrak{a}$ acts by scalar multiplication:

$$\mathsf{d}r(X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2})W^{(j,n)}_{m_1,m_2} = -(\lambda+2)W^{(j,n)}_{m_1,m_2}. \tag{5.19}$$

Combining (5.19), the action of the Lie algebra $\mathfrak{k}$ on the Wigner D-functions (3.21)-(3.23), and the Iwasawa decomposition (5.7)-(5.10) for the basis vectors v_α in $\mathfrak{p}_{\mathbb{C}}$, the formulae of the right action of v_α on $L^2(K)$ are given by

$$\begin{aligned}\mathsf{d}r(v_{\pm(\alpha_1+\alpha_2)})W^{(j,n)}_{m_1,m_2} &= \left(\mp\frac{1}{2}\mathsf{d}r(\mathsf{i}(U_0+U_3)) - \frac{1}{2}(\lambda+2)\right)W^{(j,n)}_{m_1,m_2}\\ &= \frac{1}{2}(\mp n \mp m_2 - \lambda - 2)W^{(j,n)}_{m_1,m_2} \qquad (5.20)\\ \mathsf{d}r(v_{\pm\alpha_2})W^{(j,n)}_{m_1,m_2} &= \mathsf{d}r(\mathsf{i}(U_1 \mp \mathsf{i}U_2))W^{(j,n)}_{m_1,m_2} = -\sqrt{(j\mp m_2)(j\pm m_2+1)}W^{(j,n)}_{m_1,m_2\pm 1}.\end{aligned}$$

(5.21)

We can express the left action $\mathsf{d}l(X)$ by any Lie algebra element $X \in \mathfrak{g}$ in terms of $\mathsf{d}r$ using the adjoint action of K:

$$\mathsf{d}l(X) = \mathsf{d}r(-\mathrm{Ad}^{-1}(k)X).$$

Then for any $\alpha \in \Delta^{\pm}_{nc} = \{\pm\alpha_2, \pm\alpha_1 \pm \alpha_2\}$, recalling the correspondence (5.5)-(5.6) of α with the pair of integers (m_α, n_α) and the definition of Wigner D-functions as matrix coefficients in (3.13), the left action of v_α on the functions in $C^\infty(K)$ can be expressed as a linear combination of right actions by vectors v_α with $\alpha \in \Delta^{\pm}_{nc}$, having Wigner D-functions $-W^{(\frac{1}{2},n_\alpha)}_{m_\beta,m_\alpha}(k^{-1})$ as coefficients:

$$\mathsf{d}l(v_\alpha) = \mathsf{d}r(-\mathrm{Ad}(k^{-1})v_\alpha) = \sum_{\beta\in\begin{cases}\Delta^+_{nc} \text{ if } \alpha\in\Delta^+_{nc}\\ \Delta^-_{nc} \text{ if } \alpha\in\Delta^-_{nc}\end{cases}} \mathsf{d}r(-\overline{W^{(\frac{1}{2},n_\alpha)}_{m_\beta,m_\alpha}(k^{-1})}v_\beta).$$

The same method for $SL(3,\mathbb{R})$ has been provided in [BM17]. By the unitarity of Wigner D-function matrices (3.19) and (3.20), we can change the argument from k^{-1} to k and rearrange the upper and lower indices of Wigner D-functions:

$$\mathsf{d}l(v_\alpha) = -\sum_{\beta\in\begin{cases}\Delta^+_{nc} \text{ if } \alpha\in\Delta^+_{nc}\\ \Delta^-_{nc} \text{ if } \alpha\in\Delta^-_{nc}\end{cases}} W^{(\frac{1}{2},n_\alpha)}_{m_\alpha,m_\beta}(k)\mathsf{d}r(v_\beta).$$

After listing all the $\beta' s$ in $\Delta^{\pm}_{nc}$, the sum over β above has only two terms. For $\alpha \in \Delta^{\pm}_{nc}$, we have

$$\mathsf{d}l(v_\alpha) = -\left(W^{(\frac{1}{2},\pm\frac{3}{2})}_{m_\alpha,\mp\frac{1}{2}}(k)\mathsf{d}r(v_{\pm\alpha_2}) + W^{(\frac{1}{2},\pm\frac{3}{2})}_{m_\alpha,\pm\frac{1}{2}}(k)\mathsf{d}r(v_{\pm(\alpha_1+\alpha_2)})\right).$$

Applying the formulas for the right action (5.20) and (5.21), the left action of v_α on $W^{(j,n)}_{m_1,m_2}$ can be written in terms of products of Wigner D-functions:

$$\begin{aligned}
&\mathsf{d}l(v_\alpha)W^{(j,n)}_{m_1,m_2} = \\
&-\left(-\sqrt{(j\mp m_2)(j\pm m_2+1)}W^{(\frac{1}{2},\pm\frac{3}{2})}_{m_\alpha,\mp\frac{1}{2}}W^{(j,n)}_{m_1,m_2\pm 1} + \right.\\
&\left.\frac{1}{2}(\mp n \mp m_2 - \lambda - 2)W^{(\frac{1}{2},\pm\frac{3}{2})}_{m_\alpha,\pm\frac{1}{2}}W^{(j,n)}_{m_1,m_2}\right)
\end{aligned}$$

Recall from (3.25) that the product of Wigner D-functions is in fact the linear combination of Wigner D-functions for the constituents of the tensor product representations, and the coefficients of this linear combination are products of the Clebsch-Gordan coefficients:

$$W^{(j_1,n_1)}_{m_1,m_2}W^{(j_2,n_2)}_{m_3,m_4} = \sum_{J\in\{j+\frac{1}{2},j-\frac{1}{2}\}} \left(\begin{smallmatrix} J,m_1+m_3 \\ j_1,m_1,j_2,m_3 \end{smallmatrix}\right)\left(\begin{smallmatrix} J,m_2+m_4 \\ j_1,m_2,j_2,m_4 \end{smallmatrix}\right) W^{(J,n_1+n_2)}_{m_1+m_3,m_2+m_4}$$

We can thus combine all the matrix coefficients belonging to the same J in the formula for the left action of any v_α with $\alpha \in \Delta^{\pm}_{nc}$:

$$\begin{aligned}
&\mathsf{d}l(v_\alpha)W^{(j,n)}_{m_1,m_2} = \\
&-\sum_{j_0\in\{\frac{1}{2},-\frac{1}{2}\}}\left(-\sqrt{(j\mp m_2)(j\pm m_2+1)}\left(\begin{smallmatrix} j+j_0,m_2\pm\frac{1}{2} \\ j,m_2\pm 1,\frac{1}{2},\mp\frac{1}{2} \end{smallmatrix}\right) + \frac{1}{2}(\mp n\mp m_2-\lambda-2)\left(\begin{smallmatrix} j+j_0,m_2\pm\frac{1}{2} \\ j,m_2,\frac{1}{2},\pm\frac{1}{2} \end{smallmatrix}\right)\right) \\
&\left(\begin{smallmatrix} j+j_0,m_1+m_\alpha \\ j,m_1,\frac{1}{2},m_\alpha \end{smallmatrix}\right) W^{(j+j_0,n\pm\frac{3}{2})}_{m_1+m_\alpha,m_2\pm\frac{1}{2}}.
\end{aligned} \tag{5.22}$$

If $m_2 \neq \pm j$, recall from Table 3.1 that the table of Clebsch-Gordan coefficients $\left(\begin{smallmatrix} j+j_0,m_2+m_0 \\ j,m_2,\frac{1}{2},m_0 \end{smallmatrix}\right)$ for j_0 and m_0 taking the values $\pm\frac{1}{2}$ is

$\left(\begin{smallmatrix} j+j_0,m_2+m_0 \\ j,m_2,\frac{1}{2},m_0 \end{smallmatrix}\right)$	$m_0 = -\frac{1}{2}$	$m_0 = +\frac{1}{2}$
$j_0 = -\frac{1}{2}$	$\sqrt{\frac{j+m_2}{2j+1}}$	$-\sqrt{\frac{j-m_2}{2j+1}}$
$j_0 = \frac{1}{2}$	$\sqrt{\frac{j-m_2+1}{2j+1}}$	$\sqrt{\frac{j+m_2+1}{2j+1}}$

.

Plugging these Clebsch-Gordan coefficients into the formula (5.22) for $\mathrm{d}l(v_\alpha)$, for each $\alpha \in \Delta_{nc}^{\pm}$, the action of weight vectors v_α in $\mathfrak{p}_{\mathbb{C}}$ satisfies:

$$\mathrm{d}l(v_\alpha)W_{m_1,m_2}^{(j,n)} = \frac{1}{2\sqrt{2j+1}} \sum_{j_0 \in \{\pm\frac{1}{2}\}} \left({}^{j+j_0, m_1+m_\alpha}_{j, m_1, \frac{1}{2}, m_\alpha} \right) q_{j_0,\pm}(j, m_2) \kappa_{j_0,\pm}(j, n, m_2; \lambda) W_{m_1+m_\alpha, m_2 \pm \frac{1}{2}}^{(j+j_0, n\pm\frac{3}{2})},$$

which is the formula (5.14). The expressions of the coefficients $q_{j_0,\pm}$ and $\kappa_{j_0,\pm}$ are shown in the following tables:

$q_{j_0,\pm}(j,m_2)$	-	+
$j_0 = -\frac{1}{2}$	$\sqrt{j+m_2}$	$\sqrt{j-m_2}$
$j_0 = \frac{1}{2}$	$\sqrt{j-m_2+1}$	$\sqrt{j+m_2+1}$

$\kappa_{j_0,\pm}(j,n,m_2;\lambda)$	-	+
$j_0 = -\frac{1}{2}$	$-(2j-m_2+n-\lambda)$	$2j+m_2-n-\lambda$
$j_0 = \frac{1}{2}$	$2j+m_2-n+\lambda+2$	$2j-m_2+n+\lambda+2$

We have thus finished the proof of the Proposition 5.1.

5.4 Decomposition of $I(\chi_{\delta,\lambda})$

In this section, we assume the character $\chi_{\delta,\lambda}$ satisfies $\chi_{\delta,\lambda}(H_{\alpha_i}) \in \mathbb{Z}\backslash\{0\}$ for $i = 1, 2$. In this case, we are assuming λ and δ will satisfy the condition

$$\lambda \pm \delta \in 2\mathbb{Z} \text{ and } |\lambda - \delta| \geq 2.$$

Under such assumption, would like to discuss the reducibility and compute the full composition series of the principal series $I(\chi_{\delta,\lambda})$ with $\chi_{\delta,\lambda}$ lying in different open Weyl chambers. The set $\mathtt{M}(j,n;\delta)$ consists of at most one element $m_2 = \frac{n+\delta}{3}$, hence the parameter m_2 is completely determined by n and δ in the expression of $\mathrm{d}l(v_\alpha)$. It should be pointed out that the same result can be obtained from understanding the order of zeros of the intertwining operator $A(\chi_{\delta,\lambda})$ from the next section, at those points where $\lambda \pm \delta \in 2\mathbb{Z}$.

The formulas for the coefficients $\kappa_{j_0,\pm}$ of the $\mathfrak{p}_{\mathbb{C}}$ action on $I(\chi_{\delta,\lambda})$ are displayed in the following table:

$\kappa_{j_0,\pm}$	-	+
$j_0 = -\frac{1}{2}$	$-\frac{2}{3}(3j + n - \frac{\delta}{2}) + \lambda$	$\frac{2}{3}(3j - n + \frac{\delta}{2}) - \lambda$
$j_0 = \frac{1}{2}$	$\frac{2}{3}(3j - n + \frac{\delta}{2}) + (2 + \lambda)$	$\frac{2}{3}(3j + n - \frac{\delta}{2}) + (2 + \lambda)$

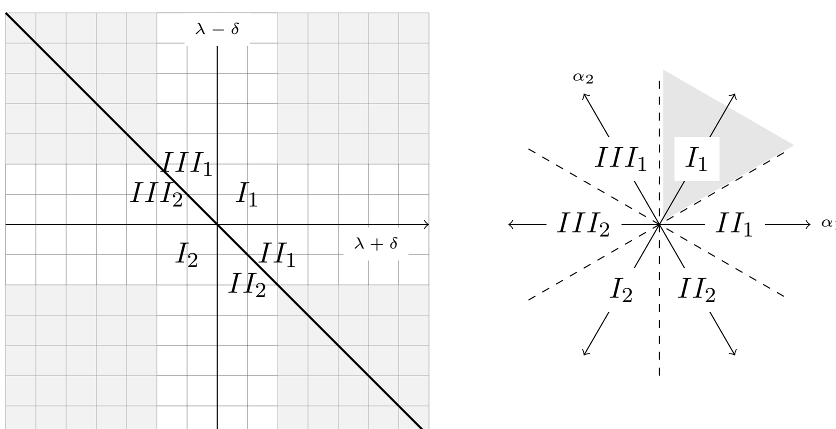
Recall from the description of the representations of $U(2)$ that $j + n \in \mathbb{Z}$ and $j \pm m_2 = j \pm \frac{n+\delta}{3} \in \mathbb{Z}$ in Section 3.2, let (k, l) be the unique pair of integers such that $n = -\delta + \frac{3}{2}l, j = \frac{k}{2}$. They live in the following cone `LatticeCond` of the lattice $\mathbb{Z}^2$:

$$(k, l) \in \texttt{LatticeCond} = \{(k, l) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} | -k \leq l \leq k \text{ and } k \equiv l \bmod 2\}.$$

The coefficients $\kappa_{j_0,\pm}$ can thus be expressed in terms of k, l, λ, δ:

$\kappa_{j_0,\pm}$	-	+
$j_0 = -\frac{1}{2}$	$-(k + l - \lambda - \delta)$	$k - l - \lambda + \delta$
$j_0 = \frac{1}{2}$	$k - l + \lambda + \delta + 2$	$k + l + \lambda - \delta + 2$

.



(a) Weyl chambers in the straight coordinate

(b) Open Weyl chambers on the root space

Figure 5.1: Weyl chambers in different coordinates

Based on the signs of $\frac{\lambda \pm \delta}{2}$ and λ, which are the values of $\chi_{\delta,\lambda}$ on coroots, the dual of Cartan subalgebra $\mathfrak{h}^*_{\mathbb{C}}$ is divided into 6 Weyl chambers as shown in the Figure 5.1. The Weyl group W acts on the characters $\chi_{\delta,\lambda}$, sending it to different Weyl chambers. The action by the simple reflections $w_{\alpha_1}, w_{\alpha_2}$ on the pair of parameters (δ, λ) and the corresponding $\chi_{\delta,\lambda}$ is:

$$w_{\alpha_1}\chi_{\delta,\lambda} = \chi_{-\frac{3\lambda+\delta}{2},\frac{\lambda-\delta}{2}}$$

$$w_{\alpha_2}\chi_{\delta,\lambda} = \chi_{\frac{3\lambda-\delta}{2},\frac{\lambda+\delta}{2}}.$$

Modules

There are 6 families of irreducible $(\mathfrak{g}, K)$-modules for the group $SU(2,1)$ depending on the parameters (δ, λ). They can be decomposed into direct sums of $\tau^{(j,n)}$'s as defined in (5.17). We are going to display these modules in diagrams of lattice points and shaded regions in (k, l)-coordinates. In these diagrams, the lattice points stand for K-types $\tau^{(j,n)}$ represented in (k, l) coordinates. The horizontal and vertical axes stand for k and l, respectively. The dashed arrows stand for a possible action that maps one K-type to another by the Lie algebra action. The irreducible subquotients are depicted by regions of different shades of gray, the darkest gray is for the finite dimensional representation V_{fin} or the holomorphic/antiholomorphic discrete series $V_{\text{disc}\pm}$, the medium gray is for $Q_\pm$ and the lightest gray is for the *quaternionic discrete series* $V_{\mathbb{H}}$. In these pictures, the lowest K-types are labeled by (j, n) instead of (k, l).

The Weyl chamber I_1

The character $\chi_{\delta,\lambda}$ in the Weyl chamber I_1 satisfies

$$\lambda - \delta \geq 2, \lambda + \delta \geq 2.$$

There exists $(\mathfrak{g}, K)$-submodules of $I(\chi_{\delta,\lambda})$ generated by finitely many K-types:

$$V_{\mathbb{H}}(\chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(\frac{\lambda}{2},\frac{\delta}{2})}$$

$$V_1(\chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(\frac{\lambda+\delta}{4},\frac{3\lambda-\delta}{4})} + U(\mathfrak{g})\tau^{(\frac{\lambda-\delta}{4},\frac{-3\lambda-\delta}{4})}$$

that form a composition series of $I(\chi_{\delta,\lambda})$:

$$V_{\mathbb{H}} \xhookrightarrow[V_1/V_{\mathbb{H}}=Q_-\oplus Q_+]{\iota_2} V_1 \xhookrightarrow[V_0/V_1=V_{\text{fin}}]{\iota_1} V_0 = I(\chi_{\delta,\lambda}).$$

The quaternionic discrete series $V_{\mathbb{H}}$, a finite dimensional representation V_{fin} and the $Q_\pm$'s are irreducible $(\mathfrak{g}, K)$ modules, which decompose into a direct sum of K-isotypic spaces:

$$V_{\mathbb{H}}(\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k-l\geq\lambda-\delta\\ k+l\geq\lambda+\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

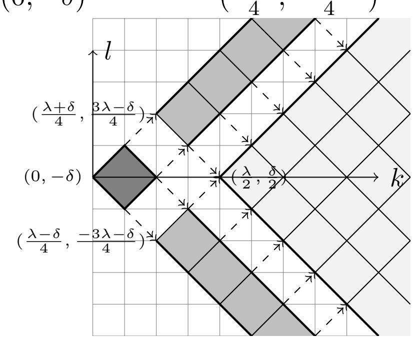
$$Q_{\pm}(\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k\mp l<\lambda\mp\delta \quad 2\\ k\pm l\geq\lambda\pm\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

$$V_{\text{fin}}(\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k+l<\lambda+\delta\\ k-l<\lambda-\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

An example when $(\delta,\lambda) = (0,4)$ is displayed in the figure below.

Figure 5.2: Weyl chamber I_1

	dark gray : V_{fin}	medium gray : $Q_{\pm}$	light gray : $V_{\mathbb{H}}$
lowest (j,n)	$(0,-\delta)$	$(\frac{\lambda\pm\delta}{4},\frac{\pm3\lambda-\delta}{4})$	$(\frac{\lambda}{2},\frac{\delta}{2})$



The Weyl chamber II_1

The character $\chi_{\delta,\lambda}$ in the Weyl chamber II_1 satisfies

$$\lambda > 0, \lambda-\delta \leq -2.$$

The two $(\mathfrak{g},K)$-submodules of $I(\chi_{\delta,\lambda})$ in the composition series are:

$$V_{\mathbb{H}}(w_{\alpha_2}\chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(\frac{\lambda+\delta}{4},\frac{3\lambda-\delta}{4})}$$

$$V_{\text{disc}-}(w_{\alpha_2}\chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(0,-\delta)},$$

where $V_{\mathbb{H}}$ is the quaternionic discrete series, and $V_{\text{disc}-}$ is the antiholomorphic discrete series. Quotienting out the direct sum of these two modules from $I(\chi_{\delta,\lambda})$, we can get

$$V_{\mathbb{H}} \oplus V_{\text{disc}-} \xhookrightarrow[V_0/(V_{\mathbb{H}}\oplus V_{\text{disc}-})=Q_-]{\iota} V_0 = I(\chi_{\delta,\lambda}).$$

The spaces $V_{\mathrm{disc}-}, V_{\mathbb{H}}$ and Q_- are irreducible $(\mathfrak{g}, K)$ modules, which are the direct sum of K-types:

$$V_{\mathbb{H}}(w_{\alpha_2}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k+l\geq\lambda+\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

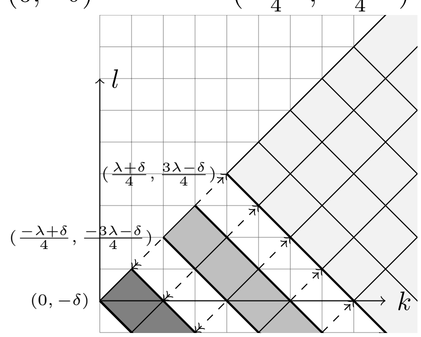
$$Q_-(w_{\alpha_2}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k+l<\lambda+\delta \\ k+l\geq-\lambda+\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

$$V_{\mathrm{disc}-}(w_{\alpha_2}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k+l<-\lambda+\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

For $(\delta, \lambda) = (6, 2)$, the regions representing the K-types of these modules are shown in the picture below:

Figure 5.3: Weyl chamber II_1

	dark gray : $V_{\mathrm{disc}-}$	medium gray : Q_-	light gray : $V_{\mathbb{H}}$
lowest (j,n)	$(0, -\delta)$	$(\frac{-\lambda+\delta}{4}, \frac{-3\lambda-\delta}{4})$	$(\frac{\lambda+\delta}{4}, \frac{3\lambda-\delta}{4})$



The Weyl chamber II_2

The character $\chi_{\delta,\lambda}$ lying in the Weyl chamber II_2 satisfies the inequality:

$$\lambda < 0, \lambda + \delta \geq 2.$$

There exists a $(\mathfrak{g}, K)$-submodule

$$Q_-(w_{\alpha_1}w_{\alpha_2}\chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(\frac{\lambda+\delta}{4},\frac{3\lambda-\delta}{4})}$$

of $I(\chi_{\delta,\lambda})$ that forms a composition series of $I(\chi_{\delta,\lambda})$:

$$Q_- \xhookrightarrow[V_0/Q_-=V_{\mathbb{H}}\oplus V_{\mathrm{disc}-}]{\iota} V_0 = I(\chi_{\delta,\lambda}).$$

The spaces $V_{\text{disc}-}, V_{\mathbb{H}}$ and Q_- are irreducible $(\mathfrak{g}, K)$ modules, which are direct sums of K-types:

$$V_{\mathbb{H}}(w_{\alpha_1} w_{\alpha_2} \chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k+l\geq -\lambda+\delta}} \tau^{(\frac{k}{2}, -\delta+\frac{3l}{2})}$$

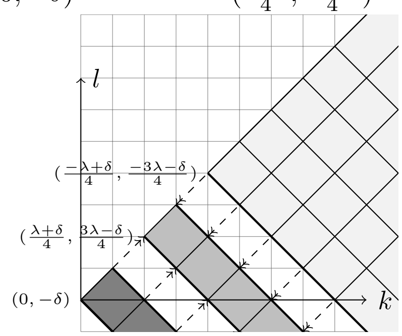
$$Q_-(w_{\alpha_1} w_{\alpha_2} \chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k+l< -\lambda+\delta \\ k+l\geq \lambda+\delta}} \tau^{(\frac{k}{2}, -\delta+\frac{3l}{2})}$$

$$V_{\text{disc}-}(w_{\alpha_1} w_{\alpha_2} \chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k+l< \lambda+\delta}} \tau^{(\frac{k}{2}, -\delta+\frac{3l}{2})}$$

For $(\delta, \lambda) = (6, -2)$, the regions representing the K-types of these modules are shown in the picture below:

Figure 5.4: Weyl chamber II_2

	dark gray : $V_{\text{disc}-}$	medium gray : Q_-	light gray : $V_{\mathbb{H}}$
lowest (j, n)	$(0, -\delta)$	$(\frac{\lambda+\delta}{4}, \frac{3\lambda-\delta}{4})$	$(\frac{-\lambda+\delta}{4}, \frac{-3\lambda-\delta}{4})$



The Weyl chamber I_2

When the character $\chi_{\delta,\lambda}$ lies in the Weyl chamber I_2,

$$\lambda - \delta \leq -2, \lambda + \delta \leq -2.$$

There exists $(\mathfrak{g}, K)$-submodules

$$V_1(w_{\alpha_1} w_{\alpha_2} w_{\alpha_1} \chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(\frac{-\lambda+\delta}{4}, \frac{-3\lambda-\delta}{4})} + U(\mathfrak{g})\tau^{(\frac{-\lambda-\delta}{4}, \frac{3\lambda-\delta}{4})}$$

$$V_{\text{fin}}(w_{\alpha_1} w_{\alpha_2} w_{\alpha_1} \chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(0,-\delta)}$$

of $I(\chi_{\delta,\lambda})$ that forms a composition series of $I(\chi_{\delta,\lambda})$:

$$V_{\text{fin}} \xhookrightarrow[V_1/V_{\text{fin}}=Q_-\oplus Q_+]{\iota_2} V_1 \xhookrightarrow[V_0/V_1=V_{\mathbb{H}}]{\iota_1} V_0 = I(\chi_{\delta,\lambda}).$$

The spaces $V_{\text{fin}}, Q_{\pm}$ and $V_{\mathbb{H}}$ are irreducible $(\mathfrak{g}, K)$ modules, which are direct sums of K-types:

$$V_{\mathbb{H}}(w_{\alpha_1} w_{\alpha_2} w_{\alpha_1} \chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k-l\geq -\lambda+\delta \\ k+l\geq -\lambda-\delta}} \tau^{(\frac{k}{2}, -\delta+\frac{3l}{2})}$$

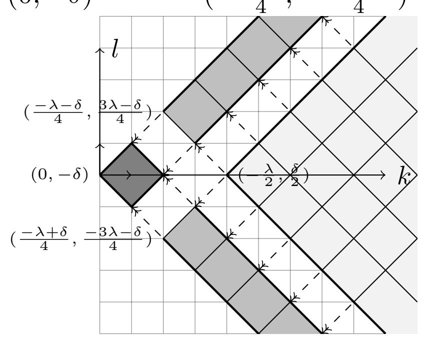
$$Q_{\pm}(w_{\alpha_1} w_{\alpha_2} w_{\alpha_1} \chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k\mp l< -\lambda\pm\delta \\ k\pm l\geq -\lambda\mp\delta}} \tau^{(\frac{k}{2}, -\delta+\frac{3l}{2})}$$

$$V_{\text{fin}}(w_{\alpha_1} w_{\alpha_2} w_{\alpha_1} \chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond} \\ k-l< -\lambda+\delta \\ k+l< -\lambda-\delta}} \tau^{(\frac{k}{2}, -\delta+\frac{3l}{2})}.$$

For $(\delta, \lambda) = (0, -4)$, the regions representing the K-types of these modules are shown in the picture below:

Figure 5.5: Weyl chamber I_2

	dark gray : V_{fin}	medium gray : $Q_{\pm}$	light gray : $V_{\mathbb{H}}$
lowest (j, n)	$(0, -\delta)$	$(-\frac{\lambda\pm\delta}{4}, -\frac{\pm 3\lambda+\delta}{4})$	$(-\frac{\lambda}{2}, \frac{\delta}{2})$



The Weyl chamber III_1

When the character $\chi_{\delta,\lambda}$ lies in the Weyl chamber III_1:

$$\lambda + \delta \leq -2, \lambda > 0.$$

Define the submodule V_2 of $I(\chi_{\delta,\lambda})$ as a direct sum of the two spaces:

$$V_{\text{disc}+}(w_{\alpha_1} \chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(0,-\delta)}$$

$$V_{\mathbb{H}}(w_{\alpha_1} \chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(\frac{\lambda-\delta}{4}, \frac{-3\lambda-\delta}{4})}.$$

These subspaces form a composition series of $I(\chi_{\delta,\lambda})$:

$$V_{\mathbb{H}} \oplus V_{\text{disc}+} \xhookrightarrow[V_0/(V_{\mathbb{H}}\oplus V_{\text{disc}+})=Q_+]{\iota} V_0 = I(\chi_{\delta,\lambda})$$

where

$$V_{\mathbb{H}}(w_{\alpha_1}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k-l\geq\lambda-\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

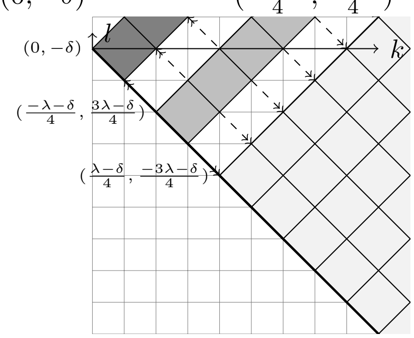
$$Q_{+}(w_{\alpha_1}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k-l<\lambda-\delta\\ k-l\geq-\lambda-\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

$$V_{\text{disc}+}(w_{\alpha_1}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k-l<-\lambda-\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}.$$

For $(\delta,\lambda) = (-6,2)$, the regions representing the K-types of these modules are shown in the picture below:

Figure 5.6: Weyl chamber III_1

	dark gray : $V_{\text{disc}+}$	medium gray : Q_+	light gray : $V_{\mathbb{H}}$
lowest (j,n)	$(0,-\delta)$	$(\frac{-\lambda-\delta}{4},\frac{3\lambda-\delta}{4})$	$(\frac{\lambda-\delta}{4},\frac{-3\lambda-\delta}{4})$



The Weyl chamber III_2

The character $\chi_{\delta,\lambda}$ lying in Weyl chamber III_2 satisfies the inequality:

$$\lambda < 0, \lambda-\delta \geq 2.$$

There exists a $(\mathfrak{g},K)$-submodule submodule Q_+ of $I(\chi_{\delta,\lambda})$ defined as:

$$Q_{+}(w_{\alpha_2}w_{\alpha_1}\chi_{\delta,\lambda}) = U(\mathfrak{g})\tau^{(\frac{\lambda-\delta}{4},\frac{-3\lambda-\delta}{4})}.$$

This subspace form a composition series of $I(\chi_{\delta,\lambda})$:

$$Q_+ \xhookrightarrow[V_1/Q_+=V_{\mathbb{H}}\oplus V_{\mathrm{disc}+}]{\iota} V_0 = I(\chi_{\delta,\lambda})$$

where

$$V_{\mathbb{H}}(w_{\alpha_2}w_{\alpha_1}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k-l\geq-\lambda-\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

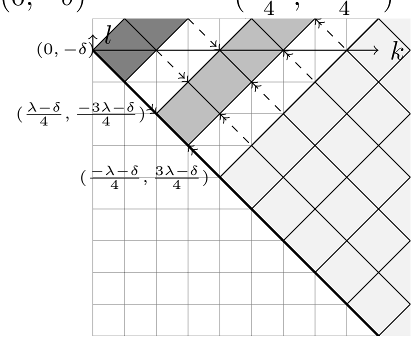
$$Q_+(w_{\alpha_2}w_{\alpha_1}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k-l<-\lambda-\delta\\ k-l\geq\lambda-\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}$$

$$V_{\mathrm{disc}+}(w_{\alpha_2}w_{\alpha_1}\chi_{\delta,\lambda}) = \bigoplus_{\substack{(k,l)\in\texttt{LatticeCond}\\ k-l<\lambda-\delta}} \tau^{(\frac{k}{2},-\delta+\frac{3l}{2})}.$$

For $(\delta,\lambda) = (-6,-2)$, the regions representing the K-types of these modules are shown in the picture below:

Figure 5.7: Weyl chamber III_2

	dark gray : $V_{\mathrm{disc}+}$	medium gray : Q_+	light gray : $V_{\mathbb{H}}$
lowest (j,n)	$(0,-\delta)$	$(\frac{\lambda-\delta}{4},\frac{-3\lambda-\delta}{4})$	$(\frac{-\lambda-\delta}{4},\frac{3\lambda-\delta}{4})$



5.5 The Intertwining Operator

We will prove the Theorem 1.3 in this section. The long intertwining operator of the principal series $I(\chi_{\delta,\lambda})$

$$A(w_0,\chi_{\delta,\lambda})f(g) = \int_{\overline{N}\cap w^{-1}Nw} f(gw_0\overline{n})\mathsf{d}\overline{n}$$

maps each vector $f\in I(\chi_{\delta,\lambda})$ to $A(w_0,w_0\chi_{\delta,\lambda})f\in I(w_0\chi_{\delta,\lambda})$. We are going to show that this operator acts diagonally on the basis elements $W^{(j,n)}_{m_1,m_2}$ with a closed-form matrix coefficient

$$[A(w,\delta,\lambda)]_{m_1} = \frac{\pi^2 2^{-\lambda-1}\Gamma(\lambda)}{\Gamma\left(1-\frac{\lambda-\delta}{2}\right)\Gamma\left(1-\frac{\lambda+\delta}{2}\right)} \frac{\Gamma\left(j+m_1-\frac{\lambda+\delta}{2}+1\right)\Gamma\left(j-m_1-\frac{\lambda-\delta}{2}+1\right)}{\Gamma\left(j+m_1+\frac{\lambda-\delta}{2}+1\right)\Gamma\left(j-m_1+\frac{\lambda+\delta}{2}+1\right)}.$$

We will start by calculating the Iwasawa decomposition of an element of $w_0\bar{n}$. Since the group $SU(2,1)$ has rank 1, there is only one Weyl group element $w_0 = \mathrm{diag}(-1,-1,1)$ as the reflection of the restricted root system $W(\mathfrak{g},\mathfrak{a})$. The intersection $\overline{N}\cap w_0^{-1}Nw_0$ is the set of matrices

$$\overline{N}\cap w_0^{-1}Nw_0 = \overline{N} = \{\mathsf{p}_{\alpha_1+\alpha_2}\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}z & 1 & 0 \\ |z|^2-2iw & \sqrt{2}\bar{z} & 1\end{pmatrix} | z\in\mathbb{C}, w\in\mathbb{R}\}. \tag{5.23}$$

The matrix $\mathsf{p}_{\alpha_1+\alpha_2}\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}z & 1 & 0 \\ |z|^2-2iw & \sqrt{2}\bar{z} & 1\end{pmatrix}$ has an Iwasawa decomposition in the Lie group $SU(2,1)$:

$$\begin{aligned}\mathsf{p}_{\alpha_1+\alpha_2}\begin{pmatrix} 1 & 0 & 0 \\ \sqrt{2}z & 1 & 0 \\ |z|^2-2iw & \sqrt{2}\bar{z} & 1\end{pmatrix} &= \begin{pmatrix} -\frac{|z|^2-2iw-1}{\sqrt{(|z|^2+1)^2+4w^2}} & -\frac{2\bar{z}}{|z|^2-2iw+1} & 0 \\ \frac{2z}{\sqrt{(|z|^2+1)^2+4w^2}} & -\frac{|z|^2+2iw-1}{|z|^2-2iw+1} & 0 \\ 0 & 0 & \frac{|z|^2-2iw+1}{\sqrt{(|z|^2+1)^2+4w^2}}\end{pmatrix} \\ &\mathsf{p}_{\alpha_1+\alpha_2}\left(\mathrm{diag}\left(\sqrt{(|z|^2+1)^2+4w^2}, 1, \frac{1}{\sqrt{(|z|^2+1)^2+4w^2}}\right)\times\right. \\ &\left.\begin{pmatrix} 1 & \frac{\sqrt{2}\bar{z}}{|z|^2-2iw+1} & \frac{|z|^2+2iw}{(|z|^2+1)^2+4w^2} \\ 0 & 1 & \frac{\sqrt{2}z}{|z|^2+2iw+1} \\ 0 & 0 & 1\end{pmatrix}\right),\end{aligned} \tag{5.24}$$

where the image of $\mathsf{p}_{\alpha_1+\alpha_2}$ on the diagonal matrix lies in $\mathfrak{a}$, and the image of $\mathsf{p}_{\alpha_1+\alpha_2}$ on the upper triangular matrix lies in N. Consider a vector $W^{(j,n)}_{m_1,m_2}$ in $I(\chi_{\delta,\lambda})$. According to the Iwasawa decomposition of an element $\bar{n}\in\bar{N}$ in (5.24), the right translation of $w_0\bar{n}$ on this vector can be simplified to

$$\begin{aligned}W^{(j,n)}_{m_1,m_2}(kw_0\bar{n}) &= \left((|z|^2+1)^2+4w^2\right)^{-\frac{\lambda+2}{2}} \\ &W^{(j,n)}_{m_1,m_2}\left(kw_0\begin{pmatrix} -\frac{|z|^2-2iw-1}{\sqrt{(|z|^2+1)^2+4w^2}} & -\frac{2\bar{z}}{|z|^2-2iw+1} & 0 \\ \frac{2z}{\sqrt{(|z|^2+1)^2+4w^2}} & -\frac{|z|^2+2iw-1}{|z|^2-2iw+1} & 0 \\ 0 & 0 & \frac{|z|^2-2iw+1}{\sqrt{(|z|^2+1)^2+4w^2}}\end{pmatrix}\right).\end{aligned} \tag{5.25}$$

Since $w_0 = \mathrm{diag}(-1,-1,1)\in K$, we can absorb w_0 by writing

$$w_0\begin{pmatrix} -\frac{|z|^2-2iw-1}{\sqrt{(|z|^2+1)^2+4w^2}} & -\frac{2\bar{z}}{|z|^2-2iw+1} & 0 \\ \frac{2z}{\sqrt{(|z|^2+1)^2+4w^2}} & -\frac{|z|^2+2iw-1}{|z|^2-2iw+1} & 0 \\ 0 & 0 & \frac{|z|^2-2iw+1}{\sqrt{(|z|^2+1)^2+4w^2}}\end{pmatrix} = \begin{pmatrix} \frac{|z|^2-2iw-1}{\sqrt{(|z|^2+1)^2+4w^2}} & \frac{2\bar{z}}{|z|^2-2iw+1} & 0 \\ -\frac{2z}{\sqrt{(|z|^2+1)^2+4w^2}} & \frac{|z|^2+2iw-1}{|z|^2-2iw+1} & 0 \\ 0 & 0 & \frac{|z|^2-2iw+1}{\sqrt{(|z|^2+1)^2+4w^2}}\end{pmatrix}.$$

If we set $z = x + \mathrm{i}y$, the Haar measure on N is given by $\mathsf{d}x\mathsf{d}y\mathsf{d}w$. Therefore, to understand the intertwining operator $A(w, \delta, \lambda)$, it suffices to compute the singular integral

$$\int_{\mathbb{C}\times\mathbb{R}} \left((|z|^2+1)^2+4w^2\right)^{-\frac{\lambda+2}{2}} W^{(j,n)}_{m_1,m_2}\begin{pmatrix} \frac{|z|^2-2\mathrm{i}w-1}{\sqrt{(|z|^2+1)^2+4w^2}} & \frac{2\bar{z}}{|z|^2-2\mathrm{i}w+1} & 0 \\ -\frac{2z}{\sqrt{(|z|^2+1)^2+4w^2}} & \frac{|z|^2+2\mathrm{i}w-1}{|z|^2-2\mathrm{i}w+1} & 0 \\ 0 & 0 & \frac{|z|^2-2\mathrm{i}w+1}{\sqrt{(|z|^2+1)^2+4w^2}} \end{pmatrix} \mathsf{d}x\mathsf{d}y\mathsf{d}w \tag{5.26}$$

for every K-type (j, n) and all indices in $-j \le m_1, m_2 \le j$. We will calculate the integral in the domain of λ where it converges, and deduce the validity of the formula in Theorem 1.3 by analytic continuation. We recall from Section 4.2.2 that there exists a number $c_\delta > 0$, such that the integral (5.26) converges if $\mathrm{Re}\lambda > c_\delta$. Based on the definition of Wigner D-functions (3.13), the integrand can be expressed as a hypergeometric sum

$$\begin{aligned}&\left((|z|^2+1)^2+4w^2\right)^{-\frac{\lambda+2}{2}} W^{(j,n)}_{m_1,m_2}\begin{pmatrix} \frac{|z|^2-2\mathrm{i}w-1}{\sqrt{(|z|^2+1)^2+4w^2}} & \frac{2\bar{z}}{|z|^2-2\mathrm{i}w+1} & 0 \\ -\frac{2z}{\sqrt{(|z|^2+1)^2+4w^2}} & \frac{|z|^2+2\mathrm{i}w-1}{|z|^2-2\mathrm{i}w+1} & 0 \\ 0 & 0 & \frac{|z|^2-2\mathrm{i}w+1}{\sqrt{(|z|^2+1)^2+4w^2}} \end{pmatrix} \\ =&c^j_{m_1}c^j_{m_2}\sum_{p=\max(0,m_1-m_2)}^{\min(j-m_2,j+m_1)} \frac{(-1)^p 2^{-m_1+m_2+2p}}{(j+m_1-p)!p!(m_2-m_1+p)!(j-m_2-p)!}\omega^{(j,n)}_{m_1,m_2}(p; z, w)\end{aligned} \tag{5.27}$$

where the function $\omega^{(j,n)}_{m_1,m_2}(p; z, w)$ is defined as a function in $z \in \mathbb{C}, w \in \mathbb{R}$

$$\begin{aligned}&\omega^{(j,n)}_{m_1,m_2}(p; z, w) = z^p\bar{z}^{-m_1+m_2+p}(-1+|z|^2+2\mathrm{i}w)^{j+m_1-p}(-1+|z|^2-2\mathrm{i}w)^{j-m_2-p} \\ &(1+|z|^2-2\mathrm{i}w)^{\frac{-2j-m_2+n-\lambda-2}{2}}(1+|z|^2+2\mathrm{i}w)^{\frac{-2j+m_2-n-\lambda-2}{2}}\end{aligned} \tag{5.28}$$

which can be factored into polynomial functions in z and w. Noting that since $1 + |z|^2 > 0$, the complex number $1 + |z|^2 \pm 2\mathrm{i}w$ lies in the right half plane, we can always take a branch cut of the power functions in (5.28) such that the value of $(1+|z|^2 \mp 2\mathrm{i}w)^{\frac{-2j\mp m_2\pm n-\lambda-2}{2}}$ when $z = 0, w = 0$ is 1.

In order to compute the integral (5.26), it suffices to integrate on each summand $\omega^{(j,n)}_{m_1,m_2}(p; z, w)$ over $\mathbb{C}\times\mathbb{R}$. We can change the rectangular coordinate $z = x+\mathrm{i}y$ to the polar coordinate $z = re^{\mathrm{i}\theta}$, and by (5.28), we have

$$\begin{aligned}
&\int_{\mathbb{C}\times\mathbb{R}} \omega^{(j,n)}_{m_1,m_2}(p;z,w)\mathsf{d}x\mathsf{d}y\mathsf{d}w \\
&= \int_0^\infty r^{-m_1+m_2+2p+1}\mathsf{d}r \int_{-\infty}^{\infty} (-1+r^2+2\mathrm{i}w)^{j+m_1-p}(-1+r^2-2\mathrm{i}w)^{j-m_2-p} \\
&(1+r^2-2\mathrm{i}w)^{\frac{-2j-m_2+n-\lambda-2}{2}}(1+r^2+2\mathrm{i}w)^{\frac{-2j+m_2-n-\lambda-2}{2}}\mathsf{d}w \int_0^{2\pi} e^{\mathrm{i}(m_1-m_2)\theta}\mathsf{d}\theta \\
&= 2\pi\delta_{m_1,m_2} \int_0^\infty r^{-m_1+m_2+2p+1} \left(\int_{-\infty}^{\infty} (-1+r^2+2\mathrm{i}w)^{j+m_1-p}(-1+r^2-2\mathrm{i}w)^{j-m_2-p}\right. \\
&\left.(1+r^2-2\mathrm{i}w)^{\frac{-2j-m_2+n-\lambda-2}{2}}(1+r^2+2\mathrm{i}w)^{\frac{-2j+m_2-n-\lambda-2}{2}}\mathsf{d}w\right)\mathsf{d}r
\end{aligned} \tag{5.29}$$

From the last line of the calculation above, we notice that the matrix

$$\left(\int_{\mathbb{C}\times\mathbb{R}} \omega^{(j,n)}_{m_1,m_2}(p;z,w)\mathsf{d}x\mathsf{d}y\mathsf{d}w\right)_{-j\le m_1,m_2\le j}$$

is diagonal due to the appearance of δ_{m_1,m_2}, i.e. its entries are nonzero if and only if $m_1 = m_2$. Therefore, the intertwining operator $A(w,\delta,\lambda)$ acts diagonally on each K-type, and write the diagonal entries as $[A(w,\delta,\lambda)]_{m_1} = \langle W^{(j,n)}_{m_1,*}, A(w,\delta,\lambda)W^{(j,n)}_{m_1,*}\rangle$. We define $\tilde{\omega}^{(j,n)}_{m_1}(r,w)$ as the inner integrand of the integral above:

$$\begin{aligned}
\tilde{\omega}^{(j,n)}_{m_1}(r,w) =&(-1+r^2+2\mathrm{i}w)^{j+m_1-p}(-1+r^2-2\mathrm{i}w)^{j-m_1-p}(1+r^2-2\mathrm{i}w)^{\frac{-2j-m_1+n-\lambda-2}{2}} \\
&(1+r^2+2\mathrm{i}w)^{\frac{-2j+m_1-n-\lambda-2}{2}},
\end{aligned}$$

so that

$$\int_{\mathbb{C}\times\mathbb{R}} \omega^{(j,n)}_{m_1,m_2}(p;z,w)\mathsf{d}x\mathsf{d}y\mathsf{d}w = 2\pi\delta_{m_1,m_2}\int_0^\infty r^{-m_1+m_2+2p+1}\left(\int_{-\infty}^{\infty}\tilde{\omega}^{(j,n)}_{m_1}(r,w)\mathsf{d}w\right)\mathsf{d}r.$$

Using the notation which we have just introduced, the diagonal elements of the intertwining operator can be expressed as

$$[A(w,\delta,\lambda)]_{m_1} = 2\pi \sum_{p=0}^{\min(j-m_1,j+m_1)} \binom{j+m_1}{p}\binom{j-m_1}{p}(-4)^p \int_0^\infty r^{2p+1}\left(\int_{-\infty}^{\infty}\tilde{\omega}^{(j,n)}_{m_1}(r,w)\mathsf{d}w\right)\mathsf{d}r \tag{5.30}$$

Apply the change of variables

$$m_1 = \tfrac{n+\delta}{3}, \quad j = \tfrac{k}{2}, \quad n = -\delta + \tfrac{3}{2}l$$

as we used when describing the structure of $SU(2,1)$ principal series in Section 5.4, the integrand $\tilde{\omega}^{(j,n)}_{m_1}(r,w)$ inside can be replaced by another function labelled by k,l

$$\begin{aligned}\tilde{\omega}_{k,l}(r,w) =&(-1+r^2+2\mathsf{i}w)^{\frac{k+l}{2}-p}(-1+r^2-2\mathsf{i}w)^{\frac{k-l}{2}-p}(1+r^2-2\mathsf{i}w)^{-\frac{k-l+\lambda+\delta+2}{2}}\\ &(1+r^2+2\mathsf{i}w)^{-\frac{k+l+\lambda-\delta+2}{2}}.\end{aligned} \tag{5.31}$$

The integral $\int_{-\infty}^{\infty}\tilde{\omega_{k,l}}(r,w)\mathsf{d}w$ converges for $\lambda > 0$. In order to calculate the integral, we need to reorganize the factors for $\tilde{\omega}_{k,l}(r,w)$ to a simpler form. By applying the change of variable from w to $\frac{1}{2}(1+r^2)w$, the integral $\int_{-\infty}^{\infty}\tilde{\omega_{k,l}}(r,w)\mathsf{d}w$ can be rewritten as follows:

$$\begin{aligned}&\int_{-\infty}^{\infty}(-1+r^2+2\mathsf{i}w)^{\frac{k+l}{2}-p}(-1+r^2-2\mathsf{i}w)^{\frac{k-l}{2}-p}(1+r^2-2\mathsf{i}w)^{-\frac{k-l+\lambda+\delta+2}{2}}\\ &(1+r^2+2\mathsf{i}w)^{-\frac{k+l+\lambda-\delta+2}{2}}\mathsf{d}w\\ =&\frac{1}{2}(1+r^2)^{-1-2p-\lambda}\int_{-\infty}^{\infty}\left(\frac{-1+r^2}{1+r^2}+\mathsf{i}w\right)^{\frac{k+l}{2}-p}\left(\frac{-1+r^2}{1+r^2}-\mathsf{i}w\right)^{\frac{k-l}{2}-p}\\ &(1-\mathsf{i}w)^{-\frac{k-l+\lambda+\delta+2}{2}}(1+\mathsf{i}w)^{-\frac{k+l+\lambda-\delta+2}{2}}\mathsf{d}w.\end{aligned} \tag{5.32}$$

Since in the original summation of Wigner D-functions, when $m_1 = m_2 = m$, $0 \le p \le \min(j-m, j+m) = \min(\frac{k-l}{2}, \frac{k+l}{2})$, the exponents of the first two factors

$$\left(\frac{-1+r^2}{1+r^2}+\mathsf{i}w\right)^{\frac{k+l}{2}-p}\left(\frac{-1+r^2}{1+r^2}-\mathsf{i}w\right)^{\frac{k-l}{2}-p}$$

in the integrand (5.32) are non-negative integers, we can thus reorganize the terms inside of the parenthesis and expand using binomial theorem:

$$\begin{aligned}&\left(\frac{-1+r^2}{1+r^2}+\mathsf{i}w\right)^{\frac{k+l}{2}-p}\left(\frac{-1+r^2}{1+r^2}-\mathsf{i}w\right)^{\frac{k-l}{2}-p}\\ =&\left(\left(\frac{-1+r^2}{1+r^2}+1\right)-1+\mathsf{i}w\right)^{\frac{k+l}{2}-p}\left(\left(\frac{-1+r^2}{1+r^2}-1\right)+1-\mathsf{i}w\right)^{\frac{k-l}{2}-p}\\ =&\sum_{K_1,K_2}(-1)^{\frac{k+l}{2}-p-K2}\binom{\frac{k-l}{2}-p}{K_1}\binom{\frac{k+l}{2}-p}{K_2}\left(\frac{-1+r^2}{1+r^2}-1\right)^{K_1}\left(\frac{-1+r^2}{1+r^2}+1\right)^{K_2}\\ &(1-\mathsf{i}w)^{k-2p-K_1-K_2}.\end{aligned} \tag{5.33}$$

Combining the factor $(1-\mathsf{i}w)^{k-2p-K_1-K_2}$ with (5.32), the integral in (5.30) which depends on w becomes

$$\begin{aligned}&\int_{-\infty}^{\infty}(1-\mathrm{i}w)^{-\frac{k-l+(\lambda+\delta+2)}{2}+k-2p-K_1-K_2}(1+\mathrm{i}w)^{-\frac{k+l+(\lambda-\delta+2)}{2}}\mathsf{d}w\\ =&\frac{2^{-K_1-K_2-2p-\lambda}\pi\Gamma(1+K_1+K_2+2p+\lambda)}{\Gamma\left(\frac{k+l+\lambda-\delta}{2}+1\right)\Gamma\left(-\frac{k+l-\lambda-\delta}{2}+1+K_1+K_2+2p\right)}.\end{aligned}\tag{5.34}$$

Then we can take care of the integral which depends on r in (5.30):

$$\begin{aligned}&\int_{0}^{\infty}(1+r^2)^{-1-2p-\lambda}\left(\frac{-1+r^2}{1+r^2}-1\right)^{K_1}\left(\frac{-1+r^2}{1+r^2}+1\right)^{K_2}r^{2p+1}\mathsf{d}r\\ =&(-1)^{K_1}2^{K_1+K_2-1}\frac{\Gamma(1+K_2+p)\Gamma(K_1+p+\lambda)}{\Gamma(1+K_1+K_2+2p+\lambda)}\end{aligned}\tag{5.35}$$

Putting (5.31)-(5.32) back into the intertwining operator $[A(w,\delta,\lambda)]_{m_1}$ integral (5.30) and applying the change of indices in j, m_1 to k, l, the summation in (5.30) becomes a sum over Γ-functions and binomial coefficients. We can utilize a trick by changing all the binomial coefficients into their Γ function expressions, and group the Γ-factors in the following way:

$$\begin{aligned}&[A(w,\delta,\lambda)]_{m_1}=2^{-\lambda-1}(-1)^{\frac{k+l}{2}}\pi^2\frac{\Gamma\left(\frac{k+l+2}{2}\right)\Gamma\left(\frac{k-l+2}{2}\right)}{\Gamma\left(\frac{k+l+\lambda-\delta}{2}+1\right)}\\ &\sum_{\substack{K_1,K_2\geq 0\\ p\geq 0}}\frac{(-1)^{K_1+K_2}}{\Gamma\left(K_1+1\right)\Gamma\left(K_2+1\right)\Gamma\left(K_1+K_2+2p-\frac{k+l-\lambda-\delta}{2}+1\right)}\\ &\frac{\Gamma\left(p+\lambda+K_1\right)\Gamma\left(1+K_2+p\right)}{\Gamma(p+1)^2\Gamma\left(\frac{k-l}{2}-p-K_1+1\right)\Gamma\left(\frac{k+l}{2}-p-K_2+1\right)}\end{aligned}$$

Reorganizing the Γ-functions into multinomial coefficients

$$\binom{n}{n_1,\ldots,n_r}=\frac{n!}{n_1!\cdots n_r!}$$

and adding auxiliary Γ factors as required, we get:

$$\begin{aligned}&[A(w,\delta,\lambda)]_{m_1}=2^{-\lambda-1}(-1)^{\frac{k+l}{2}}\pi^2\frac{\Gamma\left(\frac{k+l+2}{2}\right)\Gamma\left(\frac{k-l+2}{2}\right)\Gamma(\lambda)}{\Gamma\left(\frac{k+l-\delta+\lambda}{2}+1\right)\Gamma\left(\frac{k-l+\delta+\lambda}{2}+1\right)}\\ &\sum_{\substack{K_1,K_2\geq 0\\ p\geq 0}}(-1)^{K_1+K_2}\binom{K_1+p+\lambda-1}{\lambda-1,K_1,p}\binom{K_2+p}{p}\binom{\frac{k-l+\lambda+\delta}{2}}{\frac{k-l}{2}-p-K_1,\frac{k+l}{2}-p-K_2,-\frac{k+l-\lambda-\delta}{2}+K_1+K_2+2p}.\end{aligned}\tag{5.36}$$

Noticing that the $y^{K_1+p}z^{K_2+p}$-th multinomial coefficient of the following function in y, z:

$$y^{\frac{k-l}{2}}z^{\frac{k+l}{2}}\left(1-\frac{1}{y}-\frac{1}{z}\right)^{\frac{k-l+\lambda+\delta}{2}}$$

is the multinomial coefficient $(-1)^{k+K_1+K_2}\binom{\frac{k-l+\lambda+\delta}{2}}{\frac{k-l}{2}-p-K_1,\frac{k+l}{2}-p-K_2,-\frac{k+l-\lambda-\delta}{2}+K_1+K_2+2p}$ in the third factor of each summand, we can further take $y=(1+s)(1+t)$ and $z=1+1/t$. We consider the multinomial expansion of the function

$$
\begin{aligned}
&(1+s)^{\frac{k-l}{2}+\lambda-1}(1+t)^{\frac{k-l}{2}}(1+1/t)^{\frac{k+l}{2}}\left(1-\frac{1}{(1+s)(1+t)}-\frac{1}{(1+1/t)}\right)^{\frac{k-l+\lambda+\delta}{2}}\\
=&\sum_{\kappa_1,\kappa_2\in\mathbb{Z}}(-1)^{k+\kappa_1+\kappa_2}\binom{\frac{k-l+\lambda+\delta}{2}}{\frac{k-l}{2}-\kappa_1,\frac{k+l}{2}-\kappa_2,-\frac{k+l-\lambda-\delta}{2}+\kappa_1+\kappa_2}(1+s)^{\kappa_1+\lambda-1}\\
&(1+t)^{\kappa_1}(1+1/t)^{\kappa_2},
\end{aligned}
\tag{5.37}
$$

its coefficient of the term $s^{\lambda-1}t^0$ is given by

$$
\begin{aligned}
&\sum_{\kappa_1,\kappa_2\in\mathbb{Z}}(-1)^{k+\kappa_1+\kappa_2}\binom{\frac{k-l+\lambda+\delta}{2}}{\frac{k-l}{2}-\kappa_1,\frac{k+l}{2}-\kappa_2,-\frac{k+l-\lambda-\delta}{2}+\kappa_1+\kappa_2}\binom{\kappa_1+\lambda-1}{\kappa_1}\\
&\sum_{p\in\mathbb{Z}}\binom{\kappa_1}{p}\binom{\kappa_2}{p}
\end{aligned}
\tag{5.38}
$$

which is $(-1)^k$ times the sum in (5.36) if one changes K_i+p to κ_i. The sums in (5.36) are over non-negative integers, which is guaranteed by the non-vanishing of binomial coefficients. Putting these binomial coefficients all together, it is clear that the sum in (5.36) is the constant term coefficient of the function

$$
(-1)^{-k}s^{1+\frac{1}{2}(k-l-\lambda+\delta)}(1+s)^{\frac{\lambda-\delta}{2}-1}t^{-\frac{k+l}{2}}(1+t)^{\frac{k+l-\delta-\lambda}{2}}
\tag{5.39}
$$

which by the binomial theorem is equal to

$$
(-1)^{-k}\frac{\Gamma\left(1+\frac{k+l}{2}-\frac{\lambda+\delta}{2}\right)\Gamma\left(\frac{\lambda-\delta}{2}\right)}{\Gamma\left(\frac{k+l}{2}+1\right)\Gamma\left(\frac{k-l}{2}+1\right)\Gamma\left(1-\frac{\lambda+\delta}{2}\right)\Gamma\left(-\frac{1}{2}(k-l)+\frac{\lambda-\delta}{2}\right)}.
\tag{5.40}
$$

The matrix entries $[A(w,\delta,\lambda)]_{m_1}$ for the intertwining operator simplifies to the function

$$
[A(w,\delta,\lambda)]_{m_1}=\frac{(-1)^{\frac{k-l}{2}}\pi^2}{2^{\lambda+1}}\frac{\Gamma(\lambda)\Gamma\left(\frac{\lambda-\delta}{2}\right)}{\Gamma\left(1-\frac{\lambda+\delta}{2}\right)}\frac{\Gamma\left(\frac{k+l-\lambda-\delta+2}{2}\right)}{\Gamma\left(-\frac{k-l-\lambda+\delta}{2}\right)\Gamma\left(\frac{k+l+\lambda-\delta+2}{2}\right)\Gamma\left(\frac{k-l+\lambda+\delta+2}{2}\right)}.
\tag{5.41}
$$

If we change the indices (k,l) back to j,m_1, we have:

$$
\begin{aligned}
[A(w,\delta,\lambda)]_{m_1}=&\frac{\pi^2 2^{-\lambda-1}(-1)^{j-m_1}\Gamma(\lambda)\Gamma\left(\frac{\lambda-\delta}{2}\right)}{\Gamma\left(1-\frac{\lambda+\delta}{2}\right)}\\
&\frac{\Gamma\left(j+m_1-\frac{\lambda+\delta}{2}+1\right)}{\Gamma\left(-j+m_1+\frac{\lambda-\delta}{2}\right)\Gamma\left(j+m_1+\frac{\lambda-\delta}{2}+1\right)\Gamma\left(j-m_1+\frac{\lambda+\delta}{2}+1\right)}
\end{aligned}
$$

We can apply the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

we can move one of the Γ-factors from the denominator to the numerator and vice versa, which gives the final formula for the long intertwining operator entries:

$$[A(w,\delta,\lambda)]_{m_1} = \frac{\pi^2 2^{-\lambda-1}\Gamma(\lambda)}{\Gamma\left(1-\frac{\lambda-\delta}{2}\right)\Gamma\left(1-\frac{\lambda+\delta}{2}\right)} \frac{\Gamma\left(j+m_1-\frac{\lambda+\delta}{2}+1\right)\Gamma\left(j-m_1-\frac{\lambda-\delta}{2}+1\right)}{\Gamma\left(j+m_1+\frac{\lambda-\delta}{2}+1\right)\Gamma\left(j-m_1+\frac{\lambda+\delta}{2}+1\right)} \tag{5.42}$$

Thus we have proven Theorem 1.3.

Chapter 6

Example: $Sp(4,\mathbb{R})$

6.1 The Lie Group $Sp(4,\mathbb{R})$

6.1.1 The Structure of the Group $Sp(4,\mathbb{R})$

The real symplectic group $G = Sp(4,\mathbb{R})$ is the subgroup of $SL(4,\mathbb{R})$ consisting of all elements $g \in SL(4,\mathbb{R})$ such that $g^t J g = J$, where

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The real Lie algebra of the symplectic group is

$$\mathfrak{g} = \mathfrak{sp}(4,\mathbb{R}) = \{X \in \mathfrak{sl}(4,\mathbb{R}) | X^t J + JX = 0\}.$$

A root space decomposition for the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and its Chevalley basis are determined by the following choice of data:

1. A Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ generated by $H_1 = E_{1,1} - E_{3,3}, H_2 = E_{2,2} - E_{4,4}$;

2. The simple roots α_1 and α_2 sending $t_1H_1 + t_2H_2$ to

$$\alpha_1(t_1H_1 + t_2H_2) = t_1 - t_2, \quad \alpha_2(t_1H_1 + t_2H_2) = 2t_2$$

3. The simple coroots $\check{\alpha}_1$ and $\check{\alpha}_2$ sending $t_1H_1 + t_2H_2$ to

$$\check{\alpha}_1(t_1H_1 + t_2H_2) = t_1 - t_2, \quad \check{\alpha}_2(t_1H_1 + t_2H_2) = t_2$$

4. The fundamental weights ϖ_1 and ϖ_2 in $\mathfrak{h}_{\mathbb{C}}^*$ satisfying $\langle \varpi_i, \check{\alpha}_j \rangle = \delta_{ij}$;

5. The set of positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$;

6. $\rho_{\mathbb{C}} = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha = \frac{4\alpha_1 + 3\alpha_2}{2} = \varpi_1 + \varpi_2$;

7. A basis for positive root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$

$$X_{\alpha_1} = E_{1,2} - E_{4,3} \qquad X_{\alpha_2} = E_{2,4}$$
$$X_{\alpha_1+\alpha_2} = E_{2,3} + E_{1,4} \quad X_{2\alpha_1+\alpha_2} = E_{1,3}$$

   with the corresponding negative root vector given by $X_{-\alpha} = X_\alpha^t$;

We will also represent an element $\lambda \in \mathfrak{h}_\mathbb{C}^*$ by a pair of complex numbers (λ_1, λ_2) where $\lambda_i = \lambda(H_i)$. In this coordinate, $\rho_\mathbb{C}$ is represented by $(2, 1)$.

On the real Lie group $Sp(4, \mathbb{R})$, there is a Cartan involution $\theta(g) = (g^t)^{-1}$ which determines a subgroup of fixed points $K = G^\theta$. The Lie algebra $\mathfrak{k} \subset \mathfrak{g}$ of the maximal compact subgroup $K \subset G$ is:

$$\mathfrak{k} = \{ \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) | A \text{ antisymmetric}, B \in \mathfrak{sym}(2) \}$$

The map $\left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right) \mapsto A + iB$ from $\mathfrak{k}$ to 2×2 matrices identifies the Lie algebra $\mathfrak{k}$ with $\mathfrak{u}(2)$. The 4-dimensional Lie algebra $\mathfrak{k}$ have generators U_0, U_1, U_2, U_3 corresponding to the 4 infinitesimal generators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ of $\mathfrak{u}(2)$:

$$U_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; U_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$U_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; U_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

They satisfy a commutation relation:

$$[U_0, U_i] = 0, [U_i, U_j] = -\epsilon_{ijk} U_k$$

where ϵ_{ijk} is the Levi-Civita symbol as was defined in the formula (3.2).

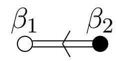
6.1.2 The Cartan Subgroups of $Sp(4, \mathbb{R})$

The Maximally Compact Cartan Subalgebra

A Cartan subalgebra of $K \cong U(2)$ is $\mathfrak{t} = \mathbb{R}(U_0 + U_3) \oplus \mathbb{R}(U_0 - U_3)$. The basis vectors $U_0 \pm U_3$ are deliberately chosen so that they can be related to H_{α_1} and H_{α_2} by Cayley transforms. Let the the simple roots β_1, β_2 act on an element $t_1(U_0+U_3)+t_2(U_0-U_3) \in \mathfrak{t}_{\mathbb{C}}$ by

$$\beta_1(t_1(U_0+U_3)+t_2(U_0-U_3)) = \mathsf{i}(t_1-t_2),$$
$$\beta_2(t_1(U_0+U_3)+t_2(U_0-U_3)) = 2\mathsf{i}t_2.$$

In the Vogan diagram (see Example 2.2), the shorter root is named to be the compact root. In this situation, all roots are imaginary:



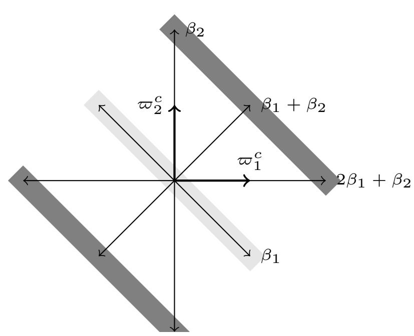
We can decompose the set of positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ into the union of the set of compact roots and noncompact roots:

$$\Delta_c^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\beta_1\}$$
$$\Delta_{nc}^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\beta_2, \beta_1+\beta_2, 2\beta_1+\beta_2\};$$

The compact root vectors are

$$v_{\pm\beta_1} = \frac{1}{\mathsf{i}}(U_1 \pm \mathsf{i}U_2).$$

These roots are displayed in the following picture, where the gray color stands for noncompact roots, and the light gray color stands for compact roots:



The subspace $\mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_{\beta_1} \oplus \mathfrak{g}_{-\beta_1}$ of $\mathfrak{g}_{\mathbb{C}}$ generated by the vectors $\{v_{\beta_1}, v_{-\beta_1}, U_0+U_3, U_0-U_3\}$ is isomorphic to $\mathfrak{gl}(2,\mathbb{C})$. The embedding of this Lie subalgebra into $\mathfrak{g}_{\mathbb{C}}$ corresponds to the embedding of $U(2)$ into G as K.

The -1 eigenspace $\mathfrak{p}_{\mathbb{C}}$ for θ is generated by the following noncompact root vectors:

$$v_{\pm(2\beta_1+\beta_2)} = \frac{1}{2}(H_1 \pm \mathsf{i}(X_{2\alpha_1+\alpha_2} + X_{-2\alpha_1-\alpha_2})) \tag{6.1}$$

$$v_{\pm\beta_2} = \frac{1}{2}(H_2 \pm \mathsf{i}(X_{\alpha_2} + X_{-\alpha_2})) \tag{6.2}$$

$$v_{\pm(\beta_1+\beta_2)} = \frac{1}{2}(X_{\alpha_1+\alpha_2} + X_{-\alpha_1-\alpha_2} \mp \mathsf{i}(X_{\alpha_1} + X_{-\alpha_1})). \tag{6.3}$$

A Cayley transform and the maximal noncompact Cartan subalgebra

There are 4 conjugacy classes of Cartan subgroups of $Sp(4,\mathbb{R})$ displayed in the diagram below, connected by Cayley transforms corresponding to the roots β_2 and $2\beta_1+\beta_2$:

$$\begin{tikzcd} & \mathbb{S}^1 \times \mathbb{S}^1 \arrow[dl, "\mathsf{c}_{\beta_2}"'] \arrow[dr, "\mathsf{c}_{2\beta_1+\beta_2}"] & \\ \mathbb{R}^\times \times \mathbb{S}^1 \arrow[dr, "\mathsf{c}_{2\beta_1+\beta_2}"'] & & \mathbb{S}^1 \times \mathbb{R}^\times \arrow[dl, "\mathsf{c}_{\beta_2}"] \\ & \mathbb{R}^\times \times \mathbb{R}^\times & \end{tikzcd} \tag{6.4}$$

Consider the *Cayley* transforms $\mathsf{c}_\beta = \operatorname{Ad}\exp(\frac{\pi}{4}(\overline{v_\beta} - v_\beta))$ for a noncompact root β, when $\beta = \beta_2$ or $2\beta_1+\beta_2$, the Cayley transform is nontrivial:

$$\mathsf{c}_{\beta_2} = \operatorname{Ad}\exp(\frac{\pi}{4}(\overline{v_{\beta_2}} - v_{\beta_2}))$$

$$\mathsf{c}_{2\beta_1+\beta_2} = \operatorname{Ad}\exp(\frac{\pi}{4}(\overline{v_{2\beta_1+\beta_2}} - v_{2\beta_1+\beta_2})).$$

The two Cayley transforms c_{β_2} and $c_{2\beta_1+\beta_2}$ commute with each other. If we start from the real Cartan subalgebra $\mathfrak{t}$, the other real Cartan subalgebras of $Sp(4,\mathbb{R})$ can be obtained by applying the Cayley transforms c_{β_2} and $\mathsf{c}_{2\beta_1+\beta_2}$:

$$\begin{tikzcd} & \mathbb{R}(U_0+U_3) \oplus \mathbb{R}(U_0-U_3) \arrow[dl, "\mathsf{c}_{\beta_2}"'] \arrow[dr, "\mathsf{c}_{2\beta_1+\beta_2}"] & \\ \mathbb{R}(U_0+U_3) \oplus \mathbb{R}\mathsf{i}H_2 \arrow[dr, "\mathsf{c}_{2\beta_1+\beta_2}"'] & & \mathbb{R}\mathsf{i}H_1 \oplus \mathbb{R}(U_0-U_3) \arrow[dl, "\mathsf{c}_{\beta_2}"] \\ & \mathbb{R}\mathsf{i}H_1 \oplus \mathbb{R}\mathsf{i}H_2 & \end{tikzcd}$$

This diagram shows how the generators of the Cartan subalgebras are mapped to each other by the Cayley transforms shown in the diagram (6.4). We define the *maximally* *noncompact Cartan subalgebra* $\mathfrak{a} = \mathbb{R}H_1 \oplus \mathbb{R}H_2 \subset \mathfrak{p}$. The roots α_i acts on $\mathfrak{h}_{\mathbb{C}} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$, and they can be related to the roots β_i on the compact Cartan subgroup by

$$\alpha_i \circ (\mathsf{c}_{\beta_2}\mathsf{c}_{2\beta_1+\beta_2}) = \beta_i.$$

Moreover, we can apply the composite Cayley transform on the root vectors v_β:

$$\mathsf{c}_{\beta_2}\mathsf{c}_{2\beta_1+\beta_2}v_\beta = \mathsf{c}_{2\beta_1+\beta_2}\mathsf{c}_{\beta_2}v_\beta = \begin{cases} \mathrm{i}X_{\beta\circ(\mathsf{c}_{\beta_2}\mathsf{c}_{2\beta_1+\beta_2})^{-1}} & \text{if } \beta \in \{\beta_2, 2\beta_1+\beta_2\} \\ X_{\beta\circ(\mathsf{c}_{\beta_2}\mathsf{c}_{2\beta_1+\beta_2})^{-1}} & \text{if } \beta \in \{\beta_1+\beta_2\} \end{cases}.$$

Weyl Group

From the embeddings $\Phi_{\alpha_1}, \Phi_{\alpha_2}$ of $SL(2,\mathbb{R})$ into $Sp(4,\mathbb{R})$ given by the simple roots α_1, α_2, the simple Weyl reflections corresponding to these two roots are:

$$w_{\alpha_i} = \Phi_{\alpha_i}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \exp\left(\frac{\pi}{2}(X_{\alpha_i} - X_{-\alpha_i})\right).$$

The Lie algebra $\mathfrak{k}$ has basis $\{X_\alpha - X_{-\alpha}\}_{\alpha\in\Delta^+}$, and the relationship between this basis and the basis $\{U_i\}$ is:

$$\begin{aligned} U_0 + U_3 &= X_{2\alpha_1+\alpha_2} - X_{-2\alpha_1-\alpha_2} & 2U_1 &= X_{\alpha_1+\alpha_2} - X_{-\alpha_1-\alpha_2} \\ 2U_2 &= X_{\alpha_1} - X_{-\alpha_1} & U_0 - U_3 &= X_{\alpha_2} - X_{-\alpha_2}. \end{aligned}$$

Therefore, the simple reflections w_{α_i} can be expressed as

$$w_{\alpha_1} = \exp(\pi U_2), w_{\alpha_2} = \exp\left(\frac{\pi}{2}(U_0 - U_3)\right). \tag{6.5}$$

Under the basis $\{\alpha_1, \alpha_2\}$, the actions of the simple reflections on an element $n_1\alpha_1 + n_2\alpha_2 \in \mathfrak{h}_{\mathbb{C}}^*$ are:

$$w_{\alpha_1}(n_1\alpha_1 + n_2\alpha_2) = (2n_2 - n_1)\alpha_1 + n_2\alpha_2$$
$$w_{\alpha_2}(n_1\alpha_1 + n_2\alpha_2) = n_1\alpha_1 + (n_1 - n_2)\alpha_2.$$

If we represent $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ by (λ_1, λ_2), the actions of w_{α_i} on (λ_1, λ_2) where $\lambda_i = \lambda(H_i)$ is

$$w_{\alpha_1}(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1)$$
$$w_{\alpha_2}(\lambda_1, \lambda_2) = (\lambda_1, -\lambda_2).$$

The action of the simple reflections on the nilpotent radical satisfies:

$$\overline{\mathfrak{n}} \cap \mathrm{Ad}(w_{\alpha_i})^{-1}\mathfrak{n} = \mathbb{R}X_{-\alpha_i}.$$

Harish-Chandra isomorphism

Since $\mathfrak{sp}(4,\mathbb{C})$ is rank 2, there are two Casimir elements Ω_2 and Ω_4 with degree 2 and degree 4 respectively. They generate the center $Z(\mathfrak{g}_\mathbb{C})$ of the complexified universal enveloping algebra. Consider the adjoint representation ad on $\mathfrak{g}_\mathbb{C}$, then the corresponding element in the center is [Yan11]:

$$\begin{aligned}\Omega_2 =& \sum_{i,j} \mathrm{Tr}(\mathrm{ad}(X_i)\mathrm{ad}(X_j))\tilde{X}_i\tilde{X}_j = \sum_{i,j} B(X_i,X_j)\tilde{X}_i\tilde{X}_j \\ =&\frac{1}{12}(H_1^2+H_2^2+4H_1+2H_2+2(X_{-\alpha_1}X_{\alpha_1}+2X_{-\alpha_2}X_{\alpha_2} \\ &+X_{-\alpha_1-\alpha_2}X_{\alpha_1+\alpha_2}+2X_{-2\alpha_1-\alpha_2}X_{2\alpha_1+\alpha_2}))\end{aligned}$$

where $\{\tilde{X}_i\}$ is a dual basis of the basis $\{X_i\}$ of $\mathfrak{g}_\mathbb{C}$ as in Section 2.1.4. The central element Ω_4 corresponding to the standard representation is a quartic element in $U(\mathfrak{g}_\mathbb{C})$ which we will only write down the $U(\mathfrak{h}_\mathbb{C})$ part explicitly:

$$\begin{aligned}\Omega_4 =& \frac{1}{5184}((H_1+H_2+3)^4-2(2H_1H_2+2H_1+4H_2+7)((H_1+2)^2+(H_2+1)^2)-11) \\ &+\text{other terms}\end{aligned}$$

Since $\rho=\varpi_1+\varpi_2$, the Harish-Chandra homomorphism γ' will map each H_i to:

$$\gamma'(H_1)=H_1-2,\gamma'(H_2)=H_2-1$$

The image of Ω_2 and Ω_4 under the Harish-Chandra isomorphism γ' are:

$$\begin{aligned}\gamma'(\Omega_2) &= \frac{1}{12}(H_1^2+H_2^2-5) \\ \gamma'(\Omega_4) &= \frac{1}{5184}(H_1^4+H_2^4+6H_1^2H_2^2-6(H_1^2+H_2^2)-11).\end{aligned}$$

6.2 Iwasawa decomposition on the group level

We would like to compute the Iwasawa decomposition $kman$ of an element $\bar{n}\in\overline{N}$. For a root α with $\chi_{-\alpha}(t)=\exp(tX_{-\alpha})\in\overline{N}$, the corresponding Lie group $SL(2,\mathbb{C})$ embedded into $G_\mathbb{C}$ has generators:

$$\chi_\alpha(t)=\exp(tX_\alpha),\chi_{-\alpha}(t)=\exp(tX_{-\alpha}),h_\alpha(t)=\exp(tH_\alpha)$$

In terms of these matrix generators of $SL(2,\mathbb{C})$, the Iwasawa decomposition of $\chi_{-\alpha}(t)$ is

$$\chi_{-\alpha}(t) = \kappa_\alpha(t) h_\alpha\left(\sqrt{1+t^2}\right) \chi_\alpha\left(\frac{t}{1+t^2}\right) \tag{6.6}$$

where

$$\kappa_\alpha(t) = \exp\left(\arctan(-t)(X_\alpha - X_{-\alpha})\right)$$

is an element in the maximal compact subgroup $K_0 \cong SO(2,\mathbb{R})$. In general, for any simple real root α for $G = Sp(4,\mathbb{R})$, we would like to study at the embedding ϕ_α of a Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ into $\mathfrak{g}$, and the corresponding homomorphism of a Lie group $\Phi_\alpha : SL(2,\mathbb{R}) \longrightarrow G$. The image of Φ_α is fixed by θ, and the embedding for the Lie algebra satisfies:

1. $\phi_\alpha\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = X_\alpha, \phi_\alpha\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = X_{-\alpha}, \phi_\alpha\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = H_\alpha$,
2. $\Phi_\alpha\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) = \chi_\alpha(t), \Phi_\alpha\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) = \chi_{-\alpha}(t), \Phi_\alpha\left(\begin{smallmatrix} t & 0 \\ 0 & 1/t \end{smallmatrix}\right) = h_\alpha(t)$.

We can see that the homomorphisms Φ_α and ϕ_α respect the Cartan involution: if we denote θ' as the Cartan involution on $SL(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$, we can see that $\Phi_\alpha \circ \theta' = \theta \circ \Phi_\alpha$ and $\phi_\alpha \circ \theta' = \theta \circ \phi_\alpha$. Therefore, $\mathrm{Im}(\Phi_\alpha) \cap K = \Phi_\alpha(K_0)$ where K_0 is the maximal compact subgroup $SO(2,\mathbb{R})$ in $SL(2,\mathbb{R})$.

Recall that under any simple reflection w_{α_i}, there is a root vector $X_{-\alpha_i}$ such that $\bar{\mathfrak{n}} \cap w_{\alpha_i}^{-1} \mathfrak{n} w_{\alpha_i} = \mathbb{R}X_{-\alpha_i}$. We can factor the nilpotent group $w_{\alpha_i}^{-1} N w_{\alpha_i}$ into

$$w_{\alpha_i}^{-1} N w_{\alpha_i} = N_{-\alpha} N' \text{ where } N' \subset N \text{ and } N_{-\alpha} = \{e^{tX_{-\alpha_i}} | t \in \mathbb{R}\}.$$

Using this factorization of the group $w_{\alpha_i}^{-1} N w_{\alpha_i}$, we can find a coordinate system on $w_{\alpha_i}^{-1} N w_{\alpha_i}$ which is consistent with the Iwasawa decomposition. Namely, for any $n \in N$, there is a $t \in \mathbb{R}$ and $n' \in N'$ such that

$$w_{\alpha_i}^{-1} n w_{\alpha_i} = \kappa_\alpha(t) h_\alpha\left(\sqrt{1+t^2}\right) \chi_\alpha\left(\frac{t}{1+t^2}\right) n'.$$

6.3 Parabolic Subgroups of $Sp(4,\mathbb{R})$

Denote P_0 as the *minimal parabolic* subgroup of $Sp(4,\mathbb{R})$, with the Lie algebra having a Levi decomposition:

$$\mathfrak{p}_0 = \mathfrak{a} \oplus \mathfrak{n}_0, \text{ where } \mathfrak{n}_0 = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha.$$

The parabolic subgroup P_0 has a Levi decomposition:

$$P_0 = MA_0N_0, \text{ where } M = \{\gamma_{\alpha_2}^{\epsilon_1}\gamma_{2\alpha_1+\alpha_2}^{\epsilon_2} | \epsilon_i = 0 \text{ or } 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where recalling the statements in the end of Section 2.2.4, the generators $\gamma_{\alpha_2}, \gamma_{2\alpha_1+\alpha_2}$ of M are defined as:

$$\gamma_{\alpha_2} = \exp(\pi(U_0 - U_3)), \gamma_{2\alpha_1+\alpha_2} = \exp(\pi(U_0 + U_3)).$$

M is the centralizer of $\mathfrak{a}$ in K, and A_0 and N_0 are the analytic subgroups formed by exponentiating $\mathfrak{a}$ and $\mathfrak{n}_0$, respectively. There are two proper standard parabolic subgroups that contain P_0. The parabolic subgroup P_1 with an abelian nilpotent radical is called a *Siegel* subgroup. The Levi decomposition $P_1 = M_1N_1$ has the form:

$$M_1 = \{\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, A \in GL(2,\mathbb{R})\} \cong GL(2,\mathbb{R})$$
$$N_1 = \{\begin{pmatrix} 1 & 0 & x_4 & x_3 \\ 0 & 1 & x_3 & x_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, x_i \in \mathbb{R}\} \cong \mathbb{R}^3.$$

Their Lie algebras have restricted root space decompositions:

$$\mathfrak{m}_1 = \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}$$
$$\mathfrak{n}_1 = \bigoplus_{\alpha\in\{\alpha_2,\alpha_1+\alpha_2,2\alpha_1+\alpha_2\}} \mathfrak{g}_\alpha.$$

There is another class of parabolic subgroup called the *Jacobi* (or *Heisenberg* and in some literature also called *Klingen*) parabolic subgroup $P_2 = M_2N_2$, having a Levi decomposition:

$$M_2 = \{\begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1/h_1 & 0 \\ 0 & c & 0 & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), h_1 \in \mathbb{R}^\times\} \cong SL(2,\mathbb{R}) \times \mathbb{R}^\times$$
$$N_2 = \{\begin{pmatrix} 1 & x_1 & x_4 & x_3 \\ 0 & 1 & x_3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x_1 & 1 \end{pmatrix}, x_i \in \mathbb{R}\}.$$

The Lie algebras of each subgroup have restricted root space decompositions:

$$\mathfrak{m}_2 = \mathfrak{a} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\alpha_2}$$
$$\mathfrak{n}_2 = \bigoplus_{\alpha \in \{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}} \mathfrak{g}_\alpha.$$

Thus $\mathfrak{n}_2 = \bigoplus_{\alpha \in \{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}} \mathbb{R}X_\alpha$. Since the root vectors X_α satisfy the commutation relations

$$[X_\alpha, X_{2\alpha_1+\alpha_2}] = 0 \text{ for } \alpha \in \{\alpha_1, \alpha_1 + \alpha_2\}$$
$$[X_{\alpha_1}, X_{\alpha_1+\alpha_2}] = 2X_{2\alpha_1+\alpha_2},$$

the group N_2 is isomorphic to the Heisenberg group $\mathcal{H}_3$.

Induction from the Minimal Parabolic

In this section consider the minimal principal series obtained by induction from the minimal parabolic subgroup P_0. A minimal principal series representation $I_{P_0}(\chi_{\delta,\lambda})$ of $Sp(4,\mathbb{R})$ is determined by the following data:

1. A continuous character $\lambda : \mathfrak{a} \longrightarrow \mathbb{C}$ represented by the pair (λ_1, λ_2) with $\lambda_i = \lambda(H_i)$. This character can be extended linearly to a character on $\mathfrak{a} \otimes \mathbb{C} = \mathfrak{h}$, also denoted by λ;

2. A character $\delta : M \longrightarrow \{\pm 1\}$ represented by a pair (δ_1, δ_2) where $\delta_i \in \{0, 1\}$ such that $\delta(\gamma_{\alpha_2}^{\epsilon_1} \gamma_{2\alpha_1+\alpha_2}^{\epsilon_2}) = (-1)^{\delta_1\epsilon_1+\delta_2\epsilon_2}$.

If the numbers λ_i are integers, with $\delta_i \equiv \lambda_i \bmod 2$, as in the second example of Example 4.1, the parameters defined above combine to an algebraic character $\chi_{\delta,\lambda}$ on the complex Cartan subalgebra $\mathfrak{a}_\mathbb{C} = \mathbb{C}H_1 \oplus \mathbb{C}H_2$. The $(\mathfrak{g}, K)$-module of the principal series representation $I_{P_0}(\chi_{\delta,\lambda})$ embeds into the space

$$C_\delta(K) = \{f : K \longrightarrow \mathbb{C} | f \text{ smooth and } f(km) = \delta(m)^{-1} f(k)\}.$$

This space is isomorphic to the space of smooth global sections of the line bundle $K \otimes_M \mathbb{C}_\delta$, where $\mathbb{C}_\delta$ is the vector space on which M acts by δ^{-1}. To get a basis for the space $C_\delta(K)$ from the Wigner D-functions, we consider the right action by $m \in M$ on an arbitrary function $f \in C_\delta(K)$. f can be written as the finite linear combination of Wigner D-functions with coefficients $a^{(j,n)}_{m_1,m_2}$:

$$f(k) = \sum_{\substack{j,n \\ -j \le m_i \le j}} a^{(j,n)}_{m_1,m_2} W^{(j,n)}_{m_1,m_2}(k).$$

The action by $m \in M$ on the right gives

$$f(km) = \sum_{\substack{j,n \\ -j \le m_i \le j}} a^{(j,n)}_{m_1,m_2} W^{(j,n)}_{m_1,m_2}(km).$$

Recall that the Wigner D-functions are matrix coefficients of $U(2)$-representations. Their values on the product of two elements km come from the multiplication of two matrices:

$$W^{(j,n)}_{m_1,m_2}(km) = \sum_{m_3} W^{(j,n)}_{m_1,m_3}(k) W^{(j,n)}_{m_3,m_2}(m).$$

The action of a general element

$$m = \gamma^{\epsilon_1}_{\alpha_2} \gamma^{\epsilon_2}_{2\alpha_1+\alpha_2} = e^{\pi(\epsilon_1+\epsilon_2)U_0} e^{\pi(-\epsilon_1+\epsilon_2)U_3} \in M$$

on the Wigner D-functions is diagonal:

$$W^{(j,n)}_{m_3,m_2}(e^{\pi(\epsilon_1+\epsilon_2)U_0} e^{\pi(-\epsilon_1+\epsilon_2)U_3}) = (-1)^{-(n-m_2)\epsilon_1-(n+m_2)\epsilon_2} \delta_{m_3,m_2}.$$

Therefore, $f(km)$ can be written as

$$f(km) = \sum_{\substack{j,n \\ -j \le m_i \le j}} a^{(j,n)}_{m_1,m_2} (-1)^{-(n-m_2)\epsilon_1-(n+m_2)\epsilon_2} W^{(j,n)}_{m_1,m_2}(k).$$

Because of the linear independence of different Wigner D-functions, the equality $f(km) = \delta(m)^{-1} f(k) = (-1)^{-\delta_1\epsilon_1-\delta_2\epsilon_2} f(k)$ holds for all $k \in K$ if and only if j, n, m_1, m_2 satisfy the compatibility condition

$$(-1)^{(n-m_2)\epsilon_1+(n+m_2)\epsilon_2} = (-1)^{\delta_1\epsilon_1+\delta_2\epsilon_2}.$$

Therefore, the space $C_\delta(K)$ can be written as the direct sum

$$C_\delta(K) = \bigoplus_{(j,n)\in \mathtt{KTypes}(\delta_1,\delta_2)} \bigoplus_{\substack{m_1 \in \{-j,-j+1,\ldots,j\} \\ m_2 \in \mathtt{M}(j,n;\delta_1,\delta_2)}} \mathbb{C} W^{(j,n)}_{m_1,m_2} \tag{6.7}$$

in which the two sets of admissible j, n, m_1, m_2 are defined as:

$$\texttt{KTypes}(\delta_1,\delta_2) = \{(j,n) \in \frac{1}{2}\mathbb{Z}_{\geq 0} \times \frac{1}{2}\mathbb{Z} | 2j \equiv 2n \equiv \delta_1 + \delta_2 \bmod 2\} \tag{6.8}$$

$$\mathtt{M}(j,n;\delta_1,\delta_2) = \{m_2 \in \{-j,-j+1,\dots,j\} | n - m_2 \equiv \delta_1 \text{ and } n + m_2 \equiv \delta_2 \bmod 2\}. \tag{6.9}$$

Similarly to the case of $SU(2,1)$, for each $(j,n) \in \texttt{KTypes}(\delta_1,\delta_2)$, we can denote

$$\tau^{(j,n)} = \bigoplus_{\substack{m_1 \in \{-j,-j+1,\dots,j\} \\ m_2 \in \mathtt{M}(j,n;\delta_1,\delta_2)}} \mathbb{C} W^{(j,n)}_{m_1,m_2} \tag{6.10}$$

as the K-isotypic subspace of $I_{P_0}(\delta,\lambda)$ which decomposes into copies of irreducible K-representations of highest weight (j,n). The restriction of the $(\mathfrak{g}, K)$-module of the principal series $I_{P_0}(\delta,\lambda)$ to K can be decomposed as a direct sum of the K-isotypic spaces $\tau^{(j,n)}$:

$$I_{P_0}(\delta,\lambda) = \bigoplus_{(j,n) \in \texttt{KTypes}(\delta_1,\delta_2)} \tau^{(j,n)}. \tag{6.11}$$

The different copies of irreducible K-representations are distinguished by the index m_2, and the action of $\mathfrak{u}(2)$ raising and lowering operators $U_1 \pm \mathrm{i} U_2$ moves each m_1 to $m_1 \pm 1$. For each K-isotypic space $\tau^{(j,n)}$, the cardinality of the set $\mathtt{M}(j,n;\delta_1,\delta_2)$ is equal to the multiplicity of K-types with highest weight (j,n). In fact, if we assume

$$(\delta_1,\delta_2) \in \{(0,0),(1,1)\},$$

the set $\mathtt{M}(j,n;\delta_1,\delta_2)$ is

$$\mathtt{M}(j,n;\delta_1,\delta_2) = \left\{ \begin{smallmatrix} \{\dots j-4, j-2, j\} & j-n+\delta_1 \equiv 0 \bmod 2 \\ \{\dots j-5, j-3, j-1\} & j-n+\delta_1 \equiv 1 \bmod 2 \end{smallmatrix} \right. \tag{6.12}$$

and its cardinality is

$$|\mathtt{M}(j,n;\delta_1,\delta_2)| = \left\{ \begin{smallmatrix} j+1 & j-n+\delta_1 \equiv 0 \bmod 2 \\ \lfloor j-\frac{1}{2} \rfloor + 1 & j-n+\delta_1 \equiv 1 \bmod 2 \end{smallmatrix} \right. . \tag{6.13}$$

We will make use of these facts to calculate the long intertwining operator for the principal series of $Sp(4,\mathbb{R})$.

6.4 The $(\mathfrak{g}, K)$-Module Structure

Normalization of basis and the Iwasawa decomposition

The set of positive and negative noncompact roots are

$$\Delta_{nc}^{+} = \{\beta_2, \beta_1+\beta_2, 2\beta_1+\beta_2\} \quad \Delta_{nc}^{-} = \{-\beta_2, -\beta_1-\beta_2, -2\beta_1-\beta_2\}.$$

We define the vectors $u_\beta \in \mathfrak{p}_{\mathbb{C}}$ by multiplying the v_β defined in (6.1)-(6.3) by a factor:

$$u_\beta = \begin{cases} \sqrt{2}\mathrm{i}v_\beta & \beta \in \{\beta_2, 2\beta_1+\beta_2, -\beta_2, -2\beta_1-\beta_2\} \\ v_\beta & \text{otherwise} \end{cases}.$$

Under this normalization, the Lie algebra Iwasawa decomposition of the basis vectors u_β is:

$$u_{\pm(2\beta_1+\beta_2)} = \frac{1}{\sqrt{2}}(\pm(U_0+U_3) + \mathrm{i}H_1 \mp 2X_{2\alpha_1+\alpha_2}) \tag{6.14}$$

$$u_{\pm\beta_2} = \frac{1}{\sqrt{2}}(\pm(U_0-U_3) + \mathrm{i}H_2 \mp 2X_{\alpha_2}) \tag{6.15}$$

$$u_{\pm(\beta_1+\beta_2)} = -(U_1 \mp \mathrm{i}U_2) + X_{\alpha_1+\alpha_2} \mp \mathrm{i}X_{\alpha_1} \tag{6.16}$$

The representation of $K = U(2)$ on $\mathfrak{p}_{\mathbb{C}}$ decomposes into two irreducible subrepresentations

$$\mathfrak{p}_{\mathbb{C}}^{+} = \oplus_{\beta\in\Delta_{nc}^{+}}\mathbb{C}u_\beta \quad \mathfrak{p}_{\mathbb{C}}^{-} = \oplus_{\beta\in\Delta_{nc}^{-}}\mathbb{C}u_\beta.$$

The u_β's are the normalized weight vectors on which the adjoint action by $U_1 \pm \mathrm{i}U_2$ acts as

$$\mathrm{ad}(U_1 \pm \mathrm{i}U_2)u_\beta = q_{\beta,\beta\pm\beta_1}u_{\beta\pm\beta_1},$$

where the coefficient $q_{\beta,\beta\pm\beta_1}$ turns out to be

$$q_{\beta,\beta\pm\beta_1} = \begin{cases} -\mathrm{i}\sqrt{2} & \text{if } \beta\pm\beta_1 \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \\ 0 & \text{otherwise} \end{cases}.$$

Left Action of $\mathfrak{p}_{\mathbb{C}}$

For any noncompact root β, define two integers:

$$m_\beta = -\mathrm{i}\beta(U_3) \in \{-1, 0, 1\} \tag{6.17}$$

$$n_\beta = -\mathrm{i}\beta(U_0) \in \{-1, 1\} \tag{6.18}$$

Each weight β is uniquely determined by the pair of integers (m_β, n_β) specified in the following chart:

	$m_\beta = -1$	$m_\beta = 0$	$m_\beta = 1$
$n_\beta = -1$	$-2\beta_1 - \beta_2$	$-\beta_1 - \beta_2$	$-\beta_2$
$n_\beta = 1$	β_2	$\beta_1 + \beta_2$	$2\beta_1 + \beta_2$

Table 6.1: Correspondence between (m_β, n_β) and noncompact roots

By embedding the principal series representation $I(\chi_{\delta,\lambda})$ into $C_\delta(K)$ (compare to the identical method in [BM17] and the $SU(2,1)$ case in Section 5), we would like to understand the action of

$$\mathsf{d}l(u_\beta) = \mathsf{d}r(-\mathrm{Ad}(k^{-1})u_\beta)$$

on any basis vector $W^{(j,n)}_{m_1,m_2}(k)$ in $C_\delta(K)$. By the definition of the principal series in (4.1), the right action on any vector $f \in I(\chi_{\delta,\lambda})$ by H_i is always a scalar multiplication by $\lambda_i + \rho(H_i)$

$$\mathsf{d}r(H_i)f = -(\lambda_i + \rho(H_i))f, \tag{6.19}$$

and the right action by any element from N annihilates f. We can therefore use the decomposition (6.14)-(6.16) and the differential operators (3.21)-(3.23) in $\mathfrak{g}_\mathbb{C}$ of u_β to calculate the right action of u_β on the basis vectors $W^{(j,n)}_{m_1,m_2}$ of $I(\chi_{\delta,\lambda})$. Also, recall from 6.1.1 that $\rho(H_1) = 2, \rho(H_2) = 1$, we have

$$\begin{aligned}
\mathsf{d}r(u_{\pm(2\beta_1+\beta_2)})W^{(j,n)}_{m_1,m_2} &= \frac{\mathsf{i}}{\sqrt{2}}(\mp n \mp m_2 - (\lambda_1 + \rho(H_1)))W^{(j,n)}_{m_1,m_2} \\
&= \frac{\mathsf{i}(\mp n \mp m_2 - (\lambda_1 + 2))}{\sqrt{2}}W^{(j,n)}_{m_1,m_2} \qquad (6.20)\\
\mathsf{d}r(u_{\pm\beta_2})W^{(j,n)}_{m_1,m_2} &= \frac{\mathsf{i}}{\sqrt{2}}(\mp n \pm m_2 - (\lambda_2 + \rho(H_2)))W^{(j,n)}_{m_1,m_2} \\
&= \frac{\mathsf{i}(\mp n \pm m_2 - (\lambda_2 + 1))}{\sqrt{2}}W^{(j,n)}_{m_1,m_2} \qquad (6.21)\\
\mathsf{d}r(u_{\pm(\beta_1+\beta_2)})W^{(j,n)}_{m_1,m_2} &= -\mathsf{i}\sqrt{(j \pm m_2)(j \mp m_2 + 1)}W^{(j,n)}_{m_1,m_2\mp 1}. \qquad (6.22)
\end{aligned}$$

Recalling the correspondence between the weight β and (m_β, n_β) discussed in (6.17)-(6.18) and Table 6.1, since the irreducible constituents of $\mathfrak{p}^\pm_\mathbb{C}$ have highest weights

$(j,n) = (1, \pm 1)$ respectively, the right action of u_β with $\beta \in \Delta_{nc}^{\pm}$ can be transferred to the left by observing

$$\mathrm{Ad}(k^{-1})u_\beta = \sum_{\nu \in \Delta_{nc}^{\pm}} \overline{W_{m_\nu, m_\beta}^{(1,\pm 1)}(k^{-1})} u_\nu = \sum_{\nu \in \Delta_{nc}^{\pm}} W_{m_\beta, m_\nu}^{(1,\pm 1)}(k) u_\nu.$$

Based on the correspondence in Table 6.1 between the weights of u_β and the pair of integers (m_β, n_β), it is clear that the left action of u_β for $\beta \in \Delta_{nc}^{\pm}$ on the Wigner D-functions $W_{m_1,m_2}^{(j,n)} \in C_\delta(K)$ can be written explicitly as follows,

$$\begin{aligned} &\mathsf{d}l(u_\beta) W_{m_1,m_2}^{(j,n)} = \\ &\left(W_{m_\beta,\mp 1}^{(1,\pm 1)} \mathsf{d}r(u_{\pm\beta_2}) + W_{m_\beta,0}^{(1,\pm 1)} \mathsf{d}r(u_{\pm(\beta_1+\beta_2)}) + W_{m_\beta,\pm 1}^{(1,\pm 1)} \mathsf{d}r(u_{\pm(2\beta_1+\beta_2)}) \right) W_{m_1,m_2}^{(j,n)}. \end{aligned} \tag{6.23}$$

According to (6.20)-(6.22), we apply the right action $\mathsf{d}r(u_\beta)$ to $W_{m_1,m_2}^{(j,n)}$ and get

$$\begin{aligned} &\mathsf{d}l(u_\beta) W_{m_1,m_2}^{(j,n)} = \\ &\mathrm{i} \left(\frac{\mp n \pm m_2 - (\lambda_2 + \rho(H_2))}{\sqrt{2}} W_{m_\beta,\mp 1}^{(1,\pm 1)} + \frac{\mp n \mp m_2 - (\lambda_1 + \rho(H_1))}{\sqrt{2}} W_{m_\beta,\pm 1}^{(1,\pm 1)} \right) W_{m_1,m_2}^{(j,n)} \\ &- \mathrm{i} \sqrt{(j \pm m_2)(j \mp m_2 + 1)} W_{m_\beta,0}^{(1,\pm 1)} W_{m_1,m_2\mp 1}^{(j,n)}. \end{aligned} \tag{6.24}$$

We can replace the products of Wigner D-functions by a linear combination of Wigner D-functions with Clebsch-Gordan coefficients as described in formula (3.25) of Section 3.3. The left action of u_β for $\beta \in \Delta_{nc}^{\pm}$ on Wigner D-functions can thus be expressed as

$$\begin{aligned} &\mathsf{d}l(u_\beta) W_{m_1,m_2}^{(j,n)} \\ =&\mathrm{i} \sum_{j_0 \in \{j-1,j,j+1\}} \left({}^{j+j_0, m_1+m_\beta}_{j,m_1,1,m_\beta} \right) \left(\frac{\mp n \mp m_2 - (\lambda_1 + \rho(H_1))}{\sqrt{2}} \left({}^{j+j_0, m_2\pm 1}_{j,m_2,1,\pm 1} \right) W_{m_1+m_\beta, m_2\pm 1}^{(j+j_0, n\pm 1)} + \right. \\ &\left(-\sqrt{(j \pm m_2)(j \mp m_2 + 1)} \left({}^{j+j_0, m_2\mp 1}_{j,m_2\mp 1,1,0} \right) + \frac{\mp n \pm m_2 - (\lambda_2 + \rho(H_2))}{\sqrt{2}} \left({}^{j+j_0, m_2\mp 1}_{j,m_2,1,\mp 1} \right) \right) \\ &\left. W_{m_1+m_\beta, m_2\mp 1}^{(j+j_0, n\pm 1)} \right) \end{aligned} \tag{6.25}$$

After computing all the Clebsch-Gordan coefficients using the formulas listed in Table 3.2, the action of weight vectors u_β of $\mathfrak{p}_{\mathbb{C}}$ on the left when $\beta \in \Delta_{nc}^{\pm}$ can be expressed as the following linear combination:

$$\mathsf{d}l(u_\beta)W^{(j,n)}_{m_1,m_2} = \frac{\mathrm{i}}{2} \sum_{\substack{j_0\in\{-1,0,1\}\\ \epsilon=\pm1}} \begin{pmatrix} j+j_0, m_1+m_\beta \\ j,m_1,1,m_\beta \end{pmatrix} C_{j+j_0} q_{j_0,\varepsilon} \kappa_{\pm,j_0,\varepsilon}(j,n,m_1;\lambda) W^{(j+j_0,n\pm1)}_{m_1+m_\beta,m_2+\varepsilon} \tag{6.26}$$

with the coefficients given in the tables:

C_{j+j_0}	
$j_0=-1$	$j^{-\frac12}(2j+1)^{-\frac12}$
$j_0=0$	$j^{-\frac12}(j+1)^{-\frac12}$
$j_0=1$	$(j+1)^{-\frac12}(2j+1)^{-\frac12}$

$q_{j_0,\varepsilon}$	$\varepsilon=-1$	$\varepsilon=1$
$j_0=-1$	$\sqrt{(j+m_2-1)(j+m_2)}$	$\sqrt{(j-m_2-1)(j-m_2)}$
$j_0=0$	$\sqrt{(j+m_2)(j-m_2+1)}$	$\sqrt{(j-m_2)(j+m_2+1)}$
$j_0=1$	$\sqrt{(j-m_2+1)(j-m_2+2)}$	$\sqrt{(j+m_2+1)(j+m_2+2)}$

$\kappa_{+,j_0,\varepsilon}$	$\varepsilon=-1$	$\varepsilon=1$
$j_0=-1$	$2+2j-m_2-n-(\lambda_2+\rho(H_2))$	$-n-m_2-(\lambda_1+\rho(H_1))$
$j_0=0$	$2-n-m_2-(\lambda_2+\rho(H_2))$	$n+m_2+(\lambda_1+\rho(H_1))$
$j_0=1$	$-2j-m_2-n-(\lambda_2+\rho(H_2))$	$-n-m_2-(\lambda_1+\rho(H_1))$

$\kappa_{-,j_0,\varepsilon}$	$\varepsilon=-1$	$\varepsilon=1$
$j_0=-1$	$n+m_2-(\lambda_1+\rho(H_1))$	$2+2j+m_2+n-(\lambda_2+\rho(H_2))$
$j_0=0$	$n+m_2-(\lambda_1+\rho(H_1))$	$-2-n-m_2+(\lambda_2+\rho(H_2))$
$j_0=1$	$n+m_2-(\lambda_1+\rho(H_1))$	$-2j+m_2+n-(\lambda_2+\rho(H_2))$

.

The $(\mathfrak{g},K)$-action on $Sp(4,\mathbb{R})$ principal series $I(\chi_{\delta,\lambda})$ is completely determined by formula (6.26) and the four tables above.

6.5 Intertwining Operators

The longest element $w_0 = w_{\alpha_2}w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}$ in the Weyl group of $Sp(4,\mathbb{R})$ corresponds to the long intertwining operator:

$$A(w_0,\lambda)f(k) = \int_{\overline{N}\cap w^{-1}Nw} f(kw\overline{n})\mathsf{d}\overline{n}.$$

Applying Langlands' Lemma (4.4), $A(w_0,\lambda)$ can be factored into 4 intertwining operators corresponding to simple reflections:

$$A(w_0,\lambda)=A(w_{\alpha_2},w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}\lambda)A(w_{\alpha_1},w_{\alpha_2}w_{\alpha_1}\lambda)A(w_{\alpha_2},w_{\alpha_1}\lambda)A(w_{\alpha_1},\lambda).$$

Proposition 6.1 *Let $w_0=w_{\alpha_2}w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}$ be the longest element in the Weyl group W of $Sp(4,\mathbb{R})$. The matrix for the long intertwining operator $A(w_0,\chi_{\delta,\lambda})$ under the basis $W^{(j,n)}_{m_1,m_2}$ of $C_\delta(K)$ has a factorization:*

$$A(w_0,\chi_{\delta,\lambda})|_{j,n}=A_4(\nu)\cdot A_3(\lambda)\cdot A_2(\lambda)\cdot A_1(\lambda)$$

If we define

$$Q(z,n)=\frac{\pi 2^{2-2z}\Gamma(2z-1)}{\Gamma(z+n)\Gamma(z-n)}, \tag{6.27}$$

and let

$$S^{j,n}_{m_3,m_2}(z)=\sum_{-j\le m_4\le j}\mathsf{i}^{-2m_4}M^{j,n}_{m_3,m_4}N^{j,n}_{m_4,m_2}Q(z,m_4)$$
$$T^n_{m_1}(z)=\mathsf{i}^{-n+m_1}Q\left(z,\frac{m_1-n}{2}\right)$$

where $N^{j,n}_{m_1,m_3}$ is the inverse matrix of $M^{j,n}_{m_1,m_3}$, with

$$M^{j,n}_{m_3,m_4}=c^j_{m_3}c^j_{m_4}\begin{cases}\frac{\mathsf{i}^{m_3-m_4}(-1)^{2j}2^{-j}}{(j-m_3)!(m_3-m_4)!(j+m_4)!}\,{}_2F_1\left(\begin{smallmatrix}-j+m_3,-j-m_4\\1+m_3-m_4\end{smallmatrix};-1\right) & m_3>m_4\\ \frac{\mathsf{i}^{m_4-m_3}(-1)^{2j}2^{-j}}{(j+m_3)!(m_4-m_3)!(j-m_4)!}\,{}_2F_1\left(\begin{smallmatrix}-j-m_3,-j+m_4\\1-m_3+m_4\end{smallmatrix},-1\right) & m_3\le m_4\end{cases}$$

then the operators $A_i(\lambda)$ act as

$$A_1(\lambda)W^{(j,n)}_{m_1,m_2}=\sum_{m_3\in\mathtt{M}(j,n;\delta_2,\delta_1)}S^{j,n}_{m_3,m_2}\left(\frac{\lambda_1-\lambda_2+1}{2}\right)W^{(j,n)}_{m_1,m_3} \tag{6.28}$$
$$A_2(\lambda)W^{(j,n)}_{m_1,m_2}=T^n_{m_2}\left(\frac{\lambda_1+1}{2}\right)W^{(j,n)}_{m_1,m_2} \tag{6.29}$$
$$A_3(\lambda)W^{(j,n)}_{m_1,m_2}=\sum_{m_3\in\mathtt{M}(j,n;\delta_2,\delta_1)}S^{j,n}_{m_3,m_2}\left(\frac{\lambda_1+\lambda_2+1}{2}\right)W^{(j,n)}_{m_1,m_3} \tag{6.30}$$
$$A_4(\lambda)W^{(j,n)}_{m_1,m_2}=T^n_{m_2}\left(\frac{\lambda_2+1}{2}\right)W^{(j,n)}_{m_1,m_2}. \tag{6.31}$$

We will prove the Proposition 6.1 in the following two sections 6.5.1 and 6.5.2.

Remark 6.1 *The simple reflection w_{α_1} sends the character $\delta=(\delta_1,\delta_2)$ on M to (δ_2,δ_1). For the w_{α_1} intertwining operator, it is important to recall that the parity condition $\mathtt{M}(j,n;\delta_2,\delta_1)$ of m_3 restricts the allowed Wigner D-functions $W^{(j,n)}_{m_1,m_3}$ in $I_{P_0}(w_{\alpha_1}\chi_{\delta,\lambda})$ as in (6.10) and (6.11).*

6.5.1 Rank 1 Intertwining Operators

Starting from any character λ on $\mathfrak{a}_{\mathbb{C}}$, the rank 1 intertwining operator $A(w_\alpha, \mu)$ associated to a simple reflection w_α can be written as

$$A(w_\alpha, \lambda) f(k) = \int_{\overline{N} \cap w_\alpha^{-1} N w_\alpha} f(k w_\alpha \bar{n}) \mathsf{d}\bar{n} = \int_{-\infty}^{\infty} f(k w_\alpha \exp(t X_{-\alpha})) \mathsf{d}t.$$

By the Iwasawa decomposition of $\exp(tX_{-\alpha})$ given in (6.6),

$$\exp(tX_{-\alpha}) = \kappa_\alpha(t) h_\alpha(\sqrt{1+t^2}) \chi_\alpha\left(\frac{t}{1+t^2}\right),$$

where $\kappa_\alpha(t) = \exp\left(\arctan(-t)(X_\alpha - X_{-\alpha})\right)$. If f is any vector in the principal series representation $I(\chi_{\delta,\lambda})$, the action of $\exp(tX_{-\alpha})$ on the right is

$$\begin{aligned} f(k w_\alpha \exp(t X_{-\alpha})) &= f\left(k w_\alpha \kappa_\alpha(t) h_\alpha(\sqrt{1+t^2}) \chi_\alpha\left(\frac{t}{1+t^2}\right)\right) \\ &= (1+t^2)^{-\frac{\langle \check{\alpha}, \lambda+\rho\rangle}{2}} f\left(k w_\alpha e^{\arctan(-t)(X_\alpha - X_{-\alpha})}\right). \end{aligned}$$

Denoting by $\theta(t) = \arctan(-t)$, since $X_{\alpha_1} - X_{-\alpha_1} = 2U_2, X_{\alpha_2} - X_{-\alpha_2} = U_0 - U_3$, and recalling from (6.5) the expressions of simple reflections w_{α_i} in terms of Euler angles, the action of $\exp(tX_{-\alpha_1})$ and $\exp(tX_{-\alpha_2})$ on a Wigner D-function $f = W^{(j,n)}_{m_1,m_2}$ is thus

$$\begin{aligned} & W^{(j,n)}_{m_1,m_2}(k w_{\alpha_1} e^{\theta(t)(X_{\alpha_1} - X_{-\alpha_1})}) \\ &= \sum_{-j \le m_3 \le j} W^{(j,n)}_{m_1,m_3}(k) W^{(j,n)}_{m_3,m_2}(0, 0, -\pi - 2\theta(t), 0) \end{aligned} \tag{6.32}$$

$$\begin{aligned} & W^{(j,n)}_{m_1,m_2}(k w_{\alpha_2} e^{\theta(t)(X_{\alpha_2} - X_{-\alpha_2})}) \\ &= \sum_{-j \le m_3 \le j} W^{(j,n)}_{m_1,m_3}(k) W^{(j,n)}_{m_3,m_2}\left(-\frac{\pi}{2} - \theta(t), \frac{\pi}{2} + \theta(t), 0, 0\right). \end{aligned} \tag{6.33}$$

Therefore, the simple intertwining operators $A(w_{\alpha_1}, \lambda) W^{(j,n)}_{m_1,m_2}$ and $A(w_{\alpha_2}, \lambda) W^{(j,n)}_{m_1,m_2}$ can be expressed in terms of integrals involving Wigner D-functions:

$$\begin{aligned} & \left(A(w_{\alpha_1}, \lambda) W^{(j,n)}_{m_1,m_2}\right)(k) = \\ & \sum_{-j \le m_3 \le j} W^{(j,n)}_{m_1,m_3}(k) \int_{-\infty}^{\infty} (1+t^2)^{-\frac{\langle \check{\alpha_1}, \lambda+\rho\rangle}{2}} W^{(j,n)}_{m_3,m_2}(0, 0, -\pi - 2\theta(t), 0) \mathsf{d}t \end{aligned} \tag{6.34}$$

$$\begin{aligned} & \left(A(w_{\alpha_2}, \lambda) W^{(j,n)}_{m_1,m_2}\right)(k) = \\ & \sum_{-j \le m_3 \le j} W^{(j,n)}_{m_1,m_3}(k) \int_{-\infty}^{\infty} (1+t^2)^{-\frac{\langle \check{\alpha_2}, \lambda+\rho\rangle}{2}} W^{(j,n)}_{m_3,m_2}\left(-\frac{\pi}{2} - \theta(t), \frac{\pi}{2} + \theta(t), 0, 0\right) \mathsf{d}t \end{aligned} \tag{6.35}$$

Observe that $\theta(t)$ is an odd function. It is important to mention that, from the formula of the Wigner D-function (3.14), we have

$$d^{(j,n)}_{m_3,m_2}(-\pi-2\theta(t)) = \sum_{p=\max(0,m_3-m_2)}^{\min(j-m_2,j+m_3)} \frac{(-1)^{2j+m_2-m_3+p}}{(j+m_3-p)!p!(m_2-m_3+p)!(j-m_2-p)!}$$
$$\cos^{m_2-m_3+2p}(\theta(t))\sin^{2j+m_3-m_2-2p}(\theta(t)).$$

If $2j+m_3-m_2 \equiv 1 \bmod 2$, the integrand of (6.34) is an odd function, which makes the integral (6.34) zero. If $W^{(j,n)}_{m_1,m_2} \in I_{P_0}(\chi_{\delta,\lambda})$, we must have

$$2j \equiv \delta_1+\delta_2 \bmod 2$$

and

$$m_2 \in \mathtt{M}(j,n;\delta_1,\delta_2).$$

The set $\mathtt{M}(j,n;\delta_1,\delta_2)$ has been defined in (6.9). In order to make the integral (6.34) nonzero, the function $d^{(j,n)}_{m_3,m_2}(-\pi-2\theta(t))$ must be an even function. In this case, the exponent $2j+m_3-m_2 \equiv 0 \bmod 2$. Therefore,

$$2j+m_3-m_2 \equiv \delta_1+\delta_2+m_3-m_2 \equiv 0 \bmod 2. \tag{6.36}$$

Thus, if m_2 satisfies $n-m_2 \equiv \delta_1 \bmod 2$ and $n+m_2 \equiv \delta_2 \bmod 2$, by (6.36), we must have

$$n-m_3 \equiv n-m_2+\delta_1+\delta_2 \equiv \delta_2 \bmod 2$$
$$n+m_3 \equiv n+m_2+\delta_1+\delta_2 \equiv \delta_1 \bmod 2.$$

Thus the parity condition m_3 is given by the set $\mathtt{M}(j,n;\delta_2,\delta_1)$, with δ_1 and δ_2 flipped from the parity condition of m_2. In particular, $A(w_{\alpha_1},\lambda)W^{(j,n)}_{m_1,m_2}$ lies in the space $I_{P_0}(w_{\alpha_1}\chi_{\delta,\lambda})$. Hence the sum in (6.34) is in fact a sum over $m_3 \in \mathtt{M}(j,n;\delta_2,\delta_1)$:

$$\left(A(w_{\alpha_1},\lambda)W^{(j,n)}_{m_1,m_2}\right)(k) =$$
$$\sum_{m_3\in\mathtt{M}(j,n;\delta_2,\delta_1)} W^{(j,n)}_{m_1,m_3}(k)\int_{-\infty}^{\infty}(1+t^2)^{-\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}}W^{(j,n)}_{m_3,m_2}(0,0,-\pi-2\theta(t),0)\mathrm{d}t. \tag{6.37}$$

Note that the Wigner D-function $W^{(j,n)}_{m_3,m_2}(0,0,-\pi-2\theta(t),0)$ in (6.32) and (6.37) has a nonzero U_2-Euler angle. In the following section, we will diagonalize the matrix of

Wigner D-functions and transform the U_2-Euler angle to a U_3- Euler angle in order to compute the intertwining operators more easily.

6.5.2 Diagonalization of the intertwining operators

From the commutation relation of Pauli matrices, U_2 and U_3 can be related in the following way:

$$U_2 = \mathrm{Ad}(e^{-\frac{3\pi}{2}U_1})U_3. \tag{6.38}$$

We can use this relation to diagonalize the action of

$$e^{(\pi+2\theta(t))U_2} = \mathrm{Ad}(e^{-\frac{3\pi}{2}U_1})e^{(\pi+2\theta(t))U_3}$$

appearing in the Iwasawa decomposition of $\exp(tX_{-\alpha_1})$ of the Wigner D-functions. By the multiplicativity of the Wigner D-function, we have

$$\begin{aligned}
&W^{(j,n)}_{m_1,m_2}(kw_{\alpha_1}e^{2\theta(t)U_2}) = W^{(j,n)}_{m_1,m_2}(ke^{(\pi+2\theta(t))U_2}) \\
&= \sum_{m_3\in \mathtt{M}(j,n;\delta_2,\delta_1)} W^{(j,n)}_{m_1,m_3}(k)W^{(j,n)}_{m_3,m_2}(e^{-\frac{3\pi}{2}U_1}e^{(\pi+2\theta(t))U_3}\exp^{\frac{3\pi}{2}U_1}) \\
&= \sum_{m_3\in \mathtt{M}(j,n;\delta_2,\delta_1)} W^{(j,n)}_{m_1,m_3}(k)\sum_{m_4,m_5} W^{(j,n)}_{m_3,m_4}(e^{-\frac{3\pi}{2}U_1})W^{(j,n)}_{m_5,m_2}(e^{\frac{3\pi}{2}U_1})W^{(j,n)}_{m_4,m_5}(e^{(\pi+2\theta(t))U_3}) \\
&= \sum_{m_3\in \mathtt{M}(j,n;\delta_2,\delta_1)} W^{(j,n)}_{m_1,m_3}(k)\sum_{m_4,m_5} W^{(j,n)}_{m_3,m_4}(e^{-\frac{3\pi}{2}U_1})W^{(j,n)}_{m_5,m_2}(e^{\frac{3\pi}{2}U_1})\times \\
&W^{(j,n)}_{m_4,m_5}(0,-\pi-2\theta(t),0,0).
\end{aligned} \tag{6.39}$$

We define the function $S'^{(j,n)}_{m_3,m_2}(z)$ by

$$\begin{aligned}
S'^{(j,n)}_{m_3,m_2}(z) = &\sum_{m_4,m_5} W^{(j,n)}_{m_3,m_4}(e^{-\frac{3\pi}{2}U_1})W^{(j,n)}_{m_5,m_2}(e^{\frac{3\pi}{2}U_1}) \\
&\int_{-\infty}^{\infty}(1+t^2)^{-z}W^{(j,n)}_{m_4,m_5}(0,-\pi-2\theta(t),0,0).
\end{aligned} \tag{6.40}$$

Change-of-basis matrix

Similarly to the relation (6.38) between U_2 and U_3, we can relate U_1 and U_2 by conjugating a multiple of U_3:

$$U_1 = \mathrm{Ad}(e^{\frac{\pi}{2}U_3})U_2. \tag{6.41}$$

Recalling the notation from from (3.13) and (3.14) that $c_m^j = \sqrt{(j+m)!(j-m)!}$, the change-of-basis matrices $W_{m_3,m_4}^{(j,n)}(e^{-\frac{3\pi}{2}U_1})$ and $W_{m_5,m_2}^{(j,n)}(e^{\frac{3\pi}{2}U_1})$ can be expressed in terms of the value of a Wigner D-function:

$$W_{m_3,m_4}^{(j,n)}(e^{-\frac{3\pi}{2}U_1}) = W_{m_3,m_4}^{(j,n)}(e^{\frac{\pi}{2}U_3}e^{-\frac{3\pi}{2}U_2}e^{-\frac{\pi}{2}U_3}) = W_{m_3,m_4}^{(j,n)}\left(-\frac{\pi}{2},\frac{3\pi}{2},\frac{\pi}{2}\right)$$
$$= c_{m_3}^j c_{m_4}^j \begin{cases} \frac{i^{m_3-m_4}(-1)^{2j}2^{-j}}{(j-m_3)!(m_3-m_4)!(j+m_4)!}\,{}_2F_1\left({-j+m_3,-j-m_4 \atop 1+m_3-m_4};-1\right) & m_3 > m_4 \\ \frac{i^{m_4-m_3}(-1)^{2j}2^{-j}}{(j+m_3)!(m_4-m_3)!(j-m_4)!}\,{}_2F_1\left({-j-m_3,-j+m_4 \atop 1-m_3+m_4},-1\right) & m_3 \le m_4 \end{cases}. \quad (6.42)$$

We define

$$M_{m_3,m_4}^j = W_{m_3,m_4}^{(j,n)}(e^{-\frac{3\pi}{2}U_1}) \quad (6.43)$$

$$N_{m_5,m_2}^j = W_{m_5,m_2}^{(j,n)}(e^{\frac{3\pi}{2}U_1}). \quad (6.44)$$

As we have seen in (3.18), the entries of M_{m_3,m_4}^j and N_{m_3,m_4}^j are related to values $P_{j-m_1}^{m_1-m_4,m_1+m_4}(0)$ of Jacobi polynomials $P_n^{\alpha,\beta}(x)$:

$$M_{m_3,m_4}^j = \frac{c_{m_4}^j}{c_{m_3}^j}(-1)^{2j} i^{-m_3+m_4} 2^{-m_4} P_{j-m_4}^{-m_3+m_4,m_3+m_4}(0) \quad (6.45)$$

$$N_{m_5,m_2}^j = \frac{c_{m_2}^j}{c_{m_5}^j}(-1)^{2j} i^{m_5-m_2} 2^{-m_2} P_{j-m_2}^{-m_5+m_2,m_5+m_2}(0). \quad (6.46)$$

After introducing these notations, the function $S_{m_3,m_2}^{\prime(j,n)}(z)$ defined in (6.40) becomes

$$S_{m_3,m_2}^{\prime(j,n)}(z) = \sum_{m_4,m_5} M_{m_3,m_4}^{(j,n)} N_{m_5,m_2}^{(j,n)} \int_{-\infty}^{\infty} (1+t^2)^{-z} W_{m_4,m_5}^{(j,n)}(0,-\pi-2\theta(t),0,0). \quad (6.47)$$

Among the two equivalent definitions (3.16) from [AS67] and (3.17) in [Res08], for the sake of simplicity in expressions we choose the definition (3.17) using hypergeometric functions

$$P_n^{\alpha,\beta}(x) = \binom{n+\alpha}{n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left({-n,-n-\beta \atop \alpha+1};\frac{x-1}{x+1}\right).$$

These Jacobi polynomials have a generating function [SM84]

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)t^n = \left(1+\frac{1}{2}(x+1)t\right)^{\alpha}\left(1+\frac{1}{2}(x-1)t\right)^{\beta} \quad (6.48)$$

for $|x| < 1$. The series on the left hand side converges absolutely for $|t| < 1$. For any meromorphic function $f(t)$, we denote $[f(t)]_0$ as the zeroth Laurent series coefficient of $f(t)$. The change-of-basis matrices $M^j_{m_3,m_4}, N^j_{m_5,m_2}$ can thus be expressed as the constant term in Laurent series,

$$M^j_{m_3,m_4} = \left[(-1)^{2j}\frac{c^j_{m_4}}{c^j_{m_3}}\mathrm{i}^{-m_3+m_4}2^{-m_4}t^{m_4-j}\left(1+\frac{t}{2}\right)^{j-m_3}\left(1-\frac{t}{2}\right)^{j+m_3}\right]_0 \tag{6.49}$$

$$N^j_{m_5,m_2} = \left[(-1)^{2j}\frac{c^j_{m_2}}{c^j_{m_5}}\mathrm{i}^{m_5-m_2}2^{-m_2}t^{m_2-j}\left(1+\frac{t}{2}\right)^{j-m_5}\left(1-\frac{t}{2}\right)^{j+m_5}\right]_0. \tag{6.50}$$

The Singular Integrals

In (4.7), we have described the procedure of calculating the matrix entries

$$[A(w,\lambda)]^{(j,n)}_{m_3,m_2} = \langle W^{(j,n)}_{m_1,m_3}, A(w,\lambda)W^{(j,n)}_{m_1,m_2}\rangle \tag{6.51}$$

of the intertwining operator $A(w,\lambda)$. The right hand side of (6.51) is independent of m_1. The calculation of simple intertwining operators reduces to the calculation of integrals (6.37) and (6.35). Combining the diagonalized operator (6.39) with (6.34) and (6.35), the problem of the calculation of the intertwining operators $A(w,\lambda)$ reduces to the following two integrals:

$$\int_{-\infty}^{\infty}(1+t^2)^{-\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}}W^{(j,n)}_{m_4,m_5}(0,-\pi-2\theta(t),0,0)\,\mathsf{d}t \tag{6.52}$$

$$\int_{-\infty}^{\infty}(1+t^2)^{-\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2}}W^{(j,n)}_{m_3,m_2}\left(-\frac{\pi}{2}-\theta(t),\frac{\pi}{2}+\theta(t),0,0\right)\mathsf{d}t \tag{6.53}$$

Recall from the definition of Wigner D-functions and their values on Euler angles that

$$W^{(j,n)}_{m_1,m_2}(\zeta,\psi,0,0) = e^{\mathrm{i}n\zeta+\mathrm{i}m_1\psi}\delta_{m_1,m_2}.$$

The Wigner D-function part of the integrands of the above two integrals (6.52) and (6.53) are

$$W^{(j,n)}_{m_4,m_5}(0,-\pi-2\theta(t),0,0) = \mathrm{i}^{-2m_4}(1+\mathrm{i}t)^{m_4}(1-\mathrm{i}t)^{-m_4}\delta_{m_4,m_5}$$

$$W^{(j,n)}_{m_3,m_2}\left(-\frac{\pi}{2}-\theta(t),\frac{\pi}{2}+\theta(t),0,0\right) = \mathrm{i}^{-n+m_2}(1+\mathrm{i}t)^{\frac{n-m_2}{2}}(1-\mathrm{i}t)^{-\frac{n-m_2}{2}}\delta_{m_3,m_2}$$

The integral $\int_{-\infty}^{\infty}(1+\mathrm{i}t)^{s_1}(1-\mathrm{i}t)^{s_2}\mathsf{d}t$ is convergent for $\mathrm{Re}(s_1+s_2)<-1$, and can be meromorphically continued to the whole complex plane as

$$\int_{-\infty}^{\infty}(1+\mathrm{i}t)^{s_1}(1-\mathrm{i}t)^{s_2}\mathsf{d}t = \pi 2^{s_1+s_2+2}\frac{\Gamma(-s_1-s_2-1)}{\Gamma(-s_1)\Gamma(-s_2)}. \tag{6.54}$$

We can use (6.54) to express the two integrals (6.52) and (6.53) in terms of Γ-functions. Thus the integral (6.52) becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty}(1+t^2)^{-\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}}W^{(j,n)}_{m_4,m_5}(0,-\pi-2\theta(t),0,0)\mathsf{d}t\\
=&\mathsf{i}^{-2m_4}\delta_{m_4,m_5}\int_{-\infty}^{\infty}(1-\mathsf{i}t)^{-\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}-m_4}(1+\mathsf{i}t)^{-\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}+m_4}\mathsf{d}t\\
=&\mathsf{i}^{-2m_4}\pi 2^{-\langle\check{\alpha}_1,\lambda+\rho\rangle+2}\frac{\Gamma(\langle\check{\alpha}_1,\lambda+\rho\rangle-1)}{\Gamma\left(\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}-m_4\right)\Gamma\left(\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}+m_4\right)}\delta_{m_4,m_5}
\end{aligned}$$

and the integral (6.53) is

$$\begin{aligned}
&\int_{-\infty}^{\infty}(1+t^2)^{-\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2}}W^{(j,n)}_{m_3,m_2}\left(-\frac{\pi}{2}-\theta(t),\frac{\pi}{2}+\theta(t),0,0\right)\mathsf{d}t\\
=&\mathsf{i}^{-n+m_2}\delta_{m_3,m_2}\int_{-\infty}^{\infty}(1-\mathsf{i}t)^{-\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2}-\frac{n-m_2}{2}}(1+\mathsf{i}t)^{-\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2}+\frac{n-m_2}{2}}\mathsf{d}t\\
=&\mathsf{i}^{-n+m_2}\pi 2^{-\langle\check{\alpha}_2,\lambda+\rho\rangle+2}\frac{\Gamma(\langle\check{\alpha}_2,\lambda+\rho\rangle-1)}{\Gamma\left(\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2}-\frac{n-m_2}{2}\right)\Gamma\left(\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2}+\frac{n-m_2}{2}\right)}\delta_{m_3,m_2}.
\end{aligned}$$

We have defined the function $Q(z,n)=\frac{\pi 2^{2-2z}\Gamma(2z-1)}{\Gamma(z+n)\Gamma(z-n)}$ in (6.27), so the integrals (6.52) and (6.53) can be expressed as:

$$\begin{aligned}
&\int_{-\infty}^{\infty}(1+t^2)^{-\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}}W^{(j,n)}_{m_4,m_5}(0,-\pi-2\theta(t),0,0)\,\mathsf{d}t\\
&\qquad\qquad=\mathsf{i}^{-2m_4}Q\left(\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2},m_4\right)\delta_{m_4,m_5} &(6.55)\\
&\int_{-\infty}^{\infty}(1+t^2)^{-\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2}}W^{(j,n)}_{m_3,m_2}\left(-\frac{\pi}{2}-\theta(t),\frac{\pi}{2}+\theta(t),0,0\right)\mathsf{d}t\\
&\qquad\qquad=\mathsf{i}^{-n+m_2}Q\left(\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2},\frac{m_2-n}{2}\right)\delta_{m_3,m_2}. &(6.56)
\end{aligned}$$

If we denote by $T^n_{m_2}(z)=\mathsf{i}^{-n+m_2}Q\left(z,\frac{m_2-n}{2}\right)$, the result of the integral in (6.56) is equal to $T^n_{m_2}(\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2})=\mathsf{i}^{-n+m_2}Q\left(\frac{\langle\check{\alpha}_2,\lambda+\rho\rangle}{2},\frac{m_2-n}{2}\right)$. By collecting the results from (6.42) and (6.55), it turns out that the $S'^{(j,n)}_{m_3,m_2}(z)$ defined in (6.40) and (6.47) is exactly the same as the $S^{(j,n)}_{m_3,m_2}(z)$ in Proposition 6.1:

$$\begin{aligned}
S^{(j,n)}_{m_3,m_2}\left(\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}\right)&=S'^{(j,n)}_{m_3,m_2}\left(\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2}\right)\\
&=\sum_{m_4}M^{(j,n)}_{m_3,m_4}N^{(j,n)}_{m_4,m_2}\mathsf{i}^{-2m_4}Q\left(\frac{\langle\check{\alpha}_1,\lambda+\rho\rangle}{2},m_4\right).
\end{aligned}$$

Therefore, by (6.37), the simple intertwining operator $A(w_{\alpha_1}, \lambda)$ acts on Wigner D-functions by

$$A(w_{\alpha_1}, \lambda)W^{(j,n)}_{m_1,m_2} = \sum_{-j\leq m_3\leq j} S^{(j,n)}_{m_3,m_2}\left(\frac{\langle\check{\alpha}_1, \lambda+\rho\rangle}{2}\right) W^{(j,n)}_{m_1,m_3}$$

From (6.35) and (6.56), we can see that the simple intertwining operator $A(w_{\alpha_2}, \lambda)$ acts by

$$A(w_{\alpha_2}, \lambda)W^{(j,n)}_{m_1,m_2} = T^n_{m_2}\left(\frac{\langle\check{\alpha}_2, \lambda+\rho\rangle}{2}\right) W^{(j,n)}_{m_1,m_2}.$$

By the Langlands' lemma (4.4)

$$A(w_0, \lambda) = A(w_{\alpha_2}, w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}\lambda)A(w_{\alpha_1}, w_{\alpha_2}w_{\alpha_1}\lambda)A(w_{\alpha_2}, w_{\alpha_1}\lambda)A(w_{\alpha_1}, \lambda),$$

we can replace λ in the formula above by $w_{\alpha_1}\lambda$, $w_{\alpha_2}w_{\alpha_1}\lambda$ and $w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}\lambda$ to calculate the other stages in the composition of simple intertwining operators. Recall that if we express λ by the pair (λ_1, λ_2) as in Section 6.1.1, where $\lambda_i = \lambda(H_i)$, the composition of actions by the Weyl group action will send the pair (λ_1, λ_2) to:

$$(\lambda_1, \lambda_2) \xrightarrow{w_{\alpha_1}} (\lambda_2, \lambda_1) \xrightarrow{w_{\alpha_2}} (\lambda_2, -\lambda_1) \xrightarrow{w_{\alpha_1}} (-\lambda_1, \lambda_2) \xrightarrow{w_{\alpha_2}} (-\lambda_1, -\lambda_2).$$

The pairing of $\lambda + \rho$ with the coroots $\check{\alpha}_1$ and $\check{\alpha}_2$ are given by

$$\langle\check{\alpha}_1, \lambda+\rho\rangle = \lambda_1 - \lambda_2 + 1$$
$$\langle\check{\alpha}_2, \lambda+\rho\rangle = \lambda_2 + 1$$

Therefore, we can conclude that the matrix coefficients for the simple intertwining operators are

$$\begin{aligned}
A(w_{\alpha_1}, \lambda)W^{(j,n)}_{m_1,m_2} &= \sum_{m_3\in\mathtt{M}(j,n;\delta_2,\delta_1)} S^{(j,n)}_{m_3,m_2}\left(\frac{\langle\check{\alpha}_1, \lambda+\rho\rangle}{2}\right) W^{(j,n)}_{m_1,m_3}\\
A(w_{\alpha_2}, w_{\alpha_1}\lambda)W^{(j,n)}_{m_1,m_2} &= T^n_{m_2}\left(\frac{\langle\check{\alpha}_2, w_{\alpha_1}\lambda+\rho\rangle}{2}\right) W^{(j,n)}_{m_1,m_2}\\
A(w_{\alpha_1}, w_{\alpha_2}w_{\alpha_1}\lambda)W^{(j,n)}_{m_1,m_2} &= \sum_{m_3\in\mathtt{M}(j,n;\delta_2,\delta_1)} S^{(j,n)}_{m_3,m_2}\left(\frac{\langle\check{\alpha}_1, w_{\alpha_2}w_{\alpha_1}\lambda+\rho\rangle}{2}\right) W^{(j,n)}_{m_1,m_3}\\
A(w_{\alpha_2}, w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}\lambda)W^{(j,n)}_{m_1,m_2} &= T^n_{m_2}\left(\frac{\langle\check{\alpha}_2, w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}\lambda+\rho\rangle}{2}\right) W^{(j,n)}_{m_1,m_2}
\end{aligned}$$

Collecting the above result with the change-of-basis matrix defined in (6.42), we have finished the proof of the Proposition 6.1.

6.5.3 Expression of $S^{j,n}_{m_1,m_4}(z)$ as Hypergeometric Functions

In this section, we assume

$$(\delta_1, \delta_2) = (0,0) \text{ or } (1,1),$$

in which case j, n are integers. We also introduce the rising factorial or the *Pochhammer symbol*

$$(a)^{(n)} = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{6.57}$$

Definition 6.1 *Part I of [Obe12].*

1. *The* Mellin transform *of a function $f(x)$ is formally defined as*

$$\mathcal{M}(f(x))(z) = \int_0^\infty f(x) x^{z-1} \mathsf{d}x. \tag{6.58}$$

2. *Consider a function $F(z)$ of one complex variable $z = \sigma + \mathsf{i}\tau$, such that*

   (a) *$F(z)$ is holomorphic on the strip $S = \{z \in \mathbb{C} | a < \sigma < b\}$, such that $F(z) \to 0$ uniformly in the strip $S_\epsilon = \{z \in \mathbb{C} | a + \epsilon < \sigma < b - \epsilon\}$ for arbitrarily small $\epsilon > 0$, and*

   (b) *$\int_{-\infty}^{\infty} |F(\sigma + \mathsf{i}\tau)| \mathsf{d}\tau < \infty$ for all $\sigma \in (a,b)$.*

   We define the inverse Mellin transform *of the function $F(z)$ as*

$$\mathcal{M}^{-1}(F(z))(x) = \frac{1}{2\pi\mathsf{i}} \int_{\gamma - \mathsf{i}\infty}^{\gamma + \mathsf{i}\infty} F(z) x^{-z} \mathsf{d}z, \tag{6.59}$$

   for $x > 0$ and some fixed $\gamma \in (a,b)$. It satisfies the property that

$$\mathcal{M}(\mathcal{M}^{-1}(F))(z) = F(z). \tag{6.60}$$

The function $F(z)$ to our interest is the ratio of Γ-functions $\frac{\Gamma(z)}{\Gamma(\frac{1+z-\nu}{2})\Gamma(\frac{1+z+\nu}{2})}$. We would like to calculate the integral

$$\frac{1}{2\pi\mathsf{i}} \int_{\gamma - \mathsf{i}\infty}^{\gamma + \mathsf{i}\infty} \frac{\Gamma(z)}{\Gamma\left(\frac{1+z-\nu}{2}\right)\Gamma\left(\frac{1+z+\nu}{2}\right)} x^{-z} \mathsf{d}z \tag{6.61}$$

along some well chosen contour on which the integral converges. Though this integral doesn't satisfy the assumption on $F(z)$ above, we still can take the inverse Mellin transform of this function, resulting in a $\mathcal{M}^{-1}(F(z))(x)$ with discontinuities. We refer to [Bat55, Page 49, Section 1.19] for a general *Mellin-Barnes integral*

$$\frac{1}{2\pi\mathsf{i}}\int_{\gamma-\mathsf{i}\infty}^{\gamma+\mathsf{i}\infty}\frac{\prod_{i=1}^{m}\Gamma(a_i+A_iz)\prod_{j=1}^{n}\Gamma(b_j-B_jz)}{\prod_{s=1}^{p}\Gamma(c_s+C_sz)\prod_{t=1}^{q}\Gamma(d_t-D_tz)}x^{-z}\mathsf{d}z, \tag{6.62}$$

where $\gamma\in\mathbb{R}$ and $A_i,B_j,C_s,D_t>0$. If $x\in\mathbb{R}$, the integrand is asymptotically $e^{-\frac{1}{2}\alpha\pi|y|}|y|^{\beta\gamma+\lambda}|\frac{x}{\rho}|^{-\gamma}$, where the numbers $\alpha,\beta,\lambda,\rho$ are defined by

$$\begin{aligned}
\alpha &= \sum_{i=1}^{m}A_i+\sum_{j=1}^{n}B_j-\sum_{s=1}^{p}C_s-\sum_{t=1}^{q}D_t\\
\beta &= \sum_{i=1}^{m}A_i-\sum_{j=1}^{n}B_j-\sum_{s=1}^{p}C_s+\sum_{t=1}^{q}D_t\\
\lambda &= \mathrm{Re}(\sum_{i=1}^{m}a_i+\sum_{j=1}^{n}b_j-\sum_{s=1}^{p}c_s-\sum_{t=1}^{q}d_j)-\frac{m+n-p-q}{2}\\
\rho &= \prod_{i=1}^{m}A_i^{A_i}\prod_{j=1}^{n}B_j{}^{-B_j}\prod_{s=1}^{p}C_s{}^{-C_s}\prod_{t=1}^{q}D_t^{D_t},
\end{aligned}$$

and $y=\mathrm{Im}(z)$. Our integral (6.61) satisfies $\alpha=\beta=0$ and $\lambda=-\frac{1}{2}$, $\rho=2$, falling into the fourth type in the description on the convergence of the Mellin-Barnes integral in [Bat55], which states that the integral (6.61) conditionally converges to an analytic function in x on the intervals $|x|<2$ and $|x|>2$, with points of discontinuity at $x=\pm2$. In [Obe12, Section II, (5.21)], for $\mathrm{Re}(z)>0$, we have an explicit formula for (6.61):

$$\mathcal{M}^{-1}\left(\frac{\Gamma(z)}{\Gamma\left(\frac{1+z-\nu}{2}\right)\Gamma\left(\frac{1+z+\nu}{2}\right)}\right)(x)=\begin{cases}2\pi^{-1}(4-x^2)^{-1/2}\cos(\nu\arccos(x/2)) & |x|<2\\ 0 & |x|>2\end{cases} \tag{6.63}$$

In the rest of this chapter, we will only interchange finite sums with this conditionally convergent integral, and thus there are no analytic issues justifying the calculation in the remaining portion of this thesis. In (6.27) we have defined the function $Q(z,n)=\frac{\pi 2^{2-2z}\Gamma(2z-1)}{\Gamma(z+n)\Gamma(z-n)}$. We would like to compute the inverse Mellin transform

$\mathcal{M}^{-1}(Q(z,m_3))(x)$ of $Q(z,m_3)$:

$$\begin{aligned}
\mathcal{M}^{-1}\left(\frac{\pi 2^{2-2z}\Gamma(2z-1)}{\Gamma(z+m_3)\Gamma(z-m_3)}\right)(x) &= \frac{4\pi}{2\pi\mathsf{i}}\int_{\gamma-\mathsf{i}\infty}^{\gamma+\mathsf{i}\infty}\frac{\Gamma(2z-1)}{\Gamma(z+m_3)\Gamma(z-m_3)}(4x)^{-z}\mathsf{d}z\\
&= \frac{2\pi}{2\pi\mathsf{i}}\int_{2\gamma-1-\mathsf{i}\infty}^{2\gamma-1+\mathsf{i}\infty}\frac{\Gamma(z)}{\Gamma(\frac{z+1+2m_3}{2})\Gamma(\frac{z+1-2m_3}{2})}(4x)^{-\frac{z+1}{2}}\mathsf{d}z\\
&= \frac{\pi}{2\pi\mathsf{i}\sqrt{x}}\int_{2\gamma-1-\mathsf{i}\infty}^{2\gamma-1+\mathsf{i}\infty}\frac{\Gamma(z)}{\Gamma(\frac{z+1+2m_3}{2})\Gamma(\frac{z+1-2m_3}{2})}(2\sqrt{x})^{-z}\mathsf{d}z\\
&= \frac{\pi}{\sqrt{x}}\mathcal{M}^{-1}\left(\frac{\Gamma(z)}{\Gamma\left(\frac{1+z-2m_3}{2}\right)\Gamma\left(\frac{1+z+2m_3}{2}\right)}\right)(2\sqrt{x})\\
&= \frac{\pi}{\sqrt{x}}\begin{cases}\pi^{-1}(1-x)^{-1/2}\cos\bigl(2m_3\arccos(\sqrt{x})\bigr) & 0<x<1\\ 0 & x>1\end{cases}\\
&= \frac{1}{\sqrt{x(1-x)}}\begin{cases}\cos\bigl(2m_3\arcsin(\sqrt{1-x})\bigr) & |x|<1\\ 0 & |x|>1\end{cases}.
\end{aligned}\tag{6.64}$$

If we define the *Heaviside step function* $\theta(x)=\left\{\begin{smallmatrix}0 & x<0\\ 1 & x>0\end{smallmatrix}\right.$, the result above can be written as

$$\begin{aligned}
\mathcal{M}^{-1}(Q(z,m_3))(x) &= \frac{1-\theta(|x|-1)}{\sqrt{(1-x)x}}\cos\left(2m_3\arcsin\left(\sqrt{1-x}\right)\right)\\
&= \frac{1-\theta(|x|-1)}{\sqrt{(1-x)x}}\left(\frac{1}{2}\left(\sqrt{x}+\mathsf{i}\sqrt{1-x}\right)^{-2m_3}+\frac{1}{2}\left(\sqrt{x}+\mathsf{i}\sqrt{1-x}\right)^{2m_3}\right)
\end{aligned}\tag{6.65}$$

Therefore, the inverse Mellin transform $\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x)$ of the matrix entries of intertwining operators $S^{j,n}_{m_3,m_2}(z)$ is

$$\begin{aligned}
\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) = \frac{1-\theta(|x|-1)}{2\sqrt{(1-x)x}}&\sum_{m_3=-j}^{j}M^j_{m_1,m_3}N^j_{m_3,m_4}\\
&\mathsf{i}^{-2m_3}\left(\left(\sqrt{x}+\mathsf{i}\sqrt{1-x}\right)^{-2m_3}+\left(\sqrt{x}+\mathsf{i}\sqrt{1-x}\right)^{2m_3}\right).
\end{aligned}\tag{6.66}$$

By (6.49) and (6.50), the change-of-basis matrix $M^j_{m_1,m_3}$ and $N^j_{m_1,m_3}$ are the zeroth Laurent coefficient of some analytic functions. By substituting the matrix entries $M^j_{m_1,m_3}$ and $N^j_{m_1,m_3}$ in (6.66) with the corresponding Laurent series (6.49) and (6.50), the sum over m_3 from $-j$ to j as in (6.66) actually gives rise to a geometric series. Recalling formulas (6.49) and (6.50) and defining

$$\begin{aligned}
M^j_{m_1,m_3}(t) &= (-1)^{2j}\frac{c^j_{m_3}}{c^j_{m_1}}\mathsf{i}^{-m_1+m_3}2^{-m_3}t^{m_3-j}\left(1+\frac{t}{2}\right)^{j-m_1}\left(1-\frac{t}{2}\right)^{j+m_1}\\
N^j_{m_3,m_4}(s) &= (-1)^{2j}\frac{c^j_{m_4}}{c^j_{m_3}}\mathsf{i}^{m_3-m_4}2^{-m_4}s^{m_4-j}\left(1+\frac{s}{2}\right)^{j-m_3}\left(1-\frac{s}{2}\right)^{j+m_3},
\end{aligned}$$

where s, t are formal variables, we define

$$\begin{aligned}
\sigma_1(s,t) =& \sum_{m_3=-j}^{j} M^j_{m_1,m_3}(t) N^j_{m_3,m_4}(s) \mathrm{i}^{-2m_3} \left(\sqrt{x} + \mathrm{i}\sqrt{1-x}\right)^{-2m_3} \\
=& \frac{c^j_{m_4}}{c^j_{m_1}} \mathrm{i}^{-m_1-m_4} 2^{-m_4} \left(1-\frac{t}{2}\right)^{j+m_1} \left(1+\frac{t}{2}\right)^{j-m_1} s^{m_4-j} t^{-j} \left(1+\frac{s}{2}\right)^j \left(1-\frac{s}{2}\right)^j \times \\
& \sum_{m_3=-j}^{j} \left(\frac{t\left(1-\frac{s}{2}\right)}{2\left(1+\frac{s}{2}\right)}\right)^{m_3} \left(\sqrt{x} + \mathrm{i}\sqrt{1-x}\right)^{-2m_3} \\
=& \frac{c^j_{m_4}}{c^j_{m_1}} \mathrm{i}^{-m_1-m_4} 2^{-m_4} \left(1-\frac{t}{2}\right)^{j+m_1} \left(1+\frac{t}{2}\right)^{j-m_1} s^{m_4-j} t^{-j} \left(1+\frac{s}{2}\right)^j \left(1-\frac{s}{2}\right)^j \times \\
& \frac{\left(\frac{(1-s/2)t}{2(1+s/2)(\mathrm{i}\sqrt{1-x}+\sqrt{x})^2}\right)^{-j} - \left(\frac{(1-s/2)t}{2(1+s/2)(\mathrm{i}\sqrt{1-x}+\sqrt{x})^2}\right)^{j+1}}{1 - \frac{(1-s/2)t}{2(1+s/2)(\mathrm{i}\sqrt{1-x}+\sqrt{x})^2}}
\end{aligned} \tag{6.67}$$

and

$$\begin{aligned}
\sigma_2(s,t) =& \sum_{m_3=-j}^{j} M^j_{m_1,m_3}(t) N^j_{m_3,m_4}(s) \mathrm{i}^{-2m_3} \left(\sqrt{x} + \mathrm{i}\sqrt{1-x}\right)^{2m_3} \\
=& \frac{c^j_{m_4}}{c^j_{m_1}} \mathrm{i}^{-m_1-m_4} 2^{-m_4} \left(1-\frac{t}{2}\right)^{j+m_1} \left(1+\frac{t}{2}\right)^{j-m_1} s^{m_4-j} t^{-j} \left(1+\frac{s}{2}\right)^j \left(1-\frac{s}{2}\right)^j \times \\
& \sum_{m_3=-j}^{j} \left(\frac{t\left(1-\frac{s}{2}\right)}{2\left(1+\frac{s}{2}\right)}\right)^{m_3} \left(\sqrt{x} + \mathrm{i}\sqrt{1-x}\right)^{2m_3} \\
=& \frac{c^j_{m_4}}{c^j_{m_1}} \mathrm{i}^{-m_1-m_4} 2^{-m_4} \left(1-\frac{t}{2}\right)^{j+m_1} \left(1+\frac{t}{2}\right)^{j-m_1} s^{m_4-j} t^{-j} \left(1+\frac{s}{2}\right)^j \left(1-\frac{s}{2}\right)^j \times \\
& \frac{\left(\frac{(1-s/2)t(\mathrm{i}\sqrt{1-x}+\sqrt{x})^2}{2(1+s/2)}\right)^{-j} - \left(\frac{(1-s/2)t(\mathrm{i}\sqrt{1-x}+\sqrt{x})^2}{2(1+s/2)}\right)^{j+1}}{1 - \frac{(1-s/2)t(\mathrm{i}\sqrt{1-x}+\sqrt{x})^2}{2(1+s/2)}}.
\end{aligned} \tag{6.68}$$

If we denote by $[\sigma_i(s,t)]_{0,0}$ for the constant term in the Laurent series expansion of $\sigma_i(s,t)$, then (6.66) can be expressed as

$$\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) = \frac{1-\theta(|x|-1)}{2\sqrt{(1-x)x}}([\sigma_1(s,t)+\sigma_2(s,t)]_{0,0}). \tag{6.69}$$

To make the calculation simpler, and since we only care about the *constant* term of the Laurent series $\sigma_i(s,t)$, we can feel free to make change of variables so that $\sigma_1(s,t)$ and

$\sigma_2(s,t)$ have the same denominators. Two such choices are

$$\begin{aligned}
&\sigma_1(-s,2/t)\\
=&\frac{c^j_{m_4}}{c^j_{m_1}}\mathsf{i}^{-m_1-m_4}2^{-j-m_4}(-1)^{m_4-j}\left(1-\frac{1}{t}\right)^{j+m_1}\left(1+\frac{1}{t}\right)^{j-m_1}s^{m_4-j}t^j\left(1+\frac{s}{2}\right)^j\left(1-\frac{s}{2}\right)^j\times\\
&\frac{\left(\frac{(1+s/2)}{t(1-s/2)(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}\right)^{-j}-\left(\frac{(1+s/2)}{t(1-s/2)(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}\right)^{j+1}}{1-\frac{(1+s/2)}{t(1-s/2)(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}}\\
=&\frac{c^j_{m_4}}{c^j_{m_1}}\mathsf{i}^{-m_1-m_4}2^{-j-m_4}(-1)^{m_4-j}(t-1)^{j+m_1}(t+1)^{j-m_1}s^{m_4-j}t^{-j}\left(1+\frac{s}{2}\right)^j\left(1-\frac{s}{2}\right)^j\times\\
&\frac{\left(\frac{(1-s/2)t(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}{(1+s/2)}\right)^{-j}-\left(\frac{(1-s/2)t(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}{(1+s/2)}\right)^{j+1}}{1-\frac{(1-s/2)t(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}{(1+s/2)}}
\end{aligned}\tag{6.70}$$

and

$$\begin{aligned}
&\sigma_2(s,2t)\\
=&\frac{c^j_{m_4}}{c^j_{m_1}}\mathsf{i}^{-m_1-m_4}2^{-j-m_4}(-1)^{j+m_1}(t-1)^{j+m_1}(1+t)^{j-m_1}s^{m_4-j}t^{-j}\left(1+\frac{s}{2}\right)^j\left(1-\frac{s}{2}\right)^j\times\\
&\frac{\left(\frac{(1-s/2)t(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}{(1+s/2)}\right)^{-j}-\left(\frac{(1-s/2)t(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}{(1+s/2)}\right)^{j+1}}{1-\frac{(1-s/2)t(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2}{(1+s/2)}}
\end{aligned}\tag{6.71}$$

After taking the constant Laurent series coefficient of the function $\sigma_1(-s,2/t)+\sigma_2(s,2t)$, we will get the following expression for $\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))$:

$$\begin{aligned}
\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) =& \frac{c^j_{m_4}}{c^j_{m_1}}\mathsf{i}^{-m_1-m_4}2^{-j-m_4}\frac{(-1)^{j+m_1}+(-1)^{m_4-j}}{2}\\
&\frac{1-\theta(|x|-1)}{\sqrt{(1-x)x}}\Bigg[\frac{1}{1-(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2\frac{(1-\frac{s}{2})t}{1+\frac{s}{2}}}\\
&\left(\left(1+\frac{s}{2}\right)^{2j}(1-t)^{j+m_1}(1+t)^{j-m_1}s^{m_4-j}t^{-2j}\left(\mathsf{i}\sqrt{1-x}+\sqrt{x}\right)^{-2j}\right.\\
&-\left(1-\frac{s}{2}\right)^{2j+1}(1-t)^{j+m_1}(1+t)^{j-m_1}\left(1+\frac{s}{2}\right)^{-1}s^{m_4-j}t\\
&\left.\left(\mathsf{i}\sqrt{1-x}+\sqrt{x}\right)^{2j+2}\right)\Bigg]_{0,0}
\end{aligned}\tag{6.72}$$

Notice that in the Laurent series expansion of the function in s,t in (6.72), the exponent of t in the Laurent expansion of the second term

$$\left(1-\frac{s}{2}\right)^{2j+1}\left(1-\frac{t}{2}\right)^{j+m_1}\left(1+\frac{t}{2}\right)^{j-m_1}\left(1+\frac{s}{2}\right)^{-1}s^{m_4-j}t$$

in the parentheses is always strictly greater than 0. Therefore it does not contribute to the zeroth Laurent series coefficient in the expansion of (6.72):

$$\begin{aligned}
\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) &= \frac{c^j_{m_4}}{c^j_{m_1}} \mathsf{i}^{-m_1-m_4} 2^{-j-m_4} \frac{(-1)^{j+m_1}+(-1)^{m_4-j}}{2} \\
&\quad \frac{1-\theta(|x|-1)}{\sqrt{(1-x)x}} \Bigg[\frac{1}{1-(\mathsf{i}\sqrt{1-x}+\sqrt{x})^2 \frac{(1-\frac{s}{2})t}{1+\frac{s}{2}}} \\
&\quad \left(\left(1+\frac{s}{2}\right)^{2j} (1-t)^{j+m_1} (1+t)^{j-m_1} s^{m_4-j} t^{-2j} \left(\mathsf{i}\sqrt{1-x}+\sqrt{x}\right)^{-2j} \Bigg]_{0,0} \right. \\
&= \frac{c^j_{m_4}}{c^j_{m_1}} \mathsf{i}^{-m_1-m_4} 2^{-j-m_4} \frac{(-1)^{j+m_1}+(-1)^{m_4-j}}{2} \frac{1-\theta(|x|-1)}{\sqrt{(1-x)x}} \\
&\quad \Bigg[\sum_{q\geq 0, q\in\mathbb{Z}} \left(\left(1+\frac{s}{2}\right)^{2j-q} \left(1-\frac{s}{2}\right)^{q} (1-t)^{j+m_1} (1+t)^{j-m_1} \times \right. \\
&\quad \left. s^{m_4-j} t^{q-2j} \left(\mathsf{i}\sqrt{1-x}+\sqrt{x}\right)^{2(q-j)} \right) \Bigg]_{0,0}, \qquad (6.73)
\end{aligned}$$

which expands to

$$\begin{aligned}
\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) &= \frac{c^j_{m_4}}{c^j_{m_1}} \mathsf{i}^{-m_1-m_4} \frac{(-1)^{2j+m_1-m_4}+1}{2} \frac{1-\theta(|x|-1)}{\sqrt{(1-x)x}} \times \\
\sum_{q,\kappa_1,\kappa_2\in\mathbb{Z}} & \binom{2j-q}{j-m_4-\kappa_1} \binom{q}{\kappa_1} \binom{j+m_1}{\kappa_2} \binom{j-m_1}{2j-q-\kappa_2} (-1)^{\kappa_1+\kappa_2} \left(\mathsf{i}\sqrt{1-x}+\sqrt{x}\right)^{2(q-j)}. \qquad (6.74)
\end{aligned}$$

The summations over κ_1 and κ_2 can be expressed as hypergeometric values

$$\sum_{\kappa_1\in\mathbb{Z}} \binom{2j-q}{\kappa_1} \binom{q}{j+m_4-\kappa_1} (-1)^{\kappa_1} = \binom{2j-q}{j-m_4} {}_2F_1\left({-q,-j+m_4 \atop 1+j-q+m_4}; -1\right) \qquad (6.75)$$

$$\sum_{\kappa_2\in\mathbb{Z}} \binom{j+m_1}{\kappa_2} \binom{j-m_1}{2j-q-\kappa_2} (-1)^{\kappa_2} = \binom{j-m_1}{2j-q} {}_2F_1\left({-2j+q,-j-m_1 \atop 1-j+q-m_1}; -1\right). \qquad (6.76)$$

The sum in (6.74) becomes

$$\sum_{0\leq q\leq 2j, q\in\mathbb{Z}} \binom{2j-q}{j-m_4} \binom{j-m_1}{2j-q} {}_2F_1\left({-q,-j+m_4 \atop 1+j-q+m_4}; -1\right) {}_2F_1\left({-2j+q,-j-m_1 \atop 1-j+q-m_1}; -1\right) \left(\mathsf{i}\sqrt{1-x}+\sqrt{x}\right)^{2(q-j)}. \qquad (6.77)$$

Again, by writing the values of hypergeometric function $2F_1$ at -1 in terms of Jacobi polynomials, we have

$${}_2F_1\left({-q,-j+m_4 \atop 1+j-q+m_4}; -1\right) = 2^{j-m_4} \frac{(j-m_4)!(j+m_4-q)!}{(2j-q)!} P^{j+m_4-q,-j+m_4+q}_{j-m_4}(0) \qquad (6.78)$$

$${}_2F_1\left({-2j+q,-j-m_1 \atop 1-j+q-m_1}; -1\right) = 2^{j+m_1} \frac{(j+m_1)!(-j-m_1+q)!}{q!} P^{j-m_1-q,-j-m_1+q}_{j+m_1}(0), \qquad (6.79)$$

the sum (6.74) is equal to

$$\sum_{0\le q\le 2j, q\in\mathbb{Z}} 2^{2j+m_1-m_4}\frac{(j-m_1)!(j+m_1)!}{q!(2j-q)!}P_{j-m_4}^{j+m_4-q,-j+m_4+q}(0)P_{j+m_1}^{j-m_1-q,-j-m_1+q}(0)\times$$
$$\left(\mathrm{i}\sqrt{1-x}+\sqrt{x}\right)^{2(q-j)}. \tag{6.80}$$

Again by (6.48), (6.74) is the constant term of the Laurent seris expansion of the following function in s and t

$$\begin{aligned}
&s^{m_4-j}t^{-j-m_1}\sum_{0\le q\le 2j, q\in\mathbb{Z}}\frac{(j-m_1)!(j+m_1)!}{q!(2j-q)!}(1+s)^{2j-q}(1-s)^q(1-t)^{2j-q}\times\\
&(1+t)^q\left(\mathrm{i}\sqrt{1-x}+\sqrt{x}\right)^{2(q-j)}\\
=&s^{m_4-j}t^{-j-m_1}\frac{(j-m_1)!(j+m_1)!}{(2j)!}\times\\
&\left(\left(\mathrm{i}\sqrt{1-x}+\sqrt{x}\right)^{-1}(1+s)(1-t)+\left(\mathrm{i}\sqrt{1-x}+\sqrt{x}\right)(1-s)(1+t)\right)^{2j}.
\end{aligned} \tag{6.81}$$

For simplicity of notations, we set $\xi=\mathrm{i}\sqrt{1-x}+\sqrt{x}$. The parenthesis of (6.81) can be reorganized as

$$\begin{aligned}
&\left(\xi^{-1}(1+s)(1-t)+\xi(1-s)(1+t)\right)^{2j}\\
=&\left(\xi+\xi^{-1}\right)+(\xi^{-1}-\xi)s-(\xi^{-1}-\xi)t-st(\xi+\xi^{-1})^{2j}\\
=&\sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4\\ \sum\nu_i=2j}}\frac{(2j)!}{\nu_1!\nu_3!\nu_2!\nu_4!}(\xi+\xi^{-1})^{\nu_1+\nu_4}(\xi^{-1}-\xi)^{\nu_3+\nu_2}(-1)^{\nu_2+\nu_4}s^{\nu_3+\nu_4}t^{\nu_2+\nu_4}.
\end{aligned} \tag{6.82}$$

Therefore, the constant term of (6.81) can be rewritten into a sum involving multinomial coefficients, noting that $2j+m_1-m_4+\nu_1-\nu_4=2j$:

$$\begin{aligned}
&(-1)^{j+m_1}\sum_{\nu_1,\nu_4}\frac{(j-m_1)!(j+m_1)!}{\nu_1!(j-m_4-\nu_4)!\nu_4!(j+m_1-\nu_4)!}(\xi+\xi^{-1})^{\nu_1+\nu_4}(\xi^{-1}-\xi)^{2j+m_1-m_4-2\nu_4}\\
=&(-1)^{j+m_1}\sum_{\nu_1,\nu_4}\frac{(j-m_1)!(j+m_1)!}{\nu_1!(j-m_4-\nu_4)!\nu_4!(j+m_1-\nu_4)!}(\xi+\xi^{-1})^{\nu_1+\nu_4}(\xi^{-1}-\xi)^{2j-\nu_1-\nu_4}.
\end{aligned} \tag{6.83}$$

Reorganizing,

$$\begin{aligned}
&\left(\frac{c^j_{m_1}}{c^j_{m_4}}\right)^2(-1)^{j+m_1}\sum_{\nu_1,\nu_4}\frac{(j-m_4)!(j+m_4)!}{\nu_1!(j-m_4-\nu_4)!\nu_4!(j+m_4-\nu_1)!}(\xi+\xi^{-1})^{\nu_1+\nu_4}(\xi^{-1}-\xi)^{2j-\nu_1-\nu_4}\\
=&\left(\frac{c^j_{m_1}}{c^j_{m_4}}\right)^2(-1)^{j+m_1}\sum_{\nu_1,\nu_4}\frac{(j-m_4)!(j+m_4)!}{\nu_1!(j-m_4-\nu_4)!\nu_4!(j+m_4-\nu_1)!}(\sqrt{x})^{\nu_1+\nu_4}(-\mathrm{i}\sqrt{1-x})^{2j-\nu_1-\nu_4}
\end{aligned} \tag{6.84}$$

We can observe that the function in the summation above can be simplified to the zeroth Laurent coefficient of a function in one variable u:

$$\begin{aligned}
&\sum_{\nu_1,\nu_4} \frac{(j-m_4)!(j+m_4)!}{\nu_1!(j-m_4-\nu_4)!\nu_4!(j+m_4-\nu_1)!}(\sqrt{x})^{\nu_1+\nu_4}(-\mathrm{i}\sqrt{1-x})^{2j-\nu_1-\nu_4} \\
=&\left[u^{-j-m_1}\sum_{\nu_1,\nu_4}\binom{j+m_4}{\nu_1}\binom{j-m_4}{\nu_4}(-\mathrm{i}\sqrt{1-x}u)^{j+m_4-\nu_1}(-\mathrm{i}\sqrt{1-x})^{j-m_4-\nu_2}u^{\nu_4}(\sqrt{x})^{\nu_1+\nu_4}\right]_0 \\
=&\left[u^{-j-m_1}(\sqrt{x}-\mathrm{i}u\sqrt{1-x})^{j+m_4}(u\sqrt{x}-\mathrm{i}\sqrt{1-x})^{j-m_4}\right]_0 \qquad (6.85)
\end{aligned}$$

Plugging the formula above back into (6.74),

$$\begin{aligned}
&\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) = \frac{c^j_{m_1}}{c^j_{m_4}}\mathrm{i}^{-m_1-m_4}\frac{(-1)^{2j+m_1-m_4}+1}{2}\frac{1-\theta(|x|-1)}{\sqrt{(1-x)x}}\times \\
&(-\mathrm{i})^{2j}(-1)^{j+m_1}\left[u^{-j-m_1}(\sqrt{x}+u\sqrt{1-x})^{j+m_4}(\mathrm{i}u\sqrt{x}+\sqrt{1-x})^{j-m_4}\right]_0, \qquad (6.86)
\end{aligned}$$

and recall that $j, m_1, m_4 \in \mathbb{Z}$ and $m_1 - m_4$ is even, we finally have

$$\begin{aligned}
\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) = &\frac{c^j_{m_1}}{c^j_{m_4}}\frac{((-1)^{2j+m_1-m_4}+1)(-1)^{\frac{m_1-m_4}{2}}(1-\theta(|x|-1))}{2\sqrt{(1-x)x}} \\
&\left[\frac{(u\sqrt{1-x}+\mathrm{i}\sqrt{x})^{j+m_4}(\sqrt{1-x}+\mathrm{i}u\sqrt{x})^{j-m_4}}{u^{j+m_1}}\right]_0. \qquad (6.87)
\end{aligned}$$

From the generating function of Jacobi polynomials which we introduced in (6.48), this inverse Mellin transform is in fact

$$\begin{aligned}
&\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) \\
=&\frac{c^j_{m_1}}{c^j_{m_4}}\frac{((-1)^{2j+m_1}\mathrm{i}^{2j}+(-1)^{j+m_4})(1-\theta(|x|-1))}{2(1-x)^{\frac{m_1+m_4+1}{2}}x^{-\frac{2j+m_1+m_4-1}{2}}}P^{(-m_1-m_4,-2j-1)}_{j+m_1}\left(\frac{2}{x}-1\right). \qquad (6.88)
\end{aligned}$$

We refer to formulas 05.06.26.0002.01 and 07.23.17.0055.01 of [Res08],

$$P^{(a,b)}_n(z) = \frac{\Gamma(a+n+1)}{\Gamma(a+1)\Gamma(n+1)}\,{}_2F_1\left({-n,a+b+n+1 \atop a+1};\frac{1-z}{2}\right) \qquad (6.89)$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1\left({a,c-b \atop c};\frac{z}{z-1}\right). \qquad (6.90)$$

Applying these two formulas to (6.88), we get

$$\begin{aligned}
&\mathcal{M}^{-1}(S^{j,n}_{m_1,m_4}(z))(x) \\
=&\frac{c^j_{m_1}}{c^j_{m_4}}\frac{((-1)^{2j}+(-1)^{m_4-m_1})(1-\theta(|x|-1))}{2(1-x)^{\frac{m_1+m_4+1}{2}}x^{\frac{m_1-m_4+1}{2}}}P^{(m_4-m_1,-m_1-m_4)}_{j+m_1}(1-2x) \qquad (6.91)
\end{aligned}$$

According to formula 9.43 in [Obe12], we can apply the Mellin transform $\mathcal{M}$ on the Jacobi polynomials,

$$\begin{aligned}&\mathcal{M}\left(\left\{\begin{smallmatrix}(b-x)^{\mu-1}P_n^{(\alpha,\beta)}(1-\gamma x) & x<b\\ 0 & x>b\end{smallmatrix}\right)(z)\\ =&\frac{\Gamma(\alpha+n+1)\Gamma(\mu)\Gamma(z)b^{z+\mu-1}}{n!\Gamma(1+\alpha)\Gamma(\mu+z)}{}_3F_2(\begin{smallmatrix}-n,1+n+\alpha+\beta,z\\1+\alpha,\mu+z\end{smallmatrix};\frac{1}{2}\gamma b)\end{aligned} \tag{6.92}$$

After applying the Mellin transform, the matrix entries $S^{j,n}_{m_1,m_4}(z)$ can be transformed to

$$\begin{aligned}&S^{j,n}_{m_1,m_4}(z)=\\ &\frac{(-1)^{2j}+(-1)^{m_4-m_1}}{2}\frac{c^j_{m_1}}{c^j_{m_4}}\frac{\Gamma\left(\frac{-m_1-m_4+1}{2}\right)\Gamma\left(z+\frac{-m_1+m_4-1}{2}\right)}{\Gamma\left(-m_1+m_4+1\right)\Gamma\left(z-m_1\right)}\\ &{}_3F_2\left(\begin{smallmatrix}-j-m_1,j-m_1+1,z-\frac{m_1}{2}+\frac{m_4}{2}-\frac{1}{2}\\ z-m_1,-m_1+m_4+1\end{smallmatrix};1\right).\end{aligned} \tag{6.93}$$

According to 07.27.17.0042.01 of [Res08], we have the following transformation property of ${}_3F_2$ value at 1:

$${}_3F_2(-n,b,c;d,e;1)=\frac{(b)^{(n)}(-b-c+d+e)^{(n)}}{(d)^{(n)}(e)^{(n)}}{}_3F_2(\begin{smallmatrix}d-b,e-b,-n\\-b-c+d+e,-b-n+1\end{smallmatrix};1). \tag{6.94}$$

Apply this transform to the formula for $S^{j,n}_{m_1,m_4}(z)$,

$$\begin{aligned}&S^{j,n}_{m_1,m_4}(z)\\ =&\frac{((-1)^{2j}+(-1)^{m_4-m_1})\pi(2j)!}{2c^j_{m_1}c^j_{m_4}}\frac{\sec\left(\frac{m_1+m_4}{2}\pi\right)\Gamma\left(\frac{2z-m_1+m_4-1}{2}\right)}{\Gamma\left(\frac{-2j-m_1+m_4+1}{2}\right)\Gamma(j+z)}\\ &{}_3F_2\left(\begin{smallmatrix}-j+z-1,-j-m_1,m_4-j\\ -2j,-j-\frac{m_1}{2}+\frac{m_4}{2}+\frac{1}{2}\end{smallmatrix};1\right).\end{aligned}$$

We also need to consider that m_1 and m_4 satisfy the same parity condition, that $m_1,m_4\in\mathtt{M}(j,n;\delta_2,\delta_1)$, and therefore $\frac{m_1+m_4}{2}$ is an integer. Hence $S^{j,n}_{m_1,m_4}$ can be expressed in a simpler form in terms of hypergeometric function ${}_3F_2$:

$$S^{j,n}_{m_1,m_4}(z)=\frac{(-1)^{\frac{m_1+m_4}{2}}(2j)!\pi}{c^j_{m_1}c^j_{m_4}}\frac{\Gamma\left(\frac{2z-m_1+m_4-1}{2}\right)}{\Gamma\left(\frac{-2j-m_1+m_4+1}{2}\right)\Gamma(j+z)}{}_3F_2\left(\begin{smallmatrix}-j+z-1,-j-m_1,m_4-j\\ -2j,-j-\frac{m_1}{2}+\frac{m_4}{2}+\frac{1}{2}\end{smallmatrix};1\right) \tag{6.95}$$

6.5.4 Normalizations

According to (4.5) and (4.6), there is a normalization factor $\gamma(w_{\alpha_1}, \lambda)$ of the intertwining operator entries $S^{j,n}_{m_1,m_4}(z)$, such that the operator $A(w_{\alpha_1}, \lambda)$ has the property

$$A(w_{\alpha_1}, w_{\alpha_1}\lambda)A(w_{\alpha_1}, \lambda) = \gamma(w_{\alpha_1}, \lambda)\gamma(w_{\alpha_1}, w_{\alpha_1}\lambda)I.$$

We define the normalized intertwining operator as $A'(w_{\alpha_1}, \lambda) = \frac{1}{\gamma(w_{\alpha_1},\lambda)}A(w_{\alpha_1}, \lambda)$. Since we have assumed that $(\delta_1, \delta_2) \in \{(0, 0), (1, 1)\}$, the j and n are integers. We can check that for $m_1, m_3 \in \mathtt{M}(j, n; \delta_2, \delta_1)$,

$$\begin{aligned}&\sum_{m_2\in\mathtt{M}(j,n;\delta_2,\delta_1)} S^{j,n}_{m_1,m_2}(z)S^{j,n}_{m_2,m_3}(1-z)\\ =&\sum_{\substack{m_2\in\mathtt{M}(j,n;\delta_2,\delta_1)\\ -j\le m_4,m_5\le j}} (-1)^{m_4+m_5}M^{j,n}_{m_1,m_4}N^{j,n}_{m_4,m_2}M^{j,n}_{m_2,m_5}N^{j,n}_{m_5,m_3}Q(z, m_4)Q(1-z, m_5). \quad (6.96)\end{aligned}$$

Since $\frac{1+(-1)^k}{2}$ is 0 for $k \equiv 1 \bmod 2$, and is 1 for $k \equiv 0 \bmod 2$, the sum on m_2 can be reduced to

$$\sum_{m_2\in\mathtt{M}(j,n;\delta_2,\delta_1)} N^{j,n}_{m_4,m_2}M^{j,n}_{m_2,m_5} = \sum_{-j\le m_2\le j} N^{j,n}_{m_4,m_2}M^{j,n}_{m_2,m_5}\frac{1+(-1)^{n-m_2-\delta_1}}{2}\frac{1+(-1)^{n+m_2-\delta_2}}{2} \quad (6.97)$$

Taking into account the condition $\delta_1 = \delta_2 \in \{0, 1\}$ and and its consequence that $m_3, n \in \mathbb{Z}$, we can expand the terms

$$\begin{aligned}\frac{1+(-1)^{n-m_2-\delta_1}}{2}\frac{1+(-1)^{n+m_2-\delta_2}}{2} =&\frac{1}{2}+\frac{(-1)^{n-\delta_1}}{2}\frac{(-1)^{m_2}+(-1)^{-m_2}}{2}\\ =&\frac{1}{2}+\frac{(-1)^{n-\delta_1}}{2}(-1)^{m_2}\\ =&\frac{1}{2}+\frac{(-1)^{n-\delta_1}}{2}e^{\mathrm{i}m_2\pi}.\end{aligned}$$

Since $W^{(j,n)}_{m_2,m_2}(0, -\pi, 0, 0) = e^{\mathrm{i}m_2\pi}$, by (6.38) and the definition (6.43) and (6.44) of $M^{j,n}_{m_2,m_5}$ and $N^{j,n}_{m_4,m_2}$, we have

$$\sum_{m_2\in\mathtt{M}(j,n;\delta_2,\delta_1)} N^{j,n}_{m_4,m_2}M^{j,n}_{m_2,m_5}W^{(j,n)}_{m_2,m_2}(0, -\pi, 0, 0) = W^{(j,n)}_{m_4,m_5}(0, 0, -\pi, 0).$$

The sum (6.97) can thus be written as

$$\sum_{m_2\in \mathtt{M}(j,n;\delta_2,\delta_1)} N^{j,n}_{m_4,m_2} M^{j,n}_{m_2,m_5} = \frac{1}{2}\delta_{m_4,m_5} + \frac{(-1)^{n-\delta_1}}{2} W^{(j,n)}_{m_4,m_5}(e^{\pi U_2})$$
$$= \frac{1}{2}\delta_{m_4,m_5} + \frac{(-1)^{j-m_4-\delta_1+n}}{2}\delta_{-m_4,m_5},$$

and the sum (6.96) becomes

$$\begin{aligned}
&\sum_{m_2\in \mathtt{M}(j,n;\delta_2,\delta_1)} S^{j,n}_{m_1,m_2}(z) S^{j,n}_{m_2,m_3}(1-z) \\
=&\sum_{-j\le m_4,m_5\le j} (-1)^{m_4+m_5} M^{j,n}_{m_1,m_4} N^{j,n}_{m_5,m_3} Q(z,m_4)Q(1-z,m_5)\frac{\delta_{m_4,m_5}}{2} + \\
&\sum_{-j\le m_4,m_5\le j} (-1)^{m_4+m_5} M^{j,n}_{m_1,m_4} N^{j,n}_{m_5,m_3} Q(z,m_4)Q(1-z,m_5)\frac{(-1)^{j-m_4-\delta_1+n}}{2}\delta_{-m_4,m_5} \\
=&\frac{1}{2}\sum_{-j\le m_4\le j} M^{j,n}_{m_1,m_4} N^{j,n}_{m_4,m_3} Q(z,m_4)Q(1-z,m_4) + \frac{1}{2}\sum_{-j\le m_4\le j} M^{j,n}_{m_1,m_4} N^{j,n}_{-m_4,m_3} \\
&(-1)^{j-m_4-\delta_1+n} Q(z,m_4)Q(1-z,-m_4).
\end{aligned} \tag{6.98}$$

We observe that the product of the Q-functions are simply products of Γ-functions:

$$Q(z,n)Q(1-z,n) = Q(z,n)Q(1-z,-n) = \pi\frac{\Gamma(z-1/2)}{\Gamma(z)}\frac{\Gamma(-z+1/2)}{\Gamma(1-z)}, \tag{6.99}$$

hence the formula (6.98) is in fact

$$\begin{aligned}
&\sum_{m_2\in \mathtt{M}(j,n;\delta_2,\delta_1)} S^{j,n}_{m_1,m_2}(z) S^{j,n}_{m_2,m_3}(1-z) \\
=&\frac{\pi}{2}\frac{\Gamma(z-1/2)}{\Gamma(z)}\frac{\Gamma(-z+1/2)}{\Gamma(1-z)} \sum_{-j\le m_4\le j}\left(M^{j,n}_{m_1,m_4} N^{j,n}_{m_4,m_3} + (-1)^{j-m_4-\delta_1+n} M^{j,n}_{m_1,m_4} N^{j,n}_{-m_4,m_3}\right) \\
=&\frac{\pi}{2}\frac{\Gamma(z-1/2)}{\Gamma(z)}\frac{\Gamma(-z+1/2)}{\Gamma(1-z)}(\delta_{m_1,m_3} + (-1)^{-m_1-\delta_1+n}\delta_{m_1,m_3}) \\
=&\pi\frac{\Gamma(z-1/2)}{\Gamma(z)}\frac{\Gamma(-z+1/2)}{\Gamma(1-z)}\frac{1+(-1)^{-m_1-\delta_1+n}}{2}\delta_{m_1,m_3}.
\end{aligned} \tag{6.100}$$

Therefore, it is natural to consider be the action of the intertwining operators on the K-types with $j=0$:

$$S^{(0,n)}_{0,0}(z) = \frac{\sqrt{\pi}\Gamma\left(z-1/2\right)}{\Gamma(z)} \tag{6.101}$$

and set the normalization factor $\gamma(w_\alpha,\lambda)$ to be

$$\gamma(w_\alpha,\lambda) = S^{(0,n)}_{0,0}\left(\frac{\langle\check{\alpha},\lambda+\rho\rangle}{2}\right).$$

The matrix entries of the normalized intertwining operator $A'(w_{\alpha_1}, \lambda)$ are given by the function

$$\mathcal{S}^{j,n}_{m_1,m_2}(z) := \frac{S^{j,n}_{m_1,m_2}(z)}{S^{(0,n)}_{0,0}(z)} = \left(\frac{\sqrt{\pi}\Gamma\left(z-1/2\right)}{\Gamma(z)}\right)^{-1} S^{j,n}_{m_1,m_2}(z), \tag{6.102}$$

and by (6.95) we can further simplify the expression of the matrix entries in terms of Pochhammer symbols:

$$\mathcal{S}^{j,n}_{m_1,m_2}(z) = \frac{(-1)^{\frac{m_1+m_2}{2}}(2j)!\sqrt{\pi}}{c^j_{m_1}c^j_{m_2}\Gamma\left(\frac{1-2j-m_1+m_2}{2}\right)} \frac{(z-1/2)^{(-\frac{m_1-m_2}{2})}}{(z)^{(j)}} {}_3F_2\left({-j+z-1,-j-m_1,-j+m_2 \atop -2j,-j-\frac{m_1-m_2}{2}+\frac{1}{2}};1\right). \tag{6.103}$$

According to (6.100), if $m_1, m_3 \in \mathtt{M}(j,n;\delta_1,\delta_2)$, the normalized intertwining operator entries $\mathcal{S}^{j,n}_{m_1,m_2}(z)$ satisfy the property

$$\sum_{m_2\in\mathtt{M}(j,n;\delta_2,\delta_1)} \mathcal{S}^{j,n}_{m_1,m_2}(z)\mathcal{S}^{j,n}_{m_2,m_3}(1-z) = \delta_{m_1,m_3}. \tag{6.104}$$

If $n \geq 0$, the n-th Laurent series coefficient of the function in terms of t is the ${}_3F_2$ hypergeometric function (see [SM84, Page 94]):

$$\left[(1-t)^{-\lambda}{}_2F_1\left({a,b \atop c};zt\right)\right]_n = \frac{\Gamma(\lambda+n)}{\Gamma(n+1)\Gamma(\lambda)} {}_3F_2\left({-n,a,b \atop c,1-n-\lambda};z\right). \tag{6.105}$$

Therefore, $\mathcal{S}^{j,n}_{m_1,m_2}(z)$ is the constant Laurent series coefficient of either of the following two functions:

$$H^j_{m_1,m_2}(z;t) = \frac{(2j)!(j+m_1)!}{c^j_{m_1}c^j_{m_2}\sqrt{\pi}}\Gamma\left(\frac{1-m_1-m_2}{2}\right)\frac{(z-1/2)^{(-\frac{m_1-m_2}{2})}}{(z)^{(j)}}\times \frac{(1-t)^{\frac{-1+m_1+m_2}{2}}}{(-t)^{j+m_1}}{}_2F_1\left({-j+m_2,-1-j+z \atop -2j};t\right) \tag{6.106}$$

$$G^j_{m_1,m_2}(z;t) = \frac{(2j)!(j-m_2)!}{c^j_{m_1}c^j_{m_2}\sqrt{\pi}}\Gamma\left(\frac{1+m_1+m_2}{2}\right)\frac{(z-1/2)^{(-\frac{m_1-m_2}{2})}}{(z)^{(j)}}\times \frac{(1-t)^{\frac{-1-m_1-m_2}{2}}}{(-t)^{j-m_2}}{}_2F_1\left({-j-m_1,-1-j+z \atop -2j};t\right). \tag{6.107}$$

6.5.5 Computing the Product Matrix

In this section we compute the product of the four matrices:

$$A'(w_0,\chi_{\delta,\lambda}) = A_4(\lambda)\cdot A_3(\lambda)\cdot A_2(\lambda)\cdot A_1(\lambda)$$

when $\delta = (0,0)$ or $(1,1)$. We would also like to normalize each individual matrices $A_i(\lambda)$ such that they satisfy the condition:

$$A_i(-\lambda)A_i(\lambda) = I.$$

Recalling the parity conditions (6.8) and (6.9) satisfied by j, n, m_1, m_2, since $(\delta_1, \delta_2) \in \{(0,0),(1,1)\}$, the pair (j,n) are integers, and m_1, m_2 satisfy

$$\substack{n-m_i+\delta_1\equiv 0 \\ n+m_i+\delta_2\equiv 0} \mod 2.$$

Despite the difficulty of calculating the product of the 4 matrices, a lot of terms in the sum can be reduced using the parity condition. We replace the function $T^n_{m_1}(z)$ by the normalized function defined by

$$\mathcal{T}^n_{m_1}(z) = \frac{\mathsf{i}^{-n+m_1}}{(z)^{(\frac{m_1-n}{2})}(z)^{(\frac{-m_1+n}{2})}}.$$

The matrix entry $[A'(\lambda)]^{j,n}_{m_1,m_2}$ of the normalized intertwining operator can be expressed as the constant term of the Laurent series in t_1, t_2 of the sum:

$$\begin{aligned}[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \mathcal{T}^n_{m_2}(\frac{\lambda_2+1}{2})\times \\ \sum_{m_3\in \mathtt{M}(j,n;\delta_1,\delta_2)} G^j_{m_2,m_3}(\frac{\lambda_1+\lambda_2+1}{2}, t_2)\mathcal{T}^n_{m_3}(\frac{\lambda_1+1}{2})H^j_{m_3,m_1}(\frac{\lambda_1-\lambda_2+1}{2}; t_1).\end{aligned} \tag{6.108}$$

Plugging (6.106) and (6.107) into (6.108), we have

$$\begin{aligned}
&[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) \\
=&\frac{((2j)!)^2}{c^j_{m_1}c^j_{m_2}\pi}\frac{\mathsf{i}^{-n+m_2}}{\left(\frac{\lambda_2+1}{2}\right)^{(\frac{m_2-n}{2})}\left(\frac{\lambda_2+1}{2}\right)^{(\frac{-m_2+n}{2})}}\frac{1}{(\frac{\lambda_1-\lambda_2+1}{2})^{(j)}(\frac{\lambda_1+\lambda_2+1}{2})^{(j)}}\times \\
&{}_2F_1\left({-j+m_1, -1-j+\frac{\lambda_1-\lambda_2+1}{2} \atop -2j}; t_1\right){}_2F_1\left({-j-m_2, -1-j+\frac{\lambda_1+\lambda_2+1}{2} \atop -2j}; t_2\right)\times \\
&\sum_{m_3\in \mathtt{M}(j,n;\delta_1,\delta_2)}\frac{\mathsf{i}^{-n+m_3}}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{m_3-n}{2})}\left(\frac{\lambda_1+1}{2}\right)^{(\frac{-m_3+n}{2})}}\Gamma\left(\frac{1+m_3+m_2}{2}\right)\Gamma\left(\frac{1-m_1-m_3}{2}\right)\times \\
&\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(\frac{m_1-m_3}{2})}\left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{m_3-m_2}{2})}\frac{(1-t_2)^{\frac{-1-m_3-m_2}{2}}}{(-t_2)^{j-m_3}}\frac{(1-t_1)^{\frac{-1+m_1+m_3}{2}}}{(-t_1)^{j+m_3}}.
\end{aligned}$$

Based on the parity of $j - m_3$, we can separate the calculation into two cases:

Summation when $j - n \equiv \delta_i$ mod 2

In the situation that $j - n \equiv \delta_i$ mod 2, the set of $m_3 \in \mathtt{M}(j, n; \delta_2, \delta_1)$'s are:

$$m_3 = j - 2p \text{ where } p \in \{0, 1, 2, \ldots, j\}.$$

Thus we can replace m_3 by $j - 2p$, and the sum reduces to a sum of finitely many Pochhammer symbols:

$$[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \frac{((2j)!)^2}{C^j_{m_1} C^j_{m_2} \pi} \frac{\mathsf{i}^{-n+m_2}}{\left(\frac{\lambda_2+1}{2}\right)^{(\frac{m_2-n}{2})} \left(\frac{\lambda_2+1}{2}\right)^{(-\frac{m_2-n}{2})}} \times$$
$${}_2F_1\left(\begin{matrix} -j+m_1, -j+\frac{\lambda_1-\lambda-1}{2} \\ -2j \end{matrix}; t_1\right) {}_2F_1\left(\begin{matrix} -j-m_2, -j+\frac{\lambda_1+\lambda-1}{2} \\ -2j \end{matrix}; t_2\right) \times$$
$$\sum_{p=0}^{j} \frac{\mathsf{i}^{-n+j-2p}}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{j-n-2p}{2})} \left(\frac{\lambda_1+1}{2}\right)^{(-\frac{j-n-2p}{2})}} \Gamma\left(\frac{1+j+m_2}{2} - p\right) \Gamma\left(\frac{1-j-m_1}{2} + p\right) \times$$
$$\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(\frac{-j+m_1}{2}+p)} \left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{j-m_2}{2}-p)} (1-t_1)^{\frac{-1+j+m_1}{2}-p} (1-t_2)^{\frac{-1-j-m_2}{2}+p} t_1^{-2j+2p} t_2^{-2p}. \tag{6.109}$$

By definition of Pochhammer symbols, we have the formula

$$(x)^{(l \pm p)} = (x)^{(l)} (x+l)^{(\pm p)} \tag{6.110}$$

which we can apply to the Pochhammer symbols in the summation (6.109) to single out the $\pm p$'s in the Pochhammer exponents. Also, by applying the formula

$$(x)^{(-p)} (1-x)^{(p)} = (-1)^p \text{ for } p \in \mathbb{Z} \tag{6.111}$$

to the Pochhammer symbols with $-p$ as exponent, we can manage to change all the exponents of the Pochhammer symbols in (6.109) to (p). The explicit set of rules is the following

$$\begin{aligned} \left(\frac{\lambda_1+1}{2}\right)^{(\frac{j-n-2p}{2})} &\to \left(\frac{\lambda_1+1}{2}\right)^{(\frac{j-n}{2})} \left(\frac{\lambda_1+1+j-n}{2}\right)^{(-p)} \\ &\to (-1)^p \left(\frac{\lambda_1+1}{2}\right)^{(\frac{j-n}{2})} \left(1 - \frac{\lambda_1+1+j-n}{2}\right)^{(p)} \end{aligned} \tag{6.112}$$

$$\left(\frac{\lambda_1+1}{2}\right)^{(-\frac{j-n-2p}{2})} \to \left(\frac{\lambda_1+1}{2}\right)^{(-\frac{j-n}{2})} \left(\frac{\lambda_1+1-j+n}{2}\right)^{(p)} \tag{6.113}$$

$$\Gamma\left(\frac{1+j+m_2}{2}-p\right) \to \Gamma\left(\frac{1+j+m_2}{2}\right)\left(\frac{1+j+m_2}{2}\right)^{(-p)}$$
$$\to (-1)^p\Gamma\left(\frac{1+j+m_2}{2}\right)\left(1-\frac{1+j+m_2}{2}\right)^{(p)} \tag{6.114}$$
$$\Gamma\left(\frac{1-j-m_1}{2}+p\right) \to \Gamma\left(\frac{1-j-m_1}{2}\right)\left(\frac{1-j-m_1}{2}\right)^{(p)} \tag{6.115}$$

$$\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(-\frac{j-m_1-2p}{2})} \to \left(\frac{\lambda_1-\lambda_2}{2}\right)^{(-\frac{j-m_1}{2})}\left(\frac{\lambda_1-\lambda_2}{2}-\frac{j-m_1}{2}\right)^{(p)} \tag{6.116}$$
$$\left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{j-m_2-2p}{2})} \to \left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{j-m_2}{2})}\left(\frac{\lambda_1+\lambda_2}{2}+\frac{j-m_2}{2}\right)^{(-p)}$$
$$\to (-1)^p\left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{j-m_2}{2})}\left(1-\frac{\lambda_1+\lambda_2}{2}+\frac{j-m_2}{2}\right)^{(p)}. \tag{6.117}$$

After applying (6.112)-(6.117) and bringing out the factors not involved in the summation, we have

$$\begin{aligned}
&[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \\
&\frac{\text{¡}^{j+m_2-2n}((2j)!)^2}{\pi c^j_{m_1} c^j_{m_2}} \frac{\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(\frac{-j+m_1}{2})}\left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{j-m_2}{2})}}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{-j+n}{2})}\left(\frac{\lambda_1+1}{2}\right)^{(\frac{j-n}{2})}} \times \\
&\frac{\Gamma\left(\frac{1+j+m_2}{2}\right)\Gamma\left(\frac{1-j-m_1}{2}\right)}{\left(\frac{\lambda_1+\lambda_2+1}{2}\right)^{(j)}\left(\frac{\lambda_1-\lambda_2+1}{2}\right)^{(j)}\left(\frac{\lambda_2+1}{2}\right)^{(\frac{n-m_2}{2})}\left(\frac{\lambda_2+1}{2}\right)^{(\frac{-n+m_2}{2})}} \times \\
&{}_2F_1\left(\begin{matrix}-j+m_1, \frac{\lambda_1-\lambda_2-2j-1}{2}\\ -2j\end{matrix};t_1\right){}_2F_1\left(\begin{matrix}-j-m_2, \frac{\lambda_1+\lambda_2-2j-1}{2}\\ -2j\end{matrix};t_2\right)\times \\
&\gamma'^{j,n}_{m_1,m_2}(\lambda;t_1,t_2),
\end{aligned} \tag{6.118}$$

where the function $\gamma'^{j,n}_{m_1,m_2}(\lambda;t_1,t_2)$ in two variables t_1, t_2 is defined as the hypergeometric sum

$$\begin{aligned}
\gamma'^{j,n}_{m_1,m_2}(\lambda;t_1,t_2) &= (1-t_1)^{\frac{-1+j+m_1}{2}}(-t_1)^{-2j}(1-t_2)^{\frac{-1-j-m_2}{2}}\times \\
&\sum_{p=0}^{j}\frac{\left(\frac{1-j-m_1}{2}\right)^{(p)}\left(\frac{-j+m_1+\lambda_1-\lambda_2}{2}\right)^{(p)}\left(\frac{1-j+n-\lambda_1}{2}\right)^{(p)}(1)^{(p)}}{\left(\frac{1-j-m_2}{2}\right)^{(p)}\left(1-\frac{j-m_2+\lambda_1+\lambda_2}{2}\right)^{(p)}\left(\frac{1-j+n+\lambda_1}{2}\right)^{(p)}}\frac{1}{p!}\left(\frac{t_1^2(1-t_2)}{t_2^2(1-t_1)}\right)^p.
\end{aligned} \tag{6.119}$$

We can use a partial sum formula for hypergeometric series from 16.2.4 of [OLBC10]:

$$\sum_{k=0}^{m} \frac{(\mathbf{a})^{(k)}}{(\mathbf{b})^{(k)}} \frac{z^k}{k!} = \frac{(\mathbf{a})^{(m)} z^m}{(\mathbf{b})^{(m)} m!} {}_{q+2}F_p \left({-m,1,1-m-\mathbf{b} \atop 1-m-\mathbf{a}} ; \frac{(-1)^{p+q+1}}{z} \right), \tag{6.120}$$

where $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_q)$ are p and q dimensional vectors of complex numbers, and the Pochhammer symbol $(\mathbf{a})^{(k)}$ and $(\mathbf{b})^{(k)}$ for these two vectors are defined as the product $(\mathbf{a})^{(k)} = (a_1)^{(k)} \ldots (a_p)^{(k)}$ and $(\mathbf{b})^{(k)} = (b_1)^{(k)} \ldots (b_q)^{(k)}$. Then the summation term in $\gamma'^{j,n}_{m_1,m_2}(\lambda; t_1, t_2)$ can be simplified into a single compact form hypergeometric function ${}_5F_4$:

$$\begin{aligned}
&\sum_{p=0}^{j} \frac{\left(\frac{1-j-m_1}{2}\right)^{(p)} \left(\frac{-j+m_1+\lambda_1-\lambda_2}{2}\right)^{(p)} \left(\frac{1-j+n-\lambda_1}{2}\right)^{(p)} (1)^{(p)}}{\left(\frac{1-j-m_2}{2}\right)^{(p)} \left(1 - \frac{j-m_2+\lambda_1+\lambda_2}{2}\right)^{(p)} \left(\frac{1-j+n+\lambda_1}{2}\right)^{(p)}} \frac{1}{p!} \left(\frac{t_1^2(1-t_2)}{t_2^2(1-t_1)}\right)^p \\
&= \frac{\left(\frac{1-j-m_1}{2}\right)^{(j)} \left(\frac{-j+m_1+\lambda_1-\lambda_2}{2}\right)^{(j)} \left(\frac{-j+n-\lambda_1+1}{2}\right)^{(j)}}{\left(\frac{1-j-m_2}{2}\right)^{(j)} \left(\frac{-j+m_2-\lambda_1-\lambda_2}{2} + 1\right)^{(j)} \left(\frac{-j+n+\lambda_1+1}{2}\right)^{(j)}} \left(\frac{t_1^2\,(1-t_2)}{t_2^2\,(1-t_1)}\right)^j \times \\
&{}_5F_4 \left({-j, 1, \frac{1-j+m_2}{2}, \frac{-j-n-\lambda_1+1}{2}, \frac{-j-m_2+\lambda_1+\lambda_2}{2} \atop \frac{1-j+m_1}{2}, \frac{1-j-n+\lambda_1}{2}, \frac{-j-m_1-\lambda_1+\lambda_2}{2} + 1, -j} ; \frac{t_2^2\,(1-t_1)}{t_1^2\,(1-t_2)} \right)
\end{aligned}$$

and if we replace the sum in (6.124) by the expression above involving ${}_5F_4$, also we recover the Pochhammer symbols from the transforms (6.112)-(6.117) based on (6.110) and (6.111), the original function (6.109) can be expressed as

$$\begin{aligned}
&[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \\
&\frac{(-1)^n((2j)!)^2}{\left(\frac{\lambda_1+1}{2}\right)^{\left(\frac{j+n}{2}\right)} \left(\frac{\lambda_1+1}{2}\right)^{\left(-\frac{j+n}{2}\right)} \left(\frac{\lambda_1-\lambda_2+1}{2}\right)^{(j)} \left(\frac{\lambda_1+\lambda_2+1}{2}\right)^{(j)}} \times \\
&\frac{1}{c^j_{m_1} c^j_{m_2}} \frac{\left(\frac{\lambda_1-\lambda_2}{2}\right)^{\left(\frac{j+m_1}{2}\right)} \left(\frac{\lambda_1+\lambda_2}{2}\right)^{\left(\frac{-j-m_2}{2}\right)}}{\left(\frac{\lambda_2+1}{2}\right)^{\left(\frac{m_2-n}{2}\right)} \left(\frac{\lambda_2+1}{2}\right)^{\left(-\frac{m_2-n}{2}\right)}} \frac{\Gamma\left(\frac{1+j-m_1}{2}\right)}{\Gamma\left(\frac{1+j-m_2}{2}\right)} \times \\
&(1-t_1)^{\frac{-1-j+m_1}{2}} (1-t_2)^{\frac{-1+j-m_2}{2}} t_2^{-2j} \times \\
&{}_2F_1\left({-j+m_1, \frac{\lambda_1-\lambda_2-2j-1}{2} \atop -2j} ; t_1 \right) {}_2F_1\left({-j-m_2, \frac{\lambda_1+\lambda_2-2j-1}{2} \atop -2j} ; t_2 \right) \times \\
&{}_5F_4 \left({-j, 1, \frac{-j+m_2+1}{2}, \frac{-j-n-\lambda_1+1}{2}, \frac{-j-m_2+\lambda_1+\lambda_2}{2} \atop -j, \frac{-j+m_1+1}{2}, \frac{-j-n+\lambda_1+1}{2}, \frac{-j-m_1-\lambda_1+\lambda_2}{2}+1} ; \frac{t_2^2\,(1-t_1)}{t_1^2\,(1-t_2)} \right).
\end{aligned} \tag{6.121}$$

The $-j$'s as parameters to the function ${}_5F_4$ don't necessarily cancel since they are non-positive. They play an important role in making the function ${}_5F_4$ rational.

Summation when $j - n \not\equiv \delta_i$ mod 2

On the other hand, if $j - n \not\equiv \delta_i$ mod 2,

$$m = j - 2p - 1 \text{ where } p \in \{0, 1, \ldots, j-1\},$$

we have

$$[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \frac{((2j)!)^2}{c^j_{m_1} c^j_{m_2} \pi} \frac{\text{¡}^{-n+m_2}}{\left(\frac{\lambda_2+1}{2}\right)^{(\frac{m_2-n}{2})} \left(\frac{\lambda_2+1}{2}\right)^{(-\frac{m_2-n}{2})}} \times$$
$${}_2F_1\left({-j+m_1, -j+\frac{\lambda_1-\lambda-1}{2} \atop -2j}; t_1\right) {}_2F_1\left({-j-m_2, -j+\frac{\lambda_1+\lambda-1}{2} \atop -2j}; t_2\right) \times$$
$$\sum_{p=0}^{j-1} \frac{\text{¡}^{-n+j-2p-1}}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{j-n-2p-1}{2})} \left(\frac{\lambda_1+1}{2}\right)^{(-\frac{j-n-2p-1}{2})}} \Gamma\left(\frac{j+m_2}{2} - p\right) \Gamma\left(\frac{2-j-m_1}{2} + p\right) \times$$
$$\left(\frac{\lambda_1 - \lambda_2}{2}\right)^{(\frac{1-j+m_1}{2}+p)} \left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{j-m_2-1}{2}-p)} (1-t_1)^{\frac{-2+j+m_1}{2}-p} \times$$
$$(1-t_2)^{\frac{-j-m_2}{2}+p} t_1^{-2j+2p+1} t_2^{-2p-1}. \tag{6.122}$$

Comparing with the even case, after applying (6.110) and (6.111), the summation will change to

$$[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) =$$
$$\frac{\text{¡}^{j+m_2-2n}((2j)!)^2}{\pi c^j_{m_1} c^j_{m_2}} \frac{\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(\frac{-j+m_1+1}{2})} \left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{j-m_2-1}{2})}}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{-j+n+1}{2})} \left(\frac{\lambda_1+1}{2}\right)^{(\frac{j-n-1}{2})}} \times$$
$$\frac{\Gamma\left(\frac{j+m_2}{2}\right) \Gamma\left(\frac{2-j-m_1}{2}\right)}{\left(\frac{\lambda_1+\lambda_2+1}{2}\right)^{(j)} \left(\frac{\lambda_1-\lambda_2+1}{2}\right)^{(j)} \left(\frac{\lambda_2+1}{2}\right)^{(\frac{n-m_2}{2})} \left(\frac{\lambda_2+1}{2}\right)^{(\frac{-n+m_2}{2})}} \times$$
$${}_2F_1\left({-j+m_1, \frac{\lambda_1-\lambda_2-2j-1}{2} \atop -2j}; t_1\right) {}_2F_1\left({-j-m_2, \frac{\lambda_1+\lambda_2-2j-1}{2} \atop -2j}; t_2\right) \times$$
$$\gamma'^{j,n}_{m_1,m_2}(\lambda; t_1, t_2), \tag{6.123}$$

where

$$\gamma''^{j,n}_{m_1,m_2}(\lambda; t_1,t_2) = (1-t_1)^{\frac{j+m_1-2}{2}} t_2^{-2j+1} t_1^{-1} (1-t_2)^{\frac{-j-m_2}{2}} \times$$
$${}_2F_1\left({-j+m_1, \frac{\lambda_1-\lambda_2-2j-1}{2} \atop -2j}; t_1\right) {}_2F_1\left({-j-m_2, \frac{\lambda_1+\lambda_2-2j-1}{2} \atop -2j}; t_2\right) \times$$
$$\sum_{p=0}^{j-1} \frac{\left(\frac{2-j-m_1}{2}\right)^{(p)} \left(\frac{-j+m_2+\lambda_1-\lambda_2}{2}\right)^{(p)} \left(1 - \frac{j-n+\lambda_1}{2}\right)^{(p)} (1)^{(p)}}{\left(\frac{2-j-m_2}{2}\right)^{(p)} \left(\frac{3}{2} - \frac{j-m_2-\lambda_1-\lambda_2}{2}\right)^{(p)} \left(\frac{2-j+n+\lambda_1}{2}\right)^{(p)}} \frac{1}{p!} \left(\frac{t_1^2(1-t_2)}{t_2^2(1-t_1)}\right)^p. \tag{6.124}$$

By the same formula (6.120) and using the relation between Pochhammer symbols and Γ-functions $(a)^{(n)} = \Gamma(a+n)/\Gamma(a)$ to absorb the extra Γ factors in $\gamma''^{j,n}_{m_1,m_2}$, the matrix entries for the long intertwining operator has an expression

$$\begin{aligned}
&[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \\
&\frac{(-1)^n((2j)!)^2}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{j+n-1}{2})}\left(\frac{\lambda_1+1}{2}\right)^{(-\frac{j+n-1}{2})}\left(\frac{\lambda_1-\lambda_2+1}{2}\right)^{(j)}\left(\frac{\lambda_1+\lambda_2+1}{2}\right)^{(j)}}\times \\
&\frac{1}{c^j_{m_1}c^j_{m_2}}\frac{\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(\frac{j+m_1-1}{2})}\left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{-j-m_2+1}{2})}}{\left(\frac{\lambda_2+1}{2}\right)^{\frac{m_2-n}{2}}\left(\frac{\lambda_2+1}{2}\right)^{-\frac{m_2-n}{2}}}\frac{\Gamma\left(\frac{j-m_1}{2}\right)}{\Gamma\left(\frac{j-m_2}{2}\right)}\times \\
&(1-t_1)^{\frac{-j+m_1}{2}}(1-t_2)^{\frac{j-m_2-2}{2}}t_2^{1-2j}t_1^{-1}\times \\
&{}_2F_1\left(\begin{matrix}-j+m_1,\frac{\lambda_1-\lambda_2-2j-1}{2}\\-2j\end{matrix};t_1\right){}_2F_1\left(\begin{matrix}-j-m_2,\frac{\lambda_1+\lambda_2-2j-1}{2}\\-2j\end{matrix};t_2\right)\times \\
&{}_5F_4\left(\begin{matrix}-j+1,1,\frac{-j+m_2+2}{2},\frac{-j-n-\lambda_1+2}{2},\frac{-j-m_2+\lambda_1+\lambda_2+1}{2}\\-j+1,\frac{-j+m_1+2}{2},\frac{-j-n+\lambda_1+2}{2},\frac{-j-m_1-\lambda_1+\lambda_2+1}{2}+1\end{matrix};\frac{t_2^2(1-t_1)}{t_1^2(1-t_2)}\right).
\end{aligned}\tag{6.125}$$

Therefore, if we set $\epsilon^{j,n}_\delta = \begin{cases}0 & j-n\equiv\delta_i \mod 2\\ 1 & j-n\not\equiv\delta_i \mod 2\end{cases}$, we can summarize the above two cases (6.121),(6.125) into one single formula:

$$\begin{aligned}
&[A(\lambda)]^{j,n}_{m_1,m_2}(t_1,t_2) = \\
&\frac{(-1)^n((2j)!)^2}{\left(\frac{\lambda_1+1}{2}\right)^{(\frac{j+n-\epsilon^{j,n}_\delta}{2})}\left(\frac{\lambda_1+1}{2}\right)^{(-\frac{j+n-\epsilon^{j,n}_\delta}{2})}\left(\frac{\lambda_1-\lambda_2+1}{2}\right)^{(j)}\left(\frac{\lambda_1+\lambda_2+1}{2}\right)^{(j)}}\times \\
&\frac{1}{c^j_{m_1}c^j_{m_2}}\frac{\left(\frac{\lambda_1-\lambda_2}{2}\right)^{(\frac{j+m_1-\epsilon^{j,n}_\delta}{2})}\left(\frac{\lambda_1+\lambda_2}{2}\right)^{(\frac{-j-m_2+\epsilon^{j,n}_\delta}{2})}}{\left(\frac{\lambda_2+1}{2}\right)^{\frac{m_2-n}{2}}\left(\frac{\lambda_2+1}{2}\right)^{-\frac{m_2-n}{2}}}\frac{\Gamma\left(\frac{1-\epsilon^{j,n}_\delta+j-m_1}{2}\right)}{\Gamma\left(\frac{1-\epsilon^{j,n}_\delta+j-m_2}{2}\right)}\times \\
&(1-t_1)^{\frac{-1+\epsilon^{j,n}_\delta-j+m_1}{2}}(1-t_2)^{\frac{-1-\epsilon^{j,n}_\delta+j-m_2}{2}}t_2^{\epsilon^{j,n}_\delta-2j}t_1^{-\epsilon^{j,n}_\delta}\times \\
&{}_2F_1\left(\begin{matrix}-j+m_1,\frac{\lambda_1-\lambda_2-2j-1}{2}\\-2j\end{matrix};t_1\right){}_2F_1\left(\begin{matrix}-j-m_2,\frac{\lambda_1+\lambda_2-2j-1}{2}\\-2j\end{matrix};t_2\right)\times \\
&{}_5F_4\left(\begin{matrix}-j+\epsilon^{j,n}_\delta,1,\frac{-j+m_2+1+\epsilon^{j,n}_\delta}{2},\frac{-j-n-\lambda_1+1+\epsilon^{j,n}_\delta}{2},\frac{-j-m_2+\lambda_1+\lambda_2+\epsilon^{j,n}_\delta}{2}\\-j+\epsilon^{j,n}_\delta,\frac{-j+m_1+1+\epsilon^{j,n}_\delta}{2},\frac{-j-n+\lambda_1+1+\epsilon^{j,n}_\delta}{2},\frac{-j-m_1-\lambda_1+\lambda_2+\epsilon^{j,n}_\delta}{2}+1\end{matrix};\frac{t_2^2(1-t_1)}{t_1^2(1-t_2)}\right).
\end{aligned}\tag{6.126}$$

Thus we have finished the proof of Theorem 1.4.

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