

MINIMAL LOGIC AND COMPUTERS

Chapter I

The Method of Subordinate Proofs

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CHAPTER I\*

THE METHOD OF SUBORDINATE PROOFS

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CHAPTER I  
THE METHOD OF SUBORDINATE PROOFS

10. Introduction

10.1 We will be using a particular form of natural deduction, known as the method of subordinate proofs, to construct proofs in the system of logic which is described in chapters II to V. There are two reasons for using the method of subordinate proofs: one is pragmatic and the other is technical. Experience has shown that the method of subordinate proofs is easy to teach and facilitates the construction of proofs. The method of subordinate proofs allows us to construct proofs which have other proofs as hypothesis in a straightforward manner.

10.2 In order to make this book self-contained, this chapter is an introduction to the method of subordinate proofs. So that the reader may concentrate on the method, we will sketch a subordinate proof formulation of a system of logic with which he is already familiar: the propositional calculus. Our discussion is quite informal. The reader is encouraged to provide the missing details using his favorite formulation of the propositional calculus.

10.3 We assume that we have a class  $S$  of (formal) sentences. We will use the italic letters

p   q   I   S

to refer to arbitrary members of  $S$ . We assume, of course, that if 'p' and 'q' are sentences, then so are

[p & q]

[p ∨ q]

[p > q]

~p

These sentences are read as "p and q", "p or q", "if p then q", and "p is false", respectively. We omit outermost square brackets when no ambiguity results from doing so.

10.4 We call '[p & q]' the conjunction of 'p' with 'q'; we call '[p ∨ q]' the disjunction of 'p' with 'q'; we call '[p > q]' the conditional of 'p' with 'q', or the implication of 'q' by 'p'; and we call '~p' the negation of 'p'.

10.5 We will introduce rules of direct consequence for the connectives '&', '∨', '>', and '~' at the appropriate points in the following discussion.

11. First-Order Proofs

11.1 The following rules of direct consequence (d.c.) may be used with first-order proofs as well as with higher-order proofs. First-order proofs are described in 11.8.

11.2 The sentence ' $p$ ' is said to be a d.c. of the sentence ' $[p \ \& \ q]$ ' by the rule of conjunction elimination (conj elim).

11.3 The sentence ' $q$ ' is likewise said to be a d.c. of the sentence ' $[p \ \& \ q]$ ' by the rule of conjunction elimination.

11.4 The sentence ' $[p \ \& \ q]$ ' is said to be a d.c. of the pair of sentences ' $p$ ', ' $q$ ' by the rule of conjunction introduction (conj int).

11.5 The sentence ' $[p \ \vee \ q]$ ' is said to be a d.c. of the sentence ' $p$ ' by the rule of disjunction introduction (dis int).

11.6 The sentence ' $[p \ \vee \ q]$ ' is likewise said to be a d.c. of the sentence ' $q$ ' by the rule of disjunction introduction.

11.7 The sentence ' $q$ ' is said to be a d.c. of the pair of sentences ' $p$ ', ' $[p \ \supset \ q]$ ' by the rule of implication elimination (imp elim). This rule is also called modus ponens (m p).

11.8 A first-order proof is a finite sequence of sentences, written as a column, and such that each sentence of the sequence is either explicitly designated as a hypothesis or else is, in fact, a direct consequence (in one of the senses defined in 11.2 to 11.7) of one or more preceding sentences of the sequence. The hypothesis of the sequence (if there are any) are designated as such by writing 'hyp' after each of them. Further, all hypothesis are grouped together at the beginning of the sequence and are separated from the other sentences in the sequence by a short horizontal line. A vertical line is drawn to the left of the sequence of sentences and the resulting expression is called a column.

11.9 The expressions in 11.10 to 11.16, 11.18, and 11.19 are proofs. Note that each sentence is numbered to the left of the vertical line and at the right there is an indication that either the sentence is a hypothesis or that it is a direct consequence of specified preceding sentences by one of the rules of direct consequence.

11.10

1	p	hyp
2	q	hyp
3	p & q	1, 2, conj int

11.11

1		$p \ \& \ q$	hyp
2		$p$	1, conj elim
3		$q$	1, conj elim

11.12

1		$p \ \& \ q$	hyp
2		$p$	1, conj elim
3		$q$	1, conj elim
4		$q \ \& \ p$	2, 3, conj int

11.13

1		$[p \ \& \ q] \ \& \ r$	hyp
2		$p \ \& \ q$	1, conj elim
3		$r$	1, conj elim
4		$p$	2, conj elim

11.14

1		$p \ \& \ [q \ \& \ r]$	hyp
2		$p$	1, conj elim
3		$q \ \& \ r$	1, conj elim
4		$q$	3, conj elim
5		$r$	3, conj elim



11.15

1		$p \ \& \ [q \ \& \ r]$	hyp
2		$p$	1, conj elim
3		$q \ \& \ r$	1, conj elim
4		$q$	3, conj elim
5		$r$	3, conj elim
6		$q \ \& \ p$	4, 2, conj int
7		$[q \ \& \ p] \ \& \ r$	6, 5, conj int

11.16

1		$s \ \& \ r$	hyp
2		$r \ \supset \ [p \ \& \ q]$	hyp
3		$r$	1, conj elim
4		$p \ \& \ q$	2, 3, m p
5		$p$	4, conj elim
6		$q$	4, conj elim
7		$q \ \& \ r$	3, 6, conj int

11.17 In 11.18 we give a proof of ' $p \vee [q \ \& \ s]$ ' from the hypotheses ' $[p \ \vee \ q] \supset [r \ \& \ s]$ ' and ' $q$ ', while 11.19 is a proof of ' $s$ ' from the hypotheses ' $p$ ', ' $[p \ \vee \ q] \supset [q \ \& \ r]$ ', and ' $[s \ \vee \ q] \supset s$ '.

## 11.18

1		$[p \vee q] \supset [r \& s]$	hyp
2		$q$	hyp
3		$p \vee q$	2, dis int
4		$r \& s$	1, 3, m p
5		$s$	4, conj elim
6		$q \& s$	2, 5, conj int
7		$p \vee [q \& s]$	6, dis int

## 11.19

1		$p$	hyp
2		$[p \vee q] \supset [q \& r]$	hyp
3		$[s \vee q] \supset s$	hyp
4		$p \vee q$	1, dis int
5		$q \& r$	2, 4, m p
6		$q$	5, conj int
7		$s \vee q$	6, dis int
8		$s$	3, 7, m p

## 11.20 Exercises

1. Give a proof of ' $[r \& s] \& [r \supset s]$ ' from the hypotheses ' $p$ ', ' $p \supset r$ ' and ' $r \supset s$ '.
2. Give a proof of ' $[p \vee q] \& r$ ' from the hypotheses ' $p \vee q$ ' and ' $[(p \vee q) \vee s] \supset r$ '.

## 12. Second-Order Proofs

12.1 All the proofs which we considered in section 11 are called first-order proofs. A first order proof is a sequence of sentences such that each sentence of the sequence is a direct consequence of preceding sentences of the sequence. We can also construct sequences out of sentences and first order proofs. Such sequences, when properly constructed, can also be regarded as proofs in a more general sense, and may be called second-order proofs. Second order proofs are described carefully in 12.6 and 12.7.

12.2 Here is an example of an second-order proof:

1	<u>r</u>	hyp
2	<u>r</u> ∨ <u>s</u>	1, dis int
3	<u>p</u> & <u>q</u>	hyp
4	<u>p</u>	3, conj elim
5	<u>p</u> ∨ <u>q</u>	4, dis int
6	<u>[p &amp; q]</u> ⊃ <u>[p ∨ q]</u>	3-5, imp int
7	<u>[r ∨ s]</u> & <u>[[p &amp; q] ⊃ [p ∨ q]]</u>	2, 6, conj int

12.3 Notice that the proof in 12.2 is not just a sequence of sentences. It is rather a sequence that consists of two sentences (lines 1 and 2) as its first two steps, a whole first-order proof (lines 3 to 5) as its third step, and two more sentences (lines 6 and 7) as its last two steps.

The whole first-order proof (3-5) is itself merely step 3 of the five step second-order proof. In other words, the whole first-order proof (3-5) is to be viewed as a single unit (just as much as a sentence is viewed as a single unit) in the construction of the second-order proof.

12.4 Line 6 of the proof in 12.2, which is actually the fourth step of the second-order proof, is a direct consequence of the whole first-order proof considered as a single unit. This kind of direct consequence has not been previously considered in this chapter. It is called implication introduction (imp int). In general, any first-order proof that has exactly one hypothesis 'p' and has a sentence 'q' as a step, can have as a direct consequence (by implication introduction), in a second-order proof, the sentence ' $p \supset q$ '. Observe that if ' $p \supset q$ ' is a direct consequence of a first-order proof by implication introduction, then 'p' must be the only hypothesis of that first-order proof, and 'q' must be a step of that proof. Usually, 'q' will be the last step of the proof. This requirement is satisfied by ' $[p \ \& \ q] \supset [p \ \vee \ q]$ ' in the proof in 12.2 above, because ' $p \ \& \ q$ ' is the only hypothesis of the first order proof 3-5, and ' $p \ \vee \ q$ ' is the last step of that same first-order proof.

12.5 Consider now the following second-order proof which has just two steps, the first of which is a four step first-order proof:

1	[p & q] & r	hyp
2	p & q	1, conj elim
3	q	2, conj elim
4	s ∨ q	3, dis int
5	[[p & q] & r] ⇒ [s ∨ q]	1-4, imp int

Notice that this proof does not have a hypothesis as its first step but rather has a whole first-order proof as its first step, and that this first-order proof does have a hypothesis as its first step. In other words, this proof, which is a second-order proof, is a sequence of just two things, neither of which is a hypothesis. The first of the two things is a four step first-order proof (which does have a hypothesis of its own and not belonging to the second-order proof), and the second of which is the sentence ' $[(p \ \& \ q) \ \& \ r] \Rightarrow [s \ \vee \ q]$ ', which is a direct consequence of the first-order proof by implication introduction.

12.6 The concept of a second-order proof can now be defined accurately as follows: A second-order proof is a sequence of things (written as a column) each of which is either a sentence or is itself a first-order proof, and if it is a sentence it is either a hypothesis (of the second-order proof) or is a direct consequence of preceding things (sentences or first-order proofs) of the sequence.

12.7 There is a special dispensation allowed to those first-order proofs that serve as steps of second-order proofs. This dispensation permits each such first-order proof to add, as steps of its own, any preceding sentences which are steps of the second-order proof of which it is a part. Such steps will be said to be reiterated, and this fact will be indicated by writing "reit" after them, followed by the number of the sentence as it originally appeared as a step of the second-order proof. The reiterated step thus appears twice, once in the second-order proof in its original location, and once in the first-order proof in its new location, and it can be used in both places like any other step. Here are some examples of second-order proofs that contain first-order proofs in which reiteration occurs:

12.8

1	$p \supset q$	hyp
2	$q \supset r$	hyp
3	$p$	hyp
4	<div style="border-left: 1px solid black; padding-left: 5px; padding-right: 5px;"> <math>p \supset q</math> </div>	1, reit
5	$q$	3, 4, m p
6	$q \supset r$	2, reit
7	$r$	5, 6, m p
8	$p \supset r$	3-7, imp int

We can say that we have given a proof of ' $p \supset r$ ' from the hypotheses ' $p \supset q$ ' and ' $q \supset r$ '. This principle, that  $p \supset r$  follows from  $p \supset q$  and  $q \supset r$ , is usually referred to as the transitivity of implication or as the principle of hypothetical syllogism.

12.9

1	[ $p \supset q$ ] & [ $p \supset r$ ]	hyp
2	— $p \supset q$	1, conj elim
3	$p \supset r$	1, conj elim
4	 $p$	hyp
5	 — $p \supset r$	3, reit
6	 $p \supset q$	2, reit
7	 $r$	4, 5, m p
8	 $q$	4, 6, m p
9	 $q$ & $r$	7, 8, conj int
10	$p \supset [q \text{ \& } r]$	4-9, imp int

Here we can say that we have proved ' $p \supset [q \text{ \& } r]$ ' from the hypothesis ' $[p \supset q] \text{ \& } [p \supset r]$ '

### 12.10 Exercises

1. Give a proof of ' $[p \supset q] \text{ \& } [p \supset r]$ ' from the hypothesis ' $p \supset [q \text{ \& } r]$ '.
2. Give a proof of ' $[p \text{ \& } [p \supset q]] \supset [q \text{ \& } r]$ ' from the hypothesis ' $r$ '.

12.11 It should be noticed that although a first-order proof can reiterate steps from a second-order proof to which it belongs, the second-order proof is not permitted to reiterate steps from any of the first-order proofs that belong to it. Reiteration is therefore a one-directional process. A first-order proof can reiterate from the second-order proof to which it belongs, but not vice versa. Furthermore, two first-order proofs cannot reiterate from each other.

12.12 A further kind of direct consequence is disjunction elimination (dis elim). This can be used in a second-order proof in such a way that a sentence 'r' is treated as being a direct consequence of the following three things: a sentence of the form 'p  $\vee$  q', a first-order proof having hypothesis 'p' and 'r' as a step, and another first-order proof having hypothesis 'q' and 'r' as a step. The two first-order proofs must have only one hypothesis apiece, and these hypotheses must be the same as the 'p' and 'q' in the sentence 'p  $\vee$  q' referred to above. Here are some second-order proofs using this further kind of direct consequence:



12.13

1	$p \vee q$	hyp
2	$p$	hyp
3	$\neg q \vee p$	2, dis int
4	$q$	hyp
5	$\neg q \vee p$	4, dis int
6	$q \vee p$	1, 2-3, 4-5, dis elim

12.14

1	$p \vee [q \ \& \ r]$	hyp
2	$p$	hyp
3	$p \vee q$	2, dis int
4	$q \ \& \ r$	hyp
5	$q$	4, conj elim
6	$p \vee q$	5, dis int
7	$p \vee q$	1, 2-3, 4-6, dis elim

12.15

1	$\underline{s}$	hyp
2	$p \vee q$	hyp
3	$p$	hyp
4	$\underline{s}$	1, reit
5	$p \ \& \ \underline{s}$	3, 4, conj int
6	$[p \ \& \ \underline{s}] \vee [q \ \& \ \underline{s}]$	5, dis int
7	$q$	hyp
8	$\underline{s}$	1, reit
9	$q \ \& \ \underline{s}$	7, 8, conj int
10	$[p \ \& \ \underline{s}] \vee [q \ \& \ \underline{s}]$	9, dis int
11	$[p \ \& \ \underline{s}] \vee [q \ \& \ \underline{s}]$	2, 3-6, 7-10, dis elim

12.16

1	$[q \vee r] \supset s$	hyp
2	$[p \ \& \ q] \vee [p \ \& \ r]$	hyp
3	$p \ \& \ q$	hyp
4	$q$	3, conj elim
5	$q \vee r$	4, dis int
6	$p \ \& \ r$	hyp
7	$r$	6, conj elim
8	$q \vee r$	7, dis int
9	$q \ \& \ r$	2, 3-5, 6-8, dis elim
10	$s$	1, 9, m p
11	$s \ \& \ [q \vee r]$	9, 10, conj int

12.17

1	$[p \ \& \ q] \vee [p \ \& \ r]$	hyp
2	$p \ \& \ q$	hyp
3	$p$	2, conj elim
4	$q$	2, conj elim
5	$q \vee r$	4, dis int
6	$p \ \& \ [q \vee r]$	3, 5, conj int
7	$p \ \& \ r$	hyp
8	$p$	7, conj elim
9	$r$	7, conj elim
10	$q \vee r$	9, dis int
11	$p \ \& \ [q \vee r]$	8, 10, conj int
12	$p \ \& \ [q \vee r]$	1, 2-6, 7-11, dis elim

12.18 The converse of 12.17 can also be established, that is, we can prove ' $\underline{p} \ \& \ \underline{r}] \vee [\underline{p} \ \& \ \underline{r}]$ ' from the hypothesis ' $\underline{p} \ \& \ [\underline{q} \ \vee \ \underline{r}]$ '. The method is to start by using conj elim to separate ' $\underline{p} \ \& \ [\underline{q} \ \vee \ \underline{r}]$ ' into its two main parts ' $\underline{p}$ ' and ' $\underline{q} \ \vee \ \underline{r}$ ', and then to proceed from the steps ' $\underline{p}$ ' and ' $\underline{q} \ \vee \ \underline{r}$ ' somewhat as 12.15 proceeds from the steps ' $\underline{s}$ ' and ' $\underline{p} \ \vee \ \underline{q}$ '. Incidentally, it is not possible to use conj elim to separate ' $[\underline{p} \ \& \ \underline{q}] \ \vee \ [\underline{p} \ \& \ \underline{r}]$ ' into parts. This is because conj elim can be applied only to a conjunction, while ' $[\underline{p} \ \& \ \underline{q}] \ \vee \ [\underline{p} \ \& \ \underline{r}]$ ' is obviously a disjunction rather than a conjunction (though its two main parts are conjunctions).

12.19

1	$[\underline{r} \supset \underline{s}] \supset [[\underline{r} \supset \underline{s}] \supset \underline{r}]$	hyp
2	$[\underline{p} \ \& \ \underline{q}] \ \vee \ [\underline{r} \supset \underline{s}]$	hyp
3	$\underline{p} \ \& \ \underline{q}$	hyp
4	$\underline{q}$	3, conj elim
5	$\underline{q} \ \vee \ \underline{s}$	4, dis int
6	$\underline{r} \supset \underline{s}$	hyp
7	$[\underline{r} \supset \underline{s}] \supset [[\underline{r} \supset \underline{s}] \supset \underline{r}]$	1, reit
8	$[\underline{r} \supset \underline{s}] \supset \underline{r}$	6, 7, m p
9	$\underline{r}$	6, 8, m p
10	$\underline{s}$	6, 9, m p
11	$\underline{q} \ \vee \ \underline{s}$	10, dis int
12	$\underline{q} \ \vee \ \underline{s}$	2, 3-4, 6-11, dis elim

12.20 In constructing proofs, the following guiding principles are useful:

- (1) It is generally best to start by using reit and elimination rules as far as possible, the elimination rules so far available being conj elim, m p (that is, imp elim), and dis elim.
- (2) If you wish to establish a conjunction, first use reit and elimination rules as far as possible, and then conj int to obtain the required conjunction.
- (3) If you wish to obtain an implication, first use reit and elimination rules as far as possible, and then use implication introduction to obtain the required implication.
- (4) If you wish to obtain a disjunction, first use reit and elimination rules as far as possible, and then use disjunction introduction to obtain the required disjunction. This method may fail, however, when one disjunction is to be proved from another disjunction, unless we are careful to proceed in the following way: First try to use dis elim with respect to the given disjunction (for example, with respect to ' $p \vee q$ ' as in 12.15), and try to establish the required disjunction by dis int within each of the first-order proofs (for example, ' $[p \ \& \ s] \vee [q \ \& \ s]$ '

in the first-order proofs in 12.15). Then obtain the required disjunction from the given disjunction by dis elim (as done in 12.15).

13. Higher Order Proofs

13.1 Third-order proofs can be designed which make use of first- and second-order proofs in exactly the same way that second-order proofs make use of first-order proofs. Here are two examples of third-order proofs:

13.2

1	$p \supset [q \supset r]$	hyp
2	$p \supset q$	hyp
3	$p \supset [q \supset r]$	1, reit
4	$p$	hyp
5	$p \supset q$	2, reit
6	$q$	4, 5, m p
7	$p \supset [q \supset r]$	3, reit
8	$q \supset r$	4, 7, m p
9	$r$	6, 8, m p
10	$p \supset r$	4-9, imp int
11	$[p \supset q] \supset [p \supset r]$	2-10, imp int

## 13.3

1	$p \supset r$	hyp
2	$q \supset r$	hyp
3	<div style="border-left: 1px solid black; padding-left: 5px;"><math>p \vee q</math></div>	hyp
4	<div style="border-left: 1px solid black; padding-left: 5px;"><math>p</math></div>	hyp
5	<div style="border-left: 1px solid black; padding-left: 5px;"><math>p \supset r</math></div>	1, reit
6	<div style="border-left: 1px solid black; padding-left: 5px;"><math>r</math></div>	4, 5, m p
7	<div style="border-left: 1px solid black; padding-left: 5px;"><math>q</math></div>	hyp
8	<div style="border-left: 1px solid black; padding-left: 5px;"><math>q \supset r</math></div>	2, reit
9	<div style="border-left: 1px solid black; padding-left: 5px;"><math>r</math></div>	7, 8, m p
10	$r$	3, 4-6, 7-9, dis elim
11	$[p \vee q] \supset r$	3-11, imp int

13.4 Notice that the first-order (or innermost) proofs in 13.3 reiterate directly from steps 1 and 2 of the main third-order proof, instead of following the procedure of 13.2 where the first-order proof reiterates from the second-order proof and the latter reiterates from the third-order proof. This direct reiteration will be allowed if it occurs from a preceding step of a higher order proof that encloses the proof doing the reiterating, as in 13.3.

13.5 The system of logic that has been described can be further extended in a similar way by allowing fourth-order proofs, fifth-order proofs, and, in general,  $n$  th-order proofs for every finite positive integer  $n$ . Here are two examples of fourth-order proofs:

13.6

1	$\underline{p \supset q}$	hyp
2	$\underline{p \vee \underline{r}}$	hyp
3	$\underline{p}$	hyp
4	$\underline{p \supset q}$	1, reit
5	$\underline{q}$	3, 4, m p
6	$\underline{q \vee \underline{r}}$	5, dis int
7	$\underline{r}$	hyp
8	$\underline{q \vee \underline{r}}$	7, dis int
9	$\underline{q \vee \underline{r}}$	2, 3-6, 7-8, dis elim
10	$[p \vee \underline{r}] \supset [q \vee \underline{r}]$	2-9, imp int
11	$[p \supset q] \supset [[p \vee \underline{r}] \supset [q \vee \underline{r}]]$	1-10, imp int

13.7

1	$\underline{q \supset \underline{r}}$	hyp
2	$\underline{p \supset q}$	hyp
3	$\underline{q \supset \underline{r}}$	1, reit
4	$\underline{p}$	hyp
5	$\underline{p \supset q}$	2, reit
6	$\underline{q}$	4, 5, m p
7	$\underline{q \supset \underline{r}}$	3, reit
8	$\underline{r}$	6, 7, m p
9	$\underline{p \supset \underline{r}}$	4-8, imp int
10	$[p \supset q] \supset [p \supset \underline{r}]$	2-9, imp int
11	$[q \supset \underline{r}] \supset [[p \supset q] \supset [p \supset \underline{r}]]$	1-10, imp int



13.8 Notice that in 13.7 the reiteration of ' $q \vee r$ ' in the innermost proof could have been directly from step 1 of the main proof instead of indirectly by way of an inner proof. Another way to shorten 13.7 would be to write it as follows:

13.9

1	$q \supset r$	hyp
2	<div style="border-left: 1px solid black; padding-left: 10px; border-bottom: 1px solid black;"><math>p \supset q</math></div>	hyp
3	<div style="border-left: 1px solid black; padding-left: 10px; border-bottom: 1px solid black;"><math>q \supset r</math></div>	1, reit
4	<div style="border-left: 1px solid black; padding-left: 10px; border-bottom: 1px solid black;"><math>p \supset r</math></div>	2, 3, trans imp
5	$[p \supset q] \supset [p \supset r]$	2-4, imp int
6	$[q \supset r] \supset [[p \supset q] \supset [p \supset r]]$	1-5, imp int

13.10 In 13.9 we have shortened 13.7 by omitting the details of the inference from ' $p \supset q$ ' and ' $q \supset r$ ' to ' $p \supset r$ '. This inference is said to be by transitivity of implication (trans imp). These details can be found in 12.8, which establishes the principle of the transitivity of implication. 13.9 can be regarded simply as an abbreviation or shorthand for 13.7, and the reference "trans imp" can be viewed as a reference to 12.8, where the details can be found for rewriting 13.9 out in full as 13.7.

13.11 We may regard transitivity of implication as a derived rule of inference, the derivation of it being given in 12.8. The other rules that we have so far been using are

underived, or primitive, rules of inference (such as conj int, m p, and so on). It is permitted to use transitivity of implication as freely in proofs as the regular underived rules.

Some other derived rules will also be given this same status below. Here is another example of the use of trans imp:

13.12

1	p	hyp
2	p $\supset$ [q $\supset$ [r $\vee$ s]]	hyp
3	[r $\vee$ s] $\supset$ r	hyp
4	q $\supset$ [r $\vee$ s]	1, 2, m p
5	q $\supset$ r	3, 4, trans imp

13.13 Another derived rule of inference is the rule of added condition (add cond) according to which 'p  $\supset$  q' may be inferred from 'q'. This rule is derived by the following proof:

13.14

1	q	hyp
2	p	hyp
3	q	1, reit
4	p $\supset$ q	2-3, imp int

The derived rule of added condition may be used in any proof. Here is an example of a proof that uses the rule of added condition:

13.15

1	$[p \supset q] \supset [p \supset r]$	hyp
2	$p$	hyp
3	$q$	hyp
4	$p \supset q$	3, add cond
5	$[p \supset q] \supset [p \supset r]$	1, reit
6	$p \supset r$	4, 5, m p
7	$p$	2, reit
8	$r$	6, 7, m p
9	$q \supset r$	3-8, imp int
10	$p \supset [q \supset r]$	2-9, imp int

Notice that this proof is the converse of 13.2.

14. Negation

14.1 In order to make the present system of logic equivalent to the standard system of two-valued propositional logic, it is necessary to add two rules of direct consequence concerned with negation. The first of these two rules of direct consequence is called double negation elimination ( $\text{neg}_2 \text{ elim}$ ) and it permits us to treat 'p' as a direct consequence of its own double negation, ' $\sim\sim p$ '. The second of these two rules of direct consequence is called negation introduction ( $\text{neg int}$ ). It permits us to treat ' $\sim p$ ' as a direct consequence of a proof that has 'p' as its only hypothesis and that has two contradictory steps 'q' and ' $\sim q$ '. Here are examples of proofs that use these two further rules of direct consequence:

## 14.2

1	$\sim q$	hyp		
2	$p \supset q$	hyp		
3	<table style="border-collapse: collapse;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px;"><math>p</math></td> <td style="padding-left: 20px;">hyp</td> </tr> </table>	$p$	hyp	
$p$	hyp			
4	$p \supset q$	1, reit		
5	q	3, 4, m p		
6	$\sim q$	1, reit		
7	$\sim p$	3-6, neg int		

14.3 The proof in 14.2 amounts to a derivation of the rule of modus tollens (m t). According to this rule, we can

regard ' $\sim p$ ' as a consequence of ' $\sim q$ ' and ' $p \supset q$ '. (This is a derived rule, like trans imp and add cond, so we may speak of it as a rule of consequence instead of a rule of direct consequence.)

## 14.4

1	$q$	hyp
2	$\sim q$	hyp
3	<div style="border-left: 1px solid black; padding-left: 5px; padding-right: 10px;"><math>\sim p</math></div>	hyp
4	<div style="border-left: 1px solid black; padding-left: 5px; padding-right: 10px;"><math>q</math></div>	1, reit
5	<div style="border-left: 1px solid black; padding-left: 5px; padding-right: 10px;"><math>\sim q</math></div>	2, reit
6	$\sim\sim p$	3-5, neg int
7	$p$	6, neg <sub>2</sub> elim

This proof amounts to a derivation of the rule of negation elimination (neg elim). According to the rule, any sentence ' $p$ ' is a consequence of any pair of contradictory sentences, ' $q$ ' and ' $\sim q$ '.

## 14.5

1	$p$	hyp
2	<div style="border-left: 1px solid black; padding-left: 5px; padding-right: 10px;"><math>\sim p</math></div>	hyp
3	<div style="border-left: 1px solid black; padding-left: 5px; padding-right: 10px;"><math>p</math></div>	1, reit
4	$\sim\sim p$	2-3, neg int

This proof amounts to a derivation of the rule of double negation introduction (neg<sub>2</sub> int). According to this rule, the double negation, ' $\sim\sim p$ ', of any sentence, ' $p$ ', is a

consequence of the sentence ' $p$ ' itself.

14.6

1	$\sim[p \vee \sim p]$	hyp
2	$p$	hyp
3	$p \vee \sim p$	2, dis int
4	$\sim[p \vee \sim p]$	1, reit
5	$\sim p$	2-4, neg int
6	$p \vee \sim p$	5, dis int
7	$\sim[p \vee \sim p]$	1-6, neg int
8	$p \vee \sim p$	7, neg <sub>2</sub> elim

14.7 The proof in 14.6 amounts to a derivation of the rule of excluded middle (ex mid). According to this rule, a sentence of the form ' $p \vee \sim p$ ', may be asserted anywhere as a step in a proof (at least in those systems in which this principle holds). Like the derived rules trans imp and add cond, the derived rules m t, neg elim, and neg<sub>2</sub> int may be used anywhere in proofs in this system logic. Here is an example of a proof that uses neg elim:

14.8

1	$\sim p$	hyp
2	$p \vee q$	hyp
3	$p$	hyp
4	$\sim p$	1, reit
5	$q$	3, 4, neg elim
6	$q$	hyp
7	$q$	2, 3-5, 6, dis elim

14.9 Notice that the second of the two first-order proofs contained in 14.8 has only one step, namely its hypothesis, ' $q$ '. This may seem odd, but it gives us, as required, a proof that has ' $q$ ' as its only hypothesis, and that contains ' $q$ ' as a step, since both first-order proofs must end with ' $q$ ' in order to establish ' $q$ ' as a step of the main proof by use of dis elim. 14.8 amounts to a derivation of the rule of modus tollendo ponens (m t p) or, as it is sometimes called, the disjunctive syllogism. According to this rule, ' $q$ ' is a consequence of ' $\sim p$ ' and ' $p \vee q$ '. There are also other forms of m t p as follows:

' $q$ ' is a consequence of ' $\sim p$ ' and ' $q \vee p$ ' by m t p.

' $q$ ' is a consequence of ' $p$ ' and ' $\sim p \vee q$ ' by m t p.

' $q$ ' is a consequence of ' $p$ ' and ' $q \vee \sim p$ ' by m t p.

14.10 Another important derived rule is the De Morgan rule (d m). According to this rule each of the following eight pairs of propositions are such that the proposition on the left is a consequence of the corresponding proposition on the right, and, conversely, the proposition on the right is a consequence of the corresponding proposition on the left:

$\sim[p \ \& \ q]$	$\sim p \ \vee \ \sim q$
$\sim[\sim p \ \& \ q]$	$p \ \vee \ \sim q$
$\sim[p \ \& \ \sim q]$	$\sim p \ \vee \ q$
$\sim[\sim p \ \& \ \sim q]$	$p \ \vee \ q$
$\sim[p \ \vee \ q]$	$\sim p \ \& \ \sim q$
$\sim[\sim p \ \vee \ q]$	$p \ \& \ \sim q$
$\sim[p \ \vee \ \sim q]$	$\sim p \ \& \ q$
$\sim[\sim p \ \vee \ \sim q]$	$p \ \& \ q$

14.11 As an example, we now derive one case of the De Morgan rule, showing that ' $\sim p \ \vee \ \sim q$ ' can be proved from the hypothesis ' $\sim[p \ \& \ q]$ '. The other cases can be derived in a similar way.



1	$\sim[p \ \& \ q]$	hyp
2	$\sim[\sim p \ \vee \ \sim q]$	hyp
3	$\sim p$	hyp
4	$\sim p \ \vee \ \sim q$	3, dis int
5	$\sim[\sim p \ \vee \ \sim q]$	2, reit
6	$\sim\sim p$	3-5, neg int
7	$p$	6, neg <sub>2</sub> elim
8	$\sim q$	hyp
9	$\sim p \ \vee \ \sim q$	8, dis int
10	$\sim[\sim p \ \vee \ \sim q]$	2, reit
11	$\sim\sim q$	8-10, neg int
12	$q$	11, neg <sub>2</sub> elim
13	$p \ \& \ q$	7, 12, conj int
14	$\sim[p \ \& \ q]$	1, reit
15	$\sim[\sim p \ \vee \ \sim q]$	2-14, neg int
16	$\sim p \ \vee \ \sim q$	15, neg <sub>2</sub> elim