UNIFORM ASYMPTOTIC APPROXIMATION OF SOLUTIONS TO THE CONDUCTIVITY PROBLEM WITH THIN OPEN FILAMENTS

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ABSTRACT OF THE DISSERTATION

Uniform asymptotic approximation of solutions to the conductivity problem with thin open filaments

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The asymptotic approximation of solutions to the conductivity problem with thin filaments is analyzed. While filaments with a closed mid-curve have been looked at previously, this thesis pays particular attention to the case of an open mid-curve, focusing on the extra singularities that can develop around the endpoints of this curve. The argument relies on a primal and dual energy argument as well as an explicit representation for the most singular part of the solution around each of the endpoints of the mid-curve. After proving the energy closeness of the reduced problem to the full problem for a constant conductivity in the inhomogeneity, related problems including those with variable conductivities, anisotropic conductivities, and curved inhomogeneities are all discussed briefly. Finally, numerical simulations are shown for many of the situations in this thesis, illustrating the convergence that has been proven here, as well as convergence that it may be possible to prove in the future.

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Chapter 1

Introduction

The problem of how electricity and electromagnetic waves travel through media has been studied by mathematicians and physicists for decades. In the case of electrostatics, this problem reduces to the conductivity equation,

$$-\nabla \cdot (\gamma \nabla u) = f,$$

where f models the source charge distribution in the domain, γ represents the conductivity of the material, and boundary conditions can be specified with either Dirichlet or Neumann conditions. One area of work in this problem, and the one we consider here, is determining how, or how much, small changes in the conductivity parameter γ affect the solution to the equation. In particular, this leads to discussing the presence of small inhomogeneities or imperfections in the material and how they affect the solution away from these inclusions.

The precise problem can be stated as follows. Consider a domain Ω and a conductivity profile γ_0 on Ω which is non-degenerate and constitutes the background object. Given a boundary condition $\varphi \in H^{1/2}(\partial\Omega)$, we can find the solution to

$$\begin{cases} -\nabla \cdot (\gamma_0 \nabla u_0) = f & \Omega \\ u_0 = \varphi & \partial\Omega, \end{cases}$$

which will be called the background solution. To model the small inhomogeneities, we choose a set $\omega_\epsilon \subset\subset \Omega$ that is 'small' in some appropriate sense, and pick a conductivity a_ϵ for this inclusion. Then, we can define the extended conductivity as

$$\gamma_\epsilon(x) = \begin{cases} \gamma_0(x) & x \in \Omega \setminus \omega_\epsilon \\ a_\epsilon & x \in \omega_\epsilon \end{cases}$$

and the solution to the 'full problem,' the problem with inhomogeneity, as the function u_ϵ satisfying

$$\begin{cases} -\nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = f & \Omega \\ u_\epsilon = \varphi & \partial\Omega. \end{cases} \tag{1.1}$$

Intuitively, if ω_ϵ is small, then the presence of this inhomogeneity should only have a small effect on the field, that is, u_0 and u_ϵ should be close in some sense. This leads to considering asymptotic estimates on the difference between these two functions, looking at how (or if) $||u_\epsilon - u_0||$ goes to zero as $\epsilon \to 0$. For certain types of ω_ϵ, that is the work being continued here.

The first case to consider in this setup is when ω_ϵ is a ball of radius ϵ contained in Ω, so that the inhomogeneity shrinks to a point as $\epsilon \to 0$. A lot of work has been done in this area, and the situation is known fairly completely. In [CFMV98], the authors develop a more general pointwise asymptotic formula for the full solution in $\Omega \subset \mathbb{R}^n$, showing that u_ϵ is the same as u_0 up to an explicit term of order ϵ^n, which depends on the conductivity of the inclusion, the solution to the background equation, and the geometry of the domain, and an error term of order $\epsilon^{n+1/2}$. These authors assume that the conductivity k_i in each of the m disjoint inhomogeneities is fixed, independent of ϵ, and that each of the pieces of the inhomogeneity shrink to a point as $\epsilon \to 0$. They also consider the limits as $k_i \to 0$ or $k_i \to \infty$, at which the problem becomes degenerate, and show that the first term in their estimate still holds. They do, however, lose the uniformity of the error term of order $\epsilon^{n+1/2}$, indicating that the degenerate problem is something that needs to be handled delicately. In the follow up paper [NV09], the authors deal with this issue, proving that the error term is uniformly valid in the conductivity of the inclusion, assuming the domain ω_ϵ shrinks to a point nicely enough as $\epsilon \to 0$. The main development there from the previous work is the use of a dual variational principle, allowing the error estimates to be connected to the background problem, which is known to be nice, as opposed to the full problem, which may degenerate as $\epsilon \to 0$.

With that case understood fairly well, a possible next direction is to look at different kinds of ω_ϵ. The simplest choice for ω_ϵ that are 'small' but do not collapse to a point as $\epsilon \to 0$ are 'thin inhomogeneities.' Let σ be a smooth hypersurface contained in Ω, and take

$$\omega_\epsilon = \{x + tn(x) : x \in \sigma,\ t \in (-\epsilon, \epsilon)\}$$

as a tubular neighborhood of σ, where $n(x)$ is a normal vector to the hypersurface σ at the point x. In [BMV01], the authors derive a formal asymptotic expansion of the solution u_ϵ in terms of the normal background solution u_0 and terms that depend on the geometry of the system, under the assumption that the conductivities in each of the inclusions are bounded and bounded away from zero. For most of the analysis, the authors here assume that the hypersurface σ is closed, and then use a mapping property to extend to open hypersurfaces. The rigorous analysis for this problem was done in [BFV03], where the authors prove an error estimate for the difference between u_0 and u_ϵ in exactly this case. By getting an explicit representation of the solution, the authors prove an error estimate of the form

$$||u_\epsilon - u_0||_{H^1(\Omega)} = O(\epsilon^{1/2})$$

in the case where σ is either closed or open, but the conductivity a_ϵ is fixed, independent of ϵ, and must be bounded and bounded away from zero. This case of thin inhomogeneities with a_ϵ independent of ϵ has also been studied in several other works, including [BF03, CV03, CV06].

The question then arises, what about uniformity of these estimates? Can the same work be carried out in the case of thin inhomogeneities where the conductivity of the inclusion is allowed to degenerate to 0 or ∞ as $\epsilon \to 0$? One of these cases, namely where $\frac{a_\epsilon}{\epsilon} \to b$ as $\epsilon \to 0$ for $0 \leq b < \infty$, was addressed in [PP13]. However, intuition says that uniformity of this convergence will not be possible in general. If the conductivity either goes to infinity or goes to zero, then the inhomogeneity will become either infinitely conductive or resistive as $\epsilon \to 0$, resulting in either a constant Dirichlet condition or a zero Neumann condition on σ. Explicitly, we see that if $\Omega \subset \mathbb{R}^2$ and $a_\epsilon = a$ is fixed (or at least controlled) so that $\epsilon a_\epsilon \to 0$ and $\frac{a_\epsilon}{\epsilon} \to \infty$ as $\epsilon \to 0$, then the solution u_ϵ to (1.1) converges to U, the solution to

$$\begin{cases} -\Delta U = f & \Omega \\ U = \varphi & \partial\Omega. \end{cases}$$

However, if a_ϵ goes to infinity fast enough so that $\epsilon a_\epsilon \to \infty$ as $\epsilon \to 0$, then the solution u_ϵ converges to u_∞, which solves

$$\begin{cases} -\Delta u_\infty = f & \Omega \setminus \sigma \\ \frac{\partial u_\infty}{\partial \tau} = 0 & \sigma \\ [u_\infty] = 0 & \sigma \\ u_\infty = \varphi & \partial\Omega, \end{cases}$$

where τ denotes the tangential derivative along σ and $[\phi]$ denotes the jump of a function across σ. Finally, if a_ϵ goes to 0 fast enough so that $\frac{a_\epsilon}{\epsilon} \to 0$ as $\epsilon \to 0$, then u_ϵ converges to u_0, which solves

$$\begin{cases} -\Delta u_0 = f & \Omega \setminus \sigma \\ \frac{\partial u_0}{\partial n} = 0 & \sigma \\ u_0 = \varphi & \partial\Omega, \end{cases}$$

where n denotes the normal vector to the curve σ. Since these are three very different problems, there should be no single ϵ-independent background solution that can handle all of these cases at the same time. The next question to be asked is if it is possible to construct different background solutions corresponding to the different behaviors of a_ϵ that will still give the asymptotic accuracy we had in the case where a_ϵ is constant.

The first step in this process was done in [DV17], where the authors consider the case $\Omega \subset \mathbb{R}^2$ and σ a closed curve. In particular, the authors define a 'reduced problem' whose solution u_ϵ^0 is asymptotically close to u_ϵ as $\epsilon \to 0$ in all cases. The problem defining u_ϵ^0 depends on a_ϵ and ϵ, so it is not a uniform background solution, but it only depends on these parameters and the curve σ, so it can still function as a reference solution. Since a uniform background solution is not possible, this is the best we can do. The next section will go into more detail as to what was done in this paper, as a lot of the results here are extensions of them. However, the arguments rely heavily on the fact that σ is a closed curve, and so does not have endpoints. Any extension to open curves must find a way to deal with the endpoints for the argument to go through.

This is the question answered here and in [CV19]. As described above, endpoints need to be dealt with, and there is a body of work already dealing with these types of problems. The endpoints of a curve in $\mathbb{R}^2$ can be thought of as corners of a polygon with angle 2π, and works like [Gri92] and [Dau88] deal with the types of singularities that can arise in elliptic equations in polygons. Under another interpretation, the endpoints of smooth curves can be thought of a transition in boundary conditions from whatever is specified on the curve to the background solution off of it. The work [CD10] deals with transitions in boundary conditions, particularly as a problem degenerates as $\epsilon \to 0$ from a smooth problem to a singular one. With the domain shrinking to a curve, this is the type of problem that we face here as well.

Beyond the scope of this thesis, other versions of this problem have been considered. The case of electromagnetic waves has been considered in [VV00, AVV01, AK03], resulting in asymptotic expansions to solutions to Maxwell's equations in the same vein. These asymptotically accurate formulas can be used to generate reconstruction algorithms to find or analyze these inhomogeneities for both the conductivity problem and electromagnetic waves [CFMV98, BHP01, BV94, AMV03, ABF04]. As the asymptotic accuracy of the background solution for the diametrically small inhomogeneities gives rise to successful approximate cloaking schemes via transformation optics [KSVW08], this type of result is also of interest for solving cloaking problems. In particular, these sorts of formulas have been used to improve approximate cloaks [ABF13].

Over the course of this thesis, we prove asymptotic accuracy of a reduced problem solution u_ϵ^0 to the full solution u_ϵ in the case of thin inhomogeneities around an open curve σ. The next section will outline the work [DV17] and describe how those authors got to their result. Chapter 2 will take the results needed from that paper and extend them to the case of an open curve. Following that, Chapter 3 will focus on the problem of dealing with the endpoints of the curve, deriving the necessary formulas to push the energy estimates through. Chapter 4 will show the energy convergence bounds that will, as established in Chapter 2, show the asymptotic accuracy of the solution to the reduced problem. The work in the Chapters 3 and 4 is also presented in our paper [CV19]. Chapter 5 will showcase other results connected to this problem. including more details about the solution near the endpoints of σ and different types of conductivities on the inhomogeneity. Chapter 6 will show numerical simulations of many of these results, illustrating the convergence in these cases, and Chapter 7 will summarize the results and look to future work.

Extension of Previous Work

A lot of the initial results here follow the work [DV17], which deals with asymptotic approximations of solutions to problems of this type, but where the curves σ are closed. Our initial goal will be to extend the base results here to open curves, and then use them to prove asymptotic exactness of our solution on open curves.

The work [DV17] starts by setting up the appropriate definitions for the domains in question, which will be done in Chapter 2, and then moves on to proving an energy lemma. This energy lemma provides the motivation for the methods in that paper; it says that if two minimizers are close in energy for all boundary data and source terms, then the solution functions are close as well, allowing energy methods and energy asymptotics to be used to prove the 'closeness' of the solution u_ϵ^0 to u_ϵ. The paper then goes into working with the energy functional for u_ϵ directly and finds a differential equation that u_ϵ^0 must solve that will allow it to be close to u_ϵ in energy as $\epsilon \to 0$. As many of these proofs rely on the fact that σ is closed, we will derive new versions of the proofs here that allow for σ to be open as well. All of this is contained in Chapter 2.

Beyond that, the paper goes on to study the solution u_ϵ^0, getting energy and regularity estimates on this solution, which will come into play in showing asymptotic closeness later. Then, the energy closeness of u_ϵ and u_ϵ^0 is proved using a primal and dual energy formulation of the problems for both u_ϵ and u_ϵ^0, which give the upper and lower bounds on the difference in the two energies, and prove that the two functions need to be close by the energy lemma from before. This is the work that will be modeled here in the next few chapters.

Chapter 2

Extensions to Open Curves

In this chapter, we extend the results of [DV17] to hold in the case where the curve σ is open. These results will provide the foundation for how we approach the problem of proving asymptotic exactness of the solution to the reduced problem to the full problem.

2.1 Notation

Let Ω be a domain in $\mathbb{R}^2$ with smooth boundary, and σ a smooth, open curve contained in Ω so that $\bar{\sigma}$ is non-self-intersecting. Particular smoothness considerations of both $\partial\Omega$ and σ will be discussed when needed. Since σ is sufficiently regular, we choose a continuous normal vector $n(x)$ along σ, and for any function u on Ω, we define u^+ and u^- to be the traces of u on the positive and negative sides, respectively, of σ with respect to the defined normal vector $n(x)$. If u is sufficiently regular, we can also define

$$\frac{\partial u^{\pm}}{\partial n}(x) = \lim_{t \to 0} \nabla u(x \pm tn(x)) \cdot n(x)$$

as the normal derivative on the positive and negative sides of σ.

With respect to our curve σ, we can define the standard distance

$$d(x, \sigma) = \min_{y \in \sigma} d(x, y),$$

where d is the Euclidean distance. The thin inhomogeneity with mid-surface σ and width 2ϵ is defined as the set

$$\omega_\epsilon := \{x \in \Omega \mid d(x, \sigma) < \epsilon\}.$$

Since σ is an open curve, this set can be divided into two parts, the first of which has the nearest point on σ, $p_\sigma(x)$, on the interior of σ and looks like the inhomogeneity from [DV17], and the second, where the nearest point on σ is one of the endpoints of the curve. From this, we define

$$\omega_\epsilon^{int} := \{x + tn(x) \mid x \in \sigma,\ t \in (-\epsilon, \epsilon)\}$$

and

$$\omega_\epsilon^{ends} = \omega_\epsilon \setminus \omega_\epsilon^{int},$$

where ω_ϵ^{ends} corresponds to the two round caps around the endpoints of σ, as shown in Figure 2.1.

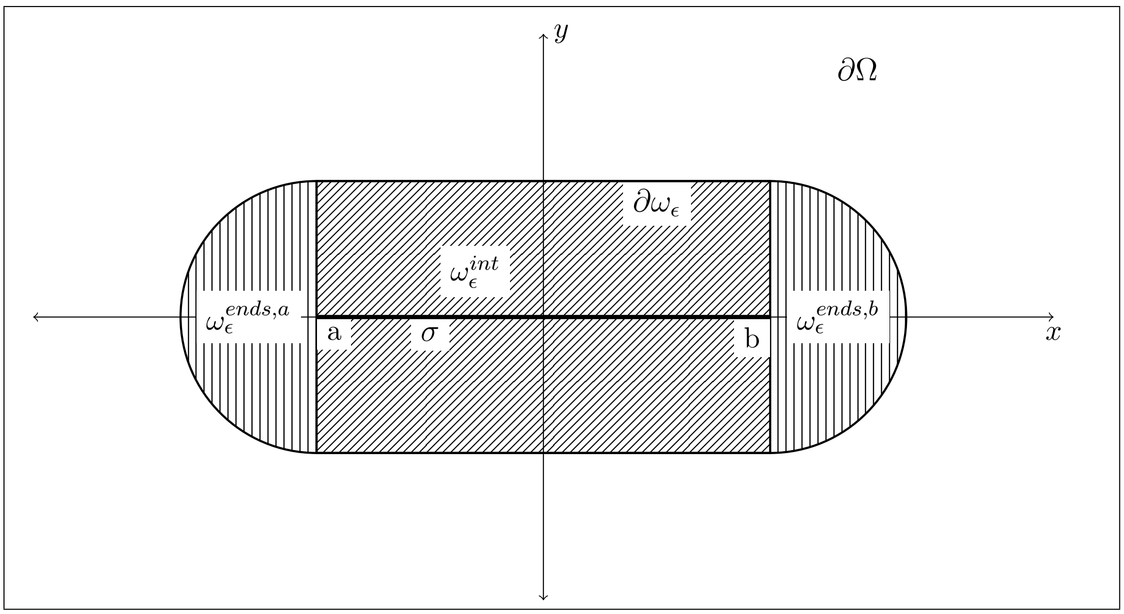

Figure 2.1: Structure of the inhomogeneity ω_ϵ

We assume that the conductivity is 1 in the background domain and it is a_ϵ in the inhomogeneity. The assumption of the background conductivity being a constant is not much of a restriction, as similar results would hold for any positive definite and symmetric conductivity. This a_ϵ is a constant, but it may depend on ϵ, allowing for the non-uniform convergence of the solution to the full problem as discussed in the introduction. Defining a conductivity in the domain by

$$\gamma_\epsilon = \begin{cases} 1 & \Omega \setminus \bar{\omega_\epsilon} \\ a_\epsilon & \omega_\epsilon, \end{cases}$$

we then have that our full problem with solution u_ϵ is given by

$$\begin{cases} -\nabla \cdot (\gamma_\epsilon \nabla u_\epsilon) = f & \Omega \\ u_\epsilon = \varphi & \partial\Omega \end{cases} \tag{2.1}$$

for some choice of $f \in L^2(\Omega)$ and $\varphi \in H^{1/2}(\partial\Omega)$, representing the charge distribution and boundary potential respectively. Since a_ϵ is a positive constant, standard arguments show that this has a unique solution $u_\epsilon \in H^1(\Omega)$. In what follows, we will require $f \in \mathcal{F}_\delta$ for some $\delta > 0$, where

$$\mathcal{F}_\delta = \{f \in L^2(\Omega) \mid \text{supp}(f) \subset \Omega \setminus \omega_\delta\},$$

instead of just $L^2(\Omega)$, as the charge density should be supported away from the inhomogeneity.

Since σ is non-self-intersecting and sufficiently regular, there exists an $\eta > 0$ such that the projection mapping

$$p_\sigma : x \mapsto \text{the unique } y \in \sigma \text{ with } d(x, y) = d(x, \sigma)$$

is well-defined on ω_η, and from this, we can define a signed distance function $d_\sigma(x)$ by the relation

$$x = p_\sigma(x) + d_\sigma(x) n(p_\sigma(x)) \qquad x \in \omega_\eta^{int}$$

that is, $d_\sigma(x) > 0$ if x is on the positive side of σ with respect to $n(x)$, and similarly for the negative side. On ω_ϵ^{ends} we do not have a well-defined normal vector, so the distance to σ is just the distance from the corresponding endpoint of σ and does not have a sign associated to it. For the remainder of this section, we will assume that $\omega_1 \subset \Omega$, and that p_σ is well-defined on ω_1. Since these facts hold for some $\eta > 0$, this is just a matter of rescaling, and serves to make the notation simpler. Based on this fact, we can extend the normal vector $n(x)$ and the curvature $\kappa(x)$ to functions on ω_1^{int} by setting $n(x) = n(p_\sigma(x))$ and $\kappa(x) = \kappa(p_\sigma(x))$. As given in [DV17], the derivatives of these functions are given by

$$\nabla d_\sigma(x) = n(x) \qquad \nabla^2(d_\sigma)(x) = \begin{pmatrix} \frac{\kappa(x)}{1+\kappa(x)d_\sigma(x)} & 0 \\ 0 & 0 \end{pmatrix} \qquad \nabla p_\sigma(x) = \begin{pmatrix} \frac{1}{1+\kappa(x)d_\sigma(x)} & 0 \\ 0 & 0 \end{pmatrix}$$

where the matrices are given in the orthonormal basis $(\tau(x), n(x))$, where τ is the 90 degree clockwise rotation of $n(x)$, which is an extension of a tangent vector field on σ to all of ω_1^{int}. The co-area formula [Cha06] then tells us that

Proposition 2.1.1. *Let $g \in L^1(\Omega)$. Then for any $\epsilon \leq 1$,*

$$\int_{\omega_\epsilon^{int}} g(x)\ dx = \int_\sigma \int_{p_\sigma^{-1}(y)\cap\omega_\epsilon^{int}} g(z)(1+\kappa(y)d_\sigma(z))d\mu^1(z)ds(y),$$

where $d\mu^1$ denotes the one-dimensional Hausdorff measure on the preimages $p_\sigma^{-1}(y) \cap \omega_\epsilon^{int}$ and $ds(y)$ is the Lebesgue measure on σ.

2.2 Distance between energy minimizers

In this section, we cite the results of [DV17], as none of this work depends on the fact that the curve σ was closed.

Lemma 2.2.1 (Lemma 3 in [DV17])**.** *Let V_ϵ and W_ϵ be two families of Hilbert spaces, and let H be another Hilbert space that continuously contains all the V_ϵ and W_ϵ. Consider $a_\epsilon : V_\epsilon \times V_\epsilon \to \mathbb{R}$ and $b_\epsilon : W_\epsilon \times W_\epsilon \to \mathbb{R}$, two families of symmetric bilinear forms that are continuous and are coercive on the corresponding homogeneous spaces (i.e., zero Dirichlet boundary condition). For any $\ell \in H'$, define the energy functionals E_ϵ^ℓ and F_ϵ^ℓ by*

$$\begin{aligned} \forall\, v \in V_\epsilon \qquad E_\epsilon^\ell(v) &= \frac{1}{2}a_\epsilon(v,v) - \ell(v) \\ \forall\, w \in W_\epsilon \qquad F_\epsilon^\ell(w) &= \frac{1}{2}b_\epsilon(w,w) - \ell(w) \end{aligned}$$

E_ϵ^ℓ and F_ϵ^ℓ admit unique minimizers $v_\epsilon^\ell \in V_\epsilon$ and $w_\epsilon^\ell \in W_\epsilon$ by the standard Lax-Milgram theory. The gap between v_ϵ^ℓ and w_ϵ^ℓ can be controlled in terms of the gap between the corresponding energies as

$$\sup_{||\ell||_{H'}\leq 1} ||v_\epsilon^\ell - w_\epsilon^\ell||_H \leq 4 \sup_{||\ell||_{H'}\leq 1} |E_\epsilon^\ell(v_\epsilon^\ell) - F_\epsilon^\ell(w_\epsilon^\ell)|$$

Lemma 2.2.2 (Lemma 4 in [DV17])**.** *Let Ω be a bounded domain in $\mathbb{R}^2$, and let V_ϵ and W_ϵ be two families of Hilbert spaces of functions defined on Ω such that, for any $\epsilon > 0$, the trace operator*

$$V_\epsilon \ni v \mapsto v|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$$

is well-defined, continuous, and has a linear, continuous right inverse $\varphi \mapsto v_\varphi$*, and the same holds for* W_ϵ*. Let* H *be another Hilbert space that continuously contains all the* V_ϵ *and* W_ϵ*. Consider* $a_\epsilon : V_\epsilon \times V_\epsilon \to \mathbb{R}$ *and* $b_\epsilon : W_\epsilon \times W_\epsilon \to \mathbb{R}$*, two families of symmetric bilinear forms that are continuous and are coercive on the corresponding homogeneous spaces (i.e., zero Dirichlet boundary condition). For any* $\varphi \in H^{1/2}(\partial\Omega)$ *and* $\ell \in H'$*, define the energy functionals* E_ϵ^ℓ *and* F_ϵ^ℓ *by*

$$\forall\, v \in V_\epsilon \qquad E_\epsilon^\ell(v) = \frac{1}{2}a_\epsilon(v,v) - \ell(v)$$
$$\forall\, w \in W_\epsilon \qquad F_\epsilon^\ell(w) = \frac{1}{2}b_\epsilon(w,w) - \ell(w)$$

and consider the minimization problems

$$\min_{\substack{v\in V_\epsilon \\ v=\varphi \text{ on } \partial\Omega}} E_\epsilon^\ell(v) \qquad\qquad \min_{\substack{w\in W_\epsilon \\ w=\varphi \text{ on } \partial\Omega}} F_\epsilon^\ell(w).$$

These problems admit unique minimizers $v_\epsilon^{\ell,\varphi} \in V_\epsilon$ *and* $w_\epsilon^{\ell,\varphi} \in W_\epsilon$ *by the standard Lax-Milgram theory. Then, for any* $s \geq 1/2$*, we have*

$$\sup_{\substack{||\ell||_{H'}\leq 1 \\ ||\varphi||_{H^s(\partial\Omega)}\leq 1}} ||v_\epsilon^{\ell,\varphi} - w_\epsilon^{\ell,\varphi}||_H \leq 4 \sup_{\substack{||\ell||_{H'}\leq 1 \\ ||\varphi||_{H^s(\partial\Omega)}\leq 1}} |E_\epsilon^\ell(v_\epsilon^{\ell,\varphi}) - F_\epsilon^\ell(w_\epsilon^{\ell,\varphi})|$$

Remark. If the spaces V_ϵ and W_ϵ are only weakly contained in H, i.e., there exist linear continuous mappings $P_\epsilon : V_\epsilon \to H$ and $Q_\epsilon : W_\epsilon \to H$ through which they may be identified as subspaces of H, the results above can still hold with modifications. The linear term in the energy functionals needs to be modified to

$$\forall\, v \in V_\epsilon \qquad E_\epsilon^\ell(v) = \frac{1}{2}a_\epsilon(v,v) - P_\epsilon^*\ell(v)$$
$$\forall\, w \in W_\epsilon \qquad F_\epsilon^\ell(w) = \frac{1}{2}b_\epsilon(w,w) - Q_\epsilon^*\ell(w)$$

and the result of the first lemma now says that

$$\sup_{||\ell||_{H'}\leq 1} ||P_\epsilon v_\epsilon^\ell - Q_\epsilon w_\epsilon^\ell||_H \leq 4 \sup_{||\ell||_{H'}\leq 1} |E_\epsilon^\ell(v_\epsilon^\ell) - F_\epsilon^\ell(w_\epsilon^\ell)|.$$

2.3 Derivation of the reduced problem

In this section, we derive the differential equation that the reduced solution u_ϵ^0 should solve in order to be an asymptotic approximation to the full solution u_ϵ. We do this by forming an asymptotic expansion of the energy functionals defining u_ϵ.

The full problem (2.1) has a solution that is given by

$$\min_{\substack{u\in H^1(\Omega)\\ u=\varphi \text{ on } \partial\Omega}} E_\epsilon(u) \qquad E_\epsilon(u) = \frac{1}{2}\int_\Omega \gamma_\epsilon |\nabla u|^2 \, dx - \int_\Omega fu \, dx \tag{2.2}$$

where we assume that $f \in \mathcal{F}_\delta$, with $\epsilon < \delta$ so that f is supported off of the inhomogeneity. To isolate the inhomogeneity and fix the size of the domain, we define a map $H_\epsilon : \omega_1 \to \omega_\epsilon$ by

$$H_\epsilon(x) = p_\sigma(x) + \epsilon(x - p_\sigma(x)).$$

On ω_1^{int}, this reduces to the map

$$p_\sigma(x) + \epsilon d_\sigma(x) n(x),$$

but on the ends, it is a dilation of factor ϵ. Thus, using the formulas from earlier, we see that

$$\begin{aligned} \nabla H_\epsilon &= \begin{pmatrix} \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma} & 0 \\ 0 & \epsilon \end{pmatrix} && \omega_1^{int} \\ \nabla H_\epsilon &= \epsilon I && \omega_1^{ends}. \end{aligned}$$

For any function $u \in H^1(\omega_\epsilon)$, we denote by $\hat{u} \in H^1(\omega_1)$ the function $u \circ H_\epsilon$, and can compute by a change of variables

$$\begin{aligned} \int_{\omega_\epsilon^{int}} |\nabla u|^2 &= \int_{\omega_1^{int}} ((\det \nabla H_\epsilon)\nabla H_\epsilon^{-1} \nabla (H_\epsilon^{-1})^T)\nabla \hat{u} \cdot \nabla \hat{u} \, dx \\ &= \epsilon \int_{\omega_1^{int}} \frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\left(\frac{\partial \hat{u}}{\partial \tau}\right)^2 dx + \frac{1}{\epsilon}\int_{\omega_1^{int}} \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}\left(\frac{\partial \hat{u}}{\partial n}\right)^2 dx \end{aligned}$$

and

$$\int_{\omega_\epsilon^{ends}} |\nabla u|^2 = \int_{\omega_1^{ends}} ((\det \nabla H_\epsilon)\nabla H_\epsilon^{-1} \nabla (H_\epsilon^{-1})^T)\nabla \hat{u} \cdot \nabla \hat{u} \, dx = \int_{\omega_1^{ends}} |\nabla \hat{u}|^2 \, dx$$

Using this change of variables, we can define a rescaled version of this energy as

$$\begin{aligned} F_\epsilon(u, v, w_0, w_1) = \frac{1}{2}\int_{\Omega\setminus\bar{\omega_\epsilon}} |\nabla u|^2 \, dx - \int_\Omega fu \, dx \\ + \frac{\epsilon a_\epsilon}{2}\int_{\omega_1^{int}} \frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\left(\frac{\partial v}{\partial \tau}\right)^2 dx \\ + \frac{a_\epsilon}{2\epsilon}\int_{\omega_1^{int}} \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2 dx \\ + a_\epsilon \int_{\omega_1^{ends,a}} |\nabla w_0|^2 \, dx + a_\epsilon \int_{\omega_1^{ends,b}} |\nabla w_1|^2 \, dx \end{aligned} \tag{2.3}$$

where a and b denote the two endpoints of σ and

$$\omega_1^{ends,a} = B_1(a) \setminus \omega_1^{int} \qquad \omega_1^{ends,b} = B_1(b) \setminus \omega_1^{int}$$

denote the two separate end caps of ω_1, where the minimization to solve this problem is taken over the space

$$V_\epsilon^0 = \{(u, v, w_0, w_1) \in H^1(\Omega \setminus \bar{\omega_\epsilon}) \times H^1(\omega_1^{int}) \times H^1(\omega_1^{ends,a}) \times H^1(\omega_1^{ends,b}) \mid CC\}$$

where CC is a set of continuity conditions given by

$$CC = \begin{cases} \forall\, x \in \sigma & u(x \pm \epsilon n(x)) = v(x \pm n(x)) \\ \forall\, \theta \in S^1 \text{ with } a + \theta \in \omega_1^{ends,a} & w_0(a+\theta) = u(a+\epsilon\theta) \\ \forall\, \theta \in S^1 \text{ with } b + \theta \in \omega_1^{ends,b} & w_1(b+\theta) = u(b+\epsilon\theta) \\ \forall\, t \in (-\epsilon, \epsilon) & w_0(a + tn(a)) = v(a + tn(a)) \\ \forall\, t \in (-\epsilon, \epsilon) & w_1(b + tn(b)) = v(b + tn(b)) \end{cases},$$

to be understood in the sense of traces, which follow from the fact that the re-scaled and combined function needs to be in $H^1(\Omega)$. If we set v, w_0 and w_1 all equal to $\hat{u}$, then we get the same value as the energy of the full problem. A sketch of the domain and how these functions connect to each other can be seen in Figure 2.2.

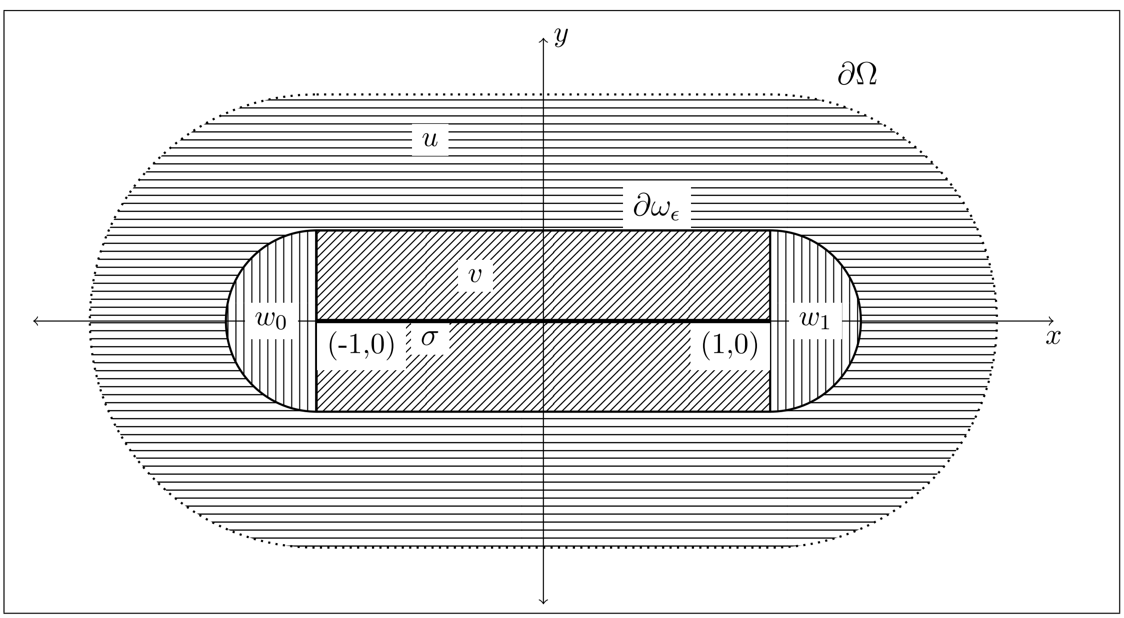

Figure 2.2: Regions for the functions in the energy functional $F(u, v, w_0, w_1)$

Since the gradient term in the 'ends' regions scales independent of ϵ, there are two ways we can formulate this energy, depending on if we want to group the w functions with the function v inside the inhomogeneity or the function u outside of it. This gives rise to two other formulations of this energy functional, $F_\epsilon^s(u,v)$ and $F_\epsilon^r(u,v)$, where the s and r denote the squared or rounded ends of the removed region. Sketches of these two regions can be found in Figures 2.3 and 2.4 respectively.

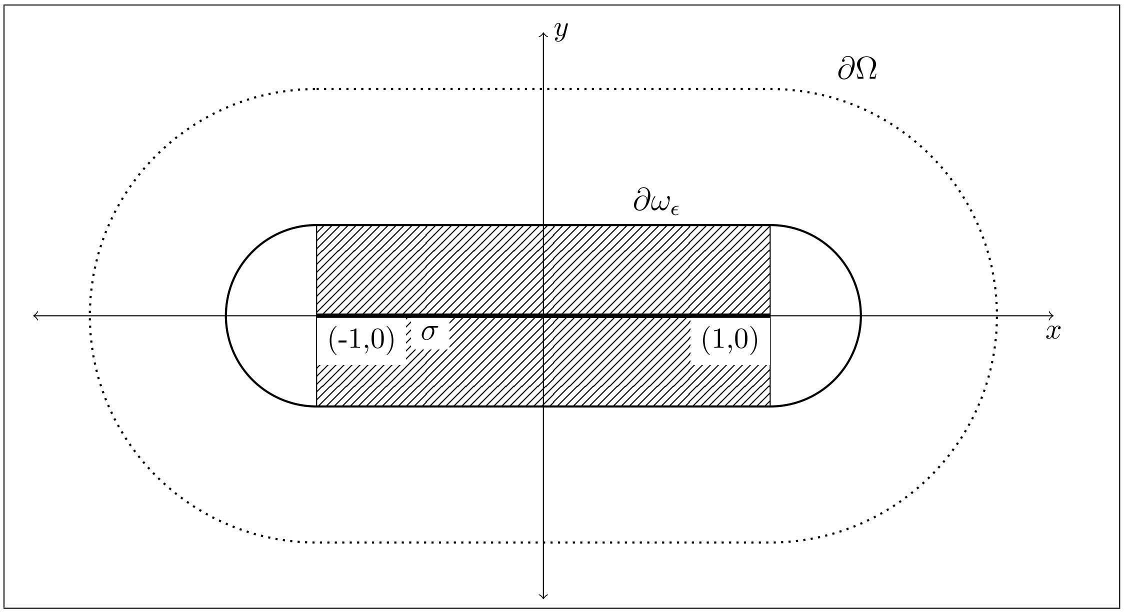

Figure 2.3: Sketch of the region ω_ϵ^s

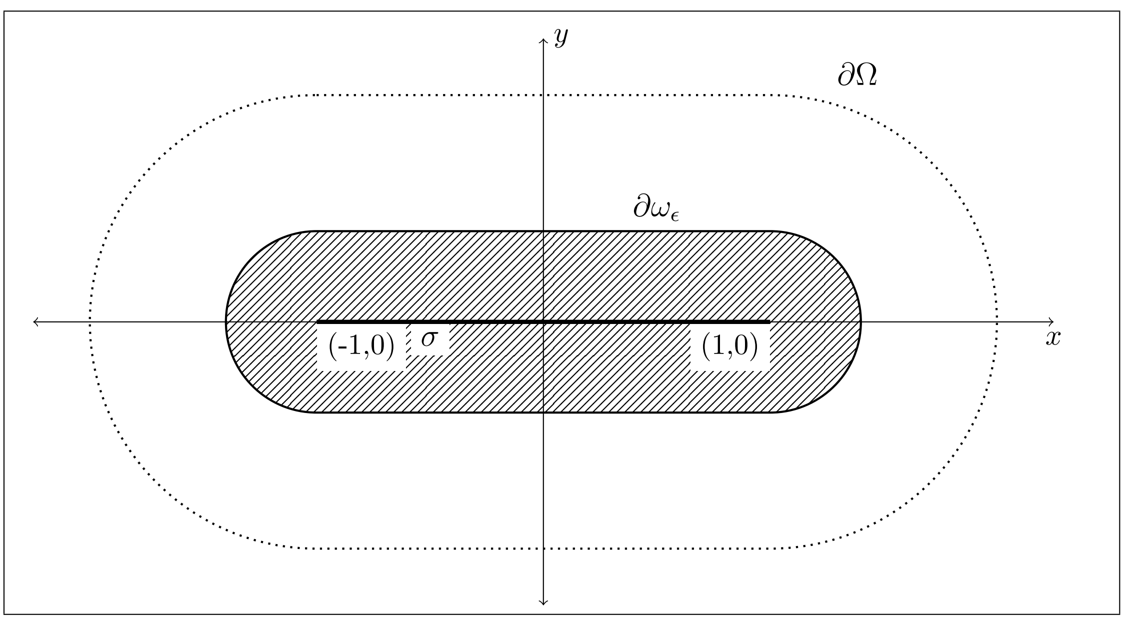

Figure 2.4: Sketch of the region ω_ϵ^r

The first of these functionals is given by

$$\begin{aligned} F_\epsilon^s(u,v) &= \frac{1}{2}\int_{\Omega\setminus\bar{\omega_\epsilon}} |\nabla u|^2\ dx + \frac{a_\epsilon}{2}\int_{\omega_\epsilon^{ends}} |\nabla u|^2\ dx - \int_\Omega fu\ dx \\ &+ \frac{\epsilon a_\epsilon}{2}\int_{\omega_1^{int}} \frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\left(\frac{\partial v}{\partial \tau}\right)^2\ dx + \frac{a_\epsilon}{2\epsilon}\int_{\omega_1^{int}} \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\ dx \end{aligned} \tag{2.4}$$

where the pair (u,v) belongs to the space

$$V_\epsilon^s = \left\{(u,v)\in H^1(\Omega\setminus\overline{\omega_\epsilon^{int}})\times H^1(\omega_1^{int})\mid CC^s\right\}$$

and CC^s is a set of continuity conditions given by

$$CC^s = \begin{cases} \forall\, x\in\sigma & u(x\pm\epsilon n(x)) = v(x\pm n(x)) \\ \forall\, t\in(-1,1) & u(a+\epsilon t n(a)) = v(a+tn(a)) \\ \forall\, t\in(-1,1) & u(b+\epsilon t n(b)) = v(b+tn(b)). \end{cases}$$

Similarly, the second functional is given by

$$\begin{aligned} F_\epsilon^r(u,v) &= \frac{1}{2}\int_{\Omega\setminus\bar{\omega_\epsilon}} |\nabla u|^2\ dx - \int_\Omega fu\ dx + \frac{a_\epsilon}{2}\int_{\omega_1^{ends}} |\nabla v|^2\ dx \\ &+ \frac{\epsilon a_\epsilon}{2}\int_{\omega_1^{int}} \frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\left(\frac{\partial v}{\partial \tau}\right)^2\ dx + \frac{a_\epsilon}{2\epsilon}\int_{\omega_1^{int}} \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\ dx \end{aligned} \tag{2.5}$$

where the pair (u,v) belongs to the space

$$V_\epsilon^r = \left\{(u,v)\in H^1(\Omega\setminus\overline{\omega_\epsilon})\times H^1(\omega_1)\mid CC^r\right\}$$

and CC^r is a set of continuity conditions given by

$$CC^r = \begin{cases} \forall\, x\in\sigma & u(x\pm\epsilon n(x)) = v(x\pm n(x)) \\ \forall\, \theta\in S^1 \text{ with } a+\theta\in\overline{\omega_1^{ends,a}} & v(a+\theta) = u(a+\epsilon\theta) \\ \forall\, \theta\in S^1 \text{ with } b+\theta\in\overline{\omega_1^{ends,b}} & v(b+\theta) = u(b+\epsilon\theta). \end{cases}$$

As both of these functionals are equivalent to the F_ϵ defined in (2.3), all of the minimizations are the same, so we are allowed to use whichever is most convenient. In order to move towards finding the appropriate reduced problem for the situation, we use the F_ϵ^s functional. To do so, we look at the leading order terms in this energy, which gives rise to the functional

$$F_\epsilon^{s,0}(u,v) = \frac{1}{2}\int_{\Omega\setminus\overline{\sigma}} |\nabla u|^2 \, dx + \frac{\epsilon a_\epsilon}{2}\int_{\omega_1^{int}} (1+\kappa d_\sigma)\left(\frac{\partial v}{\partial \tau}\right)^2 \, dx \\ + \frac{a_\epsilon}{2\epsilon}\int_{\omega_1^{int}} \frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2 \, dx - \int_\Omega fu \, dx \tag{2.6}$$

where the pair (u,v) now belongs to the space

$$V_\epsilon^{s,0} = \left\{(u,v) \in H^1(\Omega \setminus \overline{\sigma}) \times H^1(\omega_1^{int}) \mid u^\pm(x) = v(x \pm n(x)), x \in \sigma\right\}$$

where, again, this condition is understood in the sense of traces. There are two things of note about this step. First of all, we lost a boundary condition on the vertical edges of the domain ω_1^{int}, which comes from the fact that the function u is only in H^1, and so is not well-defined at the endpoints of σ. The boundary condition would need to connect v on the vertical segment to $u^\pm$ at the endpoints, which can not be done. The second point is that we removed the term

$$\frac{a_\epsilon}{2}\int_{\omega_\epsilon^{ends}} |\nabla u|^2 \, dx$$

from the expression, because

$$\lim_{\epsilon\to 0}\int_{\omega_\epsilon^{ends}} |\nabla u|^2 \, dx = 0$$

since ∇u is L^2. The full term may not go to zero as $\epsilon \to 0$ if a_ϵ is unbounded, but we will ignore this term in the same way that we ignore terms of order $\frac{a_\epsilon}{\epsilon}\epsilon$ that show up in the normal derivative term.

Now, our functional $F_\epsilon^{s,0}$ and space $V_\epsilon^{s,0}$ match exactly with the F_ϵ^0 functional and V^0 space in [DV17]. The work in that paper then carries through almost exactly to our case here. That is, the process of minimizing the function $F_\epsilon^{s,0}(u,v)$ in (2.6) can be split into two parts, fixing the u function and minimizing over all v satisfying the appropriate boundary condition on $\partial\omega_1$, and then minimizing over all u. We can then write

$$\min_{(u,v)\in V_\epsilon^{s,0}} F_\epsilon^{s,0}(u,v) = \min_{\substack{u\in H^1(\Omega\setminus\overline{\sigma}) \\ u|_{\partial\Omega}=\varphi}} \frac{1}{2}\int_{\Omega\setminus\overline{\sigma}} |\nabla u|^2 \, dx - \int_\Omega fu \, dx + G_\epsilon^0(u)$$

where

$$G_\epsilon^0(u) = \min_{\substack{v\in H^1(\omega_1)\\ v(x+n(x))=u^+(x)\\ v(x-n(x))=u^-(x)}} \frac{\epsilon a_\epsilon}{2}\int_{\omega_1}(1+\kappa d_\sigma)\left(\frac{\partial v}{\partial \tau}\right)^2\,dx + \frac{a_\epsilon}{2\epsilon}\int_{\omega_1}\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\,dx.$$

As the two terms in the expression for G_ϵ^0 are of different orders, we would expect that the behavior of the minimizer (at least to leading order) should be determined by the behavior of the term in G_ϵ^0 of higher order. Thus, we should try to find the minimizer of the term

$$\int_{\omega_1}\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\,dx.$$

Based on the Euler-Lagrange equation associated with this minimization, we know that the minimizer v must satisfy

$$\int_{\omega_1}\frac{1}{1+\kappa d_\sigma}\frac{\partial v}{\partial n}\frac{\partial w}{\partial n}\,dx = 0 \qquad \forall\, w\in H_0^1(\omega_1).$$

Using the co-area formula of Proposition 2.1.1, this becomes

$$\int_\sigma\int_{-1}^1\frac{\partial v}{\partial n}(x+tn(x))\frac{\partial w}{\partial n}(x+tn(x))\,dt\,ds(x) = 0 \qquad \forall\, w\in H_0^1(\omega_1).$$

Choosing a test function w of the form $w(x+tn(x)) = \phi(x)\psi(t)$ for an arbitrary $\phi\in C_0^\infty(\sigma)$ and $\psi\in C_0^\infty((-1,1))$ turns this expression into

$$\int_\sigma\phi(x)\int_{-1}^1\frac{\partial}{\partial t}v(x+tn(x))\psi'(t)\,dt\,ds(x) = 0.$$

Since ϕ is arbitrary, this becomes

$$\int_{-1}^1\frac{\partial}{\partial t}v(x+tn(x))\psi'(t)\,dt\,ds(x) = 0,$$

which, for ψ arbitrary, gives that the function $t\to v(x+tn(x))$ is affine. Therefore, we know that

$$v(x+tn(x)) = \frac{1+t}{2}u^+(x) + \frac{1-t}{2}u^-(x)$$

for all $x\in\sigma$ and $t\in(-1,1)$. Then, plugging this expression in for v in $G_\epsilon^0(u)$ gives that

$$G_\epsilon^0(u) \approx \frac{\epsilon a_\epsilon}{3}\int_\sigma\left(\frac{\partial u^+}{\partial\tau}\right)^2 + \left(\frac{\partial u^-}{\partial\tau}\right)^2 + \frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau}\,ds + \frac{a_\epsilon}{4\epsilon}\int_\sigma(u^+-u^-)^2\,ds.$$

In order to find an approximation u_ϵ^0 to our full solution u_ϵ, we are thus motivated to look for the solution to

$$\min_{\substack{u \in V_\sigma \\ u=\varphi \text{ on } \partial\Omega}} \quad E_\epsilon'^0(u)$$

over the space

$$V_\sigma = \{u \in H^1(\Omega \setminus \overline{\sigma}) \mid u^+|_\sigma,\ u^-|_\sigma \in H^1(\sigma)\}$$

where the energy functional is given by

$$\begin{aligned} E_\epsilon'^0(u) = \frac{1}{2}\int_{\Omega\setminus\overline{\sigma}} |\nabla u|^2 \, dx + \frac{\epsilon a_\epsilon}{3}\int_\sigma \left(\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 + \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau}\right) \, ds \\ + \frac{a_\epsilon}{4\epsilon}\int_\sigma (u^+ - u^-)^2 \, ds - \int_\Omega fu \, dx \end{aligned}$$

and the rescaled potential inside ω_1^{int} should be affine from u_ϵ^{0+} to u_ϵ^{0-} in the normal direction. Since we get to the same energy functional as [DV17], we make use of a remark later in the paper, where they provide a second functional that also provides a uniform 0^{th} order approximation to the full problem in the case of closed curves. The functional that we will focus on for the remainder of this document is

$$\begin{aligned} E_\epsilon^0(u) = \frac{1}{2}\int_{\Omega\setminus\overline{\sigma}} |\nabla u|^2 \, dx + \frac{\epsilon a_\epsilon}{2}\int_\sigma \left(\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2\right) \, ds \\ + \frac{a_\epsilon}{4\epsilon}\int_\sigma (u^+ - u^-)^2 \, ds - \int_\Omega fu \, dx \end{aligned} \tag{2.7}$$

over the same space V_σ.

2.4 Classical formulation of the reduced problem

To work towards a classical formulation of the minimization of (2.7) over V_σ, we define the homogeneous subspace

$$V_{\sigma,0} = \{v \in V_\sigma \mid v = 0 \text{ on } \partial\Omega,\ v(a) = v(b) = 0\}$$

where, again, a and b denote the endpoints of σ. With this, the variational formulation of (2.7) is given by: Find $u_\epsilon^0 \in V_\sigma$ with $u_\epsilon^0|_{\partial\Omega} = \varphi$ so that for all $v \in V_{\sigma,0}$,

$$\begin{aligned} \int_{\Omega\setminus\overline{\sigma}} \nabla u_\epsilon^0 \cdot \nabla v \, dx + \epsilon a_\epsilon \int_\sigma \left(\frac{\partial u_\epsilon^{0+}}{\partial \tau}\frac{\partial v^+}{\partial \tau} + \frac{\partial u_\epsilon^{0-}}{\partial \tau}\frac{\partial v^-}{\partial \tau}\right) \, ds \\ + \frac{a_\epsilon}{2\epsilon}\int_\sigma (u_\epsilon^{0+} - u_\epsilon^{0-})(v^+ - v^-) \, ds = \int_\Omega fv \, dx. \end{aligned} \tag{2.8}$$

By defining the norm

$$||u||^2_{V_\sigma} := ||u||^2_{L^2(\Omega)} + |u|^2_{V_\sigma}$$
$$|u|^2_{V_\sigma} := \int_{\Omega\setminus\overline{\sigma}} |\nabla u|^2 \ dx + \int_\sigma \left(\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 \right) \ ds + \int_\sigma (u^+ - u^-)^2 \ ds$$

on V_σ, we see that the blinear form on the left hand size of (2.8) is both continuous and coercive for any fixed $\epsilon > 0$ and $a_\epsilon > 0$. This coercivity is not uniform in ϵ and a_ϵ, and relies on the fact that $|\cdot|_{V_\sigma}$ is an equivalent norm to $||\cdot||_{V_\sigma}$ for all $v \in V_{\sigma,0}$.

By choosing $v \in C_0^\infty(\Omega \setminus \overline{\sigma})$, we can see that u^0_ϵ satisfies

$$-\Delta u^0_\epsilon = f \quad \text{ in } \Omega \setminus \overline{\sigma}$$

in the sense of distributions. If f and φ are smooth enough, then it can also be easily shown that u^0_ϵ is $C^{2,\alpha}$ away from σ, and so u^0_ϵ solves this equation in the classical sense. If σ is also smooth enough, this differential equation holds classically up to σ, away from the endpoints of the curve. Using (2.8) and integrating by parts both on Ω and on σ, we get that, because $v \in V_{\sigma,0}$,

$$\int_\sigma \left(-\frac{\partial u^{0+}_\epsilon}{\partial n} v^+ + \frac{\partial u^{0-}_\epsilon}{\partial n} v^- \right) \ ds - \epsilon a_\epsilon \int_\sigma \left(\frac{\partial^2 u^{0+}_\epsilon}{\partial \tau^2} v^+ + \frac{\partial^2 u^{0-}_\epsilon}{\partial \tau^2} v^- \right) \ ds$$
$$+ \frac{a_\epsilon}{2\epsilon} \int_\sigma (u^{0+}_\epsilon - u^{0-}_\epsilon)(v^+ - v^-) \ ds = 0.$$

If we choose v so that $v^- \equiv 0$ and v^+ is smooth on σ, we get that u^0_ϵ must satisfy

$$\frac{\partial u^{0+}_\epsilon}{\partial n} + \epsilon a_\epsilon \frac{\partial^2 u^{0+}_\epsilon}{\partial \tau^2} - \frac{a_\epsilon}{2\epsilon}(u^{0+}_\epsilon - u^{0-}_\epsilon) = 0 \quad \text{ on } \sigma$$

and by reversing the roles of v^+ and v^-, we see that

$$\frac{\partial u^{0-}_\epsilon}{\partial n} - \epsilon a_\epsilon \frac{\partial^2 u^{0-}_\epsilon}{\partial \tau^2} - \frac{a_\epsilon}{2\epsilon}(u^{0+}_\epsilon - u^{0-}_\epsilon) = 0 \quad \text{ on } \sigma.$$

Thus, we have the 'classical' problem that u^0_ϵ satisfies as

$$\begin{cases} -\Delta u^0_\epsilon = f & \Omega \setminus \overline{\sigma} \\ u^0_\epsilon = \varphi & \partial\Omega \\ \frac{\partial u^{0+}_\epsilon}{\partial n} + \epsilon a_\epsilon \frac{\partial^2 u^{0+}_\epsilon}{\partial \tau^2} - \frac{a_\epsilon}{2\epsilon}(u^{0+}_\epsilon - u^{0-}_\epsilon) = 0 & \sigma \\ \frac{\partial u^{0-}_\epsilon}{\partial n} - \epsilon a_\epsilon \frac{\partial^2 u^{0-}_\epsilon}{\partial \tau^2} - \frac{a_\epsilon}{2\epsilon}(u^{0+}_\epsilon - u^{0-}_\epsilon) = 0 & \sigma. \end{cases} \tag{2.9}$$

Chapter 3

Analysis of Open Curves

In order to move forward with the energy analysis, we need more detailed information about the solution to (2.9) near the endpoints of σ. To figure out how the solution behaves here, we analyze a pair of related differential equations and connect their solutions to those of (2.9).

3.1 Base problem

In order to simplify calculations, we will assume that our domain $\Omega = B_2(0)$ and that the curve σ is the line segment $\sigma = \{(x, 0) \mid x \in (-1, 1)\}$. As long as the curve σ is a line segment, this is not much of a restriction, as will be addressed later. The reduced problem (2.9) then becomes

$$\begin{cases} -\Delta u_\epsilon^0 = f & \Omega \setminus \overline{\sigma} \\ u_\epsilon^0 = \varphi & \partial\Omega \\ \frac{\partial u_\epsilon^{0+}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u_\epsilon^{0+}}{\partial x^2} - \frac{a_\epsilon}{2\epsilon}(u_\epsilon^{0+} - u_\epsilon^{0-}) = 0 & \sigma \\ \frac{\partial u_\epsilon^{0-}}{\partial y} - \epsilon a_\epsilon \frac{\partial^2 u_\epsilon^{0-}}{\partial x^2} - \frac{a_\epsilon}{2\epsilon}(u_\epsilon^{0+} - u_\epsilon^{0-}) = 0 & \sigma. \end{cases} \tag{3.1}$$

The benefit of this particular choice of domain Ω is that it is symmetric with respect to the line segment σ. Therefore, we can simplify the above equation by looking at the odd and even parts of u_ϵ^0 with respect to the line $y = 0$:

$$u^{ev}(x, y) = \frac{1}{2}(u_\epsilon^0(x, y) + u_\epsilon^0(x, -y)) \qquad u^{odd}(x, y) = \frac{1}{2}(u_\epsilon^0(x, y) - u_\epsilon^0(x, -y)).$$

By using the boundary conditions in (3.1), we have that u^{ev} satisfies

$$\begin{cases} -\Delta u^{ev} = f^{ev} & \Omega \setminus \overline{\sigma} \\ u^{ev} = \varphi^{ev} & \partial\Omega \\ \frac{\partial u^{ev\pm}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{ev\pm}}{\partial x^2} = 0 & \sigma \end{cases} \tag{3.2}$$

and u^{odd} satisfies

$$\begin{cases} -\Delta u^{odd} = f^{odd} & \Omega \setminus \overline{\sigma} \\ u^{odd} = \varphi^{odd} & \partial\Omega \\ \frac{\partial u^{odd\pm}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{odd\pm}}{\partial x^2} - \frac{a_\epsilon}{\epsilon} u^{odd\pm} = 0 & \sigma, \end{cases} \tag{3.3}$$

where $\varphi = \varphi^{ev} + \varphi^{odd}$ and $f = f^{ev} + f^{odd}$ are the decompositions of φ and f into even and odd parts respectively. Since these two solution functions are even and odd, respectively, they are determined by their values on $\Omega^+ = \Omega \cap \{y > 0\}$. Therefore, we can restrict each of these problems to Ω^+, introducing new boundary conditions on $\{y = 0\} \setminus \sigma$ from the symmetry to get that u^{ev} is defined by

$$\begin{cases} -\Delta u^{ev} = f^{ev} & \Omega^+ \\ u^{ev} = \varphi^{ev} & \partial\Omega^+ \\ \frac{\partial u^{ev}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{ev}}{\partial x^2} = 0 & \sigma \\ \frac{\partial u^{ev}}{\partial y} = 0 & \{y = 0\} \setminus \sigma \end{cases} \tag{3.4}$$

and that u^{odd} is defined by

$$\begin{cases} -\Delta u^{odd} = f^{odd} & \Omega^+ \\ u^{odd} = \varphi^{odd} & \partial\Omega^+ \\ \frac{\partial u^{odd}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{odd}}{\partial x^2} - \frac{a_\epsilon}{\epsilon} u^{odd} = 0 & \sigma \\ u^{odd} = 0 & \{y = 0\} \setminus \sigma. \end{cases} \tag{3.5}$$

3.2 Even problem regularity

Consider a solution $u \in H^1(\mathbb{R}^2_+)$ to

$$\begin{cases} -\Delta u = 0 & \mathbb{R}^2_+ \\ \frac{\partial u}{\partial y} + \alpha \frac{\partial^2 u}{\partial x^2} = 0 & \{x > 0, y = 0\} \\ \frac{\partial u}{\partial y} = 0 & \{x < 0, y = 0\} \end{cases}, \tag{3.6}$$

which is related to (3.4) through a renaming of ϵa_ϵ to α and localizing to a single endpoint of σ. Since $f \in \mathcal{F}_\delta$, localizing close enough to the endpoint will result in the source term being identically zero. In order to analyze this problem, we first define an auxiliary map $\mathcal{A}$ by

$$\mathcal{A}[f(t)] = |t| f(t).$$

Lemma 3.2.1. *The map $\mathcal{A}$ has the following properties:*

1. *$\mathcal{A}$ maps $L^2_{loc}(\mathbb{R})$ into $L^2_{loc}(\mathbb{R})$*
2. *$\mathcal{A}$ maps $H^1_{loc}(\mathbb{R})$ into $H^1_{loc}(\mathbb{R})$*
3. *$\mathcal{A}$ maps $H^2_{loc}(\mathbb{R}) \cap \{f : f(0) = 0\}$ into $H^2_{loc}(\mathbb{R}) \cap \{f : f(0) = 0\}$*
4. *$\mathcal{A}$ maps $H^{5/2}_{loc}(\mathbb{R}) \cap \{f : f(0) = 0\}$ into $H^{5/2-\beta}_{loc}(\mathbb{R}) \cap \{f : f(0) = 0\}$ for any $\beta > 0$.*

Proof. 1. This follows because $|x|$ is bounded on compact sets.

2. We compute that

$$\frac{d}{dx}(|x| f(x)) = \begin{cases} f(x) & x > 0 \\ -f(x) & x < 0 \end{cases} + |x| \frac{d}{dx} f(x)$$

which is in $L^2_{loc}(\mathbb{R})$ if f is in $H^1_{loc}(\mathbb{R})$.

3. We compute the second derivative as

$$\frac{d^2}{dx^2}(|x| f(x)) = 2f(0)\delta_0 + \begin{cases} 2f'(x) & x > 0 \\ -2f'(x) & x < 0 \end{cases} + |x| \frac{d^2}{dx^2} f(x)$$

If $f(0) = 0$, then the term with the δ_0 measure vanishes, and so the remaining function is in $L^2_{loc}(\mathbb{R})$ if f is also in $H^2_{loc}(\mathbb{R})$. The result follows from the fact that $|x| f(x)$ vanishes at 0.

4. Looking at the formula for the second derivative, as long as $f(0) = 0$, each component of the function is in at least $L^2_{loc}(\mathbb{R})$. Since we don't know if $f'(0) = 0$, the combined function may not be in $H^{1/2}_{loc}(\mathbb{R})$, but it will be in $H^{1/2-\beta}_{loc}(\mathbb{R})$ for any $\beta > 0$. ■

Corollary 3.2.1. *By interpolation of Sobolev Spaces, as in [BL76], we have that*

1. *$\mathcal{A}$ maps $H^{1/2}_{loc}(\mathbb{R})$ to $H^{1/2}_{loc}(\mathbb{R})$.*
2. *$\mathcal{A}$ maps $H^{3/2}_{loc}(\mathbb{R}) \cap \{f : f(0) = 0\}$ to $H^{3/2}_{loc}(\mathbb{R}) \cap \{f : f(0) = 0\}$.*

In polar coordinates, define a function W in the first quadrant by $W(r, \theta) = u(r^2, 2\theta)$, and since $\frac{\partial W}{\partial x} = 0$ on $\{x = 0, y > 0\}$ by the boundary condition for u, we may extend W to be a function defined on all of $\mathbb{R}^2_+$ that is even across $x = 0$. This function W now satisfies

$$\begin{cases} -\Delta W = 0 & \mathbb{R}^2_+ \\ \frac{\partial W}{\partial y} = -\alpha \frac{\partial}{\partial x}\left(\frac{1}{2|x|}\frac{\partial W}{\partial x}\right) & \{y = 0\} \end{cases} \tag{3.7}$$

Since $u \in H^1_{loc}(\mathbb{R}^2_+)$, with a bound independent of α, so is W (the converse is not true; W could have a mild singularity at the origin that becomes not integrable when moving back to u). Thus, the trace of $\frac{\partial W}{\partial y}$ on $y = 0$ is in $H^{-1/2}_{loc}(\mathbb{R})$, and so by (3.7), $\frac{1}{2|x|}\frac{\partial W}{\partial x}(x, 0)$ is in $H^{1/2}_{loc}(\mathbb{R})$, with bound controlled by $\frac{C}{\alpha}$. By the properties of the map $\mathcal{A}$, we see that $\frac{\partial W}{\partial x}(\cdot, 0)$ is in $H^{1/2}_{loc}(\mathbb{R})$, so that $W(x, 0) \in H^{3/2}_{loc}(\mathbb{R})$, and by elliptic regularity of the Dirichlet problem, we have that $W \in H^2_{loc}(\mathbb{R}^2_+)$ with bound controlled by $\frac{C}{\alpha} + C$.

Repeating this process, we then have that $\frac{\partial W}{\partial y}(\cdot, 0)$ is in $H^{1/2}_{loc}(\mathbb{R})$, so that, by (3.7) again, $\frac{1}{2|x|}\frac{\partial W}{\partial x}(\cdot, 0)$ is in $H^{3/2}_{loc}(\mathbb{R})$. Since W is even in x, $\frac{\partial W}{\partial x}$ is odd in x, and so the function $\frac{1}{2|x|}\frac{\partial W}{\partial x}(\cdot, 0)$ is odd. Since it is also continuous, it must vanish at 0. Thus, the corollary about the map $\mathcal{A}$ says that the function $\frac{\partial W}{\partial x}(\cdot, 0)$ is in $H^{3/2}_{loc}(\mathbb{R})$, implying that $W(\cdot, 0) \in H^{5/2}_{loc}(\mathbb{R})$ and $W \in H^3_{loc}(\mathbb{R}^2_+)$, where the bound is controlled by $\frac{C}{\alpha^2} + C$.

With this regularity, we then have that $\frac{\partial W}{\partial y}(\cdot, 0)$ is in $H^{3/2}_{loc}(\mathbb{R})$, so that, by (3.7), $\frac{1}{2|x|}\frac{\partial W}{\partial x}(\cdot, 0)$ is in $H^{5/2}_{loc}(\mathbb{R})$. The corollary about the map A says that the function $\frac{\partial W}{\partial x}(\cdot, 0)$ is in $H^{5/2-\beta}_{loc}(\mathbb{R})$ for any $\beta > 0$, implying that $W(\cdot, 0) \in H^{7/2-\beta}_{loc}(\mathbb{R})$ and $W \in H^{4-\beta}_{loc}(\mathbb{R}^2_+)$, with a bound controlled by $\frac{C}{\alpha^3} + C$.

Now, let V be the harmonic conjugate of W, defined so that $V(0,0) = 0$. Since $\nabla V = \nabla W^{\perp}$, we have that $V \in H^{4-\beta}_{loc}(\mathbb{R}^2_+)$ with the same bound as W. Then, the function

$$\mathbf{W}(z) = W(x,y) + iV(x,y) \qquad \text{for } z = x + iy$$

is holomorphic in a $\mathbb{C}_+ := \mathbb{C} \cap \{\mathfrak{Im}(z) \geq 0\}$ neighborhood of 0 and $C^{2,\beta}$ on the closure of this neighborhood for any $\beta < 1$ by standard Sobolev embeddings. Thus, it has a Taylor approximation of degree 2 around the origin in the sense that

$$\mathbf{W}(z) = B_0 + B_1 z + \frac{1}{2} B_2 z^2 + E(z)$$

where

$$\begin{aligned} |E(z)| &\leq \left(\frac{C}{\alpha^3} + C\right) |z|^{2+\beta} \\ \left|\frac{d}{dz} E(z)\right| &\leq \left(\frac{C}{\alpha^3} + C\right) |z|^{1+\beta} \\ \left|\frac{d^2}{dz^2} E(z)\right| &\leq \left(\frac{C}{\alpha^3} + C\right) |z|^{\beta} \end{aligned} \tag{3.8}$$

This follows from the fact that $\mathbf{w}$ has a Taylor expansion in x and y, and the $\bar{z}$ coefficients must vanish by the function being holomorphic. Since W is even in x, it follows that $B_1 = -ib_1$ and $B_2 = 2b_2$, where b_1 and b_2 are real parameters, and since $V(0,0) = 0$, $B_0 = b_0$ is also real. Since we have $C^{2,\beta}$ control on $\mathbf{W}$, we know that the coefficients b_0, b_1, and b_2 are all bounded by $\frac{C}{\alpha^3} + C$.

In order to get back to the function u, we let $\sqrt{z}$ be the complex square-root function with branch cut along the positive real axis, and define

$$\mathbf{w}(z) = \mathbf{W}(\sqrt{z})$$

with the motivation that, under this definition

$$u(x,y) = \mathfrak{Re}(w(x+iy))$$

for $(x,y) \in \mathbb{R}^2_+$. With this, we see that

$$\mathbf{w}(z) = b_0 - ib_1\sqrt{z} + b_2 z + e(z)$$

with appropriate bounds on $e(z)$ coming from (3.8), resulting in the solution u to (3.6) having the form

$$u(x,y) = b_0 + b_1 r^{1/2}\sin(\theta/2) + b_2 r\cos(\theta) + e(x,y)$$

with

$$\begin{aligned}
|e(x,y)| &\leq \left(\frac{C}{\alpha^3} + C\right) r^{1+\frac{\beta}{2}} \\
|\nabla e(x,y)| &\leq \left(\frac{C}{\alpha^3} + C\right) r^{\frac{\beta}{2}} \\
|D^2 e(x,y)| &\leq \left(\frac{C}{\alpha^3} + C\right) r^{-1+\frac{\beta}{2}}
\end{aligned}$$

in an $\mathbb{R}^2_+$ neighborhood of the origin, with (r,θ) denoting polar coordinates around $(0,0)$. For convenience, we set $u^* = b_0 + b_2 r\cos\theta + e(x,y)$, isolating the most singular term in the expression, resulting in a decomposition of the form

$$u(x,y) = b_1 r^{1/2}\sin(\theta/2) + u^*(x,y) \tag{3.9}$$

with

$$\begin{aligned}
&u^*(x,y) \in C^{1,\beta/2} \\
&|D^2 u^*| \leq \left(\frac{C}{\alpha^3} + C\right) r^{-1+\frac{\beta}{2}} \\
&u^*(0,0) = b_0 \\
&\frac{\partial u^*}{\partial x}(0,0) = b_2
\end{aligned} \tag{3.10}$$

in a $\mathbb{R}^2_+$ neighborhood of the origin.

3.3 Odd problem regularity

Now, we consider a problem of the form

$$\begin{cases}
-\Delta u = 0 & \mathbb{R}^2_+ \\
\frac{\partial u}{\partial y} = \lambda u & \{x > 0,\ y = 0\} \\
u = 0 & \{x < 0,\ y = 0\}.
\end{cases} \tag{3.11}$$

This is related to the problem (3.5) via a renaming of $\frac{\epsilon}{a_\epsilon}$ to λ, localizing to one of the endpoints of σ and removing the second derivative term. We will show later that removing this term does not affect the resulting solution too much, in the sense that these two solutions are 'close' in an appropriate sense. The analysis of this problem is very similar to that of the even problem in the last section. We define $W(r,\theta) = u(r^2, 2\theta)$ as before, but now, since $W \equiv 0$ along $x = 0$, we extend W as an odd function in x to all of $\mathbb{R}^2_+$. This function W satisfies

$$\begin{cases} -\Delta W = 0 & \mathbb{R}^2_+ \\ \frac{\partial W}{\partial y} = \lambda |x| W(x,0) & \{y = 0\}. \end{cases}$$

As before, $W \in H^1_{loc}(\mathbb{R}^2_+)$, and so its trace on $\mathbb{R}$, $W(\cdot, 0) \in H^{1/2}_{loc}(\mathbb{R})$, bounded by a constant independent of λ. By the properties of the map $\mathcal{A}$ discussed earlier, we have that $|\cdot|W(\cdot,0) \in H^{1/2}_{loc}(\mathbb{R})$, and so $\frac{\partial W}{\partial y} \in H^{1/2}_{loc}(\mathbb{R})$, with a bound controlled by $C\lambda$. Now, by using the elliptic regularity of the Neumann problem, (as opposed to the Dirichlet problem that we used before) we have that $W \in H^2_{loc}(\mathbb{R}^2_+)$, with norm controlled by $(C\lambda + C)$. We can now use the same bootstrapping argument from before: $W(\cdot, 0)$ is thus in $H^{3/2}_{loc}(\mathbb{R})$, and by properties of the map $\mathcal{A}$, so is $|\cdot|W(\cdot,0)$, because, as before, W is now a continuous function that is odd in x, and so vanishes at $(0,0)$. This regularity carries over to $\frac{\partial W}{\partial y}$, and so by the regularity of the Neumann problem, $W \in H^3_{loc}(\mathbb{R}^2_+)$. We can run the process one more time, getting to $W \in H^{4-\beta}_{loc}(\mathbb{R}^2_+)$, with norm controlled by $(C\lambda^3 + C)$.

We can again define V to be the harmonic conjugate of W, with $V(0,0) = 0$ and define the function

$$\mathbf{W}(z) = W(x,y) + iV(x,y) \qquad z = x + iy$$

Since $\mathbf{W}$ is holomorphic in a $\mathbb{C}_+ := \mathbb{C} \cap \{\mathfrak{Im}(z) \geq 0\}$ neighborhood of $(0,0)$ and is $C^{2,\beta}$ on this neighborhood for any $\beta < 1$, it has a Taylor series approximation of the form

$$\mathbf{W}(z) = B_0 + B_1 z + \frac{1}{2} B_2 z^2 + E(z)$$

where

$$
\begin{aligned}
|E(z)| &\le \left(C\lambda^3 + C\right)|z|^{2+\beta} \\
\left|\frac{d}{dz}E(z)\right| &\le \left(C\lambda^3 + C\right)|z|^{1+\beta}. \\
\left|\frac{d^2}{dz^2}E(z)\right| &\le \left(C\lambda^3 + C\right)|z|^{\beta}
\end{aligned}
\tag{3.12}
$$

Since $\mathbf{W}(0,0) = 0$, we have that $B_0 = 0$, and since W is odd in x, we have that $B_1 = b_1$ and $B_2 = -2ib_2$ where b_1 and b_2 are real parameters. After setting $\mathbf{w}(z) = \mathbf{W}(\sqrt{z})$ we get that

$$
\mathbf{w}(z) = b_1\sqrt{z} - ib_2 z + e(z)
$$

and noticing that $u(x,y) = \mathfrak{Re}(\mathbf{w}(x+iy))$ exactly as before gives the following decomposition for the solution u to (3.11):

$$
u(x,y) = b_1 r^{1/2}\cos(\theta/2) + b_2 r\sin\theta + e(x,y)
$$

with

$$
\begin{aligned}
|e(x,y)| &\le \left(C\lambda^3 + C\right) r^{1+\frac{\beta}{2}} \\
|\nabla e(x,y)| &\le \left(C\lambda^3 + C\right) r^{\frac{\beta}{2}} \\
|D^2 e(x,y)| &\le \left(C\lambda^3 + C\right) r^{-1+\frac{\beta}{2}}
\end{aligned}
$$

in an $\mathbb{R}^2_+$ neighborhood of the origin, where (r,θ) denote polar coordinates there. In a similar manner to what we did with the even problem, we will reformulate this decomposition to isolate the most singular term, giving a decomposition of the form

$$
u(x,y) = b_1 r^{1/2}\cos(\theta/2) + u^*(x,y) \tag{3.13}
$$

with

$$
\begin{aligned}
&u^*(x,y) \in C^{1,\beta/2} \\
&|D^2 u^*| \le \left(C\lambda^3 + C\right) r^{-1+\frac{\beta}{2}} \\
&\frac{\partial u^*}{\partial y}(0,0) = b_2
\end{aligned}
\tag{3.14}
$$

in a $\mathbb{R}^2_+$ neighborhood of the origin.

3.4 Localization argument

Now, we want to relate the solution to (3.6) to that of (3.2), which is directly connected to the reduced problem we are trying to solve. We will start with

$$\begin{cases} -\Delta u^{ev} = f^{ev} & \Omega^+ \\ u^{ev} = \varphi^{ev} & \partial\Omega^+ \\ \frac{\partial u^{ev}}{\partial y} + \alpha \frac{\partial^2 u^{ev}}{\partial x^2} = 0 & \sigma \\ \frac{\partial u^{ev}}{\partial y} = 0 & \{y = 0\} \setminus \sigma, \end{cases} \tag{3.15}$$

which is (3.4) with ϵa_ϵ replaced by a fixed constant α. Let (r, θ) denote polar coordinates around the point $(-1, 0)$, which is the left endpoint of σ, and assume that $f \in \mathcal{F}_\delta$ for some $\delta > 0$. Without loss of generality, we may assume that $\delta < \frac{1}{2}$. If we restrict to the domain $r < \frac{\delta}{2}$, we have that u^{ev} solves

$$\begin{cases} -\Delta u^{ev} = 0 & \{r < \frac{\delta}{2}\}_+ \\ u^{ev} = \psi^{ev} & \{r = \frac{\delta}{2}\}_+ \\ \frac{\partial u^{ev}}{\partial y} + \alpha \frac{\partial^2 u^{ev}}{\partial x^2} = 0 & \{y = 0, -1 < x < -1 + \frac{\delta}{2}\} \\ \frac{\partial u^{ev}}{\partial y} = 0 & \{y = 0,\ -1 - \frac{\delta}{2} < x < -1\} \end{cases} \tag{3.16}$$

where $\psi^{ev} = u^{ev} \mid_{r=\frac{\delta}{2}}$. By standard elliptic regularity results, we know that

$$||\psi^{ev}||_{H^{1/2}(\{r=\frac{\delta}{2}\})} \leq C \left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)} \right) \tag{3.17}$$

and we will let $C(f, \varphi)$ denote this form of constant on the right hand side of the above expression. In particular, since the curve $r = \frac{\delta}{2}$ lies in the region where $f \equiv 0$, we have that ψ^{ev} is actually smooth, and the bound in (3.17) holds in any H^s norm, not just $H^{1/2}$. We are now left with a problem that is very similar to (3.6), except we have a boundary condition at $r = \frac{\delta}{2}$. In order for the results in the previous sections to carry through, we need to show regularity of the solution u^{ev} near $(-1, 0)$. The analysis to get this regularity is almost identical to what was already done, with a few main differences.

1. When generating the function W defined by $W(r, \theta) = u(r^2, 2\theta)$, this function becomes defined on the ball of radius $\sqrt{\frac{\delta}{2}}$ instead of all of $\mathbb{R}^2$, with a boundary condition at this boundary that is related to ψ^{ev}. It shares the same norm control, up to constants that may depend on δ, which is fixed.

2. Instead of dealing with functions in $H^s_{loc}(\mathbb{R})$, the entire proof needs to use functions in $H^s(I)$, where I is an interval in $\mathbb{R}$. For $s > 0$, there is no change, but for $s < 0$, where the space is defined as a dual space of distributions, things change slightly. For $s > 0$, the space $H^{-s}(I)$ is defined as the dual space of $H^s_{00}(I)$, where this denotes the set of all functions in $H^s(I)$ that are in $H^s(\mathbb{R})$ when extended by 0 outside of I [LM72]. For $s < 1/2$, $H^s_{00}(I) = H^s(I)$ (since step functions are in H^s) and for $1/2 < s < 1$, $H^s_{00}(I) = H^s_0(I)$. The case where $s = 1/2$ is complicated, and since the final regularity result has flexibility in terms of the $\beta > 0$, we can afford to give up regularity at this first step to save the technical details. In particular, since we know that $\frac{\partial}{\partial x}\frac{1}{2|x|}\frac{\partial W}{\partial x}(x, 0) \in H^{-1/2}(I)$, we know that $\frac{\partial}{\partial x}\frac{1}{2|x|}\frac{\partial W}{\partial x}(x, 0) \in H^{-1/2-\beta}(I)$ for any $\beta > 0$, where this space is now the dual of $H_0^{1/2+\beta}(I)$. Since $\frac{1}{2|x|}\frac{\partial W}{\partial x}(x, 0)$ is odd across zero and $\frac{\partial}{\partial x}$ maps $H^{1/2-\beta}(I) \cap \{v : \int_I v = 0\}$ one-to-one and onto $H^{-1/2-\beta}(I)$ (which is proven via the adjoint map), we then know that the function $\frac{1}{2|x|}\frac{\partial W}{\partial x}(x, 0) \in H^{1/2-\beta}(I)$. The rest of the proof then works exactly as before, except all of the regularity results are reduced by some $\beta > 0$. Since our final result is that our function belongs in the space $H^{4-\beta}(I)$ for any $\beta > 0$, we can incorporate this reduced regularity from before into this to get us to the same regularity result.

3. Whenever elliptic regularity of the Dirichlet or Neumann problem is used to increase the regularity of the function, this step also needs to worry about the norm of the boundary data. However, the boundary data is smooth, and always has a norm controlled by $C(f, \varphi)$. Thus, all of the C constants that arose in the analysis become $C(f, \varphi)$, due to this extra boundary condition, but this provides no other obstacle to the analysis.

Therefore, we have that, for the even symmetry case, our solution u^{ev} has a decomposition of the form

$$u^{ev}(x,y) = b_1 r^{1/2}\sin(\theta/2) + u^{ev,*}(x,y)$$

with

$$\begin{aligned}
&u^{ev,*}(x,y) \in C^{1,\beta/2} \qquad ||u^{ev,*}||_{C^{1,\beta/2}} \leq C(f,\varphi)\left(\frac{1}{\alpha^3}+1\right)\\
&|D^2 u^{ev,*}| \leq C(f,\varphi)\left(\frac{1}{\alpha^3}+1\right) r^{-1+\frac{\beta}{2}}\\
&u^{ev,*}(-1,0) = b_0\\
&\frac{\partial u^{ev,*}}{\partial x}(-1,0) = b_2
\end{aligned} \tag{3.18}$$

for some real constants b_0, b_1, and b_2, in some neighborhood of $(-1,0)$, for any $\beta < 1$. We also know that, by the $C^{1,\beta/2}$ control, that all of the b constants are bounded by $C(f,\varphi)\left(\frac{1}{\alpha^3}+1\right)$.

The exact same analysis can be carried out for the endpoint at $(1,0)$, resulting in a similar expansion in terms of polar coordinates (r',θ'), where $\theta' = 0$ and $\theta' = 2\pi$ correspond to the curve σ. Now, let η be a radial cut-off function with $\eta(r) \equiv 1$ on $r < \frac{\delta}{2}$ and $\eta(r) \equiv 0$ on $r > \delta$. Consider the function

$$v(x,y) = b_1 r^{1/2}\sin(\theta/2)\eta(r) + b_1' r'^{1/2}\sin(\theta'/2)\eta(r')$$

and look at $v - u^{ev}$. Since $\eta(r) \equiv 1$ on $r < \frac{\delta}{2}$, the functions v and u^{ev} agree on $r < \frac{\delta}{2}$ up to a function in $C^{1,\beta/2}$ with norm controlled by $C(f,\varphi)\left(\frac{1}{\alpha^3}+1\right)$. The exact same holds for $r' < \frac{\delta}{2}$. Outside of these two regions and inside ω_δ (where $f \equiv 0$), we have that both v and u^{ev} are smooth functions with norms controlled by the same constant. Therefore, we know that in all of ω_δ, $v - u^{ev}$ is a $C^{1,\beta/2}$ function. This gives the following lemma.

Lemma 3.4.1. *Let (r,θ) and (r',θ') be polar coordinates around the two endpoints of σ, $(-1,0)$ and $(1,0)$ respectively, where in each case, the 0 and 2π angles correspond to σ. Let η denote a smooth cut-off function as above. Then the solution u^{ev} to (3.15) satisfies*

$$u^{ev} = b_1 r^{1/2}\sin(\theta/2)\eta(r) + b_1' r'^{1/2}\sin(\theta'/2)\eta(r') + u^{ev,*}$$

where $u^{ev,*}$ *is bounded in* $C^{1,\beta/2}((\omega_\delta)_+)$ *by* $C(f,\varphi)\left(\frac{1}{\alpha^3}+1\right)$, *and the constants* b_1 *and* b_1' *are bounded by* $C(f,\varphi)\left(\frac{1}{\alpha^3}+1\right)$ *as well.*

Remark. The only obstruction to extending this result to larger subdomains of Ω is the regularity of the input data. If f was at least H^2, then this result could be extended to hold in any $\tilde{\Omega} \subset\subset \Omega$ with $\partial\tilde{\Omega}$ smooth enough by standard elliptic regularity arguments, because the solution to the equation in the full domain would be at least $C^{1,\beta/2}$ for any $\beta > 0$. The norm dependence would increase from the L^2 norm of f to the H^2 norm of f in this external region. The same goes for regularity up the boundary; if φ and $\partial\Omega$ were smoother, then this could be extended to all of Ω with the corresponding increase in the norms required.

By following the exact same procedure, we can say something about problems related to (3.11). Consider a problem of the form

$$\begin{cases} -\Delta u^{odd'} = f^{odd} & \Omega^+ \\ u^{odd'} = \varphi^{odd} & \partial\Omega^+ \\ \frac{\partial u^{odd'}}{\partial y} = \lambda u^{odd'} & \sigma \\ u^{odd'} = 0 & \{y=0\} \setminus \sigma \end{cases} \tag{3.19}$$

defined on the upper-half region Ω_+. Following the same process of restricting to a neighborhood of each endpoint of σ and using the decomposition given in (3.13) and (3.14) instead of (3.9) and (3.10), gives the following lemma

Lemma 3.4.2. *Let* (r,θ) *and* (r',θ') *be polar coordinates around the two endpoints of* σ, $(-1,0)$ *and* $(1,0)$ *respectively, where in each case, the* 0 *and* 2π *angles correspond to* σ. *Let* η *denote a smooth cut-off function as above. Then the solution* $u^{odd'}$ *to* (3.19) *satisfies*

$$u^{odd'} = b_1 r^{1/2}\cos(\theta/2)\eta(r) + b_1' r'^{1/2}\cos(\theta'/2)\eta(r') + u^{odd,*}$$

where $u^{odd,*}$ *is bounded in* $C^{1,\beta/2}((\omega_\delta)_+)$ *by* $C(f,\varphi)\left(\lambda^3+1\right)$, *and the constants* b_1 *and* b_1' *are bounded by* $C(f,\varphi)\left(\lambda^3+1\right)$ *as well.*

Chapter 4

Energy Convergence

In this chapter, we use the analysis of the previous chapter to prove energy, and thus norm, convergence of the solution u_ϵ^0 to u_ϵ as $\epsilon \to 0$. In a sense, this proves that the reduced problem (2.9) is an appropriate approximation to the full problem (2.1) in the limit as $\epsilon \to 0$.

4.1 Setup

The goal for this chapter is to use Lemma 2.2.2 to show that the solution u_ϵ^0 to (2.9) is asymptotically close to the solution u_ϵ to (2.1) as $\epsilon \to 0$. To do this, we need to show that for any background f and boundary data φ, the energy of the minimizers of the corresponding energy functional are close, up to a constant depending on f and φ. To show that they are close, we make use of the dual energy maximization problem, rewriting the energy minimization as a corresponding maximization problem. This will allow us to show that

$$E_\epsilon(u_\epsilon) - E_\epsilon^0(u_\epsilon^0)$$

is small by getting a bound on

$$E_\epsilon(u) - E_\epsilon^0(u_\epsilon^0)$$

for a particular choice of u that is very similar to u_ϵ^0, and to show that

$$E_\epsilon^0(u_\epsilon^0) - E_\epsilon(u_\epsilon)$$

is small by converting both of these to maximization problems, and showing that

$$E_\epsilon^{0,c}(u_\epsilon^0) - E_\epsilon^c(u)$$

is small for an appropriately chosen u. For a more detailed description of dual energies, as well as a derivation of all of the dual energies used later in this chapter, see Appendix B.

For the next few sections, we will assume that $\Omega = B_2(0)$ and $\sigma = \{(x, 0) \mid x \in (-1, 1)\}$, and that $f \in \mathcal{F}_\delta$ for some $\delta > 0$. This will allow us to make use of the even and odd symmetric problems discussed in the last chapter. The plan for the next few sections is that we will deal with either the even or odd problem, restricting the value of a_ϵ to facilitate the necessary analysis to prove the closeness that we want. After that, we will combine these results to get a conclusion for any a_ϵ and then extend to domains that are not symmetric around σ, only assuming that σ is a straight line.

4.2 Even symmetry, $\epsilon a_\epsilon > m > 0$

Taking the domain assumptions as above and looking at the even version of the problem gives us the PDE (3.4), that is

$$\begin{cases} -\Delta u^{ev} = f^{ev} & \Omega^+ \\ u^{ev} = \varphi^{ev} & \partial\Omega^+ \\ \frac{\partial u^{ev}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{ev}}{\partial x^2} = 0 & \sigma \\ \frac{\partial u^{ev}}{\partial y} = 0 & \{y = 0\} \setminus \sigma. \end{cases}$$

If we assume that $\epsilon a_\epsilon > m > 0$, then we are in exactly the case of Lemma 3.4.1, which tells us that the u^{ev} has the form

$$u^{ev} = b_1 r^{1/2} \sin(\theta/2)\eta(r) + b_1' r'^{1/2} \sin(\theta'/2)\eta(r') + u^{ev,*}$$

where $u^{ev,*}$ is bounded in $C^{1,\beta/2}((\omega_\delta)_+)$ by $C(f, \varphi)\left(\frac{1}{(\epsilon a_\epsilon)^3} + 1\right)$, and the constants b_1 and b_1' are bounded by $C(f, \varphi)\left(\frac{1}{(\epsilon a_\epsilon)^3} + 1\right)$. Given this decomposition, we can also prove an additional fact about the remainder term $u^{ev,*}$.

Lemma 4.2.1. *For this function $u^{ev,*}$, we have*

$$\frac{\partial u^{ev,*}}{\partial x}(-1, 0) = \frac{\partial u^{ev,*}}{\partial x}(1, 0) = 0,$$

that is, the tangential derivatives of $u^{ev,*}$ vanish at the endpoints of σ.

Proof. Let T be an inverse to the trace operator on $H^{1/2}(\sigma)$ so that the lifted function vanishes on $\partial\Omega$. The variational formulation of (3.4) gives that

$$\begin{aligned}\int_{\Omega_+} \nabla u^{ev} \nabla (Tv)\ dx &= \epsilon a_\epsilon \int_\sigma \frac{\partial u^{ev}}{\partial x}\frac{\partial Tv}{\partial x}\ ds \\ &= \epsilon a_\epsilon \int_\sigma \frac{\partial u^{ev}}{\partial x}\frac{\partial v}{\partial x}\ ds \\ &= \epsilon a_\epsilon \int_\sigma \frac{\partial u^{ev,*}}{\partial x}\frac{\partial v}{\partial x}\ ds\end{aligned}$$

where the last step holds because the most singular parts of u^{ev} are identically zero along the curve σ. This equality then implies that the functional

$$v \mapsto \int_\sigma \frac{\partial u^{ev,*}}{\partial x}\frac{\partial v}{\partial x}$$

can be extended continuously to all of $H^{1/2}(\sigma)$, provided $\epsilon a_\epsilon > m > 0$. Since $u^{ev,*}$ is smooth on the interior of σ and $C^{1,\beta/2}$ up to the boundary, we can integrate by parts to get that

$$\int_\sigma \frac{\partial u^{ev,*}}{\partial x}\frac{\partial v}{\partial x}\ dx = -\int_\sigma \frac{\partial^2 u^{ev,*}}{\partial x^2} v\ dx + \frac{\partial u^{ev,*}}{\partial x} v \mid_{x=-1}^{x=1}$$

for any $v \in H^1(\sigma)$. For the first term on the right hand side, we get

$$\begin{aligned}\left|\int_\sigma \frac{\partial^2 u^{ev,*}}{\partial x^2} v\ ds\right| &\leq \int_\sigma \left|\frac{\partial^2 u^{ev,*}}{\partial x^2}\right| |v|\ ds \\ &\leq C\left(\int_\sigma d(x,\partial\sigma)^{2(-1+\beta/2+p)}\ ds\right)^{1/2}\left(\int_\sigma \frac{|v|^2}{d(x,\partial\sigma)^{2p}}\ ds\right)^{1/2} \\ &\leq C\left(\int_\sigma d(x,\partial\sigma)^{2(-1+\beta/2+p)}\ ds\right)^{1/2} ||v||_{H^p(\sigma)}\end{aligned}$$

where we have used the properties of $u^{ev,*}$ from (3.18), as well as the fact that for $p \leq 1$, $|v| \in H^p(\sigma)$, and for $0 \leq p \leq \frac{1}{2}$,

$$\int_\sigma \frac{|v|^2}{d(x,\partial\sigma)^{2p}}\ ds \leq C||v||_{H^p(\sigma)}.$$

This last statement follows from the classical Hardy's inequality, combined with interpolation of Sobolev spaces in the sense that the interpolant of $L^2(\sigma)$ and $H^1(\sigma) \cap \{f(0) = 0\}$ is $H^p(\sigma)$ for $0 \leq p \leq \frac{1}{2}$. Since β can be chosen arbitrarily close to 1, we may choose it large enough so that

$$\int_\sigma d(x,\partial\sigma)^{2(-1+\beta/2+p)}\ ds < \infty$$

for any $p > 0$. Therefore, we have that

$$\left| \int_\sigma \frac{\partial^2 u^{ev,*}}{\partial x^2} v \; ds \right| \leq C ||v||_{H^p(\sigma)}$$

for any $0 < p \leq \frac{1}{2}$, which implies that this term of the functional is continuously extendable to $H^{1/2}(\sigma)$. This then implies that

$$v \mapsto \frac{\partial u^{ev,*}}{\partial x} v \Big|_{x=-1}^{x=1}$$

must also be continuously extendable to $H^{1/2}(\sigma)$. However, we know that the trace operators at 1 and -1 are not well defined on $H^{1/2}(\sigma)$, which means that this statement can only hold if

$$\frac{\partial u^{ev,*}}{\partial x}(-1,0) = \frac{\partial u^{ev,*}}{\partial x}(1,0) = 0. \qquad \blacksquare$$

To analyze this problem, we define the even energy

$$E^{ev}(v) = \frac{1}{2} \int_{\Omega \backslash \sigma} |\nabla v|^2 \; dx + \epsilon a_\epsilon \int_\sigma \left(\frac{\partial v}{\partial x} \right)^2 \; ds - \int_{\Omega \backslash \sigma} f^{ev} v \; dx \tag{4.1}$$

for $v \in H^1(\Omega) \cap H^1(\sigma)$ with $v = \varphi^{ev}$ on $\partial\Omega$, which we want to compare to the even part

$$E_\epsilon^{ev}(v) = \frac{1}{2} \int_\Omega \gamma_\epsilon |\nabla v|^2 \; dx - \int_\Omega f^{ev} v \; dx$$

of $E_\epsilon(v)$ from (2.2), with $v \in H^1(\Omega)$ and $v = \varphi^{ev}$ on $\partial\Omega$.

In this section, we prove the following theorem:

Theorem 4.2.1. *Suppose $\epsilon a_\epsilon > m > 0$. Consider the solution u_ϵ to the problem* (2.1)*, where σ is a straight line segment and Ω is symmetric about σ, with $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$. Let f^{ev} and φ^{ev} be the even parts of f and φ respectively. Let u^{ev} be the solution to* (3.2)*, and let u_ϵ^{ev} be the even part of u_ϵ. Then*

$$|E_\epsilon^{ev}(u_\epsilon^{ev}) - E^{ev}(u^{ev})| \leq C(f, \varphi)^2 \epsilon^{\beta/4}$$

where $C(f, \varphi)$ denotes a quantity of the form

$$C(f, \varphi) = C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}).$$

This is reduced to verifying two inequalities:

$$E_\epsilon^{ev}(u_\epsilon^{ev}) - E^{ev}(u^{ev}) \leq C(f, \varphi)^2 \epsilon^{\beta/2} \tag{4.2}$$

and

$$E^{ev}(u^{ev}) - E_\epsilon^{ev}(u_\epsilon^{ev}) \leq C(f, \varphi)^2 \epsilon^{\beta/4}. \tag{4.3}$$

4.2.1 Proof of (4.2)

In order to compare $E_\epsilon^{ev}(u_\epsilon^{ev})$ to $E^{ev}(u^{ev})$, we rewrite the energy E_ϵ^{ev} as $F_\epsilon^r(u,v)$, as derived in (2.5). To get something that might be close in energy to $E^{ev}(u^{ev})$, we choose a pair of functions u^* and v^* that are related to u^{ev}. An initial guess is choosing $u^* = u^{ev}$ and $v^* = u^{ev} \circ p_\sigma$, which has energy

$$\begin{aligned} F_\epsilon^r(u^*, v^*) = & \frac{1}{2} \int_{\Omega \setminus \bar{\omega}_\epsilon} |\nabla u^*|^2 \, dx - \int_\Omega f^{ev} u^* \, dx + \frac{a_\epsilon}{2} \int_{\omega_1^{ends}} |\nabla v^*|^2 \, dx \\ & + \frac{\epsilon a_\epsilon}{2} \int_{\omega_1^{int}} \frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma} \left(\frac{\partial v^*}{\partial \tau} \right)^2 \, dx + \frac{a_\epsilon}{2\epsilon} \int_{\omega_1^{int}} \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma} \left(\frac{\partial v^*}{\partial n} \right)^2 \, dx \\ & = \frac{1}{2} \int_{\Omega \setminus \bar{\omega}_\epsilon} |\nabla u^{ev}|^2 \, dx - \int_\Omega f^{ev} u^{ev} \, dx + \epsilon a_\epsilon \int_{\omega_1^{int}} \left(\frac{\partial u^{ev}}{\partial x} \right)^2 \, dx \\ & \leq E^{ev}(u^{ev}) \end{aligned}$$

where we have used the fact that $\kappa \equiv 0$ because σ is a straight line, v^* is constant in the ends regions because the projected point is constant, and $\frac{\partial v^*}{\partial n} = 0$ because v^* is constant in the normal direction. The inequality comes from the fact that to get to E^{ev}, the energy integral needs to be extended from $\Omega \setminus \overline{\omega_\epsilon}$ to $\Omega \setminus \sigma$, while the extension on the linear term does not matter because f vanishes on ω_δ for some $\delta > 0$. This statement tells us that

$$F_\epsilon^r(u^*, v^*) - E^{ev}(u^{ev}) \leq 0, \tag{4.4}$$

which is even better than the statement we want to prove. However, this pair (u^*, v^*) does not satisfy the necessary continuity conditions to be in the space V_ϵ^r. To correct this, we introduce the function z_ϵ as the unique solution to

$$\begin{cases} -\Delta z_\epsilon = 0 & \Omega \setminus \overline{\omega_\epsilon} \\ z_\epsilon = 0 & \partial\Omega \\ z_\epsilon = u^{ev} \circ p_\sigma - u^{ev} & \partial\omega_\epsilon \end{cases}$$

We then see that the pair $(u^* + z_\epsilon, v^*)$ is now an element of V_ϵ^r, and in Appendix A.1, we prove that

$$||z_\epsilon||_{H^1(\Omega \setminus \omega_\epsilon)} \leq C(f, \varphi) \epsilon^{\beta/2}$$

for any $\beta < 1$. This result relies heavily on the specific form of the most singular part of the solution u^{ev} and the higher regularity of the $u^{ev,*}$ term. With this result, we see that

$$
\begin{aligned}
F_\epsilon^r(u^* + z_\epsilon, v^*) &= F_\epsilon^r(u^*, v^*) + \frac{1}{2}\int_{\Omega\setminus\overline{\omega_\epsilon}} |\nabla z_\epsilon|^2 \ dx + \int_{\Omega\setminus\overline{\omega_\epsilon}} \nabla z_\epsilon \nabla u^{ev} \ dx \\
&- \int_{\Omega\setminus\overline{\omega_\epsilon}} f z_\epsilon \ dx \\
&\leq F_\epsilon^r(u^*, v^*) + C(f,\varphi)^2\epsilon^\beta + C(f,\varphi)\epsilon^{\beta/2}||u^{ev}||_{H^1(\Omega\setminus\overline{\omega_\epsilon})} \\
&+ C(f,\varphi)\epsilon^{\beta/2}||f||_{L^2(\Omega)} \\
&\leq F_\epsilon^r(u^*, v^*) + C(f,\varphi)^2\epsilon^{\beta/2}
\end{aligned}
$$

Then, we have that

$$
\begin{aligned}
E_\epsilon^{ev}(u_\epsilon^{ev}) - E^{ev}(u^{ev}) &\leq F_\epsilon^r(u^* + z_\epsilon, v^*) - E^{ev}(u^{ev}) \\
&\leq F_\epsilon^r(u^*, v^*) + C(f,\varphi)^2\epsilon^{\beta/2} - E^{ev}(u^{ev}) \\
&\leq C(f,\varphi)^2\epsilon^{\beta/2}
\end{aligned}
$$

as desired by (4.2), where the last line uses (4.4).

Remark. One could also try to correct for this mismatch in boundary conditions by adding a function z_ϵ on the inside of ω_ϵ, as opposed to the outside. In this case, we would expect z_ϵ to have $r^{1/2}$ singuarities near each endpoint of σ. Computing the energy of this type of function gives something of the form

$$
a_\epsilon \int_{r<\epsilon} \left|\nabla(r^{1/2}\sin(\theta/2))\right|^2 \ dA = O(\epsilon a_\epsilon)
$$

which does not go to zero as $\epsilon \to 0$. Therefore, this has the same order of contribution to the energy as the $\frac{\partial u}{\partial \tau}$ term, and so would not be "small" enough to give this asymptotic accuracy. Surprisingly, putting the z_ϵ correction outside of ω_ϵ does not have this issue.

4.2.2 Proof of (4.3)

In order to prove (4.3), we use the dual energy formulation of E_ϵ. The point of this is to turn the minimization that normally characterizes u_ϵ into a maximization problem, so that we can replace the input to this energy with a different test function to make the energy smaller (instead of larger, like in Section 4.2.1) and get an inequality in the reverse direction. The process for doing this is described in detail in Appendix B.

The dual energy corresponding to E_ϵ^{ev}, as derived in Appendix B.2.1 is

$$E_\epsilon^{ev,c}(\xi) = \int_{\partial\Omega} \varphi\xi \cdot n \; ds - \frac{1}{2}\int_\Omega \gamma_\epsilon^{-1}|\xi|^2 \; dx$$

where $\xi \in L^2(\Omega)^2$ with $-\nabla\cdot\xi = f^{ev}$. After separating out the ω_ϵ portion of the integral and rescaling it to ω_1, and using the fact that σ is a straight line, the dual energy, as derived in Appendix B.2.1, becomes

$$\begin{aligned} F_\epsilon^{ev,c}(\xi,\eta) = \int_{\partial\Omega} \varphi\xi\cdot n \; ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\xi|^2 \; dx - \frac{1}{2\epsilon a_\epsilon}\int_{\omega_1^{int}} |\eta_1|^2 \; ds \\ - \frac{\epsilon}{2a_\epsilon}\int_{\omega_1^{int}} \; dx - \frac{1}{2a_\epsilon}\int_{\omega_1^{ends}} |\eta|^2 \; dx \end{aligned}$$

where the maximization is taken over the set

$$\begin{aligned} W_\epsilon = \{(\xi,\eta) \in L^2(\Omega\setminus\omega_\epsilon)^2 \times L^2(\omega_1)^2 \mid -\nabla\cdot\xi = f^{ev}, \; \nabla\cdot\eta = 0, \\ \eta_2(x,\pm 1) = \xi_2(x,\pm\epsilon) \quad -1 \le x \le 1, \\ \eta\cdot n = \epsilon\xi\cdot n \circ H_\epsilon \quad \partial\omega_1 \cap \partial\omega_1^{ends}\}. \end{aligned} \tag{4.5}$$

In order to make the difference

$$E^{ev}(u^{ev}) - F_\epsilon^{ev,c}(\xi,\eta)$$

small, we choose a pair (ξ,η) that behaves like ∇u^{ev}, in that we choose $\xi = \nabla u^{ev}$ and, in order to get a divergence-free vector field η, we define

$$\eta(x,y) = \begin{pmatrix} \epsilon a_\epsilon \frac{\partial}{\partial x} u^{ev}(x,0) \\ y\frac{\partial}{\partial y} u^{ev}(x,0_+) \end{pmatrix}$$

on ω_1^{int}, and $\eta \equiv 0$ on ω_1^{ends}. However, based on the decomposition of u^{ev} as

$$u^{ev} = b_1 r^{1/2}\sin(\theta/2)\eta(r) + b_1' r'^{1/2}\sin(\theta'/2)\eta(r') + u^{ev,*},$$

we know that, since $\frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial\theta}$ along σ, we have that

$$\frac{\partial}{\partial y}u^{ev}(x,0_+) = \frac{b_1}{2}r^{-1/2}\eta(r) + \frac{b_1'}{2}r'^{-1/2}\eta(r') + \frac{\partial u^{ev,*}}{\partial y}(x,0_+)$$

for $-1 < x < 1$, and this function is not in $L^2(\sigma)$. Therefore, we define a regularized version of this function

$$\begin{aligned} DU(x) = \frac{b_1}{2}\max\{x+1,\epsilon\}^{-1/2}\eta(x+1) + \frac{b_1'}{2}\max\{1-x,\epsilon\}^{-1/2}\eta(1-x) \\ + \frac{\partial u^{ev,*}}{\partial y}(x,0_+) + d_\epsilon \end{aligned}$$

where the constant d_ϵ is given by

$$\begin{aligned} d_\epsilon &= \frac{b_1}{4}\int_{-1}^{-1+\epsilon}\left[(x+1)^{-1/2}-\epsilon^{-1/2}\right]\,dx + \frac{b_1}{4}\int_{1-\epsilon}^{1}\left[(1-x)^{-1/2}-\epsilon^{-1/2}\right]\,dx \\ &= \sqrt{\epsilon}\left(\frac{b_1}{2}+\frac{b_1'}{2}\right) \end{aligned}$$

so that

$$\int_{-1}^{1} DU(x)\,dx = \int_{-1}^{1}\frac{\partial u^{ev}}{\partial y}(x,0_+)\,dx.$$

With these definitions, we see that

$$|d_\epsilon| \le C(f,\varphi)\sqrt{\epsilon}$$

because the b constants are bounded by $C(f,\varphi)$ and

$$||DU||_{L^2(-1,1)} \le |\log\epsilon|^{1/2}C(f,\varphi).$$

Finally, we define the function

$$R(x) = \int_{-1}^{x}\frac{\partial u^{ev}}{\partial y}(t,0) - DU(t)\,dt$$

on σ, which is continuous, bounded by $\sqrt{\epsilon}C(f,\varphi)$ and has $R(-1) = R(1) = 0$ by construction. With these modified derivatives of u^{ev}, we can now define our test vector field η as

$$\eta(x,y) = \begin{pmatrix} \epsilon a_\epsilon \frac{\partial}{\partial x}u^{ev}(x,0) + R(x) \\ yDU(x) \end{pmatrix}$$

in ω_1^{int}, and $\eta \equiv 0$ in ω_1^{ends}, where this time, $\eta \in L^2(\omega_1)^2$. With this choice, we have that

$$-\nabla\cdot\xi = f^{ev}$$

and

$$\nabla\cdot\eta = 0$$

because

$$\begin{aligned} \frac{\partial}{\partial x}\eta_1 &= \frac{\partial}{\partial x}\left(\epsilon a_\epsilon\frac{\partial}{\partial x}u^{ev}(x,0)+R(x)\right) \\ &= \epsilon a_\epsilon\frac{\partial^2}{\partial x^2}u^{ev}(x,0)+\frac{\partial}{\partial x}R(x) \\ &= -\frac{\partial u^{ev}}{\partial y}(x,0)+\left(\frac{\partial u^{ev}}{\partial y}(x,0)-DU(x)\right) \\ &= -DU(x) = -\frac{\partial}{\partial y}(yDU(x)) = -\frac{\partial}{\partial y}\eta_2 \end{aligned}$$

in ω_1^{int}. The inner vector field η is also divergence-free in ω_1^{ends}, and to merge the two conditions together, we use the fact that along the joint boundary of these regions

$$\frac{\partial}{\partial x}u^{ev}(-1,0) = \frac{\partial}{\partial x}u^{ev,*}(-1,0) = 0 \qquad \frac{\partial}{\partial x}u^{ev}(1,0) = \frac{\partial}{\partial x}u^{ev,*}(1,0) = 0 \tag{4.6}$$

where the derivatives are taken along σ. While this combined vector field is close to ∇u^{ev}, it doesn't satisfy the interface conditions along $\partial\omega_\epsilon$. Therefore, we define a correction vector field ξ_ϵ that satisfies

$$\begin{cases} \nabla \cdot \xi_\epsilon = 0 & \Omega \setminus \omega_\epsilon \\ (\xi_\epsilon)_2(x, \pm\epsilon) = \pm DU(x) - \frac{\partial}{\partial y}u^{ev}(x, \pm\epsilon) & -1 < x < 1 \\ \xi_\epsilon \cdot n = -\frac{\partial}{\partial n}u^{ev} & \partial\omega_\epsilon \cap \{(x,y) : |x| > 1\} \end{cases}$$

with

$$||\xi_\epsilon||_{L^2(\Omega\setminus\omega_\epsilon)} \leq \epsilon^{\beta/4}C(f,\varphi)$$

for any $\beta < 1$. The construction of this ξ_ϵ and proof of this bound is postponed to Appendix A.2. With this, we have that the pair $(\xi + \xi_\epsilon, \eta)$ is in W_ϵ, because

$$\begin{aligned} (\xi + \xi_\epsilon)_2(x, \pm\epsilon) &= \frac{\partial}{\partial y}u^{ev}(x, \pm\epsilon) \pm DU(x) - \frac{\partial}{\partial y}u^{ev}(x, \pm\epsilon) \\ &= \pm DU(x) = \eta_2(x, \pm 1) \qquad -1 < x < 1 \end{aligned}$$

and

$$(\xi + \xi_\epsilon) \cdot n = \frac{\partial}{\partial n}u^{ev} - \frac{\partial}{\partial n}u^{ev} = 0 = \eta \cdot n \qquad \partial\omega_1 \cap |x| > 1$$

which are exactly the interface conditions in (4.5). In terms of the dual energy, we see that, using the fact that $\eta \equiv 0$ in ω_1^{ends},

$$
\begin{aligned}
F_\epsilon^{ev,c}(\xi+\xi_\epsilon,\eta) = & \int_{\partial\Omega} \frac{\partial u^{ev}}{\partial n}\varphi \; ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u^{ev} + \xi_\epsilon|^2 \; dx \\
& - \frac{1}{2a_\epsilon}\int_{\omega_1^{int}} \left|\epsilon a_\epsilon \frac{\partial u^{ev}}{\partial x}(x.0) + R(x)\right|^2 \; dx \; dy \\
& - \frac{\epsilon}{2a_\epsilon}\int_{\omega_1^{int}} |yDU(x)|^2 \; dx \; dy \\
= & \int_{\partial\Omega} \frac{\partial u^{ev}}{\partial n}\varphi \; ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u^{ev} + \xi_\epsilon|^2 \; dx \\
& - \frac{1}{a_\epsilon}\int_\sigma \left|\epsilon a_\epsilon \frac{\partial u^{ev}}{\partial x}(x,0) + R(x)\right|^2 \; dx - \frac{\epsilon}{3a_\epsilon}\int_\sigma |DU(x)|^2 \; dx \\
\geq & \int_{\partial\Omega} \frac{\partial u^{ev}}{\partial n}\varphi ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u^{ev}|^2 dx - \frac{1}{a_\epsilon}\int_\sigma \left|\epsilon a_\epsilon \frac{\partial u^{ev}}{\partial x}(x,0)\right|^2 \; dx \\
& - ||\nabla u^{ev}||_{L^2(\Omega\setminus\omega_\epsilon)}||\xi_\epsilon||_{L^2(\Omega\setminus\omega_\epsilon)} - \frac{1}{2}||\xi_\epsilon||^2_{L^2(\Omega\setminus\omega_\epsilon)} \\
& - \frac{2}{\epsilon a_\epsilon}\left|\left|\epsilon a_\epsilon \frac{\partial u^{ev}}{\partial x}\right|\right|_{L^2(\sigma)} ||R||_{L^2(\sigma)} - \frac{2}{\epsilon a_\epsilon}||R||^2_{L^2(\sigma)} - \frac{\epsilon}{3a_\epsilon}||DU||^2_{L^2(\sigma)}.
\end{aligned}
\tag{4.7}
$$

From energy estimates on the reduced problem defining u^{ev}, we have that

$$
||\nabla u^{ev}||_{L^2(\Omega\setminus\omega_\epsilon)} \leq C(f,\varphi) \qquad \left|\left|\epsilon a_\epsilon \frac{\partial u^{ev}}{\partial x}\right|\right|_{L^2(\sigma)} \leq C(f,\varphi)\sqrt{\epsilon a_\epsilon}
$$

and by the work in Section B.2.2, the corresponding dual energy for u^{ev} is given by

$$
E^{ev,c}(\xi,\zeta) := \int_{\partial\Omega} \varphi^{ev}\xi\cdot n \; ds - \frac{1}{2}\int_\Omega |\xi|^2 \; dx - \epsilon a_\epsilon \int_\sigma |\zeta|^2 \; ds.
$$

Thus, we see that, after expanding integrals, the first three terms in the last expression in (4.7) are smaller than $E^{ev,c}(\nabla u^{ev}, \frac{\partial u^{ev}}{\partial x}) = E^{ev}(u^{ev})$. Using this fact combined with our estimates on R and DU gives that

$$
\begin{aligned}
F_\epsilon^{ev,c}(\xi+\xi_\epsilon,\eta) \geq & \int_{\partial\Omega} \frac{\partial u^{ev}}{\partial n}\varphi \; ds - \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u^{ev}|^2 \; dx - \frac{1}{a_\epsilon}\int_\sigma \left|\epsilon a_\epsilon \frac{\partial u^{ev}}{\partial x}(x.0)\right|^2 \; dx \\
& - C(f,\varphi)^2\epsilon^{\beta/4} - C(f,\varphi)^2\epsilon^{\beta/2} - C(f,\varphi)^2\frac{\sqrt{\epsilon}}{\sqrt{\epsilon a_\epsilon}} \\
& - \frac{C(f,\varphi)^2\epsilon}{\epsilon a_\epsilon} - \frac{C(f,\varphi)^2\epsilon^2|\log\epsilon|^2}{\epsilon a_\epsilon} \\
& \geq E^{ev}(u^{ev}) - C(f,\varphi)^2\left(\epsilon^{\beta/4} + \epsilon^{\beta/2} + \frac{\sqrt{\epsilon}}{\sqrt{\epsilon a_\epsilon}} + \frac{\epsilon}{\epsilon a_\epsilon} + \frac{\epsilon^2|\log\epsilon|^2}{\epsilon a_\epsilon}\right) \\
& \geq E^{ev}(u^{ev}) - C(f,\varphi)^2\epsilon^{\beta/4}
\end{aligned}
$$

for any $\beta > 1$, where in the last step, we use the fact that $\epsilon a_\epsilon > m > 0$ to remove the denominators and get that $\epsilon^{\beta/4}$ is the slowest term going to zero. Therefore, we have that

$$E^{ev}(u^{ev}) - F_\epsilon^{ev,c}(\xi + \xi_\epsilon, \eta) \leq C(f,\varphi)\epsilon^{\beta/4},$$

so that by the dual energy construction, we have that

$$E^{ev}(u^{ev}) - E_\epsilon^{ev}(u_\epsilon^{ev}) \leq C(f,\varphi)\epsilon^{\beta/4},$$

as desired in (4.3).

4.3 Even symmetry, $\epsilon a_\epsilon \to 0$

Now, we need to address the other half of the case with even symmetry. If $\epsilon a_\epsilon \to 0$, then our decomposition in (3.10) no longer holds, and we need a different approach. However, in this case, the only part of the equation that depends on ϵ is also going to zero, and so we can see what happens if we ignore this term. This motivates us to define the function u_0^{ev} as the solution to

$$\begin{cases} -\Delta u_0^{ev} = f^{ev} & \Omega \\ u_0^{ev} = \varphi^{ev} & \partial\Omega \end{cases}$$

which can be obtained through minimization of the functional

$$E_0^{ev}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \; dx - \int_\Omega f^{ev} v \; dx$$

over the set of all $v \in H^1(\Omega)$ with $v = \varphi^{ev}$ on $\partial\Omega$. In this section, we prove the following theorem

Theorem 4.3.1. *Assume $\epsilon a_\epsilon \to 0$ as $\epsilon \to 0$. Consider the solution u_ϵ to the problem (2.1), where σ is a straight line segment and Ω is symmetric about σ, with $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$. Let f^{ev} and φ^{ev} be the even parts of f and φ respectively. Let u^{ev} be the solution to (3.2) and u_ϵ^{ev} be the even part of u_ϵ. Then*

$$|E_\epsilon^{ev}(u_\epsilon^{ev}) - E^{ev}(u^{ev})| \leq C(f,\varphi)^2(\epsilon a_\epsilon + \epsilon^\beta)$$

for any $\beta < 1$, where $C(f,\varphi)$ denotes a quantity of the form

$$C(f,\varphi) = C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}).$$

We prove this statement by proving that both u_ϵ^{ev} and u^{ev} are close to u_0^{ev} in the sense that

$$|E^{ev}(u^{ev}) - E_0^{ev}(u_0^{ev})| \leq C(f,\varphi)^2 \epsilon a_\epsilon \tag{4.8}$$

and

$$|E_\epsilon^{ev}(u_\epsilon^{ev}) - E_0^{ev}(u_0^{ev})| \leq C(f,\varphi)^2 (\epsilon^\beta + \epsilon a_\epsilon) \tag{4.9}$$

for any $\beta < 1$, and the result above then follows by the triangle inequality.

4.3.1 Proof of (4.8)

Since u^{ev} is even across σ, it is in $H^1(\Omega)$, and not just $H^1(\Omega \setminus \sigma)$, and so is a valid test function for the energy functional E_0^{ev}. Using this fact, the minimization properties of these solutions, and the positivity of ϵa_ϵ, we see that

$$\begin{aligned}
E_0^{ev}(u_0^{ev}) \leq E_0^{ev}(u^{ev}) &= \frac{1}{2}\int_\Omega |\nabla u^{ev}|^2 \, dx \\
&\leq \frac{1}{2}\int_\Omega |\nabla u^{ev}|^2 \, dx + \epsilon a_\epsilon \int_\sigma \left|\frac{\partial u^{ev}}{\partial x}\right|^2 \, ds = E^{ev}(u^{ev}) \\
&\leq E^{ev}(u_0^{ev}) = \frac{1}{2}\int_\Omega |\nabla u_0^{ev}|^2 \, dx + \epsilon a_\epsilon \int_\sigma \left|\frac{\partial u_0^{ev}}{\partial x}\right|^2 \, ds \\
&\leq \frac{1}{2}\int_\Omega |\nabla u_0^{ev}|^2 \, dx + \epsilon a_\epsilon C(f,\varphi)^2 = E_0^{ev}(u_0^{ev}) + \epsilon a_\epsilon C(f,\varphi)^2
\end{aligned}$$

where, in moving to the last line, we used the fact that by elliptic regularity and the fact that f vanishes in a neighborhood of σ,

$$\int_\sigma \left|\frac{\partial u_0^{ev}}{\partial x}\right|^2 \, ds \leq C(f,\varphi)^2.$$

This sequence of inequalities gives us that

$$E_0^{ev}(u_0^{ev}) \leq E^{ev}(u^{ev}) \leq E_0^{ev}(u_0^{ev}) + \epsilon a_\epsilon C(f,\varphi)^2$$

from which (4.8) follows.

4.3.2 Proof of (4.9)

To establish (4.9), we use a pair of primal and dual energies in order to prove

$$E_\epsilon^{ev}(u_\epsilon^{ev}) - E_0^{ev}(u_0^{ev}) \leq \epsilon a_\epsilon C(f,\varphi)^2 \tag{4.10}$$

and

$$E_0^{ev}(u_0^{ev}) - E_\epsilon(u_\epsilon) \le \epsilon^\beta C(f,\varphi)^2 \tag{4.11}$$

from which (4.9) follows.

The same process from the previous section works to prove (4.10); we calculate that

$$\begin{aligned}
E_\epsilon^{ev}(u_\epsilon^{ev}) \le E_\epsilon(u_0^{ev}) &= \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u_0^{ev}|^2\ dx + \frac{1}{2}\int_{\omega_\epsilon} a_\epsilon |\nabla u_0^{ev}|^2\ dx \\
&\le \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u_0^{ev}|^2\ dx + \epsilon a_\epsilon C(f,\varphi)^2 \\
&\le \frac{1}{2}\int_{\Omega} |\nabla u_0^{ev}|^2\ dx + \epsilon a_\epsilon C(f,\varphi)^2 \\
&= E_0^{ev}(u_0^{ev}) + \epsilon a_\epsilon C(f,\varphi)^2
\end{aligned}$$

from which we see (4.10). For (4.11), we need to use the dual energies for both of these problems. As shown in Appendix B, the dual energy for E_ϵ is given by

$$E_\epsilon^{ev,c}(\xi) = \int_{\partial\Omega} \varphi^{ev}\xi\cdot n\ ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\xi|^2\ dx - \frac{1}{2a_\epsilon}\int_{\omega_\epsilon} |\xi|^2\ dx$$

over the set of vector fields

$$\begin{aligned}
W_\epsilon^{ev} = \{(\xi,\eta) \in L^2(\Omega\setminus\omega_\epsilon)^2 \times L^2(\omega_1)^2 \ |\ &-\nabla\cdot\xi = f^{ev},\ -\nabla\cdot\eta = 0, \\
&\eta_2(x,y) = \xi_2(x,\epsilon y)\ \ -1 \le x \le 1,\ y = \pm 1, \\
&\eta\cdot n = \epsilon(\xi\cdot n)\circ H_\epsilon \text{ on } \partial\omega_1 \cap \partial\omega_1^{ends}\}
\end{aligned}$$

and the dual energy for E_0^{ev} is

$$E_0^{ev,c}(\xi) = \int_{\partial\Omega} \varphi^{ev}\xi\cdot n\ ds - \frac{1}{2}\int_{\Omega} |\xi|^2\ dx$$

over the set

$$W_0^{ev} = \{\xi \in L^2(\Omega)^2 \ |\ -\nabla\cdot\xi = f^{ev} \text{ in } \Omega\}.$$

As before, we define a test vector field in W_ϵ^{ev} that looks like ∇u_0^{ev}. Our initial guess would be to use the vector field that is ∇u_0^{ev} in $\Omega\setminus\omega_\epsilon$ and 0 inside ω_ϵ, but this does not lie in W_ϵ^{ev}. To fix, this, we define a corrector vector field ξ_ϵ satisfying

$$\begin{cases}
\nabla\cdot\xi_\epsilon = 0 & \Omega\setminus\omega_\epsilon \\
\xi_\epsilon\cdot n = 0 & \partial\Omega \\
\xi_\epsilon\cdot n = -\nabla u_0^{ev}\cdot n & \partial\omega_\epsilon
\end{cases}$$

with norm control of

$$||\xi_\epsilon||_{L^2(\Omega\setminus\omega_\epsilon)} \leq \epsilon^\beta C(f,\varphi)$$

for any $\beta < 1$. The construction of this vector field is postponed to Appendix A.3. Then, by the definition of the space W_ϵ^{ev}, we see that the vector field

$$\tilde{\xi} = \begin{cases} \nabla u_0^{ev} + \xi_\epsilon & \Omega \setminus \omega_\epsilon \\ 0 & \omega_\epsilon \end{cases}$$

is in W_ϵ^{ev}, and so we can use it as a test function to calculate

$$\begin{aligned} E_\epsilon^{ev}(u_\epsilon^{ev}) \geq E_\epsilon^c(\tilde{\xi}) &= \int_{\partial\Omega} \varphi^{ev}\tilde{\xi}\cdot n\ ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\tilde{\xi}|^2\ dx - \frac{1}{2a_\epsilon}\int_{\omega_\epsilon}|\tilde{\xi}|^2\ dx \\ &= \int_{\partial\Omega} \varphi^{ev}\tilde{\xi}\cdot n\ ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u_0^{ev}|^2\ dx \\ &- \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\xi_\epsilon|^2\ dx - \int_{\Omega\setminus\omega_\epsilon} \nabla u_0^{ev}\cdot\xi_\epsilon\ dx \\ &\geq \int_{\partial\Omega} \varphi^{ev}\tilde{\xi}\cdot n\ ds - \frac{1}{2}\int_{\Omega} |\nabla u_0^{ev}|^2\ dx \\ &- \frac{1}{2}||\xi_\epsilon||^2_{L^2(\Omega\setminus\omega_\epsilon)} - ||\nabla u_0^{ev}||_{L^2(\Omega\setminus\omega_\epsilon)}||\xi_\epsilon||_{L^2(\Omega\setminus\omega_\epsilon)} \\ &\geq E_0^{ev}(u_0^{ev}) - C(f,\varphi)^2\epsilon^\beta \end{aligned}$$

where in the last step, we use the dual energy formula for E_0^{ev} and the norm control on ξ_ϵ. Therefore, we see

$$E_0^{ev}(u_0^{ev}) - E_\epsilon^{ev}(u_\epsilon^{ev}) \leq C(f,\varphi)^2\epsilon^\beta,$$

establishing (4.11). With this, we now have (4.10) and (4.11), which combine to give (4.9). Putting this together with (4.8) gives the full proof of Theorem 4.3.1.

4.4 Odd symmetry, $a_\epsilon < M\epsilon$

Having dealt with the even symmetry case for all ranges of a_ϵ, we now move to odd symmetry. In Chapter 3, we discussed the problem with odd symmetry, deriving that the solution u^{odd} solves (3.5)

$$\begin{cases} -\Delta u^{odd} = f^{odd} & \Omega^+ \\ u^{odd} = \varphi^{odd} & \partial\Omega^+ \\ \frac{\partial u^{odd}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{odd}}{\partial x^2} - \frac{a_\epsilon}{\epsilon} u^{odd} = 0 & \sigma \\ u^{odd} = 0 & \{y=0\} \setminus \sigma. \end{cases}$$

We will also consider a more simplified version of this problem, given by

$$\begin{cases} -\Delta u^{odd'} = f^{odd} & \Omega^+ \\ u^{odd'} = \varphi^{odd} & \partial\Omega^+ \\ \frac{\partial u^{odd'}}{\partial y} = \frac{a_\epsilon}{\epsilon} u^{odd'} & \sigma \\ u^{odd'} = 0 & \{y=0\} \setminus \sigma, \end{cases} \tag{4.12}$$

and for each of these functions, we will consider their odd extensions to all of Ω. For these differential equations, we have the corresponding energies

$$E^{odd}(v) = \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla v|^2\, dx + \epsilon a_\epsilon \int_\sigma \left|\frac{\partial v^+}{\partial \tau}\right|^2 ds + \frac{a_\epsilon}{\epsilon}\int_\sigma |v^+ - v^-|^2\, ds - \int_\Omega f^{odd} v\, dx \tag{4.13}$$

and

$$E^{odd'}(v) = \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla v|^2\, dx + \frac{a_\epsilon}{\epsilon}\int_\sigma |v^+ - v^-|^2\, ds - \int_\Omega f^{odd} v\, dx \tag{4.14}$$

where the minimzations are taken with boundary data φ^{odd} on $\partial\Omega$. If we have that $a_\epsilon < M\epsilon$ for some constant M, then the term $\frac{a_\epsilon}{\epsilon}$ will be bounded, and since the tangential derivative term has a coefficient that is now $O(\epsilon^2)$, it should not have a substantial impact on the solution. We see that this is true with the following lemma.

Lemma 4.4.1. *Under the assumption that $a_\epsilon < M\epsilon$ for some M,*

$$|E^{odd}(u^{odd}) - E^{odd'}(u^{odd'})| \le \epsilon C(f,\varphi)^2$$

Proof. Out of necessity later, the energies (4.13) and (4.14) are defined in terms of integrals on all of Ω, but it is much easier to work with regions that do not surround the curve σ. Thus, we note that

$$\begin{aligned} E^{odd}(u^{odd}) &= 2\Bigg(\frac{1}{2}\int_{\Omega^+} |\nabla u^{odd}|^2\, dx + \frac{\epsilon a_\epsilon}{2}\int_\sigma \left|\frac{\partial u^{odd,+}}{\partial \tau}\right|^2\, ds \\ &\qquad + \frac{a_\epsilon}{2\epsilon}\int_\sigma |u^{odd,+}|^2\, ds - \int_{\Omega^+} f^{odd} u^{odd}\, dx\Bigg) \\ E^{odd'}(u^{odd'}) &= 2\left(\frac{1}{2}\int_{\Omega^+} |\nabla u^{odd'}|^2\, dx + \frac{a_\epsilon}{2\epsilon}\int_\sigma |u^{odd',+}|^2\, ds - \int_{\Omega^+} f^{odd} u^{odd'}\, dx\right) \end{aligned} \tag{4.15}$$

and work with the expressions on Ω^+. For one direction of the absolute value, we see that, using minimization properties of both solutions and the positivity of ϵa_ϵ,

$$\begin{aligned} \frac{1}{2}E^{odd'}(u^{odd'}) &\le \frac{1}{2}E^{odd'}(u^{odd}) \\ &= \frac{1}{2}\int_{\Omega^+} |\nabla u^{odd}|^2\, dx + \frac{a_\epsilon}{2\epsilon}\int_\sigma |u^{odd,+}|^2\, ds - \int_{\Omega^+} f^{odd} u^{odd}\, dx \\ &\le \frac{1}{2}\int_{\Omega^+} |\nabla u^{odd}|^2\, dx + \frac{\epsilon a_\epsilon}{2}\int_\sigma \left|\frac{\partial u^{odd,+}}{\partial \tau}\right|^2\, ds \\ &+ \frac{a_\epsilon}{2\epsilon}\int_\sigma |u^{odd,+}|^2\, ds - \int_{\Omega^+} f^{odd} u^{odd}\, dx \\ &= \frac{1}{2}E^{odd}(u^{odd}) \end{aligned}$$

so that $E^{odd'}(u^{odd'}) \le E^{odd}(u^{odd})$.

For the other direction of the absolute value, we need to use the decomposition of $u^{odd'}$ developed in (3.14), which says that

$$u^{odd'} = b_1 r^{1/2}\cos(\theta/2)\eta(r) + b_1' r'^{1/2}\cos(\theta'/2)\eta(r') + u^{odd,*}$$

where $u^{odd,*}$ is bounded in $C^{1,\beta/2}((\omega_\delta)^+)$ by $C(f,\varphi)\left(\left(\frac{a_\epsilon}{\epsilon}\right)^3 + 1\right)$, and the constants b_1 and b_1' are bounded by $C(f,\varphi)\left(\left(\frac{a_\epsilon}{\epsilon}\right)^3 + 1\right)$ as well. By construction, we also have that $u^{odd,*}$ is in $H^1(\Omega)$, in particular, outside of ω_δ, controlled by $C(f,\varphi)$. Based on our assumption that $a_\epsilon < M\epsilon$, we know that all of the bounds above can be simplified to $C(f,\varphi)$. We would like to use $u^{odd'}$ as a test function to plug into the energy E^{odd} like we did in the reverse direction, but this representation of $u^{odd'}$ shows that it is not in $H^1(\sigma)$. Therefore, we define a smoothed version of this function by

$$v_\epsilon^{odd} = b_1 \max\{\epsilon, r\}^{-\rho} r^{1/2+\rho}\cos\left(\frac{\theta}{2}\right)\eta(r) + b_1' \max\{\epsilon, r'\}^{-\rho} r'^{1/2+\rho}\cos\left(\frac{\theta'}{2}\right)\eta(r') + u^{odd,*}$$

for any $\rho > 0$. We see that this function is both in $H^1(\sigma)$ and $H^1(\Omega^+)$, and so can be used as a test function in E^{odd}.

Remark. The regularized version of the solution is different here than it was in Section 4.2.2 because we have different requirements. In the previous case, we only needed $\frac{\partial u^{ev}}{\partial y}$ to be in $L^2(\sigma)$, where as here we need both that the function be in $H^1(\sigma)$ and $H^1(\Omega^+)$. Since $\cos(\theta/2)$ is not in H^1 near 0, the function needs to be modified to have extra decay near the endpoints of σ.

Using this definition of v_ϵ^{odd} and the fact that it matches $u^{odd'}$ identically outside of a ball of radius ϵ around each endpoint, we compute using the explicit formulas and the triangle inequality that

$$\begin{aligned}
\int_{\Omega^+} |\nabla v_\epsilon^{odd}|^2 \ dx &\leq \int_{\Omega^+} |\nabla u^{odd'}|^2 \ dx + \epsilon C(f,\varphi)^2, \\
\int_\sigma \left| \frac{\partial v_\epsilon^{odd}}{\partial x} \right|^2 \ ds &\leq C(f,\varphi)^2 \left[\epsilon^\rho + |\log \epsilon| + 1 + \epsilon^\rho + |\log\epsilon| + 1 + 1\right] \\
&\leq |\log\epsilon| C(f,\varphi)^2, \\
\int_\sigma |v_\epsilon^{odd}|^2 \ ds &\leq \int_\sigma |u^{odd'}|^2 \ ds + |b_1|^2 \epsilon^{-2\rho} \int_0^\epsilon t^{1+2\rho} \ dt + |b_2|^2 \epsilon^{-2\rho} \int_0^\epsilon t^{1+2\rho} \ dt \\
&\leq \int_\sigma |u^{odd'}|^2 \ ds + \epsilon^2 C(f,\varphi)^2.
\end{aligned} \tag{4.16}$$

Since we can use this v_ϵ^{odd} as a test function for E^{odd}, we see that, using all of the inequalities proved in (4.16) and the fact that $\frac{a_\epsilon}{\epsilon} < M$,

$$\begin{aligned}
\frac{1}{2} E^{odd}(u^{odd}) &\leq \frac{1}{2} E^{odd}(v_\epsilon^{odd}) \\
&= \frac{1}{2} \int_{\Omega^+} |\nabla v_\epsilon^{odd}|^2 \ dx + \frac{\epsilon a_\epsilon}{2} \int_\sigma \left| \frac{\partial v_\epsilon^{odd,+}}{\partial \tau} \right|^2 \ ds \\
&+ \frac{a_\epsilon}{2\epsilon} \int_\sigma |v_\epsilon^{odd,+}|^2 \ ds - \int_{\Omega^+} f^{odd} u^{odd} \ dx \\
&\leq \int_{\Omega^+} |\nabla u^{odd'}|^2 \ dx + \epsilon C(f,\varphi)^2 + \epsilon a_\epsilon |\log\epsilon| C(f,\varphi)^2 \\
&+ \frac{a_\epsilon}{2\epsilon} \left(\int_\sigma |u^{odd'}|^2 \ ds + \epsilon^2 C(f,\varphi)^2 \right) \\
&\leq \int_{\Omega^+} |\nabla u^{odd'}|^2 \ dx + \frac{a_\epsilon}{\epsilon} \int_\sigma |u^{odd'}|^2 \ ds + \epsilon C(f,\varphi)^2 \\
&+ \frac{M}{2} \epsilon^2 |\log\epsilon| C(f,\varphi)^2 + \frac{M}{2} \epsilon^2 C(f,\varphi)^2 \\
&\leq \frac{1}{2} E^{odd'}(u^{odd'}) + \epsilon C(f,\varphi)^2
\end{aligned}$$

which gives that

$$E^{odd}(u^{odd}) - E^{odd'}(u^{odd'}) \leq \epsilon C(f,\varphi)^2$$

giving the full result of the lemma when combined with the earlier result that

$$E^{odd'}(u^{odd'}) \leq E^{odd}(u^{odd}). \qquad \blacksquare$$

We now want to relate this solution $u^{odd'}$ to a solution with even symmetry by duality. To that end, we note that $\nabla u^{odd'}$ is a vector field in $L^2(\omega_\delta^+)^2$ that has divergence 0 (we are restricting to a set on which f is identically zero), so there exists a function $v^{ev,N} \in H^1(\omega_\delta^+)$ so that

$$\nabla u^{odd'} = \left(-\frac{\partial v^{ev,N}}{\partial y}, \frac{\partial v^{ev,N}}{\partial x}\right)$$

with $\int_{\partial\omega_\delta^+} v^{ev,N}\, ds = 0$. By duality, this function $v^{ev,N}$ will be even across $y = 0$, and we want to determine what differential equation it solves. By construction of $v^{ev,N}$ and the boundary condition on $u^{odd'}$, we see that for any $z \in H^1(\sigma)$

$$\begin{aligned} \int_\sigma \frac{\partial v^{ev,N}}{\partial x}\frac{\partial z}{\partial x}\, dx &= \int_\sigma \frac{\partial u^{odd'}}{\partial y}\frac{\partial z}{\partial x}\, dx \\ &= \frac{a_\epsilon}{\epsilon}\int_\sigma u^{odd'}\frac{\partial z}{\partial x}\, dx \\ &= -\frac{a_\epsilon}{\epsilon}\int_\sigma \frac{\partial u^{odd'}}{\partial x} z\, dx \\ &= \frac{a_\epsilon}{\epsilon}\int_\sigma \frac{\partial v^{ev,N}}{\partial y} z\, dx \end{aligned} \tag{4.17}$$

where we have also used that $u^{odd'}$ vanishes at the endpoints of σ, and that the second-to-last integral makes sense because $u^{odd'}$ is in $L^1(\sigma)$ from the representation formula and z is continuous. Since $u^{odd'}$ vanishes outside of σ, we know that $\frac{\partial v^{ev,N}}{\partial y} = -\frac{\partial u^{odd'}}{\partial x} = 0$ there, and (4.17) gives the variational formulation of a boundary condition on σ. Finally, defining $\tilde{\varphi}^{odd} = u^{odd'}\,|_{\partial\omega_\delta^+}$, we get that $v^{ev,N}$ is the solution to

$$\begin{cases} -\Delta v^{ev,N} = 0 & \omega_\delta^+ \\ \frac{\partial v^{ev,N}}{\partial y} + \frac{\epsilon}{a_\epsilon}\frac{\partial^2 v^{ev,N}}{\partial x^2} = 0 & \sigma \\ \frac{\partial v^{ev,N}}{\partial y} = 0 & \{y = 0 \cap \omega_\delta\} \\ \frac{\partial v^{ev,N}}{\partial n} = \frac{\partial \tilde{\varphi}^{odd}}{\partial \tau} & \partial(\omega_\delta)^+. \end{cases} \tag{4.18}$$

If we denote by ψ^{ev} the function $\frac{\partial \tilde{\varphi}^{odd}}{\partial \tau}$, which can be extended as an even function to all of $\partial\omega_\delta$, then the even extension of $v^{ev,N}$ to all of ω_δ (which we will still denote by

$v^{ev,N}$ solves

$$\begin{cases} -\Delta v^{ev,N} = 0 & \omega_\delta \setminus \sigma \\ \frac{\partial v^{ev,N,+}}{\partial y} + \frac{\epsilon}{a_\epsilon}\frac{\partial^2 v^{ev,N,+}}{\partial x^2} = 0 & \sigma \\ \frac{\partial v^{ev,N,-}}{\partial y} + \frac{\epsilon}{a_\epsilon}\frac{\partial^2 v^{ev,N,-}}{\partial x^2} = 0 & \sigma \\ \frac{\partial v^{ev,N}}{\partial n} = \psi^{ev} & \partial\omega_\delta \end{cases} \tag{4.19}$$

and is the minimizer of the energy

$$E^{ev,N}(v) = \frac{1}{2}\int_{\omega_\delta\setminus\sigma} |\nabla v|^2 \; dx + \frac{\epsilon}{a_\epsilon}\int_\sigma \left(\frac{\partial v}{\partial x}\right)^2 \; dx - \int_{\partial\omega_\delta} \psi^{ev} v \; ds. \tag{4.20}$$

(4.18) and (4.20) look very similar to (3.4) and (4.1), which correspond to a problem that we have already analyzed. The differences between these equations are a replacement of a_ϵ by $\frac{1}{a_\epsilon}$, removing the source term, and changing the boundary condition from a Dirichlet to a Neumann condition. Therefore, we should be able to leverage the even problem to help with getting a result here. For that, we need to relate the energy of the original problem $u^{odd'}$ to that of the rotated problem $v^{ev,N}$. Looking at each term in (4.20), we compute that

$$\begin{aligned} \int_{\partial\omega_\delta^+} \psi^{ev} v^{ev,N} \; ds &= \int_{\partial\omega_\delta^+} \frac{\partial\tilde{\varphi}^{odd}}{\partial\tau} v^{ev,N} \; ds \\ &= -\int_{\partial\omega_\delta^+} u^{odd'} \frac{\partial v^{ev,N}}{\partial\tau} \; ds \\ &= -\int_{\partial\omega_\delta^+} u^{odd'} \frac{\partial u^{odd'}}{\partial n} \; ds \\ &= \int_{\omega_\delta^+} |\nabla u^{odd'}|^2 \; dx + \frac{a_\epsilon}{\epsilon}\int_\sigma |u^{odd'}|^2 \; ds \end{aligned} \tag{4.21}$$

where the last equality comes from the variational formulation of the differential equation for $u^{odd'}$. We also have that

$$\begin{aligned} \frac{\epsilon}{2a_\epsilon}\int_\sigma \left(\frac{\partial v^{ev,N}}{\partial x}\right)^2 \; ds &= \frac{\epsilon}{2a_\epsilon}\int_\sigma \left(\frac{\partial u^{odd'}}{\partial y}\right)^2 \; ds \\ &= \frac{\epsilon}{2a_\epsilon}\int_\sigma \left(\frac{a_\epsilon}{\epsilon}u^{odd'}\right)^2 \; ds = \frac{a_\epsilon}{2\epsilon}\int_\sigma |u^{odd'}|^2 \; ds, \\ \int_{\omega_\delta^+} |\nabla v^{ev,N}|^2 \; dx &= \int_{\omega_\delta^+} |\nabla u^{odd'}|^2 \; dx. \end{aligned} \tag{4.22}$$

Using (4.21) and (4.22) in (4.20), we see that

$$
\begin{aligned}
\frac{1}{2}E^{ev,N}(v^{ev,N}) &= \frac{1}{2}\int_{\omega_\delta^+} |\nabla v^{ev,N}|^2 \; dx + \frac{\epsilon}{2a_\epsilon}\int_\sigma \left(\frac{\partial v^{ev,N}}{\partial x}\right)^2 \; dx \\
&\quad - \int_{\partial\omega_\delta} \psi^{ev} v^{ev,N} \; ds \\
&= \frac{1}{2}\int_{\omega_\delta^+} |\nabla u^{odd'}|^2 \; dx + \frac{\epsilon}{2a_\epsilon}\int_\sigma \left(\frac{\partial u^{odd'}}{\partial y}\right)^2 \; ds \\
&\quad - \int_{\omega_\delta^+} |\nabla u^{odd'}|^2 \; dx - \frac{a_\epsilon}{\epsilon}\int_\sigma |u^{odd'}|^2 \; ds \\
&= -\frac{1}{2}\int_{\omega_\delta^+} |\nabla u^{odd'}|^2 \; dx - \frac{a_\epsilon}{2\epsilon}\int_\sigma |u^{odd'}|^2 \; ds = -\frac{1}{2}E^{odd'}(u^{odd'}).
\end{aligned}
\tag{4.23}
$$

Based on the changes mentioned above, the appropriate full problem that $v^{ev,N}$ should be close to is the solution v_ϵ^N to

$$
\begin{cases}
-\nabla \cdot (\tilde{\gamma}_\epsilon \nabla v_\epsilon^N) = 0 & \omega_\delta \\
\frac{\partial v_\epsilon^N}{\partial n} = \psi^{ev} & \partial\omega_\delta
\end{cases}
$$

with the constraint $\int_{\partial\omega_\delta} v_\epsilon^N \; ds = 0$, where the coefficient $\tilde{\gamma}_\epsilon$ is defined by

$$
\tilde{\gamma}_\epsilon = \begin{cases}
1 & \omega_\delta \setminus \omega_\epsilon \\
\frac{1}{a_\epsilon} & \omega_\epsilon.
\end{cases}
$$

This solution v_ϵ^N is the minimizer of the energy

$$
E_\epsilon^N(v) = \frac{1}{2}\int_{\omega_\delta} \tilde{\gamma}_\epsilon |\nabla v|^2 \; dx - \int_{\partial\omega_\delta} \psi^{ev} v \; dx.
$$

In terms of the solution to a problem with odd symmetry, v_ϵ^N is related to u_ϵ, where the Dirichlet boundary condition of ψ^{ev} is enforced on $\partial\omega_\delta$ by the relation

$$
\gamma_\epsilon \nabla u_\epsilon = \left(-\frac{\partial v_\epsilon^N}{\partial y}, \frac{\partial v_\epsilon^N}{\partial x}\right).
$$

Thus, we see that by integration by parts again

$$
\begin{aligned}
E_\epsilon^N(v_\epsilon^N) &= \frac{1}{2}\int_{\omega_\delta} \tilde{\gamma}_\epsilon |\nabla v_\epsilon^N|^2 \; dx - \int_{\partial\omega_\delta} \frac{\partial v_\epsilon^N}{\partial n} v_\epsilon^N \; dx \\
&= -\frac{1}{2}\int_{\omega_\delta} \tilde{\gamma}_\epsilon |\nabla v_\epsilon^N|^2 \; dx = -\frac{1}{2}\int_{\omega_\delta} \gamma_\epsilon |\nabla u_\epsilon|^2 \; dx = -E_\epsilon^{odd}(u_\epsilon^{odd})
\end{aligned}
\tag{4.24}
$$

By combining (4.24) with (4.23), we see that

$$|E^{ev,N}(v^{ev,N}) - E^N_\epsilon(v^N_\epsilon)| = |E^{odd'}(u^{odd'}) - E^{odd}_\epsilon(u^{odd}_\epsilon)|$$

so that any result we have about the energy difference between even problems can be used on this odd problem. Since we are assuming $a_\epsilon < M\epsilon$ and the conductivity in the inhomogeneity in the even problem that we construct this way is $\frac{1}{a_\epsilon}$, it falls into the case of Section 4.2. All of the results in that section were proved with Dirichlet boundary conditions, but since we are looking away from the boundary of the domain, they could have also been proved with Neumann conditions. Therefore, Theorem 4.2.1 says that

$$|E^{odd'}(u^{odd'}) - E^{odd}_\epsilon(u^{odd}_\epsilon)| = |E^{ev,N}(v^{ev,N}) - E^N_\epsilon(v^N_\epsilon)| \le C(f,\varphi)^2\epsilon^{\beta/4}$$

for any $\beta < 1$, so long as we restrict all of the energies to only involve integrals over ω_δ. We would like to have the energies be defined on all of Ω, which can be done. The proof is not too technical, but since it will be used again later on (when we want to remove symmetry assumptions), it is included in Appendix A.4. At the end of that, combining this estimate with Lemma 4.4.1, we get the following theorem.

Theorem 4.4.1. *Suppose $a_\epsilon < M\epsilon$ for some $M > 0$. Consider the solution u_ϵ to the problem (2.1), where σ is a straight line segment and Ω is symmetric about σ, with $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$. Let f^{odd} and φ^{odd} be the odd parts of f and φ respectively. Let u^{odd} be the solution to (3.5), and let u^{odd}_ϵ be the odd part of u_ϵ. Then*

$$|E^{odd}_\epsilon(u^{odd}_\epsilon) - E^{odd}(u^{odd})| \le C(f,\varphi)^2\epsilon^{\beta/4}$$

where $C(f,\varphi)$ denotes a quantity of the form

$$C(f,\varphi) = C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}).$$

4.5 Odd symmetry, $\frac{a_\epsilon}{\epsilon} \to \infty$

To finish up all four cases, we prove the following theorem.

Theorem 4.5.1. *Suppose* $\frac{a_\epsilon}{\epsilon} \to \infty$ *as* $\epsilon \to 0$ *for some* $M > 0$. *Consider the solution* u_ϵ *to the problem* (2.1)*, where* σ *is a straight line segment and* Ω *is symmetric about* σ*, with* $f \in \mathcal{F}_\delta$ *and* $\varphi \in H^{1/2}(\partial\Omega)$. *Let* f^{odd} *and* φ^{odd} *be the odd parts of* f *and* φ *respectively. Let* u^{odd} *be the solution to* (3.5)*, and* u_ϵ^{odd} *be the odd part of* u_ϵ. *Then*

$$|E_\epsilon^{odd}(u_\epsilon^{odd}) - E^{odd}(u^{odd})| \leq C(f,\varphi)^2 \left(\frac{\epsilon}{a_\epsilon} + \epsilon^\beta \right)$$

where $C(f,\varphi)$ *denotes a quantity of the form*

$$C(f,\varphi) = C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}).$$

The proof of this result is similar to the result in Section 4.3, in the sense that we will show that both u_ϵ^{odd} and u^{odd} are both energy close to the function u_0^{odd}, which is the solution to

$$\begin{cases} -\Delta u_0^{odd} = f^{odd} & \Omega \\ u_0^{odd} = \varphi^{odd} & \partial\Omega \end{cases}$$

and is the minimizer of the energy

$$E_0^{odd}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega fu \, dx$$

over the set of all functions in H^1 with boundary data φ^{odd}. We will show

$$|E^{odd}(u^{odd}) - E_0^{odd}(u_0^{odd})| \leq C(f,\varphi)^2 \frac{\epsilon}{a_\epsilon} \tag{4.25}$$

and

$$|E_\epsilon^{odd}(u_\epsilon^{odd}) - E_0^{odd}(u_0^{odd})| \leq C(f,\varphi)^2 \left(\frac{\epsilon}{a_\epsilon} + \epsilon^\beta \right) \tag{4.26}$$

from which Theorem 4.5.1 follows.

4.5.1 Proof of (4.25)

In order to prove (4.25), we will prove two inequalities using primal and dual formulations of the energy. Since the solution u^{odd} may jump across σ, it is more efficient to look at half of the energy, which can be related to an energy expression on Ω^+. We define this half-energy as

$$E^{odd,+}(v) = \frac{1}{2}\int_{\Omega^+} |\nabla v|^2 \, dx + \frac{\epsilon a_\epsilon}{2}\int_\sigma \left|\frac{\partial v}{\partial x}\right|^2 \, dx + \frac{a_\epsilon}{2\epsilon}\int_\sigma |v|^2 \, dx - \int_{\Omega^+} f^{odd} v \, dx$$

over the set of all functions in $H^1(\Omega^+)$ with boundary data φ^{odd} and that are zero along $(\{y=0\}\cap\Omega)\setminus\sigma$. Using this energy, we can see that

$$\begin{aligned}\frac{1}{2}E^{odd}(u^{odd}) = E^{odd,+}(u^{odd}) &\leq E^{odd,+}(u_0^{odd})\\ &= \frac{1}{2}\int_{\Omega^+}|\nabla u_0^{odd}|^2\ dx + \frac{\epsilon a_\epsilon}{2}\int_\sigma \left|\frac{\partial u_0^{odd}}{\partial x}\right|^2\ dx\\ &+\frac{a_\epsilon}{2\epsilon}\int_\sigma |u_0^{odd}|^2\ dx - \int_{\Omega^+} f^{odd}u_0^{odd}\ dx\\ &= \frac{1}{2}\int_{\Omega^+}|\nabla u_0^{odd}|^2\ dx - \int_{\Omega^+} f^{odd}u_0^{odd}\ dx = \frac{1}{2}E_0^{odd}(u_0^{odd})\end{aligned}$$

which tells us that

$$E^{odd}(u^{odd}) - E_0^{odd}(u_0^{odd}) \leq 0. \tag{4.27}$$

For the other bound, we use the dual energy for $E^{odd,+}$, which is derived in Appendix B to be

$$E^{odd,c}(\xi,\zeta,\chi) = 2\int_{(\partial\Omega)^+}\varphi^{ev}\xi\cdot n\ ds - \int_{\Omega^+}|\xi|^2\ dx - \epsilon a_\epsilon\int_\sigma|\zeta|^2\ ds - \frac{a_\epsilon}{\epsilon}\int_\sigma \chi^2\ ds$$

where the maximization is taken over the set

$$\begin{aligned}W^{odd} = \{(\xi,\zeta,\chi)\in L^2(\Omega^+)^2\times L^2(\sigma)\times L^2(\sigma)\ |\ &-\nabla\cdot\xi = f^{odd}\\ &\xi_2 + \epsilon a_\epsilon\frac{\partial\zeta}{\partial x} - \frac{a_\epsilon}{\epsilon}\chi = 0 \text{ on } \sigma\},\end{aligned}$$

where the constraints are meant to be interpreted in the sense of distributions. By the differential equation that u_0^{odd} satisfies, we see that the triple $\left(\nabla u_0^{odd}, 0, \frac{\epsilon}{a_\epsilon}\frac{\partial u_0^{odd}}{\partial y}\right)$ is in W^{odd} and so we have

$$\begin{aligned}\frac{1}{2}E^{odd}(u^{odd}) &= E^{odd,+c}\left(\nabla u^{odd}, \frac{\partial u^{odd}}{\partial x}, \frac{\partial u^{odd}}{\partial y}\right)\\ &\geq E^{odd,+c}\left(\nabla u_0^{odd}, 0, \frac{\epsilon}{a_\epsilon}\frac{\partial u_0^{odd}}{\partial y}\right)\\ &= \int_{(\partial\Omega)^+}\varphi^{odd}\frac{\partial u_0^{odd}}{\partial n}\ ds - \frac{1}{2}\int_{\Omega^+}|\nabla u_0^{odd}|^2\ dx - \frac{\epsilon}{2a_\epsilon}\int_\sigma\left|\frac{\partial u_0^{odd}}{\partial y}\right|^2\ ds\\ &\geq \int_{(\partial\Omega)^+}\varphi^{odd}\frac{\partial u_0^{odd}}{\partial n}\ ds - \frac{1}{2}\int_{\Omega^+}|\nabla u_0^{odd}|^2\ dx - \frac{\epsilon}{a_\epsilon}C(f,\varphi)^2\\ &= E_0^{odd,c}(u_0^{odd}) - \frac{\epsilon}{a_\epsilon}C(f,\varphi)^2 = \frac{1}{2}E_0^{odd}(u_0^{odd}) - \frac{\epsilon}{a_\epsilon}C(f,\varphi)^2\end{aligned}$$

so that

$$E_0^{odd}(u_0^{odd}) - E^{odd}(u^{odd}) \leq \frac{\epsilon}{a_\epsilon}C(f,\varphi)^2 \tag{4.28}$$

which can be combined with (4.27) to establish (4.25).

4.5.2 Proof of (4.26)

The same process will allow us to prove (4.26). We are trying to compare E_0^{odd} to E_ϵ, which only differ by the factor of a_ϵ on the integral over ω_ϵ. However, since a_ϵ is large, we will not be able to use each function as an approximate minimizer for the opposite functional. In order to correct for this, we would like for our approximate minimizer to be zero in ω_ϵ, so that we can ignore the a_ϵ term. For one direction of the inequality, we take the function u_0^{odd} and build an approximate minimizer to E_ϵ in the form

$$v^{test} = \begin{cases} u_0^{odd} + v_\epsilon & \Omega \setminus \omega_\epsilon \\ 0 & \omega_\epsilon, \end{cases}$$

where v_ϵ is any function in $H^1(\Omega \setminus \omega_\epsilon)$ with $v_\epsilon = -u_0^{odd}$ on $\partial\omega_\epsilon$ and $v_\epsilon = 0$ on $\partial\Omega$ with

$$||v_\epsilon||_{H^1(\Omega\setminus\omega_\epsilon)} \leq \epsilon^\beta C(f, \varphi)$$

for any $\beta < 1$. We define this function by extending $-u_0^{odd}$ constantly along the normal direction to ω_ϵ and then multiplying by a cutoff function to match the desired boundary condition on $\partial\Omega$. Since u_0^{odd} is $H^1(\Omega)$ and vanishes along $y = 0$, we have that

$$\begin{aligned} |u_0^{odd}| &\leq \epsilon C(f, \varphi) & \partial\omega_\epsilon \\ \left|\frac{\partial u_0^{odd}}{\partial \tau}\right| &\leq C(f, \varphi) & \partial\omega_\epsilon \cap \{|x| > 1\} \\ \left|\frac{\partial u_0^{odd}}{\partial \tau}\right| &\leq \epsilon C(f, \varphi) & \partial\omega_\epsilon \cap \{|x| < 1\} \end{aligned},$$

and combining these factors together give the desired norm estimate on v_ϵ. With this, we then have that

$$\begin{aligned} E_\epsilon^{odd}(u_\epsilon^{odd}) \leq E_\epsilon^{odd}(v^{test}) &= \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla(u_0^{odd} + v_\epsilon)|^2 \, dx \\ &\leq \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u_0^{odd}|^2 \, dx + \epsilon^\beta C(f, \varphi)^2 \leq E_0^{odd}(u_0^{odd})\epsilon^\beta C(f, \varphi)^2 \end{aligned}$$

which gives

$$E_\epsilon^{odd}(u_\epsilon^{odd}) - E_0^{odd}(u_0^{odd}) \leq \epsilon^\beta C(f, \varphi)^2 \tag{4.29}$$

which is half of the desired bound.

For the other half of the inequality, we consider the dual energy corresponding to E_ϵ, which is shown in Appendix B to be

$$E_\epsilon^{odd,c}(\xi) = \int_{\partial\Omega} \varphi^{odd} \xi \cdot n \; ds - \frac{1}{2} \int_\Omega \gamma_\epsilon^{-1} |\xi|^2 \; dx.$$

Using ∇u_0^{odd} as a test field gives that

$$\begin{aligned} E_\epsilon^{odd}(u_\epsilon^{odd}) &= E_\epsilon^{odd,c}(\nabla u_\epsilon) \geq E_\epsilon^{odd,c}(\nabla u_0^{odd}) \\ &= \int_{\partial\Omega} \varphi^{odd} \frac{\partial u_0^{odd}}{\partial n} \; ds - \frac{1}{2} \int_{\Omega \setminus \omega_\epsilon} |\nabla u_0^{odd}|^2 \; dx - \frac{1}{2a_\epsilon} \int_{\omega_\epsilon} |\nabla u_0^{odd}|^2 \; dx \\ &\geq \int_{\partial\Omega} \varphi^{odd} \frac{\partial u_0^{odd}}{\partial n} \; ds - \frac{1}{2} \int_\Omega |\nabla u_0^{odd}|^2 \; dx - \frac{\epsilon}{a_\epsilon} C(f, \varphi)^2 \\ &= E_0^{odd}(u_0^{odd}) - \frac{\epsilon}{a_\epsilon} C(f, \varphi)^2 \end{aligned}$$

where we have used the fact that u_0^{odd} is smooth inside ω_ϵ, so ∇u_0^{odd} is uniformly bounded by $C(f, \varphi)$. This gives

$$E_0^{odd}(u_0^{odd}) - E_\epsilon^{odd}(u_\epsilon^{odd}) \leq \frac{\epsilon}{a_\epsilon} C(f, \varphi)^2 \tag{4.30}$$

and combining (4.29) and (4.30) gives the estimate desired for (4.26).

4.6 Results independent of a_ϵ

Now, we want to merge all of these results together to get something that works independent of a_ϵ and combines the odd and even symmetry cases together. The setup for this is the following. We have the two solutions u_ϵ^0 to (3.1) and u_ϵ to (2.1), for some $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$, where Ω is symmetric around the straight line $\sigma = \{(x, 0) \mid -1 < x < 1\}$. We split these two functions along with f and φ into their even and odd parts with respect to σ:

$$\begin{aligned} u_\epsilon^0 &= u^{ev} + u^{odd} \\ u_\epsilon &= u_\epsilon^{ev} + u_\epsilon^{odd} \\ f &= f^{ev} + f^{odd} \\ \varphi &= \varphi^{ev} + \varphi^{odd}. \end{aligned}$$

The results in Sections 4.2 and 4.3 combine to give the following.

Lemma 4.6.1.
$$\lim_{\epsilon\to 0}\sup_{P}\frac{|E^{ev}(u^{ev})-E^{ev}_{\epsilon}(u^{ev}_{\epsilon})|}{\Big(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\Big)^2}=0$$
where the parameter set P is

$$P=\{(a_\epsilon,f,\varphi)\ :\ 0<a_\epsilon\in\mathbb{R}\ \ \forall\,\epsilon>0,\ f\in\mathcal{F}_\delta,\ \varphi\in H^{1/2}(\partial\Omega)\}$$

Proof. Suppose that this statement is not true. Then there exists a sequence $\epsilon_n\to 0$ and a constant $c>0$ so that for all n,
$$c<\sup_{P}\frac{|E^{ev}(u^{ev})-E^{ev}_{\epsilon_n}(u^{ev}_{\epsilon_n})|}{\Big(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\Big)^2}.$$
This means that we can also pick a corresponding a_{ϵ_n} so that this statement holds for that a_{ϵ_n}, namely
$$c<\sup_{\substack{f\in\mathcal{F}_\delta\\ \varphi\in H^{1/2}(\partial\Omega)}}\frac{|E^{ev}(u^{ev})-E^{ev}_{\epsilon_n}(u^{ev}_{\epsilon_n})|}{\Big(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\Big)^2}. \tag{4.31}$$
Now, there are two possibilities for the sequence $\epsilon_n a_{\epsilon_n}$.

1. There exists a constant m so that $m<\epsilon_n a_{\epsilon_n}$, or

2. There exists a subsequence of $\epsilon_n a_{\epsilon_n}$ that goes to zero.

In the first case, the results of Section 4.2 apply. Thus, we know that

$$|E^{ev}(u^{ev})-E^{ev}_{\epsilon_n}(u^{ev}_{\epsilon_n})|\le\epsilon_n^{\beta/4}C(f,\varphi)^2$$

This implies that
$$\frac{|E^{ev}(u^{ev})-E^{ev}_{\epsilon_n}(u^{ev}_{\epsilon_n})|}{\Big(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\Big)^2}\le C\epsilon_n^{\beta/4}$$
which contradicts (4.31).

In the second case, the results from Section 4.3 apply to this subsequence, giving us that

$$|E^{ev}(u^{ev})-E^{ev}_{\epsilon_n}(u^{ev}_{\epsilon_n})|\le(\epsilon_n a_{\epsilon_n}+\epsilon_n^{\beta})C(f,\varphi)^2$$

which, since $\epsilon_n a_{\epsilon_n}\to 0$, the same argument gives another contradiction to (4.31). Since we have a contradiction in each case, Lemma 4.6.1 is proved. ■

In the case of odd symmetry, the analogous result holds.

Lemma 4.6.2.

$$\lim_{\epsilon\to 0}\sup_P \frac{|E^{odd}(u^{odd}) - E^{odd}_\epsilon(u^{odd}_\epsilon)|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2} = 0$$

where the parameter set P is

$$P = \{(a_\epsilon, f, \varphi) \;:\; 0 < a_\epsilon \in \mathbb{R} \;\; \forall\, \epsilon > 0,\;\; f \in \mathcal{F}_\delta,\;\; \varphi \in H^{1/2}(\partial\Omega)\}$$

Proof. Suppose that this statement is not true. Then there exists a sequence $\epsilon_n \to 0$ and a constant $c > 0$ so that for all n,

$$c < \sup_P \frac{|E^{odd}(u^{odd}) - E^{odd}_{\epsilon_n}(u^{odd}_{\epsilon_n})|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2}.$$

This means that we can also pick a corresponding a_{ϵ_n} so that this statement holds for that a_{ϵ_n}, namely

$$c < \sup_{\substack{f\in\mathcal{F}_\delta \\ \varphi\in H^{1/2}(\partial\Omega)}} \frac{|E^{odd}(u^{odd}) - E^{odd}_{\epsilon_n}(u^{odd}_{\epsilon_n})|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2}. \tag{4.32}$$

Now, there are two possibilities for the sequence $\epsilon_n a_{\epsilon_n}$.

1. There exists a constant M so that $a_{\epsilon_n} < M\epsilon_n$, or
2. There exists a subsequence of $\frac{a_{\epsilon_n}}{\epsilon_n}$ that goes to infinity.

In the first case, the results of Section 4.4 apply. Thus, we know that

$$|E^{odd}(u^{odd}) - E^{odd}_{\epsilon_n}(u^{odd}_{\epsilon_n})| \le \epsilon_n^{\beta/4} C(f,\varphi)^2$$

This implies that

$$\frac{|E^{odd}(u^{odd}) - E^{odd}_{\epsilon_n}(u^{odd}_{\epsilon_n})|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2} \le C\epsilon_n^{\beta/4}$$

which contradicts (4.32).

In the second case, the results from Section 4.5 apply to this subsequence, giving us that

$$|E^{odd}(u^{odd}) - E^{odd}_{\epsilon_n}(u^{odd}_{\epsilon_n})| \le \left(\frac{\epsilon_n}{a_{\epsilon_n}} + \epsilon_n^\beta\right) C(f,\varphi)^2$$

which by the same argument as case 1 gives another contradiction to (4.32), as $\frac{\epsilon_n}{a_{\epsilon_n}} \to 0$. Since we have a contradiction in each case, Lemma 4.6.2 is proved. ∎

Finally, we can use the fact that all of the energy expressions used in this paper split nicely with respect to even and odd symmetry across $y = 0$ in the sense that

$$E_\epsilon^0(u_\epsilon^0) = E_\epsilon^0(u^{ev} + u^{odd}) = E^{ev}(u^{ev}) + E^{odd}(u^{odd})$$

and

$$E_\epsilon(u_\epsilon) = E_\epsilon(u_\epsilon^{ev} + u_\epsilon^{odd}) = E_\epsilon^{ev}(u_\epsilon^{ev}) + E_\epsilon^{odd}(u_\epsilon^{odd})$$

and the triangle inequality to combine the results of Lemma 4.6.1 and Lemma 4.6.2 into the following theorem.

Theorem 4.6.1. *Let Ω be a domain that is symmetric around the straight line segment $\sigma = \{(x,0) \mid -1 < x < 1\}$, and let u_ϵ^0 and u_ϵ be the solutions to* (2.9) *and* (2.1) *respectively. Then*

$$\lim_{\epsilon \to 0} \sup_P \frac{|E_\epsilon^0(u_\epsilon^0) - E_\epsilon(u_\epsilon)|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2} = 0$$

where the parameter set P is

$$P = \{(a_\epsilon, f, \varphi) \;:\; 0 < a_\epsilon \in \mathbb{R} \;\; \forall \epsilon > 0, \; f \in \mathcal{F}_\delta, \; \varphi \in H^{1/2}(\partial\Omega)\}$$

Remark. The assumption that $\sigma = \{(x,0) \mid -1 < x < 1\}$ is unnecessary; the same results would go through for any straight line-segment for σ after a rotation, translation, and scaling. However, at this point, we do need σ to be a straight line segment, at least near the endpoints.

4.7 Full results

The last assumption that we want to remove is the fact that Ω is symmetric around the line segment σ. In order to address this, we will introduce a subscript on all of the energies discussed so far to refer to the domain on which the integrals are calculated. That is, we will use

$$E_{\epsilon,U}(u) = \frac{1}{2}\int_U \gamma_\epsilon |\nabla u|^2 \, dx - \int_U fu \, dx$$

and the minimization would take place over functions in $H^1(U)$ with specified boundary data on ∂U. The same goes for

$$E^0_{\epsilon,U}(u) = \frac{1}{2}\int_{U\setminus\overline{\sigma}} |\nabla u|^2 \, dx + \frac{\epsilon a_\epsilon}{2}\int_\sigma \left(\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 \right) ds$$
$$+ \frac{a_\epsilon}{4\epsilon}\int_\sigma (u^+ - u^-)^2 \, ds - \int_U fu \, dx.$$

Theorem 4.7.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1 domain, with σ a straight line segment so that $\omega_\epsilon \subset\subset \Omega$ for $\epsilon < \epsilon_0$. Fix any $\delta > 0$, and for any $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$, let u^0_ϵ and u_ϵ denote the solutions to* (2.9) *and* (2.1) *respectively. Then*

$$\lim_{\epsilon\to 0} \sup_P \frac{|E^0_{\epsilon,\Omega}(u^0_\epsilon) - E_{\epsilon,\Omega}(u_\epsilon)|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2} = 0$$

where the parameter set P is

$$P = \{(a_\epsilon, f, \varphi) \ : \ 0 < a_\epsilon \in \mathbb{R} \ \ \forall \epsilon > 0, \ f \in \mathcal{F}_\delta, \ \varphi \in H^{1/2}(\partial\Omega)\}.$$

Proof. If Ω was symmetric with respect to σ, this would exactly be the result of Theorem 4.6.1. To extend this to general Ω, we use the fact that since $\omega_\epsilon \subset\subset \Omega$ for small enough ϵ, then we can pick a domain $\tilde{\Omega} \subset \Omega$ so that $\tilde{\Omega}$ is symmetric with respect to σ. We define

$$\varphi_\epsilon = u_\epsilon \,|_{\partial\tilde{\Omega}} \qquad \varphi^0_\epsilon = u^0_\epsilon \,|_{\partial\tilde{\Omega}}$$

and find solutions to (2.9) and (2.1) on $\tilde{\Omega}$ with these boundary data. That is, we let v^0_ϵ be the solution to (2.9) on $\tilde{\Omega}$ with boundary data φ_ϵ and let v_ϵ be the solution to (2.1) on $\tilde{\Omega}$ with boundary data φ^0_ϵ. Then we define two functions on Ω

$$\bar{u}^0_\epsilon = \begin{cases} u_\epsilon & \Omega\setminus\tilde{\Omega} \\ v^0_\epsilon & \tilde{\Omega}, \end{cases} \qquad \bar{u}_\epsilon = \begin{cases} u^0_\epsilon & \Omega\setminus\tilde{\Omega} \\ v_\epsilon & \tilde{\Omega}. \end{cases}$$

By choice of boundary data, these two functions are H^1 across $\partial\tilde{\Omega}$, and based on the spaces where v_ϵ and v^0_ϵ belong, we know that

$$\bar{u}^0_\epsilon \in H^1(\Omega\setminus\sigma) \cap H^1(\sigma) \qquad \bar{u}_\epsilon \in H^1(\Omega)$$

so that we can use them in our energy formulas.

By Theorem 4.6.1, we know that the energy gaps between solutions with the same boundary data are small in an appropriate sense. This means that

$$|E^0_{\epsilon,\tilde{\Omega}}(v^0_\epsilon) - E_{\epsilon,\tilde{\Omega}}(u_\epsilon)| = o(1)C(f,\varphi) \tag{4.33}$$

and

$$|E^0_{\epsilon,\tilde{\Omega}}(u^0_\epsilon) - E_{\epsilon,\tilde{\Omega}}(v_\epsilon)| = o(1)C(f,\varphi) \tag{4.34}$$

where the $o(1)$ term goes to zero as $\epsilon \to 0$. This comes from the fact that the boundary data on $\partial\tilde{\Omega}$ is can be controlled by $C(f,\varphi)$ by standard elliptic arguments. This means that we can swap these energies in our calculations by adding in a term of the form $o(1)C(f,\varphi)$. Using these facts and the definition of $\bar{u}^0_\epsilon$, we compute that

$$\begin{aligned}
E^0_{\epsilon,\Omega}(\bar{u}^0_\epsilon) &= E^0_{\epsilon,\tilde{\Omega}}(v^0_\epsilon) + \frac{1}{2}\int_{\Omega\setminus\tilde{\Omega}} |\nabla u_\epsilon|^2 \, dx - \int_{\Omega\setminus\tilde{\Omega}} f u_\epsilon \, dx \\
&\leq E_{\epsilon,\tilde{\Omega}}(u_\epsilon) + o(1)C(f,\varphi) + \frac{1}{2}\int_{\Omega\setminus\tilde{\Omega}} |\nabla u_\epsilon|^2 \, dx - \int_{\Omega\setminus\tilde{\Omega}} f u_\epsilon \, dx \\
&= E_{\epsilon,\Omega}(u_\epsilon) + o(1)C(f,\varphi).
\end{aligned}$$

Similarly, the definition of $\bar{u}_\epsilon$ gives us

$$\begin{aligned}
E_{\epsilon,\Omega}(\bar{u}_\epsilon) &= E_{\epsilon,\tilde{\Omega}}(v_\epsilon) + \frac{1}{2}\int_{\Omega\setminus\tilde{\Omega}} |\nabla u^0_\epsilon|^2 \, dx - \int_{\Omega\setminus\tilde{\Omega}} f u^0_\epsilon \, dx \\
&\leq E^0_{\epsilon,\tilde{\Omega}}(u^0_\epsilon) + o(1)C(f,\varphi) + \frac{1}{2}\int_{\Omega\setminus\tilde{\Omega}} |\nabla u^0_\epsilon|^2 \, dx - \int_{\Omega\setminus\tilde{\Omega}} f u^0_\epsilon \, dx \\
&= E^0_{\epsilon,\Omega}(u^0_\epsilon) + o(1)C(f,\varphi).
\end{aligned}$$

However, we know that u^0_ϵ and u_ϵ are minimizers of their respective energy functionals. This gives that

$$\begin{aligned}
E_{\epsilon,\Omega}(u_\epsilon) \leq E_{\epsilon,\Omega}(\bar{u}_\epsilon) \leq E^0_{\epsilon,\Omega}(u^0_\epsilon) + o(1)C(f,\varphi) \\
E^0_{\epsilon,\Omega}(u^0_\epsilon) \leq E^0_{\epsilon,\Omega}(\bar{u}^0_\epsilon) \leq E_{\epsilon,\Omega}(u_\epsilon) + o(1)C(f,\varphi)
\end{aligned} \tag{4.35}$$

and combining the two estimates in (4.35) gives that

$$|E_{\epsilon,\Omega}(u_\epsilon) - E^0_{\epsilon,\Omega}(u^0_\epsilon)| \leq o(1)C(f,\varphi)$$

which is exactly what is asserted by Theorem 4.7.1. $\blacksquare$

Now, we can use our energy lemma to move this from energy convergence to convergence in any Sobolev norm.

Theorem 4.7.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1 domain, with σ a straight line segment so that $\omega_\epsilon \subset\subset \Omega$ for $\epsilon < \epsilon_0$. Fix any $\delta > 0$, and for any $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$, let u_ϵ^0 and u_ϵ denote the solutions to* (2.9) *and* (2.1) *respectively. Then*

$$\lim_{\epsilon \to 0} \sup_P \frac{||u_\epsilon^0 - u_\epsilon||_{H^s(\Omega \setminus \omega_{\delta'})}}{||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}} = 0$$

for any $s \geq 0$ and any $\delta' > 0$, where the parameter set P is

$$P = \{(a_\epsilon, f, \varphi) \ : \ 0 < a_\epsilon \in \mathbb{R} \ \ \forall \epsilon > 0, \ f \in \mathcal{F}_\delta, \ \varphi \in H^{1/2}(\partial\Omega)\}$$

Proof. The move from convergence in energy in Theorem 4.7.1 to convergence in $L^2(\Omega \setminus \omega_{\delta'})$ is done using Lemma 2.2.2. In this application, we take $V = H^1(\Omega)$, $W = \{v \in H^1(\Omega) \ : \ v^+, v^- \in H^1(\sigma)\}$ and $H = \mathcal{F}_{\hat{\delta}}$ for some $0 < \hat{\delta} < \min\{\delta, \delta'\}$. The projection maps P and Q into H are taken to be multiplying by the characteristic function of $\Omega \setminus \omega_{\hat{\delta}}$. The bilinear forms $a : V \times V \to \mathbb{R}$ and $b : W \times W \to \mathbb{R}$ are taken to be the forms corresponding to the energies E_ϵ and E_ϵ^0 respectively. Since $f \in \mathcal{F}_\delta \subset \mathcal{F}_{\hat{\delta}}$, it gives rise to a well-defined functional in H', and so the energy lemma can be applied. This gives us convergence involving $||u_\epsilon^0 - u_\epsilon||_{L^2(\Omega \setminus \omega_{\hat{\delta}})}$, and monotonicity of L^2 norms gives us the same result on $\Omega \setminus \omega_{\delta'}$. The fact that the lemma requires the norms of f and φ to be less than 1 is taken care of by rescaling, which removes one factor of $C(f, \varphi)$ from the denominator in this result.

The convergence in higher Sobolev norms is done using standard elliptic regularity, with the fact that $\Delta(u_\epsilon^0 - u_\epsilon) = 0$ in $\Omega \setminus \omega_{\hat{\delta}}$ and this difference vanishes on $\partial\Omega$. ∎

Chapter 5

Other Results

5.1 Computation of the stress intensity factors

With the convergence results established, we now seek to get a better idea of what these solutions look like. In particular, we will find an explicit formula for the stress intensity factors of the solution with even symmetry across the straight line σ, with $\epsilon a_\epsilon > m > 0$. As before, we will assume that $\Omega = B_2(0)$ and $\sigma = \{(x, 0) \; -1 < x < 1\}$. For convenience, Lemma 3.4.1 is stated again here.

Lemma. *Let (r, θ) and (r', θ') be polar coordinates around the two endpoints of σ, $(-1, 0)$ and $(1, 0)$ respectively, where in each case, the 0 and 2π angles correspond to σ. Let η denote a smooth cut-off function which is 1 near $r = 0$ and vanishes outside of $r = \delta/2$. Then the solution u^{ev} to (3.4) satisfies*

$$u^{ev} = b_1 r^{1/2} \sin(\theta/2)\eta(r) + b_1' r'^{1/2} \sin(\theta'/2)\eta(r') + u^{ev,*}$$

where $u^{ev,*}$ is bounded in $C^{1,\beta/2}((\omega_\delta)_+)$ by $C(f, \varphi)\left(\frac{1}{(\epsilon a_\epsilon)^3} + 1\right)$, and the constants b_1 and b_1' are bounded by $C(f, \varphi)\left(\frac{1}{(\epsilon a_\epsilon)^3} + 1\right)$ as well.

What we want to do here is find a formula for the b_1 coefficient in this expression, based on the function u. In particular, we want this formula to depend on the solution away from the corresponding singularity at $(-1, 0)$, and as much away from σ as possible. For reference, the function u^{ev} solves

$$\begin{cases} -\Delta u^{ev} = f^{ev} & \Omega^+ \\ u^{ev} = \varphi^{ev} & \partial\Omega^+ \\ \dfrac{\partial u^{ev}}{\partial y} + \epsilon a_\epsilon \dfrac{\partial^2 u^{ev}}{\partial x^2} = 0 & \sigma \\ \dfrac{\partial u^{ev}}{\partial y} = 0 & \{y = 0\} \setminus \sigma \end{cases}$$

and belongs to the space

$$V_\sigma^{ev,+} = \{u \in H^1(\Omega^+) \mid u^+|_\sigma \in H^1(\sigma), u \text{ even across } \sigma\}.$$

In this section, we prove the following.

Proposition 5.1.1. *For the decomposition above, we have that*

$$b_1\pi = \int_{\partial B_2(0)} \frac{\partial u^{ev}}{\partial n}\tilde{\psi}\ ds - \int_{\partial B_2(0)} \frac{\partial \tilde{\psi}}{\partial n} u^{ev}\ ds - u^{ev}(1,0)\sqrt{2}$$
$$+ \int_1^2 \left[\frac{\partial \tilde{\psi}}{\partial y}\right] u^{ev}\ dx + \int_{B_2(0)} f^{ev}\tilde{\psi}\ dx$$

where $\tilde{\psi} = \tilde{v} + \frac{2}{\epsilon a_\epsilon}\tilde{w}$ *for*

$$\tilde{v} = r^{-1/2}\sin\left(\frac{\theta}{2}\right) \qquad \tilde{w} = r^{1/2}\cos\left(\frac{\theta}{2}\right).$$

To start, we want to derive something similar to a variational formulation that u^{ev} must satisfy. If $w \in V_\sigma^{ev,+}$, we see that

$$\begin{aligned}
\int_{\Omega^+} f^{ev} w\ dx &= -\int_{\Omega^+} \Delta u^{ev} w\ dx \\
&= \int_{\Omega^+} \nabla u^{ev}\cdot\nabla w\ dx - \int_{\partial\Omega^+} \frac{\partial u^{ev}}{\partial n} w\ ds \\
&= \int_{\Omega^+} \nabla u^{ev}\cdot\nabla w\ dx - \int_{(\partial\Omega)^+} \frac{\partial u^{ev}}{\partial n} w\ ds - \epsilon a_\epsilon \int_\sigma \frac{\partial^2 u^{ev}}{\partial x^2} w\ ds \\
&= \int_{\Omega^+} \nabla u^{ev}\cdot\nabla w\ dx - \int_{(\partial\Omega)^+} \frac{\partial u^{ev}}{\partial n} w\ ds + \epsilon a_\epsilon \int_\sigma \frac{\partial u^{ev}}{\partial x}\frac{\partial w}{\partial x}\ ds - \frac{\partial u^{ev}}{\partial x} w\ |_a^b
\end{aligned}$$

However, in the case of even symmetry and this range of ϵa_ϵ, we know that $u^{ev} = u^{ev,*}$ along σ (since the sine terms vanish) and $\frac{\partial u^{ev,*}}{\partial x} = 0$ at the endpoints of σ, which was proven in Lemma 4.2.1. Therefore, this expression can be rearranged to

$$\epsilon a_\epsilon \int_\sigma \frac{\partial u^{ev}}{\partial x}\frac{\partial w}{\partial x}\ ds = \int_{\Omega^+} f^{ev} w\ dx - \int_{\Omega^+} \nabla u^{ev}\cdot\nabla w\ dx + \int_{(\partial\Omega)^+} \frac{\partial u^{ev}}{\partial n} w\ ds \qquad (5.1)$$

which holds for any $w \in V_\sigma^{ev,+}$.

To work towards an actual formula for b_1, we define

$$\tilde{v} = r^{-1/2}\sin(\theta/2)$$

and let $A = B_2(0)^+ \setminus B_\rho((-1,0))^+$ for some $\rho > 0$. With this, we integrate by parts to compute that

$$\int_A \nabla u^{ev}\nabla\tilde{v}\ dx = \int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n}\tilde{v}\ ds - \int_{(\partial B_\rho((-1,0)))^+} \frac{\partial u^{ev}}{\partial r}\tilde{v}\ ds + \int_A f^{ev}\tilde{v}\ dx$$

where we have used the fact that $\frac{\partial u^{ev}}{\partial n} = 0$ along $\{(x,0) \mid -2 < x < -1-\rho\}$ by even symmetry and $\tilde{v} = 0$ along $\{(x,0) \mid -1+\rho < x < 2\}$ by definition. If ρ is small enough so that f^{ev} vanishes on $B_\rho((-1,0))^+$ and $\eta(\rho) = 1$, then

$$\int_A f^{ev}\tilde{v}\ dx = \int_{\Omega^+} f^{ev}\tilde{v}\ dx$$

and we can compute that

$$\begin{aligned}\int_{(\partial B_\rho((-1,0)))^+} \frac{\partial u^{ev}}{\partial r}\tilde{v}\ ds &= \int_{(\partial B_\rho((-1,0)))^+} \frac{\partial}{\partial r}\left(b_1 r^{1/2}\sin(\theta/2)\right)\tilde{v}\ ds \\ &\quad + \int_{(\partial B_\rho((-1,0)))^+} \frac{\partial u^{ev,*}}{\partial r}\tilde{v}\ ds.\end{aligned}$$

For the first integral, we get that

$$\begin{aligned}\int_{(\partial B_\rho((-1,0)))^+} \frac{\partial}{\partial r}\left(b_1 r^{1/2}\sin(\theta/2)\right)\tilde{v}\ ds &= \frac{b_1}{2}\int_0^\pi r^{-1/2}\sin(\theta/2) r^{-1/2}\sin(\theta/2) r\ d\theta \\ &= \frac{b_1}{2}\int_0^\pi \sin^2(\theta/2)\ d\theta = \frac{b_1}{4}\pi\end{aligned}$$

and for the second, we note that because $u^{ev,*} \in C^{1,\beta/2}$ and $\nabla u^{ev,*}(-1,0) = 0$, we have that

$$\left|\frac{\partial u^{ev,*}}{\partial r}\right| \le r^{\beta/2}$$

on $B_\rho(-1,0)^+$, so that

$$\left|\int_{(\partial B_\rho((-1,0)))^+} \frac{\partial u^{ev,*}}{\partial r}\tilde{v}\ ds\right| \le \int_0^\pi \rho^{\beta/2}\rho^{-1/2}\rho\ d\theta \to 0$$

as $\rho \to 0$. For convenience in what follows, we will use $o(1)$ to denote any term that goes to zero as $\rho \to 0$. Thus, we have that

$$\int_A \nabla u^{ev}\nabla\tilde{v}\ dx = \int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n}\tilde{v}\ ds - \frac{b_1}{4}\pi + \int_A f^{ev}\tilde{v}\ dx + o(1). \tag{5.2}$$

Since $\tilde{v}$ is harmonic on A, we can integrate by parts in the other direction, giving that

$$\begin{aligned}\int_A \nabla u^{ev}\nabla\tilde{v}\ dx &= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n}u^{ev}\ ds - \int_{(\partial B_\rho((-1,0)))^+} \frac{\partial \tilde{v}}{\partial r}u^{ev}\ ds \\ &\quad - \int_{-2}^{-1-\rho} \frac{\partial \tilde{v}}{\partial y}u^{ev}\ dx - \int_{-1+\rho}^{2} \frac{\partial \tilde{v}}{\partial y}u^{ev}\ dx.\end{aligned} \tag{5.3}$$

By definition, $\frac{\partial \tilde{v}}{\partial y} = 0$ along $\{(x,0) \mid -2 < x < -1-\rho\}$, so the first term in the second line vanishes. We can also compute that

$$\begin{aligned}\int_{(\partial B_\rho((-1,0)))^+} \frac{\partial \tilde{v}}{\partial r} u^{ev}\ ds = &\int_{(\partial B_\rho((-1,0)))^+} \frac{\partial \tilde{v}}{\partial r} b_1 r^{1/2} \sin(\theta/2)\ ds \\ &+ \int_{(\partial B_\rho((-1,0)))^+} \frac{\partial \tilde{v}}{\partial r} u^{ev,*}\ ds\end{aligned}$$

where, by a similar argument as before,

$$\int_{(\partial B_\rho((-1,0)))^+} \frac{\partial \tilde{v}}{\partial r} b_1 r^{1/2} \sin(\theta/2)\ ds = -\frac{b_1}{4}\pi \tag{5.4}$$

and

$$\int_{(\partial B_\rho((-1,0)))^+} \frac{\partial \tilde{v}}{\partial r} u^{ev,*}\ ds = -\rho^{-1/2} u^{ev}(-1,0) + o(1), \tag{5.5}$$

since by the smoothness of $u^{ev,*}$, once we subtract off the value of the function at $(-1,0)$, the difference vanishes to a high enough order to cause the remainder term to go to zero with ρ. We also note that

$$\int_{-1+\rho}^{2} \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx = \int_{-1+\rho}^{1} \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx + \int_{1}^{2} \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx$$

and, since $\frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial \theta}$ along these segments, we see that

$$\left.\frac{\partial \tilde{v}}{\partial y}\right|_{y=0,x>-1} = \frac{1}{2}(x+1)^{-3/2}.$$

Noticing that

$$\frac{1}{2}(x+1)^{-3/2} = -2\frac{d^2}{dx^2}(x+1)^{1/2}$$

we can then compute that

$$\begin{aligned}\int_{-1+\rho}^{1} \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx &= \frac{1}{2}\int_{-1+\rho}^{1} u^{ev}(x+1)^{-3/2}\ dx \\ &= -2\int_{-1+\rho}^{1} u^{ev} \frac{d^2}{dx^2}(x+1)^{1/2}\ dx \\ &= 2\int_{-1+\rho}^{1} \frac{d}{dx}u^{ev}\frac{d}{dx}(x+1)^{1/2}\ dx - u^{ev}(1,0)2^{-1/2} \\ &+ u^{ev}(-1+\rho,0)\rho^{-1/2} \\ &= 2\int_{-1+\rho}^{1} \frac{d}{dx}u^{ev}\frac{d}{dx}(x+1)^{1/2}\ dx - u^{ev}(1,0)2^{-1/2} \\ &+ u^{ev}(-1,0)\rho^{-1/2} + o(1)\end{aligned} \tag{5.6}$$

where we integrated by parts in the second to last step. In the final step, we used the fact that, along σ, $u^{ev} = u^{ev,*}$, and $u^{ev,*}$ is regular enough that replacing $-1+\rho$ by -1 has a remainder term that goes to zero with ρ. Substituting the results of (5.4), (5.5), and (5.6) into (5.3), we get that

$$\begin{aligned}\int_A \nabla u^{ev} \nabla \tilde{v}\ dx = &\int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n} u^{ev}\ ds + \frac{b_1}{4}\pi + \rho^{-1/2} u^{ev}(-1,0) + o(1) \\ &- 2\int_{-1+\rho}^{1} \frac{d}{dx} u^{ev} \frac{d}{dx}(x+1)^{1/2}\ dx + u^{ev}(1,0)2^{-1/2} \\ &- u^{ev}(-1,0)\rho^{-1/2} + o(1) - \int_1^2 \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx. \\ = &\int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n} u^{ev}\ ds + \frac{b_1}{4}\pi + u^{ev}(1,0)2^{-1/2} \\ &- 2\int_{-1+\rho}^{1} \frac{d}{dx} u^{ev} \frac{d}{dx}(x+1)^{1/2}\ dx - \int_1^2 \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx + o(1).\end{aligned} \tag{5.7}$$

We can then set (5.7) equal to (5.2) via the $\int_A \nabla u^{ev} \nabla \tilde{v}\ dx$ term and solve for the b_1 term to give that

$$\begin{aligned}\frac{b_1}{2}\pi = &\int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n} \tilde{v}\ ds - \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n} u^{ev}\ ds - u^{ev}(1,0)2^{-1/2} \\ &+ 2\int_{-1+\rho}^{1} \frac{d}{dx} u^{ev} \frac{d}{dx}(x+1)^{1/2}\ dx + \int_1^2 \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx + \int_A f^{ev} \tilde{v}\ dx + o(1)\end{aligned} \tag{5.8}$$

Within this equation, only the first term in the second line is problematic, because it still depends on ρ. In order to deal with this term, we pick some $t > 0$ and define

$$\tilde{w}_\rho(r,\theta) = \begin{cases} r^{1/2}\cos(\theta/2) & r > \rho \\ \rho^{-t} r^{1/2+t}\cos(\theta/2) & r \le \rho, \end{cases}$$

$$\tilde{w}(r,\theta) = r^{1/2}\cos(\theta/2).$$

The importance of these definitions is that for all $\rho > 0$, $\tilde{w}_\rho \in H^1(B_2(0)^+) \cap H^1(\sigma)$, while $\tilde{w} \in H^1(B_2(0)^+)$, but is not in $H^1(\sigma)$. Based on these definitions, we see that

$$\begin{aligned}
\int_{-1}^{-1+\rho} \frac{d}{dx} u^{ev} \frac{d}{dx} \tilde{w}_\rho \; dx &= o(1), \\
\int_{B_2(0)^+ \cap \{r<\rho\}} \nabla u^{ev} \nabla \tilde{w}_\rho &= o(1), \\
\int_{B_2(0)^+ \cap \{r<\rho\}} \nabla u^{ev} \nabla \tilde{w} &= o(1), \\
\int_{B_2(0)^+ \cap \{r<\rho\}} f^{ev} \tilde{w}_\rho &= o(1), \\
\int_{B_2(0)^+ \cap \{r<\rho\}} f^{ev} \nabla \tilde{w} &= o(1).
\end{aligned} \tag{5.9}$$

The first of these identities comes from the fact that on σ, $u^{ev} = u^{ev,*}$, which is more regular, and $\frac{d}{dx}\tilde{w}_\rho \leq r^{-1/2}$ on this domain. The others follow from the fact that $\tilde{w}_\rho \in H^1(B_2(0)^+)$ with uniform control on the norm independent of ρ, because the changes only make the function and all derivatives smaller. Therefore, we have that

$$\begin{aligned}
\int_{-1+\rho}^{1} \frac{d}{dx} u^{ev} \frac{d}{dx} (x+1)^{1/2} \; dx &= \int_{-1+\rho}^{1} \frac{d}{dx} u^{ev} \frac{d}{dx} \tilde{w}_\rho \; dx \\
&= \int_{-1}^{1} \frac{d}{dx} u^{ev} \frac{d}{dx} \tilde{w}_\rho \; dx + o(1).
\end{aligned} \tag{5.10}$$

Since $\tilde{w}_\rho \in V_\sigma^{ev,+}$, we can use (5.1) to rewrite this as

$$\begin{aligned}
\int_{-1}^{1} \frac{d}{dx} u^{ev} \frac{d}{dx} \tilde{w}_\rho \; dx &= \frac{1}{\epsilon a_\epsilon} \int_\Omega f^{ev} \tilde{w}_\rho \; dx - \frac{1}{\epsilon a_\epsilon} \int_{B_2(0)^+} \nabla u^{ev} \nabla \tilde{w}_\rho \; dx \\
&+ \frac{1}{\epsilon a_\epsilon} \int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n} \tilde{w}_\rho \; ds \\
&= \frac{1}{\epsilon a_\epsilon} \int_\Omega f^{ev} \tilde{w} \; dx - \frac{1}{\epsilon a_\epsilon} \int_{B_2(0)^+} \nabla u^{ev} \nabla \tilde{w} \; dx \\
&+ \frac{1}{\epsilon a_\epsilon} \int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n} \tilde{w} \; ds + o(1),
\end{aligned} \tag{5.11}$$

where in the last line, we have used the estimates in (5.9) to replace $\tilde{w}_\rho$ by $\tilde{w}$ in all of the integrals by adding an $o(1)$ term. Finally, we can use the fact that $w \in H^1(B_2(0)^+)$ and is harmonic to integrate by parts to see that

$$\begin{aligned}
\int_{B_2(0)^+} \nabla u^{ev} \nabla \tilde{w} \; dx &= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{w}}{\partial n} u^{ev} \; ds - \int_{-2}^{2} \frac{\partial \tilde{w}}{\partial y} u^{ev} \; dx \\
&= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{w}}{\partial n} u^{ev} \; ds - \int_{-2}^{-1} \frac{\partial \tilde{w}}{\partial y} u^{ev} \; dx
\end{aligned} \tag{5.12}$$

where, in the second line, we used the fact that $\frac{\partial \tilde{w}}{\partial y} = 0$ along $\{(x,0) \mid -1 < x < 2\}$ by definition. Then, we can put equations (5.10), (5.11), and (5.12) together, and use them to replace the problematic term in (5.8) to give a new expression of the form

$$\begin{aligned}\frac{b_1}{2}\pi = &\int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n}\tilde{v}\ ds - \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n} u^{ev}\ ds - u^{ev}(1,0)2^{-1/2} \\ &+ \frac{2}{\epsilon a_\epsilon}\int_{B_2(0)^+} f^{ev}\tilde{w}\ dx - \frac{2}{\epsilon a_\epsilon}\int_{(\partial B_2(0))^+} \frac{\partial \tilde{w}}{\partial n} u^{ev}\ ds \\ &+ \frac{2}{\epsilon a_\epsilon}\int_{-2}^{-1} \frac{\partial \tilde{w}}{\partial y} u^{ev}\ dx + \frac{2}{\epsilon a_\epsilon}\int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n}\tilde{w}\ ds + \int_1^2 \frac{\partial \tilde{v}}{\partial y} u^{ev}\ dx \\ &+ \int_{B_2(0)^+} f^{ev}\tilde{v}\ dx + o(1)\end{aligned} \tag{5.13}$$

or, defining the function $\tilde{\psi} = \tilde{v} + \frac{2}{\epsilon a_\epsilon}\tilde{w}$, this becomes

$$\begin{aligned}\frac{b_1}{2}\pi = &\int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n}\tilde{\psi}\ ds - \int_{(\partial B_2(0))^+} \frac{\partial \tilde{\psi}}{\partial n} u^{ev}\ ds - u^{ev}(1,0)2^{-1/2} \\ &+ \int_{-2}^{-1} \frac{\partial \tilde{\psi}}{\partial y} u^{ev}\ dx + \int_1^2 \frac{\partial \tilde{\psi}}{\partial y} u^{ev}\ dx + \int_{B_2(0)^+} f^{ev}\tilde{\psi}\ dx + o(1)\end{aligned}$$

in which we can finally send $\rho \to 0$ to get

$$\begin{aligned}\frac{b_1}{2}\pi = &\int_{(\partial B_2(0))^+} \frac{\partial u^{ev}}{\partial n}\tilde{\psi}\ ds - \int_{(\partial B_2(0))^+} \frac{\partial \tilde{\psi}}{\partial n} u^{ev}\ ds - u^{ev}(1,0)2^{-1/2} \\ &+ \int_{-2}^{-1} \frac{\partial \tilde{\psi}}{\partial y} u^{ev}\ dx + \int_1^2 \frac{\partial \tilde{\psi}}{\partial y} u^{ev}\ dx + \int_{B_2(0)^+} f^{ev}\tilde{\psi}\ dx.\end{aligned} \tag{5.14}$$

This formula will be illustrated in the numerical results in Chapter 6.

Remark. As should be expected, the value for b_1 does not change if we add a constant to u^{ev}. This fact requires verifying that

$$\begin{aligned}-\int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n}\ ds + \int_1^2 \frac{\partial \tilde{v}}{\partial y} dx = 2^{-1/2} \\ -\int_{(\partial B_2(0))^+} \frac{\partial \tilde{w}}{\partial n}\ ds + \int_{-2}^{-1} \frac{\partial \tilde{w}}{\partial y}\ dx = 0\end{aligned}$$

For the second, we know that $\tilde{w}$ is harmonic in $B_2(0)^+$, which gives that

$$\begin{aligned}0 &= \int_{\partial B_2(0)^+} \frac{\partial \tilde{w}}{\partial n}\ ds \\ &= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{w}}{\partial n}\ ds - \int_{-2}^{2} \frac{\partial \tilde{w}}{\partial y}\ dx \\ &= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{w}}{\partial n}\ ds - \int_{-2}^{-1} \frac{\partial \tilde{w}}{\partial y}\ dx\end{aligned}$$

where, in the last line, we have used the fact that $\frac{\partial \tilde{w}}{\partial y} = 0$ if $x > -1$ and $y = 0$.

For the first identity, we need to be a little more careful because $\tilde{v} \notin H^1(B_2(0)^+)$. However, it is in H^1 and harmonic on the domain $B_2(0)^+ \setminus B_{1/2}((-1,0))$. Thus, we get that

$$\begin{aligned}
0 &= \int_{\partial(B_2(0)^+ \setminus B_{1/2}((-1,0)))} \frac{\partial \tilde{v}}{\partial n}\, ds \\
&= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n}\, ds - \int_{-2}^{-3/2} \frac{\partial \tilde{v}}{\partial y}\, dx - \int_{(\partial B_{1/2}((-1,0))^+} \frac{\partial \tilde{v}}{\partial r}\, ds - \int_{-1/2}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx \\
&= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n}\, ds - \int_{(\partial B_{1/2}((-1,0))^+} \frac{\partial \tilde{v}}{\partial r}\, ds - \int_{-1/2}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx,
\end{aligned}$$

where in the last line, we used the fact that $\frac{\partial \tilde{v}}{\partial y} = 0$ on $y = 0$ and $x < -1$. Based on the definition of $\tilde{v}$, we compute that

$$\begin{aligned}
\int_{(\partial B_{1/2}((-1,0))^+} \frac{\partial \tilde{v}}{\partial r}\, ds &= -\frac{1}{2}\int_0^{\pi} r^{-3/2} r \sin(\theta/2)\, d\theta \\
&= -(1/2)^{1/2} \int_0^{\pi} \sin(\theta/2)\, d\theta \\
&= -2(1/2)^{1/2} = -2^{1/2} = -(1/2)^{-1/2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{-1/2}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx &= \int_{-1/2}^{1} \frac{\partial \tilde{v}}{\partial y}\, dx + \int_{1}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx \\
&= \frac{1}{2}\int_{-1/2}^{1} (x+1)^{-3/2}\, dx + \int_{1}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx \\
&= -1(x+1)^{-1/2} \,|_{-1/2}^{1} + \int_{1}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx \\
&= -2^{-1/2} + (1/2)^{-1/2} + \int_{1}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx.
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
0 &= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n}\, ds - \int_{(\partial B_{1/2}((-1,0))^+} \frac{\partial \tilde{v}}{\partial r}\, ds - \int_{-1/2}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx \\
&= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n}\, ds + (1/2)^{-1/2} + 2^{-1/2} - (1/2)^{-1/2} - \int_{1}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx \\
&= \int_{(\partial B_2(0))^+} \frac{\partial \tilde{v}}{\partial n}\, ds + 2^{-1/2} - \int_{1}^{2} \frac{\partial \tilde{v}}{\partial y}\, dx,
\end{aligned}$$

which is exactly what we wanted to show.

Remark. The formula (5.14) could instead be computed over the domain $B_2(0)^-$, giving rise to the formula

$$\begin{aligned}\frac{b_1}{2}\pi = &\int_{(\partial B_2(0))^-} \frac{\partial u^{ev}}{\partial n}\tilde{\psi}\, ds - \int_{(\partial B_2(0))^-} \frac{\partial \tilde{\psi}}{\partial n} u^{ev}\, ds - u^{ev}(1,0)2^{-1/2} \\ &- \int_{-2}^{-1} \frac{\partial \tilde{\psi}}{\partial y} u^{ev}\, dx - \int_{1}^{2} \frac{\partial \tilde{\psi}}{\partial y} u^{ev}\, dx + \int_{B_2(0)^+} f^{ev}\tilde{\psi}\, dx,\end{aligned} \tag{5.15}$$

where the change in sign comes from the flipping of the direction of the outward normal to $B_2(0)^-$ along the horizontal segments. Adding (5.14) to (5.15) gives another formula for b_1:

$$\begin{aligned}b_1\pi = &\int_{\partial B_2(0)} \frac{\partial u^{ev}}{\partial n}\tilde{\psi}\, ds - \int_{\partial B_2(0)} \frac{\partial \tilde{\psi}}{\partial n} u^{ev}\, ds - u^{ev}(1,0)\sqrt{2} \\ &+ \int_{-2}^{-1} \left[\frac{\partial \tilde{\psi}}{\partial y}\right] u^{ev}\, dx + \int_{1}^{2} \left[\frac{\partial \tilde{\psi}}{\partial y}\right] u^{ev}\, dx + \int_{B_2(0)} f^{ev}\tilde{\psi}\, dx,\end{aligned}$$

where $[w] = w^+ - w^-$ refers to the jump of the function w across $y = 0$. Since

$$\left[\frac{\partial \tilde{\phi}}{\partial y}\right] = 0$$

along $\{(x,0) \mid -2 < x < -1\}$, the formula reduces to

$$\begin{aligned}b_1\pi = &\int_{\partial B_2(0)} \frac{\partial u^{ev}}{\partial n}\tilde{\psi}\, ds - \int_{\partial B_2(0)} \frac{\partial \tilde{\psi}}{\partial n} u^{ev}\, ds - u^{ev}(1,0)\sqrt{2} \\ &+ \int_{1}^{2} \left[\frac{\partial \tilde{\psi}}{\partial y}\right] u^{ev}\, dx + \int_{B_2(0)} f^{ev}\tilde{\psi}\, dx,\end{aligned}$$

which was the formula we set out to prove in Proposition 5.1.1.

Remark. Nothing in this derivation depended on the fact that the full domain was $B_2(0)^+$. Therefore, if we had a line segment that was not in the center of the domain, we could run the same argument. First, we would need to extend σ by straight line segments off each endpoint until they first intersect $\partial\Omega$. These extra segments would replace the integrals from -2 to -1 and 1 to 2. We would then get a formula of

$$\begin{aligned}b_1\pi = &\int_{\partial\Omega} \frac{\partial u^{ev}}{\partial n}\tilde{\psi}\, ds - \int_{\partial\Omega} \frac{\partial \tilde{\psi}}{\partial n} u^{ev}\, ds - u^{ev}(b)\sqrt{\ell} \\ &- \int_{\sigma_F} \left[\frac{\partial \tilde{\psi}}{\partial n}\right] u^{ev}\, dx + \int_{\Omega} f^{ev}\tilde{\psi}\, dx\end{aligned}$$

where σ_F denotes the line segment extending σ away from the endpoint b until it hits the boundary of Ω and ℓ is the length of σ.

5.2 Different types of conductivities

Up to this point, the conductivity a_ϵ inside the inhomogeneity ω_ϵ has been an isotropic constant throughout the region. However, most physical problems of this type, particularly those in cloaking, involve conductivities that are non-constant and/or anisotropic. In the next sections, we look at what happens to the energy functional and reduced problem when we drop these assumptions.

5.2.1 Non-constant conductivities

In the case of non-constant (but still isotropic) conductivities, we are looking to solve a problem of the form

$$\begin{cases} -\nabla \cdot (\gamma_\epsilon \nabla u) = f & \Omega \\ u = \varphi & \partial\Omega, \end{cases} \tag{5.16}$$

where

$$\gamma_\epsilon = \begin{cases} 1 & \Omega \setminus \omega_\epsilon \\ a_\epsilon(x) & \omega_\epsilon, \end{cases}$$

and ω_ϵ is a tubular neighborhood of a curve σ. For simplicity, we will assume that the curve σ is closed in order to ignore the singularities at the endpoints. The procedure for trying to find an appropriate reduced equation begins the same way, by looking at the energy formulation of the reduced problem and finding a first order approximation to it.

The energy formulation of (5.16) is minimizing

$$\frac{1}{2}\int_\Omega \gamma_\epsilon |\nabla u|^2 \, dx - \int_\Omega fu \, dx.$$

As with the work in previous chapters, we isolate the region ω_ϵ to write this functional as

$$\frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u|^2 \, dx + \frac{1}{2}\int_{\omega_\epsilon} a_\epsilon(x)|\nabla u|^2 \, dx - \int_\Omega fu \, dx.$$

We then apply a change of variables using the map H_ϵ to modify the integral over ω_ϵ to be

$$\frac{1}{2}\int_{\omega_\epsilon} a_\epsilon(x)|\nabla u|^2 \, dx = \frac{1}{2}\int_{\omega_1} a_\epsilon(H_\epsilon(y))|\nabla u|^2(H_\epsilon(y)) \det(\nabla H_\epsilon) \, dy.$$

Since the map H_ϵ satisfies

$$\nabla H_\epsilon = \begin{pmatrix} \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma} & 0 \\ 0 & \epsilon \end{pmatrix}$$

in the (τ, n) basis, we can define the functions

$$\hat{u} = u \circ H_\epsilon \qquad \hat{a}_\epsilon = a_\epsilon \circ H_\epsilon$$

and carry out the change of variables, giving that

$$\begin{aligned}
\frac{1}{2}\int_{\omega_\epsilon} a_\epsilon(x)|\nabla u|^2 \, dx &= \frac{1}{2}\int_{\omega_1} a_\epsilon(H_\epsilon(y))|\nabla u|^2(H_\epsilon(y)) \det(\nabla H_\epsilon) \, dy \\
&= \frac{1}{2}\int_{\omega_1} \hat{a}_\epsilon(y)|\nabla H_\epsilon^{-1}\nabla \hat{u}|^2 \det(\nabla H_\epsilon) \, dy \\
&= \frac{\epsilon}{2}\int_{\omega_1} \hat{a}_\epsilon(y)\frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\left(\frac{\partial \hat{u}}{\partial \tau}\right)^2 dy \\
&+ \frac{1}{2\epsilon}\int_{\omega_1} \hat{a}_\epsilon(y)\frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}\left(\frac{\partial \hat{u}}{\partial n}\right)^2 \, dy.
\end{aligned}$$

The first order approximation of each of the terms in the final line is given by

$$\frac{\epsilon}{2}\int_{\omega_1} \hat{a}_\epsilon(y)(1+\kappa d_\sigma)\left(\frac{\partial \hat{u}}{\partial \tau}\right)^2 dy + \frac{1}{2\epsilon}\int_{\omega_1} \hat{a}_\epsilon(y)\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial \hat{u}}{\partial n}\right)^2 \, dy$$

and we want to minimize this expression. Since these terms have different orders of magnitude, we try to find a minimizer of this functional by minimizing

$$\int_{\omega_1} \hat{a}_\epsilon(y)\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2 \, dy$$

over functions $v \in H^1(\omega_1)$. The Euler-Lagrange equations for this energy are that, for any $w \in H_0^1(\omega_1)$,

$$\int_{\omega_1} \hat{a}_\epsilon(y)\frac{1}{1+\kappa d_\sigma}\frac{\partial v}{\partial n}\frac{\partial w}{\partial n} \, dy = 0.$$

Applying the co-area formula means we can rewrite this condition as

$$\forall\, w \in H_0^1(\omega_1) \qquad \int_{-1}^{1}\int_\sigma \hat{a}_\epsilon(x+tn(x))\frac{\partial v}{\partial n}(x+tn(x))\frac{\partial w}{\partial n}(x+tn(x)) \, dx \, dt = 0$$

Choosing $w(x+tn(x)) = \varphi(x)\psi(t)$ for $\varphi \in C^\infty(\sigma)$ (or $C_0^\infty(\sigma)$ is σ is open) and $\psi \in C_0^\infty((-1,1))$, this expression becomes

$$\int_\sigma \varphi(x)\int_{-1}^{1} \hat{a}_\epsilon(x+tn(x))\frac{\partial v}{\partial n}(x+tn(x))\psi'(t) \, dt \, dx = 0$$

which implies that for each $x \in \sigma$,

$$\int_{-1}^{1} \hat{a}_\epsilon(x + tn(x)) \frac{\partial v}{\partial n}(x + tn(x))\psi'(t)\ dt = 0.$$

After integration by parts in t, this implies that

$$\frac{\partial}{\partial n}\left(\hat{a}_\epsilon \frac{\partial v}{\partial n}\right) = 0.$$

In the case where a_ϵ is a constant, this reduces to $\frac{\partial^2 \hat{u}}{\partial n^2} = 0$, i.e., that the map $t \to \hat{u}(x + tn(x))$ is affine for each $x \in \sigma$. In order to figure out what the potential v looks like, we need to solve the differential equation

$$\frac{d}{dt}\left(\hat{a}_\epsilon(x + tn(x))\frac{d}{dt}(v(x + tn(x)))\right) = 0$$

with the boundary conditions

$$v(x - n(x)) = u^-(x) \qquad v(x + n(v)) = u^+(x)$$

for each fixed $x \in \sigma$. The solution to this differential equation is the function

$$v(x + tn(x)) = \frac{1}{2}A_\epsilon(x)(u^+(x) - u^-(x)) \int_{-1}^{t} \frac{1}{\hat{a}_\epsilon(x + sn(x))}\ ds + u^-(x) \tag{5.17}$$

where

$$\begin{aligned} A_\epsilon(x) &= 2\left(\int_{-1}^{1} \frac{1}{\hat{a}_\epsilon(x + sn(x))}\ ds\right)^{-1} \\ &= \left(⨍_{-1}^{1} \frac{1}{\hat{a}_\epsilon(x + sn(x))}\ ds\right)^{-1} \\ &= \left(⨍_{-\epsilon}^{\epsilon} \frac{1}{a_\epsilon(x + sn(x))}\ ds\right)^{-1}. \end{aligned}$$

Using this formula, we see that

$$\frac{\partial}{\partial n}(v(x + tn(x)) = \frac{A_\epsilon(x)}{2\hat{a}_\epsilon(x + tn(x))}(u^+(x) - u^-(x))$$

so that the second term in the inner energy minimization becomes

$$
\begin{aligned}
&\frac{1}{2\epsilon}\int_{\omega_1} \hat{a}_\epsilon(y)\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\ dy \\
&\qquad = \frac{1}{2\epsilon}\int_\sigma\int_{-1}^{1} \hat{a}_\epsilon(x+tn(x))\left(\frac{\partial}{\partial n}(v(x+tn(x))\right)^2\ dt\ dx \\
&\qquad = \frac{1}{2\epsilon}\int_\sigma\int_{-1}^{1} \hat{a}_\epsilon(x+tn(x))\left(\frac{A_\epsilon(x)}{2\hat{a}_\epsilon(x+tn(x))}(u^+(x)-u^-(x))\right)^2\ dt\ dx \\
&\qquad = \frac{1}{2\epsilon}\int_\sigma\int_{-1}^{1} \hat{a}_\epsilon(x+tn(x))\frac{A_\epsilon(x)^2}{4\hat{a}_\epsilon(x+tn(x))^2}(u^+(x)-u^-(x))^2\ dt\ dx \\
&\qquad = \frac{1}{8\epsilon}\int_\sigma A_\epsilon(x)^2(u^+(x)-u^-(x))^2\int_{-1}^{1}\frac{1}{\hat{a}_\epsilon(x+tn(x))}\ dt\ dx \\
&\qquad = \frac{1}{4\epsilon}\int_\sigma A_\epsilon(x)(u^+(x)-u^-(x))^2\ dx,
\end{aligned}
$$

which looks a lot like the normal derivative term from the previous chapters, with the a_ϵ replaced by the $A_\epsilon(x)$ inside the integral.

If a_ϵ is constant, then $A_\epsilon(x) = a_\epsilon$ and the expression (5.17) reduces to the affine case discussed before. In order to get the full expression for the energy, we need to compute $\left(\frac{\partial}{\partial\tau}v(x+tn(x))\right)^2$, but the expression for the tangential derivative is much more complicated than the normal derivative. Also, there is reason to believe that the process from earlier in this document will not work for general $a_\epsilon(x)$, as we needed to use either an explicit formula for the solution or something with more than H^1 regularity in order to get uniformity of convergence. We can not guarantee more than H^1 regularity because of the possibility of extra $r^{1/2}$ singularities if A_ϵ changes wildly or is 1 on a subset of the inhomogeneity. This is why, in previous chapters, we needed to look at the corners, characterize their singularities, and use the explicit form to prove that things are still close in energy. Therefore, we look at a few specific types of $a_\epsilon(x)$ before considering the general situation. Overall, we conjecture that all of the previous results will go through in the case where a_ϵ is independent of x, but more conditions will be needed to ensure convergence when a_ϵ is allowed to vary with x. These conjectures will be stated in their respective sections.

$\hat{a}_\epsilon(x,t)$ independent of x

This situation could correspond to a physical problem with a radial inhomogeneity, where the conductivity profile is the same in every direction, but it can still be a function of distance from the curve σ. Going back to the function v on ω_1, we have

$$v(x+tn(x)) = \frac{1}{2}A_\epsilon(x)(u^+(x)-u^-(x))\int_{-1}^{t}\frac{1}{\hat{a}_\epsilon(x+sn(x))}\ ds + u^-(x)$$

where now, both A_ϵ and $\hat{a}_\epsilon$ are independent of x. This causes the expression to reduce to

$$v(x+tn(x)) = \frac{1}{2}A_\epsilon(u^+(x)-u^-(x))\int_{-1}^{t}\frac{1}{\hat{a}_\epsilon(s)}\ ds + u^-(x).$$

We can then compute that

$$\frac{\partial}{\partial\tau}v(x+tn(x)) = \left(\frac{\partial u^+}{\partial\tau}-\frac{\partial u^-}{\partial\tau}\right)\frac{1}{2}A_\epsilon\int_{-1}^{t}\frac{1}{\hat{a}_\epsilon(s)}\ ds + \frac{\partial u^-}{\partial\tau},$$

so that, if we define the function

$$\bar{a}(t) = \frac{1}{2}A_\epsilon\int_{-1}^{t}\frac{1}{\hat{a}_\epsilon(s)}\ ds,$$

then

$$\frac{\partial}{\partial\tau}(v(x+tn(x)) = \bar{a}(t)\frac{\partial u^+}{\partial\tau} + (1-\bar{a}(t))\frac{\partial u^-}{\partial\tau}.$$

Squaring this expression, we get that

$$\frac{\partial}{\partial\tau}(v(x+tn(x)))^2 = \bar{a}(t)^2\left(\frac{\partial u^+}{\partial\tau}\right)^2 + 2\bar{a}(t)(1-\bar{a}(t))\frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau} + (1-\bar{a}(t))^2\left(\frac{\partial u^-}{\partial\tau}\right)^2.$$

Now, in terms of the appropriate piece of the reduced energy, we have that

$$\begin{aligned}
\frac{\epsilon}{2}\int_{\omega_1} \hat{a}_\epsilon(y)(1+\kappa d_\sigma)\left(\frac{\partial v}{\partial \tau}\right)^2 dy &= \frac{\epsilon}{2}\int_\sigma \int_{-1}^1 \hat{a}_\epsilon(x+tn(x))(1+\kappa d_\sigma)^2\left(\frac{\partial v}{\partial \tau}\right)^2\ dt\ dx \\
&= \frac{\epsilon}{2}\int_\sigma \int_{-1}^1 \hat{a}_\epsilon(x+tn(x))\left(\frac{\partial}{\partial \tau}(v(x+tn(x))\right)^2\ dt\ dx \\
&= \frac{\epsilon}{2}\int_\sigma \int_{-1}^1 \hat{a}_\epsilon(t)\Bigg[\bar{a}(t)^2\left(\frac{\partial u^+}{\partial \tau}\right)^2 + 2\bar{a}(t)(1-\bar{a}(t))\frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau} \\
&+ (1-\bar{a}(t))^2\left(\frac{\partial u^-}{\partial \tau}\right)^2\Bigg]\ dt\ dx \\
&= \frac{\epsilon}{2}\Bigg[\int_{-1}^1 \hat{a}_\epsilon(t)\bar{a}(t)^2\ dt\int_\sigma\left(\frac{\partial u^+}{\partial \tau}\right)^2\ dx \\
&+ 2\int_{-1}^1 \hat{a}_\epsilon(t)\bar{a}(t)(1-\bar{a}(t))\ dt\int_\sigma \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau}\ dx \\
&+ \int_{-1}^1 \hat{a}_\epsilon(t)(1-\bar{a}(t))^2\ dt\int_\sigma\left(\frac{\partial u^-}{\partial \tau}\right)^2\ dx\Bigg].
\end{aligned}$$

In order to determine the coefficients of the derivative terms, we need to evaluate integrals of the form

$$\int_{-1}^1 \hat{a}_\epsilon(t)\bar{a}(t)^2\ dt$$

which, since

$$\bar{a}(t) = \frac{1}{2}A_\epsilon\int_{-1}^t \frac{1}{\hat{a}_\epsilon(s)}\ ds,$$

can be expanded to

$$\int_{-1}^1 \hat{a}_\epsilon(t)\bar{a}(t)^2\ dt = \frac{1}{4}A_\epsilon^2\int_{-1}^1 \hat{a}_\epsilon(t)\left(\int_{-1}^t \frac{1}{\hat{a}_\epsilon(s)}\ ds\right)^2\ dt,$$

with similar expressions for the other two terms involving

$$1-\bar{a}(t) = \frac{1}{2}A_\epsilon\int_t^1 \frac{1}{\hat{a}_\epsilon(s)}\ ds.$$

If the reduced equation here is supposed to be similar to the case where a_ϵ is constant like in [DV17], then this should be equal to some constant times A_ϵ. To improve the notation again, we define the following.

Definition 5.2.1. For a function $F \in C^{0,1}([a,b])$ with $F(a) = 0$ and $F' > 0$, we define the following functionals

$$\mathcal{H}_1[F] = \frac{1}{F(b)} \int_a^b \frac{F(t)^2}{F'(t)}\ dt,$$
$$\mathcal{H}_2[F] = \frac{1}{F(b)} \int_a^b \frac{F(t)(F(b)-F(t))}{F'(t)}\ dt,$$
$$\mathcal{H}_3[F] = \frac{1}{F(b)} \int_a^b \frac{(F(b)-F(t))^2}{F'(t)}\ dt,$$

where F' is defined almost everywhere. For a bounded function $f > 0$ on $[a,b]$, we define the corresponding impact factors of the function f by

$$\mathcal{I}_j[f] = \mathcal{H}_j \left[\int_a^t \frac{1}{f(s)}\ ds \right].$$

From these definitions, we see that, for the function $\hat{a}_\epsilon(t)$ defined on $(-1,1)$

$$\begin{aligned}
\mathcal{I}_1[\hat{a}_\epsilon(t)] &= \mathcal{H}_j \left[\int_{-1}^t \frac{1}{\hat{a}_\epsilon(s)}\ ds \right] \\
&= \frac{1}{\int_{-1}^1 \frac{1}{\hat{a}_\epsilon(s)}\ ds} \int_{-1}^1 \hat{a}_\epsilon(t) \left(\int_{-1}^t \frac{1}{\hat{a}_\epsilon(s)}\ ds \right)^2\ dt \\
&= \frac{A_\epsilon}{2} \int_{-1}^1 \hat{a}_\epsilon(t) \left(\int_{-1}^t \frac{1}{\hat{a}_\epsilon(s)}\ ds \right)^2\ dt
\end{aligned}$$

which is double what we want for the coefficient of the $\left(\frac{\partial u^+}{\partial \tau}\right)^2$ term. The same holds for the $\mathcal{I}_2$ and $\mathcal{I}_3$ terms. Thus, we see that, in this case, the tangential derivative term in the reduced energy looks like

$$\frac{\epsilon A_\epsilon}{4} \int_\sigma \mathcal{I}_1[\hat{a}_\epsilon] \left(\frac{\partial u^+}{\partial \tau}\right)^2 + 2\mathcal{I}_2[\hat{a}_\epsilon] \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau} + \mathcal{I}_3[\hat{a}_\epsilon] \left(\frac{\partial u^-}{\partial \tau}\right)^2\ dx$$

With this, the reduced energy takes the form

$$\begin{aligned}
E(u) = \frac{1}{2} \int_{\Omega\setminus\sigma} |\nabla u|^2\ dx - \int_\Omega fu\ dx + \frac{A_\epsilon}{4\epsilon} \int_\sigma (u^+(x) - u^-(x))^2\ dx \\
+ \frac{\epsilon A_\epsilon}{4} \int_\sigma \mathcal{I}_1[\hat{a}_\epsilon] \left(\frac{\partial u^+}{\partial \tau}\right)^2 + 2\mathcal{I}_2[\hat{a}_\epsilon] \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau} + \mathcal{I}_3[\hat{a}_\epsilon] \left(\frac{\partial u^-}{\partial \tau}\right)^2\ dx,
\end{aligned}$$

and from this, we can derive a classical form of the reduced problem as

$$\begin{cases}
-\Delta u = f & \Omega \\
u = \varphi & \partial\Omega \\
\frac{\partial u^+}{\partial n} + \frac{\epsilon A_\epsilon}{2} \left(\mathcal{I}_1[\hat{a}_\epsilon] \frac{\partial^2 u^+}{\partial \tau^2} + \mathcal{I}_2[\hat{a}_\epsilon] \frac{\partial^2 u^-}{\partial \tau^2} \right) - \frac{A_\epsilon}{2\epsilon}(u^+ - u^-) = 0 & \sigma \\
\frac{\partial u^-}{\partial n} - \frac{\epsilon A_\epsilon}{2} \left(\mathcal{I}_2[\hat{a}_\epsilon] \frac{\partial^2 u^+}{\partial \tau^2} + \mathcal{I}_3[\hat{a}_\epsilon] \frac{\partial^2 u^-}{\partial \tau^2} \right) - \frac{A_\epsilon}{2\epsilon}(u^+ - u^-) = 0 & \sigma.
\end{cases} \tag{5.18}$$

If a_ϵ is a constant, like in the previous paper, we see that $A_\epsilon = a_\epsilon$ and the impact factors can be computed as

$$\mathcal{I}_1[\hat{a}_\epsilon] = \frac{1}{2}\int_{-1}^{1} (t+1)^2 \; dt = \frac{4}{3}$$
$$\mathcal{I}_2[\hat{a}_\epsilon] = \frac{1}{2}\int_{-1}^{1} 1 - t^2 \; dt = \frac{2}{3}$$
$$\mathcal{I}_3[\hat{a}_\epsilon] = \frac{1}{2}\int_{-1}^{1} (t-1)^2 \; dt = \frac{4}{3}$$

and the classical formulation of the reduced problem that we get is

$$\begin{cases} -\Delta u = f & \Omega \\ u = \varphi & \partial\Omega \\ \frac{\partial u^+}{\partial n} + \frac{\epsilon A_\epsilon}{2}\left(\frac{4}{3}\frac{\partial^2 u^+}{\partial \tau^2} + \frac{2}{3}\frac{\partial^2 u^-}{\partial \tau^2}\right) - \frac{A_\epsilon}{2\epsilon}(u^+ - u^-) = 0 & \sigma \\ \frac{\partial u^-}{\partial n} - \frac{\epsilon A_\epsilon}{2}\left(\frac{2}{3}\frac{\partial^2 u^+}{\partial \tau^2} + \frac{4}{3}\frac{\partial^2 u^-}{\partial \tau^2}\right) - \frac{A_\epsilon}{2\epsilon}(u^+ - u^-) = 0 & \sigma, \end{cases}$$

which is identical to what was found in [DV17].

It is also fairly easy to see that if $a_\epsilon(t) = a_\epsilon(-t)$, then $\mathcal{I}_1[\hat{a}_\epsilon] = \mathcal{I}_3[\hat{a}_\epsilon]$, but this does not need to hold in general. While this does not affect closed curves, if this relation does not hold, then the even/odd symmetry argument used to analyze the open curve case will no longer be possible, because the terms will not match up and cancel. It may be possible to use some other sort of symmetry argument, but the standard one here will fail. The previous work on open curves should carry through in this case provided the conductivity is symmetric with respect to the curve σ.

Another fact to note about these problems is that the $\mathcal{I}$ coefficients may also vary with ϵ, so that the behavior of A_ϵ alone does not determine the behavior of these solutions. As will be illustrated in Chapter 6, it is the behavior of $A_\epsilon(\mathcal{I}_1 \pm \mathcal{I}_2)$ that determines how these solutions behave as $\epsilon \to 0$.

With these observations and numerical calculations, we conjecture that, provided the conductivity is symmetric with respect to σ, the results proved earlier in this paper should go through to the case where $\hat{a}_\epsilon$ is independent of x.

Conjecture. *Let $\hat{a}_\epsilon$ for $\epsilon > 0$ be any family of rescaled conductivities on $(-1, 1)$ satisfying*

1. $\hat{a}_\epsilon > 0$

2. $\hat{a}_\epsilon(t) = \hat{a}_\epsilon(-t)$ *for all* $t \in (0,1)$

3. $\hat{a}_\epsilon$ *and* $\frac{1}{\hat{a}_\epsilon} \in L^\infty(-1,1)$.

Then the solution u *to 5.18 is an asymptotically accurate approximation to the solution* u_ϵ *to 5.16 in the sense of Theorem 4.7.2.*

The main issue with proving this result will be the $\mathcal{I}$ coefficients and how their dependence on ϵ affects the convergence. It is possible that the fact that the normal derivative term is multiplied by $\frac{A_\epsilon}{\epsilon}$ and the tangential derivative term is multiplied by $\epsilon A_\epsilon \mathcal{I}_1$ could result in a situation similar to an anisotropic conductivity, which will be discussed separately in Section 5.2.2. However, since the only variance in how the terms differ is in the $\mathcal{I}$ coefficients, it should be possible to understand how these coefficients can behave and use that to produce similar results in the case where a_ϵ only depends on distance from the curve.

$\hat{a}_\epsilon(x,t)$ independent of t

The case where $\hat{a}_\epsilon(x,t)$ is independent of t could physically occur in the case of an inhomogeneity that is a blend of several different materials in the direction of the curve, but each cross-section of the inhomogeneity is uniform. Since $\hat{a}_\epsilon$ is independent of t, the average A_ϵ at each point $x \in \sigma$ is exactly $\hat{a}_\epsilon(x)$. Therefore, our function $v(x+tn(x))$ in (5.17) becomes

$$\begin{aligned} v(x+tn(x)) &= \frac{1}{2}\hat{a}_\epsilon(x)(u^+(x)-u^-(x))\int_{-1}^{t}\frac{1}{\hat{a}_\epsilon(x)}\,ds + u^-(x) \\ &= \frac{1}{2}(t+1)(u^+(x)-u^-(x)) + u^-(x) \\ &= \frac{1+t}{2}u^+(x) + \frac{1-t}{2}u^-(x), \end{aligned}$$

which is an affine function again, and identical to the function determined in [DV17]. Therefore, almost everything here works out the same as in the original work, and the same equation that was developed in previous chapters. The only difference here is that now $\hat{a}_\epsilon(x) = a_\epsilon(x)$ is a function of x, so it can not be pulled out of the integrals along σ.

With this formula for v, we know that the tangential derivative term will look like

$$\frac{1+t}{2}\frac{\partial u^+}{\partial \tau}(x) + \frac{1-t}{2}\frac{\partial u^-}{\partial \tau}(x)$$

so that after multiplying by $\hat{a}_\epsilon(x)$ and integrating in t (of which all of the functions are independent) we get an energy expression that looks like

$$\begin{aligned} E(u) = \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2 - \int_\Omega fu + \frac{1}{4\epsilon}\int_\sigma a_\epsilon(x)(u^+(x) - u^-(x))^2\, dx \\ + \frac{\epsilon}{3}\int_\sigma a_\epsilon(x)\left[\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau} + \left(\frac{\partial u^-}{\partial \tau}\right)^2\right]\, dx. \end{aligned}$$

This is identical to the reduced energy in [DV17], except for the fact that the a_ϵ terms are inside the integrals instead of outside them. Thus, we can also use the same process to get a classical formulation of the corresponding reduced problem. The variational approach to this energy minimization gives the classical formulation as

$$\begin{cases} -\Delta u = f & \Omega \\ u = \varphi & \partial\Omega \\ \frac{\partial u^+}{\partial n} + \frac{\epsilon}{3}\left(2\frac{\partial}{\partial \tau}\left(a_\epsilon(x)\frac{\partial u^+}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\left(a_\epsilon(x)\frac{\partial u^-}{\partial \tau}\right)\right) - \frac{a_\epsilon(x)}{2\epsilon}(u^+ - u^-) = 0 & \sigma \\ \frac{\partial u^-}{\partial n} - \frac{\epsilon}{3}\left(\frac{\partial}{\partial \tau}\left(a_\epsilon(x)\frac{\partial u^+}{\partial \tau}\right) + 2\frac{\partial}{\partial \tau}\left(a_\epsilon(x)\frac{\partial u^-}{\partial \tau}\right)\right) - \frac{a_\epsilon(x)}{2\epsilon}(u^+ - u^-) = 0 & \sigma. \end{cases} \tag{5.19}$$

While this may look simpler than the independent of x case, we do not expect our previous arguments to be able to prove asymptotic accuracy of this solution. The fact that a_ϵ can depend on x allows for additional $r^{1/2}$ singularities along the curve σ, which do not fit into the $C^{1,\beta/2}$ remainder term. Therefore, we would need a way to identify and handle these individually, figure out how to handle them as a group, or prevent them from occurring. Therefore, we are left with the following question.

Open Question. Under what conditions on $a_\epsilon(x)$ does our method still provide a proof of asymptotic accuracy? Beyond that, when, in general, is the solution to (5.19) asymptotically accurate and can our proof be modified to fit this new scenario?

The general case

From before, we know that the scaled function v on the interior of the inhomogeneity is given by

$$v(x+tn(x)) = \frac{1}{2}A_\epsilon(x)(u^+(x)-u^-(x))\int_{-1}^{t}\frac{1}{\hat{a}_\epsilon(x+\rho n(x))}\ d\rho + u^-(x).$$

From here forward, we will denote by $\alpha(x,t)$ the function

$$\alpha(x,t) := \int_{-1}^{t}\frac{1}{\hat{a}_\epsilon(x+\rho n(x))}\ d\rho \tag{5.20}$$

so that

$$\begin{aligned}\frac{\partial}{\partial\tau}(v(x+tn(x))) &= \frac{1}{2}\frac{\partial A_\epsilon}{\partial\tau}(x)[u](x)\alpha(x,t) + \frac{1}{2}A_\epsilon(x)\frac{\partial[u]}{\partial\tau}(x)\alpha(x,t)\\ &+ \frac{1}{2}A_\epsilon(x)[u](x)\frac{\partial\alpha}{\partial\tau}(x,t) + \frac{\partial u^-}{\partial\tau}(x)\end{aligned}$$

Following the process in [DV17], the remaining steps are to square this expression, multiply by $\hat{a}_\epsilon$, and integrate from -1 to 1 in t. Once this expression is squared out, the integrals that need to be computed only depend on $\hat{a}_\epsilon$ and α. Thus, we define

$$\mathcal{J}_{m,n}[\hat{a}_\epsilon](x) := \int_{-1}^{1}\hat{a}_\epsilon(x+tn(x))\alpha(x,t)^m\frac{\partial\alpha}{\partial\tau}(x,t)^n\ dt \tag{5.21}$$

which allows us to write

$$\begin{aligned}\int_{-1}^{1}\hat{a}_\epsilon\left(\frac{\partial}{\partial\tau}(v(x+tn(x)))\right)^2\ dt &= \left(\frac{1}{2}\frac{\partial A_\epsilon}{\partial\tau}[u] + \frac{1}{2}A_\epsilon\frac{\partial[u]}{\partial\tau}\right)^2\mathcal{J}_{2,0}[\hat{a}_\epsilon]\\ &+ \frac{1}{2}\left(\frac{\partial A_\epsilon}{\partial\tau}[u] + A_\epsilon\frac{\partial[u]}{\partial\tau}\right)A_\epsilon[u]\mathcal{J}_{1,1}[\hat{a}_\epsilon] + \left(\frac{\partial A_\epsilon}{\partial\tau}[u] + A_\epsilon\frac{\partial[u]}{\partial\tau}\right)\frac{\partial u^-}{\partial\tau}\mathcal{J}_{1,0}[\hat{a}_\epsilon]\\ &+ \frac{1}{4}A_\epsilon^2[u]^2\mathcal{J}_{0,2}[\hat{a}_\epsilon] + A_\epsilon[u]\frac{\partial u^-}{\partial\tau}\mathcal{J}_{0,1}[\hat{a}_\epsilon] + \left(\frac{\partial u^-}{\partial\tau}\right)^2\mathcal{J}_{0,0}[\hat{a}_\epsilon].\end{aligned}$$

By expanding and collecting terms, we can write this expression as

$$\begin{aligned}\int_{-1}^{1}\hat{a}_\epsilon\left(\frac{\partial}{\partial\tau}(v(x+tn(x)))\right)^2\ dt &= \frac{1}{2}\mathcal{M}_1[u]^2 + \mathcal{M}_2[u]\frac{\partial u^+}{\partial\tau} + \mathcal{M}_3[u]\frac{\partial u^-}{\partial\tau}\\ &+ \frac{1}{2}\mathcal{K}_1\left(\frac{\partial u^+}{\partial\tau}\right)^2 + \mathcal{K}_2\frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau} + \frac{1}{2}\mathcal{K}_3\left(\frac{\partial u^-}{\partial\tau}\right)^2\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_1 &= \frac{1}{2}\left(\frac{\partial A_\epsilon}{\partial \tau}\right)^2 \mathcal{J}_{2,0}[\hat{a}_\epsilon] + \frac{\partial A_\epsilon}{\partial \tau} A_\epsilon \mathcal{J}_{1,1}[\hat{a}_\epsilon] + \frac{1}{2} A_\epsilon^2 \mathcal{J}_{0,2}[\hat{a}_\epsilon] \\
\mathcal{M}_2 &= \frac{1}{2}\frac{\partial A_\epsilon}{\partial \tau} A_\epsilon \mathcal{J}_{2,0}[\hat{a}_\epsilon] + \frac{1}{2} A_\epsilon^2 \mathcal{J}_{1,1}[\hat{a}_\epsilon] \\
\mathcal{M}_3 &= -\frac{1}{2}\frac{\partial A_\epsilon}{\partial \tau} A_\epsilon \mathcal{J}_{2,0}[\hat{a}_\epsilon] - \frac{1}{2} A_\epsilon^2 \mathcal{J}_{1,1}[\hat{a}_\epsilon] + \frac{\partial A_\epsilon}{\partial \tau} \mathcal{J}_{1,0}[\hat{a}_\epsilon] + A_\epsilon \mathcal{J}_{0,1}[\hat{a}_\epsilon] \\
\mathcal{K}_1 &= \frac{1}{2} A_\epsilon^2 \mathcal{J}_{2,0}[\hat{a}_\epsilon] \\
\mathcal{K}_2 &= -\frac{1}{2} A_\epsilon^2 \mathcal{J}_{2,0}[\hat{a}_\epsilon] + A_\epsilon \mathcal{J}_{1,0}[\hat{a}_\epsilon] \\
\mathcal{K}_3 &= \frac{1}{2} A_\epsilon^2 \mathcal{J}_{2,0}[\hat{a}_\epsilon] - 2A_\epsilon \mathcal{J}_{1,0}[\hat{a}_\epsilon] + 2\mathcal{J}_{0,0}[\hat{a}_\epsilon]
\end{aligned}$$

and we have suppressed the fact that all of these are functions of x. With this term, we can now go back to the full energy expression for this problem to get that our approximate energy is given by

$$\begin{aligned}
E_\epsilon^0(u) = \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2 \, dx + \frac{\epsilon}{2}\int_\sigma \frac{1}{2}\mathcal{M}_1[u]^2 + \mathcal{M}_2[u]\frac{\partial u^+}{\partial \tau} + \mathcal{M}_3[u]\frac{\partial u^-}{\partial \tau} \\
+ \frac{1}{2}\mathcal{K}_1\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \mathcal{K}_2 \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau} + \frac{1}{2}\mathcal{K}_3\left(\frac{\partial u^-}{\partial \tau}\right)^2 \, ds \\
+ \frac{1}{4\epsilon}\int_\sigma A_\epsilon[u]^2 \, ds - \int_\Omega f u \, dx.
\end{aligned}$$

Finally, we would like to compute the classical differential equation that this energy minimizer solves. By the same tactics used before, $-\Delta u = f$ in Ω, and $u = \varphi$ on the boundary. For the boundary condition on σ, we transform this into a variational formulation and then integrate by parts. After doing so, we get that the boundary conditions on σ are

$$\begin{aligned}
&\frac{\partial u^+}{\partial n} + \frac{\epsilon}{2}\left[\frac{\partial}{\partial \tau}\left(\mathcal{K}_1\frac{\partial u^+}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\left(\mathcal{K}_2\frac{\partial u^-}{\partial \tau}\right) - \mathcal{M}_1[u]\right. \\
&\qquad \left. + \frac{\partial}{\partial \tau}\left(\mathcal{M}_2[u]\right) - \mathcal{M}_2\frac{\partial u^+}{\partial \tau} - \mathcal{M}_3\frac{\partial u^-}{\partial \tau}\right] - \frac{A_\epsilon}{2\epsilon}[u] = 0 \\
&\frac{\partial u^-}{\partial n} - \frac{\epsilon}{2}\left[\frac{\partial}{\partial \tau}\left(\mathcal{K}_2\frac{\partial u^+}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\left(\mathcal{K}_3\frac{\partial u^-}{\partial \tau}\right) + \mathcal{M}_1[u]\right. \\
&\qquad \left. + \mathcal{M}_2\frac{\partial u^+}{\partial \tau} + \frac{\partial}{\partial \tau}\left(\mathcal{M}_3[u]\right) + \mathcal{M}_3\frac{\partial u^-}{\partial \tau}\right] - \frac{A_\epsilon}{2\epsilon}[u] = 0.
\end{aligned} \tag{5.22}$$

This gives the classical equation that u_ϵ^0 should satisfy in order to approximate u_ϵ. In the case that a_ϵ is either constant, or independent of x, then this equation reduces to the earlier case. The $\mathcal{K}$ coefficients correspond to the impact factors defined earlier, and all of the $\mathcal{M}$ coefficients are zero because all of the terms in these expressions involve a tangential derivative of either a_ϵ or A_ϵ.

Variable Width Inhomogeneities

Another specific situation where we can try to apply this formulation is the case of variable-width inhomogeneities. For a function $h \in C^2(\sigma)$ with $0 < h \leq 1$, we define the variable width inhomogeneity by

$$\omega_\epsilon^h = \{x \in \omega_\epsilon \mid d(x,\sigma) < \epsilon h(p_\sigma(x))\}$$

where ω_ϵ is defined as before and p_σ is the projection onto σ, which is well-defined on ω_ϵ. For these problems, we will go back to the assumption that the conductivity inside the inhomogeneity is a constant a_ϵ, and so we can define a non-constant conductivity in the uniform domain ω_1 by

$$\hat{a}_\epsilon(x + tn(x)) = \begin{cases} a_\epsilon & |t| \leq h(x) \\ 1 & |t| > h(x) \end{cases}$$

which will correspond to the variable-width inhomogeneity ω_ϵ^h having conductivity a_ϵ.

In order to figure out the corresponding classical differential equation that the reduced solution u_ϵ^0 should satisfy, we need to find the $\mathcal{M}$ and $\mathcal{K}$ coefficient functions. To do this, we need to compute A_ϵ and all of the $\mathcal{J}$ functions. For the $\hat{a}_\epsilon$ given above, we have that

$$A_\epsilon(x) = \frac{1}{1 + h(x)\left(\frac{1}{a_\epsilon} - 1\right)}, \qquad \frac{\partial}{\partial \tau} A_\epsilon = \left(1 - \frac{1}{a_\epsilon}\right) \frac{h'(x)}{\left(1 + h(x)\left(\frac{1}{a_\epsilon} - 1\right)\right)^2},$$

and

$$\alpha(x,t) = \begin{cases} t + 1 & t < -h(x) \\ \frac{t}{a_\epsilon} + 1 + h(x)\left(\frac{1}{a_\epsilon} - 1\right) & -h(x) < t < h(x) \\ t + 1 + 2h(x)\left(\frac{1}{a_\epsilon} - 1\right) & t > h(x). \end{cases}$$

This then allows us to compute the $\mathcal{J}$ coefficients as

$$\begin{aligned}
\mathcal{J}_{2,0}[\hat{a}_\epsilon](x) &= \frac{1}{3}\Bigg[\left(1-a_\epsilon^2\right)(1-h(x))^3 + \left(a_\epsilon^2-1\right)\left(1+h(x)\left(\frac{2}{a_\epsilon}-1\right)\right)^3 \\
&\quad + 8\left(1+h(x)\left(\frac{1}{a_\epsilon}-1\right)\right)^3 \Bigg] \\
\mathcal{J}_{1,1}[\hat{a}_\epsilon](x) &= h'(x)\left(\frac{1}{a_\epsilon}-1\right)\Bigg[\left(\frac{a_\epsilon^2}{2}-1\right)\left(1+h(x)\left(\frac{2}{a_\epsilon}-1\right)\right)^2 \\
&\quad + 4\left(1+h(x)\left(\frac{1}{a_\epsilon}-1\right)\right)^2 - \frac{a_\epsilon^2}{2}(1+h(x))^2\Bigg] \\
\mathcal{J}_{0,2}[\hat{a}_\epsilon](x) &= 2h'(x)^2\left(\frac{1}{a_\epsilon}-1\right)^2(1-h(x)(1-a_\epsilon)) \\
\mathcal{J}_{1,0}[\hat{a}_\epsilon](x) &= \frac{1}{2}\Bigg[\left(1-a_\epsilon^2\right)(1-h(x))^2 + \left(a_\epsilon^2-1\right)\left(1+h(x)\left(\frac{2}{a_\epsilon}-1\right)\right)^2 \\
&\quad + 4\left(1+h(x)\left(\frac{1}{a_\epsilon}-1\right)\right)^2 \Bigg] \\
\mathcal{J}_{0,1}[\hat{a}_\epsilon](x) &= 2h'(x)\left(\frac{1}{a_\epsilon}-1\right)(1-h(x)(1-a_\epsilon)) \\
\mathcal{J}_{0,0}[\hat{a}_\epsilon](x) &= 2+2h(x)(a_\epsilon-1).
\end{aligned}$$

Plugging these in to get the $\mathcal{M}$ and $\mathcal{K}$ coefficients will provide the energy and classical expression for the reduced problem.

However, there is another way to think about these inhomogeneities that leads to an easier formulation of the problem and a fairly straight-forward way to prove a convergence result similar to what was discussed in earlier chapters. The main change in this process is that instead of separating out the domain ω_ϵ, rescaling it to ω_1, and solving the problem with that inclusion, we instead rescale the domain ω_ϵ^h to ω_1^h in the same manner, and solve this problem. The end result is that instead of specifying boundary data u^+ and u^- on the top and bottom boundaries of ω_1, we specify these conditions on the corresponding curves in $\partial\omega_1^h$.

The full problem in this situation can be written as

$$\begin{cases} -\nabla\cdot(\gamma_\epsilon\nabla u_\epsilon) = f & \Omega \\ u_\epsilon = \varphi & \partial\Omega \end{cases} \qquad \gamma_\epsilon = \begin{cases} 1 & \Omega\setminus\omega_\epsilon^h \\ a_\epsilon & \omega_\epsilon^h \end{cases} \tag{5.23}$$

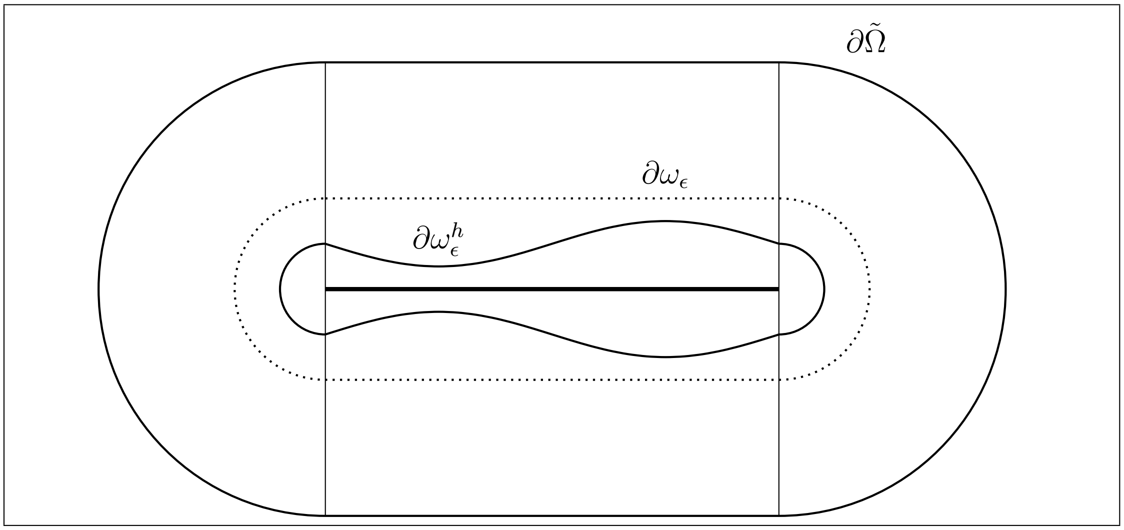

Figure 5.1: Relation between ω_ϵ and ω_ϵ^h for variable-width inhomogeneities

which has corresponding energy

$$E_\epsilon^h(u) = \frac{1}{2}\int_\Omega \gamma_\epsilon |\nabla u|^2 \, dx - \int_\Omega fu \, dx = \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon^h} |\nabla u|^2 \, dx + \frac{1}{2}\int_{\omega_\epsilon^h} a_\epsilon |\nabla u|^2 \, dx - \int_\Omega fu \, dx.$$

The process goes almost identically to the work in Chapter 2. We begin by separating the energy integral into two parts based on the conductivity profile and rescale the inhomogeneity to size 1. Taking the first order approximation of each term in the resulting expression gives an approximate energy minimization of the form

$$\min_{\substack{u\in H^1(\Omega\setminus\omega_\epsilon)\\ u|_{\partial\Omega}=\varphi}} \frac{1}{2}\int_{\Omega\setminus\overline{\sigma}} |\nabla u|^2 \, dx - \int_\Omega fu \, dx + G_\epsilon^{0,h}(u)$$

where

$$G_\epsilon^{0,h}(u) = \min_v \frac{\epsilon a_\epsilon}{2}\int_{\omega_1^h} (1+\kappa d_\sigma)\left(\frac{\partial v}{\partial \tau}\right)^2 \, dx + \frac{a_\epsilon}{2\epsilon}\int_{\omega_1^h} \frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2 \, dx, \tag{5.24}$$

where

$$v \in \{H^1(\omega_1^h) \mid v(x+h(x)n(x)) = u^+(x) \;\; v(x-h(x)n(x)) = u^-(x)\}.$$

Since a_ϵ is still isotropic, we can assume that minimizing the term of the form

$$\frac{1}{2}\int_{\omega_1^h} \frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2 \, dx$$

will given an approximate solution to the full minimization. As done previously, this gives that the function $v(x,t)$ needs to be affine in the normal direction. Since we now want to apply the boundary conditions u^+ at $t = h(x)$ and u^- at $t = -h(x)$ (as opposed to $t = \pm 1$ in previous chapters), the function v is then given by

$$v(x,t) = \frac{1}{2}\left(1 + \frac{t}{h(x)}\right) u^+(x) + \frac{1}{2}\left(1 - \frac{t}{h(x)}\right) u^-(x).$$

With this formula,

$$\frac{\partial v}{\partial t} = \frac{u^+(x) - u^-(x)}{2h(x)}$$

so that after applying the co-area formula,

$$\int_{-h(x)}^{h(x)} \frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2 \, dt = \frac{1}{2h(x)}(u^+(x) - u^-(x))^2$$

and the corresponding term in the energy is

$$\frac{a_\epsilon}{2\epsilon}\int_{\omega_1^h} \frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2 \, dx = \frac{a_\epsilon}{4\epsilon}\int_\sigma \frac{1}{h(x)}(u^+(x) - u^-(x))^2 \, ds. \tag{5.25}$$

Now, we need to look at the tangential derivative term. For this, we compute that

$$\frac{\partial v}{\partial x} = -\frac{1}{2}\frac{th'(x)}{h(x)^2}u^+(x) + \frac{1}{2}\left(1 + \frac{t}{h(x)}\right)\frac{\partial u^+}{\partial \tau}(x) + \frac{1}{2}\frac{th'(x)}{h(x)^2}u^-(x) + \frac{1}{2}\left(1 - \frac{t}{h(x)}\right)\frac{\partial u^-}{\partial \tau}(x)$$

which we can rewrite as

$$\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial \bar{u}}{\partial \tau} + \frac{t}{2h}\frac{\partial [u]}{\partial \tau} - \frac{th'}{2h^2}[u] = \frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \tau} + \frac{t}{h}\frac{\partial [u]}{\partial \tau} - \frac{th'}{h^2}[u]\right)$$

where we have used the notation $\bar{u} = u^+ + u^-$ and $[u] = u^+ - u^-$.

Now, we need to square this expression and integrate in t from $-h(x)$ to $h(x)$. When we do this integration, any term that is odd in t will vanish. Thus, we ignore these terms from the start, leaving only one cross term in the expansion. The terms that remain in $\left(\frac{\partial v}{\partial x}\right)^2$ are

$$\frac{1}{4}\left[\left(\frac{\partial \bar{u}}{\partial \tau}\right)^2 + \frac{t^2}{h^2}\left(\frac{\partial [u]}{\partial \tau}\right)^2 + \frac{t^2 h'^2}{h^4}[u]^2 - \frac{2t^2 h'}{h^3}\frac{\partial [u]}{\partial \tau}[u]\right]$$

which can be integrated in t to give

$$\begin{aligned}
\int_{-h(x)}^{h(x)} \left(\frac{\partial v}{\partial x}\right)^2 \, dt &= \frac{h}{2}\left(\frac{\partial \bar{u}}{\partial \tau}\right)^2 + \frac{h}{6}\left(\frac{\partial [u]}{\partial \tau}\right)^2 + \frac{h'^2}{6h}[u]^2 - \frac{h'}{3}\frac{\partial [u]}{\partial \tau}[u] \\
&= \frac{h}{2}\left(\frac{\partial \bar{u}}{\partial \tau}\right)^2 + \frac{1}{6}\left(\sqrt{h}\frac{\partial [u]}{\partial \tau} - \frac{h'}{\sqrt{h}}[u]\right)^2 \\
&= \frac{2h}{3}\left[\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 + \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau}\right] + \frac{h'^2}{6h}[u]^2 - \frac{h'}{3}\frac{\partial [u]}{\partial \tau}[u]
\end{aligned}$$

where the expression in the second line is given to emphasize that this is a positive quantity. This then gives the energy of the corresponding term as, after applying the co-area formula again,

$$\begin{aligned} \frac{\epsilon a_\epsilon}{2}\int_{\omega_1^h}(1+\kappa d_\sigma)\left(\frac{\partial v}{\partial\tau}\right)^2 dx &= \frac{\epsilon a_\epsilon}{12}\int_\sigma \frac{h'(x)^2}{h(x)}[u]^2\, ds - \frac{\epsilon a_\epsilon}{6}\int_\sigma h'(x)\frac{\partial[u]}{\partial\tau}[u]\, ds \\ &+ \frac{\epsilon a_\epsilon}{3}\int_\sigma h(x)\left[\left(\frac{\partial u^+}{\partial\tau}\right)^2+\left(\frac{\partial u^-}{\partial\tau}\right)^2+\frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau}\right] ds. \end{aligned} \tag{5.26}$$

Inserting (5.26) and (5.25) into (5.24) gives the approximate minimizing energy (playing the same role as E_ϵ^0) as

$$\begin{aligned} E^h(u) = &\frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2 - \int_\Omega fu + \frac{\epsilon a_\epsilon}{3}\int_\sigma h(x)\left[\left(\frac{\partial u^+}{\partial\tau}\right)^2+\left(\frac{\partial u^-}{\partial\tau}\right)^2+\frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau}\right] ds \\ &+ \frac{\epsilon a_\epsilon}{12}\int_\sigma \frac{h'(x)^2}{h(x)}[u]^2\, ds - \frac{\epsilon a_\epsilon}{6}\int_\sigma h'(x)\frac{\partial[u]}{\partial\tau}[u]\, ds \\ &+ \frac{a_\epsilon}{4\epsilon}\int_\sigma \frac{1}{h(x)}(u^+(x)-u^-(x))^2\, ds \end{aligned}$$

Note that if $h(x)\equiv 1$, then this reduces to exactly the formula proved previously for E_ϵ^0. In what follows, we will change this to an energy similar to what was used in the earlier work to perform the regularity analysis. That is, we will take the approximate energy to be

$$\begin{aligned} E_\epsilon^{0,h}(u) = &\frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2 - \int_\Omega fu + \frac{\epsilon a_\epsilon}{2}\int_\sigma h(x)\left[\left(\frac{\partial u^+}{\partial\tau}\right)^2+\left(\frac{\partial u^-}{\partial\tau}\right)^2\right] ds \\ &+ \frac{\epsilon a_\epsilon}{12}\int_\sigma \frac{h'(x)^2}{h(x)}[u]^2\, ds - \frac{\epsilon a_\epsilon}{6}\int_\sigma h'(x)\frac{\partial[u]}{\partial\tau}[u]\, ds \\ &+ \frac{a_\epsilon}{4\epsilon}\int_\sigma \frac{1}{h(x)}(u^+(x)-u^-(x))^2\, ds. \end{aligned} \tag{5.27}$$

Classical Formulation of the Reduced Problem

In order to find the classical formulation of the reduced equation corresponding to the variable-width inhomogeneity, we again compute the first variation of $E_\epsilon^{0,h}$. This tells us that

$$\begin{aligned} 0 = &\int_{\Omega\setminus\sigma}\nabla u\cdot\nabla v\, dx - \int_\Omega fv\, dx + \epsilon a_\epsilon\int_\sigma h(x)\left[\frac{\partial u^+}{\partial\tau}\frac{\partial v^+}{\partial\tau}+\frac{\partial u^-}{\partial\tau}\frac{\partial v^-}{\partial\tau}\right] ds \\ &+ \frac{\epsilon a_\epsilon}{6}\int_\sigma \frac{h'(x)^2}{h(x)}[u][v]\, ds - \frac{\epsilon a_\epsilon}{6}\int_\sigma h'(x)\left(\frac{\partial[v]}{\partial\tau}[u]+\frac{\partial[u]}{\partial\tau}[v]\right) ds + \frac{a_\epsilon}{2\epsilon}\int_\sigma \frac{1}{h(x)}[u][v]\, ds. \end{aligned}$$

for any $v\in H_0^1(\Omega)$ with $v^\pm\in H_0^1(\sigma)$.

If v is supported away from σ, we get that $-\Delta u = f$ in $\Omega \setminus \sigma$. Using this fact and integrating by parts on the first term, we see that

$$
\begin{aligned}
0 = & \int_\sigma -\frac{\partial u^+}{\partial n} v^+ + \frac{\partial u^-}{\partial n} v^- \; ds + \epsilon a_\epsilon \int_\sigma h(x) \left[\frac{\partial u^+}{\partial \tau}\frac{\partial v^+}{\partial \tau} + \frac{\partial u^-}{\partial \tau}\frac{\partial v^-}{\partial \tau} \right] \; ds \\
& + \frac{\epsilon a_\epsilon}{6} \int_\sigma \frac{h'(x)^2}{h(x)} [u][v] \; ds - \frac{\epsilon a_\epsilon}{6} \int_\sigma h'(x) \left(\frac{\partial [v]}{\partial \tau}[u] + \frac{\partial [u]}{\partial \tau}[v] \right) \; ds + \frac{a_\epsilon}{2\epsilon} \int_\sigma \frac{1}{h(x)} [u][v] \; ds.
\end{aligned}
$$

To get to a classical formulation of the boundary conditions on σ, we need to integrate by parts on all the terms that involve tangential derivatives of v. This turns the above formulation into

$$
\begin{aligned}
0 = & \int_\sigma -\frac{\partial u^+}{\partial n} v^+ + \frac{\partial u^-}{\partial n} v^- \; ds - \epsilon a_\epsilon \int_\sigma \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u^+}{\partial \tau} \right) v^+ + \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u^-}{\partial \tau} \right) v^- \; ds \\
& + \frac{\epsilon a_\epsilon}{6} \int_\sigma \frac{h'(x)^2}{h(x)} [u][v] \; ds - \frac{\epsilon a_\epsilon}{6} \int_\sigma \left(-\frac{\partial}{\partial \tau} \left(h'(x)[u] \right) + h'(x) \frac{\partial [u]}{\partial \tau} \right) [v] \; ds \\
& + \frac{a_\epsilon}{2\epsilon} \int_\sigma \frac{1}{h(x)} [u][v] \; ds.
\end{aligned}
$$

or, by simplifying the product term in the second line,

$$
\begin{aligned}
0 = & \int_\sigma -\frac{\partial u^+}{\partial n} v^+ + \frac{\partial u^-}{\partial n} v^- \; ds - \epsilon a_\epsilon \int_\sigma \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u^+}{\partial \tau} \right) v^+ + \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u^-}{\partial \tau} \right) v^- \; ds \\
& + \frac{\epsilon a_\epsilon}{6} \int_\sigma \frac{h'(x)^2}{h(x)} [u][v] \; ds + \frac{\epsilon a_\epsilon}{6} \int_\sigma h''(x)[u][v] \; ds + \frac{a_\epsilon}{2\epsilon} \int_\sigma \frac{1}{h(x)} [u][v] \; ds.
\end{aligned}
$$

Therefore, by choosing $v^+ \in H_0^1(\sigma)$ arbitrary and $v^- = 0$, we get that

$$
\frac{\partial u^+}{\partial n} + \epsilon a_\epsilon \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u^+}{\partial \tau} \right) - \frac{a_\epsilon}{2\epsilon h(x)} \left(1 + \frac{\epsilon^2}{3} h'(x)^2 + \frac{\epsilon^2}{3} h''(x) h(x) \right) [u] = 0
$$

and by reversing the roles of v^+ and v^-,

$$
\frac{\partial u^-}{\partial n} - \epsilon a_\epsilon \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u^-}{\partial \tau} \right) - \frac{a_\epsilon}{2\epsilon h(x)} \left(1 + \frac{\epsilon^2}{3} h'(x)^2 + \frac{\epsilon^2}{3} h''(x) h(x) \right) [u] = 0.
$$

This gives the classical formulation of the reduced problem as

$$
\begin{cases}
-\Delta u_\epsilon^0 = f & \Omega \setminus \overline{\sigma} \\
u_\epsilon^0 = \varphi & \partial\Omega \\
\frac{\partial u_\epsilon^{0,+}}{\partial n} + \epsilon a_\epsilon \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u_\epsilon^{0,+}}{\partial \tau} \right) - \frac{a_\epsilon}{2\epsilon h(x)} \left(1 + \frac{\epsilon^2}{3} h'(x)^2 + \frac{\epsilon^2}{3} h''(x) h(x) \right) [u_\epsilon^0] = 0 & \sigma \\
\frac{\partial u_\epsilon^{0,-}}{\partial n} - \epsilon a_\epsilon \frac{\partial}{\partial \tau} \left(h(x) \frac{\partial u_\epsilon^{0,-}}{\partial \tau} \right) - \frac{a_\epsilon}{2\epsilon h(x)} \left(1 + \frac{\epsilon^2}{3} h'(x)^2 + \frac{\epsilon^2}{3} h''(x) h(x) \right) [u_\epsilon^0] = 0 & \sigma.
\end{cases}
\tag{5.28}
$$

Unlike the formulation of the variable-width inhomogeneity that rescaled ω_ϵ as part of the derivation, this formulation can be split into even and odd parts, each of which solves a differential equation that only involves the corresponding function, that is, the differential equation for the even part only depends on the even part, and similarly for the odd part. Therefore, we can analyze the even and odd parts of this solution separately in order to prove convergence. Thus, if we assume that σ is a straight line segment, this allows us to make full use of the arguments constructed in Chapter 4 to prove convergence. For the case of a variable-width inhomogeneity, the even part of the solution to the reduced problem will satisfy the boundary condition

$$\frac{\partial u^{ev}}{\partial y} + \epsilon a_\epsilon \frac{\partial}{\partial x}\left(h(x)\frac{\partial u^{ev}}{\partial x}\right) = 0$$

on σ, while the odd part will satisfy

$$\frac{\partial u^{odd}}{\partial y} + \epsilon a_\epsilon \frac{\partial}{\partial x}\left(h(x)\frac{\partial u^{odd}}{\partial x}\right) - \frac{a_\epsilon}{\epsilon h(x)}\left(1 + \frac{\epsilon^2}{3}h'(x)^2 + \frac{\epsilon^2}{3}h''(x)h(x)\right)u^{odd} = 0$$

on σ.

Convergence Arguments

The convergence arguments used earlier in this work had two main steps,

1. Regularity estimates on the even problem and a simplified version of the odd problem, and

2. Energy closeness, calculated via a primal and dual energy formulation,

and then concluded by combining the results from different types of symmetry and asymptotic regimes of a_ϵ in order to get full convergence. As most of the work to solve this problem will mirror what was done previously, this section will only indicate how the work would change to fit this new problem, along with some model calculations where appropriate. For simplicity, we will assume that σ is the line segment from $(-1, 0)$ to $(1, 0)$, and that $h(x)$ is a $C^2(\sigma)$ function with $0 < c < h(x) \leq 1$ everywhere. This regularity assumption on h is a potential area for future study.

Regularity Results

In modifying the work of Chapter 3, the problems that we need to analyze for regularity concerns are

$$\begin{cases} -\Delta u = 0 & \mathbb{R}^2_+ \\ \frac{\partial u}{\partial y} + \alpha \frac{\partial}{\partial x}\left(\tilde{h}(x)\frac{\partial u}{\partial x}\right) = 0 & \{x > 0, y = 0\} \\ \frac{\partial u}{\partial y} = 0 & \{x < 0, y = 0\} \end{cases} \tag{5.29}$$

for the even case and

$$\begin{cases} -\Delta u = 0 & \mathbb{R}^2_+ \\ \frac{\partial u}{\partial y} = \frac{\lambda}{\tilde{h}(x)} u & \{x > 0,\ y = 0\} \\ u = 0 & \{x < 0,\ y = 0\} \end{cases} \tag{5.30}$$

for the odd case, where $\tilde{h}$ is an appropriately translated version of h. These can be compared to the reduced problems (3.6) and (3.11) in Chapter 3. Carrying out all of the same work as previously leads to one needing to analyze the regularity properties of the map $\mathcal{B}$, defined by

$$\mathcal{B}f(x) = \frac{|x|}{\tilde{h}(x^2)} f(x).$$

Previously, we needed to analyze the properties of the map $\mathcal{A}$ given by

$$\mathcal{A}f(x) = |x| f(x),$$

and these regularity properties provided the crucial steps in getting an explicit form for the most singular part of the solutions to the given differential equations. $\mathcal{B}$ plays the exact same role this time around, so proving similar properties about $\mathcal{B}$ will allow us to get the same characterization of solutions to this differential equation. If $h(x) > c > 0$ on σ and h is at least C^2, then $\tilde{h}$ also satisfies these conditions, and all of these regularity considerations can be verified. Thus, under these assumptions, we will know that the most singular part of the solution is an $r^{1/2}$ term with the same bounds on the error that were proved in Chapter 3.

Energy Arguments

While the energies involved are different, the process of getting to energy closeness is the same as in Chapter 4. Since $\omega_\epsilon^h \subset \omega_\epsilon$, the constructions of z_ϵ and ξ_ϵ can both still be done on $\Omega \setminus \omega_\epsilon$, ignoring the fact that the actual inhomogeneity is smaller. This will allow us to use the same work as before, along with the same tedious constructions and error bounds, to show that the energies of these two formulations are close in the even case with $\epsilon a_\epsilon > m > 0$. If $\epsilon a_\epsilon \to 0$, then the same arguments as before will show that both solutions will be close to the u_0 solution.

For the odd case, there are a few other issues that need to be dealt with. The first is the need to show that the solution to the reduced problem is energy close to the solution to (5.30) with $\lambda = \frac{a_\epsilon}{\epsilon}$ when this term is bounded. The calculations here go very similarly to those in Chapter 4. The only extra step we need is to use the fact that

$$\int_\sigma |u^{odd}|^2 \, ds \leq C(f, \varphi)$$

to be able to throw out the terms of order ϵa_ϵ that also depend on $|u^{odd}|^2$ in the energy calculation. This, combined with the fact that $h \in C^2(\sigma)$, will allow the same arguments to push through. Secondly, the duality argument needs to be addressed. In Chapter 4, this argument was used to relate the simplified problem with odd symmetry to a problem with even symmetry, which we already know converges in the appropriate manner. Using the same steps as before, but with our new boundary conditions, we can compute that

$$\begin{aligned} -\int_\sigma \frac{\partial}{\partial x}\left(h(x)\frac{\partial U_\epsilon}{\partial x}\right) z \, ds &= \int_\sigma h(x)\frac{\partial U_\epsilon}{\partial x}\frac{\partial z}{\partial x} \, ds \\ &= \int_\sigma h(x)\frac{\partial u^{odd'}}{\partial y}\frac{\partial z}{\partial x} \, ds \\ &= \frac{a_\epsilon}{\epsilon}\int_\sigma u^{odd'}\frac{\partial z}{\partial x} \, ds \\ &= -\frac{a_\epsilon}{\epsilon}\int_\sigma \frac{\partial u^{odd'}}{\partial x}\frac{\partial z}{\partial x} \, ds \\ &= \frac{a_\epsilon}{\epsilon}\int_\sigma \frac{\partial U_\epsilon}{\partial y} z \, ds, \end{aligned}$$

which is exactly the step needed to show that U_ϵ, which is the rotated version of the odd solution, solves the desired even-symmetric problem with conductivity $\frac{1}{a_\epsilon}$. Therefore, we can produce the same type of convergence results as before for the odd problem with $\frac{a_\epsilon}{\epsilon} < M$. The convergence for the odd problem if $\frac{a_\epsilon}{\epsilon} \to \infty$ follows in a similar manner to the original argument. Thus, we have convergence results for both even and odd symmetry, and all regimes of conductivities, and can stitch them together in the same way as before to get the full convergence result. Therefore, we have analogues of Theorem 4.7.1 and Theorem 4.7.2 in the case of a variable-width inhomogeneity ω_ϵ^h, provided that $h > c > 0$ and $h \in C^2(\sigma)$.

Theorem 5.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1 domain, with σ a straight line segment so that $\omega_\epsilon \subset\subset \Omega$ for $\epsilon < \epsilon_0$. Let h be any $C^2(\sigma)$ function with $0 < c \leq h \leq 1$. Fix any $\delta > 0$, and for any $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$, let u_ϵ^0 and u_ϵ denote the solutions to* (5.28) *and* (5.23) *respectively. Then*

$$\lim_{\epsilon \to 0} \sup_P \frac{|E_\epsilon^{0,h}(u_\epsilon^0) - E_\epsilon^h(u_\epsilon)|}{\Big(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\Big)^2} = 0$$

where the parameter set P is

$$P = \{(a_\epsilon, f, \varphi) \ : \ 0 < a_\epsilon \in \mathbb{R} \ \ \forall \epsilon > 0, \ f \in \mathcal{F}_\delta, \ \varphi \in H^{1/2}(\partial\Omega)\}.$$

Theorem 5.2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded C^1 domain, with σ a straight line segment so that $\omega_\epsilon \subset\subset \Omega$ for $\epsilon < \epsilon_0$. Let h be any $C^2(\sigma)$ function with $0 < c \leq h \leq 1$. Fix any $\delta > 0$, and for any $f \in \mathcal{F}_\delta$ and $\varphi \in H^{1/2}(\partial\Omega)$, let u_ϵ^0 and u_ϵ denote the solutions to* (5.28) *and* (5.23) *respectively. Then*

$$\lim_{\epsilon \to 0} \sup_P \frac{||u_\epsilon^0 - u_\epsilon||_{H^s(\Omega \setminus \omega_{\delta'})}}{||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}} = 0$$

for any $s \geq 0$ and any $\delta' > 0$, where the parameter set P is

$$P = \{(a_\epsilon, f, \varphi) \ : \ 0 < a_\epsilon \in \mathbb{R} \ \ \forall \epsilon > 0, \ f \in \mathcal{F}_\delta, \ \varphi \in H^{1/2}(\partial\Omega)\}.$$

5.2.2 Anisotropic Conductivities

One other generalization, and the one that also leads to many applied problems, is that of anisotropic conductivities. In the cloaking problem, the material that makes up the cloak is necessarily anisotropic (see [KSVW08]), and in the case of thin inhomogeneities, the fact that the two dimensions are scaled differently when the cloak is applied will turn an isotropic conductivity into one that is anisotropic.

In order to begin analyzing this problem, we define our conductivity a_ϵ on the inhomogeneity to be

$$a_\epsilon = \begin{pmatrix} a_\epsilon^{(1,1)} & a_\epsilon^{(1,2)} \\ a_\epsilon^{(1,2)} & a_\epsilon^{(2,2)} \end{pmatrix}$$

when written in the (τ, n) basis of ω_ϵ, which is a symmetric positive definite matrix, constant in space but whose coefficients can depend on ϵ. Then, the combined conductivity on Ω is

$$\gamma_\epsilon = \begin{cases} I & \Omega \setminus \omega_\epsilon \\ a_\epsilon & \omega_\epsilon \end{cases}$$

and we want to find an approximation to the solution to

$$\begin{cases} -\mathrm{div}(\gamma_\epsilon \nabla u_\epsilon) = f & \Omega \\ u_\epsilon = \varphi & \partial\Omega. \end{cases} \tag{5.31}$$

The procedure for doing this will follow the same ideas as in Chapter 2 for determining the energy functional corresponding to an approximation of that problem. To simplify notation, we will assume that the curve σ is closed. If it was an open curve, this same process would work, replacing ω_ϵ and ω_1 by ω_ϵ^{int} and ω_1^{int} respectively. As a starting point, we know that the energy functional corresponding to (5.31) is

$$\begin{aligned} E_\epsilon(u) &= \frac{1}{2}\int_\Omega (\gamma_\epsilon \nabla u) \cdot \nabla u \; dx - \int_\Omega fu \; dx \\ &= \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\nabla u|^2 \; dx - \int_\Omega fu \; dx + \frac{1}{2}\int_{\omega_\epsilon} (a_\epsilon \nabla u)\cdot \nabla u \; dx \end{aligned} \tag{5.32}$$

where the minimization is taken over all H^1 functions on Ω with boundary data φ on $\partial\Omega$. Next, we want to use the H_ϵ map from Chapter 2 to transform the integral over ω_ϵ to one over ω_1. The same transformation process as before gives that

$$\int_{\omega_\epsilon} (a_\epsilon \nabla u)\cdot \nabla u \; dx = \int_{\omega_1} \det(\nabla H_\epsilon)((\nabla H_\epsilon^{-1}) a_\epsilon (\nabla H_\epsilon^{-1})^T \nabla \hat{u}) \cdot \nabla \hat{u} \; dx. \tag{5.33}$$

We then compute, based on the definition of H_ϵ,

$$\begin{aligned} (\nabla H_\epsilon^{-1}) a_\epsilon (\nabla H_\epsilon^{-1})^T &= \begin{pmatrix} \frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma} & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix} \begin{pmatrix} a_\epsilon^{(1,1)} & a_\epsilon^{(1,2)} \\ a_\epsilon^{(1,2)} & a_\epsilon^{(2,2)} \end{pmatrix} \begin{pmatrix} \frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma} & 0 \\ 0 & \frac{1}{\epsilon} \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\right)^2 a_\epsilon^{(1,1)} & \frac{1}{\epsilon}\frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma} a_\epsilon^{(1,2)} \\ \frac{1}{\epsilon}\frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma} a_\epsilon^{(1,2)} & \frac{1}{\epsilon^2} a_\epsilon^{(2,2)} \end{pmatrix}. \end{aligned}$$

Since

$$\det(\nabla H_\epsilon) = \epsilon \frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}$$

we have that the integrand in (5.33) becomes

$$\begin{aligned} \det(\nabla H_\epsilon)((\nabla H_\epsilon^{-1})a_\epsilon(\nabla H_\epsilon^{-1})^T\nabla\hat{u})\cdot\nabla\hat{u} &= \epsilon a_\epsilon^{(1,1)}\frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\left(\frac{\partial\hat{u}}{\partial\tau}\right)^2 \\ + 2a_\epsilon^{(1,2)}\frac{\partial\hat{u}}{\partial\tau}\frac{\partial\hat{u}}{\partial n} + \frac{a_\epsilon^{(2,2)}}{\epsilon}\frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}\left(\frac{\partial\hat{u}}{\partial n}\right)^2 & \end{aligned} \tag{5.34}$$

and so the rescaled energy becomes

$$\begin{aligned} E_\epsilon(u,v) = {} & \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon}|\nabla u|^2\,dx - \int_\Omega fu\,dx \\ & + \frac{1}{2}\int_{\omega_1}\left[\epsilon a_\epsilon^{(1,1)}\frac{1+\kappa d_\sigma}{1+\epsilon\kappa d_\sigma}\left(\frac{\partial v}{\partial\tau}\right)^2 + 2a_\epsilon^{(1,2)}\frac{\partial v}{\partial\tau}\frac{\partial v}{\partial n}\right. \\ & \left. + \frac{a_\epsilon^{(2,2)}}{\epsilon}\frac{1+\epsilon\kappa d_\sigma}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\right]dx \end{aligned} \tag{5.35}$$

where u is a function in $H^1(\Omega\setminus\omega_\epsilon)$, v is on $H^1(\omega_1)$, with continuity conditions across the boundary, just like in Chapter 2. At this point, if we go back to assuming that a_ϵ was isotropic, we would get exactly (2.4).

Next, we want to take the highest order contribution of each term, as before, which gives an expression of the form

$$\begin{aligned} E_\epsilon(u,v) = {} & \frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2\,dx - \int_\Omega fu\,dx \\ & + \frac{1}{2}\int_{\omega_1}\left[\epsilon a_\epsilon^{(1,1)}(1+\kappa d_\sigma)\left(\frac{\partial v}{\partial\tau}\right)^2 + 2a_\epsilon^{(1,2)}\frac{\partial v}{\partial\tau}\frac{\partial v}{\partial n}\right. \\ & \left. + \frac{a_\epsilon^{(2,2)}}{\epsilon}\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\right]dx \end{aligned} \tag{5.36}$$

which, again, will match up with (2.6) if a_ϵ were isotropic, and the continuity conditions here are the same as for that equation. Finally, we want to consider this as a two-part minimization, fixing u and minimizing over the v that meet those boundary conditions, and then minimizing over u. This results in the inner energy to be minimized having the form

$$G_\epsilon^0(u) = \min_{\substack{v\in H^1(\omega_1)\\ v(x+n(x))=u^+(x)\\ v(x-n(x))=u^-(x)}} \frac{1}{2}\int_{\omega_1}\left[\epsilon a_\epsilon^{(1,1)}(1+\kappa d_\sigma)\left(\frac{\partial v}{\partial \tau}\right)^2 + 2a_\epsilon^{(1,2)}\frac{\partial v}{\partial \tau}\frac{\partial v}{\partial n}\right.$$
$$\left. + \frac{a_\epsilon^{(2,2)}}{\epsilon}\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\right] dx.$$

From here, the work in Chapter 2 goes on to say that, because we know that the terms in G_ϵ^0 are of different orders, we can assume that the form of the minimizing solution will be dictated by the largest term, which in that case was the $\frac{a_\epsilon}{\epsilon}$ term, and we could use only that one term in solving the problem. However, if a_ϵ is anisotropic, then we don't necessarily know that these terms are of different orders. For instance, if $a_\epsilon^{(1,1)}$ is $O(1)$, $a_\epsilon^{(1,2)}$ is $O(\epsilon)$, and $a_\epsilon^{(2,2)}$ is $O(\epsilon^2)$, as would be the case with a constant conductivity rescaled by a cloaking procedure, then all four terms in G_ϵ^0 are the same order. Therefore, we can not outright ignore any of these terms. We then need to find the actual minimizer of this functional in terms of u^+ and u^- (in particular, the energy of this minimizer) to get final form of the energy $E_\epsilon^0(u)$.

In the case where a_ϵ is diagonal in terms of the (τ, n) basis, we can say a bit more about this minimizer. If $a_\epsilon^{(1,2)} = 0$, then $G_\epsilon^0(u)$ takes the form

$$G_\epsilon^0(u) \min_{\substack{v\in H^1(\omega_1)\\ v(x+n(x))=u^+(x)\\ v(x-n(x))=u^-(x)}} \frac{1}{2}\int_{\omega_1}\left[\epsilon a_\epsilon^{(1,1)}(1+\kappa d_\sigma)\left(\frac{\partial v}{\partial \tau}\right)^2 + \frac{a_\epsilon^{(2,2)}}{\epsilon}\frac{1}{1+\kappa d_\sigma}\left(\frac{\partial v}{\partial n}\right)^2\right] dx.$$

To simplify the calculations, we will also assume that our mid-curve σ is the straight line from $(-1,0)$ to $(1,0)$, which further simplifies this expression to

$$G_\epsilon^0(u) \min_{\substack{v\in H^1(\omega_1)\\ v(x+n(x))=u^+(x)\\ v(x-n(x))=u^-(x)}} \frac{1}{2}\int_{\omega_1}\left[\epsilon a_\epsilon^{(1,1)}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{a_\epsilon^{(2,2)}}{\epsilon}\left(\frac{\partial v}{\partial y}\right)^2\right] dx.$$

The Euler-Lagrange equation corresponding to this functional is

$$\epsilon a_\epsilon^{(1,1)}\frac{\partial^2 v}{\partial x^2} + \frac{a_\epsilon^{(2,2)}}{\epsilon}\frac{\partial^2 v}{\partial y^2} = 0$$

on the domain $(x, y) \in [-1, 1]^2$ with boundary data u^- along $y = -1$ and u^+ along $y = 1$.

In order to solve this problem, we will abstract a little bit further, and look to solve

$$a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial^2 v}{\partial y^2} = 0$$

on $(-1,1) \times (0,2)$ with boundary conditions u^+ along $y = 2$ and u^- along $y = 0$. We will solve this using separation of variables and Fourier series. Using separation of variables, we know that if $v(x,y) = X(x)Y(y)$, then X and Y must satisfy that

$$\frac{X''}{X} = -\frac{b}{a}\frac{Y''}{Y} = -\lambda$$

for an appropriate set of eigenvalues λ. To determine the proper eigenvalues, we can expand u^+ and u^- into Fourier series on $(-1,1)$ as

$$u^{\pm}(x) = \frac{1}{2}d_0^{\pm} + \sum_{n=1}^{\infty} c_n^{\pm} \sin\left(\frac{n\pi}{2}x\right) + d_n^{\pm}\cos\left(\frac{n\pi}{2}x\right),$$

where

$$\begin{aligned} d_0^{\pm} &= \int_{-1}^{1} u^{\pm}(x)\, dx \\ d_n^{\pm} &= \int_{-1}^{1} u^{\pm}(x)\cos\left(\frac{n\pi}{2}x\right)\, dx \\ c_n^{\pm} &= \int_{-1}^{1} u^{\pm}(x)\sin\left(\frac{n\pi}{2}x\right)\, dx. \end{aligned}$$

With this expansion, we see that we want our eigenvalues to be $\lambda_n = \left(\frac{n\pi}{2}\right)^2$, which means that the function Y must either be linear ($n = 0$) or

$$Y(y) = \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) \qquad (n \geq 1).$$

We only need the hyperbolic sine terms here based on how we are going to apply boundary conditions. Applying the boundary conditions on $y = 0$ and $y = 2$, using symmetry to simplify the first of these, gives the solution v as

$$\begin{aligned} v(x,y) &= \frac{d_0^-}{2} + (d_0^+ - d_0^-)\frac{y}{4} \\ &+ \sum_{n=1}^{\infty} \frac{c_n^+ \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) + c_n^- \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)} \sin\left(\frac{n\pi}{2}x\right) \\ &= \sum_{n=1}^{\infty} \frac{d_n^+ \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) + d_n^- \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)} \cos\left(\frac{n\pi}{2}x\right) \end{aligned}$$

Remark. If the conductivity a_ϵ was isotropic, then we will have $a = \epsilon a_\epsilon$ and $b = \frac{a_\epsilon}{\epsilon}$, so that $\frac{a}{b} = O(\epsilon^2)$. For small arguments, we know that $\sinh(t) \approx t$, so this solution would reduce to

$$\begin{aligned}
v(x,y) &= \frac{d_0^-}{2} + (d_0^+ - d_0^-)\frac{y}{4} + \sum_{n=1}^{\infty} \frac{c_n^+\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) + c_n^-\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\left(n\pi\sqrt{\frac{a}{b}}\right)} \sin\left(\frac{n\pi}{2}x\right) \\
&+ \frac{d_n^+\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) + d_n^-\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\left(n\pi\sqrt{\frac{a}{b}}\right)} \cos\left(\frac{n\pi}{2}x\right) \\
&= \frac{1}{2}d_0^- + (d_0^+ - d_0^-)\frac{y}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(c_n^+ y + c_n^-(2-y))\sin\left(\frac{n\pi}{2}x\right) \\
&+ (d_n^+ y + d_n^-(2-y))\cos\left(\frac{n\pi}{2}x\right) \\
&= \left(1 - \frac{y}{2}\right)\left[\frac{1}{2}d_0^- + \sum_{n=1}^{\infty} c_n^- \sin\left(\frac{n\pi}{2}x\right) + d_n^- \cos\left(\frac{n\pi}{2}x\right)\right] \\
&+ \frac{y}{2}\left[\frac{1}{2}d_0^+ + \sum_{n=1}^{\infty} c_n^+ \sin\left(\frac{n\pi}{2}x\right) + d_n^+ \cos\left(\frac{n\pi}{2}x\right)\right],
\end{aligned}$$

which is exactly the affine combination of u^+ and u^- that was obtained as the approximate solution on ω_1 in the isotropic case.

Now, we need to find the energy of this solution to simplify the G_ϵ^0 term in the overall energy minimization. The energy is computed by

$$\frac{1}{2}\int_0^2 \int_{-1}^1 a\left(\frac{\partial v}{\partial x}\right)^2 + b\left(\frac{\partial v}{\partial y}\right)^2 \, dx \, dy.$$

To simplify computations, we define

$$\begin{aligned}
\hat{C}_n(y) &= \frac{c_n^+ \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) + c_n^- \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)} \\
\hat{D}_n(y) &= \frac{d_n^+ \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) + d_n^- \sinh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}
\end{aligned}$$

so that the solution v can be written as

$$v(x,y) = \frac{1}{2}d_0^- + (d_0^+ - d_0^-)\frac{y}{4} + \sum_{n=1}^{\infty} \hat{C}_n(y)\sin\left(\frac{n\pi}{2}x\right) + \hat{D}_n(y)\cos\left(\frac{n\pi}{2}x\right).$$

Then

$$\frac{\partial v}{\partial x} = \sum_{n=1}^{\infty} \frac{n\pi}{2}\left(\hat{C}_n(y)\cos\left(\frac{n\pi}{2}x\right) - \hat{D}_n(y)\sin\left(\frac{n\pi}{2}x\right)\right),$$

so that, by orthogonality of these trigonometric functions on $[-1,1]$,

$$\int_{-1}^1 \left(\frac{\partial v}{\partial x}\right)^2 \, dx = \sum_{n=1}^{\infty}\left(\frac{n\pi}{2}\right)^2\left(\hat{C}_n(y)^2 + \hat{D}_n(y)^2\right). \tag{5.37}$$

Similarly, we can compute

$$\frac{\partial v}{\partial y} = \frac{1}{4}(d_0^+ - d_0^-) + \sum_{n=1}^{\infty} \hat{C}_n'(y) \sin\left(\frac{n\pi}{2}x\right) + \hat{D}_n'(y) \cos\left(\frac{n\pi}{2}x\right).$$

If we define

$$\bar{C}_n(y) = \frac{c_n^+ \cosh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) - c_n^- \cosh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}$$
$$\bar{D}_n(y) = \frac{d_n^+ \cosh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}y\right) - d_n^- \cosh\left(\frac{n\pi}{2}\sqrt{\frac{a}{b}}(2-y)\right)}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}$$

so that

$$\hat{C}_n'(y) = \frac{n\pi}{2}\sqrt{\frac{a}{b}}\bar{C}_n(y) \qquad \hat{D}_n'(y) = \frac{n\pi}{2}\sqrt{\frac{a}{b}}\bar{D}_n(y)$$

we have that

$$\frac{\partial v}{\partial y} = \frac{1}{4}(d_0^+ - d_0^-) + \sqrt{\frac{a}{b}} \sum_{n=1}^{\infty} \frac{n\pi}{2} \left(\bar{C}_n(y) \sin\left(\frac{n\pi}{2}x\right) + \bar{D}_n(y) \cos\left(\frac{n\pi}{2}x\right)\right),$$

so that by orthogonality again, we have

$$\int_{-1}^{1} \left(\frac{\partial v}{\partial y}\right)^2 dx = \frac{1}{8}(d_0^+ - d_0^-)^2 + \frac{a}{b}\sum_{n=1}^{\infty} \left(\frac{n\pi}{2}\right)^2 \left(\bar{C}_n(y)^2 + \bar{D}_n(y)^2\right). \tag{5.38}$$

Combining (5.37) and (5.38) gives that the full energy can be calculated by

$$\begin{aligned} \frac{1}{2}\int_0^2 \int_{-1}^1 a\left(\frac{\partial v}{\partial x}\right)^2 + b\left(\frac{\partial v}{\partial y}\right)^2 dx\ dy = a\int_0^2 \frac{b}{16a}(d_0^+ - d_0^-)^2 \\ + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{2}\right)^2 \left[\hat{C}_n(y)^2 + \hat{D}_n(y)^2 + \bar{C}_n(y)^2 + \bar{D}_n(y)^2\right] dy. \end{aligned} \tag{5.39}$$

Finally, we need to integrate these expressions in y. The first term easily integrates to $\frac{b}{8a}(d_0^+ - d_0^-)^2$. For the rest of the terms, we will show the computations for the integrals involving the C coefficients, as the corresponding computations for the D coefficients are identical. Using the fact that

$$\sinh(a)\sinh(b) = \frac{1}{2}(\cosh(a+b) - \cosh(a-b))$$
$$\cosh(a)\cosh(b) = \frac{1}{2}(\cosh(a+b) + \cosh(a-b)),$$

we can expand the individual terms in the energy as

$$\begin{aligned}
\hat{C}_n(y)^2 &= \frac{1}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)^2}\left[\frac{(c_n^+)^2}{2}\left(\cosh\left(n\pi\sqrt{\frac{a}{b}}y\right)-1\right)\right. \\
&\quad + c_n^+c_n^-\left(\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)-\cosh\left(n\pi\sqrt{\frac{a}{b}}(y-1)\right)\right) \\
&\quad \left. + \frac{(c_n^-)^2}{2}\left(\cosh\left(n\pi\sqrt{\frac{a}{b}}(2-y)\right)-1\right)\right], \\
\bar{C}_n(y)^2 &= \frac{1}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)^2}\left[\frac{(c_n^+)^2}{2}\left(\cosh\left(n\pi\sqrt{\frac{a}{b}}y\right)+1\right)\right. \\
&\quad - c_n^+c_n^-\left(\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)-\cosh\left(n\pi\sqrt{\frac{a}{b}}(y-1)\right)\right) \\
&\quad \left. + \frac{(c_n^-)^2}{2}\left(\cosh\left(n\pi\sqrt{\frac{a}{b}}(2-y)\right)+1\right)\right].
\end{aligned} \tag{5.40}$$

These can then be integrated in y to give

$$\begin{aligned}
\int_0^2 \hat{C}_n(y)^2\ dy &= \frac{1}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)^2}\left[\frac{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)}{n\pi\sqrt{\frac{a}{b}}}\left((c_n^+)^2+(c_n^-)^2\right)\right. \\
&\quad \left. - \frac{2\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}{n\pi\sqrt{\frac{a}{b}}}c_n^+c_n^- - (c_n^+)^2 - (c_n^-)^2 + 2c_n^+c_n^-\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)\right], \\
\int_0^2 \bar{C}_n(y)^2\ dy &= \frac{1}{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)^2}\left[\frac{\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)}{n\pi\sqrt{\frac{a}{b}}}\left((c_n^+)^2+(c_n^-)^2\right)\right. \\
&\quad \left. - \frac{2\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}{n\pi\sqrt{\frac{a}{b}}}c_n^+c_n^- + (c_n^+)^2 + (c_n^-)^2 - 2c_n^+c_n^-\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)\right].
\end{aligned}$$

Adding these terms together and simplifying, we get that

$$\begin{aligned}
&\int_0^2 \hat{C}_n(y)^2 + \bar{C}_n(y)^2\ dy \\
&\qquad = \frac{2}{n\pi\sqrt{\frac{a}{b}}\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}\left[\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)\left((c_n^+)^2+(c_n^-)^2\right) - 2c_n^+c_n^-\right].
\end{aligned} \tag{5.41}$$

By combining this term with the equivalent term for the D coefficients and the other term integrated previously, we can compute the energy of this function as

$$\begin{aligned}
\frac{1}{2}\int_0^2\int_{-1}^1 a\left(\frac{\partial v}{\partial x}\right)^2+b\left(\frac{\partial v}{\partial y}\right)^2\ dx\ dy &= a\frac{b}{8a}(d_0^+-d_0^-)^2\\
&+a\sum_{n=1}^{\infty}\left(\frac{n\pi}{2}\right)^2\frac{1}{n\pi\sqrt{\frac{a}{b}}\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}\Bigg[-2c_n^+c_n^- - 2d_n^+d_n^-\\
&+\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)\left((c_n^+)^2+(c_n^-)^2+(d_n^+)^2+(d_n^-)^2\right)\Bigg]\\
&=\frac{b}{8}(d_0^+-d_0^-)^2+\sqrt{ab}\sum_{n=1}^{\infty}\frac{n\pi}{4\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}\Bigg[-2c_n^+c_n^- - 2d_n^+d_n^-\\
&+\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)\left((c_n^+)^2+(c_n^-)^2+(d_n^+)^2+(d_n^-)^2\right)\Bigg]
\end{aligned} \tag{5.42}$$

Therefore, we should want to use

$$\begin{aligned}
G_\epsilon^0(u) &= \frac{b}{8}(d_0^+-d_0^-)^2+\sqrt{ab}\sum_{n=1}^{\infty}\frac{n\pi}{4\sinh\left(n\pi\sqrt{\frac{a}{b}}\right)}\Bigg[-2c_n^+c_n^- - 2d_n^+d_n^-\\
&+\cosh\left(n\pi\sqrt{\frac{a}{b}}\right)\left((c_n^+)^2+(c_n^-)^2+(d_n^+)^2+(d_n^-)^2\right)\Bigg]
\end{aligned} \tag{5.43}$$

as our energy from the inner minimization in the anisotropic case. Since u^+ and u^- are in $H^1(\sigma)$, we know that these series will converge, and this is a valid energy to use in this problem.

Remark. Again, if we are in the isotropic case, $\sqrt{\frac{a}{b}}$ is of order ϵ, and we can use this to simplify the energy. With this fact, one-term Taylor expansions tell us that $\sinh(t)\approx t$ and $\cosh(t)\approx 1$ for t small, which gives an approximate energy of

$$\begin{aligned}
G_\epsilon^0(u) &\approx \frac{b}{8}(d_0^+-d_0^-)^2+\frac{b}{4}\sum_{n=1}^{\infty}(c_n^+-c_n^-)^2+(d_n^+-d_n^-)^2\\
&\approx \frac{b}{4}\left[\frac{1}{2}(d_0^+-d_0^-)^2+\sum_{n=1}^{\infty}(c_n^+-c_n^-)^2+(d_n^+-d_n^-)^2\right]
\end{aligned}$$

which, by the definition of the Fourier coefficients for u^+ and u^-, is exactly

$$\frac{b}{4}\int_\sigma (u^+-u^-)^2\ dx$$

which matches one of the terms in our previous energy expression with the substitution $b=\frac{a_\epsilon}{\epsilon}$.

If we want to be more accurate, we can retain one more term in the Taylor series of hyperbolic cosine, using the approximation that

$$\cosh(t) \approx 1 + \frac{t^2}{2}$$

for t small to get an energy approximation of the form

$$\begin{aligned}
G_\epsilon^0(u) &\approx \frac{b}{8}(d_0^+ - d_0^-)^2 + \frac{b}{4}\sum_{n=1}^{\infty}(c_n^+ - c_n^-)^2 + (d_n^+ - d_n^-)^2 \\
&\quad + \frac{a}{8}(n\pi)^2\left((c_n^+)^2 + (c_n^-)^2 + (d_n^+)^2 + (d_n^-)^2\right) \\
&\approx \frac{b}{4}\left[\frac{1}{2}(d_0^+ - d_0^-)^2 + \sum_{n=1}^{\infty}(c_n^+ - c_n^-)^2 + (d_n^+ - d_n^-)^2\right] \\
&\quad + \frac{a}{2}\left(\frac{n\pi}{2}\right)^2\left((c_n^+)^2 + (c_n^-)^2 + (d_n^+)^2 + (d_n^-)^2\right).
\end{aligned}$$

Using the Fourier coefficients for u^+ and u^- again, we can represent this energy as

$$G_\epsilon^0(u) \approx \frac{b}{4}\int_\sigma (u^+ - u^-)^2 \, ds + \frac{a}{2}\int_\sigma \left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 \, ds$$

which, making the substitution $a = \epsilon a_\epsilon$ and $b = \frac{a_\epsilon}{\epsilon}$, is the exact same as the reduced energy that we use in all of the other work in this paper.

For one more step of accuracy, we can also retain an additional term in the Taylor series of hyperbolic sine, giving that

$$\sinh(t) \approx t + \frac{t^3}{6}$$

for t small, or

$$\frac{1}{\sinh(t)} \approx \frac{1}{t + \frac{t^3}{6}} = \frac{1}{t}\frac{1}{1 + \frac{t^2}{6}} \approx \frac{1}{t}\left(1 - \frac{t^2}{6}\right).$$

This gives an approximate energy of

$$\begin{aligned}
G_\epsilon^0(u) \approx \frac{b}{8}(d_0^+ - d_0^-)^2 + \frac{b}{4}\sum_{n=1}^{\infty}\left(1 - \frac{(n\pi)^2 a}{6b}\right)\Bigg[- 2c_n^+ c_n^- - 2d_n^+ d_n^- \\
+ \left(1 + \frac{(n\pi)^2 a}{2b}\right)\left((c_n^+)^2 + (c_n^-)^2 + (d_n^+)^2 + (d_n^-)^2\right)\Bigg]
\end{aligned}$$

Expanding this out and collecting terms in a similar manner to previous calculations gives an expression of the form

$$\begin{aligned}
G_\epsilon^0(u) \approx \frac{b}{8}(d_0^+ - d_0^-)^2 + \frac{b}{4}\sum_{n=1}^{\infty}\left[(c_n^+ - c_n^-)^2 + (d_n^+ - d_n^-)^2\right] \\
+ \frac{a}{8}\sum_{n=1}^{\infty}(n\pi)^2\left((c_n^+)^2 + (c_n^-)^2 + (d_n^+)^2 + (d_n^-)^2\right) \\
- \frac{b}{24}\sum_{n=1}^{\infty}(n\pi)^2\left[(c_n^+ - c_n^-)^2 + (d_n^+ - d_n^-)^2\right] \\
- \frac{a^2}{48b}\sum_{n=1}^{\infty}(n\pi)^4\left((c_n^+)^2 + (c_n^-)^2 + (d_n^+)^2 + (d_n^-)^2\right).
\end{aligned}$$

Ignoring the cross term with an extra factor of $\frac{a}{b}$ and simplifying expressions as before gives that

$$\begin{aligned}
G_\epsilon^0(u) \approx \frac{b}{4}\left[\frac{1}{2}(d_0^+ - d_0^-)^2 + \frac{b}{4}\sum_{n=1}^{\infty}\left[(c_n^+ - c_n^-)^2 + (d_n^+ - d_n^-)^2\right]\right] \\
+ \frac{a}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{2}\right)^2\left((c_n^+)^2 + (c_n^-)^2 + (d_n^+)^2 + (d_n^-)^2\right) \\
- \frac{a}{6}\sum_{n=1}^{\infty}\left(\frac{n\pi}{2}\right)^2\left[(c_n^+ - c_n^-)^2 + (d_n^+ - d_n^-)^2\right]
\end{aligned}$$

which, by the definition of the Fourier series of u^+ and u^-, is the same as

$$G_\epsilon^0(u) \approx \frac{b}{4}\int_\sigma (u^+ - u^-)^2\, ds + \frac{a}{2}\int_\sigma \left(\frac{\partial u^+}{\partial\tau}\right)^2 + \left(\frac{\partial u^-}{\partial\tau}\right)^2\, ds - \frac{a}{6}\int_\sigma \left(\frac{\partial(u^+ - u^-)}{\partial\tau}\right)^2\, ds.$$

Expanding the last term and combining with the second gives the formula

$$G_\epsilon^0(u) \approx \frac{b}{4}\int_\sigma (u^+ - u^-)^2\, ds + \frac{a}{3}\int_\sigma \left(\frac{\partial u^+}{\partial\tau}\right)^2 + \left(\frac{\partial u^-}{\partial\tau}\right)^2 + \frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau} ds$$

which is exactly the same as the expression that was derived in Chapter 2 and [DV17] for the approximate energy of the function on ω_1.

Now that we have an explicit formula for the energy of the function on ω_1 in terms of the boundary data, the next step in this process would be to figure out if there is a relatively simple way to express this energy in terms of the functions u^+ and u^- for a fixed a and b when we can not assume that this ratio is small. As shown in the previous remark, when $\frac{a}{b}$ is small, we can simplify this expression in a way that makes it fairly easy to represent in terms of the boundary data. An expression of this form is necessary for getting energy estimates to start working with a convergence argument as well as for developing a numerical model to visualize how these solutions behave.

5.3 Curved inhomogeneities

Another assumption that has persisted throughout all of this work is the fact that σ needs to be a straight line segment. This assumption was dropped briefly in the previous section because we went back to assuming that σ is closed. The main issue with σ needing to be a straight segment for the open curve case is the reduction to a problem on $\mathbb{R}^2 \setminus \{(x,0) \mid x > 0\}$ in order to get the specific form of the most singular part of u^{ev}. The point was that we could translate and rotate σ so that it lies inside $\{(x,0) \mid x > 0\}$ without changing the fact that the function u_ϵ^0 was harmonic. This fact was crucial to the development of the explicit representation of the solution to the even and odd problems near each of these endpoints, which gives rise to half of the results derived earlier.

If σ is not a straight line-segment, then there are two cases for σ, resulting in different approaches to address this problem.

5.3.1 Line Segment Ends

If σ is a curve that is straight near the endpoints, that is, if the curvature vanishes in a neighborhood of each endpoint, then this issue is fairly easy to solve. The initial steps of this process involved localizing near the endpoints of σ. If σ has a portion of the boundary near each endpoint that is a straight line, then this localization can also be chosen so that the part of σ inside the region where $\eta(r) = 1$ is a straight segment. Then this region can be rotated and translated to lie within $\{(x,0) \mid x > 0\}$ and the construction of the "most singular term" in the expansion can be done as before, giving regularity estimates in this region. These results can then be extended to the entire curve σ using the interior regularity results from [DV17] to give the exact estimates that we need, proving the asymptotic accuracy of these solutions as desired.

5.3.2 Curved Ends

If σ is an open curve that does not have straight line segments near each endpoint, we can still localize near the endpoint and get a map to the domain $\mathbb{R}^2 \setminus \{(x,0) \mid x > 0\}$

so that the image of σ is contained in $\{(x,0) \mid x > 0\}$. However, since this map is not just a translation and rotation, the image of u_ϵ^0 under this transformation will no longer be harmonic. In order to analyze this problem, we need to determine what differential equation this function will solve on $\mathbb{R}^2 \setminus \{(x,0) \mid x > 0\}$.

Without loss of generality, assume that we are going to localize around the endpoint a of σ. Let $\gamma(t)$ be an arc-length parametrization of σ so that $\gamma(0) = a$, with $n(t) = \gamma'(t)^\perp$ a normal vector to σ at the point $\gamma(t)$. Then, the map

$$\Psi(x,y) = \gamma(x) + yn(x)$$

sends $\{(x,0) \mid 0 < x < \rho\}$ into σ for $\rho < \rho_0$, for some $\rho_0 > 0$. We now want to extend the parametrizations γ and n so that they are defined on $-\rho < x < 0$ so that the curvature κ is even in x, extending the curve σ through the endpoint a. This extension is uniquely defined by the point $\gamma(0)$, the tangent and normal vectors at that point, and our choice of curvature, and the extended curve is C^2 by construction. Thus, there exists a $\rho > 0$ so that $B_\rho(0)$ maps into ω_δ, the neighborhood around σ on which f vanishes, under the map Ψ, and this image contains an entire neighborhood of $\gamma(0) = a$.

Now, we can consider the function $\hat{u} = u_\epsilon^0 \circ \Psi$ on $B_\rho(0)$ and try to find the differential equation that $\hat{u}$ satisfies, as well as what the "most singular term" of this function is. To do this, we use a result stated in [DV17] that, under this geometry, we have that for any vector field ξ on ω_ϵ,

$$\operatorname{div}(\xi) = \frac{\partial}{\partial \tau}\xi_\tau + \frac{\partial}{\partial n}\xi_n + \frac{\kappa}{1+\kappa d_\sigma}\xi_n$$

where κ refers to the curvature of σ at the projected point on the curve and d_σ is the signed distance from σ. Since u_ϵ^0 is harmonic, we know that

$$0 = \operatorname{div}(\nabla u_\epsilon^0) = \frac{\partial}{\partial \tau}u_\tau + \frac{\partial}{\partial n}u_n + \frac{\kappa}{1+\kappa d_\sigma}u_n$$

and this now holds true with respect to the extended curve σ. However, by our definition of the map Ψ, any tangential derivatives of u_ϵ^0 will become x derivatives of $\hat{u}$, and similarly for normal derivatives becoming y derivatives. Therefore, this equation becomes

$$0 = \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\kappa(x)}{1+\kappa(x)y}\frac{\partial \hat{u}}{\partial y}$$

as, after the mapping, y represents the distance to the curve. So, full analysis of this problem and the limiting behavior of the solution as $\epsilon \to 0$ requires knowing the regularity of solutions to this problem on $\mathbb{R}^2 \setminus \{(x, 0) \mid x > 0\}$.

Unfortunately, our regularity results before relied on the fact that we had a harmonic function, and so could use complex analysis to get a formula for the most singular term. Since $\hat{u}$ is no longer harmonic (unless $\kappa \equiv 0$), this analysis will not go through in this case. The second order part of the differential equation defining $\hat{u}$ is just Laplace's equation, and the remaining portion is a lower order term that does not degenerate on the domain, so it will likely be possible to push the results from the harmonic problem through to this part. We would expect that we would still see an $r^{1/2}$ behavior near these endpoints, resulting in a proof of uniform validity of the estimates here as well.

Chapter 6

Numerical Results

In this chapter, we showcase some numerical results that illustrate the convergence proved in Chapter 4, as well as show numerical versions of the other results presented in Chapter 5. All of the results were computed using the FEniCS package. This package provides a way to set up domain data, generate meshes, and solve variational differential equations with the finite element method. The Python programs that were used to generate these results are included in Appendix C.

The mesh for the finite element method was chosen to be a small uniform mesh. The issue with using an adaptive mesh refinement is that if ϵ is too small compared to the mesh size, then the inhomogeneity will not register on the mesh, and the numerical solution will act as though the conductivity is 1 throughout the entire domain. Thus the adaptive method will ignore the inhomogeneity and does not work for our problem. The mesh here was chosen to be small enough so that we could pick ϵ small enough to sufficiently illustrate the convergence. The results were shown to be consistent over different mesh sizes, provided ϵ was chosen in an appropriate manner.

6.1 General convergence results

This section contains numerical results that illustrate the fact that the solution u_ϵ^0 to (2.9) is an approximation to the solution u_ϵ to (2.1) as $\epsilon \to 0$, uniformly in a_ϵ, as proved in Chapter 4. For the sake of numerical computation, we take the domain Ω to be $B_2(0)$ and the curve σ to be the straight line segment from $(-1, 0)$ to $(1, 0)$. We take the source function f to be identically zero in Ω (instead of on some smaller subset ω_δ) and specify Dirichlet boundary data as

$$\varphi(x, y) = 1 - x + x^2 y - (x + 1)y^2 + 3xy^3.$$

To carry out the computations, we make full use of the symmetry considerations from Chapter 3 to break the problem into its even and odd parts, and solve the problem on the half domain Ω^+. This results in boundary data

$$\varphi^{ev}(x,y) = 1 - x - (x+1)y^2$$

for the even part and

$$\varphi^{odd}(x,y) = x^2y + 3xy^3$$

for the odd part.

6.1.1 Even symmetry

The problem defining the even part of the solution in Ω^+ is given by (3.4), where the extra boundary conditions outside of σ are forced by the even symmetry of the problem. As the only parameter in this problem that changes with ϵ is ϵa_ϵ, the behavior of the solution as $\epsilon \to 0$ should only depend on whether ϵa_ϵ converges to 0, is bounded and strictly positive in the limit, or goes to infinity. As described previously, we consider the boundary data $\varphi^{ev}(x,y) = 1 - x - (x+1)y^2$ and compare the two solutions along the horizontal segment at $y = 0.5$, a fixed distance away from the curve σ. The plots in Figure 6.1 show both the solution u^{ev} and u^{ev}_ϵ, and in the case where $\epsilon a_\epsilon \to 0$, the solution u^{ev}_0 as well.

As expected from the results proved in this thesis, we have convergence of the two solutions as $\epsilon \to 0$ in all cases, so long as we stay a fixed distance away from the curve σ. To further analyze these results, Figure 6.2 shows the values of the solutions on the horizontal segment at $y = 0$, which includes σ.

In particular, these images show the behavior that we expected to see at the endpoints of σ due to Lemma 3.4.1, since for both $\epsilon a_\epsilon \to \infty$ and ϵa_ϵ bounded and away from zero, there is a distinct $r^{1/2}$ behavior near these endpoints. However, in the case where $\epsilon a_\epsilon \to 0$, the numerics seem to indicate that this behavior vanishes, which is consistent with the fact that u^{ev}_0, another approximating solution in this case, does not have these singularities. We also observe that the solutions seem to converge on σ, which our method of proof can not show at this point.

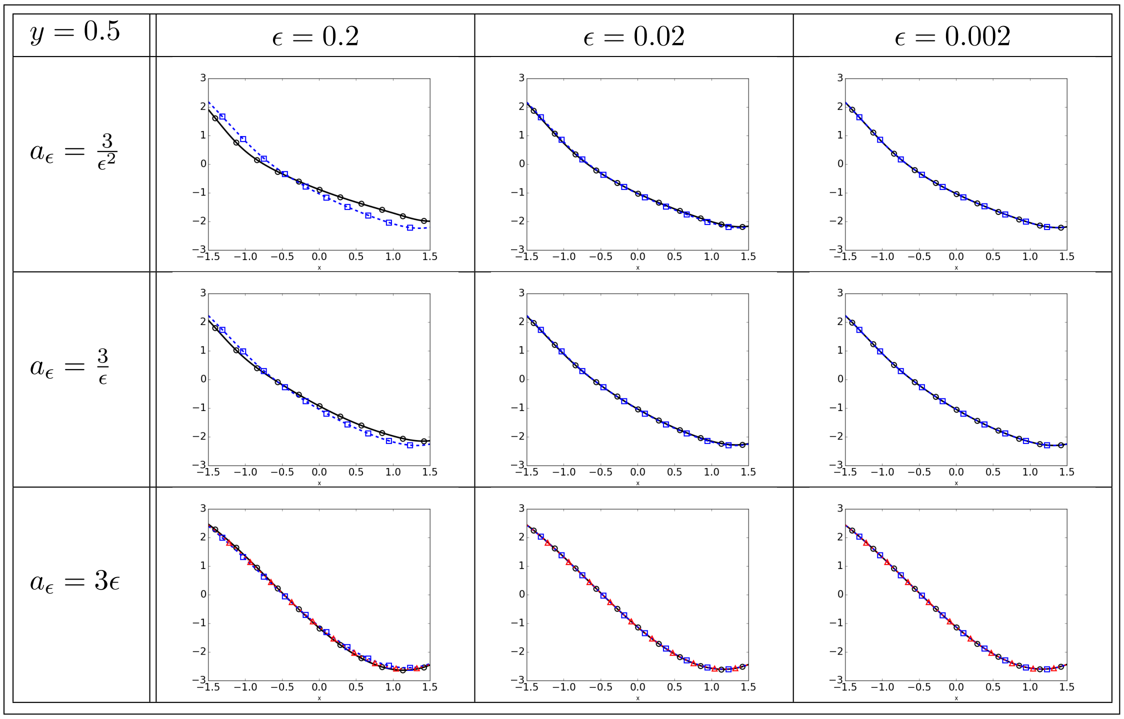

Figure 6.1: Numerical results for even symmetry along the line $y = 0.5$. In each plot, the solid line with circles represents the solution u_ϵ^{ev}, and the dashed line with squares represents the solution u^{ev}. In the last row, the dot-dash line with triangles shows the solution u_0^{ev}.

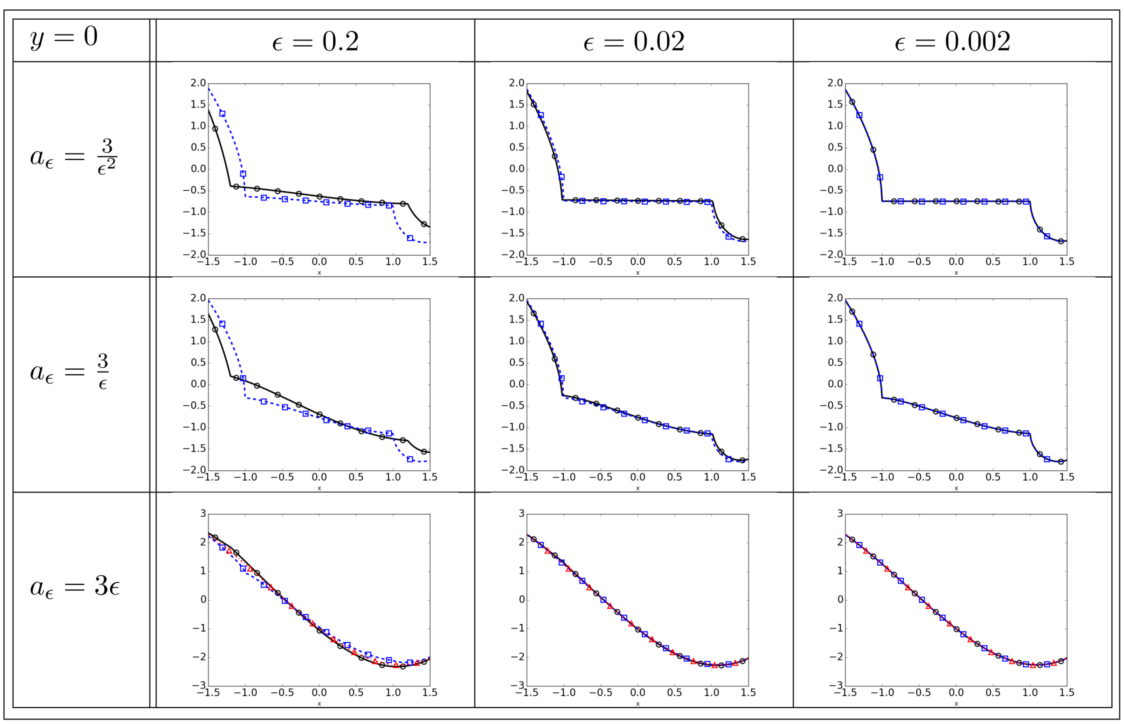

Figure 6.2: Numerical results for even symmetry along the line $y = 0$. In each plot, the solid line with circles represents the solution u_ϵ^{ev}, and the dashed line with squares represents the solution u^{ev}. In the last row, the dot-dash line with triangles shows the solution u_0^{ev}.

6.1.2 Odd Symmetry

For the odd part of the solution, the approximating problem takes the form of (3.5), which now has two changing parameters in it; ϵa_ϵ and $\frac{a_\epsilon}{\epsilon}$. However, the results of this thesis, in particular, how the convergence proof needed to be divided between Sections 4.4 and 4.5, seem to indicate that the behavior of the solutions as $\epsilon \to 0$ only depends on whether or not $\frac{a_\epsilon}{\epsilon}$ goes to infinity. As described before, we choose the boundary data $\varphi^{odd}(x, y) = x^2y + 3xy^3$ and Figure 6.3 shows the data on $y = 0$, which includes the curve σ.

Notice that due to continuity and odd symmetry, u_ϵ^{odd} is identically zero along $y = 0$ in all cases, and all solutions are zero outside of σ. However, only when $\frac{a_\epsilon}{\epsilon} \to \infty$ are u^{odd} and $u^{odd'}$ also forced to be continuous across $y = 0$, and so will vanish there as well. If $\frac{a_\epsilon}{\epsilon} < M$, then it is possible for these two solutions to jump across σ. In Lemma 4.4.1, we show that the solutions u^{odd} and $u^{odd'}$ are energy close when $\frac{a_\epsilon}{\epsilon} < M$, but the numerical results above seem to show that this may hold in more generality. The next tables show plots from the same solutions along the line $y = 0.5$ in Figure 6.4 and the vertical line $x = 0$ in Figure 6.5, including the solution u_0^{odd} when this is an appropriate approximation.

As expected, we see convergence of the solution at $y = 0.5$, and the solutions along $x = 0$ match up away from the curve σ, but if $\frac{a_\epsilon}{\epsilon}$ remains bounded, there appears to be a boundary layer forming around σ, caused by the odd symmetry of the problem with the inhomogeneity forcing a zero boundary condition.

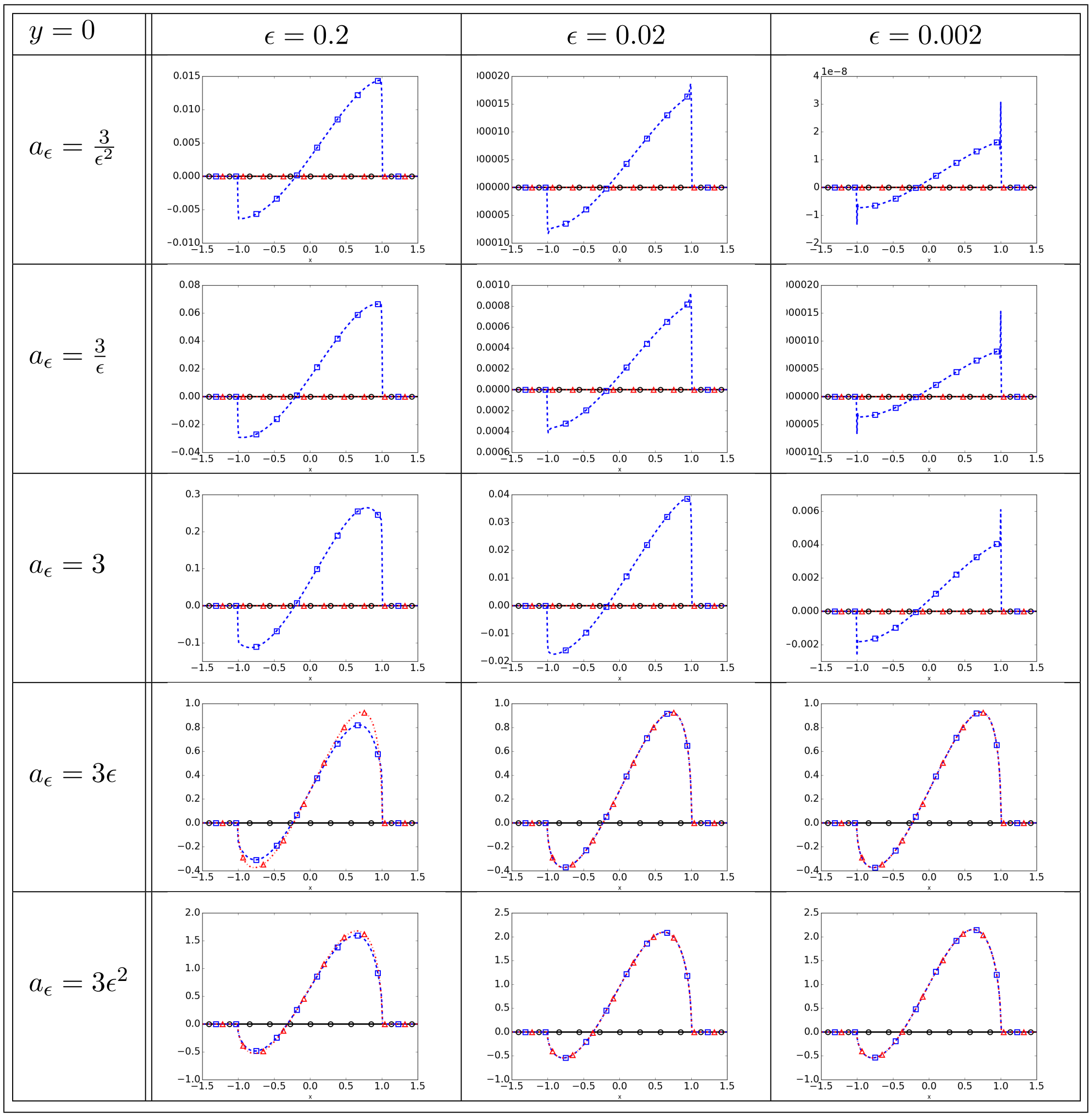

Figure 6.3: Numerical results for odd symmetry along the line $y = 0$. In each plot, the solid line with circles represents the solution u_ϵ^{odd}, the dashed line with squares represents the solution u^{odd}, and the dot-dash line with triangles represents the solution $u_{odd'}$.

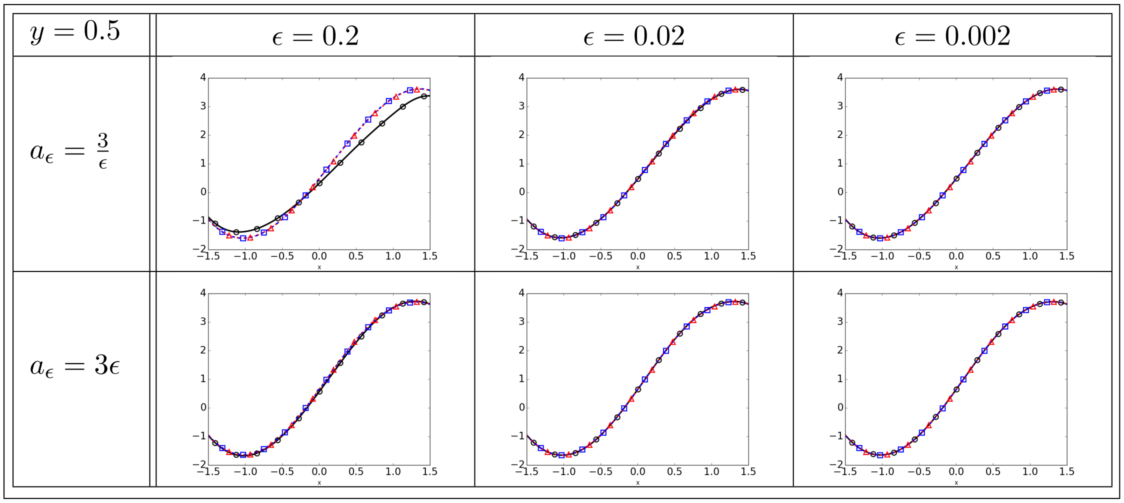

Figure 6.4: Numerical results for odd symmetry along the line $y = 0.5$. In each plot, the solid line with circles represents the solution u_ϵ^{odd} and the dashed line with squares represents the solution u^{odd}. In the first row, the dot-dash line with triangles represents the solution u_0^{odd}, while in the second row, it is the solution $u_{odd'}$.

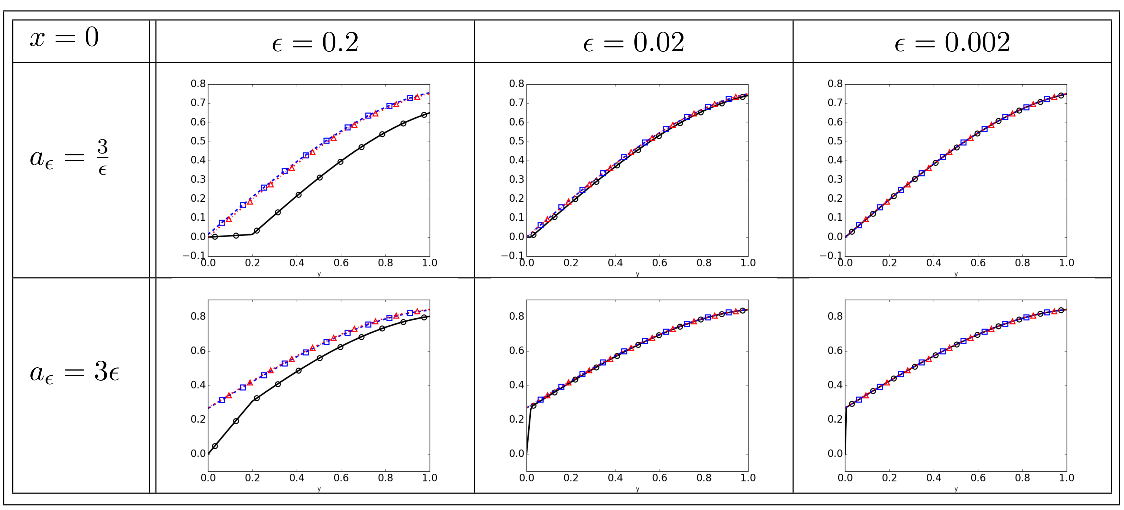

Figure 6.5: Numerical results for odd symmetry along the line $x = 0$. In each plot, the solid line with circles represents the solution u_ϵ^{odd} and the dashed line with squares represents the solution u^{odd}. In the first row, the dot-dash line with triangles represents the solution u_0^{odd}, while in the second row, it is the solution $u^{odd'}$. The kinks in the solid line solutions occur at the boundary of the inhomogeneity, and are expected.

6.1.3 Natural Conjectures

There are two main types of conjectures that these numerical computations seem to suggest that we have not yet proven. One of these is the behavior of the solutions on σ. For all values of a_ϵ in the even symmetry case, we see that the solution u^{ev} seems to approach the solution u_ϵ^{ev}, even on the curve σ, while the only results we have been able to prove thus far involve convergence in the far-field. For odd symmetry, the situation is more precarious: when $\frac{a_\epsilon}{\epsilon}$ goes to infinity, the solution u^{odd} appears to converge to 0 on σ, which is the value of u_ϵ^{odd}, except for either a boundary layer or numerical anomaly at the two endpoints. However, if $\frac{a_\epsilon}{\epsilon}$ remains bounded, we can not expect to have convergence on σ, because if the solution to (3.5) was to converge to 0 on σ, it would then have Cauchy data identically 0 on this curve, and so the solution itself would be 0, which we know is not true. Therefore, there cannot be convergence on σ in this case, but the numerical results see to show that we may have such convergence in other cases.

The second type of conjecture that is as of yet unproven is a rate estimate of convergence. Since the local convergence results of this thesis are based on a supremum argument where, in each case, only one of the two inequalities has a strict rate control on the error, the final results do not have this kind of bound. To see what might be expected for the error as $\epsilon \to 0$, we performed calculations of the L^2 error of the solution on $\omega_{0.8} \setminus \omega_{0.5}$. The plots in Figure 6.6 and Figure 6.7 show the error as a function of ϵ for several different values of a_ϵ and the even and odd parts of the solution. As before, boundary data is specified as

$$\varphi(x, y) = 1 - x + x^2 y - (x + 1)y^2 + 3xy^3,$$

so that we have

$$\varphi^{ev}(x, y) = 1 - x - (x + 1)y^2$$

for the even part of the solution and

$$\varphi^{odd}(x, y) = x^2 y + 3xy^3$$

for the odd part.

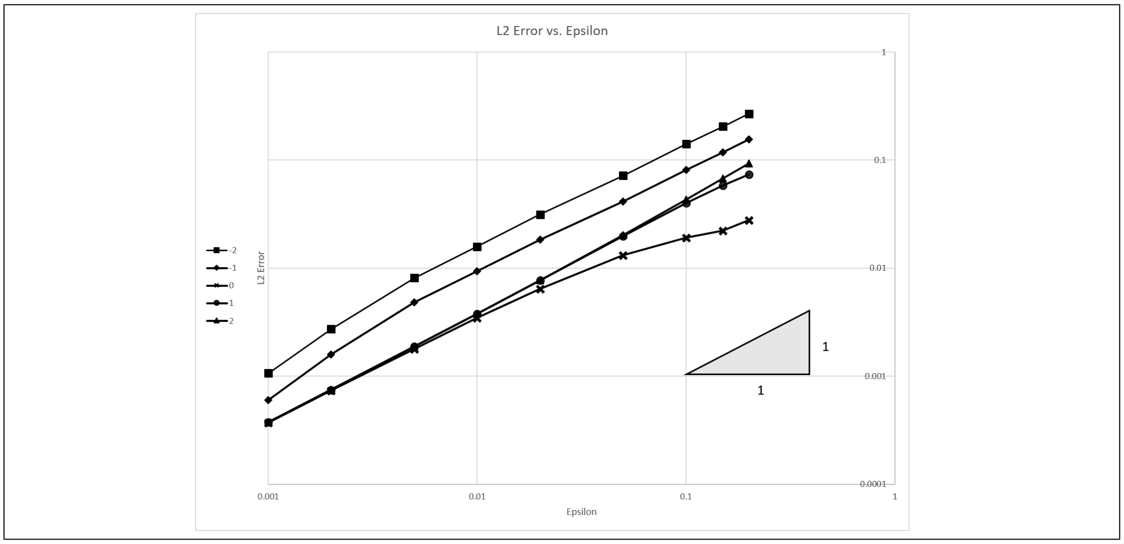

Figure 6.6: Error plot (log-log scale) for boundary data $\varphi^{ev}(x, y) = 1 - x - (x + 1)y^2$ with even symmetry. The value in the legend for each data set represents the power of ϵ in $a_\epsilon = 2\epsilon^\alpha$.

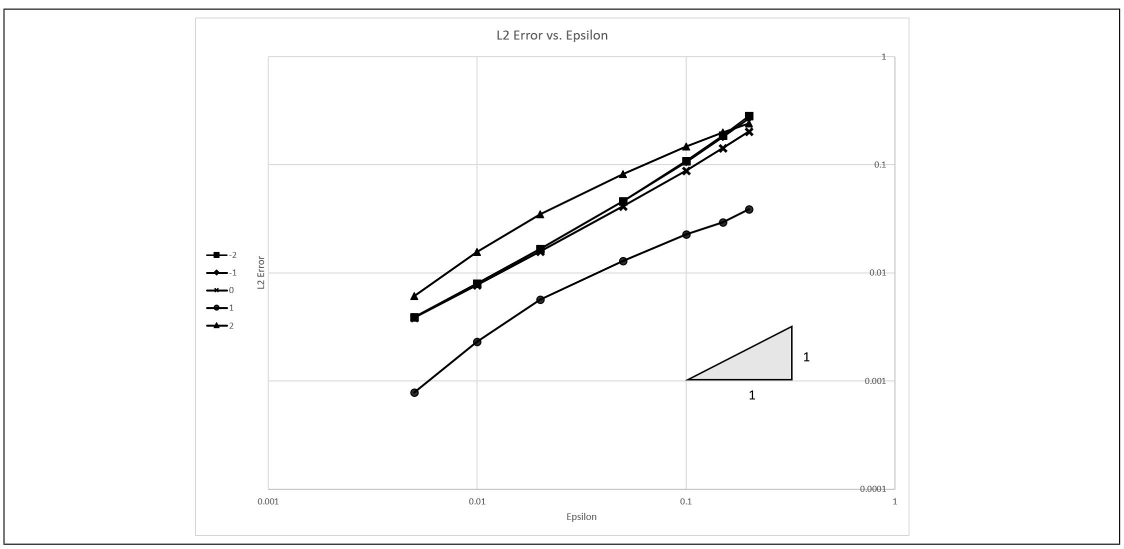

Figure 6.7: Error plot (log-log scale) for boundary data $\varphi^{odd}(x, y) = x^2y + 3xy^2$ with odd symmetry. The value in the legend for each data set represents the power of ϵ in $a_\epsilon = 2\epsilon^\alpha$.

From the results in this thesis, we expect convergence at a rate of $\epsilon^{\beta/2}$ as long as $\epsilon a_\epsilon > m > 0$ for even symmetry, and convergence at a rate of $\epsilon^{\beta/4}$ in the odd case for $a_\epsilon < M\epsilon$, for any $\beta < 1$. Our proofs, however, do not give any such rate estimates for any of the other cases. These numerical results seem to suggest that such a rate estimate exists for all cases. The slopes of the best-fit lines to these data are presented in Table 6.1.

a_ϵ	Even Symmetry	Odd Symmetry
$3\epsilon^{-2}$	1.016	1.157
$3\epsilon^{-1}$	1.016	1.144
3	0.813	1.071
3ϵ	1.006	1.014
$3\epsilon^2$	1.041	0.979

Table 6.1: Experimental rates of convergence for the approximate problem to the full problem as $\epsilon \to 0$ for a variety of conductivities a_ϵ.

These numerical results suggest that it might be possible to prove a rate estimate of order ϵ in all cases. While not shown here, the same type of results hold if the boundary data is less smooth across $y = 0$, indicating that this type of error rates estimate may hold independent of the increased regularity of this boundary data.

6.2 Stress intensity factor formula

This section illustrates the use of formula (5.14) to calculate the b_1 and b_1' coefficients in the expansion of u^{ev} in Lemma 3.4.1. This expansion should hold as long as ϵa_ϵ does not converge to zero. In each of these pictures, we show the functions u_ϵ^{ev}, u^{ev}, and the $r^{1/2}$ terms from the expansion in Lemma 3.4.1. For the graph of the $r^{1/2}$ terms, the solution on σ is plotted as the linear interpolation between the two endpoints of σ, as these $r^{1/2}$ terms only appear outside of σ.

This set of images uses boundary data $\varphi^{ev}(x, y) = 1 - x - (x+1)y^2$ and $f(x, y) = 0$. As can be seen in the images in Figures 6.8 and 6.9, the dot-dash lines with triangles fit the $r^{1/2}$ behavior that both u_ϵ and u^{ev} satisfy near the endpoints of σ. For $\epsilon a_\epsilon \to \infty$, it matches up almost exactly, but for $\epsilon a_\epsilon \to a$ for some finite number a, there is some additional error in the calculations. This comes from the fact that the numerical software is unable to correctly evaluate

$$\int_0^1 r^{-1/2}\ dr$$

because of the singularity at 0. Therefore, the integral needs to be truncated, resulting in a computed value for the b_1 and b_1' coefficients that is not exactly what it should be.

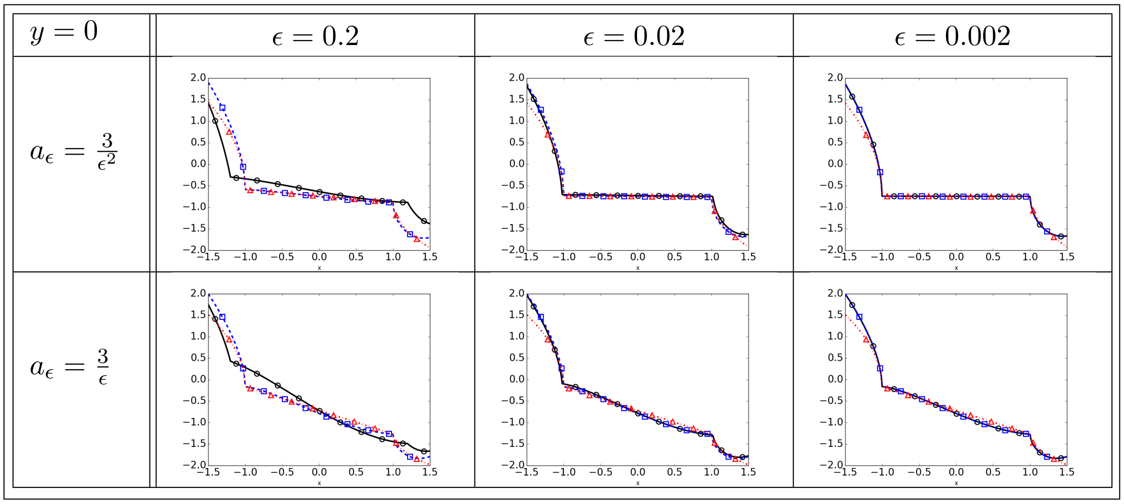

Figure 6.8: Numerical results illustrating formula (5.14) in computing the coefficient of the $r^{1/2}$ terms from Lemma 3.4.1 along $y = 0$. The solid line with circles is the solution u^{ev}_ϵ, the dashed line with squares is the solution u^{ev}, and the dot-dash line with triangles shows the $r^{1/2}$ terms.

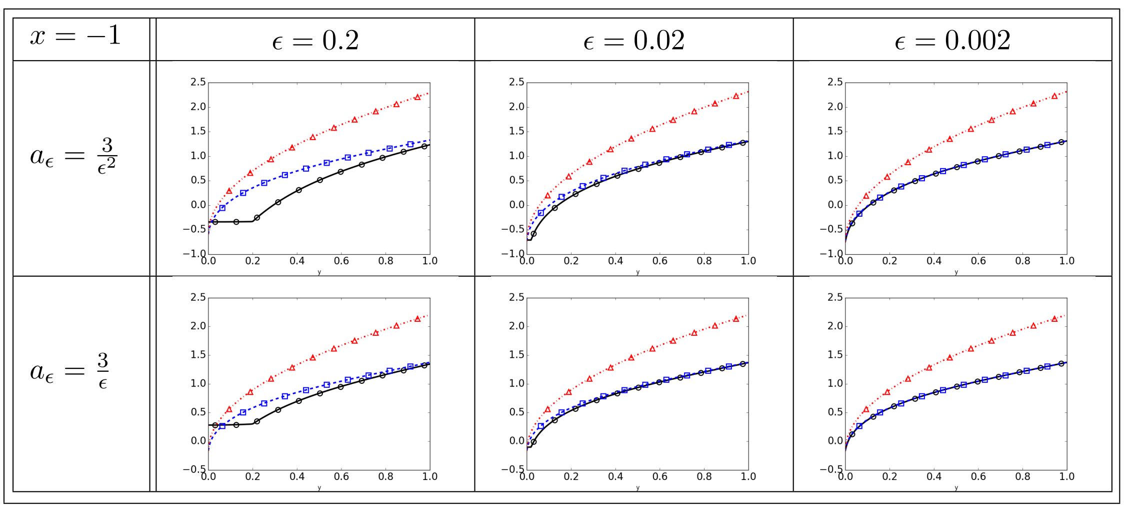

Figure 6.9: Numerical results illustrating formula (5.14) in computing the coefficient of the $r^{1/2}$ terms from Lemma 3.4.1 along $x = -1$. The solid line with circles is the solution u^{ev}_ϵ, the dashed line with squares is the solution u^{ev}, and the dot-dash line with triangles shows the $r^{1/2}$ terms.

6.3 Non-constant conductivities

Now, we illustrate numerical results for the case of non-constant conductivities as described in Section 5.2.1. The plan for these pictures is the same as before: the solid line with circles will represent the solution to the problem with the inhomogeneity and the dashed line with squares will represent the solution to the reduced problem on σ. Since the problem will be solved on the half-domain $\{y \geq 0\}$ and we want to decompose the function into its even and odd parts, we will assume that the conductivity a_ϵ is symmetric with respect to σ. The code will take the function that is used to define the conductivity and compute the impact factors as defined in Section 5.2.1 and use them to find the corresponding equation that u_ϵ^0 needs to satisfy. It will then solve both problems and plot them to show the convergence. For the even part of the solution, plots will be shown along $y = 0$, $y = 0.5$, and $x = -1$, and for the odd part, plots will be shown along $y = 0.5$ and $x = 0$. As with previous sections, Dirichlet boundary data will be specified as

$$\varphi(x, y) = 1 - x + x^2 y - (x + 1)y^2 + 3xy^3,$$

so that we have

$$\varphi^{ev}(x, y) = 1 - x - (x + 1)y^2$$

for the even part of the solution and

$$\varphi^{odd}(x, y) = x^2 y + 3xy^3$$

for the odd part.

6.3.1 Conductivity independent of x

In order to numerically simulate this problem, we first need to compute the corresponding differential equations for the even and odd parts of u_ϵ^0. The only thing that needs to be determined is the formulas for $\frac{\partial u^{ev}}{\partial y}$ and $\frac{\partial u^{odd}}{\partial y}$ on σ, which will allow us to form the variational formulation and implement it into the code. Computing this from (5.18) (after assuming that the conductivity is symmetric across σ) gives that

$$
\begin{aligned}
\frac{\partial u^{ev}}{\partial y} &= \frac{1}{2}\frac{\partial}{\partial y}\left(u(x,y)+u(x,-y)\right)|_{y=0} \\
&= \frac{1}{2}\left(\frac{\partial u^{+}}{\partial y}-\frac{\partial u^{-}}{\partial y}\right) \\
&= -\frac{\epsilon A_\epsilon}{4}\left(\mathcal{I}_1\frac{\partial^2 u^{+}}{\partial x^2}+\mathcal{I}_2\frac{\partial^2 u^{-}}{\partial x^2}+\mathcal{I}_2\frac{\partial^2 u^{+}}{\partial x^2}+\mathcal{I}_3\frac{\partial^2 u^{-}}{\partial x^2}\right) \\
&= -\frac{\epsilon A_\epsilon}{2}(\mathcal{I}_1+\mathcal{I}_2)\frac{\partial^2 u^{ev}}{\partial x^2}, \\
\frac{\partial u^{odd}}{\partial y} &= \frac{1}{2}\left(\frac{\partial u^{+}}{\partial y}+\frac{\partial u^{-}}{\partial y}\right) \\
&= \frac{A_\epsilon}{\epsilon}u^{odd}-\frac{\epsilon A_\epsilon}{4}\left(\mathcal{I}_1\frac{\partial^2 u^{+}}{\partial x^2}+\mathcal{I}_2\frac{\partial^2 u^{-}}{\partial x^2}-\mathcal{I}_2\frac{\partial^2 u^{+}}{\partial x^2}-\mathcal{I}_3\frac{\partial^2 u^{-}}{\partial x^2}\right) \\
&= \frac{A_\epsilon}{\epsilon}u^{odd}-\frac{\epsilon A_\epsilon}{2}(\mathcal{I}_1-\mathcal{I}_2)\frac{\partial^2 u^{odd}}{\partial x^2}.
\end{aligned}
$$

Two examples will be showcased here. The first has conductivity profile in the inhomogeneity given by $a_\epsilon(x,t) = 3\frac{y^2}{\epsilon^4}+\epsilon^2$. Figure 6.10 illustrates the results for even symmetry and Figure 6.11 illustrates the results for odd symmetry.

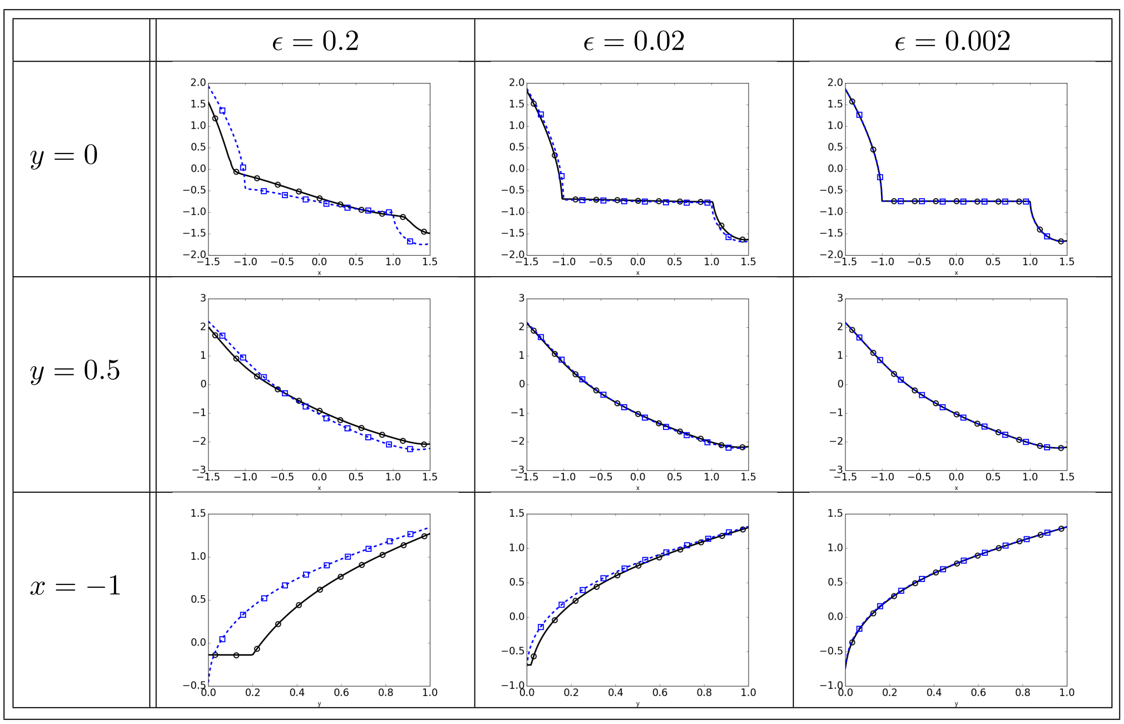

Figure 6.10: Numerical results for even symmetry with conductivity $a_\epsilon(x,y) = 3\frac{y^2}{\epsilon^4}+\epsilon^2$. In each plot, the solid line with circles represents the solution u_ϵ^{ev}, and the dashed line with squares represents the solution u^{ev}.

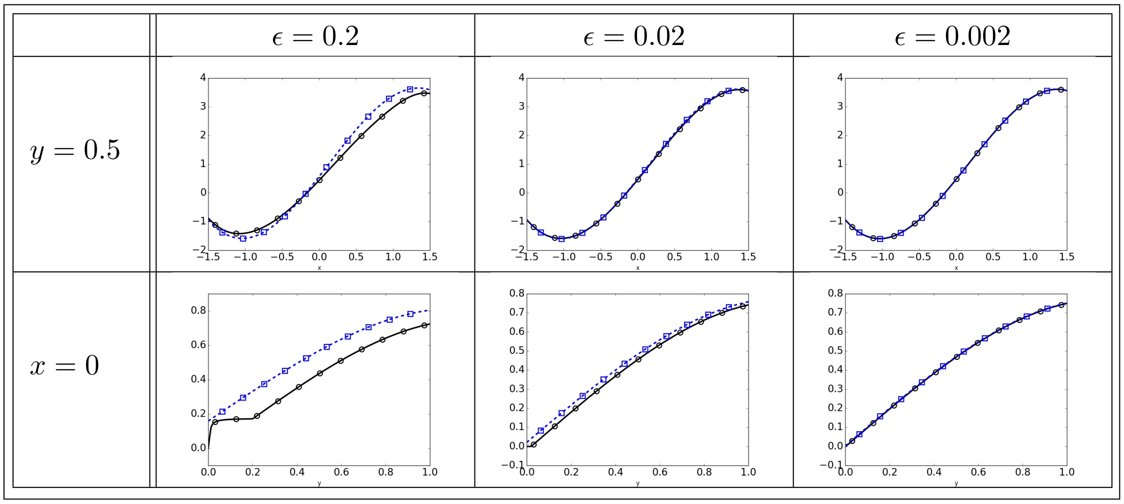

Figure 6.11: Numerical results for odd symmetry with conductivity $a_\epsilon(x, y) = 3\frac{y^2}{\epsilon^4} + \epsilon^2$. In each plot, the solid line with circles represents the solution u_ϵ^{odd}, and the dashed line with squares represents the solution u^{odd}.

The second has conductivity profile in the inhomogeneity given by $a_\epsilon(x, t) = 3\frac{y^2}{\epsilon^2} + 1$. Figure 6.12 illustrates the results for even symmetry and Figure 6.13 illustrates the results for odd symmetry.

These two examples show some of the interesting features of these problems. For each of them, we can compute the average conductivity A_ϵ. For the first, we get, after rescaling,

$$A_\epsilon = \left(\frac{1}{2} \int_{-1}^{1} \frac{dt}{3\frac{t^2}{\epsilon^2} + \epsilon^2} \right)^{-1} = \frac{\sqrt{3}}{\tan^{-1}\left(\frac{\sqrt{3}}{\epsilon^2}\right)}$$

and for the second

$$A_\epsilon = \left(\frac{1}{2} \int_{-1}^{1} \frac{dt}{3t^2 + 1} \right)^{-1} = \frac{3\sqrt{3}}{\pi},$$

both of which are finite and bounded away from zero as $\epsilon \to 0$. However, the two solutions show very different behavior as $\epsilon \to 0$. This comes from the fact that for the first conductivity $\mathcal{I}_1$ is $O(\frac{1}{\epsilon^2})$, while for the second, $\mathcal{I}_1$ is $O(1)$. Therefore, the coefficient of the $\frac{\partial^2 u}{\partial \tau^2}$ term for the first conductivity is $O(\frac{1}{\epsilon^2})$, so the solution should look like the $a_\epsilon = \frac{3}{\epsilon^2}$ case from earlier sections. In contract, this coefficient for the second conductivity is $O(1)$, so it should look like the $a_\epsilon = 3$ case from before.

It is also possible to define conductivities a_ϵ so that $\frac{A_\epsilon}{\epsilon} \to 0$, but $\epsilon A_\epsilon(\mathcal{I}_1 + \mathcal{I}_2) \to \infty$, which would represent a situation that can not be achieved by an isotropic conductivity. However, computing the integrals defining A_ϵ and $\mathcal{I}_1$ accurately becomes an issue

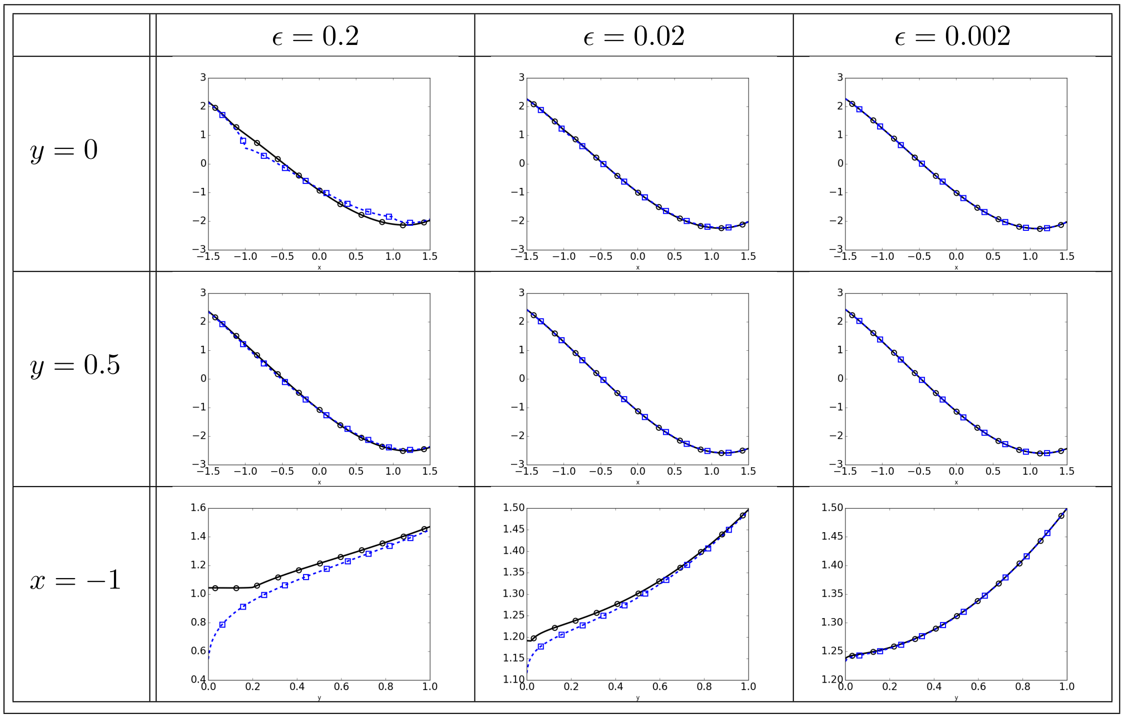

Figure 6.12: Numerical results for even symmetry with conductivity $a_\epsilon(x, y) = 3\frac{y^2}{\epsilon^2} + 1$. In each plot, the solid line with circles represents the solution u_ϵ^{ev}, and the dashed line with squares represents the solution u^{ev}.

because the functions involved are large and change rapidly. For instance, in the first example from before, the function $\frac{1}{a_\epsilon}$ is of order ϵ^2 for y away from zero, but is of order $\frac{1}{\epsilon^2}$ near $y = 0$, so numerical methods can struggle with accurately modeling this function. The conductivity profile that gives rise to a situation that has behavior distinct from any isotropic conductivity is much harder to model. The attempts made at constructing this situation numerically resulted in the reduced problem not accurately approximating the full problem with inhomogeneity because the coefficients were not computed accurately.

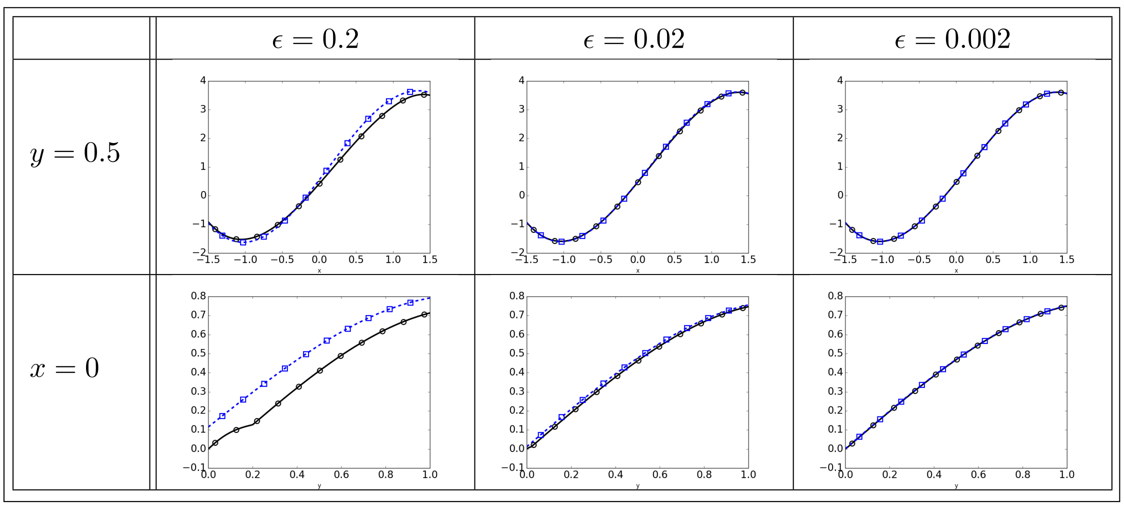

Figure 6.13: Numerical results for odd symmetry with conductivity $a_\epsilon(x, y) = 3\frac{y^2}{\epsilon^2} + 1$. In each plot, the solid line with circles represents the solution u_ϵ^{odd}, and the dashed line with squares represents the solution u^{odd}.

6.3.2 Conductivity independent of t

Again, we need to determine the problem that the even and odd parts of u_ϵ^0 will solve in the case where a_ϵ is independent of t. Doing the same computation as before, but starting from (5.19), gives

$$\begin{aligned}
\frac{\partial u^{ev}}{\partial y} &= \frac{1}{2}\left(\frac{\partial u^+}{\partial y} - \frac{\partial u^-}{\partial y}\right) \\
&= -\frac{\epsilon}{6}\left[2\frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^+}{\partial x}\right) + \frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^-}{\partial x}\right)\right. \\
&\left. + 2\frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^-}{\partial x}\right) + \frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^+}{\partial x}\right)\right] \\
&= \epsilon\frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^{ev}}{\partial x}\right), \\
\frac{\partial u^{odd}}{\partial y} &= \frac{\partial u^+}{\partial y} + \frac{\partial u^-}{\partial y} \\
&= \frac{a_\epsilon(x)}{\epsilon}u^{odd} - \frac{\epsilon}{3}\left[2\frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^+}{\partial x}\right) + \frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^-}{\partial x}\right)\right. \\
&\left. - 2\frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^-}{\partial x}\right) - \frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^+}{\partial x}\right)\right] \\
&= \frac{a_\epsilon(x)}{\epsilon}u^{odd} - \frac{\epsilon}{3}\frac{\partial}{\partial x}\left(a_\epsilon(x)\frac{\partial u^{odd}}{\partial x}\right),
\end{aligned}$$

which is implemented in variational form in the code for this simulation.

This example has boundary data $\varphi(x,y) = 1 - x + x^2y - (x+1)y^2 + 3xy^3$ with source function $f = 0$. The conductivity profile is

$$a_\epsilon(x,t) = \begin{cases} 3\epsilon^2 & x < -0.5 \\ 3\epsilon^{-2} & -0.5 < x < 0 \\ 3\epsilon^2 & 0 < x < 0.5 \\ 3\epsilon^{-2} & x > 0.5 \end{cases} \tag{6.1}$$

in ω_ϵ. The point of this conductivity profile is to illustrate that if the conductivity varies in the tangential direction, it is possible to generate additional $r^{1/2}$ singularity terms that also need to be dealt with. Figure 6.14 illustrates the results for even symmetry, and Figure 6.15 illustrates the results for odd symmetry.

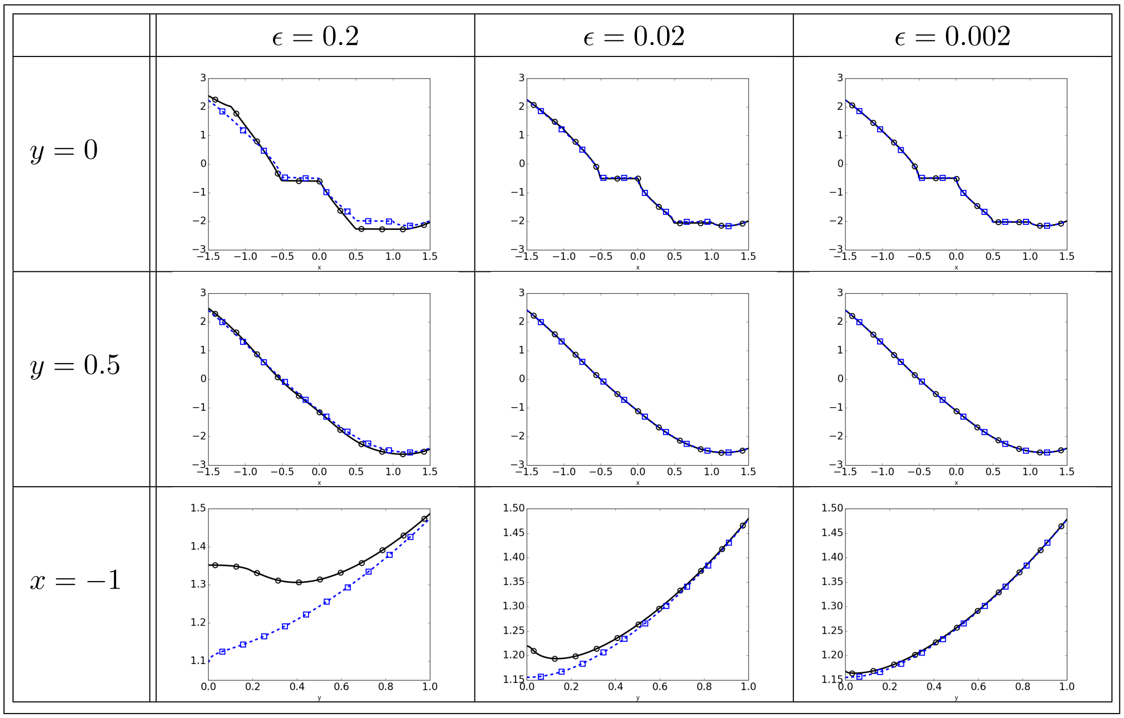

Figure 6.14: Numerical results for even symmetry with the discontinuous conductivity described in (6.1). In each plot, the solid line with circles represents the solution u_ϵ^{ev}, and the dashed line with squares represents the solution u^{ev}.

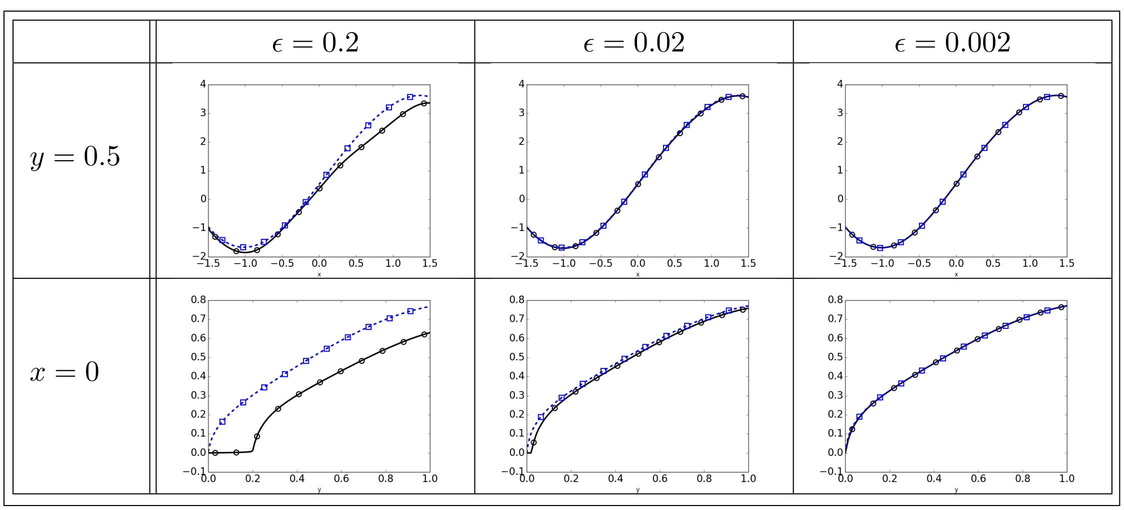

Figure 6.15: Numerical results for odd symmetry with the discontinuous conductivity described in (6.1). In each plot, the solid line with circles represents the solution u_ϵ^{odd}, and the dashed line with squares represents the solution u^{odd}.

6.3.3 Generic Conductivity

The next goal would be to model a generic variable conductivity a_ϵ within ω_ϵ. The only thing to be determined is the $\mathcal{M}$ and $\mathcal{K}$ parameters in order to plug them into the normal derivative terms in (5.22). However, we need to first determine the problems that the even and odd parts of u_ϵ^0 satisfy. We need to initially assume that $\hat{a}_\epsilon(x,t) = \hat{a}_\epsilon(x,-t)$, which will give that $\mathcal{K}_1 = \mathcal{K}_3$. Then, we can calculate

$$\begin{aligned}
\frac{\partial u^{ev}}{\partial n} &= \frac{1}{2}\left(\frac{\partial u^+}{\partial y} - \frac{\partial u^-}{\partial y}\right) \\
&= -\frac{\epsilon}{2}\left[\frac{\partial}{\partial \tau}\left(\mathcal{K}_1 \frac{\partial u^{ev}}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\left(\mathcal{K}_2 \frac{\partial u^{ev}}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\left((\mathcal{M}_2 + \mathcal{M}_3)u^{odd}\right)\right] \\
&= -\frac{\epsilon}{2}\left[\frac{\partial}{\partial \tau}\left((\mathcal{K}_1 + \mathcal{K}_2)\frac{\partial u^{ev}}{\partial \tau}\right) + \frac{\partial}{\partial \tau}\left((\mathcal{M}_2 + \mathcal{M}_3)u^{odd}\right)\right], \\
\frac{\partial u^{odd}}{\partial n} &= \frac{1}{2}\left(\frac{\partial u^+}{\partial y} + \frac{\partial u^-}{\partial y}\right) \\
&= \frac{A_\epsilon}{\epsilon} u^{odd} + \epsilon \mathcal{M}_1 u^{odd} - \frac{\epsilon}{2}\frac{\partial}{\partial \tau}\left((\mathcal{K}_1 - \mathcal{K}_2)\frac{\partial u^{odd}}{\partial \tau}\right) \\
&+ \frac{\partial}{\partial \tau}\left((\mathcal{M}_2 - \mathcal{M}_3)u^{odd}\right) - \mathcal{M}_2 \frac{\partial u^+}{\partial \tau} - \mathcal{M}_3 \frac{\partial u^-}{\partial \tau}.
\end{aligned}$$

Unfortunately, the even and odd parts of u_ϵ^0 are not decoupled in these derivatives, like they were in all of the problems in the previous sections. The only way this would happen is if all of the $\mathcal{M}$ coefficients are zero, which requires a lot of extra symmetry in the conductivity a_ϵ. In order to model this type of problem, a new numerical model would be required that does not solve the problem using symmetry, and can analyze the entire domain at once.

Chapter 7

Conclusions

The content of this thesis has three main parts. The first of these is the extension of the existing work to the case of open curves. We determine a way to take the results in [DV17] that work for closed curves and extend them to open curves. The main difficulty here is to deal with the endpoints of the curve, as there could be some issues with the regularity of the solution at these points. Our approach here is to determine the explicit form of the most singular part of the solution around each of these endpoints and use this to prove energy closeness, using the same energy lemma to move from energy closeness to norm closeness.

The second part is supplemental results to related problems. This includes the formula for computing the coefficient of $r^{1/2}$ in the expansion of the solution around the endpoints of σ, which allows for validation of the fact that this term does exist in these expansions. This section also contains explorations into adjacent problems with non-constant or anisotropic conductivities. These more general types of problems are more similar to physical problems that may arise and would have more applications.

The third part of this work is a numerical implementation of this problem, and the results that can be shown from that. The ability to generate these kinds of images can illustrate when and how the convergence occurs, as well as potentially hint at conjectures that might be able to be proven. For instance, Lemma 4.2.1 was first suggested via several numerical results that showed this behavior of the solution, and then we were able to prove this fact. The numerical simulations generated here make use of the symmetry assumptions made throughout this work to solve the problem on $B_2(0)^+$ and are run separately for even and odd symmetry. While this is slightly restrictive, it provides numerical evidence and support for all of the results that are proven here.

Where do we go from here? One clear path forward is to finish up some of the supplemental results, namely, proving the uniformly accurate asymptotic approximation of the reduced problem on σ to the full problem with the inhomogeneity in the case of a non-constant conductivity. As discussed and illustrated with numerical results, this type of argument, at least in the form presented here, will not work for this problem. In order to get the uniform approximation, the specific form of the $r^{1/2}$ term at the endpoints of σ needed to be known, but for variable conductivities, it is possible for extra $r^{1/2}$ singularities to be generated at interior points of σ. It is quite possible for a modified argument to work here, but the same one will not. In addition, attempting to find a way to represent the inner energy for anisotropic conductivities in terms of boundary data would be helpful for starting the dive into these problems. The problem for anisotropic conductvities as a whole is not very well understood, and all of the work done here assumed that the conductivity is diagonal in the (τ, n) basis, which is a reasonable start, but it still a limiting assumption. It would also be interesting to investigate what results still hold if the function f can have support on the inhomogeneity ω_ϵ. The current proofs rely on the fact that the function is harmonic near the singularity, but that may not be either necessary from the mathematical point of view or present in physical situations.

It would also be beneficial towards this analysis to develop more involved numerical models that can address the full problem in different geometries. This would allow the problem to be looked at without needing to assume even or odd symmetry and allow for curved geometries to be analyzed. This would give an idea for how the singularities do or do not change when σ is no longer a straight line segment, and give motivation for what types of results could be proved on this front. As discussed in Chapter 6, it will also allow for variable conductivities to be explored fully, as this problem can not be modeled with even and odd symmetry except in very special cases. In addition, the numerical results show that the value A_ϵ is not enough to determine the behavior of the solution, and so the impact factors as defined in Chapter 5 may not be the best way to model this situation. It would also be interesting to look into if there is a better way to solve for these parameters so that it is easier to determine the behavior of the solution to the reduced problem that is not hidden in these factors.

Most of the applications in this area, particularly those related to cloaking, require conductivities that are both anisotropic and non-constant. Solving this problem would require mixing the most difficult parts of both the anisotropic problem and the variable conductivity problem, so the approach given here may not be the best one. Techniques related to periodic homogenization may provide an easier route into analyzing these problems. Finally, once cloaking problems become a possibility, the next step would be to look at this problem in the case of non-zero frequency, i.e., for the Helmholtz equation. This would allow these results to be extended to electromagnetic imaging, and, in particular, the use of directional waves. As this problem has a non-rotationally invariant inhomogeneity, needing to cloak against directional waves may be easier because we can make use of this asymmetry to reduce the need for metamaterials in the cloak.

Appendix A

Selected Proofs

A.1 Correction function in Section 4.2.1

Lemma A.1.1. *Let z_ϵ be as in Section 4.2.1. Then*

$$||z_\epsilon||_{H^1(\Omega\setminus\omega_\epsilon)} \le C(f,\varphi)\epsilon^{\beta/2}$$

for any $\beta > 1$.

Proof. Recall that z_ϵ is the solution to

$$\begin{cases} -\Delta z_\epsilon = 0 & \Omega \setminus \overline{\omega_\epsilon} \\ z_\epsilon = 0 & \partial\Omega \\ z_\epsilon = u^{ev} \circ p_\sigma - u^{ev} & \partial\omega_\epsilon. \end{cases}$$

To prove the desired estimate, we split the function z_ϵ into two parts, $z_\epsilon = z_\epsilon^{(1)} + z_\epsilon^{(2)}$ where

$$\begin{cases} -\Delta z_\epsilon^{(1)} = 0 & \Omega \setminus \overline{\omega_\epsilon} \\ z_\epsilon^{(1)} = 0 & \partial\Omega \\ z_\epsilon^{(1)} = u^{ev,*} \circ p_\sigma - u^{ev,*} & \partial\omega_\epsilon, \end{cases}$$

and

$$\begin{cases} -\Delta z_\epsilon^{(2)} = 0 & \Omega \setminus \overline{\omega_\epsilon} \\ z_\epsilon^{(2)} = 0 & \partial\Omega \\ z_\epsilon^{(2)} = b_1 r^{1/2}\sin(\theta/2)\eta(r) + b_1' r'^{1/2}\sin(\theta'/2)\eta(r') & \partial\omega_\epsilon. \end{cases}$$

This decomposition relies on the fact that the function

$$b_1 r^{1/2}\sin(\theta/2)\eta(r) + b_1' r'^{1/2}\sin(\theta'/2)\eta(r')$$

is zero along the curve σ. We want to prove the desired norm bound for both $z_\epsilon^{(1)}$ and $z_\epsilon^{(2)}$ individually. To do so, we will show that it is possible to construct a function $Z_\epsilon^{(i)}$ that matches the boundary conditions of $z_\epsilon^{(i)}$ and has the desired norm bound for $i = 1, 2$. Then, by the fact that the variational formulation of the PDE that $z_\epsilon^{(i)}$ satisfies, we have

$$\int_{\Omega\setminus\omega_\epsilon} \nabla(z_\epsilon^{(i)} - Z_\epsilon^{(i)})\nabla z_\epsilon^{(i)}\ dx = 0$$

so that

$$\int_{\Omega\setminus\omega_\epsilon} |\nabla z_\epsilon^{(i)}|^2\ dx = \int_{\Omega\setminus\omega_\epsilon} \nabla Z_\epsilon^{(i)} \nabla z_\epsilon^{(i)}\ dx,$$

and so

$$||\nabla z_\epsilon^{(i)}||_{L^2(\Omega\setminus\omega_\epsilon)} \leq ||\nabla Z_\epsilon^{(i)}||_{L^2(\Omega\setminus\omega_\epsilon)} \leq C(f,\varphi)\epsilon^{\beta/2}.$$

This estimate can then be combined with an estimate on $Z_\epsilon^{(i)}\ |_{\partial\omega_\epsilon}$ and a version of Poincaré's inequality to get the H^1 estimates that we need.

For $Z_\epsilon^{(1)}$, we note that by the fact that $u^{ev,*}$ is $C^{1,\beta/2}$ with norm bounded by $C(f,\varphi)$, we have that on $\partial\omega_\epsilon^{int}$

$$||u^{ev,*}\circ p_\sigma - u^{ev,*}||_{L^\infty(\partial\omega_\epsilon^{int})} \leq C\epsilon^{1+\beta/2}||u^{ev,*}||_{C^{1,\beta/2}(\Omega\setminus\sigma)} \leq C(f,\varphi)\epsilon^{1+\beta/2}$$
$$\left|\left|\frac{\partial}{\partial x}(u^{ev,*}\circ p_\sigma - u^{ev,*})\right|\right|_{L^\infty(\partial\omega_\epsilon^{int})} \leq C\epsilon^{\beta/2}||u^{ev,*}||_{C^{1,\beta/2}(\Omega\setminus\sigma)} \leq C(f,\varphi)\epsilon^{\beta/2}$$

On $\partial\omega_\epsilon^{ends}$, the function $u^{ev,*}\circ p_\sigma$ is identically equal to $u^{ev,*}$ at the corresponding endpoint of σ, and so $\nabla u^{ev,*}\circ p_\sigma$ is zero on these boundaries. From Lemma 4.2.1, we know that $\frac{\partial u^{ev,*}}{\partial x} = 0$ at these endpoints and $\frac{\partial u^{ev,*}}{\partial y} = 0$ here as well by even symmetry. Therefore $\nabla u^{ev,*} = 0$ at each endpoint. Then the $C^{1,\beta/2}$ regularity of $u^{ev,*}$ gives that

$$\begin{aligned}
||u^{ev,*}\circ p_\sigma - u^{ev,*}||_{L^\infty(\partial\omega_\epsilon^{ends})} &\leq C\epsilon^{1+\beta/2}||u^{ev,*}||_{C^{1,\beta/2}(\partial\omega_\epsilon^{ends})} \\
&\leq C(f,\varphi)\epsilon^{1+\beta/2} \\
||\nabla(u^{ev,*}\circ p_\sigma - u^{ev,*})||_{L^\infty(\partial\omega_\epsilon^{ends})} &= ||\nabla u^{ev,*}||_{L^\infty(\partial\omega_\epsilon^{ends})} \\
&\leq C\epsilon^{\beta/2}||u^{ev,*}||_{C^{1,\beta/2}(\partial\omega_\epsilon^{ends})} \\
&\leq C(f,\varphi)\epsilon^{\beta/2}
\end{aligned}$$

Combining all of these estimates, we see that

$$||Z_\epsilon^{(1)}||_{W^{1,\infty}(\partial\omega_\epsilon)} = ||u^{ev,*}\circ p_\sigma - u^{ev,*}||_{W^{1,\infty}(\partial\omega_\epsilon)} \leq C(f,\varphi)\epsilon^{\beta/2}.$$

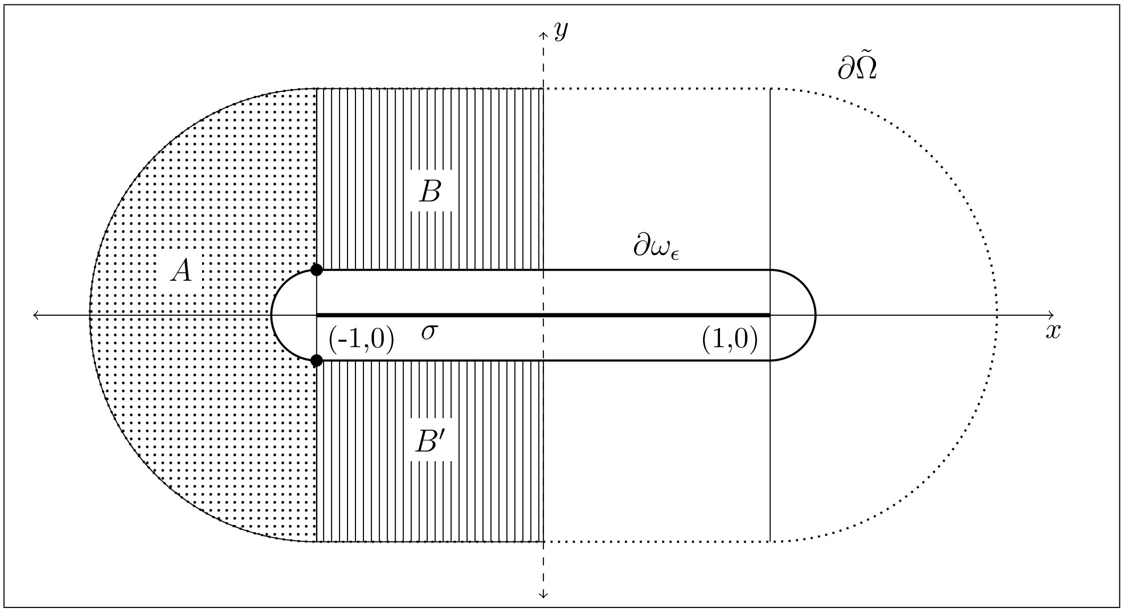

Figure A.1: Regions used in the construction of $Z_\epsilon^{(2)}$

Given this $W^{1,\infty}$ function on $\partial\omega_\epsilon$, we can extend it as a constant function along normal rays emanating from $\partial\omega_\epsilon$ and multiply by a cutoff function to get the extended $Z_\epsilon^{(1)}$ function to vanish near $\partial\Omega$. Since Ω is a fixed domain, we have that

$$||Z_\epsilon^{(1)}||_{H^1(\Omega\setminus\omega_\epsilon)} \leq C||Z_\epsilon^{(1)}||_{W^{1,\infty}(\Omega\setminus\omega_\epsilon)} \leq C||Z_\epsilon^{(1)}||_{W^{1,\infty}(\partial\omega_\epsilon)} \leq C(f,\varphi)\epsilon^{\beta/2}$$

which is exactly the desired estimate in Lemma A.1.1.

For $Z_\epsilon^{(2)}$, we carry out the same process, but we need to be more careful with the calculations. We start by defining this $Z_\epsilon^{(2)}$ function in the region $(\Omega \setminus \omega_\epsilon) \cap x < -1$, which is Region A in Figure A.1.

For ϵ small enough, the value of $Z_\epsilon^{(2)}$ on $\partial\omega_\epsilon$ is $-b_1 r^{1/2}\sin(\theta/2)$, so on this region, we extend Z_ϵ as a constant in the normal direction and multiply by a cutoff function. Thus, we have that in the region $(\Omega \setminus \omega_\epsilon) \cap \{x < 1\}$

$$Z_\epsilon^{(2)} = -b_1\epsilon^{1/2}\sin(\theta/2)\zeta(r)$$

where ζ is a cutoff function that is 1 on $r < \frac{\delta}{4}$ and 0 on $r > \frac{\delta}{2}$, where δ denotes the size of the neighborhood around σ where f vanishes. From this definition, it can be computed that

$$||Z_\epsilon^{(2)}||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{x<-1\})} \leq \epsilon^{1/2}\sqrt{|\log\epsilon|}C(f,\varphi). \tag{A.1}$$

From this, we see that the norm control we should be looking for in this case is of the form $\epsilon^{1/2}\sqrt{|\log\epsilon|}C(f,\varphi)$, which we can bound by $\epsilon^{\beta/2}$ for any $\beta<1$.

Next, we move on to the region $\{-1<x<0\}$, which is Region B (and B') in Figure A.1. To meet all of our requirements and make it easy to continue to extend this function to the right half of σ, we want a function $Z_\epsilon^{(2)}$ on $(\Omega\setminus\omega_\epsilon)\cap\{-1<x<0\}$ satisfying

$$\begin{cases} Z_\epsilon^{(2)} = -b_1\epsilon^{1/2}\frac{\sqrt{2}}{2}\zeta(y) & (\Omega\setminus\omega_\epsilon)\cap\{x=-1\} \\ Z_\epsilon^{(2)} = -b_1 r^{1/2}\sin(\theta/2)\eta(r) & \{-1<x<0,\ y=\pm\epsilon\} \\ Z_\epsilon^{(2)} = 0 & (\Omega\setminus\omega_\epsilon)\cap\{x=0\} \\ Z_\epsilon^{(2)} = 0 & \partial\Omega\cap\{-1<x<0\}, \end{cases}$$

with

$$||Z_\epsilon^{(2)}||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{-1<x<0\})}\le\epsilon^{1/2}\sqrt{|\log\epsilon|}C(f,\varphi).$$

In order to do this, we check that the boundary data define an $H^{1/2}$ function with the same norm control. Due to the cutoff functions and the edges where the function is identically zero, there are three things to check for this: the vertical segments along $\{x=-1\}$, the horizontal segments along $\{y=\pm\epsilon\}$, and the corners at $(-1,\pm\epsilon)$ [LM72].

For the vertical segments, we see that

$$||Z_\epsilon^{(2)}||_{H^{1/2}((\Omega\setminus\omega_\epsilon)\cap\{x=-1\})} = \left|\left|b_1\epsilon^{1/2}\frac{\sqrt{2}}{2}\zeta(y)\right|\right|_{H^{1/2}((\Omega\setminus\omega_\epsilon)\cap\{x=-1\})}\le\epsilon^{1/2}C(f,\varphi) \quad \text{(A.2)}$$

because ζ is smooth.

For the horizontal segments, we compute that, in rectangular coordinates,

$$\begin{aligned} Z_\epsilon^{(2)}(x,\pm\epsilon) &= -\frac{b_1\sqrt{2}}{2}r^{1/2}\sin(\theta/2)\eta(r) \\ &= \mp\frac{b_1}{2}\frac{\epsilon}{(\sqrt{(x+1)^2+\epsilon^2}+(x+1))^{1/2}}\eta(r), \\ \frac{\partial}{\partial x}Z_\epsilon^{(2)}(x,\pm\epsilon) &= \mp\frac{b_1}{2}\frac{\epsilon}{(\sqrt{(x+1)^2+\epsilon^2}+(x+1))^{1/2}}\frac{\partial}{\partial x}\eta(r) \\ &\quad \pm\frac{b_1}{4}\eta(r)\frac{\epsilon}{(\sqrt{(x+1)^2+\epsilon^2}+(x+1))^{3/2}}\left(\frac{x+1}{\sqrt{(x+1)^2+\epsilon^2}}+1\right), \end{aligned}$$

from which we compute the norms of this function on $\partial\omega_\epsilon\cap\{-1<x<0\}$. We see, after shifting coordinates, that

$$\begin{aligned}
||Z_\epsilon^{(2)}||^2_{L^2(\partial\omega_\epsilon\cap\{-1<x<0\})} &\le C(f,\varphi)^2\epsilon^2\int_0^1 \frac{dx}{\sqrt{x^2+\epsilon^2}+x}\\
&= C(f,\varphi)^2\epsilon^2\left(\int_0^\epsilon \frac{dx}{\sqrt{x^2+\epsilon^2}+x}+\int_\epsilon^1 \frac{dx}{\sqrt{x^2+\epsilon^2}+x}\right)\\
&\le C(f,\varphi)^2\epsilon^2\left(\int_0^\epsilon \frac{dx}{\epsilon}+\int_\epsilon^1 \frac{dx}{2x}\right)\\
&\le C(f,\varphi)^2\epsilon^2(1+\log\epsilon),
\end{aligned}$$

and

$$\begin{aligned}
\left|\left|\frac{\partial}{\partial x}Z_\epsilon^{(2)}\right|\right|^2_{L^2(\partial\omega_\epsilon\cap\{-1<x<0\})} &\le C(f,\varphi)^2\epsilon^2\left(\log\epsilon+\int_0^1\frac{dx}{(\sqrt{x^2+\epsilon^2}+x)^3}\right)\\
&= C(f,\varphi)^2\epsilon^2\Bigg(\log\epsilon+\int_0^\epsilon\frac{dx}{(\sqrt{x^2+\epsilon^2}+x)^3}\\
&+\int_\epsilon^1\frac{dx}{(\sqrt{x^2+\epsilon^2}+x)^3}\Bigg)\\
&\le C(f,\varphi)^2\epsilon^2\left(\log\epsilon+\int_0^\epsilon\frac{dx}{\epsilon^3}+C\int_\epsilon^1\frac{dx}{x^3}\right)\\
&\le C(f,\varphi).
\end{aligned}$$

Therefore, by logarithmic convexity of the Sobolev norms, we have that

$$||Z_\epsilon^{(2)}||^2_{H^{1/2}(\partial\omega_\epsilon\cap\{-1<x<0\})} \le \epsilon^{1/2}(|\log\epsilon|)^{1/4}C(f,\varphi) \le \epsilon^{1/2}\sqrt{|\log\epsilon|}C(f,\varphi). \tag{A.3}$$

for ϵ small.

Finally, for the corners at $(-1,\pm\epsilon)$, we compute the following integral, which is equivalent to the square of the $H^{1/2}$ norm of $Z_\epsilon^{(2)}$ across the corner.

$$\begin{aligned}
&\int_0^{\delta/2}\frac{|Z_\epsilon^{(2)}(-1,\epsilon+s)-Z_\epsilon^{(2)}(-1+s,\epsilon)|^2}{s}\,ds\\
&= \frac{|b_1|^2}{2}\int_0^{\delta/2}\left(\epsilon^{1/2}-\frac{\epsilon}{\sqrt{(s^2+\epsilon^2+s)^{1/2}}}\right)^2\frac{1}{s}\,ds\\
&= \epsilon\frac{|b_1|^2}{2}\int_0^{\delta/2\epsilon}\frac{\left[(\sqrt{t^2+1}+t)^{1/2}-1\right]^2}{t(\sqrt{t^2+1}+t)}\,dt\\
&\le \epsilon|\log\epsilon|C(f,\varphi)^2
\end{aligned} \tag{A.4}$$

where this last line comes from the fact that for t large, the integrand goes like $\frac{1}{t}$, which gives rise to the $\log\epsilon$ term, and the integrand vanishes at $t=0$, and so is bounded there.

Therefore, by combining (A.2), (A.3), and (A.4), we get that the function $Z_\epsilon^{(2)}$ has an $H^{1/2}$ norm on this boundary controlled by $\epsilon^{1/2}\sqrt{|\log\epsilon|}$, and so, by extension

$$||Z_\epsilon^{(2)}||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{-1<x<0\})} \leq \epsilon^{1/2}\sqrt{|\log\epsilon|}C(f,\varphi). \tag{A.5}$$

Since these definitions of $Z_\epsilon^{(2)}$ in regions A and B have the same trace along the boundaries $(\Omega\setminus\omega_\epsilon)\cap\{x=-1\}$, combining the estimates (A.1) and (A.5) give that this function $Z_\epsilon^{(2)}$ is in $H^1((\Omega\setminus\omega_\epsilon)\cap\{x<0\})$ with

$$||Z_\epsilon^{(2)}||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{x<0\})} \leq \epsilon^{1/2}\sqrt{|\log\epsilon|}C(f,\varphi). \tag{A.6}$$

We can perform the exact same calculations in the domain $(\Omega\setminus\omega_\epsilon)\cap\{x>0\}$ to get the same bound, and since our construction vanishes in a neighborhood of $x=0$, we can merge the two constructions together to get the function $Z_\epsilon^{(2)}$ that meets all of our requirements. It matches the boundary data of $z_\epsilon^{(2)}$ on $\partial\omega_\epsilon$, vanishes in a neighborhood of $\partial\Omega$, and has

$$||Z_\epsilon^{(2)}||_{H^1(\Omega\setminus\omega_\epsilon)} \leq \epsilon^{1/2}\sqrt{|\log\epsilon|}C(f,\varphi) \leq \epsilon^{\beta/2}C(f,\varphi)$$

for any $\beta<1$, as desired by Lemma A.1.1. ■

A.2 Correction vector field in Section 4.2.2

Lemma A.2.1. *There exists a vector field ξ_ϵ so that*

$$\begin{cases} \nabla\cdot\xi_\epsilon = 0 & \Omega\setminus\omega_\epsilon \\ (\xi_\epsilon)_2(x,\pm\epsilon) = \pm DU(x) - \frac{\partial}{\partial y}u^{ev}(x,\pm\epsilon) & -1<x<1 \\ \xi_\epsilon\cdot n = -\frac{\partial}{\partial n}u^{ev} & \partial\omega_\epsilon\cap\{(x,y):|x|>1\} \end{cases}$$

with

$$||\xi_\epsilon||_{L^2(\Omega\setminus\omega_\epsilon)} \leq \epsilon^{\beta/4}C(f,\varphi)$$

where $DU(x)$ and u^{ev} are as defined in Section 4.2.2.

Proof. Based on the fact that $\Delta u^{ev}=0$ in $\omega_\epsilon\setminus\sigma$, we know that

$$\int_{\omega_\epsilon^{ends,+}}\frac{\partial}{\partial n}u^{ev}\,ds + \int_{-1}^{1}\frac{\partial}{\partial y}u^{ev}(x,\epsilon)\,dx - \int_{y=0}\frac{\partial}{\partial y}u^{ev}(x,0_+)dx = 0$$

and

$$\int_{\omega_\epsilon^{ends,-}} \frac{\partial}{\partial n} u^{ev}\ ds - \int_{-1}^{1} \frac{\partial}{\partial y} u^{ev}(x,-\epsilon)\ dx + \int_{y=0} \frac{\partial}{\partial y} u^{ev}(x,0_-) dx = 0.$$

Since u^{ev} is even, we know that $\frac{\partial u^{ev}}{\partial y}(x,0_\pm) = 0$ on $\{y=0\} \setminus \sigma$. We can then combine the two previous equations to get that

$$\int_{\omega_\epsilon^{ends}} \frac{\partial}{\partial n} u^{ev}\ ds + \int_{-1}^{1} \frac{\partial}{\partial y} u^{ev}(x,\epsilon)\ dx - \int_{y=0} \frac{\partial}{\partial y} u^{ev}(x,0_+) dx$$
$$- \int_{-1}^{1} \frac{\partial}{\partial y} u^{ev}(x,-\epsilon)\ dx + \int_{y=0} \frac{\partial}{\partial y} u^{ev}(x,0_-) dx = 0.$$

From the definition of $DU(x)$ in Section 4.2.2, we know that

$$\int_{-1}^{1} DU(x)\ dx = \int_{-1}^{1} \frac{\partial}{\partial y} u^{ev}(x,0_+)\ dx$$

and by even symmetry of u^{ev}, we have that

$$\int_{-1}^{1} -DU(x)\ dx = \int_{-1}^{1} \frac{\partial}{\partial y} u^{ev}(x,0_-)\ dx.$$

Therefore, our calculation gives that

$$\int_{\omega_\epsilon^{ends}} \frac{\partial}{\partial n} u^{ev}\ ds + \int_{-1}^{1} \frac{\partial}{\partial y} u^{ev}(x,\epsilon) - DU(x)\ dx$$
$$- \int_{-1}^{1} \frac{\partial}{\partial y} u^{ev}(x,-\epsilon) - DU(x)\ dx = 0.$$

which implies that for the desired values of $\xi_\epsilon \cdot n$ given in Lemma A.2.1, we have that

$$\int_{\partial \omega_\epsilon} \xi_\epsilon \cdot n\ ds = 0.$$

Therefore, the function

$$V(\cdot) = \int_0^{\cdot} \xi_\epsilon \cdot n\ ds$$

is well-defined, where the integral is an arc-length integral along $\partial\omega_\epsilon$ starting at the point $(-1,\epsilon)$. If we can extend V to an H^1 function in $\Omega \setminus \omega_\epsilon$ with appropriate norm control, then the vector field

$$\xi_\epsilon = (\nabla V)^\perp = \left(\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial x} \right)$$

will be an L^2 vector field that has the appropriate normal derivatives along ω_ϵ and have the same norm bound as V. Thus, we seek to prove that this function V can be lifted to $H^1(\Omega \setminus \omega_\epsilon)$ with

$$||V||_{H^1(\Omega\setminus\omega_\epsilon)} \le C(f,\varphi)\epsilon^{\beta/4}.$$

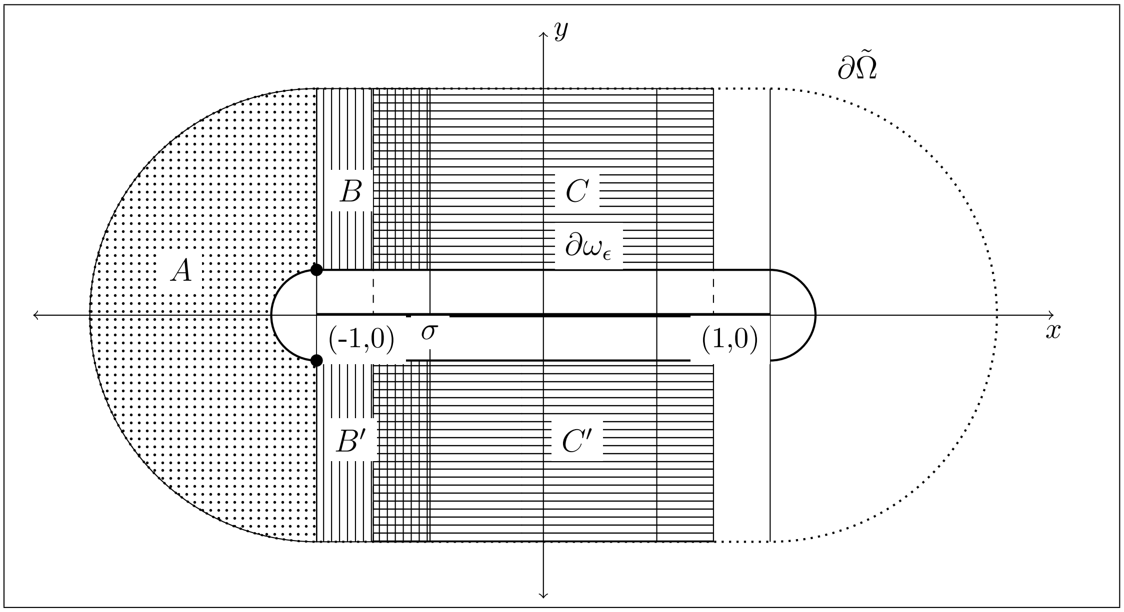

Figure A.2: Regions in the construction of V and ξ_ϵ

First, we look on the region $(\Omega \setminus \omega_\epsilon) \cap \{x < -1\}$, which is Region A in Figure A.2. On the part of $\partial\omega_\epsilon$ in this region, we have that

$$\frac{\partial}{\partial n} u^{ev} = \frac{b_1}{2}\epsilon^{-1/2}\sin(\theta/2) + \frac{\partial}{\partial r}u^{ev,*}$$

and by the regularity of $u^{ev,*}$ with the fact that the gradient vanishes at $(-1,0)$, we know that

$$\left|\frac{\partial}{\partial r}u^{ev,*}\right| \leq C\epsilon^{\beta/2}$$

on this boundary. Then, by integration, we have that, in polar coordinates around $(-1,0)$,

$$\begin{aligned} V(\epsilon,\theta) &= -\epsilon\int_{\pi/2}^{\theta}\frac{b_1}{2}\epsilon^{-1/2}\sin(\hat{\theta}/2)\ d\hat{\theta} - \epsilon\int_{\pi/2}^{\theta}\frac{\partial}{\partial r}u^{ev,*}\ d\hat{\theta} \\ &= b_1\epsilon^{1/2}\cos(\theta/2) - \frac{b_1}{2}\epsilon^{1/2} - \epsilon\int_{\pi/2}^{\theta}\frac{\partial}{\partial r}u^{ev,*}\ d\hat{\theta}. \end{aligned}$$

With the control we have on $\frac{\partial}{\partial r}u^{ev,*}$, we can extend this function as a constant function along normal rays from $\partial\omega_\epsilon$ to get a function $V(r,\theta)$ defined on all of $(\Omega\setminus\omega_\epsilon)\cap\{x < -1\}$ with

$$||V||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{x<-1\})} \leq C(f,\varphi)\epsilon^{1/2}.$$

Now, we consider the segment $\{-1 < x < -1+\rho,\ y = \epsilon\}$, where ρ is chosen small enough so that the cutoff function η is identically 1 on this segment, which is the bottom boundary of Region B in Figure A.2. Here, we know that

$$u^{ev} - u^{ev,*} = b_1 r^{1/2}\sin(\theta/2) = \frac{b_1}{\sqrt{2}}\sqrt{((x+1)^2+y^2)^{1/2}-(x+1)}.$$

Setting $s = x+1$, we then have that

$$\begin{aligned}\frac{\partial}{\partial y}u^{ev}(s,y) &= \frac{b_1}{2\sqrt{2}}\left(\sqrt{s^2+y^2}-s\right)^{-1/2}\left(\frac{y}{\sqrt{s^2+y^2}}\right) + \frac{\partial}{\partial y}u^{ev,*}\\ &= \frac{b_1}{2\sqrt{2}}\left(\frac{\left(\sqrt{s^2+y^2}+s\right)^{1/2}}{\sqrt{s^2+y^2}}\right) + \frac{\partial}{\partial y}u^{ev,*}.\end{aligned}$$

Thus, the function V (now in rectangular coordinates) has the form

$$\begin{aligned}V(x,\epsilon) &= -\int_{-1}^{x}(\xi_\epsilon)_2(s,\epsilon)\ ds = \int_{-1}^{x}\left(\frac{\partial}{\partial y}u^{ev}(s,\epsilon) - DU_+(s)\right)\ ds\\ &= \int_0^{x+1}\frac{b_1}{2\sqrt{2}}\left(\frac{\left(\sqrt{s^2+\epsilon^2}+s\right)^{1/2}}{\sqrt{s^2+\epsilon^2}}\right) + \frac{\partial}{\partial y}u^{ev,*}(s-1,\epsilon)\\ &\quad - \frac{b_1}{2}\max\{s,\epsilon\}^{-1/2} - d_\epsilon - \frac{\partial}{\partial y}u^{ev,*}(s-1,0_+)\ ds \\ &= \frac{b_1}{2}\int_0^{x+1}\frac{\left(\sqrt{s^2+\epsilon^2}+s\right)^{1/2}}{\sqrt{2}\sqrt{s^2+\epsilon^2}} - \max\{s,\epsilon\}^{-1/2}\ ds\\ &\quad + \int_0^{x+1}\frac{\partial}{\partial y}u^{ev,*}(s-1,\epsilon) - \frac{\partial}{\partial y}u^{ev,*}(s-1,0_+)\ ds - d_\epsilon(x+1).\end{aligned} \tag{A.7}$$

By the regularity of $u^{ev,*}$, we know that

$$\left|\frac{\partial}{\partial y}u^{ev,*}(s-1,\epsilon) - \frac{\partial}{\partial y}u^{ev,*}(s-1,0_+)\right| \le C(f,\varphi)\epsilon^{\beta/2}$$

so that

$$\left|\int_0^{x+1}\frac{\partial}{\partial y}u^{ev,*}(s-1,\epsilon) - \frac{\partial}{\partial y}u^{ev,*}(s-1,0_+)\ ds\right| \le C(f,\varphi)\epsilon^{\beta/2}|x+1|,$$

and by the definition of d_ϵ we know that

$$|d_\epsilon(x+1)| \le C(f,\varphi)\epsilon^{1/2}|x+1|.$$

To handle the first term, we let $g(t)$ be the function

$$g(t) = \int_0^t\left(\frac{\left(\sqrt{s^2+\epsilon^2}+s\right)^{1/2}}{\sqrt{2}\sqrt{s^2+\epsilon^2}} - \max\{s,\epsilon\}^{-1/2}\right)\ ds.$$

After a change of variables, this becomes

$$g(t) = \sqrt{\epsilon} \int_0^{t/\epsilon} \left(\frac{\left(\sqrt{s^2+1}+s\right)^{1/2}}{\sqrt{2(s^2+1)}} - \max\{s,1\}^{-1/2} \right) \, ds.$$

In looking at the integrand of g, we see that as $s \to \infty$,

$$\begin{aligned}
\frac{\left(\sqrt{s^2+1}+s\right)^{1/2}}{\sqrt{2(s^2+1)}} - \max\{s,1\}^{-1/2} &= \frac{\left(\sqrt{s^2+1}+s\right)^{1/2}}{\sqrt{2(s^2+1)}} - s^{-1/2} \\
&= \frac{\sqrt{s}\left(\sqrt{s^2+1}+s\right)^{1/2} - \sqrt{2(s^2+1)}}{\sqrt{s}\sqrt{2(s^2+1)}} \\
&= \frac{s\left(\sqrt{1+s^{-2}}+1\right)^{1/2} - s\sqrt{2(1+s^{-2})}}{s^{3/2}\sqrt{2(1+s^2)}} \\
&= s^{-1/2}\frac{\left(\sqrt{1+s^{-2}}+1\right)^{1/2} - \sqrt{2(1+s^{-2})}}{\sqrt{2(1+s^2)}} \\
&= O(s^{-5/2}),
\end{aligned}$$

and as $s \to 0$

$$\frac{\left(\sqrt{s^2+1}+s\right)^{1/2}}{\sqrt{2(s^2+1)}} - \max\{s,1\}^{-1/2} = \frac{\left(\sqrt{s^2+1}+s\right)^{1/2}}{\sqrt{2(s^2+1)}} - 1 = O(1).$$

Therefore, the integrand of g is $L^1((0,\infty))$, and so we know that

$$g(t) \leq C\sqrt{\epsilon}$$

with

$$C = \int_0^\infty \left| \frac{\left(\sqrt{s^2+1}+s\right)^{1/2}}{\sqrt{2(s^2+1)}} - \max\{s,1\}^{-1/2} \right| \, ds < \infty.$$

Therefore, the first term in the expression of $V(x,\epsilon)$ in (A.7) is also bounded by $C(f,\varphi)\sqrt{\epsilon}$ uniformly in x, giving that

$$||V||_{L^2(\{-1<x<-1+\rho, y=\epsilon\})} \leq ||V||_{L^\infty(\{-1<x<-1+\rho, y=\epsilon\})} \leq C(f,\varphi)\epsilon^{\beta/2}.$$

We want to say something about the $H^{1/2}$ norm of this function, which will let us talk about the H^1 norm of the lifted function. Thus, we look at the first derivative of

V, which by the fundamental theorem of calculus is

$$\frac{\partial}{\partial x}V(x,\epsilon) = \frac{b_1}{2}\frac{\left(\sqrt{(x+1)^2+\epsilon^2}+(x+1)\right)^{1/2}}{\sqrt{2((x+1)^2+\epsilon^2)}} - \max\{x+1,\epsilon\}^{-1/2}\ ds$$
$$+\frac{\partial}{\partial y}u^{ev,*}(x,\epsilon) - \frac{\partial}{\partial y}u^{ev,*}(x,0_+) - d_\epsilon.$$

The same change of variables as before tells us that

$$\int_{-1}^{-1+\rho}\left|\frac{\left(\sqrt{(x+1)^2+\epsilon^2}+(x+1)\right)^{1/2}}{\sqrt{2((x+1)^2+\epsilon^2)}} - \max\{x+1,\epsilon\}^{-1/2}\right|^2 dx \le C$$

where

$$C = \int_0^\infty \left|\frac{\left(\sqrt{s^2+\epsilon^2}+s\right)^{1/2}}{\sqrt{2(s^2+\epsilon^2)}} - \max\{s,\epsilon\}^{-1/2}\right|^2 ds < \infty,$$

and this constant is finite because the decay on the integrand that was proved earlier also shows that this function is $L^2((0,\infty))$. Our previous bounds on d_ϵ and $u^{ev,*}$ give that

$$\left|\left|d_\epsilon + \frac{\partial u^{ev,*}}{\partial y}(\cdot,\epsilon) - \frac{\partial u^{ev,*}}{\partial y}(\cdot,0_+)\right|\right|_{L^2(\{-1<x<-1+\rho,y=\epsilon\})} \le C(f,\varphi)\epsilon^{\beta/2}$$

which, all together, implies that

$$\left|\left|\frac{\partial}{\partial x}V(\cdot,\epsilon)\right|\right|_{L^2(\{-1<x<-1+\rho,y=\epsilon\})} \le C(f,\varphi).$$

Therefore, by logarithmic convexity of the Sobolev norms,

$$||V||_{H^{1/2}(\{-1<x<-1+\rho,y=\epsilon\})} \le C(f,\varphi)\epsilon^{\beta/4}$$

for any $\beta<1$. In order to lift this to an H^1 function on $(\Omega\setminus\omega_\epsilon)\cap\{-1<x<-1+\rho\}$ with the appropriate norm control, we need to show that it has the same $H^{1/2}$ control on the vertical segment $(\Omega\setminus\omega_\epsilon)\cap\{x=-1\}$ as well as across the corner at $(-1,\epsilon)$ [LM72]. Since we are trying to match the function V we defined on $\{x<-1\}$, we know that we want V to be zero along the vertical segment, which is clearly in $H^{1/2}$. The corner integral then simplifies to

$$\int_{-1}^{-1+\rho}\frac{|V(x,\epsilon)|^2}{|x+1|}\,dx \le \frac{|b_1|^2}{2}\int_{-1}^{-1+\rho}\frac{|g(x+1)|^2}{|x+1|}\,dx$$
$$+2\int_{-1}^{-1+\rho}\frac{\left|\int_{-1}^{x}\left(-d_\epsilon+\frac{\partial u^{ev,*}}{\partial y}(s,\epsilon)-\frac{\partial u^{ev,*}}{\partial y}(s,0_+)\right)\,ds\right|^2}{|x+1|}\,dx \tag{A.8}$$

For the second line above, we know that

$$\left| -d_\epsilon + \frac{\partial u^{ev,*}}{\partial y}(s,\epsilon) - \frac{\partial u^{ev,*}}{\partial y}(s,0_+) \right| \le C(f,\varphi)\epsilon^{\beta/2}$$

so that

$$2\int_{-1}^{-1+\rho} \frac{\left|\int_{-1}^{x}\left(-d_\epsilon + \frac{\partial u^{ev,*}}{\partial y}(s,\epsilon) - \frac{\partial u^{ev,*}}{\partial y}(s,0_+)\right)\, ds\right|^2}{|x+1|}\, dx \le C(f,\varphi)^2\epsilon^\beta \qquad \text{(A.9)}$$

by applying this bound to the inner function.

For the first term involving the function g, we have already seen that the integrand of this function is both in L^1 and L^2 on $(0,\infty)$. Therefore, we can use Hölder's inequality to show that

$$\begin{aligned} |g(t)| &\le \sqrt{\epsilon}\left(\int_0^{t/\epsilon}\left|\frac{\left(\sqrt{s^2+1}+s\right)^{1/2}}{\sqrt{2}\sqrt{s^2+1}} - \max\{s,1\}^{-1/2}\right|^p\, ds\right)^{\frac{1}{p}}\left(\int_0^{t/\epsilon} 1\, ds\right)^{\frac{1}{q}} \\ &\le C_p t^{\frac{1}{q}}\epsilon^{\frac{1}{2}-\frac{1}{q}} \end{aligned}$$

where C_p depends on the L^p norm of the integrand of g, with $1 < p < 2$, and $\frac{1}{q} = 1 - \frac{1}{p}$. From this, we get that

$$\int_{-1}^{-1+\rho}\frac{|g(x+1)|^2}{|x+1|}\, dx \le C\epsilon^{1-\frac{2}{q}}\int_0^\rho s^{\frac{2}{q}-1}\, ds \le C\epsilon^{1-\frac{2}{q}} \qquad \text{(A.10)}$$

for any $q > 2$. Putting the estimates (A.9) and (A.10) into (A.8), we get that

$$\int_{-1}^{-1+\rho}\frac{|V(x,\epsilon)|^2}{|x+1|}\, dx \le C(f,\varphi)^2\epsilon^\beta$$

for any $\beta < 1$, which is exactly the corner condition we need. Therefore, we get that this function V is $H^{1/2}$ on the entire boundary, and so we can lift it to a function $V \in H^1((\Omega\setminus\omega_\epsilon)\cap\{-1 < x < -1+\rho, y > \epsilon\})$ with

$$||V||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{-1<x<-1+\rho,y>\epsilon\})} \le C(f,\varphi)\epsilon^{\beta/4}.$$

We can perform this same construction on $(\Omega\setminus\omega_\epsilon)\cap\{-1 < x < -1+\rho, y < -\epsilon\}$, which is Region B' in Figure A.2. The only difference is that on the vertical segment, we will have constant data $\int_0^{\epsilon\pi}\xi_\epsilon\cdot n\, ds$ instead of constant data 0, and the function $V(x,\epsilon)$ will also be shifted by the same amount. Thus the identical arguments go through (once we restrict to only caring about local integrability and regularity) to give an extension V with the same norm control. Since the boundary data on both vertical segments match with those of the radial function we defined on $(\Omega \setminus \omega_\epsilon) \cap \{x < -1\}$ (Region A), we can stitch these together into a single function on $(\Omega \setminus \omega_\epsilon) \cap \{x < -1 + \rho\}$ with the desired boundary data on $\partial\omega_\epsilon$ and

$$||V||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{x<-1+\rho\}} \leq C(f,\varphi)\epsilon^{\beta/4}.$$

After shifting by the constant corresponding to $\int \xi_\epsilon \cdot n \; ds$ from $(-1,\epsilon)$ to $(1,-\epsilon)$ along $\partial\omega_\epsilon$, the same argument works on $\{x > 1-\rho\}$ to get a function with the same norm control on that region. On the interior segments $\{-1 + \frac{\rho}{2} < x < 1 - \frac{\rho}{2},\ y = \pm\epsilon\}$, which are the interior boundaries of Regions C and C' in Figure A.2, V has $W^{1,\infty}$ norm controlled by $\epsilon^{\beta/2}C(f,\varphi)$, so that V can be extended to a function in $H^1((\Omega \setminus \omega_\epsilon) \cap \{-1 + \frac{\rho}{2} < x < 1 - \frac{\rho}{2}\})$ with boundary data equal to $\int_0^\cdot \xi_\epsilon \cdot n \; ds$ and

$$||V||_{H^1((\Omega\setminus\omega_\epsilon)\cap\{-1+\frac{\rho}{2}<x<1-\frac{\rho}{2}\})} \leq C(f,\varphi)\epsilon^{\beta/2}.$$

Finally, we can paste these three constructions together with a partition of unity in x to get a function $V \in H^1(\Omega \setminus \omega_\epsilon)$ with

$$V\,|_{\partial\omega_\epsilon} = \int_0^\cdot \xi_\epsilon \cdot n \; ds \qquad ||V||_{H^1(\Omega\setminus\omega_\epsilon)} \leq C(f,\varphi)\epsilon^{\beta/4}$$

for any $\beta < 1$. Multiplying this by a cutoff function so that it vanishes near $\partial\Omega$ gives the function V that we wanted to construct.

Then, the vector field

$$\xi_\epsilon = (\nabla V)^\perp = \left(\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial x}\right)$$

satisfies

$$\begin{cases} \nabla \cdot \xi_\epsilon = 0 & \Omega \setminus \omega_\epsilon \\ (\xi_\epsilon)_2(x,\pm\epsilon) = \pm DU(x) - \frac{\partial}{\partial y}u^{ev}(x,\pm\epsilon) & -1 < x < 1 \\ \xi_\epsilon \cdot n = -\frac{\partial}{\partial n}u^{ev} & \partial\omega_\epsilon \cap \{(x,y) : |x| > 1\} \end{cases}$$

with

$$||\xi_\epsilon||_{L^2(\Omega\setminus\omega_\epsilon)} \leq \epsilon^{\beta/4}C(f,\varphi),$$

where the value of the normal derivatives comes from the fact that

$$\xi_\epsilon \cdot n = \frac{\partial V}{\partial \tau}$$

and the tangential derivatives of V are zero on $\partial\Omega$ and the desired value of $\xi_\epsilon \cdot n$ on $\partial\omega_\epsilon$. ■

A.3 Correction vector field in Section 4.3.2

Lemma A.3.1. *There exists a vector field ξ_ϵ satisfying*

$$\begin{cases} \nabla \cdot \xi_\epsilon = 0 & \Omega \setminus \omega_\epsilon \\ \xi_\epsilon \cdot n = 0 & \partial\Omega \\ \xi_\epsilon \cdot n = -\nabla u_0^{ev} \cdot n & \partial\omega_\epsilon, \end{cases}$$

with norm control of

$$||\xi_\epsilon||_{L^2(\Omega\setminus\omega_\epsilon)} \leq \epsilon^\beta C(f,\varphi)$$

for any $\beta < 1$, where u_0^{ev} is as defined in Section 4.3.2.

Proof. The construction of this ξ_ϵ is very similar to the work in Section A.2. Since $\Delta u_0^{ev} = 0$ in ω_ϵ, then we know that

$$\int_{\partial\omega_\epsilon} -\nabla u_0^{ev} \cdot n \, ds = 0,$$

so the function

$$V(\cdot) = \int_0^{\cdot} \nabla u_0^{ev} \, ds,$$

where the integral is taken to be in terms of arclength around $\partial\omega_\epsilon$ starting at $(-1,\epsilon)$ is a well-defined function. Since u_0^{ev} is C^2 in Ω, but $\partial\omega_\epsilon$ is only a $C^{1,\beta}$ curve, for any $\beta < 1$, the same calculations as before will give a similar estimate. In this case, we only need to worry about the parts of the argument that involved the $u^{ev,*}$ function, since u_0^{ev} is C^2 and does not have an $r^{1/2}$ singularity. Pushing all of the estimates through gives that this V can be lifted to all of $\Omega \setminus \omega_\epsilon$ with

$$||V||_{H^1(\Omega\setminus\omega_\epsilon)} \leq C(f,\varphi)\epsilon^\beta.$$

As before, cutting off this function near $\partial\Omega$ and taking

$$\xi_\epsilon = (\nabla V)^\perp = \left(\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial x}\right)$$

will give a solution to our problem with the norm control that we need. ■

A.4 Expanding the energy estimate to Ω in Section 4.4

In Section 4.4, we prove that

$$|E^{odd'}(u^{odd'}) - E^{odd}_\epsilon(u^{odd}_\epsilon)| = |E^{ev,N}(v^{ev,N}) - E^N_\epsilon(v^N_\epsilon)| \leq C(f,\varphi)^2\epsilon^{\beta/4}$$

as long as the energies are restricted to only involve integrals over ω_δ, the region where $f \equiv 0$. In this section, we extend this to all of Ω.

Lemma A.4.1. *The energy estimate*

$$|E^{odd'}(u^{odd'}) - E^{odd}_\epsilon(u^{odd}_\epsilon)| \leq C(f,\varphi)^2\epsilon^{\beta/4}$$

continues to hold when the domains of all of the energies are Ω, any domain symmetric around σ, instead of ω_δ.

Proof. The proof follows the same argument as in Section 4.7, and is only included here to show that it does not require future results to expand these energies to all of Ω. As in Section 4.7, we will use subscripts to denote the domains of the integrals in the energy expression, as well as the boundary curve where boundary conditions are applied.

Let u^{odd} and $u^{odd'}$ be the solutions of their respective differential equations on Ω with boundary data φ. Let ψ and ψ' be the traces of the functions u^{odd} and $u^{odd'}$ respectively on $\partial\omega_\delta$. Define v^{odd} as the minimizer of $E^{odd}_{\omega_\delta}$ with boundary data ψ' and $v^{odd'}$ as the minimizer of $E^{odd'}_{\omega_\delta}$ with boundary data ψ. Then, we define

$$\bar{u}^{odd} = \begin{cases} u^{odd'} & \Omega \setminus \omega_\delta \\ v^{odd} & \omega_\delta \end{cases} \qquad \bar{u}^{odd'} = \begin{cases} u^{odd} & \Omega \setminus \omega_\delta \\ v^{odd'} & \omega_\delta \end{cases}$$

Based on how these functions are defined and their matching traces on $\partial\omega_\delta$, $\bar{u}^{odd}$ and $\bar{u}^{odd'}$ are H^1 across $\partial\omega_\delta$, and the behavior near σ gives that

$$\bar{u}^{odd} \in V^{odd} \qquad \text{and} \qquad \bar{u}^{odd'} \in V^{odd'}$$

which allows us to use them in energy expressions.

By the result that we prove in Section 4.4, we know that

$$|E^{odd}_{\omega_\delta}(v^{odd}) - E^{odd'}_{\omega_\delta}(u^{odd'})| \leq C(f,\varphi)^2 \epsilon^{\beta/4}$$

and

$$|E^{odd}_{\omega_\delta}(u^{odd}) - E^{odd'}_{\omega_\delta}(v^{odd'})| \leq C(f,\varphi)^2 \epsilon^{\beta/4},$$

because the data ψ and ψ' are controlled by $C(f,\varphi)$. Then, by the same arguments in Section 4.7, we have that

$$\begin{aligned} E^{odd}_{\Omega}(\bar{u}^{odd}) &= E^{odd}_{\omega_\delta}(v^{odd}) + \frac{1}{2}\int_{\Omega\setminus\omega_\delta} |\nabla u^{odd'}|^2 \, dx - \int_{\Omega\setminus\omega_\delta} f u^{odd'} \, dx \\ &\leq E^{odd'}_{\omega_\delta}(u^{odd'}) + C(f,\varphi)^2 \epsilon^{\beta/4} \\ &\quad + \frac{1}{2}\int_{\Omega\setminus\omega_\delta} |\nabla u^{odd'}|^2 \, dx - \int_{\Omega\setminus\omega_\delta} f u^{odd'} \, dx \\ &= E^{odd'}_{\Omega}(u^{odd'}) + C(f,\varphi)^2 \epsilon^{\beta/4} \end{aligned}$$

and

$$\begin{aligned} E^{odd'}_{\Omega}(\bar{u}^{odd'}) &= E^{odd'}_{\omega_\delta}(v^{odd'}) + \frac{1}{2}\int_{\Omega\setminus\omega_\delta} |\nabla u^{odd}|^2 \, dx - \int_{\Omega\setminus\omega_\delta} f u^{odd} \, dx \\ &\leq E^{odd}_{\omega_\delta}(u^{odd}) + C(f,\varphi)^2 \epsilon^{\beta/4} \\ &\quad + \frac{1}{2}\int_{\Omega\setminus\omega_\delta} |\nabla u^{odd}|^2 \, dx - \int_{\Omega\setminus\omega_\delta} f u^{odd} \, dx \\ &= E^{odd}_{\Omega}(u^{odd}) + C(f,\varphi)^2 \epsilon^{\beta/4}. \end{aligned}$$

Then, since u^{odd} and $u^{odd'}$ are minimizers of their respective energy functionals over Ω, we have that

$$E^{odd}_{\Omega}(u^{odd}) \leq E^{odd}_{\Omega}(\bar{u}^{odd}) \leq E^{odd'}_{\Omega}(u^{odd'}) + C(f,\varphi)^2 \epsilon^{\beta/4},$$
$$E^{odd'}_{\Omega}(u^{odd'}) \leq E^{odd'}_{\Omega}(\bar{u}^{odd'}) \leq E^{odd}_{\Omega}(u^{odd}) + C(f,\varphi)^2 \epsilon^{\beta/4}.$$

Combining these two estimates above gives exactly the result desired in Lemma A.4.1, showing that these energy estimates extend to all of Ω. ■

Appendix B

A Primer on Dual Energies

The idea of dual energies and how they can be used in solving this type of problem goes back to at least [MS61]. The motivation for comes from the following equality of real numbers:

$$\frac{1}{2}a^2 = \max_{b\in\mathbb{R}} ab - \frac{1}{2}b^2.$$

The function on the right hand side is concave down in b, the unique maximum occurs when $b = a$, and the expression evaluates to $\frac{1}{2}a^2$. We can use this argument to turn the minimization problem that normally defines the energy formulation of an elliptic partial differential equation into a maximization problem. When the problem becomes a maximization, we know that any test function has a smaller dual energy, which thus allows us to prove both sides of an energy inequality. For instance, if we want to show that energies E_1 and E_2 are close at their minimizers, we can do that using the dual energy E_1^c to E_1. Assume that u_1 and u_2 are the minimizers of E_1 and E_2 respectively. Then, we know that

$$E_1(u_1) - E_2(u_2) \leq E_1(u^*) - E_2(u_2)$$

where u^* is any function in the appropriate space of functions that u_1 comes from. Ideally, one will be able to prove that this is small for a function u^* that may be defined to look like u_2. For the other direction of the inequality, we can use the fact that

$$E_1(u_1) - E_2(u_2) = E_1^c(u_1) - E_2(u_2) \geq E_1^c(u^*) - E_2(u_2)$$

where u^* is again any function chosen from the appropriate space that u_1 comes from. By the same logic, we then try to make this difference small by choosing u^* so it looks like u_2. The next section will show how this works for the particular case of the

Laplacian, and the rest of this appendix will show the derivation of the dual energy for all of the different types of problems that arise in the this thesis.

B.1 Dual energy for the Laplacian

Consider the solution to

$$\begin{cases} -\Delta u = f & \Omega \\ u = \varphi & \partial\Omega. \end{cases}$$

We know that this solution can be found as the minimizer of an energy functional

$$E(u) = \min_{v \in V} E(v),$$

where

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \; dx - \int_\Omega fv \; dx$$

and

$$V = \{v \in H^1(\Omega) \ : \ v = \varphi \text{ on } \partial\Omega\}.$$

We want to find a corresponding maximization problem that can be used to characterize u. To do this, we use the principle described in the introduction to this Appendix, and rewrite the squared term as a maximum

$$\frac{1}{2}\int_\Omega |\nabla v|^2 \; dx - \int_\Omega fv \; dx = \max_{\xi \in L^2(\Omega)^2} \int_\Omega \nabla v \cdot \xi \; dx - \frac{1}{2}\int_\Omega |\xi|^2 \; dx - \int_\Omega fv \; dx,$$

where the maximum is acheived at $\xi = \nabla v$. We then integrate by parts to see that

$$\begin{aligned} E(u) &= \min_{v \in V} \frac{1}{2}\int_\Omega |\nabla v|^2 \; dx - \int_\Omega fv \; dx \\ &= \min_{v \in V} \max_{\xi \in L^2(\Omega)^2} \int_\Omega \nabla v \cdot \xi \; dx - \frac{1}{2}\int_\Omega |\xi|^2 \; dx - \int_\Omega fv \; dx \\ &= \min_{v \in V} \max_{\xi \in L^2(\Omega)^2} \int_{\partial\Omega} v\xi \cdot n \; ds - \frac{1}{2}\int_\Omega |\xi|^2 \; dx - \int_\Omega (\nabla \cdot \xi + f)v \; dx \\ &\geq \max_{\xi \in L^2(\Omega)^2} \min_{v \in V} \int_{\partial\Omega} v\xi \cdot n \; ds - \frac{1}{2}\int_\Omega |\xi|^2 \; dx - \int_\Omega (\nabla \cdot \xi + f)v \; dx. \end{aligned}$$

Now, if $\nabla \cdot \xi + f \neq 0$, then v can be chosen so that the inner minimization approaches $-\infty$, which is not where the maximum in ξ is achieved. Therefore, we can add in the constraint $-\nabla \cdot \xi = f$ and not affect the final result of the calculation. Therefore, if we define the set

$$W = \{\xi \in L^2(\Omega)^2 \ \mid \ -\nabla \cdot \xi = f \text{ in } \Omega\}, \tag{B.1}$$

where this condition is interpreted in the sense of distributions, the above calculations give that

$$\min_{v \in V} \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \int_\Omega fv \, dx \geq \max_{\xi \in W} \min_{v \in V} \int_{\partial\Omega} v\xi \cdot n \, ds - \frac{1}{2} \int_\Omega |\xi|^2 \, dx.$$

Finally, v only appears in this last step in terms of its value on the boundary of Ω, which we know is φ by the definition of the set V. Replacing $v \mid_{\partial\Omega}$ by φ, we can then remove the minimization over v, which gives

$$\min_{v \in V} \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \int_\Omega fv \, dx \geq \max_{\xi \in W} \int_{\partial\Omega} \varphi\xi \cdot n \, ds - \frac{1}{2} \int_\Omega |\xi|^2 \, dx,$$

from which we can define the dual energy functional as

$$E^c(\xi) = \int_{\partial\Omega} \varphi\xi \cdot n \, ds - \frac{1}{2} \int_\Omega |\xi|^2 \, dx \tag{B.2}$$

over the set W defined in (B.1). By integration by parts, and the fact that we know where we get equality after the first step of our calculations, we see that

$$E^c(\nabla u) = E(u),$$

which makes sense because $\nabla u \in W$ as $-\Delta u = f$. Therefore, we see that

$$E(u) = \min_{v \in V} E(v) \geq \max_{\xi \in W} E^c(\xi) \geq E^c(\nabla u) = E(u), \tag{B.3}$$

so that everything in the above expression is actually an equality. Therefore, we have that

$$E(u) = E^c(\nabla u) = \max_{\xi \in W} E^c(\xi)$$

where E^c and W are defined by (B.2) and (B.1) respectively, and this is a maximization problem that can be used to characterize u, particularly if we only care about the value of the energy at u.

The rest of this appendix will show this process for all of the energies and dual energies used in this thesis. The general scheme is the same: add in a maximization, integrate by parts, and find the appropriate conditions to make the minimization, after moving it to the inside, well-defined. The final steps of the justification are also identical, since we know what the maximizer should be and at this point the primal and dual energies are equal, the steps in (B.3) work the same in each case. Therefore, the following sections will not provide all of the justification, but just show how the desired functionals are derived and show where the maximum should be achieved.

B.2 Even symmetry

B.2.1 Full problem with inhomogeneity

For the even problem with the inhomogeneity, we have that u_ϵ^{ev} solves

$$\begin{cases} -\nabla\cdot(\gamma_\epsilon\nabla u_\epsilon^{ev}) = f^{ev} & \Omega \\ u_\epsilon^{ev} = \varphi^{ev} & \partial\Omega, \end{cases}$$

which can be found as the minimizer of the functional

$$E_\epsilon^{ev}(v) = \frac{1}{2}\int_\Omega \gamma_\epsilon|\nabla v|^2 - \int_\Omega f^{ev}v$$

over the space

$$V_\epsilon^{ev} = \{v\in H^1(\Omega)\mid v \text{ is even across } y=0,\ v=\varphi^{ev} \text{ on } \partial\Omega\}.$$

From this energy functional, we can write

$$\frac{1}{2}\int_\Omega \gamma_\epsilon|\nabla v|^2 - \int_\Omega f^{ev}v = \max_{\xi\in L^2(\Omega)^2}\int_\Omega \nabla v\cdot\xi - \frac{1}{2}\gamma_\epsilon^{-1}|\xi|^2\ dx - \int_\Omega f^{ev}v\ dx$$

where the maximum will be achieved at $\xi=\gamma_\epsilon\nabla v$. By integration by parts, we get that this can be written as

$$\max_{\xi\in L^2(\Omega)^2}\int_{\partial\Omega}\varphi^{ev}\xi\cdot n - \frac{1}{2}\int_\Omega \gamma_\epsilon^{-1}|\xi|^2\ dx - \int_\Omega(\nabla\cdot\xi + f^{ev})v\ dx,$$

from which we can see that the condition we want to enforce is that $-\nabla\cdot\xi = f^{ev}$ and the dual energy is given by

$$E_\epsilon^{ev,c}(\xi) = \int_{\partial\Omega}\varphi^{ev}\xi\cdot n - \frac{1}{2}\int_\Omega \gamma_\epsilon^{-1}|\xi|^2\ dx.$$

In order to get something that we can use with our problem, we want to split up the calculation between ω_ϵ and $\Omega \setminus \omega_\epsilon$ and rescale ω_ϵ to ω_1 in order to mirror the way we simplified the primal energy in Chapter 2. Thus, we can rewrite

$$\frac{1}{2}\int_\Omega \gamma_\epsilon^{-1}|\xi|^2\ dx = \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\xi|^2\ dx + \frac{1}{2a_\epsilon}\int_{\omega_\epsilon} |\xi^{(i)}|^2\ dx$$

and consider the maximization as taking over the set $(\xi, \xi^{(i)})$ where $-\nabla \cdot \xi = f^{ev}$, $-\nabla \cdot \xi^{(i)} = 0$ and $\xi \cdot n = \xi^{(i)} \cdot n$ on ω_ϵ, assuming that $\epsilon < \delta$ so that $f^{ev} \equiv 0$ on ω_ϵ. Finally, we want to apply the map H_ϵ from Chapter 2 to transform $\xi^{(i)}$ into a vector field η defined on ω_1. We want to do this in the same way that the gradient of a function would transform, because we want to be able to say that u_ϵ^{ev} still corresponds to the maximizer in an appropriate sense. Thus, we define

$$\eta = \det{(\nabla H_\epsilon)}(\nabla H_\epsilon)^{-1}\xi^{(i)} \circ H_\epsilon$$

and if ξ_i ranges over all vector fields satisfying $-\nabla \cdot \xi^{(i)} = 0$ and $\xi \cdot n = \xi^{(i)} \cdot n$ on ω_ϵ, then η ranges over the set

$$\left\{\eta \in L^2(\omega_1)^2 \mid \nabla \cdot \eta = 0 \text{ in } \omega_1,\ \eta \cdot n = \left|\frac{\partial}{\partial \tau} H_\epsilon\right| (\xi \cdot n) \circ H_\epsilon \text{ on } \partial\omega_1\right\}.$$

Since, under this transformation,

$$\int_{\omega_\epsilon} |\xi^{(i)}|^2\ dx = \int_{\omega_1} \frac{1}{|\det \nabla H_\epsilon|}(\nabla H_\epsilon^T \nabla H_\epsilon)\eta \cdot \eta\ dx,$$

we can rewrite the dual energy as

$$\int_{\partial\Omega} \varphi^{ev}\xi \cdot n\ ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\xi|^2\ dx - \frac{1}{2a_\epsilon}\int_{\omega_1} \frac{1}{|\det \nabla H_\epsilon|}(\nabla H_\epsilon^T \nabla H_\epsilon)\eta \cdot \eta\ dx$$

where ξ and η range over the set

$$\Big\{(\xi, \eta) \in L^2(\Omega \setminus \omega_\epsilon)^2 \times L^2(\omega_1)^2 \mid -\nabla \cdot \xi = f^{ev},\ -\nabla \cdot \eta = 0$$
$$\eta \cdot n = \left|\frac{\partial}{\partial \tau} H_\epsilon\right| (\xi \cdot n) \circ H_\epsilon \text{ on } \partial\omega_1\Big\}$$

We can furthermore simplify this formula by using the specific form of H_ϵ, giving that

$$\nabla H_\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \qquad \left|\frac{\partial}{\partial \tau} H_\epsilon\right| = 1 \text{ over } \omega_1^{int},$$

$$\nabla H_\epsilon = \epsilon I \qquad \left|\frac{\partial}{\partial \tau} H_\epsilon\right| = \epsilon \text{ over } \omega_1^{ends},$$

and reducing our dual energy expression to

$$F_\epsilon^{ev,c}(\xi,\eta) = \int_{\partial\Omega} \varphi^{ev}\xi\cdot n\ ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\xi|^2\ dx - \frac{1}{2\epsilon a_\epsilon}\int_{\omega_1^{int}} |\eta_1|^2\ dx$$
$$- \frac{\epsilon}{2a_\epsilon}\int_{\omega_1^{int}} |\eta_2|^2\ dx - \frac{1}{2a_\epsilon}\int_{\omega_1^{ends}} |\eta|^2\ dx,$$

where the maximization is taken over the set

$$W_\epsilon^{ev} = \{(\xi,\eta) \in L^2(\Omega\setminus\omega_\epsilon)^2 \times L^2(\omega_1)^2 \mid -\nabla\cdot\xi = f^{ev},\ -\nabla\cdot\eta = 0,$$
$$\eta_2(x,y) = \xi_2(x,\epsilon y)\ \ -1\le x\le 1,\ y=\pm 1,$$
$$\eta\cdot n = \epsilon(\xi\cdot n)\circ H_\epsilon \text{ on } \partial\omega_1\cap\partial\omega_1^{ends}\}.$$

Based on the derivation of this functional, we know that the maximum should occur at $\xi = \nabla u_\epsilon^{ev}$ and $\eta = \nabla(u_\epsilon^{ev}\circ H_\epsilon)$, and plugging in these vector fields gives an energy that is the same as $E_\epsilon^{ev}(u_\epsilon^{ev})$.

B.2.2 Reduced problem with the curve σ

The function u^{ev} solves

$$\begin{cases} -\Delta u^{ev} = f^{ev} & \Omega \\ \frac{\partial u^{ev}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{ev}}{\partial x^2} = 0 & \sigma \\ u^{ev} = \varphi^{ev} & \partial\Omega, \end{cases}$$

which is the minimizer of the energy functional

$$E^{ev}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\ dx + \epsilon a_\epsilon \int_\sigma \left(\frac{\partial v}{\partial x}\right)^2\ ds - \int_\Omega f^{ev} v\ dx,$$

which is the energy E_ϵ^0 restricted to even functions, over the set

$$V^{ev} = \{v\in H^1(\Omega) \mid v \text{ is even across } y=0,\ v^+,v^-\in H^1(\sigma), v=\varphi^{ev} \text{ on } \partial\Omega\}.$$

To find the dual energy, we will first restrict the integrals to only take place over Ω^+, do the integration by parts on this domain, and then use symmetry to extend to all of Ω if needed. Thus, we have that we can represent the energy as

$$\int_{\Omega^+} |\nabla v|^2\ dx + \epsilon a_\epsilon \int_\sigma \left(\frac{\partial v}{\partial x}\right)^2 - 2\int_{\Omega^+} f^{ev} v\ dx.$$

Adding in the maximization term gives that

$$E^{ev}(v) = \max_{\xi,\zeta} 2\int_{\Omega^+} \nabla v\cdot\xi\ dx - \int_{\Omega^+} |\xi|^2\ dx + 2\epsilon a_\epsilon \int_\sigma \frac{\partial v}{\partial x}\zeta\ ds$$
$$- \epsilon a_\epsilon \int_\sigma |\zeta|^2\ ds - 2\int_{\Omega^+} f^{ev} v\ dx,$$

where $\xi \in L^2(\Omega^+)^2$ and $\zeta \in L^2(\sigma)$. Integration by parts turns this into an expression of the form

$$E^{ev}(v) = \max_{\xi,\zeta} 2\int_{(\partial\Omega)^+} \varphi^{ev}\xi\cdot n\ ds - 2\int_{y=0} v\xi_2\ ds$$
$$- \int_{\Omega^+} |\xi|^2\ dx + 2\epsilon a_\epsilon \int_\sigma \frac{\partial v}{\partial x}\zeta\ ds - \epsilon a_\epsilon \int_\sigma |\zeta|^2\ ds - 2\int_{\Omega^+} (\nabla\cdot\xi + f^{ev})v\ dx.$$

Once the minimum in v is moved inside the maximum over ξ and ζ, we see that the conditions on ξ and ζ in order to make the minimum finite are

$$\begin{cases} \xi_2 = 0 & \{y=0\}\setminus\sigma \\ \xi_2 + \epsilon a_\epsilon \frac{\partial\zeta}{\partial x} = 0 & \sigma \\ -\nabla\cdot\xi = f & \Omega^+. \end{cases}$$

Therefore, we define the set

$$\begin{aligned} W^{ev} = \{(\xi,\zeta) \in L^2(\Omega^+)^2 \times L^2(\sigma) \mid\ & -\nabla\cdot\xi = f, \\ & \xi_2 = 0 \text{ on } \{y=0\}\setminus\sigma, \\ & \xi_2 + \epsilon a_\epsilon \frac{\partial\zeta}{\partial x} = 0 \text{ on } \sigma\} \end{aligned} \tag{B.4}$$

and the dual energy functional for u^{ev} then becomes

$$E^{ev}(u^{ev}) = \max_{(\xi,\zeta)\in W^{ev}} 2\int_{(\partial\Omega)^+} \varphi^{ev}\xi\cdot n\ ds - \int_{\Omega^+} |\xi|^2\ dx - \epsilon a_\epsilon \int_\sigma |\zeta|^2\ ds,$$

where the maximum occurs at $\xi = \nabla u^{ev}$ and $\zeta = \frac{\partial u^{ev}}{\partial x}$. This can also be written as an energy over all of Ω by extending ξ_1 as an even function across $y=0$ and ξ_2 as an odd function. This leads to a functional of the form

$$E^{ev,c}(\xi,\zeta) := \int_{\partial\Omega} \varphi^{ev}\xi\cdot n\ ds - \frac{1}{2}\int_\Omega |\xi|^2\ dx - \epsilon a_\epsilon \int_\sigma |\zeta|^2\ ds.$$

B.2.3 Zero limit problem

The function u_0^{ev} solves

$$\begin{cases} -\Delta u_0^{ev} = f^{ev} & \Omega \\ u_0^{ev} = \varphi^{ev} & \partial\Omega, \end{cases}$$

which can be found as the minimization

$$E_0^{ev}(u_0^{ev}) = \min_{v \in V_0^{ev}} E_0^{ev}(v),$$

with

$$E_0^{ev}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \; dx - \int_\Omega f^{ev} v \; dx$$

and

$$V_0^{ev} = \{v \in H^1(\Omega) \mid v \text{ is even across } y = 0, \; v = \varphi^{ev} \text{ on } \partial\Omega\}.$$

This problem is just the Laplacian, so we can use our work from before to get that the dual energy functional should be

$$E_0^{ev,c}(\xi) = \int_{\partial\Omega} \varphi^{ev} \xi \cdot n \; ds - \frac{1}{2}\int_\Omega |\xi|^2 \; dx,$$

where the maximization is taken over the set

$$W_0^{ev} = \{\xi \in L^2(\Omega)^2 \;\mid\; -\nabla \cdot \xi = f^{ev} \text{ in } \Omega\}.$$

B.3 Odd symmetry

B.3.1 Full problem with inhomogeneity

The function u_ϵ^{odd} solves

$$\begin{cases} -\nabla \cdot (\gamma_\epsilon \nabla u_\epsilon^{odd}) = f^{odd} & \Omega \\ u_\epsilon^{odd} = \varphi^{odd} & \partial\Omega, \end{cases}$$

which can be found as the minimizer of the functional

$$E_\epsilon^{odd}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega f^{odd} v$$

over the space

$$V_\epsilon^{odd} = \{v \in H^1(\Omega) \mid v \text{ is odd across } y = 0,\ v = \varphi^{odd} \text{ on } \partial\Omega\}.$$

Up to interchanging even and odd, this is identical to the work in Section B.2.1. All of the work will go through exactly the same, resulting in a dual energy formulation

$$F_\epsilon^{odd,c}(\xi,\eta) = \int_{\partial\Omega} \varphi^{odd}\xi \cdot n\ ds - \frac{1}{2}\int_{\Omega\setminus\omega_\epsilon} |\xi|^2\ dx - \frac{1}{2\epsilon a_\epsilon}\int_{\omega_1^{int}} |\eta_1|^2\ dx \\ - \frac{\epsilon}{2a_\epsilon}\int_{\omega_1^{int}} |\eta_2|^2\ dx - \frac{1}{2a_\epsilon}\int_{\omega_1^{ends}} |\eta|^2\ dx,$$

where the maximization is taken over the set

$$\begin{aligned} W_\epsilon^{odd} = \{(\xi,\eta) \in L^2(\Omega \setminus \omega_\epsilon)^2 \times L^2(\omega_1)^2 \mid & -\nabla\cdot\xi = f^{odd},\ -\nabla\cdot\eta = 0, \\ & \eta_2(x,y) = \xi_2(x,\epsilon y)\ \ -1 \le x \le 1,\ y = \pm 1, \\ & \eta \cdot n = \epsilon(\xi\cdot n)\circ H_\epsilon \text{ on } \partial\omega_1 \cap \partial\omega_1^{ends}\}, \end{aligned}$$

and again, the maximum will occur at $\xi = \nabla u_\epsilon^{odd}$ and $\eta = \nabla(u_\epsilon^{odd}\circ H_\epsilon)$.

B.3.2 Simplified problem with the curve σ

The function $u^{odd'}$ solves

$$\begin{cases} -\Delta u^{odd'} = f^{odd} & \Omega^+ \\ u^{odd'} = \varphi^{odd} & \partial\Omega^+ \\ \frac{\partial u^{odd'}}{\partial y} = \frac{a_\epsilon}{\epsilon}u^{odd'} & \sigma \\ u^{odd'} = 0 & \{y=0\}\setminus\sigma \end{cases}$$

in Ω^+, and then can be extended as an odd function to all of Ω. It can be found as the minimizer of the energy

$$E^{odd'}(v) = \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla v|^2\ dx + \frac{a_\epsilon}{4\epsilon}\int_\sigma |v^+ - v^-|^2\ ds - \int_\Omega f^{odd} v\ dx$$

over the set

$$V^{odd'} = \{v \in H^1(\Omega\setminus\sigma) \mid v \text{ is odd across } y = 0,\ v = \varphi^{odd} \text{ on } \partial\Omega\}.$$

As before, we restrict our integrals to Ω^+, and use the fact that v is odd across $y = 0$. This gives us a new energy functional of the form

$$E^{odd'}(v) = \int_{\Omega^+ \setminus \sigma} |\nabla v|^2 \, dx + \frac{a_\epsilon}{\epsilon} \int_\sigma |v^+|^2 \, ds - 2 \int_{\Omega^+} f^{odd} v \, dx.$$

We can then convert each of the two squared terms to a maximzation in the same way that has been done previously. This gives us that

$$\begin{aligned} E^{odd'}(v) &= \max_{\xi,\chi} 2 \int_{\Omega^+ \setminus \sigma} \nabla v \cdot \xi \, dx - \int_{\Omega^+ \setminus \sigma} |\xi|^2 \\ &+ 2 \frac{a_\epsilon}{\epsilon} \int_\sigma v^+ \chi \, ds - \frac{a_\epsilon}{\epsilon} \int_\sigma \chi^2 \, ds - 2 \int_{\Omega^+} f^{odd} v \, dx \\ &= \max_{\xi,\chi} 2 \int_{(\partial\Omega)^+ \setminus \sigma} \varphi^{odd} \xi \cdot n \, ds - 2 \int_{y=0} v^+ \xi_2 \, ds - \int_{\Omega^+ \setminus \sigma} |\xi|^2 \\ &+ 2 \frac{a_\epsilon}{\epsilon} \int_\sigma v^+ \chi \, ds - \frac{a_\epsilon}{\epsilon} \int_\sigma \chi^2 \, ds - 2 \int_{\Omega^+} (\nabla \cdot \xi + f^{odd}) v \, dx, \end{aligned}$$

where $\xi \in L^2(\Omega^+)^2$ and $\chi \in L^2(\sigma)$. Since $v^+ = 0$ on $\{y = 0\} \setminus \sigma$ because v is odd, the extra conditions we need to specify are that

$$\begin{cases} -\nabla \cdot \xi = f^{odd} & \Omega^+ \\ \xi_2 - \frac{a_\epsilon}{\epsilon} \chi = 0 & \sigma. \end{cases}$$

From this, we get a dual energy functional of the form

$$E^{odd',c}(\xi, \chi) = 2 \int_{(\partial\Omega)^+ \setminus \sigma} \varphi^{odd} \xi \cdot n \, ds - \int_{\Omega^+ \setminus \sigma} |\xi|^2 - \frac{a_\epsilon}{\epsilon} \int_\sigma \chi^2 \, ds$$

where the maximization is taken over the set

$$W^{odd'} = \{(\xi, \chi) \in L^2(\Omega^+)^2 \times L^2(\sigma) \mid -\nabla \cdot \xi = f^{odd}, \ \xi_2 - \frac{a_\epsilon}{\epsilon} \chi = 0 \text{ on } \sigma\}.$$

This functional could also be extended to all of Ω if needed. The maximum here is achieved at $\xi = \nabla u^{odd'}$ and $\chi = u^{odd',+} |_\sigma$.

B.3.3 Reduced problem with the curve σ

The function u^{odd} solves

$$\begin{cases} -\Delta u^{odd} = f^{odd} & \Omega^+ \\ u^{odd} = \varphi^{odd} & \partial\Omega^+ \\ \frac{\partial u^{odd}}{\partial y} + \epsilon a_\epsilon \frac{\partial^2 u^{odd}}{\partial x^2} - \frac{a_\epsilon}{\epsilon} u^{odd} = 0 & \sigma \\ u^{odd} = 0 & \{y = 0\} \setminus \sigma, \end{cases}$$

which can be found as the minimizer of the energy

$$E^{odd}(v) = \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla v|^2 \ dx + \epsilon a_\epsilon \int_\sigma \left|\frac{\partial v^+}{\partial \tau}\right|^2 \ ds + \frac{a_\epsilon}{\epsilon}\int_\sigma |v^+ - v^-|^2 \ ds - \int_\Omega f^{odd} v \ dx$$

over the set

$$\begin{aligned} V^{odd} = \{v \in H^1(\Omega \setminus \sigma) \mid\ & v \text{ is odd across } y = 0, \\ & v^+, v^- \in H^1(\sigma), \\ & v = \varphi^{odd} \text{ on } \partial\Omega\}. \end{aligned}$$

This problem is a combination of Section B.3.2 and Section B.2.2, putting together the zeroth order term from B.3.2 with the derivative term from B.2.2. Working through both of these calculations, we get to an energy expression for E^{odd} of the form

$$\begin{aligned} E^{odd}(v) = \max_{\xi,\zeta} 2\int_{(\partial\Omega)^+} \varphi^{ev}\xi \cdot n \ ds - 2\int_{y=0} v^+\xi_2 \ ds - \int_{\Omega^+} |\xi|^2 \ dx \\ + 2\epsilon a_\epsilon \int_\sigma \frac{\partial v}{\partial x}\zeta \ ds - \epsilon a_\epsilon \int_\sigma |\zeta|^2 \ ds \\ + 2\frac{a_\epsilon}{\epsilon}\int_\sigma v^+\chi \ ds - \frac{a_\epsilon}{\epsilon}\int_\sigma \chi^2 \ ds - 2\int_{\Omega^+} (\nabla \cdot \xi + f^{odd}) v \ dx \end{aligned}$$

after integrating by parts. From this expression, we can see that for the minimum in v to be finite, we need to have (ξ, ζ, χ) satisfy

$$\begin{cases} -\nabla \cdot \xi = f^{odd} & \Omega^+ \\ \xi_2 + \epsilon a_\epsilon \frac{\partial \zeta}{\partial x} - \frac{a_\epsilon}{\epsilon}\chi = 0 & \sigma. \end{cases}$$

Therefore, we get to a dual energy for this function as

$$E^{odd,c}(\xi,\zeta,\chi) = 2\int_{(\partial\Omega)^+} \varphi^{ev}\xi \cdot n \ ds - \int_{\Omega^+} |\xi|^2 \ dx - \epsilon a_\epsilon \int_\sigma |\zeta|^2 \ ds - \frac{a_\epsilon}{\epsilon}\int_\sigma \chi^2 \ ds$$

where the maximization is taken over the set

$$\begin{aligned} W^{odd} = \{(\xi,\zeta,\chi) \in L^2(\Omega^+)^2 \times L^2(\sigma) \times L^2(\sigma) \mid\ & -\nabla \cdot \xi = f^{odd} \\ & \xi_2 + \epsilon a_\epsilon \frac{\partial \zeta}{\partial x} - \frac{a_\epsilon}{\epsilon}\chi = 0 \text{ on } \sigma\}. \end{aligned}$$

The maximum here will be achieved at $\xi = \nabla u^{odd}$, $\zeta = \frac{\partial u^{odd,+}}{\partial x} \mid_\sigma$, and $\chi = u^{odd,+} \mid_\sigma$.

B.3.4 Zero limit problem

The function u_0^{odd} solves

$$\begin{cases} -\Delta u_0^{odd} = f^{odd} & \Omega \\ u_0^{odd} = \varphi^{odd} & \partial\Omega \end{cases}$$

which again is the same as the Laplacian from the introduction to this appendix. Therefore, the dual energy is

$$E_0^{odd,c}(\xi) = \int_{\partial\Omega} \varphi^{odd} \xi \cdot n \; ds - \frac{1}{2} \int_{\Omega} |\xi|^2 \; dx,$$

where the maximization is taken over the set

$$W_0^{odd} = \{\xi \in L^2(\Omega)^2 \;\; | \;\; -\nabla \cdot \xi = f^{odd} \text{ in } \Omega\}.$$

Appendix C
Numerical Code

Here, we present snippets of the numerical code used to generate all of the images in Chapter 6. All of these calculations use the FEniCS Project packages, more information on which can be found at `http://fenicsproject.org`.

C.1 Base code

To begin, this section contains the two basic files that were used to generate the images for constant conductivities. The first one, for even symmetry, will be shown in its entirety, and the rest of the section will discuss what types of changes needed to be made in order to fit the different situation.

C.1.1 Even Symmetry

Here is the full Python code in the case of even symmetry.

```
"""
EvenSym_Base.py
  Matt Charnley, Spring 2019

Constructs the desired finite element problem with even symmetry for
    different constant values of a_epsilon and illustrates the convergence as
    epsilon goes to zero.
```

```
Plots are generated that show the solution u_eps^ev compared to u^ev along
    the horizontal line y=0, the horizontal line y = 0.5, and the vertical
    line x=-1. In the cases where u^ev_0 is an approximation to the solution,
    that graph is shown as well.

For this code, the background domain is B_2(0)^+, with the mid-curve sigma
    being the line segment from (-1,0) to (1,0). Most of the variables and
    notation here follow the work in the thesis.
"""

# Import necessary FENICS packages, as well as plotting libraries

from fenics import *
from dolfin import *
from mshr import *
import numpy as np
import matplotlib.pyplot as plt

# Define Subdomains
   # For these problems omega_eps corresponds to the inhomogeneity, denoted
       omega_epsilon in the paper.

class omega_eps(SubDomain):
   def __init__(self, eps, *args):
      self.eps = eps
      SubDomain.__init__(self)

   def inside(self,x,on_boundary):
      tol = 1E-14
      return (x[0] <= 1 + tol and x[0] >= -1 - tol and x[1] <= self.eps + tol)
          or ((x[0] + 1)**2 + x[1]**2 <= self.eps**2 + tol**2) or ((x[0]-1)**2
          + x[1]**2 <= self.eps**2 + tol**2)
```

```
  # boundary_Out is the portion of the boundary of B_2(0)^+ that is on the
      boundary of B_2(0); the region where the Dirichlet condition is
      specified.

class boundary_Out(SubDomain):
  def inside(self, x, on_boundary):
    tol = 1E-14
    return on_boundary and (x[1] >= tol or x[0] >= 2 - tol or x[0] <= -2 +
        tol)



  # boundary_Bot is the entire bottom boundary of B_2(0)^+, that is, the
      boundary along y=0

class boundary_Bot(SubDomain):
  def inside(self, x, on_boundary):
    tol = 1E-14
    return on_boundary and x[1] <= tol

  # boundary_BotL is the part of boundary_Bot that is to the left of x=-1

class boundary_BotL(SubDomain):
  def inside(self, x, on_boundary):
    tol = 1E-14
    return on_boundary and x[1] <= tol and x[0] <= -1 - tol

  # boundary_BotM is the part of boundary_Bot that is along sigma, that is,
      between x=-1 and x=1

class boundary_BotM(SubDomain):
  def inside(self, x, on_boundary):
    tol = 1E-14
    return on_boundary and x[1] <= tol and abs(x[0]) <= 1 + 2*tol
```

```
    # boundary_BotR is the part of boundary_Bot that is to the right of x=1

class boundary_BotR(SubDomain):
    def inside(self, x, on_boundary):
        tol = 1E-14
        return on_boundary and x[1] <= tol and x[0] >= 1 + tol

# Define the material properties expression. This takes a mesh function and
    sets the conductivity to be 1 outside of the inhomogeneity and a inside
    of it.

class A(Expression):
    def __init__(self, materials, a, **kwargs):
        self.materials = materials
        self.a = a

    def eval_cell(self, values, x, cell):
        if self.materials[cell.index] == 0:
            values[0] = 1
        else:
            values[0] = self.a

"""
Program Start
"""

# Define program parameters

eps_vals = [0.2, 0.02, 0.002] # Values of epsilon used in the iteration
tol = 1E-10        # Numerical tolerance

# Set up point arrays for graphing at the end.

y1 = np.linspace(-1.5 + tol, 1.5 - tol, 320)
```

```
y2 = np.linspace(0, 1, 160)

points_axis = [(y_,0) for y_ in y1]
points_horiz = [(y_, 0.5) for y_ in y1]
points_vert1 = [(-1, y_) for y_ in y2]



# Begin solving the problems

for jnd in [1,2,3,4,5]:
  # This jnd counter defines the power of epsilon in a_eps = 2 eps^(jnd-3).
      As jnd steps from 1 to 5, this will run through all of the different
      possible behaviors of eps a_eps and a_eps/eps as epsilon goes to zero

  ind = 0             # Counting variable for the epsilon values
  for eps in eps_vals:
    ind += 1
    a_eps = 3*eps**(jnd-3)  # Define the constant conductivity a_epsilon

    g = Expression('1-x[0] -(x[0]+1)*x[1]*x[1]', degree = 2)  # Dirichlet
        boundary data

    f = Expression('0', degree=2)         # Source term f

    ####
    # Solve the problem with the inhomogeneity
      # This version of the problem has the domain omega_eps with the
          differing conductivity

    UpperHalf_inhom = Circle(Point(0,0), 2) - Rectangle(Point(-2, -2),
        Point(2, 0))
      # This is the domain B_2(0)^+
```

```
innerPart1 = (Circle(Point(-1,0), eps) + Circle(Point(1,0), eps) +
    Rectangle(Point(-1,0), Point(1,eps))) - Rectangle(Point(-2,-2),
    Point(2, 0))
   # This is a geometric representation of the inhomogeneity omega_eps

innerPart = omega_eps(eps)
   # This is the Subdomain implementation of the inhomogeneity omega_eps

UpperHalf_inhom.set_subdomain(1, innerPart1)
   # We need the geometric part to generate the mesh

mesh_inhom = generate_mesh(UpperHalf_inhom, 256)  # Define the mesh
V_inhom = FunctionSpace(mesh_inhom, 'P', 1)   # Define the finite
    element space



materials = MeshFunction('size_t', mesh_inhom,2)
   # Generate a blank mesh function for materials

materials.set_all(0)
innerPart.mark(materials, 1)
   # and use the Subdomain definition of the inhomogeneity to generate
      the material differences

a = A(materials, a_eps, degree = 1)
   # using this function from before.

b_Out = boundary_Out()          # Define the external boundary
bc_inhom = DirichletBC(V_inhom, g, b_Out)  # and set the Dirichlet
    boundary condition

u_inhom = TrialFunction(V_inhom)

v_inhom = TestFunction(V_inhom)
```

```
a_inhom = a*dot(grad(u_inhom), grad(v_inhom))*dx # Set up the
    appropriate bilinear form using the
                   # material properties from before
L_inhom = f*v_inhom*dx          # and the linear term for the other side

u_inhom = Function(V_inhom)
solve(a_inhom == L_inhom, u_inhom, bc_inhom)  # Solve the variational
    problem
#
####

####
# Solve the reduced problem involving the mid-curve sigma
   # This set solves for the u^ev solution from the thesis

UpperHalf_sig = Circle(Point(0,0), 2) - Rectangle(Point(-3, -3),
    Point(3, 0))
   # This is again B_2(0)^+
sigBdry = Circle(Point(0,0), 1) - Rectangle(Point(-1,-1), Point(1,0))
UpperHalf_sig.set_subdomain(1,sigBdry)
   # While there isn't an internal domain region that needs to be dealt
       with separately, defining
   # this subdomain will make sure there are mesh points near the
       endpoints of sigma.

mesh_sig = generate_mesh(UpperHalf_sig, 256)
V_sig = FunctionSpace(mesh_sig, 'P', 1)
   # The mesh and function space are generated as before

b_marks_sig = MeshFunction('size_t', mesh_sig, 1)
b_marks_sig.set_all(0)
   # Now we define a blank marking function for the boundary
```

```
b_Out = boundary_Out()
b_Left = boundary_BotL()
b_Mid = boundary_BotM()
b_Right = boundary_BotR()
b_Bot = boundary_Bot()
   # define subdomain implementations of all of the different parts of
       the boundary

b_Out.mark(b_marks_sig, 1)
b_Left.mark(b_marks_sig, 2)
b_Mid.mark(b_marks_sig, 3)
b_Right.mark(b_marks_sig, 4)
   # and mark them with different numbers so that we can integrate over
       each of them separately.

ds_sig = Measure('ds', domain=mesh_sig, subdomain_data=b_marks_sig)
   # This defines the surface measure corresponding to the marking
       defined above.

bc_sig = DirichletBC(V_sig, g, b_Out)
   # Set the Dirichlet boundary condition

u_sig = TrialFunction(V_sig)

v_sig = TestFunction(V_sig)

e1 = interpolate(Expression('x[0]', degree=2), V_sig)
   # Define the function e1(x,y) = x. This will allow us to take x
       derivatives along the boundary
   # in order to get the variational problem that we need.

a_sig = dot(grad(u_sig), grad(v_sig))*dx + eps*a_eps*dot(grad(u_sig),
    grad(e1))*dot(grad(v_sig), grad(e1))*ds_sig(3)
```

```
  # Define the bilinear form. We use grad(u) dot (1,0) to get the
     x-derivative of u.

L_sig = f*v_sig*dx      # Define the linear form for the other side.

u_sig = Function(V_sig)
solve(a_sig == L_sig, u_sig, bc_sig) # Solve the variational problem.
#
####

####
# Solve the zero limit problem. This is the function u^ev_0 in the
   thesis, which is an approximation to
# this problem in certain cases. This problem is simpler than the
   previous ones, but follows the same form.

UpperHalf_0 = Circle(Point(0,0), 2) - Rectangle(Point(-3, -3), Point(3,
   0))

mesh_0 = generate_mesh(UpperHalf_0, 256)
V_0 = FunctionSpace(mesh_0, 'P', 1)

bc_0 = DirichletBC(V_0, g, b_Out)

u_0 = TrialFunction(V_0)

v_0 = TestFunction(V_0)

a_0 = dot(grad(u_0), grad(v_0))*dx
L_0 = f*v_0*dx

u_0 = Function(V_0)
solve(a_0 == L_0, u_0, bc_0)
#
```

```
####

####
# Data Collection and Graphing

# Gather data from the reduced problem
u_sig_axis = np.array([u_sig(point) for point in points_axis])
u_sig_horiz = np.array([u_sig(point) for point in points_horiz])
u_sig_vert = np.array([u_sig(point) for point in points_vert1])

# from the problem with the inhomogeneity
u_inhom_axis = np.array([u_inhom(point) for point in points_axis])
u_inhom_horiz = np.array([u_inhom(point) for point in points_horiz])
u_inhom_vert = np.array([u_inhom(point) for point in points_vert1])

# and from the zero problem
u_0_axis = np.array([u_0(point) for point in points_axis])
u_0_horiz = np.array([u_0(point) for point in points_horiz])
u_0_vert = np.array([u_0(point) for point in points_vert1])

  indx1H = np.arange(10, 320, 30)
indx2H = np.arange(20, 320, 30)
indx3H = np.arange(30, 320, 30)

indx1V = np.arange(5, 160, 15)
indx2V = np.arange(10, 160, 15)
indx3V = np.arange(15, 160, 15)

# Use the matlab plotting library to draw the different graphs

   # First, the one along y=0
plt.figure(1)
plt.rc('xtick', labelsize=20)
plt.rc('ytick', labelsize=20)
```

```
plt.rc('axes', labelsize=12)
plt.rc('lines', markeredgewidth=2)
plt.plot(y1, u_inhom_axis, 'k-', linewidth=3)
plt.plot(y1[indx1H], u_inhom_axis[indx1H], 'ko', MarkerSize=10,
    MarkerFaceColor='none', MarkerEdgeColor='black')
plt.plot(y1, u_sig_axis, 'k--', linewidth=3)
plt.plot(y1[indx2H], u_sig_axis[indx2H], 'ks', MarkerSize=10,
    MarkerFaceColor='none', MarkerEdgeColor='black')
if jnd>2:
   plt.plot(y1, u_0_axis, 'k-.', linewidth=3)
   plt.plot(y1[indx3H], u_0_axis[indx3H], 'k^',
      MarkerSize=10,MarkerFaceColor='none', MarkerEdgeColor='black')

plt.xlabel('x')
plt.savefig('radialgraph_axis0_ESym_'+str(jnd)+'_'+str(ind)+'.png')
plt.gcf().clear()

   # then the one along y=0.5
plt.figure(2)
plt.rc('xtick', labelsize=20)
plt.rc('ytick', labelsize=20)
plt.rc('axes', labelsize=12)
plt.rc('lines', markeredgewidth=2)
plt.plot(y1, u_inhom_horiz, 'k', linewidth=3)
plt.plot(y1[indx1H], u_inhom_horiz[indx1H], 'ko', MarkerSize=10,
    MarkerFaceColor='none', MarkerEdgeColor='black')
plt.plot(y1, u_sig_horiz, 'k--', linewidth=3)
plt.plot(y1[indx2H], u_sig_horiz[indx2H], 'ks', MarkerSize=10,
    MarkerFaceColor='none', MarkerEdgeColor='black')
if jnd>2:
   plt.plot(y1, u_0_horiz, 'k-.', linewidth=3)
   plt.plot(y1[indx3H], u_0_horiz[indx3H], 'k^',
      MarkerSize=10,MarkerFaceColor='none', MarkerEdgeColor='black')
plt.xlabel('x')
```

```
plt.savefig('radialgraph_horiz5_ESym_'+str(jnd)+'_'+str(ind)+'.png')
plt.gcf().clear()

  # and then the one along x=-1
plt.figure(3)
plt.rc('xtick', labelsize=20)
plt.rc('ytick', labelsize=20)
plt.rc('axes', labelsize=12)
plt.rc('lines', markeredgewidth=2)
plt.plot(y2, u_inhom_vert, 'k', linewidth=3)
plt.plot(y2[indx1V], u_inhom_vert[indx1V], 'ko', MarkerSize=10,
    MarkerFaceColor='none', MarkerEdgeColor='black')
plt.plot(y2, u_sig_vert, 'k--', linewidth=3)
plt.plot(y2[indx2V], u_sig_vert[indx2V], 'ks', MarkerSize=10,
    MarkerFaceColor='none', MarkerEdgeColor='black')
if jnd>2:
  plt.plot(y2, u_0_vert, 'k-.', linewidth=3)
  plt.plot(y2[indx3V], u_0_vert[indx3V], 'k^',
      MarkerSize=10,MarkerFaceColor='none', MarkerEdgeColor='black')
plt.xlabel('y')
plt.savefig('radialgraph_vert1_ESym_'+str(jnd)+'_'+str(ind)+'.png')
plt.gcf().clear()
#
####
```

C.1.2 Odd symmetry

The main thing that changes when solving this problem with odd symmetry is that we need to specify a zero Dirichlet boundary condition on the part of $y = 0$ that is outside of σ. For the even case, we got the Neumann condition for free by not including this region in our boundary integration. Thus, we need to define

```
z = Expression('0', degree=2)     # The zero function, for use as a
    boundary condition
```

and our boundary conditions change to

```
b_Out = boundary_Out()     # Define the external boundary
b_Bot = boundary_Bot()     # and the boundary at the bottom of the region

bc_inhom1 = DirichletBC(V_inhom, g, b_Out)
bc_inhom2 = DirichletBC(V_inhom, z, b_Bot)
   # and set the appropriate Dirichlet boundary conditions. The zero
       condition comes from the enforced odd symmetry.

bcs_inhom = [bc_inhom1, bc_inhom2]  # Stack the boundary conditions together
```

for the problem with the inhomogeneity,

```
b_Out = boundary_Out()
b_Left = boundary_BotL()
b_Mid = boundary_BotM()
b_Right = boundary_BotR()
b_Bot = boundary_Bot()
   # define subdomain implementations of all of the different parts of the
       boundary

b_Out.mark(b_marks_sig, 1)
b_Left.mark(b_marks_sig, 2)
b_Mid.mark(b_marks_sig, 3)
b_Right.mark(b_marks_sig, 4)
   # and mark them with different numbers so that we can integrate over
       each of them separately.

ds_sig = Measure('ds', domain=mesh_sig, subdomain_data=b_marks_sig)
   # This defines the surface measure corresponding to the marking defined
       above.
```

```
bc_sig = DirichletBC(V_sig, g, b_Out)
  # Set the Dirichlet boundary condition

bc_sig1 = DirichletBC(V_sig, z, b_Left)
bc_sig2 = DirichletBC(V_sig, z, b_Right)
  # Again, zero boundary conditions on the parts off of sigma due to odd
      symmetry.

bcs_sig = [bc_sig, bc_sig1, bc_sig2]
```

for the reduced problem on σ, and

```
bc_0 = DirichletBC(V_0, g, b_Out)
bc_01 = DirichletBC(V_0, z, b_Bot)
bcs_0 = [bc_0, bc_01]
```

for the zero problem. We also need to set up and solve the 'simplified problem,' where the second derivative term is removed. This is implemented as follows.

```
    ####
  # Solve the simpified problem involving the mid-curve sigma
    # This set solves for the u^odd' solution from the thesis

  UpperHalf_sim = Circle(Point(0,0), 2) - Rectangle(Point(-3, -3),
      Point(3, 0))
    # This is again B_2(0)^+
  sigBdry = Circle(Point(0,0), 1) - Rectangle(Point(-1,-1), Point(1,0))
  UpperHalf_sim.set_subdomain(1,sigBdry)
    # While there isn't an internal domain region that needs to be dealt
        with separately, defining
    # this subdomain will make sure there are mesh points near the
        endpoints of sigma.

  mesh_sim = generate_mesh(UpperHalf_sim, 256)
  V_sim = FunctionSpace(mesh_sim, 'P', 1)
```

```
  # The mesh and function space are generated as before

b_marks_sim = MeshFunction('size_t', mesh_sim, 1)
b_marks_sim.set_all(0)
  # Now we define a blank marking function for the boundary

b_Out = boundary_Out()
b_Left = boundary_BotL()
b_Mid = boundary_BotM()
b_Right = boundary_BotR()
b_Bot = boundary_Bot()
  # define subdomain implementations of all of the different parts of
     the boundary

b_Out.mark(b_marks_sim, 1)
b_Left.mark(b_marks_sim, 2)
b_Mid.mark(b_marks_sim, 3)
b_Right.mark(b_marks_sim, 4)
  # and mark them with different numbers so that we can integrate over
     each of them separately.

ds_sim = Measure('ds', domain=mesh_sim, subdomain_data=b_marks_sim)
  # This defines the surface measure corresponding to the marking
     defined above.

bc_sim = DirichletBC(V_sim, g, b_Out)
  # Set the Dirichlet boundary condition

bc_sim1 = DirichletBC(V_sim, z, b_Left)
bc_sim2 = DirichletBC(V_sim, z, b_Right)
  # Again, zero boundary conditions on the parts off of sigma due to
     odd symmetry.

bcs_sim = [bc_sim, bc_sim1, bc_sim2]
```

```
u_sim = TrialFunction(V_sim)

v_sim = TestFunction(V_sim)

e1 = interpolate(Expression('x[0]', degree=2), V_sim)
   # Define the function e1(x,y) = x. This will allow us to take x
       derivatives along the boundary
   # in order to get the variational problem that we need.

a_sim = dot(grad(u_sim), grad(v_sim))*dx +
    (a_eps/eps)*u_sim*v_sim*ds_sim(3)
   # Define the bilinear form. We use grad(u) dot (1,0) to get the
       x-derivative of u.

L_sim = f*v_sim*dx      # Define the linear form for the other side.

u_sim = Function(V_sim)
solve(a_sim == L_sim, u_sim, bcs_sim) # Solve the variational problem.
#
####
```

The bilinear form for the reduced problem on σ is given by

```
a_sig = dot(grad(u_sig), grad(v_sig))*dx + eps*a_eps*dot(grad(u_sig),
    grad(e1))*dot(grad(v_sig), grad(e1))*ds_sig(3) +
    (a_eps/eps)*u_sig*v_sig*ds_sig(3)
      # Define the bilinear form. We use grad(u) dot (1,0) to get the
          x-derivative of u.

L_sig = f*v_sig*dx      # Define the linear form for the other side.

u_sig = Function(V_sig)
solve(a_sig == L_sig, u_sig, bcs_sig) # Solve the variational problem.
```

and all of the other problems are similar.

C.2 Modifications for different problems

This section considers the different types of modifications that need to be made to this code in order to model the differnet types of problems addressed in Chapter 5. One of the first modifications that needed to be made for the problem was to allow for the conductivity to be written as an expression, as opposed to just a constant number. Thus, the A object defined previously was changed to

```
  # Define the material properties expression. This takes a mesh function and
      sets the conductivity to be 1 outside of the inhomogeneity and a inside
      of it.

class A_gen(Expression):
  def __init__(self, materials, a, **kwargs):
    self.materials = materials
    self.a = a

  def eval_cell(self, values, x, cell):
    if self.materials[cell.index] == 0:
      values[0] = 1
    else:
      v = np.empty(1, dtype=float)
      self.a.eval(values, x)
```

This allowed the a value input to this constructor to be an Expression as opposed to a constant. Constant conductivities could still be utilized by defining them as an expression that just has a constant value. Also, in these types of problems, the loop over power of epsilon was removed, as only a single conductivity was used for these models.

C.2.1 Variable inhomogeneity - independent of x

In the case of a conductivity independent of x, the beginning of the loop, where the conductivity is defined, becomes

```
a_eps =Expression('3*(x[1]*x[1])/(eps*eps*eps*eps) + eps*eps', eps =
    Constant(eps), degree=2)   # Define the conductivity a_epsilon on the
    appropriate domain.

  # Define a function for the rescaled conductivity.
  # This is the same as a_eps with y scaled by epsilon
def a_eps_hat(x,y):
  return 3*(y/eps)**2 + eps**2
```

where this a_eps_hat corresponds to $\hat{a}_\epsilon$ from the analytic work, and will be used in defining the impact factors. The conductivity profile for the problem with inhomogeneity is then defined by

```
a = A_gen(materials, a_eps, degree = 1)
```

where now, a_eps is an expression.

Next, we need to compute the impact factors of the conductivity from earlier in the work. Unfortunately, FEniCS does not have a great way to do these sorts of integrals numerically, so they were constructed manually.

```
# To determine the appropriate reduced equation on sigma, we need to find
    the impact factors from the thesis. As FENICS has no easy way to
    compute integrals, we will do so manually with the trapezoidal rule.

# Set up the numerical parameters
NStep = 10*int(eps**(-2))
intVals = np.linspace(-1,1,NStep+1)
stepSize = (1.0-(-1.0))/(NStep)
```

```
# Initialize Values
I1 = 0
I2 = 0
innerInt = 0
totalSum = 0

# Find the total integral of 1/a, which we need for the F(b) term in the
    definition of impact factors

for counter1 in range(NStep):
   totalSum += stepSize*1.0/2.0*(1.0/(a_eps_hat(0, intVals[counter1])) +
       1.0/(a_eps_hat(0, intVals[counter1+1])))

# Now compute the other impact factors. In order to take care of the nested
    integrals, we will
# step through the integral and increment the part that is multiplied
    inside.
for counter1 in range(NStep):
   innerInt += stepSize*1.0/2.0*(1.0/(a_eps_hat(0, intVals[counter1])) +
       1.0/(a_eps_hat(0, intVals[counter1+1])))
   I1 += (1.0/totalSum)*(1.0/2.0)*(a_eps_hat(0, intVals[counter1]) +
       a_eps_hat(0, intVals[counter1+1]))*innerInt*innerInt*stepSize
   I2 += (1.0/totalSum)*1.0/2.0*(a_eps_hat(0, intVals[counter1]) +
       a_eps_hat(0,
       intVals[counter1+1]))*innerInt*(totalSum-innerInt)*stepSize

# Find A_eps from these calculations
A_eps = 2/(innerInt)
```

Then, we are ready to set up the bilinear forms and solve the variational problems. For even symmetry, the problem is

```
 a_sig = dot(grad(u_sig), grad(v_sig))*dx +
     1.0/2.0*eps*A_eps*(I1+I2)*dot(grad(u_sig), grad(e1))*dot(grad(v_sig),
     grad(e1))*ds_sig(3)
   # Define the bilinear form. We use grad(u) dot (1,0) to get the
       x-derivative of u.

L_sig = f*v_sig*dx      # Define the linear form for the other side.

u_sig = Function(V_sig)
solve(a_sig == L_sig, u_sig, bc_sig) # Solve the variational problem.
```

while for odd symmetry, it becomes

```
a_sig = dot(grad(u_sig), grad(v_sig))*dx +
    1.0/2.0*eps*A_eps*(I1-I2)*dot(grad(u_sig), grad(e1))*dot(grad(v_sig),
    grad(e1))*ds_sig(3) + (A_eps/eps)*u_sig*v_sig*ds_sig(3)
      # Define the bilinear form. We use grad(u) dot (1,0) to get the
          x-derivative of u.
      # This comes from the equations developed in the thesis, using the
          impact factors.

L_sig = f*v_sig*dx      # Define the linear form for the other side.

u_sig = Function(V_sig)
solve(a_sig == L_sig, u_sig, bcs_sig) # Solve the variational problem.
```

C.2.2 Variable conductivity - independent of t

When the conductivity is instead independent of t, the problem is much easier. The conductivity function used here is

```
a_eps = Expression('(x[0] < -0.5 + tol) ? (3*eps*eps) : (x[0] < tol) ?
    (3/eps/eps) : (x[0] < 0.5 + tol) ? (3*eps*eps) : (3/eps/eps)', eps=eps,
    tol=tol, degree=2)
# Define the conductivity a_epsilon on the appropriate domain.
```

which sets up the piecewise defined function as in (6.1). Everything in this code works the same as the 'independent of x' code, except now we no longer need to compute the impact factors. We can use

```
# Initialize Values - These are the values we get from a constant in t
    integration
I1 = 4/3
I2 = 2/3

# Find A_eps from these calculations
A_eps = interpolate(a_eps, V_sig)
```

to find exactly what we need. The bilinear forms and problem solving are exactly as in the previous set of code; the interpolation is needed to make sure that the software interprets a_ϵ as a function in the appropriate finite element space to carry out the calculations.

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