The Undecidability of Deriving Atomic Formulas from Horn Formulas

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The following question was raised by our colleague N.S. Sridharan in connection with the work of R.A. Kowalski, especially the paper of van Emden and Kowalski [E-K]: Does there exist an algorithm to decide of an arbitrary finite set of Horn formulas and a ground literal whether or not the literal is derivable from the formulas? The answer is no; it is the purpose of this paper to show why. We show further that the derivability of open atomic formulas from Horn formulas is also undecidable. Perhaps the result is known, but we do not find it in [A],[C], [C-K], [F], [F-V], [G], [H], [M] or [McK]. However, most studies about Horn sentences seem to have been about models rather than derivations, so perhaps the question has not been raised heretofore.(footnote 1).

The proof of the result for ground literals (Section 2) is simple enough. The word problem for finitely presented semi-groups is unsolvable, as shown by Post [P]; or see texts [D] or [Y]. The axioms that represent derivations in a finitely presented semi-group as well as the defining relations for any particular finite presentation are in the form of Horn formulas. Thus, for any finite presentation of a semi-group and pair of words on its generators, that pair of words is equivalent with respect to the defining relations (i.e. they represent the same semi-group element) if and only if a particular ground literal is derivable from the Horn formulas that represent that semi-group presentation. The proof of the result for open atomic formulas (Section 3) is only slightly more complicated and depends on the first result.

Section 1: Definitions

We assume familiarity with the basic notions of first order logic. Familiarity with [E-K] and with the basic ideas of Thue systems or finitely presented semi-groups would be helpful; for details of the latter [D, chapter 6] or [Y, chapter 2]. However, the most basic definitions are given here. A <u>finite presention of a semi-group</u>, denoted by $\mathcal{S}$, is a finite set of generators and a finite set of defining relations on specific words on the generators:

$$\mathcal{S} = \langle a_1, \ldots, a_n ;\ A_i \sim A_i',\ 1 \le i \le m \rangle$$

for $A_i, A_i' \in \{a_1, \ldots, a_n\}^*$, $1 \le i \le m$. We say two words $B, B' \in \{a_1, \ldots, a_n\}^*$ are <u>equivalent in</u> $\mathcal{S}$, denoted by $B \sim_{\mathcal{S}} B'$, if there is a finite sequence of words $B_1, \ldots, B_t$ such that B_1 is B, B_t is B' and for all $1 \le j < t$, either (i) B_j is exactly the same as B_{j+1}, or (ii) B_j is of the form $B_\ell A_i B_r$ and B_{j+1} is of the form $B_\ell A_i' B_r$, or (iii) B is of the form $B_\ell B_i' B_r$ and B_{j+1} is of the form $B_\ell B_i B_r$ for some defining relation $A_i \sim A_i'$. Obviously, $\sim_{\mathcal{S}}$ is an equivalence relation on $\{a_1, \ldots, a_n\}^*$.

A formula φ of a first order language is a <u>Horn formula</u> (footnote 2) if φ is a disjunction of zero or more literals L_i where at most one L_i is an atomic formula, the rest being negations of atomic formulas. We use $\mathbf{H}$ to denote a finite set of Horn formulas. Here, as in [E-K], there are no quantifiers, and variables are thought of as being universally quantified. However, in this paper we say a formula is <u>open</u> if one or more variables occur in it and a formula is <u>ground</u> if no variables occur in it.

Let $\mathbf{H}$ be a finite set of Horn formulas and φ any formula of some first order language. We write $\mathbf{H} \vdash \varphi$ to mean that φ is derivable in the first order theory that has $\mathbf{H}$ as its non-logical axioms. For a relational system $\mathfrak{A}$ and a set of formulas Γ, we write $\mathfrak{A} \models \Gamma$ to mean each formula in Γ is true in $\mathfrak{A}$. We write $\Gamma \models \varphi$ to mean that for all relational systems $\mathfrak{A}$, if $\mathfrak{A} \models \Gamma$, then $\mathfrak{A} \models \varphi$.

Section 2: Ground atomic formulas

<u>Theorem</u> (Post): There is no algorithm to decide of an arbitrary finite presentation S and pair of words B and B' on its generators whether or not $B \underset{S}{\sim} B'$.

<u>Theorem 1</u>: There is an effective procedure $\wp$ such that for any finite presentation of a semi-group S and pair of words B and B' on its generators $\wp(S)$ is a finite set of Horn formulas and $\wp(B,B')$ is a ground atomic formula such that

$$B \underset{S}{\sim} B' \quad \text{iff} \quad \wp(S) \vdash \wp(B,B').$$

Theorem 2: There is no algorithm to decide of an arbitrary finite set of Horn formulas $\mathbf{H}$ and ground atomic formula G whether or not $\mathbf{H} \vdash G$.

The proof of Theorem 2 follows directly from the result of Post mentioned above and Theorem 1; we now prove Theorem 1.

For a given finite presentation of a semi-group, $S = \langle a_1, \ldots, a_n; A_i \sim A_i', 1 \leq i \leq m \rangle$, let $\mathcal{L}(S)$ be the first order language without equality that has exactly one two-place predicate symbol E, one two-place function symbol c, and the n+1 constant symbols $a_0, a_1, \ldots, a_n$. To each word $U \in \{a_1, \ldots, a_n\}^*$ we associate a term cU of $\mathcal{L}(S)$ that represents U: (i) if U is λ, then, cU is a_0, and (ii) if U is Va_i, then cU is $c(CV, a_i)$. Clearly, each ground term of $\mathcal{L}(S)$ either represents a word in $\{a_1, \ldots, a_n\}^*$, or a reassociation of that term does. For the same S, let $\mathcal{S}(S)$ denote the semi-group presented by S. That is, $\sim_S$ defines equivalence classes on $\{a_1, \ldots, a_n\}^*$ by $[U] = \{V : V \in \{a_1, \ldots, a_n\}^* \text{ and } U \sim_S V\}$. Then $\mathcal{S}(S) = \langle \{[U] : U \in \{a_1, \ldots, a_n\}^*\}; \bullet ; [\lambda] \rangle$ where $\bullet$ is defined by $[U] \bullet [V] = [UV]$. Then $\mathcal{S}(S)$ is a relational system for $\mathcal{L}(S)$ by associating a_0 with $[\lambda]$, a_i with $[a_i]$, $1 \leq i \leq n$, c with $\bullet$ and E with $=$.

Define $\mathcal{P}(S)$ to be the universal closure of the following set of Horn formulas:

1.	$E(x,x)$	reflexivity
2.	$E(x,y) \rightarrow E(y,x)$	symmetry
3.	$(E(x,y) \wedge E(y,z)) \rightarrow E(x,z)$	transitivity
4.	$E(c(x,c(y,z)),\ c(c(x,y),z))$	associativity
5.	$E(x,y) \rightarrow E(c(x,z),c(y,z))$	right multiplication
6.	$E(x,y) \rightarrow E(c(z,x),c(z,y))$	left multiplication
7.	$E(c(a_0,x),x)$	left identity
8.	$E(c(x,a_0),x)$	right identity
9.	$E(cA_1, cA'_1)$	defining relations
⋮		
9+m-1.	$E(cA_m, cA'_m)$	

For any pair of words $B, B' \in \{a_1, \ldots, a_n\}^*$, define $\mathcal{P}(B,B')$ to be the ground atomic formula $E(cB, cB')$. It is clear that $\mathcal{S}(S) \models \mathcal{P}(S)$; we call $\mathcal{S}(S)$ the <u>standard model of $\mathcal{P}(S)$</u>.

Let $\mathcal{G}(S)$ be the set of all ground terms of $\mathcal{L}(S)$. We define the equivalence relation $\sim_{\mathcal{P}(S)}$ on $\mathcal{G}(S)$ by $t \sim_{\mathcal{P}(S)} t'$ iff $\mathcal{P}(S) \vdash E(t,t')$. Then $\sim_{\mathcal{P}(S)}$ defines equivalence classes on $\mathcal{G}(S)$ and the equivalence class of t is denoted by [t]. Then $\mathcal{S}(\mathcal{P}(S)) = \langle \{[t] : t \in \mathcal{G}(S)\};\ +\ ;\ [a_0] \rangle$, where + is defind by $[t]+[t'] = [c(t,t')]$, is a semi-group. It is easy to see that $\mathcal{S}(\mathcal{P}(S))$ is isomorphic to $\mathcal{S}(S)$. In particular, $B \sim_S B'$ iff $[B]=[B']$ in $\mathcal{S}(S)$ iff $\mathcal{P}(S) \vdash E(cB, cB')$.

Thus Theorem 1 is proved and so Theorem 2 follows. We next use Theorem 1 to prove a result similar to Theorem 2 for open atomic formulas. (footnote 3)

Section 3: Open atomic formulas

For any language $\mathcal{L}(S)$, the universal closure of any open atomic formula E(t,t') where t and t' are distinct terms, will be called a <u>law</u>; we use L to denote a law. For any L, the <u>variety of</u> L, $\mathcal{V}(L)$ is defined by $\mathcal{V}(L) = \{\mathcal{S} : \mathcal{S}$ a semi-group and $\mathcal{S} \models L\}$.

For any law, L, and any finite presentation, S, and pair of words B and B' on the generators of S, we define the theory

$$\Phi(L,S,B,B') = \Phi(S) \cup \{E(CB,CB') \rightarrow L\}.$$

For any L we define the collection of theories $\mathcal{C}(\neg L)$ by

$$\mathcal{C}(\neg L) = \{\Phi(L,S,B,B') : \mathcal{S}(S) \notin \mathcal{V}(L)\}$$

We use $\mathbf{C}$ as a variable for members of $\mathcal{C}(\neg L)$.

<u>Theorem 3</u>: If L is a law and S_0 is a finite presentation such that $\mathcal{S}(S_0) \notin \mathcal{V}(L)$, then for all pairs of words B and B' on the generators of S_0,

$$B \sim_{S_0} B' \quad iff \quad \Phi(L,S_0,B,B') \vdash L.$$

Proof: Abbreviate $\Phi(L,S_0,B,B')$ by $\mathbf{C}_0$ and E(CB,CB') by Bs. (1) If $B \sim_{S_0} B'$ then by Theorem 1 $\Phi(S_0) \vdash Bs$ and so $\mathbf{C}_0 \vdash Bs$ and hence $\mathbf{C}_0 \vdash L$. (2) Suppose $\mathbf{C}_0 \vdash L$. By the definition of $\mathbf{C}_0$, that is the same as $\Phi(S_0), Bs \rightarrow L \vdash L$. Then, by the Deduction Theorem and some tautologies, we have $\Phi(S_0) \vdash Bs \vee L$. Therefore, in any model of $\Phi(S_0)$, either Bs or L must hold. In particular, $\mathcal{S}(S_0) \models B_S$ or $\mathcal{S}(S_0) \models L$. But, by hypothesis, $\mathcal{S}(S_0) \notin \mathcal{V}(L)$ i.e., not $\mathcal{S}(S_0) \models L$. So we must have $\mathcal{S}(S_0) \models B_S$. It then follows from the proof of Theorem 1 that $[B]=[B']$ in $\mathcal{S}(S_0)$ and hence $B \sim_{S_0} B'$.

<u>Theorem</u> <u>4</u>: Let L be a law for which there is a finite presentation S such that $\mathcal{S}(S) \notin \mathcal{V}(L)$ and S has unsolvable word problem. Then there is no algorithm to decide of an arbitrary member $\mathbf{C}$ of $\mathcal{C}(\neg L)$ whether or not $\mathbf{C} \vdash L$.

Proof: Suppose the contrary; there is an algorithm $\mathcal{A}_L$ such that for all $\mathbf{C} \in \mathcal{C}(\neg L)$, $\mathcal{A}_L(\mathbf{C}) = \begin{cases} \text{true} & \text{if } \mathbf{C} \vdash L \\ \text{false} & \text{else} \end{cases}$

Then, in particular, for those S with unsolvable word problem,

$$\mathcal{A}_L(\mathcal{P}(L,S,B,B')) = \begin{cases} \text{true} & \text{if } \mathcal{P}(L,S,B,B') \vdash L \text{ iff } B \sim_S B' \\ \text{false} & \text{if } \mathcal{P}(L,S,B,B') \nvdash L \text{ iff } B \not\sim_S B'. \end{cases}$$

But this would contradict the unsolvability of the word problem for S.

<u>Theorem</u> <u>5</u>: There is no algorithm to decide of an arbitrary finite set of Horn formulas $\mathbf{H}$ and an atomic formula A whether or not $\mathbf{H} \vdash A$.

Proof: The non-logical axioms $\mathcal{P}(S)$ and $\mathcal{P}(L,S,B,B')$ are Horn formulas. So, if A is a ground atmomic formula, use Theorem 2. Suppose A is open, that is, a law. There are well known laws such as triviality, $E(x,a_0)$ and commutativity, $E(c(x,y),c(y,x))$, which satisfy the hypothesis of Theorem 4.

Footnotes

1. One comes pretty close to the results of this paper by combining Horn's result that Horn sentences are preserved under direct power and Machover's Theorem 2 [M p. 522] that if the language includes either (a) two predicate letters, one of which has at least two-argument places, or (b) a predicate letter of more than two argument-places, then the set of sentences preserved under direct product is not recursive.

2. The definition given follows [E-K] though in other literature such as [C-K] p. 328, the definition above is for basic Horn formulas with Horn formulas being built from these by means of $\wedge, \exists, \forall$.

3. The form of these proofs is by no means new. Probably it appeared first in [L-P], see also [D, pp. 137-139], as well as in several papers in the 60's, including A. Yasuhara's thesis.

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