A Class of Variable Mesh Multistep Methods
for Solving
Simultaneous Nonlinear Equations

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ABSTRACT

In a recent paper P.T. Boggs has demonstrated the use of fixed mesh A-stable integration techniques in solving simultaneous nonlinear equations. This paper treats a class of variable mesh multistep methods. The effect of stepsize change on convergence is examined closely. It is shown that with suitable control on stepsize change a variable mesh linear multistep method is convergent if the underlying fixed mesh method satisfy certain conditions. Implicit and explicit Adams methods of order less than five and three satisfy these requirements readily. Finally the effectiveness of the proposed methods are demonstrated in numerical testings using a practical algorithm for automatic stepsize control.

1. Introduction:

The success of Newton-Raphson method in finding a root X^* of the system of nonlinear equations $F(X) = 0$ is dependent on the initial approximation to the solution. Davidenko [4], Jacovlev [10] and others proposed enlarging domain of convergence by solving a related system of differential equations with a real parameter t. Thus it is essential that efficient numerical integration techniques for this problem are investigated. Among the existing methods for instances, Kizner [11] demonstrated the use of the classical fourth order Runge-Kutta method, Meyer [12] examined a modified Newton's method, and in [1] Boggs considered a class of A-stable integration methods. From these proceeding papers it is evident that the intrinsic nonlinearity of the problem calls for flexibility in choosing mesh size at each integration step. This paper treats a class of variable mesh multistep methods [14,17, 19] applicable to nonlinear equations. With suitable control on stepsize change a variable mesh linear multistep method is convergent if the underlying fixed mesh difference equation satisfy certain conditions. Implicit and explicit Adams methods of order less than five and three satisfy these requirements readily. A practical algorithm for automatic stepsize control is proposed and numerical examples of varying difficulties are presented.

2. The differential equations and variable K-step methods

The related system of differential equations that we are interested in solving is

$$\frac{dX}{dt} = -J^{-1} F(X) \tag{2.1}$$

where $J = \frac{\partial F}{\partial X}$. Existence of a solution to (2.1) has been demonstrated in [7,12] with suitable conditions on F and J^{-1}. Along the solution curve we have $F(t) = F(0)e^{-t}$. Thus $F(t) \to 0$ as $t \to \infty$, i.e. $X(t) \to X^*$ as $t \to \infty$. Without loss of generality one may assume that $X^* = 0$.

Next we give a brief description of the numerical methods to be discussed in this paper. Given an initial value problem $Y' = f(Y,t)$, $Y(o) = Y_0$ a classical fixed k-step method is defined by

$$\sum_{i=0}^{k} (\alpha_i Y_{n-i} + h\beta_i f_{n-i}) = 0 \tag{2.2}$$

Correspond to (2.2) there exists a variable k-step method represented by

$$\sum_{i=0}^{k} (\alpha_{i,n} Y_{n-i} + h_{n-1}\beta_{i,n} f_{n-i}) = 0 \tag{2.3}$$

where $h_{n-1} = t_n - t_{n-1}$ and the α's and β's are dependent on t. To evaluate $\alpha_{i,n}$'s and $\beta_{i,n}$'s we require that (a) $\alpha_i = 0$ implies $\alpha_{i,n}=0$, $\beta_i=0$ implies $\beta_{i,n}=0$ and (b) the polynomials 1, $(t-t_{n-k})$, $\cdots$, $(t-t_{n-k})^r$ be exact for the difference equation (2.3) where $r + 1$ equal to total number of nonvanishing α_i's and β_i's. For example a variable 2-step Adams-Moulton method has the following representation for the coefficients $\alpha_{i,n}$'s and $\beta_{i,n}$'s

$$\alpha_{1,n} = -1$$

$$\beta_{0,n} = -\frac{1}{2} + \frac{h_{n-1}}{6(h_{n-1} + h_{n-2})}$$

$$\beta_{1,n} = -\frac{1}{2} - \frac{h_{n-1}}{h_{n-2}}$$

$$\beta_{2,n} = \frac{-h_{n-1}^2}{6h_{n-1}(h_{n-1} + h_{n-2})}$$

Stability and convergence of (2.3) for a well behaved f (continuous and Lipschitz bounded) has been established in Piotrowski [14], Tu [17], Gear and Tu [8].

We re-express the right side of (2.1) as $-X + G(X)$ to facilitate the convergence analysis of the associated difference equation (2.3).

Lemma 2.1 Let D be a convex region containing $X^* = 0$ such that over the region D, F is continuously differentiable $J(X)$ satisfies Lipschitz condition with constant M, $\| J^{-1}(X) \|$ exists and is bounded by A and $F(0) = 0$. Then

$$\frac{dX}{dt} = -J^{-1}(X)F(X) = -X + G(X) \tag{2.4}$$

with

$$\| G(X) \| \leq \frac{1}{2} AM \| X \|^2 \text{ on } D \tag{2.5}$$

Proof:

Since $F(0) = 0$ we can write

$$-J^{-1}(X)F(X) = X + J^{-1}(X)(F(0) - F(X) - J(X)(-X))$$

and define

$$G(X) = J^{-1}(X)\ (F(0) - F(X) - J(X)(-X))$$

then (2.5) is immediate by Ortega and Rheinboldt [13] Theorem 3.2.12 P. 73.

3. Difference equations and convergence

Apply (2.3) to (2.4) we have a nonhomogenious linear difference equation

$$\sum_{i=0}^{k} \alpha_{i,n} X_{n-i} - h_{n-1} \sum_{i=0}^{k} \beta_{i,n} X_{n-i} = \gamma_n \tag{3.1}$$

where

$$\gamma_n = -h_{n-1} \sum_{i=0}^{k} \beta_{i,n} G_{n-i} \tag{3.2}$$

The general solution of (3.1) can be written in the form $\{Y_n + Z_n\}$ with $\{Y_n\}$ being the solution of the homogeneous equation

$$\sum_{i=0}^{k} \alpha_{i,n} X_{n-i} - h_{n-1} \sum_{i=0}^{k} \beta_{i,n} X_{n-i} = 0 \tag{3.3}$$

and $\{Z_n\}$ being some particular solution of the nonhomogeneous equation. $\{Z_n\}$ is readily representable by Duhamels' principle. Here we state a representation theorem by Henrici [9].

Theorem 3.1

For $m = 0,1,2, \cdots$ let the sequence $\{y_{n,m}\}$ satisfy the condition

$$\begin{cases} y_{n,m} = 0 & n = 0,1,\cdots,m-1 \\ y_{n,m} = \dfrac{1}{(\alpha_{0,m} - h_{m-1}\beta_{0,m})} & \end{cases} \tag{3.4}$$

and let it be a solution of (3.3) for $n > m$. Then the solution of (3.1) satisfying $Z_0 = Z_1 = \cdots = Z_{k-1} = 0$ is for $n \geq k$ represented by

$$Z_n = \sum_{m=k}^{n} \gamma_m y_{n,m} \tag{3.5}$$

From (3.3) the sequence $\{y_{n,m}\}$ satisfies

$$|y_{n,m}| \leq \sum_{i=1}^{k} \left|\frac{\alpha_{i,n} - h_{n-1}\beta_{i,n}}{\alpha_{0,n} - h_{n-1}\beta_{0,n}}\right| \quad |y_{n-i,m}| \tag{3.6}$$

Hence,

$$\sum_{m=k}^{n} |y_{n,m}| \leq \sum_{i=1}^{k} \left|\frac{\alpha_{i,n} - h_{n-1}\beta_{i,n}}{\alpha_{0,n} - h_{n-1}\beta_{0,n}}\right| \sum_{m=k}^{n-i} |y_{n-i,m}|$$

It is obvious by induction that for a given variable k-step method (2.3) there exists a $K > 0$ such that

$$\sum_{m=k}^{n} |y_{n,m}| \leq K \quad \forall\, n \tag{3.7}$$

if the coefficients of the difference equations (2.3) satisfy the following condition

$$\sum_{i=1}^{k} \left|\frac{\alpha_{i,n} - h_{n-1}\beta_{i,n}}{\alpha_{0,n} - h_{n-1}\beta_{0,n}}\right| \leq 1 \quad \forall\, n \tag{3.8}$$

In [8,17] it was shown that $\alpha_{i,n}$'s and $\beta_{i,n}$'s are continuous functions of $\delta_{n-1}, \delta_{n-2}, \cdots, \delta_{n-k+1}$ in a neighborhood of $(0,0,\cdots,0)$ where

$$\delta_{n-j} = \frac{h_{n-j} - h_{n-j-1}}{h_{n-j-1}} \qquad j=1,\cdots,k-1 \tag{3.9}$$

In fact $\alpha_{i,n} = \alpha_i$ and $\beta_{i,n} = \beta_i$, $i=0,1,\cdots,k$ when $\delta_{n-1} = \delta_{n-2} = \cdots = \delta_{n-k+1} = 0 \ \forall n$. It follows that $\sum_{i=1}^{k} \left|\frac{\alpha_{i,n} - h_{n-1}\beta_{i,n}}{\alpha_{0,n} - h_{n-1}\beta_{0,n}}\right|$ is also a continuous function of $\delta_{n-1}, \delta_{n-2}, \cdots, \delta_{n-k+1}$ in a neighborhood of $(0,0,\cdots,0)$. If for some

$\bar{H} > \underline{H} > 0$ and $\underline{H} \le h \le \bar{H}$ the underlying fixed step method satisfies

$$(3.10) \qquad \sum_{i=1}^{k} \left| \frac{\alpha_i - h\beta_i}{\alpha_0 - h\beta_0} \right| < 1$$

then there exists $\underline{d} < 0$ and $\bar{d} > 0$ and the step change restriction

$$(3.11) \qquad \underline{H} \le h_{n-1} \le \bar{H} \quad \forall\ n$$

$$\underline{d} \le \delta_{n-1}, \ldots, \delta_{n-k+1} \le \bar{d} \quad \forall\ n$$

such that

$$(3.12) \qquad \sum_{i=1}^{k} \left| \frac{\alpha_{i,n} - h_{n-1}\ \beta_{i,n}}{\alpha_{0,n} - h_{n-1}\ \beta_{0,n}} \right| \le 1 \ \forall\ n$$

When $h = 0$ (3.10) becomes an equality since $\rho(1) = 0$ [9]. This necessitated a lower bound $\underline{H}$ for the stepsize h. Thus the choice of $\underline{d}$ and $\bar{d}$ depends also on $\underline{H}$ and $\bar{H}$. Also it should be noted that the requirement h_{n-1} be bounded away from zero for all n does not affect convergence since we are interested in convergence as $n \to \infty$. In practice we have used $\underline{H} = 10^{-3}$ with good results. Interestingly there are cases when the restrictions on $\underline{H}$ and $\bar{H}$ can be relaxed. For both Euler's method and Trapezoidal rule the coefficients $\alpha_{i,n}$'s and $\beta_{i,n}$'s turn out to be constants, i.e.

$$\alpha_{i,n} = \alpha_i \ , \ \beta_{i,n} = \beta_i \quad \forall\ i$$

Hence (3.12) is satisfied for h_{n-1} arbitrarily close to 0 and the lower bound $\underline{H}$ becomes unnecessary. The Trapezoidal rule goes one step further by satisfying (3.12) for arbitrarily large h_{n-1}. Thus the restrictions on $\underline{H}$ and $\bar{H}$ can be removed when using Trapezoidal rule. Indeed this sheds some light on the variable mesh version of Dalquists A-stability. However, we shall not delve into that in this paper.

It is easy to see that Adams-Bashforth method of order less than three and Adams-Moulton methods of order less than five satisfy condition (3.10) for reasonably broad ranges of $\underline{H}$ and $\bar{H}$. For example the third order Adams-Moulton method satisfies (3.10) for $0 < \underline{H} < \bar{H} < 6$. We are now ready to prove a convergence theorem similar to that given by Boggs [1] for fixed step methods.

Theorem 3.2 Assume that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\| X \| < \delta$ then $\| G(X) \| < \varepsilon \| X \|$. Consider a classical k-step method (2.2) which satisfies (3.10). Choose the corresponding variable mesh method (2.3) with step size change satisfying (3.11). Choose an ε such that $1-h_{n-1}\, \varepsilon \sum_{i=0}^{k} |\beta_{i,n}|\;\; K > 0 \;\forall\, n$ where K is a positive constant depending on the method and the initial k step sizes used. Then there exists a $\hat{\delta} \le \delta$ (δ depends on ε) such that if $\{X_n\}$ is obtained by applying this method to (2.4) from initial points $\{X_0, X_1, \cdots, X_{k-1}\} \in \{X = \| X \| \le \hat{\delta}\}$ then $\lim_{n\to\infty} \| X_n \| = 0$.

Proof:

From (3.2), for $\| X_{m-i} \| \le \delta \quad i = 0,1,\cdots,k$

$$\| Y_m \| = \| - h_{m-1} \sum_{i=0}^{k} \beta_{i,m} G_{m-i} \|$$

$$\le H \sum_{i=0}^{k} |\beta_{i,m}| \varepsilon \| X_{m-i} \|$$

$$\le \hat{\varepsilon}\, \bar{X}_m$$

where $\bar{X}_m = \sup_{0 \le i \le k} \| X_{m-i} \|$, $\hat{\varepsilon} = H \varepsilon B$ and B is a bound for $\sum_{i=0}^{k} |\beta_{i,m}| \;\forall\, m$. Thus

$$\| X_n \| = \| Y_n + Z_n \|$$

(3.14) $$\leq \| Y_n \| + \hat{\varepsilon} \sum_{m=k}^{n} \tilde{X}_m \, |y_{n,m}|$$

$$\sup_{n \leq N} \| X_n \| \leq \sup_{n \leq N} \| Y_n \| + \hat{\varepsilon} \sup_{n \leq N} \sum_{m=k}^{n} \tilde{X}_m \, |y_{n,m}|$$

$$\leq \sup_{n \leq N} \| Y_n \| + \hat{\varepsilon} K \sup_{n \leq N} \| X_n \|$$

(3.15) $$\therefore \sup_{n \leq N} \| X_n \| \leq \frac{\sup_{n \leq N} \| Y_n \|}{1 - \hat{\varepsilon} K}$$

By choosing $\hat{\delta}$ sufficiently small, $\sup_{n \leq N} \| Y_n \|$ can be made small. Hence $\| X_n \|$ will remain $< \delta$ for all n. Since $\lim_{n \to \infty} \| Y_n \| = 0$ it is clear that

(3.16) $$\lim_{N \to \infty} \sup_{n \leq N} \| X_n \| \leq 0$$

On the other hand $\| X_n \| \geq 0$ therefore we have

$$\lim_{n \to \infty} \| X_n \| = 0$$

4. Step size control

The strategy to choose an initial step size is in keeping with our objective of enlarging the domain of convergence. It is generally advisable to start the numerical integration with a small initial stepsize h_0. Let us consider the case of single nonlinear equations as an illustration. Consider the use of Euler's method with $h_0 = 1$ in Fig. 1 and Fig. 2. Both diagrams, drawn with a CALCOMP plotter, represent sixth degree polynomials $a_o + a_1 x + \cdots + a_6 x^6$. The coefficients $(a_0, a_1, \cdots, a_6)$ are $(-8, .816535, .5854298, 4.854867 \times 10^{-2}, -2.047432 \times 10^{-2}, 1.737152 \times 10^{-3}, 3.125347 \times 10^{-4})$ and $(-2, -16.28665, 18.53179, -6.882648, 1.128719, -8.448773 \times 10^{-2}, 2.365921 \times 10^{-3})$, and the initial points X_0 are 5.05 and 9.4 for Fig. 1 and Fig. 2 respectively.

In Fig. 1 the initial step size h_0 will be adjusted according to a norm reduction criteria described later in this section. Thus the numerical solution will converge to the correct root. On the other hand Fig. 2 demonstrates the need of a small h_0 even in the presence of the norm reduction criteria. Computation showed that $h_0 = 0.4$ was sufficient for correct convergence. For general purpose application we use $h_0 = 0.1$.

According to (3.10) the maximum allowable step size $\bar{H}$ varies with different methods. It is found that $\bar{H}$ for the first and second order explicit Adams method should not exceed 2 and 1, and for the third and fourth order implicit Adams method $\bar{H}$ should not exceed 6 and 3. As indicated earlier there is no restriction on $\bar{H}$ for the Trapezoidal rule. In practical computation $\underline{H}$ is needed even in the absence of (3.11). We have used $\underline{H} = 10^{-3}$ with good results in all occasions.

The stepsize h_i at t_i is controlled according to the following sequence of instructions (the norms $\| \cdot \|$ are in Euclidean norm)

a. Compute $C_s = \min \{0.05, 0.1h_{i-1}\}$, $C_b = \min \{0.05, h_{i-1}\}$.

b. If $(1 + C_s) \| F_{i-1} \| \leq \| F_i \|$ choose $h_i = \min \{\bar{H}, 1.2\, h_{i-1}\}$.

   If $(1 + C_b) \| F_{i-1} \| \leq \| F_i \|$ choose $h_i = \min \{\bar{H}, 1.5\, h_{i-1}\}$.

   Otherwise the step size is unchanged, i.e. $h_i = h_{i-1}$.

c. Compute X_{i+1} with the numerical integration method.

   Evaluate F_{i+1}.

d. If $\| F_{i+1} \| < \| F_i \|$ proceed to (f).

e. If $\| F_{i+1} \| \geq \| F_i \|$ replace h_i by $\max \{10^{-3}, 0.67\, h_i\}$

   If $h_i > 10^{-3}$ repeat from (c).

   Otherwise X_{i+1} is accepted with $h_i = 10^{-3}$.

f. To determine a new stepsize for the next integration step repeat from (a)

5. Numerical results and discussions

In our analysis thus far we have assumed that the Jacobian inverse is obtained by direct computation, i.e. find $\frac{\partial F}{\partial X}$ and invert it by a matrix inversion subroutine (DECOMP [6]). We shall also include in the numerical testings Broyden's approximation method [2] for finding the Jacobian inverse. Let H_n be the Broydon's approximation of J_n^{-1} then H_n is given by

$$H_n = H_{n-1} - \frac{[H_{n-1}(F_n - F_{n-1}) - (X_n - X_{n-1})] (H_{n-1}^T H_{n-1} F_{n-1})^T}{(H_{n-1}^T H_{n-1} F_{n-1})^T (F_n - F_{n-1})}$$

The explicit Adams method of order less than 3 and the implicit Adams method of order less than 5 with Euler predictor have been tested on the following problem:

1. Powell [15]

   $f_1 = 10\ (x_2 - x_1^2)$

   $f_2 = 1 - x_1$

   Initial value : (-2,1), Solution: (1,1)

2. Brown [3]

   $f_1 = 1/2 \sin (x_1 x_2) - x_2/4\pi - x_1/2$

   $f_2 = (1 - 1/4\pi)\ (\exp(2x_1) - e) + ex_2/\pi - 2ex_1$

   Initial value : (0.6,3.0), Approximate solution: (0.3,2.8)

3. Van Melle [18]

   $f_1 = 4 + x_1 + x_2 - x_1^2 + 2x_1x_2 + 3x_2^2$

   $f_2 = 1 + 2x_1 - 3x_2 + x_1^2 + x_1x_2 - 2x_2^2$

   Initial value: (-0.2,-0.8), Approximate solution: (3.339,-2.984)

4. Boggs [1]

$$f_1 = x_1^2 - x_2 + 1$$

$$f_2 = x_1 - \cos(\pi x_2/2)$$

Initial value: (1,0), Solution: (0,1)

5. Rosenbrock [16]

$$f_1 = 2(x_1 - 1) - 400x_1 (x_2 - x_1^2)$$

$$f_2 = 200 (x_2 - x_1^2)$$

Initial value: (-1.2,1), Solution: (1,1)

Most of the above are well-known examples for which the Newton-Raphson method fails to find the solutions. All numerical computations were performed with double precision FORTRAN on IBM system/370. In Table 1 the number of the effective integration steps NS necessary to reduce the Euclidean norm of F to less that 10^{-6} are tabulated for each problem. Abbreviated notations for various methods are EULER - Euler's method, AB2 - second order Adams-Bashforth method, AM2 - Trapezoidal rule, AM3 - third order Adams-Moultons method, AM4 - fourth order Adams-Moulton method, EULERB, AB2B, AM2B, AM3B and AM4B correspond to each of the above methods but with Broyden's approximation of the Jacobian inverse. We discuss the results of these two cases separately

(A) Direct computation of Jacobian inverse

It is observed that in most problems the implicit method converges faster than explicit methods initially. However, as the numerical integration moves close to the solution the Euler's method, with step size gradually controlled to one, is definitely superior to all other methods. The relatively large number of steps required by non-Euler's methods tend to be misleading since most of these steps occur at the final stage of convergence. A similar situation in case (B) is illustrated by Table 2. There is no substantial difference in the performance of all implicit methods with two corrections. But when only one correction is made at each step the Trapezoidal rule needs far more steps than the other implicit methods, a weak point in dealing with large systems of equations since each additional correction requires a matrix inversion.

Thus it appears to be advisable to implement a composite scheme with the initial use of EULER-AM3 predictor-corrector and the subsequent use of EULER method. The change takes place when the predictor value becomes consistently better than the corrector value in four consecutive steps. Such a scheme should combine the advantages of both implicit method and Euler's method at different stages of integration. However, the preliminary results of the composite scheme show only marginal improvement over the once-through EULER method.

(B) Broyden's approximation of Jacobian inverse

With the exception of Problem 5 we have essentially the same type of results as in case (A). We demonstrate this by summarizing the result of Problem 1 in Table 2.

Dennis [5] showed that the rate of deterioration in Broyden's approximation depends on the nonlinearity of F. More specifically a large Lipschitz bound of the Jacobian matrix in the region of interest would require exceedingly small step sizes for Euler's method. In this respect problem 5 serves as a critical test. Indeed both explicit methods fail to converge in 1000 steps. On the other hand all the implicit methods converge in less than 500 steps. It is thought that an implicit method improves the deterioration of approximation through the correction scheme.

On the whole the Trapezoidal rule with two corrections requires fewer integration steps than all other methods. Again it should be pointed out that this is not necessarily true with one correction. Nevertheless, a re-evaluation of J^{-1} by Broyden's approximation is not as laborious as the direct computation in case (A) thus making two-correction schemes easier to implement there.

Thus we propose the use of the Trapezoidal rule with two corrections for general purpose application in case (B).

We have attempted to establish a general-purpose algorithm throughout the numerical testings. Hence the same automatic stepsize control algorithm (Section 4) was applied to all problems. On that basis our numerical results compare well with those in [1,2,17]. Limited computational experience also indicate that most of our methods compare favorably with the variable mesh fourth order Runge-Kutta method [11].

In view of the fact that we have not shown (3.12) to be a necessary condition for convergence there may exist other types of variable mesh methods which are convergent without satisfying (3.12). In addition, it is felt that in a sufficiently small region containing the solution X^* there may be an optimum choice of stepsize for a non-Euler's method to have rapid convergence, as that evidenced in Euler's method with stepsize 1. These areas certainly deserve closer scrutiny in the future development.

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Table 1 Number of steps required to achieve $\|F\| < 10^{-6}$*

	Problem	Number			
Method	(1)	(2)	(3)	(4)	(5)
EULER	8	9	11	10	29
AB2	32	24	37	30	80
AM2	20	16	20	19	112
AM3	22	16	25	20	72
AM4	25	16	28	22	77
EULERB	13	11	36	14	>1000
AB2B	69	30	58	96	>1000
AM2B	15	12	39	15	262
AM3B	39	22	40	34	329
AM4B	31	17	43	26	451

* Two corrections at each step for implicit methods.

Table 2 Norm Reduction of F for Problem 1

Step No.	AM2B	AM3B	AM4B	EULERB
1	0.301496E 02	0.301496E 02	0.301496E 02	0.301496E 02
2	0.280303E 02	0.280303E 02	0.280303E 02	0.280303E 02
3	0.267802E 02	0.267802E 02	0.267802E 02	0.267802E 02
4	0.248115E 02	0.246927E 02	0.246165E 02	0.262075E 02
5	0.229825E 02	0.225789E 02	0.224700E 02	0.260346E 02
6	0.214571E 02	0.207395E 02	0.207560E 02	0.259105E 02
7	0.198164E 02	0.188467E 02	0.190724E 02	0.254875E 02
8	0.159936E 02	0.144454E 02	0.149844E 02	0.242692E 02
9	0.839799E 01	0.585719E 01	0.718614E 01	0.196720E 02
10	0.287763E 01	0.384617E 01	0.284815E 01	0.937468E 01
11	0.186777E 00	0.251551E 01	0.715218E 00	0.646333E 00
12	0.665925E-02	0.221570E 00	0.561311E 00	0.274977E-01
13	0.365476E-03	0.133669E 00	0.286457E 00	0.0
14	0.116414E-04	0.201017E-01	0.446915E-01	
15	0.728539E-06	0.756094E-02	0.416046E-01	
16		0.150762E-02	1.192503E-01	
17		0.347656E-03	0.925970E-02	
18		0.104674E-03	0.490439E-02	
19		0.102128E-03	0.239214E-02	
20		0.799202E-04	0.121560E-02	
21		0.627394E-04	0.609716E-03	
22		0.492782E-04	0.306067E-03	
23		0.387011E-04	0.153983E-03	
24		0.303947E-04	0.773258E-04	
25		0.238710E-04	0.388661E-04	
26		0.187476E-04	0.195297E-04	
27		0.147238E-04	0.981354E-05	
28		0.115636E-04	0.493149E-05	
29		0.908168E-05	0.247807E-05	
30		0.713247E-05	0.124525E-05	
31		0.560162E-05	0.625744E-06	
32		0.439934E-05		
33		0.345511E-05		
34		0.271353E-05		
35		0.213113E-05		
36		0.167372E-05		
37		0.131449E-05		
38		0.103236E-05		
39		0.810782E-06		

* Two corrections at each step for implicit methods

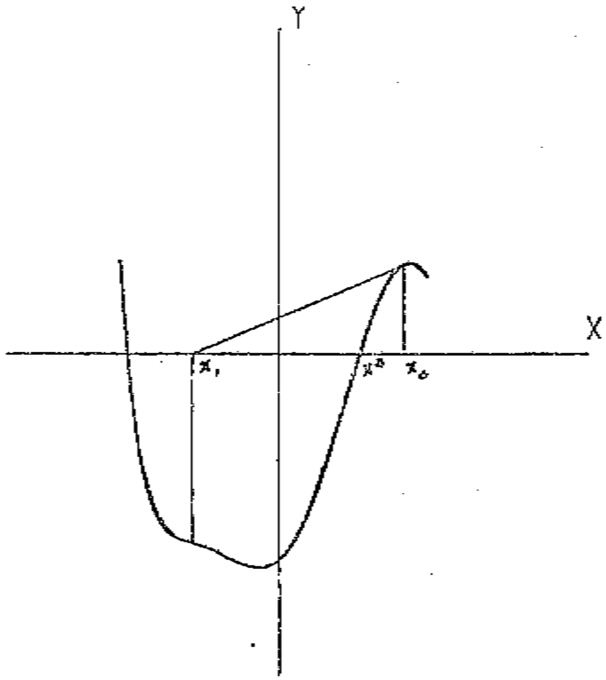

FIG. 1

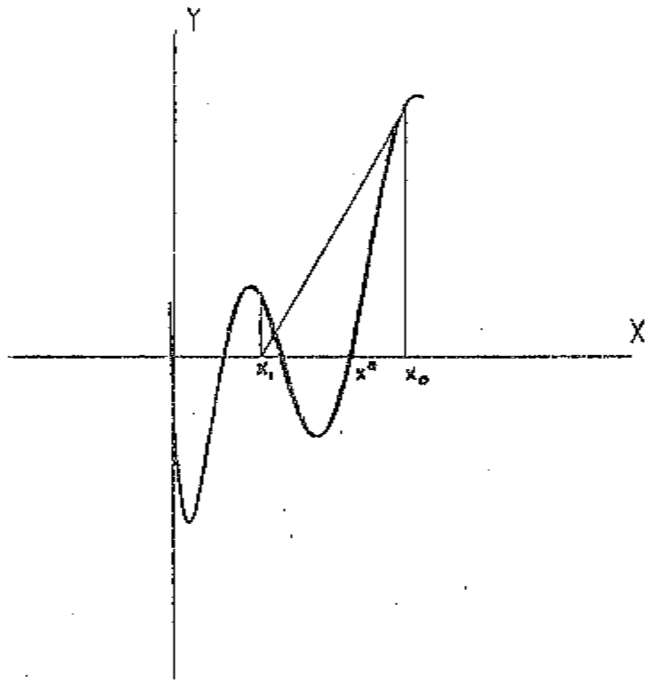

FIG. 2