RELIABILITY AND MAINTENANCE FOR SYSTEMS WITH INDIVIDUALLY REPAIRABLE COMPONENTS DEGRADING AS GAMMA PROCESSES

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ABSTRACT OF THE THESIS

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This research pertains to reliability and maintenance for a system with individually repairable components degrading as gamma processes. Life and expenses of the system are the two main factors considered in this research by analyzing the reliability and cost rate functions. The main difference of this current study from previous system reliability research is the system model has component degradation, individual component repair and different component histories. As the economy develops at an amazing speed in recent years, people care more about their experiences while doing or using something rather than basic functionality. For instance, customers concern more and more about their satisfaction with the products they buy and the services they get, so companies, producers and many other institutions attach great importance to reliability and quality of products. Moreover, when people travel, they care more about their safety and comfort. Therefore, reliability becomes increasingly more important, and it exists everywhere in our life. High reliability design practices can make products reliable and ensure their quality, their safety, their life cycles and the like. Although system reliability problems have been studied and divided into many smaller categories and these categories also

have been subdivided into even smaller parts. These problems have been studied for many years by different researchers, and quite a few studies are very complex when applied to actual systems. In general, not all of them are practical or can be solved efficiently. Conversely, they are beneficial for researchers to do further research to understand the behavior of complex systems. Because of their complexity, many models and situations in those studies are not often applied for real engineering processes. Therefore, this research aims to build a practical model widely applied to real situations, factories and industries. The reliability of many products is not always studied in detail because of their low price. For some products, such as batteries and tires which are non-repairable, engineers and products owners replace them simply because they are non-repairable. However, most industrial equipments, such as railroad tracks, gas pipelines and huge mechanical facilities, are repairable, and it is economical to repair the failed components rather than to discard the whole system. Meanwhile, rebuilding a new system is very costly. Taking all these factors into consideration, this research aims to build a model of a system with individually repairable components, each of them degrading as a gamma process. The component degradation paths can also be probabilistically dependent due to shared exposure to shocks. Studying the reliability and cost rate function of such systems can lead us to obtain the optimal maintenance policy. The results and conclusions of this study can be widely used in practical engineering work in future research.

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1 Introduction

This thesis describes research activities to construct a general system reliability model for practical engineering processes and further reliability research. This research pertains to a system with individual repairable components that are degrading as gamma processes. Considering real situations, random shocks are subjected to the system. To study this model, analyses of reliability and cost rate are the main goals that need to be achieved. This research presents a new system reliability model with individually repairable components. Challenging examples are constructed for the purpose of understanding the whole process and extension of previous research.

Many devices, products and machines are repairable and it is not economical to abandon the whole system if it fails. In many cases, wearout of these types of systems is due to a monotonically increasing degradation, such as wear, corrosion, etc. The gamma process is a very common and useful process to model the degradation of components whose degradations are monotonically increasing. The component degradation models are then used to obtain the system reliability and cost rate of a system with individually repairable components.

1.1 Motivation

In the field of industrial engineering, reliability is an important cornerstone for product quality, maintenance and performance. Successful products should have high reliability to satisfy customers' expectations. Therefore, more and more industries and manufacturers begin to design, develop and produce new products with superior reliability, and sellers choose to sell highly reliable products which bring goodwill and benefits to them in return. Moreover, products with high reliability make customers feel safe and secure while selecting these products. Considering a system with many individually repairable components, such as railroad track systems, gas pipeline systems, and many others, engineers can repair each failed component individually instead of replacing the whole system, which is inefficient and may cost too much. Therefore, it is of great importance to analyze system reliability and decide when to inspect and preventively repair each component to prevent failures. Moreover, many systems in industries and factories have already existed in the real working environment for a very long time, like the systems mentioned above. At each inspection, it is very important to acknowledge the status of the system and decide the maintenance methods towards each individual component.

In industrial fields, Condition-based Maintenance (CBM) is a commonly applied method to maintain a system and prevent failures. As described in Jong-Ho and Hong-Bae [12], for many systems, CBM is a good choice of a maintenance method that can increase the effectiveness of applying specific maintenance policies to systems. In their case studies, there are four examples which are oil analysis, crack propagation analysis, field operation data analysis and vibration analysis. CBM is one of the most attractive maintenance tools in industries or factories. It is a predictive maintenance method, which is used in this research. In this research, CBM is used for a system with individually repairable components. Then on-condition thresholds for each component are introduced and used as a predictive critical value to prevent the component's failure.

This research aims to construct a system with individually repairable components degrading as dependent gamma processes and also considering shocks coming from some external environment, and the expense of maintenance and failure. Zhao et al. [13] introduce a new mixed shock model and study the failure process of a single-component system, i.e., a fibrous carbon composite. Their system model is considered to consider both interval degradation process and external shocks accelerating the failure process. Moreover, cost functions are accomplished and compared to meet the optimum maintenance policy. However, their model has many limitations and assumptions and is too specific to be applied widely. This is more motivation to perform this research, chosen to construct a system model degrading as a simple gamma process and damaged from shocks whose size is normally distributed, and arriving as a Poisson process.

Many previous reliability research and its corresponding maintenance policies focus on a single-component system, like Zhao et al. [13] and Che et al. [14] who analyze an electrohydraulic servo valves (EHSVs). This is not common in industrial fields because big systems tend to have multiple components. Motivated by these studies, the proposed model is completed and can be used as a standard model and applied to real industrial situations and many other research problems.

1.2 Problem Description

The research focuses on a system with individually repairable components degrading as gamma processes. This system is built as a model and simulated in MATLAB. Considering the expenses of inspection, maintenance and failure, the main task is to determine reliability and cost rate by applying relevant knowledge of the gamma process extended to the whole system model. As for the gamma process, it is a cumulative degradation process whose difference between individual degradation values in a time interval follows the gamma distribution. The gamma distribution is a probability distribution which has two parameters to determine the shape of the distribution. The shape parameter α controls the rate of jump arrivals [1] (incremental increases in degradation) and when α equals to some specific value, gamma distribution can be a special case of another specific distribution like exponential distribution, Erlang distribution, etc. Gamma distribution is flexible and an appropriate model for simulations of many different processes. The scale parameter β controls the range of the gamma distribution. A gamma distribution with $\beta = 1$ is known as the standard gamma distribution. For a gamma process, the shape parameter is a function of the starting and ending time of the time period being considered.

Maintenance for a system can be carried out into following three actions: inspection, predictive and replacement maintenance. In the systems being studied, when the degradation path passes a defined threshold, the component is considered to fail and is subsequently repaired. Moreover, another threshold can also be defined to prevent components from failure, called on-condition threshold, which is a predictive maintenance policy. At each inspection, if the component's degradation passes the on-condition threshold, the predictive maintenance policy instructs engineers to replace them in order to prevent a system failure and idle time. This policy is common in reality, like replacing tires when there is noticeable wear, etc. In order to lower the expense of maintaining the whole system, the cost rate function is considered as an objective to be minimized. Determination of the values of on-condition threshold and inspection maintenance interval decisions are the most important part of the research work.

Available degradation models are different in many different ways, and the system can also vary accordingly, and have many different degradation patterns. This research mainly focuses on the gamma process models, since monotonic degradation processes associated with mechanical components are being considered. The research plan first considers if it is a linear or nonlinear expected degradation process. If it changes as a linear function, the process degrades probabilistically in a constant trend; otherwise the process can quickly degrade if it changes as a nonlinear function. Also, the research considers incoming shocks from an external environment, and the failure process varies accordingly. The shock magnitude and damage size have some specified distributions. In this research, the normal distribution is considered, for the reason that normal distribution is widely used in the previous research, and is often more applicable than others for actual examples.

The first factor to evaluate the whole system performance is the cost rate function. When the minimum cost rate is achieved, and the criteria used by engineers to decide to replace or repair components, are decided by minimizing a cost rate function. By changing the cost of failure and preventive maintenance (C_F and C_{PM}) in an initial cost rate function, some meaningful conclusions can be achieved concerning the minimum cost rate and optimum time. Meanwhile, changing parameters in an alternative maintenance cost rate function, which includes the cost of inspection, cost of replacement and penalty cost per unit idle time or downtime (C_I, C_R and C_ρ), results in many interesting and meaningful results to be achieved. Moreover, by comparing the results in each different cost rate function, the answer to which model is better with different parameters can be achieved.

Finally, a real case study is completed to help researchers better understand the model constructed in this research. Different situations are considered and an optimum model is chosen to fit the real system.

1.3 Notation

The notation used in formulating the reliability and cost rate models in Sections 3 and 4 is now listed. CDF means cumulative distribution function, while PDF means probability density function.

$X(t)$ = Degradation at time t

α_0 = Shape parameter of a gamma distribution

β = Scale parameter of a gamma distribution

$\alpha(t)$ = Shape parameter of gamma distribution at time t for a gamma process

b = Parameter of nonlinear degradation

H_i^1 = Failure threshold of component i

H_i^2 = On-condition threshold of component i

U_i = Initial degradation of component i at the beginning of an inspection interval

$f_{U_i}(u)$ = Probability density function of initial degradation of component i

$\hat{a}$ = Slope parameter of simulated generalized linear model of initial degradation

$\hat{b}$ = Intercept parameter of simulated generalized linear model of initial degradation

$g(\cdot)$ = Gamma distribution probability density function

$G\{\cdot\}$ = Cumulative distribution function of gamma distribution

$R(t)$ = Probability of no failure during time interval 0 to t

$R_r(\tau)$ = Probability of no replacement during time interval τ

$R_f(\tau)$ = Probability of no failure during time interval τ

τ = Inspection time interval

CRT = Cost rate

C_{PM} = Cost of preventive maintenance

C_F = Cost of failure

C_I = Cost of inspection

C_R = Cost of replacement for a system

C_{R_i} = Cost of replacement for component i

C_ρ = Penalty cost per unit idle time or downtime

$E[\rho]$ = Average idle time or downtime

$f_{T_{H-U}}(t)$ = PDF of component failure time given initial degradation U_i

$F_T(v)$ = Probability that the component fails during a time interval 0 to v

$F_{X(t)}(H)$ = Probability that $X(t) < H$ at time t

λ = Parameter or rate of Poisson process

m = Number of shocks

μ_m = Mean of normal distribution when shocks arrive m times

σ_m = Standard deviation of normal distribution when shocks arrive m times

$Y_{n_m}(t)$ = Cumulative shock damage size occurred in the degradation of component n at time t

$f_{Y_i}^{<m>}(y)$ = PDF of m shock damage sizes for component i

t_{i_0} = Starting time of component i

2 Background and Literature Review

There has been many important and famous research focusing on building a degradation model and analyzing the reliability of its components or system. These papers chiefly present and discuss models with component degradation and shocks, as well as some maintenance policies to minimize the cost rate. However, some models in their research are very complex for many practicing engineers and too specific to be applied to common situations in reality. The objective of this research is to build a new system model which can be widely used for future study by oncoming researchers and applied in real practical situations.

Lu and Meeker [2] focus on the degradation process of linear or nonlinear fixed random effects. They introduce and discuss some previous life tests that record only time-to-failure which may lead to the difficulty of estimating reliability. Thus, by analyzing degradation data instead, they use methods based on Monte Carlo simulation to assess reliability by estimating the degradation trend and a closed form equation of a time-to-failure distribution. Based on their research, many reliability models were constructed in a simpler way and more researchers found a new key towards the gate of analyzing the reliability degradation path. In a word, their research is extremely significant which provides insights and lays a solid foundation for the author and many other researchers to build their own systems and conduct their research. Also, Lu et al. [16] analyze the linear degradation data to get the cumulative distribution function using a constructed model with random regression coefficients and a standard deviation function. In real situations, according to Boulanger and Escobar [17], traditional reliability assessment methods, which are based on accelerated life tests are not suitable for products with high reliabilities, since their qualities are almost perfect at the beginning. Recording performance changes, which is also named degradation changes, is one alternative approach to obtain the reliability assessment. Due to the short time of products' development, Meeker et al. [18] constructs another well-performed method to assess the reliability more conveniently compared with previous time-to-failure records. Based on this method, a relationship between components failure and amount of degradation are built and failure time distribution can be estimated.

There are many other degradation models studied by previous researchers. Gorjian et al. [3] reviews and summarizes the most common degradation models and synthesizes these models and classifies them in different categories. Therefore, for each type of degradation, researchers can find an optimum known degradation model to fit it.

Degradation $X(t)$ is defined as the total wearout of a component or a system. For this research, monotonic increasing degradation is considered due to its applicability for mechanical components and systems not only in research studies, but also many practical industrial fields. The difference between two degradation measures in a time interval follows a specific distribution. Figure 1 shows a simulated one-component gamma process, Figure 2 is for a system, and Figure 3 is the sample degradation process with 3 components. In this research, the gamma distribution is taken into consideration. Threshold H_i^1 is a standard value which determines whether the component or system fails. It can be the number of total defects of a machine, the number of total abrasion of a tire and so on. When the degradation process of components or a system reaches the threshold, repairs or replacements need to be taken. In this research, the policy of repairing an individual component is to replace it with a brand new one.

Figure 1 shows a typical simple degradation process of a component simulated in MATLAB (the parameters are: α_0 = 0.1, $\alpha(t) = \alpha_0 t$, β = 0.5 and H^1 = 30 for a gamma process). The component degradation is monotonic and failure occurs when it passes the threshold of H^1 = 30. As it is shown in Figure 1 for one simulation, the degradation passes the threshold at approximately 1.7 unit time, and then a failure occurs.

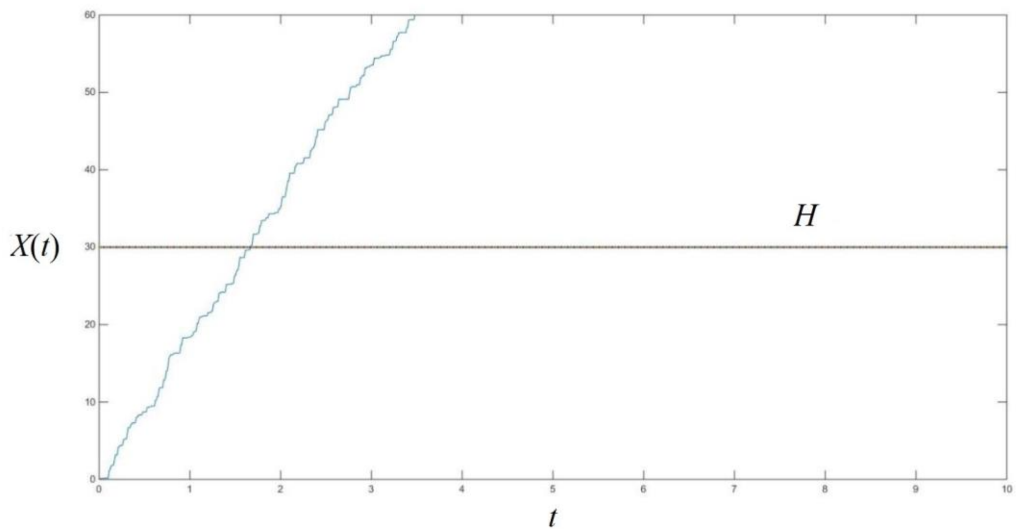

Figure 1 – Simulated sample process

There are many general degradation models. Based on the process which the degradation follows, the trend can be different. Some of them are monotonic while others are non-monotonic. In the real world, many devices or systems have a monotonic increasing degradation. Thus, this research mainly focuses on monotonic increasing degradation trends due to its universality. From previous research and studies, degradation models can also be divided into three types: linear, concave and convex degradation. Respectively, their degradation rates are constant, decreasing and increasing. Figure 2 [4] shows the trend of each general degradation model.

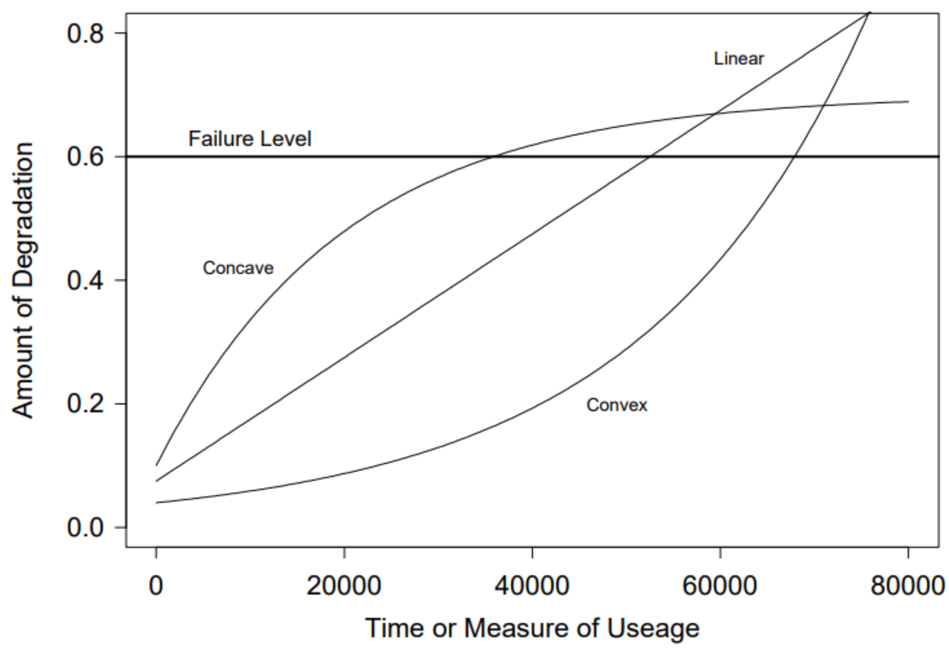

Figure 2 – Possible shapes for univariate degradation curves [4]

Noortwijk [5] generally reviews the degradation model of gamma process and applies it to maintenance planning. This research concludes that the gamma process is well suited for degradation models and useful for determining optimal inspection and maintenance policies. This is because the gamma process is a monotonically increasing degradation process. Failures come when degradation paths pass the critical value.

In order to focus on one common and specific degradation model, the degradation process in this research is simulated to follow a gamma process because the research pertains to mechanical components and systems. The component degradation is monotonic increasing and the gamma process is a relevant and useful model. It is a degradation process which the difference between two individual degradation levels in a time interval follows gamma distribution whose function is:

$$X(t_2) - X(t_1) \sim g(x; \alpha(t_2) - \alpha(t_1), \beta)$$

$$g(x;\alpha,\beta) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}$$

The shape parameter for a degradation depends on the interval, $\alpha(t_2) - \alpha(t_1)$, and β is the scale parameter. For an expected linear degradation, the shape parameter = $\alpha(t_2) - \alpha(t_1) = \alpha_0(t_2 - t_1) = \alpha_0 \Delta t = \alpha_0 \tau$ for a constant inspection interval τ. As for an expected nonlinear degradation, the shape parameter is different, which can be expressed as: $\alpha(t_2) - \alpha(t_1) = \alpha_0 t_2^b - \alpha_0 t_1^b = \alpha_0 (t_2^b - t_1^b)$. b is the power of time interval t which is dependent on real situations. In general, b is very unlikely to be greater than 2 because the reliability of normal industrial products has no reason to degrade so quickly except defective goods. Gamma process can be described as an incremental process, see the example in Figure 3. Assume there are three components and the parameters for this sample gamma process are: α_0 = 4 and β = 1. These three components have their own degradation path even if they have the same parameters of gamma process.

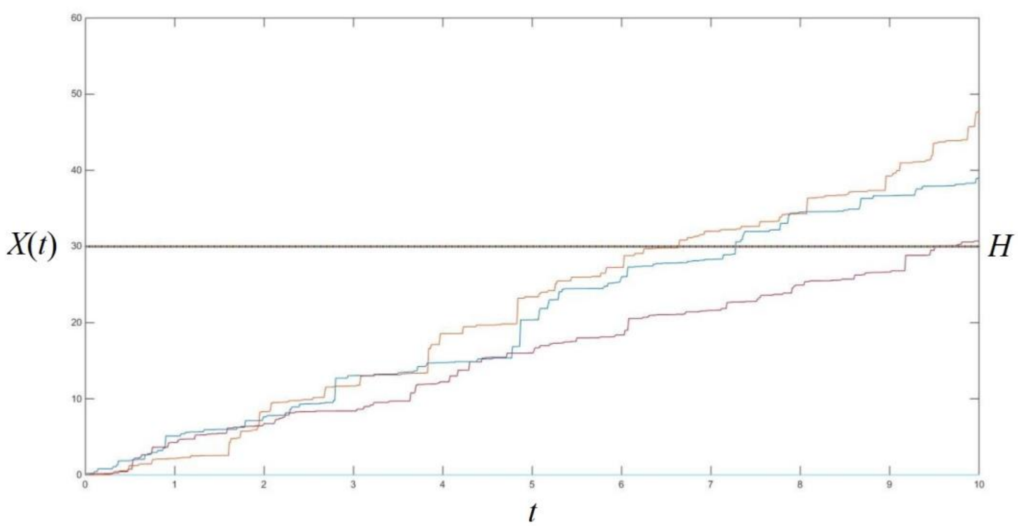

Figure 3 – Sample gamma process

The component can also be subjected to external shocks ([6], [7], [8]), which directly impact the degradation path. If shocks arrive as a Poisson process, and shock damage is a normally distributed random variable, the gamma process with shocks is indicated in Figure 4. The total degradation is the sum of gradual degradation and cumulative shock damages.

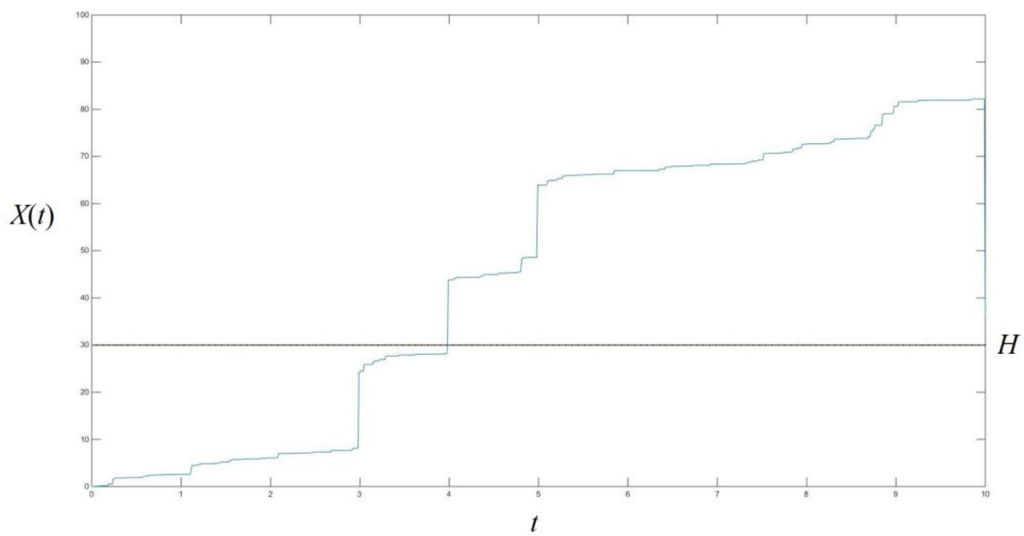

Figure 4 – Sample gamma process with shocks

Due to the variety of external shocks, some of the research assumes that the shock magnitude follows some specific distributions, while others consider only normal distributions. Also, shocks arrive at random times and these time intervals could have different interarrival times which follow different distributions. Most previous research assumes these shocks arrive as a Poisson process, which is the most common arrival situation, so the time between shocks is an exponential random variable. The probability of k shocks arriving in a time interval τ is presented in the equation below:

$$\Pr\{(N(t+\tau)-N(t))=k\}=\frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!}$$

$$\Pr\{(N(t+\tau)-N(t))=0\}=\frac{e^{-\lambda\tau}(\lambda\tau)^0}{0!}=e^{-\lambda\tau}$$

Li and Pham [6] assume that the shock damage size in their example is based on a similar exponential distribution. Moreover, Li and Pham [9] focuses on a generalized multi-state degraded system reliability model subject to multiple competing failure processes. Their system is characterized by a finite number of states. Specifically, they focus on a system with components in two states, operating or failed. Shocks are also considered in their research. On the condition of different assumptions, situations are variable. The model in this research includes shocks arriving as a Poisson process and the shock size follows its corresponding normal distribution. For the purpose of easily understanding and gradual improvement, this research mainly focuses on a simple model degrading as a gamma process with or without shocks arriving as Poisson process. Reliability functions are different with arrival shocks and become more complicated as well. Many methods are applied to solve this difficulty, such as reliability simulations. A series system consisting of individual components is previously modeled by Li et al. [19]. Due to the difficulty of taking integrals of a joint normal probability density function to obtain reliability functions, they apply mathematical model to derive the reliability functions.

With influences of external shocks, previous research often assumes systems degrade as two competing failure processes, a hard failure process due to shock itself and soft failure process due to the total degradation ([7], [8]). Each failure process has their own failure threshold. Any of them can cause the system to fail if they reach their own thresholds. Moreover, their models are based on a system with non-repairable components. If any component or the system fails, the whole system is abandoned. Hard failure and soft failure are two common failure processes considered in many previous research. Jiang et al. [20] presents system reliability and maintenance models degrading with these two processes. Another analysis, done by Huang and Askin [21], gives researchers another practical example of how to assess reliability for electronic devices with multiple competing failure modes involving performance aging degradation. Two major failure modes considered in their research are catastrophic failure and degradation failure, which is also called hard failure and soft failure.

In order to build the system reliability model, stochastic modeling methods are widely used in previous research. A stochastic model is a mathematical tool to estimate the probability distribution with one or more random variables. Bian and Gebraeel [22] study a system whose components have degradation rate interactions. In their research, a stochastic modeling method is applied to analyze the interactions among the degradation processes of every component. Mean residual lifetime distributions are predicted. Another stochastic model for degradation-based reliability is presented by Kharoufeh and Cox [23]. This model is constructed on the basis of degradation data as well. However, their model is for single-unit systems. Another research is about a single-unit system whose cumulative degradation over time is a continuous wear process. It is analyzed by Kharoufeh [24] and the failure time distribution is derived.

As for system modeling, many of the models only considered the situation of non-repairable systems so that there must be a failure level H_i^1 for each component. There are many other problems considering an inspection level H_i^2 for checking whether the component needs to be maintained. Peng, et al. [7] discuss a system model with components degrading as multiple dependent competing failure processes. The model is built on the assumption that failures occur if one of the degradation processes, which are soft failure process and hard failure process, passes a threshold. Meanwhile, an inspection time τ is taken into consideration. Failures are only detected when the system is inspected. Because of this inspection level, the total function of reliability and cost rate are complicated. The probability of replacing a component reaches the on-condition threshold is correlated to the inspection level H_i^2. However, the probability of a failure occurring is related to the failure threshold H_i^1. In their research, a system with just one component is considered. Tseng et al. [25] uses degradation data of fluorescent lamps to improve the reliability (or lifetime) of them. The threshold considered in their research is the luminosity or light intensity. When it is below some value or level, a failure occurs. Their model is suitable for highly reliable products.

Song et al. [8] focuses on the same model, but extended to the system level with multiple components while systems are non-repairable. When the system fails, due to failures of any components or any shock in the hard failure process higher than the threshold, then the system is replaced. This is a common model for many consumer products like cell phones or tablet computers, but not appropriate for industrial applications. Their research also discusses systems in the situation of parallel and series types. However, in their research model, the system is firstly considered as a simple model without inspection level H_i^2. The components are arranged in series so each failure of the components leads to the failure of the whole system. Then, a parallel situation is considered and variable reliability functions are derived in order to build their model. Moreover, Song et al. [26] builds another system model whose components degrade as multiple dependent competing failure processes. Interactions between soft failure and catastrophic failure, as well as interactions among components cannot be ignored based on their assumptions. Optimum maintenance policies are determined by applying their model to numerical examples. One different research of Rafiee et al. [27] uses dependent competing failure degradation processes as the degradation processes followed by the system models they built. However, originally, the degradation rate can vary due to the influences caused by shocks.

To apply reliability models and assessment methods to real situations, maintenance policies are extremely important. Previous research provides many meaningful and novel examples of maintenance policies to researchers to construct their own maintenance policies for corresponding system reliability models. Wang [28] gives a summary, comparisons and classifications of different maintenance policies applied in previous research, and emphasizes policies on one-component systems. Similarly, Lu et al. [29] compares many maintenance policies based on predicted failure probabilities, corresponding maintenance expenses and the profit losses. In this case, a stochastic dynamic process with continuous degradation is considered.

Condition-based maintenance (CBM) is common and widely used in reliability research. Many research completed before focuses on determining the optimum CBM policy for a system consisting of only one individual component. Tian and Liao [30] choose to build a system model with multiple components and use a CBM policy based on a proportional hazards model. It is complex to evaluate the cost rate when the system has many components. They use an original numerical algorithm to achieve their optimization. Similarly, to apply CBM policies to a multi-component system, this research presents another numerical algorithm. Based on the constructed system reliability model, cost rate evaluations are accomplished. Another model constructed by Liao et al. [31] is applied to a numerical example to meet the optimum maintenance policy. Also, CBM policies and gamma process degradations are used in their model, a predictive maintenance policy is one of the CBM policies and also widely applied in system reliability assessment. Grall et al. [32] focuses on predictive maintenance policies. They use a gradually deteriorating one-component system as their model to do analysis. The preventive replacement threshold and the inspection schedule are two main maintenance variables, which is called on-condition threshold and inspection time interval in the author’s research. Zhou et al. [33] creates a new predictive maintenance scheduling policy which is named as reliability-centered predictive maintenance. This policy is tested on a system subject to the degradation with imperfect maintenance. Also, this studied maintenance policy can be added to CBM in order to make it perfect.

A cost rate function is one method to evaluate whether a predictive maintenance policy is effictive or not. They are different cost rate functions based on the situation in each previous research and each function has different cost of failure, maintenance and so on. Variables in cost rate functions play different roles in evaluations and each of them contributes to the cost rate function differently. Chiao and Yu [34] develop an optimal systematic approach to identify the most important factors which affects product reliability, such as inspection frequency or termination time. Also, their test plan aims to minimize the total experiment cost, which is highly similar to the author’s research. Like previous research, this research plan focuses on the degradation of reliability function and the approaches to minimize the cost rate.

3 General System Reliability Model with Repairable Components

This section describes a general system reliability model and introduces the definitions of steady state and some policies of component replacements. The initial degradation distribution, and when to repair the system by replacing the failed component, are two main topics in this part. General functions and plots are presented in this section.

3.1 Steady State System Behavior

For systems with repairable components, assume these components are new and unused at the very beginning of system life. Suppose these systems are then inspected at the end of each time interval, and failed components are only detected by inspection. The component which has already failed at the end of each time interval is repaired by being replaced by a new one while the remaining operating components continue to work properly in the next time interval, but have aged. For systems having maintenance policy with inspection level, components which reach the on-condition threshold are replaced with a new one. The fact that whether a component has failed or not is only known at each end of the time interval so replacements can only be made after each inspection.

After the system has operated for a very long time, it can be considered that each component has been replaced multiple times in the replacement cycle. Thus, at the system level, the frequency of component replacements can be assumed to reach a steady state behavior after some time. Here, the initial degradation for component i is U_i, which is the degradation at the beginning of steady state interval. In a word, U_i is the initial degradation of component i. It can have many different distributions due to different assumptions of the model types, but initially it is constructed to follow a uniform distribution whose parameters are 0 and H_i^1 for a system degraded as expected linear gamma process without shocks. It cannot be higher than H_i^1 because failed components are replaced at the end of each time interval.

Suppose there is a system with n components in series and inspected every τ times. After $k\tau$, with k large, the system reaches steady state and each component has an initial degradation U_i, as indicated in Figure 5. U_i is a random variable between 0 and H_i^1. Also, if initial degradation U_i is simulated, the observed form of its distribution can be obtained as shown in Figure 6.

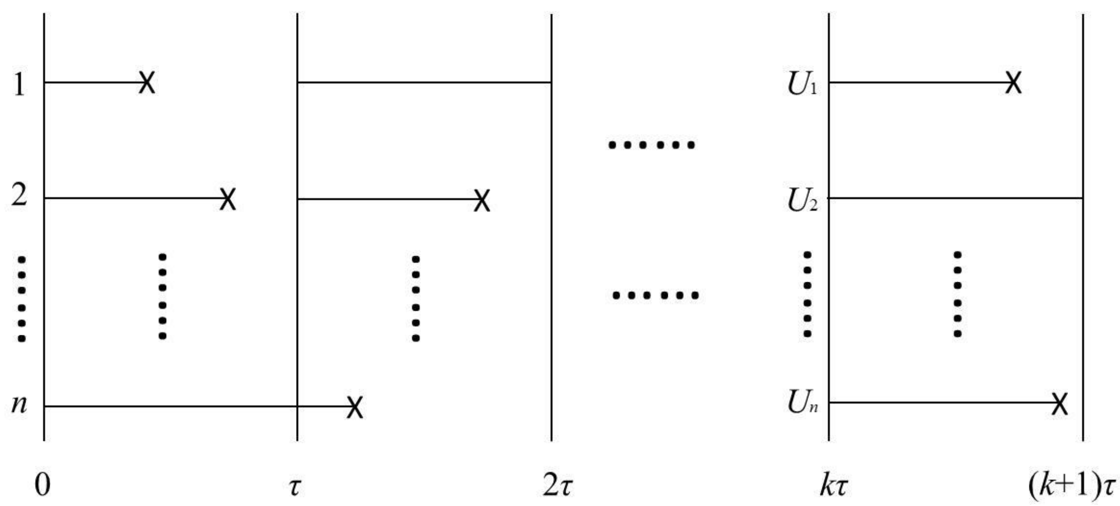

Figure 5 – Sample steady state of n components' system

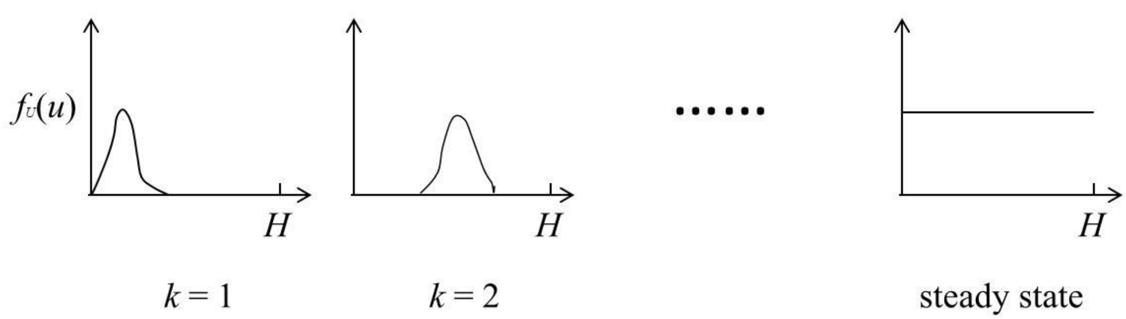

Figure 6 – Simulated distribution of initial degradation

As an example, assume these components are degrading as an expected linear gamma process with no shocks, the shape and scale parameters are $\alpha_0\tau$ and β (assume α_0 = 4 and β = 1). The degradation $X(t)$ for time interval 0 to t follows a gamma process, with $\alpha(t) = \alpha_0 \times t$ and β = 1. The total degradation is the cumulative value of $X(t)$. The degradation $X(t)$ should be less than the threshold H_i^1 or the component fails. Equations below show the definitions and the relationships among them.

$$X(t_1) \sim g(x;\alpha(t_1),\beta)\,,\ X(t_2) \sim g(x;\alpha(t_2),\beta)\,,$$

$$X(t_2) - X(t_1) \sim g(x;\alpha(t_2)\text{-}\alpha(t_1),\beta) \text{ with } \alpha(t_2) - \alpha(t_1) = \alpha_0 \times \Delta t = \alpha_0\tau$$

As an example,

$$U_1 \sim Uniform(0, H_1^1)\,,\ U_n \sim Uniform(0, H_n^1)\,,\ f_{U_i}(u) = \frac{1}{H_i^1 - 0}\,, \text{ for } 0 \le u \le H_i^1$$

The probability that a series system survives the steady state interval of τ duration, with independent $X_i(t)$ and U_i, is given by:

$$\begin{aligned} R(\tau) &= \Pr\{U_1 + X_1(\tau) < H_1^1 \cap \ldots \cap U_n + X_n(\tau) < H_n^1\} \\ &= \Pr\{X_1(\tau) < H_1^1 - U_1\} \cap \ldots \cap \Pr\{X_n(\tau) < H_n^1 - U_n\} \\ &= \int_0^{H_1^1} \Pr\{X_1(\tau) < H_1^1 - u_1\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^1} \Pr\{X_n(\tau) < H_n^1 - u_n\} f_{U_n}(u)du \\ &= \int_0^{H_1^1} G\{H_1^1 - u_1;\alpha_0\tau,\beta\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^1} G\{H_n^1 - u_n;\alpha_0\tau,\beta\} f_{U_n}(u)du \\ &= \prod_{i=1}^{n} \int_0^{H_i^1} G\{H_i^1 - u_i;\alpha_0\tau,\beta\} f_{U_i}(u)du \end{aligned}$$

Therefore, the probability of surviving time interval or inspection interval τ for a 2-component system in steady state is given by:

$$R(\tau) = \prod_{i=1}^{2} \frac{1}{H_i^1} \int_0^{H_i^1} G\{H_i^1 - u;\alpha_0\tau,\beta\} du$$

These two probabilities represent a conditional reliability that is, the probability of surviving the interval conditional on initial random degradation amounts U_i.

As for linear degraded system with arrival shocks, the initial degradation distribution is simulated in MATLAB and approximately to follow a generalized linear model (shown later in Figure 55). More importantly, the slope parameter $\hat{a}$ and the intercept parameter $\hat{b}$ of the model are only affected by the mean and standard deviation of the shock size, which is: μ and σ. Thus, the initial degradation for all components is the same.

$$f_{U_i}(u) = \hat{a}u + \hat{b}\,,\ \text{for}\ 0 \le u \le H_i^1\ \ (\text{or}\ H_i^2\,)$$

In this situation, due to the difference of initial degradation distribution, the probability of surviving time interval or inspection interval τ is quite different. This probability function is extended in the main part of Section 4.

3.2 Component Replacement

The most common way to repair a system is to replace the failed component. One failure of a component in a series system causes the whole system to fail. In this case, replacement needs to be done. This section introduces a simple cost rate function of replacing one component in different situations no matter whether there is initial degradation, whether the degradation process is linear or not, whether there are shocks that cause damages to the system and accelerate the degradation process.

3.2.1 Cost Rate of the Process on Account of Linear Degradation without Shocks

The cost rate for one component degrading as a gamma process is the following if the component is replaced preventively at time v (and random failure time is T). Consider there is no initial degradation. If the system is operating properly, there is an expense for preventive maintenance. If the system fails, there is a penalty cost for financial loss when system is down.

$$\text{cost rate} = \frac{C_{PM}(1 - F_T(v)) + C_F F_T(v)}{\int_0^v 1 - F_T(v)dv} \text{ [10]}$$

$$\begin{aligned} F_T(v) &= \Pr\{T \le v\} = \Pr\{X(v) \ge H\} \\ &= 1 - F_{X(t)}(H) \end{aligned}$$

$$F_{X(v)}(H) = \int_0^H g(x; \alpha_0, v, \beta)dx$$

The value of C_F and C_{PM} is selected from [10] which C_F = 100 and C_{PM} = 50. So while applying these equations to MATLAB, the plot of cost rate is evaluated in order to find the optimum time v which leads to the lowest cost rate. Figure 7 shows the graph of cost rate versus time point.

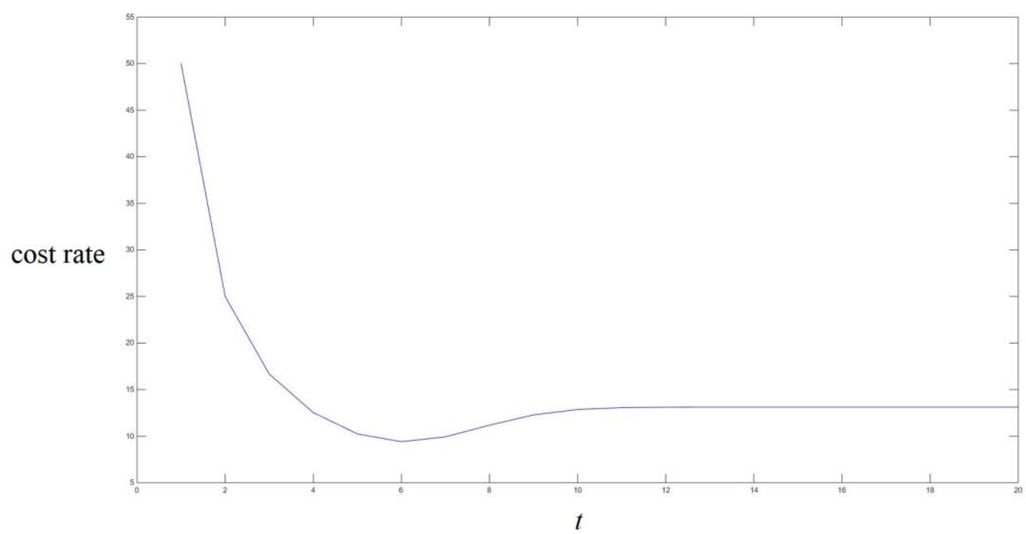

Figure 7 – Cost rate vs. Time of the process on account of linear degradation without shocks

The optimum time v is approximately 6. If the minimum time interval increment to find each cost rate is 0.1, the optimum time v equals to 6.1 which leads to the lowest cost rate: 9.391908.

To further investigate the model, C_F can be changed from 100 to 50 which equals to C_{PM}, so the cost rate should be monotonously decreasing because the costs of repairing and failing are the same. It makes no difference whether engineers repair the failed components or just let it fail, so the optimal policy is to let it fail to get maximum life, as shown in Figure 8.

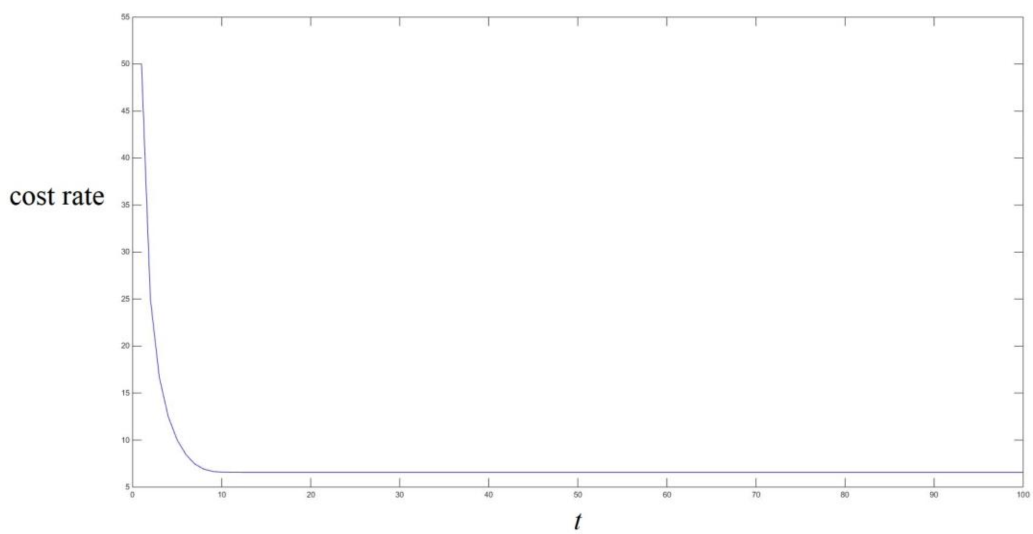

Figure 8 – Cost rate vs. Time when $C_F = C_{PM}$

This cost rate function does not consist of any initial degradation U_i, so the cost rate is now computed taking initial degradation into consideration, so as to find out whether there are some differences. Adding the factor of initial degradation U_i, the cost rate graph is shown in Figure 9:

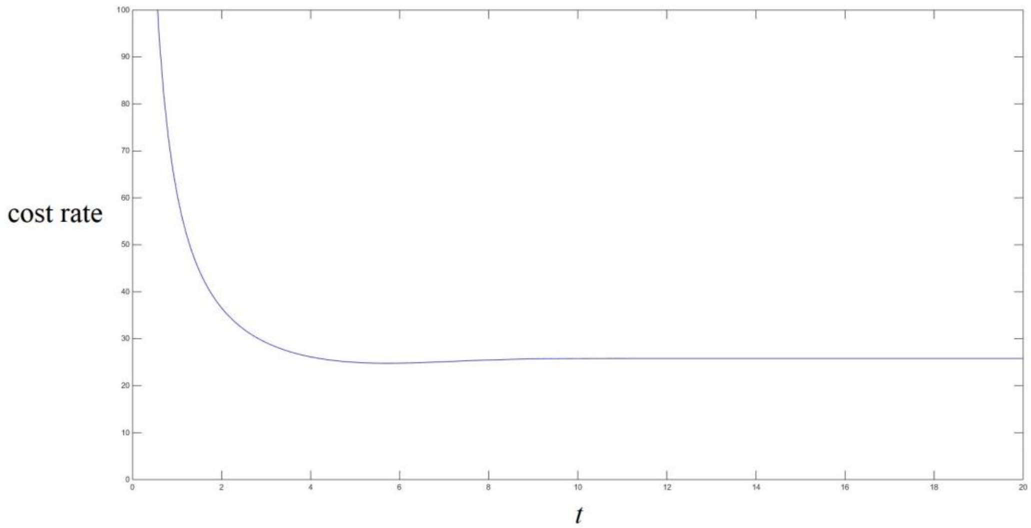

Figure 9 – Cost rate vs. Time of the process on account of linear degradation considering initial degradation U_i

In this case, the minimum cost rate is 24.783324 when time equals approximately 5.7. Compared with other results, it is logical that the optimum replacement time occurs earlier, and the minimum cost rate becomes larger. This is because there is an initial degradation at the beginning of each maintenance interval which makes the process more rapidly to pass the threshold and fail.

3.2.2 Cost Rate of the Process on Account of Nonlinear Degradation without Shocks

Cost rate of this nonlinear gamma process is different from the cost rate of previous linear gamma process because of the different probability density function. In this case, $F_T(v) = 1 - F_{X(v)}(H)$ but $F_{X(v)}(H)$ is not the same. In this case, the initial time is assumed to be 0. Thus, the shape parameter is $\alpha_0 v^b$ and the failure probability function is:

$$F_{X(v)}(H) = \int_0^H g(x; \alpha_0, b, v, \beta) dx$$

In Figure 10, when time increases to 3 or more, the cost rate becomes larger and then remains constant. The optimum time *v* is approximately 2.5. As a result, the optimum time *v* equals to 2.4, and the lowest cost rate is 23.617799.

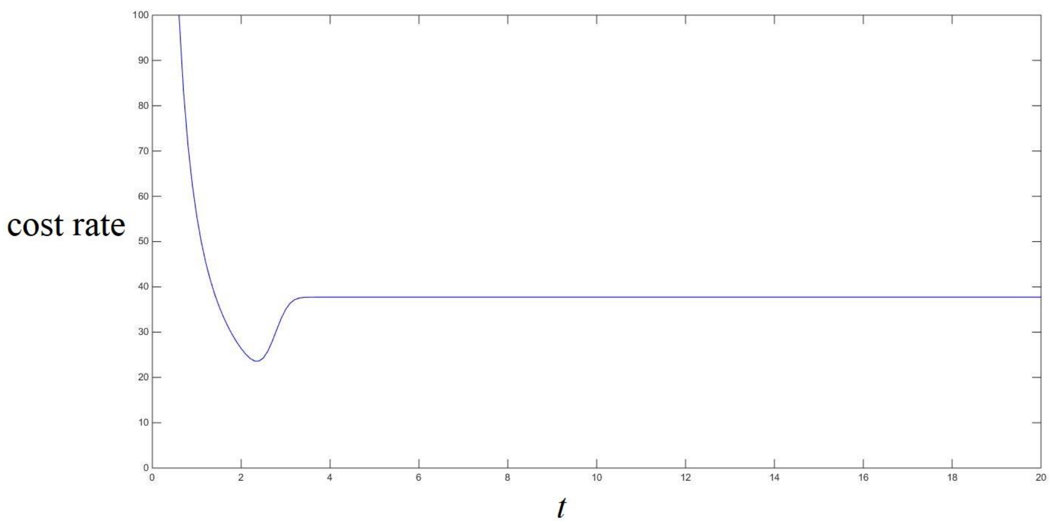

Figure 10 – Cost rate vs. Time of the process on account of nonlinear degradation

At the same time, considering initial degradation U_i, the graph is shown in Figure 11. As an interesting result, it seems that there is no optimum minimum cost rate in this figure. In fact, in this case, the lowest cost rate is 144.58248 when time equals to 1.8. Compared with the previous answer, it is clear that the optimum time occurs earlier, and the minimum cost rate becomes larger considering initial degradation U_i.

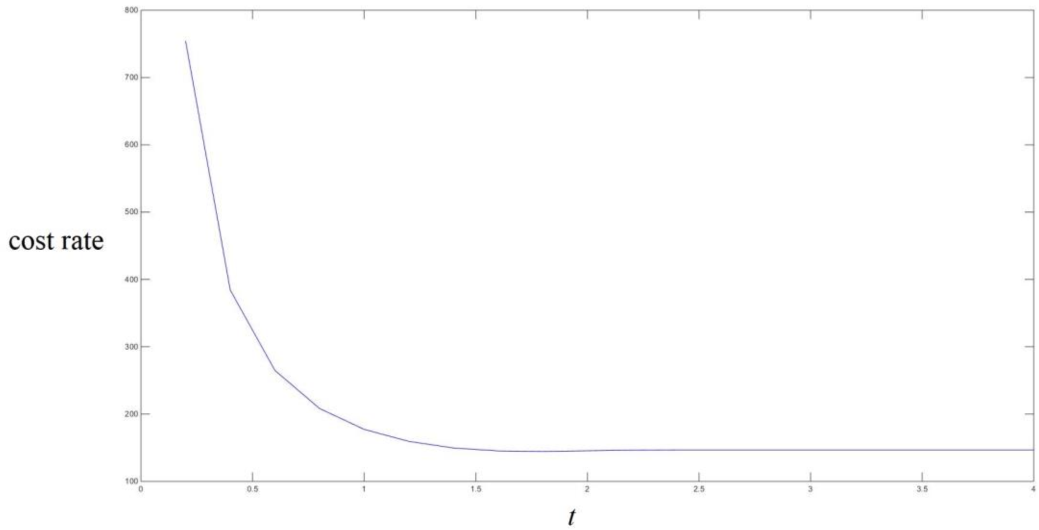

Figure 11 – Cost rate vs. Time of the process on account of nonlinear degradation with initial degradation U_i and shocks

3.2.3 Cost Rate of the Process on Account of Linear Degradation with Shocks

Now considering a shock arrival process, the method to determine the minimum cost rate is the same as processes without shocks. Shocks arrive as a Poisson process, and each shock has shock damage Y_i, so the total degradation is the sum of gradual degradation, which is gamma distributed, and cumulative shock damage. Generally, $F_T(v)$ can be determined as follows:

$$F_T(v) = \Pr\{T \le v\} = \Pr\{X(v) + \sum_{i=0}^{m} Y_i \ge H^1\} = 1 - \Pr\{X(v) \le H^1 - \sum_{i=0}^{m} Y_i\}$$
$$= 1 - \sum_{m=0}^{\infty} \int_0^{H^1} G\{H^1 - y; \alpha_0 v, \beta\} f_Y^{<m>}(y) dy \times \frac{e^{-\lambda v}(\lambda v)^m}{m!}$$

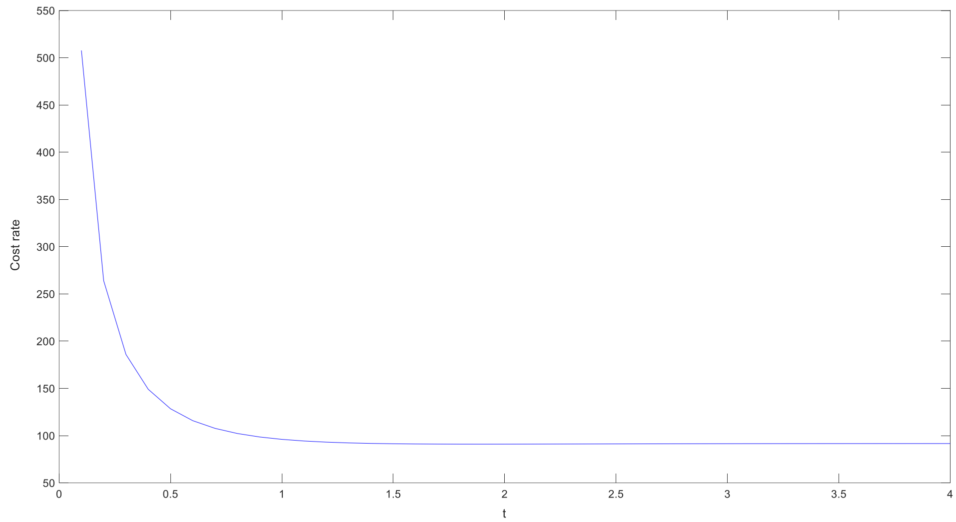

Figure 12 – Cost rate vs. Time of the process on account of linear degradation with shocks

As is shown in cost rate graph in Figure 12, the cost rate drops sharply at the beginning and then goes up to a higher value slowly. Obtained in MATLAB, the minimum cost rate in this case is 91.005737 when time equals to 1.9. Also, this cost rate function does not consider initial degradation U_i.

3.2.4 Cost Rate of the Process on Account of Nonlinear Degradation with Shocks

Changing equations in MATLAB, the graph of cost rate with the parameter of C_F = 100 and C_{PM} = 50 is shown in Figure 13. As it is discussed before, reliability reduces to 0 quickly. So when time goes past the point when cost rate is the smallest, because the difference between C_F and C_{PM} is small, these two parameters, especially C_F, do not have a noticeable effect on cost rate because $F_T(v)$ starts to dominate the trend when times goes by. However, in this case, when time equals to 1.5, the cost rate has a minimum value of 91.665627.

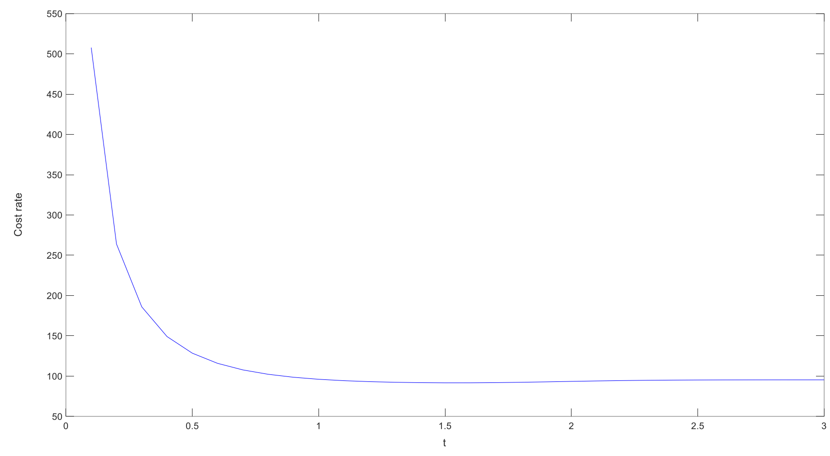

Figure 13 – Cost rate vs. Time of the process on account of nonlinear degradation with shocks

When C_F is changed from 100 to 1000 to observe whether there is a difference, it can be observed in Figure 14, there is a sharp difference. Because C_F is much higher than C_{PM}, when time t passes the time of the lowest cost rate, $C_F \times F_T(v)$ is much larger than $C_{PM} \times (1 - F_T(v))$, and the probability of paying a high expense of penalty goes up. The trend is noticeable after time passing the lowest cost rate. In this case, the lowest cost rate is 441.279974 when time equals to 0.3.

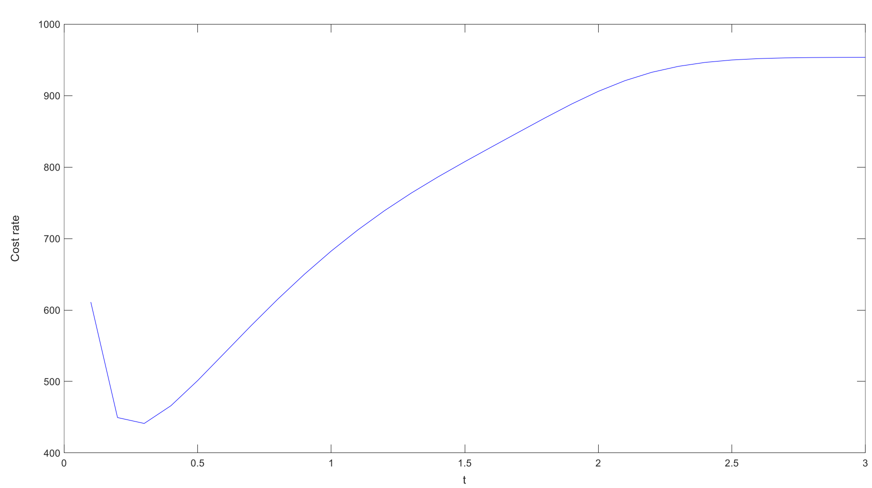

Figure 14 – Cost rate vs. Time if $C_F = 1000$

4 System Reliability Models

In this section, new system level models are proposed and research results are described. Cost rate models have been developed and tested for simple linear and nonlinear degraded systems with and without shocks.

General system models can be divided into two main parts, repairable and non-repairable. For systems such as railroad track systems, gas pipeline systems or many other expensive systems, engineers choose to design them as a repairable system due to the expensive penalty cost of discarding the whole system. This kind of system has the individually repairable components often connected in series. Each failure of the component causes the failure of the whole system. Consider one of the railway tracks is broken or one of the pipes is shut down due to a failure, no trains are allowed to run on it and no gas should be carried via that pipeline system.

This section mainly discusses system reliability models and the reliability of the component degradation process models and focuses on one specific cost rate as an example. For the situations where the degradation is linear or not, and whether shocks are coming from the external environment, different reliability functions are constructed. To fully introduce the distribution for the initial degradation, two simulated experiments are operated including process with and without shocks.

For analyzing the system performance with multiple components, the most important factor that every researcher would not ignore is reliability. Based on the constructed model, the reliability functions are different with each other due to the different conditions considered in the model. In this part, the research plan focuses on the model specifically by adding different conditions one by one into the system. The first condition is whether this process can be modeled as an expected linear degradation process. The second condition is whether the degradation processes can be modeled with or without shocks. Resulting degradation processes have different reliability functions.

4.1 Reliability and Cost Rate of Expected Linear Degraded Gamma Process

4.1.1 Linear Degradation without Shocks

4.1.1.1 Linear Degradation without Shocks Regardless of On-condition Threshold

The first model considered is for a series system without shocks. All components have linear expected degradation paths. Consider a system with five individually repairable components. When the system has been operating for a long time, as described in Section 3, the system is in steady state and each component has its own initial degradation at the beginning of a time interval. Different components have different initial degradation U_i, so the initial degradation of component 1 to n at time $k\tau$ is U_1 to U_n. U_i follows some distribution $f_{Ui}(u)$, supported by the simulation result, a uniform distribution. Thus, the initial degradation U_i is added into the system reliability function.

Equations below show the definitions and the relationships among them, for independent $X_i(t)$ and U_i. Because the beginning time of the system is not 0 but some time point in steady state, this reliability function actually represents the probability of system survival during time interval τ. It is called conditional reliability in this research.

$$U_1 \sim Uniform(0, H_1^1), \ldots, U_n \sim Uniform(0, H_n^1), \quad f_{U_i}(u) = \frac{1}{H_i^1 - 0}, \text{ for } 0 \le u_i \le H_i^1$$

$$
\begin{aligned}
R(\tau) &= \Pr\{U_1 + X_1(\tau) < H_1^1 \cap \ldots \cap U_n + X_n(\tau) < H_n^1\} \\
&= \Pr\{X_1(\tau) < H_1^1 - U_1\} \cap \ldots \cap \Pr\{X_n(\tau) < H_n^1 - U_n\} \\
&= \int_0^{H_1^1} \Pr\{X_1(\tau) < H_1^1 - u_1\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^1} \Pr\{X_n(\tau) < H_n^1 - u_n\} f_{U_n}(u)du \\
&= \int_0^{H_1^1} G\{H_1^1 - u_1; \alpha_0\tau, \beta\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^1} G\{H_n^1 - u_n; \alpha_0\tau, \beta\} f_{U_n}(u)du \\
&= \prod_{i=1}^{n} \int_0^{H_i^1} G\{H_i^1 - u_i; \alpha_0\tau, \beta\} f_{U_i}(u)du
\end{aligned}
$$

$G\{\cdot\}$ is the cumulative distribution function of a gamma distribution $g(\cdot)$. The initial shape parameter and scale parameter are the same with Section 3 that $\alpha_0 = 4$ and $\beta = 1$. Degradation $X(t)$ at time t follows the gamma process, which $\alpha(t)$ depends on the time.

$$X(t_1) \sim g(x; \alpha(t_1), \beta), \; X(t_2) \sim g(x; \alpha(t_2), \beta),$$

$$X(t_2) - X(t_1) \sim g(x; \alpha(t_2) - \alpha(t_1), \beta) \text{ with } \alpha(t_2) - \alpha(t_1) = \Delta\alpha(t) = \alpha_0 \times \Delta t = \alpha_0 \tau$$

The total degradation is the cumulative value of $X(t)$, which is required to be less than the failure threshold = 30, and the time interval τ equals to 0.5. Figure 15 shows 5 simulated component degradation paths.

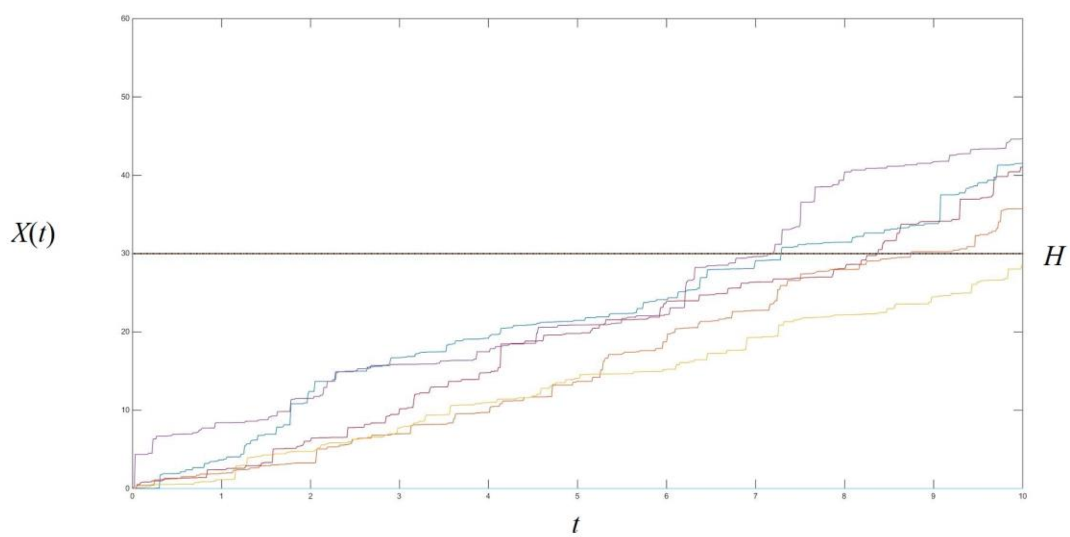

Figure 15 – Gamma process on account of linear degradation without shocks

The probability of surviving without failure, $R(\tau)$ is a conditional reliability. From the MATLAB results, it can be concluded that $R(0.5) = 0.8333$ for 1 component and $R_s(0.5) = 0.4019$ for 5 components in series.

For the cost rate in each time interval, it is the value of the total expenses divided by time interval duration τ. So the cost rate function depends on time interval τ. Different time intervals have different cost rate values. The minimum cost rate can be obtained by calculating the cost rate function and evaluations for different intervals.

As for this situation, there are three expenses which are C_I, cost of inspection, C_R, cost of replacement for a system, and C_ρ, penalty cost per unit idle time. So the total cost equals to the summation of C_I, C_R times the probability of replacement, (1-$R(\tau)$), and C_ρ times the expected idle time in a time interval, $E[\rho]$. Here are the functions for the steady state cost rate.

$$CRT = \frac{C_I + C_R(1 - R(\tau)) + C_\rho E[\rho]}{\tau}$$

$$E[\rho] = \int_0^\tau (\tau - t) f_{T_{H-U}}(t)dt$$

$f_{T_{H-U}}(t)$ is the probability density function representing that the component fails at time t during the time interval τ, given initial degradation U_i. If the component fails during the time interval τ then there is idle time or down time.

$f_{T_{H-U}}(t)$ is deduced in the following steps.

$$
\begin{aligned}
f_{T_{H-U}}(t) &= \frac{d}{dt}F_{T_{H-U}}(t) = \frac{d}{dt}\Pr\{T_{H-U} < t\} = \frac{d}{dt}\Pr\{X(t) > H-U\} \\
&= \frac{d}{dt}\int_0^H \Pr\{X(t) > H-U \mid U = u\} f_U(u)du \\
&= \frac{d}{dt}\int_0^H (1 - F_{X(t)}(H-u)) f_U(u)du \\
&= -\int_0^H \frac{d}{dt}F_{X(t)}(H-u) f_U(u)du
\end{aligned}
$$

$F_{X(t)}(H-u)$ is the cumulative distribution function of a gamma distribution $f_{T_{H-U}}(t)$, and is solved using numerical MATLAB methods. In order to evaluate $f_{T_{H-U}}(t)$, a plot is obtained in the MATLAB, as shown in Figure 16, for typical values (α_0 = 4 and β = 1).

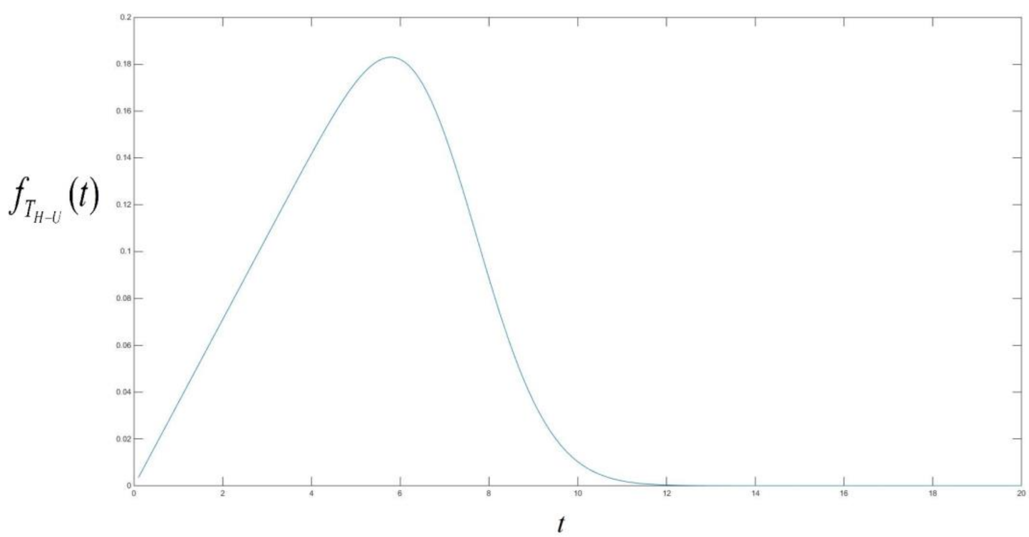

Figure 16 – Probability density function of component fails during a time interval τ

In order to simplify the calculating process and get results more quickly, $E[\rho]$ is deducted in the following steps:

$$
\begin{aligned}
E[\rho] &= \int_0^{\tau} (\tau - t)\, f_{T_{H-U}}(t)dt \\
&= \int_0^{\tau} \tau \cdot f_{T_{H-U}}(t)dt - \int_0^{\tau} t \cdot f_{T_{H-U}}(t)dt \\
&= \tau \cdot \int_0^{\tau} f_{T_{H-U}}(t)dt - \int_0^{\tau} t dF_{T_{H-U}}(t) \\
&= \tau \cdot F_{T_{H-U}}(t)\,|_0^{\tau} - (t \cdot F_{T_{H-U}}(t)\,|_0^{\tau} - \int_0^{\tau} F_{T_{H-U}}(t)dt) \\
&= \tau \cdot F_{T_{H-U}}(\tau) - \tau \cdot F_{T_{H-U}}(0) - \tau \cdot F_{T_{H-U}}(\tau) + 0 \cdot F_{T_{H-U}}(0) + \int_0^{\tau} F_{T_{H-U}}(t)dt \\
&= \tau \cdot F_{T_{H-U}}(\tau) - \tau \cdot 0 - \tau \cdot F_{T_{H-U}}(\tau) - 0 + \int_0^{\tau} F_{T_{H-U}}(t)dt \\
&= \int_0^{\tau} F_{T_{H-U}}(t)dt
\end{aligned}
$$

In this case, $F_{T_{H-U}}(t)$ represents the probability that the component with initial degradation u is failed at time t which is the same as one minus the probability that the degradation path does not pass the threshold H:

$$
\begin{aligned}
F_{T_{H-U}}(t) &= 1 - R_{T_{H-U}}(t) \\
&= \Pr\{T_{H-U} > t\} \\
&= \Pr\{X(t) < H - U\}
\end{aligned}
$$

$$
E[\rho] = \int_0^{\tau} (1 - R_{T_{H-U}}(t))dt
$$

Considering example parameter C_I = 100, C_R = 200 and C_ρ = 200, and Figure 17 shows the trend of cost rate. The minimum cost rate arrives at approximately 3.8. To observe how C_I and C_ρ affect the total cost rate, they are varied to obtain Figures 18 and 19, which show C_I and C_ρ effects on cost rate.

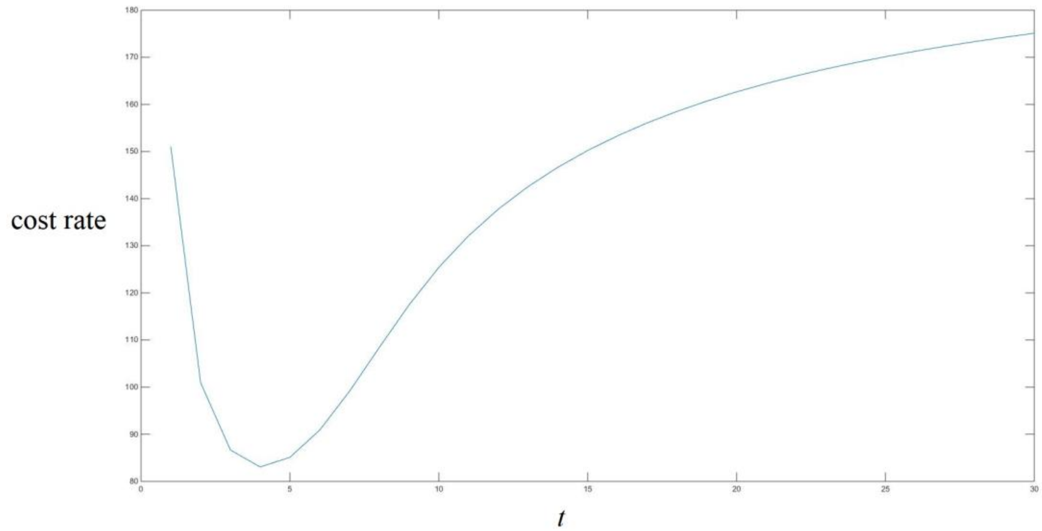

Figure 17 – Cost rate considering C_I = 100, C_R =200 and C_ρ = 200

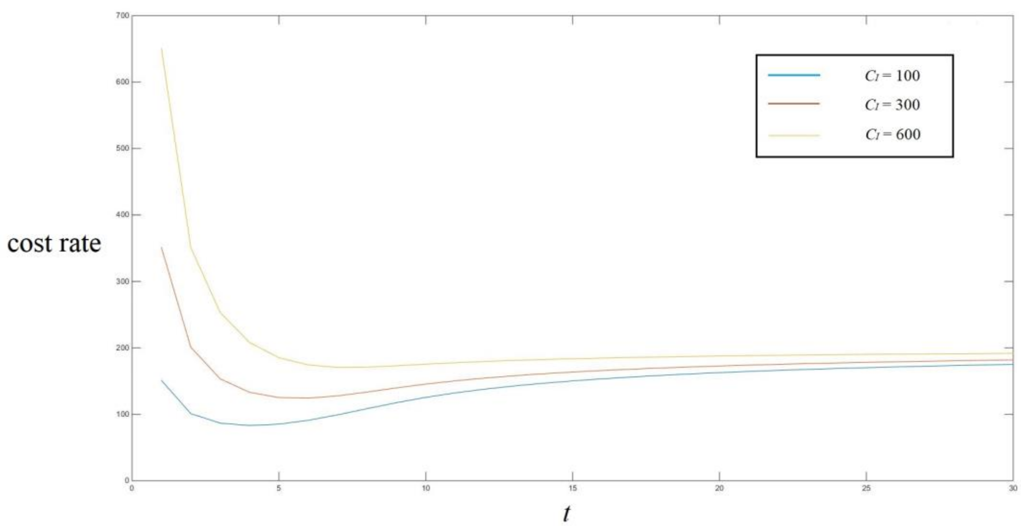

Figure 18 – Cost rate comparisons when C_I changing

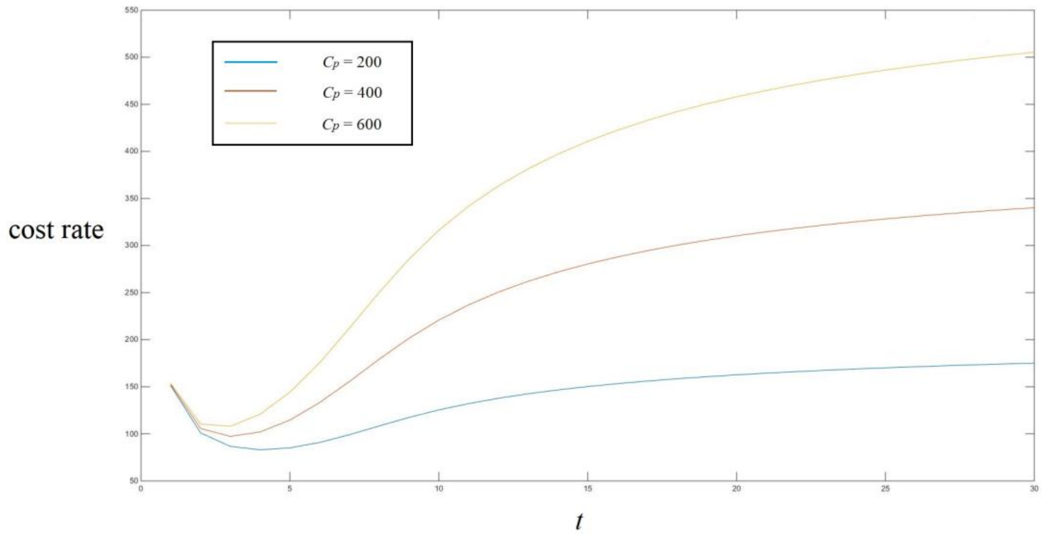

Figure 19 – Cost rate comparisons when C_ρ changing

From the graphs, C_I varies from 100 to 600 while C_ρ varies from 200 to 600. The trends reflect that both C_I and C_ρ lead to a larger cost rate if they become larger. The larger C_I makes the lowest cost rate come later. Engineers choose to inspect later because the cost of inspection is expensive. On the contrast, the larger C_ρ makes the smallest cost rate come earlier. In this situation, low cost of inspection makes engineers decide to inspect more frequently to replace the failed component for the purpose of avoiding a system failure and lower the expected idle time.

4.1.1.2 Linear Degradation without Shocks Considering On-condition Threshold

As for linear degradation without shocks, the previous cases show the policy of how to obtain the optimum minimum cost rate. In those cases, there is only one threshold which determine the failure of the component. If any of the component's degradation passes the threshold, it is considered as a failure and replaced by a new one. Differently, in many real situations in both factories and industrial fields, engineers choose to have a previously determined threshold for each component in order to avoid a failure or protect an important system. Imagine a car tire should be replaced when wear reaches an unacceptable level. There would be a serious damage towards the car and passengers if the tire is not replaced. Engineers choose to replace the component if the degradation passes the on-condition threshold H^2 during the inspection action. Therefore, in this case, idle time can only exist if engineers find that the component or the system has failed (degradation path passes the failure threshold H^1) during the inspection action.

Considering initial degradation U_i for the conditional reliability function; it still follows a uniform distribution observed from the simulation done in MATLAB. Because any component is replaced if its degradation path passes the on-condition threshold, the range for initial degradation is between 0 and H^2. Hence, there are two conditioned reliabilities to compute the cost rate function, the probability of not replacing a component and the probability of no failure. Adding initial degradation distribution and on-condition threshold to these two conditional reliability functions, the following shows the final equations in this case. $R_r(\tau)$ represents the probability of no replacement and $R_f(\tau)$ represents the probability of no failure:

$$U_1 \sim Uniform(0, H_1^2),\ \ldots,\ U_n \sim Uniform(0, H_n^2),\ \ f_{U_i}(u) = \frac{1}{H_i^2 - 0},\ \text{for}\ 0 \le u_i \le H_i^2$$

$$\begin{aligned}
R_r(\tau) &= \Pr\{U_1 + X_1(\tau) < H_1^2 \cap \ldots \cap U_n + X_n(\tau) < H_n^2\} \\
&= \Pr\{X_1(\tau) < H_1^2 - U_1\} \cap \ldots \cap \Pr\{X_n(\tau) < H_n^2 - U_n\} \\
&= \int_0^{H_1^2} \Pr\{X_1(\tau) < H_1^2 - u_1\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^2} \Pr\{X_n(\tau) < H_n^2 - u_n\} f_{U_n}(u)du \\
&= \int_0^{H_1^2} G\{H_1^2 - u_1; \alpha_0\tau, \beta\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^2} G\{H_n^2 - u_n; \alpha_0\tau, \beta\} f_{U_n}(u)du \\
&= \prod_{i=1}^{n} \int_0^{H_i^2} G\{H_i^2 - u_i; \alpha_0\tau, \beta\} f_{U_i}(u)du
\end{aligned}$$

$$
\begin{aligned}
R_f(\tau) &= \Pr\{U_1 + X_1(\tau) < H_1^1 \cap \ldots \cap U_n + X_n(\tau) < H_n^1\} \\
&= \Pr\{X_1(\tau) < H_1^1 - U_1\} \cap \ldots \cap \Pr\{X_n(\tau) < H_n^1 - U_n\} \\
&= \int_0^{H_1^2} \Pr\{X_1(\tau) < H_1^1 - u_1\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^2} \Pr\{X_n(\tau) < H_n^1 - u_n\} f_{U_n}(u)du \\
&= \int_0^{H_1^2} G\{H_1^1 - u_1; \alpha_0\tau, \beta\} f_{U_1}(u)du \times \ldots \times \int_0^{H_n^{21}} G\{H_n^1 - u_n; \alpha_0\tau, \beta\} f_{U_n}(u)du \\
&= \prod_{i=1}^{n} \int_0^{H_i^2} G\{H_i^1 - u_i; \alpha_0\tau, \beta\} f_{U_i}(u)du
\end{aligned}
$$

As before, $G\{\cdot\}$ is the cumulative function of gamma distribution. As a numerical example, parameters for the gamma process are the same: $\alpha_0 = 4$ and $\beta = 1$. Expenses which are C_I, cost of inspection and C_ρ, penalty cost per unit idle time are the same as the previous case except C_{R_i}, cost of replacement of component i. To calculate the expected idle time, $R_f(\tau)$ is the critical value which needs to be taken into the function. Here are the functions for the expected idle time $E[\rho]$ and steady state cost rate CRT.

$$CRT = \frac{C_I + \sum_{i=1}^{n} C_{R_i}(1 - R_{r_i}(\tau)) + C_\rho E[\rho]}{\tau} \text{ where:}$$

$$E[\rho] = \int_0^{\tau} (1 - R_f(t))dt$$

Considering a system consisting of 4 components in series, $C_I = 100$, $C_{R_i} = 200$ for every component, $C_\rho = 200$. C_I and C_ρ are the same for each individual component because they are for system level. Here is a numerical example shown as Figure 20 for that system. Each line represents different maintenance policy which has different on-condition threshold H^2.

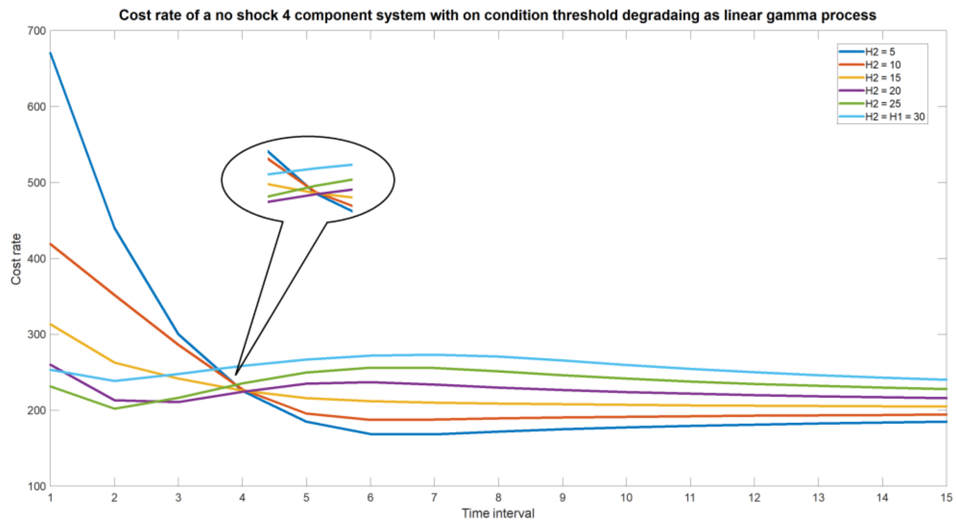

Figure 20 – Cost rate of a no shock 4 component system with on-condition threshold

From Figure 20, it is easy to observe many interesting results. When time interval between inspections is short, the probability of replacement is close to 0 and the expected idle time is very short. The cost of inspection is expensive so it is wise to higher the on-condition threshold to enlarge the lifetime of the component to avoid a frequently component replacement and its expense. Thus, the cost rate of maintaining systems with on-condition thresholds H^2 equal to 25, 30 and 20 are three lowest. When inspection interval is approximately 2, the optimum maintenance on-condition threshold is 25 and the minimum cost rate is 200. If time interval between two inspections goes higher, the cost of inspection no longer dominates the total cost rate. First, the policy with on-condition threshold H^2 equaling 20 is the best. At 3 unit time interval, the minimum cost rate is approximately 210 with maintenance policy which on-condition threshold equals to 20. Eventually, the policy with the lowest on-condition threshold, 5, wins the competition. This is because engineers choose to replace components more frequently in order to make sure conditional reliability is high, which means the probability of replacement is low and the expected idle time is short. Overall, when inspection interval is fixed, a different maintenance policy should be taken. When τ is larger (like 6, 7 or 8), small on-condition thresholds should be the maintenance policy. When τ is small, large on-condition thresholds are suggested.

Considering different situations in real world applications, sometimes the cost of replacement for one component is much higher while the cost of idle time penalty is higher at some other time. Changing C_{R_i} and C_ρ in two different ways, the optimum maintenance policies are different. Figure 21 shows the trend of cost rate if C_{R_i} increases generally.

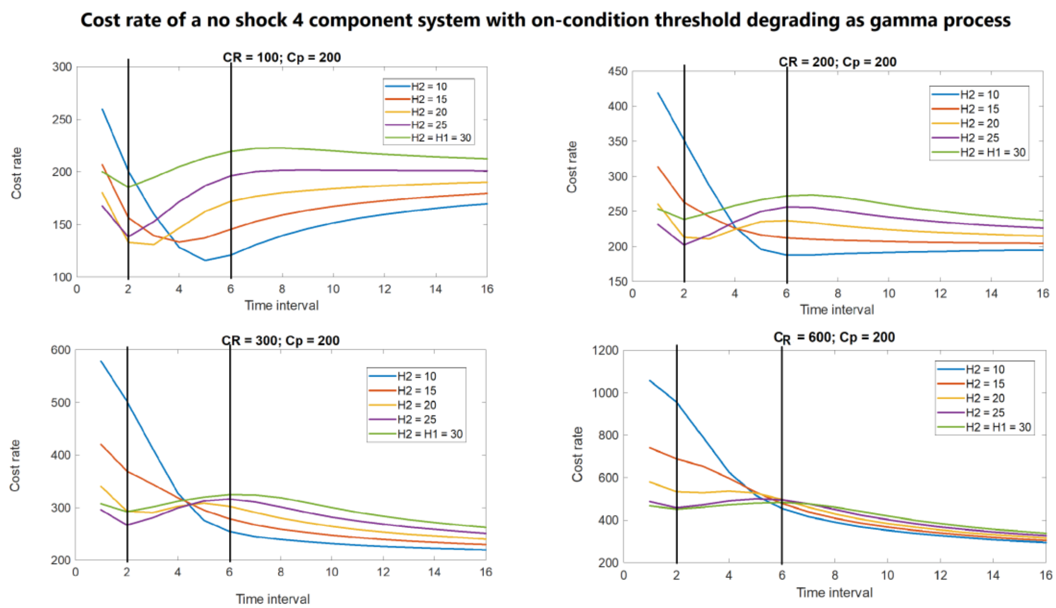

Figure 21 – Cost rate of a no shock 4 component system with on-condition threshold when C_R increasing

As C_{R_i} is increasing generally, the cost rate of each maintenance policy converges closely to each other. For the reason that C_{R_i} is higher than C_ρ, the probability of replacement is significant in the cost rate function at the beginning, which means higher maintenance threshold H^2 has lower cost rate. When inspection interval τ gets longer, every maintenance policy has the probability of replacement which is close to 1. Then the expected idle time dominates the cost rate function. When inspection time interval is fixed, for example, 2 or 6 unit time interval, the optimum on-condition threshold is different. Influenced by cost of replacement, the optimum maintenance policy varies. If C_{R_i} = 100, C_ρ = 200 and τ = 2, the optimum on-condition threshold equals 20. As inspection time interval is increasing (like 6), the optimum on-condition threshold equals 10. There are 4 different situations in Figure 21. Based on multiple combinations of inspection time interval and cost of replacement, the optimum on-condition threshold is different. Taking a specific situation, when C_{R_i} = 2000 and C_ρ = 10, a more specific analysis is done.

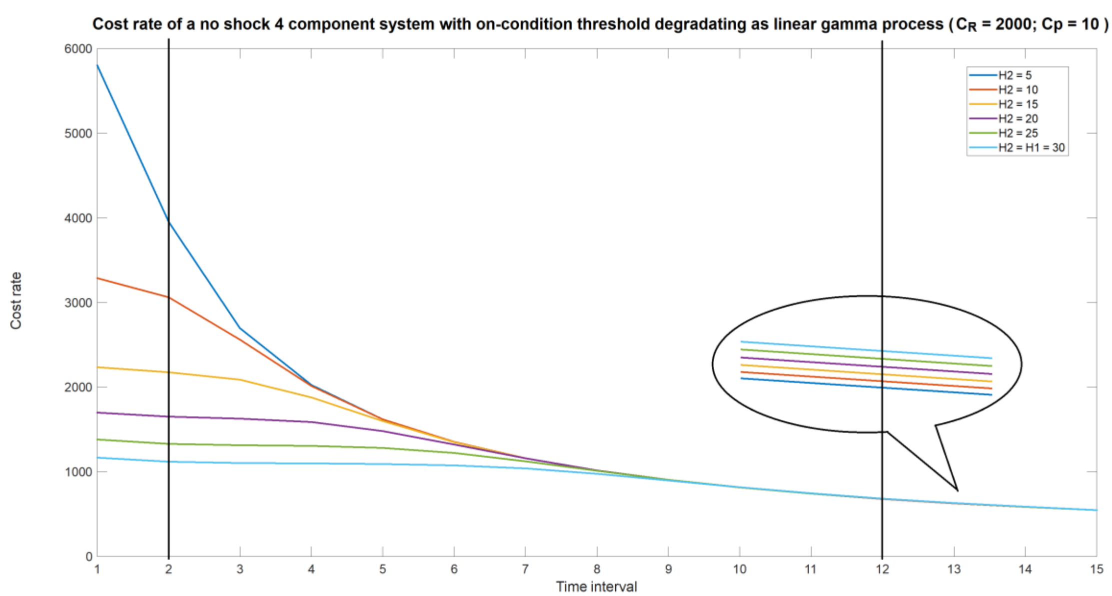

Figure 22 – Cost rate of a no shock 4 component system with on-condition threshold when C_R is high

With extremely high cost of replacement, the key to reduce cost rate is to lower the probability of replacement. Therefore, if the on-condition threshold is higher, the probability of replacement is lower. As it is shown in Figure 22, at the very beginning, the cost rate is low if the on-condition threshold is high. When the time interval becomes larger, the probability of replacement for each component gets close to 1. Cost of inspection and cost of replacement become almost certain values and divided by time interval. The only factor that influences the cost rate is the expected idle time. Maintenance policies with small on-condition thresholds force engineers to replace components more frequently to make the expected idle time shorter. Therefore, smaller on-condition threshold leads to a lower cost rate at the end, as shown in Figure 22. When the time interval goes to infinity, there is no optimum minimum cost rate. This is because the probability of replacement is close to 1 and the expected idle time gets close to τ. Therefore, the cost rate function converges to:

$$\lim_{\tau \to \infty} CRT = \frac{C_I + C_R}{\tau} + C_\rho$$

When inspection time interval τ becomes longer and longer, the cost rate starts to decrease monotonically. In practice, this is undesirable and impractical because the interval between engineers checking a system will never be that long. For fixed inspection time interval, there are different corresponding optimum maintenance policies. If $\tau = 2$, the optimum on-condition threshold is 30. If $\tau = 12$, the optimum on-condition threshold is 5.

For the purpose of comparing the situation when increasing C_{R_i} with the situation when deceasing C_ρ, cost rate functions are shown in Figure 23. In this figure, C_ρ is decreasing in order for C_{R_i} to influence the function.

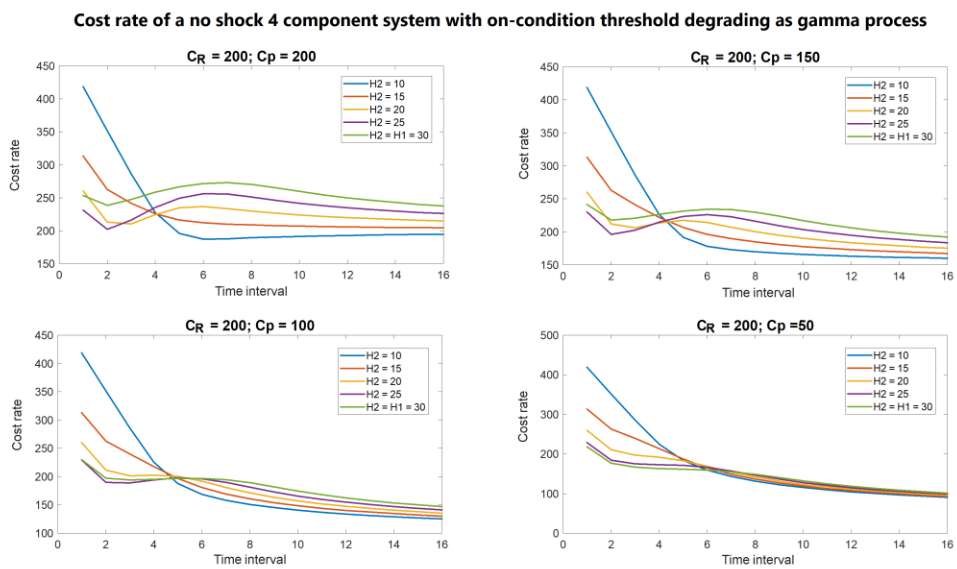

Figure 23 – Cost rate of a no shock 4 component system with on-condition threshold when C_ρ is decreasing

The cost rate trend is almost the same as increasing C_{R_i} when decreasing C_ρ. At first, smaller on-condition threshold has a higher cost rate. However, maintenance policy with larger on-condition threshold tend to have a higher cost rate. Thus, in situations when C_{R_i} is much larger than C_ρ, it is wise to set a large on-condition threshold if inspection time interval is short, while a small on-condition threshold is better when the inspection time interval is long.

However, for many real systems, the cost of expected idle time is much higher than the cost of replacement. Consider a situation when a train transit system is down, or a gas pipeline system is broken, or an electric transmission line is shut down. Hundreds or thousands of people and companies are affected. This kind of failure spawns a great number of economic loss both in direct or indirect ways. In general, this kind of system is extremely important in society that people, companies and governments deeply depend on. Once the system is down, the cost of replacement is far lower than the cost of expected idle time. For situations like these, it is practical to assume C_ρ is much higher than the cost of replacement, C_{R_i} . Figure 24 shows the cost rate function based on that assumption.

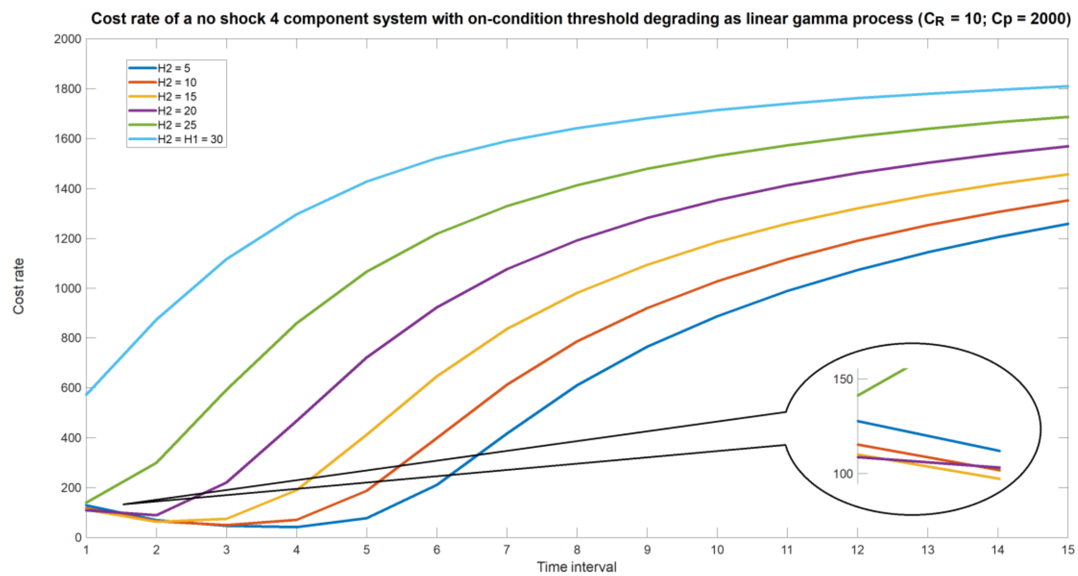

Figure 24 – Cost rate of a no shock 4 component system with on-condition threshold when C_ρ is high

Figure 24 shows an alternative situation. When C_ρ is far larger than the cost of replacement, the expected idle time plays an important role in the cost rate function. As inspection time interval is small, the situation is complicated due to the cost of inspection. However, smaller on-condition thresholds tend to have lower cost rate because the corresponding systems tend to have little chance to have idle time. When the time interval becomes larger, since the probability of replacement approaches 1, the main factor that still influences the cost rate is the expected idle time. Therefore, a maintenance policy with small on-condition threshold is better. The global optimum minimum cost rate is approximately 50 when inspection time interval approximately equal to 4 and on-condition threshold equal to 5.

Meanwhile, in real situations, the cost of inspection and expected idle time focus on the whole system. However, the cost of replacement focuses on individual components in this models. Figure 25 shows the cost rate function with different replacement expenses.

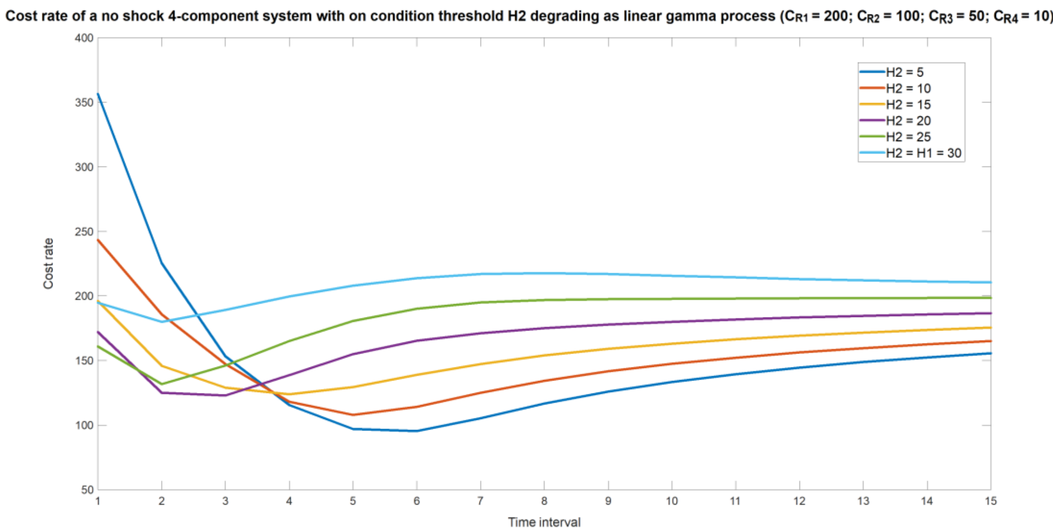

Figure 25 – Cost rate of a no shock 4 component system with on-condition threshold having different C_R

Compared with the cost rate function with the same replacement expenses, there are 4 optimum minimum cost rates based on different inspection intervals and on-condition thresholds. From Figure 25, if inspection interval τ is less than approximately 1.5 unit time interval, the optimum on-condition threshold is 25. If inspection interval τ is between approximately 1.5 and 3.3 unit time interval, the optimum on-condition threshold is 20. When the inspection interval τ is between approximately 3.3 and 3.7 unit time interval, the optimum on-condition threshold changes to 15. Finally, with inspection interval τ getting longer, the optimum on-condition threshold becomes lower and 5 is the best choice with the overall optimum minimum cost rate which is approximately 80. However, in real factories or industries, inspection interval is often fixed, based on practical considerations. Based on that truth, choosing an optimum on-condition threshold to make cost rate low is what reliability engineers need to decide.

Many factories or companies have their own determined inspection time intervals, with some helpful previous engineering experiences or standard rules. When inspection time interval is fixed, on-condition threshold becomes the only factor in maintenance policy. Figure 26 shows 4 plots of the trend of cost rate function if inspection interval is fixed. When inspection time interval is short, like shown in subplot 1, the optimum on-condition threshold is approximately 24.4. If inspection time interval equals 3 unit time interval, the optimum on-condition threshold is approximately 16. When inspection time interval gets longer, the situation becomes more complex. The optimum on-condition threshold is the lowest one when inspection time interval is 6 while the optimum on-condition threshold is the highest one when inspection time interval equals 9. Based on different situations, different inspection time intervals determine the optimum on-condition thresholds.

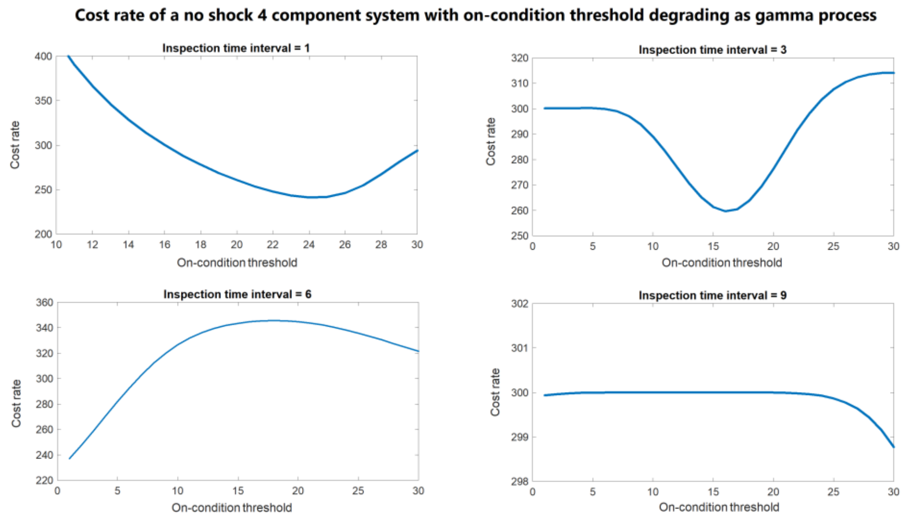

Figure 26 – Cost rate of a no shock 4 component system with on-condition threshold when τ is fixed

In order to select an optimum on-condition threshold based on different inspection time interval, 3-D plots are accomplished in MATLAB. This visualized method is a direct way to find the optimum minimum cost rate. Parameters for this case are all the same as the system in Section 4.1.1.1. The 3-D plot is shown in Figure 27.

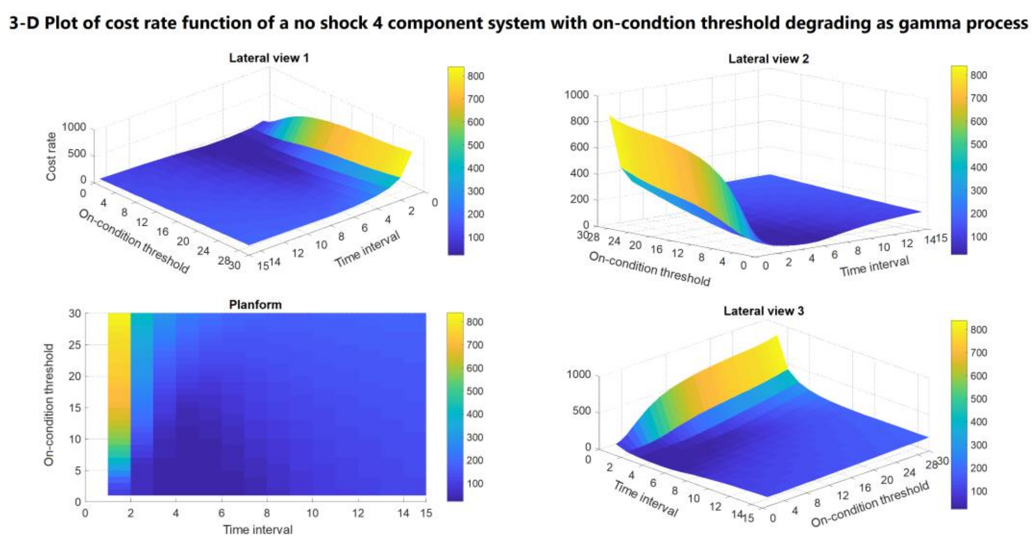

Figure 27 – 3-D plot of cost rate of a no shock 4 component system with on-condition threshold if C_ρ is 200

Actually, this 3-D plot shown in Figure 27 is a specific example of the system model constructed in MATLAB. Based on different situations, parameters should be changed to meet the real requirements. In this case, the optimum minimum cost rate is approximately 150 with the inspection time interval equaling approximately 5 and on-condition threshold equaling approximately 1. To fully have a look at the 3-D plot, a contour plot is shown in Figure 28.

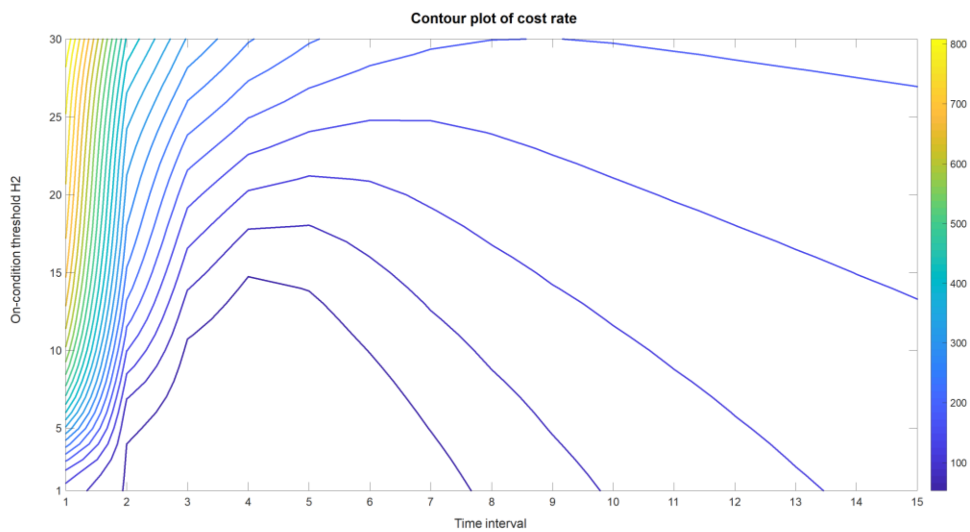

Figure 28 – Contour plot of cost rate if C_ρ is 200

As shown in contour plot, the optimum combination of on-condition threshold and inspection time interval is approximately 1 and 4.5 unit time interval, under the assumption that $C_I = 100$, $C_{R_i} = 200$ and $C_\rho = 200$. When inspection time interval, τ, goes to infinity, the cost rate is going to become smaller and smaller. There will never be an optimum minimum cost rate. However, this situation is uncommon in practice.

Changing C_ρ from 200 to 100, another 3-D plot is presented in Figure 29. Since the units of replacement cost and expected idle time cost are not the same, one is cost per replacement, another is cost per unit time, changing any expenses of the cost rate function leads to a different example plot.

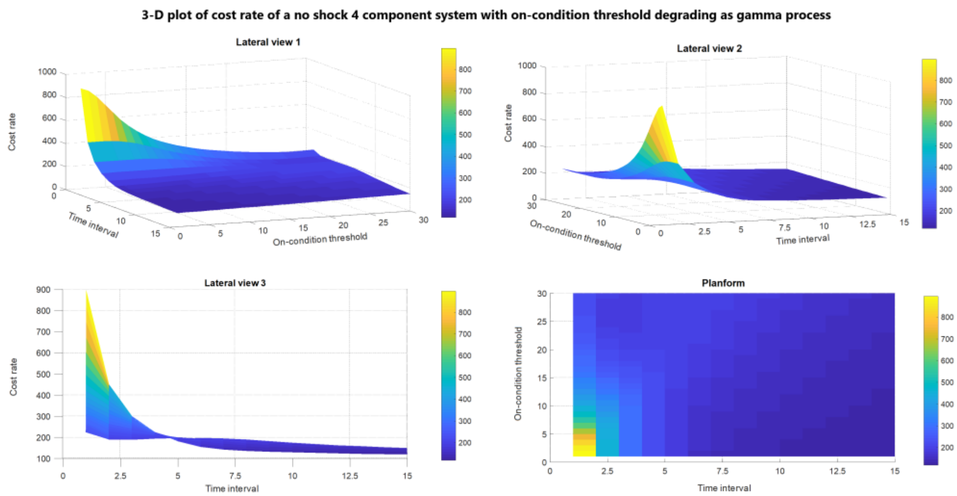

Figure 29 – 3-D plot of cost rate of a no shock 4 component system with on-condition threshold if C_p is 100

According to this 3-D plot in Figure 25, the optimum minimum cost rate is difficult to observe when inspection time interval is short. As discussed before, when time interval goes to infinity, the cost rate decreases. In order to find the optimum minimum cost rate, a contour plot is shown in Figure 30.

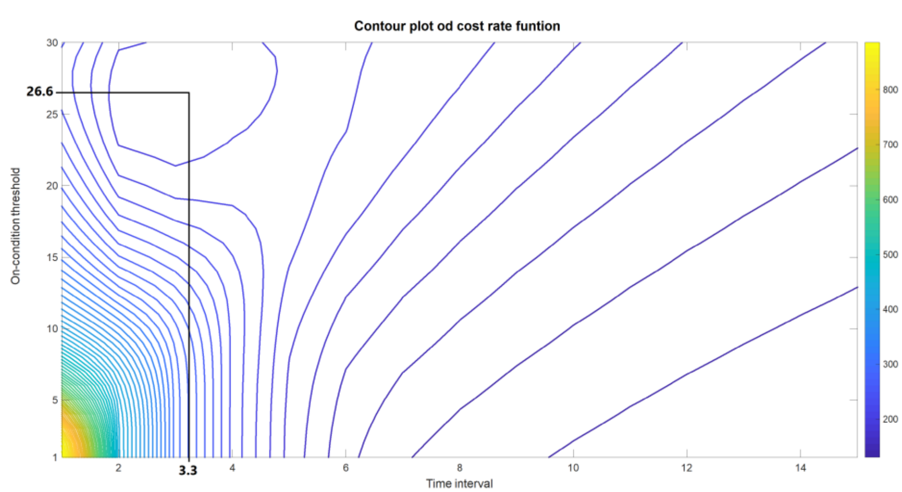

Figure 30 – Contour plot of cost rate if C_p is 100

It is clear that the optimum minimum cost rate is at the center of the circle. To reach this optimum cost rate, the inspection time interval is approximately 3.3 unit time interval and the on-condition threshold is approximately 26.6. When inspection time interval gets longer and longer, as *x*-axis shows, the trend of cost rate monotonically decreases and become smaller than this optimum cost rate.

4.1.2 Linear Degradation with Shocks

4.1.2.1 Linear Degradation with Shocks Regardless of On-condition Threshold

Reliability with shocks is distinctly different from reliability without shocks. Regarding damages or shocks coming from the external environment, they accelerate the degradation process and the whole system fails more quickly. The whole system experiences a shock, so when the system is shocked, all components experienced the shock at the same time. Different environments may have different values of shock sizes and different time intervals between each shock. In general, shock sizes often follow a normal distribution, and shock arrivals occur as a Poisson process.

For the existing failure model, the total degradation is the sum of the gamma process and cumulative shock damages. Considering the value of shocks and the initial gamma process, the function of reliability in this case can be deducted in the following part:

$$
\begin{aligned}
R(\tau) &= \Pr\{U_1 + Y_1 + X_1(\tau) < H_1^1 \cap ... \cap U_n + Y_n + X_n(\tau) < H_n^1\} \\
&= \sum_{m=0}^{\infty} \Pr\{X_1(\tau) < H_1^1 - U_1 - Y_1 \cap \\
&\quad ... \cap X_n(\tau) < H_n^1 - U_n - Y_n \mid N(\tau) = m\} \times \Pr(N(\tau) = m) \\
&= \sum_{m=0}^{\infty} (\int_0^{H_1^1} \int_0^{H_1^1 - u} \Pr\{X_1(\tau) < H_1^1 - u_1 - y\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dydu \times \\
&\quad ... \times \int_0^{H_n^1} \int_0^{H_n^1 - u} \Pr\{X_n(\tau) < H_n^1 - u_n - y\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dydu) \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!} \\
&= \sum_{m=0}^{\infty} (\int_0^{H_1^1} \int_0^{H_1^1 - u} G\{H_1^1 - u_1 - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dydu \times \\
&\quad ... \times \int_0^{H_n^1} \int_0^{H_n^1 - u} G\{H_n^1 - u_n - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dydu) \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!} \\
&= \sum_{m=0}^{\infty} \prod_{i=1}^{k} \int_0^{H_i^1} \int_0^{H_i^1 - u} G\{H_i^1 - u - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{Ui}(u) dydu \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!}
\end{aligned}
$$

$G\{H_i^1 - u - y; \alpha_0\tau, \beta\}$ is the cumulative distribution function of a gamma distribution with parameter $\alpha_0\tau$ and β. Y_i is the total shock damage from *m* shocks for component *i*. $f_{Y_i}^{<m>}(y)$ is the probability density function of the sum of *m* shock damage sizes, which follow normal distributions whose parameter are: mean = μm and variance = $\sigma^2 m$. μ and σ^2 are the initial parameters of a normal distribution for shock damage size while *m* is the number of shocks. $f_{U_i}(u)$ is the probability density function of initial degradation which may have many different distributions. Initially, it is simulated in MATLAB and it can be approximately to follow a generalized linear model:

$$f_{U_i}(u) = \hat{a}u + \hat{b}, \text{ for } 0 \le u \le H_i^1 \text{ (or } H_i^2\text{)}$$

Assuming each component has the same threshold, initial degradation and shock sizes, the graph in Figure 32 shows the trend of the reliability of one component. In this case, it is assumed that shock sizes follow a normal distribution: whose mean, μ, equals 5, and standard deviation, σ, equals 1. The shocks arrive as a Poisson process whose parameter, λ, equals 3. By simulating the degradation in MATLAB, a totally different process is obtained in Figure 31. The first degradation passes the threshold at approximately 2.3, and the last one is about 5.1. Figure 32 shows the reliability graph.

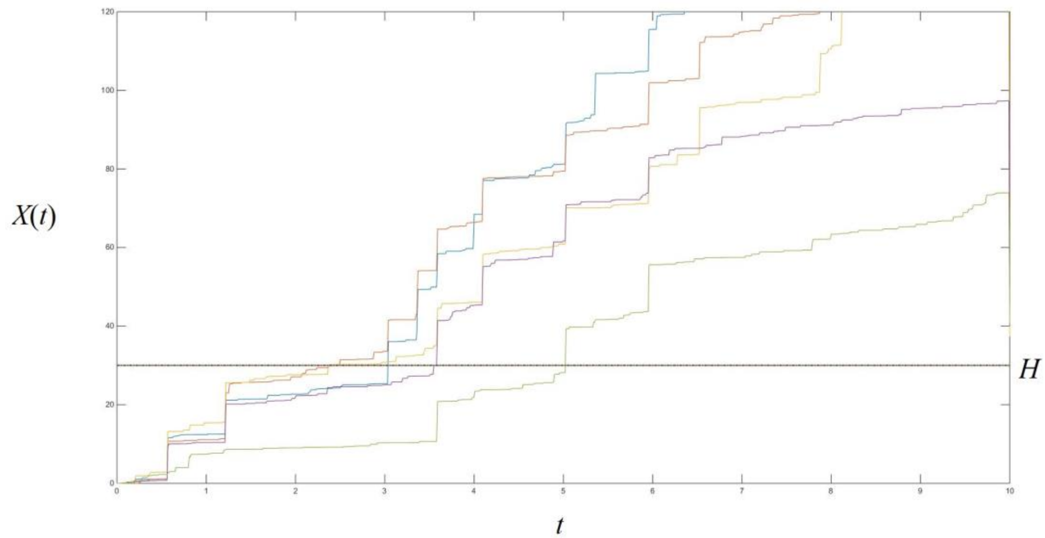

Figure 31 – Gamma process on account of linear degradation with shocks

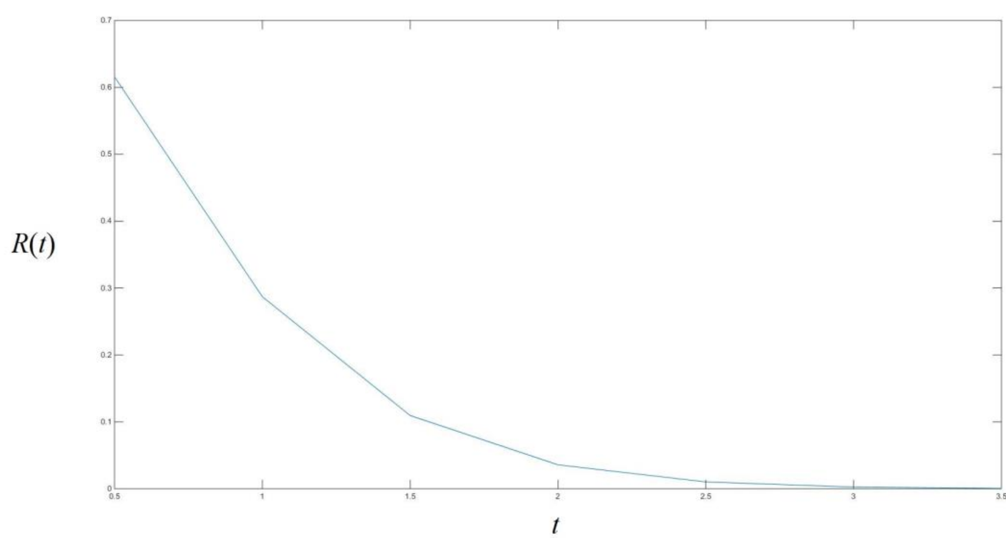

Figure 32 – Reliability of gamma process on account of linear degradation with shocks

Adding shocks to the conditional reliability function, the cost rate function in this case is different from the cost rate function of the system model in Section 4.1.1.1 due to the difference of the probability of no replacement and the expected idle time. Cost rate functions is presented in the following:

$$CRT = \frac{C_I + C_R(1 - R(\tau)) + C_\rho E[\rho]}{\tau}$$

$$E[\rho] = \int_0^\tau (1 - R_{T_{H-U}}(t))dt$$

In this case, the initial parameter for gamma process is: α_0 = 2, β = 1. The mean and the standard deviation for upcoming shock size is: μ = 2, σ = 1, with the Poisson process rate: λ = 1. These parameters were selected for the purpose of increasing the lifetime of the system to fully analyze the cost rate function and avoid a fast failure period compared with the previous system and its parameters. Considering the same example parameters for cost rate function: C_I = 100, C_R = 200 and C_ρ = 200, a cost rate function plot is presented in Figure 33.

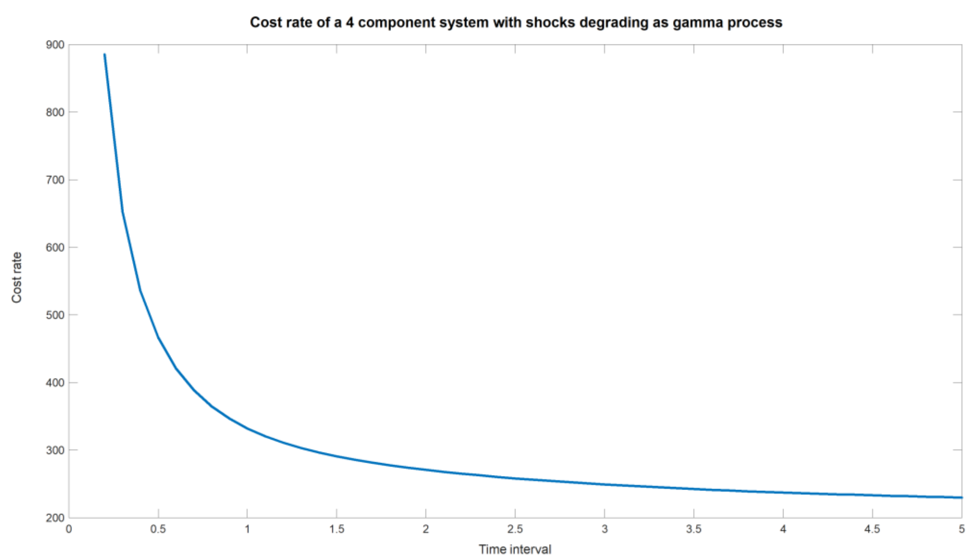

Figure 33 – Cost rate of a 4 component system with shocks if C_R is 200

There is no optimum minimum cost rate in Figure 33. The cost rate goes down when the time interval gets longer. This is because the unit of cost of replacement is cost per replacement while the unit of expected idle time penalty expense is cost per unit time. If the value of C_R is almost the same as the value of C_ρ, when the time interval becomes longer, the total cost rate decreases because C_ρ is not important compared with the summation of the first two expenses in the objective function. Thus, the cost rate decreases as shown in Figure 33. In this case, engineers decide to have a longer component lifetime and not decrease component preventively. In order to make C_ρ significant, another example with parameters: C_I = 100, C_R = 200 and C_ρ = 2000, a cost rate function plot is presented in Figure 34.

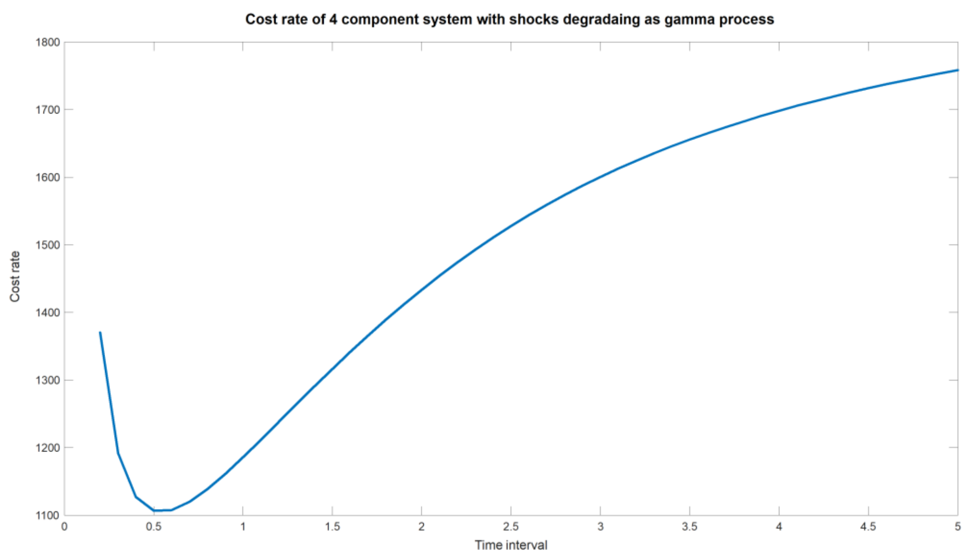

Figure 34 – Cost rate of a 4 component system with shocks if C_R is 2000

The optimum minimum cost rate comes at approximately 0.5 unit time interval, it is about 1110. When C_ρ is large, engineers decide to avoid the expected idle time of the system. Thus, the optimum inspection time interval is very short.

In order to find out how C_I and C_ρ make effect on cost rate function, two sensitivity plots are made in MATLAB. For the purpose of having comparisons, the basic combination of cost parameters is: C_I = 100, C_R = 200 and C_ρ = 2000, which are the same as parameters of cost rate function in Figure 34. One sensitive analysis results from changing C_I from 100 to 700, another one is changing C_ρ from 2000 to 6000.

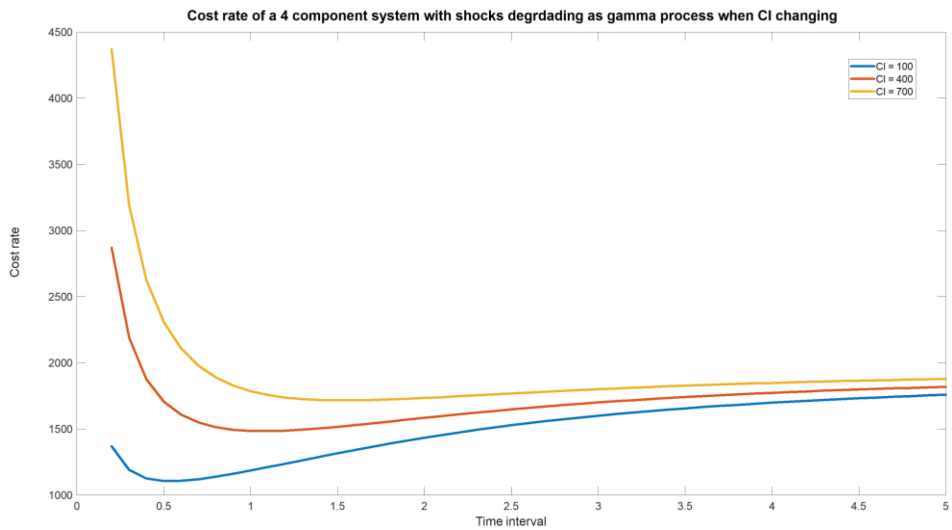

Figure 35 – Cost rate of a 4 component system with shocks when C_I increasing

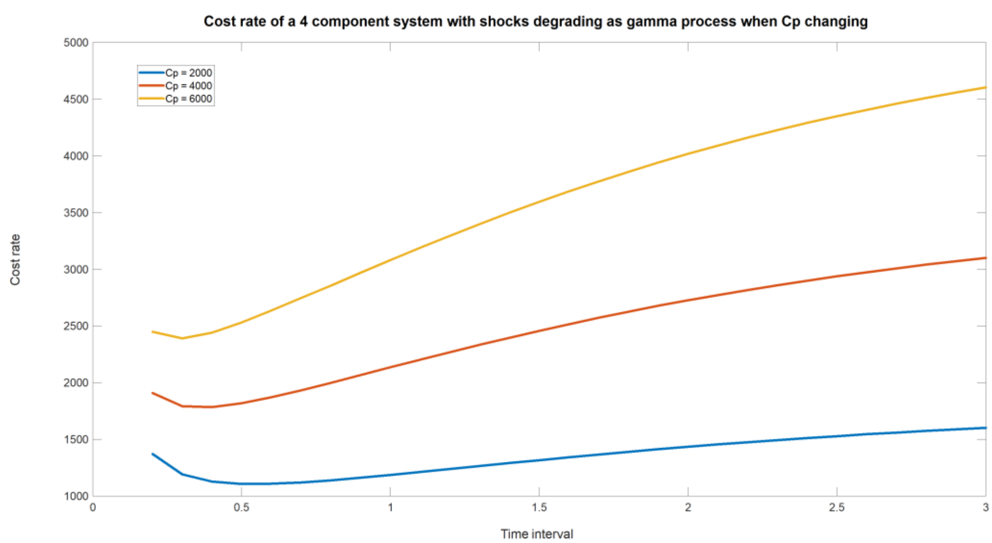

Figure 36 – Cost rate of a 4 component system with shocks when C_ρ increasing

As C_I increasing from 100 to 700, the optimum minimum cost rate comes later and becomes higher. Engineers choose to inspect the system later to lower the cost of inspection. However, in Figure 36, if C_ρ is increasing, the optimum minimum cost rate is earlier. This is because the failed system or components can be replaced more frequently to avoid a high cost of expected idle time. Both sensitive analysis results are similar to the sensitive analysis results completed in Section 4.1.1.1.

4.1.2.2 Linear Degradation with Shocks Considering On-condition Threshold

Section 4.1.2.1 presents a case of linear degradation process with shocks regardless of on-condition threshold. In this section, on-condition thresholds are added to the model. Similarly, parameters for gamma process, shock size and Poisson process are all the same as parameters in Section 4.1.2.1. The cost rate function is distinctly different due to conditional reliability functions and the expected idle time function. The probability of no replacement represents the probability that the degradation path does not pass the on-condition threshold H^2, which means there is no replacement. However, the expected idle time only exists in the situation that the degradation path passes the failure threshold H^1 which means the failed component causes the whole system to fail. The integral limits for initial degradation u is between 0 and H^2 because any components whose degradation paths pass the on-condition threshold is replaced. Based on different situations, the integral limitation for shock size is different. It is between 0 and H^2 for $R_r(\tau)$ while is between 0 and H^1 for $R_f(\tau)$. Here are deductions of conditional reliability functions $R_r(\tau)$ and $R_f(\tau)$.

$$R_r(\tau) = \Pr\{U_1 + Y_1 + X_1(\tau) < H_1^2 \cap \ldots \cap U_n + Y_n + X_n(\tau) < H_n^2\}$$
$$= \sum_{m=0}^{\infty} \Pr\{X_1(\tau) < H_1^2 - U_1 - Y_1 \cap$$
$$\ldots \cap X_n(\tau) < H_n^2 - U_n - Y_n \mid N(\tau) = m\} \times \Pr(N(\tau) = m)$$
$$= \sum_{m=0}^{\infty} (\int_0^{H_1^2} \int_0^{H_1^2 - u} \Pr\{X_1(\tau) < H_1^2 - u_1 - y\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dy du \times$$
$$\ldots \times \int_0^{H_n^2} \int_0^{H_n^2 - u} \Pr\{X_n(\tau) < H_n^2 - u_n - y\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dy du) \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!}$$
$$= \sum_{m=0}^{\infty} (\int_0^{H_1^2} \int_0^{H_1^2 - u} G\{H_1^2 - u_1 - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dy du \times$$
$$\ldots \times \int_0^{H_n^2} \int_0^{H_n^2 - u} G\{H_n^2 - u_n - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dy du) \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!}$$
$$= \sum_{m=0}^{\infty} \prod_{i=1}^{k} \int_0^{H_i^2} \int_0^{H_i^2 - u} G\{H_i^2 - u - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_i}(u) dy du \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!}$$
$$R_f(\tau) = \Pr\{U_1 + Y_1 + X_1(\tau) < H_1^1 \cap \ldots \cap U_n + Y_n + X_n(\tau) < H_n^1\}$$
$$= \sum_{m=0}^{\infty} \Pr\{X_1(\tau) < H_1^1 - U_1 - Y_1 \cap$$
$$\ldots \cap X_n(\tau) < H_n^1 - U_n - Y_n \mid N(\tau) = m\} \times \Pr(N(\tau) = m)$$
$$= \sum_{m=0}^{\infty} (\int_0^{H_1^2} \int_0^{H_1^1 - u} \Pr\{X_1(\tau) < H_1^1 - u_1 - y\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dy du \times$$
$$\ldots \times \int_0^{H_n^2} \int_0^{H_n^1 - u} \Pr\{X_n(\tau) < H_n^1 - u_n - y\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dy du) \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!}$$
$$= \sum_{m=0}^{\infty} (\int_0^{H_1^2} \int_0^{H_1^1 - u} G\{H_1^1 - u_1 - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dy du \times$$
$$\ldots \times \int_0^{H_n^2} \int_0^{H_n^1 - u} G\{H_n^1 - u_n - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dy du) \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!}$$
$$= \sum_{m=0}^{\infty} \prod_{i=1}^{k} \int_0^{H_i^2} \int_0^{H_i^1 - u} G\{H_i^1 - u - y; \alpha_0\tau, \beta\} f_{Y_i}^{<m>}(y) f_{U_i}(u) dy du \times \frac{e^{-\lambda\tau} (\lambda\tau)^m}{m!}$$

As for cost rate functions, the cost of replacement focuses on individual component. Other expenses are the same as previous system model and they focus on the whole system. Following is the cost rate function and when C_I = 100, C_{R_i} = 200 and C_ρ =

200, as shown in Figure 36.

$$CRT = \frac{C_I + \sum_{i=1}^{n} C_{R_i}(1 - R_{r_i}(\tau)) + C_\rho E[\rho]}{\tau}$$

$$E[\rho] = \int_0^\tau (1 - R_f(t))dt$$

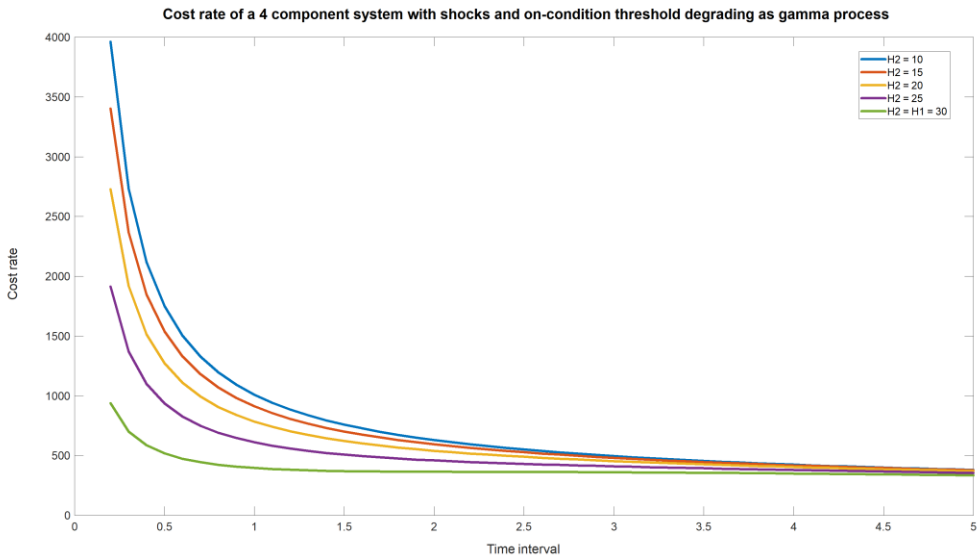

Figure 37 – Cost rate of a 4 component system with shocks and on-condition threshold

In Figure 37, no matter whether inspection time interval is long or short, the optimum maintenance policy is the one whose on-condition threshold is the largest. This is totally different from the maintenance policy for previous system model without shocks. In this case, the global optimum minimum cost rate is approximately 400 with on-condition threshold equaling 30 and inspection time interval equaling approximately 1.75. Due the complexity of conditional reliability functions $R_r(\tau)$ and $R_f(\tau)$, a simulation method for calculating the cost rate function and numerically approximating the integrals is added to the research.

In order to verify the correctness of the numerical method in MATLAB, a simulation method with same parameters is done. The total degradation at time t, $X(t)$, the shock size, Y_i, the initial degradation, U_i, and the number of upcoming shocks, m, are randomly produced in MATLAB using the inverse function of each corresponding cumulative distribution function. Thus, conditional reliability is the number of the summation of these values which is smaller than threshold divided by the total numbers of simulation rounds. Figure 38 shows the plot of cost rate function using simulation method.

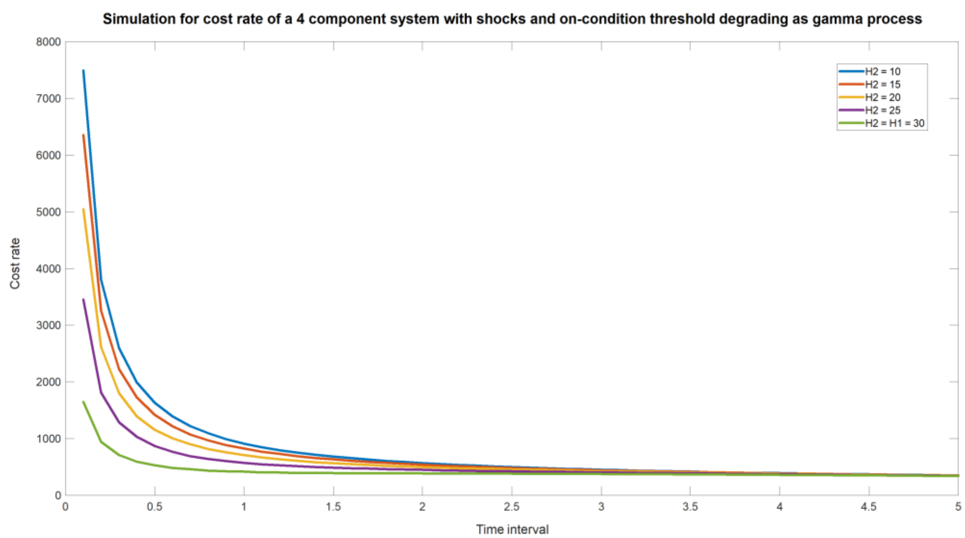

Figure 38 – Simulation for cost rate function

The number of simulation runs is 20,000. The results are similar to the numerical results. However, there is still a little difference between simulation results and numerical results because this kind of error in simulation method cannot be totally eliminated. When the number of simulation runs goes to infinity, the simulation results can be extremely close to the real numerical results. Therefore, with the simulation results, the numerical model in MATLAB can be verified. As shown in Figure 38, the optimum minimum cost rate is approximately 450 with inspection time interval equaling 1.5 unit time interval and on-condition threshold equaling 30. It is a little higher than the optimum minimum cost rate obtained from numerical results. Also, the optimum inspection interval is shorter. The reason for these differences is that the probability of replacement and the probability of failure are not in aggreement which may lead to an error in cost rate function. Thus, numerical results are guaranteed to be correct and numerical calculations are chosen for the following analyses.

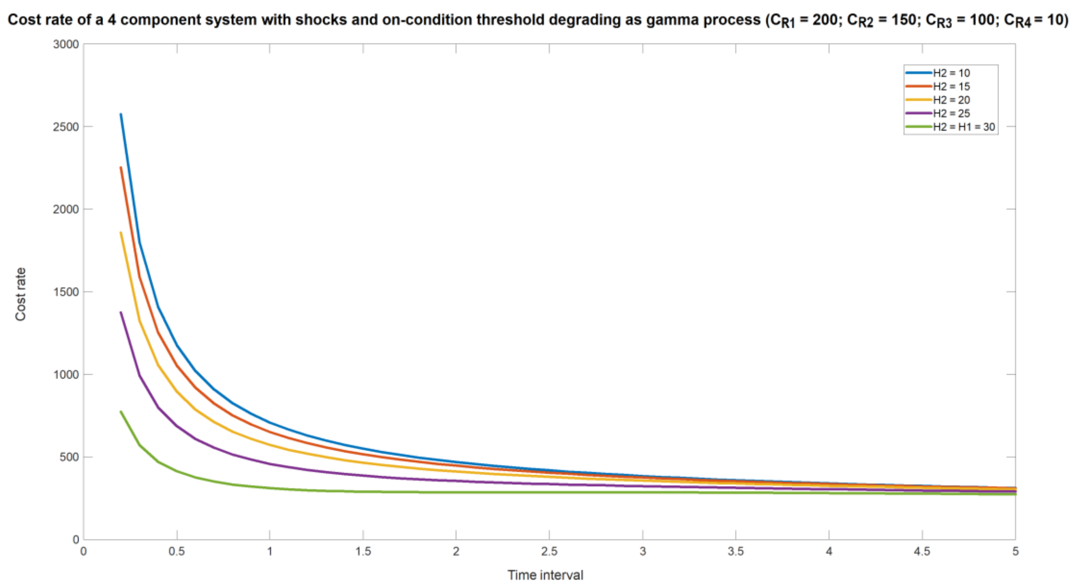

Figure 39 – Cost rate of a 4 component system with shocks and on-condition threshold having different C_R

When components have different cost of replacement, which is common in real situations, the cost rate function is visualized as in Figure 39. As shown in the figure, the optimum minimum cost rate is approximately 200 with on-condition threshold H^2 equaling 30. The best inspection time interval is approximately 1.72 unit time interval. Compared with the system whose components have the same cost of replacement, the optimum on-condition threshold is the same, which is 30. Also, the optimum inspection time interval is almost the same. The only difference between two cost rate models is the value of optimum minimum cost rate. Based on the fact that cost of each replacement is different and is not higher than 200, the total cost rate is a little bit lower.

To deepen the research, more sensitivity analyses are completed using numerical calculations in MATLAB. Based on different purposes, varying the cost of replacement and varying the cost of expected idle time are two main goals in the following steps.

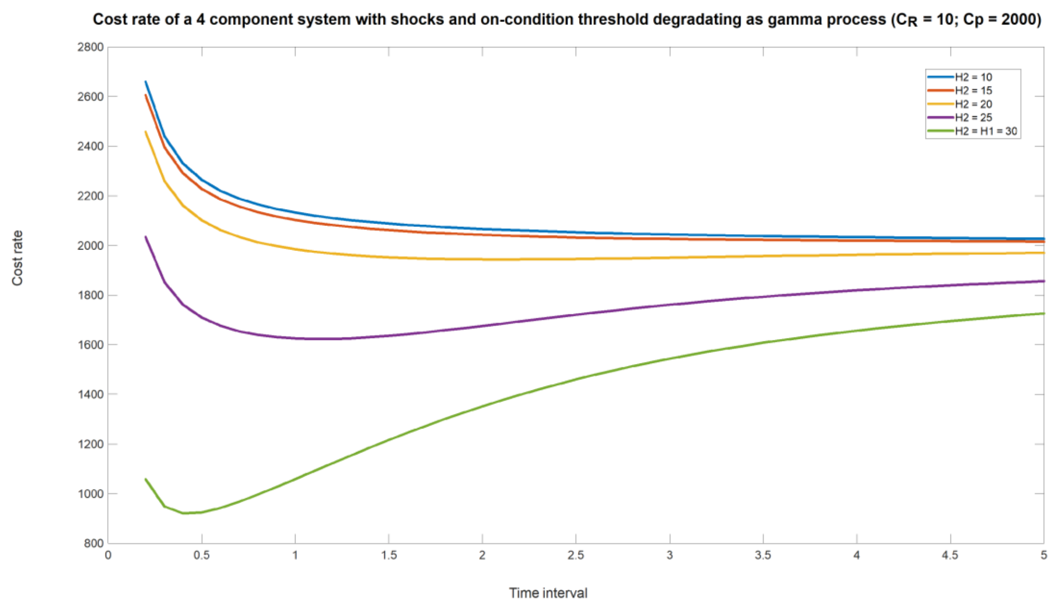

Figure 40 – Cost rate of a 4 component system with shocks and on-condition threshold when C_p is high

Changing cost of expected idle time to an extremely high value, the cost rate trends varies a lot. In this case, the optimum on-condition threshold is 30. The minimum cost rate is approximately 910 with inspection time interval equaling 1.75 unit time interval. Moreover, gaps between each cost rate line becomes wider compared with the previous cost rate plot. Even the cost of expected idle time is extremely high, the highest on-condition threshold gives the system a lowest cost rate, which is on the opposite side of the previous cost rate model in Section 4.1.1.2. This is mainly because the system fails much faster due to external shocks. Also, during a very short inspection time interval, the influence caused by expected idle time is too small to be considered. Thus, in order to lower the cost rate, the most important work is to reduce the probability of replacement. The maintenance policies with higher on-condition thresholds tend to make the system have lower cost rates.

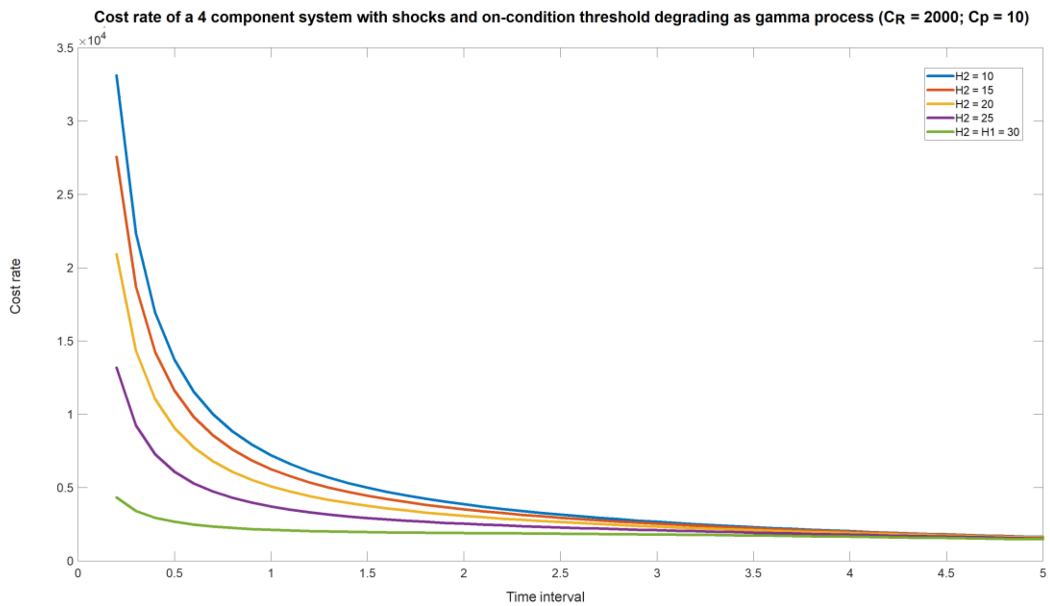

Figure 41 – Cost rate of a 4 component system with shocks and on-condition threshold when C_R is high

On the contrast, if the cost of replacement is changed from 200 to 2,000, the cost rate plot changes, as shown in Figure 41. The optimum cost rate is approximately 2,500 with on-condition threshold equaling 30 and inspection time interval equaling approximately 2. Higher on-condition thresholds have the meaning that engineers choose to fully extend the lifetime of components untill a failure happens. This maintenance policy is to avoid a high cost of replacement and lower the cost rate of the system, which is exactly reflected on Figure 41.

Figure 42 shows the comparison of cost rate function of two system models; linear degrading without shocks and linear degrading with shocks. All costs are the same except the initial shape parameter α_0 and scale parameter β.

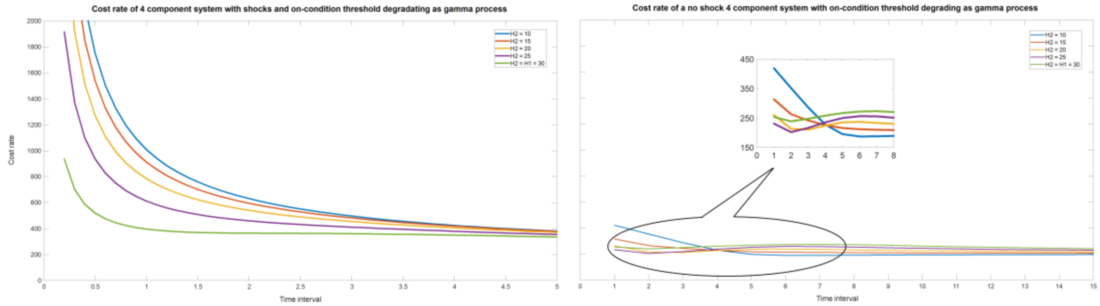

Figure 42 – Comparison of cost rate functions

The difference between two cost rate models are very noticeable. The optimum minimum cost rate of expected linear degraded model with shocks is higher than without shocks. For system models with shocks, for these model parameters, it is wise for engineers to choose a higher on-condition threshold to fully extend the lifetime of components. On the contrast, for system models without shocks, because the degradation path reaches the threshold slowly and the reliability deceases with a low speed, the optimum maintenance policy are more difficult to choose. Based on actual inspection time interval, values of different maintenance expenses and the type of system model, the on-condition threshold varies a lot.

4.2 Reliability and Cost Rate of Expected Nonlinear Degraded Gamma Process

4.2.1 Nonlinear Degradation without Shocks

4.2.1.1 Nonlinear Degradation without Shocks Regardless of On-condition Threshold

For systems degrading as an expected nonlinear gamma process, their degradation processes are different from processes degrading as expected linear gamma process. The parameter $\alpha(t)$ does not linearly depend on time t. It depends on a time power function of t. The whole degradation process of the system has a faster failure rate, for $b > 1$.

For the proposed nonlinear model, $\alpha(t) = \alpha_0 \times t^b$ so $\Delta\alpha(t) = \alpha_0 \times (t_2{}^b - t_1{}^b)$. Therefore, the degradation $X(t_2) - X(t_1)$ is distributed as a more complicated gamma distribution, i.e., $g(x; \alpha(t_2) \text{-} \alpha(t_1), \beta) = g(x; \alpha_0 \times (t_2{}^b - t_1{}^b), \beta)$. In general, because of the power of time, b (b>1), the degradation path becomes steep. As for systems in industries and factories, the probability of products degrading as fast as shown in Figure 43 is unlikely, but possible. This is because all components or systems are well designed with highly reliable quality and is unlikely to fail so quickly. Thus, from our test in MATLAB, the value of b is less than or equals to 2 in most times in order to satisfy the reliability requirement. Figure 43 is a degradation path with the same parameters of gamma process as in Section 4.1.1.1. In addition, the value of power, b, is 2. With this assumption, the first component fails at approximately 2.7 unit time.

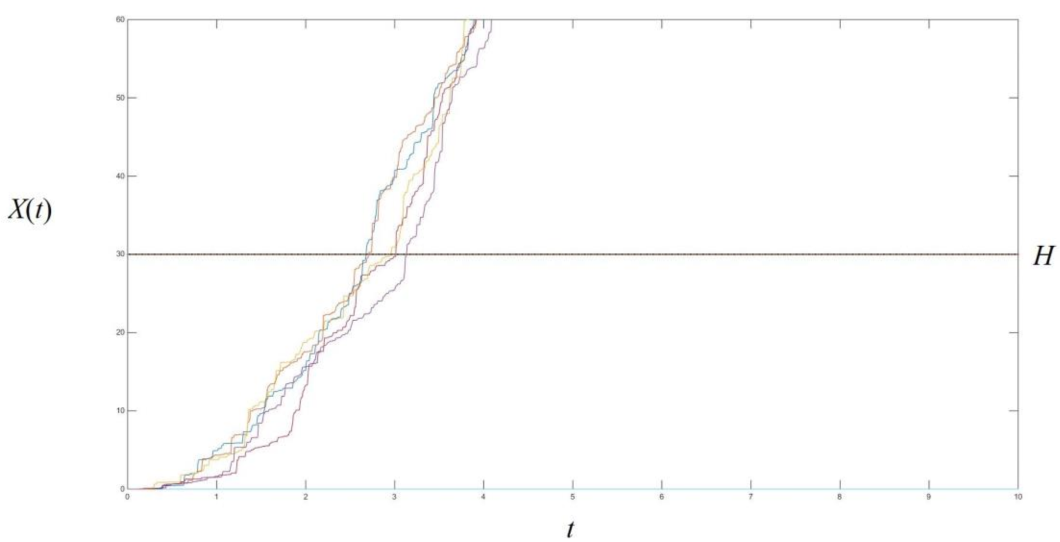

Figure 43 – Gamma process on account of nonlinear degradation without shocks

To determine the probability of the system surviving the interval of duration τ, the following equations are used and applied in MATLAB.

$$
\begin{aligned}
R(\tau) &= \Pr\{U_1 + X_1(N\tau) - X_1((N-1)\tau) < H_1^1 \cap ... \cap U_n + X_n(N\tau) - X_n((N-1)\tau) < H_n^1\} \\
&= \Pr\{X_1(N\tau) - X_1((N-1)\tau) < H_1^1 - U_1\} \cap ... \cap \Pr\{X_n(N\tau) - X_n((N-1)\tau) < H_n^1 - U_n\} \\
&= \int_0^{H_1^1} \Pr\{X_1(N\tau) - X_1((N-1)\tau) < H_1^1 - u_1\} f_{U_1}(u)du \times \\
&\quad ... \times \int_0^{H_n} \Pr\{X_n(N\tau) - X_n((N-1)\tau) < H_n^1 - u_n\} f_{U_n}(u)du \\
&= \int_0^{H_1^1} G\{H_1^1 - u_1; \alpha_0 (N\tau)^b - \alpha_0 [(N-1)\tau]^b, \beta\} f_{U_1}(u)du \times \\
&\quad ... \times \int_0^{H_n^1} G\{H_n^1 - u_n; \alpha_0 (N\tau)^b - \alpha_0 [(N-1)\tau]^b, \beta\} f_{U_n}(u)du \\
&= \prod_{i=1}^{n} \int_0^{H_i^1} G\{H_i^1 - u_i; \alpha_0 (N\tau)^b - \alpha_0 [(N-1)\tau]^b, \beta\} f_{U_i}(u)du
\end{aligned}
$$

In order to reach steady states, N should be extremely large. Actually, based on the simulation results in Section 4.3 for systems degrading as expected linear gamma process, N should be at least or greater than 200. In previous constructed system models, due to its linear degraded type, the difference between two time points is the inspection time interval. Random numbers can be produced based on one simple variable, τ. Without the results of initial degradation simulations, $f_{U_i}(u)$, the conditional reliability cannot be obtained. Thus, in this research, assumptions for starting time point and the value of initial degradation is applied to compute the conditional reliability function. Cost rate functions can be constructed on the basis of these assumptions. Here, considering two individual components are included in the system. In this situation, the starting point and the value of initial degradation are different for different components. However, in expected linear degradation section, the incremental degradation only depends on the inspection time interval τ. This is because conditional reliability is built on the inspection time interval, τ, not starting time point and has the distribution of initial degradation,

$f_{U_i}(u)$, not the value of initial degradation. Hence, it is assumed that the value of the first initial degradation is a specified value of 2.4 while the other one is 3.7. The first starting time point is 0.5 unit time and the other one is 0.7 unit time. Thus:

$$R(\tau) = \prod_{i=1}^{n} G\{H_i^1 - u_i; \alpha_0 (t_{i_0} + \tau)^b - \alpha_0 t_{i_0}{}^b, \beta\}$$

Taking assumed values into the conditional reliability function, it becomes:

$$R(\tau) = G\{H_i^1 - 2.4; \alpha_0 (0.5+\tau)^b - \alpha_0 (0.5)^b, \beta\} \times G\{H_i^1 - 3.7; \alpha_0 (0.7+\tau)^b - \alpha_0 (0.7)^b, \beta\}$$

Then, the cost rate function can be built with this conditional reliability function. Applying previous maintenance expenses, which C_I = 100, C_R = 200 and C_ρ = 200, to the cost rate function, the follow plot is a visualized cost rate function.

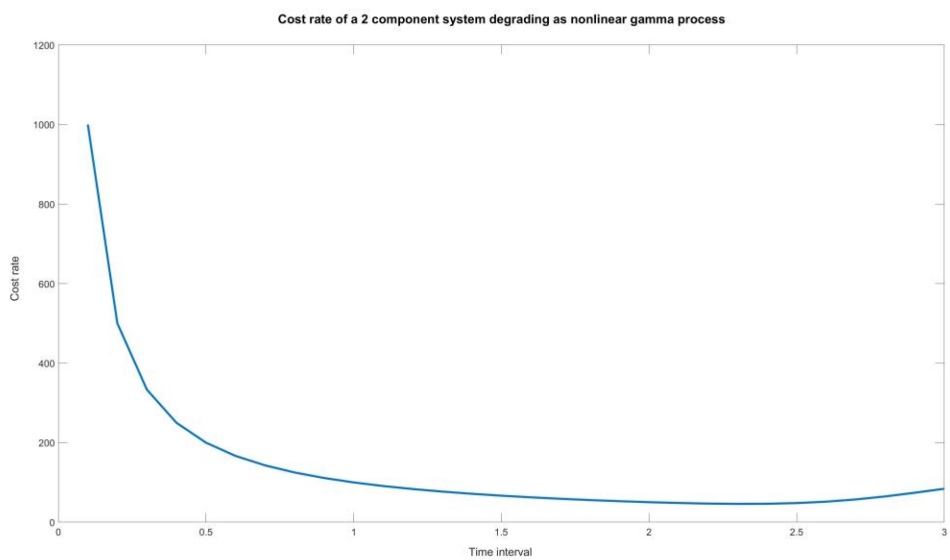

Figure 44 – Cost rate of a 2 component system degrading as nonlinear gamma process without shocks

The optimum minimum cost rate is approximately 60 with the optimum inspection time interval equaling approximately 2.4. Moreover, when varying C_I and C_ρ, sensitive analyses are completed. As shown in Figure 45 and Figure 46, when increasing C_I, the optimum cost rate comes later. When increasing C_ρ, the optimum cost rate comes earlier. These results are the same as previous results, due to different maintenance policies will be carried out by engineers when changing different costs.

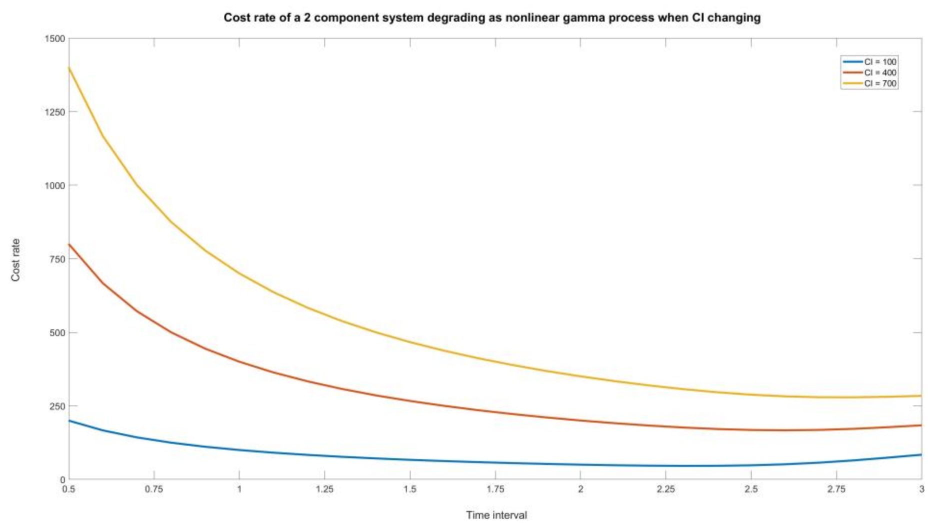

Figure 45 – Cost rate of a 2 component system degrading as nonlinear gamma process without shocks if C_I changing

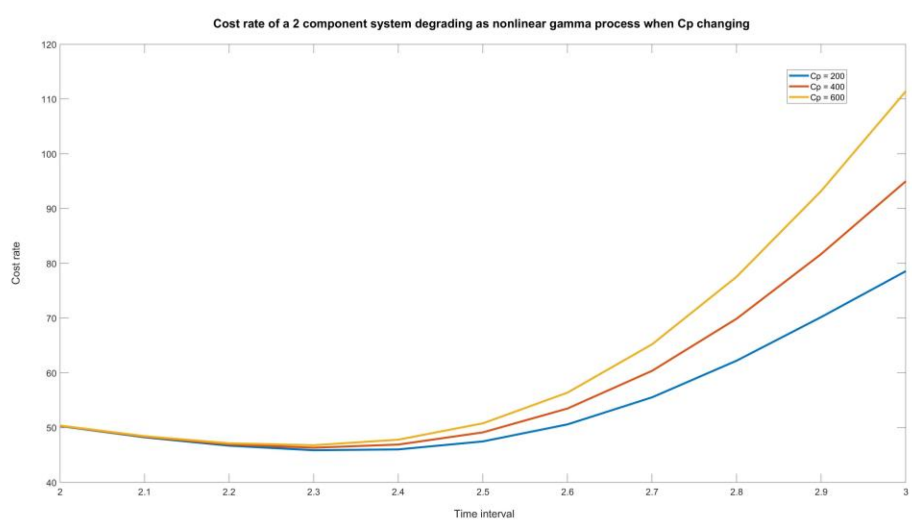

Figure 46 – Cost rate of a 2 component system degrading as nonlinear gamma process without shocks if C_ρ changing

4.2.1.2 Nonlinear Degradation without Shocks Considering On-condition Threshold

When the on-condition threshold is added to the maintenance policy, the conditional reliability is changed. The limitation for initial degradation U_i is from 0 to H^2. Here is the deduction of conditional reliability function:

$$
\begin{aligned}
R(\tau) &= \Pr\{U_1 + X_1(N\tau) - X_1((N-1)\tau) < H_1^1 \cap \ldots \cap U_n + X_n(N\tau) - X_n((N-1)\tau) < H_n^1\} \\
&= \Pr\{X_1(N\tau) - X_1((N-1)\tau) < H_1^1 - U_1\} \cap \ldots \cap \Pr\{X_n(N\tau) - X_n((N-1)\tau) < H_n^1 - U_n\} \\
&= \int_0^{H_1^2} \Pr\{X_1(N\tau) - X_1((N-1)\tau) < H_1^1 - u_1\} f_{U_1}(u)du \times \\
&\quad \ldots \times \int_0^{H_n^2} \Pr\{X_n(N\tau) - X_n((N-1)\tau) < H_n^1 - u_n\} f_{U_n}(u)du \\
&= \int_0^{H_1^2} G\{H_1^1 - u_1; \alpha_0(N\tau)^b - \alpha_0[(N-1)\tau]^b, \beta\} f_{U_1}(u)du \times \\
&\quad \ldots \times \int_0^{H_n^2} G\{H_n^1 - u_n; \alpha_0(N\tau)^b - \alpha_0[(N-1)\tau]^b, \beta\} f_{U_n}(u)du \\
&= \prod_{i=1}^{n} \int_0^{H_i^2} G\{H_i^1 - u_i; \alpha_0(N\tau)^b - \alpha_0[(N-1)\tau]^b, \beta\} f_{U_i}(u)du
\end{aligned}
$$

Since $f_{U_i}(u)$ cannot be simulated easily, on-conditioned thresholds cannot be meaningfully determined when the value of initial degradation and the starting time point are assumed. Cost rate functions and maintenance policies are the same as the results in Section 4.2.1.1.

4.2.2 Nonlinear Degradation with Shocks

4.2.2.1 Nonlinear Degradation with Shocks Regardless of On-condition Threshold

For systems with individually repairable components degrading as nonlinear gamma process with shocks, the reliability function is complex. Considering incoming shocks and initial high rate of degrading, the degradation process is accelerated.

Equations for this part are almost the same as equations in Section 4.1.2.1. However, time t has the power of b which equals a specific number. In most situations, it cannot be greater than 2, which is explained in Section 2.

$$
\begin{aligned}
R(\tau) = & \Pr\{U_1 + Y_1 + X_1(N\tau) - X_1((N-1)\tau) < H_1^1 \cap \\
& \ldots \cap U_n + Y_n + X_n(N\tau) - X_n((N-1)\tau) < H_n^1\} \\
= & \sum_{m=0}^{\infty} \Pr\{X_1(N\tau) - X_1((N-1)\tau < H_1^1 - U_1 - Y_1 \cap \\
& \ldots \cap X_n(N\tau) - X_n((N-1)\tau < H_n^1 - U_n - Y_n \mid N(\tau) = m\} \times \Pr(N(\tau) = m) \\
= & \sum_{m=0}^{\infty} (\int_0^{H_1^1} \int_0^{H_1^1 - u} \Pr\{X_1(N\tau) - X_1((N-1)\tau < H_1^1 - u_1 - y\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dydu \times \\
& \ldots \times \int_0^{H_n^1} \int_0^{H_n^1 - u} \Pr\{X_n(N\tau) - X_n((N-1)\tau < H_n^1 - u_n - y\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dydu) \times \frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!} \\
= & \sum_{m=0}^{\infty} (\int_0^{H_1^1} \int_0^{H_1^1 - u} G\{H_1^1 - u_1 - y; \alpha_0 (N\tau)^b - \alpha_0[(N-1)\tau]^b, \beta\} f_{Y_i}^{<m>}(y) f_{U_1}(u) dydu \times \\
& \ldots \times \int_0^{H_n^1} \int_0^{H_n^1 - u} G\{H_n^1 - u_n - y; \alpha_0 (N\tau)^b - \alpha_0[(N-1)\tau]^b, \beta\} f_{Y_i}^{<m>}(y) f_{U_n}(u) dydu) \times \frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!} \\
= & \sum_{m=0}^{\infty} \prod_{i=1}^{k} \int_0^{H_i^1} \int_0^{H_i^1 - u} G\{H_i^1 - u - y; \alpha_0 (N\tau)^b - \alpha_0[(N-1)\tau]^b, \beta\} f_{Y_i}^{<m>}(y) f_{U_i}(u) dydu \times \frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!}
\end{aligned}
$$

Applying changed equations to codes in MATLAB, the degradation plot and the conditional reliability plot is indicated in Figures 47 and Figure 48. With shocks coming into the process, the degradation has a high speed of passing through the threshold and quickly wears out. Compared with components' degradation processes without shocks, components of this process fail quickly because of random shocks. At approximately 1.4, the first failure is observed and the last one fails at approximately 2.4. Conditional reliability is indicated in Figure 48, with the assumption that the starting time is 0.

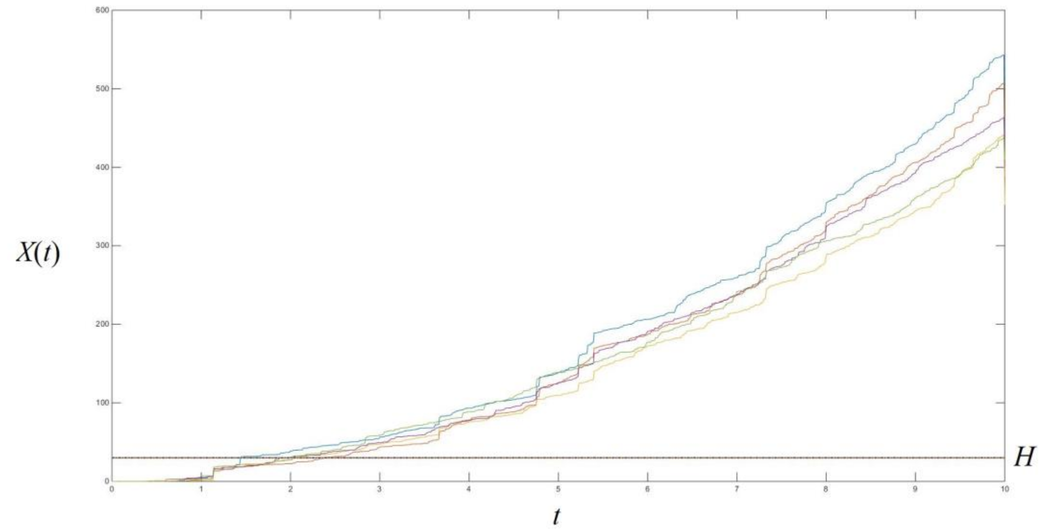

Figure 47 – Gamma process on account of nonlinear degradation with shocks

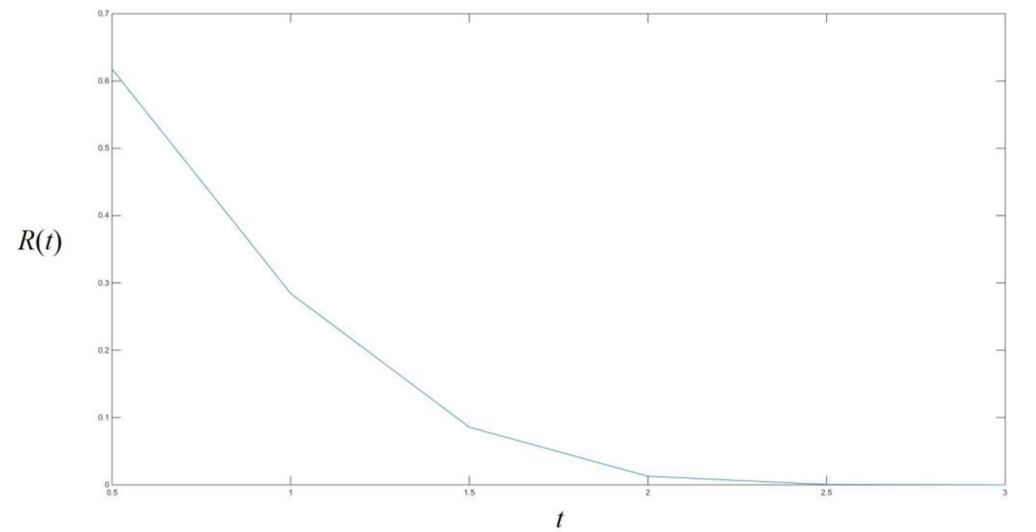

Figure 48 – Reliability of gamma process on account of nonlinear degradation with shocks

Adding assumed value of initial degradation and the starting time into the conditional reliability, it becomes:

$$R(\tau)=\sum_{m=0}^{\infty}\prod_{i=1}^{k}\int_0^{H_i^1-u_i} G\{H_i^1-u_i-y;\alpha_0(t_{i_0}+\tau)^b-\alpha_0(t_{i_0})^b,\beta\} f_{Y_i}^{<m>}(y)dy\times\frac{e^{-\lambda\tau}(\lambda\tau)^m}{m!}$$

Therefore, taking the same costs of maintenance policies of Section 4.2.1.1, a visualized cost rate function is shown in Figure 49.

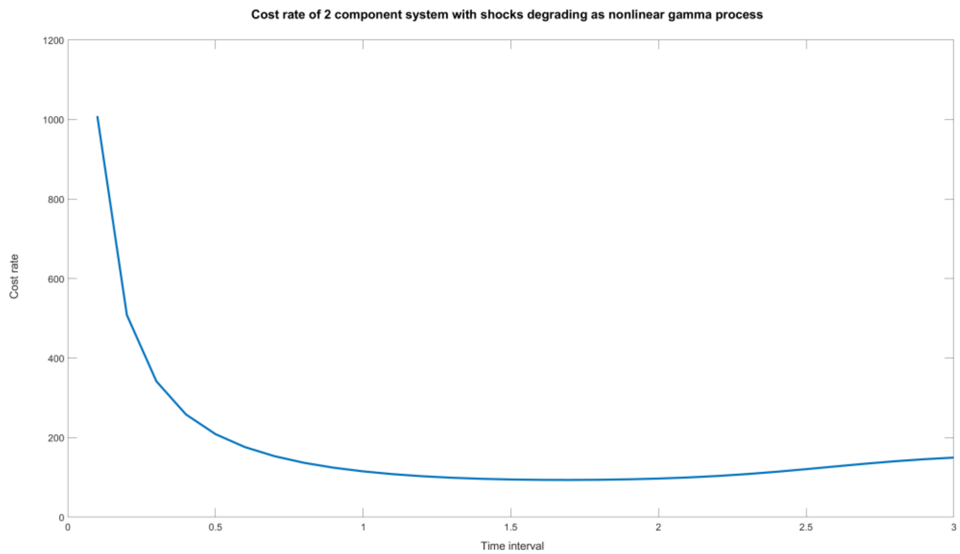

Figure 49 – Cost rate of a 2 component system degrading as nonlinear gamma process with shocks

In this case, the optimum cost rate is approximately 120 and the optimum inspection time interval is approximately 1.7. Compared with the model in Section 4.2.1.1, which does not have any shocks, the optimum inspection time interval is shorter and the lowest cost rate is higher. This is because shocks force the system to fail more quickly. Engineers decide to inspect more frequently to avoid failures.

Changing C_I and C_ρ separately, sensitive analyses are accomplished. Figure 50 and Figure 51 show the results of them. When C_I becomes larger, the optimum inspection time interval is longer to meet the best cost rate. If C_ρ is large, the optimum inspection time interval is short. Results are similar to the previous sensitive analyses.

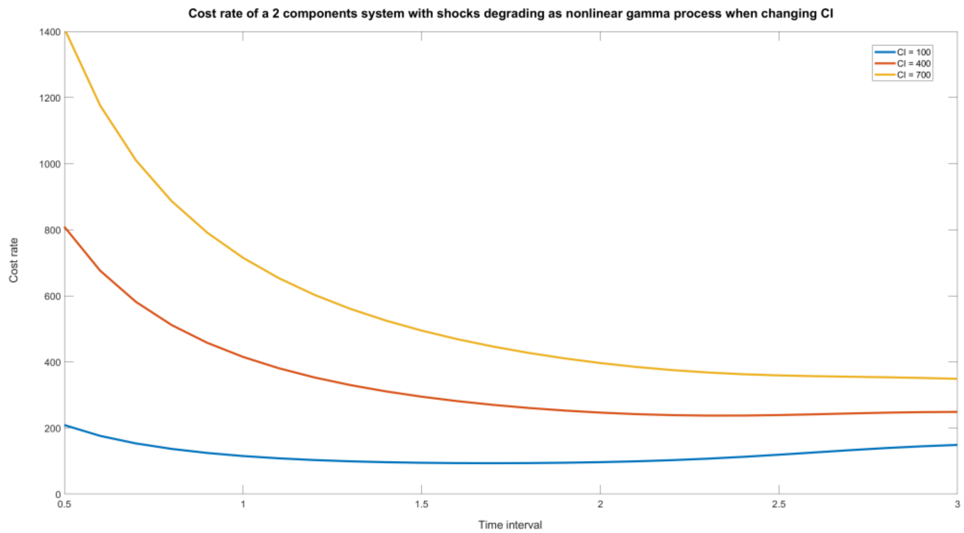

Figure 50 – Cost rate of a 2 component system degrading as nonlinear gamma process with shocks if C_I changing

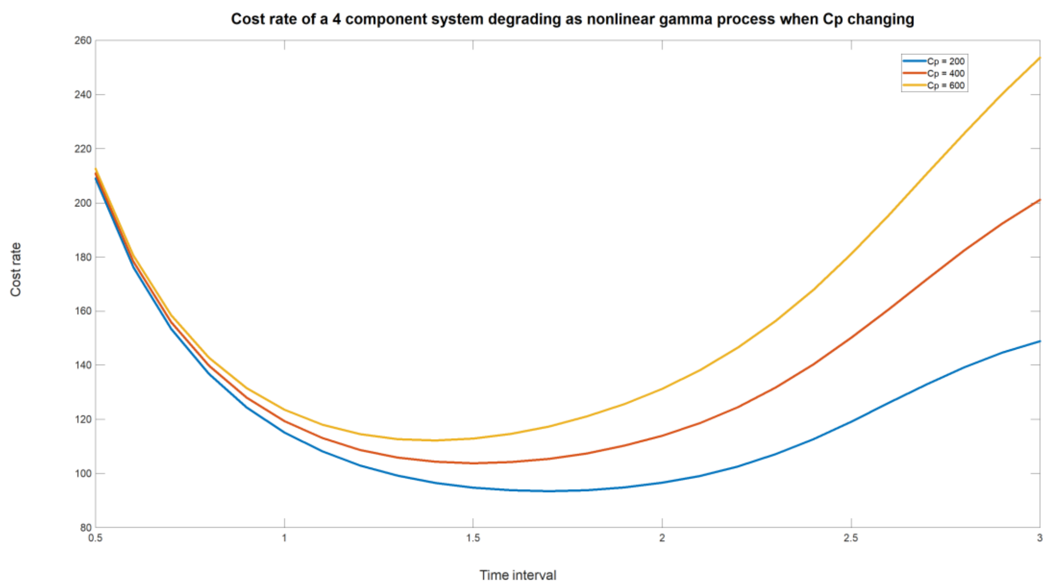

Figure 51 – Cost rate of a 2 component system degrading as nonlinear gamma process with shocks if C_ρ changing

4.2.2.2 Nonlinear Degradation with Shocks Considering On-condition Threshold

Adding on-condition thresholds to the system is a difficult challenge. The limitation for initial degradation U_i is from 0 to H^2. Thus, the limitation for random shock size is from 0 to H^2 - u_i. There are two conditional reliability functions for different purposes. One is for calculating the probability of no failure, $R_f(\tau)$, while the other one is for calculating the probability of no replacement, $R_r(\tau)$. With the assumed values, two conditional reliability functions can be accomplished. Thus, the cost rate function is constructed based on these two reliability functions. As a numerical example in this section, C_I = 100, C_R = 200 and C_ρ = 2000. The cost rate plot is shown in Figure 52.

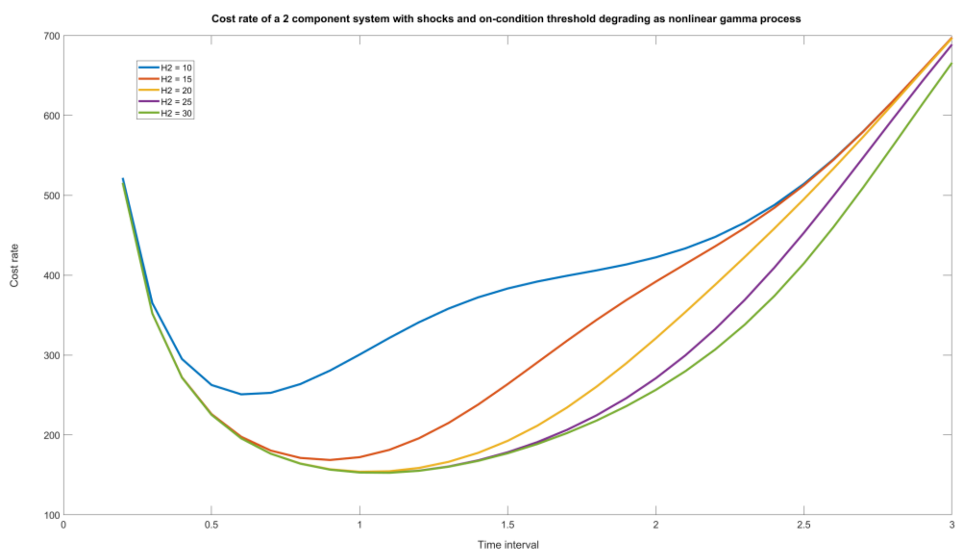

Figure 52 – Cost rate of a 2 component system degrading as nonlinear gamma process with shocks and on-condition threshold

From the plot, the optimum on-condition threshold is 30, which is the same as the failure threshold. The minimum cost rate is approximately 145 when the system is inspected every 1.1 unit time interval.

4.3 Determination of Initial Degradation Distribution

This section introduces the method of obtaining the simulated distribution of initial degradation U_i. When interval τ equals to some values, if the state of the system is greater than threshold, it means the system has failed and the failed component is replaced with a new one. If the maintenance policy has no on-condition threshold, the limitation of initial degradation is between 0 and H_i^1. For situation that the maintenance policy has on-condition threshold, the limitation of initial degradation is between 0 and H_i^2. This is because any components whose degradation is higher than on-condition threshold will be replaced. In other words, their initial degradation cannot be greater than H_i^2.

Figure 53 shows the result of simulation that indicates that after the 200th inspection interval, the component replacement reaches the steady state. Indeed, the average of total degradation does not change anymore and it is the same for all the next inspection intervals.

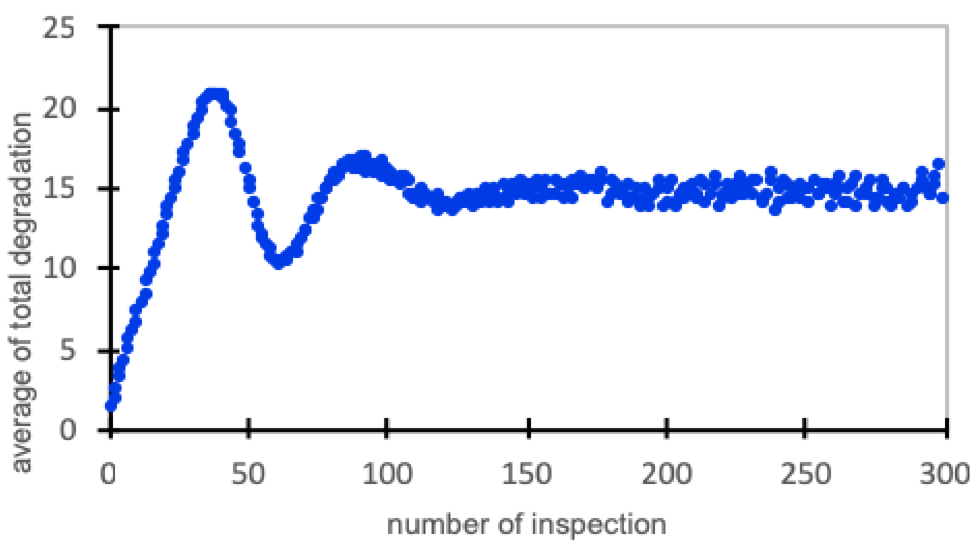

Figure 53 – Simulation of initial degradation

By using the simulation of multi-component system with different individual repairable components, the initial degradation at steady state for each component can be found. Figure 54 represents the histogram plot for the component initial degradation at steady state without any shock arrivals.

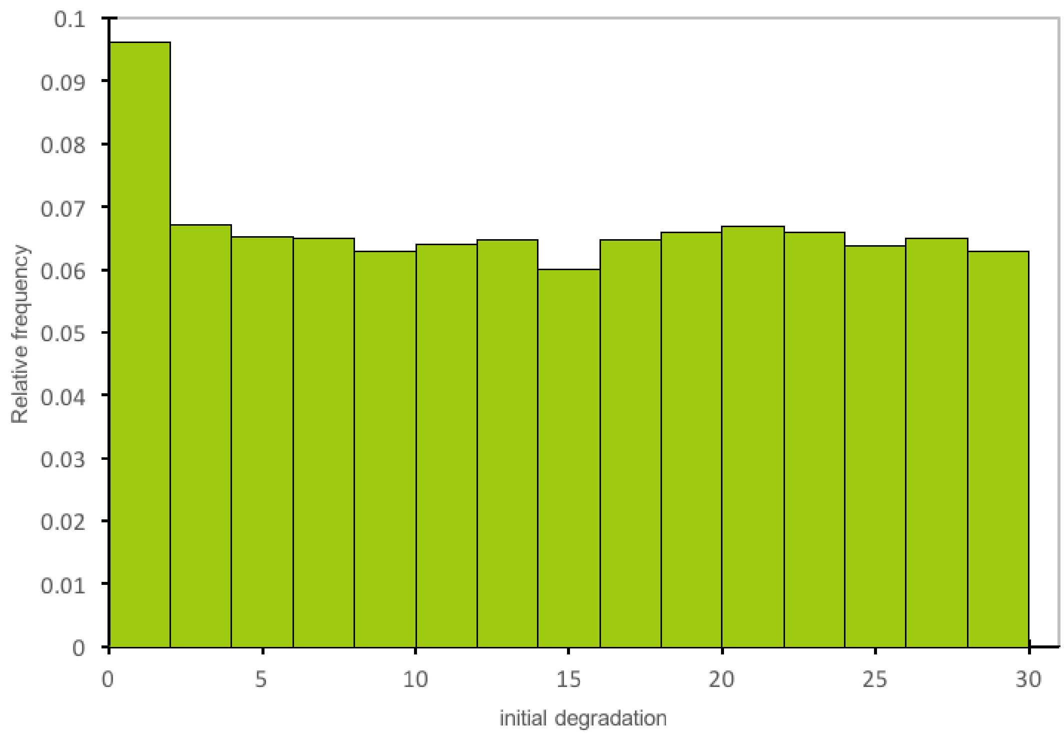

Figure 54 – Simulated histogram of initial degradation U_i when degradation is linear without shocks

Figure 54 and Figure 55 is a visual way to present the distribution of initial degradation. The result of simulation in Figure 54 is: $f_{U_n}(u) = \frac{1}{30-0}$, for $0 \le u \le 30$. Based on several times of simulations, the parameter for all uniform distributions are all the same if there is no on-condition threshold. On the contrast, if there is on-condition threshold, the result of simulation is: $f_{U_n}(u) = \frac{1}{H_i^2 - 0}$, for $0 \le u \le H_i^2$. It is only influenced by the initial shape parameter, α_0, and scale parameter, β, of the gamma process.

Another simulation for initial degradation is based on linear degradation with shocks. In this part, parameters are changed for the purpose of extending the lifetime of components. Thus, it is clear to analyze the cost rate function. For gamma distribution, the initial shape parameter, α_0, is 2 and the scale parameter, β, is 1. Mean and standard deviation of 1 random shock, μ and σ, is 2 and 1. As for the number of upcoming shocks in time interval τ, λ, which is also the parameter of Poisson process, equals 2. After several rounds of simulation, the distribution of initial degradation is shown in Figure 55. Obviously, it follows a generalized linear model.

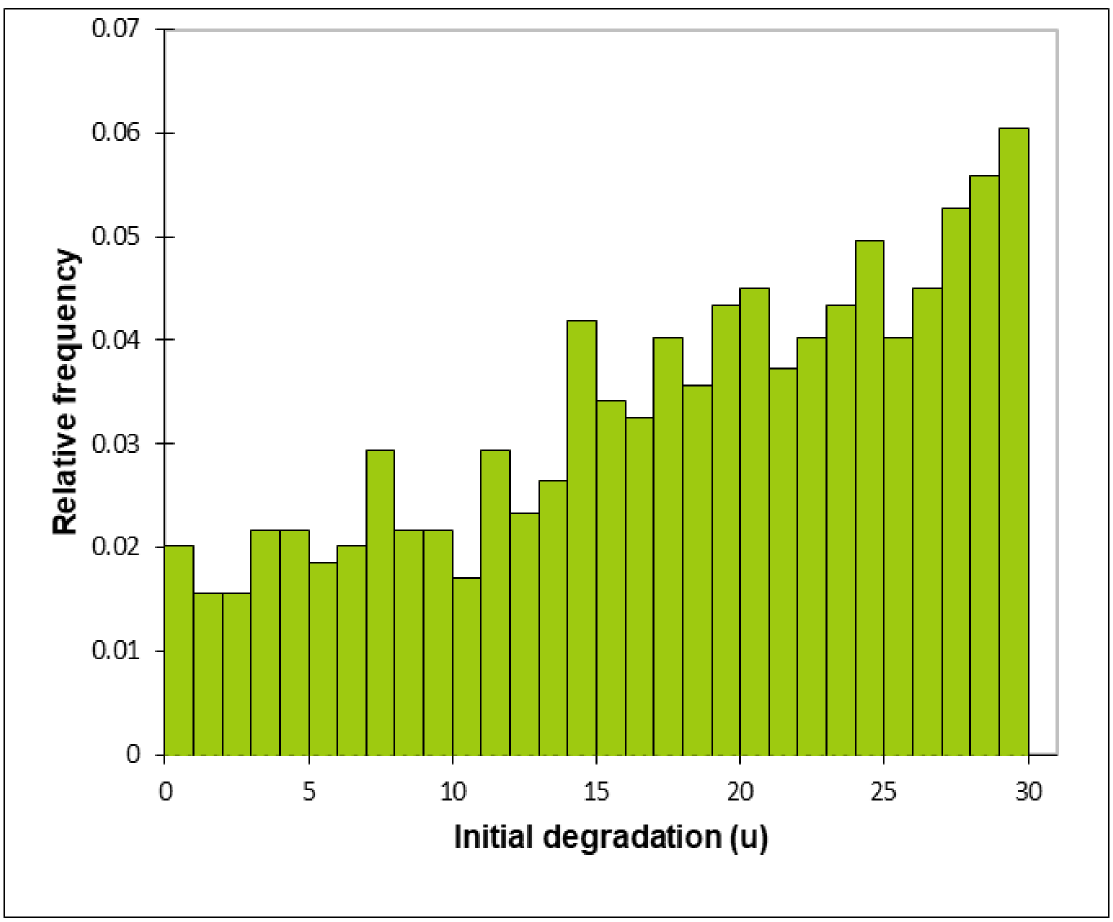

Figure 55 – Simulated histogram of initial degradation U_i when degradation is linear with shocks

Apparently, the distribution of initial degradation with shocks are different from the previous one. It follows a generalized linear model. The close function for this simulation is: $f_{U_n}(u) = \hat{a}u + \hat{b}$. For a case with initial shape parameter, α_0 = 2 and scale parameter, β = 1 when shock arrival rate is 1 and it follows normal distribution with mean = 2 and standard deviation = 1, then $\hat{a}$ equals 0.0011 and $\hat{b}$ equals 0.015. Thus, the distribution of initial degradation in this situation is: $f_{U_n}(u) = 0.0011 \times u + 0.015$.

Based on all simulation results got from this section, different distributions of initial degradation are accomplished for different degradation types. With these distributions, conditional reliability functions of linear degradation processes with and without shocks and corresponding cost rate functions can be deducted.

5 Case Study

In this section, the models constructed in Section 4 is chosen to fit a real system in industry. A previous article written by Lin et al. [15] describes a research focusing on reliability assessment towards a multi-component system, which is similar to this research. In their paper, a series system which consists of a pneumatic valve and a centrifugal pump is considered, as shown in Figure 56 [15]. In a nuclear power plant, this system is also described to be as a subsystem of a Residual Heat Removal System (RHRS). During a shutdown operation, as well as the time after it, RHRS has the function of cooling the reactor. By transferring heat from the core to the outside environment, the safety to the whole system is guaranteed. According to Lin's [15] finding, this kind of system setting is widely used, for different purposes.

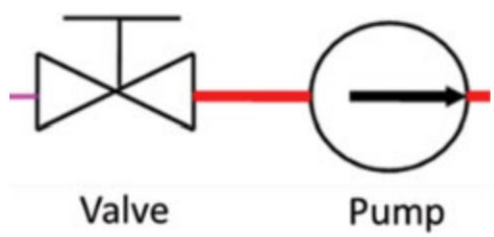

Figure 56 – Test case system [15]

5.1 System Similarities and Differences

First, in their research, there are two components which are pneumatic valve and centrifugal pump. For a system, they are connected in series. In this research, the system model consists of 4 components in series.

Second, their reliability calculations are constructed under the assumptions of specific degradation paths. In other words, it is modeled by a fourstate, continuous-time, homogeneous Markov chain. In this research, the degradation paths are modeled by gamma process degradation.

Third, the degradation process in their model is affected by external shocks, which are water hammers and internal thermal shocks. Different components are influenced by different shocks. Moreover, shock arrivals follow a Poison process, which is similar to model constructed in this research. The shock size is normal distributed with specific mean and standard deviation.

Based on these similarities and available datasets in Lin's [15] model, a real case study is accomplished in Section 5.2.

5.2 Model Application

In this part, with the data presented, a cost rate plot is shown in Figure 58. Due to the limitation of available dataset, the fitted reliability model is chosen to degrade as an expected linear gamma process with shocks but without on-condition threshold. This is because the only available numerical dataset is reliability values at different time points. Thus, the numerical reliability function is hard to get. With this assumption, the only fitted cost rate model is constructed. Figure 57 shows the only available dataset of reliability values and its corresponding reliability plot in Lin's [15] paper.

Table 2. Quantitative comparison of the results obtained by MC3 and FV3.

	Method		
Time (seconds)	*MC3*	*FV3*	*Relative error (%)*
100	0.9611	0.9607	0.0438
200	0.9021	0.9011	0.1162
300	0.8230	0.8205	0.3027
400	0.7285	0.7263	0.2974
500	0.6284	0.6271	0.2109
600	0.5312	0.5300	0.2394
700	0.4395	0.4397	0.0365
800	0.3576	0.3591	0.4157
900	0.2467	0.2459	0.3204
1000	0.0335	0.0332	0.8955

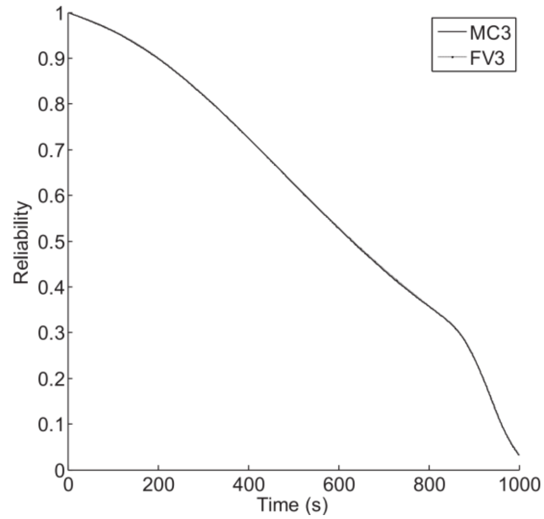

Figure 57 – Test case dataset and corresponding reliability plot [15]

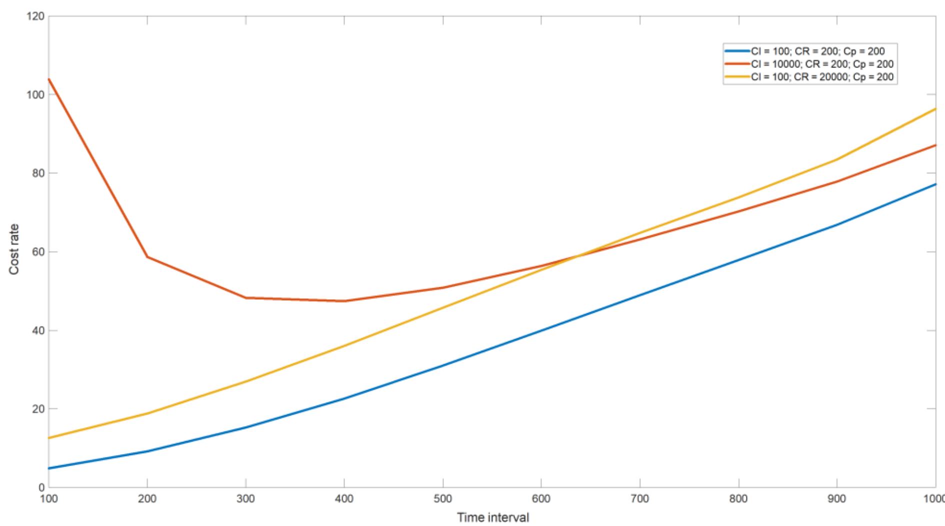

Figure 58 – Cost rate of test case system with 3 possible situations

With the fitted cost rate model being visualized in MATLAB, three different combinations of cost are shown in Figure 58. As an example, if the inspection expense is 100, the cost of replacement is 200 and the cost of expected idle time is 200, the optimum minimum cost rate is approximately 5 which requires a frequent inspection. This is similar to the situation when the cost of replacement is extremely high. If inspection expense is high, the optimum minimum cost rate is approximately 48 with inspection time interval equaling approximately 400s. These are only three possible situations of cost rate functions. However, based on real expenses of inspection, replacement and expected idle time, the situations are totally differently from each other. Also, as summarized in Section 4, inspection time interval is determined by engineers in real factories or industries for most of the time. Therefore, for real systems in industries and factories, the model should be carefully selected under the constraints of real degradation data, actual expenses of maintenance policies and suggestions from experienced reliability engineers.

6 Further Extensions

Further extension involves four tasks, extending and improving the model, distribution of initial degradation, optimization, and more test cases. Each task will extend the research in one specific direction individually. In order to make researchers deeply understand this research and be applied in practical situations, further study will be conducted in the following four directions.

6.1 Extending and Improving the Model

The model in this research is imperfect for system degrading as expected nonlinear gamma process and needs to be improved and extended. In this research, for expected nonlinear gamma process, since the distribution of initial degradation cannot be simulated, numerical conditional reliability functions cannot be deducted. To present an example, the values of two initial degradations and two starting time points are assumed. Then, conditional reliability functions and relative cost rate functions can be deducted. In further studies, the first task will be to extend and improve the degradation model of expected nonlinear gamma process. Cost rate functions will be developed for all cases, for example, the maintenance policies can be different for different components. Thus, the failure thresholds are different depending on what type of the component is. In this research, due to the assumption that all components are the same, the initial shape parameters, the scale parameters of gamma process, the failure thresholds are the same. As for further study, the diversity of this model can be extended. Moreover, the cost rate models for a system with components degrading as expected nonlinear gamma process with and without shocks can be constructed for further study.

For the purpose of predictive maintenance, an inspection level H_i^2 has already been added to the model. Based on the assumption that all components are the same, the on-condition thresholds are the same for each component. The model is useful and can be applied to the system whose components are all the same. Such as railway track system and gas pipeline system. For systems with multiple different components, reliability functions and cost rate functions should be changed and these functions calculated previously are also need to be re-calculated.

6.2 Distributions of Initial Degradation

The next task will be studying the distribution of U_i. Both empirical and parametric distributions will be determined. The i^{th} initial degradation can have many distributions. Initially, for system degrading as expected linear gamma process without shocks, it is simulated to follow a uniform distribution whose parameter is H_i^1。 Therefore, it is straightforward to analyze the reliability in this case. Figure 59 showing the probability density function of initial degradation U_i of one component: $f_{U_i}(u) = \frac{1}{H_i^1 - 0}$, for $0 \le U_i \le H_i^1$ with $H_i^1 = 30$.

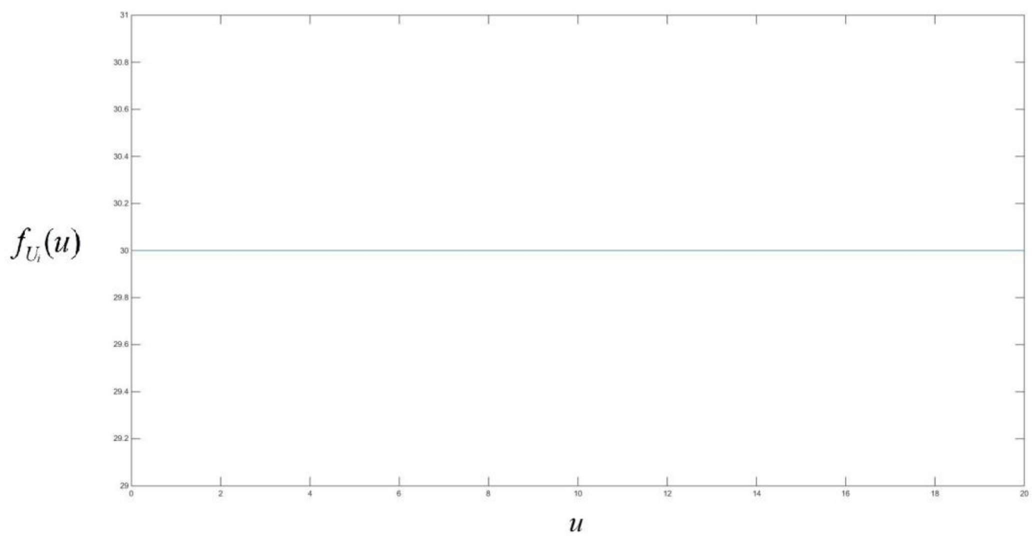

Figure 59 – PDF of U_i if it follows a uniform distribution

Meanwhile, for systems degrading as expected linear gamma process with shocks, the initial degradation distribution is different. As analyzed in Section 4.3, it is simulated to follow a generalized linear model. The parameter for this case is $\hat{a}$ and $\hat{b}$. Values of these two parameters are only determined and influenced by the initial shape parameter of gamma process, α_0, the scale parameter of gamma process, β, mean of initial shock size, μ, and standard deviation of initial shock size, σ.

In the future, for further study, the research will be to change the distribution functions of initial degradation. Simulations will be completed for initial degradation of expected nonlinear gamma process with and without shocks. Results may have many variations such as some exponential distributions, geometric distributions, simulated distributions and many other distributions. Here is a sample plot of PDF of initial degradation U_i if it follows an exponential distribution. The function is: $f_{U_i}(u) = \lambda e^{-\lambda u}$ whose parameter $\lambda = 1.5$.

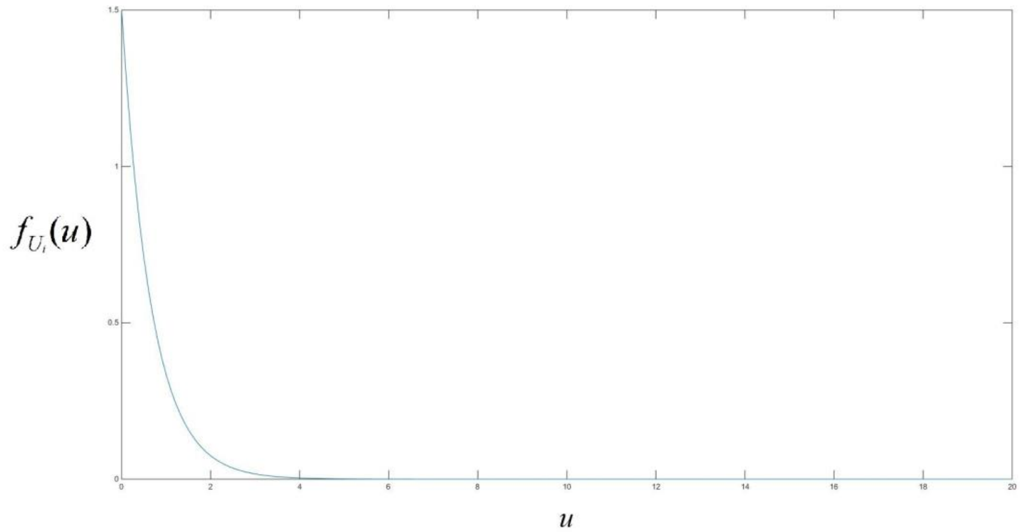

Figure 60 – PDF of U_i if it follows an exponential distribution

6.3 Optimization

After finishing the comparison works and sensitive analyses, the research finds the optimum inspection level H_i^2 and inspection time interval τ to search for the lowest cost rate in the end. However, the optimum model is selected only in system models degrading as expected linear gamma process. In order to accomplish more tasks, the model of expected nonlinear gamma process with and without shocks and on-condition thresholds will be built. In that case, the cost rate function will change in response to parameter changes. The expected idle time will vary a lot due to variations of conditional reliability functions. Also, different maintenance policies with different cost rate functions should be considered since real situations are more specific. One or two cost rate functions are not enough. For cost rate functions in this research, if a component fails in time interval τ, it has penalty cost, C_ρ, and replacement cost, C_R. This has been mentioned in Section 4. For further study, more cost rate function and expenses should be considered due to different situations. For example, as shown in Figure 61, component 2 and 4 fail, and the penalty cost of idle time is based on individual component. $\rho_i(\tau)$ represents the expected idle time of component i if it is inspected every τ. Thus, the total penalty cost is: $\rho_2(\tau)\times C_{p_2}+\rho_4(\tau)\times C_{p_4}$, which can be added to the cost rate function.

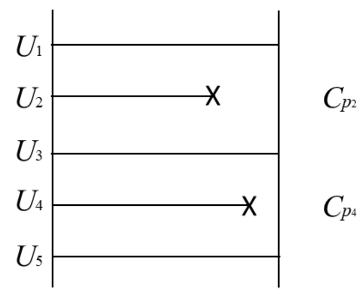

Figure 61 – Steady state of each component at the beginning of each interval

6.4 Model Evaluation

The model constructed in this research should be tested more to determine whether the model is useful and meaningful. In theory, there are two methods to obtain the assessment results. One is to apply this model to a simulation example and compare the results with the previous conclusions. Another method is to take a real example into consideration. If the model has the ability to predict the failure or minimize the cost rate in that situation, the model is meaningful. In this research, one real test case is completed and the result of optimum minimum cost rate is good.

This model constructed can be widely applied to many further studies. To test the accuracy of this model, and for the purpose of making sure it is a meaningful model, this system with individually repairable components will be tested. Considering many similar research before, if this model can obtain some similar results when applied to the same situation, the whole model is meaningful. When a degradation process can be simulated and predicted by methodologies of this model, failed components or parts can be fixed and the failure of the whole system can be avoided. Also, lower the cost rate is one of the objectives need to be achieved.

There is a real example representing the model constructed in this research perfectly. There is a research [11] about a railroad track system which is a series system with individually repairable tracks. Each failure of a track will cause the whole system to fail. Tracks in this system can be preventively repaired when engineers inspect the system at the end of a time period. The failure occurs due to many factors in reality. It is important to predict when will the yellow tag mileposts (i.e. on-condition) turn into red and cause a failure in a railroad track system. They assume the time to failure follows a Weibull distribution and use Maximum Likelihood Estimation (MLE) method to estimate the parameters of the distribution. By applying degradation data to their model, the research plan will be to obtain the results in the MATLAB and use the degradation rate as a tool to monitor whether the system has reached red tags. However, they have not considered expenses and the cost rate function in the model. Due to the limitation of accessing their dataset, this suitable example can only be analyzed in theory.

7 Conclusion

This research constructs models of different systems with individually repairable components degrading as gamma processes. Uniquely, this research combines the soft failure process and the hard failure process together in a different way. Based on this model, the degradation process, reliability and cost rate functions are clearly described both in equations and visual plots. With the deepening of the research and the increasing of the complexity, there will be more and more factors that can be added into the model and corresponding analysis towards the whole system can be accomplished.

As for degradation, the components degrading as nonlinear processes fail more quickly than the components degrading as linear processes. They quickly pass the threshold and then fail. Even some components degrading as the same process, the time for each component to failure is different. Focusing on the reliability, it is clear that compared with the component degrading as expected linear process, these components degrading as nonlinear process have a lower reliability at the same time. After adding shocks into the system, the degradation takes only a little time to pass the threshold. Adding shocks and the initial degradation to the system model, the time to failure or replacement becomes much shorter. The conditional reliability with these two factors has a sharp trend to fall down to zero. Therefore, in practice, engineers need to avoid external shocks hitting the system and use high-quality components in order to avoid high initial degradation. Then the reliability of the whole system can be higher than other systems at the same time.

Taking expenses into consideration, if the component has a higher probability to fail quickly, the maintenance should be done earlier. For systems with and without shocks, the minimum cost rate at the optimum time is different. Like the previous conclusion, if the component experiences the process with shocks, it will fail more quickly, so the optimum maintenance time should be earlier, and the cost rate is generally higher. Initial degradation U_i is important in the cost rate function. For the same process and the same components, if the component has an initial degradation U_i at the beginning of each interval, the optimum time occurs earlier, and the minimum cost rate becomes larger. Also, values of each expense are different in different real situations. Sensitive analyses conclude that the optimum on-condition thresholds and inspection time interval vary a lot depending on what costs are included in the cost rate function and which expenses are high. In theory, for the purpose of having low cost rate, when C_I is high, it is wise to inspect later. If C_ρ is high, engineers should inspect earlier. In industrial fields and factories, since inspection time interval is fixed and expenses are determined by the market, engineers should make a best decision after thinking about the real situation and then find the optimum lowest cost rate.

For further research purpose, more variable factors can be added into the model and a wider comparison can also be completed. Moreover, based on the finding, separately changing more parameters of conditional reliability functions or cost rate functions in the model will be another interesting challenging sensitive analysis in the future. More importantly, testing the accuracy of the model and applying the whole system into reality will bring much more meaningful results in return.

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