TCHEBYCHEFF'S INEQUALITY IS BEST-POSSIBLE

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Tchebycheff's Inequality is Best-Possible

Given $\underline{x}(n) = (x_1,\ldots,x_n) \in R^n$ form the average component $m = \sum_{i=1}^{n} x_i/n$, and, for integer $k \geq 1$, the k^{th} absolute moment $\mu_k = \sum_{i=1}^{n} |x_i - m|^k/n$; $\|\underline{x}(n)\|_k \equiv \mu_k^{1/k}$ is a norm for R^n. The μ_k describe how closely the x_i cluster about m.

An elementary and widely known result (see even [1]) is Tchebycheff's theorem: denoting by I_A the function that is 1 when proposition A is true and 0 otherwise, and by A_i the proposition "$|x_i - m| \geq t \, \|\underline{x}(n)\|_k$" for $t > 0$ and $\underline{x}(n) \in R^n$, we have $N_k(\underline{x}(n);t) = \#\{\text{true } A_i\} = \sum_{i=1}^{n} I_{A_i} \leq \sum_{i=1}^{n} |x_i - n|^k I_{A_i}/(t^k \mu_k)$ $\leq n/t^k$ so that

$$F_k(\underline{x}(n);t) \leq 1/t^k; \tag{1}$$

$F_k \equiv N_k/n$ is the fraction $\{x_i : |x_i - m| \geq t \, \|\underline{x}(n)\|_k\}$.

Actually, (1) arises from the celebrated Tchebycheff inequality [2] whereby, for a random variable X with $E(X) = m$ and $E(|X-m|^k) = \mu_k$, $\|X\|_k = \mu_k^{1/k}$,

$$P\{|X - m| \geq t \, \|X\|_k\} \leq 1/t^k; \tag{2}$$

when $P\{X = x_i\} = 1/n$, $j = i, \ldots, n$, (2) collapses to (1).

Given m, k, μ_k and $t \geq 1$, take $P\{X=m + t\,\|X\|_k\} = 1/(2t^k) = P\{X=m - t\,\|X\|_k\}$ and $P\{X=m\} = (t^k-1)/t^k$. In this case $P\{|X-m| \geq t\,\|X\|_k\} = 1/t^k$ which shows that (2) is sharp: that is,

(3) $$\min\,[1/t^k - P\{|X-m| \geq t\,\|X\|_k\}] = 0, \text{ all } t \geq 1,$$

the min being taken over all random variables X with E(X) = m.

In view of the way in which (1) arises as a special case of (2) one might not expect it to be as tight an inequality as (2). Nevertheless, it is best possible in the following sense.

Theorem. Given $t \geq 1$ and an integer $k \geq 1$.

(4) $$\lim_{n\to\infty} \inf\,[1/t^k - F_k(\underline{x}(n);t)] = 0,$$

the inf being taken over all $\underline{x}(n) \in R^n$ with $\frac{1}{n}\sum_{i=1}^{n} x_i = m$; i.e., you can't reduce the right hand side of (1) and still have it hold for all n.

Proof. If $t^k \geq 1$ is rational, say R/S in lowest terms, let i/j denote 2 R/S in lowest terms and choose $\underline{x}(i) \in R^i$ such that $x_1 = \ldots = x_j = t$, $x_{j+1} = \ldots = x_{2j} = -t$, and $x_\ell = 0$, $\ell = 2j + 1, \ldots, i$ (note that $i/2j = t^k \geq 1$).

Then $m = 0$, $\|\underline{x}(i)\|_k = 1$ and $F_k(\underline{x}(i);t) = 2j/i = 1/t^k$ so the expression in square brackets in (4) is 0 for $n = i$. Otherwise take a sequence of rationals $r_i \to t^k$. The above construction applies to each r_i and this establishes (4) in general. □

Thus, without further information about those $\underline{x}(n) \in R^n$ under consideration, one cannot in general improve upon (1) in describing how frequently large $|x_i - m|$'s can occur. It may therefore be reassuring to consumers of (1) to know that this "rough" frequency bound is, in fact, quite excellent.

As a final remark, (1) can be generalized. In fact, if $g > 0$ is a non-decreasing function on R^+, then for $t > 0$

$$(5) \qquad \liminf_{n\to\infty} \left[\sum_{i=1}^{n} g(|x_i - m|)/g(t\|\underline{x}(n)\|_k) - N_k(\underline{x}(n);t)\right] = 0.$$

If $g(u) = u^k$, (5) becomes (4). An interesting extension is the case where $g(u) = \log(1 + u)$.

Acknowledgement. I learned about (1) through Jonah Levy.

References

[1] M. P. Dolciani, W. Wooton, E. F. Beckenbach, R. C. Jurgensen, and A. J. Donnelly, Modern School Mathematics, Algebra I, Houghton-Miflin Company, Boston (1970).

[2] W. Feller, An Introduction to Probability Theory and Its Applications, Vol. I, 3rd Edition, John Wiley, New York, (1968).