BRANE TRANSPORT BEYOND CALABI-YAU: A TALE OF GEOMETRY AND LOCALIZATION

By

JOEL AARON CLINGEMPEEL

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ABSTRACT OF THE DISSERTATION

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By JOEL AARON CLINGEMPEEL

Dissertation Director: Gregory Moore

The goal of this thesis is to study the behavior of branes preserving B-type supersymmetry in two-dimensional N = (2,2) theories as parameters are varied across the (quantum-corrected) Kähler moduli space. These branes may naturally be organized into a category which is equivalent to the derived category of coherent sheaves on the target geometry. For theories constructed from an abelian gauged linear sigma model, this wall-crossing is studied using the analytic continuation of the hemisphere partition function for which an explicit integral formula is known by work on localization of Hori-Romo. This leads to an understanding of transport functors which exhibit splitting of branes between the Higgs and Coulomb branches, thus going beyond the work of Hori-Herbst-Page in the Calabi-Yau case.

As a case study, explicit brane transport is worked out for Hirzebruch-Jung surfaces, extending the work of Moore-Martinec at the level of K-theory. Furthermore, the hemisphere partition function is evaluated with suitable regularization and shown to match to leading order the known formula for central charge in terms of characteristic classes. This is based on the arXiv hep-th 1811.12385 which was joint with B. Le Floch and M. Romo.

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Chapter 1

Introduction

String theory and supersymmetry have in recent years had a profound impact on both the physics and mathematics communities. On the physics side, string theory provides a leading candidate for unifying quantum mechanics with gravity by replacing point particles with tiny, vibrating strings which obey the laws of quantum mechanics. One consequence is the existence of supersymmetry, a symmetry that mixes fermionic and bosonic fields. Even without string theory, supersymmetry could still exist, and its existence would shed light on a number of phenomological puzzles. From the theoretical side, supersymmetry provides a powerful toolbox for exact results that would be unattainable in ordinary quantum field theories. From the mathematics side, strings and supersymmetry have been and will continue to be a great source of inspiration for new insights into pure mathematics, ranging from knots to the topology of three and four dimensional manifolds to mirror symmetry. Regardless of the status of strings and supersymmetry in terms of describing nature, they will remain interesting in terms of the underlying mathematics.

One particular area that has been incredibly rich for the interplay between mathematics and physics has been in 2-dimensional (2,2) supersymmetric quantum field theories which describe the quantum dynamics of a string propagating in a spacetime geometry X, often called a *(supersymmetric) sigma model with target* X. The quantum field theory itself contains a lot of information about the geometry of X which can be extracted from studying objects such as branes that preserve certain subsets of the supersymmetry or equivalently by a procedure called topological twisting in which one is able to remove dynamical aspects and is left with a theory whose details can be described purely in terms of the geometry of X. On the quantum field theory side, there

are two ways that one can establish a correspondence between this and a quantum field theory on a different manifold $\hat{X}$ which then leads to a bridge between geometrical questions about X and $\hat{X}$.

One such bridge is via mirror symmetry. The first vague hint of mirror symmetry was the existence of a curious involution of the supersymmetry algebra, but this alone didn't seem to point to anything quite so fantastic. In [15], given that a theory with target X has two naturally associated rings, one of which turns out to be isomorphic to the cohomology of X, it was conjectured that the second ring should be the cohomology of a mirror manifold $\hat{X}$ and that some sort of duality should exist between the two theories. The classic paper [6] then established mirror symmetry for a quintic projective Calabi-Yau threefold, establishing concrete evidence for this proposal. Later in [12] mirror symmetry was systematically derived from T-duality for a large class of theories, and subsequent work by many others showed that incorporating branes would lead to far more robust versions of mirror symmetry. This subsequently inspired a number of mathematical papers on rigorously proving results in mirror symmetry although a number of the physical insights still remain unproven.

The other way to get a different geometry is via a phase transition of the theory which does not involve invoking T-duality. One can vary certain parameters of a theory with target X such that for a generic small perturbation, the geometry of X slightly changes, but when one crosses certain exceptional walls in the parameter space, one suddenly obtains a different manifold. Such a drastic jump across such a wall is called wall-crossing. This approach was greatly clarified in the famous paper [19] in which it was shown that by gauging (in a supersymmetric way) a theory whose target is $\mathbb{C}^d$, giving what is called a *gauged linear sigma model* (GLSM), one can obtain at low energies ungauged theories with a variety of non-linear targets. The result geometry depends on real parameters of the UV theory called FI parameters which naturally take values in a real vector space. This vector space may be subdivided into a fan of polyhedral cones which correspond to distinct target manifolds, and crossing the borders between adjacent polyhedral cones gives rise to the wall-crossing described above. Strictly speaking, the positions of the walls are modified by quantum corrections which asymptote to the

classical walls at infinity, but the essential idea of wall-crossing remains the same. This was actually done in a bit more generality as one can turn on a superpotential, but the focus of this thesis shall be in the case of vanishing superpotential. We will also restrict to the case of an abelian gauge group though it would be interesting to remove this restriction in subsequent work.

A striking fact is that many aspects of the underlying quantum field theory remain invariant despite there perhaps having been a major change in the geometry. This is because of the role of theta angles: these have no geometric interpretation in terms of the target geometry and in fact behave like turning on background electric fields. However, these serve to complexify the parameter space: the FI parameters and theta angles naturally combine to form holomorphic coordinates, and the walls that were once of real codimension one are now in fact of complex codimension one. This means that there are now paths that interpolate between phases, and anything that is invariant under a small perturbation of parameters should be invariant under transport along such a path between phases. There is, however, monodromy phenomena associated with the choice of path.

A natural question one might ask is why an equivalence between theories on different target manifolds should lead to precise correspondences between purely geometric invariants of the two manifolds. This is because of a procedure called *topological twisting* in which the theory is modified so as to have no kinematic information and in particular to be invariant under any perturbations of the metric; the quantities that remain then purely depend on the geometry of the underlying manifold. An alternate perspective is that even without twisting one can find such geometric invariants from examining *BPS objects:* objects which preserve some fraction of the supersymmetry operators. Studying how BPS branes are mapped under mirror symmetry leads to the famous and striking *homological mirror symmetry conjecture* of Kontsevich [14] relating the derived category of coherent sheaves on one manifold to the derived Fukaya category on another.

Instead of studying how BPS branes are mapped under mirror symmetry, one could instead study the behavior of such branes under phase transitions. In [8], this was considered for the particular case of a certain subset of the supersymmetry charges

called B-type supersymmetry. For an (ungauged) sigma model, the category of branes preserving this supersymmetry is precisely the derived category of coherent sheaves on the target geometry. They analyzed how the GLSM viewpoint the transformations between derived categories that naturally arise when crossing walls in a parameter space. Given a brane in a phase, one can lift it to the GLSM where it can be expressed in terms of line bundles which are topologically trivial but may be charged under the gauge group. Naively, one can then push this down to a different phase, but unfortunately, such a map fails to be well-defined. The key insight of [8] is that in fact only line bundles whose charges lie within a certain band in the charge lattice can be transported, a principle called the *grade restriction rule*. This extends to arbitrary branes as one can always construct a lift to a complex line bundles obeying the charge condition, thus leading to well-defined transport functors between chambers associated to paths. The choice of theta angle can change the homotopy class of the path which leads to a different transport functor and hence opens the door to monodromy behavior. This can be viewed morally as an isotrivial bundle of categories with a flat connection on the parameter space, but making this precise is currently an open question.

The hemisphere partition function, given by integrating over all field configurations on a hemisphere with boundary conditions set by a brane, is a powerful tool for gaining deep insights into these theories. This function is believed to be equal to the geometric central charge with instanton corrections [11]. In [11], a formula for the partition function is derived via supersymmetric localization as an integral over a suitable contour which for abelian models lies in the Lie algebra of the gauge group. In this thesis, we show that by studying the analytic continuation of the partition function of a UV lift a brane between phases of a GLSM, one can obtain a more robust version of the grade restriction rule which applies to anomalous models. The IR target geometry in a phase is the *Higgs branch* of vacua, characterized by having all vector multiplets Higgsed while the chiral multiplets may fluctuate in certain ways. When crossing a wall, a *Coulomb branch* may open which is characterized by one or more vector multiplets flucuating, and branes which cross the wall split into branes supported on these two branches. This wall-crossing obeys a generalization of the grade restriction rule in which we actually

obtain two bands in the charge lattice: a wider band corresponding to all charged line bundles that one can be transported to the other phase, possibly with a Coulomb branch contribution, and a narrower band corresponding to charged line bundles which may be transported to give solely a contribution to the geometry in the other phase. Moreover, we show how one can determine the precise Coulomb branch contribution for a charged line bundle in the wide window, thus completely specifying how the brane is transported. This can be viewed as the physical counterpart to mathematical work in [2].

We also analyze convergence aspects of the hemisphere partition function. The localization formula as derived in [11] requires R-charges to lie in $(0, 2)$, but our application of interest is in the case in which all R-charges vanish. One can try to turn on tiny R-charges and then take the limit as they tend to zero which will give a simplified formula should the contour not be pinched between colliding poles. However, this is in many cases unavoidable so such a simplified formula is not universally valid. We show that for the case of compactly supported branes, exactly the right cancellation of poles occurs for this to work. Moreover, if we restrict further to branes which become trivial in the IR, then there is enough cancellation of poles so that no residue can contribute, yielding that the partition function vanishes as expected. For branes which are not compactly supported, it seems that the R-charges serve as a kind of equivariant regularization; it would be interesting to understand this more deeply and to see if any meaningful information can be extracted which is independent of the choice of R-charges. This is a bit reminiscent of the Nekrasov instanton moduli space appearing in four dimensional supersymmetric gauge theories; the volume formally diverges though one can regularize it through $U(1)^2$-equivariant parameters and from this extract information about the original (non-equivariant) theory.

As a test case, we consider the various (partial) resolutions of $\mathbb{C}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts diagonally with charges $(1, p)$. The Calabi-Yau case is precisely when $p = n - 1$. The number of exceptional divisors needed to be blown up in order to resolve this singularity is given by the length of a minimal continued fraction expansion of $\frac{n}{p}$, and the details of this continued fraction encode information about the resulting geometry. The fully resolved complex surface is called a *Hirzebruch-Jung surface.* The category of branes

is given by the derived representation category in the fully singular phase, the derived category of coherent sheaves in the fully resolved phase (in which case it is generated by sheaves wrapping the different exceptional divisors together with the structure sheaf), and a hybrid between the two for intermediate phases - namely sheaves with additional representation-theoretic data at the orbifold points. In the mathematical literature, it is known that certain representations of $\mathbb{Z}_n$ called special representations correspond to the sheaves wrapping the different exceptional divisors; see [18] and [13]. The paper of [16] shed light on how this arises from a physical viewpoint but worked with the K-theory rather than using the full machinery of derived categories; they clarified in particular the role of the non-special representations: these are related to the massive vacua of a Coulomb branch that opens when interpolating between the singular geometry and the resolved geometry.

For our particular class of examples, we were able to extend the brane transport of [16] to the level of categories. We also showed that the leading order part of the localization formula, given by taking a residue at the origin, gives as expected the result of a geometric integral over a formula involving characteristic classes where the latter can be evaluated using combinatorial machinery as explained in [4] to study the differential topology of toric varieties.

This thesis is organized as follows. We review the basics of 2d N = (2,2) supersymmetric theories in chapter 2, and we cover the basics of gauged linear sigma models in chapter 3 as was understood in [19]. In chapter 4, we introduce the Hirzebruch-Jung geometries which will serve as our core set of examples for everything that follows, and then in chapter 5 after a quick review of branes in general, we explain the role of the derived category as describing the branes preserving B-type supersymmetry and build up to the picture in [8] of how transporting branes between GLSM phases leads in the Calabi-Yau case to functors between derived categories. Chapter 6 contains a review of the basic idea of deriving (finite dimensional) integral formulas for partition functions from localization as well as the particular formula from [11] that we will need. In chapter 7, after first discussing the $U(1)$ case for intuition, we proceed to show the role of compactly supported branes: while the formula as derived in [11] requires positive

R-charges, compactly supported branes give precisely the right cancellation of poles in the integrand to enable one to take the R-charges to zero without the contour being pinched between poles. We also show that the partition function vanishes for empty branes as expected. After briefly describing the structure of the derived category on Hirzebruch-Jung models in terms of line bundles, we then calculate from the localization formula the leading (0-instanton) contribution to the partition function. In chapter 8, we present machinery from [4] that enables one to do explicit calculations with cohomology, K-theory, and their compactly supported variants for toric varieties in terms of the combinatorial data defining them and subsequently apply this to evaluate for fully-resolved Hirzebruch-Jung geometries a geometric formula for central charge in terms of an integral involving characteristic classes, obtaining agreement to leading order with the result obtained by localization. We turn in chapter 9 to understanding the generalization of the grade restriction rule beyond the Calabi-Yau case. We demonstrate this first for the $U(1)$ case through studying the analytic continuation of the localization formula and subsequently reframe this in a more functorial manner. In particular, we give a procedure for determining the exact contribution of transported branes to the Coulomb branch. We then argue that through Higgsing one can obtain a generalization to higher rank abelian theories. Finally, chapter 10 explores explicit brane transport for Hirzebruch-Jung geometries using the results obtained in the previous chapter.

Chapter 2

Background on 2-dimensional Supersymmetry

In this section, we review the basics of $N = (2, 2)$ supersymmetry in two dimensions. An excellent reference for this subject covering much more than we say here is the book [10]. We initially consider a flat "worldsheet" (which one can think of as the trajectory of a moving string - analogous to the worldline of a particle in general relativity) $\mathbb{R}^{1,1}$ with spatial coordinate x and time coordinate t. This can be generalized to other surfaces though additional subtleties will be involved in what follows. Then the Poincare group is the group of transformations generated by boosts (hyperbolic rotations) and translations in space and time, and the corresponding Lie algebra is generated by the Hamiltonian H, the momentum P, and the angular momentum M which are the respective Noether currents for time translation, spatial translation, and Lorentz boosts. Poincare invariance then is then equivalent to being relativistic. It is possible to extend the Poincare Lie algebra to a Lie superalgebra with additional generators called *supersymmetry generators* $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$ obeying the relations

$$\begin{aligned}
\mathcal{Q}_+^2 = \mathcal{Q}_-^2 = \overline{\mathcal{Q}}_+^2 = \overline{\mathcal{Q}}_-^2 &= 0 \\
\{\mathcal{Q}_\pm, \overline{\mathcal{Q}}_\pm\} &= H \pm P \\
\{\overline{\mathcal{Q}}_+, \overline{\mathcal{Q}}_-\} = \{\mathcal{Q}_+, \mathcal{Q}_-\} &= 0 \\
\{\mathcal{Q}_-, \overline{\mathcal{Q}}_+\} = \{\mathcal{Q}_+, \overline{\mathcal{Q}}_-\} &= 0 \\
[iM, \mathcal{Q}_\pm] = \mp\mathcal{Q}_\pm, [iM, \overline{\mathcal{Q}}_\pm] &= \mp\overline{\mathcal{Q}}_\pm \\
[iF_V, \mathcal{Q}_\pm] = -i\mathcal{Q}_\pm, [iF_V, \overline{\mathcal{Q}}_\pm] &= i\overline{\mathcal{Q}}_\pm \\
[iF_A, \mathcal{Q}_\pm] = \mp i\mathcal{Q}_\pm, [iF_A, \overline{\mathcal{Q}}_\pm] &= i \pm \overline{\mathcal{Q}}_\pm
\end{aligned} \tag{2.1}$$

with the usual Hermiticity property $\mathcal{Q}_\pm^\dagger = \overline{\mathcal{Q}}_\pm$ and where F_V and F_A are respectively the generators of the vector and axial symmetries which may or may not be broken

depending on the theory, a point we shall return to shortly. It is possible to slightly generalize this by introducing central charges to the algebra, but we shall not consider this here.

In order to write down expressions invariant under supersymmetry, we introduce the superspace formalism which provides a convenient way of writing down expressions invariant under supersymmetry. We first introduce "supercoordinates" which consist of our ordinary (bosonic) coordinates x, t together with Grassmann (fermionic) parameters $\theta_\pm$ and $\bar{\theta}_\pm$ which anticommute and square to zero. A Lorentz boost which acts on the bosonic coordinates via

$$\begin{pmatrix} x \\ t \end{pmatrix} \mapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{2.2}$$

also acts on the fermionic coordinates via

$$\theta^\pm \mapsto e^{\pm\gamma/2}\theta^\pm \tag{2.3}$$

$$\bar{\theta}^\pm \mapsto e^{\pm\gamma/2}\bar{\theta}^\pm. \tag{2.4}$$

A superfield is then a function on superspace. The Taylor expansion in the Grassmann parameters is automatically finite by nilpotency. It is then easy to write down the supersymmetry operators in terms of the actions on the superfields.

$$\mathcal{Q}_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm \tag{2.5}$$

$$\overline{\mathcal{Q}}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm \tag{2.6}$$

To evaluate Grassman integrals, we use the rule $\int(a\theta + b)d\theta = a$ for θ a real Grassman parameter; this result works for general functions because the Taylor expansion in a Grassman variable must stop at the linear term by nilpotency. Complex Grassman integrals can then be calculated by writing complex Grassman parameters in terms of real and imaginary parts. Thus these integrals can be viewed as essentially projecting onto certain coefficients in the θ-expansion. We can then introduce the vector and axial actions for a superfield $\mathcal{F}$ by

$$\begin{aligned} e^{i\alpha F_V} &: \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\alpha q_V}\mathcal{F}(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm) \\ e^{i\alpha F_A} &: \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\beta q_V}\mathcal{F}(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm) \end{aligned} \tag{2.7}$$

for integers q_V and q_A called vector and axial R-charges respectively.

We have three basic types of expressions which are invariant under supersymmetry; more complicated supersymmetric actions can be expressed as a mixture of these. By carrying out the θ integrals, we end up with an explicit expression for the Lagrangian. A key advantage of superfield notation is that one can write rather succinct expressions which when expanded would look rather complicated; this is possible because the constraint of being supersymmetric is quite tight.

- An expression of the form
$$\int d^2x d^4\theta K(\mathcal{F}) \tag{2.8}$$
for $\mathcal{F}$ a superfield is called a D-term.

- A chiral superfield is a superfield Φ which can be expressed as
$$\Phi(\dots) = \phi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm) \tag{2.9}$$
where $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$. Then an expression of the form
$$\int d^2x d^2\theta W(\Phi) \tag{2.10}$$
is called an F-term.

- A twisted chiral superfield is a superfield U which can be expressed as
$$U(\dots) = \nu(\tilde{y}^\pm) + \theta^+\bar{\chi}_-(\tilde{y}^\pm) + \bar{\theta}^-\chi_-(\tilde{y}^\pm) + \theta^+\bar{\theta}^- E(\tilde{y}^\pm) \tag{2.11}$$
where $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$. Then an expression of the form
$$\int d^2x d^2\tilde{\theta}\widetilde{W}(U) \tag{2.12}$$
is called a twisted F-term.

As an alternative perspective, if we introduce differential operators acting on superfields via
$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm \tag{2.13}$$
$$\overline{D}_\pm = -\frac{\partial}{\partial\theta^\pm} + i\theta^\pm\partial_\pm, \tag{2.14}$$

then chiral and twisted chiral superfields can be defined by the equations $\overline{D}_{\pm}\Phi = 0$ and $D_{+}\overline{U} = \overline{D}_{-}\overline{U} = 0$ respectively. Note that the expressions for $D_{\pm}$ and $\overline{D}_{\pm}$ are almost the same as those for $\mathcal{Q}_{\pm}$ and $\overline{\mathcal{Q}}_{\pm}$ except that the signs of their second terms are flipped.

To construct a quantum field theory, one begins by writing down a Lagrangian L which is then integrated over the worldsheet to give the action S. In a classical theory (i.e. if we wished to describe the classical motion of a string), we would find field configurations that extremize this functional. To promote this to a quantum theory, one has to consider a functional integral over all field configurations.

$$\frac{1}{Z}\int D(fields)\ldots e^{\int d^2xL} \tag{2.15}$$

Here the "$\ldots$" is an expression in the fields corresponding to something physical that would be observed, and the net result of this integral is to compute its expectation value. Here Z is called the partition function and is given by computing the integral in the above equation *without* the "$\ldots$" part. One can roughly think of the exponential part as a kind of (unnormalized) probability density function on an infinite-dimensional space of field configurations, and Z is then the normalization constant. Strictly speaking, this integral (and in fact quantum field theory as a whole) is *not* mathematically well-defined, but it has proved again and again to be a tremendous source of mathematical insight. To recover the classical theory, one rescales the action by $\frac{1}{\hbar}$ and takes $\hbar \to 0$, effectively reducing to an integral against a delta function supported on the (classical) solution to the variational problem. Note that physically $\hbar$ corresponds to the scale on which quantum effects are relevant, and taking $\hbar \to 0$ is mathematically equivalent to holding $\hbar$ fixed and "zooming out" which one would expect to reduce to everyday physics.

As a first prototype, we consider a Lagrangian equal to a D-term of the form

$$L = \int d^4\theta \sum_i \bar{\Phi}_i\Phi_i = \sum_i |\partial_0\phi_i|^2 - \sum_i |\partial_1\phi_i|^2 + |F|^2 + \ldots \tag{2.16}$$

where $\Phi_1, \ldots, \Phi_N$ are chirals and where "$\ldots$" refers to the fermionic terms. The derivative terms are standard kinetic terms, and the field F is non-dynamical and may be eliminated via its equations of motion. Though presently we end up with $F = 0$, this need not continue to be the case when we add more terms to the Lagrangian. This can

essentially viewed as starting with a (bosonic) theory of a string propagating freely in $\mathbb{C}^d$ and adding the minimum terms needed for it to be supersymmetric.

Chapter 3

Gauged Linear Sigma Models

It is quite desirable to be able to construct theories of strings propagating on non-linear backgrounds. As string theory requires ten dimensions, for our universe to be described by a string theory one necessarily must compactify, leading to a (possibly rather complicated) curved manifold. Also from the viewpoint of extracting insights into pure mathematics from physics, there are far more interesting manifolds than $\mathbb{C}^d$!

So how does one proceed in constructing a theory with non-linear target? The most obvious approach is to cover the manifold with charts, write down a Lagrangian like the flat space one for each chart and somehow "glue" these together. In practice, however, this can get quite tricky. In [19], E. Witten introduced a beautiful alternative: by gauging in a way compatible with supersymmetry a theory on flat space, one can obtain at low energies theories with a wide variety of non-linear targets.

As a motivation, consider the (non-supersymmetric) theory of complex scalars $\phi_1, \ldots, \phi_N$ with Lagrangian $-\sum_i |\partial_\mu \phi_i|^2 - U(\phi)$ where the potential $U(\phi)$ is given by $U(\phi) = \frac{e^2}{2}\left(\sum_i |\phi_i|^2 - \zeta\right)^2$. For $\zeta > 0$, the vacuum manifold obtained by minimizing U is S^{2N-1}, and at low energies this theory would reduce to a theory in which the string is confined to this S^{2N-1}. It turns out to be possible to construct a supersymmetric analog of this, and thus we can gain additional leverage in study the resulting low energy theory by using powerful tools from supersymmetry.

3.1 Lagrangian Description

It turns out that one can construct a wide array of low energy target manifolds from our supersymmetric theory on $\mathbb{C}^d$ simply by gauging the theory. For simplicity, we temporarily restrict to a single chiral Φ. Under the transformation $\phi \mapsto e^{i\alpha}\phi$, our

Lagrangian above is invariant under a global transformation (i.e. when α is constant) but is not invariant under a local transformation (i.e. when α is permitted to vary over the worldsheet) because of the derivative terms. To remedy this, we promote the transformaion to a superfield transformation $\Phi \mapsto e^{iA}\Phi$. This doesn't quite work because now $\bar{\Phi}\Phi \mapsto \bar{\Phi}e^{-i(\bar{A}-A)}\Phi$, but the expression $\bar{\Phi}e^{V}\Phi$ *is* invariant provided we constrain V to be real and impose the transformation law $V \to V + i(\bar{A} - A)$. Such a symmetry that we make exist locally is called a gauge symmetry.

The superfield V is called a vector multiplet and using the above gauge symmetry above can be expressed in the *Wess-Zumino* gauge, meaning that it can be constructed from a vector field ν, a complex scalar field σ, fermions $\lambda_\pm$ and $\bar{\lambda}_\pm$ and an auxiliary real field D. There is still a residual gauge symmetry fixing this form, and this corresponds to ordinary (non-supersymmetric) gauge transformations of ν. As the vector multiplet is dynamical, it should have its own kinetic term and possibly a potential. For this, we define the field strength as $\nu_{01} = \partial_0\nu_1 - \partial_1\nu_0$ and then consider a corresponding twisted chiral superfield Σ whose expansion begins with ν_{01}. Then the gauge kinetic part of the Lagrangian is given by $\sum_\alpha \frac{1}{2e^2}\int d^4\theta\bar{\Sigma}_\alpha\Sigma_\alpha + c.c.$ where c.c. stands for complex conjugate, analogous to the kinetic term for the chiral Φ. Here e is the gauge coupling whose role will become clear later. We also have the freedom to add a twisted F-term depending on Σ which will play a crucial role in what roles: for this, we shall only consider those of the form $\sum_\alpha -t_\alpha \int d^2\tilde{\theta}\Sigma_\alpha + c.c.$ Though we could be more general, this already as we shall see leads to a lot of extremely robust wall-crossing phenomena. Here $t_\alpha = \zeta_\alpha - i\theta_\alpha$ where ζ_α and θ_α are the FI parameter and theta angle respectively.

Now we can introduce a potential in a way that preserves supersymmetry by adding an F-term to the Lagrangian (or twisted F though by symmetry it would have essentially the same consequence) which takes the form $\int d^2\theta W(\Phi_1, \ldots, \Phi_N) + c.c$ After doing the superfield calculus, we see that it changes the value of the auxiliary field F in the chiral multiplet, leading to a potential term of the form $-|W'(\phi_1, \ldots, \phi_N)|$. This term then acts as a potential and makes it energetically expensive for the string to move in a region where W' is large. When such a term is added, W is called the *superpotential.* Similarly, the term introduced before depending on Σ is called a *twisted superpotential.*

Putting everything together, we have a Lagrangian of the form

$$L = \int d^4\theta \bar{\Phi} e^V \Phi + \int d^4\theta \frac{1}{2e^2} \bar{\Sigma}\Sigma + c.c. + -t \int d^2\tilde{\theta}\Sigma + c.c. + \int d^2\theta W(\Phi) + c.c. \quad (3.1)$$

$$= L_{kin} + L_{gauge} + L_{FI,\theta} + L_W. \quad (3.2)$$

We can generalize this to have chirals $\Phi_1, \ldots, \Phi_N$ and vector multiplets $V_1, \ldots, V_r$.

$$L = \int d^4\theta \left(\sum_{i=1}^{N} e^{\sum_\alpha Q_{i\alpha} V_\alpha} \bar{\Phi}_i \Phi_i - \frac{1}{2e^2} \sum_\alpha \bar{\Sigma_\alpha} \Sigma_\alpha + c.c. \right) + \quad (3.3)$$

$$\int d^2\tilde{\theta} \sum_\alpha (-t_\alpha \Sigma_\alpha) + c.c. + \int d^2\theta W(\Phi_1, \ldots, \Phi_N) + c.c. \quad (3.4)$$

Here the overall gauge symmetry is now $U(1)^r$ and $Q_{i\alpha}$ is the matrix giving the charges for how each gauge factor $U(1)_\alpha$ acts on each chiral Φ_i. More generally, one could consider a non-abelian gauge group together with a chiral representation, but for our purposes these representations will always be abelian and hence reduce to the above description.

It is natural to ask whether a theory of this form is invariant under part or all of the $U(1)_V \times U(1)_A$ group of vector and axial rotations. It can be shown that such an action is always invariant under $U(1)_A$, and $U(1)_V$ invariance holds if an only if the superpotential is quasi-homogeneous by which we mean that $W(\lambda^{q^i}\Phi^i) = \lambda^2 W(\Phi^i)$ for some choice $\{q^i\}$ of R-charges. This does not, however, imply invariance of the quantum theory due to the presence of anomalies: the measure on the space of fields may fail to be invariant, rendering correlation functions *not* invariant. An important result for N=(2,2) theories is that the vector symmetry is never anomalous, and the axial symmetry is anomalous if and only if $\sum_i Q_{i\alpha} \neq 0$ for some i.

3.2 Classical Phases in the Infrared

If we expand the superfields, carry out the θ-integrals, and eliminate auxiliary fields via equations of motion, then the bosonic part consists of the kinetic energy (given by the terms involving derivatives of fields) minus the potential energy. The latter has the following form.

$$U = \sum_i |Q_{i\alpha}\sigma_\alpha|^2 |\phi_i|^2 + \frac{e^2}{2} \sum_\alpha \left(\sum_i Q_{i\alpha} |\phi_i|^2 - \zeta_\alpha \right)^2 + \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 \quad (3.5)$$

Like in the non-supersymmetric toy example, the string is effectively confined to the moduli of vacua, i.e. the vanishing locus of $U(\phi)$. To find this, we see we can eliminate the first term by either i) setting all σ_α's to 0 which leads to the *Higgs branch* or ii) setting all ϕ_i's to 0 which leads to the *Coulomb branch*. In addition, there are also hybrid scenarios called *mixed branches*. For now, we focus on the Higgs branch except to point out that Coulomb / mixed branches can only occur for exceptional choices of ζ_α's as otherwise we will not be able to eliminate the second term.

To eliminate the second term, we impose the equations

$$\sum_i Q_{i\alpha}|\phi_i|^2 = \zeta_\alpha \text{ for all } \alpha. \tag{3.6}$$

Furthermore, there is an additional redundancy correspond to the action of the gauge group on this solution set, and we should thus quotient the solution set by the gauge action. The last term of the potential then simply imposes further equations

$$\frac{\partial W}{\partial \phi_i} = 0 \text{ for all } i. \tag{3.7}$$

Moreover, the fields in the vector multplet are eliminated by a supersymmetric analog of the Higgs mechanism: σ_α becomes heavy with mass of order $e|\zeta_\alpha|^{1/2}$ and at low energies should be integrated out.

Remark 1. *It is possible that certain fluctuations transverse to the equations coming from (3.7) may be massless. In this case, the resulting IR theory will allow fluctuations in these theories but will have a nontrivial superpotential. Even when this is not the case, it is possible up to some subtleties (see [8]) to go to an intermediate energy scale in which equation 3.7 is not imposed, but one instead has a nontrivial superpotential.*

From the mathematics side, this construction for $W = 0$ is precisely the construction of a toric variety from symplectic reduction corresponding to an action of a compact abelian Lie group G on a complex vector space V with equation (3.6) giving the moment maps. Here the ζ_α's are coordinates on $\mathfrak{g} = \mathrm{Lie}(G)$, i.e. we have collectively that $\zeta \in \mathfrak{g}^\vee$. Concretely, G is isomorphic to $U(1)^r$ for some r, and the representation is given by its collection of weights $Q^j : \mathfrak{g} \to \mathbb{C}$ specifying how $\mathfrak{g}$ acts infinitesimally on each of the

coordinates $\phi_1, \ldots, \phi_N$ for V. Note that in general the resulting geometry may have orbifold singularities.

From now on, we shall adhere to the following convention.

Convention 1. *In what follows, we shall assume $W = 0$ except to occasionally point out what generalizations one can obtain from the introduction of a nontrivial superpotential.*

Denote by X_ζ the result of the above symplectic reduction where ζ without an index refers to our collective choice of ζ_α's. As a complex orbifold, X_ζ is equivalent to taking a suitable *geometric invariant theory (GIT)* quotient, meaning that we remove a Zariski-closed subset called a deleted set (in these cases, a hyperplane arrangement) and then quotient by the complexification of G (isomorphic to $(\mathbb{C}^*)^r$). The correct choice of deleted set needed to reproduce the symplectic reduction construction depends on the choice of $\zeta \in \mathfrak{g}^\vee$. If we regard ζ and ζ' as equivalent in the case that X_ζ and $X_{\zeta'}$ give isomorphic toric stacks, then the subdivision of $\mathfrak{g}^\vee$ into equivalence classes gives a natural subdivision into a complete fan of polyhedral cones which we call *classical phases.* Physically, this construction is relevant for ζ deep within a cone as the positions of the walls receive quantum corrections which are suppressed when we run off to infinity deep inside a cone. The Kähler metric, however, *will* be sensitive to changing ζ within a cone, and in fact, the cone containing ζ is naturally identified with the Kähler cone of X_ζ. The union of the interiors of all such cones is called the *classical Kähler moduli space,* denoted by $\mathcal{M}_K^{cl}$. We next turn to how this is modified by quantum effects.

3.3 Quantum Effects

There are several different quantum effects that are relevant. First we have the renormalization of the FI parameters which is given by

$$\zeta_\alpha(\mu) = \zeta_\alpha^{UV} + \left(\sum_i Q_{i\alpha}\right) \log(\frac{\mu}{M^{UV}}) \tag{3.8}$$

for ζ_α^{UV} a UV FI parameter, μ an energy scale, and M^{UV} a UV mass scale. We thus see that the axial anomaly is cancelled precisely in the case that the FI parameters do not run under RG flow. In this case, the theory is actually invariant under the *N=2,2*

superconformal algebra which contains both the N = (2,2) supersymmetry algebra and all worldsheet conformal transformations. Conformal invariance is extremely powerful for constraining quantum field theories and enabling exact calculations though since we will not be using it explicitly, we merely refer the interested reader to the book [10]. Geometrically, this happens if and only if the target geometry is Calabi-Yau: this is because having $\sum_i Q_{i\alpha} \neq 0$ for some α is equivalent to having c_1 integrate nontrivially along a complex codimension one cycle.

For anomalous theories, we see that renormalization is actually playing two roles at once: going from the GLSM to a theory in a phase and moving in the FI parameter space, possibly cutting between phases. A potential source of concern is that if RG flow would take us from one phase to another, say from phase I to phase II, then it might not be possible to start in the UV and flow to phase I. Perhaps RG flow always bypasses phase I and goes directly from the UV to phase II. We can avoid this problem by taking the gauge coupling $e \to \infty$ for fixed energy scale: this (just like flowing to the IR) will produce a theory with target the vacuum manifold modulo gauge, but it keeps with the energy scale fixed which then enables us to send the energy scale to zero as a further limit, thus ensuring that we visit phase I before phase II. The $e \to \infty$ limit also has the effect of decoupling the vector multiplet.

There are also quantum corrections to the positions of the walls in the moduli space. A classical wall corresponds to where a Coulomb (or mixed) branch can open. If we write $\sigma = \sigma_0 u_0$ for u_0 a vector normal to the boundary wall, then taking σ_0 to be large forces chirals Φ_i for which $Q_i \cdot u_0 \neq 0$ to become heavy. Integrating these out then yields a low energy effective theory on this branch with a twisted superpotential for σ_0.

$$\widetilde{W}_{eff}(\sigma_0) = -(t \cdot u_0)\sigma_0 - \sum_{i:Q_i \cdot u_0 \neq 0} (Q_i \cdot u_0)\sigma_0 \Big(\log \frac{(Q_i \cdot u_0)\sigma_0}{\mu} - 1 \Big) \tag{3.9}$$

The corresponding vacua, given by the critical points of the above twisted superpotential, are given by

$$t \cdot u_0 = -(Q^{tot} \cdot u_0) \log \left(\frac{\sigma_0}{\mu} \right) - \sum_{i:Q_i \cdot u_0 \neq 0} (Q_i \cdot u_0) \log(Q_i \cdot u_0) \tag{3.10}$$

There are then two distinct cases to consider.

- In the non-anomalous case where $Q^{tot} \cdot u_0 = 0$, the first term vanishes, and we see that the position of the wall is simply shifted by a constant amount.
- In the anomalous case where $Q^{tot} \cdot u_0 \neq 0$, we have no wall - merely a finite set of solutions corresponding to Coluomb (or mixed) branch vacua which join the Higgs branch when crossing between two phases

One should actually consider the FI parameters together with real lifts of theta angles as describing complex coordinates $t_\alpha = \zeta_\alpha - i\theta_\alpha$ on $\mathfrak{g} \otimes \mathbb{C}$, and after removing the quantum-corrected walls, we obtain the *complexified Kähler moduli space* $\mathcal{M}_K$. From the above analysis, we see that hitting a wall requires taking a particular value for a theta angle so the walls are in fact *complex* codimension one. This implies that one can continuously interpolate between phases, giving a conceptual reason for why certain properties would be wall-crossing invariant.

Remark 2. *It may seem puzzling as to why one should take real lifts of theta angles when the underlying theory only depends on their values modulo* 2π. *This is because taking real lifts allows for topologically distinct paths connecting phases which, as will be shown later, lead to distinct transport functors. This is related to the existence of nontrivial monodromy around the singular loci.*

Chapter 4

Hirzebruch-Jung Geometry

The McKay correspondence builds a beautiful bridge between algebraic geometry and representation theory. Let $\Gamma \subset SU(2)$ be a finite subgroup acting on $\mathbb{C}^2$ via linear transformations. Then Γ follows an ADE classification and is either cyclic (A_n), dihedral (D_n), or one of three exceptional types (E_n for $n \in \{6, 7, 8\}$). The quotient $\mathbb{C}^2$ has an orbifold singularity at the origin with stabilizer Γ and is smooth elsewhere. Taking a minimal resolution of the singularity, we obtain a finite set of exceptional divisors isomorphic to copies of $\mathbb{CP}^1$. If we construct a graph by drawing a vertex for each exceptional divisor and connecting pairs of vertices when the corresponding exceptional divisors intersect, then we obtain precisely the Dynkin diagram of the corresponding Lie algebra! Moreover, the intersection matrix for the exceptional divisors equals minus the Cartan matrix associated to this Lie algebra.

One possible way to explore this physically is to construct a sigma model to $\mathbb{C}^2/\Gamma$. Then B-branes correspond to equivariant vector bundles and hence to representations of Γ. If we consider wall-crossing corresponding to blowing up exceptional divisors, then these branes should map to sheaves wrapping exceptional divisors (with appropriate equivariant structure at any remaining singularities). Going to the fully resolved phase in particular should map the derived representation category of Γ to the derived category of the resolved geometry.

In [16], this kind of brane transport was considered at the level of K-theory for a natural generalization of the A_n case: one can allow Γ to be any subgroup of $U(2)$ as opposed to just $SU(2)$. This allows for the Calabi-Yau condition to be violated and for a Coulomb branch containing massive vacua to open when crossing from the resolved to singular phase. They argued that the counting of branes on both sides is consistent

provided one takes into account the Coulomb branch vacua together with the branes wrapping cycles of the resolved geometry. In [8], this brane transport for the A_n model was carried out at the categorical level but without the generalization allowing for massive vacua. There has also been relevant mathematical work as in [18] and [13]. We will show how using a generalization of the grade restricted rule from [8] to the anomalous case enables one to carry out both generalizations simultaneously.

4.1 Background

We consider the generalization of the A_n McKay correspondence given by the quotient of $\mathbb{Z}_n$ acting on $\mathbb{C}^2$ via $(X, Y) \mapsto (\omega X, \omega^p Y)$ for ω an n-th root of unity which we denote by $\mathbb{C}^2/\mathbb{Z}_{n(p)}$. Taking $p = n - 1$ yields the usual A_n quotient which is Calabi-Yau; otherwise we obtain a massive generalization.

We now turn to a review of continued fractions; all of the material here is pulled from [16]. We may write $\frac{n}{p}$ as a continued fraction as follows

$$\frac{n}{p} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\cdots \frac{1}{a_r}}}} \tag{4.1}$$

where $a_i \geq 1$ are integers. We will sometimes denote such a continued fraction as $[a_1, \ldots, a_r]$. Such a representation is *not* unique as is easily observed via the formula

$$[x + 1, 1, y + 1] = [x, y]. \tag{4.2}$$

However, there is a unique *minimal* representative for which all entries are ≥ 2. In these notes, we will focus on minimal representatives although many of our remarks will apply more generally. The generalized Cartan matrix for a continued fraction expansion $[a_1, \ldots, a_r]$ is given by

$$C_{\alpha\beta} = a_\alpha \delta_{\alpha\beta} - \delta_{\alpha+1\beta} - \delta_{\alpha-1\beta}. \tag{4.3}$$

Note that in the Calabi-Yau case ($r = n - 1$), we have $\frac{n}{n-1} = [2, \ldots, 2]$ with $n - 1$ entries, thus recovering the usual A_n Cartan matrix. In what follows, it will be convenient to introduce the recursively defined sequences p_α and q_α for $\alpha = 0, \ldots, r + 1$. We define

$$\frac{p_{j-1}}{p_j} = [a_j, \ldots, a_r] \tag{4.4}$$

$$\frac{q_{j+1}}{q_j} = [a_j, \ldots, a_1] \tag{4.5}$$

for $1 \leq j \leq r$ with the initial conditions $p_{r+1} = 0$, $q_0 = 0$, $p_r = 1$, and $q_1 = 1$. Note that the sequences p_α and q_α are decreasing and increasing respectively and that $p_0 = n$. From this, one can write a nice formula for the inverse of $C_{\alpha\beta}$.

$$(C^{-1})_{\alpha\beta} = \begin{cases} \frac{1}{n} q_\alpha p_\beta & 1 \leq \alpha \leq \beta \leq r \\ \frac{1}{n} p_\alpha q_\beta & 1 \leq \beta \leq \alpha \leq r \end{cases} \tag{4.6}$$

We now note several relations which follow easily and which will prove useful later.

$$\frac{p_\alpha}{q_\alpha} > \frac{p_{\alpha+1}}{q_{\alpha+1}} \tag{4.7}$$

$$q_\alpha p_{\alpha-1} - q_{\alpha-1} p_\alpha = n \tag{4.8}$$

$$p_{\alpha-1} - a_\alpha p_\alpha + p_{\alpha+1} = 0 \tag{4.9}$$

$$q_{\alpha-1} - a_\alpha q_\alpha + q_{\alpha+1} = 0 \tag{4.10}$$

This setup admits a generalization based on minors of the Cartan matrix which is useful for analyzing partially resolved phases. Define

$$d_{ij} = -d_{ji} = \det(C_{\alpha\beta})_{i<\alpha,\beta<j} \qquad \text{for } 0 \leq i < j \leq r+1 \tag{4.11}$$

$$d_{ii} = 0 \qquad \text{for } 0 \leq i \leq r+1. \tag{4.12}$$

Doing cofactor expansion on the first or last row of a minor gives the recursion relations

$$d_{i(j-1)} + d_{i(j+1)} = a_j d_{ij} \tag{4.13}$$

$$d_{(i-1)j} + d_{(i+1)j} = a_i d_{ij} \tag{4.14}$$

from which a simple induction argument yields

$$[a_i, \ldots, a_{j-1}] = \frac{d_{(i-1)j}}{d_{ij}} \tag{4.15}$$

$$[a_j, \ldots, a_{i+1}] = \frac{d_{i(j+1)}}{d_{ij}}. \tag{4.16}$$

As the continued fractions are ≥ 1, we must have $d_{ij} < d_{(i-1)j}$ and $d_{ij} < d_{i(j+1)}$ for $i < j$. We then recover our above setup by taking $p_j = d_{j(r+1)}$ and $q_j = d_{0j}$. We may also use (4.13) and (4.14) recursively to obtain

$$p_i q_j - p_j q_i = n d_{ij} \text{ for } 0 \leq i, j \leq r+1 \tag{4.17}$$

from which can then prove that

$$d_{ij}d_{kl} - d_{ik}d_{jl} + d_{il}d_{jk} = 0. \tag{4.18}$$

Alternatively, one can start with the matrix

$$\begin{pmatrix} p_i & p_j & p_k & p_l \\ q_i & q_j & q_k & q_l \\ p_i & p_j & p_k & p_l \\ q_i & q_j & q_k & q_l \end{pmatrix} \tag{4.19}$$

whose determinant trivially vanishes. If one asks how after two steps of cofactor expansion (going along the rows in order) to reduce to the determinant that computes nd_{kl}, one sees that one must either pick p_i and then q_j OR p_j and then q_i, leading to a net contribution of

$$(p_i q_j - p_j q_i) n d_{kl} = n^2 d_{ij} d_{kl}. \tag{4.20}$$

The sum over all analogous contributions to the determinant give a result proportional the left hand side of our identity.

Next following [16], we introduce two equivalent GLSMs to study the geometry of $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ and its (partial) resolutions.

Model I

	P	X_1	X_2	X_3	$\dots$	X_r	Q
$U(1)_1$	1	$-a_1$	1	0	$\dots$	0	0
$U(1)_2$	0	1	$-a_2$	1	$\dots$	0	0
$U(1)_3$	0	0	1	$-a_3$	$\dots$	0	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$U(1)_r$	0	0	0	0	$\dots$	$-a_r$	1

Here P is charged 1 under $U(1)_1$ and is neutral under all other $U(1)$'s while Q is charged 1 under $U(1)_r$ and is neutral under all other $U(1)$'s. The charges of the X_i's are given by minus the generalized Cartan matrix. In the fully resolved phase, the number of exceptional divisors is r, precisely the size of the continued fraction expansion of $\frac{n}{p}$. We

may diagonalize the charges using $n(C^{-1})_{\alpha\beta}$ to obtain the following second (equivalent) model.

Model II

	P	X_1	X_2	X_3	$\dots$	X_r	Q
$U(1)_1$	p_1	$-n$	0	0	$\dots$	0	q_1
$U(1)_2$	p_2	0	$-n$	0	$\dots$	0	q_2
$U(1)_3$	p_3	0	0	$-n$	$\dots$	0	q_3
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$U(1)_r$	p_r	0	0	0	$\dots$	$-n$	q_r

Convention 2. *In the context of Hirzebruch-Jung models, we write $t_\alpha = \zeta_\alpha - i\theta_\alpha$ when referring to Model I and $t'_\alpha = \zeta'_\alpha - i\theta'_\alpha$ when referring to Model II.*

Here the geometric interpretation is more transparent: increasing the α-th FI parameter ζ'_α increases the size of the α-th exceptional divisor in the regime where $\zeta'_\alpha > 0$ for fixed values of the neighboring FI parameters; this can also impact the size of the neighboring exceptional divisors. Following the discussion on section 6 of [16], we can make this more concrete by looking at allowed values of $|P|^2$ and $|Q|^2$. The region of admissible values is given by the region in the first quadrant of the $(|P|^2, |Q|^2)$ plane which is above the line

$$p_\alpha |P|^2 + q_\alpha |Q|^2 = \zeta'_\alpha \tag{4.21}$$

for each α. Note that the FI parameter ζ'_α corresponds to the position of a line of slope $\frac{p_\alpha}{q_\alpha}$, and these slopes are decreasing in α by virtue of equation (4.7). Moving the αth line so that it cuts through the region of admissible values (and therefore that region contains as part of its boundary a line segment on this line) corresponds to a blow-up introducing an exceptional divisor whose size is now controlled by ζ'_α. A full resolution corresponds to arranging the lines so that each line contains a segment which is part of the boundary of the admissible region. The resulting geometry is called a Hirzebruch-Jung surface and contains r exceptional divisors - one for each FI parameter.

4.2 Local Models

Let $E_i = \{X_i = 0\}$ for $0 \leq i \leq r+1$, and let A be all $i \in \{1, \ldots, r\}$ for which E_i is nonempty. Then E_0 and E_{r+1} are noncompact while E_i for $i \in A$ is an exceptional divisor and is isomorphic to $\mathbb{CP}^1$.

Suppose $E_i \cap E_j$ is nonempty. Then no indices between i and j may be contained in A as otherwise E_i and E_j would be separated by one or more exceptional divisors. Near the intersection point all chirals except for X_i and X_j are frozen to vevs.

We now turn to analyzing the residual gauge symmetry in the basis of GLSM I. Let $(g_1, \ldots, g_r) \in U(1)^r$ be such an element. Then using that $X_0, \ldots, X_{i-1}$ are fixed, we can iteratively show $g_1 = \cdots = g_i = 1$. Similarly, using that $X_{r+1}, \ldots, X_{j+1}$ are fixed iteratively gives that $g_{r+1} = \cdots = g_{j+1} = 1$. Now for each α we have $g_{\alpha-1} = g_\alpha^{a_\alpha} g_{\alpha+1}^{-1}$, and we obtain through induction that

$$g_\alpha = g_{j-1}^{d_{\alpha j}} \text{ for } i \leq \alpha \leq j, \tag{4.22}$$

by using our recursion relations. This finally forces $g_{j-1}^{d_{ij}} = 1$, leaving us with the residual gauge group $\mathbb{Z}_{d_{ij}}$. From looking at the action on the two unfrozen chirals, we see that we can describe the intersection via the local model

	X_i	X_j
$\mathbb{Z}_{d_{ij}}$	$d_{(i+1)j}$	1

Next we study the geometry of a single exceptional divisor E_j. Choose $i < j$ and $k > j$ such that E_i and E_k are neighboring E_j, i.e. they each intersect E_j at a point. Similar to the previous case, we work in the basis of GLSM I and iteratively show that $g_1 = \ldots g_i = 1$ and $g_{r+1} = \ldots g_k = 1$ and then deduce by induction that $g_\alpha = g_{i+1}^{d_{i\alpha}}$ and $g_\alpha = g_{k-1}^{d_{\alpha k}}$. Then $g_{i+1}^{d_{ij}} = g_j = g_{k-1}^{d_{jk}}$ so therefore

$$g_{i+1} = h^{d_{jk}/m}\omega^u \text{ and } g_{k-1} = h^{d_{ij}/m}\omega^v \text{ for } (h, \omega) \in U(1) \times \mathbb{Z}_m \tag{4.23}$$

where $m \in \mathbb{Z}_{\geq 1}$ and $u, v \in \mathbb{Z}$ satisfying $m = \gcd(d_{ij}, d_{jk}) = ud_{ij} - vd_{jk}$. We then obtain by working out explicitly the action on X_i, X_j, and X_k the following local model

	X_i	X_j	X_k
$U(1)$	d_{jk}/m	$-d_{ik}/m$	d_{ij}/m
$\mathbb{Z}_m$	u	s	v

where

$$l = -d_{ik} \tag{4.24}$$

$$s \equiv d_{i(j-1)}u - d_{(j-1)k}v \bmod m. \tag{4.25}$$

This corresponds geometrically to a Zariski-open neighborhood of a single exceptional divisor corresponding to $X_j = 0$. This exceptional divisor is geometrically given by $\mathbb{WCP}^1_{a,b}$.

As it will prove useful later, we note that this easily yields a formula for the complexified FI parameter of the local model:

$$t_{loc} = \sum_{\alpha=i+1}^{k-1} \left(\frac{d_{i\min(\alpha,j)} d_{\max(\alpha,j)k}}{m} \right) t_\alpha \tag{4.26}$$

where the coefficient of t_α comes from writing g_α as an exponent of h.

We turn to understanding the geometry of sheaves on $\mathbb{WCP}^1_{a,b}$. We can construct this geometry by gluing $\mathbb{C}/\mathbb{Z}_a$ with coordinate x to $\mathbb{C}/\mathbb{Z}_b$ with coordinate y. Then x^a and y^b are single-valued, and we may glue by identifying $x^a = y^{-b}$. If we are given a line bundle on $\mathbb{WCP}^1_{a,b}$, then its restriction to $\mathbb{C}/\mathbb{Z}_a$ is characterized by its charge under $\mathbb{Z}_a$ which we denote by $-\bar{\gamma} \in \mathbb{Z}_a$. Similarly, its restriction to $\mathbb{C}/\mathbb{Z}_b$ is characterized by its charge under $\mathbb{Z}_b$ which we denote by $\bar{\delta} \in \mathbb{Z}_b$. We can describe the restriction of a section of this line bundle to our cones via some $f_N : \mathbb{C}^* \to \mathbb{C}$ such that $f_N(x) = \omega_a^{-\bar{\gamma}} f_N(\omega_a x)$ and some $f_S : \mathbb{C}^* \to \mathbb{C}$ such that $f_S(y) = \omega_b^{\bar{\delta}} f_N(\omega_b y)$ which are together related by a transition map $f_N(x) = (\dots) f_S(y)$ for $x^a = y^{-b}$. Assuming a transition map of the form $x^\gamma y^\delta$, we see that to obtain the correct restrictions we must have $-\bar{\gamma} \equiv \gamma (\bmod\ a)$ and $\bar{\delta} \equiv \delta (\bmod\ b)$. Then the dependence on (γ, δ) is only up to its class in $\mathbb{Z}^a/((a,-b)\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}_m$ for $m = \gcd(a,b)$ so we may view this group as parameterizing all such line bundles. Note that in the case $m = 1$ of ordinary weighted projective space, this group is just $\mathbb{Z}$, indicating that all line bundles are (possibly negative) tensor powers of a single bundle $\mathcal{O}(1)$.

Chapter 5

B-Branes in a GLSM

If we generalize our worldsheet to allow for boundary components, then it is necessary to impose boundary conditions which correspond to manifolds equipped with bundles and connections or possibly a bound state of these. Quite remarkably, the boundary conditions themselves become dynamical objects leading to the notion of *branes*, and their study leads to a rich mixture of physics and mathematics. After briefly reviewing branes in general, we will focus on a particular class of branes called B-branes whose study leads to the algebraic geometer's notion of a derived category. These will then be the main focus of this thesis. The main reference on which this chapter is based is [8].

5.1 From Strings to B-Branes

If we consider our theory on the worldsheet $\Sigma = \mathbb{R} \times \mathbb{R}_{\geq 0}$ instead of $\mathbb{R}^{1,1}$, then it is necessary to impose boundary conditions. For the moment, we discuss boundary conditions in general and then extend to the supersymmetric case. If the target space was $\mathbb{R}$, there would be two types of boundary conditions we could impose

- *Dirichlet:* $\phi|_{\partial\Sigma} = \phi_0 \in \mathbb{R}$
- *Neumann:* $n^\mu \partial_\mu \phi|_{\partial\Sigma}$ for n^μ a normal vector

The respective physical interpretations are fixing an endpoint and prohibiting any momentum from flowing off the end of the string. If we extend to $\mathbb{R}^n$, then we have the freedom to make an independent choice for each coordinate direction. One can further extend this to a more global setting by considering a target X with submanifold Y. To restrict the end of the string to Y is to effectively impose Dirichlet boundary conditions in all normal directions, and one can furthermore impose Neumann boundary conditions

(infinitesimally) for tangential directions to Y. The Neumann portion of the boundary condition may be generalized via

$$G_{IJ}\partial_n\phi^J = F_{IJ}\partial_t\phi^J \tag{5.1}$$

where we write G_{IJ} for the spacetime metric, F_{IJ} for the spacetime field strength of the gauge field, and the subscripts n and t for the normal and tangential directions to the boundary on the worldsheet. This in turn requires adding to the action a boundary term so that the variation of the (total) action has no boundary part. The boundary term may be given explicitly via $-\int_{\partial\Sigma}\phi^*(A)$ for $A = A_I d\phi^I$. The gauge field is a connection on a (possibly nontrivial) vector bundle on Y; including multiple gauge fields is equivalent to summing the corresponding bundles.

A natural question therefore is if the introduction of such boundary conditions breaks supersymmetry. Though it impossible to preserve all four supercharges of the $N = (2,2)$ supersymmetry algebra, it is, however, possible to preserver at most half. If we let $Q_A = \bar{Q}_+ + Q_-$ and $Q_B = Q_+ + Q_-$, then it is possible to preserve either i) Q_A and $\bar{Q}_A$ (called *A-type supersymmetry*) or ii) Q_B and $\bar{Q}_B$ (called *B-type supersymmetry*). One can further deform either of these by rotating Q_- but not Q_+ or vice versa, but we shall not consider this here. The case of interest to us for the remainder of this thesis will be B-type supersymmetry in which a delicate analysis of the boundary conditions (see [9]) leads to branes as a complex submanifold together with a complex vector bundle or more generally a bound state of these which we shall later define more precisely.

More formally, we may express a brane $\mathcal{B}$ as $\mathcal{B} = (E, A, Q)$ for E a graded vector bundle, A a connection on E, and Q a degree one operator on E satisfying $Q^2 = 0$. The grading corresponds physically to the R-charge, and different ways of forming complexes ammount to different ways of binding branes together. In particular, binding a pair of branes amounts to taking a mapping cone. As we will soon see, there are at low energies many relations between branes, and even though a priori E could be a bundle on any submanifold, in fact every brane is equivalent to a complex of vector bundles on the entire space. This is related to the phenomenon of *tachyon condensation* and is discussed in great detail in [8]. Thus in the discussion that follows we will assume E to

be a vector bundle on spacetime.

If we change the worldsheet and now consider the case when $\Sigma = [0,1] \times \mathbb{R}$, we acquire the freedom to choose a different brane for each boundary component. Quantization leads to a space of string states associated with the pair of branes. One can show that the ground states are given by the $\mathcal{Q}_B$ cohomology of such states by essentially the same arguments used in supersymmetric quantum mechanics to match ground states with cohomology classes. Namely, one notes that $H = \{\mathcal{Q}_B, \overline{\mathcal{Q}}_B\}$ as can easily be verified using the N = (2,2) supersymmetry algebra as in (2.1). It follows that for any state of positive energy, $\mathcal{Q}_B$-closed implies $\mathcal{Q}_B$-exact so these states do not contribute to the $\mathcal{Q}_B$ cohomology. Meanwhile, a positivity argument shows that $H = 0$ for a state if and only if $\mathcal{Q}_B = \overline{\mathcal{Q}}_B = 0$ so therefore the cohomology gives exactly the ground states.

Now suppose we choose branes $\mathcal{B}_1 = (E_1, A_1, Q_1)$ and $\mathcal{B}_2 = (E_2, A_2, Q_2)$. The computation of the $\mathcal{Q}_B$-cohomology is simplified by virtue of the facts that i) the fields $\psi^i_\pm$ and $\partial_z x^i, \partial_{\bar{z}} x^i$ are paired under Q and thus give no net contribution and ii) $\bar{\psi}_+ - \bar{\psi}_- = 0$ by virtue of the boundary conditions. If one identifies $\bar{\psi}^j$ with $d\bar{x}$ and $g_{i\bar{j}}\dot{x}^i$ with $-i\partial_{\bar{j}} + A_{\bar{j}}$, this reduces to computing the cohomology of

$$\mathcal{H}_{zero}(\mathcal{B}_1, \mathcal{B}_2) = \bigoplus_{i=1}^{n} \Omega^{0,i}(X, Hom(E_1, E_2)). \tag{5.2}$$

By using field an expression for $\mathcal{Q}_B$ in fields as given in [8], we see that

$$i\mathcal{Q}_B\phi = \bar{\partial}_{A_{1,2}}\phi + Q_2\phi - (-1)^{|\phi|}\phi Q_1 \tag{5.3}$$

where $\bar{\partial}_{A_{1,2}}$ is the Cauchy-Riemann operator given by

$$\bar{\partial}_{A_{1,2}}\phi = dz^{\bar{j}} \wedge (\partial_{\bar{j}}\phi + iA_{2,\bar{j}}\phi - iA_{1,\bar{j}}\phi). \tag{5.4}$$

Here $|\phi|$ is the R-charge which is given as the sum of the gradings from $Hom(E_1, E_2)$ and from forms. It is known that one has isomorphisms

$$H^{0,p}_{\bar{\partial}_{A_{1,2}}}(X, Hom(E_1, E_2)) \cong H^p(X, Ext(E_1, E_2)) \cong Ext^p(E_1, E_2) \tag{5.5}$$

relating the Dolbeault cohomology to the Ext cohomology, allowing for a purely algebraic interpretation.

5.2 The Derived Category and Its Physical Interpretation

It is natural to organize branes into a category. We take the objects to be branes, morphisms to be spaces of ground string states of vanishing R-charge, and composition of morphisms to be "gluing" strings which quantum mechanically gives a pairing of wavefunctions. We wish moreover to consider two branes which flow to the same object in the infrared to in fact be the same in the category. This leads to two types of relations: D-term deformations and brane-antibrane annihilate. Amazingly, as is shown in [8], this is the same category as the (bounded) derived category which is famous in the mathematical literature!

To construct the derived category of a variety X, we begin by considering the *homotopy category* $Ho(X)$ whose objects are bounded complexes of coherent sheaves and whose morphisms are chain maps modulo chain homotopies with the obvious notion of composition. We call a morphism a *quasi-isomorphism* if it induces isomorphisms on the cohomology, and we define the derived category $D^b(X)$ to be the category obtained from $Ho(X)$ by appending (formal) inverses to quasi-isomorphisms. A morphism is then an alternating sequence of morphisms in $Ho(X)$ and these formal inverses for quasi-isomorphisms.

This leads to the notion of a morphism in the derived category as a sequence of "roof diagrams" where each such diagram consists of a formally inverted quasi-isomorphism followed by an ordinary morphisms (i.e. in $Ho(X)$).

$$\begin{tikzcd}
 & Y_1 \arrow[dl, "q_1"'] \arrow[dr, "f_1"] & & & & Y_{n-1} \arrow[dl, "q_{n-1}"'] \arrow[dr, "f_{n-1}"] & \\
X_1 & & X_2 & \cdots & X_{n-1} & & X_n
\end{tikzcd} \tag{5.6}$$

It turns out by a theorem in homological algebra (as described for instance in [7]) that any such sequence can be expressed as a single "roof diagram."

$$\begin{tikzcd} & \tilde{Y} \arrow[dl, "\tilde{q}"'] \arrow[dr, "\tilde{f}"] & \\ X_1 & & X_n \end{tikzcd} \tag{5.7}$$

As is discussed in [7] or [8], given $\mathcal{F}, \mathcal{G} \in D^b(X)$, one has that $Hom_{D^b(X)}(\mathcal{F}, \mathcal{G}[n]) \cong Ext^n(\mathcal{F}, \mathcal{G})$ where the latter can alternatively be computed as the cohomology of the complex obtained by applying the Hom functor to an appropriate resolution. Furthermore, the derived category comes equipped with a natural collection of functors $[n]$ for $n \in \mathbb{Z}$ called *shift functors* where $[n]$ shifts the grading of a complex $\mathcal{C}$ n units to the right, i.e. $\mathcal{C}[n]_j = \mathcal{C}_{j+n}$. The shift functors then enable one to obtain the higher Ext groups via $Hom_{D^b(X)}(\mathcal{F}, \mathcal{G}[n]) = Ext^n(\mathcal{F}, \mathcal{G})$ which corresponds physically by the discussion in section 5.1 to ground string states with varying R-charges.

Though in principle coherent sheaves can be quite complicated, it turns out that they can always be resolved by vector bundles and are hence always equivalent (in the derived category) to complexes of vector bundles. Furthermore, for toric varieties which shall be the focus of this thesis, one can actually resolve by line bundles; see [5] for details. This makes it substantially easier to do explicit calculations in the derived category.

5.3 The Role of the GLSM

If we now think of varying ζ, then we have for each point in $\mathcal{M}_K$ a brane category given by $D^b(X_\zeta)$. A natural question to ask is how to define a transport functor that relates for a particular choice of path the brane category at one endpoint to the brane category at the other. More precisely, we can associate paths to transport functors between derived categories. Small deformations of the path will not change the functor, but there can be interesting monodromy phenomena associated with the choice of homotopy class of paths. Ideally, one would like to have a notion of a bundle of categories with a flat connection on $\mathcal{M}_K$, but it is not yet known how to realize this sharply. In fact, due to difficulty with quantum/stringy effects, we will work far away from the center so as

to suppress the difficult-to-control corrections.

Since the GLSM was used to construct the theories in all phases in the first place, it is natural to suspect that it should play a pivotal role in understanding the transport between phases, and indeed [8] showed this to be the case. To understand this, we first take a short digression to understand the brane category for the GLSM.

Since the target of the GLSM is topologically trivial, so is necessarily any vector bundle on it. One might naively be inclined to say that a brane is therefore a complex of vector spaces, but this would fail to take the gauge group into account. The proper way to account for this is to study a complex of pairs consisting of a vector bundle together with a fiber-preserving action of the gauge group. The isomorphism type of the fiber representation cannot change as we vary the fiber so we can identify each pair with simply a gauge representation. If we explicitly identify $G = U(1)^r$, then any representation decomposes into characters, and the lattice of such characters is $Hom(U(1)^r, U(1)) \cong \mathbb{Z}^k$. Thus if we define $\mathcal{W}(q_1, \ldots, q_k)$, called a *Wilson line brane*, to be a line bundle with character corresponding to $(q_1, \ldots, q_k) \in \mathbb{Z}^k$, then in fact any object in the brane category of the GLSM can be written as a complex of such objects. We denote the resulting category by $D^b(\mathbb{C}^d, U(1)^r)$.

Remark 3. *In more mathematical terms, we are studying the* equivariant *derived category for $\mathbb{C}^d$ with its $U(1)^r$ action.*

Remark 4. *The origin of the term Wilson line brane comes from comes from the fact that the effect of this brane is equivalent to starting with a trivial brane (corresponding to the trivial line bundle on the target) and inserting a Wilson line on the worldsheet boundary. See [8].*

Given a brane in the UV GLSM theory, we can for a choice of phase flow down to get a brane on the Higgs branch theory which gives rise to a functor $F_\zeta : D^b(\mathbb{C}^d, U(1)^r) \to D^b(X_\zeta)$. This map is naturally viewed from the GIT viewpoint as follows. Consider $(\mathbb{C}^d \backslash \Delta_\zeta) \times \mathbb{C}$ where Δ_ζ is the (phase-dependent) deleted set. If we consider the gauge acting on the first factor by its usual action on the matter fields and on the section factor as specified by $(q_1, \ldots, q_k)$, then the quotient is the total space for a line bundle over

$X_\zeta = (\mathbb{C}^d \backslash \Delta_\zeta)/(\mathbb{C}^*)^k$ which we denote by $\mathcal{O}(q_1, \ldots, q_k)$. Then F_ζ maps $\mathcal{W}(q_1, \ldots, q_k)$ to $\mathcal{O}(q_1, \ldots, q_k)$, and since F_ζ acts on a complex degree-by-degree and commutes with taking direct sums, this is sufficient to uniquely specify it.

5.4 Derived Categories in Phases from Generators and Relations

A useful perspective is to start with the UV brane category, denoted by $D^b(\mathbb{C}^d, U(1)^r)$, which is given by complexes of Wilson lines. We think of this as a $U(1)^r$-equivariant version of $D^b(\mathbb{C}^d)$. From the GIT viewpoint, after choosing a phase, we have for each $U(1)_\alpha$ a corresponding deleted set Δ_α. We can take a Koszul resolution of $\mathcal{O}_{\Delta_\alpha}$ by equivariant vector bundles, and by our earlier remark, this can in fact be expressed purely in terms of equivariant *line bundles,* i.e. Wilson lines. Then upon flowing to the IR, $\mathcal{O}_{\Delta_\alpha}$ becomes trivial, giving an exact sequence of Wilson lines which we can view as a relation in the derived category. If we write $\mathcal{O}(\overrightarrow{q})$ for the Higgs branch image of $\mathcal{W}(\overrightarrow{q})$, then we can think of $\mathcal{O}(\overrightarrow{q})$ as a set of generators with one relation for each deleted set.

For explicit calculations, it is helpful to think in terms of modules. Schematically, if we have variables $x_1, \ldots, x_m, y_1, \ldots, y_n$ with the deleted set given by $y_1 = \cdots = y_n = 0$, then we take a Koszul resolution of $\mathbb{C}[x_1, \ldots, x_m]$, viewed as a $\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$-module. Note that we are working with *graded* modules so it is important to keep up with the charges at each stage of the resolution. In practice, one starts with a charge-neutral copy of $\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ surjecting to $\mathbb{C}[x_1, \ldots, x_m]$ and then at each stage finds the charges by looking at the connecting map.

5.5 Brane Transport following Hori-Herbst-Page

We now seek to understand how to transport branes from one phase to another. If we are given adjacent phases separated by a codimension-one wall and containing ζ and ζ' respectively, then we seek a functor $F_{\zeta,\zeta'} : D^b(X_\zeta) \to D^b(X_{\zeta'})$. A naive attempt to define such as brane transport functor would be as follows: given a brane in a phase, lift it to a complex of Wilson lines in the UV, and then push down to the brane category of

a different phase. Unfortunately, this fails to even be well-defined, as different choices of UV lifts give different results.

$$\begin{tikzcd} & D^b(\mathbb{C}^d, U(1)) \arrow[dl, "F_\zeta"'] \arrow[dr, "F_{\zeta'}"] & \\ D^b(X_\zeta) \arrow[rr, dashed, "F_{\zeta,\zeta'}"] & & D^b(X_{\zeta'}) \end{tikzcd} \tag{5.8}$$

A key result of [8] is that for the non-anomalous case, only certain Wilson lines may be transported through the wall. By working asymptotically at infinity in $\mathcal{M}_K$, the so-called *large volume limit,* and crossing a codimension one wall, the gauge group is always Higgsed to no bigger than a single $U(1)$, allowing one to focus solely on the $U(1)$ case. In this case, we define

$$\mathcal{S} = \sum_{Q_i > 0} Q_i = -\sum_{Q_i < 0} Q_i \tag{5.9}$$

where the second equality follows because in the non-anomalous case, $\sum Q_i = 0$. Then a Wilson line $\mathcal{W}(q)$ may only be transported through the wall if the window condition is satisfied:

$$-\frac{\mathcal{S}}{2} < \frac{\theta}{2\pi} + q < \frac{\mathcal{S}}{2}. \tag{5.10}$$

For a complex involving Wilson lines not obeying this condition, one can use the relations in the derived category to find an equivalent complex which does obey the window condition and then transport that. The idea is to "cancel" Wilson lines by replacing a complex with the mapping cone of a map from this complex to an exact sequence. From the viewpoint of the derived category, as the exact sequence is trivial, this construction doesn't change the object. Furthermore, by arranging to have the identity map from a Wilson line to itself, one can then remove the pair which also preserves the corresponding object in the derived category. For branes far outside of the window, one may have to do this iteratively with each iteration removing the charges farthest from the window.

For example, consider the Hirzebruch-Jung GLSM for $\mathbb{C}^2/\mathbb{Z}_{2(1)}$ which has two chirals charged 1 and one chiral charged -2 under a single $U(1)$. This flows for $\zeta \ll 0$ to

$\mathbb{C}^2/\mathbb{Z}_2$ and for $\zeta \gg 0$ to a resolution which has a single exceptional divisor. Note that the latter geometry retracts onto $\mathbb{CP}^1$ and thus admits the same classification of line bundles as the familiar story for $\mathbb{CP}^1$. In the $\zeta \gg 0$ phase, one has that the complex $\mathcal{W}(-1) \to \mathcal{W}(0)^{\oplus 2} \to \mathcal{W}(1)$ becomes trivial in the IR. For a slightly negative theta angle, the window condition only allows one to transport complexes constructed from $\mathcal{W}(0)$ and $\mathcal{W}(1)$. If we wanted to transport $\mathcal{W}(-1)$, we could taking a mapping cone of $\mathcal{W}(-1)$ with our IR trivial complex to get $\mathcal{W}(-1) \to \mathcal{W}(-1) \oplus \mathcal{W}(0)^{\oplus 2} \to \mathcal{W}(1)$. By arranging so that the first map is projection onto the $\mathcal{W}(-1)$ summand, we can cancel the pair of $\mathcal{W}(-1)$'s to obtain $\mathcal{W}(0)^{\oplus 2} \to \mathcal{W}(1)$ which now obeys the window condition and can be transported to the $\zeta \ll 0$ phase. Far more examples of binding to cancel Wilson lines and obtain a complex obeying a window condition can be found in [8].

For a given choice of θ, the Wilson lines obeying the window condition are sufficient to generate the derived category in the sense that an arbitrary brane is equivalent to a complex of sums of Wilson lines. This then enables one to transport more general branes by first rewriting them in terms of branes in the window. A main goal of this thesis is to understand how this framework should generalize to anomalous models.

This also highlights the crucial role played by the theta angle. Changing the theta angle would correspond to changing the window condition which in turn would modify the transport functor. This is related to the fact that since the quantum-correct walls are removed, $\mathcal{M}_K$ is no longer path-connected (though it is still connected); different (homotopy classes of) paths between two phases can give different transport functors.

A more functorial picture is that the theta-dependent window $\mathbf{w}$ determines a *window category* $\mathcal{T}_{\mathbf{w}}$ which is the full subcategory of $D^b(\mathbb{C}^d, U(1))$ generated by the Wilson lines obeying the window condition, and its projections to $D^b(X_\zeta)$ and $D^b(X_{\zeta'})$ generate the respective derived categories and in fact exhibit equivalences of categories. This is not, however, compatible with forming tensor products; in fact, the window category is not even closed under tensor products. To determine the image of some $\mathcal{B} \in D^b(X_\zeta)$ under $F_{\zeta,\zeta'}$, one chooses a lift $\widetilde{\mathcal{B}} \in \mathcal{T}_{\mathbf{w}}$ with $F_\zeta(\widetilde{\mathcal{B}}) = \mathcal{B}$, and then one has $F_{\zeta,\zeta'}(\mathcal{B}) = F_{\zeta'}(\widetilde{\mathcal{B}})$.

$$
\begin{tikzcd}
 & D^b(\mathbb{C}^d, U(1)) \arrow[dl, "F_\zeta"'] \arrow[dr, "F_{\zeta'}"] & \\
 & \mathcal{T}_{\mathbf{w}} \arrow[ul, phantom, "\cup"] \arrow[dl, "\cong"] \arrow[dr, "\cong"] & \\
D^b(X_\zeta) \arrow[rr, dashed, "F_{\zeta,\zeta'}"] & & D^b(X_{\zeta'})
\end{tikzcd}
\tag{5.11}
$$

Remark 5. *In [8], the result given is in fact a bit more general. In the case of a nontrivial superpotential W, instead of considering complexes of Wilson lines, one considers matrix factorizations. A* matrix factorization *is given by a $\mathbb{Z}_2$-graded sum of Wilson lines together with an odd endomorphism Q satisfying $Q^2 = W\,Id$. At the level of the transport, virtually nothing changes: one still transports a brane by expressing it through binding to IR trivial branes purely in terms of Wilson lines obeying the grade restriction rule. This generalization will not, however, be a focus of this thesis.*

Chapter 6

Localization Background

In this chapter, we review the basics of supersymmetric localization and then introduce the particular case that we shall need: the disk partition function for GLSMs as introduced by [11]. Since the details of localization are quite technical, we will make no attempt to give a full derivation; rather we shall be content to sketch the key ideas underlying localization and to examine what physical properties can be gleaned from the specific localization formula relevant to our case of interest.

6.1 The Idea of Localization

This review section is roughly based on the lectures of Benini in [3]. We begin with a Euclidean supersymmetric quantum field theory formulated on a compact manifold, possibly with boundary provided one imposes appropriate boundary conditions. If instead given a Lorentzian theory, one can Wick rotate via $t \to i\tau$ with τ real to Euclidean signature, compute a localization result, and then analytically continue back. To take a supersymmetric theory that was originally formulated on Euclidean space and rewrite it on a curved manifold, it is necessary to couple to a background supergravity multiplet which in particular contains conformally constant spinors that enable us to realize the supersymmetry transformations; the details are quite technical so we refer the interested reader to [3]. The reason for assuming compactness is to circumvent delicate issues with infinity.

The partition function is computed by a path integral of the form

$$Z = \int D\phi e^{-S(\phi)} \tag{6.1}$$

where ϕ refers collectively to all fields. Consider the following deformation

$$Z(t) = \int D\phi e^{-(S(\phi)+Q\mathcal{V})}. \tag{6.2}$$

Then we have

$$\frac{d}{dt}Z(t) = \int D\phi Q(\dots) = 0 \tag{6.3}$$

via a version of Stokes theorem for field space, implying that $Z(t)$ is independent of t. Hence we can evaluate our original partition function, initially given by $t = 0$, via instead taking the limit $t \to \infty$. This reduces to an integral over the fields obeying $Q\mathcal{V} = 0$.

We can further improve upon this by parameterizing ϕ via

$$\phi = \phi_0 + t^{-1/2}\hat{\phi}. \tag{6.4}$$

The choice of t-dependency implies that the action for large t we have

$$S + tQ\mathcal{V} = S(\phi_0) + (Q\mathcal{V})^{quad}(\hat{\phi}) + \mathcal{O}(t^{-1/2}) \tag{6.5}$$

where $(Q\mathcal{V})^{quad}(\hat{\phi})$ is the quadratic part of $Q\mathcal{V}$. This then reduces our localization formula to

$$\int D\phi_0 e^{-S(\phi_0)} \frac{1}{\mathrm{SDet}'(Q\mathcal{V})^{quad}_{\phi_0}} \tag{6.6}$$

where SDet refers to the superdeterminant which is given as the ratio of bosonic to fermionic determinants with the prime symbol indicating that one should remove zero modes to get a sensible value for the partition function. The calculation of the superdeterminant is often achieved via index theorems.

By making the choice

$$\mathcal{V} = \sum_{\text{fermions } \psi} (Q\psi)^{\ddagger} \tag{6.7}$$

where $\ddagger$ is any anti-linear operator for which $Q\mathcal{V}$ is non-negative for the fields over which we are integrating and for which the bosonic variation of $\mathcal{V}$ vanishes, we reduce to an integral over the moduli space of solutions to $Q\psi = 0$. These are often called *BPS equations*.

6.2 Hori-Romo Review

Following [11] and specializing to the case of abelian gauge group, we have the following formula for the hemisphere partition function as an integral over an appropriate cycle in the Coulomb branch moduli.

$$Z_{D_2}(\mathcal{B}) = C(r\Lambda)^{\hat{c}/2} \int_{\gamma \subset \mathfrak{t}_{\mathbb{C}}} d^{l_G}\sigma \prod_i \Gamma\left(iQ_i(\sigma) + \frac{R_i}{2}\right) e^{it(\sigma)} f_{\mathcal{B}}(\sigma) \tag{6.8}$$

Here the dependence on the brane $\mathcal{B}$ is solely through the brane factor $f_{\mathcal{B}}(\sigma')$ which is defined via

$$f_{\mathcal{B}}(\sigma) = \mathrm{Tr} e^{\pi i \mathbf{r}} e^{2\pi\rho(\sigma)} \tag{6.9}$$

for $\mathbf{r}$ the R-charge and ρ the matter representation. We must assume that the R-charges satisfy $0 < R_i < 2$ for all i. One can send the R-charges to 0 to obtain a simplified formula *provided* one does not encounter contour pinching; however, contour pinching *can* occur for general models. We later analyze when contour pinching is avoided and show that it corresponds to $\mathcal{B}$ being compactly supported. Furthermore, we have the freedom to rescale by an overall constant as this does not affect physical observables and can therefore choose a normalization according to convenience. Also recall that t depends on the energy scale.

The brane factor for a single Wilson line $\mathcal{W}(\vec{q})$ is $e^{2\pi\vec{q}\cdot\sigma}$, and for a more general brane, the brane factor can be easily computed as a sum over all Wilson lines, weighted by signs determined by R-charges. Note that the *only* dependence of the partition function on the brane is through the brane factor.

The reason that the dependence on t is holomorphic is discussed in chapter 4 of [11]. Essentially, changing $\bar{t}$ would amount to an antiholomorphic deformation of the twisted superpotential, and such a deformation is $\mathcal{Q}$-exact.

Solving the BPS equations in the localization calculation (see [11]) would naively lead to γ being the real locus, but it is necessary to consider a continuous deformation so as to ensure convergence. As there are codimension one poles from the Gamma functions, care must be taken so as not to cross the poles during such a deformation. Since the integration cycle is supposed to be a valid A-brane [11], we should further

assume that the contour is Lagrangian. For $U(1)$ models, as the integration cycle is in $\mathbb{C}$, the Lagrangian condition is vacuous.

Convention 3. *Except when explicitly mentioned otherwise, we redefine the partition function so as to omit the factor of* $(r\Lambda)^{\hat{c}/2}$.

The partition function gives an easy way to observe monodromy under theta angle shifts. Indeed, if we take $\theta_\alpha \to \theta_\alpha + 2\pi$ for some α, then the exponential is multiplied by $e^{2\pi}$ which may in turn be absorbed into the brane factor if we shift the α-th gauge charge of each Wilson line (in some fixed resolution of the brane) by 1. This is equivalent to tensoring the complex with $\mathcal{W}(0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is in the α-th position. This was pointed out without localization in [8] via the observation that θ only enters the GLSM Lagrangian through the expression $\theta + 2\pi q$.

Chapter 7

Central Charge from Localization

Having the localization formula at hand for the hemisphere partition function, one must ask what it actually computes, drawing inspiration from physical string theory. Given a brane $\mathcal{B}$ on a Calabi-Yau threefold target X, we can consider type II string theory on $\mathbb{R}^{3,1} \times X$ with a brane of the form $\mathbb{R} \times \mathcal{B}$ for $\mathbb{R}$ a particle worldline in $\mathbb{R}^{3,1}$. After dimensional reduction on X, we obtain a BPS particle in a 4-dimensional supersymmetric theory on $\mathbb{R}^{3,1}$ which has a well-defined central charge under the 4d N=2 supersymmetry algebra. It is known that this is given by

$$Z^{4d}(\mathcal{B}) = Q \cdot \Pi \tag{7.1}$$

where Q is the RR charge given by

$$Q(\mathcal{B}) = \sqrt{\hat{A}(X)} ch^c(\mathcal{B}) \tag{7.2}$$

as derived in [17] while Π is a certain period integral. Putting these together, one may write down the formula

$$Z^{4d}(\mathcal{B}) = \int_X \sqrt{\hat{A}(X)} e^{\tau} ch^c(\mathcal{B}) + \dots \tag{7.3}$$

where $\dots$ refers to higher order corrections and where $\tau = \omega - iB$ is the *complexified Kähler class* for ω the ordinary Kahler class and B a *B field*; note that τ will be related to t by a simpler linear transformation. For details on this central charge formula, see [1] and also the book [10]. Note that in the Calabi-Yau case, the $\hat{A}$ class is the same as the Todd class; it is common in this case to see the above formula with $\sqrt{Td}(X)$ in the literature.

An important claim of [11] is that this central charge in the four dimensional theory can be determined as the hemisphere partition function of $\mathcal{B}$ in the sigma model to X,

i.e.

$$Z^{4d}(\mathcal{B}) = Z_{D^2}(\mathcal{B}). \tag{7.4}$$

Furthermore, even in the anomalous case in which the definition of brane central charge in terms of a 4-dimensional N=2 theory breaks down, the integral over geometric data still obeys the same relation to the hemisphere partition function. Since the localization formula for $Z_{D^2}(\mathcal{B})$ may be expanded in residues, one might ask how much of (7.3) is captured by the leading residue. It actually captures more than the leading term in the central charge that we have shown. If one replaces $\sqrt{\hat{A}(X)}$ by the Gamma class Γ_X in the above integral, then this captures an additional piece of the expansion (which would be negligible in the large volume limit) and matches the leading residue. Explicitly, we have

$$Z^{0-inst}_{D^2,res}(\mathcal{B}) = \int_X \Gamma_X e^{\tau} ch^c(\mathcal{B}). \tag{7.5}$$

After clarifying general results about the hemisphere partition function, we will use it to compute the leading residue contribution to the central charges for branes in Hirzebruch-Jung models. Later in thesis in chapter 8 we explicitly check that this matches with the geometric formula.

7.1 The Partition Function for $U(1)$ Models

Here we study in detail central charge calculations for a one-parameter Hirzebruch-Jung model which is constructed from a gauge group $U(1)$ and chirals X_0, X_1, and X_2 having respective charges 1, $-n$, and 1. Here the $\zeta \ll 0$ phase gives rise to the orbifold target $\mathbb{C}^2/\mathbb{Z}_{n(1)}$ while the $\zeta \gg 0$ phase gives rise to the minimal resolution of this singularity which is isomorphic to the total space of $\mathcal{O}(-n) \to \mathbb{CP}^1$.

Turning on R-charges R_j for $j = 0, 1, 2$ in the interval $(0, 2)$, the localization formula reduces to

$$Z_{D^2}(\mathcal{B}) = \int_L \frac{d\sigma}{2\pi} \Gamma(i\sigma + R_0/2)\Gamma(-in\sigma + R_1/2)\Gamma(i\sigma + R_2/2)e^{it\sigma} f_{\mathcal{B}}(\sigma). \tag{7.6}$$

where we have used our freedom of rescaling to choose a convenient normalization for

what follows. The poles coming from the Gamma functions are given explicitly by

$$i\sigma = -R_j/2 - k < 0 \qquad \text{for } j \in \{0,2\} \text{ and } k \geq 0 \tag{7.7}$$

$$i\sigma = \frac{1}{n}(R_1/2 + k) > 0 \qquad \text{for } k \geq 0. \tag{7.8}$$

Through mixing R-symmetry with gauge symmetry, we can set $R_1 = 0$ and also shift L slightly so that for each Gamma function, all poles are on the same side of the contour. As we shall see, the remaining R-charges are crucial for regularization though will turn out to be irrelevant if we restrict to compact branes. If we try to take either of the remaining R-charges to zero, then for generic $\mathcal{B}$, the contour gets pinched between two poles, and the resulting integral blows up.

To see how this diverges arises, note that given a holomorphic function f:

$$\int_{\mathbb{R}} \frac{f(x)dx}{(x-i\epsilon)(x+i\epsilon)} = 2\pi i \frac{f(i\epsilon)}{2i\epsilon} + \int_C \frac{f(x)dx}{(x-i\epsilon)(x+i\epsilon)} \tag{7.9}$$

where C is the contour obtained by bending $\mathbb{R}$ slightly to go over the pole at $x = i\epsilon$. If we take $\epsilon \to 0$, we obtain a divergence of the form $\frac{\pi f(0)}{\epsilon}$.

We now turn to calculating the partition function for specific branes. First we consider the case of a single Wilson line which descends to $\mathcal{O}(q)$ and has with brane factor $e^{2\pi q\sigma}$. Taking $R_0, R_2 \to 0$, we get

$$Z^{0-inst}_{D^2,res}(\mathcal{O}(q)) = \frac{1}{n}\frac{\Gamma(R_0/2)}{-2\pi i}\frac{\Gamma(R_2/2)}{-2\pi i} + \mathcal{O}(1) = -\frac{1}{n\pi^2 R_0 R_2} + \mathcal{O}(\frac{1}{R_0}) + \mathcal{O}(\frac{1}{R_2}). \tag{7.10}$$

The divergent part is a factor of $\frac{1}{n}$ times the partition function for two free chirals corresponding to the noncompact directions and is in particular independent of q. This can also be viewed as computing the $U(1)^2$ equivariant volume of $\mathbb{C}^2$ with an additional factor of $\frac{1}{n}$ coming from quotienting by $\mathbb{Z}_n$.

Next we examine the structure sheaf of the exceptional divisor E which may be lifted to $\mathcal{W}(n) \xrightarrow{X_1} \mathcal{W}$. This has brane factor $f_1(\sigma) = 1 - e^{2\pi n\sigma}$ which serves to cancel the pole of $\Gamma(-in\sigma)$; more explicitly, we have by the reflection formula

$$\Gamma(-in\sigma)f_1(\sigma) = \frac{-2\pi i e^{\pi n\sigma}}{\Gamma(1+in\sigma)} \tag{7.11}$$

which clearly has no pole near the origin. Then we can take the remaining R-charges to zero in the localization formula free of any contour pinching, and thus the leading

contribution to the partition function may be calculated directly via

$$Z^{0-inst}_{D^2,res}(\mathcal{O}_E) = \int_0 \frac{d\sigma}{2\pi} \frac{-2\pi i e^{\pi n\sigma}}{\Gamma(1+in\sigma)} \Gamma(i\sigma)^2 e^{it\sigma} = \frac{t - i\pi n + (n-2)\gamma}{-2\pi i} \tag{7.12}$$

where $\int_0$ means that the integral is taken around the pole at the origin. Even if we were to put in the original R_1, it would modify the brane factor in exactly the right way to cancel the pole coming from $\Gamma(-in\sigma + R_1/2)$, leading to the same conclusion. We could also twist by for instance considering $\mathcal{W}(q+n) \xrightarrow{X_1} \mathcal{W}(q)$ which flows to $\mathcal{O}_E(q)$. In this case, the brane factor would be multiplied by $e^{2\pi q\sigma}$, but the argument about canceling poles to avoid contour pinching would still work the same way.

Turning to noncompact branes, we will show that the divergences we obtain do in fact depend on more than just the brane factor with R-charges turned off. Indeed, let us consider the brane $\mathcal{B}_{k_0,k_1,k_2}$ arising from the UV resolution $\mathcal{W}(0) \xrightarrow{X_0^{k_0} X_1^{k_1} X_2^{k_2}} \mathcal{W}(k_0 - nk_1 + k_2)$ for some $k_i \geq 0$. This brane is supported on $\{X_0 = 0\} \cup \{X_1 = 0\} \cup \{X_2 = 0\}$ which contains the base as well as two noncompact components. It has brane charge $f_{\mathcal{B}_{k_0,k_1,k_2}}(\sigma) = 1 - e^{2\pi(k_0 - nk_1 + k_2)\sigma - i\pi(k_0 R_0 + k_2 R_2)}$. Though the $R_j \to 0$ limit only depends on $k_0 - nk_1 + k_2$, the partition function is given by

$$\begin{aligned} Z^{0-inst}_{D^2,res}(\mathcal{B}_{k_0,k_1,k_2}) &= \frac{1 - e^{-i\pi(k_0 R_0 + k_2 R_2)}}{(-2\pi i)^2} \left(\frac{1}{n} \Gamma(\frac{R_0}{2}) \Gamma(\frac{R_2}{2}) + \mathcal{O}(1) \right) \\ &= \frac{1}{n(2\pi i)} \left(k_0 \Gamma(\frac{R_0}{2}) + k_2 \Gamma(\frac{R_2}{2}) \right) + \dots \end{aligned} \tag{7.13}$$

This gives a geometric interpretation of the divergences as encoding the multiplicities of the noncompact components.

7.2 Compact Branes in Higher Rank Models

To evaluate the partition function for higher rank models, we follow the Jeffrey-Kirwan (JK) prescription which yields the following sum over residues:

$$Z_{D^2,res}(\mathcal{B}) = \sum_{J|\zeta \in \mathrm{Cone}_J} \sum_{k|J \to \mathbb{Z}_{\geq 0}} \pm \mathrm{Res}_{i\sigma = i\sigma_{J,k}} \left(\prod_j \Gamma(iQ^j \cdot \sigma + \frac{R_j}{2}) e^{it\cdot\sigma} f_{\mathcal{B}}(\sigma) \right). \tag{7.14}$$

Here the choice of ζ corresponds to what is usually called a JK parameter, and in fact, only the choice of phase containing ζ matters in determining which poles contribute. The sum over J is interpreted as summing over all subsets $J \subset \{1, \dots, n\}$ such that

the positive-linear span of the Q^j's for $j \in J$ contains ζ. A shortcut is that the sum of residues we take should be such that the factor $e^{it\cdot\sigma}$ must have all positive powers of $e^{-\zeta k_j}$ to ensure convergence; this does not, however, fix the signs.

For $K \subset \{1, \ldots, n\}$, we define $E_K = \{\phi_j = 0 \text{ for } j \in K\}$. We first point out that the brane factor for $\mathcal{O}_K$ will cancel the poles of Gamma functions corresponding to chirals with indices in K. This will be useful in what follows for determining when we are able to cancel certain poles in the JK expansion.

We first suppose that E_K is contained in the deleted set which corresponds to the case in which the brane $\mathcal{O}_{E_K}$ which becomes trivial on the Higgs branch. Then by definition there is no solution to the D-term equations when $\phi_j = 0$ for all $j \in K$, but such solutions correspond exactly to expressing ζ as an element of Cone_{K^c} so therefore we have $\zeta \notin \mathrm{Cone}_{K^c}$. For any J in the above expansion, since $\zeta \in \mathrm{Cone}_J$, we must have $J \not\subset K^c$ so then $J \cap K$ is nontrivial. But this means that that one of the poles whose residue we are taking is canceled by the brane factor, implying that the contribution is trivial. As this holds for all terms, we see that the central charge vanishes as one should expect. This easily extends to twists, and moreover, since the brane factor is additive, this further generalizes to arbitrary empty branes (as these can be constructed from twists of the structure sheaves of different components of the deleted set).

We now turn to when E_K is compact. Then the JK prescription becomes a sum over all $J \subset K^c$ because other terms involve taking a residue of a pole that has been canceled and thus do not contribute.

Lemma 7.1. *E_K is compact if and only if there exists s such that $Q^j \cdot s > 0$ for all $j \in K$.*

If such an s exists, then the norm squared of points in E_K is bounded above by $\frac{\zeta\cdot s}{\min_{i\notin K} Q_i\cdot s}$. If no such s exists, then Cone_{K^c} does not lie in a half-space and hence contains a line through the origin. This forces $\sum_i \lambda^i Q_i$ for some positive coefficients λ^i, but then given $X \in E_K$ one can construct arbitrarily large solutions by shifting each $|X_i|^2$ by the same multiple of λ^i, thus implying that E_K is not compact. $\square$

Using the reflection formula repeatedly, we may write

$$Z_{D^2,res} = \sum_{\substack{J\subset K^c \\ \zeta\in \mathrm{Cone}_J}} \sum_{k:J\to\mathbb{Z}_{\geq 0}} \pm \mathrm{Res}_{i\sigma=i\sigma_{J,k}} \left(\frac{e^{it\cdot\sigma} e^{\sum_{j\in K}(-\pi Q^j\dot{\sigma}+i\pi R_j/2)} \prod_{j\notin K}\Gamma(iQ^j\cdot\sigma+R_j/2)}{(-2\pi i)^{\#(K^c\setminus J)}\prod_{j\in K}\Gamma(1-iQ^j\cdot\sigma-R_j/2)} \right). \tag{7.15}$$

The only potentially problematic Gamma functions are those in the numerator. If we shift σ by $\sigma - i\epsilon s$ where s is as given by the lemma, then the numerator Gamma functions become $\Gamma(iQ^j\cdot\sigma+\epsilon(Q^j\cdot s)+R_j/2)$. Because for these $Q^j\cdot s>0$, we can now take $R_j\to 0$ for all j without hitting any of the corresponding poles. Though we need not have $Q^j\cdot s>0$ for the poles from the denominator Gamma functions, these poles will not lead to contour pinching anyhow. This immediately generalizes to twists because the brane factor gets multiplied by an overall factor which doesn't alter our argument about canceling poles, and this then further generalizes to arbitrary compactly supported branes (which are necessarily supported on a union of exceptional divisors) by the additivity of the brane factor.

7.3 Line Bundles on Hirzebruch-Jung Models

We now turn to explicitly understand sheaves on a (possibly partially resolved) Hirzebruch-Jung model X. This essentially comes from specializing the results of section 5.4. The derived category category is generated by line bundles of the form $\mathcal{O}(b_1,\dots,b_r)$ for a rank r model where such a line bundle is given as the Higgs branch image of the Wilson line $\mathcal{W}(b_1,\dots,b_r)$. A section of the structure sheaf $\mathcal{O}$ is a G-equivariant function on $\mathbb{C}^d\setminus\Delta_\zeta$. As multiplication by X_j maps this to $\mathcal{O}(0,\dots,0,1,-a_j,1,0,\dots,0)$, the latter sheaf is of the form $\mathcal{O}(E_j)$ where $E_j=\{X_j=0\}$. To be explicit, we have

$$\begin{aligned} &\mathcal{O}(E_0)=\mathcal{O}(1,0,\dots), \qquad \mathcal{O}(E_1)=\mathcal{O}(-a_1,1,0,\dots), \\ &\mathcal{O}(E_\alpha)=\mathcal{O}(\dots,0,1,-a_\alpha,1,0,\dots) \text{ for } 1<\alpha<r, \\ &\mathcal{O}(E_r)=\mathcal{O}(\dots,0,1,-a_r), \qquad \mathcal{O}(E_{r+1})=\mathcal{O}(\dots,0,1). \end{aligned} \tag{7.16}$$

Note that we need all such E_α's to generated the derived category and not just those for $1\leq\alpha\leq r$. A brane in the GLSM which is supported on the deleted set descends on

the Higgs branch to a trivial brane, thereby giving a relation in the derived category. We obtain from this two kinds of relations. (All twists of such relations are of course trivial in the derived category as well.)

1. If the α-th divisor is not blown-up, then Δ_ζ contains $\{X_\alpha = 0\}$ as a component. Passing a resolution of its structure sheaf to the derived category gives $\mathcal{O}(-E_\alpha) \xrightarrow{X_\alpha} \mathcal{O}$ as a relation.

2. If the α-th and β-th divisors are blown up, and there is no $\alpha < \gamma < \beta$ for which the γ-th divisor is also blown up, then Δ_ζ contains $\{X_\alpha = X_\beta = 0\}$. Passing a resolution of its structure sheaf to the derived category gives

$$\mathcal{O}(-E_\alpha - E_\beta) \xrightarrow{(X_\alpha, X_\beta)} \mathcal{O}(-E_\beta) \oplus (-E_\alpha) \xrightarrow{(X_\beta, -X_\alpha)} \mathcal{O} \tag{7.17}$$

as a relation.

We now turn to understanding line bundles geometrically in terms of their pullbacks to local models. We first consider the intersection of two neighboring exceptional divisors E_i and E_j. The residual gauge group $\mathbb{Z}_{d_{ij}}$ embeds in $U(1)^r$ via

$$\mathbb{Z}_{d_{ij}} \ni 1 \mapsto (1, \ldots, 1, \omega^{d_{(i+1)j}}, \omega^{d_{(i+2)j}}, \ldots, \omega^{d_{(j-1)j}}, 1, \ldots, 1) \tag{7.18}$$

where $\omega = e^{2\pi i/d_{ij}}$ and where the entries of the form $\omega^{d_{\alpha j}}$ are in positions $i+1, \ldots, j-1$. We thus obtain an equivariant line bundle on a neighborhood of $E_i \cap E_j$ given by

$$\mathcal{W}(q) \text{ on } \mathbb{C}^2/\mathbb{Z}_{d_{ij}(d_{(i+1)j})} \text{ with } \mathbb{Z}_{d_{ij}}\text{-charge } q = \sum_{\alpha=i+1}^{j-1} d_{\alpha j} b_\alpha. \tag{7.19}$$

We next consider a neighborhood of a single blown up exceptional divisor E_j which intersects E_i and E_j for $0 \le i < j < k \le r+1$. The gauge group of the local model is $U(1) \times \mathbb{Z}_m$ where $m = \gcd(d_{ij}, d_{jk})$, and $(h, \omega) \in U(1) \times \mathbb{Z}_m$ corresponds to $(g_\alpha) \in U(1)^r$ where

$$g_\alpha = \begin{cases} (h^{d_{jk}/m}\omega^u)^{d_{i\alpha}} & \text{for } i \le \alpha \le j \\ (h^{d_{ij}/m}\omega^v)^{d_{\alpha k}} & \text{for } j \le \alpha \le k \\ 0 & \text{otherwise} \end{cases} \cdot \tag{7.20}$$

From this, one can compute the charges in the local model to be

$$\begin{aligned} \frac{d_{jk}}{m}\sum_{\alpha=i+1}^{j-1} d_{i\alpha}b_\alpha + \frac{d_{ij}d_{jk}b_j}{m} + \frac{d_{ij}}{m}\sum_{\alpha=j+1}^{k-1} d_{\alpha k}b_\alpha \text{ under } U(1), \\ u\sum_{\alpha=i+1}^{j-1} d_{i\alpha}b_\alpha + v\sum_{\alpha=j+1}^{k-1} d_{\alpha k}b_\alpha \text{ under } \mathbb{Z}_m. \end{aligned} \tag{7.21}$$

In the fully resolved case this simplifies considerably: $\mathcal{O}(b_1,\ldots,b_r)$ pulls back to give $\mathcal{O}(b_j)$ on E_j.

7.4 Central Charges for Hirzebruch-Jung Models

We now turn to using this machinery to explicitly compute the central charges for compactly supported branes in Hirzebruch-Jung geometries. We first consider the central charge of an intersection $E_i \cap E_j$ of adjacent exceptional divisors which is a $\mathbb{Z}_{d_{ij}}$ orbifold point. For a Wilson line $\mathcal{W}(\rho)$, we have the brane factor

$$f(\sigma) = (1 - e^{2\pi i(iQ^i\cdot\sigma + R_i/2)})(1 - e^{2\pi i(iQ^j\cdot\sigma + R_j/2)})e^{2\pi\rho\cdot\sigma}. \tag{7.22}$$

As the brane is compact, the brane factor will cancel enough poles to avoid contour pinching; in particular, the poles from the Gamma functions associated with X_i and X_j are canceled. The JK prescription requires us to sum over choices of r chirals not containing X_i and X_j, but since there are only $r+2$ total chirals, this means we get one term in the expansion labeled by $J = \{i,j\}^c$. We obtain

$$Z^{0-inst}_{D^2,res} = \pm\text{res}_{i\sigma=i\sigma_{\{i,j\}}}\left(\frac{e^{(it+2\pi\rho)\cdot\sigma}}{(-2\pi i)^2}\prod_{\ell=i,j}\frac{-2\pi i e^{-\pi Q^\ell\cdot\sigma + i\pi R_\ell/2}}{\Gamma(1-iQ^\ell\cdot\sigma - R_\ell/2)}\prod_{\ell\neq i,j}\Gamma(iQ^\ell\cdot\sigma + R_\ell/2)\right) \tag{7.23}$$

where $\sigma_{\{i,j\}}$ is given by solving $iQ^\ell\cdot\sigma + R_\ell/2 = 0$. The sign cancels with the sign of the residue with the latter being given by $1/\det(Q^\ell)_{\ell\neq i,j}$. Now we have

$$\det Q^\ell_{\ell\neq i,j} = \begin{pmatrix} U & N_1 & 0 \\ 0 & -C_{(ij)} & 0 \\ 0 & N_2 & L \end{pmatrix} = \det(-C_{(ij)}) = (-1)^{j-i}d_{ij} \tag{7.24}$$

for U and L upper and lower triangular matrices with 1s on the diagonal, N_1 and N_2 matrices with a single 1 in the corner closest to the diagonal of the main matrix and

the rest 0s, and $C_{(ij)}$ the partial Cartan which then leads to an overall factor of $1/d_{ij}$. We thus have

$$Z^{0-inst}_{D^2,res} = \lim_{R\to 0}\frac{1}{d_{ij}}\left(\frac{e^{(it+2\pi\rho)\cdot\sigma}}{(-2\pi i)^2}\prod_{\ell=i,j}\frac{-2\pi i e^{-\pi Q^\ell\cdot\sigma+i\pi R_\ell/2}}{\Gamma(1-iQ^\ell\cdot\sigma-R_\ell/2)}\prod_{\ell\neq i,j}\right)_{i\sigma=i\sigma_{\{i,j\}}} = \frac{1}{d_{ij}}. \quad (7.25)$$

Note that this depends on the fact that $\sigma_{\{i,j\}}\to 0$ as $R\to 0$ which in turn leads to a result *independent* of t and ρ.

We now turn to computing the central charge of an exceptional divisor E_j for $1\leq j\leq r$. We let $i<j<k$ denote the indices of the neighboring exceptional divisors. The brane factor is given by

$$f(\sigma) = (1-e^{2\pi i(iQ^j\cdot\sigma+R_j/2)})e^{2\pi\rho\cdot\sigma} \quad (7.26)$$

and as the brane is compact, we will manage to avoid contour pinching through canceling poles. In the JK expansion, we must sum over sets J of r chirals with $j\notin J$ and $\zeta\in\mathrm{Cone}_J$. Now $\zeta\in\mathrm{Cone}_J$ exactly when E_{J^c} is nonempty which occurs for $J^c=\{i,j\}$ or $\{j,k\}$.

We first compute the contribution from $J^c=\{i,j\}$. The leading residue comes from solving $Q^\ell\cdot\sigma = iR_\ell/2$ for $\ell\neq i,j$. Explicitly, we have

$$i\sigma_{\{i,j\}} = \begin{cases} -\sum_{\ell=0}^{\alpha-1} d_{\ell\alpha}\frac{R_\ell}{2} & 1\leq\alpha\leq i \\ -\frac{1}{d_{ij}}\sum_{\ell=0}^{r+1} d_{i\min(\alpha,\ell)}d_{j\max(\alpha,\ell)}\frac{R_\ell}{2} & i\leq\alpha\leq j \\ -\sum_{\ell=\alpha+1}^{r+1} d_{\alpha\ell}\frac{R_\ell}{2} & j\leq\alpha\leq r \end{cases} \quad (7.27)$$

which can be verified using our identities for the d_{ij}'s. From the recursion relations, one can show that

$$iQ^i\cdot\hat{\sigma}_{\{i,j\}} + \frac{R_i}{2} = \sum_{\ell=0}^{r+1}\frac{d_{\ell j}}{d_{ij}}\frac{R_\ell}{2} \quad (7.28)$$

and similarly

$$iQ^j\cdot\hat{\sigma}_{\{i,j\}} + \frac{R_j}{2} = \sum_{\ell=0}^{r+1}\frac{d_{i\ell}}{d_{ij}}\frac{R_\ell}{2}. \quad (7.29)$$

We then compute the residue of the partition function to be

$$\begin{aligned}
&\operatorname{sign}\big(\det(Q^\ell)_{\ell\neq i,j}\big)\operatorname{res}_{i\hat\sigma=i\hat\sigma_{\{i,j\}}}\left(\frac{e^{it\cdot\hat\sigma-\pi Q^j\cdot\hat\sigma+i\pi R_j/2}\prod_{\ell\neq j}\Gamma\big(iQ^\ell\cdot\hat\sigma+R_\ell/2\big)}{-2\pi i\,\Gamma\big(1-iQ^j\cdot\hat\sigma-R_j/2\big)}\right)\\
&\quad=\frac{1}{d_{ij}}\frac{e^{it\cdot\hat\sigma_{\{i,j\}}+i\pi\sum_{\ell=0}^{r+1}(d_{i\ell}/d_{ij})(R_\ell/2)}\Gamma\big(\sum_{\ell=0}^{r+1}\frac{d_{\ell j}}{d_{ij}}\frac{R_\ell}{2}\big)}{-2\pi i\,\Gamma\big(1-\sum_{\ell=0}^{r+1}\frac{d_{i\ell}}{d_{ij}}\frac{R_\ell}{2}\big)}\\
&\quad=\frac{i}{2\pi}\left(\frac{2}{\sum_{\ell=0}^{r+1}d_{\ell j}R_\ell}+\left(\frac{2it\cdot\hat\sigma_{\{i,j\}}}{\sum_{\ell=0}^{r+1}d_{\ell j}R_\ell}+\frac{(i\pi-\gamma)\sum_{\ell=0}^{r+1}d_{i\ell}R_\ell}{d_{ij}\sum_{\ell=0}^{r+1}d_{\ell j}R_\ell}-\frac{\gamma}{d_{ij}}\right)+O(R)\right)
\end{aligned}\tag{7.30}$$

where we obtained the factor of $\frac{1}{d_{ij}}$ from the determinant of Q^ℓ for $\ell\neq i,j$. For the $J^c=\{j,k\}$ contribution, we follow a similar process and obtain

$$\begin{aligned}
&\operatorname{sign}\big(\det(Q^\ell)_{\ell\neq j,k}\big)\operatorname{res}_{i\hat\sigma=i\hat\sigma_{\{j,k\}}}\left(\frac{e^{it\cdot\hat\sigma-\pi Q^j\cdot\hat\sigma+i\pi R_j/2}\prod_{\ell\neq j}\Gamma\big(iQ^\ell\cdot\hat\sigma+R_\ell/2\big)}{-2\pi i\,\Gamma\big(1-iQ^j\cdot\hat\sigma-R_j/2\big)}\right)\\
&\quad=\frac{i}{2\pi}\left(\frac{2}{\sum_{\ell=0}^{r+1}d_{j\ell}R_\ell}+\left(\frac{2it\cdot\hat\sigma_{\{j,k\}}}{\sum_{\ell=0}^{r+1}d_{j\ell}R_\ell}+\frac{(i\pi-\gamma)\sum_{\ell=0}^{r+1}d_{\ell k}R_\ell}{d_{jk}\sum_{\ell=0}^{r+1}d_{j\ell}R_\ell}-\frac{\gamma}{d_{jk}}\right)+O(R)\right).
\end{aligned}\tag{7.31}$$

Then if we add the two, the divergences cancel! Then using

$$i\hat\sigma^\alpha_{\{i,j\}}-i\hat\sigma^\alpha_{\{j,k\}}=\begin{cases}0 & \text{for } 1\le\alpha\le i,\\ (d_{i\alpha}/d_{ij})\sum_{\ell=0}^{r+1}d_{\ell j}R_\ell/2 & \text{for } i\le\alpha\le j,\\ (d_{\alpha k}/d_{jk})\sum_{\ell=0}^{r+1}d_{\ell j}R_\ell/2 & \text{for } j\le\alpha\le k,\\ 0 & \text{for } k\le\alpha\le r,\end{cases}\tag{7.32}$$

and our identities for the $d_{\beta\gamma}$'s, we obtain the finite $R\to 0$ limit

$$Z^{\text{0-instanton}}_{D^2,\text{residue}}=(\mathfrak{r}\Lambda)^{\hat c/2}\frac{i}{2\pi}\left(\left(\sum_{\alpha=i+1}^{j-1}\frac{d_{i\alpha}}{d_{ij}}t_\alpha\right)+t_j+\left(\sum_{\alpha=j+1}^{k-1}\frac{d_{\alpha k}}{d_{jk}}t_\alpha\right)-i\pi\frac{d_{ik}}{d_{ij}d_{jk}}+\frac{d_{ik}-d_{jk}-d_{ij}}{d_{ij}d_{jk}}\gamma\right).\tag{7.33}$$

In the case where $i=j-1$ and $k=j+1$ in which there are no singularities on the exceptional divisor, this reduces to

$$Z^{0-inst}_{D^2,res}=\frac{i}{2\pi}(t_j-i\pi a_j+(a_j-2)\gamma).\tag{7.34}$$

More generally, the central charge for an exceptional divisor is the same as what one gets by computing its central charge in the corresponding local model:

$$Z^{0-inst}_{D^2,res}=\frac{i}{2\pi}\frac{1}{m}\frac{t_{loc}-i\pi d_{ik}/m+(d_{ik}/m-d_{jk}/m+d_{jk}/m)\gamma}{(d_{ij}/m)(d_{jk}/m)}\tag{7.35}$$

where the formula for t_{loc} was derived in the section on local models.

Chapter 8

Aspects of K-theory

We now turn to explicitly evaluating the leading contribution to the central charge for Hirzebruch-Jung surfaces following the geometric formula (7.5). Fortunately, it turns out that the (compactly supported) cohomology and (compactly supported) K-theory of any toric geometry can be computed explicitly in terms of generators and relations which in turn can be obtained from the defining combinatorial data. For far more details, consult [4].

8.1 K-theory and Cohomology of Toric Varieties

We first introduce an alternative perspective on toric geometry which is more common in the mathematical literature. Consider a rank n lattice N and a fan Σ of polyhedral cones in $N \otimes \mathbb{R}$. Let $\Sigma(1) = \{v_1, \ldots, v_d\}$ be the rays of Σ and $I = \{1, \ldots, d\}$ be the corresponding indices.

Consider the map $\phi : Hom(\Sigma(1), \mathbb{C}^*) \to Hom(N^*, \mathbb{C}^*)$ defined by sending $f : \Sigma(1) \to \mathbb{C}^*$ to $m \mapsto \prod_{v \in \Sigma(1)} f(v)^{\langle m, v \rangle}$ for $m \in N^*$. If we let $G_{\mathbb{C}} = ker(\phi)$, then G acts on $\mathbb{C}^d$ via

$$g \cdot (\phi_1, \ldots, \phi_d) = (g(v_1)x_1, \ldots, g(v_d)x_d) \tag{8.1}$$

where we identify coordinates of $\mathbb{C}^d$ with elements of $\Sigma(1)$. If we let

$$\Delta(\Sigma) = \bigcup_{\substack{\mathcal{S} \subset \Sigma(1) \text{ not spanning} \\ \text{a cone of } \Sigma}} \{\phi_j = 0 \text{ for } j \in \mathcal{S}\}, \tag{8.2}$$

then one can show that the $G_{\mathbb{C}}$-action preserves $\mathbb{C}^d \backslash \Delta(\Sigma)$, and we obtain a toric variety from taking the quotient by this action which we denote by $\mathbb{P}_\Sigma$. This is precisely the same quotient one would take in constructing this via a GIT quotient from the GLSM viewpoint.

For computing the cohomology in the case of orbifold singularities, we have an untwisted sector which gives the ordinary topological cohomology (i.e. what we get if we just consider the underlying topological space and "forget" about our singularities) together with twisted sectors which contain information about the singularities. Physically, it is this enhanced cohomology whose dimension correctly calculates the number of vacua. In the case of a nonsingular toric variety, the untwisted sector is the full cohomology, and there are no twisted sectors.

The untwisted sector cohomology is given by

$$H_0^*(\mathbb{P}_\Sigma) = \frac{\mathbb{C}[D_1, \ldots, D_d]}{\{\Sigma_i m(v_i) D_i | m \in N^*\}, \mathcal{I}_{SR}} \tag{8.3}$$

where $\mathcal{I}_{SR}$ is the Stanley-Reisner ideal spanned by all products $\prod_{i \in J} D_i$ for which $J \subset I$ does not span a cone of Σ. The generators D_i have degree 2 and correspond to toric divisors given by (in the GLSM picture) $\phi_i = 0$. In addition, the twisted sectors correspond to nonzero $\gamma = \sum_i \gamma_i v_i \in N$ with $\gamma_i \in [0, 1)$. For $\sigma \in \Sigma$, define $\mathrm{Star}(\sigma) = \{\sigma' \in \Sigma | \sigma \subset \sigma'\}$. Then the corresponding pieces of the cohomology are given by

$$H_\gamma^*(\mathbb{P}_\Sigma) = \frac{\mathbb{C}[\bar{D}_i]_{i \in S_\gamma(1)}}{\{\Sigma_i m(v_i) \bar{D}_i | m \in \mathrm{Ann}(v_i \in \sigma(\gamma))\}, \mathcal{I}_{SR}^\gamma}, \qquad S_\gamma := \mathrm{Star}(\sigma(\gamma)) - \sigma(\gamma) \tag{8.4}$$

where here the Stanley-Reisner ideal $\mathcal{I}_{SR}^\gamma$ is given by all $\prod_{i \in J} \bar{D}_i$ for J not a cone in $Star(\Sigma(\gamma))$. The full cohomology $H^*(\mathbb{P}_\Sigma)$ is then given as the direct sum over all sectors.

The compactly supported cohomology is similarly constructed from an untwisted sector and (in the case of orbifold singularities) twisted sectors. The untwisted sector is given as a quotient of the free module over $H^*(\mathbb{P}_\Sigma)$ with generators F_J. Explicitly, we have

$$H_{c,0}^*(\mathbb{P}_\Sigma) = \bigoplus_{\sigma_J^\circ \subset \Sigma^\circ} \frac{\mathbb{C}[D_1, \ldots, D_d]}{\langle H_1, H_2 \rangle} \tag{8.5}$$

where H_1 and H_2 are the relations defined by

$$\begin{aligned} H_1 &= \{D_i F_J = F_{J \cup \{i\}} \text{ for } i \notin J, J \cup \{i\} \in \Sigma\}, \\ H_2 &= \{D_i F_J = 0 \text{ for } i \notin J, J \cup \{i\} \notin \Sigma\}. \end{aligned} \tag{8.6}$$

The twisted sectors $H_{c,\gamma}^*(\mathbb{P}_\Sigma)$ are given by

$$H_{c,\gamma}^*(\mathbb{P}_\Sigma) = \bigoplus_{\sigma_J^\circ \subseteq \Sigma_\gamma^\circ} \frac{\mathbb{C}[\overline{D}_i]_{i \in S_\gamma} \overline{F}_J}{\langle H_1^\gamma, H_2^\gamma \rangle} \tag{8.7}$$

with relations H_1^γ and H_2^γ given by

$$\begin{aligned} H_1^\gamma &= \{\overline{D}_i\overline{F}_J = \overline{F}_{J\cup\{i\}} \text{ for } i \notin J, J\cup\{i\} \in \Sigma_\gamma\}, \\ H_2^\gamma &= \{\overline{D}_i\overline{F}_J = 0 \text{ for } i \notin J, J\cup\{i\} \notin \Sigma_\gamma\} \end{aligned} \tag{8.8}$$

and where $\Sigma_\gamma = \Sigma/\sigma(\gamma)$ is the quotient fan. Then the compactly supported cohomology H_c^* is given as the direct sum of the untwisted and twisted sectors.

The K-theory is given by

$$K_0(\mathbb{P}_\Sigma) = \frac{\mathbb{C}[R_i^\pm]_{i\in I}}{\left\{\prod_{i\in I} R_i^{m(v_i)} - 1 \,\middle|\, m \in N^*\right\}, \mathcal{I}_K}, \qquad \mathcal{I}_K = \left\langle \prod_{i\in J}(1-R_i) | J \notin \Sigma \right\rangle. \tag{8.9}$$

The compactly supported K-theory is given by taking the free $K_0(\mathbb{P}_\Sigma)$-module generated by G_J for $\sigma_J^\circ \subset \Sigma^\circ$ and quotienting by the relations

$$\begin{aligned} &\{(1-R_i^{-1})G_J = G_{J\cup\{i\}}, \text{ for } \ i \notin J, J\cup\{i\} \in \Sigma\} \\ &\{D_iF_J = 0, \ \text{ for } \ i \notin J, J\cup\{i\} \notin \Sigma\}. \end{aligned} \tag{8.10}$$

With this machinery in place, the Chern character

$$ch : K_0(\mathbb{P}_\Sigma) \to H^*(\mathbb{P}_\Sigma) \tag{8.11}$$

can be explicitly computed by

$$\begin{aligned} ch_\gamma(R_i) &= 1, && i \in Star(\sigma(\gamma)), \\ ch_\gamma(R_i) &= e^{\overline{D}_i}, && i \in S_\gamma, \\ ch_\gamma(R_i) &= e^{2\pi i\gamma_i} \prod_{j\notin\sigma(\gamma)} ch(R_j)^{m_i(v_j)}, && i \in \sigma(\gamma). \end{aligned} \tag{8.12}$$

where ch_γ denotes the component of the Chern character in the (un)twisted sector labeled by γ. Also the compactly supported Chern character can be explicitly calculated via

$$\begin{aligned} &ch_\gamma^c\Big(\prod_i R_i^{k_i} G_I\Big) \\ &= \begin{cases} 0 & \text{for } I \nsubseteq Star(\sigma(\gamma)), \\ \left(\prod_i ch_\gamma(R_i)^{k_i} \prod_{i\in I, i\notin\sigma(\gamma)} \frac{1-e^{-\overline{D}_i}}{\overline{D}_i}\right) \prod_{i\in I\cap\sigma(\gamma)} (1-ch_\gamma(R_i)^{-1})\overline{F}_I, & \text{for } I \subseteq Star(\sigma(\gamma)), \end{cases} \end{aligned} \tag{8.13}$$

where similarly ch_γ denotes the component of the compactly supported Chern character in the (un)twisted sector labeled by γ and where $\bar{F}_I$ for $I \subset \mathrm{Star}(\sigma(\gamma))$ denotes the projection to S_γ.

Finally, we define the integration map

$$\int : H_c^*(\mathbb{P}_\Sigma) \to \mathbb{C} \tag{8.14}$$

in a sector by

$$\int \bar{F}_I = \frac{1}{|\mathrm{Vol}_I|} \tag{8.15}$$

for $|I| = \mathrm{rank}_{N_\gamma}$ and zero otherwise. Here Vol_I means the index of the sublattice of N_γ spanned by I, and N_γ is defined by $N/\mathrm{Span}(\sigma(\gamma))$.

To be able to apply this framework to a GLSM, we must convert from the GLSM description of the target toric geometry to the primary fan. This construction is well known so we simply review it here. If we have n chirals and k gauge fields, then the charge matrix $Q_{i\alpha}$ can be regarded as a collection of vectors $Q_1, \ldots, Q_r$ in $\mathbb{Z}^n$. Then we can obtain the desired lattice via $N = \mathbb{Z}^d/\mathrm{Span}_{\mathbb{Z}}(Q_\alpha)$ viewed as a lattice inside $\mathbb{R}^d/\mathrm{Span}_{\mathbb{Z}}(Q_\alpha)$ If we let $v_1, \ldots, v_d$ be the images of the basis vectors in $\mathbb{Z}^n$ under this quotient, then we can take $\Sigma(1)$ to be all v_i's for which $\phi_i = 0$ is *not* part of the deleted set (i.e. for which $\phi_i = 0$ contains a solution to the D-term equations).

To use this in practice, one still needs to find the generators of N. If we define the map

$$f : N^* \to Hom((\mathbb{C}^*)^d, \mathbb{C}^*) \tag{8.16}$$

$$m \mapsto (\lambda \mapsto \prod_{i=1}^{d} \lambda_i^{m(v)}), \tag{8.17}$$

then N^* is given by the points in $\mathbb{R}^d$ for which $G = ker(f)$. One can also realize N as the dual of the lattice of gauge-invariant monomials.

8.2 Geometric Central Charges for Hirzebruch-Jung Geometries

For a compactly supported sheaf $\mathcal{F}$ on a variety X, we define the central charge to be

$$Z(\mathcal{F}) = \int_X e^\tau \hbar^{-\frac{i}{2\pi}c_1(X)} \hat{\Gamma}(X) ch^c(\mathcal{F}) \tag{8.18}$$

where as before τ is the complexified Kahler class. The gamma class $\Gamma(X)$, $c_1(X)$ and τ are all elements of $H^*(X)$ and act on $\mathcal{F}$ which as viewed as an element of $ch^c(\mathcal{F}) \in H_c^*(X)$ to produce for the integrand an element of $H_c^*(X)$ which may be integrated according to the rule given earlier. Note that this only depends on $\mathcal{F}$ through its class in $K_0^c(X)$.

We now turn to determining the central charges geometrically for the fully-resolved Hirzebruch-Jung geometries. For the purpose of doing computations with Hirzebruch-Jung geometries, it doesn't matter if one uses GLSM I or II. Therefore we shall use GLSM II as these computations are much simpler. Since $Q_j = (p_j 0 \dots 010 \dots 0 q_j)$ with a 1 in the j-th position, the lattice $\mathbb{Z}^{r+2}/(Q_j)_{j=1}^r$ can easily be expressed solely in terms of elements of the form $(a0 \dots 0b)$. Moreover, it is easily seen that minus the basis elements descend to (p_j, q_j). Thus we may take

$$S = \{v_j\}_{j=0}^{r+1} \qquad v_j = (p_j, q_j). \tag{8.19}$$

Moreover, since no deleted sets in the fully resolved phase are given by the vanishing of a single chiral, we may take $\Sigma(1) = S$. In fact, contracting the j-th exceptional divisor would correspond exactly to removing v_j from $\Sigma(1)$. From our relations for p_j and q_j, we must have $v_{j-1} - a_j v_j + v_{j+1} = 0$ so that the lattice spanned by all the v_j's is in fact spanned by v_0 and v_1, and thus we can take the lattice to be

$$N = \mathrm{Span}_{\mathbb{Z}}\{(n, 0), (p, 1)\} \tag{8.20}$$

for which the dual is given by

$$N^* = \mathrm{Span}_{\mathbb{Z}}\{(1/n, -p/n), (0, 1)\}. \tag{8.21}$$

From here we restrict ourselves to the fully resolved phase. Then the deleted sets are of the form $\{X_i = X_j = 0\}$ for $i \neq j \pm 1$ so therefore the two-dimensional cones take the form $\{j, j+1\}$, and we have $\Sigma(1) = S$. As $\Sigma(1)$ spans the entire lattice, there are no twisted sectors, a fact to be expected from the geometry being absent of orbifold singularities. The cohomology is given by

$$H_0^*(\mathbb{P}_\Sigma) = \frac{\mathbb{C}[D_0, \dots, D_{r+1}]}{\{\sum_i p_i D_i = \sum_i q_i D_i = 0\}, \mathcal{I}_{SR}} \tag{8.22}$$

where $\mathcal{I}_{SR} = \langle D_i D_j | |i-j| \geq 2 \rangle$.

From this, one can obtain seemingly stronger relations. Observe that

$$0 = (\sum_i p_i D_i) D_i = p_i D_i^2 + p_{i-1} D_i D_{i-1} + p_{i+1} D_i D_{i+1} \tag{8.23}$$

where we define D_i to vanish for i outside the allowed range. Then an induction argument starting with $D_{-1} = 0$ shows that

$$p_i D_i^2 + p_{i+1} D_i D_{i+1} = 0, \tag{8.24}$$

and one can similarly show that

$$q_i D_i^2 + q_{i+1} D_i D_{i+1} = 0. \tag{8.25}$$

Then taking a suitable linear combination of these two relations, we obtain

$$0 = \frac{1}{n}(p_i q_{i+1} - p_{i+1} q_i) D_i^2 = d_{ii+1} D_i = D_i^2 \tag{8.26}$$

which implies that $D_i D_{i+1}$ also vanishes and hence that $D_i D_j$ in fact vanishes for *all* i, j. This ought to be expected as the resolved geometry is homotopy equivalent to a wedge sum of (two-dimensional) exceptional divisors so therefore $H^4(\mathbb{P}_\Sigma) = 0$. Also using that $\frac{1}{n}(p_i q_j - p_j q_i) = d_{ij}$, one obtains

$$\sum_j d_{ij} D_j = \frac{p_i}{n}(\sum_j q_j D_j) - \frac{q_i}{n}(\sum_j p_j D_j) = 0. \tag{8.27}$$

We introduce a change of variables for the cohomology based on our change of variables between GLSM's I and II:

$$D_i = \sum_\alpha Q^i_\alpha \eta^\alpha, \qquad \eta^\alpha = \sum_\beta -(C^{-1})^{\alpha\beta} D_\beta. \tag{8.28}$$

We may expand the Kähler class as

$$\tau = \sum_\alpha \tau_\alpha \eta^\alpha. \tag{8.29}$$

To compute the Chern class, we use the Euler sequence

$$0 \to \mathcal{O}^r \to \bigoplus_{j=0}^{r+1} \mathcal{O}(D_j) \to T\mathbb{P}_\Sigma \to 0 \tag{8.30}$$

from which we obtain

$$ch(\mathbb{P}_\Sigma) = 2 + \sum_\alpha (2 - a_\alpha)\eta^\alpha. \tag{8.31}$$

We can then obtain the Gamma class as

$$\hat{\Gamma}(\mathbb{P}_\Sigma) = 1 - \frac{i\gamma}{2\pi}\sum_\alpha (2 - a_\alpha)\eta^\alpha \tag{8.32}$$

where γ denotes the Euler-Mascheroni constant.

The compactly supported cohomology and K-theory are respectively generated by G_I's and F_I's for $I = \{\alpha\}$ or $\{j, j+1\}$ where $\alpha = 1, \ldots, r$ while $j = 0, \ldots, r$. The generators of the compactly supported K-theory correspond geometrically to branes wrapping exceptional divisors, and one can show that

$$ch^c(G_\alpha) = \left(1 - \frac{1}{2}D_\alpha\right) F_\alpha. \tag{8.33}$$

Observe that

$$0 = (\sum_j d_{ij}D_j)F_i = d_{ii-1}F_{ii-1} + d_{ii}D_iF_i + d_{ii+1}F_{ii+1} = (-1)F_{ii-1} + (0)D_iF_i + (1)F_{ii+1} \tag{8.34}$$

and thus $F_{ii+1} = F_{01}$ for all i. Then we have

$$D_iF_i = (-\sum_{j\neq i} p_jD_i)F_i = (-p_{i-1} - p_{i+1})D_i = a_iD_i \tag{8.35}$$

which may be combined with our previously known relations to obtain the relation

$$D_\alpha F_\beta = (\delta_{\alpha,\beta-1} - a_\alpha\delta_{\alpha,\beta} + \delta_{\alpha,\beta+1})F_{\{0,1\}} = -C_\alpha^\beta F_{\{0,1\}} \tag{8.36}$$

which implies easily that $\eta_\alpha F_\beta = \delta_{\alpha,\beta}F_{\{0,1\}}$.

We then evaluate the integrand of the central charge integral:

$$\begin{aligned} &e^\tau \hbar^{-\frac{i}{2\pi}c_1(\mathbb{P}_\Sigma)}\hat{\Gamma}(\mathbb{P}_\Sigma)ch^c(G_\alpha) = \\ &F_\beta + \left(-\frac{i\gamma}{2\pi}(2 - a_\beta) + \tau^\beta - \frac{i}{2\pi}(2 - a_\beta)\log(\hbar) + \frac{a_\beta}{2}\right)F_{\{0,1\}}. \end{aligned} \tag{8.37}$$

Since $\{v_0, v_1\}$ span the entire lattice N, integrating over X picks out the coefficient of $F_{\{0,1\}}$. Therefore

$$Z(\mathcal{O}_{E_\beta}) = Z(G_\beta) = -\frac{i\gamma}{2\pi}(2 - a_\beta) + \tau^\beta - \frac{i}{2\pi}(2 - a_\beta)\log(\hbar) + \frac{a_\beta}{2}. \tag{8.38}$$

If we identify

$$t_\beta r^{-1} = -2\pi i \tau_\beta - (2 - a_\beta) \log \hbar \text{ where } \hbar = r\Lambda, \tag{8.39}$$

then this matches the localization calculation up to a factor of $(r\Lambda)^{\hat{c}/2}$.

Chapter 9

Grade Restriction Rule for Anomalous Models

We now turn to understanding B-brane transport for anomalous models. We first work out the $U(1)$ case using the analytic continuation of the partition function between phases and then results the conclusion in a more functorial light. Finally, the general abelian case is reduced to the $U(1)$ case by showing that when crossing a codimension-one wall at infinity, the theory is at all times Higgsed to at most a $U(1)$ theory (possibly times an irrelevant finite factor). Our conclusion parallels the mathematical theory of GIT wall-crossing as explained in [2].

9.1 Analytic Continuation of the Partition Function for $U(1)$ Models

We first demonstrate how to derive the grade restriction rule for $U(1)$ models. We consider a $U(1)$ GLSM with chirals Q^j for $1 \leq j \leq n$, and we let $N_+ = \sum_{Q_j>0} Q_j$ and $N_- = \sum_{Q_j<0} -Q_j$ so that $N_+ - N_- = \sum_j Q_j$. For convenience, we let $t' = t + \sum_j Q_j \log Q_j$. Geometrically, the theory in a phase will be a sigma model with target a total space of line bundles over a weighted projective space.

In the case that $N_+ = N_-$, our model is Calabi-Yau, and there is a well-defined quantum-corrected wall separating two phases: $\zeta \ll 0$ and $\zeta \gg 0$. For $N_+ \neq N_-$, we may without loss of generality assume that $N_+ > N_-$ in which case for fixed complexified energy scale μ, the theory for $\zeta \gg 0$ is described by a sigma model to a Higgs branch, and the theory for $\zeta \ll 0$ is described by a sigma model to a different Higgs branch together with $\sum_j Q_j$ massive vacua on the Coulomb branch corresponding to

$$\sigma \sim \mu \exp\left(\frac{-t' + 2\pi i k}{N_+ - N_-}\right), \qquad k \in \mathbb{Z}_{N_+ - N_-}. \tag{9.1}$$

Our main goal is to understand how when passing from $\zeta \gg 0$ to $\zeta \ll 0$ a brane can

split into Coulomb and Higgs branch components.

We approach this through the hemisphere partition function. For the $U(1)$ case, the choice of admissible contour amounts to a choice of deformation of $\mathbb{R} \subset \mathbb{C}$ which avoids poles and guarantees convergence. The Lagrangian condition is trivially satisfied for any such choice. We may write

$$\log |integrand| = -A_q(\sigma) + \mathcal{O}(\log \sigma), \tag{9.2}$$

$$\begin{aligned} A_q(\sigma) &= (\zeta' + (N_+ - N_-)(\log|\tau + i\upsilon| - 1))\upsilon + \\ &\left(\frac{\pi}{2}(N_+ + N_-) + (N_+ - N_-)\arctan\frac{\upsilon}{|\tau|} - \mathrm{sign}(\tau)(\theta + 2\pi q)\right)|\tau| \end{aligned} \tag{9.3}$$

where $\sigma = \tau + i\upsilon$ and where we have introduced the *boundary potential* $A_q(\sigma)$. We have convergence as long as A_q tends to infinity quickly at the ends of the contour. If we assume that the model is Calabi-Yau, then for any $\zeta' \neq 0$, we just have to take our contour to be given by $\nu = \pm\tau^2$ with the sign given by the sign of ζ' to ensure that the first term is dominant, and then convergence is clear. At $\zeta' = 0$, we have $A_q(\sigma) = (\pi N_+ - \mathrm{sign}\tau(\theta + 2\pi q))|\tau|$. Convergence holds here as well *provided* we assume

$$-\frac{N_+}{2} < \frac{\theta}{2\pi} + q < \frac{N_+}{2}. \tag{9.4}$$

Outside of this interval, we lose control over the convergence when ζ' goes to zero. The upshot is we obtain for the Calabi-Yau case the grade restriction rule that was originally derived in [8] albeit through different means: they derived the boundary potential $A_q(\sigma)$ through explicit calculations using mode expansions rather than through a localization formula.

For the phase $\zeta \gg 0$ with $N_+ \geq N_-$ (the case $N_+ < N_-$ being similar), the integral with our choice of contour picks up poles on the positive imaginary axis of the form $\sigma = i(R_j/2 + k)/Q^j$ for $Q^j > 0$ and $k \in \mathbb{Z}_{\geq 0}$. For generic values of the R_j's, all poles are simple, and we can expand in a sum of residues to obtain

$$\begin{aligned} &Z_{D^2}(\mathcal{B}) = \\ &C(\mathfrak{r}\Lambda)^{\hat{c}/2} \sum_{j|Q^j>0} \sum_{k=0}^{\infty} \frac{2\pi(-1)^k}{k!\,Q^j} e^{-t(R_j/2+k)/Q^j} f_{\mathcal{B}}\left(\frac{i(R_j/2+k)}{Q^j}\right) \prod_{i\neq j} \Gamma\left(\frac{R_i}{2} - \frac{Q^i}{Q^j}\left(\frac{R_j}{2} + k\right)\right). \end{aligned} \tag{9.5}$$

The asymptotics of a summand can be shown to be

$$\begin{aligned}\log|\text{summand}| &= -\frac{\zeta}{Q^j}k - \sum_i \frac{Q^i k}{Q^j}\log\left|\frac{Q^i k}{eQ^j}\right| + \mathcal{O}(\log k)\\ &= -\frac{N_+ - N_-}{Q^j}(k\log k - k - k\log|Q^j|) - \frac{\zeta + \sum_i Q^i \log|Q^i|}{Q^j}k + \mathcal{O}(\log k).\end{aligned} \tag{9.6}$$

For $N_+ > N_-$, the sum always converges while for $N_+ = N_-$, the sum converges only when $\zeta' > 0$. For the case $N_+ < N_-$, we get an asymptotic series in fractional powers of e^{-t}.

We now focus specifically on the case of a Wilson line $\mathcal{W}(q)$ for $\zeta \ll 0$ with $N_+ > N_-$. We can expand in residues as an asymptotic series in $\lambda = \exp(-\frac{\zeta'}{N_+ - N_-})$. The terms shrink until λ is on the order of $\frac{k}{|Q^j|}$ at which point they grow again. The contour integral can correspondingly be decomposed into two parts: a) a sum over residues bounded above by $\mathrm{Im}(-\sigma) < \lambda$ and b) an integral over a contour L_λ which passes through the imaginary axis at $\sigma = -i\lambda$. We can render this integral convergent if we can deform it into a piece of a saddle contour passing through a critical point together with a piece that stays in a region where the A_q is quickly growing. The ability to do this depends crucially on q. There are three cases to consider

1. $\mathcal{W}(q)$ is in the small window, i.e.

$$\left|\frac{\theta}{2\pi} + q\right| < \frac{1}{2}\min(N_-, N_+) \tag{9.7}$$

   Then we can bound from below the coefficient of $|\tau|$ by some positive constant, say $|\tau| > c$. Then

$$A_q(\tau - i\lambda) \geq A_q(-i\lambda) + c|\tau| = (N_+ - N_-)\lambda + c|\tau|. \tag{9.8}$$

   If we simply take $L_\lambda = \mathbb{R} - i\lambda$, then the contribution behaves like $\exp^{-(N_+ - N_-)\lambda}$, leading to

$$Z_{D^2}(\mathcal{W}(q)) = \left(\sum_{j:Q^j<0} \sum_{k=0}^{-R_j/2 - Q^j\lambda} (\dots)\lambda^{-\frac{N_+ - N_-}{|Q^j|}k}\right) + \mathcal{O}(\exp^{-(N_+ - N_-)\lambda}). \tag{9.9}$$

2. $\mathcal{W}(q)$ is in the big window, i.e.

$$\left|\frac{\theta}{2\pi} + q\right| < \frac{1}{2}\max(N_-, N_+). \tag{9.10}$$

Away from the imaginary axis, one has

$$\log(\text{integrand}(\sigma)) = \begin{cases} (N_+ - N_-)i\sigma(\log(\sigma) - \ell_+ - 1) + \mathcal{O}(\log|\sigma|) \\ (N_+ - N_-)i\sigma(\log(-\sigma) - \ell_- - 1) + \mathcal{O}(\log|\sigma|) \end{cases} \tag{9.11}$$

where

$$\ell_\pm = -\frac{\zeta}{N_+ - N_-} + i\frac{\theta + 2\pi q \mp \frac{\pi}{2}(N_+ + N_-)}{N_+ - N_-} \tag{9.12}$$

with $\mathrm{Re}(\ell_\pm) = \log\lambda$. Then we obtain a saddle point at $\log(\sigma) = \ell_+$ for $|\mathrm{Im}(\ell_+)| < \pi/2$, a saddle point at $\log(-\sigma) = \ell_-$ for $|\mathrm{Im}(\ell_-)| < \pi/2$, and no other saddle points. The saddle points for fixed θ correspond to the Coulomb branch vacua and are given by solutions to the equations

$$(i\sigma)^{N_+ - N_-} = \exp(-t'), \qquad t' = t + \sum_i Q_i \log Q_i \bmod 2\pi i. \tag{9.13}$$

This is because the imaginary part of the effective twisted superpotential with appropriate boundary term is proportional to $A_q(\sigma)$ (see [11]).

The existence of such a critical point detects when a brane has a contribution to a massive vacuum whose exact location is then given by 9.13. The saddle integral then gives a contribution with the asymptotic

$$\left|\int_{L_{saddle}} d\sigma \text{integrand}\right| \sim \exp^{\left((N_+ - N_-)\cos\left(\frac{N_+\pi - (\theta + 2\pi q)}{N_+ - N_-}\right)\lambda\right)}. \tag{9.14}$$

3. $\mathcal{W}(q)$ is in neither window.

   In this case, it is impossible to deform the contour so as to keep $A_q(\sigma)$ positive. Hence there is no analytic continuation so we are unable to transport $\mathcal{W}(q)$.

9.2 Anomalous $U(1)$ Brane Transport

We continue to assume $N_+ \geq N_-$ so that we are transporting from the phase $\zeta \gg 0$ to the phase $\zeta \ll 0$. In the phase $\zeta \gg 0$, the complex

$$\mathcal{K}_{\gg 0}(q) = \mathcal{W}(q) \otimes \bigotimes_{j|Q_j>0} \left(\mathcal{W}(-Q_j) \xrightarrow{X_j} \mathcal{W}(0)\right) \tag{9.15}$$

is a resolution of a twist of the structure sheaf of the deleted set and hence has trivial image on the Higgs branch. Thus $Z_{D^2}(\mathcal{K}_{\gg 0}) = 0$. For $N_+ > N_-$, this brane can be transported through the window, and hence we obtain that by analytic continuation $Z_{D^2}(\mathcal{K}_{\gg 0})$ in fact vanishes in all phases. In the non-anomalous case of $N_+ = N_-$, this transport argument fails, and the empty branes in the two phases can be entirely different.

Given a complex $\mathcal{B}_1$ of Wilson lines, one can bind to copies of $\mathcal{K}_{\gg 0}(q)$ for different values of q to systematically remove charges outside of the wide window until one obtains a brane $\mathcal{B}_2$ which obeys the wide window condition and has the same image on the Higgs branch as $\mathcal{B}_1$. How this is achieved is exactly like the single window case discussed in section 5.5. At the level of brane factors, this corresponds to a decomposition of the form

$$f_{\mathcal{B}_1}(\sigma) = f_{\mathcal{B}_2}(\sigma) + P(e^{2\pi\sigma}) f_{\mathcal{K}_{\gg 0}}(\sigma) \tag{9.16}$$

for P a Laurent polynomial. This is effectively polynomial division, and the "remainder" $f_{\mathcal{B}_2}(\sigma)$ is unique.

We may now transport $\mathcal{B}_2$ to where $\zeta \ll 0$, and the image on the Higgs branch may be determined via our descent procedure. However, it still remains to determine the Coulomb branch contribution. We may write

$$f_{\mathcal{B}_2}(\sigma) = \sum_{q=q_{\min}}^{q_{\max}} a_q e^{2\pi q\sigma} \tag{9.17}$$

where the sum is taken over q in the wide window. Then each q in the wide window but not in the narrow window gives $|a_q|$ massive vacua to the Coulomb branch. This can be shown by analyzing the critical points of the partition function integrand where by linearity it suffices to analyze when the brane factor is $e^{2\pi q}$ for some q. Now consider the complex

$$\mathcal{K}_{\ll 0}(q) = \mathcal{W}(q) = \bigotimes_{j|Q_j<0} \left(\mathcal{W}(-Q_j) \xrightarrow{X_j} \mathcal{W}(0) \right). \tag{9.18}$$

If $\mathcal{K}_{\ll 0}(q)$ is grade-restricted, then the image on the Higgs branch is trivial in the phase $\zeta \ll 0$. Then by binding $\mathcal{B}_2$ to copies of $\mathcal{K}_{\ll 0}(q)$'s, we can find a brane $\mathcal{B}_3$ with the same image on the Higgs branch as $\mathcal{B}_2$. Once again this is analogous to the binding discussed in section 5.5

However, $\mathcal{K}_{\ll 0}(q)$ may give contributions to several Coulomb branch vacua, and so the Coulomb branch contributions of our brane transport come from looking at the "difference" between $\mathcal{B}_2$ and $\mathcal{B}_3$, necessarily constructed from copies of $\mathcal{K}_{\ll 0}(q)$ for various q. This corresponds at the level of brane factors to a decomposition

$$f_{\mathcal{B}_2}(\sigma) = f_{\mathcal{B}_3}(\sigma) + \widetilde{P}(e^{2\pi\sigma}) f_{\mathcal{K}_{\ll 0}}(\sigma) \tag{9.19}$$

for $\widetilde{P}$ a Laurent polynomial.

The specific Coulomb branch vacua to which this contributes come from looking at the critical points of the localization integrand with brane factor $-\widetilde{P}(e^{2\pi\sigma}) f_{\mathcal{K}_{\ll 0}}(\sigma)$. To construct complexes descending solely to a single Coulomb branch vacuum, one must take linear combinations of $e^{2\pi q\sigma} f_{\mathcal{K}_{\ll 0}}(\sigma)$ for various q so that there is only one $e^{2\pi q}$ term for q in the wide window but not the narrow window. Then result is then the brane factor for the desired complex.

It is illuminating to recast this in a more functorial manner. Let $\mathcal{T}_{\mathbf{W}_{\text{wide}}}$ and $\mathcal{T}_{\mathbf{W}_{\text{narrow}}}$ denote respectively the full subcategories of $D^b(\mathbb{C}^d, U(1))$ generated by the Wilson lines obeying the wide and narrow window conditions. Here their respective projections generate the derived categories for the phases when $\zeta \ll 0$ and $\zeta \gg 0$. Like before, the projections from window categories exhibit equivalences of categories but are not compatible with tensor products, an operation under which the window categories are not even closed. Then if we denote by $\mathcal{C}$ the category of Coulomb branch branes, then the RG flow functor F_{RG} maps $D^b(\mathbb{C}^d, U(1))$ to $\langle \mathcal{C}, D^b(X_\zeta) \rangle$, the latter being the category given as a semiorthogonal decomposition of the categories of Coulomb and Higgs branch branes for the phase containing ζ.

$$\begin{tikzcd}
 & D^b(\mathbb{C}^d, U(1)) \arrow[dl, "F_{\mathrm{RG}}"'] \arrow[dr, "F_{\zeta \gg 0}"] & \\
\langle C, D^b(X_{\zeta \ll 0}) \rangle & \mathcal{T}_{\mathbf{w}_{\text{wide}}} \arrow[l, "\cong"'] \arrow[dr, "\cong"] & \\
D^b(X_{\zeta \ll 0}) \arrow[u, phantom, "\cup"] & \mathcal{T}_{\mathbf{w}_{\text{narrow}}} \arrow[l, "\cong"'] \arrow[u, phantom, "\cup"] & D^b(X_{\zeta \gg 0})
\end{tikzcd} \tag{9.20}$$

That we have a semiorthogonal decomposition is a consequence of the mathematical paper [2], but as of now we do not have a physical understanding for why this occurs or

if there is a mechanism by which the IR image of a brane can be a non-trivial bound state of a Higgs branch brane and a Coulomb branch brane. It would be extremely interesting to understand this in future work.

Remark 6. *The conclusions here should also apply to matrix factorizations in the presence of a nontrivial superporential since the introduction of a superpotential does not alter the localization formula.*

9.3 Higgsing

A general description of wall crossing at the heart of $\mathcal{M}_K$ would be rather difficult. However, if we restrict to only crossing codimension one walls and remain far from the center, then at all times the gauge group is Higgsed to something no larger than $U(1)$ times a finite factor. Such a finite factor will not affect the partition function and thus will not alter the brane transport so the results of the previous sections will then suffice to define transport maps. As we will argue below, the functor for crossing such a wall can be obtained by reducing to the grade restriction rule for the case of a single $U(1)$. When interpreted in terms of the full charge lattice, it will become a *band restriction rule* as we now impose restrictions on a certain linear combination of the charges which depends on the particular wall being crossed.

Recall that the real FI parameters take values in $\mathfrak{g}^\vee$. Then the Coulomb branch moduli may naturally be identified with $\mathfrak{g}$, and we have a pairing

$$\mathfrak{g} \times \mathfrak{g}^\vee \to \mathbb{C}. \tag{9.21}$$

We stress that neither space individually possesses a natural metric. Suppose we wish to cross a given wall. Then the wall is given as the positive linear space of $k-1$ charges, say $Q_{i_2}, \ldots, Q_{i_r} \in \mathfrak{g}^\vee$. We assume all walls we cross are codimension one which guarantees that the charges are linearly independent.

For the Higgsing, we look at the D-term equations

$$\sum_i Q_i^\alpha |\phi_i|^2 = \zeta^\alpha \tag{9.22}$$

in a regime where $\zeta_2, \ldots, \zeta_r \gg 0$, i.e. in a regime in which we differ by motion of ζ_1 from a point on the wall far from any boundary. We allow ζ_1 to evolve with the RG flow so that we go from $\zeta_1 \ll 0$ to $\zeta_1 \gg 0$. For a wall which is parallel to the direction of the RG flow, we cannot cross the wall in this way, but we may fix an energy scale and move an FI parameter to cross this wall. Essentially the same argument as below applies with now ζ_1 corresponding to the FI parameter which is moving. Note that in a Calabi-Yau model, all walls are of the latter type.

If ζ is written exclusively in terms of $Q_{i_2}, \ldots, Q_{i_k}$, then the corresponding chirals all get VEVs. In the case where this does not happen, then as ζ is generic within the wall, it must be that ζ is expressed in terms of at least r total Q_i's which would then completely Higgs the gauge group. For otherwise it would be possible to re-express the wall as a *different* set of $r-1$ chirals. In either case, the gauge group is Higgsed to no larger than $U(1)$.

For ζ on the wall (i.e. with $\zeta_1 = 0$), since we can take $\zeta_2, \ldots, \zeta_r$ as large as desired, we can ensure that in fact these VEVs are arbitrarily large. The Lagrangian contains mass terms of the form $|\phi_i|^2 |Q_i \cdot \sigma|^2$ for all i so therefore $Q_{i_j} \cdot \sigma$ for $j = 2, \ldots, k$ is very heavy and effectively frozen at low energies. Note that even though the RG flow eventually goes to a region in which the gauge group is completely Higgsed, our ability to take $\zeta_2, \ldots, \zeta_r$ large *independently* of the energy scale allows us to ensure that in the wall crossing regime, σ is effectively frozen to the line

$$\mathcal{L} = \{\sigma \ : \ Q_{i_2} \cdot \sigma = \cdots = Q_{i_r} \cdot \sigma = 0\} \subset \mathfrak{g}. \tag{9.23}$$

Now the effective (bulk) twisted superpotential restricted to $\mathcal{L}$ is completely determined by $Q_i \cdot \sigma$ for all i and for all $\sigma \in \mathcal{L}$, i.e. by how the chirals are charged under the $U(1)$ subgroup corresponding to $\mathcal{L}$. If we include the boundary contribution from a Wilson line brane, this term is completely determined by $q \cdot \sigma$ for q the brane charges and for $\sigma \in \mathcal{L}$. The brane contribution can then be found explicitly by noting that q and θ only appear in the Lagrangian through the combination $q + \frac{\theta}{2\pi}$. This leads to the equation

$$\widetilde{W}_{eff}(\sigma_1) = -t \cdot \sigma_1 - \sum_i (Q_i \cdot \sigma_1) \left(\log \left(\frac{Q_i \cdot \sigma_1}{\mu} \right) - 1 \right) + 2\pi i q \cdot \sigma_1. \tag{9.24}$$

Note that discarding the boundary term $2\pi i q \cdot \sigma_1$ and reparameterizing σ_1 in terms of a basis vector yields precisely 3.9. In this way, we are able to reduce to the $U(1)$ case. We caution that this approximation is only valid for crossing a single wall in an asymptotic regime. If we wish to cross multiple walls, each wall crossing will be in a different asymptotic regime and thus involve a different $U(1)$ subgroup, hence leading to a different grade restriction rule. Once reduced to the $U(1)$ case, it is this twisted effective superpotential which is then utilized to derived the associated window conditions.

More explicitly, for a wall crossing parameter u, define

$$N_{\pm} = \sum_i (Q_i \cdot u)^{\pm} \tag{9.25}$$

where the sum is taken over only the positive or negative terms, depending on the subscript of N. Assume without loss of generality that $N_+ > N_-$ as the reverse case is analogous. Then we may transport branes according to the wide window condition

$$\left| \frac{\theta \cdot u}{2\pi} + q \cdot u \right| < \frac{1}{2} N_+ \tag{9.26}$$

and the narrow window equation

$$\left| \frac{\theta \cdot u}{2\pi} + q \cdot u \right| < \frac{1}{2} N_-. \tag{9.27}$$

As before, to transport a general brane, we resolve as a complex of Wilson lines, bind to empty branes in order to express the complex using only windows satisfying the wide window condition, and then push down into the phase on the other side. The Wilson lines that do not also satisfy the narrow window condition will in addition give rise to Coulomb branch branes.

Chapter 10

Wall-crossing for Hirzebruch-Jung Models

As an application of the general wall-crossing story from chapter 9, we studying wall-crossing in Hirzebruch-Jung models for the case that the exceptional divisors are blown up in order, obtaining agreement with the K-theoretic statements in [16]. For illustrative purposes, we give two concrete examples: $\mathbb{C}^2/\mathbb{Z}_{5(1)}$ and $\mathbb{C}^2/\mathbb{Z}_{5(2)}$.

10.1 Through the walls

From GLSM II, we see a clear path towards obtaining a sequence of wall crossings which go from the fully singular phase to the fully resolved phase by iterative blow-ups. Though there are many possible orders in which one could proceed, it is simplest to move the lines in order of increasing α. It may be helpful at this point for the reader to read through the general procedure shown below in parallel with section 3 where we work through an explicit example with lots of pictures. Thus for the first wall crossing, we move the first line so that a portion of it passes through the first quadrant. The position of this wall corresponds to when the line passes through the origin, and this clearly corresponds to when ζ_1' changes sign from negative to positive. Thus a normal vector to this wall is given by $(1, 0, \ldots, 0)$.

For subsequent wall crossings, we are moving the αth line until it cuts through the intersection point of the $(\alpha - 1)$th line with the $|P|^2$ axis. We note that the fact that the slopes are decreasing as α increases ensures that this is the first instance for which the αth line will cut through the admissible region. The intersection of the $(\alpha - 1)$th and αth lines for any values of $\zeta'_{\alpha-1}$ and ζ'_α are given by solving the system

$$p_{\alpha-1}|P|^2 + q_{\alpha-1}|Q|^2 = \zeta_{\alpha-1} \tag{10.1}$$

$$p_\alpha |P|^2 + q_\alpha |Q|^2 = \zeta_\alpha \tag{10.2}$$

which upon specializing to $|Q|^2 = 0$ easily leads to

$$\zeta_{\alpha-1} p_\alpha - \zeta_\alpha p_{\alpha-1} = 0. \tag{10.3}$$

Thus we obtain $(0, \ldots, 0, p_\alpha, -p_{\alpha-1}, 0, \ldots, 0)$ as our normal vector where the two non-zero entries occur in the $\alpha - 1$th and αth positions.

From our normal vectors, we see that the charge vector relevant to the first wall crossing is given by

	P	X_1	X_2	$\ldots$	X_r	Q
$U(1)$	p_1	$-n$	0	$\ldots$	0	1

,

(using the fact that $q_1 = 1$) and that the charge relevant to the αth wall crossing is given by

	P	X_1	$\ldots$	$X_{\alpha-2}$	$X_{\alpha-1}$	X_α	$X_{\alpha+1}$	$\ldots$	X_r	Q
$U(1)$	0	0	$\ldots$	0	$-p_\alpha n$	$p_{\alpha-1} n$	0	$\ldots$	0	$-n$

,

where for the latter we have used equation (4.8). If we consider a Wilson line $\mathcal{W}(b_1, \ldots, b_r)$, then for the first wall crossing, we have the grade restriction rule,

$$|\tilde{b}_1| < \frac{1}{2}(p_1 + 1) \tag{10.4}$$

while for the αth wall-crossing for $\alpha > 1$, we have

$$| - p_\alpha \tilde{b}_{\alpha-1} + p_{\alpha-1} \tilde{b}_\alpha | < \frac{1}{2}(p_\alpha + 1) n \tag{10.5}$$

where we have defined $\tilde{b}_\alpha = b_\alpha + \frac{\theta_\alpha}{2\pi}$.

Remark 7. *The charges one obtains for the α-th wall-crossing after Higgsing correspond precisely to the local model of the α-th exceptional divisor as described in section 4.2.*

Since the wall crossing in this model appears rather simple, it is tempting to try to directly calculate the brane transport from this directly. Though this should still be possible in principle, it is rendered more difficult by a crucial subtlety. Namely, a

redundancy in the gauge symmetry leads to constraints on the allowed Wilson lines. This is due to the fact that this model arises from Model I via a matrix with non-unit determinant so it is necessary to restrict to Wilson lines which emerge from Model I. Parameterizing these by writing them explicitly as the images of Wilson lines from Model I is effectively the same as just converting everything back to Model I, and so we shall pursue this latter strategy. Therefore we use $n(C^{-1}_{\alpha\beta})$ to convert the above into a sequence of wall crossings for Model I. We will fix the theta angles throughout what follows so that our windows start at zero.

Given a Wilson line $\mathcal{W}(b_1, \ldots, b_r)$ in Model II, the corresponding Wilson line in Model I is given by $\mathcal{W}(c_1, \ldots, c_r)$ where $b_\alpha = \sum_\beta n(C^{-1})_{\alpha\beta} c_\beta$. If we define for model I $\tilde{c}_\beta = c_\beta + \frac{\theta'_\beta}{2\pi}$ where θ'_β is a theta angle in Model I, then we also have $\tilde{b}_\alpha = \sum_\beta n(C^{-1})_{\alpha\beta}\tilde{c}_\beta$. For $\alpha > 1$, we compute

$$\begin{aligned} &- p_\alpha \tilde{b}_{\alpha-1} + p_{\alpha-1}\tilde{b}_\alpha = \\ &- p_\alpha \left(\sum_{\beta \leq \alpha-1} p_{\alpha-1} q_\beta + \sum_{\beta \geq \alpha} p_\beta q_{\alpha-1} \right) \tilde{c}_\beta + p_{\alpha-1} \left(\sum_{\beta \leq \alpha-1} p_\alpha q_\beta + \sum_{\beta \geq \alpha} p_\beta q_\alpha \right) \tilde{c}_\beta. \end{aligned} \tag{10.6}$$

Note that the first and third sum cancel, yielding

$$\sum_{\beta \geq \alpha} p_\beta \tilde{c}_\beta \left(p_{\alpha-1} q_\alpha - p_\alpha q_{\alpha-1} \right) = n \sum_{\beta \geq \alpha} p_\beta \tilde{c}_\beta \tag{10.7}$$

where we have used equation (4.8). For $\alpha = 1$, we have a simpler computation

$$\tilde{b}_1 = \sum_\beta p_\beta q_1 \tilde{c}_\beta = \sum_\beta p_\beta \tilde{c}_\beta \tag{10.8}$$

where we have used that $q_1 = 1$. We therefore arrive at the following grade restriction rule for Model I.

$$\left| \sum_{\beta=\alpha}^{r} p_\beta \tilde{c}_\beta \right| < p_\alpha + 1 \text{ for } \alpha = 1, \ldots, r \tag{10.9}$$

Note that α starts at 1 instead of 2 as we have combined the results for $\alpha > 1$ with the result for $\alpha = 1$.

For ease of computation, we restrict to the case where the theta angles satisfy $\sum_{\alpha=1}^{r} p_\alpha \theta'_\alpha = -\pi(p_\alpha + 1) + 2\pi\epsilon$ for small $\epsilon > 0$. Then the j-th band simply requires

$$0 \leq \sum_{\alpha=j}^{r} p_\alpha c_\alpha \leq p_j. \tag{10.10}$$

It is easily observed that the Wilson lines

$$\mathcal{W}(0,\dots 0), \mathcal{W}(1,0,\dots,0), \mathcal{W}(0,1,0,\dots,0), \dots, \mathcal{W}(0,\dots,0,1) \tag{10.11}$$

fit through all of the walls.

Claim 1. *Any Wilson line brane which fits through all of the walls is necessarily one of the above.*

Proof. We proceed by induction. First it is obvious from taking $\alpha = r$ in the window condition that $c_r \in \{0,1\}$. Now suppose that out of the charges $c_{\alpha+1}, \dots, c_r$, at most one may be equal to 1, and the rest must be 0. Then $\sum_{\beta=\alpha+1}^{r} p_\beta c_\beta$ may have at most one non-zero term which (if it exists) is necessarily equal to p_β for some $\beta > \alpha$. Since $p_\alpha > p_\beta$ for any $\beta > \alpha$, if $c_\alpha < 0$, then we must have

$$\sum_{\beta=\alpha}^{r} p_\beta c_\beta = p_\alpha c_\alpha + \left(\sum_{\beta=\alpha+1}^{r} p_\beta c_\beta \right) < 0. \tag{10.12}$$

This contradicts our window condition, thereby establishing that $c_\alpha \geq 0$. Again using the window condition, it is clear that c_α must actually be zero unless $c_\beta = 0$ for all $\beta > \alpha$ in which case c_α may be either 0 or 1. By induction, we see that the above Wilson line branes are the only branes which simultaneously satisfy all the window conditions. □

We now turn to analyzing the IR limits of these Wilson lines in the fully singular phase. To determine the corresponding representation of $\mathbb{Z}_n$, it is first necessary to determine how $\mathbb{Z}_n$ embeds in $U(1)^r$. Using equation 4.9, it is easily observed that $(\omega^{p_1}, \dots, \omega^{p_r})$ stabilizes the X_i's and therefore gives the desired embedding. It follows then that $\mathcal{W}(0,\dots,0,1,0,\dots,0)$ with the 1 in the αth position maps to ρ^{p_α} in the IR, and obviously $\mathcal{W}(0,\dots,0)$ maps to the trivial representation. Thus we see that the branes which can be passed through all the walls to the large volume phase are precisely the fractional branes corresponding to (i) the trivial representation and (ii) the "special" representations (see [13], [16], and [18]).

The derived category for the fully singular phase is generated by $\rho^0, \rho^1, \dots, \rho^{n-1}$ for ρ a "fundamental representation" of $\mathbb{Z}_n$. We thus see how each fractional brane may be lifted uniquely to a Wilson line brane which passes through all the windows and into the

fully resolved phase. In particular, the trivial representation goes to the structure sheaf, and the special representations go to the Higgs branch and become coherent sheaves wrapping exceptional divisors as described in section 7.3, precisely in agreement with [16]. The remaining branes split between the Higgs and Coulomb branches. While we don't have a closed form expression for this, it can be computed for specific values of n and p using the procedure discussed in section 9.2.

10.2 $\mathbb{C}^2/\mathbb{Z}_{5(1)}$

We first consider a one parameter example in which there is an interesting splitting between the Higgs and Coulomb branches.

	P	X	Q
$U(1)$	1	-5	1

If we consider a Wilson line $\mathcal{W}(k)$ and take the theta angle to be slightly negative, then we obtain the narrow window

$$0 \leq k \leq 1 \tag{10.13}$$

and the wide window

$$-2 \leq k \leq 2. \tag{10.14}$$

Clearly the branes $\mathcal{W}(0)$ and $\mathcal{W}(1)$ map to line bundles on the resolved geometry which when restricted to the exceptional divisor look like $\mathcal{O}$ and $\mathcal{O}(1)$ respectively. It remains to determine the splittings of $\mathcal{W}(q)$ for $q \in \{-2, -1, 2\}$. For convenience, when analyzing brane factors, we set $z = e^{2\pi\sigma}$.

In the case of $\mathcal{W}(2)$, one can see that it goes to a single Coulomb branch vacuum whose location is given by 9.13. Meanwhile, the image on the Higgs branch is $\mathcal{O}(2)$ which can be written in terms of the generating set $\mathcal{O}(0), \mathcal{O}(1)$ by binding to the brane $\mathcal{O}(0) \to \mathcal{O}(1)^{\oplus 2} \to \mathcal{O}(2)$ to obtain $\mathcal{O}(0) \to \mathcal{O}(1)^{\oplus 2}$. Note that the analogous GLSM brane $\mathcal{K}_{\zeta \ll 0}(2) : \mathcal{W}(0) \to \mathcal{W}(1)^{\oplus 2} \to \mathcal{W}(2)$ has a nontrivial image on the Coulomb branch. This is reflected by the following calculation involving brane factors:

$$f_{\mathcal{W}(2)}(z) = z^2 = (z^2 - 2z + 1) + (2z - 1) = f_{\mathcal{K}_{\zeta \ll 0}(2)}(z) + (-1 + 2z) \tag{10.15}$$

where $2z-1$ is the brane factor of the pure Higgs part while $f_{\mathcal{K}_{\zeta\ll 0}(2)}(z)$ is the brane factor of the pure Coulomb part.

The case of $\mathcal{W}(-1)$ follows a similar analysis. First one can see that it goes to a single Coulomb branch vacuum whose location is given by 9.13. Meanwhile, the image on the Higgs branch is $\mathcal{O}(2)$ which can be written in terms of the generating set $\mathcal{O}(0), \mathcal{O}(1)$ by binding to the brane $\mathcal{O}(-1) \to \mathcal{O}(0)^{\oplus 2} \to \mathcal{O}(1)$ to obtain $\mathcal{O}(0)^{\oplus 2} \to \mathcal{O}(1)$. Note that the analogous GLSM brane $\mathcal{K}_{\zeta\ll 0}(1) : \mathcal{W}(-1) \to \mathcal{W}(0)^{\oplus 2} \to \mathcal{W}(1)$ has a nontrivial image on the Coulomb branch as is reflected by the following calculation involving brane factors:

$$f_{\mathcal{W}(-1)}(z) = z^{-1} = (z^{-1} - 2 + z) + (2 - z) = f_{\mathcal{K}_{\zeta\ll 0}(1)}(z) + (2 - z) \tag{10.16}$$

where $2-z$ is the brane factor of the pure Higgs part while $f_{\mathcal{K}_{\zeta\ll 0}(1)}(z)$ is the brane factor of the pure Coulomb part.

The case of $\mathcal{W}(-2)$ is more interesting. Analogous to as before, one first notes that it goes to a single Coulomb branch vacuum whose location is given by 9.13. Also the Higgs branch image is $\mathcal{O}(-2)$, but this time to express it in terms of the generating set $\mathcal{O}(0), \mathcal{O}(1)$, we must first bind to $\mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}(0)$ to obtain $\mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}(0)$ and then bind to two copies of $\mathcal{O}(-1) \to \mathcal{O}(0)^{\oplus 2} \to \mathcal{O}(1)$ to obtain $\mathcal{O}(0)^{\oplus 4} \to \mathcal{O}(0) \oplus \mathcal{O}(1)^{\oplus 2}$. Note that the analogous GLSM branes $\mathcal{K}_{\zeta\ll 0}(0) : \mathcal{W}(-2) \to \mathcal{W}(-1)^{\oplus 2} \to \mathcal{W}(0)$ and $\mathcal{K}_{\zeta\ll 0}(1) : \mathcal{W}(-1) \to \mathcal{W}(0)^{\oplus 2} \to \mathcal{W}(1)$ have nontrivial images on the Coulomb branch as is demonstrated by the following calculation involving brane factors:

$$\begin{aligned} &f_{\mathcal{W}(-2)}(z) = z^{-2} = (z^{-2} - 2z^{-1} + 1) + (2z^{-1} - 1) = \\ &(z^{-2} - 2z^{-1} + 1) + 2(z^{-1} - 2 + z) + (3 - 2z) = f_{\mathcal{K}_{\zeta\ll 0}(0)}(z) + 2f_{\mathcal{K}_{\zeta\ll 0}(1)}(z) + (3 - 2z) \end{aligned} \tag{10.17}$$

with $3-2z$ being the pure Higgs brane factor while $f_{\mathcal{K}_{\zeta\ll 0}(0)}(z) + 2f_{\mathcal{K}_{\zeta\ll 0}(1)}(z)$ is the brane factor of the pure Coulomb part. Here while $\mathcal{K}_{\zeta\ll 0}(1)$ flows to a single Coulomb branch, this is not the case for $\mathcal{K}_{\zeta\ll 0}(0)$. However, in the expression

$$f_{\mathcal{K}_{\zeta\ll 0}(0)}(z) + 2f_{\mathcal{K}_{\zeta\ll 0}(1)}(z) = z^{-2} + z - 3, \tag{10.18}$$

the z^{-1} dependency cancels, reflecting that one can bind two copies of $\mathcal{K}_{\zeta\ll 0}(1)$ to $\mathcal{K}_{\zeta\ll 0}(0)$ to construct a brane flowing only to a single Coulomb branch vacuum.

In summary, we have that $\mathcal{W}(0)$ and $\mathcal{W}(1)$ flow respectively to only the Higgs branch and together generate the derived category while $\mathcal{W}(-2)$, $\mathcal{W}(-1)$, and $\mathcal{W}(2)$ go to both a Coulomb branch vacuum *and* the corresponding Higgs branch line bundle. However, the branes $\mathcal{K}_{\zeta\ll 0}(2)$, $\mathcal{K}_{\zeta\ll 0}(1)$, and $\mathcal{K}_{\zeta\ll 0}(1)^{\oplus 2} \to \mathcal{K}_{\zeta\ll 0}(0)$ flow to the three distinct Coulomb branch vacua and have no Higgs branch image.

10.3 $\mathbb{C}^2/\mathbb{Z}_{5(2)}$

We next turn to the simplest example which is neither Calabi-Yau nor a single parameter model. Specializing our two models to this example, we obtain the following.

Model I:

	P	X_1	X_2	Q
$U(1)_1$	1	-3	1	0
$U(1)_2$	0	1	-2	1

Model II:

	P	X_1	X_2	Q
$U(1)_a$	2	-5	0	1
$U(1)_b$	1	0	-5	3

Since these models have two parameters, it is straightforward to directly find the wall crossings by inspecting the four "cones" in the plane (in either model) as shown in Figures 10.1 and 10.2.

However, it will be instructive to still follow our general procedure introduced in section 1. Since the phase boundaries in a two parameter model are (asymptotically) given by the charge vectors for the different chirals (instead of more generally a cone spanned by such vectors), a useful consistency check (for both models) comes from noting that the normal vectors we obtain below from our general procedure are each normal to a specific vector of charges as expected. As there are two FI parameters, we have two lines in the $(|P|^2, |Q|^2)$ plane:

$$2|P|^2 + |Q|^2 = \zeta_1' \tag{10.19}$$

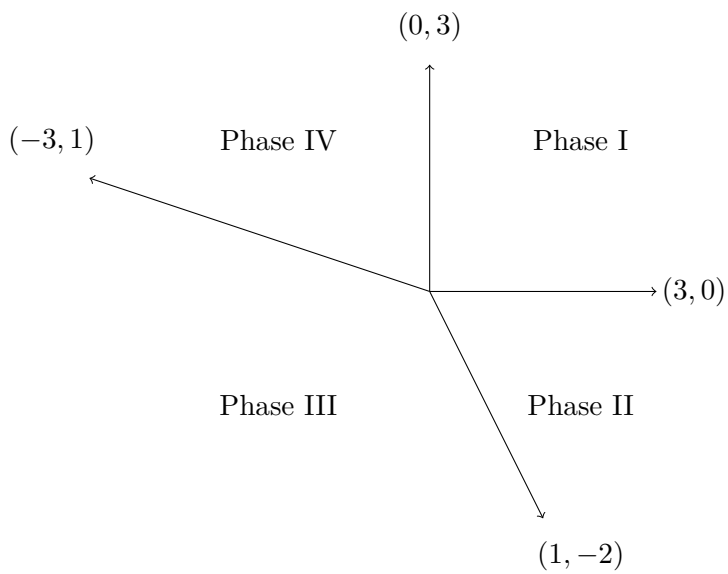

Figure 10.1: Model I for $\mathbb{C}^2/\mathbb{Z}_{5(2)}$

$$|P|^2 + 3|Q|^2 = \zeta_2'. \tag{10.20}$$

We begin in the fully resolved phase (Phase III), corresponding to $\zeta_1', \zeta_2' \ll 0$, as shown in Figure 10.3. We then start increasing ζ_1'. We see that the first line enters the first quadrant when ζ_1' becomes positive as shown in Figure 10.4, yielding the normal vector $(0, 1)$. We are now in a partially resolved phase (Phase II) corresponding to the set-up shown in Figure 10.5. If we now fix a positive value of ζ_2' and increase ζ_1' until the admissible region has boundary components on both lines (corresponding to the fully resolved phase), we see that we must solve

$$2|P|^2 = \zeta_1' \tag{10.21}$$

$$|P|^2 = \zeta_2' \tag{10.22}$$

where we use the fact that lines 1 and 2 have slopes 2 and $\frac{1}{3}$ respectively to conclude that this occurs when both lines intersection at a point on the positive $|P|^2$ axis as shown in Figure 10.6. We thus see that

$$-\zeta_1' + 2\zeta_2' = 0. \tag{10.23}$$

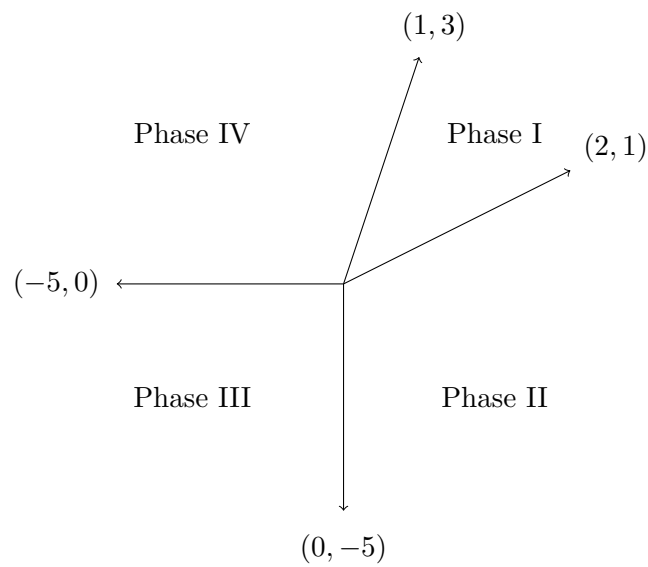

Figure 10.2: Model II for $\mathbb{C}^2/\mathbb{Z}_{5(2)}$

and therefore obtain the normal vector $(-1, 2)$. The fully resolved phase (Phase I) is shown in Figure 10.7.

The respective $U(1)$'s relevant for the wall-crossings are given by

	P	X_1	X_2	Q
$U(1)$	2	-5	0	1

and

	P	X_1	X_2	Q
$U(1)$	0	-5	10	-5

,

and from this we obtain the window conditions

$$|\tilde{b}_1| < \frac{3}{2} \tag{10.24}$$

and

$$|-\tilde{b}_1 + 2\tilde{b}_2| < 5 \tag{10.25}$$

where we follow the convention of section 10.1 in which a tilde means with the theta angle correction added. We may convert this to the first model via the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

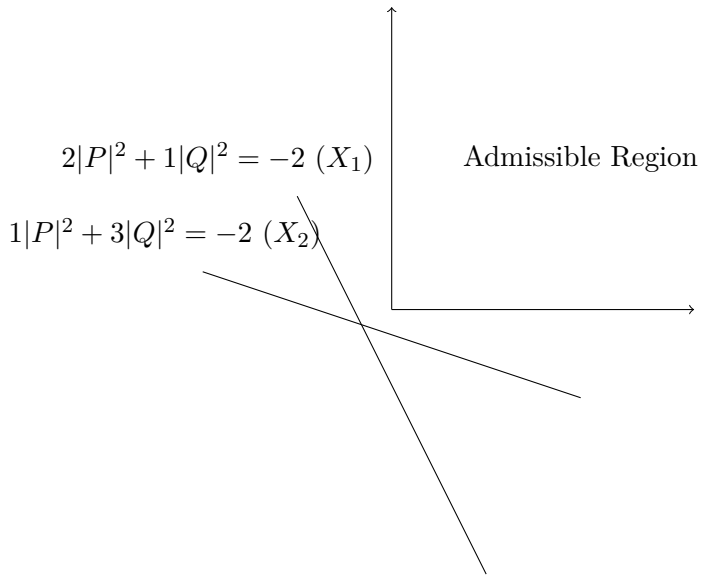

Figure 10.3: Starting in the fully singular phase (Phase III)

which yields

$$|2\tilde{c}_1 + \tilde{c}_2| < \frac{3}{2} \tag{10.26}$$

and

$$|\tilde{c}_2| < 1. \tag{10.27}$$

Taking θ_1', θ_2' to be slightly negative, we get the windows

$$-1 \leq 2c_1 + c_2 \leq 1 \tag{10.28}$$

and

$$0 \leq \tilde{c}_2 \leq 1 \tag{10.29}$$

from which we see that the only solutions are $\mathcal{W}(0,0)$, $\mathcal{W}(0,1)$, and $\mathcal{W}(-1,1)$ which map respectively to ρ^0, ρ^2, and ρ^4 for ρ a "fundamental representation" of $\mathbb{Z}_5$. If we retrace the calculations using the large window bounds instead of the small window bounds (for the same theta angle values), we get

$$-2 \leq 2c_1 + c_2 \leq 2 \tag{10.30}$$

and

$$0 \leq \tilde{c}_2 \leq 1. \tag{10.31}$$

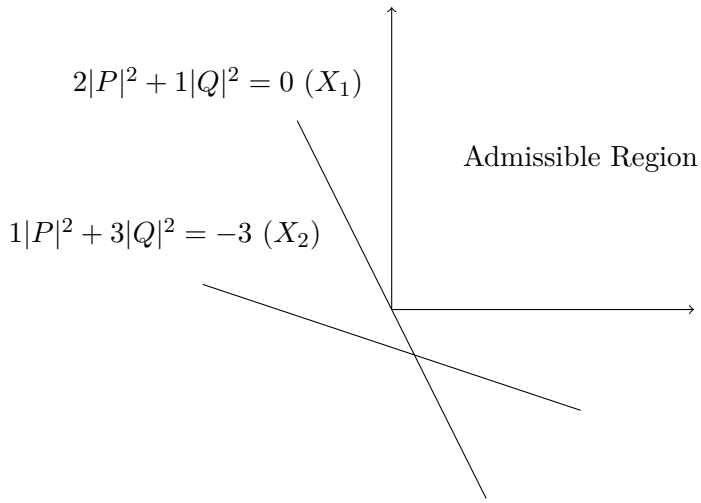

Figure 10.4: Resolving X_1 (Phase III –> Phase II)

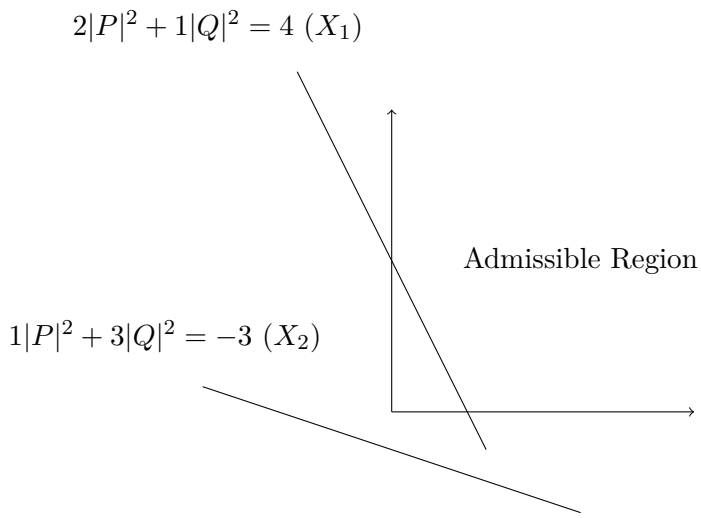

Figure 10.5: Intermediate phase with X_1 resolved (Phase II)

Now two additional branes obey the window conditions, namely $\mathcal{W}(1,0)$ and $\mathcal{W}(-1,0)$ which split between the Higgs and Coulomb branch according to the discussion in section 9.2. Note that the theta angle choices we are making here are different from those in section 10.1 so the resulting functor will differ by monodromy.

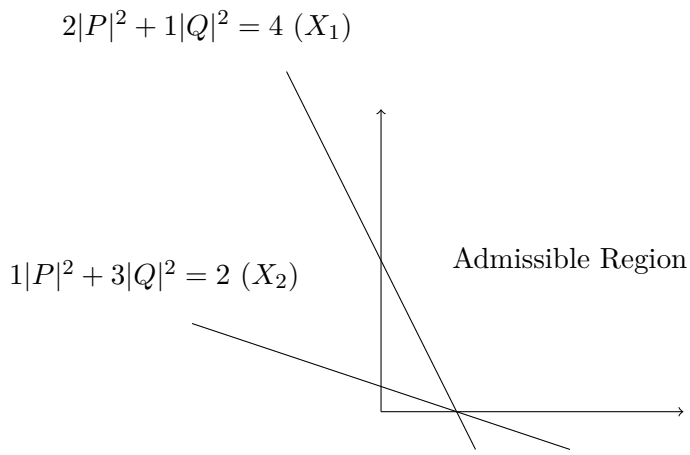

Figure 10.6: Resolving X_2 (Phase II –> Phase I)

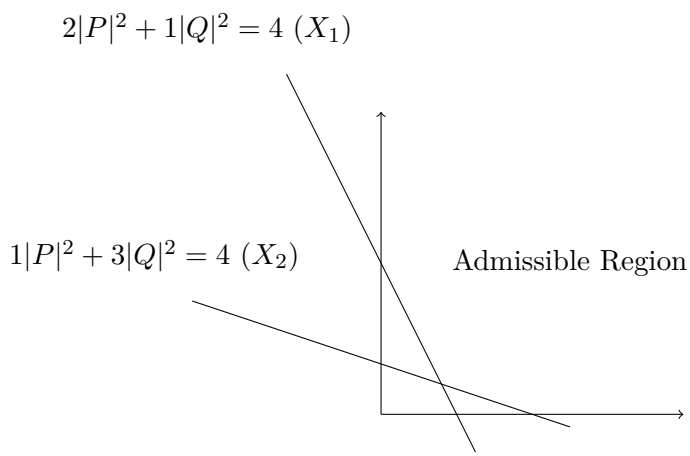

Figure 10.7: Ending in the fully resolved phase (Phase I)

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