

Extended Basic Logic and Ordinal Numbers

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## 1. Introduction

1.1 It is assumed that the reader is familiar with the definition of the system  $\underline{K}$  of basic logic [5,8,11,12] and the system  $\underline{K}'$  of extended basic logic [7,10] as described by Fitch. In order to avoid confusion between various formulations of basic logic and extended basic logic, a sketch of a definition of these systems which is similar to the definitions given by Hermes [15] is given here.

1.2 This essay is organized as follows. In section 2 there is a definition of the system  $\underline{K}$  together with a statement of a number of theorems about the system. The subject of section 3 is  $\underline{K}'$ . Section 4 contains some results which are used in section 6. The relation of identity of the system  $\underline{R}$  of combinatory logic [19,20] is defined and it is shown that this relation is completely represented in  $\underline{K}'$ . There is an outline of a proof that the identity relation of  $\underline{R}$  and, consequently, its theory of numbers and functions of numbers are available in  $\underline{K}'$ . Section 5 contains an informal discussion of constructible ordinals in the second number class. This is preparation for the two main sections:

1.3 The concept of a  $\underline{U}$ -expression representing an ordinal is introduced in section 6. It is shown that each Church-Kleene [1] ordinal is represented by a  $\underline{U}$ -expression and that there is a  $\underline{U}$ -expression which completely represents the set of  $\underline{U}$ -expressions which represent ordinals. Schütte [21] introduces ordinals by defining another order relation on the natural numbers. After this is done, it is easy to establish connections between constructive ordinals and the natural numbers in Schütte's ordering. In section 7 it is shown that Schütte's order relation is completely represented in  $\underline{K}'$  and that transfinite induction on these ordinals is a derived rule of  $\underline{K}'$ .

1.4 The remainder of this section contains a statement of notational conventions and a definition of the set  $\mathcal{U}$  of well-formed formulas (wffs) of the systems  $\mathcal{K}$  and  $\mathcal{K}'$ . The conventions used here are taken from the Dictionary of Symbols of Mathematical Logic [4].

1.5 Convention. An expression consisting of single quotation marks together with the expression enclosed by them serves as a metalinguistic name for the expression thus enclosed. For example, the following expression

'a'

is used as a name for the following expression

a .

1.6 Convention. If an expression contains occurrences of the following italic letters, with or without subscripts, 'a', 'b', 'c', 'd', 'x', 'y', 'z', or is itself one of these letters, then the total expression serves as a metalinguistic variable referring to one or more expressions obtainable from the expression by replacing these letters by wffs in various ways. Numerical subscripts attached to these letters are, of course, counted as parts of the letters. For example, the expression

'b'

serves as a metalinguistic variable which refers to wffs.

1.7 Convention. The lower case italic letters

i j k m n

are used as metavariables whose range is natural numbers. Furthermore, as a general convention, if an expression in Roman type occurs, the object which it represents will appear in italics. For example, '0', '1', '2', ... are  $\mathcal{U}$ -expressions which represent the natural numbers 0, 1, 2, ..., respectively.

1.8 Definition. The set U of well-formed formulas (also referred to as the set of U-expressions) is defined inductively as follows:

- (1) '#' is a wff.
- (2) If 'a' and 'b' are wffs, then '(ab)' is a wff.
- (3) The only expressions which are wffs are those which can be shown to be such by virtue of (1) and (2).

1.9 Convention. Outermost parentheses of wffs may be omitted if no ambiguity results from doing so. When internal parentheses are omitted from wffs, the omitted parentheses are to be thought of as inserted as far to the left as is consistent with the expression being well-formed. For example, 'abc' is an abbreviation for '(ab)c' which is an abbreviation for '((ab)c)'. Also, '[b a c]', with spaces left on each side of the central expression, serves as an abbreviation for 'abc'; if there is no possibility of confusion, the outer square brackets are omitted.

1.10 '#(##)', '#(###)', '#(####)', and '#(#####)' are abbreviated as '=', 'v', '&', and 'A', respectively. Also, we assign the following abbreviations:

- 'λ<sub>1</sub>' for '#(#####)'
- 'λ<sub>2</sub>' for '#(#####)'
- 'λ<sub>3</sub>' for '#(#####)'
- ⋮
- 'λ<sub>n</sub>' for '#( # ... # )'  

$$\underbrace{\hspace{2em}}_{2n+4 \text{ times}}$$

2. The System K of Basic Logic

2.1 The set  $\underline{K}$  is a subset of the set  $\underline{U}$  of wffs.  $\underline{K}$  is defined by means of rules for identity, disjunction, conjunction, and abstraction. Fitch [11,12] has shown that existence and negation can be defined for the system  $\underline{K}$ .

2.2 Definition. The set  $\underline{K}$  of wffs is defined by means of the following four rules:

Rule [=] 'a = b' is in  $\underline{K}$  if 'a' and 'b' are the same wff.

Rule [ $\vee$ ] 'a  $\vee$  b' is in  $\underline{K}$  iff (if and only if) either 'a' or 'b' is in  $\underline{K}$ .

Rule [ $\&$ ] 'a  $\&$  b' is in  $\underline{K}$  iff both 'a' and 'b' are in  $\underline{K}$ .

Rule [ $\lambda$   $\underline{n}$ ] For all  $\underline{n} > 1$ , ' $\lambda_{\underline{n}-1} \dots \lambda_{\underline{n}} (---x_{\underline{1}}, \dots, x_{\underline{n}} ---) a_{\underline{1}} \dots a_{\underline{n}}$ ' is in  $\underline{K}$  iff ' $(--- a_{\underline{1}}, \dots, a_{\underline{n}} ---)$ ' is in  $\underline{K}$ .

The last three rules may be stated with "if" replacing "iff". Then, using the inversion principle of Lorenzen [16], it can be shown that they are valid as stated.

2.3 It is straightforward to verify that  $\underline{K}$  is consistent. There are wffs 'a' such that 'a' is not in  $\underline{K}$ . In particular, identities of the form 'b = c', where 'b' and 'c' are different wffs, are not in  $\underline{K}$ .

2.4 The following theorems provide additional rules for  $\underline{K}$ . Since Fitch has given proofs of these theorems, the proofs are omitted and cited.

2.5 Theorem. (Fitch [12].) There is a wff 'E' such that the following rule is valid for  $\underline{K}$ .

Rule [E] 'Ea' is in  $\underline{K}$  iff there is a wff 'b' such that 'ab' is in  $\underline{K}$ .

2.6 Theorem. (Fitch [11].) There is a wff 'F' such that the following rule is valid.

Rule [F] 'Fa(bc)' is in  $\underline{K}$  iff 'abc' is in  $\underline{K}$ .

2.7 Theorem. (Fitch [11].) There is a wff ' $\sim$ ' such that the following rules are valid.

Rule [ $\sim$ ]= ' $\sim[a = b]$ ' is in  $\mathcal{K}$  if ' $a$ ' and ' $b$ ' are different wffs.

Rule [ $\sim\vee$ ] ' $\sim[a \vee b]$ ' is in  $\mathcal{K}$  iff ' $\sim a$ ' and ' $\sim b$ ' are in  $\mathcal{K}$ .

Rule [ $\sim\&$ ] ' $\sim[a \& b]$ ' is in  $\mathcal{K}$  iff either ' $\sim a$ ' or ' $\sim b$ ' (or both) is in  $\mathcal{K}$ .

Rule [ $\sim\lambda_n$ ] For all  $n \geq 1$ ;

$'\sim(\lambda_{\underline{n-1}}x_1 \dots x_n (\text{---}x_1, \dots, x_n \text{---})a_1 \dots a_n)'$  is in  $\mathcal{K}$  iff ' $\sim(\text{---}a_1, \dots, a_n \text{---})'$  is in  $\mathcal{K}$ .

Rule [ $\sim F$ ] ' $\sim(F(a(bc)))'$  is in  $\mathcal{K}$  iff ' $\sim(abc)'$  is in  $\mathcal{K}$ .

Rule [ $\sim\sim$ ] ' $\sim(\sim a)'$  is in  $\mathcal{K}$  iff ' $a$ ' is in  $\mathcal{K}$ .

Rule [ $\sim$ ] There is no wff ' $a$ ' such that ' $a$ ' and ' $\sim a$ ' are both in  $\mathcal{K}$ .

2.8 The following properties of  $\mathcal{K}$  will be used below.

2.9 Definition. A wff ' $a$ ' is said to represent in  $\mathcal{K}$  an  $n$ -ary relation  $R$  among wffs just in the case that ' $ax_1 \dots x_n$ ' is in  $\mathcal{K}$  if  $R$  relates ' $x_1$ ' ..., ' $x_n$ ' in that order ( $n \geq 1$ ). A wff ' $a$ ' is said to completely represent in  $\mathcal{K}$  an  $n$ -ary relation  $R$  among wffs just in the case that ' $a$ ' represents  $R$  and ' $\sim(ax_1 \dots x_n)$ ' is in  $\mathcal{K}$  if  $R$  does not relate ' $x_1$ ', ..., ' $x_n$ ' in that order ( $n \geq 1$ ).

2.10 Theorem. (Fitch [11], paragraph 6.7) A relation among wffs is recursively enumerable iff it is represented in  $\mathcal{K}$ .

2.11 Theorem. (Fitch [11], paragraph 6.10) For every relation  $R$  among wffs, the following conditions are equivalent:

- (1)  $R$  is general recursive
- (2)  $R$  is completely represented in  $\mathcal{K}$
- (3)  $R$  and its complement are both represented in  $\mathcal{K}$ .

2.12 ' $(\exists x)(\dots x \dots)'$  is an abbreviation for ' $E\lambda_1 x(\dots x \dots)'$  and ' $(\exists x_1 \dots x_n)(\text{---}x_1, \dots, x_n \text{---})'$ . This definition and rule [E] gives a rule

for existential quantification in  $\mathcal{K}$ . Also, if ' $(\dots \underline{x} \dots \underline{y} \dots)$ ' obeys excluded middle in  $\mathcal{K}$ , then by rules [F] and [ $\sim$ F], ' $(\exists \underline{xy}) [ \underline{a} = \underline{xy} ] \& (\dots \underline{x} \dots \underline{y} \dots)$ ' obeys excluded middle in  $\mathcal{K}$ .

3. The System  $K'$  of Extended Basic Logic

3.1 The set  $K'$  is a subset of the set  $U$  of wffs which is more inclusive than  $K$ .  $K'$  provides ways for dealing with universal quantification which are not available in  $K$ . However,  $K'$  is not a constructively definable class of wffs. The rules for  $K'$  are the four rules used to define  $K$  plus a rule for universality. Fitch [12,10] has shown that existence and negation can be defined in  $K'$ .

3.2 Definition. The set  $K'$  is defined by means of the following five rules.

Rule [=] 'a = b' is in  $K'$  if 'a' and 'b' are the same wff.

Rule [v] 'a v b' is in  $K'$  iff either 'a' or 'b' (or both) is in  $K'$ .

Rule [&] 'a & b' is in  $K'$  iff both 'a' and 'b' are in  $K'$ .

Rule [ $\lambda_n$ ] For all  $n \geq 1$ ,

' $\lambda_{\underline{n-1}} \dots \lambda_{\underline{n}} (---x_1, \dots, x_{\underline{n}} ---) \underline{a_1} \dots \underline{a_n}$ ' is in  $K'$  iff ' $(--- \underline{a_1}, \dots, \underline{a_n} ---)$ ' is in  $K'$ .

Rule [A] 'Aa' is in  $K'$  iff all wffs 'b' are such that 'ab' is in  $K'$ .

As in the definition of  $K$ , the last four rules may be stated with "if" replacing "iff". Then, using the inversion principle of Lorenzen [16], it can be shown that they are valid as stated.

3.3 The proof of the consistency of  $K'$  is the same as the proof of the consistency of  $K$ . The following theorems provide additional rules for the system  $K'$ . As in section 2, the proofs are omitted.

3.4 Theorem. (Fitch [12].) There is a wff 'E' such that the following rule is valid in  $K'$ .

Rule [E] 'Ea' is in  $K'$  iff there is a wff 'b' such that 'ab' is in  $K'$ .



3.5 Theorem. (Fitch [10].) There is a wff ' $\sim$ ' such that the following rules are valid in  $\mathcal{K}'$ .

Rule [ $\sim$ ]=] ' $\sim[a = b]$ ' is in  $\mathcal{K}'$  if ' $a$ ' and ' $b$ ' are different wffs.

Rule [ $\sim$  $\vee$ ] ' $\sim[a \vee b]$ ' is in  $\mathcal{K}'$  iff ' $\sim a$ ' and ' $\sim b$ ' are in  $\mathcal{K}'$ .

Rule [ $\sim$  $\&$ ] ' $\sim[a \& b]$ ' is in  $\mathcal{K}'$  if either ' $\sim a$ ' or ' $\sim b$ ' (or both) is in  $\mathcal{K}'$ .

Rule [ $\sim$  $\lambda_{\underline{n}}$ ] For all  $\underline{n} \geq 1$ ,

$'\sim(\lambda_{\underline{n}} x_{\underline{1}} \dots x_{\underline{n}} (\text{--- } x_{\underline{1}}, \dots, x_{\underline{n}} \text{---}) a_{\underline{1}} \dots a_{\underline{n}})'$  is in  $\mathcal{K}'$  iff ' $\sim(\text{--- } a_{\underline{1}}, \dots, a_{\underline{n}} \text{---})'$  is in  $\mathcal{K}'$ .

Rule [ $\sim$ A] ' $\sim(Aa)$ ' is in  $\mathcal{K}'$  iff there is a wff ' $b$ ' such that ' $\sim(ab)$ ' is in  $\mathcal{K}'$ .

Rule [ $\sim$ E] ' $\sim(Ea)$ ' is in  $\mathcal{K}'$  iff ' $\sim(ab)$ ' is in  $\mathcal{K}'$  for all wffs ' $b$ '.

Rule [ $\sim$ ] ' $\sim(\sim a)$ ' is in  $\mathcal{K}'$  iff ' $a$ ' is in  $\mathcal{K}'$ .

Rule [ $\sim$ ] There is no wff ' $a$ ' such that ' $a$ ' and ' $\sim a$ ' are both in  $\mathcal{K}'$ .

3.6 Theorem. (Fitch [10].) There is a wff ' $*$ ' such that the following rules are valid.

Rule [ $*$ ] ' $*xab$ ' is in  $\mathcal{K}'$  iff for some  $\underline{n} > 1$ , there is a finite sequence

$'c_{\underline{1}}', \dots, 'c_{\underline{n}}'$  such that ' $c_{\underline{1}}$ ' is ' $a$ ', ' $c_{\underline{n}}$ ' is ' $b$ ', and ' $xc_{\underline{1}}c_{\underline{2}}', \dots, 'xc_{\underline{n-1}}c_{\underline{n}}'$  are all in  $\mathcal{K}'$ .

Rule [ $\sim*$ ] ' $\sim(*xab)$ ' is in  $\mathcal{K}'$  iff for every finite  $\underline{n} > 1$ , every sequence

$'c_{\underline{1}}', \dots, 'c_{\underline{n}}'$  such that ' $c_{\underline{1}}$ ' is ' $a$ ' and ' $c_{\underline{n}}$ ' is ' $b$ ' is such that at least one of ' $\sim(xc_{\underline{1}}c_{\underline{2}})', \dots, \sim(xc_{\underline{n-1}}c_{\underline{n}})$ ' is in  $\mathcal{K}'$ .

3.7 The following properties of  $\mathcal{K}'$  will be used in section 4.

3.8 Definition. A wff ' $a$ ' is said to represent in  $\mathcal{K}'$  an  $\underline{n}$ -ary relation  $R$  among wffs just in the case that ' $ax_{\underline{1}} \dots x_{\underline{n}}'$  is in  $\mathcal{K}'$  if  $R$  relates ' $x_{\underline{1}}', \dots, 'x_{\underline{n}}'$  in that order ( $\underline{n} \geq 1$ ). A wff ' $a$ ' is said to completely represent in  $\mathcal{K}'$  an  $\underline{n}$ -ary relation  $R$  among wffs just in the case that ' $a$ ' represents  $R$  and ' $\sim(ax_{\underline{1}} \dots x_{\underline{n}})'$  is in  $\mathcal{K}'$  if  $R$  does not relate ' $x_{\underline{1}}', \dots, 'x_{\underline{n}}'$  in that order ( $\underline{n} \geq 1$ ).

3.9 Theorem. (Fitch [7].) Every recursively enumerable relation among wffs is completely represented in  $\mathcal{K}'$ . In particular, the class  $\mathcal{K}$  is completely represented in  $\mathcal{K}'$ .

3.10 Theorem. (Myhill [17].) If the set of Gödel numbers of a class  $\mathcal{R}$  of wffs is hyperarithmetical, then  $\mathcal{R}$  is completely represented in  $\mathcal{K}'$ .

3.11 ' $(\exists \underline{x})(\dots \underline{x} \dots)$ ' is an abbreviation for ' $E\lambda_{\underline{1}\underline{x}}(\dots \underline{x} \dots)$ ' and ' $(\exists \underline{x}_{\underline{1}} \dots \underline{x}_{\underline{n}})(\dots \underline{x}_{\underline{1}}, \dots, \underline{x}_{\underline{n}} \dots)$ ' is an abbreviation for ' $(\exists \underline{x}_{\underline{1}}) \dots (\exists \underline{x}_{\underline{n}})(\dots \underline{x}_{\underline{1}}, \dots, \underline{x}_{\underline{n}} \dots)$ '. This definition and rules [E] and [ $\sim$ E] give rules for existential quantifiers in  $\mathcal{K}'$ . Similarly, ' $(\underline{x})(\dots \underline{x} \dots)$ ' is an abbreviation for ' $A\lambda_{\underline{1}\underline{x}}(\dots \underline{x} \dots)$ ' and ' $(\underline{x}_{\underline{1}}, \dots, \underline{x}_{\underline{n}})(\dots \underline{x}_{\underline{1}}, \dots, \underline{x}_{\underline{n}} \dots)$ ' is an abbreviation for ' $(\underline{x}_{\underline{1}}) \dots (\underline{x}_{\underline{n}})(\dots \underline{x}_{\underline{1}}, \dots, \underline{x}_{\underline{n}} \dots)$ '. This definition and rules [A] and [ $\sim$ A] give rules for universal quantifiers in  $\mathcal{K}'$ .

#### 4. The Identity Relation of System R

4.1 The system  $\underline{R}$  of combinatory logic was defined in [19,20]. The relation of identity of the system  $\underline{R}$  may be thought of as a relation which relates two  $\underline{R}$ -formulas (wffs of the system  $\underline{R}$ ) just in the case that they are names for the same object. In 3.2 of [20], this relation is defined by giving an inductive definition of a set  $\Delta$  of identities; definition 4.3, below is equivalent.

4.2 In this essay, ' $\varepsilon$ ' is used as a metalinguistic name for a relation among members of the class  $\underline{U}$  of wffs of  $\underline{K}$  and  $\underline{K}'$ . This relation is the same as the relation of identity of the system  $\underline{R}$ . In order to avoid confusion, the relation named by ' $\varepsilon$ ' will be called equality.

4.3 Definition. Let ' $K$ ' serve as an abbreviation for ' $\lambda_2 xy(x)$ ' and let ' $S$ ' serve as an abbreviation for ' $\lambda_3 xyz(xz(yz))$ '. The relation  $\varepsilon$  of equality among  $\underline{U}$ -expressions is defined inductively as follows:

- (1) 'a'  $\varepsilon$  'a'
- (2) 'Kab'  $\varepsilon$  'a'
- (3) 'Sabc'  $\varepsilon$  'ac(bc)'
- (4) If 'b' is the result of substituting 'c' for one or more occurrences of 'd' in 'a' and if 'c'  $\varepsilon$  'd', then 'a'  $\varepsilon$  'b'.
- (5) The only wffs which are equal are those which are equal by virtue of (1) to (4).

4.4 It will now be shown that there is a wff ' $q$ ' which represents  $\varepsilon$  in  $\underline{K}$ . In order to show this, it is necessary to show that two relations among wffs are completely represented in  $\underline{K}$ .

4.5 Lemma. There is a wff ' $I$ ' such that 'Iab' is in  $\underline{K}$  (is in  $\underline{K}'$ ) if 'a' occurs in 'b' and otherwise ' $\sim(Iab)$ ' is in  $\underline{K}$  (is in  $\underline{K}'$ ).

Proof

Let 'I' serve as an abbreviation for a wff such that

$$\underline{Iab} \leftrightarrow [[\underline{a} = \underline{b}] \vee (\exists xy)[[\underline{b} = \underline{xy}] \& [\underline{Iax} \vee \underline{Iay}]]]$$

It is straightforward to verify that 'Iab' is in  $\mathcal{K}$  (in  $\mathcal{K}'$ ) if 'a' occurs in 'b'.

The combination measure of a wff 'a' is defined to be the number of times definition 1.8(2) must be used in order to show that 'a' is a wff.

Using 2.12 and induction on the combination measure of 'b', it is straightforward to verify that for all wffs 'a' and 'b' either 'Iab' or ' $\sim(\underline{Iab})$ ' is in  $\mathcal{K}$ .

Using rules [E] and [ $\sim$ E] for  $\mathcal{K}'$  and induction on the combination measure of 'b', it is straightforward to verify that for all wffs 'a' and 'b' either 'Iab' or ' $\sim(\underline{Iab})$ ' is in  $\mathcal{K}'$ .

4.6 Lemma. There is a wff 'T' such that 'Tabcd' is in  $\mathcal{K}$  (is in  $\mathcal{K}'$ ) if 'a' is the result of substituting 'b' for an occurrence of 'c' in 'd' and otherwise ' $\sim(\underline{Tabcd})$ ' is in  $\mathcal{K}$  (is in  $\mathcal{K}'$ ).

Proof

Let 'T' serve as an abbreviation for a wff such that

$$\begin{aligned} \underline{Tabcd} \leftrightarrow & [[[\underline{b} = \underline{c}] \vee \sim(\underline{Icd})] \& [\underline{a} = \underline{d}]] \\ & \vee [[\underline{c} = \underline{d}] \& [\underline{a} = \underline{b}]] \\ & \vee (\exists xy)[[\underline{d} = \underline{xy}] \& (\exists \underline{a_1} \underline{a_2})[[\underline{a} = \underline{a_1 a_2}] \\ & \& \underline{Ta_1 bcx} \& \underline{Ta_2 bcy}]]] . \end{aligned}$$

It is straightforward to verify that 'Tabcd' is in  $\mathcal{K}$  (is in  $\mathcal{K}'$ ) just in the case that 'a' is the result of substituting 'b' for an occurrence of 'c' in 'd'.

Using 2.12 and double induction on the combination measure (lemma 4.5) of 'a' and 'd', it is straightforward to verify that for all wffs 'a', 'b', 'c', 'd', either 'Tabcd' or ' $\sim(\underline{Tabcd})$ ' is in  $\mathcal{K}$ .

Using rules [E] and [ $\sim$ E] for  $\mathcal{K}'$  and double induction on the combination measure (lemma 4.5) of 'a' and 'd', it is straightforward to verify that for all

wffs 'a', 'b', 'c', 'd', either 'Tabcd' or '~(Tabcd)' is in  $\mathbb{K}$ '.

4.7 Theorem. There is a wff 'q' which represents in  $\mathbb{K}$  the relation  $\equiv$  of equality among wffs.

Proof

If we let 'q' serve as an abbreviation for a wff such that

$$\begin{aligned} q_{ab} \leftrightarrow & [[a = b] \vee (\exists c)[a = Kbc] \\ & \vee (\exists cdx)[[a = Scdx \ \& \ b = cx(dx)]] \\ & \vee (\exists cd)[Tbcda \ \& \ qcd]] \end{aligned}$$

then it is straightforward, using definitions 2.9 and 4.3, to verify that 'q' represents  $\equiv$  in  $\mathbb{K}$ .

4.8 Theorem. There is a wff 'Q' which completely represents in  $\mathbb{K}$ ' the relation  $\equiv$  of equality among wffs.

Proof

By theorem 3.9, there is a wff 'k' which completely represents in  $\mathbb{K}$ ' the class  $\mathbb{K}$  of wffs. By theorem 4.7, there is a wff 'q' which represents  $\equiv$  in  $\mathbb{K}$ . Therefore, if we define 'Q' as an abbreviation for a wff such that

$$Q_{ab} \leftrightarrow k(q_{ab}) \quad ,$$

then 'Q' completely represents in  $\mathbb{K}$ ' the relation  $\equiv$  of equality among wffs. Hereafter, let ' $\equiv$ ' serve as an abbreviation for 'Q'.

4.9 By theorems 2.10 and 4.7, the relation  $\equiv$  is recursively enumerable. It can be shown that  $\equiv$  is not effectively decidable. Assume, for reductio, that it is decidable. It can be shown that this gives a decision procedure for the identity

$$\underline{f(i)} = \underline{j}$$

where f is any partial recursive function from natural numbers to natural numbers and i and j are any natural numbers. This is known to be undecidable. Therefore,  $\equiv$  is not decidable.

4.10 Since the relation of equality of the system  $\mathbb{R}$  is completely represented in  $\mathbb{K}'$ , the theory of this relation in the system  $\mathbb{R}$  is available in  $\mathbb{K}'$ . In particular, the theorems concerning the representation of natural numbers and functions of natural numbers in the system  $\mathbb{R}$  is available in  $\mathbb{K}'$ . This will greatly simplify the discussion in sections 6 and 7 below. The remainder of this section contains a summary of some of these properties. Proofs of these statements for the system  $\mathbb{R}$ , which apply in  $\mathbb{K}'$ , are given in [19,20].

4.11 Define a countable set of relations of identity among  $\mathbb{U}$ -expressions as follows:

$$\underline{a} =_1 \underline{b} \leftrightarrow (\underline{x}) [\underline{ax} = \underline{bx}]$$

$$\underline{a} =_2 \underline{b} \leftrightarrow (\underline{x}, \underline{y}) [\underline{axy} = \underline{bxy}]$$

and so forth. Obviously, each of these relations is completely represented in  $\mathbb{K}'$ . A  $\mathbb{U}$ -expression ' $\underline{a}$ ' is said to represent the natural number  $\underline{n}$  if

$$\underline{axy} = \underbrace{x(x(x \dots (xy) \dots))}_{\underline{n} \text{ times}}$$

is in  $\mathbb{K}'$ . Each natural number is represented by an  $\mathbb{R}$ -numeral; let '0', '1', '2', ... serve as abbreviations for these  $\mathbb{U}$ -expressions. There is a  $\mathbb{U}$ -expression, 'N', which completely represents the set of  $\mathbb{U}$ -expressions which represent natural numbers.

A  $\mathbb{U}$ -expression, ' $\underline{f}$ ', is said to represent an  $\underline{n}$ -ary function  $\underline{f}$  in  $\mathbb{K}'$  iff

$$\underline{fa_1 \dots a_n} =_2 \underline{a}$$

is in  $\mathbb{K}'$  iff  $f(\underline{x_1}, \dots, \underline{x_n}) = \underline{x}$  where ' $\underline{a_1}$ ', ' $\underline{a_2}$ ', ..., ' $\underline{a_n}$ ' and ' $\underline{a}$ ', respectively represent  $x_1, x_2, \dots, x_n$  and  $x$ . Each partial recursive function is represented by a  $\mathbb{U}$ -expression. A  $\mathbb{U}$ -expression, ' $\underline{a}$ ' is said to completely represent a relation  $A$  among natural numbers iff ' $\underline{axy}$ ' is in  $\mathbb{K}'$  if  $A(x,y)$  and ' $\sim(\underline{axy})$ ' is in  $\mathbb{K}'$  if not  $A(x,y)$  where ' $\underline{x}$ ' and ' $\underline{y}$ ', respectively, represent  $x$  and  $y$ . Each partial recursive relation among natural numbers has a complete representation in  $\mathbb{K}'$ .

4.12 The abstraction rules of  $\underline{R}$  are available in  $\underline{K}'$ . For all  $n \geq 1$ , there is an effective procedure to define ' $\underline{\Lambda}_{\underline{n-1}} \underline{x}_1 \dots \underline{x}_n (\dots \underline{x}_1, \dots, \underline{x}_n \dots)$ ' so that

$$\underline{\Lambda}_{\underline{n-1}} \underline{x}_1 \dots \underline{x}_n (\dots \underline{x}_1, \dots, \underline{x}_n \dots) \underline{a}_1 \dots \underline{a}_n = (\dots \underline{a}_1, \dots, \underline{a}_n \dots)$$

is in  $\underline{K}'$  and the rule of equality elimination applies to  $\underline{U}$ -expressions and sub-expressions of the form ' $\underline{w}ab$ '. This means that the obvious analog of Church's  $\lambda$ -K-calculus [3] is available in  $\underline{K}'$ .

## 5. Ordinal Numbers

5.1 This section contains an informal description of some of the ordinals which will be shown to have representations in  $\mathcal{K}'$ . This informal discussion is followed by a brief summary of two different ways of providing notations for these ordinals. Sections 6 and 7, below, contain outlines of proofs of the existence of  $\mathcal{U}$ -expressions which represent these ordinals as well as additional ordinals.

5.2 The ordinals in the first number class are the finite ordinals: 0, 1, 2, 3, . . . . The limit of this sequence of ordinals, called  $\omega$  is the first ordinal in the second number class. An ordinal which is the limit of a sequence of ordinals is said to be a limit ordinal;  $\omega$  is the first limit ordinal. If we are speaking in terms of von Neumann ordinals,  $\omega$  is the set of all finite ordinals. Fitch has shown that each natural number has a representation in  $\mathcal{K}'$ . Further, there is a  $\mathcal{U}$ -expression, 'N', which completely represents the set of  $\mathcal{U}$ -numerals. Consequently,  $\mathcal{K}'$  contains the ordinal  $\omega$  (in a way which will be made precise below).

5.3 Once the first ordinal of the second number class is available, we can continue using the successor function to obtain additional ordinals:  $\omega, \omega+1, \omega+2, \omega+3, \dots$ . The limit of this sequence is  $\omega \times 2$ . Beginning with the limit ordinal  $\omega \times 2$ , we can again apply the successor function to obtain the sequence:  $\omega \times 2, (\omega \times 2)+1, (\omega \times 2)+2, (\omega \times 2)+3, \dots$ . The limit of this sequence is  $\omega \times 3$ . Proceeding in this way, it is possible to define the ordinal  $\omega \times \alpha$  for any finite ordinal  $\alpha$ . Thus, the sequence  $\omega, \omega \times 2, \omega \times 3, \dots$  can be defined. The limit of this sequence is  $\omega \times \omega$  or  $\omega^2$ .

5.4 The successor function can again be applied to  $\omega^2$ . The limit of the following sequence is  $\omega^3$ :



$\dots, \omega^2, \omega^{2+1}, \omega^{2+2}, \omega^{2+3}, \dots, \omega^{2+\omega}, (\omega^{2+\omega})+1, (\omega^{2+\omega})+2, (\omega^{2+\omega})+3,$   
 $\dots, \omega^{2+(\omega \times 2)}, (\omega^{2+(\omega \times 2)})+1, (\omega^{2+(\omega \times 2)})+2, (\omega^{2 \times (\omega \times 2)})+3, \dots,$   
 $\omega^{2+(\omega \times 3)}, (\omega^{2+(\omega \times 3)})+1, (\omega^{2+(\omega \times 3)})+2, (\omega^{2+(\omega \times 3)})+3, \dots$

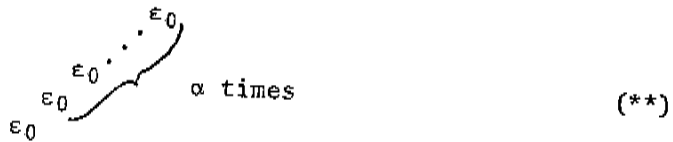
In this way, it is possible to define  $\omega^\alpha$  for any finite ordinal  $\alpha$ .  
 The limit of the sequence  $\omega, \omega^2, \omega^3, \dots$  is the ordinal  $\omega^\omega$ . It is  
 possible to continue as before to define  $(\omega^\omega)^\alpha$  for any finite ordinal  $\alpha$   
 so that it is possible to define the sequence  $\omega^\omega, (\omega^\omega)^2, (\omega^\omega)^3, \dots$   
 The limit of this sequence is  $(\omega^\omega)^\omega$ . Continuing further, it is possible  
 to define  $\omega^{(\omega^\omega)}$  or  $\omega^{\omega^\omega}$ . This kind of limiting process can be continued  
 to define

$$\underbrace{\omega^{\omega^{\omega^{\dots}}}}_{\omega \text{ times}} \quad (*)$$

where  $\alpha$  is any finite ordinal.

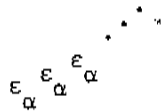
5.5 Note that each of the limits described in 5.2 to 5.4 is a  
 limit of a sequence of order type  $\omega$  (that is, a sequence which can be  
 placed in 1-1 correspondence with the finite ordinals or non-negative  
 integers). The limit of the sequence obtained by letting  $\alpha$  be each of  
 the finite ordinals in (\*) of 5.4 is called the first epsilon number,  
 $\epsilon_0$ . It is also the first fixed point of the monotonically increasing  
 continuous function  $f$  defined by the identity  $f(\alpha) = \omega^\alpha$ .

5.6 At  $\epsilon_0$ , it is, of course, possible to continue using the successor  
 function and limits of sequences of order type  $\omega$  and define ordinals such  
 as  $\epsilon_0+1, \epsilon_0+\omega, \epsilon_0+\omega^2, \epsilon_0+\omega^\omega, \epsilon_0+\epsilon_0, \epsilon_0^{\epsilon_0}, \epsilon_0^{(\epsilon_0^{\epsilon_0})}$ , or  $\epsilon_0^{\epsilon_0^{\epsilon_0}}$ , and so  
 forth. Thus, by means of the successor function and limits of sequences  
 of order type  $\omega$ , it is possible to define



where  $\alpha$  is an ordinal less than  $\epsilon_0$ . The limit of the sequence obtained by letting  $\alpha$  be any ordinal less than  $\epsilon_0$  in (\*\*) is the second epsilon number,  $\epsilon_1$ . It is also the second fixed point of the function  $f$  defined by  $\underline{f}(\alpha) = \omega^\alpha$ .

5.7 For any ordinal,  $\alpha$ ,  $\epsilon_\alpha$  is defined in the same way. That is,  $\epsilon_\alpha$  is the  $(\alpha+1)$ st fixed point of the function  $\underline{f}$  defined by  $\underline{f}(\alpha) = \omega^\alpha$ .  $\epsilon_{\alpha+1}$  is also the limit of the sequence



Note that if  $\alpha$  is a constructible ordinal (that is, an ordinal which is defined using the successor function and limits of sequences of order type  $\omega$ ), then  $\epsilon_\alpha$  is a constructible ordinal.

5.8 It is possible to continue beyond the epsilon numbers defined above and to define the critical epsilon numbers. The critical epsilon numbers are the fixed points of the monotonically increasing function  $\underline{f}$  defined by the identity  $\underline{f}(\alpha) = \epsilon_\alpha$ . The first critical epsilon number,  $\chi_0$ , is the first fixed point of  $\underline{f}$ . It is also the limit of the sequence  $\epsilon_0, \epsilon_{\epsilon_0}, \epsilon_{\epsilon_{\epsilon_0}}, \dots$

It is not possible to constructively define all ordinals in the second number class. The first ordinal of the third number class,  $\omega_1$ , is the cardinal of the continuum. If an ordinal is constructible, then

there is a notation which describes the construction of the ordinal. This set of notations is enumerable. Consequently, there is a least ordinal  $\xi$  of the second number class which is not constructible.

5.9 Two systems of notations for these ordinals will be considered in sections 6 and 7. Using each system of notation, it will be shown that there are  $\underline{U}$ -expressions which represent the ordinals as well as  $\underline{U}$ -expressions which completely represent the well-ordering relation.

5.10 Church and Kleene [1,2] introduced the concept of a formally definable ordinal. They define  $\lambda$ -expression which represent the finite ordinals. This definition is quite similar to Church's definition of  $\lambda$ -numerals. In addition, they define a  $\lambda$ -expression which represents a limit ordinal in terms of the order type of the sequence and the function used to define the sequence. They are able to provide  $\lambda$ -expressions which represent the ordinals described above as well as additional ordinals. In addition, they provide a constructive proof that their set of ordinals is simply ordered. It is possible to closely parallel their proofs in  $\underline{K}'$ .

5.11 Schütte develops a theory of ordinal numbers by defining another well-ordering on the natural numbers. His order relation is primitive recursive. Schütte has representations for the ordinals described above as well as other ordinals. This well-ordering relation is completely represented in  $\underline{K}'$  and it is possible to derive a rule of induction for these ordinals in  $\underline{K}'$ . These results, which are described in section 7, below, can be combined with Fitch's methods for representing calculi [6,11] to argue that  $\underline{K}'$  provides a demonstrably consistent metalanguage for the proof theory described by Schütte.

6. Church-Kleene Ordinals in K'

6.1 Church and Kleene [1,2] use the following abbreviations to define their ordinals:

'0' for 'Aa(a1)'

'S<sub>0</sub>' for 'A2ab(b2a)'

'L' for 'A3abc(c3ab)'

'1', '2', '3', ..., respectively, are abbreviations for 'S<sub>0</sub>0', 'S<sub>0</sub>1', 'S<sub>0</sub>2', ... When necessary to avoid confusion, the subscript 'o' is used to refer to ordinals and functions of ordinals. Here is how these U-expressions represent finite ordinals:

$$0b = b1$$

$$1b = b20$$

$$2b = b21$$

$$3b = b22$$

Observe that the representing formula contains a coding of just how the ordinal was constructed using the successor function. The following definition of a U-expression representing an ordinal is a modification of the Church-Kleene definition.

6.2 Definition. A U-expression is said to represent an ordinal just in the case that it can be shown to represent an ordinal by virtue of one of the following rules:

- (1) If 'a' represents the ordinal  $\alpha$  and

$$\underline{a} = \underline{1} \underline{b}$$

then 'b' also represents  $\alpha$ .

- (2) '0' represents the ordinal zero.

- (3) If 'a' represents the ordinal  $\alpha$ , then ' $S_{\underline{0}a}$ ' represents the successor of  $\alpha$ .
- (4) If  $\alpha$  is the limit of an increasing sequence of ordinals,  $\alpha_0, \alpha_1, \alpha_2, \dots$  of order type  $\omega$  and if ' $\underline{r}$ ' is a  $\underline{U}$ -expression such that the U-expressions ' $\underline{r}0$ ', ' $\underline{r}1$ ', ' $\underline{r}2$ ', ... represent the ordinals  $\alpha_0, \alpha_1, \alpha_2, \dots$ , respectively, then ' $LO\underline{r}$ ' represents  $\alpha$ .

An ordinal of the first or second number class is said to be K' definable if there is a  $\underline{U}$ -expression which represents the ordinal. If a  $\underline{U}$ -expression represents an ordinal, it is said to be a  $\underline{U}$ -ordinal.

6.3 The least ordinal,  $\omega$ , of the second number class is the limit of the sequence 0, 1, 2, ... By definition 6.2(4), ' $LOI$ ' represents the ordinal  $\omega$ . Let ' $\omega$ ' serve as an abbreviation for ' $LOI$ '. The following illustrates how ' $\omega$ ' represents the set of finite ordinals:

$$\begin{aligned}\omega 0 &= LOI0 \\ &= 310I \\ \omega 1 &= LOI1 \\ &= 3200I \\ \omega 2 &= LOI2 \\ &= 3210I\end{aligned}$$

Notice that the concept of a  $\underline{U}$ -expression representing an ordinal differs from the concept of a  $\underline{U}$ -expression representing a set of  $\underline{U}$ -expressions. Also, observe that ' $\omega$ ' is defined in such a way that the  $\underline{U}$ -expression ' $\omega\underline{a}$ ', where ' $\underline{a}$ ' represents a finite ordinal  $\alpha$ , contains a description of how  $\alpha$  was built up from 0 using the successor function and the identity function and of how  $\omega$  is the limit of a sequence of ordinals, namely, 0, 1, 2, 3, ...

6.4 Definition. (Church-Kleene, [1].) A sequence of ordinals of order type  $\omega$  is  $\mathcal{K}'$  defined as a function of ordinals by 'r' if, for every  $\mathcal{U}$ -ordinal 'a' which represents a finite ordinal  $\alpha$ , 'ra' represents the  $(1+\alpha)$ th ordinal of the sequence.

6.5 If the definition of normal form for the  $\lambda$ -calculus is modified in the obvious way to deal with the abstraction of  $\mathcal{R}$  which is available in  $\mathcal{K}'$ , the Church-Kleene constructive proofs of the following theorems apply to  $\mathcal{U}$ -ordinals.

6.6 Theorem. (Church-Kleene, [1].) Every  $\mathcal{U}$ -ordinal has a normal form.

6.7 Theorem. (Church-Kleene, [1].) If a  $\mathcal{U}$ -ordinal, 'a', represents an ordinal  $\alpha$ , then 'a' cannot represent an ordinal distinct from  $\alpha$ .

6.8 In order to consider the question of which ordinals are represented by  $\mathcal{U}$ -ordinals, it is necessary to define the concept of a  $\mathcal{U}$ -expression representing a function of ordinals. The following is a minor modification of the Church-Kleene definition.

6.9 Definition. A function is said to be a function in the first and second number class if the domain and range of the function are ordinals in the first and second number class. A  $\mathcal{U}$ -expression, 'f', is said to represent the n-ary function  $f$  in the first and second number class if 'fa<sub>1</sub> ... a<sub>n</sub>' is a  $\mathcal{U}$ -ordinal and if 'fa<sub>1</sub> ... a<sub>n</sub> = a' is in  $\mathcal{K}'$  just in the case that  $f(\alpha_1, \dots, \alpha_n) = \alpha$  where 'a<sub>1</sub>', ..., 'a<sub>n</sub>', and 'a', respectively, represent the ordinals  $\alpha_1, \dots, \alpha_n$  and  $\alpha$ .

6.10 The next two paragraphs contain an outline of the essential details of the proof of the statement that all ordinals which are definable as limits of sequences of order type  $\omega$  are represented by  $\mathcal{U}$ -ordinals. In particular, the functions (ordinal) addition, multiplication, and

exponentiation are represented in  $\mathbb{K}$ '. In this discussion, it will be claimed that there are  $\mathbb{U}$ -expressions with certain properties. The existence of these  $\mathbb{U}$ -expressions is a direct consequence of:

6.11 Theorem. (Church and Kleene, [1].) If 'a', 'b' and 'c' are  $\mathbb{U}$ -expressions built up out of 'S' and 'K', it is possible to find eight  $\mathbb{U}$ -expressions, 'f<sub>ijk</sub>' (where the subscripts i, j, k take the values 1 and 2) with the following properties:

$$\begin{array}{ll} f_{ijk}^0 = \underline{n} \underline{a} & f_{2jk}^0 = \underline{n} \underline{a} f_{2jk} \\ f_{i1k}(S_o a) = \underline{n} \underline{ba} & f_{i2k}(S_o a) = \underline{n} \underline{c} f_{i2k} \underline{a} \\ f_{ij1}(Lar) = \underline{n} \underline{car} & f_{ij2}(Lar) = \underline{n} \underline{c} f_{ij2} \underline{ar} \end{array}$$

The subscript n in the sign '=' depends on the  $\mathbb{U}$ -expressions a, b, c only.

Proof. The proof of this theorem is a straightforward modification of the proof of theorem 3 of [1]. The proof is related to the Church-Kleene proof in the same way that the proof of theorem 7.6 of Chapter II of [19] is related to Church's proof that primitive recursive functions are  $\lambda$ -definable [20].

6.12 There is a  $\mathbb{U}$ -expression 'f' such that:

$$\begin{aligned} f0b &= \underline{1} \underline{Ib} \\ &= \underline{1} \underline{b} \\ f(S_o a)b &= \underline{1} \underline{S_o (fab)} \\ f(Lar)b &= \underline{1} \underline{La(\underline{\Lambda m}(f(\underline{rm})b))} \end{aligned}$$

It will now be shown that if 'a' and 'b', respectively, represent ordinals  $\alpha$  and  $\beta$ , then 'fab' represents the sum of  $\alpha$  and  $\beta$ . If  $\alpha$  is not a limit ordinal, then the first two properties of 'f' can be used to show:

$$\underline{fab} = \underline{1} S_0(S_0 \dots (S_0(\underline{fdb}) \dots))$$

γ times

where 'd' represents the largest limit ordinal,  $\delta$ , less than  $\alpha$  and  $\alpha = \delta + \gamma$ . At this point, the third property of 'f' is applicable. This is precisely what is required. Now suppose  $\alpha$  is a limit ordinal. As an example, consider  $\omega$ , the limit of the sequence 0, 1, 2, 3, ... In this case, 'a' is 'LOI' so the third property of 'f' applies:

$$f(\text{LOI})\underline{b} = \underline{1} \text{LO}(\text{Am}(f\text{Im}))\underline{b}$$

That is, by definition 6.2 (4), the result is the limit of the sequence  $\beta, \beta+1, \beta+2, \dots$ . A similar argument applies if 'a' represents some other limit ordinal. Therefore, 'f' represents ordinal addition. Let '[b +<sub>o</sub> a]' serve as an abbreviation for 'fab'.

6.13 By a similar modification of the primitive recursive definitions, it is possible to exhibit U-expressions which represent multiplication, exponentiation, and the constant function. Let 'x<sub>o</sub>', 'exp<sub>o</sub>', and 'K<sub>o</sub>' serve as abbreviations for these U-expressions. Further, let 'b<sup>a</sup>' serve as an abbreviation for 'exp<sub>o</sub> ba'. The statements in 6.12 and 6.13 are the essential details of the proof of:

6.14 Theorem. Each of the finite ordinals and each ordinal definable by addition, multiplication, exponentiation and as the limit of a sequence of order type  $\omega$  has a representation in K'.

Proof. If an ordinal  $\delta$  is the sum, product, or exponentiation of two ordinals  $\alpha$  and  $\beta$ , then by theorem 6.11 and 6.12 and 6.13, there is a U-ordinal which represents  $\delta$ . Similarly, if  $\delta$  is the limit of the sequence f( $\alpha, 0$ ), f( $\alpha, 1$ ), f( $\alpha, 2$ ), ... then, by definition 6.2(4), 'Lfa' represents  $\delta$  where 'f' represents f and 'a' represents  $\alpha$ .



6.15 By theorem 6.14, there are  $\underline{U}$ -expressions which represent the epsilon numbers described in 5.5 and 5.6. There is another more direct way of showing that these ordinals are represented by  $\underline{U}$ -ordinals. The epsilon numbers are the fixed points of the continuous, monotonically increasing function  $\underline{f}$  defined by the identity  $\underline{f}(\alpha) = \omega^\alpha$ . It will now be shown that there are  $\underline{U}$ -ordinals which represent the fixed points of  $\underline{f}$  and that these  $\underline{U}$ -expressions also represent the limits of the appropriate sequences. A number of preliminary results, which are modifications of results of Church and Kleene [1] are required. The proofs closely follow the proofs in [1] so they are omitted.

6.16 Lemma. There is a  $\underline{U}$ -expression 'P' such that:

$$\begin{aligned} P0 &= \underline{1} 0 \\ P(S_{\underline{0}} a) &= \underline{1} a \\ P(\underline{Lar}) &= \underline{1} \underline{Lar} \end{aligned}$$

'P' represents the predecessor function on ordinals. (The predecessor of 0 is 0, the predecessor of a non-limit ordinal is the ordinal which precedes it, the predecessor of a limit-ordinal is the limit ordinal).

6.17 Lemma. There is a  $\underline{U}$ -expression, 'f', such that:

$$\begin{aligned} f0 &= \underline{1} 0 \\ f(S_{\underline{0}} a) &= \underline{1} a \\ f(\underline{Lar}) &= \underline{1} K_0 1(\underline{Lar}) \\ &= \underline{1} 1 \end{aligned}$$

'f' represents a function from ordinals to ordinals defined as follows:

$$\underline{f}(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a finite ordinal} \\ 1 & \text{if } \alpha \text{ is an infinite ordinal} \end{cases}$$

6.18 Lemma. There is a  $\mathcal{U}$ -expression, 't' such that:

$$t0 = \underline{1} 0$$

$$t(S_{\underline{0}}a) = \underline{1} K_{\underline{0}} 1a$$

$$= \underline{1} 1$$

$$t(Lar) = \underline{1} K_{\underline{0}} 2 a$$

$$= \underline{1} 2$$

't' represents the function  $\underline{t}$ :

$$\underline{t}(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is the ordinal } 0 \\ 1 & \text{if } \alpha \text{ is a successor ordinal} \\ 2 & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

6.19 Lemma. The  $\mathcal{U}$ -expression ' $\underline{\Lambda x}(P(xS_{\underline{0}}0))$ ' represents (using the appropriate definition) a function from natural numbers to finite ordinals such that its value for argument  $\underline{n}$  is the  $\underline{n}$ th finite ordinal. There is a  $\mathcal{U}$ -expression, 'T' such that

$$T0 = \underline{2} 1$$

$$T(S_{\underline{0}}a) = \underline{2} +1(Ta)$$

$$T(Lar) = \underline{2} K1(Lar)$$

'T' represents a function,  $\underline{T}$ , from ordinals to natural numbers with the following properties. If  $\alpha$  is a finite ordinal, then  $\underline{T}(\alpha)$  is the  $\alpha$ -th natural number. If  $\alpha$  is an infinite ordinal, then  $\underline{T}(\alpha)$  is the integer  $\underline{n}$  such that if  $\beta$  is the largest limit ordinal less than or equal to  $\alpha$ , then  $\alpha = \underline{S}_{\underline{0}}^{\underline{n}}(\beta)$ .

6.20 Theorem. There is a  $\mathcal{U}$ -expression 'e' such that if  $\underline{a}$  is a  $\mathcal{U}$ -ordinal which represents  $\alpha$ , then ' $\underline{ea}$ ' is a  $\mathcal{U}$ -ordinal which represents



6.21 In a completely analogous way, it is possible to show that there are  $\underline{U}$ -ordinals which represent the critical epsilon numbers. In particular, in the definition of 'e' in the proof of theorem 6.20, replace occurrences of ' $\underline{\text{exp}}_0 \omega$ ' with the  $\underline{U}$ -expression which represents the function  $\underline{f}$  defined by the identity  $\underline{f}(\alpha) = \varepsilon_\alpha$ . By theorem 6.20, there is such a  $\underline{U}$ -expression.

6.22 Theorem. Each critical epsilon number is represented by a  $\underline{U}$ -ordinal.

6.23 Theorem. There is a  $\underline{U}$ -expression 'Ord' which completely represents the set of  $\underline{U}$ -ordinals.

Proof. Definition 6.2 (1), (2), and (3) obviously define a recursive set of  $\underline{U}$ -expressions. This is the set of  $\underline{U}$ -expressions which represent finite ordinals and by theorem 3.9 there is a  $\underline{U}$ -expression, '0' which completely represents this set. Now, use the function  $\underline{T}$  of lemma 6.19 to express the order condition of 6.2 (4) in  $\underline{K}'$ :

$$\underline{\&Lr} \leftrightarrow (\underline{x}) [0\underline{x} = \underline{T}(\underline{rx}) < \underline{T}(\underline{r}(S_0\underline{x}))]$$

where '<' completely represents the relation less than among natural numbers. By the closure rules for excluded middle, ' $\underline{\&Lr}$ ' is definite. In fact,  $\underline{\&Lr}$  is primitive recursive. Use Fitch's procedure for defining self-referential relations [9] to define a  $\underline{U}$ -expression 'Ord' with the following property:

$$\begin{aligned} \underline{\text{Ord}} \underline{a} \leftrightarrow \underline{a} = 0_0 \vee (\exists \underline{b}) [\underline{\text{Ord}} \underline{b} \ \&\ \underline{a} = \underline{1} \underline{b}] \\ \vee (\exists \underline{b}) [\underline{\text{Ord}} \underline{b} \ \&\ \underline{a} = \underline{1} S_0 \underline{b}] \\ \vee (\exists \underline{r}) [\underline{\&Lr} \ \&\ \underline{a} = \underline{1} \underline{1} 0 \underline{r}] \end{aligned}$$

Note that each of the existentially quantified subexpressions are definite and that the remaining clause uses the identity of  $\underline{K}'$ . Con-

sequently, by induction on finite ordinals, it can be shown that for all  $\mathcal{U}$ -expressions ' $\underline{a}$ ', either ' $\text{Ord } \underline{a}$ ' or ' $\sim(\text{Ord } \underline{a})$ ' (but not both) is in  $\mathcal{K}'$ . Therefore, ' $\text{Ord}$ ' completely represents the set of  $\mathcal{U}$ -ordinals.

6.24 It has been shown that all ordinals which are definable by means of the successor function, addition, multiplication, and exponentiation as well as ordinals which are limits of sequences of order type  $\omega$  are represented by  $\mathcal{U}$ -ordinals. Further, the set of  $\mathcal{U}$ -ordinals is completely represented in  $\mathcal{K}'$ . It is possible to continue this development and show that the binary relation less than which well-orders the  $\mathcal{U}$ -ordinals is completely represented in  $\mathcal{K}'$ . A proof of this result is given for Schutte's ordinals in section 7.

## 7. Schütte Ordinals in $\mathcal{K}'$

7.1 Many of the proof theoretic results in Schütte's book, Proof Theory [12], are established by transfinite induction to constructible ordinals. Schütte develops a theory of constructible ordinals by defining a decidable order relation on the natural numbers. This well-ordering has the property that prime numbers and prime numbers with prime exponents correspond to limit ordinals. For example, 3 corresponds to  $\omega$ ,  $\underline{3^3}$  corresponds to  $\omega^\omega$ , and  $\underline{5}$  corresponds to  $\varepsilon_0$ . Schütte's ordinals as well as the set of ordinals are clearly represented by  $\mathcal{U}$ -expressions. In this section, it will be shown that Schütte's ordinals are well ordered in  $\mathcal{K}'$  and that transfinite induction on these ordinals is a derived rule of  $\mathcal{K}'$ . With this result and Fitch's results concerning the representation of calculi [5,6,7,11] in  $\mathcal{K}$  and  $\mathcal{K}'$ , it can be asserted that  $\mathcal{K}'$  provides a consistent metalanguage for Schütte's proof theory.

7.2 As was shown in section 4, the arithmetic of natural numbers developed in the system  $\mathcal{R}$  is available in  $\mathcal{K}'$  and the results of [19] will be used when required. ' $\underline{p}_0$ ', ' $\underline{p}_1$ ', ' $\underline{p}_2$ ', ... are used as names for the prime numbers  $\underline{2}$ ,  $\underline{3}$ ,  $\underline{5}$ , ... By convention 1.7, ' $\underline{p}_0$ ', ' $\underline{p}_1$ ', ' $\underline{p}_2$ ', ... are abbreviations for  $\mathcal{U}$ -expressions which represent these primes.

7.3 After some technical preliminaries, Schütte's definition of two binary relations, under ( $\subset$ ) and before ( $\prec$ ), on natural numbers is stated in 7.7. Paragraph 7.11 contains a proof that these relations are completely represented in  $\mathcal{K}'$ . After this, it is shown that  $\prec$  well-orders the natural numbers in  $\mathcal{K}'$ . Then, in 7.29, it is shown that transfinite induction for the relation  $\prec$  is a derived rule of  $\mathcal{K}'$ .

7.4 If  $\underline{i}$  is a natural number, then  $\underline{i}_j$  is the exponent of the  $j$ th prime in the (unique) prime decomposition of  $\underline{i}$ , e.g.,  $\underline{i} = p_0^{\underline{i}_0} \cdot p_1^{\underline{i}_1} \cdot \dots \cdot p_n^{\underline{i}_n}$ . It is well known that there exists a  $\underline{k}$  such that for all  $\underline{j} > \underline{k}$ ,  $\underline{i}_j = 0$ . Continuing this definition,  $\underline{i}_{jk}$  is the exponent of the  $k$ th prime in the prime decomposition of  $\underline{i}_j$ , e.g.  $\underline{i}_j = p_0^{\underline{i}_{j0}} \cdot p_1^{\underline{i}_{j1}} \cdot \dots \cdot p_m^{\underline{i}_{jm}}$ . This convention is extended to any finite number of subscripts. For all natural numbers  $\underline{i}, \underline{j}, \underline{k}, \underline{n}, \dots$

$$\underline{i} > \underline{i}_j > \underline{i}_{jk} > \underline{i}_{jkn} > \dots \geq 0$$

Numbers such as  $\underline{i}_j$ ,  $\underline{i}_{jk}$ , and  $\underline{i}_{jkn}$  are called unary, binary, and ternary indices of  $\underline{i}$ , respectively. A similar statement applies for more than three subscripts.

7.5 Lemma. There is a  $\mathcal{U}$ -expression, ' $\ell$ ', which represents the function  $\ell$  whose value for argument  $\underline{i}$  is the integer  $\underline{k}$  such that  $\underline{i}_j = 0$  for all  $\underline{j} > \underline{k}$ . There is a  $\mathcal{U}$ -expression, ' $G\ell$ ', which represents the binary function  $G\ell$  defined by the identity  $G\ell(\underline{i}, \underline{j}) = \underline{i}_j$  provided  $\underline{j} \leq \ell(\underline{i})$ . Further, for all  $\underline{n} \geq 1$ , there is a function which maps an integer onto its  $\underline{n}$ -ary indices; these functions are represented by  $\mathcal{U}$ -expressions.

Proof

Gödel [13] has given primitive recursive definitions of  $\ell$  and  $G\ell$ . By 4.11, there are  $\mathcal{U}$ -expressions which represent these functions. Let the function  $\iota$  be defined as follows:

$$\iota(\underline{i}, \underline{j}) = \begin{cases} G\ell(\underline{i}, \underline{j}) & \text{if } \underline{j} \leq \ell(\underline{i}) \\ 0 & \text{if } \underline{j} > \ell(\underline{i}) \end{cases}$$

Since  $\ell$  and  $G\ell$  are primitive recursive functions,  $\iota$  is a primitive recursive function [14]. Therefore, there is a  $\mathcal{U}$ -expression which represents  $\iota$ . For binary subscripts,  $\iota_2(\underline{i}, \underline{j}, \underline{k}) = \iota(\iota(\underline{i}, \underline{j}), \underline{k})$  is the appropriate function. A similar argument applies for  $\underline{n}$ -ary indices. Hereafter, straightforward proofs of this kind will be omitted.

7.6 Definition. Stated informally, the rank of a natural number,  $\underline{i}$ , is the largest  $\underline{n}$  such that there is a non zero  $\underline{n}$ -ary index of  $\underline{i}$ . The rank,  $\rho(\underline{i})$ , of a natural number is defined recursively as follows:

$$\rho(\underline{i}) = \begin{cases} \underline{0} & \text{if } \underline{i} = \underline{0} \\ \left[ \max_{0 \leq j \leq \ell(\underline{i})} \rho(\underline{i}_j) \right] + \underline{1} & \text{if } \underline{i} \neq \underline{0} \end{cases}$$

Since  $\rho(\underline{i}) > \rho(\underline{i}_j)$ , this is a recursive definition of  $\rho$  and, therefore, there is a  $\underline{U}$ -expression, ' $\rho$ ', which represents  $\rho$ . For example, only  $\underline{0}$  has rank  $\underline{0}$ ,  $\underline{1}$  is the only natural number with rank  $\underline{1}$ ; only primes and pairwise products of primes have rank  $\underline{2}$ .

7.7 Definition. (Schütte, [21].) The binary relations under ( $\subset$ ) and before ( $\prec$ ) are defined by simultaneous induction as follows:

- (1) If  $\underline{i} \neq \underline{0}$ , then  $\underline{0} \subset \underline{i}$  and  $\underline{0} \prec \underline{i}$ .
- (2) If  $\underline{i} \neq \underline{0}$  and  $\underline{j} \neq \underline{0}$ , then  $\underline{i} \subset \underline{j}$  if there exists a  $\underline{k}$  such that  $\underline{i}_k \prec \underline{j}_k$  and for all  $\underline{n} > \underline{k}$ ,  $\underline{i}_n = \underline{j}_n$ .
- (3) If  $\underline{i} \neq \underline{0}$  and  $\underline{j} \neq \underline{0}$ , then  $\underline{i} \prec \underline{j}$  if for all  $\underline{k} < \ell(\underline{i})$ ,  $\underline{i}_k \prec \underline{j}_k$ .
- (4) If  $\underline{i} \neq \underline{0}$  and  $\underline{j} \neq \underline{0}$ , then  $\underline{i} \prec \underline{j}$  if  $\underline{j} \subset \underline{i}$  and if for at least one  $\underline{k}$ ,  $\underline{i} \prec \underline{j}_k$  or  $\underline{i} = \underline{j}_k$ .

7.8 Some illustrations of the properties of  $\prec$  and  $\subset$  are useful at this point. By 7.7(1),  $\underline{0} \prec \underline{i}$  and  $\underline{0} \subset \underline{i}$  if  $\underline{i} \neq \underline{0}$ . By 7.7(2),  $\underline{1} \subset \underline{i}$  for all  $\underline{i} > \underline{1}$ . By 7.7(3),  $\underline{1} \prec \underline{i}$  for all  $\underline{i} > \underline{1}$ . It can be shown that the order relation  $\prec$  is such that the successor function,  $\sigma$ , is defined by the identity  $\sigma(\underline{i}) = \underline{2}^{\underline{i}}$ . The initial segment of this ordering is  $\underline{0}, \underline{1}, \underline{2}, \underline{2}^{\underline{2}}, \underline{2}^{\underline{2}^{\underline{2}}}, \dots$  (Hereafter, association right is used for exponents so that  $\underline{2}^{\underline{2}^{\underline{2}}}$  is an abbreviation for  $\underline{2}^{\underline{2}^{\underline{2}}}$ .) 7.7(2) and (3) can be used to show  $\underline{0} \prec \underline{1} \prec \underline{2} \prec \underline{2}^{\underline{2}} \prec \underline{2}^{\underline{2}^{\underline{2}}} \prec \dots$ . Furthermore, for any  $\underline{n}$ ,



$$\underbrace{2^{\underbrace{2^{\dots^2}}_n}}_n \text{ times} \tag{*}$$

is before  $\underline{3}$  by 7.7(2) and (3). By 7.7(2) and (4),  $\underline{3} < \underline{2^m}$  where  $m$  is a prime. By a similar argument,  $\underline{3} < \underline{2^m}$  if  $m$  is not of the form (\*). Consequently,  $0, \underline{1}, \underline{2}, \underline{2^2}, \underline{2^{2^2}}, \dots$  are the finite ordinals and  $\underline{3}$  is  $\omega$ .

7.9 The successor of  $\underline{3}$  is  $\underline{2^3}$  and the ordering continues  $\underline{3}, \underline{2^3}, \underline{2^{2^3}}, \dots$  using definition 7.7, it can be shown that only natural numbers of the form

$$\underbrace{2^{\underbrace{2^{\dots^2}}_n}}_n \tag{**}$$

are after  $\underline{3}$  and before  $\underline{3^2}$ . Thus this sequence corresponds to  $\omega, \omega+1, \omega+2, \dots$  and  $\underline{3^2}$  corresponds to  $\omega + \omega$  or  $\omega \times 2$ . Continuing with the successor function, we get natural numbers of the form

$$\underbrace{2^{\underbrace{2^{\dots^2}}_n}}_n$$

Each of these numbers has the property that it is after  $\underline{3^2}$  and before  $\underline{3^{2^2}}$ . In addition, there are no other integers after  $\underline{3^2}$  and before  $\underline{3^{2^2}}$ . Therefore,  $\underline{3^{2^2}}$  corresponds to  $\omega \times 3$ . It is easy to see that  $\underline{3}, \underline{3^2}, \underline{3^{2^2}}, \underline{3^{2^{2^2}}}, \dots$  corresponds to the ordinals  $\omega, \omega \times 2, \omega \times 3, \omega \times 4, \dots$ , respectively. Continuing with a similar argument, it can be shown that  $\underline{3^3}$  corresponds to  $\omega^\omega$ . Further,  $\underline{5}$  is after

$$\underbrace{3^{\underbrace{3^{\dots^3}}_n}}_n$$

so  $\underline{5}$  corresponds to  $\epsilon_0$ .

7.10 The relation  $<$  has the property that  $\underline{2^i} \cdot \underline{5}$  where  $i < \underline{2^i} \cdot \underline{5}$  is the  $i$ th epsilon number.  $\underline{3} \cdot \underline{5}$  corresponds to  $\chi_0$ , the first critical epsilon number and  $\underline{2^i} \cdot \underline{3} \cdot \underline{5}$ , where  $i < \underline{2^i} \cdot \underline{3} \cdot \underline{5}$ , corresponds to  $\chi_i$ , the  $i$ th critical epsilon number.

7.11 Theorem. There are  $\mathbb{U}$ -expressions ' $\subset$ ' and ' $\prec$ ' which completely represent the relations  $\subset$  and  $\prec$  in  $\mathbb{K}'$ .

Proof

Using complete induction on the sum of the ranks of  $\underline{i}$  and  $\underline{j}$ , Schütte [21], (theorem 11.1) proves that  $\prec$  and  $\subset$  are effectively decidable. Therefore, there is a (total) recursive function  $f_{\subset}$  such that

$$f_{\subset}(\underline{i}, \underline{j}) = \begin{cases} 0 & \text{if } \underline{i} \prec \underline{j} \\ 1 & \text{otherwise} \end{cases}$$

By 4.11, there is a  $\mathbb{U}$ -expression ' $f_{\subset}$ ' which represents  $f_{\subset}$ . Let ' $\subset$ ' serve as an abbreviation for ' $\lambda_{\underline{x}\underline{y}}[f_{\subset}\underline{i}\underline{y} = \underline{2} 0]$ '. Then ' $\subset$ ' completely represents the relation under. A similar argument applies for ' $\prec$ '.

7.12 In 7.13 to 7.28 it is shown that induction on the natural numbers ordered by the relations  $\prec$  is available in  $\mathbb{K}'$ . Induction for the relation  $\subset$  is established in the metalanguage only because it is needed to establish induction for  $\prec$  in  $\mathbb{K}'$ . These proofs are completed by induction on natural numbers (with their usual order.)

7.13 Lemma. For all natural numbers,  $\underline{i}$ , ' $\sim[i \subset i]$ ' and ' $\sim[i \prec i]$ ' are in  $\mathbb{K}'$  (antireflexive).

Proof

The proof is by induction on the rank of  $\underline{i}$ . If  $\rho(\underline{i}) = \underline{0}$  then, by definition 7.6,  $\underline{i} = \underline{0}$ . By definition 7.7, it is not the case that  $\underline{i} \subset \underline{i}$  or  $\underline{i} \prec \underline{i}$ . Therefore, by theorem 7.11, ' $\sim[i \subset i]$ ' and ' $\sim[i \prec i]$ ' are in  $\mathbb{K}'$ .

The inductive step is to set  $\rho(\underline{i}) = \underline{n} > 0$ . Thus,  $\underline{i} \neq \underline{0}$ . The inductive hypothesis is: for all  $\underline{j}$  such that  $\rho(\underline{j}) < \underline{n}$ , ' $\sim[j \subset j]$ ' and ' $\sim[j \prec j]$ ' are in  $\mathbb{K}'$ . Only 7.7(2) can be used to show that  $\underline{i} \subset \underline{i}$ . Therefore, there must be a  $\underline{k}$

such that ' $\sim[i_k < i_k]$ ' is in  $\mathcal{K}'$  and  $\rho(\underline{i}_k) < \rho(\underline{i})$ . This contradicts the inductive hypothesis so, by theorem 7.11, ' $\sim[i < i]$ ' is in  $\mathcal{K}'$ . Only 7.7(3) and (4) can be used to show that  $\underline{i} < \underline{i}$ . Both of these clauses require that ' $i < i$ ' is in  $\mathcal{K}'$ . This has already been shown to be false. Therefore, by theorem 7.11, ' $\sim[i < i]$ ' is in  $\mathcal{K}'$ . This argument can be carried out entirely inside  $\mathcal{K}'$ .

7.14 Lemma. If ' $\sim[i =_2 j]$ ' is in  $\mathcal{K}'$ , then either ' $i < j$ ' or ' $j < i$ ' and either ' $i < j$ ' or ' $j < i$ ' are in  $\mathcal{K}'$  (totality).

Proof

The proof is by induction on the sum of the ranks of  $\underline{i}$  and  $\underline{j}$ . If  $\rho(\underline{i}) + \rho(\underline{j}) = 0$ , then  $\underline{i} = \underline{j}$  so ' $i =_2 j$ ' is in  $\mathcal{K}'$ .

The inductive step is to let  $\rho(\underline{i}) + \rho(\underline{j}) = \underline{n} > 0$  and to assume the theorem for all  $\underline{i}$  and  $\underline{j}$  such that  $\rho(\underline{i}) + \rho(\underline{j}) < \underline{n}$ .

If ' $\sim[i =_2 j]$ ' and ' $i =_2 0$ ' are in  $\mathcal{K}'$ , then by definition 7.7 and theorem 7.11, ' $i < j$ ' and ' $i < j$ ' are in  $\mathcal{K}'$ . A similar argument applies if ' $\sim[i =_2 j]$ ' and ' $j =_2 0$ ' are in  $\mathcal{K}'$ .

The remaining case is ' $\sim[i = j]$ ', ' $\sim[i = 0]$ ', and ' $\sim[j = 0]$ ' are in  $\mathcal{K}'$ . There are two subcases:

1. There is a  $\underline{k}$  such that  $\underline{i}_k \neq \underline{j}_k$  and  $\underline{i}_m = \underline{j}_m$  for all  $\underline{m} > \underline{k}$ .  $\rho(\underline{i}_k) + \rho(\underline{j}_k) < \underline{n}$  so the lemma holds for  $\underline{i}_k$  and  $\underline{j}_k$ . Therefore, by definition 7.7 and theorem 7.11, ' $i < j$ ' or ' $j < i$ ' is in  $\mathcal{K}'$ .
2. Without loss of generality, assume ' $i < j$ ' is in  $\mathcal{K}'$ . Therefore, for all  $\underline{k} < \ell(\underline{i})$ , ' $i_k < j$ ' is in  $\mathcal{K}'$ . By definition 7.7,  $\underline{i} < \underline{j}$ ; by theorem 7.11, ' $i < j$ ' is in  $\mathcal{K}'$ . In case 7.7(4) was used, there is a  $\underline{k}$  such that not  $\underline{i}_k < \underline{j}$ . By the inductive hypothesis, the theorem holds for  $\underline{j} < \underline{i}_k$ . Therefore, by 7.7(4)  $\underline{j} < \underline{i}$  and ' $j < i$ ' is in  $\mathcal{K}'$  by theorem 7.11. This completes the proof.

7.15 Lemma. If ' $i < j$ ' and ' $j < k$ ' are in  $\mathbb{K}'$  then ' $i < k$ ' is in  $\mathbb{K}'$ . If ' $i < j$ ' and ' $j < k$ ' are in  $\mathbb{K}'$ , then ' $i < k$ ' is in  $\mathbb{K}'$  (transitivity). [The proof which is a modification of Schütte's proof of theorem 11.4 [21] in the same way that the proof of lemma 7.14 is a modification of Schütte's proof of theorem 11.3, is omitted.]

7.16 Definition (Schütte, [12], p. 112). A property  $A$  of natural numbers is said to be  $\leftarrow$ -progressive ( $c$ -progressive) if  $A(i)$  for all  $i < j$  ( $i < j$ ) implies  $A(j)$ . Transfinite  $\leftarrow$ -induction ( $c$ -induction) asserts that every  $\leftarrow$ -progressive ( $c$ -progressive) property holds for all natural numbers.

7.17 Here is an outline of the proof that transfinite  $\leftarrow$ -induction is a derived rule of  $\mathbb{K}'$ . Let  $\underline{I}$  be a property of natural numbers which holds if  $\leftarrow$ -induction holds in  $\mathbb{K}'$  to the natural number  $i$ . It will be shown that  $\underline{I}$  is  $\leftarrow$ -progressive. This result is used to show that transfinite  $\leftarrow$ -induction is a derived rule of  $\mathbb{K}'$ . Only induction on natural numbers (in their usual order) and the fact that  $<$  and  $c$  are completely represented in  $\mathbb{K}'$  are used.

7.18 Convention. In the metalanguage, an expression of the form

$$(\forall \underline{a} \in \underline{b}) [(\dots \underline{a} \dots) \rightarrow (\dots \underline{a} \dots)]$$

will be used to express the statement "For all  $\underline{b}$ -expressions ' $\underline{a}$ ' such that ' $\underline{ba}$ ' is in  $\mathbb{K}'$ , if ' $(\dots \underline{a} \dots)$ ' is in  $\mathbb{K}'$  then ' $(\dots \underline{a} \dots)$ ' is in  $\mathbb{K}'$ ."

7.19 Definition. Let  $\underline{P}(\underline{a})$  be a metalinguistic abbreviation for  $(\forall i \in \mathbb{N}) [(\forall j \in \mathbb{N}) [j < i \rightarrow \underline{aj}] \rightarrow \underline{ai}]$ . That is,  $\underline{P}(\underline{a})$  asserts that ' $\underline{a}$ ' represents a property of natural numbers which is  $\leftarrow$ -progressive in  $\mathbb{K}'$ . Let  $\underline{I}'(\underline{a}, i)$  serve as a metalinguistic abbreviation for:

$$\text{if } \underline{P}(\underline{a}) \text{ then } (\forall j \in \mathbb{N}) [j < i \rightarrow \underline{aj}]$$

That is,  $\underline{I}'(\underline{a}, i)$  asserts that if ' $\underline{a}$ ' represents a progressive property of natural numbers in  $\mathbb{K}'$  then for all integers,  $j$ , before or equal to  $i$ , ' $\underline{aj}$ '

is in  $\mathbb{K}'$ . Stated in another way,  $\underline{I}(a, i)$  asserts that transfinite  $\leftarrow$ -induction holds in  $\mathbb{K}'$  for ' $a$ ' up to ' $i$ '. Finally, let  $\underline{I}(i)$  serve as a metalinguistic abbreviation for the statement:

$$(\forall a \in \mathbb{U}) [\text{if } \underline{P}(a) \text{ then } (\forall j \in \mathbb{N}) [j < i \rightarrow a_j]]$$

That is, for the statement that for all  $\mathbb{U}$ -expressions ' $a$ ', is ' $a$ ' represents a progressive property of natural numbers in  $\mathbb{K}'$ , then  $\leftarrow$ -induction to  $i$  is a derived rule of  $\mathbb{K}'$ .

7.20 Lemma. If  $\leftarrow$ -induction to some integer  $j$  is a (derived) rule of  $\mathbb{K}'$ , then it is a (derived) rule of  $\mathbb{K}'$  for any integer  $i < j$ . That is, if  $\underline{I}(j)$  and  $i < j$ , then  $\underline{I}(i)$ .

Proof

By the definition of  $\underline{I}$ .

7.21 Lemma. If  $\underline{P}(a)$ , then ' $a_0$ ' is in  $\mathbb{K}'$ .

Proof

Let  $i = 0$  in the definition of  $\underline{P}$ :

$$(\forall j \in \mathbb{N}) [j < 0 \rightarrow a_j] \rightarrow a_0$$

There is no natural number  $j$  such that  $j < 0$  by definition 7.7. Consequently, since  $\leftarrow$  is completely represented in  $\mathbb{K}'$ , the antecedant is vacuously true so ' $a_0$ ' is in  $\mathbb{K}'$ . [Remark: Lemmas 7.20 and 7.21 are also provable if they refer to a property,  $\underline{A}$ , of natural numbers; Schütte provides the proof ([12], p. 104).]

7.22 Lemma.  $\underline{I}(i)$  is  $\leftarrow$ -progressive.

Proof

On the hypothesis that  $\underline{I}(j)$  for all  $j < i$ , it will be shown that  $\underline{I}(i)$ . By the hypothesis, for all  $\mathbb{U}$ -expressions which represent  $\leftarrow$ -progressive properties of natural numbers,  $\leftarrow$ -induction for all  $j < i$  is a rule of  $\mathbb{K}'$ , i.e.,  $(\forall a \in \mathbb{U}) [\underline{P}(a) \rightarrow (\forall j \in \mathbb{N}) [j < i \rightarrow a_j]]$ .

Since 'a' represents a progressive property, this means that 'ai' is in  $\mathcal{K}'$ .

7.23 Lemma.  $\underline{I}(0)$  is a derived rule of  $\mathcal{K}'$ . [The proof, which is an obvious revision of the proof of lemma 7.21, is omitted.]

7.24 It remains to be shown that  $\underline{I}(i)$  is a derived rule of  $\mathcal{K}'$  for all natural numbers  $i$ . This will be done in several steps. First, a particular restriction of a property of natural numbers will be defined. Then it will be shown that this restriction,  $\underline{I}^*$ , of  $\underline{I}$  is  $\epsilon$ -progressive. Then it will be shown that transfinite  $\epsilon$ -induction is a derived rule of  $\mathcal{K}'$ . Finally, this result is used to show that transfinite  $\epsilon$ -induction is a rule of  $\mathcal{K}'$ .

7.25 Let  $\underline{O}(A)$  serve as a metalinguistic abbreviation for the statement that 'A' is a  $\epsilon$ -progressive property of natural numbers, i.e., for  $(\forall i)[(\forall j)[j < i \text{ implies } A(j)] \text{ implies } A(i)]$ .

7.26 Definition. The restriction,  $\underline{A}^*$ , of a property A of natural numbers is defined as follows:

$$\underline{A}^*(i) = \begin{cases} \underline{A}(0) & \text{if } i = 0 \\ (\forall j)[\underline{I}(i, j) \text{ implies } \underline{A}(j)] & \text{if } i \neq 0 \end{cases}$$

The second part of the definition of  $\underline{A}^*$  asserts that if  $\epsilon$ -induction to the indicies (7.4) of  $i$  is a rule of  $\mathcal{K}'$ , then  $\underline{A}^*(i)$  is true.

7.27 Lemma.  $\underline{I}^*$  is  $\epsilon$ -progressive, i.e.,  $\underline{O}(\underline{I}^*)$

Proof

On the hypothesis

$$(\forall j)[j < i \text{ implies } \underline{I}^*(j)] \tag{1}$$

it must be shown that  $\underline{I}^*(i)$ .

By lemma 7.23 and definition 7.26,  $\underline{I}^*(\underline{0})$ .

If  $\underline{i} \neq \underline{0}$ , then by definition 7.26, it must be shown that  $\underline{I}(\underline{i})$  is a consequence of

$$\underline{I}(\underline{i}_{\underline{k}}) \text{ for all indices } \underline{i}_{\underline{k}} \text{ of } \underline{i} \quad (2)$$

Since, by lemma 7.22,  $\underline{I}$  is  $\triangleleft$ -progressive, it is sufficient to show that  $\underline{I}(\underline{k})$  is a consequence of

$$\underline{k} \triangleleft \underline{i} \quad (3)$$

By induction on the rank of  $\underline{k}$ , it will be shown that  $\underline{I}(\underline{k})$ .

If  $\rho(\underline{k}) = \underline{0}$  then  $\underline{k} = \underline{0}$ .  $\underline{I}(\underline{0})$  by lemma 7.23.

If  $\rho(\underline{k}) > \underline{0}$  then  $\underline{k} > \underline{0}$ . The inductive hypothesis is:

$$(\forall \underline{n}) [\rho(\underline{n}) < \rho(\underline{k}) \text{ and } \underline{n} \triangleleft \underline{i} \text{ implies } \underline{I}(\underline{n})] \quad (4)$$

There are two cases to consider:

1.  $\underline{i} \subset \underline{k}$ . In this case, (3) can be true only by definition 7.7(4) so there is a  $\underline{m}$  such that  $\underline{k} \triangleleft \underline{i}_{\underline{m}}$ . In this case,  $\underline{I}(\underline{k})$  is a consequence of lemma 7.20.

2.  $\underline{k} \subset \underline{i}$ . By (1),  $\underline{I}^*(\underline{k})$  and also

$$(\forall \underline{m}) [\underline{I}(\underline{k}_{\underline{m}}) \text{ implies } \underline{I}(\underline{k})] \quad (5)$$

By 7.4,  $\underline{k}_{\underline{m}} \triangleleft \underline{k}$ . With (3) we have  $\underline{k}_{\underline{m}} \triangleleft \underline{i}$ . Further,  $\rho(\underline{k}_{\underline{m}}) < \rho(\underline{k})$  so, by (4) we have  $\underline{I}(\underline{k}_{\underline{m}})$  and by (5)  $\underline{I}(\underline{k})$ .

7.28 Lemma. For every property,  $\underline{A}$ , of natural numbers, transfinite  $\subset$ -induction holds for  $\underline{A}^*$ , i.e.,

$$\underline{0}(\underline{A}^*) \text{ implies } (\forall \underline{i}) \underline{A}^*(\underline{i})$$

Proof

On the hypothesis  $\underline{0}(\underline{A}^*)$  it will be shown that for all  $\underline{i}$ ,  $\underline{A}^*(\underline{i})$ . For  $\underline{i} = \underline{0}$ , this is a consequence of a minor modification of lemma 7.20. The essential idea of the proof for  $\underline{i} \neq \underline{0}$  is this: The first  $\ell(\underline{i})$  primes are used in the prime decomposition of  $\underline{i}$ . Select an arbitrary

integer  $\underline{j}$  (possibly  $\underline{i}$ ) which can be decomposed into the first  $\ell(\underline{i})$  primes. For all  $\underline{k} < \underline{j}$ , suppose that  $\underline{A}^*(\underline{k})$ . On this hypothesis, show  $\underline{A}^*(\underline{j})$ . By lemma 7.14,  $<$  is total. The proof summarized above is correct for all natural numbers. Consequently,  $(\forall \underline{i}) \underline{A}^*(\underline{i})$ .

Let  $\ell(\underline{i}) = \underline{n}$  and select arbitrary natural numbers  $\underline{j}_0, \underline{j}_1, \dots, \underline{j}_n$  as the indicies of  $\underline{j}$ , i.e., let

$$\underline{j} = \underline{p}_0^{\underline{j}_0} \cdot \underline{p}_1^{\underline{j}_1} \cdot \dots \cdot \underline{p}_n^{\underline{j}_n}$$

For all  $\underline{k} < \underline{j}$ , suppose  $\underline{A}^*(\underline{k})$ . By definition 7.7(1) and (3),  $\underline{k} < \underline{j}$  if there is an  $\underline{m}$  such that  $\underline{k}_m < \underline{j}_m$  and for all  $\underline{q} > \underline{m}$ ,  $\underline{k}_q = \underline{j}_q$  or if  $\underline{k} = \underline{0}$ . For all  $\underline{m}$  such that  $\underline{0} \leq \underline{m} \leq \underline{n}$ :

$$(\forall \underline{k}_m) [\underline{k}_m < \underline{j}_m \text{ implies } (\forall \underline{k}_0) \dots (\forall \underline{k}_{\underline{m}-1}) \underline{A}^*(\underline{p}_0^{\underline{k}_0} \cdot \underline{p}_1^{\underline{k}_1} \cdot \dots \cdot \underline{p}_{\underline{m}-1}^{\underline{k}_{\underline{m}-1}} \cdot \underline{p}_m^{\underline{k}_m} \cdot \underline{p}_{\underline{m}+1}^{\underline{j}_{\underline{m}+1}} \cdot \dots \cdot \underline{p}_n^{\underline{j}_n})] \quad (*)$$

In order to simplify the following discussion, let ' $\underline{D}(\underline{j}_m, \dots, \underline{j}_n)$ ' serve as a metalinguistic abbreviation for (\*), above, and let ' $\underline{B}(\underline{j}_m, \dots, \underline{j}_n)$ ' serve as a metalinguistic abbreviation for:

$$(\forall \underline{k}_0) \dots (\forall \underline{k}_{\underline{m}-1}) \underline{A}^*(\underline{p}_0^{\underline{k}_0} \cdot \dots \cdot \underline{p}_{\underline{m}-1}^{\underline{k}_{\underline{m}-1}} \cdot \underline{p}_m^{\underline{j}_m} \cdot \dots \cdot \underline{p}_n^{\underline{j}_n})$$

of course, these abbreviations apply for  $\underline{m} = \underline{0}, \underline{1}, \dots, \underline{n}$ . Note that  $\underline{B}(\underline{j}_0, \dots, \underline{j}_n)$  is  $\underline{A}^*(\underline{j})$ .

It is straightforward to verify that

$$\underline{D}(\underline{j}_0, \dots, \underline{j}_n) \text{ and } \dots \text{ and } \underline{D}(\underline{j}_n) \text{ implies } \underline{B}(\underline{j}_0, \dots, \underline{j}_n)$$

This is true because  $\underline{A}^*$  is  $<$ -progressive, by definition 7.7(2), the antecedent of the implication is equivalent to  $(\forall \underline{k} \{ \underline{k} < \underline{j} \text{ implies } \underline{A}^*(\underline{j}) \})$ .

By induction on  $\underline{m}$ , it will now be shown that for all  $\underline{m}$  such that  $\underline{0} \leq \underline{m} \leq \underline{n}$ :

$$\underline{D}(\underline{j}_m, \dots, \underline{j}_n) \text{ and } \dots \text{ and } \underline{D}(\underline{j}_n) \text{ implies } \underline{B}(\underline{j}_0, \dots, \underline{j}_n) \quad (**)$$



It has already been shown for  $\underline{m} = \underline{0}$ . The hypothesis of the induction is that (\*\*) is true for  $\underline{m}$  and the inductive step is to show that (\*\*) is true for  $\underline{m} + \underline{1}$ . This is equivalent to giving a proof of  $\underline{B}(\underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})$  on the hypothesis:

$$\underline{D}(\underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}}) \text{ and } \dots \text{ and } \underline{D}(\underline{j}_{\underline{n}}) \text{ implies } [(\forall r)[\underline{D}(r, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}}) \text{ implies } \underline{B}(r, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})]]$$

It is only necessary to give a proof of  $\underline{B}(\underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})$  on the hypothesis

$$(\forall r)[\underline{D}(r, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}}) \text{ implies } \underline{B}(r, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})]$$

Using the definition of  $\underline{D}$ , this hypothesis is:

$$(\forall r)[(\forall k)[k \prec r \text{ implies } \underline{B}(k, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})] \text{ implies } \underline{B}(r, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})]$$

which asserts that  $\underline{B}$  is  $\prec$ -progressive in its first argument. If  $\prec$ -induction to some number  $\underline{s}_{\underline{m}}$  is available, we can conclude  $\underline{B}(\underline{s}_{\underline{m}}, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})$ . The definition of  $\underline{A}^*$  which is used in the definition of  $\underline{B}$  is such that  $\prec$ -induction to any index of  $\underline{j}$  holds. Therefore,  $(\forall \underline{s}_{\underline{m}})\underline{B}(\underline{s}_{\underline{m}}, \underline{j}_{\underline{m}+1}, \dots, \underline{j}_{\underline{n}})$ . This was to be shown to complete the proof of (\*\*).

Therefore, we have  $\underline{D}(\underline{j}_{\underline{n}}) \text{ implies } \underline{B}(\underline{j}_{\underline{n}})$ , that is,

$(\forall r_{\underline{n}})[r_{\underline{n}} \prec \underline{j}_{\underline{n}} \text{ implies } \underline{B}(r_{\underline{n}})] \text{ implies } \underline{B}(\underline{j}_{\underline{n}})$ . for all  $\underline{j}_{\underline{n}}$ . This is the  $\prec$ -progressiveness of  $\underline{B}$ . Again, since an essential part of the definition of  $\underline{A}^*$  is that  $\prec$ -induction holds to any index  $\underline{j}_{\underline{n}}$  of  $\underline{j}$ , we have  $(\forall \underline{s})\underline{B}(\underline{s})$ . Therefore, we have  $\underline{A}^*(\underline{i})$ . That is, the  $\prec$ -progressive property  $\underline{A}^*$  applies to all natural numbers. This completes the proof of the lemma.

7.29 Theorem  $\prec$ -induction to any natural number is a derived rule of  $\underline{K}'$ , i.e.,  $(\forall i)\underline{I}(i)$ .

Proof

By lemma 7.26,  $I^*$  is  $\leftarrow$ -progressive. By lemma 7.28,  $(\forall i) I^*(i)$ . It remains to be shown that  $I$  itself is a property of all natural numbers. For all  $\underline{i} \neq \underline{0}$ , if  $I(\underline{i}_j)$  for all indicies  $\underline{i}_j$  of  $\underline{i}$ , then  $I(\underline{i})$ . By induction on the rank of  $\underline{i}$ , it will now be shown that  $(\forall i) I(i)$ .

Case 1.  $\rho(\underline{i}) = \underline{0}$ . Then  $\underline{i} = \underline{0}$  and  $I(\underline{0})$  by lemma 7.22.

Case 2.  $\rho(\underline{i}) > \underline{0}$  and  $(\forall k)[\rho(k) < \rho(\underline{i}) \text{ implies } I(k)]$ .

Since  $\rho(\underline{i}_j) < \rho(\underline{i})$ ,  $I(\underline{i}_j)$  for all indicies of  $\underline{i}$  and therefore  $I(\underline{i})$ .

7.30 This completes the proof of the statement that transfinite induction to constructible ordinals is a derived rule of  $\mathbb{K}'$ . With this result, it is possible to develop the arithmetic of ordinals in  $\mathbb{K}'$  and to prove in  $\mathbb{K}'$ , for example, that  $\underline{5}$  is the first fixed point of the function  $\underline{f}$  defined by the identity  $\underline{f}(\underline{i}) = \exp_{\underline{0}}(\underline{i}, \underline{3})$ . This is straightforward since the proofs in section 13 of Chapter IV of Schütte [21] can be completed in  $\mathbb{K}'$  using the derived rule of  $\leftarrow$ -induction (theorem 7.29).

8. Summary

8.1 This essay contains two proofs of the statement that the theory of constructible ordinals can be developed in  $\mathbb{K}'$ . Furthermore, it has been shown that transfinite induction to constructible ordinals is a derived rule of  $\mathbb{K}'$ . This result was established using only induction on the natural numbers (in their usual order).

8.2 In section 6, it was shown that the Church-Kleene ordinals are represented in  $\mathbb{K}'$  and that the set of these ordinals is completely represented in  $\mathbb{K}'$ . In addition, the arithmetic of constructible ordinals was developed in  $\mathbb{K}'$ . This development includes the fixed points of monotone increasing continuous functions of ordinals.

8.3 In section 7, it was shown that Schütte's ordinals can be dealt with in  $\mathbb{K}'$ . These ordinals are obtained by defining another order relation on the natural numbers. This order relation was shown to be completely represented in  $\mathbb{K}'$ . Then, it was shown that transfinite induction on these ordinals is a derived rule of  $\mathbb{K}'$ . This result suggests a proof of the statement that the addition of a consistent form of material implication would provide a proper extension of  $\mathbb{K}'$ .

8.4 The derived rule of induction may be stated using the conventions of 7.19 as follows:

$$(\forall a \in U) [(\forall j \in N) [(\forall j \in N) [j \prec i \rightarrow a_j] \rightarrow a_i] \text{ implies } (\forall k \in N) a_k \text{ is in } \mathbb{K}']$$

If a rule for material implication, which is sufficiently restrictive to avoid the Curry paradox were available in  $\mathbb{K}'$ , essentially the proof given in section 7 could be carried out inside  $\mathbb{K}'$  to give a proof of the following formula:

$$(\forall a) [(\forall i) [N_i \supset [(\forall j) \{N_j \supset [j \prec i \supset a_j]\}] \supset a_i]] \supset (\forall k) \{N_k \supset a_k\}$$

Thus, with the addition of implication, this formula would be a theorem of the system. It appears that it cannot be formulated in terms of the primitives of  $K'$ . This question will be considered in a future paper.

8.5 The system  $K'$  is clearly sufficient to deal with the metamathematical operations which Schütte used to develop his proof theory. For example, the assignment of Gödel numbers to formulas, the assignment of ordinals to formulas, etc. can be completed with  $K'$ . Using the derived rule of transfinite induction, it appears to be possible to do all of Schütte's proof theory within  $K'$ . Thus  $K'$  provides a consistent metalanguage for this proof theory. This matter will be discussed further in another paper.

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