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SEMI-DISCRETIZATIONS OF HYPERBOLIC EQUATIONS

By: R. Vichnevetsky

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Department of Computer Science
Rutgers, The State University
New Brunswick, N.J. 08903

PROPAGATION PROPERTIES OF SEMI-DISCRETIZATIONS OF HYPERBOLIC EQUATIONS

R. Vichnevetsky, Rutgers University
New Brunswick, N.J. 08903, USA

ABSTRACT

A simple conservative semi-discretization of hyperbolic equations is analyzed in terms of its response to time-varying boundary conditions.

By performing this analysis in the Fourier domain, it is shown, among other results, that there is a limit or "cut-off" frequency beyond which sinusoidal solutions are affected with spurious amplitude decay. This cut-off frequency is that for which the group velocity of the semi-discrete approximation vanishes.

1. INTRODUCTION

Consider the semi-discretization

$$\frac{du_n}{dt} = -c\left(\frac{u_{n+1} - u_{n-1}}{2\cdot h}\right) \tag{1}$$

of the simple advection equation

$$\frac{\partial U}{\partial t} + c\,\frac{\partial U}{\partial x} = 0 \tag{2}$$

taken as a model of hyperbolic equations. Here,

$$x_n = n\cdot h \quad ; \; n = 0, 1, 2, \ldots\ldots$$

is a regular discretization of the x axis, and

$$u_n \simeq U(x_n, t)$$

are the approximations of $U(x,t)$ in the discrete points $\{x_n\}$

Interesting aspects of the error intro-duced by this semi-discretization are revealed by considering the propagation of the influence of a variable boundary condition $U(0,t)$ into the domain $x > 0$ of the equation. (The domain of the equation is taken to be $0 \leq x < \infty$).

To pursue this, let $\hat{U}(x,\Omega)$ be the Fourier transform of $U(x,t)$ for x fixed:

$$\hat{U}(x,\Omega) = \int_{-\infty}^{\infty} U(x,t)\cdot e^{-i\Omega t}\,dt \tag{3}$$

Upon substitution into (2) we find, using the derivative rule for Fourier transforms

$$i\Omega\,\hat{U} + c\,\frac{\partial \hat{U}}{\partial x} = 0 \tag{4}$$

This equation may be integrated in x giving:

$$\hat{U}(x,\Omega) = \hat{U}(0,\Omega)\cdot e^{-i\frac{\Omega\cdot x}{c}} \tag{5}$$

where $\hat{U}(0,\Omega)$ is the Fourier transform of the given boundary condition.

Returning from (7.4) to the time domain gives:

$$U(x,t) = U\left(0, t - \frac{x}{c}\right) \tag{6}$$

which is an expression of the known translation property of solutions of the advection equation (2).

The ratio of Fourier transforms

$$\frac{\hat{U}(x,\Omega)}{\hat{U}(0,\Omega)} = e^{-i\frac{\Omega\cdot x}{c}} \tag{7}$$

is independent of U and expresses the propagation characteristics of solutions of the advection equation. Such a ratio is called a <u>transfer function</u> in operational calculus.

We may now ask what the equivalent of (5) or (7) becomes in the semi-discrete numerical approximation (1).

Let:

$$\hat{u}_n(\Omega) = \int_{-\infty}^{\infty} u_n(t)\cdot e^{-i\Omega t}\,dt \tag{8}$$

be the Fourier transform of the numerical solutions on the t-continuous lines $x_n = n\cdot h$.

Also let $\hat{E}(\Omega)$ be the (yet unknown)ratio of those Fourier transforms for the two points h apart in the x direction:

$$\hat{E}(\Omega) = \frac{\hat{u}_{n+1}(\Omega)}{\hat{u}_n(\Omega)} \tag{9}$$

This function we shall logically call the "cell transfer function" of the semi-discretization (1).

Note that (9) turns out to be the analog of (7) in numerical solutions, specifically:

$$\hat{E}(\Omega) \circeq e^{-i\frac{\Omega h}{c}} \tag{10}$$

(where $\circeq$ means "is the numerical approximation of"). Using the definition (9), we may write the Fourier transform of the semi-discretization (1) as:

$$i\Omega\,\hat{u}_n = -c\left(\frac{\hat{E} - \hat{E}^{-1}}{2\cdot h}\right)\cdot\hat{u}_n \tag{11}$$

I.e.

$$i\Omega = -c\left(\frac{\hat{E} - \hat{E}^{-1}}{2\cdot h}\right) \tag{12}$$

or

$$\hat{E}^2 + i\,\frac{2\Omega h}{c}\cdot\hat{E} - 1 = 0 \tag{13}$$

which is the characteristic equation for $\hat{E}(\Omega)$. The solution is:

$$\hat{E}(\Omega) = -i\frac{\Omega h}{c} \pm \sqrt{-\left(\frac{\Omega h}{c}\right)^2 + 1} \tag{14}$$

The root we are interested in is that corresponding to the + sign. (The other root corresponds the kind of spurious , oscillatory solutions described in references [13] and [14])

An equivalent form for $\hat{E}(\Omega)$ which sometimes turns out to be more useful is obtained as follows; let:

$$\hat{E}(\Omega) = e^{\beta} \tag{15}$$

We may then write (12) as:

$$i\Omega = -c\left(\frac{e^{\beta}-e^{-\beta}}{2\cdot h}\right) = -c\cdot\frac{\sinh(\beta)}{h}$$

(16)

(where $\sinh(\beta)$ is the hyperbolic sine function) and, returning to $\hat{E}$:

$$\hat{E}(\Omega) = e^{-\arg\sinh\left(\frac{i\Omega h}{c}\right)}$$

(17)

When $(\Omega\cdot h/c) \leqslant 1$, this becomes:

$$\hat{E}(\Omega) = e^{-i\,\arcsin(\Omega\cdot h/c)}$$

(18)

and, when $(\Omega\cdot h/c) > 1$:

$$\hat{E}(\Omega) = e^{-i\pi/2}\cdot e^{-\arg\cosh(\Omega h/c)}$$
$$= -i\cdot e^{-\arg\cosh(\Omega h/c)}$$

(19)

This function is of course identical to (14).

2. RESPONSE TO A SINUSOIDAL BOUNDARY CONDITION

We let the boundary condition $U(o,t)$ be the harmonic function:

$$U(o,t) = u_o(t) = e^{i\Omega t} \tag{20}$$

It is clear that the stationary numerical solution (or solution of (1)) shall also be harmonic in time for all n. We may write, separating amplitude and phase:

$$u_n(t) = |u_n(t)| \cdot e^{i(\Omega t + \alpha_n)} \tag{21}$$

It then becomes an easy matter to relate this to (9). One finds:

$$\alpha_{n+1} - \alpha_n = \angle \hat{E}(\Omega) \tag{22}$$

and

$$\frac{|U_{n+1}|}{|U_n|} = |\hat{E}(\Omega)| \tag{23}$$

where $|\hat{E}(\Omega)|$ and $\angle \hat{E}(\Omega)$ are the amplitude and phase of the transfer function $\hat{E}$

When $(\Omega \cdot h / c \leqslant 1)$, then (18) gives

$$|\hat{E}(\Omega)| = |e^{i \arcsin(\Omega h / c)}| = 1 \tag{24}$$

and

$$\angle \hat{E}(\Omega) = \arcsin(\Omega h / c) \tag{25}$$

This means that the amplitude of sinusoidal solutions is the same for all n, i.e. that the sinusoidal boundary condition propagates in the x direction without decay in amplitude. (The semi-discretization (1) is called conservative for that reason).

For $x = x_n$ we have

$$\alpha_n = -n \cdot \arcsin\,(\Omega h / c)$$

or:

$$u_n = e^{i(\Omega t - n \cdot \arcsin(\Omega h / c))} \tag{26}$$

to be compared with the exact solution for the same problem:

$$(\alpha_n)_{exact} = n \cdot \Omega h / c$$

or

$$U(x_n, t) = e^{i(\Omega t - n \Omega h / c)} \tag{27}$$

For $(\Omega h / c) \to 0$, (26) converges to (27), as expected.

3. NUMERICAL PHASE VELOCITY

The difference in phases in between (26) and (28) expresses a discrepancy in the velocity of propagation in the numerical solution: We derive from (26) the velocity at which numerical solutions propagate in the x direction. Let

$$c^*(\Omega) = \frac{\Omega h}{\arcsin(\Omega h/c)} \tag{28}$$

so that(26) may be rewritten in a form analogous to (27):

$$u_n(t) = e^{i(\Omega t - n\Omega h/c^*(\Omega))} \tag{29}$$

We thus see that $c^*(\Omega)$ is the numerical velocity of propagation, and is seen (unlike for exact solutions) to be frequency dependent.

Figure 1 shows this dependence graphically . We note that for all $(\Omega h/c)$ in $[0, 1]$, $c^*(\Omega)$ is less than c , i.e. numerical solutions travel slower than they should.

The frequency dependence of this velocity introduces what is called spurious dispersion in the propagation of signals by the discrete approximation. (propagation as described by equation (2) with c =constant is non dispersive)

4. CUT-OFF FREQUENCY OF THE APPROXIMATION

An interesting situation develops when the frequency is increased: When $\Omega h/c > 1$,

then according to (19):

$$\begin{aligned} \hat{E}(\Omega) &= -i\, e^{-\arg\cosh(\Omega h/c)} \\ &= -i\left(\frac{\Omega h}{c} - \sqrt{\left(\frac{\Omega h}{c}\right)^2 - 1}\right) \end{aligned} \tag{30}$$

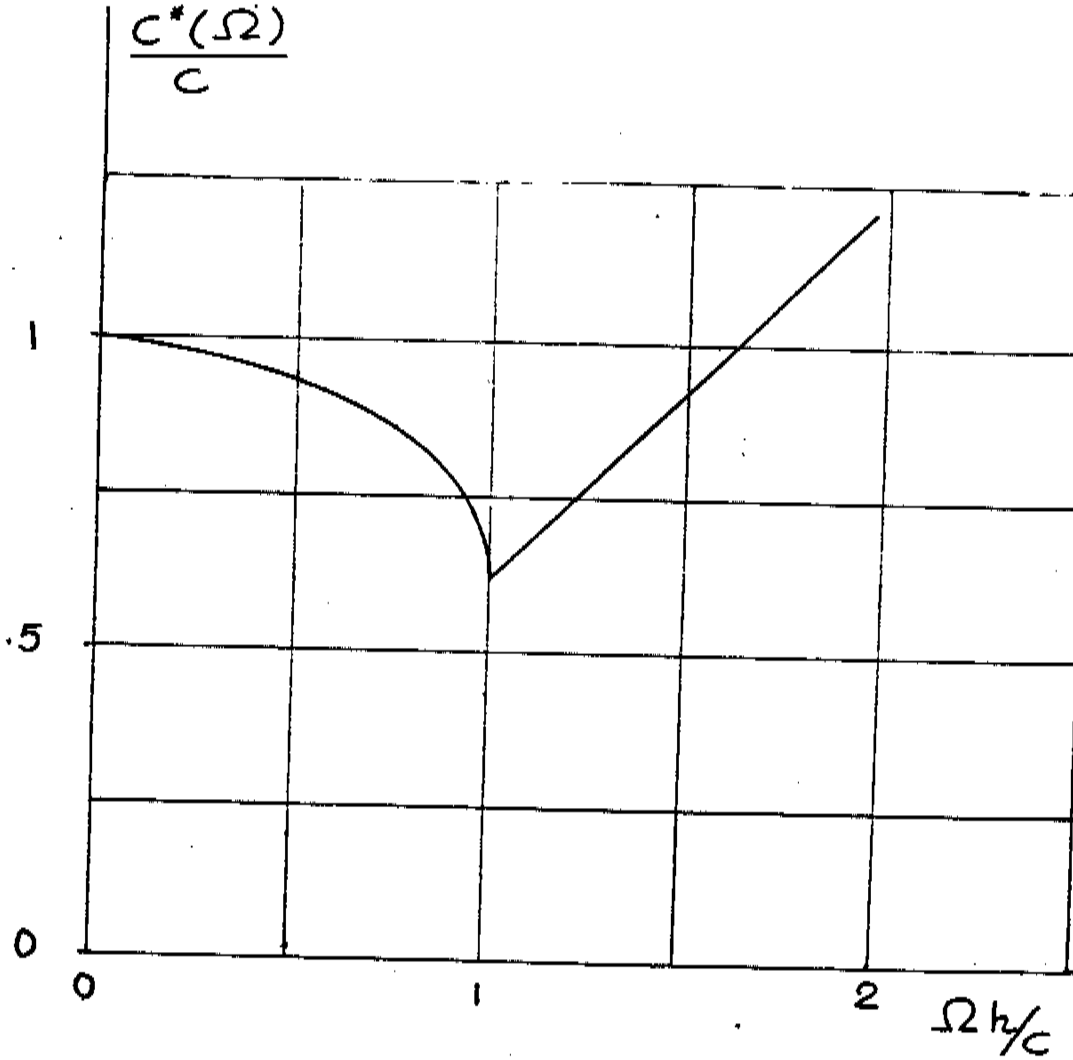

Figure 1 · Numerical phase velocity distorsion as a function of the time-frequency Ω at the boundary.

The phase $\angle \hat{E}(\Omega)$ is now constant:

$$\angle \hat{E}(\Omega) = \frac{\pi}{2} \tag{31}$$

and the amplitude

$$|\hat{E}(\Omega)| = \frac{\Omega h}{c} - \sqrt{\left(\frac{\Omega h}{c}\right)^2 - 1} \tag{32}$$

<u>is less than one, i.e. the amplitude of sinusoidal components decays with increasing</u> <u>n</u>. This is illustrated in Figures <u>2</u> and <u>3</u>

The frequency $\Omega_c = c/h$ is thus a <u>cut off frequency</u> : Sinusoidal components in the boundary condition at frequencies greater than Ω_c are affected with spurious decay in their propagation away from the exciting boundary.

We have the interesting property that if we want to generate into the domain of the equation a sinusoidal wave by applying a time-sinusoidal boundary condition with frequency Ω, and if $\Omega > \frac{c}{h}$, then the amplitude decays in space at a rate which increases monotonically with the excess of frequency:

$$\Omega - \frac{c}{h}$$

The phase velocity when $\Omega h/c > 1$ is given by:

$$\begin{aligned} c^*(\Omega) &= \Omega h/(-\angle \hat{E}(\Omega)) \\ &= 2\Omega h/\pi \end{aligned} \tag{33}$$

It varies linearly with Ω, as shown in Figure <u>1</u>

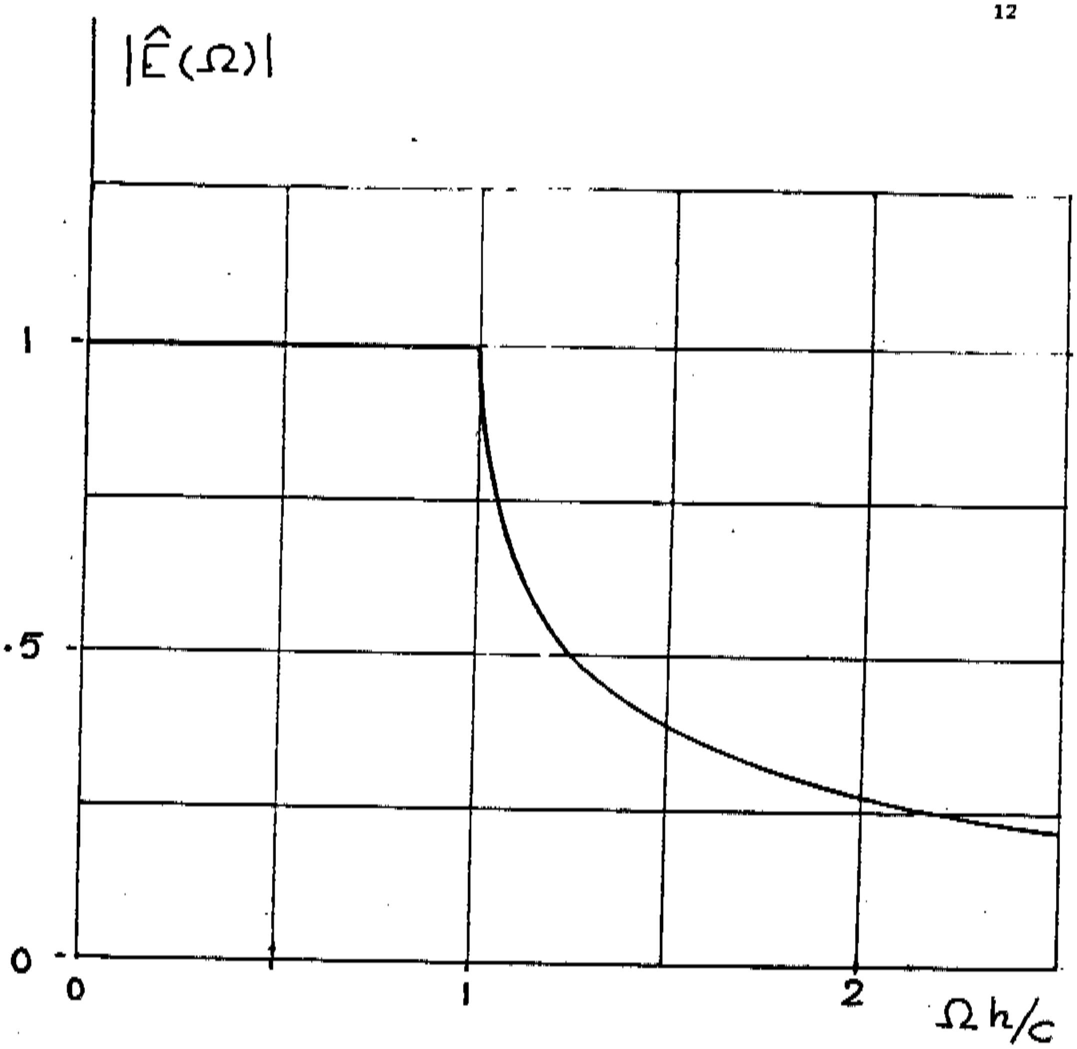

Figure 2 Amplitude ratio across one h in the x direction as a function of the time frequency Ω at the boundary.

Wavelength

Another property of sinusoidal numerical solutions when $\Omega h/c \gg 1$ is this:

The wavelength is the same for all frequencies and this wavelength is equal to $4h$.
To demonstrate this interesting result, simply observe from (31) that the phase angle of the numerical solution u_n is:

$$\alpha_n = n \cdot \pi/2 \tag{34}$$

The wavelength is the distance over which α varies by 2π, i.e.:

$$\text{wavelength} = h \cdot 2\pi/(n \cdot \pi/2) = 4h \tag{35}$$

Numerical illustrations of this interesting case may be found in [Roache, (1972) pp 56-58].

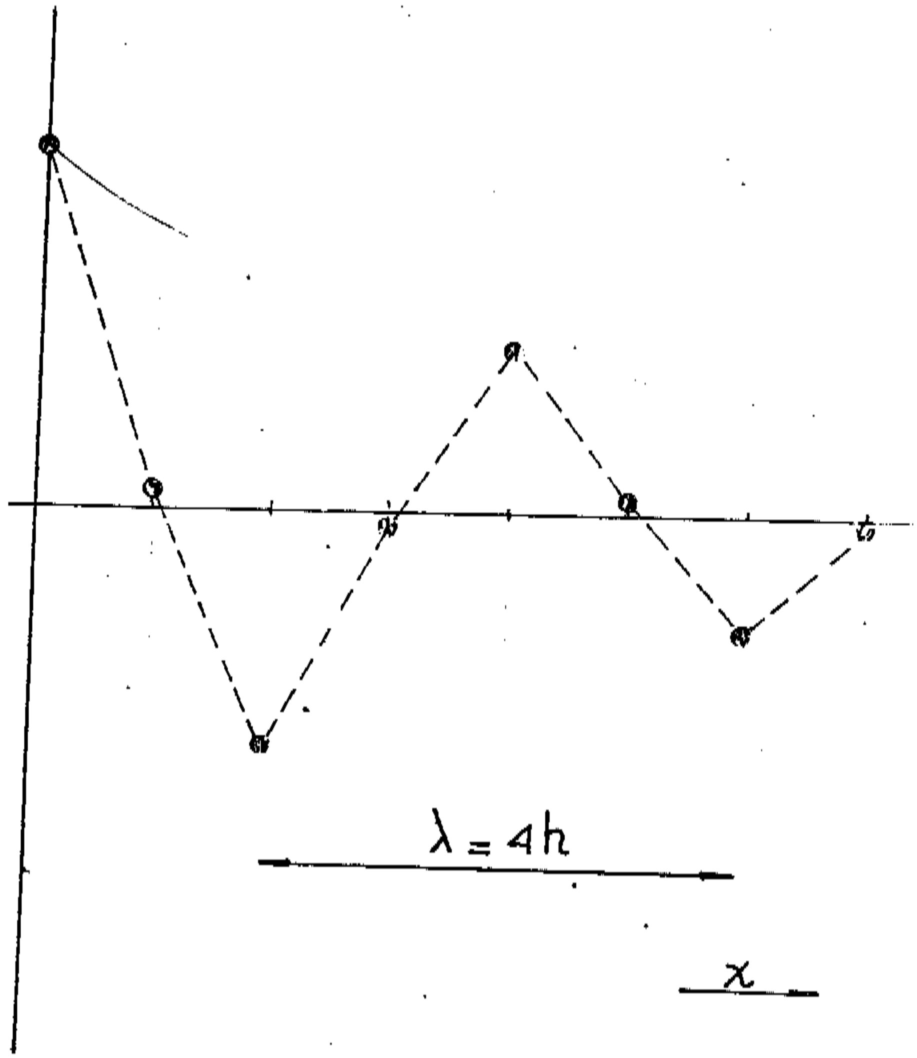

Figure 3

Numerical sinusoidal response when $\Omega h / c > 1$

The wavelength λ is constant = $4h$ and there is amplitude decay with x as $e^{-n \, \mathrm{arg\,cosh}\,(\Omega h / c)}$

5. RELATION TO THE GROUP VELOCITY

There is an interesting relationship between the precedings results and the concept of group velocity:

If one considers the dispersive propagation of a function which is sinusoidal with a slowly varying envelope, then one finds that the envelope is displaced without change at the "group" velocity $\mathcal{V}(\Omega)$, related to the phase velocity $c^*(\Omega)$ by:*

$$\frac{1}{\mathcal{V}} = \frac{d(\Omega/c^*)}{d\Omega} \tag{36}$$

In the present case where $c^*(\Omega)$ is the phase velocity of sinusoidal numerical solutions of (1) we have:

$$\frac{1}{\mathcal{V}} = \frac{d}{d\Omega}\left(\frac{1}{h}\ \mathrm{arc\ sin}\left(\Omega h/c\right)\right)$$

$$= \frac{1}{c\sqrt{1-(\Omega h/c)^2}} \tag{37}$$

or (figure 4):

$$\mathcal{V}(\Omega) = c\cdot\sqrt{1-(\Omega h/c)^2} \tag{38}$$

<u>This group velocity goes to zero when Ω reaches the "cut-off" frequency $\Omega_c = c/h$.</u>

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* See e.g. Brillouin (1946).

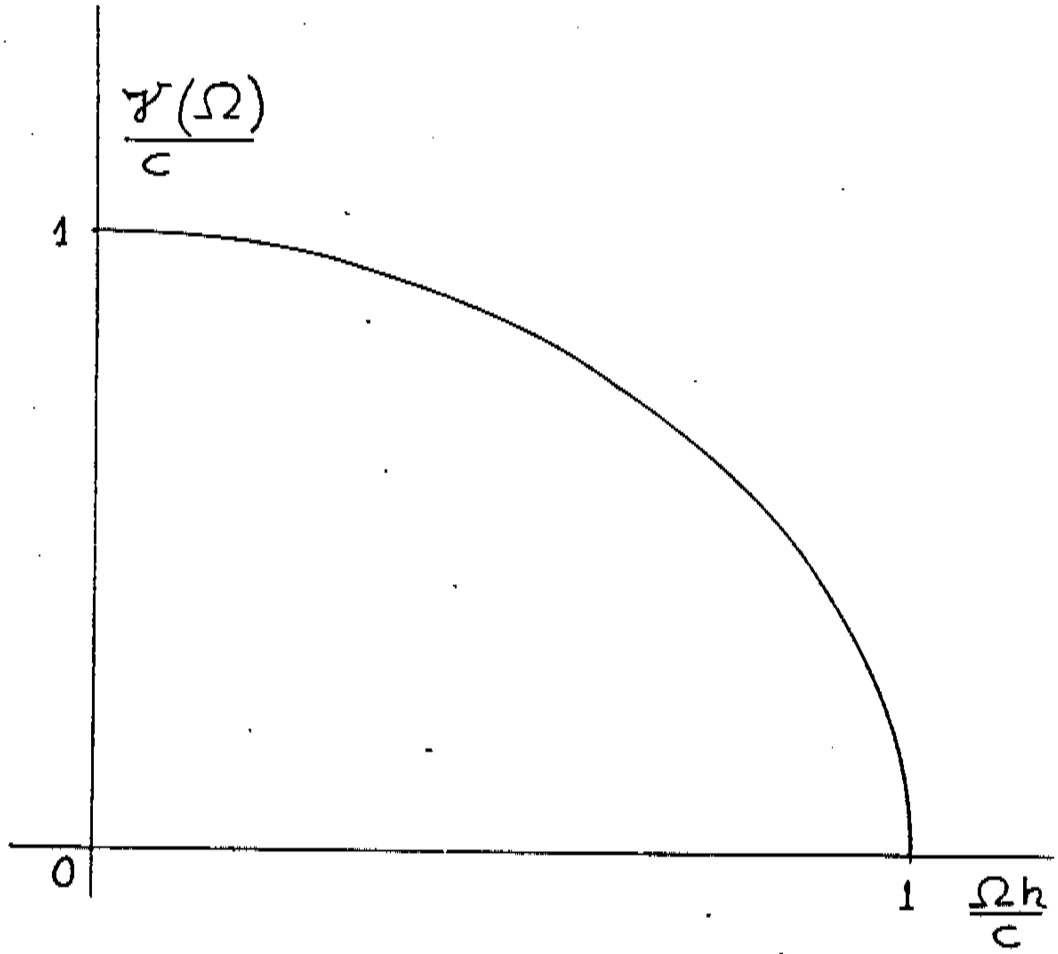

Figure 4 : Group velocity as a function of the frèquency Ω

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