SPECTRAL AND SCATTERING THEORY FOR DISPERSIVE THREE-BODY SYSTEMS

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ABSTRACT OF THE DISSERTATION

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In quantum scattering theory, one seeks to characterize the spectrum of and asymptotic evolution by an N-body Schrödinger Hamiltonian H. For instance, one may prove asymptotic completeness, which states that an N-body system separates into freely evolving subsystems, each of which is in a bound state.

The Mourre estimate $E_{\triangle}(H)[H, iA]E_{\triangle}(H) \geq \theta E_{\triangle}(H)$ has been an indispensable tool in the analysis of N-body systems. It implies certain propagation estimates including local decay estimates and minimal velocity bounds. These have been used to analyze Hamiltonians $p^2 + V$ for a broad class of N-body potentials V.

In this dissertation, we extend the Mourre theory, the subsequent propagation estimates, and the asymptotic completeness (for negative energy) to a Hamiltonian $H = p^2 + |k| + V$ designed to capture some aspects of the photoelectric effect. Modifications are needed; the necessary inequalities are achieved by examining the spectrum and by using a different formula to compute commutators with $|k|$.

Acknowledgements

Much of this dissertation is also part of the preprint [5], which is under review. I am indebted to my advisor, Dr. Avraham Soffer, for his mentorship through this process.

Dedication

For Danny, Mom, and Dad.

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Chapter 1

Introduction

1.1 Background

In a microscopic game of billiards, we'd like to be certain of all the possible ways the game can end. But how?

More specifically, in quantum scattering theory, one investigates the long-time behavior of N-body quantum systems. Since its development in the 1920s, many advances have been made in understanding the solutions to Schrödinger's equation for N particles interacting pairwise:

$$i\frac{\partial}{\partial t}\psi = H\psi$$

where $\psi \in \mathcal{H} = L^2(\mathbb{R}^{3N})$ is a wavefunction on configuration space $\mathbb{R}^{3N}$ and

$$H = -\triangle + \sum_{i\neq j} V_{ij}$$

is a Schrödinger N-body Hamiltonian for some interaction potential functions V_{ij}. In 1951, Tosio Kato established the self-adjointness of these operators, placing them in the realm of the abstract quantum theory developed by Von Neumann and others. In particular, the dynamics of the solutions are given by a strongly continuous one-parameter unitary group

$$\psi_t(x) = e^{-itH}\psi_0(x)$$

Exactly what behavior these solutions could exhibit would remain an open question for decades. In 1969, Ruelle showed that under certain modest conditions, the solution space could be divided into bound states $\psi \in \mathcal{H}_b$, which are spanned by eigenfunctions exhibiting the spatial decay property

$$\lim_{R\to\infty} \|(1-\chi_{B_R}(x))e^{-iHt}\psi\| = 0$$

(where B_R is the ball of radius R centered at the origin) and continuous-spectrum states $\psi \in \mathcal{H}_c$, which are orthogonal to the bound states and exhibit mean ergodic decay

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|\chi_{B_R}(x)e^{-iHs}\psi\|^2 ds = 0$$

where $R \geq 0$ is arbitrary. This characterization would later be generalized by Amrein, Georgescu, and Enss. One highly sought-after conjecture during this period was asymptotic completeness. To state asymptotic completeness, we need the notion of a scattering state. This can be easily understood with reference to a specific 3-body problem.

Let $\mathcal{H} = L^2(\mathbb{R}^6)$ be the configuration space of a restricted 3-body system, where $x \in \mathbb{R}^3$ gives the coordinates of one particle, $y \in \mathbb{R}^3$ gives the coordinates of another particle, and a particle of large mass stays fixed at the origin. The 3-body Hamiltonian for this system can be expressed as

$$H = p^2 + k^2 + V_{12}(x) + V_{13}(y) + V_{23}(x-y)$$

where the momentum p is the Fourier conjugate of x, k is the Fourier conjugate of y, and V_{ij} are potential functions. One expects that in a scattering situation, one or two of these particles might go far away from the others, so their interaction would become negligible. Thus, the dynamics would asymptotically become the dynamics of one of the truncated Hamiltonians:

$$H_{(123)} = H$$

$$H_{(12)} = p^2 + k^2 + V_{12}(x)$$

$$H_{(13)} = p^2 + k^2 + V_{13}(y)$$

$$H_{(23)} = p^2 + k^2 + V_{23}(x - y)$$

$$H_{(0)} = p^2 + k^2$$

where the cluster decompositions $a \in \{(123), (12), (13), (23), (0)\}$ represent which particles stay together asymptotically. Specifically, an *outgoing scattering state* can be defined as a state ψ such that

$$\lim_{t\to\infty} e^{-iHt}\psi = \lim_{t\to\infty} \sum_a e^{-iH_a t}\phi^a$$

where the ϕ^a are essentially bound states of the resulting subsystems

$$\phi^{(123)} \text{ is an eigenfunction of } H$$

$$\phi^{(12)} \text{ is an eigenfunction of } H^{(12)} = p^2 + V_{(12)}(x)$$

$$\phi^{(13)} \text{ is an eigenfunction of } H^{(13)}k^2 + V_{(13)}(y)$$

$$\phi^{(23)} \text{ is an eigenfunction of } H^{(23)}(p - k)^2 + V_{(23)}(x - y)$$

except for $\phi^{(0)}$ which exhibits free dynamics, i.e. evolution by e^{itH_0}. More accurately, ϕ^a is a member of $L^2(x_a) \otimes \mathcal{H}_{bound}(H^a)$, where x^a is the position of the particle external to the cluster and $\mathcal{H}_{bound}(H^a)$ is the subspace of bound states of H^a. There is a notion of an incoming scattering state involving the limit as $t \to -\infty$, and one can show that the incoming scattering states are precisely the outgoing scattering states.

The conjecture of N-body *asymptotic completeness* states that every state is a linear combination of a bound state and a scattering state. This offers a effective geometric characterization of all solutions to the equation.

This conjecture was proven under certain assumptions (for example by Lavine in 1971 under the assumption of purely repulsive potentials), and then under very general assumptions by Sigal and Soffer in 1987. The key to the proof of Sigal and Soffer is developing a microlocal positive commutator method to construct *propagation observables* that decay along the flow in classically forbidden regions of the phase space of the N-body system.

Consider a Schrödinger Hamiltonian H, another self-adjoint operator A, and an interval Ω in the continuous spectrum of H. Let $E_\Omega(H)$ be the spectral projection onto Ω. An abstract result due to Mourre in 1981 shows that (under some additional assumptions required to make this statement well-defined), the *Mourre inequality*

$$E_\Omega(H)[H, iA]E_\Omega(H) \geq \alpha E_\Omega(H)$$

for some $\alpha > 0$ implies that for any closed $[a, b] \in \Omega$, there is a $c_0 \in \mathbb{R}$ such that (see [19])

$$\sup_{Re(z)\in[a,b], Im(z)\neq 0} \| |A + i|^{-1}(H - z)^{-1}|A + i|^{-1}\| \leq c_0$$

which in turn implies that (see [21])

$$\int_{-\infty}^{\infty} \| |A + i|^{-1} e^{-itH} E_\Omega(H)\psi\|^2 dt \lesssim \|\psi\|^2$$

Thus there is a general framework for inequalities of this type from the Mourre inequality, including *local decay estimates* of the form

$$\int_{-\infty}^{\infty} \|(x^2 + 1)^{-\mu} e^{-itH} E_\Omega(H)\psi\|^2 dt \lesssim \|\psi\|^2$$

and *minimal velocity estimates* [22] of the form

$$\int_{-\infty}^{\infty} \|\chi_{(0,\epsilon)}\Big(\frac{(x^2 + 1)}{t}\Big) e^{-itH} E_\Omega(H)\psi\|^2 dt \lesssim \|\psi\|^2$$

Once these types of propagation estimates are established, the game is to use them to show that the part of $e^{-itH}\psi$ that does *not* separate into freely evolving subsystems (as in the statement of asymptotic completeness) converges to 0.

While these methods were successful for N-body Hamiltonians of the form $p^2 + V$, they don't immediately generalize to Hamiltonians with different kinetic parts, such as the relativistic $\sqrt{p^2 + m^2}$, without some extra work. Thus, for instance, asymptotic completeness in the case of the photoelectric effect for the hydrogen atom (the scattering of one proton, one electron, and N photons) remains open.

In 2002, Galtbayar, Jensen, and Yajima proved a Mourre estimate for a version of the Nelson model of an atom restricted to less than two photons, but remarked that the estimate achieved was insufficient to obtain the local decay estimates needed for a proof of asymptotic completeness.

In 2016, Soffer proved maximal velocity estimates for the Hamiltonian $|p| + V(x)$, extending positive commutator estimates to this case with a singular symbol.

This dissertation establishes asymptotic completeness for the restricted 3-body problem with one proton, one electron, and one "photon":

$$H = p^2 + |k| + V_{12}(x) + V_{13}(y) + V_{23}(x - y)$$

under certain assumptions; specifically negative energy and spectral assumptions on the subsystems.

1.2 Intuition for three massive particles

Here we will outline how the techniques discussed thus far work for the restricted three-body Hamiltonian

$$H = p^2 + k^2 + V_{12}(x) + V_{13}(y) + V_{23}(x - y)$$

It is enlightening to see what is exactly the same and what is genuinely a property of the singular Hamiltonian of interest.

First, one needs to prove the Mourre estimate. There are several nice things about the kinetic part of the Hamiltonian that make this process easy. One is that commutators of p^2 with functions $j_a(x, y)$ are nice, so it is quick to prove that $[p^2, j_a(x, y)]$ is relatively H-compact for certain j_a. These j_a will arise as a partition of unity on configuration space. Any kinetic part that is less nice or even singular may cause difficulty here. Another is that the kinetic parts of each particle individually are the same, which allows each cluster decomposition to be treated in the same way. Finally, something special happens when the kinetic parts are all the same and all p^2; when you treat the (23) cluster, the external and internal momentum $p_a = p + k$ and $p^a = p - k$ separate:

$$\begin{aligned} H_a = p^2 + k^2 + V_{23}(x-y) &= \frac{1}{4}(p_a + p^a)^2 + \frac{1}{4}(p_a - p^a)^2 + V_{23}(x^a) \\ &= \frac{1}{2}(p_a)^2 + \frac{1}{2}(p^a)^2 + V_{23}(x^a) \end{aligned}$$

An important part of proving the Mourre estimate on this cluster is considering what happens when you project onto the bound state of the subsystem:

$$\frac{1}{2}(p_a)^2 + \frac{1}{2}(p^a)^2 + V_{23}(x^a) \approx \frac{1}{2}(p_a)^2 + \lambda$$

When you cannot separate out the Hamiltonian of a subsystem, it is trickier to balance the contribution of the external momentum p_a and the contribution of the subsystem's energy λ to the positive commutator.

The proof of the Mourre estimate in this case can be found in e.g. [9]. The idea is to first localize onto subsystems using a partition of unity $\sum j_a^2 = 1$ so that one can prove a Mourre estimate for each subsystem H_a separately. Then, one can consider each value of the external momentum separately. When the external momentum is small, then since the total energy is fixed away from threshold energies, one can use the fact that the subsystem is on its continuous spectrum. When the total momentum is large, one can exploit that to obtain a positive commutator.

The Mourre estimate is known to imply local decay estimates and minimal velocity estimates, but specifically those involving the Mourre conjugate operator A. So to obtain local decay instead in terms of the position X, one must prove e.g. Lemma 8.2 in [20], which depends on the structure of H.

Once these auxiliary estimates are established, one wishes to prove the existence of candidates for scattering states. This is established by proving the existence of Deift-Simon wave operators

$$s - \lim_{t \to \pm\infty} e^{itH_a} j_a(x,y) e^{-itH} P_{\mathrm{ac}}(H) E_{\triangle}(H)$$

where P_{ac} projects onto the absolutely continuous spectrum, and j_a is part of a partition of unity projecting onto a cone in configuration space where one would expect evolution by H_a to propagate. Putting these together for each cluster decomposition a yields an asymptotic decomposition into scattering states for any ψ in the continuous spectrum of H. We call this asymptotic clustering, and asymptotic completeness follows by an induction argument.

The choice of a configuration space partition of unity $j_a(x,y)$ could be substituted for a phase space partition of unity $j_a(x,y)f_a(p,k)$ or something else. However, for the simple three-body Hamiltonian in question, these wave operators will do.

The existence of these wave operators is established using Cook's method, where the limiting quantity is expressed as an integral of a derivative. Thus we must prove that the Heisenberg derivative

$$D_t j_a(X)\psi := [p^2 + k^2, j_a] + I_a j_a$$

is integrable in time, where I_a is the external potential energy of the cluster decomposition in question. Under certain assumptions on the potential, $I_a j_a$ is easily controlled by local decay. Moreover, since commutators $[p^2, j_a]$ are nice, these terms are also easily controlled by local decay. It is the difficulties arising from $[|k|, j_a]$ that may necessitate a use of a phase space partition of unity or some additional assumptions to prove asymptotic completeness for our singular Hamiltonian of interest.

Chapter 2

Definitions and potential assumptions

Let $\mathbb{R}^6_X = \mathbb{R}^3_x \oplus \mathbb{R}^3_y$ be the configuration space for a restricted three-particle system, where coordinates have been selected so that a first particle (a "proton", treated as having infinite mass) is at the origin, x is the position of the second particle (an "electron"), and y is the position of the third particle (a "photon"). Where $X = (x, y)$ represents a point in configuration space, we let $P = (p, k)$ be the Fourier conjugate of X, so that k is the momentum of the photon and p is the momentum of the electron. We define the **free Hamiltonian**

$$H_0 := p^2 + |k|$$

as an operator on $L^2(\mathbb{R}^6_P)$. The free Hamiltonian models the kinetic energies p^2 and $|k|$ of the electron and of a photon, respectively. The proton is fixed at the origin and so makes no contribution, and creation and annihilation of photons is ignored in this simplified model. As a multiplier operator, H_0 is self-adjoint on the domain $D(H_0) := \{\psi \in L^2(\mathbb{R}^6) : H_0\psi \in L^2(\mathbb{R}^6)\}$. The graph norm on $D(H_0)$ is equivalent to the weighted L^2 norm with weight $(1 + p^4 + k^2)\ dp\ dk$. As such, the operator H_0 is essentially self-adjoint on the dense set $C_0^\infty(\mathbb{R}^6)$, the class of smooth functions with compact support; this class is dense in the weighted space $L^2(\mathbb{R}^6_P; (1 + p^4 + k^2)\ dp\ dk)$.

We introduce a three-body potential function $V = V_{12}(x) + V_{13}(y) + V_{23}(x - y)$, where V_{12}, V_{13}, and V_{23} are functions on $\mathbb{R}^3$. V_{12} is the electron-proton interaction, V_{13} is the photon-proton interaction, and V_{23} is the electron-photon interaction. We define the **full Hamiltonian** on $L^2(\mathbb{R}^6_X)$:

$$H = H_0 + V_{12}(x) + V_{13}(y) + V_{23}(x-y)$$

We assume that the potentials satisfy the assumptions of the Kato-Rellich theorem so that H is self-adjoint on $D(H) = D(H_0)$ and bounded below. In particular, we make assumption (RB) below.

Next we define the five **cluster decompositions**

$$a \in \{(xy0), (y)(x0), (x)(y0), (xy)(0), (x)(y)(0)\}$$

These are used as indices for quantities representing the system in the following situations respectively: when all three particles are close together, when just the photon is far away, when just the electron is far away, when the photon and electron are close together but far away from the proton, and when all three particles are far apart. The number of clusters for a particular decomposition is denoted $\#(a)$, e.g. $\#((x)(y0)) = 2$.

When we are analyzing a particular cluster decomposition, it's useful to have a uniform notation to describe the internal and external cluster coordinates. We write x^a and p^a for internal coordinates and x_a and p_a for external coordinates, e.g. for $a = (y)(x0)$ we write $y = x_a$, $x = x^a$, $k = p_a$, and $p = p^a$. The rest of the notation is contained in this chart.

	(xy0)	(y)(x0)	(x)(y0)
x_a	-	y	x
x^a	X	x	y
p_a	-	k	p
p^a	P	p	k
$I^a(x^a)$	V	$V_{12}(x)$	$V_{13}(y)$
I_a	0	$V_{13}(y) + V_{23}(x-y)$	$V_{12}(x) + V_{23}(x-y)$

	(xy)(0)	(x)(y)(0)
x_a	$x+y$	X
x^a	$x-y$	-
p_a	$p+k$	P
p^a	$p-k$	-
$I^a(x^a)$	$V_{23}(x-y)$	0
I_a	$V_{12}(x)+V_{13}(y)$	V

The assumption (RB) is that the following operators are are relatively H_0-bounded with relative bound less than 1: for all partitions a,

$$\begin{cases} I^a \\ x^a \cdot \bigtriangledown I^a \\ x^a \cdot \bigtriangledown (x^a \cdot \bigtriangledown I^a) \end{cases} \tag{RB}$$

One simple way to satisfy (RB) is to have each potential be smooth and decay at infinity, although weaker conditions will suffice.

We define the **truncated Hamiltonians** $H_a = H_0 + I^a = H - I_a$. These represent the approximate energy of the system when it has separated into clusters in the manner suggested by a, so we can neglect certain potentials. The I_a are the **intercluster potentials**, and the I^a are the internal potentials.

We focus our attention on the three **2-cluster decompositions** such that $\#(a) = 2$. For each of these cluster decompositions, the truncated Hamiltonians H_a can be written as a direct integral (cf. [14]) over the **reduced Hamiltonians** $H_a(s)$ defined below. This is justified because H_a commutes with p_a in each case. Essentially, we get to identify p_a with a number s.

$H_{(y)(x0)}(s) := p^2 + |s| + V_{12}(x)$ is an operator on $L^2(\mathbb{R}^3_x)$, so that $H_{(y)(x0)} = \int_{\mathbb{R}^3_s}^{\oplus} H_{(y)(x0)}(s)\, ds$ is an operator on $L^2(\mathbb{R}^6) = \int_{\mathbb{R}^3_s}^{\oplus} L^2(\mathbb{R}^3_x)\, ds$.

$H_{(x)(y0)}(s) = s^2 + |k| + V_{13}(y)$ is an operator on $L^2(\mathbb{R}^3_y)$, so that

$H_{(x)(y0)} = \int_{\mathbb{R}^3_s}^{\oplus} H_{(x)(y0)}(s)\ ds$ is an operator on $L^2(\mathbb{R}^6) = \int_{\mathbb{R}^3_s}^{\oplus} L^2(\mathbb{R}^3_y)\ ds$.

$H_{(xy)(0)}(s) = \frac{1}{4}(p^a + s)^2 + \frac{1}{2}|p^a - s| + V_{23}(x^a)$ is an operator on $L^2(\mathbb{R}^3_{x^a})$ so that $H_{(xy)(0)} = \int_{\mathbb{R}^3_s}^{\oplus} H_{(xy)(0)}(s)\ ds$ is an operator on $L^2(\mathbb{R}^6) = \int_{\mathbb{R}^3_s}^{\oplus} L^2(\mathbb{R}^3_{x^a})\ ds$.

In order to prove the Mourre estimate, we use an additional spectral assumption on the reduced Hamiltonians $H_{(xy)(0)}(s)$ for the photon-electron cluster. Specifically, we assume

$$H_{(xy)(0)}(s) \text{ has no eigenvalues embedded in its continuous spectrum for any } s \in \mathbb{R}^3 \tag{SPEC}$$

and

$$\text{For some } \theta > 0 \text{ the eigenfunctions of } B(tv) \text{ satisfy } \left(-\frac{p^a}{|p^a|} \cdot t\right) \geq \theta \text{ in expectation.} \tag{SPEC2}$$

This is because eigenvalues can behave poorly as s varies, where the continuous spectrum is concerned. Isolated eigenvalues, on the other hand, have a well-understood structure (cf. [16]). It seems likely that one can remove the spectral assumptions on the subsystem by proving appropriate Mourre estimates.

We define the **subsystem Hamiltonians** as

$$h_{(y)(x0)} = H_{(y)(x0)}(0) = p^2 + V_{12}(x)$$

$$h_{(x)(y0)} = H_{(x)(y0)}(0) = |k| + V_{13}(y)$$

$$h_{(xy)(0)} = H_{(xy)(0)}(0) = \frac{1}{4}(p^a)^2 + \frac{1}{2}|p^a| + V_{23}(x^a)$$

We further assume the following relative boundedness and compactness properties.

$$\begin{cases} V_{13} \text{ and } y \cdot \bigtriangledown V_{13} \text{ are relatively } |k|\text{-compact as operators on } L^2(\mathbb{R}^3_y) \\ V_{13} \text{ and } y \cdot \bigtriangledown V_{13} \text{ have relative } |k|\text{-bound less than } 1 \\ V_{12} \text{ and } x \cdot \bigtriangledown V_{12} \text{ are relatively } p^2\text{-compact as operators on } L^2(\mathbb{R}^3_x) \\ V_{12} \text{ and } x \cdot \bigtriangledown V_{12} \text{ have relative } p^2\text{-bound bound less than } 1 \\ V_{23}(x^a) \text{ and } x^a \cdot \bigtriangledown V_{23}(x^a) \text{ are relatively } (p^a + s)^2\text{-compact} \\ \text{ as operators on } L^2(\mathbb{R}^3_{x^a}) \text{ for all } s \in \mathbb{R}^3 \\ V_{23}(x^a) \text{ and } x^a \cdot \bigtriangledown V_{23}(x^a) \text{ have relative } (p^a + s)^2\text{-bound} \\ \text{ less than 1 for all } s \in \mathbb{R}^3 \end{cases} \tag{RC1}$$

These hold if, for instance, each potential function is continuous and decays at infinity. The Kato-Rellich theorem and (RC1) imply that the reduced and subsystem Hamiltonians are self-adjoint on their respective domains $D(|k|)$, $D(p^2)$, and $D((p^a)^2 + |p^a|)$.

The eigenvalues of the subsystem Hamiltonians, along with zero, form the set of **thresholds**. The threshold energies are significant; if the entire system is given a threshold energy, then a subsystem may form a bound state without any kinetic energy left over to separate it from the remaining particle. This presents complications in sections below.

2.1 Definitions involving commutators

Since we will frequently need to commute unbounded operators, but commutators of unbounded operators are a priori only defined as quadratic forms, the following is convenient. For self-adjoint operators H and A, it is known that if H satisfies the $C^1(A)$ property (cf. [2]), then the commutator $[H, iA]$ is well-defined and in fact the virial theorem holds. For more, see Proposition II.1 of [19] and conditions (M) and (M') from [11].

Lemma 2.1 (Formal commutators are well defined). *Suppose we have two (possibly unbounded) self-adjoint operators H and A, where H is bounded below. A priori there exists a quadratic form $[H, iA]_0$ on $D(H) \cap D(A)$. Suppose that $[H, iA]_0$ evaluated on a space S of test vectors agrees (on S) with the closed quadratic form associated with a self-adjoint operator C defined on an operator domain $D(C)$. Then, under the following conditions:*

$$e^{itA} \text{ maps both } D(H) \text{ and } S \text{ into themselves.} \tag{FC1}$$

$$S \subset D(H) \cap D(A) \tag{FC2}$$

$$S \text{ is a core for } D(H) \tag{FC3}$$

$$D(H) \subset D(C) \tag{FC4}$$

We have that H is $C^1(A)$, $[H, iA]_0$ is closeable, the self-adjoint operator associated to its closure is C, and the virial theorem

$$\langle \psi, C\psi \rangle = 0 \text{ whenever } \psi \text{ is an eigenvector of } H.$$

holds.

We always write $E_\triangle(H)$ for the spectral projection of the operator H onto the interval $\triangle \subset \mathbb{R}$.

Definition 2.2 (Mourre estimate). *Let H and A be self-adjoint operators on a Hilbert space, let $E \in \mathbb{R}$, and let $\alpha > 0$. Suppose that H and A satisfy the hypotheses of Lemma 2.1, so that $[H, iA]$ is a well-defined self-adjoint operator with domain containing $D(H)$. H is said to satisfy a Mourre estimate at E with conjugate operator A, constant α, and width δ if there exists an open interval $\triangle = (E - \delta, E + \delta)$ and a compact operator K such that*

$$E_\triangle(H)[H, iA]E_\triangle(H) \geq \alpha E_\triangle(H) + K$$

We define the Mourre conjugate operators

$$A^a := \frac{1}{2}\left(p^a \cdot x^a + x^a \cdot p^a\right)$$

$$A_a := \frac{1}{2}\left(p_a \cdot x_a + x_a \cdot p_a\right)$$

$$A := A^a + A_a = \frac{1}{2}\left(P \cdot X + X \cdot P\right)$$

Since $C_0^\infty(\mathbb{R}^n)$ is invariant under dilations $\psi(r) \mapsto e^{-nt/2}\psi(e^t r)$ and these form a strongly continuous unitary group, all these operators (the generators of the unitary dilation groups) are known to be essentially self-adjoint on their respective C_0^∞ spaces (cf. Theorem VIII.10 of [21]).

Lemma 2.1 applies to our H and A, with S being the class of C_0^∞ functions. Condition (FC1) is satisfied since the e^{itA} are dilations, which map both $L^2(\mathbb{R}^6; (1+k^2+p^4)\ dk\ dp)$ and $C_0^\infty(\mathbb{R}^6)$ into themselves. Condition (FC2) is satisfied as well from elementary properties of smooth functions with compact support. Condition (FC3) is satisfied by the Kato-Rellich theorem; since S is a core for H_0, S is also a core for H. To show condition (FC4), we examine the formal commutator

$$C = 2p^2 + |k| - \sum_{\#(a)=2} x^a \cdot \bigtriangledown I^a$$

From another application of Kato-Rellich, this has the same domain as H. Thus the lemma applies, and $[H, iA]$ is thought of as extending to the operator C. Similarly, we may compute the commutators:

$$[H_a, iA] = 2p^2 + |k| - x^a \cdot \bigtriangledown I^a \text{ for each } a$$

$$[h_{(y)(x0)}, iA^a] = 2p^2 - x \cdot V_{12}(x)$$

$$[h_{(x)(y0)}, iA^a] = |k| - y \cdot V_{13}(y)$$

$$[h_{(xy)(0)}, iA^a] = \frac{1}{2}(p^a)^2 + \frac{1}{2}|p^a| - x^a \cdot V_{23}(x^a)$$

$$[H_{(xy)(0)}(s), iA^a] = \frac{1}{2}(p^a)^2 + \frac{1}{2}p^a \cdot s + \frac{1}{2}\frac{(p^a)^2 - s \cdot p^a}{|p^a - s|} - x^a \cdot V_{23}(x^a)$$

Since the constant involved in the Mourre estimate for our H depends on the distance from thresholds, before stating it we must describe the spectra of the subsystem Hamiltonians. The subsystem Hamiltonians all satisfy a Mourre estimate at all nonzero energies E (with conjugate operators $\mathrm{sgn}(E)A^a$). The arguments are standard and use little more than the functional calculus, all following ([19], Theorem I.1). Because of this, we would like to use Mourre theory to conclude that eigenvalues of each h_a may only accumulate at 0, and each h_a has no continuous singular spectrum. Moreover, since the essential spectrum of each h_a must be $[0,\infty)$ by Weyl's theorem (using (RC1)), there is no continuous spectrum below 0. But to draw these conclusions from the result in [19], we need to verify also a condition on the second commutators. Self-adjoint operators H and A are said to satisfy condition (2COMM) if

H and A satisfy the hypotheses of Lemma 2.1, and given the operator C extending $[H,iA]$, we have that the hypotheses are also satisfied by C and A, so that $[C,iA]$ extends to a a self-adjoint operator with domain containing $D(H)$.

(2COMM)

Computing the second commutators

$$[[h_{(y)(x0)}, iA^a], iA^a] = 4p^2 + x\cdot\bigtriangledown(x\cdot\bigtriangledown V_{12}(x))$$

$$[[h_{(x)(y0)}, iA^a], iA_y] = |k| + y\cdot\bigtriangledown(y\cdot\bigtriangledown V_{13}(y))$$

$$[[h_{(xy)(0)}, iA^a], iA^a] = (p^a)^2 + \frac{1}{2}|p^a| + x^a\cdot\bigtriangledown(x^a\cdot\bigtriangledown V_{23}(x^a))$$

we confirm that condition (2COMM) is satisfied in each case if we assume (RB2):

$$\begin{cases} y\cdot\bigtriangledown(y\cdot\bigtriangledown V_{13}(y)) \text{ has relative } |k|\text{-bound less than 1} \\ x\cdot\bigtriangledown(x\cdot\bigtriangledown V_{12}(x)) \text{ has relative } p^2\text{-bound bound less than 1} \\ x^a\cdot\bigtriangledown(x^a\cdot\bigtriangledown V_{23}(x^a)) \text{ has relative } (p^a)^2+\frac{1}{2}|p^a|\text{-bound} \\ \text{less than 1} \end{cases} \tag{RB2}$$

and apply Kato-Rellich. Thus the conditions of Mourre's paper [19] are satisfied, and we may invoke the result that eigenvalues of each h_a may only accumulate at 0, and each h_a has no continuous singular spectrum (the other part about Weyl's theorem holds irregardless). We can now use our understanding of the spectra.

For each of the 2-cluster decompositions a define

$$G_a = \inf\{\{\lambda : \lambda \text{ eigenvalue of } h_a\} \cup \{0\}\}$$

This is either an eigenvalue of h_a or 0, because eigenvalues of h_a may only accumulate at 0. Then define $d(E, a)$ to be

$$d(E,a) = \begin{cases} (E - \sup\{\lambda : \lambda \text{ zero or eigenvalue of } h_a \ , \ \lambda < E\}) : \text{ E not} \\ \qquad\qquad\qquad \text{zero nor eigenvalue of } h_a \ , \ E > G_a \\ 0 : \ \text{E zero or eigenvalue of } h_a \\ b : \ E < G_a \end{cases}$$

where b can be any positive constant one wishes. Roughly speaking, $d(E, a)$ is the distance from E to the nearest eigenvalue of h_a to the left of E. Then, let $d(E) = \min_a d(E, a)$. Then $d(E)$ is, roughly, the distance from E to the nearest threshold of any type to the left of E.

Finally we are in a position to state the first main theorem.

Theorem 2.3. *Suppose that the potential functions satisfy (RB), (RB2), (RC1), and (SPEC) and (SPEC2), as well as (RC2) below. Then for any $\epsilon > 0$ and any nonthreshold energy E, the full Hamiltonian H satisfies a Mourre estimate at energy E with conjugate operator A and constant $\alpha > 0$.*

The second main theorem is the statement of asymptotic completeness. The unitary evolutions e^{itH} and e^{itH_a} are all well-defined by the functional calculus. Following [22], we say the evolution defined by H is **asymptotically complete** if it is asymptotically clustering at all nonthreshold energies. That is, if for every nonthreshold energy E, there exists an interval $\triangle$ containing E so that whenever $\psi \in ran(E_\triangle(H))$ is orthogonal to the eigenfunctions of H, there exists $\{\phi_a\}_{\#(a)>1}$ such that

$$\lim_{t\to\infty} \|e^{-itH}\psi - \sum_{\#(a)>1} e^{-itH_a}\phi_a\| = 0$$

We prove asymptotic completeness under the additional assumptions of negative energy, short range, and exponential decay of eigenfunctions.

Theorem 2.4. *Suppose the potential functions satisfy the same assumptions as Theorem 2.3, and (SR), and (FDE) below. Then, for every* negative *nonthreshold energy E, there exists an interval $\triangle$ containing E so that whenever $\psi \in ran(E_\triangle(H))$ is orthogonal to the eigenfunctions of H, there exists $\{\phi_a\}_{\#(a)>1}$ such that*

$$\lim_{t\to\infty} \|e^{-itH}\psi - \sum_{\#(a)>1} e^{-itH_a}\phi_a\| = 0$$

Due to the Kato-Rosenblum theorem, one can also add an arbitrary trace class perturbation to H and retain the result. This should enable the study of models such as in [10].

In Section 3 we prove Theorem 2.3. In section 4, the Mourre theory is used to prove local decay and minimal velocity estimates. In Section 5, these prove Theorem 2.4.

Chapter 3

Proof of the Mourre estimate

3.1 Configuration space partition of unity

We can break up the proof of the Mourre estimate for H into problems involving each H_a by deploying a configuration space partition of unity (e.g. [9], but originally due to Deift and Simon). We need functions $\{j_a(x,y)\}_a$ on $\mathbb{R}^6$ satisfying the following requirements:

- $\sum_a j_a^2 = 1$ and each j_a is C^∞.
- Each j_a is homogeneous of degree 0 outside of the unit ball; in particular, derivatives of any j_a of any order are relatively H_0-compact since they decay in all directions.
- The following multiplier operators are relatively H_0-compact (and they remain so if j_a is replaced with any derivative of j_a).

$$\begin{cases} j_{(xy0)} \\ I_a j_a \\ [I_a, iA] j_a \end{cases} \tag{RC2}$$

In the case that the potentials are continuous and decaying in all directions, these relative compactness properties can be achieved by selecting the partition of unity so that all of the above functions decay in all directions in $\mathbb{R}^6$. The assumption (RC2) on the potentials is that these operators are indeed relatively H_0-compact for the j_a thus constructed. We proceed with the construction.

It will suffice to construct j_a that satisfy the following support conditions:

For some positive constants $C_0, \ldots, C_{11}$

- $j_{(x)(y0)}$ is supported in $\{X = (x,y) \in \mathbb{R}^6 : |x| > C_0|X|, |y| < C_1|X|, |x-y| > C_2|X|\}$.
- $j_{(y)(x0)}$ is supported in $\{X = (x,y) \in \mathbb{R}^6 : |y| > C_3|X|, |x| < C_4|X|, |x-y| > C_5|X|\}$.
- $j_{(xy)(0)}$ is supported in $\{X = (x,y) \in \mathbb{R}^6 : |x| > C_6|X|, |y| > C_7|X|, |x-y| < C_8|X|$.
- $j_{(x)(y)(0)}$ is supported in $\{X = (x,y) \in \mathbb{R}^6 : |x| > C_9|X|, |y| > C_{10}|X|, |x-y| > C_{11}|X|\}$

If these support conditions hold, then in the case that the potential functions are smooth and decay in all directions in $\mathbb{R}^3$, all the functions in (RC2) indeed decay in all directions in $\mathbb{R}^6$. We now construct such functions $\{j_a\}_a$. Define the following sets on the unit sphere $\mathbb{S}^5 \in \mathbb{R}^6$:

$$\begin{aligned}
U_{(x)(y0)} &:= \{X = (x,y) : |y| < \frac{1}{20}, |x| > \frac{1}{10}\} \\
U_{(y)(x0)} &:= \{X = (x,y) : |x| < \frac{1}{20}, |y| > \frac{1}{10}\} \\
U_{(xy)(0)} &:= \{X = (x,y) : |x| > \frac{1}{30}, |y| > \frac{1}{30}, |x-y| < \frac{1}{10}\} \\
U_{(x)(y)(0)} &:= \{X = (x,y) : |x| > \frac{1}{30}, |y| > \frac{1}{30}|x-y| > \frac{1}{20}\}
\end{aligned}$$

By the parallelogram law $2|X|^2 = |x+y|^2 + |x-y|^2$ and the Pythagorean theorem $X^2 = |x|^2 + |y|^2$, these sets are nonempty and form an open cover of the unit sphere. We are guaranteed the existence of a traditional partition of unity $\{\chi_a\}_a$ subordinate to the four sets in this open cover. By assuming homogeneity of degree 0, we can extend these functions to a traditional partition of unity on $\mathbb{R}^6 \setminus \{0\}$.

We can also partition unity in $\mathbb{R}^6$ into two functions χ_0 and χ_1 with the following properties: χ_0 is supported entirely within the unit sphere, and χ_1 is supported entirely away from 0 while being equal to 1 outside the unit sphere.

Now, we can define $\widetilde{j}_{(xy0)} := \chi_0$ and $\widetilde{j}_a := \chi_1\chi_a$ for the other clusters a. Then, by declaring $j_a := \widetilde{j}_a/\sqrt{\sum_a(\widetilde{j}_a)^2}$ we get the $\{j_a\}_a$ with all the desired properties.

3.2 Breaking apart the main estimate

We claim that for any $f \in C_0^\infty(\mathbb{R})$,

$$\sum_a f(H)[H,iA]f(H)j_a^2 = (\text{compact operators}) + \sum_{\#(a)>1} j_a f(H_a)[H_a,iA]f(H_a)j_a \quad (3.1)$$

Fix an $f \in C_0^\infty(\mathbb{R})$. The $a = (xy0)$ term is compact; the proof is reserved for the next section. Then, for each cluster decomposition $a \neq (xy0)$, we can commute around the associated term.

$$\begin{aligned}
f(H)[H,iA]f(H)j_a^2 &= f(H)[H,iA]f(H)j_a^2 - f(H)[H,iA]f(H_a)j_a^2 \\
&\quad + f(H)[H,iA]f(H_a)j_a^2 - f(H)[H,iA]j_af(H_a)j_a \\
&\quad + f(H)[H,iA]j_af(H_a)j_a - f(H)j_a[H,iA]f(H_a)j_a \\
&\quad + f(H)j_a[H,iA]f(H_a)j_a - f(H_a)j_a[H,iA]f(H_a)j_a \\
&\quad + f(H_a)j_a[I_a,iA]f(H_a)j_a + f(H_a)j_a[H_a,iA]f(H_a)j_a \\
&\quad - j_af(H_a)[H_a,iA]f(H_a)j_a + j_af(H_a)[H_a,iA]f(H_a)j_a \\
&= f(H)[H,iA]\left(f(H) - f(H_a)\right)j_a^2 + f(H)[H,iA][f(H_a),j_a]j_a \\
&\quad + f(H)[[H,iA],j_a]f(H_a)j_a + \left(f(H) - f(H_a)\right)j_a[H,iA]f(H_a)j_a \\
&\quad + f(H_a)j_a[I_a,iA]f(H_a)j_a + [f(H_a),j_a][H_a,iA]f(H_a)j_a \\
&\quad + j_af(H_a)[H_a,iA]f(H_a)j_a
\end{aligned}$$

This is an equality of bounded operators. The operator $[I_a,iA]$ should be understood not as a closure of a formal commutator $[I_a,iA]_0$ in its own right, but rather as $[H,iA]-[H_a,iA]$ which is a self-adjoint operator having domain $D(H)$.

It remains to show that all of these terms except for $j_af(H_a)[H_a,iA]f(H_a)j_a$ are compact.

3.3 Compactness

We repeatedly apply the following basic tool (indeed this was already used to describe some compactness properties of the partition of unity).

Lemma 3.1 (Elementary compactness lemma)**.** *Suppose that $f(x)$ and $g(p)$ are continuous functions $\mathbb{R}^n \to \mathbb{C}$ such that $f(x)$ and $g(p)$ decay at infinity. Then, $f(x)g(p)$ is a compact operator on $L^2(R^n)$.*

The proof is omitted.

At this point, we fix a cluster decomposition $a \neq (xy0)$. It is helpful to isolate the following.

Lemma 3.2 (Some compactness)**.** *The following operators are compact for all decompositions a.*

$$[f(H_a), j_a] \tag{3.2}$$

$$(f(H) - f(H_a))\, j_a \tag{3.3}$$

$$f(H)[[H, iA], j_a]f(H_a) \tag{3.4}$$

$$j_a[I_a, iA]f(H_a) \tag{3.5}$$

Note that this completes the proof of (3.1). To prove these operators are compact, it is convenient to replace the functions f with resolvents. To this end we prove:

Lemma 3.3 (Auxiliary compactness)**.** *The following operators are compact for all decompositions a.*

$$[\frac{1}{H_a + i}, j_a] \tag{3.6}$$

$$\left(\frac{1}{H + i} - \frac{1}{H_a + i}\right) j_a \tag{3.7}$$

$$\frac{1}{H + i}[[H, iA], j_a]\frac{1}{H_a + i} \tag{3.8}$$

$$j_a[I_a, iA]\frac{1}{H_a + i} \tag{3.9}$$

We start with the compactness of (3.6). We compute

$$\begin{aligned}\frac{1}{H_a+i}j_a - j_a\frac{1}{H_a+i} &= \frac{1}{H_a+i}j_a(H_a+i)\frac{1}{H_a+i} - \frac{1}{H_a+i}(H_a+i)j_a\frac{1}{H_a+i}\\ &= \frac{1}{H_a+i}\left(j_aH_0 - H_0j_a\right)\frac{1}{H_a+i}\end{aligned}$$

Since $\frac{1}{H_a+i}$ has range in $D(H)$ and j_a maps $D(H)$ into itself, the above are equalities of bounded operators. Since

$$\begin{aligned}&\frac{1}{H_a+i}\left(j_aH_0 - H_0j_a\right)\frac{1}{H_a+i}\\ &= \frac{1}{H_a+i}(H_0+i)\frac{1}{H_0+i}\left(j_aH_0 - H_0j_a\right)\frac{1}{H_0+i}(H_0+i)\frac{1}{H_a+i}\end{aligned}$$

and $(H_0+i)\frac{1}{H_a+i}$ is bounded by the potential assumptions, we just need to show that the operator $\frac{1}{H_0+i}\left(j_aH_0 - H_0j_a\right)\frac{1}{H_0+i}$ is compact. We obtain the following:

$$\begin{aligned}\frac{1}{H_0+i}\left(j_aH_0 - H_0j_a\right)\frac{1}{H_0+i} &= \frac{1}{H_0+i}2(\triangledown_x j_a)\cdot\triangledown_x\frac{1}{H_0+i} + \frac{1}{H_0+i}(\triangle_x j_a)\frac{1}{H_0+i}\\ &\quad + \frac{1}{H_0+i}\left(j_a|k| - |k|j_a\right)\frac{1}{H_0+i}\end{aligned}$$

Since $\frac{1}{H_0+i}(\triangle_x j_a)$ is compact by the fact that $\triangle_x j_a$ decays in all directions, the term $\frac{1}{H_0+i}(\triangle_x j_a)\frac{1}{H_0+i}$ is compact. Similarly, we have that $\frac{1}{H_0+i}2(\frac{d}{dx_\ell}j_a)$ is compact for $\ell \in \{1,2,3\}$. Moreover, $\frac{d}{dx_\ell}\frac{1}{H_0+i}$ is bounded for $\ell \in \{1,2,3\}$. Thus the term $\frac{1}{H_0+i}2(\triangledown_x j_a)\cdot\triangledown_x\frac{1}{H_0+i}$ is compact. We focus on the only remaining term, $\frac{1}{H_0+i}\left(j_a|k| - |k|j_a\right)\frac{1}{H_0+i}$, which takes more effort. We will use the 'square root lemma' $|k| = \frac{1}{\pi}\int_0^\infty \frac{s^{-\frac{1}{2}}}{s+k^2}k^2 ds$.

We assert that for all Schwartz functions $\psi(q,r)$:

$$\begin{aligned}&\frac{1}{H_0+i}\left(j_a|k| - |k|j_a\right)\frac{1}{H_0+i}\psi\\ &= \frac{1}{\pi}\int_0^\infty \frac{1}{H_0+i}\left(j_a\frac{s^{-\frac{1}{2}}}{s+k^2}k^2 - \frac{s^{-\frac{1}{2}}}{s+k^2}k^2 j_a\right)\frac{1}{H_0+i}\psi\ ds\end{aligned} \tag{3.10}$$

We need only prove that for ψ Schwartz class, $\frac{1}{\pi}\int_0^\infty \frac{s^{-\frac{1}{2}}}{s+k^2}k^2\psi\ ds$ as a $L^2(\mathbb{R}^6)$-valued integral converges to $|k|\psi$. It evidently converges pointwise to $|k|\psi$; one can check that it is convergent as a Bochner integral.

For Schwartz ψ, we may apply j_a, $\frac{s^{-1/2}}{s+k^2}$, and k^2 in any order without encountering domain issues, since each operator preserves the Schwartz class. Therefore on the Schwartz class the following hold:

$$[j_a, k^2] = (\triangle_y j_a) + 2(\triangledown_y j_a)\cdot\triangledown_y \tag{3.11}$$

$$[j_a, \frac{s^{-\frac{1}{2}}}{s+k^2}k^2] = [j_a, \frac{s^{-\frac{1}{2}}}{s+k^2}]k^2 + \frac{s^{-\frac{1}{2}}}{s+k^2}[j_a, k^2] \tag{3.12}$$

$$\begin{aligned}[j_a, \frac{s^{-\frac{1}{2}}}{s+k^2}] &= \frac{s^{-\frac{1}{2}}}{s+k^2}\frac{s+k^2}{s^{-\frac{1}{2}}}j_a\frac{s^{-\frac{1}{2}}}{s+k^2} - \frac{s^{-\frac{1}{2}}}{s+k^2}j_a\frac{s+k^2}{s^{-\frac{1}{2}}}\frac{s^{-\frac{1}{2}}}{s+k^2}\\ &= -\frac{s^{-\frac{1}{2}}}{s+k^2}(s^{\frac{1}{2}})((\triangle_y j_a) + 2(\triangledown_y j_a)\cdot\triangledown_y)\frac{s^{-\frac{1}{2}}}{s+k^2} \end{aligned}\tag{3.13}$$

Therefore, combining (3.11)-(3.13) we obtain:

$$\begin{aligned}&[j_a, \frac{s^{-\frac{1}{2}}}{s+k^2}k^2]\\ &= -\frac{s^{-\frac{1}{2}}}{s+k^2}(s^{\frac{1}{2}})((\triangle_y j_a) + 2(\triangledown_y j_a)\cdot\triangledown_y)\frac{s^{-\frac{1}{2}}}{s+k^2}k^2 + \frac{s^{-\frac{1}{2}}}{s+k^2}((\triangle_y j_a) + 2(\triangledown_y j_a)\cdot\triangledown_y)\end{aligned}\tag{3.14}$$

Moreover,

$$\begin{aligned}(2(\triangledown_y j_a)\cdot\triangledown_y) &= 2[(\triangledown_y j_a), \triangledown_y] + 2(\triangledown_y\cdot(\triangledown_y j_a))\\ &= -2(\triangle_y j_a) + 2(ik\cdot(\triangledown_y j_a))\end{aligned}$$

so that by (3.14)

$$[j_a, \frac{s^{-\frac{1}{2}}}{s+k^2}k^2]$$
$$= \frac{s^{-\frac{1}{2}}}{s+k^2}(s^{\frac{1}{2}})((\triangle_y j_a) - 2(ik\cdot(\bigtriangledown_y j_a)))\frac{s^{-\frac{1}{2}}}{s+k^2}k^2 - \frac{s^{-\frac{1}{2}}}{s+k^2}((\triangle_y j_a) - 2(ik\cdot(\bigtriangledown_y j_a))) \tag{3.15}$$

Using (3.15), we can break up the integral in (3.10).

$$\frac{1}{\pi}\int_0^\infty \frac{1}{H_0+i}\left(j_a\frac{s^{-\frac{1}{2}}}{s+k^2}k^2 - \frac{s^{-\frac{1}{2}}}{s+k^2}k^2 j_a\right)\frac{1}{H_0+i}\psi\ ds$$

$$\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \frac{1}{H_0+i}\left(-\frac{s^{-\frac{1}{2}}}{s+k^2}(\triangle_y j_a)\right)\frac{1}{H_0+i}\psi\ ds \\
&\quad + \frac{1}{\pi}\int_0^\infty \frac{1}{H_0+i}\left(\frac{s^{-\frac{1}{2}}}{s+k^2}(2ik\cdot(\bigtriangledown_y j_a))\right)\frac{1}{H_0+i}\psi\ ds \\
&\quad + \frac{1}{\pi}\int_0^\infty \frac{1}{H_0+i}\left(\frac{s^{-\frac{1}{2}}}{s+k^2}(s^{\frac{1}{2}})(\triangle_y j_a)\frac{s^{-\frac{1}{2}}}{s+k^2}k^2\right)\frac{1}{H_0+i}\psi\ ds \\
&\quad + \frac{1}{\pi}\int_0^\infty \frac{1}{H_0+i}\left(-\frac{s^{-\frac{1}{2}}}{s+k^2}(s^{\frac{1}{2}})(2ik\cdot(\bigtriangledown_y j_a))\frac{s^{-\frac{1}{2}}}{s+k^2}k^2\right)\frac{1}{H_0+i}\psi\ ds
\end{aligned}$$

Motivated by this, we compute the following operator norms for fixed s and show that they are integrable functions of s on $[0,\infty)$.

$$\|\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(\triangle_y j_a)\frac{1}{H_0+i}\|_{L^2\to L^2} \tag{3.16}$$

$$\|\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(k\cdot(\bigtriangledown_y j_a))\frac{1}{H_0+i}\|_{L^2\to L^2} \tag{3.17}$$

$$\|\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(s^{\frac{1}{2}})(\triangle_y j_a)\frac{s^{-\frac{1}{2}}}{s+k^2}(k^2)\frac{1}{H_0+i}\|_{L^2\to L^2} \tag{3.18}$$

$$\|\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(s^{\frac{1}{2}})(k\cdot(\bigtriangledown_y j_a))\frac{s^{-\frac{1}{2}}}{s+k^2}(k^2)\frac{1}{H_0+i}\|_{L^2\to L^2} \tag{3.19}$$

This would mean that the operator-valued integral

$$\frac{1}{\pi}\int_0^\infty \frac{1}{H_0+i}\left(j_a\frac{s^{-\frac{1}{2}}}{s+k^2}k^2 - \frac{s^{-\frac{1}{2}}}{s+k^2}k^2 j_a\right)\frac{1}{H_0+i}\ ds \tag{3.20}$$

converges in norm to a bounded operator $L^2(\mathbb{R}^6) \to L^2(\mathbb{R}^6)$ ([4]). Then, by Lemma 3.1, each of the four operators in (3.16)-(3.19) is in fact a compact operator for almost every s, so the integral (3.20) converges to a compact operator. Consequently, the bounded operator $\frac{1}{H_0+i}(j_a|k| - |k|j_a)\frac{1}{H_0+i}$ from (3.10) must extend uniquely from the dense Schwartz class to be this compact operator. This will conclude the proof for (3.6).

So, we prove that each of the operator norms $(3.16)-(3.19)$ are integrable functions of s.

Consider the operator in (3.16). We have,

$$\begin{aligned}\|\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(\triangle_y j_a)\frac{1}{H_0+i}\|_{L^2\to L^2} &\le \|\frac{s^{-\frac{1}{2}}}{s+k^2}\|_\infty \|\triangle_y\, j_a\|_\infty \\ &\lesssim s^{-3/2}\end{aligned}$$

so at least this is integrable near $s=\infty$. Furthermore, we have for any $0<\epsilon<1$ and $0<\delta<1$ that

$$\begin{aligned}&\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(\triangle_y j_a)\frac{1}{H_0+i} \\ &= \frac{1}{H_0+i}\left(\frac{s^{-\frac{1}{2}}}{s+k^2}|k|^{1+\epsilon}\right)\left(\frac{1}{|k|^{1+\epsilon}}\frac{1}{(1+|y|^{1+\delta})}\right)\left((1+|y|^{1+\delta})(\triangle_y j_a)\right)\frac{1}{H_0+i}\end{aligned}$$

as bounded operators. We fix (non-optimally) $\epsilon = \frac{1}{5}$ and $\delta = \frac{2}{5}$. Evidently $\left((1+|y|^{1+\delta})(\triangle_y j_a)\right)$ is a bounded operator, because $\triangle_y j_a$ is homogeneous of degree -2. By Hölder's inequality, $\frac{1}{(1+|y|^{1+\delta})}$ is a bounded operator taking $L^2(\mathbb{R}^3_y)$ into $L^{10/9}(\mathbb{R}^3_y)$. Then by a Hardy-Littlewood-Sobolev estimate (Corollary 5.10 in [18]), $\frac{1}{|k|^{6/5}}$ is a bounded operator taking $L^{10/9}(\mathbb{R}^3_y)$ into $L^2(\mathbb{R}^3_y)$. Since $\left(\frac{1}{|k|^{1+\epsilon}}\frac{1}{(1+|y|^{1+\delta})}\right)$ is therefore a bounded operator $L^2(\mathbb{R}^3_y) \mapsto L^2(\mathbb{R}^3_y)$, it extends to a bounded operator on the tensor product $L^2(\mathbb{R}^3_y \oplus \mathbb{R}^3_z)$. Lastly, the operator norm

$$\|\frac{s^{-\frac{1}{2}}}{s+k^2}|k|^{6/5}\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)} = \|\frac{s^{-\frac{1}{2}}}{s+k^2}|k|^{6/5}\|_\infty \lesssim s^{-9/10}$$

so that

$$\|\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(\triangle_y j_a)\frac{1}{H_0+i}\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)} \lesssim s^{-9/10}$$

Since the operator norm (3.16) is both $\lesssim s^{-3/2}$ and $\lesssim s^{-9/10}$, it is integrable in s.

Now we turn to (3.17). We have

$$\begin{aligned}
&\|\frac{1}{H_0+i}\frac{s^{-\frac{1}{2}}}{s+k^2}(-ik\cdot(\bigtriangledown_y j_a))\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)}\\
&\le \sum_{\ell=1}^{3}\|\frac{-ik_\ell}{i+p^2+|k|}\frac{s^{-\frac{1}{2}}}{s+k^2}(\bigtriangledown_y j_a)_\ell\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)}\\
&\le \sum_{\ell=1}^{3}\|\frac{-ik_\ell}{i+p^2+|k|}\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)}\|\frac{s^{-\frac{1}{2}}}{s+k^2}(\frac{\partial}{\partial y_\ell}j_a)\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)}\\
&\le \sum_{\ell=1}^{3}\|\frac{-ik_\ell}{i+p^2+|k|}\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)}\|\frac{s^{-\frac{1}{2}}}{s+k^2}\|_{L^\infty(\mathbb{R}^6)}\|(\frac{\partial}{\partial y_\ell}j_a)\|_{L^\infty(\mathbb{R}^6)}\\
&\lesssim s^{-3/2}
\end{aligned}$$

Now we need to deal with s small. We obtain

$$\begin{aligned}
&\|\frac{1}{i+p^2+|k|}\frac{s^{-\frac{1}{2}}}{s+k^2}(-ik\cdot(\bigtriangledown_y j_a))\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)}\\
&\le \sum_{\ell=1}^{3}\|\frac{1}{i+p^2+|k|}\frac{s^{-\frac{1}{2}}|k|}{s+k^2}(-i\frac{k_\ell}{|k|})(\frac{\partial}{\partial y_\ell}j_a)\|_{L^2(\mathbb{R}^6)\to L^2(\mathbb{R}^6)}
\end{aligned}$$

Now for each fixed x,

$$\|(\frac{\partial}{\partial y_\ell}j_a)\psi(x,y)\|_{L^{14/9}(y)} \le \|(\frac{\partial}{\partial y_\ell}j_a)\|_{L^7(y)}\|\psi(x,y)\|_{L^2(y)} \lesssim \|\psi(x,y)\|_{L^2(y)}$$

with a constant that does not depend on x. The Fourier transform is bounded $L^{14/9} \to L^{14/5}$. Since $\frac{s^{-\frac{1}{2}}|k|}{s+k^2}$ is an operator $L^{14/5}(y) \to L^2(y)$ with operator norm $\|\frac{s^{-\frac{1}{2}}|k|}{s+k^2}\|_{L^7(k)} \lesssim s^{-11/14}$, the above operator norm $L^2(\mathbb{R}^6) \to L^2(\mathbb{R}^6)$ is $\lesssim s^{-11/14}$.

Thus (3.17) is also an integrable function of s. The proof that (3.18) and (3.19) are integrable functions of s evidently reduces to the proofs for (3.16) and (3.17). Therefore the integral in (3.20) converges to a compact operator, so (3.6) is compact.

Now we prove (3.7) is compact. By the second resolvent identity

$$\left(\frac{1}{H+i} - \frac{1}{H_a+i}\right) j_a = \frac{1}{H+i}\left(I_a\right)\frac{1}{H_a+i} j_a$$

This is valid since $D(H_a) \subset D(I_a)$ by assumption. By commuting we get

$$= \frac{1}{H+i} I_a j_a \frac{1}{H_a+i} + \frac{1}{H+i} I_a [\frac{1}{H_a+i}, j_a]$$

These terms are both bounded operators a priori so this is a straightforward equality of bounded operators (We have $\frac{1}{H+i}\left(I_a\right)$ extending to a bounded operator by the relative boundedness assumptions on the potential). The first term $I_a j_a \frac{1}{H_a+i}$ is compact because of the relative compactness properties of our $I_a j_a$. The compactness of the second term $\frac{1}{H+i} I_a [\frac{1}{H_a+i}, j_a]$ then reduces to compactness result (3.6).

Next, consider (3.8). By relative boundedness assumptions it suffices to prove $\frac{1}{H_0+i}[[H, iA], j_a]\frac{1}{H_0+i}$ is compact. This is equal to $\frac{1}{H_0+i}[2p^2 + |k|, j_a]\frac{1}{H_0+i}$, which is compact by the proof for (3.7).

Finally, consider (3.9). By relative boundedness assumptions, it suffices to prove $j_a[I_a, iA]\frac{1}{H_0+i}$ is compact, which is true from the relative compactness properties of the j_a. This concludes the proof of Lemma 3.3, so we turn to the proof of Lemma 3.2.

If a subset $\mathcal{F} \subseteq C_\infty(\mathbb{R})$, the continuous functions vanishing at infinity, has the following properties:

1. $\mathcal{F}$ contains resolvents $\frac{1}{x+\xi}$ for all ξ in some open set $u \subset \mathbb{C}$.

2. $\mathcal{F}$ forms a vector space.

3. $\mathcal{F}$ is closed under convergence in the L^∞ norm.

Then $\mathcal{F} = C_\infty(\mathbb{R})$ (Appendix to ch. 3 in [6]).

Let $\mathcal{F}$ be the class of such functions f so that (3.2) is compact. We have proven the first property in Lemma 3.3 (the choice of $\xi = i$ was arbitrary), and the second and third properties are evident. Since f was taken to be C_0^∞ in Lemma 3.1, this is sufficient for (3.2).

Now let $\mathcal{F}$ be the class of such functions f so that (3.3) is compact. Again, we have proven the first property, and the second and third properties are evident.

For (3.4) and (3.5), one can prove compactness by multiplying and dividing by resolvents to reduce to (3.8) and (3.9).

Thus, we do indeed have the breaking apart of the main estimate as in (3.1).

3.4 The cluster $(xy0)$

Directly from the previous section we have that $f(H)j_{(xy0)}$ is compact:

$$f(H)j_{(xy0)} = f(H)(H+i)\frac{1}{H+i}(H_0+i)\frac{1}{H_0+i}j_{(xy0)}$$

which is compact since $f(H)(H+i)$ is bounded, $\frac{1}{H+i}(H_0+i)$ is bounded, and $\frac{1}{H_0+i}j_{(xy0)}$ is compact by Lemma 3.1. Therefore $j_{(xy0)}f(H)[H,iA]f(H)j_{(xy0)}$ is compact as claimed in (3.1).

3.5 The cluster $(x)(y)(0)$

In this section, we fix $a = (x)(y)(0)$, the cluster decomposition corresponding to free dynamics.

Lemma 3.4. *Fix $\epsilon > 0$ and an energy $E \neq 0$. Then there exists $\delta > 0$ so that H_a satisfies a Mourre estimate at E with conjugate operator A, width δ, and constant $\alpha_{(x)(y)(0)}$, where*

$$\alpha_{(x)(y)(0)} = \begin{cases} E - \epsilon : \; E > 0 \\ c : \; E < 0 \end{cases}$$

and c is any positive constant one wishes.

Proof for $E < 0$. Since $E_\triangle(H_a) = 0$ the desired operator inequality is trivial. $\square$

Proof for $E > 0$. Fix $\epsilon > 0$. Pick $\delta < \min(E, \epsilon)$, and select $\triangle = (E - \delta, E + \delta)$. We have

$$\begin{aligned} E_\triangle(H_a)[H_a, iA]E_\triangle(H_a) &= E_\triangle(p^2 + |k|) \left(2p^2 + |k|\right) E_\triangle(p^2 + |k|) \\ &\geq (E - \delta)E_\triangle(p^2 + |k|) \\ &\geq (E - \epsilon)E_\triangle(p^2 + |k|) \end{aligned}$$

from the functional calculus. $\square$

3.6 The cluster $(x)(y0)$

In this section, fix $a = (x)(y0)$, the cluster decomposition corresponding to the photon-proton cluster. Our aim is to prove the following.

Lemma 3.5. *Fix $\epsilon > 0$ and an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a. Then there exists $\delta > 0$ so that H_a satisfies a Mourre estimate at E with conjugate operator A, width δ, and constant $\alpha_{(x)(y0)}$, where*

$$\alpha_{(x)(y0)} = \begin{cases} \min(2d(E, a) - \epsilon, E - \epsilon) : \; E > 0 \\ 2d(E, a) - \epsilon : \; E < 0 \end{cases}$$

The strategy of proof is to consider each value of the electron's relative momentum r separately. Because r commutes with the operator $[H_a, iA]$, we can defined the **fibered commutator**

$$[H_a, iA](s) := 2s^2 + |k| - y \cdot \triangledown V_{13}(y)$$

which is a self-adjoint operator on $L^2(\mathbb{R}^3_y)$ by Kato-Rellich. We can then write as a direct integral:

$$\begin{aligned}
&E_\triangle(H_a)[H_a, iA]E_\triangle(H_a) \\
&= E_\triangle(p^2 + |k| + V_{13}(y)) \left(2p^2 + |k| - y \cdot \triangledown_y V_{13}(y)\right) E_\triangle(p^2 + |k| + V_{13}(y)) \\
&= \int_{\mathbb{R}^3_s}^{\oplus} E_\triangle(s^2 + |k| + V_{13}(y)) \left(2s^2 + |k| - y \cdot \triangledown_y V_{13}(y)\right) E_\triangle(s^2 + |k| + V_{13}(y)) ds \\
&= \int_{\mathbb{R}^3_s}^{\oplus} E_\triangle(H_a(s))[H_a, iA](s) E_\triangle(H_a(s)) ds
\end{aligned}$$

It is then sufficient to prove a uniform Mourre estimate on each fiber. For fibers where the relative momentum s is large, the positive contribution to the commutator comes from $2s^2$. For fibers where the relative momentum is small, we rely on the energy E being away from thresholds for the positive contribution. With this in mind, we prove Lemma 3.6 and Lemma 3.7, which show that the remaining terms are small.

Lemma 3.6. *Fix $\epsilon > 0$, an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a, and a value of $s_0 \neq 0$. Then there exists $\delta > 0$ and an open set $U \subset \mathbb{R}^3$ containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:*

$$E_\triangle(H_a(s))[h_a, iA^a]E_\triangle(H_a(s)) \geq -\epsilon E_\triangle(H_a(s))$$

Proof. Fix ϵ, E and s_0 as above. We may select $\delta_0 > 0$ so that for $\triangle_0 = (E-\delta_0, E+\delta_0)$:

$$E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0)) \geq -\frac{\epsilon}{4} E_{\triangle_0}(H_a(s_0)) + K \tag{3.21}$$

where K is a compact operator. To see why, we break up into three cases. First, if $E - s_0^2$ is a negative eigenvalue of h_a, then choose δ_0 so small that $\triangle_0 - s_0^2$ contains no continuous spectrum of h_a. Then by the virial theorem for $[h_a, iA^a]$ we have

$$
\begin{aligned}
&E_{\triangle}(H_a(s_0))[h_a, iA^a]E_{\triangle}(H_a(s_0)) \\
&= E_{\triangle}(s_0^2 + |k| + V_{13}(y))[h_a, iA^a]E_{\triangle}(s_0^2 + |k| + V_{13}(y)) \\
&= E_{\triangle - s_0^2}(h_a)[h_a, iA^a]E_{\triangle - s_0^2}(h_a) \\
&= 0
\end{aligned}
$$

Second, if $E - s_0^2$ is negative but in the resolvent set of h_a, we can pick δ_0 so small that $\triangle_0 - s_0^2$ contains no spectrum of h_a at all, and the desired estimate (3.21) follows because the projections $E_{\triangle - s_0^2}(h_a)$ are zero. Third, if $E - s_0^2$ is nonnegative, we can take $0 < \delta_0 < E - s_0^2 + \frac{\epsilon}{4}$. Then we compute

$$
\begin{aligned}
&E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0)) \\
&= E_{\triangle_0}(s_0^2 + |k| + V_{13}(y))\Bigg(|k| - y \cdot \triangledown V_{13}(y)\Bigg)E_{\triangle_0}(s_0^2 + |k| + V_{13}(y)) \\
&= E_{\triangle_0 - s_0^2}(|k| + V_{13}(y))\Bigg(|k| - y \cdot \triangledown V_{13}(y)\Bigg)E_{\triangle_0 - s_0^2}(|k| + V_{13}(y)) \\
&= E_{\triangle_0 - s_0^2}(|k| + V_{13}(y))\Bigg(|k| + V_{13}(y)\Bigg)E_{\triangle_0 - s_0^2}(|k| + V_{13}(y)) \\
&\quad + E_{\triangle_0 - s_0^2}(|k| + V_{13}(y))\Bigg(- V_{13}(y) - y \cdot \triangledown V_{13}(y)\Bigg)E_{\triangle_0 - s_0^2}(|k| + V_{13}(y)) \\
&= (E - s_0^2 - \frac{\epsilon}{4})E_{\triangle_0 - s_0^2}(|k| + V_{13}(y)) + K \\
&\geq -\frac{\epsilon}{4}E_{\triangle_0}(H_a(s_0)) + K
\end{aligned}
$$

by using the functional calculus, where

$$
K = E_{\triangle_0 - s_0^2}(h_a)\Bigg(- V_{13}(y) - y \cdot \triangledown V_{13}(y)\Bigg)E_{\triangle - s_0^2}(h_a)
$$

is compact. Thus we can always find δ_0 so that (3.21) holds. All that remains is to prove Lemma 3.6 is to remove the compact K. For another proof in this vein, see (Eqn. (3.4) in [9]).

Fix such a δ_0 as in (3.21). We write e.g. $E_{pp\triangle_0}(H_a(s_0)) := E_{pp}(H_a(s_0))E_{\triangle_0}(H_a(s_0))$, where $E_{pp}(H_a(s_0))$ represents the projection onto the pure point spectrum. The next claim is that we have:

$$E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0))$$
$$\geq -\frac{\epsilon}{2}E_{\triangle_0}(H_a(s_0)) + (1 - E_{pp\triangle_0}(H_a(s_0)))K_1(1 - E_{pp\triangle_0}(H_a(s_0))) \tag{3.22}$$

for some compact K_1. Assuming (3.22), then we could select δ_1 so small that for $\triangle_1 = (E - \delta_1, E + \delta_1)$, we could multiply both sides of the above by $E_{\triangle_1}(H_a(s_0))$ and use

$$(E_{\triangle_1}(H_a(s_0)) - E_{pp\triangle_1}(H_a(s_0)))K_1(E_{\triangle_1}(H_a(s_0)) - E_{pp\triangle_1}(H_a(s_0))) \geq -\frac{\epsilon}{2}E_{\triangle_1}(H_a(s_0))$$

to conclude that:

$$E_{\triangle_1}(H_a(s_0))[h_a, iA^a]E_{\triangle_1}(H_a(s_0)) \geq -\epsilon E_{\triangle_1}(H_a(s_0)) \tag{3.23}$$

Then we would select $\delta = \delta_1/2$ and define $U := \{s \in \mathbb{R}^3 : s_0^2 - \delta < s^2 < s_0^2 + \delta\}$. This would give us the conclusion of the lemma, letting $\triangle = (E - \delta, E + \delta)$; for any $s \in U$, we could prove the desired inequality by taking (3.23) and multiplying on both sides by $E_{\triangle - s^2}(h_a)$. This is due to the fact that as defined, $\triangle - s^2 \subset \triangle_1 - s_0^2$ for any $s \in U$. So it remains to show (3.22).

Since K is compact, we may select a finite-dimensional projection F with range contained in that of $E_{\triangle_0 pp}(H_a(s_0))$ so that

$$\|(1 - F)K(1 - F) - (1 - E_{\triangle_0 pp}(H_a(s_0)))K(1 - E_{\triangle_0 pp}(H_a(s_0)))\| \leq \frac{\epsilon}{2}$$

Then, from multiplying (3.21) on both sides by $(1 - F)$, we obtain

$$(E_{\triangle_0}(H_a(s_0)) - F)[h_a, iA^a](E_{\triangle_0}(H_a(s_0)) - F)$$
$$\geq -\frac{\epsilon}{4}(E_{\triangle_0 pp}(H_a(s_0)) - F) + (1 - F)K(1 - F)$$

By using the property by which F was obtained, we glean from this that

$$
\begin{aligned}
&(E_{\triangle_0}(H_a(s_0)) - F)[h_a, iA^a](E_{\triangle_0}(H_a(s_0)) - F) \\
&\geq -\frac{\epsilon}{2}(E_{\triangle_0 pp}(H_a(s_0)) - F) + (1 - E_{\triangle_0 pp}(H_a(s_0)))K(1 - E_{\triangle_0 pp}(H_a(s_0)))
\end{aligned}
$$

We can multiply out the left-hand side of the above, and apply the virial theorem to get

$$
\begin{aligned}
&E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0)) - C^*F - F^*C \\
&\geq -\frac{\epsilon}{2}(E_{\triangle_0 pp}(H_a(s_0)) - F) + (1 - E_{\triangle_0 pp}(H_a(s_0))K(1 - E_{\triangle_0 pp}(H_a(s_0))
\end{aligned}
$$

where we have let $C = F[h_a, iA^a]E_{\triangle_0}(H_a(s_0))(1 - E_{\triangle_0 pp}(H_a(s_0)))$. To obtain (3.22) from this is a matter of showing that for some compact K_2,

$$C^*F + F^*C \geq (1 - E_{\triangle_0 pp})K_2(1 - E_{\triangle_0 pp}) + \frac{\epsilon}{2}F^*F$$

Yet this follows from

$$C^*F + F^*C \geq -(\frac{2}{\epsilon}C^*C + \frac{\epsilon}{2}F^*F)$$

letting $K_2 = -\frac{2}{\epsilon}C^*C$. So let $K_1 = (1 - E_{\triangle_0 pp})(K + K_2)(1 - E_{\triangle_0 pp})$. Note that it doesn't matter how K_1 depends on ϵ, because of the discussion surrounding (3.22) and (3.23). That completes the analysis. □

We will also use the following.

Lemma 3.7. *Fix $\epsilon > 0$, an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a, and a value of $s_0 \neq 0$ so that $E - s_0^2$ is not an eigenvalue of h_a. Then there exists $\delta > 0$ and an open set $U \subset \mathbb{R}^3$ containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:*

$$E_{\triangle}(H_a(s))(-V_{13}(y) - y \cdot \bigtriangledown V_{13}(y))E_{\triangle}(H_a(s)) \geq -\epsilon E_{\triangle}(H_a(s))$$

Proof. Fix ϵ, E, and s_0 as above. Then there is a width $\delta_0 > 0$ so that $(E - \delta_0 - s_0^2, E + \delta_0 - s_0^2)$ contains no eigenvalues of h_a. Then, there exists $\delta_1 \leq \delta_0$ small enough so that for $\triangle_1 = (E - \delta_1, E + \delta_1)$:

$$E_{\triangle_1}(s_0^2 + |k| + V_{13}(y)(-V_{13}(y) - y \cdot \bigtriangledown V_{13}(y))E_{\triangle_1}(s_0^2 + |k| + V_{13}(y))$$
$$\geq -\epsilon E_{\triangle_1}(s_0^2 + |k| + V_{13}(y))$$

We select $\delta = \delta_1/2$, and let $U := \{s \in \mathbb{R}^3 : s_0^2 - \delta < s^2 < s_0^2 + \delta\}$. As before, this concludes the proof. $\square$

These two lemmas give us sufficient control over the junk terms in the fibered commutator $[H_a, iA](s)$, so we can now attack it.

Lemma 3.8. *Fix $\epsilon > 0$, an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a and a value $s_0 \in \mathbb{R}^3$. Then there exists $\delta > 0$ and an open set U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$ we have:*

$$E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) \geq \alpha_{(x)(y0)}E_{\triangle}(H_a(s))$$

Note that the dependence on ϵ comes from the way $\alpha_{(x)(y0)}$ was defined.

For the proof, we fix ϵ, E and s_0 as above. Select δ_0 so that $\triangle_0 = [E - \delta_0, E + \delta_0]$ does not contain 0 or any thresholds, and also so $\delta_0 < \epsilon/2$. Then select $\tau > 0$ so that $[E - \delta_0 - t, E + \delta_0 - t]$ does not contain 0 or any thresholds for $0 \leq t \leq \tau$. Furthermore τ can be selected so $\tau \geq d(E, a) - \epsilon$. The choice of τ serves to separate our analysis into ‘small external momentum’ and ‘large external momentum’.

We handle the cases of $E < 0$ and $E > 0$ separately.

Proof for $E < 0$. Consider the small-momentum case, $s_0^2 \leq \tau$. Then, the projection $E_{\triangle_0}(H_a(s_0))$ is evidently zero. We may select $\delta = \delta_0/2$ and $U := \{s \in \mathbb{R}^3 : s_0^2 - \delta < s^2 < s_0^2 + \delta\}$. Then for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \geq (2\tau - \epsilon)E_\triangle(H_a(s))$$

for all $s \in U$, since the projections $E_\triangle(H_a(s))$ are zero for all such s, so we may have in fact any constant we wish (in place of $2\tau - \epsilon$).

Now consider the large-momentum case, $s_0^2 \geq \tau$. By Lemma 3.6, we may select $\delta < \delta_0$ and U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$E_\triangle(H_a(s))[h_a, iA^a]E_\triangle(H_a(s)) \geq -\epsilon E_\triangle(H_a(s))$$

Then, we have

$$\begin{aligned}
&E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \\
&= E_\triangle(H_a(s))\Bigg(2s^2 + [h_a, iA^a]\Bigg)E_\triangle(H_a(s)) \\
&\geq 2\tau E_\triangle(H_a(s)) - \epsilon E_\triangle(H_a(s)) \\
&\geq (2\tau - \epsilon)E_\triangle(H_a(s))
\end{aligned}$$

for all $s \in U$.

So, for any fiber s_0: there exists $\delta > 0$ and a U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$\begin{aligned}
E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) &\geq (2\tau - \epsilon)E_\triangle(H_a(s)) \\
&\geq (2d(E, a) - 3\epsilon)E_\triangle(H_a(s))
\end{aligned}$$

where the last line holds because $\tau \geq d(E,a) - \epsilon$. By a renaming of ϵ we have our conclusion. $\square$

Proof for $E > 0$. Consider the small-momentum case, $s_0^2 \leq \tau$. By Lemma 3.7, we may select $\delta < \delta_0$ and U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$E_{\triangle}(H_a(s))(-V_{13}(y) - y \cdot \bigtriangledown V_{13}(y))E_{\triangle}(H_a(s)) \geq -\frac{\epsilon}{2}E_{\triangle}(H_a(s))$$

Then, we have

$$\begin{aligned}
&E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) \\
&= E_{\triangle}(H_a(s))\bigg(2s^2 + |k| + V_{13}(y) - V_{13}(y) - y \cdot \bigtriangledown V_{13}(y)\bigg)E_{\triangle}(H_a(s)) \\
&\geq (E - \delta_0)E_{\triangle}(H_a(s)) + E_{\triangle}(H_a(s))\bigg(-V_{13}(y) - y \cdot \bigtriangledown V_{13}(y)\bigg)E_{\triangle}(H_a(s)) \\
&\geq (E - \delta_0 - \frac{\epsilon}{2})E_{\triangle}(H_a(s)) \\
&\geq (E - \epsilon)E_{\triangle}(H_a(s))
\end{aligned}$$

for all $s \in U$, where the second-to-last step is by the functional calculus..

The large-momentum case for $E > 0$ is handled the same way as the large-momentum case for $E < 0$; the same estimate with constant $(2\tau - \epsilon)$ holds.

So, for any fiber s_0^2: there exists a $\delta > 0$ and a U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$\begin{aligned}
E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) &\geq \min(E - \epsilon, 2\tau - \epsilon)E_{\triangle}(H_a(s)) \\
&\geq \min(E - \epsilon, 2d(E,a) - 3\epsilon)E_{\triangle}(H_a(s))
\end{aligned}$$

where the last line holds because $\tau \geq d(E,a) - \epsilon$. By a renaming of ϵ we have our conclusion. $\square$

Now we can proceed with the proof of Lemma 3.5.

Proof. Fix $\epsilon > 0$ and an energy E. If $E < G_a$, then by taking δ small enough, the projections $E_\triangle(H_a)$ are zero and the conclusion is trivial, so we may assume $E \geq G_a$. Let M be a number so that $M >> E - G_a$. Take the compact set $\{s \in \mathbb{R}^3 : s^2 \leq M\}$ and use Lemma 3.8 to cover it with sets U_i, so that there exists a $\delta_i > 0$ so that for each $s \in U_i$, taking $\triangle_i = (E - \delta_i, E + \delta_i)$:

$$E_{\triangle_i}(H_a(s))[H_a, iA](s)E_{\triangle_i}(H_a(s)) \geq \alpha_{(x)(y0)}E_{\triangle_i}(H_a(s))$$

Extract a finite subcover and let δ be the minimum over the finite collection of δ_i associated to the subcover. It is then the case that for all s such that $s^2 \leq M$, taking $\triangle = (E - \delta, E + \delta)$:

$$E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \geq \alpha_{(x)(y0)}E_\triangle(H_a(s))$$

Since the projections are 0 when $s^2 > E - G_a$, the above inequality is also true for $s^2 > M$. We conclude that the inequality holds for all $s \in \mathbb{R}^3$. Thus we have an inequality on the whole direct integral

$$\int_{\mathbb{R}^3_s}^{\oplus} E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s))ds \geq \alpha_{(x)(y0)} \int_{\mathbb{R}^3_s}^{\oplus} E_\triangle(H_a(s))ds$$

which is exactly Lemma 3.5. □

3.7 The cluster $(y)(x0)$

In what follows, fix $a = (y)(x0)$, the cluster decomposition corresponding to the photon-proton cluster. The analysis is much the same as the previous cluster but is outlined for the sake of completeness. Our aim is to prove the following.

Lemma 3.9. *Fix $\epsilon > 0$ and an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a. Then there exists $\delta > 0$ so that H_a satisfies a Mourre estimate at E with conjugate operator A, width δ, and constant $\alpha_{(y)(x0)}$, where*

$$\alpha_{(y)(x0)} = \begin{cases} \min(d(E,a)-\epsilon, E-\epsilon): & E>0 \\ d(E,a)-\epsilon: & E<0 \end{cases}$$

Because q commutes with the operator $[H_a, iA]$, we can define the fibered commutator

$$[H_a, iA](s) = |s| + 2p^2 - x \cdot \bigtriangledown V_{12}(x)$$

and then writing as a direct integral:

$$\begin{aligned}
&E_{\triangle}(H_a)[H_a, iA]E_{\triangle}(H_a) \\
&= E_{\triangle}(p^2 + |k| + V_{12}(x)) \left(2p^2 + |k| - x \cdot \bigtriangledown V_{12}(x)\right) E_{\triangle}(p^2 + |k| + V_{12}(x)) \\
&= \int_{\mathbb{R}^3_s}^{\oplus} E_{\triangle}(|s| + p^2 + V_{12}(x)) \left(|s| + 2p^2 - x \cdot \bigtriangledown V_{12}(x)\right) E_{\triangle}(|s| + p^2 + V_{12}(x)) ds \\
&= \int_{\mathbb{R}^3_s}^{\oplus} E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) ds
\end{aligned}$$

It will be sufficient to prove a Mourre estimate of each fiber. The positive contribution will come from $|s|$ for large values of s, and from the energy E being nonthreshold for small values of s. Lemmas 3.10 and 3.11 show that the remaining terms are small.

Lemma 3.10. *Fix $\epsilon > 0$, an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a, and a value of $s_0 \neq 0$, there exists $\delta = \delta(s_0, \epsilon) > 0$ and and open set $U \subset \mathbb{R}^3$ containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:*

$$E_{\triangle}(H_a(s))[h_a, iA^a]E_{\triangle}(H_a(s)) \geq -\epsilon E_{\triangle}(H_a(s))$$

Proof. Fix ϵ, E, and s_0 as above. We may select $\delta_0 > 0$ so that for $\triangle_0 = (E-\delta_0, E+\delta_0)$:

$$E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0)) \geq -\frac{\epsilon}{4} E_{\triangle_0}(H_a(s_0)) + K \qquad (3.24)$$

where K is a compact operator. To see why, break up into three cases. If $E-|s_0|$ is a negative eigenvalue of h_a, then choose δ_0 so small that $\triangle_0 - |s_0|$ contains no continuous spectrum of h_a. Then by the virial theorem for $[h_a, iA^a]$, we have

$$E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0)) = 0$$

Second, if $E - |s_0|$ is negative but in the resolvent set of h_a, we can pick δ_0 so small that $\triangle_0 - |s_0|$ contains no spectrum of h_a at all, whereby the desired estimate (3.24) follows because the projections $E_{\triangle_0}(H_a(s_0))$ are zero. Third, if $\triangle_0-|s_0|$ is nonnegative, we can take $0 \leq \delta_0 < E - |s_0| + \frac{\epsilon}{4}$. Then we compute

$$\begin{aligned}
&E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0)) \\
&= E_{\triangle_0-|s_0|}(p^2 + V_{12}(x))\Bigg(p^2 + V_{12}(x)\Bigg)E_{\triangle_0-|s_0|}(p^2 + V_{12}(x)) \\
&\quad + E_{\triangle_0-|s_0|}(p^2 + V_{12}(x))\Bigg(-V_{12}(x) - x\cdot\bigtriangledown V_{12}(x)\Bigg)E_{\triangle_0-|s_0|}(p^2 + V_{12}(x)) \\
&\geq (E - |s_0| - \frac{\epsilon}{4})E_{\triangle_0-|s_0|}(p^2 + V_{12}(x)) + K \\
&\geq -\frac{\epsilon}{4}E_{\triangle_0-|s_0|}(p^2 + V_{12}(x)) + K
\end{aligned}$$

by using the functional calculus, where

$$K = E_{\triangle_0-|s_0|}(h_a)\Bigg(-V_{12}(x) - x\cdot\bigtriangledown V_{12}(x)\Bigg)E_{\triangle_0-|s_0|}(h_a)$$

is compact. Thus we can always find δ_0 so that (3.24) holds. It remains to remove the compact K.

Fix such a δ_0 as in (3.24). We claim that

$$\begin{aligned}
&E_{\triangle_0}(H_a(s_0))[h_a, iA^a]E_{\triangle_0}(H_a(s_0)) \\
&\geq -\frac{\epsilon}{2}E_{\triangle_0}(H_a(s_0)) + (1 - E_{pp\triangle_0}(H_a(s_0)))K_1(1 - E_{pp\triangle_0}(H_a(s_0)))
\end{aligned} \tag{3.25}$$

for some compact K_1. Assuming (3.25), we could then select δ_1 so small that for $\triangle_1 = (E - \delta_1, E + \delta_1)$, we could multiply both sies of the above by $E_{\triangle_1}(H_a(s_0))$ and use

$$(E_{\triangle_1}(H_a(s_0)) - E_{pp\triangle_1}(H_a(s_0)))K_1(E_{\triangle_1}(H_a(s_0)) - E_{pp\triangle_1}(H_a(s_0))) \geq -\frac{\epsilon}{2}E_{\triangle_1}(H_a(s_0))$$

to conclude that

$$E_{\triangle_1}(H_a(s_0))[h_a(y), iA^a]E_{\triangle_1}(H_a(s_0)) \geq -\epsilon E_{\triangle_1}(H_a(s_0)) \tag{3.26}$$

Then we would select $\delta = \delta_1/2$ and define $U =: \{s \in \mathbb{R}^3 : |s_0| - \delta < |s| < |s_0| + \delta$. This would give us the conclusion of the lemma, letting $\triangle = (E - \delta, E + \delta)$; for any $s \in U$, we could prove the desired inequality by taking (3.26) and multiplying on both sides by $E_{\triangle - |s|}(h_a)$. So it remains to show (3.25). But this follows from the same argument as the proof of (3.22) from the previous section. □

We will also use the following.

Lemma 3.11. *Fix $\epsilon > 0$, an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a, and a value of $s_0 \neq 0$ so that $E - |s_0|$ is not an eigenvalue of h_a. Then there exists $\delta > 0$ and an open set U containing s_0 so that for all $s_0 \in U$, letting $\triangle = (E - \delta, E + \delta)$:*

$$E_\triangle(H_a(s)) + (-V_{12}(x) - x \cdot \triangledown V_{12}(x))E_\triangle((H_a(s)) \geq -\epsilon E_\triangle((H_a(s))$$

Proof. Fix ϵ, E, and s_0 as above. Then there is a width $\delta_0 > 0$ so that $(E - \delta_0 - |s_0|, E + \delta_0 - |s_0|)$ contains no eigenvalues of h_a. Then, there exists $\delta_1 \leq \delta_0$ so that for $\triangle_1 = (E - \delta_1, E + \delta_1)$:

$$E_{\triangle_1}(|s| + p^2 + V_{12}(x) + (-V_{12}(x) - x \cdot \triangledown V_{12}(x))E_{\triangle_1}(|s| + p^2 + V_{12}(x))$$
$$\geq -\epsilon E_{\triangle_1}(p^2 + |s| + V_{12}(x))$$

We select $\delta = \delta_1/2$, and let $U = \{s \in \mathbb{R}^3 : |s_0| - \delta < |s| < |s_0| + \delta\}$. □

Lemma 3.12. *Fix $\epsilon > 0$, an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian h_a, and a value of $s_0 \neq 0$. Then there exists a $\delta > 0$ and an open set $U \subset \mathbb{R}^3$ containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:*

$$E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \geq \alpha_{(y)(x0)}E_\triangle(H_a(s))$$

Fix $\epsilon > 0$, E, and s_0 as above. Select δ_0 so that $\triangle_0 = [E-\delta_0, E+\delta_0]$ does not contain 0 or any thresholds, and also so $\delta_0\epsilon/2$. Then select $\tau > 0$ so that $[E - \delta_0 - t, E + \delta_0 - t]$ does not contain 0 or any thresholds for $0 \leq t \leq \tau$. Furthermore τ can be selected to be $\geq d(E, a) - \epsilon$.

We handle the cases of $E < 0$ and $E > 0$ separately.

Proof for $E < 0$. Now consider $E < 0$.

Consider $|s_0| \leq \tau$. The projection $E_{\triangle_0}(H_a(s_0))$ is evidently 0. We may select $\delta = \delta_0$ and $U := \{s \in \mathbb{R}^3 : |s_0| - \delta < |s| < |s_0| - \delta\}$. Then for all $s \in U$, letting $\triangle = (E - \delta, E_+\delta)$:

$$E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \geq (\tau - \epsilon)E_\triangle(H_a(s))$$

since the projections $E_\triangle(H_a(s))$ are zero for all such s so we may have any constant we wish (in place of $(\tau - \epsilon)$).

Now consider $|s_0| \geq \tau$. By Lemma 3.10, we may select $\delta < \delta_0$ and U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$E_\triangle(H_a(s))[h_a, iA^a]E_\triangle(H_a(s)) \geq -\epsilon E_\triangle(H_a(s))$$

Then, we have

$$E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) = E_{\triangle}(H_a(s))\left(|s| + [h_a, iA^a]\right)E_{\triangle}(H_a(s))$$
$$\geq \tau E_{\triangle}(H_a(s)) - \epsilon E_{\triangle}(H_a(s))$$
$$= (\tau - \epsilon)E_{\triangle}(H_a(s))$$

for all $s \in U$.

So, for any fiber s_0: there exists a U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) \geq (\tau - \epsilon)E_{\triangle}(H_a(s))$$
$$\geq (d(E, a) - 2\epsilon)E_{\triangle}(H_a(s))$$

where the last line holds because $\tau \geq d(E, a) - \epsilon$. By a renaming of ϵ we have our conclusion.

□

Proof for $E > 0$. Consider $s_0 \leq \tau$. By Lemma 3.11, we may select $\delta < \delta_0$ and U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$E_{\triangle}(H_a(s))(-V_{12}(x) - x \cdot \bigtriangledown V_{12}(x))E_{\triangle}(H_a(s)) \geq -\frac{\epsilon}{2}E_{\triangle}(H_a(s))$$

Then, we have

$$
\begin{aligned}
&E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \\
&= E_\triangle(H_a(s))\Big(2p^2 + |s| + V_{12}(x) - V_{12}(x) - x \cdot \bigtriangledown V_{12}(x)\Big)E_\triangle(H_a(s)) \\
&\geq (E - \delta_0)E_\triangle(H_a(s)) + E_\triangle(H_a(s))\Big(- V_{12}(x) - x \cdot \bigtriangledown V_{12}(x)\Big)E_\triangle(H_a(s)) \\
&\geq (E - \delta_0 - \frac{\epsilon}{2})E_\triangle(H_a(s)) \\
&\geq (E - \epsilon)E_\triangle(H_a(s))
\end{aligned}
$$

for all $s \in U$.

The case $|s_0| \geq \tau$ for $E > 0$ is handled in the same way as the for $E < 0$; the same estimate with constant $(\tau - \epsilon)$ holds.

So, for any fiber s_0: there exists a $\delta > 0$ and a U containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$:

$$
\begin{aligned}
E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) &\geq \min(\tau - \epsilon, E - \epsilon)E_\triangle(H_a(s)) \\
&\geq \min(d(E, a) - 2\epsilon, E - \epsilon)E_\triangle(H_a(s))
\end{aligned}
$$

By a renaming of ϵ we have our conclusion. $\square$

Using this, the proof of Lemma 3.9 is the same as the proof of Lemma 3.5.

3.8 The cluster $(xy)(0)$

In what follows, fix $a = (xy)(0)$, the cluster decomposition corresponding to the electron-photon cluster. We aim to prove the following.

Lemma 3.13. *Fix $\epsilon > 0$ and an energy $E \neq 0$ not an eigenvalue of the subsystem Hamiltonian $h_a = \frac{1}{4}(p^a)^2 + \frac{1}{2}|p^a| + V_{23}(x^a)$. Then there exists a $\delta > 0$ so that H_a satisfies a Mourre estimate at E with conjugate operator A, width δ, and constant $\alpha_{(xy)(0)} > 0$.*

As done for the other 2-cluster decompositions, we can write the commutator $[H_a, iA]$ as a direct integral over the fibered commutators

$$[H, iA](s) = \frac{1}{2}(p^a + s)^2 + \frac{1}{2}|p^a - s| - x^a \cdot V_{23}(x^a)$$

We note that s does not separate out from the fibered commutators, which means that the positive commutator estimate in the large s case will require a different strategy than for the other decompositions (The small s case will again exploit the choice of E away from thresholds). This different strategy requires us to know a little more about the spectrum of $H_a(s)$.

Lemma 3.14 (The spectrum of $H_{(xy)(0)}(s)$)**.** *Under the spectral assumptions (SPEC) and (SPEC2) the continuous spectrum of $H_a(s)$ is $[\min_{p^a}(\frac{1}{4}(p^a + s)^2 + \frac{1}{2}|p^a - s|), \infty)$, and the eigenvalues of $H_a(s)$ below that are isolated, increasing as $|s|$ increases.*

Proof. The essential spectrum of $H_a(s)$ is contained in $[\min_{p^a}(\frac{1}{4}(p^a+s)^2+\frac{1}{2}|p^a-s|), \infty)$ by Weyl's theorem. This minimum can be computed as

$$\min_{p^a}(\frac{1}{4}(p^a + s)^2 + \frac{1}{2}|p^a - s|) = \begin{cases} s^2 : & |s| \leq \frac{1}{2} \\ |s| - \frac{1}{4} : & |s| > \frac{1}{2} \end{cases}$$

By the assumption (SPEC), the essential spectrum is exactly the absolutely continuous spectrum; there are no eigenvalues of $H_a(s)$ in the continuous spectrum region $[\min_{p^a}(\frac{1}{4}(p^a + s)^2 + \frac{1}{2}|p^a - s|), \infty)$. We concern ourselves now with the eigenvalues.

The operator $H_a(s)$ is unitarily equivalent to the operator $B(s) := \frac{1}{4}(p^a + 2s)^2 + \frac{1}{2}|p^a| + V_{23}(x^a)$. The unitary equivalence is given by $U(s) := e^{-isp^a}$, so that

$$U(-s)H_a(s)U(s) = B(s)$$

Thus $H_a(s)$ and $B(s)$ have the same spectrum for any choice of s.

Pick a unit vector $v \in \mathbb{R}^3$. We consider the family of operators $B(tv)$ for $t \in \mathbb{R}$. These form a self-adjoint holomorphic family of operators in the sense of Kato.

Specifically, fix a domain $D_0 \in \mathbb{C}$ symmetric with respect to the real axis. We have that $B(tv)$ is a closed, densely defined operator for all $t \in D_0$. As a function of t, $B(tv)$ is holomorphic for $t \in D_0$, and $B(\bar{t}v) = B(tv)^*$. We know it is holomorphic because we can compute its derivatives (either in the weak sense or strong sense):

$$\frac{dB(tv)}{dt} = 2t + p^a \cdot v$$

Because this is a holomorphic family of operators, its isolated eigenvalues and their eigenvectors can be thought of as varying holomorphically in a certain sense. Fix an isolated eigenvalue λ_0 of $B(0)$. We know from the Mourre estimate that all eigenvalues below 0 are simple and do not accumulate, so we can draw a curve Γ around λ_0 that is entirely contained in the resolvent set and encloses no other points of the spectrum of $B(0)$. It is then known that for small t, all eigenvalues of $B(tv)$ inside Γ can be described by a function $\lambda(t)$ that is analytic in a region about $t = 0$, satisfies $\lambda(0) = \lambda_0$, and gives a real eigenvalue of $H_a(tv)$ for each real t in its domain- also there are no other eigenvalues of $B(tv)$ for any small enough t in the interior of Γ. Even better, there exists at least one analytic family of real-valued eigenvectors $\psi_t(p^a)$ for $\lambda(t)$, as long as this family $\lambda(t)$ continues to exist.

We want to compute $\lambda(t)$ for $t > 0$. What follows is an application of the Feynman-Hellman theorem.

$$\begin{aligned}
\frac{d\lambda(t)}{dt} &= \frac{d}{dt}\langle \psi_t, B(tv)\psi_t \rangle \\
&= \langle \psi_t, \frac{dB(tv)}{dt}\psi_t \rangle + \langle \frac{d\psi_t}{dt}, B(tv)\psi_t \rangle + \langle \psi_t, B(tv)\frac{d\psi_t}{dt} \rangle \\
&= \langle \psi_t, \frac{dB(tv)}{dt}\psi_t \rangle + \lambda(t)\frac{d}{dt}\langle \psi_t, \psi_t \rangle \\
&= \langle \psi_t, \frac{dB(tv)}{dt}\psi_t \rangle \\
&= \langle \psi_t, (2t + p^a \cdot v)\psi_t \rangle
\end{aligned}$$

Due to the virial theorem that

$$[B(tv), x^a] = \frac{1}{2}(p^a) + tv + \frac{1}{2}\frac{p^a}{|p^a|} = 0$$

on eigenfunctions, this is where we obtain a nice sufficient condition on the eigenfunctions of $B(s)$, that is easily checked if eigenfunctions are known.

For some $\theta > 0$ the eigenfunctions of $B(tv)$ satisfy $(-\frac{p^a}{|p^a|} \cdot v) \geq \theta$ in expectation. (SPEC2)

There is reason to believe such a condition holds, but we will assume this for our purposes.

Then $\lambda(t) \geq \theta t + \lambda_0$.

While the function $\lambda(t)$ may not exist for all t, this process can be analytically continued as long as $\lambda(t)$ remains below the continuous spectrum of $B(tv)$. Here is why: suppose that $\lambda(t_0)$ is below the continuous spectrum of $B(tv)$, and we can define an operator $B((t-t_0)v)$ and compute its derivatives in the same way. So the spectrum of $H_a(tv)$ below the continuous spectrum consists only of eigenvalues that move upwards as tv moves away from the origin. □

At this point we have everything we need to handle the $E < 0$ case.

Proof of Lemma 3.13 for $E < 0$. Fix ϵ and $E < 0$ as in the lemma. We may select δ so small that $(E - \delta, E + \delta)$ contains no eigenvalues of h_a, and also so that $\delta \leq \frac{\epsilon}{2}$. Fix τ so small that $\triangle = (E - \delta, E + \delta)$ contains no eigenvalues of $H_a(s)$ for $|s| \leq \tau$.

Consider the small-momentum case, $|s| \leq \tau$. We have $E_\triangle(H_a(s)) = 0$ for all such s,

$$E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \geq (\theta\tau)E_\triangle(H_a(s))$$

Since the projections were 0, we could have put any constant where $\theta\tau$ is.

Now consider the large momentum case, $|s| \geq \tau$. We know that for all such s, $E_{\triangle}(H_a(s))$ is a (possibly zero) projection onto a finite-dimensional subspace of the pure point spectrum of $H_a(s)$. Because of this, we can make use of the virial theorem (specifically, that $[H_a(s), iA^a] = 0$ on eigenvectors of $H_a(s)$).

$$
\begin{aligned}
& E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) \\
& = E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) \\
& = E_{\triangle}(H_a(s))\left([H_a, iA](s) - [H_a(s), iA^a]\right)E_{\triangle}(H_a(s)) \\
& = E_{\triangle}(H_a(s))\left(\frac{1}{2}s^2 + \frac{1}{2}s \cdot p^a + \frac{1}{2}\frac{s^2 - s \cdot p^a}{|s - p^a|}\right)E_{\triangle}(H_a(s)) \\
& = E_{\triangle}(H_a(s))\left(s \cdot \triangledown_s H_a(s)\right)E_{\triangle}(H_a(s)) \\
& \geq \theta|s|E_{\triangle}(H_a(s)) \\
& \geq \theta\tau E_{\triangle}(H_a(s))
\end{aligned}
$$

where second-to-last last inequality comes from the Feynman-Hellman theorem. Finally, we have

Then the following is immediate:

$$
\begin{aligned}
E_{\Delta}(H_a)[H_a, iA]E_{\Delta}(H_a) &= \int_{s\in\mathbb{R}^3}^{\oplus} E_{\Delta}(H_a(s))[H_a, iA](s)E_{\Delta}(H_a(s))ds \\
&\geq \int_{s\in\mathbb{R}^3}^{\oplus} \theta\tau E_{\triangle}(H_a(s))ds \\
&= \theta\tau E_{\triangle}(H_a)
\end{aligned}
$$

By a renaming of ϵ we have the conclusion. $\square$

To attack the $E > 0$ case, we will need to be more judicious in our selection of width δ for different fibers s. As a result, we need to make a covering argument over $\mathbb{R}^3_s$. First we prove what we need for individual s.

Lemma 3.15. *Fix $\epsilon > 0$, an energy $E > 0$ (which is not an eigenvalue of the subsystem Hamiltonian h_a), and a choice of $s \in \mathbb{R}^3$. Then there there exists $\delta = \delta(s) > 0$ so that, taking $\triangle = (E - \delta, E + \delta)$, we have:*

$$E_{\triangle}(H_a(s))[H_a, iA](s)E_{\triangle}(H_a(s)) \geq \alpha_{(xy)(0)}E_{\triangle}(H_a(s))$$

Proof. Fix ϵ and $E > 0$ as in the lemma. We may select δ_0 so small that $(E-\delta_0, E+\delta_0)$ contains no eigenvalues of h_a, and also that $\delta_0 < \frac{\epsilon}{2}$. Fix τ so small that $\triangle_0 = (E - \delta_0, E + \delta_0)$ contains no eigenvalues of $H_a(s)$ for $|s| \leq \tau$.

Consider the small-momentum case, $|s| \leq \tau$. On these fibers:

$$\begin{aligned}
&E_{\triangle_0}(H_a(s))[H_a, iA](s)E_{\triangle_0}(H_a(s)) \\
&= E_{\triangle_0}(H_a(s)) \left(\frac{1}{4}(p^a + s)^2 + H_a(s) - V_{23}(x^a) - x^a \cdot \bigtriangledown V_{23}(x^a) \right) E_{\triangle_0}(H_a(s)) \\
&\geq (E - \delta_0)E_{\triangle_0}(H_a(s)) + E_{\triangle_0}(H_a(s))KE_{\triangle_0}(H_a(s))
\end{aligned}$$

by the functional calculus, where $K = -V_{23}(x^a) - x^a \cdot \bigtriangledown V_{23}(x^a)$ is relatively $H_a(s)$-compact. Since $\triangle_0$ contains no eigenvalues of $H_a(s)$, $E_{\triangle}(H_a(s)) \to 0$ in the strong operator topology as $\triangle \searrow 0$. Therefore $E_{\triangle}(H_a(s))KE_{\triangle}(H_a(s)) \to 0$ in norm as $\triangle \searrow 0$. So, by choosing $\delta_1 \leq \delta_0$ small enough, and letting $\triangle_1 = (E - \delta_1, E + \delta_1)$, we can ensure that

$$E_{\triangle_1}(H_a(s))[H_a, iA](s)E_{\triangle_1}(H_a(s)) \geq (E - \delta_0)E_{\triangle_1}(H_a(s)) - \frac{\epsilon}{2}E_{\triangle_1}(H_a(s))$$

and therefore that

$$E_{\triangle_1}(H_a(s))[H_a, iA](s)E_{\triangle_1}(H_a(s)) \geq (E - \epsilon)E_{\triangle_1}(H_a(s))$$

This is all we need to do in our analysis of fibers $|s| \leq \tau$.

Now we consider the large-momentum case, $|s| \geq \tau$. We need to be very careful in selecting the width, to satisfy a whole host of auxiliary inequalities. Let $\delta_2 < \delta_0$ be so small that, letting $\triangle_2 = (E - \delta_2, E + \delta_2)$:

$$\|\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)K\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)\| \leq \frac{\epsilon}{4} \tag{3.27}$$

where $K = -x^a - x^a \cdot \bigtriangledown V_{23}(x^a)$ is relatively $H_a(s)$-compact.

Because $E_{\triangle_2}(H_a(s))KE_{\triangle_2}(H_a(s))$ is compact, we may select a finite-dimensional projection F so that

$$\begin{aligned}\|\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)K\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)\\ - \Big(E_{\triangle_2}(H_a(s)) - F\Big)K\Big(E_{\triangle_2}(H_a(s)) - F\Big)\| \leq \frac{\epsilon}{4}\end{aligned} \tag{3.28}$$

Now define $C := F[H_a, iA](s)(E_{\triangle_2}(H_a(s)) - E_{\triangle pp}(H_a(s)))$. Evidently

$K_1 = -\epsilon^{-1}C^*C$ is a compact operator. If we select $\delta_3 < \delta_2$ small enough small enough, then letting $\triangle_3 = (E - \delta_3, E + \delta_3)$, we have:

$$\|\Big(E_{\triangle_3}(H_a(s)) - E_{\triangle_3 pp}(H_a(s))\Big)K_1\Big(E_{\triangle_3}(H_a(s)) - E_{\triangle_3 pp}(H_a(s))\Big)\| \leq \epsilon \tag{3.29}$$

From the same argument as the $E < 0$ case, we have that

$$E_{\triangle_0 pp}(H_a(s))[H_a, iA](s)E_{\triangle_0 pp}(H_a(s)) \geq \theta\tau E_{\triangle_0 pp}(H_a(s)) \tag{3.30}$$

Moreover, we can compute

$$\begin{aligned}&E_{\triangle_2}(H_a(s))[H_a, iA](s)E_{\triangle_2}(H_a(s))\\ &= E_{\triangle_2}(H_a(s))\left(\frac{1}{4}(p^a + s)^2 + H_a(s) + K\right)E_{\triangle_2}(H_a(s))\\ &\geq (E - \delta_0)E_{\triangle_2}(H_a(s)) + E_{\triangle_2}(H_a(s))KE_{\triangle_2}(H_a(s))\end{aligned} \tag{3.31}$$

where $K = -V_{23}(x^a) - x^a \cdot \bigtriangledown_s V_{23}(x^a)$ is $H_a(s)$-compact. We can multiply both sides of (3.31) on the left and right by $(1-F)$ to obtain

$$\begin{aligned}
&(E_{\triangle_2}(H_a(s)) - F)[H, iA](s)(E_{\triangle_2}(H_a(s)) - F) \\
&\geq (E-\delta_0)(E_{\triangle_2}(H_a(s)) - F) + (E_{\triangle_2}(H_a(s)) - F)K(E_{\triangle_2}(H_a(s)) - F)
\end{aligned}$$

Then we use (3.27) and (3.28):

$$\begin{aligned}
&(E_{\triangle_2}(H_a(s)) - F)[H, iA](s)(E_{\triangle_2}(H_a(s)) - F) \\
&\geq (E-\delta_0)(E_{\triangle_2}(H_a(s)) - F) - \frac{\epsilon}{2}
\end{aligned} \tag{3.32}$$

Additionally:

$$\begin{aligned}
&2F[H_a, iA](s)F - E_{\triangle_{2pp}}(H_a(s))[H_a, iA](s)F - F[H_a, iA](s)E_{\triangle_{2pp}}(H_a(s)) \\
&= (E_{\triangle_{2pp}}(H_a(s)) - F)[H_a, iA](s)(E_{\triangle_{2pp}}(H_a(s)) - F) + F[H_a, iA](s)F \\
&\qquad - E_{\triangle_{2pp}}(H_a(s))[H_a, iA](s)E_{\triangle_{2pp}}(H_a(s)) \\
&= s\cdot(\triangle\lambda(s))(E_{\triangle_{2pp}}(H_a(s)) - F) + s\cdot(\triangle\lambda(s))F - s\cdot(\triangle\lambda(s))E_{\triangle_{2pp}}(H_a(s)) \\
&= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
&(E_{\triangle_2}(H_a(s)) - F)[H_a, iA](s)F + F[H_a, iA](s)(E_{\triangle_2}(H_a(s)) - F) \\
&= \Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_{2pp}}(H_a(s))\Big)[H_a, iA](s)F \\
&\qquad + F[H_a, iA](s)\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_{2pp}}(H_a(s))\Big)
\end{aligned} \tag{3.33}$$

since the difference between the left hand side and the right hand side was just calculated to be zero.

From the inequality $(\epsilon^{-1/2}C + \epsilon^{1/2}F)^*(\epsilon^{-1/2}C + \epsilon^{1/2}F) \geq 0$ we obtain

$$\begin{aligned}
&\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)[H_a, iA](s)F \\
&\qquad + F[H_a, iA](s)\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big) \\
&\geq -\epsilon F + \Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)K_1\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)
\end{aligned}$$

but then, applying (3.33) to this, we can substitute out the left-hand side:

$$\begin{aligned}
&(E_{\triangle_2}(H_a(s)) - F)[H_a, iA](s)F + F[H_a, iA](s)(E_{\triangle_2}(H_a(s)) - F) \\
&\geq -\epsilon F + \Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)K_1\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)
\end{aligned} \tag{3.34}$$

We are ready to tackle the main estimate. We have

$$\begin{aligned}
&E_{\triangle_2}(H_a(s))[H_a, iA](s)E_{\triangle_2}(H_a(s)) \\
&= F[H_a, iA](s)F \\
&\quad + (E_{\triangle_2}(H_a(s)) - F)[H_a, iA](s)F \\
&\quad + F[H_a, iA](s)(E_{\triangle_2}(H_a(s)) - F) \\
&\quad + (E_{\triangle_2}(H_a(s)) - F)[H_a, iA](s)(E_{\triangle_2}(H_a(s)) - F)
\end{aligned}$$

Applying (3.30) to the first term, (3.32) to the last term, and (3.34) to the middle two terms:

$$\begin{aligned}
&\geq \theta\tau F + (E - \delta_0)(E_{\triangle_2}(H_a(s)) - F) - \frac{\epsilon}{2} - \epsilon F \\
&\qquad + \Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)K_1\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big) \\
&\geq \min(\theta\tau, E - \delta_0)E_{\triangle_2}(H_a(s)) - \frac{\epsilon}{2} \\
&\qquad + \Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)K_1\Big(E_{\triangle_2}(H_a(s)) - E_{\triangle_2 pp}(H_a(s))\Big)
\end{aligned}$$

Multiplying on both sides by $E_{\triangle_3 pp}(H_a(s))$ and using (3.29):

$$\geq \Big(\min(\theta\tau, E-\epsilon) - \epsilon \Big) E_{\triangle_3 pp}(H_a(s))$$

The conclusion follows by a renaming of ϵ.

□

Building on the inequality for individual fibers s, we can find a single width that works for all fibers, thus finishing the analysis for this cluster.

Proof of Lemma 3.13 for $E > 0$. Fix ϵ and $E > 0$ as in the lemma. Fix any $s_0 \in \mathbb{R}^3$. By Lemma 3.15, there exists δ_0 so that letting $\triangle_0 = (E - \delta - 0, E + \delta - 0)$:

$$E_{\triangle_0}(H_a(s_0))[H_a, iA](s_0)E_{\triangle_0}(H_a(s_0)) \geq \alpha_{(xy)(0)} E_{\triangle_0}(H_a(s_0))$$

Let f be a smoothed version of $E_\triangle$. First we prove that $f(H_a(s))$ is norm continuous in s at s_0. For simplicity we write

$$F(s) := \Big(p^a \cdot (s - s_0) + (s^2 - s_0)^2 + (|p^a - s| - |p^a - s_0|) \Big)$$

in what follows.

$$\begin{aligned}
& f(H_a(s)) - f(H_a(s_0)) \\
&= \int \widehat{f}(\lambda) e^{i\lambda H_a(s)} \int_0^\lambda e^{-irH_a(s)} \Big(H_a(s) - H_a(s_0) \Big) e^{-irH_a(s_0)} dr \; d\lambda \\
&\approx \int \widehat{f}(\lambda) e^{i\lambda H_a(s)} \int_0^\lambda e^{-irH_a(s)} \Big(F(s) \Big) e^{-irH_a(s_0)} dr \; d\lambda \\
&\approx \int \widehat{f}(\lambda) e^{i\lambda H_a(s)} \frac{H_a(s)}{H_a(s) + i} \int_0^\lambda e^{-irH_a(s)} \Big(F(s) \Big) e^{-irH_a(s_0)} dr \; d\lambda \\
&+ \int \widehat{f}(\lambda) e^{i\lambda H_a(s)} \frac{i}{H_a(s) + i} \int_0^\lambda e^{-irH_a(s)} \Big(F(s) \Big) e^{-irH_a(s_0)} dr \; d\lambda
\end{aligned}$$

The latter of these integrals converges to 0 in norm as $s \to s_0$ because

$$\frac{i}{H_a(s) + i} \Big(p^a \cdot (s - s_0) + (s^2 - s_0)^2 + (|p^a - s| - |p^a - s_0|) \Big)$$

converges to 0 uniformly as $s \to 0$. We investigate the former, rewriting it as

$$\int \widehat{f}(\lambda)\frac{d}{d\lambda}\Big(e^{i\lambda H_a(s)}\Big)\frac{-i}{H_a(s)+i}\int_0^\lambda e^{-irH_a(s)}\Big(F(s)\Big)e^{-irH_a(s_0)}dr\ d\lambda$$

Applying integration by parts, this is equal to

$$\int \widehat{f}'(\lambda)e^{i\lambda H_a(s)}\frac{-i}{H_a(s)+i}\int_0^\lambda e^{-irH_a(s)}\Big(F(s)\Big)e^{-irH_a(s_0)}dr\ d\lambda$$
$$+\int \widehat{f}(\lambda)\frac{-i}{H_a(s)+i}e^{-irH_a(s)}\Big(F(s)\Big)e^{-i\lambda H_a(s_0)}d\lambda$$

both of which converge to 0 in norm as $s \to s_0$. Therefore $f(H_a(s))$ is norm continuous at $s = s_0$.

Since additionally $f(H_a(s_0))[H,iA](s)F(H_a(s_0))$ is norm continuous in s by a similar argument, it follows by a 3ϵ argument that $f(H_a(s))[H,iA](s)F(H_a(s))$ is norm continuous.

Since this works for any $f \in C^\infty$ supported in $\triangle_0$, we fix such an f that is equal to 1 on a smaller interval $(E-\delta, E+\delta) = \triangle \subset \triangle_0$. Let U be an open set containing s_0 so that for all $s \in U$, we have

$$\|f(H_a(s))[H_a,iA](s)f(H_a(s)) - f(H_a(s_0))[H_a,iA](s_0)f(H_a(s_0))\| \leq \epsilon$$

and $\|f(H_a(s_0)) - f(H_a(s))\| \leq \epsilon$

Then, since

$$E_{\triangle_0}(H_a(s_0))[H_a,iA](s_0)E_{\triangle_0}(H_a(s_0)) \geq \alpha_{(xy)(0)}E_{\triangle_0}(H_a(s_0))$$

we have that, by multiplying:

$$f(H_a(s_0))[H_a,iA](s_0)f(H_a(s_0)) \geq \alpha_{(xy)(0)}f(H_a(s_0))$$

And then for all $s \in U$, we must have

$$f(H_a(s))[H_a, iA](s)f(H_a(s)) \geq \alpha_{(xy)(0)} f(H_a(s)) - 2\epsilon$$

Finally, multiplying through by $E_\triangle(H_a(s))$ and renaming the constant:

$$E_\triangle(H_a(s))[H_a, iA](s)E_\triangle(H_a(s)) \geq \alpha_{(xy)(0)} E_\triangle(H_a(s))$$

Therefore given any ϵ, an energy $E > 0$ not an eigenvalue of the subsystem Hamiltonian h_a, and a value of $s_0 \neq 0$, there exists $\delta > 0$ and an open set $U \in \mathbb{R}^3$ containing s_0 so that for all $s \in U$, letting $\triangle = (E - \delta, E + \delta)$, the above Mourre estimate holds. Thus the same covering argument as in the proof of Lemma 3.5 works, and by a renaming of ϵ we have our conclusion.

□

3.8.1 An informal argument for assumption (SPEC2)

Here is why (SPEC2) should be a good assumption, at least for certain potentials. Letting g_ϵ be a smoothed version approximating the Mourre conjugate operator $\frac{1}{2}\frac{p^a}{|p^a|} \cdot x^a + (sym.)$ (for instance, replace $\frac{p^a}{|p^a|}$ with $\frac{p^a}{((p^a)^2+\epsilon)^{1/2}}$). Then, on eigenfunctions of $B(s)$, we have that $[B(s), ig_\epsilon] = 0$ by the virial theorem. Written out, this approximately says that

$$\frac{p^a}{|p^a|} \cdot \left(\frac{1}{2}p^a + s + \frac{1}{2}\frac{p^a}{|p^a|}\right) + [g_\epsilon, V_{23}] = 0$$

Notice that if $[g_\epsilon, V_{23}] > 0$ then we have our conclusion (SPEC2). Furthermore, the above can be approximately rewritten

$$-s \cdot \frac{p^a}{|p^a|} = [B(0), ig_\epsilon]$$

So it suffices to determine if $[B(0), ig_\epsilon]$ is positive. Any state can be broken down into an eigenstate of $B(0)$ and a continuous spectrum state of $B(0)$. The eigenstate satisfies $[B(0), ig_\epsilon] = 0$ in expectation by the virial theorem, so it remains to consider if

$[B(0), ig_\epsilon] > 0$ on continuous spectrum states of $B(0)$. One may expect this to be true because $B(0) = H_a(0) = \frac{1}{4}(p^a)^2 + \frac{1}{2}|p^a| + V_{23}$.

3.9 Completing the Mourre estimate

Proof. Given a nonzero, nonthreshold energy E we may select a single δ small enough to invoke lemmas 3.4, 3.5, 3.9, and 3.13. We can select ϵ so small that all the constants α_a are positive. Then we can select a C_0^∞ function f that is 0 outside of $(E-\delta, E+\delta)$ and is equal to 1 on a smaller interval $\triangle$ containing E. We employ the localization Lemma 3.1 using this f:

$$\begin{aligned}\sum_a f(H)[H, iA]f(H) &= (\text{compact operators}) + \sum_{a \neq (xy0)} j_a f(H_a)[H_a, iA]f(H_a)j_a \\ &\geq \sum_{a \neq (xy0)} \alpha_a j_a f(H_a) j_a + (\text{compact operators}) \\ &\geq (\min_{a \neq (xy0)} \alpha_a) f(H)^2 + (\text{compact operators})\end{aligned}$$

This last step is by e.g. Lemma 4.20 in [6]. Multiplying on both sides by $E_\triangle(H)$ concludes the proof of Theorem 2.3. $\square$

In order to invoke Mourre's result, we check that H and A also satisfy (2COMM). Since $D(C) = D(H)$, a computation using Lemma 2.1 reveals that $[C, iA]$ extends to

$$4p^2 + |k| + x \cdot \triangledown(x \cdot \triangledown V_{12}(x)) + y \cdot \triangledown(y \cdot \triangledown V_{13}) + V_{23}(x-y) \cdot \triangledown((x-y) \cdot \triangledown V_{23}(x-y))$$

, which is a bounded operator on $D(H)$ by Kato-Rellich and the potential assumptions (RB2). Therefore by [19] or Theorem 1.1 in [20], we have that the point spectrum of H consists of simple eigenvalues which only may accumulate at thresholds, and that there is no singular continuous spectrum.

Chapter 4

Local decay and minimal velocity estimates

4.1 Local decay

A major consequence of the Mourre estimate is local decay estimates. An abstract result originally due to Mourre can be found in ([20], Thm 7.8), which we recreate here.

Lemma 4.1. *Suppose that H, H_0, and A are three self-adjoint operators so that $D(H) = D(H_0)$ and H and H_0 are both bounded from below. Assume that hypotheses (FC1)-(FC4) and (2COMM) hold for H and A. Moreover, assume that (FC1)-(FC4) hold for H_0 and A so that $[H_0, iA]$ extends to an operator defined on $D(H)$. Finally, assume that the core of test vectors S used to define the operator $[H_0, iA]$ is mapped into itself by A. Then, let $\triangle$ be an interval in which a Mourre estimate holds for H with conjugate operator A, so that $\triangle$ does not contain any eigenvalues of A. We have*

$$\sup_{0<\epsilon<1} \|(|A|+1)^{-\mu}(H-\lambda-i\epsilon)^{-1}(|A|+1)^{-\mu}\| < \infty$$

for any fixed $\mu > \frac{1}{2}$, where this holds uniformly as λ runs through compact subsets of $\triangle$.

Since these extra conditions hold under our assumptions, we are able to invoke this lemma for our H, H_0, and A. Next, this estimate can be modified to remove the reference to the operator A and instead say something about the position X. Define the notation $\langle X\rangle := \sqrt{X^2+1}$. Specifically, we want to prove that for any interval $\triangle$ where the Mourre estimate holds for H,

$$\sup_{0<\epsilon<1} \|\langle X\rangle^{-\mu}(H-\lambda-i\epsilon)^{-1}\langle X\rangle^{-\mu}\| < \infty \tag{4.1}$$

for any fixed $\mu > \frac{1}{2}$, where this holds uniformly as λ runs through compact subsets of $\triangle$. The main fact used to perform this swap is that

$$(|A|+1)^{\mu}(H+i)^{-1}\langle X\rangle^{-\mu} \tag{4.2}$$

is a bounded operator for any $0 \le \mu \le 1$. Assuming (4.2) is indeed bounded for any $0 \le \mu \le 1$, we proceed to prove (4.1). Let L^2_μ be the weighted L^2 space $\{f \in L^2(\mathbb{R}^6) : \langle X\rangle^{\mu} f \in L^2(\mathbb{R}^6)\}$. Then (4.1) is equivalent to saying that $(H-\lambda-i\epsilon)^{-1}$ is bounded from L^2_μ to $L^2_{-\mu}$ uniformly as λ and ϵ vary over the required sets. Since

$$(H-\lambda-i\epsilon)^{-1} = (H+i)^{-1}+(\xi+i)^{-1}(H+i)^{-2}+(\xi+i)^{2}(H+i)^{-1}(H-\lambda-i\epsilon)^{-1}(H+i)^{-1}$$

(where $\xi = \lambda + i\epsilon$) it remains to show that $(H+i)^{-1}(H-\lambda-i\epsilon)^{-1}(H+i)^{-1}$ is bounded from L^2_μ to $L^2_{-\mu}$ uniformly as λ and ϵ vary over the required sets. But for this we can rewrite

$$\begin{aligned}&\langle X\rangle^{-\mu}(H+i)^{-1}(H-\lambda-i\epsilon)^{-1}(H+i)^{-1}\langle X\rangle^{-\mu} = \\ &T^*\Big((|A|+1)^{-\mu}(H-\lambda-i\epsilon)^{-1}(|A|+1)^{-\mu}\Big)T\end{aligned}$$

where $T = \Big((|A|+1)^{\mu}(H+i)^{-1}\langle X\rangle^{-\mu}\Big)$ and then using (4.2) and Lemma 4.1 gives us (4.1). It remains to prove that (4.2) is bounded for any $\mu > \frac{1}{2}$. Without loss of generality we may assume also that $\mu \le 1$. In fact, we will prove that (4.2) is bounded for $\mu = 0$ and for $\mu = 1$, and then use Stein's interpolation theorem for analytic families of operators in order to draw the conclusion. The case $\mu = 0$ is evident. We consider the case $\mu = 1$. We need only bound

$$(P\cdot X)(H+i)^{-1}\langle X\rangle^{-1}$$

The equalities that follow come from restricting our attention to the dense domain of Schwartz functions.

$$(P \cdot X)(H+i)^{-1}\langle X\rangle^{-1} = S_1 + S_2$$

where

$$S_1 = (P(H+i)^{-1}) \cdot (X\langle X\rangle^{-1})$$

which is bounded, and

$$\begin{aligned} S_2 &= P \cdot [X, (H+i)^{-1}]\langle X\rangle^{-1} \\ &= P \cdot \Big((H+i)^{-1}[H_0, X](H+i)^{-1}\Big)\langle X\rangle^{-1} \\ &= P \cdot \Big((H+i)^{-1}(-2ip, -2i\frac{k}{|k|})(H+i)^{-1}\Big)\langle X\rangle^{-1} \end{aligned}$$

which is also bounded (thinking of $(-2ip, -2i\frac{k}{|k|})$ as a vector in $\mathbb{C}^6$, so the dot product makes sense). Since (4.2) is then shown to be bounded for all required μ, we have the desired estimate (4.1).

An operator B on $L^2(\mathbb{R}^6)$ is said to be H-smooth if for all $\phi \in L^2(\mathbb{R}^6)$, we have $e^{-itH} \in D(B)$ a.e. t and

$$\int_{-\infty}^{\infty} \|Be^{-itH}\phi\|^2 dt \lesssim \|\phi\|^2 \tag{4.3}$$

We say B is H-smooth on $\overline{\Omega}$ if $BE_{\overline{\Omega}}(H)$ is H-smooth.

This can be interpreted as the observable B decaying along the flow. By the general theory ([21], Theorems XIII.25 and XIII.30), the estimate (4.1) implies that for any interval Ω not containing eigenvalues or thresholds of H, $\langle X\rangle^{-\mu}$ is H-smooth on $\overline{\Omega}$ for any $\mu > \frac{1}{2}$. This fact is 'local decay'.

4.2 Minimal velocity estimates

Another important consequence of the Mourre estimate is minimal velocity estimates, which we use in what follows. In all that follows, we write $F(S)$ to signify a smoothed characteristic function of the set defined by S in configuration space. It is known (cf. [27]) that the Mourre estimate implies the following:

Lemma 4.2. *For all ψ such that the right-hand side makes sense, and t sufficiently far from 0, we have*

$$\|F(\frac{A}{|t|} < b)e^{-iHt}E_{\triangle}(H)\psi\| \lesssim |t|^{-5/4}(\|\psi\|^2 + \||A|^{5/4}\psi\|^2)^{\frac{1}{2}} \tag{4.4}$$

where $\triangle$ is any interval in the continuous spectrum of H, and b is any constant less than θ (the constant appearing in the Mourre estimate for that interval). The exponent $\frac{5}{4}$ is not optimal.

Proof. The estimate (4.4) follows immediately from the Mourre estimate and the abstract theory in [27]. The constant $-5/4$ is not the best attainable, but it's sufficient for our purposes. □

The goal is to swap out the reference to the auxiliary operator A with a reference to x. To this end, we will prove:

Lemma 4.3. *For all ψ such that the right-hand side of (4.4) makes sense, we have*

$$\lim_{t\to\pm\infty} F(\frac{X^2}{|t|^{2-\epsilon}} < \delta)e^{-iHt}E_{\triangle}(H)\psi = 0 \tag{MV}$$

where $0 < \epsilon << 2$ is a small positive constant, $\triangle$ is any interval in the continuous spectrum of H, and δ is any positive constant less than θ (the constant appearing in the Mourre estimate for $\triangle$).

We need to take a few steps before we can prove this. The operator $F(\frac{X^2}{t^{2-\epsilon}} < \delta)$ can be written as

$$F(\frac{A}{|t|} < b)F(\frac{X^2}{|t|^{2-\epsilon}} < \delta) + F(\frac{A}{|t|} \geq b)F(\frac{X^2}{|t|^{2-\epsilon}} < \delta)$$

Since

$$\|F(\frac{A}{|t|} < b)F(\frac{X^2}{|t|^{2-\epsilon}} < \delta)e^{-iHt}E_\triangle(H)\psi\| \lesssim |t|^{-5/4}(\|\psi\|^2 + \||A|^{5/4}\psi\|^2)^{\frac{1}{2}}$$

by (4.4), it will be sufficient to prove that

$$\|F(\frac{A}{|t|} \geq b)F(\frac{X^2}{|t|^{2-\epsilon}} < \delta)e^{-iHt}E_\triangle(H)\psi\| \lesssim |t|^{-\epsilon}\|\psi\|$$

For expedience of notation, define the following:

$$F_1 := F(\frac{X^2}{|t|^{2-\epsilon}} < \delta)$$

$$F_2(A) := F(\frac{A}{|t|} \geq b)$$

$$g := g(H)$$

where g is a smoothed version of the energy cutoff function $E_\triangle$ so that $g(H)E_\triangle(H) = E_\triangle(H)$.

$$\widetilde{A} := F_1 g A g F_1$$

$$F_2(\widetilde{A}) := F(\frac{\widetilde{A}}{|t|} \geq b)$$

Thus, the thing to be estimated is:

$$\|F_2(A)F_1 e^{-iHt}E_\triangle(H)\psi\|$$

Ensuing computations are much simplified by understanding some commutators of these operators. Here, s is a constant, and $O(|t|^n)$ represents an operator with norm bounded by a constant times $|t|^n$.

Lemma 4.4.

$$[\frac{A}{t}, F_1] = O(|t|^{-1}) \tag{4.5}$$

$$[\frac{A}{|t|}, g] = O(|t|^{-1}) \tag{4.6}$$

$$[e^{-is\frac{A}{t}}, F_1 g] = e^{-is\frac{A}{t}} s O(|t|^{-1}) \tag{4.7}$$

$$[g, X] = O(1) \tag{4.8}$$

$$H[g, X] = O(1) \tag{4.9}$$

$$F_1 g \frac{A}{|t|} = O(t^{-\epsilon/2}) \tag{4.10}$$

Proof. For (4.5)

$$[\frac{A}{t}, F_1] \approx \frac{X^2}{|t|^{3-\epsilon}} F'(\frac{X^2}{|t|^{2-\epsilon}} < \delta) = O(|t|^{-1})$$

For (4.6), by e.g. Lemma 4.12 in [6] $[A, g]$ is bounded, so

$$[\frac{A}{|t|}, g] = O(t^{-1})$$

Then (4.7) follows from (4.5) and (4.6) applied to the Fourier transform formula:

$$[e^{-is\frac{A}{t}}, F_1 g] \approx e^{-is\frac{A}{t}} \int_0^s e^{ir\frac{A}{t}} [\frac{A}{t}, F_1 g] e^{ir\frac{A}{t}} dr$$

For (4.8), we compute as follows.

$$[g(H), X] \approx \int_{-\infty}^{\infty} \widehat{g}(\lambda) e^{i\lambda H} \int_0^{\lambda} e^{-isH} [H, X] e^{isH} \; ds \; d\lambda$$

Now, $[H, x] = 2p + \frac{k}{|k|}$. Since $\frac{k}{|k|}$ is bounded after all, we may focus our analysis on p. It remains to estimate:

$$\int_{-\infty}^{\infty} \widehat{g}(\lambda) e^{i\lambda H} \int_0^{\lambda} e^{-isH} p e^{isH} \; ds \; d\lambda$$

It suffices to bound

$$\int_{-\infty}^{\infty} \widehat{g}(\lambda) \Big(\frac{d}{d\lambda} e^{i\lambda H}\Big) \frac{1}{H+i} \int_0^{\lambda} e^{-isH} p e^{isH} \; ds \; d\lambda$$

because the difference between this and the desired expression is bounded. Integrating by parts, we find that this equals

$$\int_{-\infty}^{\infty} \widehat{g}'(\lambda) e^{i\lambda H} \frac{1}{H+i} \int_0^{\lambda} e^{-isH} p e^{isH} \; ds \; d\lambda + \int_{-\infty}^{\infty} \widehat{g}(\lambda) \frac{1}{H+i} p e^{i\lambda H} \; d\lambda$$

which is certainly bounded.

For (4.9)

$$[g(H), X]H \approx \int_{-\infty}^{\infty} \widehat{g}(\lambda)\frac{d}{d\lambda}(e^{i\lambda H})\int_0^{\lambda} e^{-isH}[H, X]e^{isH}\ ds\ d\lambda$$

Integrating by parts, we find that this equals

$$\int (\widehat{g}')e^{i\lambda H}\int_0^{\lambda} e^{-isH}[X, H]e^{isH}ds\ d\lambda$$
$$+ \int \widehat{g}[X, H]e^{i\lambda H}d\lambda$$

The former term can be handled by the same technique used to prove (4.8) bounded. The latter term is equal to

$$(2p + \frac{k}{|k|})g(H)$$

which is bounded.

For (4.10), it suffices to prove

$$F_1 g\frac{X \cdot P}{|t|} = O(|t|^{-\epsilon/2}) \tag{4.11}$$

We write

$$\begin{aligned}
&F_1 g\frac{X \cdot P}{|t|} \\
&= \frac{1}{|t|}F_1 gX(H + i) \cdot \frac{1}{H + i}P \\
&= \frac{1}{|t|}F_1[g, X](H + i) \cdot \frac{1}{H + i}P \\
&+ \frac{X}{|t|}F_1(g)(H + i) \cdot \frac{1}{H + i}P
\end{aligned}$$

The latter term is $O(t^{-\epsilon/2})$, so we estimate the former. It suffices to prove that $[g, X](H + i)$ is bounded, but we have already done this.

□

Note that $\frac{\widetilde{A}}{t}$ therefore decays in time. Therefore $F_2(\widetilde{A})$ actually equals 0 for sufficiently large t. So to estimate $\|F_2(A)F_1 e^{-iHt}E_\triangle(H)\psi\|$, it is sufficient to estimate $\|\Big(F_2(\widetilde{A}) - F_2(A)\Big)F_1 e^{-iHt}E_\triangle(H)\psi\|$.

Next, we prove the **localization lemma**.

Lemma 4.5. $\|F_2(A)F_1E_\triangle(H)\| \to 0$ *as* $t \to \infty$.

Proof. It's sufficient to prove that

$$\lim_{t\to\infty} \|\Big(F_2(\widetilde{A}) - F_2(A)\Big)F_1 g E_\triangle(H)e^{-iHt}\psi\| = 0$$

We (formally) express F_2 using a Fourier transform:

$$\begin{aligned}
&\Big(F_2(\widetilde{A}) - F_2(A)\Big)(F_1 g)\\
&\approx \int \widehat{F_2}(\lambda)\Big(e^{-i\lambda\widetilde{A}/t} - e^{-i\lambda A/t}\Big)\, d\lambda\ (F_1 g)\\
&= \int \widehat{F_2}(\lambda)e^{-i\lambda\widetilde{A}/t}\int_0^\lambda e^{i\lambda\widetilde{A}/t}\Big(\frac{\widetilde{A}}{t} - \frac{A}{t}\Big)e^{-i\lambda A/t}\, ds\, d\lambda\ (F_1 g)\\
&= \int \widehat{F_2}(\lambda)e^{-i\lambda\widetilde{A}/t}\int_0^\lambda e^{is\widetilde{A}/t}\Big(\frac{\widetilde{A}}{t} - \frac{A}{t}\Big)[e^{-isA/t}, (F_1 g)]\, ds\, d\lambda\\
&\qquad + \int \widehat{F_2}(\lambda)e^{-i\lambda\widetilde{A}/t}\int_0^\lambda e^{is\widetilde{A}/t}\Big(\frac{\widetilde{A}}{t} - \frac{A}{t}\Big)(F_1 g)e^{-isA/t}\, ds\, d\lambda\\
&= \int \widehat{F_2}(\lambda)e^{-i\lambda\widetilde{A}/t}\int_0^\lambda e^{is\widetilde{A}/t}\Big(\frac{\widetilde{A}}{t} - \frac{A}{t}\Big)e^{-isA/t}sO(|t|^{-1})\, ds\, d\lambda\\
&\qquad + \int \widehat{F_2}(\lambda)e^{-i\lambda\widetilde{A}/t}\int_0^\lambda e^{is\widetilde{A}/t}\Big(\frac{\widetilde{A}}{t} - \frac{A}{t}\Big)(F_1 g)e^{-isA/t}\, ds\, d\lambda
\end{aligned}$$

The first integral converges to an operator that is $O(|t|^{-1})$ (applying integration by parts once to the inner integral). The second integral is shown to also converges to an operator that is $O(t^{-\epsilon/2})$ by (4.10). □

Now, (MV) is an immediate corollary of the localization lemma 4.5.

Chapter 5

Asymptotic completeness

5.1 Existence of the wave operators

In this section our aim is to prove existence of a collection of Deift-Simon wave operators arising from operators $\{F_a : \#(a) = 2\}$ that form a partition of unity. Let $\triangle$ be an interval not containing eigenvalues or thresholds of H, such that $\triangle < 0$. Let $\delta > 0$ be selected (based on $\triangle$) to make the minimal velocity estimates hold. Then, let $\delta' > 0$ and $\delta'' > 0$ be constants so that $\delta'' >> \delta'$ and $\delta := \delta'' + \delta'$. Let $0 < \epsilon << \frac{1}{2}$. We define time-dependent partitions of unity for each $\#(a) = 2$ using smoothed characteristic functions F by

$$F_a = F(\frac{(x^a)^2}{|t|^{2-\epsilon}} < \delta') \tag{5.1}$$

Then, we define the **wave operators** as the strong limits

$$\Omega_a^{\pm} := s - \lim_{t \to \pm\infty} W_a(t) \tag{5.2}$$

where

$$W_a(t) := E_{\triangle}(H_a) e^{iH_a t} F_a e^{-iHt} E_{\triangle}(H) \tag{5.3}$$

The objective is to prove that the strong limits $\Omega_a^{\pm}$ exist. This is accomplished by Cook's method (cf. Theorem XI.4 in [21]):

Lemma 5.1. *Suppose there is a set D of wavefunctions ψ dense in the absolutely continuous spectrum of H so that for each $\psi \in D$, there exists a $T > 0$ so that*

$$\int_{|t|>T} \left\| E_{\triangle}(H_a) e^{iH_a t} \left(I_a F_a + \frac{d}{dt} F_a + [H_0, F_a] \right) e^{-iHt} E_{\triangle}(H) \psi \right\| dt < \infty \tag{5.4}$$

Then the strong limits $\Omega_a^{\pm}$ exist.

Proof. Since $W_a(t)\psi$ is strongly differentiable and

$$W_a'(t)\psi \approx E_{\triangle}(H_a) e^{iH_a t} \left(I_a F_a + \frac{d}{dt} F_a + [H_0, F_a] \right) e^{-iHt} E_{\triangle}(H) \psi \tag{5.5}$$

for $t > s > T$,

$$\| W_a(t)\psi - W_a(s)\psi \| \leq \int_t^s \| W_a'(u)\psi \| du \tag{5.6}$$

and by assumption this can be made arbitrarily small by choice of T, we have that $W_a(t)$ is Cauchy at $t \to \pm\infty$. Since $W_a(t)$ constitute a uniformly bounded family of operators, the existence of limits on a dense set implies existence of the strong limits. $\square$

Therefore our goal is to prove

Lemma 5.2. *Let ψ be a wavefunction such that $(\|\psi\|^2 + \||A|^{5/4}\psi\|^2)^{\frac{1}{2}} < \infty$. There exists a $T > 0$ so that*

$$\int_{|t|>T} \left\| E_{\triangle}(H_a) e^{iH_a t} \left(I_a F_a + \frac{d}{dt} F_a + [H_0, F_a] \right) e^{-iHt} E_{\triangle}(H) \psi \right\| dt < \infty$$

and therefore the strong limits $\Omega_a^{\pm}$ exist.

The necessary facts for the proof of Lemma 5.2 are the minimal velocity estimate (MV)), the **short range assumptions** for $\#(a) = 2$:

$$\| F(\frac{(x^a)^2}{|t|^{2-\epsilon}} > c) I^a \| \text{ converges to 0 as } t \to \pm\infty \text{ and is integrable over } t \text{ away from 0} \tag{SR}$$

where $c > 0$ is any constant, and **fast decay of the eigenfunctions** for $\#(a) = 2$:

$$E_{\triangle}(H_a)\langle x^a\rangle^4 \text{ is a bounded operator.}$$
$$E(H_a)I_a\langle X\rangle^{\mu} \text{ is bounded for some } \mu > 1 \tag{FDE}$$

where $\triangle$ is an interval below 0. The fast decay of the eigenfunctions is where we are using the negativity of $\triangle$; the above energy cutoffs are then projections onto eigenvalues of subsystems, so (FDE) merely alleges that these eigenfunctions decay rapidly.

We proceed by considering the three terms I_aF_a, $\frac{d}{dt}F_a$, and $[H_0, F_a]$ in separate lemmas.

Lemma 5.3. *Let ψ be a wavefunction such that $(\|\psi\|^2 + \||A|^{5/4}\psi\|^2)^{\frac{1}{2}} < \infty$. There exists a $T > 0$ so that*

$$\int_{|t|>T} \left\| E_{\triangle}(H_a)e^{iH_at}\left(I_aF_a\right)e^{-iHt}E_{\triangle}(H)\psi\right\| dt < \infty$$

Proof. By the fast decay of eigenfunctions (FDE) it suffices to prove that

$$\|\langle X\rangle^{-(1+\epsilon)}e^{-iHt}E_{\triangle}(H)\psi\| \tag{5.7}$$

is integrable in t away from 0, for arbitrary $\epsilon > 0$. So it is sufficient to see if

$$\|\langle X\rangle^{-(1+\epsilon)}F(\frac{A}{|t|} < b)E_{\triangle}(H)e^{-iHt}\psi\| \tag{5.8}$$

$$\|\langle X\rangle^{-(1+\epsilon)}F(\frac{A}{|t|} > b)E_{\triangle}(H)e^{-iHt}\psi\| \tag{5.9}$$

are both integrable in t. Evidently (5.8) is integrable in t, from Lemma 4.2. Then (5.9) is proven integrable in t as follows.

We can prove $\|\langle X\rangle^{-(\alpha)}F(\frac{A}{|t|} > b)E_{\triangle}(H)\|$ is $O(|t|^{-\alpha})$ for $\alpha = 1, 2$ and then use complex interpolation to conclude. We write

$$\|\langle X\rangle^{-\alpha}F(\frac{A}{|t|} > b)E_{\triangle}(H)\| \lesssim \frac{1}{|t|^{\alpha}}\|\langle X\rangle^{-\alpha}F(\frac{A}{|t|} > b)A^{\alpha}E_{\triangle}(H)\|$$

Since the difference between $\frac{1}{2}(\tanh(b-\frac{A}{|t|})+1)$ and $F(\frac{A}{|t|}>b)$ decays fast at infinity and therefore

$$\big(\frac{1}{2}(\tanh(b-\frac{A}{|t|})+1)-F(\frac{A}{|t|}>b)\big)A^{\alpha}$$

is bounded, it is sufficient to prove that

$$\|\langle X\rangle^{-\alpha}\frac{1}{2}\big(\tanh(b-\frac{A}{|t|})+1\big)A^{\alpha}E_{\triangle}(H)\|$$

is bounded. We let $F_3=F_3(A)=\frac{1}{2}\big(\tanh(b-\frac{A}{|t|})+1\big)$. Take the expression A^{α} and commute all the P to the right and all the X to the left (the canonical commutation relations make sure the commutator terms are even nicer, so we need only estimate the result after commuting). Then since $<P>^{\alpha}E_{\triangle}(H)$ is bounded, it is sufficient to prove

$$\|\langle X\rangle^{-\alpha}F_3X^{\alpha}\|$$

is a bounded operator. In what follows we use the fact that F_3 is analytic in a strip of width greater than 2 and containing the real line.

$$\begin{aligned}\langle X\rangle^{-\alpha}F_3X^{\alpha}&=(\mathit{bounded})+[\langle X\rangle^{-\alpha},F_3]X^{\alpha}\\&=\langle X\rangle^{-\alpha}[X^{\alpha},F_3]\\&\approx\langle X\rangle^{-\alpha}\int\int_0^{\lambda}e^{isA/t}[X^{\alpha},\frac{A}{|t|}]e^{-isA/t}\,ds\,e^{i\lambda A/t}\widehat{F_3}(\lambda)\,d\lambda\end{aligned}$$

Then, since $[X^{\alpha},A]\approx X^{\alpha}$,

$$\approx\langle X\rangle^{-\alpha}\int\int_0^{\lambda}e^{isA/t}\frac{X^{\alpha}}{|t|}e^{-isA/t}\,ds\,e^{i\lambda A/t}\widehat{F_3}(\lambda)\,d\lambda$$

Using next the fact that $e^{-s\frac{A}{|t|}}$ are dilations,

$$\begin{aligned}
&\approx \langle X\rangle^{-\alpha} \int\int_0^\lambda e^{\alpha s/t}\frac{X^\alpha}{|t|}\, ds\, e^{i\lambda A/t}\widehat{F_3}(\lambda)\, d\lambda \\
&\approx \langle X\rangle^{-\alpha} \int (e^{\alpha\lambda/t}-1)X^\alpha\, e^{i\lambda A/t}\widehat{F_3}(\lambda)\, d\lambda \\
&\approx (\mathit{bounded})\cdot \int (e^{\alpha\lambda/t}-1)\, e^{i\lambda A/t}\widehat{F_3}(\lambda)\, d\lambda \\
&\approx F_3(A+\alpha i) - F_3(A)
\end{aligned}$$

This is bounded, so using Stein's interpolation theorem, we get the desired result.

□

Next we consider the terms in Lemma 5.2 containing $\frac{d}{dt}F_a$.

Lemma 5.4. *Let ψ be a wavefunction. There exists a $T>0$ so that*

$$\int_{|t|>T}\left\|E_\triangle(H_a)e^{iH_at}\left(\frac{d}{dt}F_a\right)e^{-iHt}E_\triangle(H)\psi\right\|dt<\infty$$

Proof. This term is a constant multiple of $\frac{(x^a)^2}{|t|^{3-\epsilon}}F'(\frac{(x^a)^2}{|t|^{2-\epsilon}}<\delta')$. Since from (FDE) $E_\triangle(H_a)$ is a bounded operator times $\langle x^a\rangle^{-4}$, the term

$$\|E_\triangle(H_a)\left(\frac{(x^a)^2}{|t|^{3-\epsilon}}F'(\frac{(x^a)^2}{|t|^{2-\epsilon}}<\delta')\right)e^{-iHt}E_\triangle(H)\psi\|$$

decays in t at least as fast as $|t|^{-(5-2\epsilon)}$ (and so is integrable). This is because $F'(\frac{(x^a)^2}{|t|^{2-\epsilon}}<\delta')$ is supported only on the configuration space region where $(x^a)^2\approx |t|^{2-\epsilon}$. □

Finally, we consider the terms in Lemma 5.2 containing $[H_0,F_a]$. We split this up into the easier term $[p^2,F_a]$ and the term of interest, $[|k|,F_a]$.

Lemma 5.5. *Let ψ be a wavefunction. There exists a $T>0$ so that*

$$\int_{|t|>T}\left\|E_\triangle(H_a)e^{iH_at}[p^2,F_a]e^{-iHt}E_\triangle(H)\psi\right\|dt<\infty$$

Proof. This term is nonzero for the 2-clusters $a = (y)(x0)$ and $a = (xy)(0)$. For these, computing the commutator, we need only estimate the following to be integrable in t.

$$\|E_\triangle(H_a)\left(p \cdot F'(\frac{(x^a)^2}{|t|^{2-\epsilon}} < \delta')\frac{2x^a}{|t|^{2-\epsilon}}\right)e^{-iHt}E_\triangle(H)\psi\|$$

$$\|E_\triangle(H_a)\left(F''(\frac{(x^a)^2}{|t|^{2-\epsilon}} < \delta'))\frac{4(x^a)^2}{|t|^{2(2-\epsilon)}}\right)e^{-iHt}E_\triangle(H)\psi\|$$

$$\|E_\triangle(H_a)\left(F'(\frac{(x^a)^2}{|t|^{2-\epsilon}} < \delta')\frac{2}{|t|^{2-\epsilon}}\right)e^{-iHt}E_\triangle(H)\psi\|$$

They are indeed integrable by by (FDE) and, once again, the ability to exchange (x^a) for t. □

Finally, we consider the term containing $[|k|, F_a]$. This is only nonzero for the 2-clusters $(x)(y0)$ and $(xy)(0)$. For simplicity of notation we write out the proof only for $(x)(y0)$ but it is the same for $(xy)(0)$, replacing references with x or y with references to $(x + y)$ and $(x - y)$ respectively.

Lemma 5.6. *Let ψ be a wavefunction. There exists a $T > 0$ so that*

$$\int_{|t|>T} \left\|E_\triangle(H_a)e^{iH_at}[|k|, F_a]e^{-iHt}E_\triangle(H)\psi\right\|dt < \infty$$

The approach here is to use the square root expression for $|k|$. We build up to this with a series of lemmas.

Lemma 5.7. *The following operator-valued integral converges in norm to a bounded operator.*

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}ds$$

Proof. We have that $\langle y\rangle^{-4}$ is a bounded operator, and

$$\|\frac{s^{-1/2}}{s+k^2}\|_{op} \approx s^{-3/2}$$

so the integrand has sufficient decay near $s = \infty$. To estimate the integrand near $s = 0$, we use Hardy-Littlewood-Sobolev (where $6/5$ is a non-optimal choice):

$$\langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2} = \langle y\rangle^{-4}\frac{1}{|k|^{6/5}}|k|^{6/5}\frac{s^{-1/2}}{s+k^2}$$

Since $\langle y\rangle^{-6/5}\frac{1}{|k|^{6/5}}$ is a bounded operator on $L^2(\mathbb{R}^3)$ and $\||k|^{6/5}\frac{s^{-1/2}}{s+k^2}\|_{op} \lesssim s^{-9/10}$, we have our conclusion. $\square$

Lemma 5.8. *The following operator-valued integrals converge to bounded operators.*

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}2y\cdot qE_\triangle(H)ds \tag{5.10}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}y^2E_\triangle(H)ds \tag{5.11}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}E_\triangle(H)ds \tag{5.12}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}2y\cdot k\frac{k^2}{s+k^2}E_\triangle(H)ds \tag{5.13}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}\left(y^2\right)\frac{k^2}{s+k^2}E_\triangle(H)ds \tag{5.14}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}\frac{k^2}{s+k^2}E_\triangle(H)ds \tag{5.15}$$

Proof. In all cases, commute all y all the way to the left and use Hardy-Littlewood-Sobolev if necessary, as in the proof of Lemma 5.7. $\square$

Lemma 5.9. *The following operator-valued integrals converge to bounded operators, which are integrable in t near $t=\infty$.*

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}F'(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{2y}{|t|^{2-\epsilon}}\cdot kE_\triangle(H)ds \tag{5.16}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}\left(F''(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{4y^2}{|t|^{2(2-\epsilon)}}+F'(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{2}{|t|^{2-\epsilon}}\right)E_\triangle(H)ds \tag{5.17}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}F'(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{2y}{|t|^{2-\epsilon}}\cdot k\frac{k^2}{s+k^2}E_\triangle(H)ds \tag{5.18}$$

$$\int_0^\infty \langle y\rangle^{-4}\frac{s^{-1/2}}{s+k^2}\left(F''(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{4y^2}{|t|^{2(2-\epsilon)}}+F'(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{2}{|t|^{2-\epsilon}}\right)\frac{k^2}{s+k^2}E_\triangle(H)ds \tag{5.19}$$

Proof. Applying Lemma 5.8 and the fact that $F'(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{1}{|t|^{2-\epsilon}}$ and $F''(\frac{y^2}{|t|^{2-\epsilon}}<\delta')\frac{1}{|t|^{2(2-\epsilon)}}$ are bounded operators that are integrable in t near $t=\infty$, this is immediate.

$\square$

Lemma 5.10. *The following operator-valued integral converges to a bounded operator that is integrable in t near $t = \infty$.*

$$\int_0^\infty \langle y \rangle^{-4} [\frac{s^{-1/2}}{s + k^2} k^2, F(\frac{y^2}{|t|^{2-\epsilon}} < \delta')] E_\triangle(H) ds$$

Proof. Expanding this commutator, we get several terms as in (5.16)-(5.19). □

Finally, the proof of Lemma 5.6 arrives.

Proof. Since $E_\triangle(H_a)$ is a bounded operator times $\langle y \rangle^{-4}$ and we can use the square-root representation of $|k|$, Lemma 5.6 is a corollary of 5.10. □

And now we have accounted for all the terms in Lemma 5.2, so we have proven this as well.

The next goal is to show that the energy cutoffs $E_\triangle(H_a)$ appearing on the left hand side of $W_a(t)$ can be removed and the existence of the wave operator still holds. Define the following operators $\widetilde{W_a}(t)$ for $\#(a) = 2$ by

$$\widetilde{W_a}(t) := e^{iH_a t} F_a e^{-iHt} E_\triangle(H) \tag{5.20}$$

Lemma 5.11. *Let ψ be a wavefunction such that $(\|\psi\|^2 + \||A|^{5/4}\psi\|^2)^{\frac{1}{2}} < \infty$. The following limits exist for $\#(a) = 2$:*

$$\lim_{t \to \pm\infty} \widetilde{W_a} \psi \tag{5.21}$$

Once this is established, then because such ψ form a dense set, it is immediate that the strong limits

$$s - \lim_{t \to \pm\infty} e^{iH_a t} F_a e^{-iHt} E_\triangle(H)$$

exist. That is the fact that we will use in the proof of asymptotic completeness.

Proof. It is sufficient to show that

$$\lim_{t \to \pm\infty} \chi(H_a) e^{iH_a t} F_a e^{-iHt} f(H) \psi \tag{5.22}$$

exists, where $\chi(H_a)$ is a smoothed version of $(1 - E_{\triangle}(H))$ and $f(H)$ is a smoothed version of $E_{\triangle}(H)$, so that these two functions have disjoint support and $\triangle$ contained in the support of f. This in mind, one wishes to estimate

$$e^{iH_a t}[\chi(H_a), F_a]e^{-iHt} f(H)\psi \tag{5.23}$$

$$e^{iH_a t} F_a \Big(\chi(H_a) - \chi(H)\Big) e^{-iHt} f(H)\psi \tag{5.24}$$

and show that these converge to 0 as $t \to \infty$; this is sufficient to prove Lemma 5.11.

We can estimate (5.23) via

$$\begin{aligned}
&[\chi(H_a), F_a]f(H) = \\
&\int \widehat{\chi}(\lambda)e^{-iH_a\lambda} \int_0^\lambda e^{iH_a s}[H_0, F_a]e^{-iH_a s} ds \; d\lambda \; f(H)
\end{aligned}$$

By the same estimates used to prove Lemma (5.5) and Lemma (5.6), $[H_0, F_a]f(H)$ is bounded with norm converging to 0 in t. While we lack the use of (FDE) in this case, note that this estimate is strictly easier to prove because we only need convergence to 0 in t, not integrability in t.

Similarly, we estimate (5.24) via

$$\begin{aligned}
&e^{iH_a t} F_a \Big(\chi(H_a) - \chi(H)\Big) f(H) e^{-iHt} f(H)\psi \\
&= e^{iH_a t} F_a \int \int_0^\lambda e^{-iH_a s} I_a f(H) e^{iHs} \; ds \; \widehat{\chi}(\lambda) e^{-i\lambda H} \; d\lambda \; e^{-iHt} f(H)\psi \\
&= e^{iH_a t} \int \int_0^\lambda [F_a, e^{-iH_a s}] I_a f(H) e^{iHs} \; ds \; \widehat{\chi}(\lambda) e^{-i\lambda H} \; d\lambda \; e^{-iHt} f(H)\psi \\
&+ e^{iH_a t} \int \int_0^\lambda e^{-iH_a s} F_a I_a f(H) e^{iHs} \; ds \; \widehat{\chi}(\lambda) e^{-i\lambda H} \; d\lambda \; e^{-iHt} f(H)\psi
\end{aligned}$$

Since $I_a f(H)$ is bounded, and $[F_a, e^{-iH_a s}] = e^{-iH_a s}\int_0^s e^{iH_a r}[H_0, F_a]e^{-iH_a r} dr\ d\lambda$, the analysis of the first integral reduces to the same estimates as for (5.23). It remains to estimate

$$e^{iH_a t}\int\int_0^\lambda e^{-iH_a s}F_a I_a f(H)e^{iHs}\ ds\ \widehat{\chi}(\lambda)e^{-i\lambda H}\ d\lambda\ e^{-iHt}f(H)\psi \tag{5.25}$$

We write

$$\begin{aligned} &F_a I_a f(H)\\ &= \left(F(\frac{(x^a)^2}{|t|^{2-\epsilon}} < \delta')F(\frac{(x_a)^2}{|t|^{2-\epsilon}} > \delta'')\right)I_a f(H)\\ &+ \left(F(\frac{(x^a)^2}{|t|^{2-\epsilon}} < \delta')F(\frac{(x_a)^2}{|t|^{2-\epsilon}} < \delta'')\right)I_a f(H) \end{aligned} \tag{5.26}$$

The latter term is taken care of by the minimal velocity estimate; the term $\left(F(\frac{(x^a)^2}{|t|^{2-\epsilon}} < \delta')F(\frac{(x_a)^2}{|t|^{2-\epsilon}} < \delta'')\right)$ has phase space support in $X^2 < \delta|t|^{2-\epsilon}$, so commuting this out to the right gives a term that decays in time. The former term is taken care of by the short range assumption (SR).

□

We have proven the existence of the wave operators that we need, and are now in a position to prove asymptotic clustering at energy E.

5.2 Proof of the theorem

Given a state ψ, we define $\phi_a := \lim_{t\to\pm\infty}\widetilde{W}_a(t)\psi$.

We are ready to prove Theorem 2. Let ψ be a state on the range of $E_\triangle(H)$. Following [22], we write:

$$\begin{aligned} e^{-iHt}\psi &= \sum_{\#(a)=2} F_a e^{-iHt}\psi + rem. \\ &= \sum_{\#(a)=2} e^{-iH_a t}\widetilde{W}_a(t)\psi + rem. \end{aligned}$$

where $rem.$ converges to 0 in t. Taking limits, we arrive at the statement of the theorem. It remains to show that the remainder does indeed converge to 0 for a dense set of ψ.

$$rem. = (1 - F(\frac{x^2}{|t|^{2-\epsilon}} < \delta') - F(\frac{y^2}{|t|^{2-\epsilon}} < \delta') - F(\frac{(x-y)^2}{|t|^{2-\epsilon}} < \delta'))e^{-iHt}E_\triangle(H)\psi$$

By minimal velocity it is free to add:

$$\begin{aligned} rem. = (1 - F(\frac{x^2}{|t|^{2-\epsilon}} < \delta')F(\frac{y^2}{|t|^{2-\epsilon}} > \delta'') - F(\frac{y^2}{|t|^{2-\epsilon}} < \delta')F(\frac{x^2}{|t|^{2-\epsilon}} > \delta'') \\ - F(\frac{(x-y)^2}{|t|^{2-\epsilon}} < \delta')F(\frac{(x+y)^2}{|t|^{2-\epsilon}} > \delta''))e^{-iHt}E_\triangle(H)\psi \end{aligned}$$

since the rest converges to 0 in t. Note the operator

$$\begin{aligned} T := (1 - F(\frac{x^2}{|t|^{2-\epsilon}} < \delta')F(\frac{y^2}{|t|^{2-\epsilon}} > \delta'') - F(\frac{y^2}{|t|^{2-\epsilon}} < \delta')F(\frac{x^2}{|t|^{2-\epsilon}} > \delta'') \\ -F(\frac{(x-y)^2}{|t|^{2-\epsilon}} < \delta')F(\frac{(x+y)^2}{|t|^{2-\epsilon}} > \delta'')) \end{aligned}$$

is supported in the phase space region where $x^2 > \delta'|t|^{2-\epsilon}$, $y^2 > \delta'|t|^{2-\epsilon}$, and $(x-y)^2 > \delta'|t|^{2-\epsilon}$.

We may write

$$rem. = T\Big(E_\triangle(H) - E_\triangle(H_0)\Big)e^{-iHt}E_\triangle(H)\psi$$

because we are projecting onto negative energy. Due to the phase space support of T, this converges to 0 by the same reasoning following (5.24). This concludes the proof.

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