NUMERICAL SOLUTION OF INTEGRAL EQUATIONS OF THE FIRST KIND

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ABSTRACT

A piecewise polynomial least squares method is examined for integral equations of the first kind which have stable Sobelev space settings. Error estimates are obtained, and the stability of the approximation procedure is assessed.

1. Introduction

The prototypic integral equation of the first kind is

$$\int_D k(x,y)\phi(y)dy = f(x), \; x \in D,$$

where the integral transformation $K[\;\;] = \int_D k(x,y)\,[\;\;]dy$ is a compact operator on $L^2(D)$. It is well known that this problem is apt to be difficult to solve numerically since small L^2 changes in the data f may produce arbitrarily large L^2 changes in the solution ϕ. Moreover, this L^2 setting for the problem corresponds closely with the context in which integral equations of the first kind are usually encountered at the operational level.

In general, one's chances of producing a good approximate solution depend on the amount of smoothing done by the integral tranformation K. The less smoothing, the less the disparity between the unstable L^2 setting and the function space setting in which the problem is stable, and the more tractable the problem is likely to be. Thus integral equations of the first kind with nonsmooth kernels are often more amenable to numerical solution.

In many applications, for example, numerical differentiation [2], inverse problems for differential equations [6], and simple layer potential formulations of elliptic boundary value problems [5], the integral transformation may be viewed as a stable operator from $L^2(D)$ onto some Sobelev space $H^{\ell}(D)$, $\ell > 0$, consisting of functions whose derivatives of order zero through ℓ are quadratically integrable. Moreover, when the index ℓ is small, say one or two, the amount of smoothing done by the integral operator is relatively minor. For such problems, it is appropriate to ask whether good numerical solutions can be obtained without recourse to regularization devices, especially when the data is known fairly precisely.

In this paper, a piecewise polynomial least squares method is examined for integral equations of the first kind which fit into this Sobelev space setting. The method is similar to that described by Hanson and Phillips [4], for example, except that regularization is not employed. In addition, the stable Sobelev space setting is used for analysis purposes.

In Section 2, the least squares method is developed in an abstract setting in which K is assumed to be a stable operator from one Hilbert space onto another, and shown to be capable of producing accurate numerical solutions depending on the precision of the data. Condition number estimates are also obtained for the coefficient matrix of the discretized problem. In Section 3, these results are applied to an illustrative problem, the Dirichlet problem for Laplace's equation in a disk, formulated as a Fredholm integral equation of the first kind. Finally, in Section 4, some general inferences are drawn about the relation between the amount of smoothing done by the integral transformation and the effectiveness of the approximation procedure.

2. The Approximation Procedure

Let K be a one-to-one bounded linear operator from a real Hilbert space S onto another real Hilbert space T. Then by the open mapping theorem [3], K^{-1} is bounded from T back to S. The operator is thus stable in the S-T setting, that is, $||K|| \cdot ||K^{-1}|| < \infty$.

Now consider the problem $K\phi = f$ where $f \in T$. To allow for the possibility of imprecise data, we assume that f is available in approximate form as $\tilde{f} \in T$. We now define a least squares approximation procedure as follows. Let S^h be some finite dimensional subspace of S, and let $\phi^h \in S^h$ be that member of S^h for which

$$||K\phi^h - \tilde{f}||_T$$

is minimized, where $||\cdot||_T$ denotes the norm on T. This can be reduced to a set of linear algebraic equations by choosing a basis $\{\psi_1, \ldots, \psi_n\}$ for S^h, representing ϕ^h in the form

$$\phi^h = \sum_{j=1}^{n} c_j \psi_j, \tag{1}$$

and then differentiating the resulting quadratic form with respect to the coefficients c_j. This leads to the linear system $Ac=b$ with

$$\begin{aligned} A_{ij} &= (K\psi_j, K\psi_i)_T \qquad 1 \le i,j \le n \\ b_i &= (\tilde{f}, K\psi_i)_T \qquad 1 \le i \le n. \end{aligned} \tag{2}$$

The matrix A is symmetric and positive definite; hence there exists a unique approximate solution $\phi^h \in S^h$.

To facilitate assessment of the accuracy of ϕ^h, we denote the image of S^h under K by T^h, and define P_{T^h} as the orthogonal projection of T onto T^h. Using this terminology, we then have

$$K\phi^h = P_{T^h}\tilde{f} \tag{3}$$

as the operator equation which the approximate solution satisfies. The residual, which we define by $K\phi^h-f$, may also be written as follows:

$$\begin{aligned} K\phi^h-f &= P_{T^h}\tilde{f}-f \\ &= (P_{T^h}-I)f+P_{T^h}(\tilde{f}-f) \\ &= (P_{T^h}-I)K(\phi-\psi^h)+P_{T^h}(\tilde{f}-f), \quad \psi^h \varepsilon S^h, \end{aligned} \tag{4}$$

where the fact that $(P_{T^h}-I)K\psi^h=0$ for all $\psi^h \varepsilon S^h$ has been used in the last equality. Application of K^{-1} then yields

$$\phi^h-\phi = K^{-1}(P_{T^h}-I)K(\phi-\psi^h)+K^{-1}P_{T^h}(\tilde{f}-f). \tag{5}$$

Finally, upon taking norms in (4) and (5), we obtain

$$||K\phi^h-f||_T \leq ||K||\cdot||\phi-\psi^h||_S + ||\tilde{f}-f||_T \tag{6}$$

$$||\phi^h-\phi||_S \leq \kappa(K)||\phi-\psi^h||_S + ||K^{-1}||\cdot||\tilde{f}-f||_T. \tag{7}$$

where $\kappa(K) = ||K||\cdot||K^{-1}||$ is the condition number of K. Apart from data imprecision, we are thus guaranteed that the approximate solution ϕ^h will be of the same order of accuracy as the best possible approximation to ϕ in the subspace S^h. We summarize these results as follows:

Theorem 1: Let $K\phi=f$ where K is a one-to-one bounded linear operator from S onto T and $f\varepsilon T$. The solution $\phi^h \varepsilon S^h$ of the minimization problem

$$\min_{\psi^h \varepsilon S^h} ||K\psi^h - \tilde{f}||_T ,$$

where $S^h \subset S$ and $\tilde{f}\varepsilon T$, then satisfies

$$\text{(i)} \quad ||\phi^h-\phi||_S \le \kappa(K)||\phi-\psi^h||_S+||K^{-1}||\cdot||\tilde{f}-f||_T$$

$$\text{(ii)} \quad ||K\phi^h-f||_T \le ||K||\cdot||\phi-\psi^h||_S+||\tilde{f}-f||_T$$

for all $\psi^h \varepsilon S^h$.

We now consider the stability of the discretized problem Ac=b (2), the relevant parameter here being the condition number $\kappa(A)$ of A, the ratio of the largest to smallest eigenvalues of A. Denoting by B the Gram matrix of the basis functions $\{\psi_i\}_{i=1}^n$ for S^h $(B_{ij}=(\psi_j,\psi_i)_S)$, we show the following:

Theorem 2:

$$\kappa(A) \le (\kappa(K))^2\kappa(B) \tag{8}$$

Proof: First note that

$$c^T Ac = \sum_{i,j=1}^{n} c_j(K\psi_j,K\psi_i)_T c_i = ||K\sum_{j=1}^{n} c_j\psi_j||_T^2 ,$$

and that

$$\frac{||\psi||}{||K^{-1}||} \le ||K\psi||_T \le ||K||\cdot||\psi||_S .$$

We then have

$$c^TAc \le ||K||^2||\sum_{j=1}^{n} c_j\psi_j||_S^2 = ||K||^2c^TBc \le ||K||^2\lambda_{max}(B)||c||_2^2$$

and $$c^TAc \ge \frac{1}{||K^{-1}||^2}\cdot||\sum_{j=1}^{n} c_j\psi_j||_S^2 = \frac{c^TBc}{||K^{-1}||^2} \ge \frac{\lambda_{min}(B)}{||K^{-1}||^2}\cdot||c||_2^2$$

Thus $$\frac{\lambda_{min}(B)}{||K^{-1}||^2} \le \frac{c^TAc}{c^Tc} \le ||K||^2\lambda_{max}(B),$$

from which it follows that

$$\lambda_{max}(A) \le ||K||^2\lambda_{max}(B)$$

$$\lambda_{min}(A) \ge \frac{\lambda_{min}(B)}{||K^{-1}||^2}$$

Hence $$\kappa(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)} \le [\kappa(K)]^2\kappa(B),$$

which completes the proof.

We thus have a uniform bound on the stability of the coefficient matrix A in terms of that of B, the latter being potentially estimable in practice. Also, it is appropriate to ask what effect errors in the right hand vector b have upon the accuracy of ϕ^h, and the governing parameter here is $\sqrt{\kappa(A)}$, rather than $\kappa(A)$ as one might anticipate.

<u>Theorem 3</u>: Let $\tilde{\phi}^h$ be the approximate solution obtained upon replacement of b by $\tilde{b}$. Then

$$\frac{||\tilde{\phi}^h-\phi^h||_S}{||\phi^h||_S} \le \kappa(K)\sqrt{\kappa(A)}\frac{||\tilde{b}-b||_2}{||b||_2}. \tag{9}$$

<u>Proof:</u> The expansion coefficients c_j of ϕ^h satisfy $Ac=b$, while those of $\tilde{\phi}^h$ -- call them $\tilde{c}_j$ -- satisfy $A\tilde{c}=\tilde{b}$. Thus $\tilde{\phi}^h-\phi^h = \sum_{j=1}^{n}(\tilde{c}_j-c_j)\psi_j$, where $\tilde{c}-c=A^{-1}(\tilde{b}-b)$. Moreover,

$$||K(\tilde{\phi}^h-\phi^h)||_T^2 = \left(\sum_{j=1}^{n}(\tilde{c}_j-c_j)K\psi_j, \sum_{i=1}^{n}(\tilde{c}_i-c_i)K\psi_i\right)_T$$

$$= (\tilde{c}-c)^T A(\tilde{c}-c)$$

$$= (\tilde{b}-b)^T A^{-1}(\tilde{b}-b)$$

Hence

$$||K(\tilde{\phi}^h-\phi^h)||_T \le \frac{||\tilde{b}-b||_2}{\sqrt{\lambda_{min}(A)}}$$

(10) and

$$||\tilde{\phi}^h-\phi^h||_S \le \frac{||K^{-1}||\cdot||\tilde{b}-b||_2}{\sqrt{\lambda_{min}(A)}}.$$

In an analagous manner, it may be shown that

$$\|\phi^h\|_S \geq \frac{\|b\|_2}{\|K\| \cdot \sqrt{\lambda_{max}(A)}} , \tag{11}$$

whence, upon division of (10) by (11), the desired result follows.

To make use of these results for a given integral equation of the first kind, it is first necessary to find a stable function space setting for the problem. Assuming the integral transformation K is an invertible compact operator on L^2, it is always possible to trivially stabilize the problem by measuring the size of functions f in the range of K by $\|f\|_R \equiv \|K^{-1}f\|_{L^2}$. In the event that the norm $\|\cdot\|_R$ is compatible with that of some Sobelev space H^ℓ, one then has a potentially workable setting, and it is this situation with which we shall be concerned. It should be noted that some applications of integral equations of the first kind are amenable to this type of stabilization (e.g., numerical differentiation, simple layer potential formulations for elliptic partial differential equations, inverse problems for differential equations) while others are not (e.g., the backward heat equation). Moreover, in comparison with differential equation problems, which involve derivatives explicitly, it is usually less obvious what the stable Sobelev space setting is for an integral equation of the first kind for which one exists.

Rather than proceed with the development along general lines, we now consider a specific illustrative problem, the Dirichlet problem for Laplace's equation in a disk, formulated as a Fredholm integral equation of the first kind. The analysis for this problem can readily be extended in a parallel manner to accommodate other integral equations of the first kind which have stable Sobelev space settings.

3. An Illustrative Example

Let Ω be a two dimensional region bounded by a closed contour Γ of finite length, and consider the problem

$$\Delta u = 0 \quad \text{in } \Omega \tag{12}$$

$$u = f \quad \text{on } \Gamma$$

To obtain the desired integral equation formulation of this problem, the solution is represented as a simple layer potential

$$u(P) = \int_{\Gamma} \log \left\{ \frac{1}{\rho\left(P, P(s')\right)} \right\} \phi(s')ds , \tag{13}$$

where s' denotes are length along Γ and $\rho(P,P(s'))$ is the length of the chord connecting the points $P \in \Omega$ and $P(s')\epsilon\Gamma$. Letting $P \to P(s)\epsilon\Gamma$, the following integral equation for the density ϕ is obtained [5]:

$$\int_{\Gamma} \log \left\{ \frac{1}{\rho(P(s),P(s'))} \right\} \phi(s')ds' = f(s), \quad s\epsilon\Gamma. \tag{14}$$

This formulation of the problem thus reduces by one the dimensionality of the operator equation to be solved numerically.

We now consider the selection of function spaces for the special case in which Γ is a circle of radius a. Here the kernel function becomes

$$\log \left\{ \frac{1}{(P(s),P(s'))} \right\} = \log \left\{ 2a^2 - 2a^2 \cos \left(\frac{s' - s}{a}\right) \right\}^{-1/2}, \tag{15}$$

which has as its Fourier series [8]

$$(16) \qquad -\log a + \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \cos\frac{ks}{a} \cos\frac{ks'}{a} + \sin\frac{ks}{a} \sin\frac{ks'}{a} \right\}.$$

From this we infer that for $a \neq 1$, the corresponding integral operator K is one-to-one and bounded from $H^0(\Gamma)$ onto $H^1(\Gamma)$, where the inner product on $H^\ell(\Gamma)$ is

$$(17) \qquad (f,g) = \int_\Gamma \sum_{k=0}^{\ell} \left[D^\ell f(s) \cdot D^\ell g(s) \right] ds \text{ and } ||f||_\ell = \sqrt{(f,f)_\ell}.$$

It can also be shown that for more general geometric configurations, the stablizing spaces may still be chosen as $H^0(\Gamma)$ and $H^1(\Gamma)$, but here, for illustrative purposes, we focus our attention on the special case in which Γ is a non-unit circle.

Suppose, for concreteness, that the approximation subspace S^h consists of linear splines over a uniform grid of size h on Γ, for which we use the usual tent function basis

$$(18) \qquad \psi_i(s) = \begin{cases} 1 - \dfrac{|s-ih|}{h}, & s \in [(i-1)h,(i+1)h] \\ 0, & s \notin [(i-1)h,(i+1)h] \end{cases} \quad i=1,\ldots,n,$$

where nh = length of Γ. Using the $H^0 \to H^1$ setting with f approximated by $\tilde{f}$, the least squares solution $\phi^h = \sum_{j=1}^{n} c_j \psi_j$ is then obtained by solving $Ac=b$ (2) where

$$(19) \qquad \begin{aligned} A_{ij} &= (K\psi_j, K\psi_i)_0 + (DK\psi_j, DK\psi_i)_0 & 1 \le i,j \le n \\ b_i &= (\tilde{f}, K\psi_i)_0 + (D\tilde{f}, DK\psi_i)_0 & 1 \le i \le n \end{aligned}$$

If we assume that $\phi \in H^2(\Gamma)$, then its L^2 orthogonal projection $\zeta^h \in S^h$ satisfies $||\phi-\zeta^h||_0 = O(h^2)$ [1], and it follows from Theorem 1 that

$$||\phi-\phi^h||_0 \leq O(h^2) + ||K^{-1}|| \cdot ||\tilde{f}-f||_1. \tag{20}$$

Moreover, the condition number of the Gram matrix B (8) for the tent function basis, as measured in the H^0 sense, is uniformly bounded by 6 [7]. Thus, by Theorem 2, the condition number of the coefficient matrix A is uniformly bounded in h -- a highly desirable feature. When applied in this manner, the least squares procedure therefore produces a stable discretization of the problem and a convergent approximation if the data is precise. There are liabilities, however, which result from the computation of inner products in the H^1 sense ; derivative information is required to evaluate the elements of A and b, and to estimate the effect of imprecision in f.

We now revise the function space setting for the problem to accommodate the case where the inner products are taken in the more customary L^2 sense. It may be shown from (16) that the integral transformation under consideration is a stable operator from $H^{-1}(\Gamma)$ onto $H^0(\Gamma)$, where $H^{-1}(\Gamma)$ consists of distributional derivations of $H^0(\Gamma)$ functions. As implemented in this setting with the same linear spline subspace, the least squares method leads to the linear system Ac=b with

$$\begin{aligned} A_{ij} &= (K\psi_j, K\psi_i)_0 \qquad 1 \leq i,j \leq n \\ b_i &= (f, K\psi_i)_0 \qquad 1 \leq i \leq n. \end{aligned} \tag{21}$$

If we again assume that $\phi \varepsilon H^2(\Gamma)$, the L^2 projection $\zeta^h \varepsilon S^h$ of ϕ satisfies $||\phi-\zeta^h||_{-1} = O(h^3)$ [1], and using this with Theorem 1, we obtain

(22) $$||\phi-\phi^h||_{-1} \leq O(h^3) + ||K^{-1}|| \cdot ||\tilde{f}-f||_0 .$$

This estimate is not particularly useful, though, since the H^{-1} norm is rather insensitive to high frequency oscillations. To obtain an L^2 error estimate, we first use (5) with $\psi^h = \zeta^h$:

(23) $$\phi-\phi^h = (\phi-\zeta^h) + K^{-1}P_{T^h} K(\phi-\zeta^h)+K^{-1}P_{T^h}(\tilde{f}-f).$$

Now the last two terms of the right hand side are members of S^h, for which the inverse inequality [1]

(24) $$||\psi^h||_0 \leq O\left(\frac{1}{h}\right)||\psi^h||_{-1} \quad \psi^h \varepsilon S^h$$

is satisfied. Hence, upon taking norms, we obtain

(25) $$||\phi-\phi^h||_0 \leq ||\phi-\zeta^h||_0 + O\left(\frac{1}{h}\right)\left[\kappa(K)||\phi-\zeta^h||_{-1} + ||K^{-1}P_{T^h}(\tilde{f}-f)||\right]$$
$$\leq O(h^2)+O\left(\frac{1}{h}\right) ||\tilde{f}-f||_0$$

To be sure, this sort of asymptotic analysis has its practical limitations, but nevertheless it does permit one to draw some potentially useful inferences. For example, the optimal rate of convergence of ϕ^h to ϕ is achieved barring data imprecision, and for $\tilde{f} \neq f$ an accurate result can be obtained when ϕ may be approximated well by an S^h for which h is large in comparison to $||\tilde{f}-f||_0$.

We can also use (8) to estimate the condition number of the coefficient matrix A (21). We first note that $||\sum_{j=1}^{n} c_j\psi_j||_{-1}^2 = c^T Bc$ where $B_{ij} = (\psi_j,\psi_i)_{-1}$, and that $||\sum_{j=1}^{n} c_j\psi_j||_0^2 \geq ||\sum_{j=1}^{n} c_j\psi_j||_{-1}^2 \geq 0(h^2)||\sum_{j=1}^{n} c_j\psi_j||_0^2$, using the inverse inequality again. We then have

$$c^T B' c \geq c^T Bc \geq 0(h^2)c^T B' c \tag{26}$$

where $B'_{ij} = (\psi_j,\psi_i)_0$, and since B is uniformly stable, $\kappa(B) = 0\left(\frac{1}{h}\right)^2$, which is a tolerable rate of growth. Furthermore, by Theorem 3, an imprecise approximation $\tilde{b}$ to b produces an approximate solution ϕ^h satisfying

$$||\tilde{\phi}^h-\phi^h||_{-1} \leq 0\left(\frac{1}{h}\right)\frac{||\tilde{b}-b||}{||b||_2}2, \tag{27}$$

which becomes

$$||\tilde{\phi}^h-\phi^h||_0 \leq 0\left(\frac{1}{h}\right)^2 \frac{||\tilde{b}-b||_2}{||b||_2} \tag{28}$$

upon application of the inverse inequality. It is interesting to note that if one were to use (9) to estimate the effect of imprecision in f, the predicted L^2 error in ϕ^h would be $0\left(\frac{1}{h}\right)^2||\tilde{f}-f||_0$ instead of $0\left(\frac{1}{h}\right)||\tilde{f}-f||_0$ as in (25). The reason for this is that the perturbation $\tilde{b}-b$ resulting from an L^2 change in f lies primarily in the direction of the low frequency eigenvectors of A, whereas the oppositive is required for sharpness of the condition number bound (19).

The latter typically occurs, however, for "irregular" perturbations due to roundoff error.

It is also important to note that, upon replacement of ϕ by ϕ^h in (13), the resulting error in u depends on the size of the residual $K\phi^h - f$, according to the maximum principle for harmonic functions. Moreover, this can be estimated from Theorem 1 as follows:

$$||K\phi^h - f||_0 = O(h^3) + ||\tilde{f} - f||_0 . \tag{29}$$

We therefore gain an extra power of h, h^3 as opposed to h^2, keeping in mind the final objective for approximating the solution of (13). Furthermore, the overall process of computing the solution to the boundary value problem via the integral equation approach is a stable one, and the instability in the transformation from f to ϕ is reversed when the latter, essentially an intermediate quantity, is used to approximate the solution to the boundary value problem.

4. Concluding Remarks

Here we briefly indicate how the analysis of the preceding example can be generalized. In particular, we consider the least squares approximation of the integral equation of the first kind $K\phi = f$ under the following assumptions:

(i) K is a stable operator from $H^{-\ell}(D)$, $\ell > 0$, to $H^0(D)$.

(ii) There exists a $\zeta^h \varepsilon S^h$ for which $||\phi-\zeta^h||_0 = O(h^k)$, $||\phi-\zeta^h||_{-\ell} = O(h^{k+\ell})$.

(iii) The Gram matrix $B\left(B_{ij} = (\psi_j,\psi_i)_0\right)$ of the basis $\{\psi_1,\ldots,\psi_n\}$ for S^h is uniformly stable.

(iv) S^h satisfies the inverse inequality $||\psi^h||_0 \le O\left(\frac{1}{h}\right)^{\ell} ||\psi^h||_{-\ell}$.

(Properties (ii), (iii) and (iv) are satisfied, for example, by the standard piecewise polynomial subspaces over equally spaced grids and by trigonometric polynomials).

It may then be shown, in a manner analagous to that used in Section 3, that the least squares solution $\phi^h \varepsilon S^h$ for which $||K\phi^h - f||_0$ is minimized, satisfies:

(i) $||\phi-\phi^h||_0 = O(h^k)$, $\quad ||\phi-\phi^h||_{-\ell} = O(h^{k+\ell})$

(ii) $||K\phi^h-f||_0 = O(h^{k+\ell})$.

In addition, the condition number of the coefficient matrix A for the discretized problem $(A_{ij} = (K\psi_j,K\psi_i)_0$ grows at a rate $O\left(\frac{1}{h}\right)^{2\ell}$. Moreover, if the method is implemented with an approximation $\tilde{f}$ to f, the above error estimate for $||\phi-\phi^h||_0$ can be augmented by a term $O\left(\frac{1}{h}\right)^{\ell} ||\tilde{f}-f||_0$ to accomodate this.

Thus, the greater the smoothing effect of K (i.e., the larger the index ℓ), the less stable the discretized problem will be, and the more difficult it becomes to assess the effect of errors in the data. The preceding estimates provide a quantitative measure of this relationship, which in turn has a direct bearing on the numerical tractability of integral equations of the first kind.

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