SUPERCONVERGENCE OF PIECEWISE POLY-NOMIAL GALERKIN APPROXIMATIONS FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND.

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Abstract

Piecewise polynomial Galerkin approximations for Fredholm integral equations of the second kind are shown to possess superconvergence properties in some circumstances.

1. Introduction

Galerkin's method is one of the standard numerical techniques for solving Fredholm integral equations of the second kind [3]. Commonly the approximation subspace is chosen to consist of piecewise polynomials, for which the global rate of convergence may be shown to be optimal under quite general conditions.

Here we ask whether there is some distinguished set of points at which the convergence rate exceeds the global optimum. That this is the case for two-point boundary value problems is already well documented (see for example Douglas & Dupont [4]), and one might well expect some similar phenomenon to occur for Fredholm integral equations of the second kind.

In the next section, we show under appropriate regularity assumptions that at points where the orthogonal projection of the solution exhibits an accelerated rate of convergence, so too will the Galerkin approximation. Thus the superconvergence question for integral equations is reduced to its least squares counterpart.

We then establish some superconvergence properties for piecewise polynomial least squares approximations over a uniform grid. At the "Gaussian" points of the subintervals, an extra power of the grid size h is gained in the case where no interelement continuity is imposed. This is also shown to be the case for continuous piecewise polynomials of odd degree.

We include corroborative computational results for Galerkin's method with a linear spline subspace, as applied to an illustrative integral equation.

2. L_∞ error estimates for Galerkin's method

We denote by

(1) $$\phi = f+\lambda K\phi$$

the Fredholm integral equation of the second kind

(2) $$\phi(x) = f(x)+\lambda\int_a^b k(x,y)\phi(y)dy, \quad a\le x\le b.$$

Assuming that K is a compact operator on C[a,b], and that λ is not among its eigenvalues, this problem has a unique solution $\phi\in C[a,b]$ for any $f\in C[a,b]$.

To approximate the solution via Galerkin's method, one chooses a finite dimensional subspace $S^h = \text{span}\,\{\psi_1,\ldots,\psi_n\}$ and determines that $\phi^h\in S^h$ for which

(3) $$(\phi^h,\psi^h) = (f,\psi^h)+\lambda(K\phi^h,\psi^h), \text{ all } \psi^h\in S^h$$

This leads to the linear algebraic equations

$$\sum_{j=1}^{n} \left[(\psi_j,\psi_i)-\lambda(K\psi_j,\psi_i)\right] c_j = (f,\psi_i),\ i=1,\ldots,n$$

for the expansion coefficients of the approximate solution:

$$\phi^h = \sum_{j=1}^{n} c_j\psi_j.$$

We restrict our considerations here to piecewise polynomial subspaces having the following properties:

(A) S^h consists of piecewise polynomials of degree k over a uniform partition of [a,b]:

$$a=x_o<x_1<\ldots<x_m=b;\ x_{i+1}-x_i=h.$$

(B) For any $f\in C^{(\ell)}[a,b]$, $\ell\in\{1,\ldots,k+1\}$, there exists an "interpolant" $P_I^h f\in S^h$ for which

$$||f-P_I^h f|| \le c_\ell h^\ell ||f^{(\ell)}||$$

where c_ℓ is independent of f. Here and throughout the paper, we use the $L_\infty[a,b]$ norm exclusively:

$$||f|| = \sup_{x\in[a,b]} |f(x)|$$

(C) The orthogonal projection P^h: $L_\infty[a,b]\to S^h$ is bounded independently of h in the $L_\infty[a,b]$ sense. This assumption is satisfied by the standard piecewise polynomial subspaces encompassed by (A) and (B). We defer further discussion of this point until later.

We now characterize the error in least squares approximation in terms of the $L_\infty[a,b]$ norm:

Lemma 1: For any $\ell\in\{1,\ldots,k+1\}$

(i) If $\phi^{(\ell)}(x)\in C[a,b]$, $||\phi-P^h\phi||=O(h^\ell)$

(ii) If $\frac{\partial^\ell}{\partial x^\ell}\{k(x,y)\}\in C([a,b]\times[a,b])$, $||(I-P^h)K||=O(h^\ell)$

(iii) If $\frac{\partial^\ell}{\partial y^\ell}\{k(x,y)\}\in C([a,b]\times[a,b])$, $||K(I-P^h)||=O(h^\ell)$

Proof: We prove only (i); (ii) and (iii) may be established in a similar manner. Write

$$||\phi-P^h\phi|| \le ||\phi-P_I^h\phi|| + ||P^h(P_I^h\phi-\phi)||$$
$$= (1+||P^h||)\cdot||P_I^h\phi-\phi||$$

and (i) follows from (B) and (C).

We are now ready to establish a basic result on the $L_\infty[a,b]$ convergence of Galerkin's method.

Theorem 1: Assuming, in addition to (A), (B) and (C), that

(D) $||(I-P^h)K||\to 0$ as $h\to 0$, and

(E) $\phi^{(\ell_1)}(x)$ and $\frac{\partial^{\ell_2}}{\partial y^{\ell_2}}\{k(x,y)\}$ are continuous, $1\le\ell_1,\ell_2\le k+1$; then

(i) For h sufficiently small, there exists a unique approximate solution $\phi^h \epsilon S^h$, for which

(ii) $||\phi^h - P^h\phi|| = O(h^{\ell_1+\ell_2})$, and

(iii) $||\phi^h - P^h\phi|| = O(h^{\ell_1})$.

Proof: First note that the approximate solution is characterized by

(4) $$(I-\lambda P^h K)\phi^h = P^h f,$$

which follows from (3) upon writing $\psi^h = P^h\psi$, $\psi \epsilon L_2[a,b]$, and using the self-adjointness of P_h. There results $((I-\lambda P^h K)\phi^h - P^h f, \psi) = 0$ for all $\psi \epsilon L_2[a,b]$, which is equivalent to the above.

Writing

$$I-\lambda P^h K = (I-\lambda K) - \lambda(P^h - I)K,$$

it follows from (D) that $I-P^h K$ has a uniformly bounded inverse as $h \to 0$:

$$||(I-\lambda P^h K)^{-1}|| \leq ||(I-\lambda K)^{-1}|| \cdot \{\frac{1}{1-||(I-\lambda K)^{-1}|| \cdot |\lambda| \cdot ||(P^h - I)K||}\}$$

$$= ||(I-\lambda K)^{-1}|| \cdot \{1+o(h)\} \text{ as } h \to 0.$$

Thus (i) is true.

Now rewrite (1) as

$$(I-\lambda K)P^h\phi = f - (I-\lambda K)(\phi - P^h\phi)$$

and apply P^h to both sides:

(5) $$(I-\lambda P^h K)P^h\phi = P^h f + \lambda P^h K(\phi - P^h\phi).$$

Subtraction of (5) from (4) then yields

$$(I-\lambda P^h K)(\phi^h - P^h\phi) - \lambda P^h K(I-P^h)(\phi - P^h\phi),$$

where an extra factor of the idempotent operator $I-P^h$ has been added on the right. Hence

(6) $$||\phi^h - P^h\phi|| \leq |\lambda| \cdot ||(I-\lambda P^h K)^{-1}|| \cdot ||K(I-P^h)|| \cdot ||\phi - P^h\phi||$$

From (6), (E) and the lemma, part (ii) of the theorem follows directly. Application of the triangle inequality

$$||\phi^h-\phi|| \le ||\phi^h-P^h\phi|| + ||P^h\phi-\phi||$$

yields (iii).

The main import of the theorem for our purposes is that the Galerkin solution ϕ^h may tend to the orthogonal projection $P^h\phi$ much faster than $P^h\phi$ approaches ϕ. Thus if the "least squares" approximation $P^h\phi$ exhibits an accelerated rate of convergence at some distinguished set of points, so too will ϕ^h. We state this more precisely:

Corollary 1: Assuming (A)-(E), and that

(F) At a discrete set of points $X^h=\{\bar{x}^h\}\subset[a,b]$, $\max\limits_{X^h} |\phi-P^h\phi|=O(h^{\ell_3})$,

then $\max\limits_{X^h} |\phi-\phi^h| = O(h^{\ell})$, where $\ell=\min\{\ell_1+\ell_2,\ell_3\}$.

Proof: Apply the triangle inequality

$$|\phi-\phi^h| \le |\phi-P^h\phi| + |P^h\phi-\phi^h|$$

over X^h, and the result follows directly.

We are thus led to consider possible superconvergence properties of piecewise polynomial least squares approximations.

3. Superconvergence of piecewise polynomial least squares approximations

We consider here two classes of piecewise polynomial subspaces in which least squares approximations will be shown to exhibit superconvergence phenomena:

$$(7)\qquad \begin{aligned} S^h_{k,-1} &= \{\psi^h \mid \psi^h \text{ is a polynomial of degree} \le k \text{ on each subinterval } [x_{i-1},x_i],\ 1\le i\le m\} \\ S^h_{k,0} &= S^h_{k,-1} \cap C[a,b]. \end{aligned}$$

It is well known that these subspaces have the assumed approximation property (B). That orthogonal projection is bounded in the L_∞ sense (property (C)) has been shown by de Boor [2] for $S^h_{k,0}$, and, for the other subspace, $S^h_{k,-1}$, we include a proof of this as an appendix.

For each subinterval $[x_{i-1},x_i]$ of our partition of $[a,b]$, we denote by $x_{i,1},\ldots,x_{i,k+1}$ the "Gauss points" gotten by a linear mapping of the roots of the $(k+1)^{\underline{st}}$ degree Legendre polynomial into $[x_{i-1},x_i]$. For the discontinuous piecewise polynomial subspace $S^h_{k,-1}$, one gains an extra power of h in the convergence rate of least squares approximations at the set of points

$$G^h_k = \bigcup_{\substack{1\le i\le m \\ 1\le \ell\le k+1}} x_{i,\ell}$$

Lemma 2: For $S^h = S^h_{k,-1}$,

$$\max_{G^h_k} |f-P^hf| \le C\|f^{(k+2)}\|h^{k+2}$$

where C is independent of f.

Proof: Let $f_I \epsilon S^h_{k,-1}$ be the interpolant of $f(x)$ at the Gauss points G^h_k. Using the error formula for polynomial interpolation, we may write

$$f=f_I+p_{k+1}+r$$

where

$$p_{k+1}(x) = \{\prod_{\ell=1}^{k+1} (x-x_{i,\ell})\} f^{(k+1)}\left(\frac{x_{i-1}+x_i}{2}\right), \; x\epsilon[x_{i-1},x_i]$$

$$||r||\leq C^1||f^{(k+2)}||h^{k+2}, \; C^1 \text{ independent of } f.$$

Using the L_∞ boundedness of the orthogonal projection and the fact that $P^h p_{k+1}=0$, we obtain

$$||P^h f-f_I||\leq C||f^{(k+2)}||h^{k+2}, C=||P^h||C^1.$$

Applying this inequality at the Gauss set G^h_k where f_I and f coincide yields the desired result.

We now show that the second subspace $S^h_{k,0}$ produces the same superconvergence phenomenon provided m is odd.

Lemma 3: Let $S^h=S^h_{k,0}$ where k is odd. Then

$$\max_{G^h_k} |f-P^h f|\leq C_1||f^{(k+2)}||h^{k+2}$$

where C_1 is independent of f.

Proof: We start with

$$f=f_I+p_{k+1}+r$$

as in the proof of the previous lemma, but now $f_I \notin S^h$ because it is in general discontinuous at the interior knots $x_1,\ldots,x_{m-1}$. However, the magnitude of the jump in f_I at these points is the same as that in r, since p_{k+1} is continuous throughout [a,b]. Using the fact that $||r||\le\cdot C^1||f^{(k+2)}||h^{k+2}$, we may thus construct a function $g(x)\in S^h_{k,-1}$ for which $f_I+g\in S^h_{k,0}$ and $||g||\le||r||$.

Applying P^h to

$$f = (f_I+g)+p_{k+1}+(r-g),$$

and noting that $P^h(f_I+g)=f_I+g$ and $P^h p_{k+1}=0$, we obtain

$$||P^h f-f_I||\le||g||+||P^h||\cdot||r-g||\le(1+2||P^h||)||r||,$$

which then leads to the superconvergence result.

From Corollary 1 and Lemmas 1 and 2, it follows that an extra power of h is gained in the convergence rate of the Galerkin approximation at the Gauss points when the subspaces $S^h_{k,0}$, k odd, and $S^h_{k,-1}$ are used. No such systematic superconvergence phenomenon occurs for $S^h_{k,0}$, k even.

4. A computational example

Galerkin's method was applied to the integral equation

$$\phi(x) = 1+2\int_0^1 e^{x-y}\phi(y)\,dy$$

using a linear spline subspace over a uniform grid of subinterval size h. The exact solution to this problem is

$$\phi(x) = 1-2\left(\frac{e-1}{e}\right)e^x.$$

In the adjoining table, the maximum error at the n+1 knots

$$x_i = ih,\ i=0,\ldots,n,$$

is compared to that at the 2n Gauss points

$$x_i+\frac{h}{2}\left(1\pm\frac{1}{\sqrt{3}}\right),\ i=0,\ldots,n-1.$$

The last two columns indicate that the error reduction produced by halving h approaches 1/4 at the knots and 1/8 at the Gauss points, thus corroborating the theoretical prediction.

$\frac{1}{h}$	Maximum Error		Reciprocal of Error Reduction	
	at knots	at Gauss points	at knots	at Gauss points
2	.627 E-1	.535 E-2		
4	.168 E-1	.815 E-3	3.73	6.56
8	.434 E-2	.116 E-3	3.87	7.03
16	.110 E-2	.155 E-4	3.95	7.48
32	.278 E-3	.201 E-5	3.96	7.71
64	.697 E-4	.255 E-6	3.99	7.88

Table 1

APPENDIX

Lemma 4: For $S^h = S^h_{k,-1}$ as defined in (7), the orthogonal projection $P^h: L_2[a,b] \to S^h_{k,-1}$ is bounded independently of h as a map on $L_\infty[a,b]$.

Proof: We first establish the analagous result for the orthogonal projection, call it Q^h, whose range is the set of polynomials of degree $\leq k$ over $[0,h]$.

Let $||f||_{L_\infty[0,h]} = 1$. Then $||Q^h f||_{L_2[0,h]} \leq \sqrt{h}$. Now change variables from $x \in [0,h]$ to $y \in [-1,1]$:

$$x = \frac{h}{2}(1+y)$$

$$g(y) \equiv Q^h F(x)$$

and expand g(y) in terms of the Legendre polynomials:

$$g(y) = \sum_{\ell=0}^{k} c_\ell P_\ell(y).$$

Then use the following properties [1].

(i) $||P_\ell||_{L_\infty[-1,1]} = 1$

(ii) $||P_\ell||_{L_2[-1,1]} = \sqrt{\frac{2}{2\ell+1}}$

to write

$$||g||_{L_\infty[-1,1]} \leq \sum_{\ell=0}^{k} |c_\ell| = \left(\sqrt{}\,|c_0|, \ldots, \sqrt{\frac{2}{2k+1}}\,|c_k|\right)\cdot\left(\frac{1}{\sqrt{2}}, \ldots, \sqrt{\frac{2k+1}{2}}\right)^T.$$

Application of the Schwarz inequality yields

$$||g||_{L_\infty[-1,1]} \leq ||g||_{L_2[-1,1]} \cdot \sqrt{\frac{1+3+\ldots+(2k+1)}{2}}$$

It now follows that

$$||Q^h f||_{L_\infty[0,h]} = ||g||_{L_\infty[-1,1]} \le ||g||_{L_2[-1,1]} \frac{n+1}{\sqrt{2}} = n+1$$

since

$$||g||_{L_2[-1,1]} = \sqrt{\frac{2}{h}}\ ||Q^h f||_{L_2[0,h]} \le \sqrt{2}\ .$$

Thus

$$||Q^h||_{L_\infty[0,h]} \le n+1.$$

We now consider the orthogonal projection P^h in the statement of the lemma:

$$\max_{x\epsilon[a,b]} |P^h f| = \max_{x\epsilon[x_{i-1},x_i]} |P^h f| \quad \text{for some} \quad i\epsilon\{1,\ldots,m\}$$

$$\le (k+1) \max_{x\epsilon[x_{i-1},x_i]} |f|$$

$$\le (k+1) \max_{x\epsilon[a,b]} |f|$$

Thus in the $L_\infty[a,b]$ sense, $||P^h|| \le k+1$.

References

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