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ON THE SOLVABILITY OF SINGULAR INTEGRAL EQUATIONS VIA GAUSS-JACOBI QUADRATURE

by

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Abstract

It is shown that the coefficient matrix obtained by the discretization of singular integral equations using Gauss-Jacobi quadrature formulae is non-singular for any n, if λ is not an eigenvalue of the equation.

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1. Introduction:

Direct methods for the solution of the singular integral equation of Cauchy type, viz.

$$(1.1)\qquad a\,g(s) + \frac{b}{\pi}\int_{-1}^{1}\frac{g(t)dt}{t-s} + \lambda\int_{-1}^{1}K(s,t)\,g(t)\,dt = f(s) \qquad -1<s<1$$

are becoming increasingly popular. These methods have two common features: (a) the singular behavior of the solution at the end-points is explicitly built into the solution. Thus $g(t)$ is replaced by $w(t)y(t)$, where

$$(1.2)\qquad w(t) = (1-t)^{\alpha}\,(1+t)^{\beta}$$

and α,β are determined by the index theory [7]. (b) A quadrature formula is then used to replace the integrals by sums involving the unknown values of $y(t)$ at certain predetermined nodes followed by a collocation scheme to yield a linear system of algebraic equations.

In 1969, Erdogan [2] proposed the use of orthogonal polynomials leading to an algebraic system of equations in the coefficients of the polynomial expansion of $y(t)$. Subsequently it was shown by Erdogan and Gupta [3] that Gauss-Chebyshev quadrature formulae can be used with advantage. Krenk [7] extended the Erdogan-Gupta analysis to the Gauss-Jacobi quadrature to deal with the case $a \neq o$ in (1.1). Orthogonal polynomials have been used also by Dow and Elliott [1] to solve the singular integral equation numerically. Piecewise polynomial approximations to the auxiliary function $y(t)$ have been found satisfactory and competitive in [5] and [4]. The question of convergence of such methods has received little attention.

Perhaps prompted by Linz [8], recently Ioakimides and Theocaris [6] have claimed the convergence of Gaussian direct methods which appears to be based on two tacit assumptions. First, as the number of abscissae of integration is increased, the system of linear algebraic equations obtained through collocation remains non-singular. Second, the values of $y(t)$ at the nodes are the solution of the system, something which can happen only if there is no error in quadrature. If this is not so, then the question of convergence becomes more intricate and would require an analysis of the error in Gaussian quadrature and its effect on the interpolatory polynomial in detail.

In this paper we show that unless λ is an eigenvalue of equation (1.1), the system of linear equations obtained through Gaussian quadrature and collocation is non-singular.

The paper is organized as follows: Section 2 contains some preliminary mathematical results and identities involving the zeros of Jacobi polynomials. Section 3 contains an explicit representation of the inverse of the coefficient matrix. Using this explicit form, a closed form expression for the determinant is obtained. Finally we show that the magnitude of the determinant of the coefficient matrix is bounded below away from zero. In section 4 it is shown

2. Preliminary Mathematics:

Weight Function

For simplicity assume that the index k of equation (1.1) is equal to 1. Then we know that the solution of (1.1) is not unique [7]. In physical problems, to get a unique solution an extra condition has to be imposed. Usually it is of the form:

$$\frac{1}{\pi}\int_{-1}^{1} g(t)dt = N. \tag{2.1}$$

Since $\alpha+\beta=-k$ and $k=1$, we may set $\alpha=-\sigma$ and express (1.2) as

$$w(t) = (1-t)^{-\sigma} (1+t)^{-1+\sigma}, \tag{2.2}$$

where σ is determined by the equation

$$\cot(\sigma\pi) = \frac{a}{b} \tag{2.3}$$

Certain Identities

In the identity ([7], p. 226, (2.7))

$$P_{j-k}^{(-\alpha,-\beta)}(s_\ell) = \sum_{i=1}^{n} \frac{P_{n-k}^{(-\alpha,-\beta)}(t_i)}{P_n^{(\alpha,\beta)'}(t_i)} \frac{P_j^{(\alpha,\beta)}(t_i)}{t_i-s_\ell} \tag{2.4}$$

setting k=1, and

$$(2.5)\qquad w_i = -2^{-1}\,\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\pi}\,\frac{P_{n-1}^{(-\alpha,-\beta)}(t_i)}{P_n^{(\alpha,\beta)'}(t_i)}$$

we find that

$$(2.6)\qquad P_{j-1}^{(-\alpha,-\beta)}(s_\ell) = -\frac{2\pi}{\Gamma(\alpha)\Gamma(1-\alpha)}\sum_{i=1}^{n} w_i\,\frac{P_j^{(\alpha,\beta)}(t_i)}{t_i-s_\ell}$$

where $P_n^{(\alpha,\beta)}(t_i) = 0 \qquad i = 1,2,\ldots,n$

$P_{n-1}^{(-\alpha,-\beta)}(s_\ell) = 0 \qquad \ell = 1,2,\ldots,n-1$

and $\Gamma(z)$ is the Gamma function.

For $j=0$, $P_{-1}^{(-\alpha,-\beta)}(s_\ell) = 0$ (by convention: see Szegö [9], p. 43), and (2.6) reduces to

$$(2.7)\qquad \sum_{i=1}^{n}\frac{w_i}{t_i-s_\ell} = 0.$$

Further ([7], p. 226, (2.3); j=0, k=1)

$$(2.8)\qquad \frac{P_{n-1}^{(-\alpha,-\beta)'}(x)}{P_n^{(\alpha,\beta)}(x)} = \frac{2\pi}{\Gamma(\alpha)\Gamma(1-\alpha)}\sum_{i=1}^{n}\frac{w_i}{(t_i-x)}$$

Differentiating (2.8) and putting $x=s_\ell$, we get

$$(2.13) \qquad b\sum_{i=1}^{n} w_i \frac{y(t_i)}{t_i - s_\ell} + \lambda\pi \sum_{i=1}^{n} w_i K(s_\ell, t_i) y(t_i) = f(s_\ell), \quad \ell=1,\ldots,n-1$$

$$\sum_{i=1}^{n} w_i \, y(t_i) = N$$

for determining the approximate value of $y(t_i)$.

3. The coefficient matrix and its inverse

For $\lambda=0$, the coefficient matrix of (2.13) is given by:

(3.1) $$A_n = (a_{i,j}), = b\, w_j/(t_j-s_i),\ i=1,\dots,n-1,\ j=1,\dots,n$$

$$a_{n,j} = w_j,\ j=1,\dots,n$$

LEMMA: The matrix $B_n = (b_{i,j})$, where

(3.2) $$b_{i,j} = \frac{b}{a^2+b^2}\ \frac{w_j^*}{t_i-s_j},\ j=1,\dots,\ n-1,\ i=1,\dots,n$$

$$b_{i,n} = 1,\ i=1,\dots,n$$

is the inverse of A_n.

Proof

We have to verify that $A_n B_n = I$, i.e.

(3.3) $$\frac{b^2}{a^2+b^2}\ w_\ell^* \sum_{i=1}^{n} \frac{w_i}{(t_i-s_\ell)(t_i-s_m)} = \begin{cases} 0 & \text{if } \ell\neq m \\ 1 & \text{if } \ell=m \end{cases}$$

which is easy to do using (2.7), (2.12) and noting that,

$$\frac{1}{(t_i-s_\ell)(t_i-s_m)} = \frac{1}{(s_\ell-s_m)}\left[\frac{1}{t_i-s_\ell} - \frac{1}{t_i-s_m}\right].$$

Note: For the rest of this section we shall assume, as we may without loss of generality, that $a^2+b^2=1$ (normalization of (1.1)).

Theorem The determinant of the coefficient matrix A_n is nonzero for any integer $n\geq 2$. Moreover, its magnitude is bounded below away from zero.

Proof

Since

(3.4) $$\det(B_n) = \det(A_n) \frac{\prod_{j=1}^{n-1} w_j^*}{\prod_{i=1}^{n} w_i}$$

and $\det(B_n) \det(A_n) = 1$, we have

(3.5) $$(\det A_n)^2 = \frac{\prod_{i=1}^{n} w_i}{\prod_{j=1}^{n-1} w_j^*}$$

$$= \frac{(\Gamma(\alpha+1)\Gamma(\alpha+2))^2}{\pi\, 4^{n-1} n^3} \left\{\frac{\Gamma(2n)}{\Gamma(n)}\right\}^2 \frac{\Gamma(n+\beta+1)}{(\Gamma(n+\alpha+1))^3}$$

(The derivation is given in the appendix).

Since α and β lie between -1 and 0 for k=1, and $\Gamma(z)$ is an increasing function for all $z \geq 2$, it follows easily from (3.5) using standard estimation and Legendre's duplication formula

(3.6) $$\sqrt{\pi}\, \Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2),$$

that

(3.7) $$(\det A_n)^2 > \left[\frac{\Gamma(\alpha+1)\Gamma(\alpha+2)}{4\pi}\right]^2, \text{ for } n \geq 2$$

which proves the theorem.

A case of special interest is $\alpha = \beta = -1/2$. Equation (3.5) is reduced to a simple form

(3.8) $$(\det A_n)^2 = \frac{4^{n-1}}{n^3}$$

4. Existence of the numerical solution

In the matrix notation (2.13) is written as

$$(A_n + \lambda C_n) \underline{Y} = \underline{f} \tag{4.1}$$

where A_n is defined in (3.1)

$$\underline{Y} = [y(t_1), \ldots, y(t_n)]^T, \; \underline{f} = [f(s_1), \ldots, f(s_{n-1}), N]^T$$

and

$$C_n = (c_{i,j}), \; c_{i,j} = \pi w_j K(s_i, t_j), \; i = 1,2,\ldots,n-1, \; j=1,\ldots,n \tag{4.2}$$

$$c_{n,j} = 0, \; j=1,\ldots,n$$

Let

$$h(s) = f(s) - \lambda \int_{-1}^{1} K(s,x) \, w(x) \, y(x) \, dx \tag{4.3}$$

Equation (1.1) may be converted into the Fredholm integral equation:

$$y(t) = \frac{1}{a^2+b^2} \left[a \frac{h(t)}{w(t)} - \frac{b}{\pi} \int_{-1}^{1} \frac{h(s)}{w(s)} \frac{ds}{s-t} \right] + N. \tag{4.4}$$

Once again, if we set

$$w_1(t) = \frac{1}{w(t)} = (1-t)^{\alpha_1} (1+t)^{\beta_1}$$

where $\alpha_1 = -\alpha$, $\beta_1 = -\beta$,

and apply the Gauss-Jacobi formula with n replaced by n-1 and k=-1

(Krenk [7], (5.8)),

we get

$$(4.5) \qquad y(t_i) = \frac{1}{a^2+b^2} \sum_{j=1}^{n-1} w_j^* \left[\frac{-b}{s_j-t_i}\right] h(s_j) + N$$

and $\quad P_{n-1}^{(\alpha_1,\beta_1)}(s_j) = 0, \qquad j = 1,\ldots,n-1$

$\quad P_n^{(-\alpha_1,-\beta_1)}(t_i) = 0, \qquad i = 1,\ldots,n.$

Also we approximate (4.3) using the classical Gauss-Jacobi quadrature,

$$(4.6) \qquad h(s_j) = f(s_j) - \lambda\pi \sum_{r=1}^{n} w_r \quad K(s_j,x_r)\, y\,(x_r)$$

and $\quad P_n^{(\alpha,\beta)}(x_r) = 0 \qquad r = 1,\ldots,n.$

From (4.5), (4.6) and since $x_i = t_i$, we get

$$(4.7) \qquad y(t_i) + \lambda\pi \sum_{r=1}^{n} y(t_r)\, w_r \sum_{j=1}^{n-1} \frac{b}{a^2+b^2} w_j^* \frac{K(s_j,t_r)}{t_i-s_j} =$$

$$= -\frac{b}{a^2+b^2} \sum_{j=1}^{n-1} w_j^* \frac{f(s_j)}{s_j-t_i} + N$$

In the matrix notation (4.7) becomes

$$(4.8) \qquad (I + \lambda Q_n)\, \underline{Y} = \underline{F}$$

where Q_n is the matrix whose (i,j)th element is

$$(4.9) \qquad q_{i,j} = \frac{\pi b}{a^2+b^2} w_j \sum_{m=1}^{n-1} \frac{K(s_m,t_j)w^*_m}{t_i-s_m},$$

$$\underline{F} = [\, F(t_1),\ldots\ldots,F(t_n)]^T,$$

$$F(t_i) = -\frac{b}{a^2+b^2} \sum_{m=1}^{n-1} w_m^* \frac{f(s_m)}{s_m - t_i} + N \tag{4.10}$$

We know from the Fredholm theory that except for those λ which are eigenvalues of (4.4) the matrix

$$(I + \lambda Q_n) \tag{4.11}$$

possesses a bounded inverse for any integer n, ([8] p. 335, [1]).

Since

$$Q_n = A_n^{-1} C_n \tag{4.12}$$

it follows that $I + \lambda A_n^{-1} C_n$ is nonsingular, which implies that

$$A_n (I + \lambda A_n^{-1} C_n) = A_n + \lambda C_n \tag{4.13}$$

nonsingular for any integer n.

The last relation (4.13) and the fact that equations (1.1), (4.4) have the same eigenvalues shows that the Gauss-Jacobi method possesses a unique solution for any n, as long as λ is not an eigenvalue of (1.1).

Appendix:

The alternative form for the weights given by Szegö ([9], p. 352, (15.3.1)),

$$w_i = \frac{2^{\alpha+\beta+1}}{\pi} \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)} (1-t_i^2)^{-1} \{P_n^{(\alpha,\beta)\prime}(t_i)\}^{-2} \tag{A.1}$$

is more convenient to derive a compact expression for the determinant.

If we assume that $\alpha+\beta = -1$ and use equations ([9], p. 143, (6.71.5), (6.71.8)) and (A.1) we get,

(A.2)
$$\prod_{i=1}^{n} w_i = \left\{\frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{\Gamma(n+1)\ \Gamma(n)}\right\}^n \prod_{i=1}^{n} (1-t_i^2)^{-1} [\ell_n^{(\alpha,\beta)}]^{2n-4} [D_n^{(\alpha,\beta)}]^{-2}$$

where

(A.3)
$$\ell_n^{(\alpha,\beta)} = 2^{-n} \binom{2n+\alpha+\beta}{n} = \frac{\Gamma(\alpha+\beta+2n+1)}{2^n n!\,\Gamma(n+1+\alpha+\beta)} = \frac{\Gamma(2n)}{2^n n!\,\Gamma(n)}$$

(A.4)
$$D_n^{(\alpha,\beta)} = 2^{-n(n-1)} \prod_{\nu=1}^{n} \nu^{\nu-2n+2} (\nu+\alpha)^{\nu-1} (\nu+\beta)^{\nu-1} (\nu+n+\alpha+\beta)^{n-\nu}.$$

Similarly

(A.5)
$$\prod_{j=1}^{n-1} w_j^* = 4^{n-1} \left\{\frac{\Gamma(n-\alpha)\ \Gamma(n-\beta)}{\Gamma(n)\ \Gamma(n+1)}\right\}^{n-1} \prod_{j=1}^{n-1} (1-s_j^2)^{-1} [\ell_{n-1}^{(-\alpha,-\beta)}]^{2n-6} [D_{n-1}^{(-\alpha,-\beta)}]^{-2},$$

where

$$\ell_{n-1}^{(-\alpha,-\beta)} = 2^{1-n} \frac{\Gamma(2n)}{\Gamma(n) n!}$$

and $D_{n-1}^{(-\alpha,-\beta)}$ is obtained from (A.4) by replacing n with n-1, α with $-\alpha$ and β with $-\beta$.

Further

$$\prod_{i=1}^{n} (1-t_i^2) = \left(\prod_{i=1}^{n} (x-t_i)\right)^2_{x=1}$$

which is the monic Jacobi polynomial evaluated at x=1.

Hence

(A.6)
$$\prod_{i=1}^{n} (1-t_i^2) = \left[\frac{2^n n!\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)} P_n^{(\alpha,\beta)}(1)\right]^2$$

$$= \frac{4^n}{\{\Gamma(1+\alpha)\}^2} \left\{\frac{\Gamma(n)\ \Gamma(n+\alpha+1)}{\Gamma(2n)}\right\}^2 ;$$

for, $$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} .$$

Likewise

(A.7) $$\prod_{j=1}^{n-1} (1-s_j^2) = \frac{4^{n-1}}{(\Gamma(1-\alpha))^2} \left[\frac{n!\Gamma(n-\alpha)}{\Gamma(2n)}\right]^2 .$$

Moreover, after cancellation of common factors,

(A.8) $$\frac{D_{n-1}^{(-\alpha,-\beta)}}{D_n^{(\alpha,\beta)}} = \frac{4^{n-1}}{n} \Gamma(\alpha+2)\Gamma(\beta+2) \frac{((n-1)!)^2}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} .$$

The final product after some more cancellations turns out to be

(A.9) $$\frac{\prod_{i=1}^{n} w_i}{\prod_{j=1}^{n-1} w_j^*} = \frac{(\Gamma(\alpha+1)\Gamma(\alpha+2))^2}{\pi\ 4^{n-1} n^3} \left\{\frac{\Gamma(2n)}{\Gamma(n)}\right\}^2 \frac{\Gamma(n+\beta+1)}{(\Gamma(n+\alpha+1))^3} .$$

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