Constructing Piecewise Linear Homeomorphisms

Diane Souvaine*
and Rephael Wenger†

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Abstract

Let \( P = \{p_1, \ldots, p_n\} \) and \( Q = \{q_1, \ldots, q_n\} \) be two point sets lying in the interior of rectangles in the plane. We show how to construct a piecewise linear homeomorphism of size \( O(n^3) \) between the rectangles which maps \( p_i \) to \( q_i \) for each \( i \). This bound is optimal in the worst case; i.e., there exist point sets for which any piecewise linear homeomorphism has size \( \Omega(n^3) \).

Introduction

A homeomorphism is a 1-1, onto, continuous map with continuous inverse. Problems of constructing homeomorphisms arise in cartography, animation and computational fluid dynamics. A cartographer may wish to merge two similar maps, perhaps slightly distorting one, so that common landmarks coincide. A computer animator may want to transform one shape into the another, while preserving certain features. An aeronautical engineer using computational fluid dynamics may need to map a mesh onto the region surrounding the wing of a plane.

Each application places different requirements on the choice of homeomorphism. Simplicity, robustness, and complexity of the construction, as well as smoothness, angular distortion and ease of modification of the resulting homeomorphism, are factors of varying importance in the various algorithms and techniques used in constructing homeomorphisms.

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†Ohio State University, Columbus, OH 43210, U.S.A. (wenger@cis.ohio-state.edu). Supported in part by NSA grant MDA904-93-H-3026 and by the NSF Regional Geometry Institute (Smith College, July 1993) grant DMS-90 13220.
Figure 1: Additional vertices are required to construct isomorphic triangulations of $P$ and $Q$.

In this paper we consider “piecewise linear” homeomorphisms in which two regions are partitioned into corresponding pieces, linear maps are defined between the pieces, and the linear maps are combined to give a homeomorphism between the original regions. Such “piecewise linear” homeomorphisms are simple to construct and modify and their complexity can be easily quantified and measured. Their primary drawback is their lack of smoothness between pieces and the possibility of unnecessarily introducing large angular distortion. Nevertheless, their innate simplicity recommends them for constructing homeomorphisms between complex regions or for constructing homeomorphisms under many constraints.

In [3], Saalfeld proposed using piecewise linear homeomorphisms for map conflation, the process of merging cartographic maps. Given two cartographic maps of the same geographic area with identified corresponding point landmarks $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$, define a homeomorphism between the two which sends $p_i$ to $q_i$. If such a homeomorphism does not introduce too much distortion, it will identify each point in one cartographic map with a reasonably close duplicate in the other.

Saalfeld constructed his homeomorphism by partitioning the cartographic maps into corresponding triangles, defining a linear homeomorphism between the triangles, and combining the triangles to give a piecewise linear homeomorphism. The challenging step is partitioning the cartographic maps into corresponding triangles. Ideally, all vertices of the triangles would come from the original point sets $P$ and $Q$. However, this is not always possible.
and additional vertices may be required. (See Figure 1 where any triangulation with vertex set $P$ contains triangle $p_3, p_4, p_6$ while any triangulation with vertex set $Q$ contains triangle $q_3, q_4, q_5$ and not triangle $q_3, q_4, q_6$.) By Euler's formula, the number of triangles used in defining a piecewise linear homeomorphism and hence its "complexity" is a proportional to the number of additional vertices. Finding the minimum number of additional vertices required is an open problem. This paper shows that in the worst case at most $O(n^2)$ vertices are required and gives an $O(n^2)$ algorithm for constructing a homeomorphism of that size. By a result of Pach, Shahrokhi and Szegedy, these bounds are asymptotically tight [2].

Formal Definitions and Statement of Goals

A homeomorphism $h$ from region $R_p$ to region $R_q$ is piecewise linear if there is some triangulation of $R_p$ such that $h$ is linear on each triangle in the triangulation. Given a piecewise linear homeomorphism $h$ there are many such triangulations of $R_p$. Define the size of a piecewise linear homeomorphism $h$ between compact regions $R_p$ and $R_q$ as the fewest number of vertices, edges and triangles among all such triangulations of $R_p$. If $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ are point sets in $R_p$ and $R_q$, respectively, we wish to construct the smallest piecewise linear homeomorphism from $R_p$ to $R_q$ which maps $p_i$ to $q_i$ for each $i$.

Let $h$ be a piecewise linear homeomorphism from $R_p$ to $R_q$ and let $\tau_p$ be a triangulation of $R_p$ such that $h$ is linear on each triangle in $\tau_p$. The map $h$ and triangulation $\tau_p$ induce a triangulation $\tau_q$ on $R_q$ where each vertex, edge and triangle in $\tau_p$ maps to a corresponding vertex, edge and triangle of $\tau_q$.

Conversely, assume $R_p$ and $R_q$ have isomorphic triangulations $\tau_p$ and $\tau_q$ where every vertex, edge and triangle, $v, e, t \in \tau_p$ corresponds to a unique vertex, edge and triangle $v', e', t' \in \tau_q$ and this correspondence preserves incidence relations. Such isomorphic triangulations of $R_p$ and $R_q$ are called joint triangulations in [3] and compatible triangulations in [1]. Triangulations $\tau_p$ and $\tau_q$ define a piecewise linear homeomorphism $h$ between $R_p$ and $R_q$ as follows. For every vertex $v \in \tau_p$ corresponding to vertex $v' \in \tau_q$, let $h(v) = v'$.

For every point $p \in R_p$ lying in triangle $v_1, v_2, v_3$ of $\tau_p$, express $p$ in barycentric coordinates as $p = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ and let $h(p) = \alpha_1 v'_1 + \alpha_2 v'_2 + \alpha_3 v'_3$. The map $h$ is a piecewise linear homeomorphism from $R_p$ to $R_q$.

In [3], Saalfeld gives a method for constructing isomorphic triangulations between the convex hull of a point set $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$.
such that \( p_i \) and \( q_i \) are corresponding vertices for each \( i \). Of course, no such triangulation is possible if the vertices in clockwise order around the convex hull of \( P \) do not correspond to the vertices in clockwise order around the convex hull of \( Q \). Saalfeld's construction uses an exponential number of additional vertices, edges and triangles.

In [1], Aronov, Seidel and Souvaine show how to construct isomorphic triangulations of two simple polygons with vertices \( P = \{p_1, \ldots, p_n\} \) and \( Q = \{q_1, \ldots, q_n\} \) in clockwise order such that \( p_i \) and \( q_i \) are corresponding vertices for each \( i \). Their construction has \( O(n^2) \) vertices, edges and triangles. Moreover, they prove their construction asymptotically optimal in the worst case by giving examples of polygons which require \( \Omega(n^2) \) triangles to construct isomorphic triangulations. Any piecewise linear homeomorphism between such polygons has \( \Omega(n^2) \) size.

In this paper we show how to construct isomorphic triangulations of size \( O(n^2) \) of rectangles with interior points \( P = \{p_1, \ldots, p_n\} \) and \( Q = \{q_1, \ldots, q_n\} \) such that \( p_i \) and \( q_i \) are corresponding vertices for each \( i \). Such triangulations induce homeomorphisms of size \( O(n^2) \). A recent result of Pach, Shahrokhi and Szegedy implies that such a construction is optimal in the worst case; i.e., there exist point sets for which any isomorphic triangulations and piecewise linear homeomorphism have size \( \Omega(n^2) \) [2]. These results also give \( \Theta(n^2) \) bounds for the problem considered by Saalfeld.

### Constructing Isomorphic Triangulations

We start with a slight generalization of a theorem from [1].

**Theorem 1 (Aronov, Seidel, Souvaine [1])** Let \( P = \{p_1, \ldots, p_n\} \) and \( Q = \{q_1, \ldots, q_n\} \) be sets of points in clockwise order around the boundaries of simple polygons \( R_p \) and \( R_q \) such that \( P \) and \( Q \) contain the vertices of \( R_p \) and \( R_q \) respectively. There exist isomorphic triangulations of \( R_p \) and \( R_q \) of size \( O(n^2) \) such that \( p_i \) and \( q_i \) are corresponding triangulation vertices for each \( i \). Moreover, the boundaries of \( R_p \) and \( R_q \) contain no triangulation vertices other than those in \( P \) and \( Q \).

**Outline of proof:** The paper [1] contains a proof when \( P \) and \( Q \) are exactly the vertex sets of \( R_p \) and \( R_q \), respectively, but the proof for this generalized version is the same. The authors construct isomorphic triangulations of \( R_p \) and \( R_q \) using spiderweb patterns of \( \lceil (n - 5)/2 \rceil \) nested polygons, each with \( n \) Steiner points as vertices, a single Steiner point in the interior of
the innermost polygon and connected to each vertex of that polygon, spokes joining corresponding corners of neighbor polygons and canonical diagonals. (See Figure 2.) The spiderweb patterns are constructed by starting from the original polygons, shrinking them slightly to form smaller nested polygons, and cutting off two ears (a triangle containing two polygon edges which lies completely in the polygon) from each of the interior polygons. The process is repeated until the innermost polygons have five or fewer vertices at which point a single interior vertex is connected to the vertices of the innermost polygon. The resulting structure has size $O(n^2)$ and can be constructed in $O(n^2)$ time.

We are now ready for the main theorem of this paper.

**Theorem 2** Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be sets of $n$ distinct points in the interior of rectangles $R_p$ and $R_q$, respectively. Isomorphic triangulations of $R_p$ and $R_q$ of size $O(n^2)$ where $p_i$ corresponds to $q_i$, $i \leq n$, can be constructed in $O(n^2)$ time.
Figure 3: Constructing $\gamma_q$.

**Proof:** We first consider the case where the points in $P$ and in $Q$ lie on vertical lines $l_p$ and $l_q$ through the center of each rectangle. (See Figure 3.) Relabel the points in $P$ and $Q$ so that $p_1, \ldots, p_n$ lie in order along $l_p$. Obviously, the corresponding points $q_1, \ldots, q_n$ will not necessarily be in order along $l_q$. We will construct a piecewise linear simple (non self-intersecting) curve $\gamma_q$ of $R_q$ through the points in $Q$ in the order $q_1, \ldots, q_n$.

Choose a point on the boundary of $R_q$ left of $l_q$ and connect it to $q_1$ crossing over $l_q$. Recross $l_q$ and connect to $q_2$, again crossing over $l_q$. Recross $l_q$ and connect to $q_3$ dipping to the right of $q_1$, if necessary, and to the left of all other intervening points. In general, connect $q_i$ to $q_{i+1}$ by dipping to the left of all points $q_j$ where $j > i$ and to the right of all points $q_j$ where $j < i$. Cross $l_q$ at any point of intersection. Connect $q_n$ to a point to the
right of $l_q$. The curve between any points $q_i$ and $q_{i+1}$ requires at most $O(n)$ crossings and so can be realized by $O(n)$ line segments. Thus $\gamma_q$ contains $O(n^2)$ line segments. These line segments can all be drawn parallel to the coordinate axes. Note also that $\gamma_q$ intersects $l_q$ at $O(n^2)$ points and requires only three line segments between any two intersection points with $l_q$.

Curve $\gamma_q$ can be constructed in $O(n^3)$ time as follows. Coordinate the rectangle $R_q$ so that the left and right edges of the rectangle lie on the line $x = n - 1$ and $x = n + 1$, respectively. Line $l_q$ lies on the line $x = 0$. Let $\gamma_q^i$ be the portion of $\gamma_q$ connecting $q_i$ to $q_{i+1}$. Draw the vertical line segments of $\gamma_q$ on the lines $x = -n + i$ and $x = i$. If $\gamma_q^i$ crosses $l_q^j$ between $q_j$ and $q_{j'}$, $j \leq i < j'$, draw a horizontal line segment $i/n$ of the distance from $q_j$ to
Connect $\gamma_q^i$ to its endpoints $q_i$ and $q_{i+1}$ with horizontal line segments at the same vertical altitude as $q_i$ and $q_{i+1}$. Connect the boundary to $q_1$ by a vertical segment on the line $x = -n$ and a horizontal segment at the altitude of $q_1$. Similarly, connect $q_n$ to the boundary by a horizontal segment at the altitude of $q_n$ followed by a vertical segment on the line $x = n$. The positioning of these segments ensures that $\gamma_q$ never crosses itself. Since the $O(n^2)$ vertical and horizontal coordinates of the segments composing $\gamma_q$ can be determined in constant time, the curve $\gamma_q$ can be constructed in $O(n^2)$ time.

Let $Q'$ be the set of intersection points of $\gamma_q$ and $l_q$. Of course, $Q \subseteq Q'$. Label the points in $Q'$ with the labels $q'_1, q'_2, \ldots, q'_m$ in order from top to bottom along $l_q$. The original points in $Q$ now have two labels. Label $q'_0$ the top endpoint of $l_q$. Choose a corresponding point set $P' = \{p'_1, \ldots, p'_m\}$ from $l_p$ such that $p'_j = p_i$ if $q'_j = q_i$, and points of $P'$ lie on $l_p$ in the same order as corresponding points of $Q'$ on $\gamma_q$. (See Figure 4.) We will construct a piecewise linear simple (non self-intersecting) curve $\gamma_p$ through the points in $P'$ in the order $p'_1, \ldots, p'_m$. Choose a point $p'_0$ on the top boundary of $R_p$ right of $l_p$ and connect it to $p'_1$ using a vertical and then a horizontal line segment. Connect $p'_2$ to $p'_3$ by two horizontal and one vertical segment to the left of $l_p$. Repeat this for each $p'_i$, connecting it to $p'_{i+1}$ with two horizontal and one vertical segment lying alternately left and right of $l_p$. Connect $p'_m$ by a horizontal and vertical segment to the boundary of $R_p$ (see Figure 5).

We claim that the algorithm described above will successfully draw $\gamma_p$ through the points $p'_1, \ldots, p'_m$. Consider the $i$th step, where $\gamma_p$ has been drawn up through point $p'_i$. Let $\gamma_p^i$ be the portion of $\gamma_p$ from $p'_0$ to $p'_i$. Similarly let $l_q^i$ be the portion of $l_q$ from $q'_0$ to $q'_i$. The curves $\gamma_p^i$ and $l_q^i$ form a planar subdivision of $R_p$. The curves $l_q^i$ and $\gamma_q$ form a planar subdivision of $R_q$. A simple induction argument shows that these planar subdivisions are isomorphic and that this isomorphism maps each vertex $p'_j$, $j \leq i$, to the corresponding vertex $q'_j$, $j \leq i$.

Finally, we need to show that $p'_i$ can be connected to $p'_{i+1}$ using only two horizontal and one vertical segment. Let $f_p$ be some face in the subdivision formed by $\gamma_p^i$ and $l_p$. By induction, each $p'_j$ is connected to $p'_{j+1}$, $j < i$, by two horizontal and one vertical segment. Thus $f_p$ is a rectangle with a distinguished side $s$ and disjoint rectangles adjacent to $s$ removed. (See Figure 6.) Clearly any point on $f_p \cap s$ can be connected to any other point on $f_p \cap s$ by two horizontal and one vertical segment. Thus $p'_i$ can be connected to $p'_{i+1}$ by two horizontal and one vertical segment. By a similar argument, $p'_m$ can be connected to the boundary of $R_p$ with one horizontal and one horizontal.
vertical segment. This completes the verification of the construction of $\gamma_p$.

Curve $\gamma_p$ can be constructed in $O(n^2)$ time. Coordinatize the rectangle $R_p$ so that the left and right edge lie on the lines $x = -m$ and $x = m$, respectively. Note $m \leq n^2$. Choose a point set $P'$ along $l_p$ in $O(n^2)$ time by simultaneously walking along $l_p$ and $\gamma_q$ and creating a new point $p'_i$ on $l_p$ for each point of $\gamma_q \cap l_q$ which is not in $Q$. Each of the horizontal segments of $\gamma_p$ crosses $l_p$ at some point $p'_i \in P'$ and its $y$-coordinate is inherited from the $y$-coordinate of $p'_i$. Draw each vertical segment on the curve connecting $p'_i$ to $p'_{i+1}$ on the vertical lines $x = m - j$ or $x = -m + j$ where $p'_i$ is the $j$'th point of $P'$ from the bottom and $p'_i$ lies below $p'_{i+1}$ or $p'_i$ is the $j$'th point of $P'$ from the bottom and $p'_{i+1}$ lies below $p'_i$. By this choice of coordinates,
nested curves will avoid intersections. Draw the vertical lines on the curves connecting $p'_1$ and $p'_m$ to the boundaries on the vertical lines $x = m$ and $x = -m$. respectively. As in the construction of $\gamma_q$, the $O(n^2)$ vertical and horizontal coordinates of the segments composing $\gamma_q$ can be determined in constant time, so the curve $\gamma_q$ can be constructed in $O(n^2)$ time.

Currently, the labelled vertices $p'_0, p'_1, \ldots$, that appear in order along $\gamma_p$ exactly match the labelled vertices $q'_0, q'_1, \ldots$, as they appear along $l_p$. Similarly, the labelled vertices that appear in order along $\gamma_q$ match the labelled vertices as they appear on $l_p$. There is currently no correspondence, however, between the corners of $\gamma_p$ and points on $l_q$, nor between the corners of $\gamma_q$ and points on $l_p$. It is this incompatibility that must now be rectified.

Let $P'' = \{p''_1, p''_2, \ldots, \}$ be the vertices of $\gamma_p$ and the points $\gamma_p \cap l_p$ labelled in order along $\gamma_p$. Clearly $P' \subseteq P''$. Choose a corresponding point set $Q'' = \{q''_0, q''_1, \ldots, \}$ on $l_q$ such that $q''_j = q'_j$ if $p''_j = p'_j$ and points of $Q''$ lie on $l_q$ in the same order as corresponding points on $\gamma_p$. Similarly, let $Q''' = \{q'''_1, q'''_2, \ldots, \}$ be the vertices of $\gamma_q$ and the points of $\gamma_q \cap l_q$ labelled in order along $\gamma_q$ and choose a corresponding point set $P'''$ on $l_p$. Note that at

Figure 6: Face $f_p$. 
most two new points are added between any two points \( q_i \) and \( q_{i+1} \) on \( l_q \) or between \( p_i \) and \( p_{i+1} \) on \( l_p \).

As shown above, with the addition of these new vertices along \( l_p \) and \( l_q \), the planar subdivision of \( R_p \) induced by \( \gamma_p \) and \( l_p \) is isomorphic to the subdivision of \( R_q \) induced by \( l_q \) and \( \gamma_q \). Moreover, the corresponding points of \( P'' \cup P''' \) and \( Q'' \cup Q''' \) lie on the boundary of corresponding faces in matching order. By Theorem 1, isomorphic triangulations can be constructed for each pair of faces using \( O(k^2) \) triangles and \( O(k^2) \) time, where \( k \) is the number of points on the boundaries of the faces. Piecing together these triangulations gives the desired isomorphic triangulation between \( R_p \) and \( R_q \).

How large are these isomorphic triangulations and how much time overall is necessary to achieve them? Let \( F_p \) be the set of faces in the planar subdivision of \( R_p \). Let \( \text{k}_f \) be the number of points of \( P'' \cup P''' \) on the boundary of each face \( f \in F_p \). Each triangulation of \( f \) has size \( O(k_f^2) \). We wish to show that \( \sum_{f \in F_p} k_f^2 \) is \( O(n^2) \).

The point set \( P' \) divides \( l_p \) into a set of \( O(n^2) \) segments. Partition \( F_p \) into the set \( F'_p \) of faces which intersect \( l_p \) in two or fewer segments and the set \( F''_p \) of faces which intersect \( l_p \) in more than two segments. If a face intersects \( l_p \) in more than two segments, it must intersect \( l_p \) below a point \( p_i \in P \), between \( p_i \) and \( p_{i+1} \in P \), and above \( p_{i+1} \). At most one face to the left of \( l_p \) intersects \( l_p \) below \( p_i \), between \( p_i \) and \( p_{i+1} \), and above \( p_{i+1} \). Similarly at most one face to the right of \( l_p \) intersects \( l_p \) below \( p_i \), between \( p_i \) and \( p_{i+1} \), and above \( p_{i+1} \). Thus at most \( 2n-2 \) faces are in \( F''_p \). Moreover, the faces in \( F''_p \) intersect \( l_p \) in a total of at most \( 6n-6 \) segments.

Since each segment of \( l_p \) induced by \( P' \) contains at most two additional points of \( P''' \) and the subpath of \( \gamma_p \) which connects \( p_i \) to \( p_{i+1} \) has at most two corners, the number \( k_f \) of points of \( P'' \cup P''' \) on the boundary of \( f \) is at most three times the number of points of \( P'' \) on the boundary of \( f \) or at most six times the number of segments in which \( f \) intersects \( l_p \). Thus,

\[
\sum_{f \in F'_p} k_f^2 \leq \sum_{f \in F_p} 12^2 = O(n^2)
\]

and

\[
\sum_{f \in F''_p} k_f^2 \leq (\sum_{f \in F_p} k_f)^2 \leq (36n)^2 = O(n^2).
\]

Thus the triangulations of \( R_p \) and \( R_q \) have size \( O(n^2) \). A similar argument shows that the isomorphic triangulations can be constructed in \( O(n^2) \) time.
The preceding construction generalizes to constructing isomorphic triangulations between point sets which do not lie on $l_p$ and $l_q$. By a small perturbation we may assume that no two points of $P$ or of $Q$ share the same $y$-coordinate. Draw polygonal lines $l'_p$ and $l'_q$ connecting $P$ and $Q$ in top to bottom order in $R_p$ and $R_q$, respectively. Draw the curves $\gamma_p$ and $\gamma_q$ as described before, always drawing the vertical line segments left or right of all the points in $P$ or in $Q$. Construct isomorphic triangulations between corresponding faces and piece them together to form an isomorphic triangulations of size $O(n^2)$. Details are left to the reader.  

The desired result on piecewise linear homeomorphisms is an immediate corollary of Theorem 2.

**Corollary 1** Let $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ be sets of $n$ distinct points in the interior of rectangles $R_p$ and $R_q$, respectively. A piecewise linear homeomorphism $h$ between $R_p$ and $R_q$ of size $O(n^2)$ where $h(p_i) = q_i$ can be constructed in $O(n^2)$ time.

**Lower Bounds**

The bounds in Theorem 2 are optimal in the worst case by an argument of Pach, Shahrokhi and Szegedy [2]. Let $P = \{p_1, \ldots, p_n\}$ be a set of $n$ points on a vertical line $l_p$ and let $Q = \{q_1, \ldots, q_n\}$ be a set of $n$ points on a vertical line $l_q$ with the labels $q_i$ randomly permuted. Consider the graph $G$ with edges $(q_i, q_{i+1})$ and $(q_i, q_j)$ where $q_i$ lies adjacent to $q_j$ in top to bottom order. With high probability, every embedding of $G$ in the plane has $\Omega(n^2)$ edge crossings. Thus the image of $l_p$ in $R_q$ must cross $\Omega(n^2)$ times the line $l_q$. Each of these crossings must be contained in a separate triangle of isomorphic triangulations so the size of these triangulations is $\Omega(n^2)$.

**Conclusion and Open Problems**

We have given an $O(n^2)$ algorithm for constructing a piecewise linear homeomorphism identifying corresponding sets of $n$ points which is asymptotically optimal in the worst case. However, the required size of such a homeomorphism may vary from $\Omega(n)$ to $\Omega(n^2)$. An open problem is to give an algorithm which constructs the minimum size homeomorphism. We do not know if this problem is NP-complete or if it can be solved in polynomial time. If the problem is NP-complete or even if it has a polynomial time algorithm but the polynomial has high degree, we would like to know if
there are faster approximation algorithms which produce a homeomorphism whose size is a constant times the optimal.

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References

