A PROBLEM IN OPTIMAL MERGING

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1. Introduction.

Let $A_m = a_1<a_2<\ldots<a_m$ and $B_n = b_1<b_2<\ldots<b_n$ be disjoint sets of distinct elements. It is an interesting and difficult problem to determine the least number of binary comparisons which will always suffice to merge the sets A_m and B_n. Call this the (m,n) merging problem and let x be any algorithm to solve it. Define $M_x(m,n)$ as the maximum number of comparisons which x will ever require to merge A_m and B_n. Define

$$M(m,n) = \min_x\{M_x(m,n)\}. \quad (1)$$

M(m,n) is the best worst case, or mini-max number of comparisons needed to merge A_m and B_n. Any algorithm a for which $M(m,n) = M_a(m,n)$ is optimal in the mini-max sense. For the remainder of this report "optimal" will be used exclusively in this sense and "comparison" will always mean "binary comparison". M(m,n) will also be referred to as the <u>cost</u> of the optimal (m,n) merging algorithm.

When m=1 the merging problem is identical to that of inserting an item into an ordered list, hence binary search/insertion is the optimal algorithm. Therefore,

$$M(1,n) = 1+[\log_2 n] \quad (2)$$

where [a] denotes "the greatest integer not exceeding a." The (2,n) problem was solved by Hwang and Lin [1] and by Graham [unpublished, c.f. 2]. We will discuss this result in the next section. In section 3 of this note we present a solution to the (3,n) merging problem.

M(m,n) has proven very hard to solve for general m and n. In a subsequent report we will present several significant extensions to the present knowledge of this problem. There are a number of special cases for which (1) has been solved completely. Knuth's text [2] discusses many of these results and also contains much useful introductory material.

2. The Determination of M(2,n).

One obvious way to solve the (2,n) problem is to first merge a_1 into B_n using binary search/insertion. Suppose, after k steps we find $b_i < a_1 < b_{i+1}$, a_2 could then be inserted into the remaining n-i elements of B_n. This is a (1,n-i) problem which can be solved optimally giving a total cost of $k+[\log_2(n-i)]+1$ steps. This method would have a maximum worst case cost of $2 + 2[\log_2 n]$ comparisons (since i might be zero), however, we can often do better.

Let

$$1+S_{2k} = [\tfrac{17}{7} 2^{k-1}], \quad \text{and}$$

$$1+S_{2k+1} = [\tfrac{12}{7} 2^{k}], \quad \text{for } k > 0. \tag{3}$$

Equation (3) generates the sequence whose initial terms are given in table 1.

i	2	3	4	5	6	7	8	9	10	11	12
$1+S_i$	2	3	4	6	9	13	19	27	38	54	77

Table 1. Initial terms in the sequence $1+S_i$.

Hwang and Lin proved the following two theorems relating the sequence (3) to $M(2,n)$.

Theorem 1. $$M(2,S_i) \leq i. \quad (4)$$

Theorem 2. $$M(2,1+S_i) > i. \quad (5)$$

Theorem 1 was proven by giving an inductive algorithm to solve the (2,n) problem which never requires more than i comparisons for $n < 1+S_i$. Theorem 2 was proven by showing that there is no algorithm to solve the $(2,1+S_i)$ problem in less than i+1 steps. As a consequence of these theorems we have

Theorem 3. $$M(2,n) = [\log_2(\tfrac{14}{17} n)]+[\log_2(\tfrac{7}{12} n)]+2, \; n > 0. \quad (6)$$

Furthermore, as a consequence of theorem 2, the algorithm stated in theorem 1 is optimal.

3. The (3,n) Merging Problem.

Our work in this section follows closely the ideas developed by Hwang and Lin in the work cited above. We have modified and extended their definitions and notation in many cases but the underlying principles are the same. The work reported here was initially motivated by exercise 5.3.2.14 in the Knuth text and constitutes a negative answer to that problem.

Let

$$1+T_{3k-1} = 2[\tfrac{17}{7} 2^{k-2}],$$
$$1+T_{3k} = [\tfrac{19}{7} 2^{k-2}] + [\tfrac{12}{7} 2^{k-1}], \qquad (7)$$
$$1+T_{3k+1} = [\tfrac{19}{7} 2^{k-1}] + [\tfrac{12}{7} 2^{k-2}] + 2^{k-3},$$

for all $k \geq 3$, except that $1+T_8 = 9$.

Equation (7) generates the sequence whose initial terms are given in Table 2.

i	8	9	10	11	12	13	14	15	16	17	18
$1+T_i$	9	11	14	18	23	29	38	48	60	76	98

Table 2. Initial terms in the sequence $1+T_i$.

We have proven theorems analogous to theorems 1 and 2 above for the (3,m) problem based on (7). They are

Theorem 4. $$M(3,T_i) \leq i. \qquad (8)$$

Theorem 5. $$M(3,1+T_i) > i. \qquad (9)$$

The proof of theorem 4 is an inductive algorithm which solves the (3,n) problem in i or less steps for $n < 1+S_i$. The proof of theorem 5 shows that there is no algorithm to solve the $(3,1+T_i)$ in less than i+1 steps. In the following sections of this report we will prove these theorems.

3.1 Preliminary Notions.

By assumption the sets A_m and B_n are disjoint, ordered, and contain only distinct elements. In order to describe the proofs of theorems 4 and 5 it will be necessary to represent the partial orderings of $(A_m \cup B_n)$ that occur during various stages of a merge.

Definition. A *partial ordering* of a set is binary relation R on elements of the set that is transitive (xRy and yRz -> xRz), irreflexive (xRx), and anti-symmetric (xRy -> yRx).

A partial ordering can be represented in many ways, each of the following are equivalent;

i. a set of ordered pairs specifying those elements of the set for which the relation obtains,

ii. a *directed acyclic graph* (DAG),

iii. a set of *permissible* permutations (i.e. a left to right ordering of the elements of the set so that if xRy then x is to left of y).

Of course the relation must be finite for any of these representations to make sense.

The equivalence of the above representations can be seen by considering that any set of ordered pairs is a binary relation and can be represented as a directed graph. Since the relation in this case is irreflexive its graph can have no cycles. Any DAG can be embedded on a line segment so that all arcs point in one direction. Each such embedding is a permissible permutation. Any set of permutations of the elements of a set specify a (not necessarily unique) partial ordering of the set. Conversely, a DAG specifies a unique set of permutations. It is most convenient to think of a partial ordering as a set of permutations.

Before any information is known about the final order of ($A_m \cup B_n$) an algorithm to merge the sets must be prepared to deal with each of the $\binom{m+n}{m}$ initial permutations that are possible complete orderings of the two sets. When the sets have been merged there can be only one complete ordering. These two cases or states, initial and terminal, are the trivial partial orderings of $(A_m \cup B_n)$. Every merging algorithm based on comparisons defines a finite sequence of partial orderings $\{p_r\}$ beginning with the first and ending with the last. Each partial ordering p_i in this sequence is a refinement of the partial orderings which precede it in the sequence. That is, p_i contains only those permutations consistent with the results of the first i comparisons.

Suppose, for arbitrary partial ordering p_i and p_j, p_i is a refinement of p_j. We write $p_i \leq p_j$ and say p_i <u>dominates</u> p_j.

<u>Claim</u>. If $p_i \leq p_j$ and if every comparison algorithm which completely orders p_i requires at least k steps, then there is no algorithm which completely orders p_j in less than k steps.

<u>Proof</u>. Suppose that the claim is false; then there is an algorithm to order p_j in less steps than any algorithm can order p_i. But this algorithm for p_j must also order p_i since $p_i \leq p_j$, a contradiction.

Let $p_k(a_i<b_j)$ denote the subset of the k^{th} partial ordering induced on A_m and B_n by some merging algorithm which is consistent with the relation $a_i < b_j$.

<u>Claim</u>. $p_k = p_k(a_i<b_j) \cup p_k(a_i>b_j)$.

<u>Proof</u>. For every permutation in p_k either a_i precedes b_j $(a_i < b_j)$ or it does not. Therefore $p_k(a_i<b_j)$ and $p_k(a_i>b_j)$ are disjoint and together cover p_k.

Since $p_k(a_i<b_j)$ is a refinement of p_k the claim above asserts that $p_k(a_i<b_j)$ dominates p_k. Let $\hat{p}_k$ denote the mini-max cost of completely ordering p_k. These ideas lead to the next lemma.

Lemma 6. Hwang and Lin [1]. If for any p_k, for each index i of A_m there exists an x_i such that,

$$\hat{p}_k(a_i > b_{x_i}) \geq h, \text{ and}$$
$$\hat{p}_k(a_i < b_{x_{i+1}}) \geq h, \text{ then}$$
$$\hat{p}_k \geq 1+h.$$

Since it is necessary to consider the effect of a sequence of comparisons in order to prove theorem 5, this lemma is the means by which the cost of a partial ordering is shown to satisfy the theorem.

A diagramatic or graphic representation of a partial ordering, called a configuration, is a 2m component vector $V = (v_1,\ldots,v_{2m})$ in which v_i $(1 \leq i \leq m)$ is the index j of the largest element of B_n for which $(a_i > b_j)$ or 0 if there is no such j, and v_{i+m} is the index of the smallest element h of B_n for which $(a_i < b_h)$ or $n+1$ if there is no such h. Clearly $v_i < v_{i+m}$. As an example of a configuration, suppose $n=2^k$, $m=3$, and the order relations are $a_1<b_{i+1}$ for $i=2^{k-2}$, and $b_j<a_3$ for $j=2^{k-1}$; the configuration is

$$V = (0,0,2^{k-1},2^{k-2}+1,2^k+1,2^k+1).$$

Figure 1 is an example of a configuration diagram. In this example the indices of row 2 have been partitioned to illustrate the relative positions in the set B_n for which there is no information about a_1 and a_3. Without loss of generality, this can be done for any configuration diagram since the partition is simply an expanded form of the row length. This expansion will be used whenever it helps to clarify the geometry of a configuration.

Configuration diagrams can be related in a natural and intuitive way. Observe that if C_i is the configuration representing p_i and C_j is the configuration representing p_j and $p_i \prec p_j$, the diagram of C_i

$$\begin{array}{|lr|} 2^{k-2} & \\ 2^{k-2} + 2^{k-2} & + \; 2^{k-1} \\ & 2^{k-1} \end{array}$$

Figure 1. A Configuration Example.

"fits" inside the diagram of C_j.

<u>Definition</u>. The <u>cost</u> of a configuration is the mini-max number of comparisons required to completely order the partial ordering it describes (i.e. $\hat{p}$).

<u>Definition</u>. A configuration is <u>disjoint</u> if there exist components v_i, v_j, and v_k of V such that $(i < j < k)$ and $(v_i+1 < v_{i+m})$ and $(v_k+1 < v_{k+m})$ and $(v_j+1 = v_{j+m})$. A configuration that is not disjoint is <u>overlapped</u>.

The cost of a configuration is the sum of the costs of its disjoint components. A disjoint configuration diagram is disconnected.

Henceforth no distinction will be made between partial orderings, configurations, and diagrams of configurations.

3.2 Addition Lemmas.

It is convenient for the work which follows to have a method for dealing with least integer functions of fractions of powers of 2. The following simplification and addition rules provide this mechanism.

<u>Lemma</u> 7. If a is an integer then $[a+b] = a+[b]$.

<u>Proof</u>. Immediate.

Lemma 8. Terms in equations (3) and (7) can be rewritten using lemma 7 in the following ways:

$$[\tfrac{12}{7}\, 2^{k-1}] = 2^{k-1} + [\tfrac{5}{7}\, 2^{k-1}], \qquad (10)$$

$$[\tfrac{17}{7}\, 2^{k-1}] = 2^{k} + [\tfrac{3}{7}\, 2^{k-1}], \qquad (11)$$

$$[\tfrac{19}{7}\, 2^{k-1}] = 2^{k-1} + [\tfrac{12}{7}\, 2^{k-1}]. \qquad (12)$$

Proof. Immediate.

For large k the process can be continued so that successive powers of 2 can be removed (or added). Equations (10)-(12) are just a few of the possibilities. Any fraction can be expanded in this way.

Lemma 9. For any integer $r > 0$, if $r|7 \neq 0$, then

(Read $x|y$ as "the remainder of x divided by y.")

a. $[\frac{3}{7} r] + [\frac{4}{7} r] + 1 = r$,

b. $[\frac{2}{7} r] + [\frac{5}{7} r] + 1 = r$,

c. $[\frac{1}{7} r] + [\frac{6}{7} r] + 1 = r$.

Proof. We will do case (a), the other cases may be proven in the same manner. The lemma is clearly true if $r < 7$. If $r > 7$ then there exist positive integers p and q such that $7p+q=r$. Since $r|7 \neq 0$, $q \neq 0$. Then,

$$[\tfrac{3}{7}(7p+q)] + [\tfrac{4}{7}(7p+q)] + 1 =$$
$$[3p + \tfrac{3}{7} q] + [4p + \tfrac{4}{7} p] + 1 =$$
$$7p + [\tfrac{3}{7} q] + [\tfrac{4}{7} q] + 1 = 7p+q, \text{ by lemma 7.}$$

Since $q<7$ the lemma is proved.

Lemma 10. Each of the following sums and relations follows from the above lemmas. Each proof is similar to the proof of lemma 9.

$$2([\tfrac{17}{7}\, 2^{k-2}]+[\tfrac{9}{7}\, 2^{k-3}]) \leq 1+T_{3k} \leq 2([\tfrac{17}{7}\, 2^{k-2}]+[\tfrac{9}{7}\, 2^{k-3}])+1 \qquad (13)$$

$$[\frac{9}{7} 2^{k-2}] - [\frac{6}{7} 2^{k-2}] < 2^{k-3} \tag{14}$$

$$2^{k-4} + [\frac{9}{7} 2^{k-3}] + [\frac{5}{7} 2^{k-1}] - 1 \leq S_{2k-2} \tag{15}$$

$$2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] < T_{3k-4} \tag{16}$$

3.3 Proof of Theorem 4.

Theorem 4. $$M(3,T_i) \leq i. \tag{8}$$

This theorem is proven by giving a explicit algorithm to solve the $(3,T_i)$ merging problem. The sequence T_i was defined in (7). There are three cases to consider, i=3k-1, 3k, and 3k+1. Additionally, there are three configurations of A_2 and B_n that arise during the algorithm which are treated separately. The algorithm decomposes each case into a series of configurations which are smaller than the original problem. These configurations are then solved optimally. The notations M(1,n) and M(2,n) denote both the use of the optimal 1 and 2 merge algorithms discussed above, and their complexity. The notation M(3,n) denotes an inductive application of this algorithm. We consider each case in turn, dealing only with the problem of merging A_3 into T_i elements. For $T_{i-1} < n \leq T_i$ we can always add T_i-n elements to the set B_n so that the total is T_i. The addition of these elements cannot affect the number of comparisons required by this algorithm in the worst case.

Proof. By induction. The theorem can be verified for k = 3 using the methods of the previous chapter. This is illustrated in exhibit II. Therefore, assume the theorem is true for all integers less than k and that k > 3.

Case a. i = 3k-1.

$$T_{3k-1} = 2[\frac{17}{7} 2^{k-2}]-1. \quad (17)$$

Let $x = [\frac{17}{7} 2^{k-2}]$,

$y = 2^{k-1}$.

Step I. $a_2 : b_x$

{Read "a_2 is compared with b_x." We will always deal with both results ('<' or '>') after a comparison step, indenting to distinguish subsequent steps and inserting these notes to clarify a point.}

1. $a_2 > b_x$ ->

   Step II. $a_1 : b_y$

   1. $a_1 < b_y$ -> $M(1,2^{k-1}-1) + M(2,S_{2k-2}) = 3k-3$.

      {At this point the problem has been decomposed into two subproblems. Since $a_1 < b_y$ and $a_2 > b_x$ the configuration is disjoint and thus these problems are independent. We therefore apply the optimal (1,n) binary insertion algorithm to merge a_1 into b_1 through b_{y-1}, and the optimal (2,n) algorithm to merge $a_2 < a_3$ into b_{x+1} through b_n. The notation $M(1,y-1)$ and $M(2,x-1)$ could have been used in place of the above. This will be done whenever it is convenient. Also, the grouping of elements of A_3 in this sum is from left to right, starting with row 1.}

   2. $a_1 > b_y$ ->

      Step III. $a_1 : b_{x+1}$

      1. $a_1 < b_{x+1}$ -> $M(1,[\frac{3}{7} 2^{k-2}]) + M(2,S_{2k-2}) = 3k-5$.
      2. $a_1 > b_{x+1}$ -> $M(3,T_{3k-4}) = 3k-4$ (Induction).

2. $a_2 < b_x$ ->

{This result is symmetric with the result in a.I.1 because the element a_2 is the center element of A_3 and the element b_x is the center element of B_n (since n must be odd).}

The results of all comparisons lead to a completion of the merge in 3k-1 or less steps, as was to be shown.

Case b. $i = 3k$.

$$T_{3k} = [\frac{19}{7} 2^{k-2}] + [\frac{12}{7} 2^{k-1}] - 1. \qquad (18)$$

Let $x = [\frac{17}{7} 2^{k-2}] + [\frac{9}{7} 2^{k-3}]$,

$y = T_{3k} - (2^{k-3} + 2^{k-4} - 1)$,

$z = T_{3k} - [\frac{5}{7} 2^{k-1}] + 1$,

$w = 2^{k-1}$,

$v = 2^{k-1} + 2^{k-2}$.

Step I. $a_2 : b_x$

1. $a_2 > b_x$ ->

Step II. $a_3 : b_y$

1. $a_3 > b_y$ ->

Step III. $a_2 : b_z$

1. $a_2 > b_z$ -> $M(1,T_{3k}) + C_1(2k-4) = 3k-3$.

{An algorithm to merge the special configuration C_1 induced on $a_2 < a_3$ and $b_{z+1} < \ldots < b_n$ in 2k-4 steps is given below as case (d). The configuration in case (d) is rotated 180 degrees for convenience.}

2. $a_2 < b_z$ -> $C_2(2k-1) + M(1,2^{k-3}+2^{k-4}-1) = 3k-3$.

{An algorithm to merge the special configuration C_2 induced on $a_1 < a_2$ and $b_1 < \ldots < b_{z-1}$ in 2k-1 steps is given below as case (e).}

2. $a_3 < b_y$ ->

<u>Step III</u>. $a_1 : b_w$

1. $a_1 < b_w \rightarrow M(1, 2^{k-1}-1) + M(2, S_{2k-2}) = 3k-3$.

{Note that this result uses (15).}

2. $a_1 > b_w \rightarrow$

<u>Step IV</u>. $a_1 : b_v$

1. $a_1 < b_v \rightarrow M(1, 2^{k-2}-1) + M(2, S_{2k-2}) = 3k-4$.

{Note that this result uses (15).}

2. $a_1 > b_v \rightarrow M(3, T_{3k-4}) = 3k-4$ (Induction).

{Note that this result uses (16).}

2. $a_2 < b_x$

{Note that x is at most half of T_{3k} by lemma 10. Therefore if this result occurs it is at worst a sub-configuration of the result in b.I.1 and the algorithm treats both identically.}

In each case the algorithm can complete the merge in a total of 3k steps or less, as was to be shown.

<u>Case c</u>. $i = 3k+1$.

$$T_{3k+1} = 2^{k-1} + 2^{k-3} + [\tfrac{12}{7} 2^{k-1}] + [\tfrac{12}{7} 2^{k-2}] - 1. \qquad (19)$$

Let $x = 2^{k-3} + 2^{k-2}$,

$y = T_{3k+1} - S_{2k}$,

$z = T_{3k+1} - 2^k + 1$.

<u>Step I</u>. $a_1 : b_x$

{Note that $T_{3k+1} - x = T_{3k}$ and that $T_{3k+1} - 2^k \leq T_{3k-2}$.}

1. $a_1 > b_x \rightarrow M(3, T_{3k}) = 3k$ (Induction).

2. $a_1 < b_x \rightarrow$

<u>Step II</u>. $a_2 : b_y$

1. $a_2 > b_y$ -> $M(1,2^{k-2}+2^{k-3}-1) + M(2,S_{2k}) = 3k-1$.

2. $a_2 < b_y$ ->

   Step III. $a_3:b_z$

   1. $a_3 < b_z$ -> $M(3,z) = 3k-2$ (Induction).

   2. $a_3 > b_z$ -> $C_3(2k-2) + M(1,2^k-1) = 3k-2$.

   {The resulting configuration is disjoint. The order relations of a_1 and a_2 induce configuration C_3 of case (f) below.}

In all cases the merge can be finished in a total of 3k+1 steps as was to be shown.

Case d. The configuration C_1 below can be ordered in at most 2k-4 steps.

$$C_1: \quad \begin{array}{|l|} 2^{k-4} + 2^{k-3} - 1 \\ 2^{k-4} + [\frac{12}{7} 2^{k-4}] + 2^{k-4} + 2^{k-3} - 1 \end{array} \leq 2k-4$$

Let $x = 2^{k-3} + [\frac{12}{7} 2^{k-4}]$,

$y = 2^{k-4}$,

$z = [\frac{12}{7} 2^{k-4}] + 2^{k-4}$

Step I. $a_2:b_x$

1. $a_2 > b_x$ -> $M(1,2^{k-3}+2^{k-4}-1) + M(1,2^{k-3}-1) = 2k-5$.

2. $a_2 < b_x$ ->

   Step II. $a_1:b_y$

   1. $a_1 < b_y$ -> $M(1,2^{k-4}-1) + M(1,x-1) = 2k-6$.

   2. $a_1 > b_y$ ->

      Step III. $a_2:b_z$

1. $a_2 < b_z \rightarrow M(2, S_{2k-7}) = 2k-7$.

2. $a_2 > b_z \rightarrow M(1, 2^{k-3}-1) + M(1, 2^{k-4}-1) = 2k-7$.

In each case the configuration can be ordered in at most $2k-4$ steps, as was claimed.

Case e. The configuration C_2 below can be ordered in at most $2k-1$ steps. This configuration arises from b.III.2 and equation (12).

$$C_2: \quad \begin{array}{l} | \; 2^{k-2} + [\frac{9}{7}\, 2^{k-3}] \qquad | \\ | \; [\frac{12}{7}\, 2^{k-2}] + 2^{k-2} + 2^{k-1} - 1 \; | \leq 2k-1 \end{array}$$

Let $x = [\frac{12}{7}\, 2^{k-2}] + 2^{k-2}$,

$y = [\frac{12}{7}\, 2^{k-2}]$.

Step I. $a_2 : b_x$

1. $a_2 > b_x \rightarrow M(1, 2^{k-2} + [\frac{9}{7}\, 2^{k-3}]) + M(1, 2^{k-1}-1) = 2k-2$.

2. $a_2 < b_x \rightarrow$

Step II. $a_2 : b_y$

1. $a_2 < \; \rightarrow M(2, S_{2k-3}) = 2k-3$.

2. $a_2 > \; \rightarrow M(1, 2^{k-2} + [\frac{9}{7}\, 2^{k-3}]) + M(1, 2^{k-2}-1) = 2k-3$.

In each case the configuration C_2 can be ordered in at most $2k-1$ comparisons as was claimed.

Case f. The configuration C_3 below can be ordered in $2k-2$ steps or less.

$$C_3: \quad \begin{array}{l} | \; 2^{k-3} + 2^{k-2} - 1 \qquad | \\ | \; 2^{k-3} + 2^{k-2} + [\frac{3}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-2}] + [\frac{1}{7}\, 2^{k-2}] \; | \leq 2k-2 \end{array}$$

Let $x = 2^{k-3} + [\frac{3}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-2}] + [\frac{1}{7}\, 2^{k-2}] + 1$,

$y = 2^{k-3}$,

$z = [\frac{12}{7} 2^{k-3}]$.

Step I. $a_2 : b_x$

1. $a_2 > b_x \to M(1, 2^{k-3} + 2^{k-2}-1) + M(1, 2^{k-2}-1) = 2k-3$.

2. $a_2 < b_x \to$

   Step II. $a_1 : b_y$

   1. $a_1 < b_y \to M(1, 2^{k-3}-1) + M(1, x) = 2k-4$

   2. $a_1 > b_y \to$

      Step III. $a_2 : b_z$

      1. $a_2 < b_z \to M(2, S_{2k-5}) = 2k-5$.

      2. $a_2 > b_z \to M(1, 2^{k-2}-1) + M(1, x-(y+z)) = 2k-5$.

      {Note that

$$[\frac{3}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-2}] + [\frac{1}{7} 2^{k-2}] \leq [\frac{9}{7} 2^{k-2}] \qquad (20)$$

      by lemmas 8-10.}

In all cases this configuration can be ordered in at most 2k-2 steps as was claimed.

This completes the proof of theorem 4.

3.4 Proof of Theorem 5.

Theorem 5. $$M(3, 1+T_i) > i. \qquad (9)$$

The proof of this theorem is complex but not deep. To show that no algorithm can solve the $(3, 1+T_i)$ merging problem in less than i+1 steps means saying something about all algorithms that solve this problem. To do this a method of representing algorithms, or better, classes of algorithms, is needed. Using lemma 6, there is a natural and obvious equivalence relation which allows this to be done in a (relatively) simple way. Given any configuration of A_3 and B_n, we divide algorithms into 3 classes based on the first comparison each algorithm makes. If the first comparison is a_i to b_x for some x, then the algorithm is in class i. Of course, x should be chosen such that lemma 6 can be applied. The advantage of this method is that it is only necessary to worry about configurations that result from one of these comparisons.

As in theorem 4, the proof works by induction, considering the cases i = 3k-1, 3k, and 3k+1. Starting with the initial (totally unordered) configuration, the sub-configurations that result from one of the three possible comparisons are analyzed to see if dominant sub-configurations can be shown to have the proper lower bound. If so, then lemma 6 is applied. Otherwise subsequent comparisons in the resulting configuration are analyzed until a dominating or set of dominating sub-configurations is identified.

The method of proof will be described in detail based on figure 2. This example is taken from a middle section of the proof of theorem 5 and is used here only for the purpose of explanation. Each stage of the proof or a lemma associated with a stage of the proof is represented by a configuration diagram (a). The top left corner indicates the index number of the configuration. The minimum cost (less one) claimed for the configuration to be completely ordered is represented in the lower right of the configuration diagram (b). Next follow the quantities x, y, and z representing the element indices x_i ($1 \leq i \leq 3$) of lemma 6 (c). These quantities divide the configuration in six ways representing each possible result of one of the three possible comparisons (d). If the cost following a possible comparison can

5.2.4:

$$\left|\begin{array}{l} 2^{k-1} + 2^{k-1} + [\frac{5}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-1}] \\ \qquad 2^{k-1} + [\frac{3}{7}\, 2^{k-2}] + [\frac{9}{7}\, 2^{k-3}] \\ \qquad\qquad [\frac{9}{7}\, 2^{k-3}] \end{array}\right| > 3k-2$$

Let $x = 2^{k-1}$,

$y = 1+T_{3k} - [\frac{5}{7}\, 2^{k-1}]$,

$z = 1+T_{3k} - 2^{k-4}$.

$a_1 \leq b_x$ -> 5.2.5

$a_1 > b_x$ -> $M(1,2^{k-1}) + M(1,2^{k-1}) + M(1,2^{k-3}) = 3k-2$.

$a_2 \leq b_y$ -> $M(1,2^k)$ + Lemma 12 (with k-1 for k) = 3k-2.

$a_2 > b_y$ -> Lemma 14.1 + $M(1,2^{k-3}) = 3k-2$.

$a_3 \leq b_z$ -> Lemma 15.1 + $M(1,2^{k-4}) = 3k-2$.

$a_3 > b_z$ -> 5.2.6

Figure 2. A Stage of the Proof of Theorem 5.

be shown to meet that claimed for the configuration then this fact (showing the decomposition) is indicated immediately (e). Otherwise, the index of the new configuration is given (f). This latter figure will be the refinement of the current configuration corresponding to the additional relation indicated on the left (g). Occassionally, the cost following a comparison will meet that claimed for the configuration but the decomposition will involve the use of a lemma. When this occurs the lemma's configuration index will be given (h).

Proof. By induction. The theorem can be verified for all $k \leq 5$ by direct calculation. The relevant cases are displayed in exhibit II. Therefore, assume the theorem is true for all integers less than k and that $k > 5$. There are three cases to consider. The first case is straight-forward, it is the second and third cases which present the major problem. Since each generates a large number of subproblems, wherever possible these configurations are broken down into dominant sub-configurations which can then be analyzed individually. We will first prove two lemmas (3.4.1), each dealing with a separate subproblem. These lemmas are used in some way by all the subproblems in the second and third sections of the proof. The case $i = 3k-1$ is then proven next (3.4.2), followed in turn by the sub-configurations of the second case (3.4.3), the second case proper (3.4.4), the sub-configurations of the third case (3.4.5), and finally the third case (3.4.6).

3.4.1 Two Lemmas.

Lemma 11. The configuration 11.1 below has a minimum cost of 2k+2, for all $k \geq 0$.

Proof. By induction. The lemma can be verified for $k \leq 3$ by direct calculation using the techniques of the previous chapter. Therefore, assume the lemma is true for all integers less than k and that $k > 3$.

11.1:

$$\begin{array}{l} | \ [\frac{11}{7} 2^k] \ | \\ | \ [\frac{12}{7} 2^k] \ | \ > 2k+1 \end{array}$$

Let $x = 2^{k-1}$,

$y = [\frac{17}{7} 2^{k-1}]$.

$a_1 \leq b_x$ -> 11.2

$a_1 > b_x$ -> $M(1,2^{k-1}) + M(1,2^k) = 2k+1$.

$a_2 \leq b_y$ -> $M(1,2^k) + M(1,2^{k-1}) = 2k+1$.

$a_2 > b_y$ -> $M(2,1+S_{2k}) = 2k+1$.

11.2:

$$\begin{array}{l} | \ 2^{k-1} + [\frac{8}{7} 2^{k-1}] \ | \\ | \ [\frac{17}{7} 2^{k-1}] \ | \ > 2k \end{array}$$

Let $x = 2^{k-2} + [\frac{2}{7} 2^{k-2}]$,

$y = [\frac{12}{7} 2^{k-1}]$.

$a_1 \leq b_x$ -> 11.3

$a_1 > b_x$ -> 11.4

$a_2 \leq b_y$ -> 11.5

$a_2 > b_y \rightarrow M(2, 1+S_{2k-1}) = 2k.$

11.3:

$| \; 2^{k-2} + 2^{k-1} \; |$

$| \; [\frac{5}{7} 2^{k-2}] + 2^{k-1} + [\frac{3}{7} 2^{k-1}] \; | \quad > 2k-1$

{Note that a constant of 1 removed from bottom row.}

Let $x = 2^{k-2}$,

$y = [\frac{17}{7} 2^{k-2}]$.

$a_1 \leq b_x \rightarrow 11.6$

$a_1 > b_x \rightarrow M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1.$

$a_2 \leq b_y \rightarrow M(1,2^{k-1}) + M(1,2^{k-2}) = 2k-1.$

$a_2 > b_y \rightarrow M(2,1+S_{2k-2}) = 2k-1.$

11.4:

$| \; 2^{k-2} + [\frac{2}{7} 2^{k-2}] \; |$

$| \; 2^{k} + [\frac{3}{7} 2^{k-1}] \; | \quad > 2k-1$

Let $x = 2^{k-3}$,

$y = [\frac{10}{7} 2^{k-1}]$.

$a_1 \leq b_x \rightarrow 11.7$

$a_1 > b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k}) = 2k-1.$

$a_2 \leq b_y \rightarrow M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1.$

$a_2 > b_y \rightarrow 11.8$

11.5:

$$\left| \begin{array}{r} 2^{k-1} + 2^{k-2} + [\frac{3}{7}\, 2^{k-2}] + 1 + [\frac{4}{7}\, 2^{k-2}] + [\frac{2}{7}\, 2^{k-2}] \\ 1 + [\frac{4}{7}\, 2^{k-2}] + [\frac{3}{7}\, 2^{k-1}] \end{array} \right| > 2k-1$$

Let $x = 2^{k-1}$,

$y = 2^k + [\frac{5}{7}\, 2^{k-3}]$.

$a_1 \leq b_x$ -> 11.9

$a_1 > b_x$ -> $M(1,2^{k-1}) + M(1,2^{k-2}) = 2k-1$.

$a_2 \leq b_y$ -> $M(1,2^{k}) + M(1,2^{k-3}) = 2k-1$.

$a_2 > b_y$ -> 11.10

11.6:

$$\left| \begin{array}{l} 2^{k-1} \\ 2^{k-2} + [\frac{5}{7}\, 2^{k-2}] + [\frac{3}{7}\, 2^{k-1}] \end{array} \right| > 2k-2$$

Let $x = 2^{k-3}$,

$y = [\frac{12}{7}\, 2^{k-2}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-1}) + M(1,2^{k-3}) = 2k-2$.

$a_2 > b_y$ -> $M(2,1+S_{2k-3}) = 2k-2$.

11.7:

$$\left| \begin{array}{l} 2^{k-3} + [\frac{4}{7}\, 2^{k-3}] \\ 2^{k} + [\frac{5}{7}\, 2^{k-3}] \end{array} \right| > 2k-2$$

Let $x = 2^{k-4}$,

$y = 2^{k-1} + [\frac{5}{7} 2^{k-3}]$.

$a_1 \le b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_1 > b_x \rightarrow M(1,2^{k-4}) + M(1,2^{k}) = 2k-2$.

$a_2 \le b_y \rightarrow M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$

$a_2 > b_y \rightarrow 11.11$

11.8:

$$\begin{vmatrix} 2^{k-2} + [\frac{2}{7} 2^{k-2}] \\ [\frac{3}{7} 2^{k-1}] + 2^{k-1} \end{vmatrix} > 2k-2$$

Let $x = 2^{k-3}$,

$y = [\frac{3}{7} 2^{k-1}] + 2^{k-2}$.

$a_1 \le b_x \rightarrow 11.11$

$a_1 > b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_2 \le b_y \rightarrow M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_2 > b_y \rightarrow 11.12$

11.9:

$$\begin{vmatrix} 2^{k-2} + [\frac{3}{7} 2^{k-2}] + 1 + [\frac{4}{7} 2^{k-2}] + [\frac{2}{7} 2^{k-2}] \\ 1 + [\frac{4}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-1}] \end{vmatrix} > 2k-2$$

Let $x = 2^{k-2}$,

$y = 2^{k-1} + [\frac{5}{7} 2^{k-3}]$.

$a_1 \le b_x \rightarrow 11.13$

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-1}) + M(1,2^{k-3}) = 2k-2$.

$a_2 > b_y$ -> 11.14

11.10:

$$\left| \begin{matrix} 2^{k-1} + 2^{k-2} + [\frac{3}{7} 2^{k-2}] + 1 + [\frac{4}{7} 2^{k-2}] + [\frac{2}{7} 2^{k-2}] \\ 1 + [\frac{4}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-3}] \end{matrix} \right| > 2k-2$$

Let $x = 2^{k-1}$,

$y = 2^k + [\frac{3}{7} 2^{k-4}]$.

$a_1 \leq b_x$ -> 11.14

$a_1 > b_x$ -> $M(1,2^{k-1}) + M(1,2^{k-3}) = 2k-2$.

$a_2 \leq b_y$ -> $M(1,2^k) + M(1,2^{k-4}) = 2k-2$.

$a_2 > b_y$ -> $M(1,2^{k-1}) + M(1,2^{k-3}) = 2k-2$.

11.11:

$$\left| \begin{matrix} 2^{k-3} + [\frac{4}{7} 2^{k-3}] \\ 2^{k-1} + [\frac{5}{7} 2^{k-3}] \end{matrix} \right| > 2k-3$$

Let $x = 2^{k-4}$,

$y = 2^{k-2} + [\frac{5}{7} 2^{k-3}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(1,2^{k-1}) = 2k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_2 > b_y$ -> 11.15

11.12:

$$\begin{vmatrix} 2^{k-2} + [\frac{4}{7} 2^{k-3}] \\ 2^{k-2} + [\frac{12}{7} 2^{k-3}] \end{vmatrix} > 2k-3$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7} 2^{k-3}]$.

$a_1 \leq b_x$ -> 11.15

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3$.

$a_2 > b_y$ -> $M(2,1+S_{2k-4}) = 2k-3$.

11.13:

$$\begin{vmatrix} [\frac{3}{7} 2^{k-2}] + 1 + [\frac{4}{7} 2^{k-2}] + [\frac{2}{7} 2^{k-2}] \\ 1 + [\frac{4}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-1}] \end{vmatrix} > 2k-3$$

Let $x = 2^{k-3}$,

$y = 2^{k-2} + [\frac{5}{7} 2^{k-3}]$.

$a_1 \leq b_x$ -> 11.15

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3$.

$a_2 > b_y$ -> 11.16

11.14:

$$\begin{vmatrix} 2^{k-2} + [\frac{3}{7} 2^{k-2}] + 1 + [\frac{4}{7} 2^{k-2}] + [\frac{2}{7} 2^{k-2}] \\ 1 + [\frac{4}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-3}] \end{vmatrix} > 2k-3$$

Let $x = 2^{k-2}$,

$y = 2^{k-1} + [\frac{3}{7} 2^{k-4}]$.

$a_1 \leq b_x$ -> 11.16

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-1}) + M(1,2^{k-4}) = 2k-3$.

$a_2 > b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3$.

11.15:

$$\left| \begin{array}{l} 2^{k-3} + [\frac{4}{7} 2^{k-3}] \\ 2^{k-2} + [\frac{5}{7} 2^{k-3}] \end{array} \right| > 2k-4$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7} 2^{k-3}]$,

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(1,2^{k-2}) = 2k-4$.

$a_2 \leq b_y$ -> $M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4$.

$a_2 > b_y$ -> 11.17

11.16:

$$\left| \begin{array}{r} [\frac{3}{7} 2^{k-2}] + 1 + [\frac{4}{7} 2^{k-2}] + [\frac{2}{7} 2^{k-2}] \\ 1 + [\frac{4}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-3}] \end{array} \right| > 2k-4$$

Let $x = 2^{k-3}$,

$y = 2^{k-2} + [\frac{3}{7} 2^{k-4}]$.

$a_1 \leq b_x$ -> 11.17

$a_1 > b_x \to M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4.$

$a_2 \leq b_y \to M(1,2^{k-2}) + M(1,2^{k-4}) = 2k-4.$

$a_2 > b_y \to M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4.$

11.17:

$$\left| \left[\frac{11}{7} 2^{k-3}\right] \right|$$
$$\left| \left[\frac{12}{7} 2^{k-3}\right] \right| \quad > 2k-5$$

By the induction hypothesis this configuration has a minimum cost of at least 2k-4 comparisons. Since every possible comparison or sequence of comparisons ends in a configuration which requires a total of 2k+2 steps to completely order, as was to be shown, the lemma is proved.

Because the proof of this lemma is inductive it is in a sense circular. This statement is made not to impugn the validity of the proof or of mathematical induction in general, but rather to point out that any configuration in the chain could have been used as the starting configuration of the lemma (of course using the appropriate bound). Consequently, without loss of generality we are justified in citing any of these configurations as lemmas in their own right choosing values of k as seems appropriate, and will do so as the need arises. The next lemma is a case in point.

Lemma 12. The configuration 12.1 below has a minimum cost of 2k-1 steps.

Proof.

12.1:

$| \ [\frac{9}{7}\ 2^{k-2}]\ |$

$| \ [\frac{5}{7}\ 2^{k}] \quad | \ > 2k-2$

This configuration is identical with that of 11.8 in the previous lemma. As was shown there, it has a minimum cost of 2k-1, as claimed. In light of the remarks concluding the proof of lemma 11 and since the proof of this lemma depends directly on the proof of that lemma, substitution instances of this lemma may and will be used as needed.

3.4.2 Theorem 5: The First Case.

The first case is to show

$$M(3, 1+T_{3k-1}) > 3k-1.$$

Where

$$1+T_{3k-1} = 2[\frac{17}{7}\ 2^{k-2}]. \qquad (21)$$

The initial configuration is 5.1.1 below. Because of the symmetry of regular configurations there are only two possible first comparisons. Using the addition lemmas 8-10, it is easy to show that

$$T_{3k-1} - T_{3k-2} \geq 2^{k-2}. \qquad (22)$$

5.1.1:

$| \ [\frac{17}{7}\ 2^{k-2}] + [\frac{17}{7}\ 2^{k-2}]\ |$

$| \ [\frac{17}{7}\ 2^{k-2}] + [\frac{17}{7}\ 2^{k-2}]\ |$

$| \ [\frac{17}{7}\ 2^{k-2}] + [\frac{17}{7}\ 2^{k-2}]\ | \ > 3k-1$

Let $x = 2^{k-2}$,

$y = [\frac{17}{7}\ 2^{k-2}]$.

$a_1 \leq b_x \rightarrow M(3, 1+T_{3k-2}) = 3k-1$ (Induction).

$a_1 > b_x \rightarrow M(1,2^{k-2}) + M(2,1+S_{2k-1}) = 3k-1.$

$a_2 \le b_y \rightarrow M(1,2^{k-1}) + M(2,1+S_{2k-2}) = 3k-1.$

In the configuration that results from any possible initial comparison there exists a set of dominating sub-configurations with a minimum cost of 3k-1 as claimed. Therefore the initial configuration has a minimum cost of at least 3k steps, as was to be shown.

3.4.3 Lemmas for the Second Case.

Lemma 14. The configuration 14.1 (below) has a minimum cost of 2k comparisons to completely order.

14.1:

$$\left| \begin{array}{l} 2^{k-2} + [\frac{9}{7}\, 2^{k-3}] \\ 2^{k-2} + [\frac{12}{7}\, 2^{k-2}] + 2^{k-1} \end{array} \right| > 2k-1$$

Let $x = 2^{k-3}$,

$y = 2^{k-2} + [\frac{12}{7}\, 2^{k-2}]$.

$a_1 \le b_x \rightarrow M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1.$

$a_1 > b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k}) = 2k-1.$

$a_2 \le b_y \rightarrow M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1.$

$a_2 > b_y \rightarrow$ 14.2

14.2:

$$\left| \begin{array}{l} 2^{k-2} + [\frac{9}{7}\, 2^{k-3}] \\ 2^{k-2} + [\frac{5}{7}\, 2^{k-2}] + 2^{k-2} \end{array} \right| > 2k-2$$

Let $x = 2^{k-3}$,

$y = [\frac{12}{7} 2^{k-2}]$.

$a_1 \le b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_2 \le b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_2 > b_y$ -> 14.3

14.3:

$| 2^{k-2} + [\frac{9}{7} 2^{k-3}] |$

$| 2^{k-2} + [\frac{5}{7} 2^{k-2}] | \quad > 2k-3$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7} 2^{k-3}]$.

$a_1 \le b_x$ -> Lemma 11.2 (k-3 for k-1) = 2k-3.

{Note. The cited lemma dominates the configuration that arises from this result. The length of row 1 is (given the substitution used) is $2^{k-3}+[\frac{8}{7} 2^{k-3}]$ in that lemma.}

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_2 \le b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3$.

$a_2 > b_y$ -> $M(2,1+S_{2k-4}) = 2k-3$.

Every comparison results in a configuration or set of configurations that have a total cost of at least 2k steps, as was to be shown. Therefore, the lemma is proved.

Lemma 15. The configuration 15.1 (below) has a minimum cost of 2k+1 steps to completely order.

15.1:

$$\begin{vmatrix} 2^{k-1} + [\frac{3}{7} 2^{k-2}] + [\frac{2}{7} 2^{k-3}] + 2^{k-5} \\ 2^{k-1} + 2^{k-2} + 2^{k-3} + 2^{k-5} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] \end{vmatrix} > 2k$$

Let $x = 2^{k-2}$,

$y = 2^{k-2} + 2^{k-3} + 2^{k-5} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}]$.

$a_1 \leq b_x$ -> 15.2

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(1,2^k) = 2k$.

$a_2 \leq b_y$ -> $M(1,2^{k-1}) + M(1,2^{k-1}) = 2k$.

$a_2 > b_y$ -> 15.3

15.2:

$$\begin{vmatrix} 2^{k-2} + [\frac{3}{7} 2^{k-2}] + [\frac{2}{7} 2^{k-3}] + 2^{k-5} \\ 2^{k-1} + 2^{k-3} + 2^{k-5} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] \end{vmatrix} > 2k-1$$

Let $x = 2^{k-3}$,

$y = 2^{k-3} + 2^{k-5} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^k) = 2k-1$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1$.

$a_2 > b_y$ -> 15.4

15.3:

$$\left| 2^{k-1} + [\tfrac{3}{7}\, 2^{k-2}] + [\tfrac{2}{7}\, 2^{k-3}] + 2^{k-5} \right|$$
$$\left| 2^{k-2} + 2^{k-3} + 2^{k-5} + [\tfrac{5}{7}\, 2^{k-2}] + [\tfrac{5}{7}\, 2^{k-1}] \right| > 2k-1$$

Let $x = 2^{k-2}$,

$y = 2^{k-3} + 2^{k-5} + [\tfrac{5}{7}\, 2^{k-2}] + [\tfrac{5}{7}\, 2^{k-1}]$.

$a_1 \leq b_x$ -> 15.4

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1$.

$a_2 \leq b_y$ -> $M(1,2^{k-1}) + M(1,2^{k-2}) = 2k-1$.

$a_2 > b_y$ -> $M(2,1+S_{2k-2}) = 2k-1$.

15.4:

$$\left| 2^{k-2} + [\tfrac{3}{7}\, 2^{k-2}] + [\tfrac{2}{7}\, 2^{k-3}] + 2^{k-5} \right|$$
$$\left| 2^{k-3} + 2^{k-5} + [\tfrac{5}{7}\, 2^{k-2}] + [\tfrac{5}{7}\, 2^{k-1}] \right| > 2k-2$$

Let $x = 2^{k-3}$,

$y = [\tfrac{12}{7}\, 2^{k-2}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_2 > b_y$ -> Lemma 11.1 (k-2 for k) = 2k-2.

{Note. The cited lemma dominates the configuration which arises if this result occurs. In this case row 1 is strictly greater than $[\tfrac{11}{7}\, 2^{k-2}]$.}

Every comparison results in a configuration or set of configurations that have a total cost of at least 2k+1 steps, as was to be shown. Therefore, the lemma is proved.

3.4.4 Theorem 5: The Second Case.

The second case is to show

$$M(3,1+T_{3k}) > 3k, \tag{23}$$

where

$$1+T_{3k} = [\tfrac{19}{7}\, 2^{k-2}] + [\tfrac{12}{7}\, 2^{k-1}]. \tag{24}$$

By lemma 10,

$$T_{3k} - T_{3k-1} \geq 2^{k-2}, \quad \text{and,} \tag{25}$$

$$2([\tfrac{17}{7}\, 2^{k-2}]+[\tfrac{9}{7}\, 2^{k-3}]) \leq 1+T_{3k} \leq 2([\tfrac{17}{7}\, 2^{k-2}]+[\tfrac{9}{7}\, 2^{k-3}])+1, \tag{13}$$

hence there are only two initial comparisons which are unique, and only three distinct outcomes. The initial configuration is

5.2.1:

$$\begin{array}{l} \mid [\tfrac{19}{7}\, 2^{k-2}] + [\tfrac{12}{7}\, 2^{k-1}] \mid \\ \mid [\tfrac{19}{7}\, 2^{k-2}] + [\tfrac{12}{7}\, 2^{k-1}] \mid \\ \mid [\tfrac{19}{7}\, 2^{k-2}] + [\tfrac{12}{7}\, 2^{k-1}] \mid \; > 3k \end{array}$$

Let $x = 2^{k-2}$,

$y = [\tfrac{17}{7}\, 2^{k-2}] + [\tfrac{9}{7}\, 2^{k-3}] + 1.$

$a_1 \leq b_x$ -> $M(3,1+T_{3k-1}) = 3k$ (Induction).

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(2,1+S_{2k}) = 3k$.

$a_2 \leq b_y$ -> 5.2.2

5.2.2:

$$\left| \begin{array}{l} 2^{k-1} + 2^{k-2} + 2^{k-2} + [\frac{5}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-1}] \\ 2^{k-1} + [\frac{3}{7}\, 2^{k-2}] + [\frac{9}{7}\, 2^{k-3}] \\ 2^{k-1} + [\frac{3}{7}\, 2^{k-2}] + [\frac{9}{7}\, 2^{k-3}] \end{array} \right| > 3k-1$$

Let $x = 2^{k-1}$,

$y = 1+T_{3k} - [\frac{12}{7}\, 2^{k-2}]$.

$z = 1+T_{3k} - [\frac{9}{7}\, 2^{k-3}]$.

$a_1 \le b_x$ -> 5.2.3

$a_1 > b_x$ -> $M(1,2^{k-1}) + M(2,1+S_{2k-2}) = 3k-1$.

$a_2 \le b_y$ -> $M(1,2^{k}) + M(2,1+S_{2k-3}) = 3k-1$.

$a_2 > b_y$ -> $M(1,2^{k-1}) + M(1,2^{k-2}) + M(1,2^{k-1}) = 3k-1$.

$a_3 \le b_z$ -> 5.2.4

$a_3 > b_z$ -> $M(1,2^{k-1}) + M(2,1+S_{2k-2}) = 3k-1$.

5.2.3:

$$\left| \begin{array}{l} 2^{k-2} + 2^{k-2} + [\frac{5}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-1}] \\ 2^{k-1} + [\frac{3}{7}\, 2^{k-2}] + [\frac{9}{7}\, 2^{k-3}] \\ 2^{k-1} + [\frac{3}{7}\, 2^{k-2}] + [\frac{9}{7}\, 2^{k-3}] \end{array} \right| > 3k-2$$

Let $x = 2^{k-2}$,

$y = 1+T_{3k} - (2^{k-1} + [\frac{12}{7}\, 2^{k-2}])$,

$z = 1+T_{3k} - (2^{k-1} + [\frac{9}{7}\, 2^{k-3}])$.

$a_1 \le b_x$ -> $M(3,1+T_{3k-3}) = 3k-2$ (Induction).

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-2}) = 3k-2$.

$a_2 \le b_y$ -> $M(1,2^{k-1}) + M(2,1+S_{2k-3}) = 3k-2$.

$a_2 > b_y \to M(1,2^{k-2}) + M(1,2^{k-2}) + M(1,2^{k-1}) = 3k-2.$

$a_3 \le b_z \to 5.2.5$

$a_3 > b_z \to M(1,2^{k-2}) + M(2,1+S_{2k-2}) = 3k-2.$

5.2.4:

$$\begin{vmatrix} 2^{k-1} + 2^{k-1} + [\tfrac{5}{7}\, 2^{k-2}] + [\tfrac{5}{7}\, 2^{k-1}] \\ 2^{k-1} + [\tfrac{3}{7}\, 2^{k-2}] + [\tfrac{9}{7}\, 2^{k-3}] \\ [\tfrac{9}{7}\, 2^{k-3}] \end{vmatrix} > 3k-2$$

Let $x = 2^{k-1}$,

$y = 1+T_{3k} - [\tfrac{5}{7}\, 2^{k-1}]$,

$z = 1+T_{3k} - 2^{k-4}$.

$a_1 \le b_x \to 5.2.5$

$a_1 > b_x \to M(1,2^{k-1}) + M(1,2^{k-1}) + M(1,2^{k-3}) = 3k-2.$

$a_2 \le b_y \to M(1,2^{k})$ + Lemma 12 (with k-1 for k) = 3k-2.

$a_2 > b_y \to$ Lemma 14.1 + $M(1,2^{k-3}) = 3k-2.$

$a_3 \le b_z \to$ Lemma 15.1 + $M(1,2^{k-4}) = 3k-2.$

$a_3 > b_z \to 5.2.6$

5.2.5:

$$\begin{vmatrix} 2^{k-2} + 2^{k-2} + [\tfrac{5}{7}\, 2^{k-2}] + [\tfrac{5}{7}\, 2^{k-1}] \\ 2^{k-1} + [\tfrac{3}{7}\, 2^{k-2}] + [\tfrac{9}{7}\, 2^{k-3}] \\ [\tfrac{9}{7}\, 2^{k-3}] \end{vmatrix} > 3k-3$$

Let $x = 2^{k-2}$,

$y = 1+T_{3k} - [\tfrac{12}{7}\, 2^{k-1}]$,

$z = 1+T_{3k} - (2^{k-1} + 2^{k-4})$.

$a_1 \leq b_x$ -> $M(1,S_{2k-2}) + M(1,2^{k-3}) = 3k-3$.

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) + M(1,2^{k-3}) = 3k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-1})$ + Lemma 12.1 (with k-1 for k) = 3k-3.

$a_2 > b_y$ -> Lemma 14.2 + $M(1,2^{k-3}) = 3k-3$.

$a_3 \leq b_z$ -> Lemma 15.3 + $M(1,2^{k-4}) = 3k-3$.

$a_3 > b_z$ -> 5.2.7

5.2.6:

$$\left| \begin{array}{r} [\frac{12}{7} 2^{k-2}] + [\frac{12}{7} 2^{k-1}] + 2^{k-3} + 2^{k-4} \\ 2^{k-1} + [\frac{3}{7} 2^{k-2}] + [\frac{11}{7} 2^{k-4}] \\ [\frac{2}{7} 2^{k-3}] + 2^{k-4} \end{array} \right| > 3k-3$$

Let $x = 2^{k-1}$,

$y = 1+T_{3k} + 2^{k-4} - [\frac{5}{7} 2^{k-1}]$,

$z = 1+T_{3k} - (2^{k-4} + 2^{k-5})$.

$a_1 \leq b_x$ -> 5.2.7

$a_1 > b_x$ -> $M(1,2^{k-1}) + M(1,2^{k-1}) + M(1,2^{k-4}) = 3k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k})$ + Lemma 11.12 (k-3 for k-2) = 3k-3.

$a_2 > b_y$ -> Lemma 14.1 + $M(1,2^{k-4}) = 3k-3$.

$a_3 \leq b_z$ -> Lemma 15.1 + $M(1,2^{k-5}) = 3k-3$.

$a_3 > b_z$ -> $M(1,2^{k-1}) + M(1,2^{k-1}) + M(1,2^{k-4}) = 3k-3$.

5.2.7:

$$\begin{array}{|r|}[\frac{12}{7}2^{k-2}] + [\frac{5}{7}2^{k-1}] + 2^{k-3} + 2^{k-4} \\ 2^{k-1} + [\frac{3}{7}2^{k-2}] + [\frac{11}{7}2^{k-4}] \\ [\frac{2}{7}2^{k-3}] + 2^{k-4}\end{array} > 3k-4$$

Let $x = 2^{k-2}$,

$y = 1+T_{3k} - [\frac{12}{7} 2^{k-1}]$,

$z = 1+T_{3k} - (2^{k-1} + 2^{k-4} + 2^{k-5})$.

$a_1 \le b_x$ -> $M(2,1+S_{2k-2}) + M(1,2^{k-4}) = 3k-4$.

$a_1 > b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) + M(1,2^{k-4}) = 3k-4$.

$a_2 \le b_y$ -> $M(1,2^{k-1})$ + Lemma 11.12 (k-3 for k-2) = 3k-4.

$a_2 > b_y$ -> Lemma 14.2 + $M(1,2^{k-4})$ = 3k-4.

$a_3 \le b_z$ -> Lemma 15.3 + $M(1,2^{k-5})$ = 3k-4.

$a_3 > b_z$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) + M(1,2^{k-4}) = 3k-4$.

Every possible comparison results in a configuration or set of configurations that require a total of 3k steps to completely order, as was to be shown. Therefore, we conclude that equation 23 is correct.

3.4.5 Lemmas for the Third Case.

Lemma 16. The configuration 16.1 (below) has a minimum cost of 2k-1.

Using lemmas 8-10 it can be shown that

$$[\frac{5}{7} 2^{k-3}] \le 1 + [\frac{2}{7} 2^{k-3}] + [\frac{3}{7} 2^{k-3}] \qquad (26)$$

16.1:

$$\begin{array}{|l|}2^{k-2} + 2^{k-3} \\ 2^{k-1} + [\frac{12}{7}2^{k-4}] + [\frac{5}{7}2^{k-3}]\end{array} > 2k-2$$

Let $x = 2^{k-3}$,

$y = 2^{k-2} + [\frac{12}{7} 2^{k-4}] + [\frac{5}{7} 2^{k-3}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_2 > b_y$ -> Lemma 12.3 (k-2 for k-1) = 2k-2

{Note. The lemma cited dominates the configuration that arises should this result occur.}

Since all possibilities must take a minimum of 2k-1 steps, as was to be shown, the lemma is proved.

Lemma 17. The configuration 17.1 (below) has a minimum cost of 2k-1 steps.

17.1:

$$\begin{array}{|l|l} 2^{k-4} + [\frac{5}{7} 2^{k-2}] & \\ 2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{12}{7} 2^{k-1}] & > 2k-2 \end{array}$$

{Note for $k \geq 5$, Row 2 is $\geq 2^k + 2^{k-4}$.}

Let $x = 2^{k-4}$,

$y = 2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(1,2^{k}) = 2k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_2 > b_y$ -> 17.2

17.2:

$$\left| \begin{array}{l} 2^{k-4} + [\frac{5}{7}\, 2^{k-2}] \\ 2^{k-4} + [\frac{5}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-1}] \end{array} \right| > 2k-3$$

Let $x = 2^{k-4}$,

$y = 2^{k-4} + [\frac{5}{7}\, 2^{k-2}] + [\frac{3}{7}\, 2^{k-2}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(1,2^{k-1}) = 2k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_2 > b_y$ -> 17.3

17.3:

$$\left| \begin{array}{l} 2^{k-4} + [\frac{5}{7}\, 2^{k-2}] \\ 2^{k-4} + [\frac{5}{7}\, 2^{k-2}] + [\frac{3}{7}\, 2^{k-2}] \end{array} \right| > 2k-4$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7}\, 2^{k-3}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(1,2^{k-2}) = 2k-4$.

$a_2 \leq b_y$ -> $M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4$.

$a_2 > b_y$ -> $M(2,1+S_{2k-5}) = 2k-4$.

{Note that these last two cases use lemma 10 and depend on the assumption that $k > 5$.}

Since all possibilities require at least 2k-1 steps, as was to be shown, the lemma is proved.

Lemma 18. The configuration 18.1 (below) has a minimum cost of 2k steps.

18.1:

$$\left| \begin{matrix} 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] & \\ 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{12}{7} 2^{k-1}] \end{matrix} \right| > 2k-1$$

Let $x = 2^{k-3}$,

$y = 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}]$.

$a_1 \leq b_x$ -> Lemma 17.1 = 2k-1.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k}) = 2k-1$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1$.

$a_2 > b_y$ -> 18.2

18.2:

$$\left| \begin{matrix} 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] & \\ 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] \end{matrix} \right| > 2k-2$$

Let $x = 2^{k-3}$,

$y = 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-2}]$.

$a_1 \leq b_x$ -> Lemma 17.2 = 2k-2.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2$.

$a_2 > b_y$ -> 18.3

18.3:

$$\begin{vmatrix} 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] & \\ 2^{k-3} + 2^{k-4} + [\frac{5}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-2}] \end{vmatrix} > 2k-3$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7} 2^{k-3}]$.

$a_1 \leq b_x$ -> Lemma 17.3 = 2k-3.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3$.

$a_2 > b_y$ -> $M(2,1+S_{2k-4}) = 2k-3$.

Since all possibilities require at least 2k steps, as was to be shown, the lemma is proved.

Lemma 19. The configuration 19.1 (below) has a minimum cost of 2k-3 steps.

19.1:

$$\begin{vmatrix} 2^{k-2} & \\ 2^{k-2} + [\frac{3}{7} 2^{k-2}] \end{vmatrix} > 2k-4$$

Let $x = 2^{k-4}$,

$y = 2^{k-3} + 2^{k-4} + [\frac{3}{7} 2^{k-2}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(1,2^{k-2}) = 2k-4$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-4}) = 2k-4$.

$a_2 > b_y$ -> $M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4$.

Since all possibilities require at least 2k-3 steps, as was to be shown, the lemma is proved.

Lemma 20. The configuration 20.1 (below) has a minimum cost of 2k-2 steps.

20.1:

$$\left| \begin{array}{l} 2^{k-3} + 2^{k-2} \\ 2^{k-3} + [\frac{17}{7} 2^{k-3}] + [\frac{5}{7} 2^{k-4}] \end{array} \right| > 2k-3$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7} 2^{k-3}]$.

$a_1 \leq b_x \rightarrow$ 20.2

$a_1 > b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3.$

$a_2 \leq b_y \rightarrow M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3.$

$a_2 > b_y \rightarrow M(2,1+S_{2k-4}) = 2k-3.$

20.2:

$$\left| \begin{array}{l} 2^{k-2} \\ [\frac{17}{7} 2^{k-3}] + [\frac{5}{7} 2^{k-4}] \end{array} \right| > 2k-4$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7} 2^{k-3}]$.

$a_1 \leq b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4.$

$a_1 > b_x \rightarrow M(1,2^{k-4}) + M(1,2^{k-2}) = 2k-4.$

$a_2 \leq b_y \rightarrow M(1,2^{k-2}) + M(1,2^{k-4}) = 2k-4.$

$a_2 > b_y \rightarrow M(2,1+S_{2k-5}) = 2k-4.$

Since all possibilities require at least 2k-2 steps, as was to be shown, the lemma is proved.

<u>Lemma</u> 21. The configuration 21.1 (below) has a minimum cost of 2k steps.

21.1:

$$\begin{array}{|l|} 2^{k-2} + 2^{k-3} \\ 2^{k} + 2^{k-3} + 2^{k-4} \end{array} > 2k-1$$

Let $x = 2^{k-3}$,

$y = 2^{k-1}$.

$a_1 \leq b_x \to M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1.$

$a_1 > b_x \to M(1,2^{k-3}) + M(1,2^{k}) = 2k-1.$

$a_2 \leq b_y \to M(1,2^{k-2}) + M(1,2^{k-1}) = 2k-1.$

$a_2 > b_y \to$ 21.2

21.2:

$$\begin{array}{|l|} 2^{k-2} + 2^{k-3} \\ 2^{k-1} + 2^{k-3} + 2^{k-4} \end{array} > 2k-2$$

Let $x = 2^{k-3}$,

$y = 2^{k-2}$.

$a_1 \leq b_x \to M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2.$

$a_1 > b_x \to M(1,2^{k-3}) + M(1,2^{k-1}) = 2k-2.$

$a_2 \leq b_y \to M(1,2^{k-2}) + M(1,2^{k-2}) = 2k-2.$

$a_2 > b_y$ -> 21.3

21.3:

$$\left| \begin{matrix} 2^{k-2} + 2^{k-3} \\ 2^{k-2} + 2^{k-3} + 2^{k-4} \end{matrix} \right| > 2k-3$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7} 2^{k-3}]$.

$a_1 \leq b_x$ -> 21.4

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) = 2k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) = 2k-3$.

$a_2 > b_y$ -> $M(2,1+S_{2k-4}) = 2k-3$.

21.4:

$$\left| \begin{matrix} 2^{k-2} \\ 2^{k-2} + 2^{k-4} \end{matrix} \right| > 2k-4$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7} 2^{k-3}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-3}) = 2k-4$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(1,2^{k-2}) = 2k-4$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-4}) = 2k-4$.

$a_2 > b_y$ -> $M(2,1+S_{2k-5}) = 2k-4$.

Since all possibilities must take a minimum of $2k$ steps, as was to be shown, the lemma is proved.

3.4.6 Theorem 5: The Third Case.

The third case is to show

$$M(3,1+T_{3k+1}) > 3k+1, \qquad (27)$$

where

$$1+T_{3k+1} = 2^{k-1} + 2^{k-3} + [\tfrac{12}{7} 2^{k-1}] + [\tfrac{12}{7} 2^{k-2}]. \qquad (28)$$

The initial configuration is 5.3.1 (below). There are only two distinct initial comparisons, and only three distinct outcomes.

5.3.1:

$$\begin{array}{l} |\ 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-1}] + [\frac{12}{7} 2^{k-2}]\ | \\ |\ 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-1}] + [\frac{12}{7} 2^{k-2}]\ | \\ |\ 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-1}] + [\frac{12}{7} 2^{k-2}]\ | \quad > 3k+1 \end{array}$$

Let $x = 2^{k-2} + 2^{k-3}$,

$y = 2^k$.

$a_1 \leq b_x \rightarrow M(3,1+T_{3k}) = 3k+1$ (Induction).

$a_1 > b_x \rightarrow$ 5.3.2

$a_2 \leq b_y \rightarrow M(1,2^k) + M(2,1+S_{2k-1}) = 3k+1.$

{Note that $2^{k+1} > 1+T_{3k+1}$.}

5.3.2:

$$\begin{array}{l} |\ 2^{k-2} + 2^{k-3} \qquad\qquad\qquad\qquad\qquad\quad | \\ |\ 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-1}] + [\frac{12}{7} 2^{k-2}]\ | \\ |\ 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-1}] + [\frac{12}{7} 2^{k-2}]\ | \quad > 3k \end{array}$$

Let $x = 2^{k-3}$,

$y = 1+T_{3k+1} - [\frac{17}{7} 2^{k-1}]$,

$z = 1+T_{3k+1} - 2^{k-1}$.

$a_1 \le b_x$ -> $M(1,2^{k-2}) + M(2,1+S_{2k}) = 3k$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(2,1+S_{2k+1}) = 3k$.

$a_2 \le b_y$ -> $M(1,2^{k-2}) + M(2,1+S_{2k}) = 3k$.

$a_2 > b_y$ -> Lemma 16.1 + $M(1,2^{k}) = 3k$.

$a_3 \le b_z$ -> $M(1,2^{k-2}) + M(1,2^{k}) + M(1,2^{k-1}) = 3k$.

$a_3 > b_z$ -> 5.3.3

5.3.3:

$$| 2^{k-2} + 2^{k-3} |$$

$$| 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] |$$

$$| 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] | > 3k-1$$

Let $x = 2^{k-3}$,

$y = 2^{k-3} + [\frac{12}{7} 2^{k-2}]$,

$z = 1+T_{3k+1} - (2^{k-1} + [\frac{9}{7} 2^{k-2}])$.

$a_1 \le b_x$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-1}) = 3k-1$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(2,1+S_{2k}) = 3k-1$.

$a_2 \le b_y$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-1}) = 3k-1$.

$a_2 > b_y$ -> 5.3.4

$a_3 \le b_z$ -> 5.3.12

$a_3 > b_z$ -> 5.3.14

5.3.4:

$$\left|\begin{array}{l} 2^{k-2} + 2^{k-3} \\ 2^{k-2} + 2^{k-3} + [\frac{5}{7} 2^{k-2}] \\ 2^{k-2} + 2^{k-3} + [\frac{5}{7} 2^{k-2}] + [\frac{12}{7} 2^{k-1}] \end{array}\right| > 3k-2$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7} 2^{k-3}]$,

$z = 1+T_{3k+1} - 2^k$.

$a_1 \leq b_x$ -> 5.3.5

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) + M(1,2^k) = 3k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) + M(1,2^{k-1}) = 3k-2$.

$a_2 > b_y$ -> $M(2,1+S_{2k-4}) + M(1,2^k) = 3k-2$.

$a_3 \leq b_z$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) + M(1,2^{k-1}) = 3k-2$.

$a_3 > b_z$ -> 5.3.6

5.3.5:

$$\left|\begin{array}{l} 2^{k-2} \\ 2^{k-2} + [\frac{5}{7} 2^{k-2}] \\ 2^{k-2} + [\frac{5}{7} 2^{k-2}] + [\frac{12}{7} 2^{k-1}] \end{array}\right| > 3k-3$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7} 2^{k-3}]$,

$z = 2^{k-2} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3})$ + Lemma 17.1 = $3k-3$.

$a_1 > b_x$ -> $M(1,2^{k-4})$ + Lemma 18.1 = $3k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) + M(1,2^{k-1}) = 3k-3$.

$a_2 > b_y \rightarrow M(2,1+S_{2k-5}) + M(1,2^k) = 3k-3.$

$a_3 \leq b_z \rightarrow M(1,2^{k-2}) + M(1,2^{k-3}) + M(1,2^{k-1}) = 3k-3.$

$a_3 > b_z \rightarrow$ 5.3.7

5.3.6:

$$\begin{vmatrix} 2^{k-2} + 2^{k-3} \\ 2^{k-2} + 2^{k-3} + [\frac{5}{7}\, 2^{k-2}] \\ 2^{k-2} + 2^{k-3} + [\frac{5}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-1}] \end{vmatrix} > 3k-3$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7}\, 2^{k-3}]$,

$z = 1+T_{3k+1} - (2^k + 2^{k-2})$.

$a_1 \leq b_x \rightarrow$ 5.3.7

$a_1 > b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k-2}) + M(1,2^{k-1}) = 3k-3.$

$a_2 \leq b_y \rightarrow M(1,2^{k-2}) + M(1,2^{k-2}) + M(1,2^{k-2}) = 3k-3.$

$a_2 > b_y \rightarrow M(2,1+S_{2k-4}) + M(1,2^{k-1}) = 3k-3.$

$a_3 \leq b_z \rightarrow M(1,2^{k-2}) + M(1,2^{k-2}) + M(1,2^{k-2}) = 3k-3.$

$a_3 > b_z \rightarrow$ 5.3.8

5.3.7:

$$\begin{vmatrix} 2^{k-2} \\ 2^{k-2} + [\frac{5}{7}\, 2^{k-2}] \\ 2^{k-2} + [\frac{5}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-1}] \end{vmatrix} > 3k-4$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7}\, 2^{k-3}]$,

$z = 2^{k-2} + [\frac{5}{7}\ 2^{k-2}] + [\frac{3}{7}\ 2^{k-2}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3})$ + Lemma 17.2 = $3k-4$.

$a_1 > b_x$ -> $M(1,2^{k-4})$ + Lemma 18.2 = $3k-4$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) + M(1,2^{k-2}) = 3k-4$.

$a_2 > b_y$ -> $M(2,1+S_{2k-5}) + M(1,2^{k-1}) = 3k-4$.

$a_3 \leq b_z$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) + M(1,2^{k-2}) = 3k-4$.

$a_3 > b_z$ -> 5.3.9

5.3.8:

$$\begin{vmatrix} 2^{k-2} + 2^{k-3} \\ 2^{k-2} + 2^{k-3} + [\frac{5}{7}\ 2^{k-2}] \\ 2^{k-2} + 2^{k-3} + [\frac{5}{7}\ 2^{k-2}] + [\frac{3}{7}\ 2^{k-2}] \end{vmatrix} > 3k-4$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7}\ 2^{k-3}]$,

$z = 2^{k-2} + [\frac{5}{7}\ 2^{k-2}] + [\frac{3}{7}\ 2^{k-2}]$.

$a_1 \leq b_x$ -> 5.3.9

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(1,2^{k-2}) + M(1,2^{k-2}) = 3k-4$.

$a_2 \leq b_y$ -> $M(1,2^{k-2})$ + Lemma 19.1 = $3k-4$.

$a_2 > b_y$ -> $M(2,1+S_{2k-4}) + M(1,2^{k-2}) = 3k-4$.

$a_3 \leq b_z$ -> $M(1,2^{k-2}) + M(1,2^{k-2}) + M(1,2^{k-3}) = 3k-4$.

$a_3 > b_z$ -> 5.3.10

5.3.9:

$$\left| \begin{array}{l} 2^{k-2} \\ 2^{k-2} + [\frac{5}{7} 2^{k-2}] \\ 2^{k-2} + [\frac{5}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-2}] \end{array} \right| > 3k-5$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7} 2^{k-3}]$,

$z = 2^{k-3} + [\frac{5}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-2}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3})$ + Lemma 17.3 = 3k-5.

$a_1 > b_x$ -> $M(1,2^{k-4})$ + Lemma 18.3 = 3k-5.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) + M(1,2^{k-3}) = 3k-5$.

$a_2 > b_y$ -> $M(2,1+S_{2k-5}) + M(1,2^{k-2}) = 3k-5$.

$a_3 \leq b_z$ -> $M(1,2^{k-2}) + M(1,2^{k-3}) + M(1,2^{k-3}) = 3k-5$.

$a_3 > b_z$ -> 5.3.11

5.3.10:

$$\left| \begin{array}{l} 2^{k-2} + 2^{k-3} \\ 2^{k-2} + [\frac{5}{7} 2^{k-2}] + [\frac{6}{7} 2^{k-3}] \\ 2^{k-2} + [\frac{5}{7} 2^{k-2}] + [\frac{6}{7} 2^{k-3}] \end{array} \right| > 3k-5$$

Let $x = 2^{k-4}$,

$y = [\frac{17}{7} 2^{k-3}]$,

$z = 2^{k-2} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-4}]$.

$a_1 \leq b_x$ -> $M(1,2^{k-3}) + M(2,1+S_{2k-4}) = 3k-5$.

$a_1 > b_x$ -> $M(1,2^{k-4}) + M(2,1+S_{2k-3}) = 3k-5$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-5}) = 3k-5$.

$a_2 > b_y \rightarrow M(2,1+S_{2k-4}) + M(1,2^{k-3}) = 3k-5.$

$a_3 \leq b_z \rightarrow$ Lemma 20.1 $+ M(1,2^{k-4}) = 3k-5.$

$a_3 > b_z \rightarrow M(2,1+S_{2k-4}) + M(1,2^{k-3}) = 3k-5.$

5.3.11:

$$\begin{vmatrix} 2^{k-2} \\ 2^{k-3} + [\frac{5}{7} 2^{k-2}] + [\frac{6}{7} 2^{k-3}] \\ 2^{k-3} + [\frac{5}{7} 2^{k-2}] + [\frac{6}{7} 2^{k-3}] \end{vmatrix} > 3k-6$$

Let $x = 2^{k-4}$,

$y = [\frac{12}{7} 2^{k-3}]$,

$z = 2^{k-3} + [\frac{5}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-4}]$.

$a_1 \leq b_x \rightarrow M(1,2^{k-3}) + M(2,1+S_{2k-5}) = 3k-6.$

$a_1 > b_x \rightarrow M(1,2^{k-4}) + M(2,1+S_{2k-4}) = 3k-6.$

$a_2 \leq b_y \rightarrow M(1,2^{k-2}) + M(2,1+S_{2k-6}) = 3k-6.$

$a_2 > b_y \rightarrow M(2,1+S_{2k-5}) + M(1,2^{k-3}) = 3k-6.$

$a_3 \leq b_z \rightarrow$ Lemma 20.2 $+ M(1,2^{k-4}) = 3k-6.$

$a_3 > b_z \rightarrow M(1,2^{k-3}) + M(2,1+S_{2k-5}) = 3k-6.$

5.3.12:

$$\begin{vmatrix} 2^{k-2} + 2^{k-3} \\ 2^{k-1} + 2^{k-3} + [\frac{12}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] \\ [\frac{9}{7} 2^{k-2}] \end{vmatrix} > 3k-2$$

Let $x = 2^{k-3}$,

$y = 1+T_{3k+1} - [\frac{17}{7} 2^{k}]$,

$z = 1+T_{3k+1} - (2^{k-1} + 2^{k-3})$.

$a_1 \le b_x \rightarrow M(1,2^{k-2}) + M(1,2^{k-1}) + M(1,2^{k-2}) = 3k-2$.

$a_1 > b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k}) + M(1,2^{k-2}) = 3k-2$.

$a_2 \le b_y \rightarrow M(1,2^{k-2})$ + Lemma 12.1 = $3k-2$.

$a_2 > b_y \rightarrow$ Lemma 16.1 + $M(1,2^{k-2}) = 3k-2$.

$a_3 \le b_z \rightarrow M(1,2^{k-2}) + M(1,2^{k}) + M(1,2^{k-3}) = 3k-2$.

$a_3 > b_z \rightarrow$ 5.3.13

5.3.13:

$$\left|\begin{array}{l} 2^{k-2} + 2^{k-3} \\ 2^{k-1} + 2^{k-3} + [\frac{12}{7}\, 2^{k-2}] + [\frac{5}{7}\, 2^{k-4}] + 2^{k-3} + 2^{k-4} \\ \hfill 2^{k-3} + [\frac{4}{7}\, 2^{k-3}] \end{array}\right| > 3k-3$$

Let $x = 2^{k-3}$,

$y = 1+T_{3k+1} - [\frac{17}{7}\, 2^{k-1}]$,

$z = 1+T_{3k+1} - (2^{k-1} + 2^{k-3} + 2^{k-4})$.

$a_1 \le b_x \rightarrow M(1,2^{k-2}) + M(1,2^{k-1}) + M(1,2^{k-3}) = 3k-3$.

$a_1 > b_x \rightarrow M(1,2^{k-3}) + M(1,2^{k}) + M(1,2^{k-3}) = 3k-3$.

$a_2 \le b_y \rightarrow M(1,2^{k-2})$ + Lemma 12.11 = $3k-3$.

$a_2 > b_y \rightarrow$ Lemma 16.1 + $M(1,2^{k-3}) = 3k-3$.

$a_3 \le b_z \rightarrow$ Lemma 21.1 + $M(1,2^{k-4}) = 3k-3$.

{Note. The lemma cited dominates the configuration that arises when this result occurs.}

$a_3 > b_z \rightarrow M(1,2^{k-2}) + M(1,2^{k-1}) + M(1,2^{k-3}) = 3k-3$.

5.3.14:

$$\begin{vmatrix} 2^{k-2} + 2^{k-3} \\ 2^{k-3} + [\frac{12}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] + [\frac{5}{7} 2^{k-2}] \\ 2^{k-3} + [\frac{12}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-1}] + [\frac{5}{7} 2^{k-2}] \end{vmatrix} > 3k-2$$

Let $x = 2^{k-3}$,

$y = 2^{k-3} + 2[\frac{5}{7} 2^{k-2}]$,

$z = 1+T_{3k+1} - (2^{k-1} + [\frac{9}{7} 2^{k-2}] + 2^{k-2})$.

$a_1 \leq b_x$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-2}) = 3k-2$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(2,1+S_{2k-1}) = 3k-2$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-2}) = 3k-2$.

$a_2 > b_y$ -> Lemma 21.3 + $M(12^{k-1}) = 3k-2$.

$a_3 \leq b_z$ -> $M(1,2^{k-2}) + M(1,2^{k-1}) + M(1,2^{k-2}) = 3k-2$.

$a_3 > b_z$ -> 5.3.15

5.3.15:

$$\begin{vmatrix} 2^{k-2} + 2^{k-3} \\ 2^{k-3} + [\frac{12}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-2}] \\ 2^{k-3} + [\frac{12}{7} 2^{k-2}] + [\frac{3}{7} 2^{k-2}] + [\frac{5}{7} 2^{k-2}] \end{vmatrix} > 3k-3$$

Let $x = 2^{k-3}$,

$y = [\frac{17}{7} 2^{k-3}]$,

$z = 1+T_{3k+1} - (2^{k-1} + [\frac{9}{7} 2^{k-2}] + 2^{k-2} + 2^{k-3})$.

$a_1 \leq b_x$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-3}) = 3k-3$.

$a_1 > b_x$ -> $M(1,2^{k-3}) + M(2,1+S_{2k-2}) = 3k-3$.

$a_2 \leq b_y$ -> $M(1,2^{k-2}) + M(2,1+S_{2k-3}) = 3k-3$.

$a_2 > b_y \rightarrow M(2,1+S_{2k-4}) + M(1,2^{k-1}) = 3k-3.$

$a_3 \leq b_z \rightarrow$ Lemma 21.2 $+ M(1,2^{k-3}) = 3k-3.$

$a_3 > b_z \rightarrow M(1,2^{k-2}) + M(2,1+S_{2k-3}) = 3k-3.$

Since all possibilities terminate in at least 3k+2 steps, as was to be shown, we conclude that equation (27) is correct.

Happily, this completes the proof of theorem 5.

Except for the application of these results made in the last chapter, theorems 4 and 5 appear to have little immediate usefulness. They probably represent the limits of what can be accomplished in this field without significant machine assistance. An automated extension of these results presents some interesting problems that could prove to be a fruitful area of inquiry.

References:

[1] F. K. Hwang and S. Lin, "Optimal Merging of 2 Elements with n Elements", Acta Informatica 1, 145-178 (1971).

[2] D. E. Knuth, _The Art of Computer Programming Vol III: Sorting and Searching_, Addison-Wesley, 1973.

A Problem in Optimal Merging

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Abstract

In [1] Hwang and Lin solved the problem of determining the minimum number of binary comparisons which will always suffice to merge an ordered set of size 2 with any ordered set of size n. This is called the (2,n) merging problem. In [2] Knuth gives as an exercise (due to the above authors) problem 5.3.2.14 which, if solved affirmatively, would be a solution to the corresponding (3,n) merging problem.

In this report we present a solution for the (3,n) problem that solves problem 5.3.2.14 in the negative. This solution follows the same general line of attack as that used by Hwang and Lin but introduces several new concepts. This work is basically an enumeration of merging algorithms to solve the (3,n) problem and opens the way for mechanically carrying out the extension of this result for the (4,n), (5,n), etc, problems.

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[1] F. K. Hwang and S. Lin, "Optimal Merging of 2 Elements with n Elements", Acta Informatica 1, 145-178 (1971).

[2] D. E. Knuth, The Art of Computer Programming Vol III: Sorting and Searching, Addison-Wesley, 1973.