A BASIC FAMILY OF ITERATION FUNCTIONS FOR POLYNOMIAL ROOT FINDING AND ITS CHARACTERIZATIONS

Bahman Kalantari *

Department of Computer Science

Rutgers University, New Brunswick, NJ 08903

Iraj Kalantari

Department of Mathematics

Western Illinois University, Macomb, IL, 61455

Rahim Zaare-Nahandi†

Department of Mathematics

University of Tehran, Tehran, Iran

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Abstract

Let $p(x)$ be a polynomial of degree $n \geq 2$ with coefficients in a subfield K of the complex numbers. For each natural number $m \geq 2$, let $L_m(x)$ be the $m \times m$ lower triangular matrix whose diagonal entries are $p(x)$ and for each $j = 1, \ldots, m-1$, its j-th subdiagonal entries are $p^{(j)}(x)/j!$. For $i = 1, 2$, let $L_m^{(i)}(x)$ be the matrix obtained from $L_m(x)$ by deleting its first i rows and its last i columns. $L_1^{(1)}(x) \equiv 1$. Then, the function $B_m(x) = x - p(x)\ det(L_{m-1}^{(1)}(x))/det(L_m^{(1)}(x))$ is a member of $S(m, m+n-2)$, where for any $M \geq m$, $S(m, M)$ is the set of all rational iteration functions such that for all roots θ of $p(x)$, $g(x) = \theta + \sum_{i=m}^{M} \gamma_i(x)(\theta - x)^i$, with $\gamma_i(x)$'s also rational and well-defined at θ. Given $g \in S(m, M)$, and a simple root θ of $p(x)$, $g^{(i)}(\theta) = 0$, $i = 1, \ldots, m-1$, and $\gamma_m(\theta) = (-1)^m g^{(m)}(\theta)/m!$. For $B_m(x)$ we obtain $\gamma_m(\theta) = (-1)^m det(L_{m+1}^{(2)}(\theta))/det(L_m^{(1)}(\theta))$. For $m = 2$ and 3, $B_m(x)$ coincides with Newton's and Halley's, respectively. If all roots of $p(x)$ are simple, $B_m(x)$ is the unique member of $S(m, m+n-2)$. By making use of the identity $0 = \sum_{i=0}^{n} [p^{(i)}(x)/i!](\theta - x)^i$, we arrive at two recursive formulas for constructing iteration functions within the $S(m, M)$ family. In particular the B_m's can be generated using one of these formulas. Moreover, the other formula gives a simple scheme for constructing a family of iteration functions credited to Euler as well as Schröder, whose m-th order member belong to $S(m, mn)$, $m > 2$. The iteration functions within $S(m, M)$ can be extended to arbitrary smooth functions f, with the automatic replacement of $p^{(j)}$ with $f^{(j)}$ in g as well as $\gamma_m(\theta)$.

1. INTRODUCTION.

Let $p(x)$ be a polynomial of degree $n \geq 2$ with coefficients in a subfield K of the complex numbers. For each natural number $m \geq 2$, let $L_m(x)$ be the $m \times m$ lower triangular matrix whose diagonal entries are $p(x)$ and for each $j = 1, \ldots, m-1$, its j-th subdiagonal entries are $p^{(j)}(x)/j!$, i.e.

$$L_m(x) = \begin{pmatrix} p(x) & 0 & 0 & \ldots & 0 \\ p'(x) & p(x) & 0 & \ldots & 0 \\ \frac{p''(x)}{2!} & p'(x) & p(x) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{p^{(m-1)}(x)}{(m-1)!} & \frac{p^{(m-2)}(x)}{(m-2)!} & \frac{p^{(m-3)}(x)}{(m-3)!} & \ldots & p(x) \end{pmatrix}. \tag{1.1}$$

For $i = 1, 2$, let $L_m^{(i)}(x)$ be the $(m-i) \times (m-i)$ matrix obtained from $L_m(x)$ by deleting its first i rows, and its last i columns, i.e.

$$L_m^{(1)} = \begin{pmatrix} p'(x) & p(x) & 0 & \ldots & 0 \\ \frac{p''(x)}{2!} & p'(x) & p(x) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{p^{(m-1)}(x)}{(m-1)!} & \frac{p^{(m-2)}(x)}{(m-2)!} & \frac{p^{(m-3)}(x)}{(m-3)!} & \ldots & p'(x) \end{pmatrix}, \quad L_m^{(2)} = \begin{pmatrix} \frac{p''(x)}{2!} & p'(x) & \ldots & 0 \\ \frac{p'''(x)}{3!} & \frac{p''(x)}{2!} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^{(m-1)}(x)}{(m-1)!} & \frac{p^{(m-2)}(x)}{(m-2)!} & \ldots & \frac{p''(x)}{2!} \end{pmatrix},$$

with $L_1^{(1)}(x) \equiv 1$. The function

$$B_m(x) = x - p(x) \ \frac{det(L_{m-1}^{(1)}(x))}{det(L_m^{(1)}(x))}, \tag{1.2}$$

defines a fixed-point iteration function which has local m-th order convergence for simple roots of $p(x)$ (see Remark 1.1 below). Here det is the determinant function. We shall refer to the set of all $B_m(x)$'s, $m = 2, 3, \ldots$, as the *Basic Family* of fixed-point iteration functions for polynomial root finding. Specific members of the family are

$$B_2(x) = x - \frac{p(x)}{p'(x)},$$

which is the well-known Newton's function, and

$$B_3(x) = x - \frac{2p'(x)p(x)}{2(p'(x))^2 - p''(x)p(x)},$$

which is Halley's function. For $m = 4$, one obtains

$$B_4(x) = x - \frac{6p'(x)^2p(x) - 3p''(x)p(x)^2}{p'''(x)p(x)^2 + 6p'(x)^3 - 6p''(x)p'(x)p(x)}.$$

The function $B_2(x)$ is derived in almost all the numerical analysis textbooks with the usual geometric interpretations (see e.g. Atkinson [2]). The Function $B_3(x)$, credited to the astronomer Halley [12], is perhaps the best-known third-order method, and has been rediscovered and/or derived through various interesting means, see e.g. Bateman [3], Wall [25], Bodewig [4], Hamilton [13], Stewart [22], Frame [7, 8, 9], Traub [23], Hansen and Patrick [14], Popovski [17], Gander [10]. For the interesting history of Halley's function see the recent paper of Scavo and Thoo [18].

Halley's function can also be obtained by applying Newton's method to the function $p(x)/\sqrt{p'(x)}$, see Bateman [3], as well as Brown [5], and Alefeld [1]. Most recently, this was rediscovered by Gerlach [11] who also generalized it to get high order methods : Let $F_1(x) = p(x)$, and define inductively for $i = 2, 3, \ldots$, $F_i(x) = F_{i-1}(x)/[F_{i-1}(x)]^{1/i}$. Then for $m = 2, 3, \ldots$, the function $G_m(x) = x - F_{m-1}(x)/F'_{m-1}(x)$, under some very restrictive conditions, will have m-th order convergence rate. Interestingly, $G_4(x)$ coincides with $B_4(x)$ and possibly they are all identical. Gerlach however offers no closed formula for $G_m(x)$. Moreover, the domain of G_m's is very restrictive. Indeed the computation of $G_m(x)$ is too cumbersome and Gerlach only obtains $G_4(x)$ for $p(x) = x^3 - \alpha$. In [15], the closed formula for $B_m(x)$ corresponding to this cubic polynomial as well as to $p(x) = x^2 - \alpha$ was obtained. Indeed for general functions, $B_4(x)$ had been obtained previously and through different schemes by Snyder [21], Traub [23](page 91), and others. Interestingly, Hamilton [13], made use of determinants in deriving Halley's function. In fact Hamilton does make use of the determinant of the matrix $L_m^{(1)}(x)$, to arrive at high order methods, but he does not obtain $B_m(x)$ for any $m \geq 4$. For instance he obtains a forth-order method which is different than $B_4(x)$ (see page 521, [13]).

The algebraic characterization and the existence of the Basic Family was proved in [16]. The present paper gives further characterizations, closed formula for its members, their corresponding asymptotic constant, and thus offers a deeper understanding of each of its individual members, including Newton's and Halley's functions.

Definition 1.1. For each pair of nonnegative integers m and M, $M \geq m$, let $S(m, M)$

be the set of all function $g(x) \in K(x)$, i.e. a rational function with coefficient in K, so that for all roots θ of $p(x)$ we have

$$g(x) = \theta + \sum_{i=m}^{M} \gamma_i(x)(\theta - x)^i, \tag{1.3}$$

where $\gamma_i(x) \in K(x)$, for $i = m, \ldots, M$, and well-defined for any simple root θ, i.e. $\lim_{x \to \theta} \gamma_i(x) \equiv \gamma_i(\theta)$ exists. Moreover, $\gamma_m(x)$ and $\gamma_M(x)$ are not identically zero.

Remark 1.1. Given $g \in S(m, M)$, from the continuity of $\gamma_i(x)$'s at a simple root θ, and since $m \geq 2$ it follows that there exists a neighborhood of θ for which $|g(x) - \theta| \leq C|x - \theta|^m$, for some constant C. This implies the existence of a neighborhood I_θ of θ for which for any $x_0 \in I_\theta$ the fixed-point iteration

$$x_{k+1} = g(x_k), \quad k \geq 0, \tag{1.4}$$

converges to θ, and clearly

$$\lim_{k \to \infty} \frac{(\theta - x_{k+1})}{(\theta - x_k)^m} = -\gamma_m(\theta), \tag{1.5}$$

i.e. the rate of convergence of the sequence $\{x_k\}_{k=0}^{\infty}$ is of order m. Also note that if x_0 is in K, then so are all the iterates in (1.4).

Denoting $\gamma_m(x)$ corresponding to $B_m(x)$ by $\gamma_m^{(m)}(x)$, we show that for a simple root θ, the asymptotic constant of convergence for $B_m(x)$ is

$$\gamma_m^{(m)}(\theta) = (-1)^m \frac{det(L_{m+1}^{(2)}(\theta))}{det(L_m^{(1)}(\theta))} = \frac{(-1)^m}{p'(\theta)^{m-1}} det(L_{m+1}^{(2)}(\theta)). \tag{1.6}$$

In particular for $m = 2$ and $m = 3$ we get

$$\gamma_2^{(2)}(\theta) = \frac{p''(\theta)}{2p'(\theta)}, \qquad \gamma_3^{(3)}(\theta) = -\frac{2p'(\theta)p'''(\theta) - 3p''(\theta)^2}{12p'(\theta)^2}.$$

We show that if all roots of p are simple, $B_m(x)$ is the unique member of $S(m, m+n-2)$. The following definition provides means by which members from two different $S(m, M)$'s can be compared.

Definition 1.2. Given $g \in S(m, M)$, we define the *order* of g to be m. The *coeffient vector* of g is the vector $\Gamma(x) = (\gamma_m(x), \ldots, \gamma_M(x))$. The *leading coefficient* of g is γ_m. The *width* of g is $M - m + 1$. We define the *depth* of g to be d if the formula for g depends on $p^{(j)}$, $j = 0, \ldots, d$. Similarly we define the *depth* of the leading coefficient of g. The *simple-root-depth* of g is defined to be ρ, if for any simple root θ of $p(x)$, $\gamma_m(\theta)$ depends on $p^{(j)}(\theta)$, $j = 0, \ldots, \rho$.

Remark 1.2. If g and h are in $S(m, M)$, so is any affine combination, $\alpha g + \beta h$, where $\alpha + \beta = 1$. As, $B_m(x)$ will be shown to belong to $S(m, n + m - 2)$, it follows that for any $m \geq 2$, and $M \geq m + n - 2$, the set $S(m, M)$ is nonempty. If $p(x)$ has simple roots, as $B_m(x)$ will be shown to be the unique element of $S(m, m + n - 2)$, it follows that $S(m, M)$ is empty if $M < n + m - 2$. Thus, $B_m(x)$ is also *minimal* with respect to width. Traub [23], has shown that any m-th order method must have depth at least equal to $m - 1$. Thus, in the sense of depth too, $B_m(x)$ is minimal.

Remark 1.3. Given $g \in S(m, M)$, from Taylor's Theorem and the continuity of γ_i's at a simple root θ, it is easy to conclude that that $g^{(i)}(\theta) = 0$, for all $i = 1, \ldots, m - 1$, and that

$$\gamma_m(\theta) = (-1)^m \frac{g^{(m)}(\theta)}{m!}. \tag{1.7}$$

It should be noted that as functions $\gamma_m(x)$ and $(-1)^m g^{(m)}(x)/m!$ are not identical. For example take $g(x) = x - p(x)/p'(x)$. In other words $g^{(m)}(x)$ and $\gamma_m(x)$ may have different depths, but they have the same simple-root-depth.

Definition 1.3. Let $S^\circ(m, M)$ be the set of all $g(x) \in K(x)$ such that $p(y)$ divides

$$\gamma_0(y) = -g(x) + y + \sum_{i=m}^{M} \gamma_i(x)(y - x)^i, \tag{1.8}$$

in $K(x)[y]$, i.e. polynomials in y with coefficients in $K(x)$, such that $\gamma_i(x)$ is well-defined for any simple root θ of $p(x)$, and neither $\gamma_m(x)$ nor $\gamma_M(x)$ are identically zero.

Remark 1.4. If θ is a root of p, then $\gamma_0(\theta) = 0$. Thus, it follows that $S^\circ(m, M)$ is a subset of $S(m, M)$. Also, if all the roots of $p(x)$ are simple (in particular if it is irreducible), then $S^\circ(m, M) = S(m, M)$. To see this pick $g \in S(m, M)$, and define a corresponding $\gamma_0(y)$. Then there exists $q(y), r(y) \in K(x)[y]$ such that

$$\gamma_0(y) = p(y)q(y) + r(y), \tag{1.9}$$

where the degree of $r(y)$ is less than n. Since $\gamma_0(\theta) = 0$, for all roots of p, it follows that $r(y) \equiv 0$ (see e.g. [24]). Note that in particular we must have

$$g(x) = x - \gamma_0(x). \tag{1.10}$$

In general however, $S(m, M)$ and $S^\circ(m, M)$ are not the same. For example consider $p(x) = x^2$. Then $g_1(x) = x/2$ and $g_2(x) = x^2$ both lie in $S(2, 2)$, but $g_2(x)$ is not in $S^\circ(2, 2)$.

Definition 1.4. Let $\overline{S}(m, M)$ be the set of all $g(x)$ which can be represented as (1.3) but with the relaxation of the condition that $\gamma_i(\theta)$'s are well-defined.

We will give two formulas for generating new members of $S(m, M)$ with higher orders. Both formulas make use of the following relationship which is a consequence of Taylor's Theorem, but can also be derived using purely algebraic means (see [16])

$$0 = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}(\theta - x)^i. \tag{1.11}$$

From (1.11) we have

$$-p(x) = \sum_{i=1}^{n} \frac{p^{(i)}(x)}{i!}(\theta - x)^i. \tag{1.12}$$

Writing

$$x = \theta - (\theta - x), \tag{1.13}$$

and adding corresponding sides to (1.12) we get

$$B_1(x) \equiv x - p(x) = \theta + (p'(x) - 1)(\theta - x) + \sum_{i=2}^{n} \frac{p^{(i)}(x)}{i!}(\theta - x)^i \in S(1, n). \tag{1.14}$$

From (1.11) we also immediately get Newton's function as a member of $S(2, n)$

$$B_2(x) = x - \frac{p(x)}{p'(x)} = \theta + \sum_{i=2}^{n} \frac{p^{(i)}(x)}{i!p'(x)}(\theta - x)^i. \tag{1.15}$$

As we shall see from $B_1(x)$ and $B_2(x)$ together with the application of the two formulas one can construct a large class of iterative solutions which include the Basic Family, and the Euler-Schröder family. Finally, we show that the iteration functions within $S(m, M)$ can be extended to arbitrary smooth functions f, with the automatic replacement of $p^{(j)}$ with $f^{(j)}$ in g, and the asymptotic constant of convergence $\gamma_m(\theta)$.

In Section 2, we give an algebraic proof of the existence of $B_m(x)$. The result will be used in Section 3 for deriving a closed formula for $B_m(x)$, and in Section 4, which describes the two formulas for generation of new iteration functions. In Section 5, we consider the extension of iteration function within $S(m, M)$ to arbitrary smooth functions.

2. ALGEBRAIC PROOF OF EXISTENCE OF THE BASIC FAMILY

In this section we prove algebraically the existence of an iteration function in $\overline{S}(m, m+n-2)$ as well as its uniqueness under the assumption that $p(x)$ has simple roots. We shall only be concerned with the existence of $B_m(x)$ as opposed to its closed form. The theorem of this section will be used in the subsequent sections in deriving the closed form, as well as proving the equivalence of $S(m, m+n-2)$, $S^\circ(m, m+n-2)$, and $\overline{S}(m, m+n-2)$, given that $p(x)$ has simple roots. The proof of the theorem also motivates the definition of $S^\circ(m, m+n-2)$ which in turn gives rise to the closed formula for $B_m(x)$. Moreover the theorem will be used to prove that the Basic Family can also be obtained recursively using one the formulas derived in Section 4.

Theorem 2.1. For any natural number $m \geq 2$, $\overline{S}(m, m+n-2)$ is nonempty, and has a unique element if $p(x)$ has simple roots.

To prove Theorem 2.1 we first need an auxiliary lemma. Given two natural numbers s and t let

$$\binom{s}{t} \equiv \frac{s(s-1)\dots(s-t+1)}{t!}, \tag{2.1}$$

the binomial coefficient for $s \geq t$, and zero, for $s < t$. For a given pair of natural numbers

m and r define the following $r \times r$ matrix

$$U_{m,r} = \begin{pmatrix} \binom{m}{1} & \binom{m+1}{1} & \dots & \binom{m+r-1}{1} \\ \binom{m}{2} & \binom{m+1}{2} & \dots & \binom{m+r-1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m}{r} & \binom{m+1}{r} & \dots & \binom{m+r-1}{r} \end{pmatrix}. \tag{2.2}$$

Lemma 2.1. For any natural numbers m and r, $U_{m,r}$ is invertible (in fact $det(U_{m,r}) = \binom{m+r-1}{r}$).

Proof. To prove the lemma we will show that by performing elementary operations on $U_{m,r}$, we can reduce it to the following matrix

$$W_{m,r} = \begin{pmatrix} m & (m+1) & \dots & (m+r-1) \\ m^2 & (m+1)^2 & \dots & (m+r-1)^2 \\ \vdots & \vdots & \ddots & \vdots \\ m^r & (m+1)^r & \dots & (m+r-1)^r \end{pmatrix}.$$

Since $W_{m,r} = V^T D$, where V is an invertible Vandermonde matrix, and $D = diag(m, m+1, \dots, m+r-1)$, it follows that $W_{m,r}$ is invertible. To obtain this reduction, we first multiply the j-th row of $U_{m,r}$ by $j!$. Let U^1 denote the new matrix. The first column of U^1 has entries which can be written as the following polynomials in m: $f_1(m) = m$, $f_2(m) = m^2 - m, \dots$, $f_r(m) = m^r + \alpha_{r-1}m^{r-1} + \alpha_{r-2}m^{r-2} + \dots + \alpha_1 m$, for some coefficients α_i, $i = 1, \dots, r-1$. The second column of U^1 can be written as the same polynomials evaluated at $(m+1)$, and so on. Thus, U^1 is the $r \times r$ matrix whose i-th row vector is given by

$$U^1 = \begin{pmatrix} f_1(m) & f_1(m+1) & \dots & f_1(m+r-1) \\ f_2(m) & f_2(m+1) & \dots & f_2(m+r-1) \\ \vdots & \vdots & \ddots & \vdots \\ f_r(m) & f_r(m+1) & \dots & f_r(m+r-1) \end{pmatrix}.$$

By adding scalar multiples of the first row of U^1 to other rows, we obtain a new matrix U^2 whose first and second rows are the corresponding rows of $W_{m,r}$, and whose remaining rows are polynomials free of the linear terms. Next, by adding scalar multiples of the second row of U^2 to its i-th rows, $i = 3, 4, \dots, r$, we obtain U^3 whose first three rows are those of $W_{m,r}$ and whose remaining rows are polynomials free of the linear and quadratic terms. Clearly, repeating this process we arrive at $W_{m,r}$. □

Proof of Theorem 2.1. Let θ be any root of $p(x)$. Recalling (1.3) and replacing γ_k by $(-1)^k\gamma_k$ for convenience, we will prove the existence of rational functions $g(x)$ and $\gamma_k(x)$, $k = m, \ldots, (m+n-2)$ with coefficient over K so that

$$-g(x) + \theta + \sum_{k=m}^{m+n-2} \gamma_k(x)(x-\theta)^k = 0. \tag{2.3}$$

Using that $p(\theta) = 0$, for any $k \geq n$ we can write

$$\theta^k = \sum_{l=0}^{n-1} \alpha_l \theta^l, \tag{2.4}$$

where $\alpha_l \in K$. From (2.4) it is easy to see that for each $j = 0, \ldots, (n-2)$,

$$(x-\theta)^{m+j} = \sum_{i=0}^{n-1} \theta^i P_i^{m+j}(x), \tag{2.5}$$

where $P_i^{m+j}(x)$'s are polynomials with coefficients in K. Furthermore, for $i \leq m+j$,

$$P_i^{m+j}(x) = \binom{m+j}{i} x^{m+j-i} + \textit{lower order terms}, \tag{2.6}$$

and for $i > m+j$, $P_i^{m+j}(x) \equiv 0$, which from (2.1) can be written as

$$P_i^{m+j}(x) = 0 = \binom{m+j}{i} x^{m+j-i}. \tag{2.7}$$

Using (2.5), we can rewrite (2.3) as

$$-g(x) + \sum_{i=0}^{n-1} \theta^i h_i(x) = 0, \tag{2.8}$$

where

$$h_1(x) = 1 + \sum_{k=m}^{m+n-2} \gamma_k(x) P_1^k(x), \tag{2.9}$$

$$h_i(x) = \sum_{k=m}^{m+n-2} \gamma_k(x) P_i^k(x), \quad i = 0,\ i = 2, \ldots, n. \tag{2.10}$$

Thus, $\overline{S}(m, m+n-2)$ is nonempty if the system of equations $h_i(x) \equiv 0$, $i = 1, \ldots, (n-1)$, is solvable, in which case

$$g(x) = h_0(x) = \sum_{k=m}^{m+n-2} \gamma_k(x) P_0^k(x) \in \overline{S}(m, m+n-2). \tag{2.11}$$

Equivalently, $\overline{S}(m, m+n-2)$ is nonempty if the system of nonlinear equations

$$\Pi_m(x)\Gamma_m(x) = e, \tag{2.12}$$

is solvable where

$$\Gamma_m(x) = [\gamma_m(x), \gamma_{m+1}(x), \ldots, \gamma_{m+n-2}(x)]^T, \quad e = [-1, 0, \ldots, 0]^T, \tag{2.13}$$

and where $\Pi_m(x)$ is the following $(n-1) \times (n-1)$ matrix

$$\Pi_m(x) = \begin{pmatrix} P_1^m(x) & P_1^{m+1}(x) & \ldots & P_1^{m+n-2}(x) \\ P_2^m(x) & P_2^{m+1}(x) & \ldots & P_2^{m+n-2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n-1}^m(x) & P_{n-1}^{m+1}(x) & \ldots & P_{n-1}^{m+n-2}(x) \end{pmatrix}. \tag{2.14}$$

All that remains to be done is to show that the determinant of $\Pi_m(x)$ which is a polynomial in the indeterminate x, is not identically zero, in which case

$$\Gamma_m(x) = \Pi_m^{-1}(x)e. \tag{2.15}$$

To show that $det(\Pi_m(x))$ is not identically zero, it suffices to prove that its highest term is not zero. From (2.6) and (2.7), the highest term of $det(\Pi_m(x))$ is the determinant of $H_m(x)$, the $(n-1) \times (n-1)$ given by

$$H_m(x) = \begin{pmatrix} \binom{m}{1}x^{m-1} & \binom{m+1}{1}x^{m} & \ldots & \binom{m+n-2}{1}x^{m+n-3} \\ \binom{m}{2}x^{m-2} & \binom{m+1}{2}x^{m-1} & \ldots & \binom{m+n-2}{2}x^{m+n-4} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{m}{n-1}x^{m-n+1} & \binom{m+1}{n-1}x^{m-n+2} & \ldots & \binom{m+n-2}{n-1}x^{m-1} \end{pmatrix}. \tag{2.16}$$

It is easy to check that

$$det(H_m(x)) = x^{(m-1)(n-1)} det(U_{m,n-1}), \tag{2.17}$$

hence not identically zero by Lemma 2.1.

We must show that $\gamma_m(x)$ and $\gamma_{m+n-2}(x)$ are not identically zero. To prove this first suppose that $p(x)$ has simple roots and consider the expression $-g(x) + \sum_{i=0}^{n-1} \theta^i h_i(x)$ as a polynomial over $K(x)$ in the indeterminate θ. Every root of $p(x)$ is a root of this polynomial. On the other hand its degree is $n-1$. It follows that $-g(x)+h_0(x)=0$, and $h_i(x)=0$, for $i=1,\ldots,(n-1)$. In other words (2.15) must be satisfied implying the uniqueness of $g(x)$. Let us denote this unique solution by

$$B_m(x)=\theta+\sum_{i=m}^{m+n-2}\gamma_i^{(m)}(x)(\theta-x)^i. \tag{2.18}$$

Next we prove that $B_m(x)$ and $B_{m+1}(x)$ are distinct. Otherwise, we must have $\Gamma_m(x)=\Gamma_{m+1}(x)$. From (2.12) the following must hold

$$\Pi_m(x)\Gamma_m(x)=\Pi_{m+1}(x)\Gamma_m(x). \tag{2.19}$$

Equivalently,

$$(\Pi_{m+1}(x)-\Pi_m(x))\Gamma_m(x)=0. \tag{2.20}$$

But by the same reasoning as before, the matrix of the leading coefficient of (2.20) is $H_{m+1}(x)$, whose determinant is nonzero. This implies $\Gamma_m(x)\equiv 0$, a contradiction. Hence the proof of distinctness

Next we prove that for all $m=2$ the leading coefficient, $\gamma_m^{(m)}(x)$, and the last coefficient, $\gamma_{m+n-2}^{(m)}(x)$, are nonzero functions. As we have seen for $m=2$, the function $x-p/p'$ (see (1.15)) lies in $\overline{S}(2,n)$. From uniqueness we must have $B_2(x)=x-p/p'$. Now assume $m>2$ and consider $B_{m-1}(x)$, $B_m(x)$, and $B_{m+1}(x)$. If the leading coefficient of $B_m(x)$ is identically zero, then

$$\frac{1}{2}B_m(x)+\frac{1}{2}B_{m+1}(x)\in\overline{S}(m+1,m+n-1),$$

contradicting uniqueness. If the last coefficient of $B_m(x)$ is identically zero, then

$$\frac{1}{2}B_m(x)+\frac{1}{2}B_{m-1}(x)\in\overline{S}(m-1,m+n-3),$$

again a contradiction.

To complete the proof we must show that the leading coefficient of $B_m(x)$, as well as the last coefficient remain to be nonzero even if $p(x)$ has multiple roots. But in this case too we can arrive at the same set of polynomials $P_i^{m+j}(x)$ and hence the same equation (2.11). $\square$

3. DERIVING CLOSED FORM FOR MEMBERS OF THE BASIC FAMILY

In this section we shall derive the closed form for members of the Basic Family. In doing so, it is more convenient to consider the functions in $\overline{S}(m, M)$, with $m \geq 2$, $M \geq m+n-2$. Recall that $g \in \overline{S}(m, M)$ implies there exists $\gamma_i(x) \in K(x)$, $i = m, \dots, M$, such that $p(y)$ divides

$$\gamma_0(y) = -g(x) + y + \sum_{l=m}^{M} \gamma_l(x)(y-x)^l, \tag{3.1}$$

in $K(x)[y]$. Equivalently, if we let $z = y - x$, then $f(z) = p(z+x)$ must divide $h(z) = \gamma_0(z+x)$. From Taylor's Theorem as applied to the polynomial $p(y)$ we have

$$f(z) = \sum_{i=0}^{n} \frac{p^i(x)}{i!} z^i. \tag{3.2}$$

Since $\gamma_0(x) = x - g(x)$,

$$h(z) = \gamma_0(x) + z + \sum_{l=m}^{M} \gamma_l(x) z^l. \tag{3.3}$$

From the divisibility condition, there exist a polynomial

$$q(z) = \sum_{j=0}^{M-n} v_j z^j, \tag{3.4}$$

with $v_j \in K(x)$ such that

$$h(z) = f(z)q(z). \tag{3.5}$$

Equivalently, while suppressing x for brevity we get the system of equations

$$pv_0 = \gamma_0, \tag{3.6}$$

$$pv_0 + p'v_1 = 1, \tag{3.7}$$

$$\sum_{i+j=l} \frac{p^{(i)}}{i!} v_j = 0, \quad l = 2, \dots, m-1, \tag{3.8}$$

$$\sum_{i+j=l} \frac{p^{(i)}}{i!} v_j = \gamma_l(x), \quad l = m, \dots, M. \tag{3.9}$$

The matrix corresponding to the first m equations, i.e. (3.6), (3.7), and (3.8) is precisely the matrix L_m (see (1.1)). Letting b to be the m-vector $(\gamma_0, 1, 0, \dots, 0)^T$ and v the m-vector $(v_0, \dots, v_{m-1})^T$, we get

$$L_m v = b. \tag{3.10}$$

Lemma 3.1. For any natural number k the matrix

$$Z_{n,k} = \begin{pmatrix} \binom{n}{1} & \binom{n}{0} & 0 & \dots & 0 \\ \binom{n}{2} & \binom{n}{1} & \binom{n}{0} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{k} & \binom{n}{k-1} & \binom{n}{k-2} & \dots & \binom{n}{1} \end{pmatrix},$$

is invertible.

Proof. All that is needed is to reduce it the a matrix of the type $U_{m,r}$ (see (2.2)). Using the identity $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$, starting with the last column of $Z^1 = Z_{n,k}$ we add to each column its previous column to get a new matrix Z^2. We repeat this process to columns of Z^2 except for its 2-nd column and get a new matrix Z^3. Next we repeat the process to columns of Z^3 except for its 3-rd column. This process will eventually result in a matrix of the type $U_{m,r}$. $\Box$.

From Lemma 3.1, analogous to the proof of invertibility of $H_m(x)$, we can immediately conclude

Corollary 3.1. For each natural number $m \geq 2$, the matrix $L_m^{(1)}$ is invertible. $\Box$

Theorem 3.1. Let $B_m(x)$ be the function in $\overline{S}(m, m+n-2)$ whose existence was proved in Theorem 2.1. Then

$$B_m(x) = x - p(x)\ \frac{det(L_{m-1}^{(1)}(x))}{det(L_m^{(1)}(x))}.$$

Moreover, it lies in $S(m, m+n-2)$, and if θ is a simple root of $p(x)$ then

$$\gamma_m^{(m)}(\theta) = (-1)^m \frac{det(L_{m+1}^{(2)}(\theta))}{det(L_m^{(1)}(\theta))}.$$

Proof. Letting Δ_j to be the determinant of L_m with the j-th column replaced with b, from Cramer's rule as applied to the system (3.10) we get

$$v_j = \frac{\Delta_j}{det(L_m)}, \quad j = 0, \dots, m-1. \tag{3.11}$$

Since p is nonzero, $det(L_m)$ is not identically zero, thus each v_j is well-defined and unique. Since for $B_m(x)$, $M = m + n - 2$, we must have $v_{m-1} = 0$. In particular by expanding the determinant of Δ_{m-1} along its last column, and then expanding the second resulting determinant along its first row, we get

$$v_{m-1} = \Delta_{m-1} = \gamma_0 det(L_m^{(1)}) - p(x) det(L_{m-1}^{(1)}) = 0. \qquad (3.12)$$

From Corollary 3.1, $L_m^{(1)}$ is invertible so that (3.12) can be solved for γ_0 which results in the closed formula for $B_m(x)$. To show $B_m(x)$ also lies in $S(m, m + n - 2)$, we will compute $\gamma_l^{(m)}(\theta)$, $l = m, \ldots, m + n - 2$.

First consider $l = m$. The $m + 1$ equations corresponding to (3.6), (3.7), (3.8), and the first equation of (3.9) at $x = \theta$. Since the first of these equations is zero, if we let $b^{(m)}(\theta)$ denote the m-vector $(1, 0, \ldots, 0, \gamma_m^{(m)}(\theta))$, we get the following $m \times m$ system

$$L_{m+1}^{(1)}(\theta) v(\theta) = b^{(m)}(\theta). \qquad (3.13)$$

The above system has a unique solution since the coefficient matrix is a lower triangular matrix with diagonal entries $p'(\theta) \neq 0$. Using the fact that $v_{m-1}(\theta) = 0$, the Cramer's rule, and by expansion along the last column of the matrix $L_{m+1}^{(1)}(\theta)$ with its last column replaced by $b^{(m)}(\theta)$, we get

$$\gamma_m^{(m)}(\theta) det(L_m^{(1)}(\theta)) - (-1)^{m-1} det(L_{m+1}^{(2)}(\theta)) = 0. \qquad (3.14)$$

The above can be solved for $\gamma_m^{(m)}(\theta)$ which results in the stated equation of the theorem. For $l = m + 1, \ldots, m + n - 2$, the corresponding equation in $\gamma_i^{(m)}$ is of the form

$$\gamma_l^{(m)}(\theta) det(L_m^{(1)}(\theta)) + w(\theta), \qquad (3.15)$$

for some $w(\theta)$, hence are well-defined proving that $B_m(x)$ lies in $S(m, m + n - 2)$. □

Let us treat $\gamma_0(x)$ in equation (3.6) also as a variable, and consider the the m equation in $m + 1$ unknowns corresponding to equation (3.6) – (3.8), i.e.

$$L_m v - \gamma_0 e_1 = e_2, \qquad (3.16)$$

where e_i is the column m-vector with the i-th component equal to 1, and all other components equal to 0. A *basic solution* of the above system is a solution with at most m

nonzero components. The following is evident

Theorem 3.2. The basic solution of the system of equations in (3.16) corresponding to $v_i = 0$, $i = 0, \ldots, m-1$ are $B_{i+1}(x)$, where $B_0(x) = x$. □

Remark 3.1. If $g \in S(m, M)$ has the property that its corresponding equation in (3.10) satisfies $v_{m-1} = 0$, and $v_m = 0$ (a condition that is necessarily satisfied for $M = m + n - 2$), then $\gamma_m(x)$ depends only on $p^{(i)}(x)$, $i = 0, \ldots, m$, i.e. the simple-root-depth of g will be m.

Remark 3.2. Since Taylor's Theorem holds in any field of characteristic zero, the existence of $B_m(x)$ extends to the case where K is such a field.

4. TWO FORMULAS AND THEIR APPLICATIONS.

In this section we derive two formulas that will result in the generation of very large class of iteration functions. Both will all rely on (1.12).

To derive the first formula, suppose that $g \in S(m, M)$, with $m \geq 1$, i.e.

$$g = \theta + \sum_{i=m}^{M} \gamma_i (\theta - x)^i. \qquad (4.1),$$

where for the sake of simplicity we have suppressed the argument x. From (1.12) we have

$$(-p)^m = \sum_{i=m}^{M'} \mu_i^{(m)} (\theta - x)^i, \qquad (4.2)$$

where

$$\mu_i^{(m)} = \sum_{i_1 + \ldots + i_m = i, i_j \geq 1} \frac{p^{(i_1)} \cdots p^{(i_m)}}{i_1! \cdots i_m!}, \qquad i = m, \ldots, mn. \qquad (4.3)$$

Multiplying (4.2) by $-\gamma_m / \mu_m^{(m)} = -\gamma_m / p'^m$ and adding to (4.1) we get

$$h = g - (-1)^m \gamma_m \left(\frac{p}{p'}\right)^m = \theta + \sum_{i=m+1}^{M'} \eta_i (\theta - x)^i, \qquad (4.4)$$

where

$$M' = \max\{M, mn\},$$

$$\eta_i = \gamma_i - \gamma_m \frac{\mu_i^{(m)}}{(p')^m}, \quad i = m+1, \ldots, M', \tag{4.5}$$

with $\gamma_i \equiv 0$ for $i > M$, and $\mu_i^{(m)} \equiv 0$ for $i > mn$.

Theorem 4.1. Assume that the leading coefficient of h is not identically zero. Then, h lies in $S(\overline{m}, \overline{M})$, for some $\overline{m} \geq m+1$, $\overline{M} \leq M'$. If $\overline{m} = m+1$, then

$$\eta_{m+1}(\theta) = \gamma_{m+1}(\theta) - \gamma_m(\theta) \frac{m}{2} \frac{p''(\theta)}{p'(\theta)}. \tag{4.6}$$

Proof. Let θ be a simple root of $p(x)$. Since $g \in S(m, M)$ for $i = m, \ldots, M$, $\gamma_i(\theta)$ is well-defined. Clearly, for $i = m+1, \ldots, mn$, $\eta_i(\theta)$ is well-defined. The equation (4.6) follows from (4.5) and that $\mu_{m+1}^{(m)} = (m/2)(p'' p'^m)$. □

Next we derive another formula. Let g be in $S(m-1, M_{m-1})$, and h in $S(m, M_m)$. Let $M_{m+1} = \max\{M_m, M_{m-1}, m+n-1\}$. We write

$$g = \theta + \sum_{i=m-1}^{M_{m+1}} \gamma_i (\theta - x)^i, \tag{4.7}$$

where

$$\gamma_i \equiv 0, \quad \forall i > M_{m-1},$$

$$h = \theta + \sum_{i=m}^{M_{m+1}} \eta_i (\theta - x)^i, \tag{4.8}$$

where

$$\eta_i \equiv 0, \quad \forall i > M_m.$$

From (4.7) and (4.8) we get

$$g - h = \gamma_{m-1} (\theta - x)^{m-1} + \sum_{i=m}^{M_{m+1}} (\gamma_i - \eta_i)(\theta - x)^i. \tag{4.9}$$

Also, multiplying (1.12) by $(\theta - x)^{m-1}$, and since $p^j \equiv 0$, for $j > n$, we get

$$0 = p(\theta - x)^{m-1} + \sum_{i=m}^{M_{m+1}} \frac{p^{(i+1-m)}}{(i+1-m)!} (\theta - x)^i. \tag{4.10}$$

Multiplying (4.9) by p, and (4.10) by $-\gamma_{m-1}$ and adding the resulting equations we get

$$p(g - h) = \sum_{i=m}^{M_{m+1}} \Big(p(\gamma_i - \eta_i) - \gamma_{m-1} \frac{p^{(i+1-m)}}{(i+1-m)!} \Big) (\theta - x)^i. \tag{4.11}$$

Multiplying (4.11) by $-\eta_m/D$, where

$$D = [p(\gamma_m - \eta_m) - p'\gamma_{m-1}], \tag{4.12}$$

and adding the result to (4.8) we get

$$f = h - \eta_m \frac{p(g-h)}{D} = \theta + \sum_{i=m+1}^{M_{m+1}} \phi_i (\theta - x)^i, \tag{4.13}$$

with

$$\phi_i = \eta_i - \eta_m \frac{p(\gamma_i - \eta_i) - \gamma_{m-1}\frac{p^{(i+1-m)}}{(i+1-m)!}}{D}, \quad \forall i = m+1, \ldots, M_{m+1}. \tag{4.14}$$

Theorem 4.2. Suppose that D is not identically zero. Then $f \in S(\overline{m}, \overline{M})$, where $\overline{m} \geq m+1$, $\overline{M} \leq M_{m+1}$. If $\overline{m} = m+1$, then

$$\phi_{m+1}(\theta) = \eta_{m+1}(\theta) - \eta_m(\theta) \frac{p''(\theta)}{2p'(\theta)}. \tag{4.15}$$

Proof. Let θ be a simple root of $p(x)$. Clearly f lies in $\overline{S}(\overline{m}, \overline{M})$ for some $m+1 \leq \overline{m} \leq \overline{M} \leq M_{m+1}$. Since γ_{m-1} is not identically zero, and since it is a rational function over K, together with the fact that $p'(\theta) \neq 0$ it follows from (4.14) that for $i = m+1, \ldots, M_{m+1}$ we have

$$\lim_{x \to \theta} \phi_i(x) = \eta_i(\theta) - \eta_m(\theta) \frac{p^{(i+1-m)}(\theta)}{(i+1-m)! p'(\theta)},$$

i.e. $\phi_i(\theta)$ is well-defined. In particular we get (4.15). $\square$

Next we consider two application of these formulas.

Theorem 4.3. Suppose that in (4.13), $g = B_{m-1}$ and $h = B_m$, then $f = B_{m+1}$, for all $m \geq 2$.

Proof. It suffices to prove this under the assumption that p has only simple roots. In order to prove the theorem we first need to show that the function D is not identically zero. From (4.11) we have

$$p(g-h) = D + \sum_{i=m+1}^{m+n-1} \left(p(\gamma_i - \eta_i) - \gamma_{m-1} \frac{p^{(i+1-m)}}{(i+1-m)!} \right) (\theta - x)^i. \tag{4.16}$$

If D is identically zero, then

$$B_{m+1} + p(g-h) \in S(m+1, m+n-1). \tag{4.17}$$

But this contradicts the uniqueness result proved in Theorem 2.1. Thus, D is not identically zero, implying that $f \in S(m+1, m+n-1)$. Again by Theorem 2.1, we must have that $f = B_{m+1}$. $\square$

Theorem 4.4. Consider the family $E_1, E_2, E_3, \ldots,$ of iteration functions obtained from the repeated application of (4.4) starting with $E_1 = B_1 = x - p(x)$. For $m \geq 3$, $E_m \in S(m, mn)$. Moreover, the depth of E_m is $m-1$.

Proof. The fact that E_m lies in $S(m, mn)$ is a trivial consequence of Theorem 4.1. The claim on the depth can be proved inductively using that theorem and (4.5). $\square$

For a given i let us denote the γ_i's corresponding to E_m by $\gamma_i^{(m)}$'s. We now obtain $\gamma_m^{(m)}$ for $m = 1, 2, 3, 4,$ and 5. From (4.4) this will give E_2, E_3, E_4, and E_5. From (1.14) we have $\gamma_1^{(1)} = (p' - 1)$. This gives $E_2 = x - p/p'$. It follows that $\gamma_2^{(2)} = p''/2p'$, $\gamma_3^{(2)} = p'''/3!p'$, so

$$\gamma_3^{(3)} = \frac{p'''}{6p'} - \frac{2p''}{4p'^2}.$$

Next we need $\gamma_4^{(4)} = \gamma_4^{(3)} - \gamma_3^{(3)}(3/2)(p''/p')$. This in turn requires the computation of $\gamma_4^{(3)} = \gamma_4^{(2)} - \gamma_2^{(2)}\mu_4^{(2)}/p'^2$. Note that $\gamma_4^{(2)} = p^{(4)}/4!p'$, and $\mu_4^{(2)} = 2p'p'''/3! + p''^2/4$. Substituting these and simplifying we get

$$\gamma_4^{(4)} = \frac{p^{(4)}}{4!p'} - \frac{5p''p'''}{12p'^2} + \frac{5p''^3}{8p'^3}.$$

Thus,

$$E_5 = x - (\frac{p}{p'}) + (\frac{p}{p'})^2 \frac{p''}{2p'} - (\frac{p}{p'})^3 (\frac{p'''}{6p'} - \frac{2p''}{4p'^2}) + (\frac{p}{p'})^4 (\frac{p^{(4)}}{4!p'} - \frac{5p''p'''}{12p'^2} + \frac{5p''^3}{8p'^3}),$$

and $E_2 - E_5$ coincide with the first few members of a family credited both to Euler as well as Schröder. For historical comments as well as other approaches for deriving this family see Shub and Smale [19].

5. EXTENSIONS TO NON-POLYNOMIAL ROOT FINDING

In the previous sections we have shown the existence of many fixed point iteration functions for finding roots of a given polynomial $p(x)$, namely functions within $S(m, M)$, any of whose members, say $g(x)$, has m-th order rate of convergence to any simple root θ of $p(x)$. In this section we will prove that these high order methods derived for polynomials extend to high order methods for arbitrary smooth functions $f(x)$, having the same order of convergence to simple roots, where in the formulas for $g(x)$, one simply replaces $p(x)$ and its higher derivatives with $f(x)$ and its corresponding derivatives. Moreover, for a simple root θ, its asymptotic constant of convergence $\gamma_m(\theta)$ can be obtained by simply replacing $p^{(j)}(\theta)$ with $f^{(j)}(\theta)$. In particular, both depth and simple-root-depth will be unchanged.

Theorem 5.1. Assume that there exists $h : \Re^m \to \Re$, $m \geq 2$, so that for any polynomial $p(x)$ of degree at most $(2m-1)$ the following property holds: If θ is a simple root of $p(x)$, then h is in C^m in a neighborhood of the point $z_p(\theta) = (\theta, p(\theta), p'(\theta), \ldots, p^{(m-1)}(\theta))$, and the function $g_p(x) = h(x, p(x), p'(x), \ldots, p^{(m-1)}(x))$, satisfies

$$g_p(\theta) = \theta, \tag{i}$$

$$g_p^{(i)}(\theta) = 0, \quad i = 1, \ldots, m-1. \tag{ii}$$

Then, the above properties can be extended to any function $f(x)$ which is in C^{2m-1} in a neighborhood of a simple root θ in the sense that $p^{(j)}$ can be replaced with $f^{(j)}$. Moreover

$$g_f^{(m)}(\theta) = g_{p_n}^{(m)}(\theta), \tag{iii}$$

where $p_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(\theta)}{i!}(x-\theta)^i$, the Taylor polynomial of degree $n = 2m-1$ at θ.

Proof. Since f is in C^n, $f^{(j)}(x) = p_n^{(j)}(x) + r_n^{(j)}(x)$, $j = 0, \ldots, n$, where $r_n^{(0)}(x)$ is the remainder term. Since $f^{(j)}(\theta) = p_n^{(j)}(\theta)$ for all $j = 0, \ldots, n$ we have

$$r_n^{(j)}(\theta) = 0, \quad j = 1, \ldots, n. \tag{5.1}$$

In particular, θ is a simple root of $p_n(x)$, and properties (i) and (ii) apply to $p_n(x)$. Also, the fact that h is in C^n in a neighborhood of the point $z_f(\theta)$ follows immediately. From the chain rule we have

$$g'_f = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial f} f' + \frac{\partial h}{\partial f'} f'' + \ldots + \frac{\partial h}{\partial f^{(m-1)}} f^{(m)}. \tag{5.2}$$

If we now substitute for $f(x)$ and $f^{(j)}(x)$ in (5.2), using (5.1) and the continuity of the partial derivatives of h we conclude that $g'_f(x)$ is continuous in an interval containing $z_f(\theta)$. From the repeated application of the chain rule, the assumption that h is in C^m, that f is in C^{2m-1}, and the use of (5.1), it follows by induction that for all $j = 0, \ldots, m$, $g_f^{(j)}(x)$ is continuous in an interval containing $z_f(\theta)$. Moreover $g_f^{(j)}(\theta) = g_{p_n}^{(j)}(\theta)$, for all $j = 0, \ldots, m$. □

Corollary 5.1. The depth of $B_m(x)$ with respect to $f \in C^{2m-1}$ remains to be $m-1$. Moreover, the simple-root-depth is m.

As an application of the above consider the replacement of f in Newton's method with $f(x)/f'(x)$ which results in a new iteration function $g(x)$ credited to Schröder (e.g. see Scavo and Thoo [18]). Then, from the corollary $g''(\theta)/2$ can easily be computed in terms of f and its higher derivatives.

Remark 5.1. The assumption $f \in C^{2m-1}$ in the above corollary can be relaxed for $B_m(x)$. It suffice to have $f \in C^m$. For example for $B_2(x) = x - p(x)/p'(x)$, it is enough to assume that $f \in C^2$, despite the fact that $B''_2(x)$ relies on the third derivative of f.

Concluding Remarks.

In this paper we have characterized a family of rational iteration functions for polynomial root-finding having an m-th order rate of convergence to simple roots. Although individual members of the family like Newton's and Halley's iteration functions have been known, no closed formula for the general member of the Basic Family, nor their asymptotic constant had previously been known. These formulas hold in the more general case where the underlying field K is any field of characteristic zero. We have shown that the members of this family are "optimal" with respect to the notions of depth, width, simple-root-depth (see Definition 1.2, and Remark 1.2). Under the assumption of the simplicity of the roots, the m-th order iteration function was shown to be the unique member of the class $S(m, m+n-2)$. We also described two simple recursive formulas for generation of new iteration functions which in particular resulted in the generation of the Euler-Schröder family whose m-th order member belongs to $S(m, mn)$, $m > 2$. Shub and Smale [19] refer to an "incremental" version of Euler-Schröder family as the "the most appropriate for practically computing zeros of complex polynomials" (see Example 5_k, page 110). In view of the development of the Basic Family, the closed forms, and properties established in this paper, it is fair to venture that their incremental version would be at least as good as the Euler-Schröder family. In particular, the closed form of the Basic Family suggests parallel implementations (see e.g. [15] for a theoretical analysis of parallel implementation of the Basic Family as applied to square-root computation). Another theoretical and practical problem of interest is the study of global convergence of the Basic Family.

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References

[1] Alefeld, G.(1981). On the convergence of Halley's Method. *Amer. Math. Monthly* 88, 530-536.

[2] Atkinson, K.E.(1989). *An Introduction to Numerical Analysis.* Second Edition, John Wiley and Sons, Inc.

[3] Bateman, H.(1938). Halley's method for solving equations. *Amer. Math. Monthly* 45, 11-17.

[4] Bodewig, E.(1949). On types of convergence and on the behavior of approximations in the neighborhood of a multiple root of an equation. *Quart. App. Math.* 7, 325-333.

[5] Brown, G.H., Jr.(1977). On Halley's variation of Newton's method. *Amer. Math. Monthly* 84, 726-728.

[6] Davies, M and Dawson, B.(1975). On the global convergence of Halley's iteration formula. *Numer. Math.* 24, 133-135.

[7] Frame, J.S.(1944). A variation of Newton's method. *Amer. Math. Monthly* 51, 36-38.

[8] Frame, J.S.(1945). Remarks on a variation of Newton's method. *Amer. Math. Monthly* 52, 212-214.

[9] Frame, J.S.(1953). The solution of equations by continued fraction. *Amer. Math. Monthly* 60, 293-305.

[10] Gander, W.(1985). On Halley's iteration method. *Amer. Math. Monthly* 92, 131-134.

[11] Gerlach, J.(1994). Accelerated convergence in Newton's method. *SIAM Review.* 36, 272-276.

[12] Halley, E. (1694). A new, exact, and easy method of finding roots of any equations generally, and that without any previous reduction (Latin). *Philos. Trans. Roy. Soc. London* 18 136-148. [English translation: *Philos. Trans. Roy. Soc. London*(abridged) 3, 640-649 (1809).]

[13] Hamilton, H.J.(1950). A type of variation of Newton's method. *Amer. Math. Monthly* 57,517-522.

[14] Hansen, E. and Patrick, M.(1977). A family of root finding methods. *Numer. Math.* 27, 257-269.

[15] Kalantari, B. and Kalantari, I.(1993). High order iterative methods for approximating square and cube roots. Technical Report LCSR-TR-213, Department of Computer Science, Rutgers University, New Brunswick, NJ.

[16] Kalantari, B. and Kalantari, I.(1993). High order iterative methods for approximating roots of polynomials. Technical Report LCSR-TR-214, Department of Computer Science, Rutgers University, New Brunswick, NJ.

[17] Popovski, D.B.(1980). A family of one-point iteration formulae for finding roots. *Internat.. J. Computer Math.* 8, 85-88.

[18] Scavo, T.R. and Thoo, J.B.(1995). On the geometry of Halley's method. *Amer. Math. Monthly* 102, 417-426.

[19] Shub, M. and Smale, S.(1985). On the geometry of polynomials and a theory of cost: Part I. *Ann. Scient. Ec. Norm. Sup*, 18, 107-142.

[20] Smale, S.(1994). Private communications with the first author.

[21] Snyder, R.W.(1955). One more correction formula. *Amer. Math. Monthly* 62, 722-725.

[22] Stewart, J.K.(1951). Another variation of Newton's method. *Amer. Math. Monthly* 58, 331-334.

[23] Traub, J.F.(1964). *Iterative Methods for the Solution of Equations.* New Jersey, Prentice Hall.

[24] van der Waerden, B.L.(1970) *Algebra*, Volume 1. F. Ungar Publishing Co., New York.

[25] Wall, H.S.(1948). A modification of Newton's method. *Amer. Math. Monthly* 55, 90-94.