Combinatorial Complexity of Signed Discs

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Abstract

Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be two collections of topological discs of arbitrary radii. The collection of discs is 'topological' in the sense that their boundaries are Jordan curves and each pair of Jordan curves intersect at most twice. We prove that the region $\cup\mathcal{C}^+ - \cup\mathcal{C}^-$ has combinatorial complexity at most $10n-30$ where $p = |\mathcal{C}^+|$, $q = |\mathcal{C}^-|$ and $n = p+q \geq 5$. Moreover, this bound is achievable. We also show less precise bounds that are stated as functions of p and q.

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1 Introduction

Analysis of the combinatorial complexity of geometric or topological arrangements is often a prelude to the complexity analysis of algorithms. There have been several recent papers on the combinatorial complexity of arrangements of planar curves. In particular, Kedem et al. [4] shows that if $\mathcal{C}$ is a collection of $n \geq 0$ *topological discs* then the union $\cup\mathcal{C}$ of these topological discs has boundary complexity at most $\max\{n, 6n-12\}$, and that this bound is tight for all $n \geq 0$. A *geometric disc* is a standard disc which is defined by its center and radius. A *topological disc* is a bounded planar region whose boundary is a Jordan curve, called a *topological circle.* In an arrangement of topological circles, two distinct topological circles, like their geometric counterparts, may only intersect in at most two points.

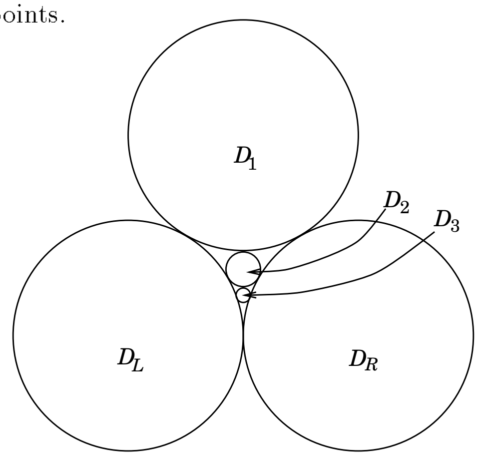

Figure 1. Basic construction.

We introduce the notation $\beta(n)$ to denote $\max\{n, 6n-12\}$. The construction to achieve the bound $\beta(n)$ for $n \geq 3$ is simple, even for geometric discs: begin by forming placing 3 unit discs whose boundaries intersect pairwise at two points each. See Figure 1, where the discs are labeled D_1, D_L, D_R. The three discs form an interstitial space in which we place a disc D_2: we make D_2 maximal in size so it touch the other 3 discs. We can now successively place maximal discs $D_3, D_4, \ldots, D_{n-2}$ into the interstitial spaces formed by the previous discs. It is not hard to verify the bound of $6n-12$ maximal arcs as well as the bound of $6n-12$ vertices where adjacent arcs meet. The lower and upper bounds for $n = 0, 1, 2$ are obvious. Clearly, the boundary complexity of the null set is null. The boundary of a single disc has 1 maximal arc and no vertices. If two discs intersect in exactly 2 points, the boundary consists of 2 maximal arcs and 2 vertices.

In this paper, we extend the result of [4] to a new situation, where the topological discs are colored either *positive* or *negative*. We want the determine the combinatorial complexity of the union of the positive discs minus the union of the negative discs. The literature has some results about bichromatic arrangements of discs, under the so-called "red-blue combination lemmas" [3,1], but these results do not apply to our situation which seems to be fundamentally different. Throughout this paper, the terms "disc", "circle", and "arc" mean "topological disc", "topological circle" and "topological arc" unless otherwise stated.

Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the collections of positive and negative discs, respectively. Let $p = |\mathcal{C}^+|$, $q = |\mathcal{C}^-|$ and $n = p + q \geq 1$. We are interested in the region

$$R = R(\mathcal{C}^+, \mathcal{C}^-) := \cup\mathcal{C}^+ - \cup\mathcal{C}^-$$

defined as the union of the positive discs minus the union of the negative discs. The boundary of a positive disc is called a *positive circle* and a connected portion of a positive circle is called a *positive arc*; similarly, we speak of *negative circles* and *negative arcs*. Clearly the boundary of R can be decomposed into a collection of *maximal (circular) arcs*; each maximal arc is either *positive* or *negative*. The *vertices* of R are the endpoints of these maximal arcs. It is worth noting that an intersection of a positive circle with a negative circle could appear on the boundary of R as a non-vertex.

For simplicity, we always make the following

> *Regularity Assumption*: two Jordan curves in a collection are either disjoint or intersect transversally (at two distinct points). Three Jordan curves do not pass through a common point.

In constructions where we say "two discs touch" it is understood that they should be slightly expanded to satisfy the regularity assumption.

It also turns out to be more convenient to count vertices rather than maximal arcs. The number of maximal arcs in an arrangement is not necessarily the same as the number of vertices: a positive disc that does not intersect any other discs gives rise to one maximal arc but no vertices. But if there is more than one disc, then we may increase the number of maximal arcs by modifying such a disc to intersect other discs. Since we are interested in the largest possible number of maximal arcs, we may assume the number of maximal arcs equals the number of vertices in our arrangements provided $p \geq 1, q \geq 1, n = p + q \geq 2$.

Let $B(\mathcal{C}^+, \mathcal{C}^-)$ denote the *combinatorial complexity* of the boundary of R, i.e., the number of maximal arcs of R. Let $B(p, q)$ denote the maximum value attained by $B(\mathcal{C}^+, \mathcal{C}^-)$ over all choices of $\mathcal{C}^+, \mathcal{C}^-$ where $p = |\mathcal{C}^+|, q = |\mathcal{C}^-|$. We have

$$B(0, q) = \beta(0) = 0, \qquad B(p, 0) = \beta(p)$$

The first result is obvious and the second is a restating of the result of [4]. Henceforth, we assume that $p \geq 1$ and $q \geq 1$ in considering $B(p, q)$.

Let $B(n)$ be the maximum value attained by $B(p,q)$ where $n = p + q$. We always assume $n \geq 2$ in this context. Let $B^*(p,q), B^*(n)$ be analogously defined for geometric discs. Clearly,

$$B^*(p,q) \leq B(p,q), \qquad B^*(n) \leq B(n).$$

The first inequality is an equality when $q = 0$ (from [4]) and when $p = 0$ (obviously). It is an open question as to whether the first inequality is strict for some values of p, q.

Applications. Goodrich and Kravets [2] studied the problem of matching a pattern P to a background B where P, B are finite point sets in the plane. An obvious scenario for this is in astronomy. The problem is to translate P so as to obtain an ϵ-match with a subset of B. Let us define B_ϵ to be the union of ϵ-discs about points of B. For each p, the set of translation vectors that puts p inside B_ϵ is denoted $T(p)$. So $T(p) = B_\epsilon - p$. They were interested in constructing the intersection of all $T(p)$'s as p range over P. Suppose now we have certain circular regions that must NOT be matched to p. (This could be epsilon-discs about points of B.) Then $T(p)$ becomes a region of the form $R(\mathcal{C}^+, \mathcal{C}^-)$.

We can also reformulate our problem as follows. Suppose we have a collection of opaque planar discs. Each disc is colored black or white. We place each black disc on the plane $z = 1$ and each white disc on the plane $z = 0$. Now we view the configuration from an infinite distance, vertically above the discs. Then the complexity of the white region we see is $B(\mathcal{C}_{white}, \mathcal{C}_{black})$.

2 A $10n$-Lower Bound Construction for $B(n)$.

The reader might be tempted to conclude from the result of [4] that $B(n) \leq \beta(n)$. This proves to be false:

Lemma 1 *For all $q \geq 1$,*

$$B(2,q) \geq \max\{6q, 10q - 10\}.$$

Hence $B(n) \geq 10n - 30$ for all $n \geq 3$.

This lemma can be seen directly if $q \leq 2$ or $n \leq 4$. (In particular, the reader should verify that $B(2,2) \geq 12$.) So assume $q \geq 3$ and $n \geq 5$. This result follows from an explicit construction. We begin with geometric discs. See Figure 2. Let A^+, A^- be two very large discs of equal radii whose centers lie on the positive y-axis, and such that their south-poles passes through the points $(0, 1-\varepsilon)$ and $(0,1)$, respectively. Here $0 < \varepsilon \ll 1$. Moreover their radii are $\gg 1$ so that the southernmost portions of their boundaries are well-approximated by the horizontal lines $y = 1 - \varepsilon$ and $y = 1$, respectively.

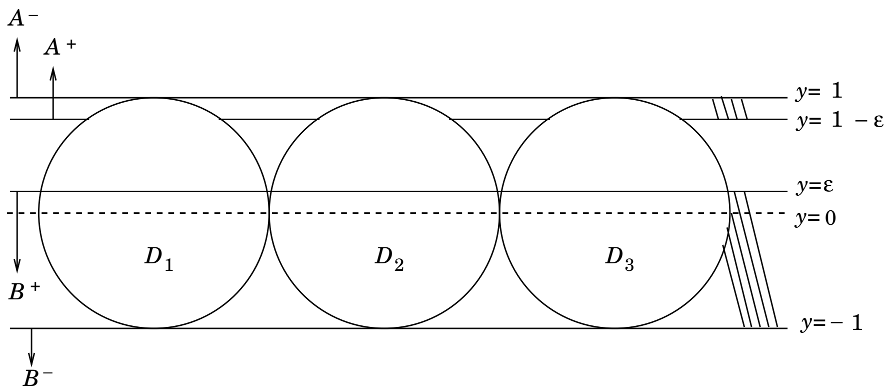

Figure 2. Arrangement with large complexity for R.

Similarly, let B^+, B^- be two very large discs with equal radii whose centers lie on the negative y-axis and whose north-poles pass through the points $(0, \varepsilon)$ and $(0, -1)$. Again their large radii implies that the northernmost portions of their boundaries are well-approximated by the horizontal lines $y = \varepsilon$ and $y = -1$, respectively.

Let $m = q - 2 \geq 1$. Let $D_1, D_2, \ldots, D_m$ be discs of radii $1 + \varepsilon^2$, and whose centers lie on the x-axis, laid out in a row centered about the origin. Each D_i touches A^- and B^-; it also touches D_{i-1} (provided $i \geq 2$) and D_{i+1} (provided $i \leq m - 1$).

Let $\mathcal{C}^+ = \{A^+, B^+\}$ and $\mathcal{C}^- = \{A^-, B^-, D_1, \ldots, D_m\}$. This gives

$$B^*(\mathcal{C}^+, \mathcal{C}^-) = 10(m - 1) + 12 = 10n - 38. \tag{1}$$

Distortion. We now distort the above construction so that the discs are no longer geometric. Let the discs A^- and A^+ form the vertices a_L and a_R that are to the left and right (respectively) of the discs $D_1, \ldots, D_m$. Similarly, the discs B^- and B^+ form the vertices b_L and b_R. Refer to Figure 3.

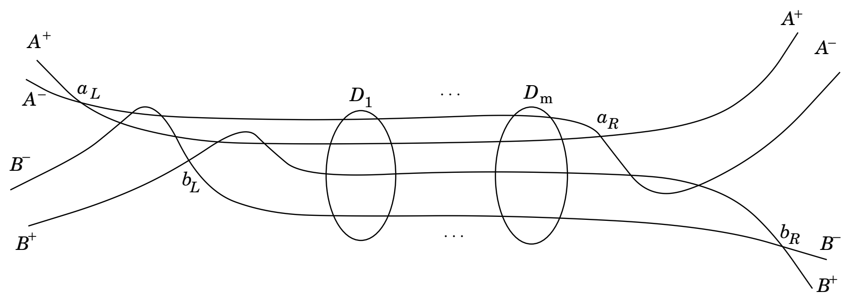

Figure 3. Distortion of construction.

(i) First we "tilt" A^+ and B^+ so that they intersect to form two new vertices ab_1 and ab_2. The new vertices lie to the right of a_L and b_L but to the left of the D_i's.

(ii) Add a protrusion to B^- to the left of b_L so as to intersect both A^- and A^+ to the right of a_L and left of ab_1, ab_2 to form four new vertices.

(iii) Add a protrusion to A^- to the right of a_R to intersect B^+ in two new vertices. These two vertices are to the right of a_R and left of b_R.

We have added a total of 8 new vertices, improving Equation (1) to

$$B(\mathcal{C}^+, \mathcal{C}^-) = 10(m-1) + 20 = 10n - 30.$$

The following will be useful later, and it follows from a simple extension of the above construction.

Lemma 2 *For all $q \geq 1$,*

$$B(3, q) \geq \max\{8q + 4, 10q - 2\}.$$

Proof. For $q = 1$, it is easy to achieve boundary complexity 12. The reader should verify that $B(3, 2) \geq 20$. For $q \geq 3$, we need to take the corresponding construction from the lemma above and insert a single positive circle which increases the boundary complexity by 8. Any positive circle which touches D_i, D_{i+1}, A^+ and B^+ without covering any extant vertices achieves this increase. **Q.E.D.**

In fact, we could insert up to $q - 3$ additional positive circles, where each added circle yields 8 new vertices. This proves for $p \geq 3, q \geq 3$,

$$B(p, q) \geq 8p + 10q - 26 \tag{2}$$

provided $p \leq q - 1$.

3 A Simple Upper Bound for $B(n)$

In order to prove upper bounds for $B(n)$ and $B(p,q)$ we classify vertices of the region $R = R(\mathcal{C}^+, \mathcal{C}^-)$ into three types (see Figure 4):

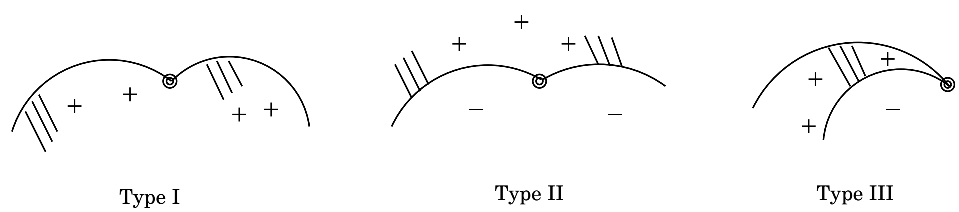

Figure 4. Three Types of Vertices.

A *type I* vertex is one incident on two positive arcs of R; a *Type II* vertex is one incident on two negative arcs of R; a *Type III* vertex is one incident on a negative and a positive arc.

It is clear that the number of type I vertices is at most $\beta(p)$ using the result of [4]. Similarly, the number of type II vertices is at most $\beta(q)$. The next section gives an upper bound on the number of type III vertices. But for now, we make a simple observation that leads to a non-trivial upper bound on $B(n)$:

> *CLAIM: the number of type I and type III vertices combined is at most $\beta(p+q)$.*

To see this, let $U = (\bigcup \mathcal{C}^+) \cup (\bigcup \mathcal{C}^-)$ be the union of the positive and negative discs. The claim follows because there is a natural injection (1-1 map) from the set of types I and III vertices to the vertices of U.

It follows that

$$\begin{aligned} B(p,q) &\leq \beta(q) + \beta(p+q) \\ &= \begin{cases} 6p + 12q - 24 & \text{if } q \geq 3 \\ 6p + 7q - 12 & \text{if } q = 2 \\ 6p + 6q - 12 & \text{if } q = 1, p \geq 2 \\ p + q & \text{if } q = 1, p = 1 \end{cases} \end{aligned} \quad (3)$$

Since we assume that p and q are both positive, we may further deduce that

$$B(n) \leq 12n - 28, \qquad n \geq 3.$$

This upper bound leaves a substantial gap between it and the $10n$-lower bound from the last section. In order to eliminate that gap, we first need to establish some exact bounds on $B(p,q)$, especially for small values of p,q.

Lemma 3

$$
\begin{aligned}
B(1,q) &= \beta(1+q), && (4)\\
B(2,q) &= \max\{6q, 10q-10\}, && (5)\\
B(p,1) &= 6(p+q)-12, \qquad (p\geq 3), && (6)\\
B(p,2) &= 6(p+q)-10, \qquad (p\geq 3), && (7)\\
B(p,q) &= 6p+12q-24, \qquad (p\geq 3, 3\leq q\leq p-1), && (8)
\end{aligned}
$$

(i) $B(1,q)$: To prove the upper bound, assume you are given an arrangement of q negative circles and 1 positive circle C_0. Suppose they define the usual region $R=\cup\mathcal{C}^+ - \cup\mathcal{C}^-$ with maximum complexity $B(1,q)$. We transform this arrangement as follows. First turn all the q negative circles into positive ones. Next, turn the circle C_0 inside out, i.e., the exterior of C_0 is now regarded as a positive region R_0 (such a region is not quite a disc, but we fix it below). Let R^* be the union of these $1+q$ positive regions. Note that R^* is precisely the complement of R and has exactly the same boundary. If we can transform R_0 into a disc without changing the boundary complexity of R^* then we achieve the upper bound $B(1,q)\leq B(1+q,0)=\beta(1+q)$, as desired.

We next transform R_0 into a disc. This is simple if C_0 (the boundary of R_0) contains a point v that is exterior to the union of the q negative discs. In this case, we enclose all the $q+1$ discs inside a large circle C^*. Then we connect v to some point on C^* along a path π. We can create a new disc by cutting the annular region between C_0 and C^* along π. This new arrangement of $q+1$ discs has the same boundary complexity $B(1,q)$ as before, but it is now clear that that $B(1,q)\leq\beta(1+q)$. Suppose that no such v exists. But then every point on C_0 is contained within at least one negative disc. Consequently, at least two negative circles intersect outside of C_0 and outside of all other negative discs. The complexity of R would be increased if C_0 were expanded to cover this intersection point, contradicting the fact that R has maximum complexity.

The lower bound is trivial for $q=1,2$ and straightforward for $q\geq 3$: Form an arrangement of q negative discs whose (unsigned) complexity is $\beta(q)$. It is easy to add a positive disc to cover all the vertices of this arrangement so as to get $\beta(q)$ type II vertices while also touching 3 negative circles to produce 6 type III vertices.

(ii) $B(2,q)$: The lower bound appeared in section 2. As for the upper bound, we may dispose of the case where $q=1$ and $q=2$ directly: the bounds here are $B(2,q)=6q$. For $q\geq 3$, we need to show $B(2,q)\leq 10q-10$. There are, however, at most 2 type I vertices, at most $\beta(q)$ type II vertices, and[2] at most

[2]This bound on the number of type III vertices is given as the main result in the next section.

$4q$ type III vertices. Since $\beta(q) = 6q - 12$ for $q \geq 3$, the sum is $10q - 10$.

(iii) $B(p, 1)$, $p \geq 3$: The upper bound results from the fact that there are no type II vertices, and the number of types I and III is at most $6(p+1) - 12$ as argued above. This proves $B(p, 1) \leq 6(p+1) - 12$. Moreover, an arrangement achieving this bound is easily seen to be achievable.

(iv) $B(p, 2)$, $p \geq 3$: For the upper bound, there are at most 2 type II vertices, and at most $6(p+2) - 12$ types I and III vertices. Hence

$$B(p, 2) \leq 6(p+2) - 10.$$

We show that this bound is achieveable. First we do a construction with geometric discs: begin with the geometric construction of Figure 1 in the introductory section where we have three equal-sized discs D_L, D_R, D_1 in a mutually touching arrangement, and D_2, D_3 are maximal discs placed in interstitial spaces. We may assume that D_1, D_2, D_3 have centers on the y-axis. Assume that D_L, D_R, D_2 are positive, D_1, D_3 are negative. This arrangement yields $6(p+q) - 12 = 6(3+2) - 12$ vertices. Now, we turn D_1, D_3 into topological discs by slightly enlarging their intersections with D_2, then "protruding" this enlarged intersection until the center of D_2 is covered by D_1 and D_3. Note that we have now now created two new type II vertices without affecting the types I and III vertices. This proves $B(3, 2) = 6(3+2) - 10$.

We can extend this construction for all $p \geq 3$ as following: for each additional positive disc, we put it into an interstitial space defined by three discs (the signs of these discs is immaterial), and touching all three. This prove $B(p, 2) = 6(p+2) - 10$.

(v) The upper bound has been established as part of Equation (3). To prove the lower bound, we expand the example used in (iv). We first assume that $p - 1 = q \geq 3$. Start with the diagram D_L, D_R, D_1, D_2, D_3. Now add discs $D_4, D_5, D_6, \ldots, D_{n-2}$ such that D_i is in the institial space defined by D_L, D_R and D_{i-1} but touching all three. D_{2i} is positive and D_{2i-1} is negative, $i \geq 1$. At the outset, when all discs are geometric, the complexity is $6(p+q)-12$. Then deform D_{2i-1} and D_{2i+1} as explained in part (iv) to cover the center of D_{2i}, introducing $2(q-1)$ new type II vertices. Now deform the intersection of D_5 with D_L to connect with the intersection of D_L and D_1 adding 2 more vertices. Now deform the intersection of D_{4i+3} (D_{4i+5}, respectively) with D_R (resp. D_L) to intersect the intersection of D_R (resp. D_L) with D_{4i-1} (resp. D_{4i+1}) and also to intersect the intersection of D_R (resp. D_L) with D_1, thus adding $4(q-3)$ intersection points. Thus the total complexity of the arrangement is:

$$\begin{aligned} B(p, q) &\geq [6(p+q) - 12] + [2(q-1) + 2 + 4(q-3)] \\ &\geq [6p + 6q - 12] + [6q - 12] \\ &\geq 6p + 12q - 24. \end{aligned}$$

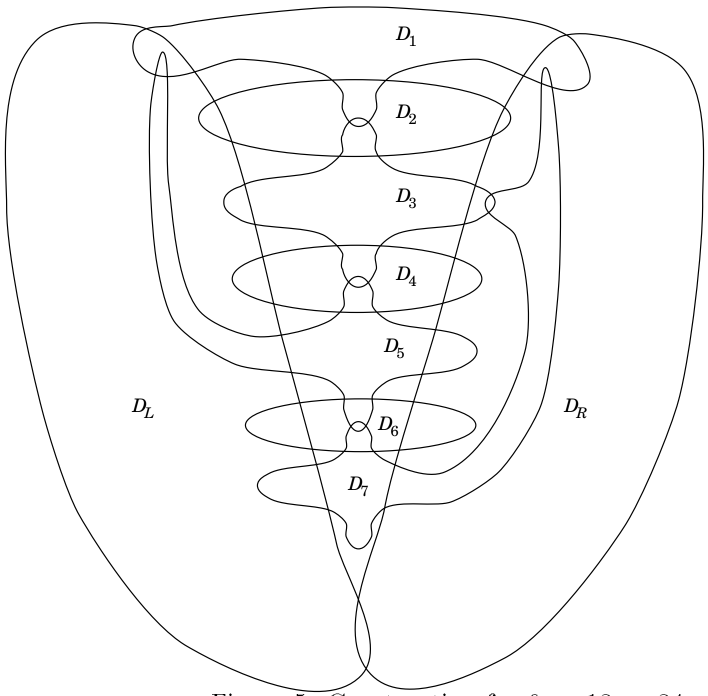

Figure 5. Construction for $6p + 12q - 24$.

In case $p-1 > q$, we first do the above construction with $q+1$ positive discs and q negative discs. Then we add the remaining $p - q - 1$ positive discs, with each additional disc contributing 6 new vertices. This concludes the proof of the lemma.

4 Exact bound for $B(n)$

The main result of this section will be an exact bound for $B(n)$. To achieve this result, we need a tighter bound on the number of type III vertices in an arbitrary arrangement.

Lemma 4 (Main Lemma)
(i) In any arrangement of $p + q \geq 3$ discs where $p \geq 1$ are positive and $q \geq 1$ are negative, the number of type III vertices is at most

$$4(p + q) - 8.$$

In case $p = 1$ or $q = 1$, this upper bound can be improved to $2(p + q) - 2$.
(ii) These upper bounds are achievable for all p, q.

First we prove the upper bound (part (i)). It is sufficient to consider only collections of signed discs $\mathcal{C}^+, \mathcal{C}^-$ that maximize the number of type III vertices for given $p = |\mathcal{C}^+|, q = |\mathcal{C}^-|$. Given that a type III vertex occurs only when a positive circle intersects a negative circle at a point v which does not lie in the interior of any disc (positive or negative), no disc D of the collection can be contained in the union of the remaining discs of $\mathcal{C}^- \cup \mathcal{C}^-$.

The proof has two stages. In the initial stage, we prove the Main Lemma under the the following *simplifying assumption*. The *covering number* of a point v in the plane is the number of discs in $\mathcal{C}^+ \cup \mathcal{C}^-$ that contains v.

> **Simplifying assumption.** No point in the plane has covering number greater than two.

In the second stage, we prove that any collection of signed discs can be transformed by shrinking each disc until the collection satisfies our simplifying assumption. Moreover, this transformation does not decrease the number of type III vertices.

Stage 1. Under the simplying assumption, for each disc D in $\mathcal{C}^+ \cup \mathcal{C}^-$, we may pick a point $v(D)$ in the interior of D such that $v(D)$ has covering number 1. We then define an *embedded* planar graph G as follows: the vertices of G are the points $v(D)$; an edge $e(D, D')$ consisting of a simple curve in $D \cup D'$ connects $v(D)$ and $v(D')$ in the graph G if and only if D and D' intersect; no two edges intersect except in sharing a common endpoint; and $e(D, D')$ avoids all other discs in $\mathcal{C}^+ \cup \mathcal{C}^-$. An edge $e(D, D')$ is *dichromatic* if D and D' have different signs. The number of type III vertices is exactly twice the number of dichromatic edges. Thus, part (i) of the Main Lemma amounts to the following result about graphs:

Lemma 5 *Given an embedded planar graph G on $n = p + q \geq 3$ vertices where $p \geq 1$ are colored positive and $q \geq 1$ are colored negative, the number of dichromatic edges is at most $2(p+q)-4$. If $p = 1$ or $q = 1$, the bound can be improved to $p + q - 1$.*

Proof. The improved bound when $p = 1$ or $q = 1$ is trivial; hence assume $p \geq 2, q \geq 2$. Since G is embedded in the plane, it defines a finite set $S(G)$ of connected open regions; all these regions are simply-connected and bounded, with the exception of a unique unbounded, non-simply-connected region. By introducing additional edges if necessary, we may assume that each connected region (including the infinite region) is bounded by exactly three edges. Each edge e gives rise to two *borders*, i.e., pairs of the form (e, c_1) and (e, c_2) where $c_1 \neq c_2$ are the two connected regions separated by e. Since G has $3n - 6$ edges, there are $6n-12$ borders. For any region $c \in S(G)$, at most 2 of the three borders of c are dichromatic. Hence there are at most $2(6n-12)/3 = 4n-8$ dichromatic borders. This translates to at most $2n - 4$ dichromatic edges, **Q.E.D.**

Stage 2. It remains to justify the simplifying assumption. We do this by giving a terminating procedure to transform the discs in $\mathcal{C}^+ \cup \mathcal{C}^-$ such that the number of type III vertices is not decreased, although it may increase. At termination, the arrangement will satisfy our simplifying assumption. First, note that we can assume that the number of type III vertices is maximized for the given number of positive and negative discs. This clearly implies that the boundary of each disc intersects the boundary of some other discs.

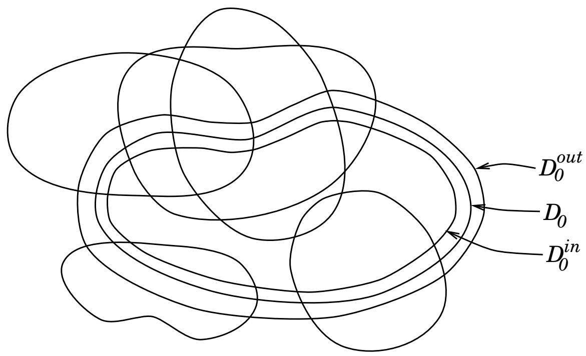

Figure 6. The annular regions for D_0.

In the following, D_i denotes a disc with boundary C_i, for various indices i. Now fix some disc D_0 and define the arcs $\alpha_i := C_0 \cap D_i$. Suppose the non-empty arcs are α_i for $i = 1, \ldots, k$. We will describe how to "shrink" each D_i for $i = 1, \ldots, k$. (By shrinking a disc, we mean that the modified disc is contained in the original disc.) To do this, we need two auxiliary discs,

$$D_0^{in}, D_0^{out}$$

such that $D_0^{in} \subseteq D_0 \subseteq D_0^{out}$, and both $D_0^{out} - D_0$ and $D_0 - D_0^{in}$ are annular regions such that each $C_1, \ldots, C_k$ intersects C_0^{in} (respectively, C_0^{out}) transversally at two points, and no two C_i's intersect inside $D_0^{out} - D_0^{in}$.

The two properties can be ensured by making the annular regions sufficiently narrow. Now we shrink each D_i in turn. There are two cases:
(i) If α_i is contained in the union of the remaining α_j's then we shrink D_i while holding $D_i - D_0^{out}$ invariant, until $D_i \cap D_0 = \emptyset$.
(ii) Otherwise, we shrink D_i while holding $D_i - D_0$ invariant, until $D_i \cap D_0^{in} = \emptyset$.
In case (i), the new arc α_i becomes empty; in case (ii), α_i is unchanged.

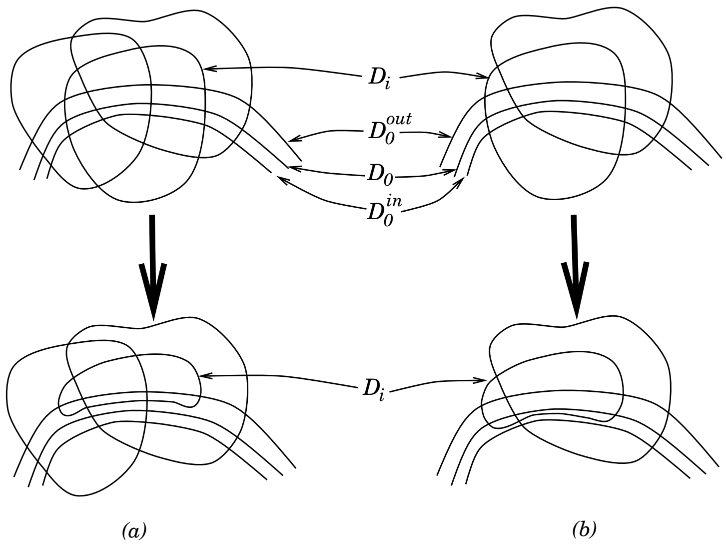

Figure 7. Shrinking disk D_i: the two cases.

Moreover, the union of the new α_j's equal the union of the old α_j's. Also, at the end of this shrinking process, we get $D_i \cap D_j \cap D_0 = \emptyset$ if and only if $\alpha_i \cap \alpha_j = \emptyset$. Of course, we have removed intersections between many pairs of circles. But since each removed intersection is covered by some other disc, they could not possibly be type III vertices. We conclude that the number of type III vertices is unchanged. For reference, call this process *cleaning-up* of D_0.

We repeat this cleaning-up process for each choice of $D_0 \in \mathcal{C}^+ \cup \mathcal{C}^-$. At the end of this, we have the property that no point in the plane has covering number strictly greater than 3. [In proof: suppose $D_0 \cap D_1 \cap D_2 \cap D_3$ is non-empty and the boundary of one of these discs (say, D_0) contains a point of $D_0 \cap D_1 \cap D_2 \cap D_3$. Then the arcs $\alpha_1, \alpha_2, \alpha_3$ (defined as before) have a point in common. This implies that one of these arcs (say, α_1) is contained in the union of the other two ($\alpha_2 \cup \alpha_3$). This contradicts a basic property when we shrank D_1 under case (i).]

Our final goal is to ensure no point is triply-covered. So suppose the triple intersection $D_0 \cap D_1 \cap D_2$ is non-empty. There are basically two cases to consider, as shown in Figure 8:

(a) Suppose the boundary of $D_0 \cap D_1 \cap D_2$ is composed of arcs from only two of the three discs. Without loss of generality, let $D_0 \cap D_1$ be contained in D_2. We then shrink D_0 so that $D_0 \cap D_1 \cap D_2$ becomes empty as in the figure. It is easy to verify that regardless of the signs of the discs, the number of type III vertices

is unchanged. [Note: actually, this case disappears after we do a cleaning-up of D_0 or D_1.]
(b) Suppose the boundary of $D_0 \cap D_1 \cap D_2$ is composed of arcs from all three discs. We can shrink D_0 so that $D_0 \cap D_1 \cap D_2$ becomes empty, and exactly three new vertices appear in the union $D_0 \cup D_1 \cup D_2$, as in Figure 8. Depending on signs of these three discs, number of type III intersections is either unchanged or increased by two.

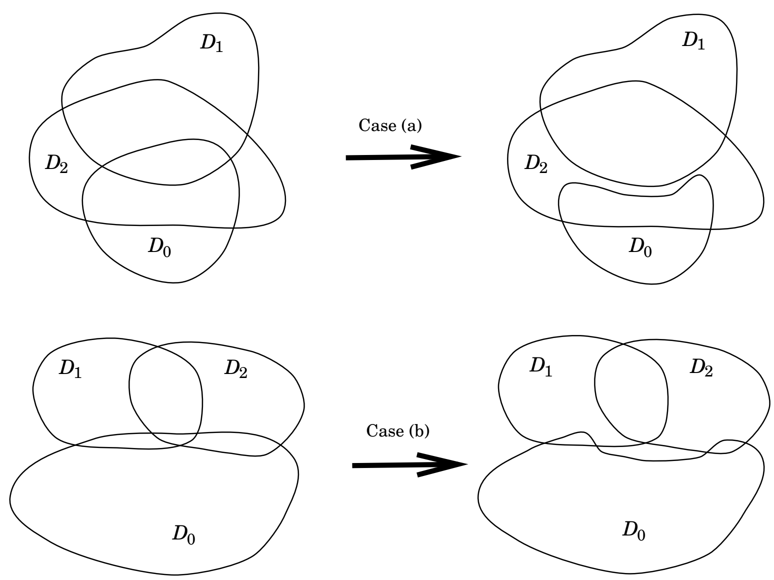

Figure 8. Removing a triple intersection: two cases.

Since this modification does not increase the covering number of any point, this process can be repeated until no point has covering number more than two. This concludes our justification of the simplifying assumption.

Remark: It is interesting to note the present argument is inherently an inductive argument for topological discs. So it is a case of strengthening the induction hypothesis. A similar argument, in which one repeatedly transforms an arrangement of topological discs, appeared in [4]. It seems difficult here to find transformations to ensure that the number of vertices do not decrease (which is a property in [4]). Indeed, our transformations may decrease the number of type II vertices; but they do not decrease the number of type I vertices.

Lower bound on the number of Type III vertices. To show part (ii) of the Main Lemma, we construct an arrangement of signed discs with $4(p+q)-8$ Type III vertices, for all $p \geq 2, q \geq 2$. (The case where $p=1$ or $q=1$ is trivial.) First introduce two positive discs, D_L and D_R, with unit radii and centered at $(-1,0),(1,0)$, respectively. See Figure 1 from an earlier construction. Now introduce q negative discs

$$D_1, D_2, \ldots, D_q,$$

all centered on the positive y-axis, such that each disc D_i touches both D_L and D_R, and also touches D_{i-1} (provided $i \geq 2$) and touches D_{i+1} (provided $i \leq q-1$). Clearly this creates $4q$ type III vertices.

Finally, introduce $p-2$ positive discs

$$D'_1, \ldots, D'_{p-2}$$

such that each D'_i touches both D_1 and D_2, also touches D'_{i-1} (provided $i \geq 2$) and touches D'_{i+1} (provided $i \leq p-3$). This creates another $4(p-2)$ type III vertices.

Thus we have a total of $4(p+q)-8$ type III vertices.

Main Result. We are ready to show the main result of this paper:

Theorem 6 *For $n \geq 5$, $B(n) = 10n - 30$.*

Proof. The lower bound $B(n) \geq 10n-30$ has been shown in section 2. As for the upper bound, we have several cases: When $p=0$, we check that $B(n) = B(0,q) = 0 \leq 10n-30$. When $q=0$, $B(n) = B(p,0) = \max\{p, 6p-12\} = 6n-12 < 10n-30$. When $p=1$, by Lemma 3, $B(n) = \beta(n) = 6n-12 < 10n-30$. When $q=1$, the same lemma gives $B(n) = 6n-12$.

Finally assume $p \geq 2$ and $q \geq 2$. Since $p+q \geq 5$, either p or q exceeds 2. Bounding the number of types I, II and III vertices separately as usual, we get

$$\begin{aligned} B(p,q) &\leq \max\{p, 6p-12\} + \max\{q, 6q-12\} + 4(p+q) - 8 \\ &\leq \begin{cases} 10(p+q)-32 & \text{if } p \geq 3, q \geq 3 \\ 10p+5q-20 & \text{if } p \geq 3, q = 2 \\ 5p+10q-20 & \text{if } p = 2, q \geq 3 \end{cases} \\ &\leq \begin{cases} 10(p+q)-32 & \text{if } p \geq 3, q \geq 3 \\ 10(p+q)-30 & \text{if } p = 2 \text{ or } q = 2 \end{cases} \qquad (9) \\ &\leq 10(p+q)-30. \end{aligned}$$

Q.E.D.

5 Bounds for $B(p,q)$

We seek bounds on $B(p,q)$ that are exact, up to additive constants. Unlike the case of $B(n)$, the picture here is incomplete, even if we restrict p or q to be sufficiently large. Here is what we know: from Lemma 3, we have exact bounds for $B(p,q)$ when $p \le 2$ or $q \le 2$ and when $3 \le q \le p-1$. Combining Lemma 2 with Equation (9), we get another the exact bound,

$$B(3,q) = 10q - 2, \qquad q \ge 3. \tag{10}$$

The difficulty in getting tight general bounds stems from the fact that we must use different techniques depending on the relative sizes of p and q. For instance, we get the following hybrid upper bound by combining Lemma 3 and Equation (9):

Lemma 7 *For $p \ge 3, q \ge 3$,*

$$\begin{aligned} B(p,q) &\le 6p + 10q + \min\{4p-32, 2q-24\} \\ &\le \begin{cases} 10(p+q) - 32 & \text{if } 2p-4 \le q \\ 6p + 12q - 24 & \text{if } 2p-4 \ge q \end{cases} \end{aligned}$$

One suspects that the bound at the cross-over value, $q = 2p-4$, is not tight.

Similarly, one can derive lower bounds using hybrid constructions:

$$B(p,q) \ge 6p + 10q - 22, \qquad p \ge 5, q \ge 3.$$

This inequality comes from

$$B(p,q) \ge B(2,q) + B(p-2,0) = (10q-10) + (6p-12),$$

corresponding to doing the $10n$-construction to achieve the bound $B(2,q)$, combined with the $6n$-construction for the remaining $p-2$ positive discs. For $3 \le p \le q-1$, Equation (2) provided a somewhat better lower bound of

$$B(p,q) \ge 8p + 10q - 26.$$

We can extend the bound of Equation (8) in Lemma 3 somewhat. Let's take the construction in that proof, and revise it one more time. Counterintuitively, we begin by doubling the number of positive circles. Previously, we had the discs $D_L, D_R, D_1, D_2, \ldots, D_{2k}, D_{2k+1}$, where the negative discs are D_{2i+1} ($i = 0, \ldots, k$). We now duplicate each D_{2i} ($i = 1, \ldots, k$), denoting the new discs by D_{2i}^a and D_{2i}^b. Now there are roughly twice as many positive circles as negative ones. These discs are initially geometric. With the exception of D_L, D_R, the rest of the discs are centered on the y-axis and linearly ordered as

$$D_1, D_2^a, D_2^b, D_3, D_4^a, D_4^b, \ldots, D_{2k}^a, D_{2k}^b, D_{2k+1}.$$

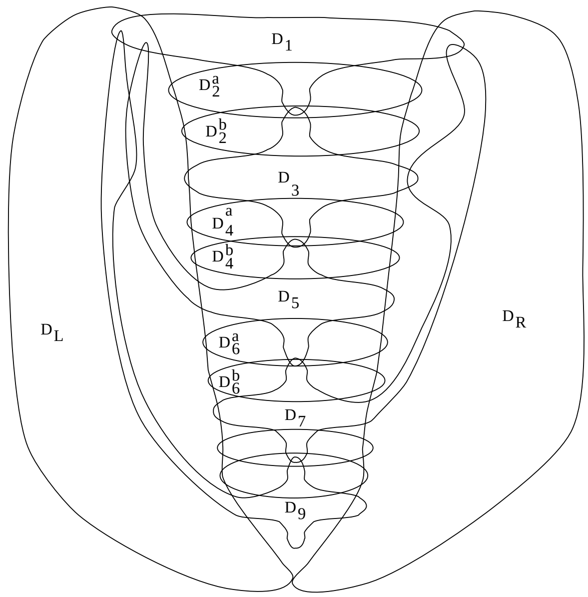

Figure 9. Doubling the number of positive circles.

Moreover, each D_x touches D_L, D_R and its predecessor (unless $x = 1$) and its successor (unless $x = 2k + 1$). The initial complexity is $6(p + q) - 12$, where $p = 2k + 2$ and $q = k + 1$. Then deform D_{2i-1} and D_{2i+1} as explained in the proof of Lemma 3 to cover some point in the intersection of D^a_{2i} and D^b_{2i}, introducing $2(q - 1)$ new vertices. Now deform the intersection of D_5 with D_L to connect with the intersection of D_L and D_1 adding 2 more vertices. Now deform the intersection of D_{4i+3} (D_{4i+5}, respectively) with D_R (resp. D_L) to intersect the intersection of D_R (resp. D_L) with D_{4i-1} (resp. D_{4i+1}) and also to intersect the intersection of D_R (resp. D_L) with D_1, thus adding $4(q - 3)$ intersection points. Thus the total complexity of the arrangment is:

$$[6(p+q)-12]+[2(q-1)+2+4(q-3)] = [6p+6q-12]+[6q-12] = 6p+12q-24,$$

the same figure arising in Lemma 2.

Now we start adding more negative discs. In each interstitial space formed by forgetting the presence of D_L and bounded by $D_{4i-3}, D^a_{4i-2}, D^b_{4i-2}$, D_{4i-1}, D^a_{4i}, D^b_{4i}, D_{4i+1}, for $i = 1$ to $k/2$, insert a topological disc D^a_{Li} which intersects D_{4i-3} and D_{4i+1} in two vertices each inside D_L, which then intersects D_L in two vertices in between D^a_{4i-2} and D^b_{4i-2}, which intersects D^a_{4i-2} in two vertices, and which intersects D^b_{4i-2} in two vertices, but also intersects D_{4i-1} in two vertices while it is inside D^b_{4i-2}. Now insert a topological disc D^b_{Li} which intersects D^a_{Li} and D_{4i+1} in two vertices each inside D_L, which intersects D_L in two vertices in between D_{4i}^a and D_{4i}^b, which intersects D_{4i}^b in two vertices, and which intersects D_{4i}^a in two vertices, but also intersects D_{4i-1} in two vertices while it is inside D_{4i}^a.

Repeat symmetrically for the interstitial spaces, forgetting D_R, bounded by $D_{4i-1}, D_{4i}^a, D_{4i}^b, D_{4i+1}, D_{4i+2}^a, D_{4i+2}^b, D_{4i+3}$, for $i = 1$ to $k/2 - 1$.

In $k-1$ spaces, we have inserted 2 new negative discs each having complexity 12, thus maintaining the total complexity $6p+12q-24$ over a broader spectrum of values of q. Since $p = 2k+2$ and $q \leq 3k-1$, the last part of Lemma 3 can be extended to read:

$$B(p,q) = 6p + 12q - 24, \qquad (p \geq 3, \quad 3 \leq q \leq 1.5p - 4).$$

Unfortunately, we leave a gap for $B(p,q)$ when $1.5p - 4 \leq q \leq 2p - 4$.

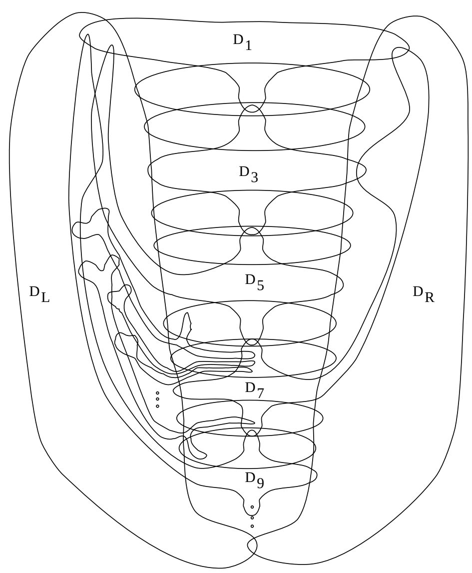

Figure 10. Tight lower bound for $B(p,q)$ for $p/2 \leq q \leq 1.5p - 4$

Another goal is to achieve a lower bound to match the upper bound

$$B(p,q) \leq 10(p+q) - 32, \quad (2p - 4 \leq q).$$

How close can we get to that goal by expanding the previous figure? That figure achieves a complexity of $9.6(p+q) - O(1)$ for $q = 1.5p - 4$. Let's start adding more negative discs. Unfortunately, the highest complexity negative disc which we can insert into the current construction is of complexity 10. But we can insert arbitrarily many of them! Insert a disc D_{Li}^c which intersects both

D^a_{Li} and D^b_{Li} inside D_L, which intersects D_L between D_{4i-1} and D^b_{4i-2}, which intersects D^b_{4i-2} and also intersects D_{4i-1} while inside D^b_{4i-2}. The next insertion D^d_{Li} has the same pattern, but intersects D^c_{Li} instead of D^a_{Li}. This process can be iterated arbitrarily often, increasing the average complexity of a single disc from 9.6 to arbitrarily close to 10.

It remains an open problem to find a concrete example which for arbitrary values of p and for $q \geq 2p - 4$ can achieve the bound of $10(p+q) - 32$.

6 Final Remarks

1. In the introduction, we view the signed disc problem as placing the discs in two parallel planes in 3-space: each disc is colored either black or white; each black disc lies in the plane $z = 1$ and each white disc in the plane $z = 0$; we view the configuration from an infinite distance vertically above the discs and compute the complexity of the visible white region. Suppose instead that we have a total of $k \leq n$ distinct horizontal planes and the n discs are placed on these planes (but no plane may have both black and white discs). What is the complexity $B_k(n)$ of the white region viewed from vertically above?

2. Although we have focused on topological discs, geometric discs are interesting as well. Note that from section 2, we know that

$$B^*(n) = 10n - c$$

where c is a constant between 30 and 36. The most pressing problem is to give an explicit instance such that $B^*(p, q) < B(p, q)$. According to Lemma 3,

$$B(2,2) = 12, \quad B(2,3) = 20, \quad B(2,4) = 30.$$

It is not obvious from the construction of section 2 hat these bounds can be achieved geometrically. But it turns out that they are indeed geometric. For example, Figure 11 proves that $B^*(2, 3) = 20$.

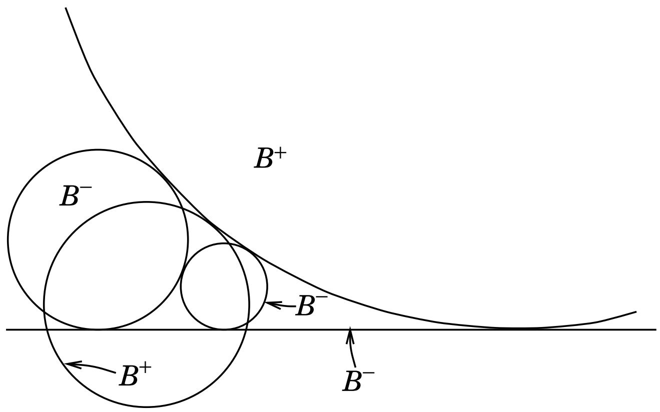

Figure 11. Construction for $B^*(2,3) \geq 20$.

According to Equation (10), $B(3,3) = 28$. An arrangement achieving this bound must simultaneously achieve $\beta(3) = 6$ type I vertices, $\beta(3) = 6$ type II vertices and $4(3+3)-8 = 16$ type III vertices. Surprisingly, this can be achieved geometrically too: $B^*(3,3) = 28$. So any proof showing a difference between topology and geometry must prove an impossible configuration for 7 or more geometric discs.

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