HYPERGRAPH NIM

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ABSTRACT OF THE DISSERTATION

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NIM is a game in which two players take turns removing tokens from piles. There are usually several non overlapping piles each of which can have any amount of tokens in it. In a turn a player selects a nonempty pile and removes any positive number of tokens from it. The player that removed the last token(s) wins the game.

This thesis focused on a generalization of the game NIM in which players on their turn instead of selecting a particular pile, they select a subset of piles and then proceed to remove and positive amount of tokens from each pile i.e. at least on token from each pile. Not all subsets of piles can be selected though, the available options are usually given as a hypergraph $\mathcal{H}$. Also if any of the piles are empty then the corresponding moves that include that pile are no longer legal. If not specifically said otherwise we assume the base set of $\mathcal{H}$ is V i.e. $\mathcal{H} \subseteq 2^V$ This generalization we named Hypergraph NIM and it is a very broad generalization as it also includes other NIM generalizations like Moore's NIM [25].

The main goal of the study of such games is to find the best strategy to win. If both players play optimally then each position of the game is either winning or a losing position. The winning position always has a legal move to a losing position while losing position only has legal moves to winning positions. The strategy for the player in the winning position is to play the move that lead to a losing position for the other player, and the second player can only make a legal move back to a winning position for the first player. This continues until the first player makes the last legal move and wins the game. In this regard, the empty pile position is by default a losing position as well as all the positions that have no legal moves.

Given this strategy we only need to identify the winning/losing positions. A generalization of the winning/losing position is the so called Sprague-Grundy value. You may think of winning positions having value 1 and losing having 0 before. But now, the Sprague-Grundy value is still 0 for losing positions, but it maybe be any positive integer for the winning positions. Given a finite subset $S \subseteq \mathbb{Z}_{\geq}$, let $mex(S) = min(\mathbb{Z}_{\geq} \setminus S)$ (the *minimum excluded value*) be the smallest $k \in \mathbb{Z}_{\geq}$ that is not in S. In particular, $mex(\emptyset) = 0$, by the definition.

Given an impartial game $\Gamma = (X, E)$, the SG function $\mathcal{G}_\Gamma : X \to \mathbb{Z}_{\geq}$ is defined recursively, as follows: $\mathcal{G}_\Gamma(x) = mex(\emptyset) = 0$ for any terminal x and, in general, $\mathcal{G}_\Gamma(x) = mex(\{\mathcal{G}_\Gamma(x') \mid x \to x'\})$. The use of the Sprague-Grundy value comes into play when we consider a sum of games in which winning/losing positions of the base games are not enough to find the winning/losing position of the sum.

The tetris function $\mathcal{T}_{\mathcal{H}}(x)$ is the maximum number of consequtive moves starting from x in $\mathrm{NIM}_{\mathcal{H}}$. It follows that $\mathcal{T}_{\mathcal{H}} \geq \mathcal{G}_{\mathcal{H}}$ and in some cases they are equal. Calculating the tetris values is the same thing as calculating the maximum b-matching, which is polynomial for graphs and NP-hard for dimension 3 or more. We have found a fast algorithm for calculating $\mathcal{T}_{\binom{[n]}{k}}$. We sort x then move over all tokens from the $n-k$ smallest piles over to the largest k piles one at a time to the smallest pile (at the time). This in turn gives a us a new position whose tetris is obvious (size of its smallest pile). This algorithm can be sped up to $O(n \cdot log(n))$ time.

Games where $\mathcal{T}_{\mathcal{H}} = \mathcal{G}_{\mathcal{H}}$ are called SG-decreasing. We found a necessary condition for $\mathcal{H}$ to be SG-decreasing:

♠ $\forall S \subseteq V$ with $\mathcal{H}_S \neq \emptyset\, \exists H \in \mathcal{H}_S$ such that $H \cap H' \neq \emptyset\, \forall H' \in \mathcal{H}_S$.

The sufficient conditions for $\mathcal{H}$ to be SG-decreasing are:

(1) $\mathcal{H}$ is hypergraph with dimension ≤ 3 (no edge has size bigger then 3) with property ♠.

(2) $\mathcal{H}$ is an intersecting hypergraph, i.e. every edge intersects every other edge.

(3) $\mathcal{H}$ is a graph with an intersecting edge.

We generalize the concept of sum of games. Given games $\Gamma_i = (X_i, E_i)$, $i \in V = \{1, ..., n\}$, and a hypergraph $\mathcal{H} \subseteq 2^V$, we define the $\mathcal{H}$-combination $\Gamma_{\mathcal{H}} = (X, E)$ of these games by setting

$$X \;=\; \prod_{i \in V} X_i, \text{ and}$$

$$E \;=\; \left\{ (x, x') \in X \times X \;\middle|\; \exists\, H \in \mathcal{H} \text{ such that } \begin{array}{ll} (x_i, x'_i) \in E_i & \forall i \in H, \\ x_i = x'_i & \forall i \notin H \end{array} \right\}.$$

In other words the game is played by choosing $H = \{i, j, ...\} \in \mathcal{H}$ and the making a move in games Γ_i,$\Gamma_j, ...$

Lets take a look at the following equation (which is false in general):

$$\mathcal{G}_{\Gamma_{\mathcal{H}}} \;=\; \mathcal{G}_{NIM_{\mathcal{H}}}\left(\mathcal{G}_{\Gamma_1}, ..., \mathcal{G}_{\Gamma_n}\right). \tag{1}$$

SG theorem says its true for $\mathcal{H} = \{\{1\}, \{2\}, ..., \{n\}\}$ (which is the sum of games).

We found that equality holds for arbitrary hypergraph $\mathcal{H} \subseteq 2^V$ and SG-decreasing games (games where SG value can only decrease by moves, a.k.a. SG-decreasing games) Γ_i, $i \in V$.

Given games Γ_i, $i \in V$, and a hypergraph $\mathcal{H} \subseteq 2^V$ we have the equality

$$\mathcal{T}_{\Gamma_{\mathcal{H}}} \;=\; \mathcal{T}_{NIM_{\mathcal{H}}}\left(\mathcal{T}_{\Gamma_1}, ..., \mathcal{T}_{\Gamma_n}\right).$$

The above two theorems immediately imply the following statement. If $\mathcal{H}$ is a SG-decreasing hypergraph then $\mathcal{H}$-combination of SG-decreasing games is SG-decreasing. In particular, a SG-decreasing combination of SG-decreasing hypergraphs is SG-decreasing.

The main result of our research is the discovery of an infinite family of graph for which an explicit formula for their SG-value has been found. While all members of the family are yet to be discovered, infinitely many have been found, and all of them belong to the family of connected minimal transversal-free hypergraphs.

To state our main result we need to introduce some additional notation.

To a position $x \in \mathbb{Z}_{\geq}^V$ of $\text{NIM}_{\mathcal{H}}$ let us associate the following quantities:

$$m(x) = \min_{i \in V} x_i \tag{2a}$$

$$y_{\mathcal{H}}(x) = \mathcal{T}_{\mathcal{H}}(x - m(x)e) + 1 \tag{2b}$$

$$v_{\mathcal{H}}(x) = \binom{y_{\mathcal{H}}(x)}{2} + \left(\left(m(x) - \binom{y_{\mathcal{H}}(x)}{2} - 1 \right) \mod y_{\mathcal{H}}(x) \right), \tag{2c}$$

where e is the n-vector of full ones. Finally, we define

$$\mathcal{U}_{\mathcal{H}}(x) = \begin{cases} \mathcal{T}_{\mathcal{H}}(x) & \text{if } m(x) \leq \binom{y_{\mathcal{H}}(x)}{2} & \text{(3a)} \\ v_{\mathcal{H}}(x) & \text{otherwise.} & \text{(3b)} \end{cases}$$

With this notation the results of [8, 9, 23] can be stated as the SG function of the considered games is defined by (3a)-(3b), that is, $\mathcal{G} = \mathcal{U}$. It was a surprise to see that the "same" formula works for seemingly very different games. In view of this, we call the expression (3a)-(3b) the *JM formula*, in honor of the results of Jenkyns and Mayberry [23]. We call a hypergraph $\mathcal{H}$ a *JM hypergraph* if this formula describes the SG function of $\text{NIM}_{\mathcal{H}}$.

Let us add that the formula looks the same but it depends on $\mathcal{T}_{\mathcal{H}}$ and, hence, the actual values depend on the hypergraph $\mathcal{H}$. In fact, function $\mathcal{T}_{\mathcal{H}}$ may be difficult to compute [10], even for cases when the JM formula is valid.

The results are as follows.

(i) We can calculate the winning-losing partition of JM hypergraphs, without the knowledge of its tetris function. In short, x is a losing position if and only if $y_{\mathcal{H}}(x) = 1$. While this may look like we use the tetris function after all, we actually only need to check whether any move exists.

(ii) A JM hypergraph is minimal transversal-free.

(iii) A graph (that is, a 2-uniform hypergraph) is JM if and only if it is connected and minimal transversal-free. We provide a complete list of JM graphs, see figure 8.

(iv) A matroid hypergraph is defined as the basis of some matroid. It is JM if and only if it is transvesal-free. This implies that all self-dual matroid hypergraphs are JM.

(v) Hypergraphs defined by connected k-edge subgraphs of a given graph are JM under certain conditions. Namely, a hypergraph satisfying the following properties is JM:

(A1) $\mathcal{H}$ is minimal transversal-free.

(D1) For every pair of hyperedges $H, H' \in \mathcal{H}$ there exists a chain $\mathcal{C} = \{H_0, H_1, \ldots, H_p\} \subseteq \mathcal{H}$ such that $H = H_0$ and $H' = H_p$.

(D2) For every subhypergraph $\mathcal{F} \subseteq \mathcal{H} \subseteq 2^V$ such that $V(\mathcal{F}) \neq V$ there exist hyperedges $F \in \mathcal{F}$ and $S \in \mathcal{H}$ such that $\emptyset \neq (S \setminus F) \subseteq V \setminus V(\mathcal{F})$.

(vi) A Symmetric hypegraph with size sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{Z}^k_{\geq}, \lambda_1 < \lambda_2, < \cdots < \lambda_k)$ is defined to be the $\mathcal{H}(\lambda) = \bigcup_{i=1}^k \binom{V}{\lambda_i}$. It is JM if and only if its size sequence satisfies

(i) $\lambda_{i+1} - \lambda_i \leq \lambda_1$ for all $i \in [k-1]$.

(ii) $\lambda_1 + \lambda_k = n \geq 3$.

(vii) For every integer k, the number of vertices of a k-uniform JM hypergraph is bounded by $k\binom{2k}{k}$.

(viii) We can obtain JM Hypergraphs from ”non-saturated” JM hypergraphs in the following way. Let $\mathcal{H}$ be a JM hypergraph and let $H \notin \mathcal{H}$ be a set such that H can be obtained as union of some edges in $\mathcal{H}$. If H is not a transversal in $\mathcal{H} \cup \{H\}$, then $\mathcal{H} \cup \{H\}$ is also JM.

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1 Preliminaries

1.1 Set notations

We denote by $\mathbb{Z}_{\geq}$ the set of nonnegative integers $\mathbb{Z}_{\geq} = \{0, 1, 2, \dots\}$, by $\mathbb{Z}_{>}$ the set of positive integers $\mathbb{Z}_{>} = \{1, 2, \dots\}$. For $n \in \mathbb{Z}_{>}$ we denote $[n] = \{1, 2, \dots, n\}$.

1.2 Game theory

There are innumerable kinds of activities and situations that might be described as games. In fact any two or more persons interaction can be considered a game. A large portion of these may be studied within the context of classical game theory. However, the games I wish to consider are more specialized than what is traditionally addressed by game theory. More specialized, because I shall only be concerned with determining the winner of a game, and not with other issues of interest in game theory such as maximizing payoff, studying cooperative strategies, etc.

1.2.1 Combinatorial games

A combinatorial game is defined to be a two-player, perfect-information game with no chance elements.

1.2.2 Impartial games

In combinatorial game theory , an impartial game [1, 3] is a game in which the allowable moves depend only on the position and not on which of the two players is currently moving, and where the payoffs are symmetric. In other words, the only difference between player 1 and player 2 is that player 1 goes first. For example, the games like Tictactoe and Go are not impartial because the player can only add his own pieces. Furthermore, impartial games are played with perfect information and no chance moves. Meaning all information about the game and operations for both players are visible to both players.

An impartial game can be modeled by a directed graph $\Gamma = (X, E)$, in which a vertex $x \in X$ represents a *position*, while a directed edge $(x, x') \in E$ represents a *move* from position x to x', which we will also denote by $x \to x'$. The graph Γ may be infinite, but we will always assume that any sequence of successive moves (called a *play*) $x \to x'$, $x' \to x''$, $\cdots$ is finite. In particular, this implies that Γ has no directed cycles. The game is played by two players with a token placed at an initial position. They alternate in moving the token along the directed edges of the graph. The game ends when the token reaches a *terminal*, that is, a vertex with no outgoing edges. The player who made the last move wins, equivalently, the one who is out of moves, loses. In the rest of the thesis we consider only impartial games and call them simply games. Let us emphasize that the word "game" and notation, like Γ, refer to the family of games in which the initial position can be chosen arbitrarily.

1.2.3 Wining and losing positions

It is not difficult to characterize the winning strategies of an impartial game. The subset $\mathcal{P} \subseteq X$ is called the set of $\mathcal{P}$-positions (losing positions) if the following two properties hold:

(i) $\mathcal{P}$ is *independent*, that is, for any $x \in \mathcal{P}$ and move $x \to x'$ we have $x' \notin \mathcal{P}$;

(a) $\mathcal{P}$ is *absorbing*, that is, for any $x \notin \mathcal{P}$ there is a move $x \to x'$ such that $x' \in \mathcal{P}$.

In graph theory such a set is also called a kernel [37]. A *terminal position* is a position in which we cannot move forward i.e. there are no moves available and hence a position in which the game ends. The player who moved to such a position wins the game and the player who was handed in such a position loses the game. In some cases the winner and loser are reversed but not in our case.

It is easily seen that the set $\mathcal{P}$ can be obtained by the following simple recursive algorithm: include in $\mathcal{P}$ all terminal positions of Γ; delete from Γ all positions from which there is a move to a terminal position, together with all terminal positions, and repeat the above.

It is also clear that any move $x \to x'$ of a player to a *$\mathcal{P}$-position* $x' \in \mathcal{P}$ is a winning move. Indeed, by (i), the opponent must leave $\mathcal{P}$ by the next move (or cannot move), and then, by (a), the player can reenter $\mathcal{P}$. Since, by definition, all plays of Γ are finite and all terminals are in $\mathcal{P}$, sooner or later the opponent will be out of moves.

In combinatorial game theory positions $x \in \mathcal{P}$ and $x \notin \mathcal{P}$ are usually called a $\mathcal{P}$- and $\mathcal{N}$-positions, respectively. The next player wins in an $\mathcal{N}$-position, while the previous one wins in a $\mathcal{P}$-position.

1.2.4 Sum of games

Given two games Γ_1 and Γ_2, their sum (also called disjunctive compound, see [16]) $\Gamma_1 + \Gamma_2$ is played as follows: On each turn, a player chooses either Γ_1 or Γ_2 and plays in it, leaving the other game unchanged. The game ends when no move is possible, neither in Γ_1 nor in Γ_2. Obviously, this operation is commutative and associative and, hence, it allows us to define the sum $\Gamma_1 + \cdots + \Gamma_n$ of n summand games for any integer $n \geq 2$.

To play optimally the sum, it is not enough to know the winning-losing partitions of all the games Γ_i, $i \in [n]$. Sprague and Grundy [32, 33, 18] resolved this problem.

1.2.5 NIM$_n$

NIM$_n$ in an example of a impartial game in which the player take turns picking one of the n piles and removing a positive amount of tokens from it. The player that cannot do so loses the game. By definition, NIM$_n$ is the sum of n games, each of which (a single pile NIM$_1$) is trivial. Yet, NIM$_n$ itself is not. It was solved by Bouton in his seminal paper [14] as follows. The NIM-sum $x_1 \oplus \cdots \oplus x_n$ of nonnegative integers is defined as the bitwise binary sum. For example,

$$3 \oplus 5 = 011_2 \oplus 101_2 = 110_2 = 6,\ 3 \oplus 6 = 5,\ 5 \oplus 6 = 3, \text{ and } 3 \oplus 5 \oplus 6 = 0.$$

It was shown in [14] that $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n_{\geq}$ is a $\mathcal{P}$-position of NIM$_n$ if and only if $x_1 \oplus \cdots \oplus x_n = 0$.

To play the sum $\Gamma = \Gamma_1 + \Gamma_2$, it is not sufficient to know $\mathcal{P}$-positions of Γ_1 and Γ_2, since $x = (x^1, x^2)$ may be a $\mathcal{P}$-position of Γ even when x^1 is not a $\mathcal{P}$-position of Γ_1 and x^2 is not a $\mathcal{P}$-position of Γ_2. For example, $x = (x_1, x_2)$ is a $\mathcal{P}$-position of the two pile NIM$_2$ if and only if $x_1 = x_2$, while only $x_1 = 0$ and $x_2 = 0$ are the unique $\mathcal{P}$-positions of the corresponding single pile games.

1.2.6 Sprague-Grundy theory

To play the sums we need the concept of the Sprague-Grundy (SG) function, which is a refinement of the concept of $\mathcal{P}$-positions.

Given a finite subset $S \subseteq \mathbb{Z}_{\geq}$, let $mex(S) = min(\mathbb{Z}_{\geq} \setminus S)$ (the *minimum excluded value*) be the smallest $k \in \mathbb{Z}_{\geq}$ that is not in S. In particular, $mex(\emptyset) = 0$, by the definition.

Given an impartial game $\Gamma = (X, E)$, the SG function $\mathcal{G}_\Gamma : X \to \mathbb{Z}_{\geq}$ is defined recursively, as follows: $\mathcal{G}_\Gamma(x) = mex(\emptyset) = 0$ for any terminal x and, in general, $\mathcal{G}_\Gamma(x) = mex(\{\mathcal{G}_\Gamma(x') \mid x \to x'\})$.

It can be seen easily that the following two properties define the SG function uniquely.

(1) No move keeps the SG value, that is, $\mathcal{G}_\Gamma(x) \neq \mathcal{G}_\Gamma(x')$ for any move $x \to x'$.

(2) The SG value can be arbitrarily (but strictly) reduced by a move, that is, for any integer v such that $0 \leq v < \mathcal{G}(x)$ there is a move $x \to x'$ such that $\mathcal{G}_\Gamma(x') = v$.

The definition of the SG function implies several other important properties:

(3) The $\mathcal{P}$-positions are exactly the zeros of the SG function: $\mathcal{G}_\Gamma(x) = 0$ if and only if x is a $\mathcal{P}$-position of Γ.

(4) The SG function of NIM$_n$ is the NIM-sum of the cardinalities of its piles, that is, $\mathcal{G}_{NIM_n}(x) = x_1 \oplus \cdots \oplus x_n$ for all $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n_{\geq}$; see [14, 32, 33, 18].

(5) In general, the SG function of the sum of n games is the NIM-sum of the n SG functions of the summands. More precisely, let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ be the sum of n games and $x = (x^1, \ldots, x^n)$ be a position of Γ, where x^i is a position of Γ_i for $i \in V = [n]$, then $\mathcal{G}_\Gamma(x) = \mathcal{G}_{\Gamma_1}(x^1) \oplus \cdots \oplus \mathcal{G}_{\Gamma_n}(x^n)$; see [32, 33, 18].

SG theory shows that playing a sum of games $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ may be effectively replaced by NIM_n in which each summand game Γ_i is replaced by a pile of $x_i = \mathcal{G}_{\Gamma_i}(x)$ stones, for $i \in V$. decreased by every move.

1.3 Graphs and hypergraphs

1.3.1 Graph

In the most common sense of the term [6, 37, 30], a graph is an ordered pair $G = (V, E)$ comprising a set V of vertices or nodes or points together with a set E of edges or arcs or lines, which are 2-element subsets of V (i.e. an edge is associated with two vertices, and that association takes the form of the unordered pair comprising those two vertices). To avoid ambiguity, this type of graph may be described precisely as undirected and simple.

1.3.2 Hypergraph

In mathematics, a hypergraph is a generalization of a graph in which an edge can join any number of vertices. Formally, a hypergraph $\mathcal{H}$ is a pair $\mathcal{H} = (V, E)$ where V is a set of elements called nodes or vertices, and E is a set of non-empty subsets of V called hyperedges or edges. Therefore, E is a subset of $2^V \setminus \{\emptyset\}$, where 2^V is the power set of V. When we say an edge $H \in \mathcal{H}$ we in fact mean $H \in E$. Often we shall not explicitly state V, but its assumed that $V = \bigcup_{H \in \mathcal{H}} H$, when we wish to specify V we shall denote this by $\mathcal{H} \subset 2^V$. This make sense since isolated vertices do not contribute to the game outcome whatsoever. A hypergraph whose edges are all of size $k \in \mathcal{N}$ is called *k-uniform*, or just *uniform* if k is not important. A hypergraph's *dimension*, denoted by $dim(\mathcal{H})$, is the size of its largest edge. A *k-transversal* of $\mathcal{H} \in 2^V$ is a set $S \in V$ such that $|H \cap S| \geq k$ for all $H \in \mathcal{H}$. The 1-transversals are usually referred to as transversals.

1.3.3 Subhypergraph

If $\mathcal{H} = (V, E)$, $\mathcal{H}' = (V', E')$ are two hypergraphs with $V' \subseteq V$ and $E' \subseteq E$ then $\mathcal{H}'$ is called a *subhypergraph* of $\mathcal{H}$ and we shall denote this with $\mathcal{H}' \subseteq \mathcal{H}$. If we further have $\mathcal{H}' \neq \mathcal{H}$ then $\mathcal{H}'$ is called a *proper subhypergraph* of $\mathcal{H}$.

A subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ is called *induced* if $E' = \{H \in E | H \subseteq V'\}$, and we shall denote this by $\mathcal{H}_{V'}$ or by $\mathcal{H} \setminus S$ if $V' = V \setminus S$.

A hypergraph $\mathcal{H}$ is called *transversal-free* if no hyperedge $H \in \mathcal{H}$ is a transvesal of $\mathcal{H}$. Finally, we say that $\mathcal{H}$ is *minimal transversal-free* if it is transversal-free and every nonempty proper induced subhypergraph of it is not.

A *singleton* denotes and edge of size 1, or (sub)hypergraph consisting of such an edge.

1.3.4 Degree and neighborhood

Given a hypergraph $\mathcal{H} = (V, E)$ and a subset $F \subseteq E$ of the edges, we denote by $d_F(v)$ the *degree* of vertex $v \in V$ with respect to the subhypergraph (V, F). In other words, $d_F(v)$ is the number of edges in F that are incident with vertex v. If $d_F(v) > 0$ then we call v a *supporting vertex* of subset F, and we denote by $V(F) \subseteq V$ the set of supporting vertices of F, that is, $V(F) = \{v \in V \mid d_F(u) > 0\}$. Denote the *neighborhood* of $i \in V$ as usual $N_{\mathcal{H}}(i) = \{k \in V | \exists H \in \mathcal{H} : \{i, k\} \subseteq H\}$. A *leaf* is a vertex v with $d_E(v) = 1$.

1.3.5 Path, cycle, tree and connectivity

Given a hypergraph $\mathcal{H} = 2^V$, a *path* is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that $\{v_{i-1}, v_i\} \subseteq H_i$ for some $H_i \in \mathcal{H}$, $i \in \mathbb{Z}_k$ and $H_i \neq H_j$ for $i \neq j$. The *path size* is k. A *shortest path* is a path of minimum size. A *cycle* is a path $v_0, v_1, \ldots, v_k$ with $v_0 = v_k$. A hypergraph is *connected* if for every two vertices there exists a path between them. An alternative definition is that a hypergraph $\mathcal{H} \subseteq 2^V$ is *not connected* if V can be partitioned into two nonempty subsets V_1 and V_2 such that every hyperedge of $\mathcal{H}$ is contained in either V_1 or V_2. Otherwise $\mathcal{H}$ is called *connected*. A *connected component* of a hypergraph $\mathcal{H}$ is a nonempty subgraph $\mathcal{H} \subseteq \mathcal{H}$ that is maximally connected i.e. it contains some vertex and all vertices reachable by a path from it.

A *tree* is a connected graph without a cycle. A bipartite graph is a graph without any cycle of odd size.

1.3.6 Standard graph names

The complete graph on n vertices K_n is a graph consisting of every pair as an edge, also known as a clique when it is a subgraph. A complement of a graph $G = (V, E)$ is the graph $\bar{G} = (V, E(K_n) \setminus E(G))$. The complete bipartite graph $K_{m,n}$ is the complement of $K_n \cup K_m$. A cycle graph C_n is a graph consisting of a single cycle of length n. A star is a graph consisting of $n-1$ leaves and an additional vertex to which all of them are connected. Some additional choices of graph names follow the one found at http://www.graphclasses.org/smallgraphs.html.

The complete k-uniform hypergraph $\binom{[n]}{k}$ consist of all hyperedges of size k on n vertices. Sometimes we instead use $\binom{S}{k}$ and specify a set S in case where multiple (sub)hypergraphs are under discussion and there might be some notation overlap.

1.4 Miscellaneous

These are not the primary subject we are going to talk about, but they occasionally pop up.

1.4.1 Characteristic vector

Let us denote by $\chi(S)$ the *characteristic vector* of a set S. That is $\chi(S)_j = 1$ if $j \in S$ and $\chi(S)_j = 0$ if $j \notin S$.

1.4.2 Matroid

In terms of independence, a finite *matroid* M is a pair $(E, \mathcal{I})$, where E is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called the independent sets) with the following properties:

(1) The empty set is independent, i.e., $\emptyset \in \mathcal{I}$. Alternatively, at least one subset of E is independent, i.e., $\mathcal{I} \neq \emptyset$.

(2) Every subset of an independent set is independent, i.e., for each $A' \subset A \subset E$, if $A \in \mathcal{I}$ then $A' \in \mathcal{I}$. This is sometimes called the hereditary property.

(3) If A and B are two independent sets (i.e., each set is independent) and A has more elements than B, then there exists $x \in A \backslash B$ such that $B \cup \{x\}$ is in $\mathcal{I}$. This is sometimes called the augmentation property or the independent set exchange property.

The first two properties define a combinatorial structure known as an independence system.

Maximal independent sets are called bases. Let $B(M)$ denote the set of all bases of M. M is uniquely defined by $B(M)$ simply by taking all possible subsets of elements in $B(M)$. A dual M^* of the matroid M is a matroid whose bases are the complements of the bases of M: $B(M^*) = \{I \setminus b | b \in B(M)\}$. A Matroid is self dual if $M = M^*$. A matroid hypegraph is defined to be the bases of some matroid M. A matroid hypergraph is *self-dual* if $V \setminus H \in \mathcal{H}$ for all $H \in \mathcal{H}$, that is, if the corresponding matroid is self-dual. Let us remark that in some papers self-dual matroids are called *identically self-dual*, see [30, 7, 35].

1.4.3 Support

For a hypergraph $\mathcal{H} \in 2^V$ and a position $x \in \mathbb{Z}^V_{\geq}$ we define the *support* of x as $supp(x) = \{v \in V | x_v \neq 0\}$.

2 Generalized NIM

Let us extend the notion of the game NIM_n in three different ways, however the most general extension called hypergraph $\mathrm{NIM}_{\mathcal{H}}$ contains the other two as special cases, so for the rest of the thesis we shall consider hypergraph $\mathrm{NIM}_{\mathcal{H}}$ by default.

2.1 Hypergraph $\mathrm{NIM}_{\mathcal{H}}$

We generalize the game NIM_n over a hypergraph $\mathcal{H}$ in the following way. The piles are the vertices of the hypergraph, however the legal moves are now the hyperedges of the hypergraph, i.e. on your turn you have to pick an hyperedge and remove at least one token from each vertex on that hyperedge, then pass the turn the opponent. The choice of the edge is therefore only allowed if all vertices on that edge contain at least one token. If no such edge exist on your turn then you lose the game. Also $\emptyset$ is never in the hypergraph, i.e. we cannot pass the turn without making a move.

We denote this game as $\mathrm{NIM}_{\mathcal{H}}$. The game is played from a starting position $x = (x_1, \dots, x_n) \in \mathbb{Z}^n_{\geq}$ where $V = [n]$ and $x_i \in \mathbb{Z}_{\geq}$ for all $i \in V$. Coordinate x_i denotes the number of stones in pile $i \in V$. Provided a legal move exist the player will pick an edge $H \in \mathcal{H}$ and *move* to a new position x', denoted $x \to x'$. Note that by choice of H the following properties must hold: $x_i = x'_i$ for $i \notin H$ and $x_i > x'_i \geq 0$ for $i \in H$. We denote by $\mathcal{G}_{\mathcal{H}}(x)$ the SG value of x in $\mathrm{NIM}_{\mathcal{H}}$.

Given a hypergraph $\mathcal{H} \subseteq 2^V$, a hyperedge $H \in \mathcal{H}$, and a position $x \in \mathbb{Z}^V_{\geq}$, we call a move $x \to x'$ an H-move if $\{i \in V \mid x'_i < x_i\} = H$. Furthermore, for positions $x \geq \chi(H)$ we shall consider two special H-moves from x:

Slow H-move: $x \to x^{s(H)}$ defined by $x^{s(H)}_i = x_i - 1$ for $i \in H$, and $x^{s(H)}_i = x_i$ for $i \notin H$, that is by decreasing every piles in H by exactly one unit;

Fast H-move: $x \to x^{f(H)}$ defined by $x^{s(H)}_i = 0$ for $i \in H$, and $x^{s(H)}_i = x_i$ for $i \notin H$, that is by decreasing the size of every piles in H to zero.

The following holds for any H-move $x \to x'$: $x_i \geq x^{s(H)} \geq x'_i \geq x^{f(H)} \geq 0$.

2.2 Moore's $\mathrm{NIM}^{\leq}_{n,k}$

Moore's [25] generalization of the game of NIM_n is played as follows. Given two integer parameters n, k such that $1 < k < n$, and n piles of tokens. Two players take turns. By one move a player reduces at least one and at most k piles. The player who makes the last move wins. The $\mathcal{P}$-positions of this game were characterized by Moore in 1910 and an explicit formula for its Sprague-Grundy function was given by Jenkyns and Mayberry [23] in 1980, for the case $n = k+1$ only.

Moore's $\mathrm{NIM}^{\leq}_{n,k}$ is a special case of $\mathrm{NIM}_{\mathcal{H}}$ with the hypergraph $\mathcal{H} = \bigcup_{i=1}^{k} \binom{[n]}{i}$.

2.3 Exact $\mathrm{NIM}^{=}_{n,k}$

Similar to Moore's $\mathrm{NIM}^{\leq}_{n,k}$ we are given two integer parameters n, k such that $1 < k < n$, and n piles of tokens. By one move a player reduces all k piles by at least one token each. The player who makes the last move wins. Exact $\mathrm{NIM}^{=}_{n,k}$ is essentially just $\mathrm{NIM}_{\mathcal{H}}$ over a complete k-uniform hypergraph $\mathcal{H} = \binom{[n]}{k}$. This generalization was introduced in [9] and solved for $2k \geq n$. In Section 5.7 we show a family of games that includes $\mathrm{NIM}^{=}_{2k,k}$ and $\mathrm{NIM}^{\leq}_{n,n-1}$ has Sprague-Grundy value equal to (25a)-(25b). This formula is based on another value called the tetris value that can be calculated efficiently as described in Section 3.2. For the case of $2k > n$, $\mathrm{NIM}^{=}_{n,k}$ is a game of $\mathrm{NIM}_{\mathcal{H}}$ on an intersecting hypergraph whose Sprague-Grundy function equal its tetris function (see Sections 4.4 and 3.2).

3 Tetris function

Given a hypergraph $\mathcal{H} \in 2^V$, the *tetris value* of a position $x \in \mathbb{Z}_{\geq}^V$ is defined as the maximum number of consecutive moves one can take starting with x. We denote this value as $T_{\mathcal{H}}(x)$. It is easy to see the the tetris value of a position $x = (x_1, \ldots, x_n)$ in Moore's $\text{NIM}_{n,k}^{\leq}$ equals to $\sum_{i=1}^n x_i$, in fact any hypergraph containing all singletons has the same value. More examples are given in Section 3.4.

Let us call a sequence of moves on $\mathcal{H}$ that corresponds to the tetris value of x a *tetris sequence* of moves for $\mathcal{T}_{\mathcal{H}}(x)$. Note that for every such sequence there exists a tetris sequence of slow moves which includes the same hyperedges but the moves themselves are all slow. This is because edges that contain other edges do not contribute to the tetris value, see Lemma 4.

Let us first observe some basic properties of the tetris function that will be instrumental in our proofs. We assume in the sequel that a hypergraph $\mathcal{H} \subseteq 2^V$ is fixed, and all positions mentioned are assumed to be from $\mathbb{Z}_{\geq}^V$.

Lemma 1 *For every position $x \in \mathbb{Z}_{\geq}^V$ we have*

$$\mathcal{G}_{\mathcal{H}}(x) \leq \mathcal{T}_{\mathcal{H}}(x).$$

Proof: By the definition of the SG function, for every position x we have a move $x \to x'$ such that $\mathcal{G}_{\mathcal{H}}(x') = \mathcal{G}_{\mathcal{H}}(x) - 1$. Thus, starting from x we can make $\mathcal{G}_{\mathcal{H}}(x)$ consecutive moves (in each move decreasing the SG function exactly by 1.) On the other hand, the Tetris function value is the maximum number of such consecutive moves, proving thus the above inequality. $\square$

Lemma 2 *If $x \geq x'$ are two positions, then*

$$\mathcal{T}_{\mathcal{H}}(x) \;\geq\; \mathcal{T}_{\mathcal{H}}(x') \;\geq\; \mathcal{T}_{\mathcal{H}}(x) - \sum_{i=1}^n (x_i - x_i').$$

Proof: Any sequence of moves starting with x' can be repeated from x, and hence $\mathcal{T}_{\mathcal{H}}(x) \;\geq\; \mathcal{T}_{\mathcal{H}}(x')$. Furthermore decreasing one of the piles by one unit can decrease the tetris value by at most one. From this the second inequality follows. $\square$

Corollary 1 *If $x, x' \in \mathbb{Z}_{\geq}^V$ are two positions such that*

$$x_i' \geq x_i \text{ for all } i \neq j, \text{ and } x_j' = x_j - 1$$

for some index j, then we have $\mathcal{T}(x') \geq \mathcal{T}(x) - 1$.

Proof: Follows from Lemma 2. $\square$

Lemma 3 (Contiguity property of the tetris function) *Assume $x \to x'$ is an H-move. Then for every integer z such that $\mathcal{T}_{\mathcal{H}}(x') \leq z \leq \mathcal{T}_{\mathcal{H}}(x^{s(H)})$ there exists an H-move $x \to x''$ for which $\mathcal{T}_{\mathcal{H}}(x'') = z$ and $x' \leq x'' \leq x^{s(H)}$.*

Proof: Consider a sequence of positions $x^0 = x^{s(H)} \geq x^1 \geq \cdots \geq x^p = x'$, where $\sum_{j=1}^n (x_j^{i-1} - x_j^i) = 1$ for all $i = 1, \ldots, p$. By Lemma 2 we have $\mathcal{T}(x^{i-1}) \geq \mathcal{T}(x^i) \geq \mathcal{T}(x^{i-1}) - 1$ and all of these positions are reachable from x by an H-move. Thus, the statement follows. $\square$

Corollary 2 *For an arbitrary hypergraph $\mathcal{H} \subseteq 2^V$ and positions $x', x'' \in \mathbb{Z}_{+}^V$ the inequality $x'' \leq x'$ implies $\mathcal{T}_{\mathcal{H}}(x'') \leq \mathcal{T}_{\mathcal{H}}(x')$ and we have*

$$\{\mathcal{T}_{\mathcal{H}}(y) \mid x'' \leq y \leq x'\} \;=\; \{t \mid \mathcal{T}_{\mathcal{H}}(x'') \leq t \leq \mathcal{T}_{\mathcal{H}}(x')\}.$$

Furthermore, if for a position $x \in \mathbb{Z}_{+}^V$ we have both $x \to x'$ and $x \to x''$ as H-moves for some $H \in \mathcal{H}$, then all moves $x \to y$ for $x'' \leq y \leq x'$ are H-moves.

Lemma 4 *Let $\mathcal{H} \subseteq \widetilde{\mathcal{H}} \subseteq 2^V$ be two hypergraphs. Then for every position $x \in \mathbb{Z}_{\geq}^V$ we have $\mathcal{T}_{\mathcal{H}}(x) \leq \mathcal{T}_{\widetilde{\mathcal{H}}}(x)$.*

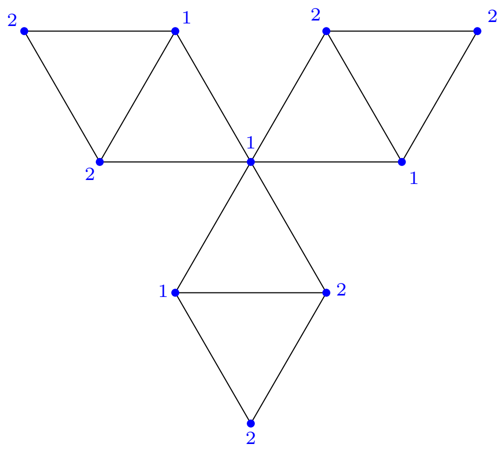

Figure 1: Counterexample to the equality in Lemma 5

Proof: Since $\mathcal{H} \subseteq \widetilde{\mathcal{H}}$, any move in $\mathrm{NIM}_{\mathcal{H}}$ is also a move in $\mathrm{NIM}_{\widetilde{\mathcal{H}}}$. Thus, the claim follows by the definition of the Tetris function. $\square$

3.1 B-matching and hardness

Given a graph $G = (V, E)$ and a vector $b \in \mathbb{Z}_{\geq}^{V}$, a b-matching is a function $f : E \to \mathbb{Z}_{\geq}$, such that $\sum_{e \in E, e \ni v} f(e) \leq b_v, \forall v \in V$. In other words a b-matching can be viewed as a multiset of slow moves from $x = b$, and the maximum b-matching problem is the same problem as finding the tetris value. A non polynomial formula known as the Tutte-Berge formula for computing this for graph is given as

$$\mathcal{T}_G(x) = \min_{U \subseteq V}(x(U) + \sum_{K \text{ is a connected component of } G \setminus U} \lfloor \frac{x(K)}{2} \rfloor),$$

where $x(S) = \sum_{s \in S} x_s$.

For graphs the maximum b-matching is known to be computable in polynomial time (see [17, 34, 30]). However for hypergraphs of dimension 3 or more this problem is NP-hard.

Lemma 5 *The tetris value in a hypergraph $\mathcal{H}$ is bounded above by*

$$\mathcal{T}_{\mathcal{H}}(x) \leq \min_{S \subseteq V \text{ is a } d\text{-(minimal) transversal of } \mathcal{H},\ d=1,\ldots,\dim(\mathcal{H})} \lfloor \frac{\sum_{i \in S} x_i}{d} \rfloor$$

Proof: It is enough to show that each d-transversal S satisfies the inequality $\mathcal{T}_{\mathcal{H}}(x) \leq \lfloor \frac{\sum_{i \in S} x_i}{d} \rfloor$. By definition of a d-transversal each slow tetris move must decrease $\sum_{i \in S} x_i$ by at least d. Hence $d\mathcal{T}_{\mathcal{H}}(x) \leq \sum_{i \in S} x_i$. $\square$

It often turns out that we have an equality above for small hypergraphs. This formula can be very explicit way to compute the tetris value. For example Transversal hypergraphs, K_4 and C_5 (see Section 3.4). Here is a counterexample for when they are not equal:

This graph and positions x above has $T(x) = 7$ (the selection of U as the middle vertex gives the Tutte-Berge value of $7 = 1 + 3 \cdot \lfloor \frac{1+2+2}{2} \rfloor$) but all 1-transversals have size ≥ 9 since they must include ≥ 2 points from each of the outer 3 triangles and the 2-transversal gives us $\lfloor \frac{16}{2} \rfloor = 8$

3.2 Tetris value of $\binom{[n]}{k}$

Let us focus on $\mathrm{NIM}_{\mathcal{H}}$ on k-uniform hypergraph $\mathcal{H} = \binom{[n]}{k}$ consisting of all edges size k and vertex size n. Let us recall this game is the same as $\mathrm{NIM}^{=}_{n,k}$. Note that due to extreme symmetry of $\binom{[n]}{k}$ we can reorder x any way we like without affecting the tetris value.

The following observation will be used several times.

Lemma 6 *Given a position $x = (x_1, \ldots, x_n)$ and $i, j \in N$ such that $x_i < x_j$, then for the position $x' = (x'_1, \ldots, x'_n)$ defined by*

$$x'_l = \begin{cases} x_i + 1, & \textit{if } l = i, \\ x_j - 1, & \textit{if } l = j, \\ x_l, & \textit{otherwise}, \end{cases} \tag{4}$$

we have $\mathcal{T}_{n,k}(x) \leq \mathcal{T}_{n,k}(x')$. In other words, the tetris function is not increasing when we move a token from a larger pile to a smaller one.

Proof: Consider any sequence of slow moves from x resulting in x''. If $x''_j > 0$ then the same sequence of slow moves can be made from x' since $x'_l \geq x_l$ for $l \neq j$. If $x''_j = 0$ then since $x_j > x_i$, this sequence contains a slow move reducing x_j but not x_i. Let us modify this move reducing x_i rather than x_j and keeping all other moves of the sequence unchanged. The obtained sequence has the same length and consists of slow moves from x'. $\square$

Notice that we can generalize Lemma 6 replacing ± 1 in (4) by $\pm\Delta$ for any integer $\Delta \in [0, x_j - x_i]$.

Lemma 7 *The slow move that reduces the k largest piles of x reduces the tetris value $\mathcal{T}_{\binom{[n]}{k}}(x)$ by exactly one.*

Proof: Let x' be the position obtained from x by such a slow move, and let x'' be another position obtained by some slow move. By applying (4) repeatedly, we can obtain x' from x'' with $\mathcal{T}_{\binom{[n]}{k}}(x'') \leq \mathcal{T}_{\binom{[n]}{k}}(x')$ by Lemma 6. This implies that x' has the highest tetris value among all positions each reachable from x by a slow move. By Lemma 2, each slow move reduces the tetris value by at least one and there exists a slow move reducing it by exactly one. Hence, $\mathcal{T}_{\binom{[n]}{k}}(x') = \mathcal{T}_{\binom{[n]}{k}}(x) - 1$. $\square$

This lemma provides a pseudo-polynomial algorithm of calculating the tetris value by repeatedly decreasing by 1 each of the k currently largest piles until one of them becomes empty. The number of such reductions is the tetris value of x.

Lemma 8 *Given the hypergraph $\mathcal{H} = \binom{[n]}{k}$ and a position $x \in \mathbb{Z}_+^{Z_n}$ such that $x_1 \geq x_2 \geq \cdots \geq x_n > 0$, let us consider the following hyperedge $H_0 = \{1, 2, 3, \ldots, k-1\} \cup \{n\}$. Then we have $\mathcal{T}_{\mathcal{H}}(x^{s(H_0)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$.*

Proof: Let us now consider a longest sequence of consecutive slow moves with hyperedges H^1, H^2, $\ldots$, $H^{\mathcal{T}_{\mathcal{H}}(x)}$ where $H^1 = \{1, \ldots, k\}$. Such a sequence exists due to Lemma 7. Assume first that there exists a hyperedge $H^i \ni n$. Then $\mathcal{T}_{\mathcal{H}}(x^{s(H^i)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$ and thus we can apply Lemma 4 and conclude that $\mathcal{T}_{\mathcal{H}}(x^{s(H_0)}) = \mathcal{T}_{\mathcal{H}}(x^{s(H^i)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$. Otherwise, if no such hyperedge exists, then H_0, H^2, $\ldots$, $H^{\mathcal{T}_{\mathcal{H}}(x)}$ also forms a longest sequence of consecutive slow moves, proving our claim. $\square$

3.2.1 Computing the tetris function in polynomial time for $\mathrm{NIM}_{\binom{[n]}{k}}$

Given a non-decreasing position x, let us construct $\bar{x}$ from x by emptying the first $n-k$ piles and adding these $\sum_{i=1}^{n-k} x_i$ tokens, one by one, to the last k piles as follows. By each step add one token to the smallest of these k piles. If there are several such piles, break the tie by adding this token to the pile of the largest index, to keep the resulting x vector nondecreasing. It is easy to see that $\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) = \min(\bar{x}_i | n-k+1 \leq i \leq n) = \bar{x}_{n-k+1}$.

Proposition 1 *The above construction keeps the tetris value, $\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) = \mathcal{T}_{\binom{[n]}{k}}(x)$.*

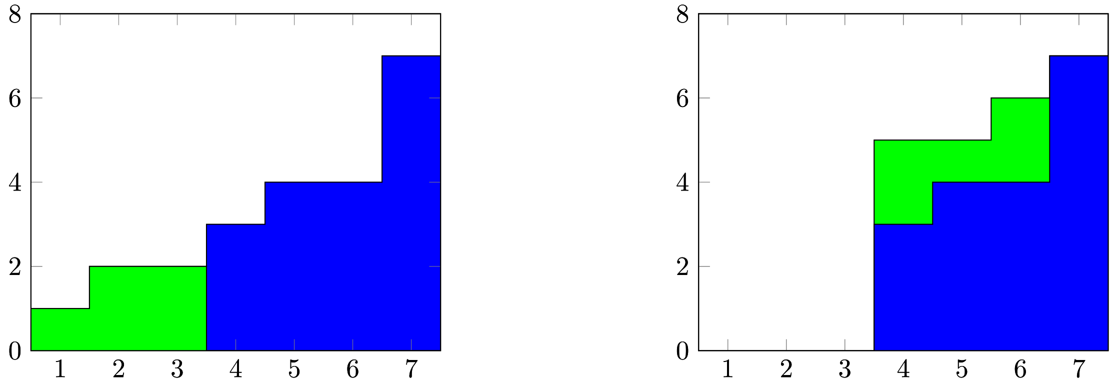

Figure 2: $k = 4, n = 7$, $x = (1, 2, 2, 3, 4, 4, 7)$, and $\bar{x} = (0, 0, 0, 5, 5, 6, 7)$.

Proof: Let us note that $\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) \le \mathcal{T}_{\binom{[n]}{k}}(x)$ by Lemma 6. By the definition of $\bar{x}$, none of tokens from the smallest $n-k$ piles of x is moved to any pile of size larger than $\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) + 1$ and so we have

$$V(x) := \sum_{i=1}^{n} \min(x_i, \mathcal{T}_{\binom{[n]}{k}}(\bar{x}) + 1) = \sum_{i=n-k+1}^{n} \min(\bar{x}_i, \mathcal{T}_{\binom{[n]}{k}}(\bar{x}) + 1). \tag{5}$$

Since $\bar{x}_{n-k+1} = \mathcal{T}_{\binom{[n]}{k}}(\bar{x})$, we get by (5) that

$$V(x) \le k - 1 + k\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) < k(\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) + 1).$$

Assume now indirectly that $\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) < \mathcal{T}_{\binom{[n]}{k}}(x)$. Then it is possible to construct a sequence of $\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) + 1$ slow moves from x. By such sequence any pile would be reduced at most $\mathcal{T}_{\binom{[n]}{k}}(\bar{x}) + 1$ times, and therefore the total number of the removed tokens is at most $V(x)$, implying $k(\mathcal{T}_{\binom{[n]}{k}}(\bar{x})+1) \le V(x)$, contradicting the above inequality. The obtained contradiction implies that $\mathcal{T}_{\binom{[n]}{k}}(x) = \mathcal{T}_{\binom{[n]}{k}}(\bar{x})$. $\square$

Let us note that position $\bar{x}$ is defined above by a non-polynomial algorithm. However $\bar{x}$ and consequently $\mathcal{T}_{\binom{[n]}{k}}(x) = \bar{x}_{n-k+1}$ can be computed in a more efficient way.

Proposition 2 *Given a nondecreasing position x, the corresponding $\bar{x}$ can be obtained in linear time in n.*

Proof: Let $s = \sum_{i=1}^{n-k} x_i$ be the number of tokens we shift on top of the largest k piles; see Figure 2. By the definition of $\bar{x}$ we know that for some $\ell < k$ the first $\ell + 1$ columns of $\bar{x}$ will have almost the same number of tokens (at most one difference.) To determine this index ℓ and the height of the resulting piles, we use simple volume based arguments. We need to compute first the following parameters.

For each $i = \in [k-1]$, we denote by $y_i = x_{n-k+i+1} - x_{n-k+i}$ the difference of the sizes of consecutive piles. Set $s_0 = 0$, $s_k = \infty$, and for $i = 1, \ldots, k-1$, set $s_i = s_{i-1} + i \cdot y_i$ (i.e., the number of tokens we need to shift on top of the first i piles $(n-k+1), \ldots, (n-k+i)$ to make them all equal to $x_{n-k+1+i}$.) We define a unique ℓ by $s_\ell \le s < s_{\ell+1}$. We define $a = s - s_\ell$, $\alpha = \lfloor \frac{a}{\ell+1} \rfloor$, and $\beta = a \mod (\ell + 1)$. We fill up the first $\ell + 1$ columns to level $x_{n-k+\ell+1}$ using s_ℓ tokens. Then, we place the remaining a tokens by increasing each of the first $\ell + 1$ columns (indexed $n-k+1, \ldots, n-k+\ell+1$) by α and the last β of these by one more, as in the following expression.

$$\bar{x}_i = \begin{cases} 0, & \text{if } i = 1, \ldots, n-k; \\ x_{n-k+\ell+1} + \alpha, & \text{if } i = n-k+1, \ldots, n-k+1+\ell; \\ x_{n-k+\ell+1} + \alpha + 1, & \text{if } i = n-k+2+\ell-\beta, \ldots, n-k+1+\ell; \\ x_i, & \text{if } i = n-k+2+\ell, \ldots, n. \end{cases}$$

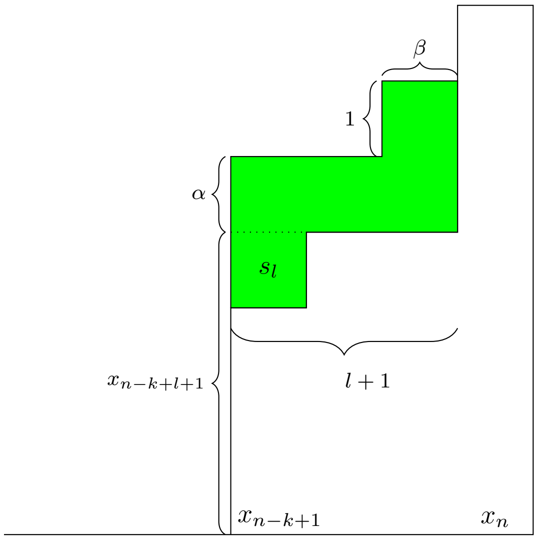

Figure 3: An example of calculating $\bar{x}$ for $x = (1, 2, 2, 3, 4, 4, 7)$ with $k = 4$.

It is easy to see that this defines $\bar{x}$ correctly, and that all these parameters can be computed in $O(n)$ time, if x is a nondecreasing vector. □

Remark 1 *Technically, only the computation of $s = \sum_{i=1}^{n-k} x_i$ depends on n. All other computations in the previous proof can be done in $O(k)$ time.*

Proposition 1 has the following consequence that we need to use in the sequel several times.

Corollary 3 *Any position x such that*

$$k\delta \le \sum_{i=1}^{n} \min(x_i, \delta) \quad \text{and} \quad \sum_{i=1}^{n} \min(x_i, \delta + 1) < k(\delta + 1) \tag{6}$$

has the tetris value of $\mathcal{T}_{\binom{[n]}{k}}(x) = \delta$.

Proof: The sequence $g(\delta) = \frac{1}{\delta} \sum_{i=1}^{n} \min(x_i, \delta)$ is monotone non-increasing for $\delta \in \mathbb{Z}_{\ge}$, and hence, the inqualities (6) can hold for only one δ. Proposition 1 implies that (6) holds for $\delta = \mathcal{T}_{\binom{[n]}{k}}(x) = \mathcal{T}_{\binom{[n]}{k}}(\bar{x})$. □

3.2.2 Polynomial computation of a move to a given tetris value in $\mathrm{NIM}_{\binom{[n]}{k}}$

Let us now consider moves from a given position x in the game $\mathrm{NIM}_{\binom{[n]}{k}}$. Given an integer value δ, we are interested to find efficiently a move $x \to x'$ such that $\mathcal{T}_{\binom{[n]}{k}}(x') = \delta$. First, let us note that this may not be possible for certain values of δ. By Lemma 6 the position x' that has the smallest such δ value is the one in which the largest k piles of x are reduced to zero. By Lemma 3 we can conclude that for every value $\mathcal{T}_{\binom{[n]}{k}}(x') \le \delta < \mathcal{T}_{\binom{[n]}{k}}(x)$ there exists a move $x \to x''$ such that $x \ge x'' \ge x'$ and $\mathcal{T}_{\binom{[n]}{k}}(x'') = \delta$.

Proposition 3 *Such an x'' can be determined in $O(n \log(\sum_{i=n-k+1}^{n} x_i))$ time.*

Proof: Let us start with $x^\ell = x'$ and x^u obtained from x by decreasing the largest k piles by one unit each. Then we have $\mathcal{T}_{\binom{[n]}{k}}(x^\ell) \le \delta \le \mathcal{T}_{\binom{[n]}{k}}(x^u)$. Using the monotonicity of the tetris function we perform a binary search in the space of positions between x^ℓ and x^u. In a general step we compute $L = \sum_{i=n-k+1}^{n} x_i^\ell$ and $U = \sum_{i=n-k+1}^{n} x_i^u$, set $M = \lfloor \frac{L+U}{2} \rfloor$ and compute $y_i = int(\frac{x_i^\ell + x_i^u}{2})$ for $i = n - k + 1$, where $int(\cdot)$ is a rounding to a nearest integer value in such a way that $\sum_{i=n-k+1}^{n} y_i = M$. Finally, we set $y_i = x_i$ for $i < n-k+1$. If $\mathcal{T}_{\binom{[n]}{k}}(y) < \delta$ then we replace x^ℓ by y, otherwise we replace x^u by y.

Clearly these computations can be done in each step in $O(n)$ time, and computing the tetris value of y can also be done in $O(n)$ time by Proposition 2. □

Remark 2 *Similarly to the proof of Proposition 2, we could improve the complexity of the above algorithm to $O(n)$.*

3.2.3 Tetris functions and degree sequences of graphs and hypergraphs

A related problem is the hypergraph realization of a given degree sequence. Let us fix $V = \{1, 2, \ldots, n\}$ as the set of vertices. A multi-hypergraph $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ is a family of subsets (called hyperedges) of V, i.e., $H_j \subseteq V$ for all $j = 1, \ldots, m$. We allow the same subset to appear multiple times in $\mathcal{H}$.

Given an integer vector $x \in \mathbb{Z}^n_{\geq}$, one can ask if there exists a k-uniform multi-hypergraph $\mathcal{H}$ on the vertex set V such that $d_{\mathcal{H}}(i) = x_i$ for all $i \in V$. Equivalently, we ask the existence of a bipartite graph $G = (X, Y, E)$ such that $|X| = n$, $|Y| = m$, $d_G(i) = x_i$ for all $i \in X$, and $d_G(j) = k$ for all $j \in Y$. For the latter we can apply the classical Gale-Ryser theorem [30, 37] claiming that the answer is yes if and only if

(*) $\sum_{i=1}^{n} x_i = km$

(**) $\sum_{i=1}^{n} \min(x_i, \delta) \geq k\delta$ for all $\delta = 1, \ldots, m$.

Let us note that checking these conditions may not be polynomial in x and k, since $m = \sum_{i=1}^{n} x_i/k$ according to (*). Let us also note that following a sequence of slow moves starting from position x, each time the set of columns that we decrease by 1 can be considered as a hyperedge of $\mathcal{H}$. Thus a maximal sequence of slow moves will construct $\mathcal{H}$ if the tetris function achieves its trivial bound $k\mathcal{T}_{\binom{[n]}{k}}(x) = \sum_{i=1}^{n} x_i$. This equality is in fact equivalent with (*), since we must have $m = \mathcal{T}_{\binom{[n]}{k}}(x)$ in this case. Our results in this section thus prove that for the above degree sequence realization problems the most efficient answer is to compute the tetris function value in linear time, and then compare it to its trivial upper bound. If these are the same then the answer is yes.

Havel (1955) [22] and Hakimi (1962) [20] provided a simple greedy algorithm based characterization for the recognition of degree sequences of bipartite graphs. For the above case their criterion states that x is a degree sequence of a k-uniform multi-hypergraph if and only if the position x' is, where x' is obtained from x by decreasing the k largest components of x by 1. Note that this implies a recursive process that is one of the definitions we used for the tetris function.

Let us remark finally that in general x is not the degree sequence of a k-uniform multi-hypergraph. In this case however a move $x \to x'$ such that $\mathcal{T}_{\binom{[n]}{k}}(x') = 0$ provides us with a minimal modification such that $x'' = x - x'$ becomes the degree sequence of such a hypergraph.

3.3 Other tricks for computing the tetris value

We have seen in Section 3.1 that caluclating the tetris value is not so straightforward. Tutte's algorithm can be time consuming and it is not very explicit. A much more explicit formula is in Lemma 5 , but the equality is mostly there for small graphs. We will see in Section 3.4 on that proving equality is not always trivial. There are also other much simpler ways to get a more explicit formula if the hypergraph has certain properties. In this section we will describe these simpler ways.

Lemma 9 *Let $\mathcal{H} \subset 2^V$ be a hypergraph with $V = \{1, 2, \ldots, n\}$. If $i < j, i \notin N_{\mathcal{H}}(j)$ and $N_{\mathcal{H}}(i) = N_{\mathcal{H}}(j)$ then the tetris value of $\mathcal{H}$ can be cacluated as*

$$\mathcal{T}_{\mathcal{H}}(x = (x_1, \ldots, x_n)) = \mathcal{T}_{\mathcal{H}\setminus\{j\}}(x_1, \ldots, x_{i-1}, x_i + x_j, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

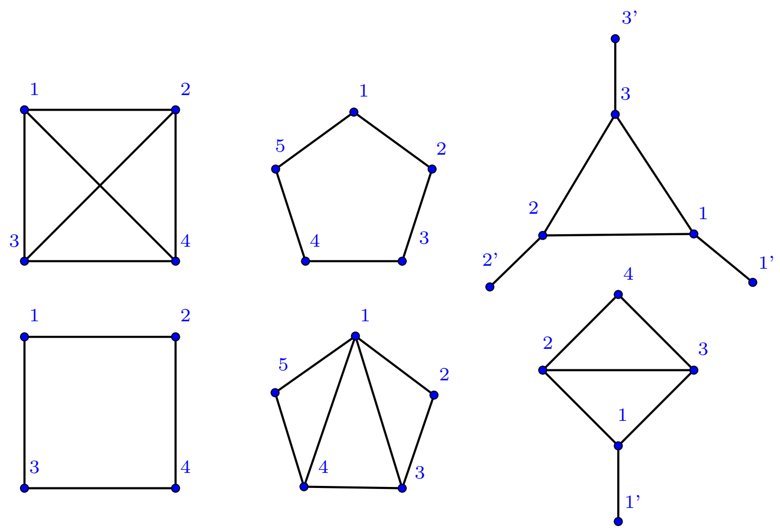

Figure 4: Graphs K_4, C_5, Net, C_4, Gem and Kite

Proof: Let $H_1 \in \mathcal{H}$ be a hyperedge with $i \in H_1$. Then by assumption there exist edge $H_2 = H_1 \setminus \{i\} \cup \{j\} \in \mathcal{H}$. Suppose H_2 is part of a tetris sequence of slow moves for $\mathcal{T}_{\mathcal{H}}(x)$. Then we can move a token from pile j to pile i and change a single occurence of H_2 to H_1. Hence the new position cannot have lower tetris value then the old one. To prove equality just reverse the argument. Same argument applies to a different choice of H_1 and H_2. And if j is not part of any tetris sequence of moves then the same must apply to i hence we can move tokens between the two without affecting the tetris value. The final hypergraph and position are simply obtained by moving all tokens from pile j to i. $\square$

Lemma 10 *Let $\mathcal{H} \subset 2^V$ be a hypergraph with $V = \{1, 2, \ldots, n\}$. If $i < j$ and $N_{\mathcal{H}}(j) = \{i\}$ then the tetris value $\mathcal{T}_{\mathcal{H}}(x_1, \ldots, x_n)$ can be calculated as*

$$\min(x_i, x_j) + \mathcal{T}_{\mathcal{H}\setminus j}(x_1, \ldots, x_{i-1}, x_i - \min(x_i, x_j), x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

Proof: Lets start with any tetris sequence of slow moves for $\mathcal{T}_{\mathcal{H}}(x)$. We know i must appear in at least $\min(x_i, x_j)$ edges otherwise we could add more edges $\{i, j\}$ and increase the tetris value. Now we can change some of the moves that include i but not j to be $\{i, j\}$ and we can do this until we have the edge $\{i, j\}$ appear $\min(x_i, x_j)$ times. This is also a tetris sequence of slow moves for $\mathcal{T}_{\mathcal{H}}(x)$. Hence when calculating $\mathcal{T}_{\mathcal{H}}(x)$ we can always first start of with the second sequence and then calulate the tetris value for the rest of the hypergraph. $\square$

Lemma 11 *Let $\mathcal{H} \supseteq \binom{[n]}{k}$ be a hypergraph. If all edges of $\mathcal{H}$ are of size $\geq k$ then $\mathcal{T}_{\mathcal{H}} = \mathcal{T}_{\binom{[n]}{k}}$.*

Proof: By Lemma 4 we have $\mathcal{T}_{\mathcal{H}} \geq \mathcal{T}_{\binom{[n]}{k}}$. To show $\mathcal{T}_{\mathcal{H}} \leq \mathcal{T}_{\binom{[n]}{k}}$ start with any tetris sequence of moves in $\mathcal{T}_{\mathcal{H}}(x)$. Since all edges are size $\geq k$ we can create another tetris sequence by reducing all such moves to any of their subsets of size k. Since all these edges are in $\binom{[n]}{k}$ this implies that this is a tetris sequence for $\mathcal{T}_{\binom{[n]}{k}}(x)$. $\square$

3.4 Tetris value examples

We give explicit formulas of the tetris function for certain hypergraphs and families of hypergraphs.

3.4.1 Transversal hypergraphs

Let $V = \{1, 2, \ldots, 2k\}$ be the vertex set of the hypergraph T_{2k} whose hyperedges consist of one element from each of the pairs $\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}$. We shall call this hypergraph a $2k$-Transversal hypergraph.

Lemma 12 *The tetris value of T_{2k} can be calculated as*

$$\mathcal{T}_{T_{2k}}(x) = \min(x_1 + x_2, x_3 + x_4, \ldots, x_{2k-1} + x_{2k})$$

Proof: Each slow move decreases $\min(x_1 + x_2, x_3 + x_4, \ldots, x_{2k-1} + x_{2k})$ by exactly one and if $\min(x_1 + x_2, x_3 + x_4, \ldots, x_{2k-1} + x_{2k}) = 0$ then there are no moves possible. □

3.4.2 C_4

The cycle graph on 4 vertices known as C_4 is defined as

$$C_4 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}.$$

Lemma 13 *The tetris value for C_4 can be easily calulated as*

$$\mathcal{T}_{C_4} = \min(x_1 + x_3, x_2 + x_4).$$

Proof: It easy to see C_4 is a Transversal graph, therefore we can use the formula from Lemma 12. □

3.4.3 C_5

The cycle graph on 5 vertices known as C_5 is defined as

$$C_5 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}.$$

Lemma 14 *The tetris value for C_5 can be easily calulated as*

$$\mathcal{T}_{C_5} = \min(x_1 + x_3 + x_4, x_2 + x_4 + x_5, x_3 + x_5 + x_1, x_4 + x_1 + x_2, x_5 + x_2 + x_3, \lfloor \frac{\sum_{i=1}^{5} x_i}{2} \rfloor)$$

Proof: The right hand side of the formula is an upper bound to $\mathcal{T}_{C_5}(x)$ by Lemma 5. Suppose $\mathcal{T}_{C_5}(x) < \lfloor \frac{\sum_{i=1}^{5} x_i}{2} \rfloor = \lfloor \frac{x_1+x_2+x_3+x_4+x_5}{2} \rfloor$. Since each slow move uses up 2 tokens out of the total $x_1 + x_2 + x_3 + x_4 + x_5$, then there must be at least 2 tokens left over after any tetris sequence of slow moves. We consider three cases.

(1) All leftover tokens are in a single pile. Without loss of generality say this pile is 1. We now claim that $\{2, 3\}$ and $\{4, 5\}$ cannot be both tetris moves. If they were, we can get a higher tetris value by removing one repetition of $\{2, 3\}$ and $\{4, 5\}$ and adding one repetition of $\{1, 2\}, \{5, 1\}, \{3, 4\}$. If $\{2, 3\}$ is not a tetris move then $x_5 + x_2 + x_3$ is indeed the tetris value since each tetris move decreases this value by 1 and all of their tokens were used (note $x_1 + x_4$ cannot be since we have leftover tokens at 1). And if $\{4, 5\}$ is not a tetris move then $x_2 + x_4 + x_5$ is the tetris value by same argument.

(2) The leftover tokens are distributed in two piles of (Hamming) distance 2, at least one in each pile. Without loss of generality let these piles be 5 and 2. Now we claim that $\{3, 4\}$ is not a tetris move. Otherwise we could increase the tetris value by removing a single repetition of $\{3, 4\}$ and adding a single repetition of $\{4, 5\}$ and $\{2, 3\}$. Since $\{3, 4\}$ is not a tetris move, then $x_1 + x_3 + x_4$ is equal to the tetris value.

(3) Any other distibution of leftover tokens automatically implies tetris value is larger that assumed, since leftover tokens would be in two adjacent piles and an additional move can be immediately found.

In cases (1) and (2) a different choice of the initial pile(s) simply leads to a rotated value $x_i + x_{i+2} + x_{i+3}$ which is in the lemma's formula. $\square$

3.4.4 The Net graph

The Net graph $\mathcal{H}_{net}$ is defined as

$$\mathcal{H}_{net} = \{\{1,1'\},\{2,2'\},\{3,3'\},\{1,2\},\{2,3\},\{1,3\}\}.$$

Lemma 15 *The tetris of the Net graph can be characterized as*

$$\begin{aligned} \mathcal{T}_{\mathcal{H}_{net}}(x) &= \min(x_1,x_{1'}) + \min(x_2,x_{2'}) + \min(x_3,x_{3'}) + \\ &+ \mathcal{T}_{\binom{[3]}{2}}(x_1 - \min(x_1,x_{1'}), x_2 - \min(x_2,x_{2'}), x_3 - \min(x_3,x_{3'})) \end{aligned}$$

Proof: Trim all three leaves $1', 2', 3'$ using Lemma 10 and the remainder of the graph is $\binom{[3]}{2}$. $\square$

3.4.5 The Kite graph

The Kite graph $\mathcal{H}_{kite}$ is defined as

$$\mathcal{H}_{kite} = \{\{1,1'\},\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\}\}\,.$$

Lemma 16 *The tetris value of the graph can be characterized as*

$$\mathcal{T}_{\mathcal{H}_{kite}}(x) = \min(x_1,x_{1'}) + \mathcal{T}_{\binom{[3]}{2}}(x_1 - \min(x_1,x_{1'}) + x_4, x_2, x_3).$$

Proof: Trim the leaf $1'$ using Lemma 10, then merge the remainder of x_1 and x_4 together using Lemma 9 and then we are left with the graph $\binom{[3]}{2}$.

3.4.6 The SG-decreasing graphs

For $\alpha, \beta, \gamma \in \mathbb{Z}_{\geq}$, let G be a graph like in Figure 6 with

$$G = \left\{ \{u,v\} \cup \bigcup_{i=1}^{\alpha}\{u,u_i\} \cup \bigcup_{i=1}^{\beta}\{v,v_i\} \cup \bigcup_{i=1}^{\gamma}\{u,w_i\} \cup \bigcup_{i=1}^{\gamma}\{v,w_i\} \right\}.$$

Lemma 17 *The tetris value for G can be calculated as*

$$\begin{aligned} \mathcal{T}_G(x) &= \min(\sum_{i=1}^{\alpha} x_{u_i}, x_u) + \min(\sum_{i=1}^{\beta} x_{v_i}, x_v) + \\ &+ \mathcal{T}_{\binom{[3]}{2}}(x_u - \min(\sum_{i=1}^{\alpha} x_{u_i}, x_u), x_v - \min(\sum_{i=1}^{\beta} x_{v_i}, x_v), \sum_{i=1}^{\gamma} x_{w_i}) \end{aligned}$$

Proof: We can apply Lemma 9 and merge all the u_i into a single vertex u_0. Repeat the process to merge all v_i into v_0 and all w_i into w_0. We obtain a new graph $G' = \{\{u,v\},\{u,u_0\},\{v,v_0\},\{u,w_0\},\{v,w_0\}\}$. Then we can trim the leaves u_0, v_0 by Lemma 10, and we are left with the graph $\binom{[3]}{2}$. $\square$

Note that if we have a singleton on say u the formula can be simplified to $x_u + \min(x_v, \sum_{i=1}^{\beta} x_{v_i} + \sum_{i=1}^{\gamma} x_{w_i})$ and if both have singletons then its just $x_u + x_v$.

4 $\mathcal{H}$ combinations and SG-decreasing hypergraphs

Given a hypergraph $\mathcal{H} \subseteq 2^V$, $\text{NIM}_\mathcal{H}$ can also be defined as the $\mathcal{H}$-combination of n single pile NIM_1 games. The family of $\text{NIM}_\mathcal{H}$ games is closed under hypergraph combinations. In fact this operation can be extended for hypergraphs in the following way. Let $\mathcal{F}_i \subseteq 2^{U_i}$, $i \in V$ be hypergraphs such that U_i, $i \in V$ are pairwise disjoint. Then for a hypergraph $\mathcal{H} \subseteq 2^V$ we define the $\mathcal{H}$-combination of $\mathcal{F}_i$, $i \in V$ by

$$\mathcal{F} = \left\{ \bigcup_{i \in H} F_i \;\middle|\; H \in \mathcal{H} \text{ and } F_i \in \mathcal{F}_i \text{ for all } i \in H \right\}.$$

Note that $\text{NIM}_\mathcal{F}$ is the $\mathcal{H}$-combination of the games $\text{NIM}_{\mathcal{F}_i}$, $i \in V$.

Let us recall the Sprague-Grundy theorem, which tells us that for any position $x = (x_1, \ldots, x_n)$, $x_i \in X_i$ for $i \in V$ and $\mathcal{H} = \{\{1\}, \{2\}, \ldots, \{n\}\}$ we have the equality:

$$\mathcal{G}_{\Gamma_\mathcal{H}}(x) \;=\; \mathcal{G}_{NIM_\mathcal{H}}\left(\mathcal{G}_{\Gamma_1}(x_1), \ldots, \mathcal{G}_{\Gamma_n}(x_n)\right). \tag{7}$$

In this thesis we would like to find other classes of games for which Equality (7) holds.

A game Γ is called *SG-decreasing* if the SG value is strictly decreased by every move. The single pile NIM_1 is the simplest example of an SG-decreasing game. Other examples are $\text{NIM}^{=}_{n,k}$ for $n < 2k$, [9]. We call a hypergraph $\mathcal{H}$ *SG-decreasing* if the game $\text{NIM}_\mathcal{H}$ is SG-decreasing.

Theorem 1 *Equation* (7) *holds for an arbitrary hypergraph $\mathcal{H} \subseteq 2^V$ and SG-decreasing games Γ_i, $i \in V$.*

Let us remark that Equation (7) does not always hold. Consider, for example, $\Gamma_1 = NIM_1$, $\Gamma_2 = NIM_2$ and $\mathcal{H} = \{\{1, 2\}\}$. In this case Γ_2 is not SG-decreasing and Equation (7) may fail. Indeed, if we consider the position $x = (1, (1, 1))$ of the compound game then the right hand side is 0, while the left hand side is 1.

While computing the SG function for games seems to be very hard, in general, the above theorem allows us to outline new cases when the problem is tractable.

If we replace in Equation (7) the SG function by the Tetris function we always get equality.

Theorem 2 *Given games $\Gamma_i = (X_i, E_i)$, $i \in V$, and a hypergraph $\mathcal{H} \subseteq 2^V$ we have the equality*

$$\mathcal{T}_{\Gamma_\mathcal{H}}(x) \;=\; \mathcal{T}_{NIM_\mathcal{H}}\left(\mathcal{T}_{\Gamma_1}(x_1), \ldots, \mathcal{T}_{\Gamma_n}(x_n)\right), \tag{8}$$

for $x = (x_1, \ldots, x_n)$, where $x_i \in X_i$ for $i \in V$.

The above two theorems immediately imply the following statement.

Corollary 4 *If $\mathcal{H}$ is an SG-decreasing hypergraph then the $\mathcal{H}$-combination of SG-decreasing games is SG-decreasing. In particular, the $\mathcal{H}$-combination of SG-decreasing hypergraphs is SG-decreasing.*

While recognizing if a given hypergraph is SG-decreasing is a hard decision problem, we can provide a necessary and sufficient condition for hypergraphs of dimension at most 3.

Theorem 3 *A hypergraph $\mathcal{H}$ of dimension at most* 3 *is SG-decreasing if and only if*

$$\forall S \subseteq V \quad \textit{with} \quad \mathcal{H}_S \neq \emptyset \quad \exists H \in \mathcal{H}_S \quad \textit{such that} \quad \forall H' \in \mathcal{H}_S \;:\; H \cap H' \neq \emptyset. \tag{9}$$

Furthermore, this condition can be tested in polynomial time for hypergraphs of any fixed dimension.

Let us remark that computing the Tetris value (as well as the SG) function values for $\text{NIM}_\mathcal{H}$ is an NP-hard problem, even for hypergraphs of dimension at most 3. Even under Condition (9), the problem remains NP-hard for hypergraphs of dimension 4 or larger. While computing the Tetris value of a position is polynomial for hypergraphs of dimension 2, the problem is still open for hypergraphs of dimension 3 under Condition (9).

4.1 Hypergraph Combinations of Games

Given games $\Gamma_i = (X_i, E_i)$, $i \in V = \{1, \ldots, n\}$, and a hypergraph $\mathcal{H} \subseteq 2^V$, we define the $\mathcal{H}$-combination $\Gamma_\mathcal{H} = (X, E)$ of these games by setting

$$X = \prod_{i \in V} X_i, \text{ and}$$

$$E = \left\{ (x, x') \in X \times X \;\middle|\; \exists\, H \in \mathcal{H} \text{ such that } \begin{array}{ll} (x_i, x'_i) \in E_i & \forall i \in H, \\ x_i = x'_i & \forall i \notin H \end{array} \right\}.$$

Let us remark that hypergraph combinations generalize conjunctive and selective compounds [16, 31]. Let us denote by $\mathcal{P}_\mathcal{H}$ the set of P-positions of $\text{NIM}_\mathcal{H}$.

Proof of Theorem 1

Consider the $\mathcal{H}$-combination $\Gamma_\mathcal{H} = (X, E)$ as defined above and recall that $X = X_1 \times \cdots \times X_n$. We show that the function defined by the right hand side of Equation (7) satisfies the defining properties of the SG function.

First, consider a position $x = (x_1, \ldots, x_n) \in X$ and denote by $g(x) = (\mathcal{G}_{\Gamma_i}(x_i) \mid i \in V) \in \mathbb{Z}_\geq^V$ the vector of SG values in the n given games. Notice that $g(x) \in \mathbb{Z}_\geq^V$ is a position in the game $\text{NIM}_\mathcal{H}$ for every $x \in X$.

Let us define a function $f : X \to \mathbb{Z}_\geq$ by

$$f(x_1, \ldots, x_n) = \mathcal{G}_\mathcal{H}(g(x)).$$

Consider first a move $(x, x') \in E$, where $(x_i, x'_i) \in E_i$ for $i \in H$ for some hyperedge $H \in \mathcal{H}$. By the definition of E we must have $x'_i = x_i$ for all $i \notin H$. Denote by $g' \in \mathbb{Z}_\geq^V$ the corresponding vector of SG values. Note that $g'_i < g_i$ for $i \in H$, since Γ_i is an SG-decreasing game for all $i \in V$, and $g'_i = g_i$ for all $i \notin H$ since $x_i = x'_i$ for these indices. Consequently, $g \to g'$ is a move in $\text{NIM}_\mathcal{H}$ and therefore $f(x) = \mathcal{G}_\mathcal{H}(g) \neq \mathcal{G}_\mathcal{H}(g') = f(x')$. Thus, we proved that every move in $\Gamma_\mathcal{H}$ changes the value of function f.

Next, let us consider an integer $0 \leq v < f(x)$. We are going to show that there exists a move $x \to x'$ in $\Gamma_\mathcal{H}$ such that $f(x') = v$. Let us consider again the corresponding integer vector $g = g(x) \in \mathbb{Z}_\geq^V$, for which we have $f(x) = \mathcal{G}_\mathcal{H}(g)$. By the definition of the SG function of $\text{NIM}_\mathcal{H}$, there must exists a move $g \to g'$ such that $\mathcal{G}_\mathcal{H}(g') = v$. Assume that this move is an H-move for some $H \in \mathcal{H}$, that is that $g'_i < g_i$ for $i \in H$ and $g'_i = g_i$ for $i \notin H$. Then, $\mathcal{G}_{\Gamma_i}(x_i) = g_i > g'_i$ for all $i \in H$ and, thus, we must have moves $x_i \to x'_i$ in Γ_i, $i \in H$, such that $\mathcal{G}_{\Gamma_i}(x'_i) = g'_i$ for all $i \in H$. Then with $x'_i = x_i$ for $i \notin H$, we get a move $x \to x'$ in the $\mathcal{H}$-combination such that $f(x') = v$. Thus, each smaller SG value can be realized by a move in the combination game.

These arguments can be completed by a simple induction to show that $\mathcal{G}_{\Gamma_\mathcal{H}}(x) = f(x)$ for all $x \in X$. □

Since the Tetris function $\mathcal{T}$ is defined as the length of a longest path in the directed graph of the game, $\mathcal{T}$ is uniquely characterized by the following three properties:

(a) Every move decreases its value.

(b) If $\mathcal{T}$ is positive in a position then there exists a move from this position that decreases $\mathcal{T}$ by exactly one.

(c) In each terminal position $\mathcal{T}$ takes value zero.

Proof of Theorem 2

Similarly to the proof of Theorem 1, we shall show that the function defined by the right hand side of Equation (8) satisfies the above properties (a), (b) and (c).

Consider a position $x = (x_1, \ldots, x_n) \in X$ and denote by $t(x) = (\mathcal{T}_{\Gamma_i}(x_i) \mid i \in V) \in \mathbb{Z}_\geq^V$ the vector of Tetris values in these n games. Notice that t is a position in the game $\text{NIM}_\mathcal{H}$. Let us denote by

$$f(x_1, \ldots, x_n) = \mathcal{T}_\mathcal{H}(t(x))$$

the function defined by the right hand side of Equation (8).

Consider first a move $(x, x') \in E$, where $(x_i, x_i') \in E_i$ for $i \in H$ for some hyperedge $H \in \mathcal{H}$. By the definition of E we must have $x_i' = x_i$ for all $i \notin H$. Denote by $t' \in \mathbb{Z}_{\geq}^V$ the corresponding vector of Tetris values, and note that $t_i' < t_i$ for $i \in H$ since T_{Γ_i} satisfies property (a) for all $i \in V$, and $t_i' = t_i$ for all $i \notin H$ since $x_i = x_i'$ for these indices. Consequently, $t \to t' = t(x')$ is a move in $\mathrm{NIM}_{\mathcal{H}}$, and therefore $f(x) = \mathcal{T}_{\mathcal{H}}(t(x)) > \mathcal{T}_{\mathcal{H}}(t(x')) = f(x')$, since $\mathcal{T}_{\mathcal{H}}$ satisfies property (a). Thus we proved that every move in $\Gamma_{\mathcal{H}}$ decreases the value of function f.

Consider next an arbitrary position $x \in X$ such that $0 < f(x) = \mathcal{T}_{\mathcal{H}}(t(x))$. Since $\mathcal{T}_{\mathcal{H}}$ satisfies property (b), there exists a move $t(x) \to t'$ in $\mathrm{NIM}_{\mathcal{H}}$ such that $\mathcal{T}_{\mathcal{H}}(t') = \mathcal{T}_{\mathcal{H}}(t(x)) - 1$. Then, by the definition of $\mathrm{NIM}_{\mathcal{H}}$ we must have $H = \{i \in V \mid t_i > t_i'\} \in \mathcal{H}$. Since $\mathcal{T}_{\Gamma_i}$ satisfies property (b), there must exist moves $x_i \to x_i'$ such that $\mathcal{T}_{\Gamma_i}(x_i') = \mathcal{T}_{\Gamma_i}(x_i) - 1 = t_i - 1$ for $i \in H$. Define $x_i' = x_i$ for $i \notin H$. Then we have $\mathcal{T}_{\mathcal{H}}(t(x)) - 1 \geq \mathcal{T}_{\mathcal{H}}(t(x')) \geq \mathcal{T}_{\mathcal{H}}(t')$ by the definition of $\mathrm{NIM}_{\mathcal{H}}$. Consequently we have $f(x') = f(x) - 1$.

Finally, to see property (c), let us consider a terminal position $x \in X$ and its corresponding Tetris vector $t(x)$. By the definition of $\mathrm{NIM}_{\mathcal{H}}$ this is a terminal position if and only if $\{i \in V \mid t_i = 0\}$ intersects all hyperedges of $\mathcal{H}$, in which case we must have $f(x) = T_{\mathcal{H}}(t(x)) = 0$. □

4.2 A necessary condition for $\mathcal{T}_{\mathcal{H}} = \mathcal{G}_{\mathcal{H}}$

Let us start by observing that for every game $\Gamma = (X, E)$ and position $x \in X$ we have the inequality

$$\mathcal{G}_{\Gamma}(x) \;\leq\; \mathcal{T}_{\Gamma}(x). \tag{10}$$

Let us continue with some basic properties of SG-decreasing hypergraphs.

Lemma 18 *Given a hypergraph $\mathcal{H} \subseteq 2^V$, the following three statements are equivalent:*

(i) *$\mathcal{H}$ is a SG-decreasing hypergraph;*

(ii) $\mathcal{G}_{\mathcal{H}} = \mathcal{T}_{\mathcal{H}}$;

(iii) *for all positions $x \in \mathbb{Z}_{\geq}^V$ and for all integers $0 \leq v < \mathcal{T}_{\mathcal{H}}(x)$ we have a move $x \to x'$ in $\mathrm{NIM}_{\mathcal{H}}$ such that $\mathcal{T}_{\mathcal{H}}(x') = v$.*

Proof: These equivalences follow directly from the definitions of SG-decreasing hypergraphs, Tetris and SG functions, and SG-decreasing games. □

Let us associate to a hypergraph $\mathcal{H} \subseteq 2^V$ the set of positions $\mathbb{Z}_{\mathcal{H}} \subseteq \mathbb{Z}_{\geq}^V$ which have zero Tetris value:

$$\mathbb{Z}_{\mathcal{H}} \;=\; \{x \in \mathbb{Z}_{\geq}^V \mid T_{\mathcal{H}}(x) = 0\}.$$

Obviously, we have

$$\mathbb{Z}_{\mathcal{H}} \;\subseteq\; \mathcal{P}_{\mathcal{H}}, \tag{11}$$

since there is no move from x by the definition of the Tetris function. We shall show next that in fact all P-positions of $\mathrm{NIM}_{\mathcal{H}}$ are in $\mathbb{Z}_{\mathcal{H}}$ if and only if Condition (9) holds.

Theorem 4 *Given a hypergraph $\mathcal{H} \subseteq 2^V$, $\emptyset \notin \mathcal{H}$, we have $\mathbb{Z}_{\mathcal{H}} = \mathcal{P}_{\mathcal{H}}$ if and only if $\mathcal{H}$ satisfies Condition (9).*

Proof: We always have $\mathbb{Z}_{\mathcal{H}} \subseteq \mathcal{P}_{\mathcal{H}}$ by Formula (11).

Assume first that we also have $\mathcal{P}_{\mathcal{H}} \subseteq \mathbb{Z}_{\mathcal{H}}$, and consider a subset $S \subseteq V$ such that $\mathcal{H}_S \neq \emptyset$. For a position $x \in \mathbb{Z}_{\geq}^V$ we denote by $supp(x) = \{i \mid x_i > 0\}$ the set of its support. Let us then choose a position $x \in \mathbb{Z}_{\geq}^V$ such that $supp(x) = S$. Since $\mathcal{H}_S \neq \emptyset$, we have $\mathcal{T}_{\mathcal{H}}(x) > 0$ implying $\mathcal{G}_{\mathcal{H}}(x) > 0$ by our assumption. Then, by the definition of the SG function we must have a hyperedge $H' \in \mathcal{H}$ and an H'-move $x \to x'$ such that $0 = \mathcal{G}_{\mathcal{H}}(x') \geq \mathcal{G}_{\mathcal{H}}(x^{f(H')})$, implying again by our assumption that $\mathcal{T}_{\mathcal{H}}(x^{f(H')}) = 0$. Thus, $H' \subseteq supp(x) = S$ must intersect all hyperedges of $\mathcal{H}_S$. Since this argument works for an arbitrary subset $S \subseteq V$ with $\mathcal{H}_S \neq \emptyset$, property (9) follows.

For the other direction assume $\mathcal{H}$ satisfies property (9), and consider a position $x \in \mathbb{Z}_{\geq}^V$ for which $\mathcal{T}_{\mathcal{H}}(x) > 0$. Then $\mathcal{H}_{supp(x)} \neq \emptyset$, and thus, by property (9) we have a hyperedge

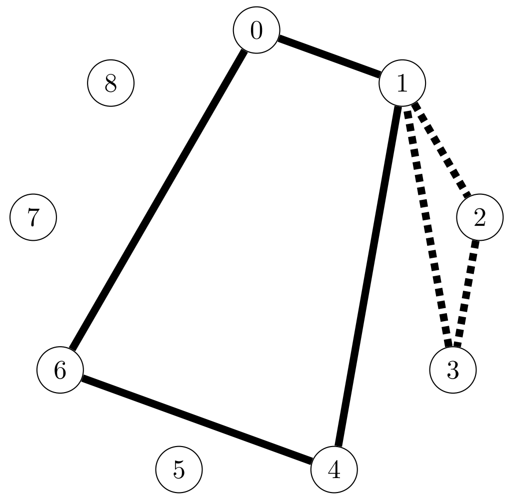

Figure 5: A hypergraph $\mathcal{H}$ on the ground set $V = \mathbb{Z}_9$, with hyperedges $T_i = \{i, i+1, i+2\}$ and $F_i = \{i, i+1, i+4, i+6\}$ for $i \in \mathbb{Z}_9$, where additions are modulo 9, that is, $\mathcal{H} = \{T_i,\ F_i \mid i \in \mathbb{Z}_9\}$. The figure shows T_1 (dotted) and F_0 (solid.) This hypergraph satisfies (9), yet, it is not SG-decreasing.

$H \in \mathcal{H}_{supp(x)}$ that intersects all hyperedges of this induced subhypergraph, that is, for which $\mathcal{T}_{\mathcal{H}}(x^{f(H)}) = 0$, implying $\mathcal{G}_{\mathcal{H}}(x^{f(H)}) = 0$ by (10). Since $x \to x^{f(H)}$ is an H-move, $\mathcal{G}_{\mathcal{H}}(x) \neq 0$ is implied by the definition of the SG function. Since this follows for all positions x with $\mathcal{T}_{\mathcal{H}}(x) > 0$, we conclude that $\mathcal{P}_{\mathcal{H}} \subseteq \mathbb{Z}_{\mathcal{H}}$, as claimed. □

Corollary 5 *Condition* (9) *is necessary for a hypergraph to be SG-decreasing.* □

The following example demonstrates that Condition (9) alone is not enough, generally, to guarantee that a hypergraph is SG-decreasing, or equivalently by (ii) of Lemma 18, to ensure the equality of the SG and Tetris functions.

Lemma 19 *The hypergraph $\mathcal{H}$ in Figure 5 satisfies condition* (9)*, but does not satisfy the equality $\mathcal{G}_{\mathcal{H}} = \mathcal{T}_{\mathcal{H}}$.*

Proof: To see this, let us set T_j and F_j for $j \in \mathbb{Z}_9$ as in the caption of Figure 5, where additions are modulo 9. Let us observe first that

$$T_j \cap F_i \neq \emptyset \text{ and } F_j \cap F_i \neq \emptyset \text{ for all } i, j \in \mathbb{Z}_9.$$

An easy analysis shows that $\mathcal{H}$ satisfy condition (9).

On the other hand, for the position $x = (1, 1, \ldots, 1) \in \mathbb{Z}_+^V$ we have $\mathcal{T}_{\mathcal{H}}(x) = 3$. Furthermore, $\mathcal{T}_{\mathcal{H}}(x - \chi(T_j)) = 2$ and $\mathcal{T}_{\mathcal{H}}(x - \chi(F_j)) = 0$ for all $j \in \mathbb{Z}_9$. Thus, there exists no move $x \to x'$ with $\mathcal{T}_{\mathcal{H}}(x') = 1$, which by (iii) of Lemma 18 implies that $\mathcal{G}_{\mathcal{H}} \neq \mathcal{T}_{\mathcal{H}}$. □

Note that the above hypergraph is of dimension 4. In Section 4.3 we show that there are no such examples among the hypergraphs of dimension at most 3, as claimed in Theorem 3. It is also interesting to note that for hypergraphs of dimension 2, Condition (9) can be substantially simplified.

Lemma 20 *Assume that $\mathcal{H} \subseteq 2^V$ is a hypergraph of* $\dim(\mathcal{H}) = 2$ *and such that it has at least one edge $H \in \mathcal{H}$ with $|H| = 2$. Then $\mathcal{H}$ satisfies Condition* (9) *if and only if there is a hyperedge $H \in \mathcal{H}$ such that $H \cap H' \neq \emptyset$ for all $H' \in \mathcal{H}$.*

Proof: Figure 6 below shows the possible structure of such (hyper)graphs. On the left H is a singleton (black thin circle), while on the right it is a 2-element set (black thin edge). Circles in both pictures indicate possible singletons (1-element hyperedges.) □

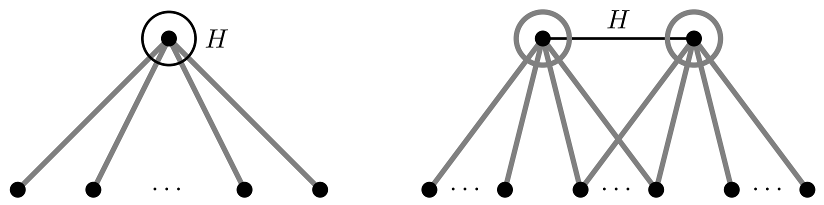

Figure 6: The SG-decreasing graphs

4.3 A sufficient condition for $\mathcal{T}_{\mathcal{H}} = \mathcal{G}_{\mathcal{H}}$

Given a hypergraph $\mathcal{H} \subseteq 2^V$, and a position $x \in \mathbb{Z}_{\geq}^V$, let us call an integer vector $m \in \mathbb{Z}_{\geq}^{\mathcal{H}}$ an x-vector, if

$$\sum_{H \in \mathcal{H}} m_H \chi(H) \ \leq \ x \quad \text{and} \quad \sum_{H \in \mathcal{H}} m_H \ = \ \mathcal{T}_{\mathcal{H}}(x). \tag{12}$$

Let us denote by $M(x) \subseteq \mathbb{Z}_{\geq}^{\mathcal{H}}$ the family of x-vectors. Let us further define

$$\mathcal{H}^{x-pack} \ = \ \{H \in \mathcal{H} \mid \exists m \in M(x) \text{ s.t. } m_H > 0\}, \tag{13}$$

that is $\mathcal{H}^{x-pack}$ is the subfamily of $\mathcal{H}$ of those hyperedges that participate with a positive multiplicity in some maximal $\mathcal{T}_{\mathcal{H}}(x)$-packing of $\mathcal{H}$. Every vector $m \in M(x)$ corresponds to such a maximal $\mathcal{T}_{\mathcal{H}}(x)$-packing of $\mathcal{H}$.

Let us consider $\mathcal{H}_{\mathrm{supp}(x)}$ the subhypergraph induced by the support of x, and define a subhypergraph of $\mathcal{H}_{\mathrm{supp}(x)}$ as

$$\mathcal{H}^{x-all} \ = \ \{H \in \mathcal{H}_{\mathrm{supp}(x)} \mid \forall H' \in \mathcal{H}_{\mathrm{supp}(x)} : H \cap H' \neq \emptyset\}, \tag{14}$$

consisting of those hyperedges that intersect all others in this subhypergraph.

Lemma 21 *Consider a hypergraph $\mathcal{H} \subseteq 2^V$ that satisfies condition (9), and a position $x \in \mathbb{Z}_{\geq}^V$ such that $\mathcal{T}_{\mathcal{H}}(x) > 0$. Then we have $\mathcal{H}^{x-all} \neq \emptyset$ and $\mathcal{H}^{x-pack} \neq \emptyset$. Furthermore, we have*

(i) $\mathcal{T}_{\mathcal{H}}(x) \ > \ \mathcal{T}_{\mathcal{H}}(x^{s(H)}) \ \geq \ \mathcal{T}_{\mathcal{H}}(x^{f(H)}) \ \geq \ 0$ *for all* $H \in \mathcal{H}_{\mathrm{supp}(x)}$;

(ii) $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) \ \geq \ \mathcal{T}_{\mathcal{H}}(x) - |H|$ *for all* $H \in \mathcal{H}_{\mathrm{supp}(x)}$;

(iii) $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) \ = \ \mathcal{T}_{\mathcal{H}}(x) - 1$ *for all* $H \in \mathcal{H}^{x-pack}$;

(iv) $\mathcal{T}_{\mathcal{H}}(x^{f(H)}) \ = \ 0$ *if and only if* $H \in \mathcal{H}^{x-all}$;

(v) $\mathcal{T}_{\mathcal{H}}(x) \ \geq \ \mathcal{T}_{\mathcal{H}}(x - \chi(\{k\})) \ \geq \ \mathcal{T}_{\mathcal{H}}(x) - 1$ *for all* $k \in \mathrm{supp}(x)$.

Proof: Trivial by the definitions. □

We are now ready to prove Theorem 3.

Proof of Theorem 3.

By Corollary 5, condition (9) is necessary for a hypergraph of any dimension to be SG-decreasing. We prove next that for hypergraphs of dimension at most 3 condition (9) is also sufficient.

Assume to the contrary that there exists a hypergraph $\mathcal{H} \subseteq 2^V$ of $\dim(\mathcal{H}) \leq 3$ satisfying condition (9) such that $\mathcal{G}_{\mathcal{H}} \neq \mathcal{T}_{\mathcal{H}}$. By Lemma 18 this implies the existence of a position $x \in \mathbb{Z}_{\geq}^V$ and a value $\mathcal{T}_{\mathcal{H}}(x) > v \geq 0$ such that there exists no move $x \to x'$ with $\mathcal{T}_{\mathcal{H}}(x') = v$. Since condition (9) applies to all induced subhypergraphs, we can assume without any loss of generality that

$$V = \mathrm{supp}(x). \tag{15}$$

Then, by Lemma 3 it follows that for all $H \in \mathcal{H}$ we must have

$$\begin{aligned} &\text{either} && \mathcal{T}_{\mathcal{H}}(x^{s(H)}) \leq v - 1, && \text{(16a)} \\ &\text{or} && \mathcal{T}_{\mathcal{H}}(x^{f(H)}) \geq v + 1. && \text{(16b)} \end{aligned}$$

By (i) of Lemma 21 we cannot have both (16a) and (16b) hold for a hyperedge $H \in \mathcal{H}$. Thus, the above defines a unique partition of the hyperedges of $\mathcal{H}$:

$$\begin{aligned}\mathcal{H}_1 &= \{H \in \mathcal{H} \mid \mathcal{T}_{\mathcal{H}}(x^{s(H)}) \leq v-1\} \quad \text{and} \\ \mathcal{H}_2 &= \{H \in \mathcal{H} \mid \mathcal{T}_{\mathcal{H}}(x^{f(H)}) \geq v+1\}.\end{aligned} \tag{17}$$

Thus, for $H \in \mathcal{H}_1$ we get by (ii) of Lemma 21 that

$$\mathcal{T}_{\mathcal{H}}(x) - 3 \;\leq\; \mathcal{T}_{\mathcal{H}}(x^{s(H)}) \;\leq\; v-1,$$

while for $H \in \mathcal{H}_2$ we get by (i) and (iii) of Lemma 21 that

$$\mathcal{T}_{\mathcal{H}}(x) - 1 \;\geq\; \mathcal{T}_{\mathcal{H}}(x^{s(H)}) \;\geq\; \mathcal{T}_{\mathcal{H}}(x^{f(H)}) \;\geq\; v+1.$$

These inequalities together imply that we must have $v = \mathcal{T}_{\mathcal{H}}(x) - 2 > 0$, and that

$$\text{for } H \in \mathcal{H}_1 \text{ we have} \qquad \mathcal{T}_{\mathcal{H}}(x^{s(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 3, \text{ and} \tag{18a}$$

$$\text{for } H \in \mathcal{H}_2 \text{ we have} \qquad \mathcal{T}_{\mathcal{H}}(x^{f(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1. \tag{18b}$$

The next series of claims help us to prove that we must have $\mathcal{T}_{\mathcal{H}}(x) = 3$, and that we have $x_i = 1$ for all $i \in H \in \mathcal{H}_1$.

Lemma 22 *We have $\mathcal{H}^{x-all} \subseteq \mathcal{H}_1$.*

Proof: For all $H \in \mathcal{H}^{x-all}$ we have by definition $\mathcal{T}_{\mathcal{H}}(x^{f(H)}) = 0 < \mathcal{T}_{\mathcal{H}}(x) - 1$. Thus, $H \in \mathcal{H}_1$ follows. □

Lemma 23 *For all $H \in \mathcal{H}_1$ we have $|H| = 3$.*

Proof: The claim follows by the definition of $\mathcal{H}_1$, (ii) of Lemma 21, and the assumption that $\dim(\mathcal{H}) \leq 3$. □

Lemma 24 *We have $\mathcal{H}^{x-pack} = \mathcal{H}_2$.*

Proof: By definition, for all $H \in \mathcal{H}^{x-pack}$ we have $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1 > \mathcal{T}_{\mathcal{H}}(x) - 3$, implying $H \in \mathcal{H}_2$. For $H \in \mathcal{H}_2$ by (i) of Lemma 21 it follows that $\mathcal{T}_{\mathcal{H}}(x) > \mathcal{T}_{\mathcal{H}}(x^{s(H)}) \geq \mathcal{T}_{\mathcal{H}}(x^{f(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$, implying $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$. Let us choose an arbitrary $m \in M(x^{s(H)})$ and define $m'_H = m_H + 1$ and $m'_{H'} = m_{H'}$ for all $H' \neq H$. Then we have $m' \in M(x)$ and $m'_H > 0$ implying $H \in \mathcal{H}^{x-pack}$ by (13). □

Lemma 25 *For all $m \in M(x)$ and $H \in \mathcal{H}$ we have $m_H \leq 1$.*

Proof: If $m_H \geq 2$ for some $H \in \mathcal{H}$ then for position $x' = x - 2\chi(H)$ we have that $\mathcal{T}_{\mathcal{H}}(x') = \mathcal{T}_{\mathcal{H}}(x) - 2$ and $x \to x'$ is a move, contradicting our assumption that there exists no such move. □

Lemma 26 *For all $H_1 \in \mathcal{H}^{x-all}$ and $H_2 \in \mathcal{H}^{x-pack} (= \mathcal{H}_2)$ we have $|H_1 \cap H_2| = 1$.*

Proof: Let us assume to the contrary that $|H_1 \cap H_2| \geq 2$. By Lemma 22 we have that $|H_1| = 3$. Assume w.l.o.g. that $H_1 = \{i,j,k\}$ and $\{i,j\} \subseteq H_2$. Let us then define position x' by $x'_\ell = x_\ell$ for $\ell \notin \{i,j\}$ and $x'_\ell = x_\ell - 1$ for $\ell \in \{i,j\}$. Then we have $x' \geq x^{s(H_2)}$, implying

$$\mathcal{T}_{\mathcal{H}}(x') \geq \mathcal{T}_{\mathcal{H}}(x^{s(H_2)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$$

by the monotonicity of $\mathcal{T}_{\mathcal{H}}$, (iii) of Lemma 21, and Lemma 24. Furthermore, we have $x' - \chi(\{k\}) \leq x^{s(H_1)}$ implying by (v) of Lemma 21 that

$$\mathcal{T}_{\mathcal{H}}(x') - 1 \;\leq\; \mathcal{T}_{\mathcal{H}}(x' - \chi(\{k\})) \;\leq\; \mathcal{T}_{\mathcal{H}}(x^{s(H_1)}).$$

From the above $\mathcal{T}_{\mathcal{H}}(x^{s(H_1)}) \geq \mathcal{T}_{\mathcal{H}}(x)-2$ follows, contradicting (18a). This contradiction proves that we must have $|H_1 \cap H_2| \leq 1$, while the definition of $\mathcal{H}^{x-all}$ implies $H_1 \cap H_2 \neq \emptyset$, concluding the proof of our claim. $\Box$

For a multiplicity vector $m \in M(x)$ let us associate the corresponding position $x(m)$ defined by

$$x(m) \;=\; \sum_{H \in \mathcal{H}} m(H)\chi(H). \tag{19}$$

Lemma 27 *For all $m \in M(x)$ and $i \in H^* \in \mathcal{H}^{x-all}$ we have $x(m)_i = x_i$.*

Proof: Clearly, we must have $x(m) \leq x$ for all $m \in M(x)$, by the definition of $M(x)$. Assume to the contrary that there exists $m \in M(x)$ an index $i \in H^* = \{i,j,k\}$ such that $x(m)_i < x_i$. Then we have $x(m) \leq x - \chi(\{i\})$, implying by (v) of Lemma 21 that

$$\mathcal{T}_{\mathcal{H}}(x) \;\geq\; \mathcal{T}_{\mathcal{H}}(x-\chi(\{i\})) \;\geq\; \sum_{H \in \mathcal{H}} m(H) \;=\; \mathcal{T}_{\mathcal{H}}(x),$$

from which $\mathcal{T}_{\mathcal{H}}(x-\chi(\{i\})) = \mathcal{T}_{\mathcal{H}}(x)$ follows. Thus, again by (v) of Lemma 21, we would get

$$\mathcal{T}_{\mathcal{H}}(x^{s(H^*)}) \;=\; \mathcal{T}_{\mathcal{H}}((x-\chi(\{i\}))-\chi(\{j,k\})) \;\geq\; \mathcal{T}_{\mathcal{H}}(x-\chi(\{i\}))-2 \;=\; \mathcal{T}_{\mathcal{H}}(x)-2,$$

contradicting (18a) and Lemma 22. This contradiction proves our claim. $\Box$

Corollary 6 *For all $H^* \in \mathcal{H}^{x-all}$ we have $\mathcal{T}_{\mathcal{H}}(x) = \sum_{i \in H^*} x_i$.*

Proof: Applying Lemma 26 for an $m \in M(x)$, and noting that $m(H) > 0$ implies $H \in \mathcal{H}_2$ by Lemma 24, we can write

$$\begin{aligned}
\mathcal{T}_{\mathcal{H}}(x) &= \sum_{H \in \mathcal{H}} m(H) \\
&= \sum_{H \in \mathcal{H}_2} m(H) \\
&= \sum_{H \in \mathcal{H}_2} m(H)|H \cap H^*| \\
&= \sum_{i \in H^*} \sum_{\substack{H \in \mathcal{H}_2 \\ H \ni i}} m(H) \\
&= \sum_{i \in H^*} x(m)_i \\
&= \sum_{i \in H^*} x_i,
\end{aligned}$$

where the last equality follows by Lemma 27. $\Box$

Lemma 28 *For all $H^* \in \mathcal{H}^{x-all}$ and all $i \in H^*$ we have $x_i = 1$.*

Proof: Let us fix a hyperedge $H^* = \{i,j,k\} \in \mathcal{H}^{x-all}$ and note that Lemmas 26 and 27 imply the existence of a hyperedge $H_2 \in \mathcal{H}_2$ with $H_2 \cap H^* = \{i\}$. Let us then consider an arbitrary multiplicity vector $m \in M(x^{f(H_2)})$. Let us note that for all $H \in \mathcal{H}$ with $m(H) > 0$ we must have $H \subseteq \mathrm{supp}(x^{f(H_2)}) \subseteq \mathrm{supp}(x)$, and thus $H \cap (H^* \setminus H_2) \neq \emptyset$ by the definition of $\mathcal{H}^{x-all}$. Thus, using (18b) we can write

$$\begin{aligned}
\mathcal{T}_{\mathcal{H}}(x) - 1 \;=\; \mathcal{T}_{\mathcal{H}}(x^{f(H_2)}) &= \sum_{H\in\mathcal{H}} m(H) \\
&\le \sum_{H\in\mathcal{H}} m(H)|H \cap (H^* \setminus H_2)| \\
&= x(m)_j + x(m)_k \\
&\le x_j^{f(H_2)} + x_k^{f(H_2)} \\
&= x_j + x_k \\
&= \mathcal{T}_{\mathcal{H}}(x) - x_i.
\end{aligned}$$

From the above $x_i \le 1$ follows, while $H^* \subseteq \mathrm{supp}(x)$ implies $x_i \ge 1$. □

Corollary 7 *We have $\mathcal{T}_{\mathcal{H}}(x) = 3$.*

Proof: Corollary 6 and Lemma 28 imply $\mathcal{T}_{\mathcal{H}}(x) = 3$. □

Corollary 8 *We have $\mathcal{H}^{x-all} = \mathcal{H}_1$.*

Proof: By Corollary 7 and (18a) we have $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) = 0$ for every $H \in \mathcal{H}_1$, implying $\mathcal{H}_1 \subseteq \mathcal{H}^{x-all}$. Thus the claim follows by Lemma 22. □

As a consequence of the above, we can restate (18a) - (18b) as follows:

$$\mathcal{T}_{\mathcal{H}}(x^{s(H)}) = 0 \;\text{ for all } H \in \mathcal{H}_1 = \mathcal{H}^{x-all}, \tag{20a}$$

$$\mathcal{T}_{\mathcal{H}}(x^{f(H)}) = 2 \;\text{ for all } H \in \mathcal{H}_2 = \mathcal{H}^{x-pack}. \tag{20b}$$

Furthermore, by Lemma 28 and Corollary 8 we have

$$x_i = 1 \quad \text{for all} \quad i \in \bigcup_{H\in\mathcal{H}_1} H. \tag{21}$$

Lemma 29 *For every $H \in \mathcal{H}_1$ and for every $i \in H$ there exists $H' \in \mathcal{H}_2$ such that $H \cap H' = \{i\}$.*

Proof: Since $\mathcal{T}_{\mathcal{H}}(x) = 3$ by Corollary 7, the equalities in (21) and Lemma 24 imply the claim. □

In the rest of the proof we show that $\mathcal{H}_1$ and $\mathcal{H}_2$ have some special structure, from which we can derive a contradiction at the end. To this end we show first that $\mathcal{H}_1$ includes three hyperedges such that any two of those intersect in exactly one point.

Lemma 30 *For all $H^* \in \mathcal{H}_1$ and $i \in H^*$ there exists $H^{**} \in \mathcal{H}_1$ such that $i \notin H^{**}$.*

Proof: By Lemma 23 we have $|H^*| = 3$, and therefore we must have a point $j \in H^* \setminus \{i\}$. Lemma 29 imply the existence of a hyperdege $H \in \mathcal{H}_2$ such that $H^* \cap H = \{j\}$, and therefore $i \notin H$. This implies $\mathcal{H}_{V\setminus\{i\}} \neq \emptyset$, since this induced subhypergraph contains H. Therefore, by condition (9) there exists a hyperedge $H^{**} \in \mathcal{H}_{V\setminus\{i\}}$ that intersects all others in this induced subhypergraph. Consequently, for all hyperedges $H' \in \mathcal{H}$ such that $H' \cap H^{**} = \emptyset$ we must have $i \in H'$. Therefore, $\mathcal{T}_{\mathcal{H}}(x^{f(H^{**})}) \le x_i = 1$. By the definition of $\mathcal{H}_2$ and (20b) we get $H^{**} \in \mathcal{H}_1$, as claimed. □

Lemma 31 *Consider $H_1, H_2 \in \mathcal{H}_1$ such that $i \in H_1 \cap H_2$. Then there exist no $H_3 \in \mathcal{H}_1$ such that $H_3 \subseteq (H_1 \cup H_2) \setminus \{i\}$.*

Proof: By Lemma 29 there exists a hyperedge $H' \in \mathcal{H}_2$ such that $H_1 \cap H' = \{i\}$. By Lemma 26 we also must have $|H_2 \cap H'| = 1$, thus by $i \in H_2$ we get $H' \cap ((H_1 \cup H_2) \setminus \{i\}) = \emptyset$. Since by Lemma 26 we must have $H' \cap H_3 \neq \emptyset$ for all $H_3 \in \mathcal{H}_1$, the claimed relation is implied. $\square$

Lemma 32 *There exist hyperedges $H_1, H_2 \in \mathcal{H}_1$ such that $|H_1 \cap H_2| = 1$.*

Proof: By Lemma 30 we have $|\mathcal{H}_1| \geq 2$, and no point belongs to all edges of $\mathcal{H}_1$. Since $\mathcal{H}_1 = \mathcal{H}^{x-all}$ by Corollary 7, it is an intersecting family. Since $\dim(\mathcal{H}_1) = 3$, any two distinct hyperedges of $\mathcal{H}_1$ intersect in one or two points.

Assume to the contrary that any two (distinct) hyperedges of $\mathcal{H}_1$ intersect in two points. Pick arbitrary two hyperedges of $\mathcal{H}_1$, say $H_1 = \{i, j, k\}$ and $H_2 = \{i, j, \ell\}$, and let $H_3 \in \mathcal{H}_1$ such that $i \notin H_3$. By Lemma 30 such an H_3 exists. Then, $|H_1 \cap H_3| \geq 2$ and $|H_2 \cap H_3| \geq 2$ together with $i \notin H_3$ imply $H_3 = \{j, k, \ell\}$, that is, $H_3 \subseteq (H_1 \cup H_2) \setminus \{i\}$, contradicting Lemma 31. This contradiction proves the claim. $\square$

Lemma 33 *There exist hyperedges $H_1, H_2, H_3 \in \mathcal{H}_1$ such that $|H_p \cap H_q| = 1$ for all $1 \leq p < q \leq 3$.*

Proof: By Lemma 32, we have $H_1, H_2 \in \mathcal{H}_1$ such that $H_1 \cap H_2 = \{i\}$ for some $i \in V$. Then by Lemma 30, there exists $H_3 \in \mathcal{H}_1$ such that $i \notin H_3$. We also have $H_3 \not\subseteq (H_1 \cup H_2) \setminus \{i\}$ by Lemma 31. Thus the claim follows. $\square$

Corollary 9 *There exist six distinct points $X = \{a, b, c, d, e, f\} \subseteq V$ such that $H_1 = \{a, b, f\}$, $H_2 = \{b, c, d\}$ and $H_3 = \{c, a, e\}$ are all hyperedges in $\mathcal{H}_1$.* $\square$

We show next that $\mathcal{H}_2$ has also a special form with respect to these six points.

Lemma 34 *For all $H \in \mathcal{H}_2$ we have one of the following: $\{a, d\} \subseteq H$, $\{b, e\} \subseteq H$, or $\{c, f\} \subseteq H$.*

Proof: By Lemmas 24, 26, and Corollary 8 we have $|H \cap H_p| = 1$ for all $p = 1, 2, 3$. Then either H has the form as claimed, or $H = \{d, e, f\}$. In the latter case however, let us consider $H' \in \mathcal{H}_2$ such that $H' \cap H = \emptyset$. Such an H' must exist by the facts $\mathcal{T}_{\mathcal{H}}(x) = 3$ and $H \in \mathcal{H}_2 = \mathcal{H}^{x-pack}$. This set also must intersect H_p, $p = 1, 2, 3$ in exactly one point, however this is now impossible without intersecting H, too. Thus, only the claimed forms remain feasible for sets of $\mathcal{H}_2$. $\square$

Corollary 10 *Using $\alpha = \{a, d\}$, $\beta = \{b, e\}$ and $\gamma = \{c, f\}$, we can conclude that the subhypergraphs*

$$\begin{aligned}
\mathcal{H}_{2,\alpha} &= \{H \in \mathcal{H}_2 \mid \alpha \subseteq H\}, \\
\mathcal{H}_{2,\beta} &= \{H \in \mathcal{H}_2 \mid \beta \subseteq H\}, \text{ and} \\
\mathcal{H}_{2,\gamma} &= \{H \in \mathcal{H}_2 \mid \gamma \subseteq H\}
\end{aligned}$$

form a partition of $\mathcal{H}_2$. In particular, none of these families are empty.

Proof: The first claim follows directly from Lemma 34. By Equation (21) we have $x_a = x_b = x_c = x_d = x_e = x_f = 1$, and thus for any $m \in M(x)$ and $\mu \in \{\alpha, \beta, \gamma\}$ we must have $\sum_{H \in \mathcal{H}_{2,\mu}} m(H) \leq 1$ by Lemma 27. On the other hand we have $\mathcal{T}_{\mathcal{H}}(x) = 3$ by Corollary 7, and thus for all $m \in M(x)$ and for all $\mu \in \{\alpha, \beta, \gamma\}$ we must have a hyperedge $H \in \mathcal{H}_{2,\mu}$ with $m(H) = 1$, completing the proof of the claim. $\square$

Lemma 35 *These exists no hyperedge $H \in \mathcal{H}_1$ that would contain μ for $\mu \in \{\alpha, \beta, \gamma\}$.*

Proof: Assume to the contrary that e.g., $H = \{a, d, u\} \in \mathcal{H}_1$. Then by Lemma 26 we must have $u \in H'$ for all $H' \in \mathcal{H}_{2,\beta} \cup \mathcal{H}_{2,\gamma}$, and thus, in particular, $u \notin X$. Since $\mathcal{T}_{\mathcal{H}}(x) = 3$, we must have $x_u \geq 3$. Let us then consider the H-move $x \to x'$, where $x'_i = x_i$ for $i \notin H$, $x'_a = 0$, $x'_d = 0$, and $x'_u = 1$. Then all hyperedges of $\mathcal{H}_2$ that are subsets of $\mathrm{supp}(x')$ contain u, and thus we must have $\mathcal{T}_{\mathcal{H}}(x') = 1$, contradicting Equation (20a). □

Let us next introduce $N_\mu = \bigcup\{H \setminus \mu \mid H \in \mathcal{H}_{2,\mu}\}$ for $\mu \in \{\alpha, \beta, \gamma\}$. Note that these sets are disjoint from $X = H_1 \cup H_2 \cup H_3$, defined in Corollary 9, by Lemma 26.

Lemma 36 *Let $\mu, \nu \in \{\alpha, \beta, \gamma\}$, $\mu \neq \nu$ and consider two sets $H \in \mathcal{H}_{2,\mu}$ and $H' \in \mathcal{H}_{2,\nu}$ such that $H \cap H' = \emptyset$. Then there exists a hyperedge $H'' \in \mathcal{H}_{2,\mu} \cup \mathcal{H}_{2,\nu}$ that intersects both H and H'.*

Proof: By Condition (9), we must have a set $H'' \subseteq H \cup H'$, $H'' \in \mathcal{H}$ that intersects all sets in the non-empty induced subhypergraph $\mathcal{H}_{H \cup H'}$. If $H'' \in \mathcal{H}_1$ then $|H''| = 3$ by Lemma 23, and thus we must have either $|H'' \cap H| \geq 2$ or $|H'' \cap H'| \geq 2$, contradicting Lemma 26. Thus, we must have $H'' \in \mathcal{H}_2$, and therefore $H'' \in \mathcal{H}_{2,\mu} \cup \mathcal{H}_{2,\nu}$ by Lemma 34, as claimed. □

Corollary 11 *For $\mu, \nu \in \{\alpha, \beta, \gamma\}$, $\mu \neq \nu$, we either have $N_\mu \subseteq N_\nu$ or $N_\nu \subseteq N_\mu$.*

Proof: If there are points $u \in N_\mu \setminus N_\nu$ and $v \in N_\nu \setminus N_\mu$, then by Lemma 36 we have either $\mu \cup \{v\} \in \mathcal{H}_{2,\mu}$, or $\nu \cup \{v\} \in \mathcal{H}_{2,\nu}$ contradicting $u \notin N_\nu$ or $v \notin N_\mu$. □

Lemma 37 *Let $\mu, \nu \in \{\alpha, \beta, \gamma\}$, $\mu \neq \nu$. Then, there exists no two distinct points $u, v \in V \setminus X$ such that all four sets $\mu \cup \{u\}$, $\mu \cup \{v\}$, $\nu \cup \{u\}$, and $\nu \cup \{v\}$ are hyperedges of $\mathcal{H}$.*

Proof: Assume to the contrary that such points do exist. Then by Lemma 35 these sets are all from $\mathcal{H}_2$. By Condition (9) we must have a hyperedge $H \subseteq \{\mu\} \cup \{\nu\} \cup \{u, v\}$ in $\mathcal{H}$ that intersects all these sets. Since H must intersect some of these four sets in two points, $H \in \mathcal{H}_2$ holds by Lemma 26. Then, by Corollary 10, we have $H \in \mathcal{H}_{2,\mu} \cup \mathcal{H}_{2,\nu}$. This is however impossible since there exists no such subset of size at most 3 that would either contain μ or ν and intersect all these four sets. □

Corollary 12 *For all $\mu, \nu \in \{\alpha, \beta, \gamma\}$, $\mu \neq \nu$ we have $|N_\mu \cap N_\nu| \leq 1$.*

Proof: Immediate from Lemma 37. □

Corollary 13 *Up to a relabeling of the vertices, we have $N_\alpha \subseteq N_\beta \subseteq N_\gamma$, and $|N_\alpha| \leq |N_\beta| \leq 1$.*

Proof: Immediate by Corollaries 11 and 12. □

Lemma 38 *At most one of α, β, and γ is a hyperedge of $\mathcal{H}$.*

Proof: If e.g., $\alpha, \beta \in \mathcal{H}$ then by Condition (9) we must have a hyperedge $H \in \mathcal{H}$ such that $H \subseteq \alpha \cup \beta$ and it intersects both α and β. Since $\alpha, \beta \in \mathcal{H}_2$, by Lemmas 26 and 23, we have that $H \notin \mathcal{H}_1$ and hence $H \in \mathcal{H}_2$. Then, by Corollary 10, we must have $H \in \mathcal{H}_{2,\alpha}$ or $H \in \mathcal{H}_{2,\beta}$. Since H must intersect both α and β, $|H| = 3$ follows, from which we derive a contradiction by Lemma 26, due to the structure of $\mathcal{H}_1$ sets within the set X. □

Lemma 39 $N_\alpha \neq \emptyset$.

Proof: Assume to the contrary that $N_\alpha = \emptyset$. This implies that $\mathcal{H}_{2,\alpha} = \{\alpha\}$. Let us now consider an arbitrary $m \in M(x)$. Since $\mathcal{T}_{\mathcal{H}}(x) = 3$ by Corollary 7, we must have hyperedges $H_\mu \in \mathcal{H}_{2,\mu}$ for all $\mu \in \{\alpha, \beta, \gamma\}$ with $m(H_\mu) = 1$ by Equation (21). In particular, we must have $m(\alpha) = 1$ and $m(H) = 1$ for some $H \in \mathcal{H}_{2,\beta}$. Since $\alpha \cap H = \emptyset$, by Condition (9) we must have a hyperedge $H' \in \mathcal{H}$ that intersects both α and H such that $H' \subseteq \alpha \cup H$. If $H' \in \mathcal{H}_1$ then we get a contradiction by Lemma 26. Thus we must have $H' \in \mathcal{H}_2$. Then by Corollary 10 and equality

$\mathcal{H}_{2,\alpha} = \{\alpha\}$ imply that $H' \in \mathcal{H}_{2,\beta}$ which contradicts the fact that α is disjoint from all sets of $\mathcal{H}_{2,\beta}$. □

Thus, by Corollary 13 and Lemma 39, we have $|N_\alpha| = |N_\beta| = 1$, that is, for some $u \in V$ we have $N_\alpha = N_\beta = \{u\} \subseteq N_\gamma$ and, therefore $H = \gamma \cup \{u\} \in \mathcal{H}_{2,\gamma}$. Let $x' = x^{f(H)}$, and consider $m \in M(x')$. By Lemma 26, we have $m(H^*) = 0$ for all $H^* \in \mathcal{H}_1$. Furthermore, for any $H' \in \mathcal{H}_2$ such that $u \in H'$ we also must have $m(H') = 0$. Consequently, only $H' \in \mathcal{H}_{2,\alpha} \cup \mathcal{H}_{2,\beta}$, $u \notin H'$ can have $m(H') = 1$ (and not more by Equation (21).) Since by Lemma 38 at most one of α and β can belong to $\mathcal{H}_2$, we have $\mathcal{T}_\mathcal{H}(x') \leq 1$, which contradicts Equation (20b).

This completes the proof of the first claim of the theorem.

Let us next prove that Condition (9) can be tested in polynomial time for hypergraphs of constant dimension k.

Let us call a hyperedge $H \in \mathcal{H}$ *intersecting* if it intersects all hyperedges of $\mathcal{H}$.

We will show the following implication: If Condition (9) does not hold for $\mathcal{H}$ then we can find a subset $U \subseteq V$ such that $|U| \leq k\binom{2k}{k}$, $\mathcal{H}_U \neq \emptyset$ and there is no intersecting hyperedge in $\mathcal{H}_U$. Since k is a fixed constant, we will need to check only polynomially many induced subhypergraphs to find such U. This clearly can be accomplished in polynomial time.

To prove the above implication, let us note that if Condition (9) does not hold then there is a minimal subset $U \subseteq V$ such that $\mathcal{H}_U \neq \emptyset$ and $\mathcal{H}_U$ has no intersecting hyperedge. Such a subhypergraph $\mathcal{H}_U$ satisfies the conditions of the following lemma.

Lemma 40 ([4]) *Given a hypergraph $\mathcal{F} \subseteq 2^U$ of dimension at most k such that $\mathcal{F}$ has no intersecting hyperedge, but all nonempty proper induced subhypergraphs of $\mathcal{F}$ do have one, then,*

$$|U| \;\leq\; k\binom{2k}{k}.$$

Proof: Our assumptions imply that for every $i \in U$ we have a hyperedge $F_i \in \mathcal{F}$ such that $F_i \cap F' \neq \emptyset$ for all $F' \in \mathcal{F}$ with $i \notin F'$. Let us denote by $\mathcal{F}' = \{F_i \mid i \in U\}$ the family of these hyperedges. Since $\mathcal{F}$ does not have an intersecting hyperedge, for every $F \in \mathcal{F}'$ there exists a hyperedge $B(F) \in \mathcal{F}$ disjoint from F. Let us choose a minimal subhypergraph $\mathcal{B} \subseteq \mathcal{F}$ such that

$$\forall F \in \mathcal{F}' \;\; \exists B \in \mathcal{B} : \quad F \cap B = \emptyset. \tag{22}$$

Let us note first that such a $\mathcal{B}$ must form a cover of U, that is, $U = \bigcup_{B \in \mathcal{B}} B$. This is because for all $F_i \in \mathcal{F}'$ there exists a $B \in \mathcal{B}$ such that $F_i \cap B = \emptyset$ and, consequently, $i \in B$. Let us observe next that for all $B \in \mathcal{B}$ there exists at least one $A(B) \in \mathcal{F}'$ such that $A(B) \cap B = \emptyset$ and $A(B) \cap B' \neq \emptyset$ for all $B' \in \mathcal{B} \setminus \{B\}$. This is because we choose $\mathcal{B}$ to be a minimal family with respect to (22). Let us now define $\mathcal{A} = \{A(B) \in \mathcal{F}' \mid B \in \mathcal{B}\}$. The pair $\mathcal{A}$, $\mathcal{B}$ of hypergraphs now satisfies the conditions of a classical theorem of Bollobás [5], which then implies that

$$|\mathcal{A}| = |\mathcal{B}| \leq \binom{2k}{k}.$$

Since $\dim(\mathcal{B}) \leq k$ and it covers U, our claim follows. □ □

4.4 Another sufficient condition for $\mathcal{T}_\mathcal{H} = \mathcal{G}_\mathcal{H}$

We can strengthen Condition (9) by requiring that any two hyperedges intersect, i.e,

$$\text{for all } H, H' \in \mathcal{H} \text{ we have } H \cap H' \neq \emptyset. \tag{23}$$

We call such hypergraphs *intersecting*.

Theorem 5 *If $\mathcal{H} \subseteq 2^V$ is an intersecting hypergraph then equality $\mathcal{G}_\mathcal{H}(x) = \mathcal{T}_\mathcal{H}(x)$ holds for all positions $x \in \mathbb{Z}^V_{\geq}$.*

Proof: Let us consider an arbitrary position $x \in \mathbb{Z}^V_{\geq}$. If $\mathcal{T}_\mathcal{H}(x) = 0$ then the claim holds by definition. Assume that $\mathcal{T}_\mathcal{H}(x) > 0$ and consider a hyperedge $H \in \mathcal{H}$ such that $\mathcal{T}_\mathcal{H}(x - \chi_H) =$

$\mathcal{T}_\mathcal{H}(x)-1$. Such a hyperedge exists, since $\mathcal{T}_\mathcal{H}(x)>0$. Let us consider positions $x^{s(H)}$ and $x^{f(H)}$. By our choice of H, we have $\mathcal{T}_\mathcal{H}(x^{s(H)}) = \mathcal{T}_\mathcal{H}(x)-1$. Since the hypergraph is intersecting, we also have $\mathcal{T}_\mathcal{H}(x^{f(H)}) = 0$. Thus, by Lemma 3, for all values $0 \le v \le \mathcal{T}_\mathcal{H}(x)-1$ there exists an H-move $x \to x'$ such that $\mathcal{T}_\mathcal{H}(x') = v$. Since this holds for all positions, we get $\mathcal{G}_\mathcal{H}(x) = \mathcal{T}_\mathcal{H}(x)$, by Lemma 18. □

4.5 Computing the Tetris function

Theorem 6 *Given a hypergraph $\mathcal{H} \subseteq 2^V$ and a position $x \in \mathbb{Z}^V_\geq$, computing $\mathcal{T}_\mathcal{H}(x)$ is*

(i) *NP-hard for intersecting hypergraphs, already for dimension 4;*

(ii) *NP-hard for hypergraphs of dimension at most 3;*

(iii) *polynomial for hypergraphs of dimension at most 2 (i.e., for graphs).*

Proof: Let us consider an arbitrary hypergraph $\mathcal{H} \subseteq 2^V$. Its matching number $\mu(\mathcal{H})$ is the maximum number of pairwise disjoint hyperedges of $\mathcal{H}$ and is known to be NP-hard to compute already for the hypergraphs of dimension 3 [24].

Let us consider $w \notin V$ and define $\mathcal{H}^* = \{H \cup \{w\} \mid H \in \mathcal{H}\}$. Also consider position $x \in \mathbb{Z}^{V\cup\{w\}}_\geq$ defined by $x_i = 1$ for $i \in V$ and $x_w = |\mathcal{H}|$. Then $\mathcal{H}^*$ is an intersecting hypergraph and we have $\mathcal{T}_{\mathcal{H}^*}(x) = \mu(\mathcal{H})$. This equality still holds when $\mathcal{H}$ is of dimension 3 and $x_i = 1$ for all $i \in V$.

Yet, if $\mathcal{H}$ is of dimension at most 2 then $T_\mathcal{H}(b)$ for a position $b \in \mathbb{Z}^V_\geq$ is the so called b-matching number of the underlying graph and is known to be computable in polynomial time [17, 34]. □

Corollary 14 *Given a hypergraph $\mathcal{H} \subseteq 2^V$ and a position $x \in \mathbb{Z}^V_\geq$, computing $\mathcal{G}_\mathcal{H}(x)$ is NP-hard, even for intersecting hypergraphs.*

Proof: Since intersecting hypergraphs satisfy Condition (9), Theorem 3 implies $\mathcal{T}_\mathcal{H} = \mathcal{G}_\mathcal{H}$. Thus, the claim follows, by Theorem 6 (i). □

Let us finally remark that the complexity of computing the Tetris value of a position for hypergraphs of dimension 3 is open under Condition (9). It remains open even under the more restrictive condition (23).

5 JM hypergraphs

The main result of our research is the discovery of an infinite family of graph for which an explicit formula for their SG-value has been found. While all members of the family are yet to be discovered, infinitely many have been found, and all of them belong to the family of connected minimal transversal-free hypergraphs.

To state our main result we need to introduce some additional notation.

To a position $x \in \mathbb{Z}^V_{\geq}$ of $\text{NIM}_{\mathcal{H}}$ let us associate the following quantities:

$$m(x) = \min_{i \in V} x_i \tag{24a}$$

$$y_{\mathcal{H}}(x) = \mathcal{T}_{\mathcal{H}}(x - m(x)e) + 1 \tag{24b}$$

$$v_{\mathcal{H}}(x) = \binom{y_{\mathcal{H}}(x)}{2} + \left(\left(m(x) - \binom{y_{\mathcal{H}}(x)}{2} - 1\right) \mod y_{\mathcal{H}}(x)\right), \tag{24c}$$

where e is the n-vector of full ones. Finally, we define

$$\mathcal{U}_{\mathcal{H}}(x) = \begin{cases} \mathcal{T}_{\mathcal{H}}(x) & \text{if } m(x) \leq \binom{y_{\mathcal{H}}(x)}{2} & \text{(25a)} \\ v_{\mathcal{H}}(x) & \text{otherwise.} & \text{(25b)} \end{cases}$$

With this notation the results of [8, 9, 23] can be stated as the SG function of the considered games is defined by (25a)-(25b), that is, $\mathcal{G} = \mathcal{U}$. It was a surprise to see that the "same" formula works for seemingly very different games. In view of this, we call the expression (25a)-(25b) the *JM formula*, in honor of the results of Jenkyns and Mayberry [23]. We call a hypergraph $\mathcal{H}$ a *JM hypergraph* if this formula describes the SG function of $\text{NIM}_{\mathcal{H}}$.

Let us add that the formula looks the same but it depends on $\mathcal{T}_{\mathcal{H}}$ and, hence, the actual values depend on the hypergraph $\mathcal{H}$. In fact, function $\mathcal{T}_{\mathcal{H}}$ may be difficult to compute [10], even for cases when the JM formula is valid.

We provide some necessary and some sufficient conditions for a hypergraph to be JM. This section's main results are as follows.

(i) We provide the winning-losing partition of JM hypergraphs, without the knowledge of its tetris function.

(ii) A JM hypergraph is minimal transversal-free.

(iii) A graph (that is, a 2-uniform hypergraph) is JM if and only if it is connected and minimal transversal-free. We provide a complete list of JM graphs.

(iv) A matroid hypergraph is JM if and only if it is transvesal-free. This implies that all self-dual matroid hypergraphs are JM.

(v) Hypergraphs defined by connected k-edge subgraphs of a given graph are JM under certain conditions.

(vi) A Symmetric hypegraph is JM if and only if its size sequence satisfies certain properties.

(vii) For every integer k, the number of vertices of a k-uniform JM hypergraph is bounded by $k\binom{2k}{k}$.

For instance, $\binom{V}{k} = \{H \subseteq V \mid |H| = k\}$ is a self-dual matroid hypergraph if $n = 2k$. This example shows that (iii) generalizes the main result of [9]. Another example for a self-dual matroid with $n = 2k$ is the hypergraph $\mathcal{H}_{2^k} = \{H \subseteq V \mid |H \cap \{i, i+k\}| = 1,\ 1 \leq i \leq k\}$, that is, the family of 2^k minimal transversals of a family of k pairs. It was proved in [7] that any self-dual matroid on $n = 2k$ elements must have at least 2^k bases. Thus, the latter construction is extremal in this respect.

We remark that [7] showed also the existence of self-dual matroids on $n = 2k$ elements whenever certain type of symmetric block designs exists on k points. Since many families of such block designs are known, the above cited result shows that numerous other families of self-dual matroids (and JM hypergraphs) exist.

For (iv) we can mention the following circulant hypergraphs defined by consecutive k edges of simple cycles on $n = 2k$ or $n = 2k + 1$ vertices. Another example is defined by connected k-edge subgraphs of a rooted tree, where the root has degree $k + 1$ and each of the subtrees connected to the root have exactly k edges.

For a positive integer $\eta \in \mathbb{Z}_{\geq}$ let us associate the set

$$Z(\eta) \;=\; \left\{ i \in \mathbb{Z}_{\geq} \,\middle|\, \binom{\eta}{2} \leq i < \binom{\eta+1}{2} \right\}. \tag{26}$$

It is immediate to see the following properties:

Lemma 41 *If $\eta \neq \eta'$ then $Z(\eta) \cap Z(\eta') = \emptyset$. Furthermore, we have*

$$\mathbb{Z}_{\geq} \;=\; \bigcup_{\eta=1}^{\infty} Z(\eta).$$

□

Let us recall next that by the definitions of the quantities in (24) the value $v_{\mathcal{H}}(x)$ depends only on the pair of integers $m(x)$ and $y_{\mathcal{H}}(x)$.

Lemma 42 *For an arbitrary positive integer $\eta \in \mathbb{Z}_{\geq}$ we have*

$$\left\{ v_{\mathcal{H}}(x) \;\middle|\; x \in \mathbb{Z}_{\geq}^V,\; y_{\mathcal{H}}(x) = \eta \right\} \;=\; Z(\eta).$$

Proof: Follows by (24c) and the fact that $m(x)$ can take arbitrary integer values modulo $y_{\mathcal{H}}(x) = \eta$. □

Lemma 43 *For an arbitrary position $x \in \mathbb{Z}_{\geq}^V$ and move $x \to x'$ in $\mathrm{NIM}_{\mathcal{H}}$ we have $(m(x), y_{\mathcal{H}}(x)) \neq (m(x'), y_{\mathcal{H}}(x'))$.*

Proof: If $m(x) = m(x')$ and $x \to x'$ is an H-move for a hyperedge $H \in \mathcal{H}$, then we have the inequality $x - \chi(H) \geq x'$, where $\chi(H)$ is the characteristic vector of H. This implies

$$x - m(x)e \geq \chi(H) + x' - m(x')e$$

from which $y_{\mathcal{H}}(x') \geq y_{\mathcal{H}}(x) + 1$ follows. □

Lemma 44 *A position $x \in \mathbb{Z}_{\geq}^V$ is long if and only if $v_{\mathcal{H}}(x) \geq m(x)$.*

Proof: Note first that if x is long then $m(x) \leq \binom{y_{\mathcal{H}}(x)}{2}$ by (25a), and thus, by Lemma 42 it follows that $m(x) \leq v_{\mathcal{H}}(x)$. On the other hand, if x is short then we have $m(x) > \binom{y_{\mathcal{H}}(x)}{2}$ by (25b), and thus, $m(x) - \binom{y_{\mathcal{H}}(x)}{2} - 1 \geq 0$, implying

$$m(x) - \binom{y_{\mathcal{H}}(x)}{2} - 1 \geq \left(\left(m(x) - \binom{y_{\mathcal{H}}(x)}{2} - 1 \right) \mod y_{\mathcal{H}}(x) \right)$$

from which by (25b) it follows that $m(x) - 1 \geq v_{\mathcal{H}}(x)$. □

Lemma 45 *For an arbitrary position $x \in \mathbb{Z}_{\geq}^V$ and move $x \to x'$ such that $m(x) \leq \mathcal{U}_{\mathcal{H}}(x') < \mathcal{T}_{\mathcal{H}}(x)$ position x' is long.*

Proof: If x' were short then by Lemma 44 we would get $\mathcal{U}_{\mathcal{H}}(x') = v_{\mathcal{H}}(x') < m(x') \leq m(x)$, contradicting $m(x) \leq \mathcal{U}_{\mathcal{H}}(x')$. □

Lemma 46 *Let $\mathcal{H} \subseteq 2^V$ and $\widetilde{\mathcal{H}} \subseteq 2^V$ be two hypergraphs, and $x, \widetilde{x} \in \mathbb{Z}_{\geq}^V$ be two positions such that $m(x) = m(\widetilde{x})$ and $y_{\mathcal{H}}(x) = y_{\widetilde{\mathcal{H}}}(\widetilde{x})$. Then we have $v_{\mathcal{H}}(x) = v_{\widetilde{\mathcal{H}}}(\widetilde{x})$. Furthermore, x is short in $\mathrm{NIM}_{\mathcal{H}}$ if and only if $\widetilde{x}$ is short in $\mathrm{NIM}_{\widetilde{\mathcal{H}}}$.*

Proof: Recall that by (25b) the $v(x)$-value of a position x depends only on the $m(x)$ and $y(x)$ values, and do not depend on any other parameters of the underlying hypergraphs. Similarly, the type of a position (long or short) also depends only on these two integer values. $\square$

Lemma 47 *Let $\mathcal{H}, \widetilde{\mathcal{H}} \subseteq 2^V$ be two hypergraphs such that $\widetilde{\mathcal{H}}$ contains a hyperedge different from V. Then for every position $x \in \mathbb{Z}_{\geq}^V$ there exists a position $\widetilde{x} \in \mathbb{Z}_{\geq}^V$ such that $m(x) = m(\widetilde{x})$ and $y_{\mathcal{H}}(x) = y_{\widetilde{\mathcal{H}}}(\widetilde{x})$.*

Proof: Choose a minimal hyperedge $H \in \widetilde{\mathcal{H}}$, and consider the position $\widetilde{x} = m(x)e + (y_{\mathcal{H}}(x) - 1)\chi(H)$. Since $\widetilde{\mathcal{H}}$ is assumed to have a hyperedge different from V, we can choose H such that $H \neq V$. Therefore, we have $m(\widetilde{x}) = m(x)$, and $\widetilde{x} - m(\widetilde{x})e = (y_{\mathcal{H}}(x) - 1)\chi(H)$, implying $y_{\mathcal{H}}(x) = y_{\widetilde{\mathcal{H}}}(\widetilde{x})$. $\square$

Lemma 48 *Consider an arbitrary position $x \in \mathbb{Z}_{\geq}^V$ and hyperedge $H \in \mathcal{H}$ such that $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$ and $m(x^{s(H)}) = m(x) - 1$. Then we have $y_{\mathcal{H}}(x^{s(H)}) \geq y_{\mathcal{H}}(x)$.*

Proof: Since $m(x^{s(H)}) = m(x) - 1$ we have the inequality $x^{s(H)} - m(x^{s(H)})e \geq x - m(x)e$ and thus the claim follows from Lemma 2. $\square$

5.1 Winning and losing positions of JM hypergraphs

While $T_{\mathcal{H}}$ might be difficult to compute, the winning and losing position are much easier to characterize.

Lemma 49 *For a JM hypergraph $\mathcal{H} \in 2^V$, $x \in \mathbb{Z}_{\geq}^V$ is a losing position if and only if*

- *$m(x) = 0$ and $T_{\mathcal{H}}(x) = 0$, or*
- *$m(x) > 0$ and $\mathcal{T}_{\mathcal{H}}(x - m(x)e) = 0$.*

The lemma implies that optimal play, given that we are not in one of the two above positions, is to move to one of those two positions. Namely if $m(x) = 0$ find the transversal edge in *supp*(x) and do a fast move on x. And if $m(x) > 0$ then find the transversal edge in *supp*$(x - m(x)e)$ and remove all but $m(x)$ tokens from each vertex on it. We should note that the decision problem whether $T_{\mathcal{H}}(x) = 0$ is $O(|V| \cdot |\mathcal{H}|)$ as one only need to check every hyperedge $H \in \mathcal{H}$ and figure out if at least one of them covers only nonempty piles. Calculating supp(x) is also $O(|V| \cdot |\mathcal{H}|)$, as we are scanning through hyperedge and removing those that cover any empty piles. Finding a transversal hyperedge on the other hand is $O(|V|^2 \cdot |\mathcal{H}|^2)$ since we need to check pairs of edges whether they intersect.

Proof: Recall that losing positions are exactly the positions with $\mathcal{G}_{\mathcal{H}}(x) = 0$. Hence for a JM hypergraph $\mathcal{G}_{\mathcal{H}}(x) = \mathcal{U}_{\mathcal{H}}(x) = 0$ if either

(1) $T_{\mathcal{H}}(x) = 0$, and $m(x) \leq \binom{y_{\mathcal{H}}(x)}{2}$

(2) $v_{\mathcal{H}}(x) = 0$ and $m(x) > \binom{y_{\mathcal{H}}(x)}{2}$.

In case (1) $m(x) \leq \binom{y_{\mathcal{H}}(x)}{2}$ can be omitted. This comes from the fact that $T_{\mathcal{H}}(x) = 0$ implies $m(x) = 0$. In case (2) we have

$$0 = v_{\mathcal{H}}(x) = \binom{y_{\mathcal{H}}(x)}{2} + \left(\left(m(x) - \binom{y_{\mathcal{H}}(x)}{2} - 1\right) \mod y_{\mathcal{H}}(x)\right),$$

which implies $\binom{y_{\mathcal{H}}(x)}{2} = 0$ and $\left(\left(m(x) - \binom{y_{\mathcal{H}}(x)}{2} - 1\right) \mod y_{\mathcal{H}}(x)\right) = 0$. The former equality implies $y_{\mathcal{H}}(x) = 1$ (or equivalently $\mathcal{T}_{\mathcal{H}}(x - m(x)e) = 0$). With this we can forget about the second equality since something modulo 1 always equals 0. The inequality in case (2) can be simplified to $m(x) > \binom{y_{\mathcal{H}}(x)}{2} = \binom{1}{2} = 0$. $\square$

5.2 Necessary Conditions

In this section we prove some properties of JM hypergraphs.

For every hypergraph $\mathcal{H} \subseteq 2^V$, we assume that $V = \bigcup_{H \in \mathcal{H}} H$.

Lemma 50 *A JM hypergraph $\mathcal{H}$ is connected.*

Proof: Let us assume indirectly that $\mathcal{H}$ is not connected. Let $H_1, H_2 \in \mathcal{H}$ be two minimal hyperedges in two different connected components of $\mathcal{H}$. Consider the position x defined as follows:

$$x_i = \begin{cases} 1 & \text{if } i \in H_1 \\ 3 & \text{if } i \in H_2 \\ 0 & \text{otherwise.} \end{cases}$$

This position has $\mathcal{G}_{\mathcal{H}}(x) = 2$ (the NIM sum of 1 and 3). It has $m(x) \leq 1$ and $3 \leq y_{\mathcal{H}}(x) \leq 4$. Therefore x is long. Since $\mathcal{T}_{\mathcal{H}}(x) = 4$ we can conclude $\mathcal{U}_{\mathcal{H}}(x) \neq \mathcal{G}_{\mathcal{H}}(x)$, which contradicts the assumption. □

Given a hypergraph $\mathcal{H} \subseteq 2^V$ and a subset $S \subseteq V$, we denote by $\mathcal{H}_S$ the *induced subhypergraph*, defined as

$$\mathcal{H}_S = \{H \in \mathcal{H} \mid H \subseteq S\}.$$

Lemma 51 *If $\mathcal{H}$ is a JM hypergraph, then it is minimal transversal-free.*

Proof: Let us assume first indirectly that $H_0 \in \mathcal{H}$ is a transversal of $\mathcal{H}$, say $H_0 \in \mathcal{H}$ intersects all hyperedges of $\mathcal{H}$. Consider the position $x \in \mathbb{Z}_{\geq}^V$ defined by $x_i = 1$, $i \in V$. For this position we have $m(x) = 1$, $y_{\mathcal{H}}(x) = 1$, and $v_{\mathcal{H}}(x) = 0$. Thus, x is short and $\mathcal{U}_{\mathcal{H}}(x) = 0$. On the other hand, a slow H_0-move $x \to x'$ takes us into a position with $x'_i = 0$, $i \in H_0$. Since H_0 intersects all hyperedges of $\mathcal{H}$ we must have $\mathcal{T}_{\mathcal{H}}(x') = 0$, which implies by property (A) of the SG function that $\mathcal{G}_{\mathcal{H}}(x) \neq 0$, or in other weords that $\mathcal{G}_{\mathcal{H}}(x) \neq \mathcal{U}_{\mathcal{H}}(x)$, which implies on its turn that $\mathcal{H}$ is not JM. This contradiction implies that $\mathcal{H}$ is transversal-free.

To see minimality with respect the transversal-freeness, let us consider an arbitrary proper subset $S \subset V$ for which the induced subhypergraph $\mathcal{H}_S$ is not empty, and a position with $x_i = 0$ for all $i \in V \setminus S$, and $x_i > 0$ for all $i \in S$. Every move from x is an H-move for some $H \in \mathcal{H}_S$. Furthermore, $\mathcal{T}_{\mathcal{H}}(x) > 0$, $m(x) = 0$ (since $V \setminus S \neq \emptyset$), and thus, all such positions are long, implying (by our assumption that $\mathcal{H}$ is JM) that $\mathcal{G}_{\mathcal{H}}(x) = \mathcal{U}_{\mathcal{H}}(x) = \mathcal{T}_{\mathcal{H}}(x) > 0$ for all such positions. Thus, we must have a move $x \to x'$ such that $\mathcal{G}_{\mathcal{H}}(x') = \mathcal{T}_{\mathcal{H}}(x') = 0$. This is possible only if this move is an H-move for a hyperedge $H \in \mathcal{H}_S$ that intersects all hyperedges of $\mathcal{H}$. □

In the rest of this subsection, we study further properties of transversal-free hypergraphs. This is used to obtain sufficient conditions for a hypergraph to be JM.

Lemma 52 *If $\mathcal{H}$ is transversal-free and $x \to x'$ is a move in* $\text{NIM}_{\mathcal{H}}$, *then we have $\mathcal{T}_{\mathcal{H}}(x') \geq m(x)$.*

Proof: Since $\mathcal{H}$ is transversal-free, for every hyperedge $H \in \mathcal{H}$ there exists an $H' \in \mathcal{H}$ such that $H \cap H' = \emptyset$. Consequently, for every move $x \to x'$ we must have $\mathcal{T}_{\mathcal{H}}(x') \geq m(x)$. This is because if $x \to x'$ is an H-move and $H' \in \mathcal{H}$ is disjoint from H, then we have $x'_i = x_i \geq m(x)$ for all $i \in H'$. □

Lemma 53 *If $\mathcal{H}$ is a transversal-free hypergraph, $x \in \mathbb{Z}_{\geq}^V$ is a long position, and $x \to x'$ is a move such that $0 \leq \mathcal{U}_{\mathcal{H}}(x') < m(x)$, then x' is a short position, for which we have $m(x') \leq m(x)$ and $m(x) - m(x') + 1 \leq y_{\mathcal{H}}(x') < y_{\mathcal{H}}(x)$.*

Proof: By Lemma 52 we have $\mathcal{T}_{\mathcal{H}}(x') \geq m(x)$. Now, if x' were long then $\mathcal{U}_{\mathcal{H}}(x') = \mathcal{T}_{\mathcal{H}}(x') \geq m(x)$ would follow, contradicting our assumption that $\mathcal{U}_{\mathcal{H}}(x') < m(x)$. The inequality $m(x') \leq m(x)$ holds for any move $x \to x'$. Since x is long and x' is short, we have the inequalities

$$\binom{y_{\mathcal{H}}(x)}{2} \;\geq\; m(x) \;\geq\; m(x') \;>\; \binom{y_{\mathcal{H}}(x')}{2}$$

from which $y_{\mathcal{H}}(x') < y_{\mathcal{H}}(x)$ follows. Assume next that $x \to x'$ is an H-move for some hyperedge $H \in \mathcal{H}$. Since $\mathcal{H}$ is transversal-free by Lemma 51, there exists $H' \in \mathcal{H}$ such that $H \cap H' = \emptyset$. Then we have $x'_i = x_i \geq m(x)$ for all $i \in H'$, and thus, $y_{\mathcal{H}}(x') \geq m(x) - m(x') + 1$ follows by (24b). $\square$

Lemma 54 *If $\mathcal{H}$ is a transversal-free hypergraph, $x \in \mathbb{Z}^V_{\geq}$ is a short position, and $x \to x'$ is a move such that $0 \leq \mathcal{U}_{\mathcal{H}}(x') < \mathcal{U}_{\mathcal{H}}(x)$, then x' must also be a short position, for which we have $m(x') \leq m(x)$ and $m(x) - m(x') + 1 \leq y_{\mathcal{H}}(x') \leq y_{\mathcal{H}}(x)$ with $(m(x), y_{\mathcal{H}}(x)) \neq (m(x'), y_{\mathcal{H}}(x'))$.*

Proof: Since x is short, we have $\mathcal{U}_{\mathcal{H}}(x) = v_{\mathcal{H}}(x) < m(x)$ by Lemma 44. We also have $\mathcal{T}_{\mathcal{H}}(x') \geq m(x)$ by Lemma 52. Thus, $\mathcal{U}_{\mathcal{H}}(x') < \mathcal{U}_{\mathcal{H}}(x) = v_{\mathcal{H}}(x) < m(x) \leq \mathcal{T}_{\mathcal{H}}(x')$ follows by our assumption, implying $\mathcal{U}_{\mathcal{H}}(x') \neq \mathcal{T}_{\mathcal{H}}(x')$. Thus, x' is not long. The inequality $m(x') \leq m(x)$ holds for any move $x \to x'$. Lemma 42 and $v_{\mathcal{H}}(x') < v_{\mathcal{H}}(x)$ implies $y_{\mathcal{H}}(x') \leq y_{\mathcal{H}}(x)$. Furthermore, $v_{\mathcal{H}}(x') \leq v_{\mathcal{H}}(x)$ also implies $(m(x), y_{\mathcal{H}}(x)) \neq (m(x'), y_{\mathcal{H}}(x'))$ since the pair of integers $(m(x), y_{\mathcal{H}}(x))$ determines uniquely the value $v_{\mathcal{H}}(x)$. Assume next that $x \to x'$ is an H-move for some hyperedge $H \in \mathcal{H}$. Since $\mathcal{H}$ is transversal-free by Lemma 51, there exists $H' \in \mathcal{H}$ such that $H \cap H' = \emptyset$. Then we have $x'_i = x_i \geq m(x)$ for all $i \in H'$, and thus, $y_{\mathcal{H}}(x') \geq m(x) - m(x') + 1$ follows by (24b). $\square$

Lemma 55 *If a hypergraph $\mathcal{H} \subseteq 2^V$ is transversal-free, then the function $\mathcal{U}_{\mathcal{H}}$ satisfies property (A), that is, for all moves $x \to x'$ in $\mathrm{NIM}_{\mathcal{H}}$ we have $\mathcal{U}_{\mathcal{H}}(x) \neq \mathcal{U}_{\mathcal{H}}(x')$.*

Proof: To prove this statement, we consider four cases, depending on the types of the positions x and x', which can be long or short.

If both x and x' are long then $\mathcal{U}_{\mathcal{H}}(x) = \mathcal{T}_{\mathcal{H}}(x) \neq \mathcal{T}_{\mathcal{H}}(x') = \mathcal{U}_{\mathcal{H}}(x')$, since every move strictly decreases the Tetris value by its definition.

If x is long and x' is short then we have $\mathcal{U}_{\mathcal{H}}(x) = \mathcal{T}_{\mathcal{H}}(x) > m(x) \geq m(x') > v_{\mathcal{H}}(x') = \mathcal{U}_{\mathcal{H}}(x')$, proving the claim. Here the first strict inequality is implied by the fact that every move strictly decreases the Tetris value and by Lemma 52 yielding $\mathcal{T}_{\mathcal{H}}(x) > \mathcal{T}_{\mathcal{H}}(x') \geq m(x)$. The inequality $m(x) \geq m(x')$ holds for every move $x \to x'$. Finally $m(x') > v_{\mathcal{H}}(x')$ is implied by Lemma 44.

If x is short and x' is long then we have $\mathcal{U}_{\mathcal{H}}(x') = \mathcal{T}_{\mathcal{H}}(x') \geq m(x)$ by Lemma 52, and $m(x) > v_{\mathcal{H}}(x) = \mathcal{U}_{\mathcal{H}}(x)$ by Lemma 44, which together imply the claim.

Finally, if both x and x' are short then we have $(m(x), y_{\mathcal{H}}(x)) \neq (m(x'), y_{\mathcal{H}}(x'))$ by Lemma 43. If $y_{\mathcal{H}}(x) \neq y_{\mathcal{H}}(x')$, then $v_{\mathcal{H}}(x) \neq v_{\mathcal{H}}(x')$ follows by Lemma 42, since in this case $Z(y_{\mathcal{H}}(x)) \cap Z(y_{\mathcal{H}}(x')) = \emptyset$. If $y_{\mathcal{H}}(x) = y_{\mathcal{H}}(x')$ then by (25b) we have $v_{\mathcal{H}}(x) = v_{\mathcal{H}}(x')$ if and only if $m(x') = m(x) - \alpha y_{\mathcal{H}}(x)$ for some positive integer α. Thus, we must have $m(x') \leq m(x) - y_{\mathcal{H}}(x)$. This implies, by (24b), that $y_{\mathcal{H}}(x') \geq y_{\mathcal{H}}(x) + 1$, which contradicts $y_{\mathcal{H}}(x) = y_{\mathcal{H}}(x')$, completing the proof of our statement. To see the last implication, recall that $\mathcal{H}$ is transversal-free. Thus, if $x \to x'$ is an H-move, then there is a hyperedge $H' \in \mathcal{H}$ such that $H \cap H' = \emptyset$. For this hyperedge we have $x'_i = x_i \geq m(x)$ for all $i \in H'$, from which the claim follows by (24b). $\square$

The above lemma has the following consequence.

Theorem 7 *Let $\mathcal{H}$ be a JM hypergraph, $H, H' \in \mathcal{H}$, $H \cap H' = \emptyset$, and $H \subseteq S \subseteq V \setminus H'$. Then $\mathcal{H}^+ = \mathcal{H} \cup \{S\}$ is also a JM hypergraph with $\mathcal{G}_{\mathcal{H}^+} = \mathcal{G}_{\mathcal{H}}$.*

Proof: Let note first that $H \subseteq S$ implies that $\mathcal{T}_{\mathcal{H}^+} = \mathcal{T}_{\mathcal{H}}$ and consequently $y_{\mathcal{H}^+} = y_{\mathcal{H}}$, $v_{\mathcal{H}^+} = v_{\mathcal{H}}$, and thus $\mathcal{U}_{\mathcal{H}^+} = \mathcal{U}_{\mathcal{H}}$. Furthermore, any move in $\mathrm{NIM}_{\mathcal{H}}$ is still a move in $\mathrm{NIM}_{\mathcal{H}^+}$. Therefore, for every $0 \leq v < \mathcal{U}_{\mathcal{H}^+}(x)$ there exists a move $x \to x'$ in $\mathrm{NIM}_{\mathcal{H}^+}$ such that $\mathcal{U}_{\mathcal{H}^+}(x') = v$. Finally, by $S \cap H' = \emptyset$ the hypergraph $\mathcal{H}^+$ is also transversal-free, and thus by Lemma 55 we have $\mathcal{U}_{\mathcal{H}^+}(x') \neq \mathcal{U}_{\mathcal{H}^+}(x)$ for all moves $x \to x'$ of $\mathrm{NIM}_{\mathcal{H}^+}$. $\square$

5.3 Sufficient conditions

Let us first recall that properties (A) and (B) characterize the SG function of an impartial game. We can reformulate these now for $\mathrm{NIM}_{\mathcal{H}}$, and obtain the following necessary and sufficient condition for $\mathcal{H}$ to be JM:

Lemma 56 *A hypergraph $\mathcal{H} \subseteq 2^V$ is JM if and only if the following conditions hold:*

(A0) $\mathcal{H}$ *is transversal-free.*

(B1) For every long position $x \in \mathbb{Z}^V_{\geq}$ and integer $m(x) \leq z < \mathcal{T}_{\mathcal{H}}(x)$ there exists a move $x \to x'$ such that x' is long and $\mathcal{T}_{\mathcal{H}}(x') = z$.

(B2) For every long position $x \in \mathbb{Z}^V_{\geq}$ and integer $0 \leq z < m(x)$ there exists a move $x \to x'$ such that x' is short and $v_{\mathcal{H}}(x') = z$.

(B3) For every short position $x \in \mathbb{Z}^V_{\geq}$ and integer $0 \leq z < v_{\mathcal{H}}(x)$ there exists a move $x \to x'$ such that x' is short and $v_{\mathcal{H}}(x') = z$.

Proof: It is easy to see by Lemmas 45, 53 and 54 that conditions (B1), (B2), and (B3) are simple and straightforward reformulations of condition (B) for the case of $\text{NIM}_{\mathcal{H}}$ and the function $g = \mathcal{U}_{\mathcal{H}}$ defined in (25a) - (25b).

Finally, Lemma 55 shows that condition (A0) implies (A), while Lemma 51 shows that if $\mathcal{H}$ is JM, then it is also transversal-free. □

5.3.1 General sufficient conditions

Let us next replace conditions (B2) and (B3) with somewhat simpler sufficient conditions.

Lemma 57 *If a hypergraph $\mathcal{H} \subseteq 2^V$ satisfies the following two conditions then it also satisfies (B2) and (B3).*

(C2) For every position $x \in \mathbb{Z}^V_{\geq}$ and integer $1 \leq \eta < y_{\mathcal{H}}(x)$ there exists a move $x \to x'$ such that $m(x') = m(x)$ and $y_{\mathcal{H}}(x') = \eta$.

(C3) For every position $x \in \mathbb{Z}^V_{\geq}$ and integers $0 \leq \mu < m(x)$ and $m(x) - \mu + 1 \leq \eta \leq y_{\mathcal{H}}(x)$ there exists a move $x \to x'$ such that $m(x') = \mu$ and $y_{\mathcal{H}}(x') = \eta$.

Proof: Let us consider first another hypergraph $\widetilde{\mathcal{H}} = \{S \subseteq V \mid 1 \leq |S| \leq n-1\}$. Then by the earlier cited result of Jenkyns and Mayberry [23] $\widetilde{\mathcal{H}}$ is a JM hypergraph. Note also that the games $\text{NIM}_{\mathcal{H}}$ and $\text{NIM}_{\widetilde{\mathcal{H}}}$ are both played over the same set of positions $\mathbb{Z}^V_{\geq}$.

Let us now consider a position $x \in \mathbb{Z}^V_{\geq}$. By Lemma 47 there exists a position $\widetilde{x} \in \mathbb{Z}^V_{\geq}$ such that $m(x) = m(\widetilde{x})$ and $y_{\mathcal{H}}(x) = y_{\widetilde{\mathcal{H}}}(\widetilde{x})$. Now, let us observe that since $\widetilde{\mathcal{H}}$ is JM, properties (B2) and (B3) are satisfied by Lemma 56. Let us also note that if $\widetilde{x} \to \widetilde{x}'$ is a move in $\text{NIM}_{\widetilde{\mathcal{H}}}$ guaranteed to exist by properties (B2) and (B3) then Lemmas 53 and 54 show that $(m(\widetilde{x}'), y_{\widetilde{\mathcal{H}}}(\widetilde{x}')) \in S(m(\widetilde{x}), y_{\widetilde{\mathcal{H}}}(\widetilde{x})) = S(m(x), y_{\mathcal{H}}(x))$, where the set $S(\alpha, \beta)$ is defined as

$$S(\alpha, \beta) = \left\{ (\mu, \eta) \,\middle|\, \begin{array}{c} 0 \leq \mu \leq \alpha, \\ \alpha - \mu + 1 \leq \eta \leq \beta \end{array} \right\} \setminus \{(\alpha, \beta)\}.$$

Let us observe next that properties (C2) and (C3) imply that for every $(\mu, \eta) \in S(m(x), y_{\mathcal{H}}(x))$ there exists a move $x \to x'$ in $\text{NIM}_{\mathcal{H}}$ such that $m(x) = \mu$ and $y_{\mathcal{H}}(x) = \eta$. Thus, for every move $\widetilde{x} \to \widetilde{x}'$ in $\text{NIM}_{\widetilde{\mathcal{H}}}$ that validates properties (B2) and (B3) for $\widetilde{\mathcal{H}}$ we have a corresponding move $x \to x'$ in $\text{N}\textsc{im}_{\mathcal{H}}$ such that $m(\widetilde{x}') = m(x')$ and $y_{\mathcal{H}}(x') = y_{\widetilde{\mathcal{H}}}(\widetilde{x}')$, implying by Lemma 46 that $v_{\mathcal{H}}(x') = v_{\widetilde{\mathcal{H}}}(\widetilde{x}')$ and that x' and $\widetilde{x}'$ are of the same type, that is, both are long or both are short.

Consequently, properties (C2) and (C3) do imply properties (B2) and (B3), as claimed. □

Corollary 15 *If a hypergraph $\mathcal{H}$ satisfies properties (A0), (B1), (C2) and (C3), then it is JM.*

Proof: The claim follows by Lemmas 57 and 56. □

5.3.2 Simplified sufficient conditions

The conditions in Corollary 15 still involve the existence of moves with certain properties. In this section we further weaken those conditions, and replace them with easier to check properties of the hypergraph itself.

Given a hypergraph $\mathcal{H} \subseteq 2^V$, we say that a subfamily $\{H_0, H_1, \ldots, H_p\} \subseteq \mathcal{H}$ forms a *chain* if

$$H_{k+1} \cap H_k \neq \emptyset \quad \text{and} \quad |H_{k+1} \setminus H_k| = 1 \;\text{ for all } k = 0, \ldots, p-1. \tag{27}$$

For convenience, $p = 0$ is possible, that is, a single set is considered to be a chain.

For a subhypergraph $\mathcal{F} \subseteq \mathcal{H} \subseteq 2^V$ we denote by $V(\mathcal{F})$ the set of vertices of $\mathcal{F}$, that is, $V(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} F$. In particular we have $V(\mathcal{H}) = V$.

We shall consider the following properties:

(A1) $\mathcal{H}$ is minimal transversal-free.

(D1) For every pair of hyperedges $H, H' \in \mathcal{H}$ there exists a chain $\mathcal{C} = \{H_0, H_1, \ldots, H_p\} \subseteq \mathcal{H}$ such that $H = H_0$ and $H' = H_p$.

(D2) For every subhypergraph $\mathcal{F} \subseteq \mathcal{H} \subseteq 2^V$ such that $V(\mathcal{F}) \neq V$ there exist hyperedges $F \in \mathcal{F}$ and $S \in \mathcal{H}$ such that $\emptyset \neq (S \setminus F) \subseteq V \setminus V(\mathcal{F})$.

Our main claim in this subsection is that the above properties are sufficient for a hypergraph to be JM.

Theorem 8 *If a hypergraph $\mathcal{H}$ satisfies properties (A1), (D1), and (D2), then it is JM.*

To arrive to a proof of this theorem, we need to prove several consequences of the above properties. In particular, property (D2) we need only to show that it implies the existence of a particular tetris move in $\mathrm{NIM}_{\mathcal{H}}$.

Lemma 58 *If a hypergraph $\mathcal{H} \subseteq 2^V$ satisfies property (D2), then it satisfies the following property as well:*

(D3) For every position $x \in \mathbb{Z}_{\geq}$ with $m(x) > 0$ there exists a hyperedge $H \in \mathcal{H}$ such that $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$ and $m(x^{s(H)}) = m(x) - 1$.

Proof: Let us fix a position $x \in \mathbb{Z}_{\geq}$ for which $m(x) > 0$ holds. An equivalent way of saying property (D3) is that there exists a hyperedge $H \in \mathcal{H}$ such that $x \to x^{s(H)}$ is a tetris move and that for some $i \in H$ we have $x_i = m(x)$.

To prove the lemma let us assume indirectly that this is not the case. In other words, let us introduce

$$\mathcal{F} \;=\; \{H \in \mathcal{H} \mid \mathcal{T}_{\mathcal{H}}(x^{s(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1\},$$

and assume indirectly that $V(\mathcal{F}) \neq V$ (in particular, if $x_i = m(x)$ then $i \notin V(\mathcal{F})$).

By property (D2) we have hyperedges $F \in \mathcal{F}$ and $S \in \mathcal{H}$ such that $S \setminus F \neq \emptyset$ and $S \setminus F \subseteq V \setminus V(\mathcal{F})$. Let us now consider a mapping $\mu : \mathcal{H} \to \mathbb{Z}_{\geq}$ that defines a tetris sequence for position x, that is, we have $\sum_{H \in \mathcal{H}} \mu(H) = \mathcal{T}_{\mathcal{H}}(x)$ and $x' = \sum_{H \in \mathcal{H}} \mu(H)\chi(H) \leq x$. Since $F \in \mathcal{F}$, we can choose μ such that $\mu(F) > 0$. Note that by our assumption, we have for any $i \in V \setminus V(\mathcal{F})$ that $x'_i = 0 < m(x) \leq x_i$. Thus, if we define

$$\mu'(H) \;=\; \begin{cases} \mu(F) - 1 & \text{if } H = F \\ 1 & \text{if } H = S \\ \mu(H) & \text{otherwise,} \end{cases}$$

then μ' also defines a tetris sequence, with $\mu'(S) > 0$ in contradiction with the fact that $S \not\subseteq V(\mathcal{F})$. This contradiction proves that our indirect assumption is not true, that is, $V(\mathcal{F}) = V$, from which the claim of the lemma follows. $\square$

Let us remark that condition (D3) may be necessary for a hypergraph to be JM, but we cannot prove this.

In particular, we are going to show that properties (A1), (D1), and (D3) imply properties (A0), (B1), (C2) and (C3), and thus, Theorem 8 will follow by Corollary 15.

Clearly (A1) implies (A0). The other three implications we will show separately.

Lemma 59 *If a hypergraph $\mathcal{H} \subseteq 2^V$ satisfies properties (A1), (D1), and (D3), then it also satisfies property (B1), that is, for all long positions $x \in \mathbb{Z}^V_{\geq}$ and for all integer values $m(x) \leq z < \mathcal{T}_{\mathcal{H}}(x)$ there exists a move $x \to y$ such that y is long and $\mathcal{T}_{\mathcal{H}}(y) = z$.*

Proof: Let us fix a position $x \in \mathbb{Z}^V_{\geq}$. By Lemma 58 there exist $j \in H \in \mathcal{H}$ such that $\mathcal{T}_{\mathcal{H}}(x^{s(H)}) = \mathcal{T}_{\mathcal{H}}(x) - 1$ and $x_j = m(x)$. By property (A1) the subhypergraph $\mathcal{H}_{V \setminus \{j\}}$ contains a transversal $H' \in \mathcal{H}$. By property (D2) we have then a chain $\{H_0, H_1, \ldots, H_p\}$ such that $H_0 = H$, $H_p = H'$ and $|H_{k+1} \setminus H_k| \leq 1$ for all $k = 0, 1, \ldots, p-1$. Let us then define positions $x^{\alpha,k}$ and $x^{\omega,k}$ for $k = 1, \ldots, p$ by

$$x_i^{\alpha,k} = \begin{cases} 0 & \text{if } i \in H_{k-1} \cap H_k \\ x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \\ x_i & \text{if } i \notin H_k, \end{cases} \qquad x_i^{\omega,k} = \begin{cases} 0 & \text{if } i \in H_k \\ x_i & \text{if } i \notin H_k. \end{cases}$$

Set $x^{\alpha,0} = x^{s(H)}$, and define $x^{\mu,0}$ and $x^{\omega,0}$ by

$$x_i^{\mu,0} = \begin{cases} 0 & \text{if } i = j \\ x_i - 1 & \text{if } i \in H_0 \setminus \{j\} \\ x_i & \text{if } i \notin H_0, \end{cases} \qquad x_i^{\omega,0} = \begin{cases} 0 & \text{if } i \in H_0 \\ x_i & \text{if } i \notin H_k. \end{cases}$$

We claim first that all positions $x^{\alpha,k} \geq y \geq x^{\omega,k}$ are long and are reachable from x by an H_k-move $x \to y$, for $k = 1, \ldots, p$. This is because $m(y) = 0$ for all these positions since they have a zero (namely $y_i = 0$ for all $i \in H_{k-1} \cap H_k$, which is not an empty set by (27)). The analogous claim holds for positions $x^{\mu,0} \geq y \geq x^{\omega,0}$ since $y_j = 0$ for all these positions. We also claim that positions $x^{\alpha,0} \geq y \geq x^{\mu,0}$ are also long whenever x is long (and they are reachable from x by an H_0-move $x \to y$). The last claim is true because we have $m(y) \leq m(x^{\alpha,0}) = m(x) - 1$, and $y_{\mathcal{H}}(y) \geq y_{\mathcal{H}}(x^{\alpha,0}) \geq y_{\mathcal{H}}(x)$ by Lemma 48, and thus, the fact that x is long implies $m(y) < m(x) \leq \binom{y_{\mathcal{H}}(x)}{2} \leq \binom{y_{\mathcal{H}}(y)}{2}$.

Let us observe next that the sets of Tetris values for these ranges of positions form intervals by Lemma 3.2.2. Namely, we have

$$\begin{aligned} \left\{\mathcal{T}_{\mathcal{H}}(y) \mid x^{\alpha,0} \geq y \geq x^{\mu,0}\right\} &= \left[\mathcal{T}_{\mathcal{H}}(x^{\mu,0}), \mathcal{T}_{\mathcal{H}}(x) - 1\right], \\ \left\{\mathcal{T}_{\mathcal{H}}(y) \mid x^{\mu,0} \geq y \geq x^{\omega,0}\right\} &= \left[\mathcal{T}_{\mathcal{H}}(x^{\omega,0}), \mathcal{T}_{\mathcal{H}}(x^{\mu,0})\right], \text{ and} \\ \left\{\mathcal{T}_{\mathcal{H}}(y) \mid x^{\alpha,k} \geq y \geq x^{\omega,k}\right\} &= \left[\mathcal{T}_{\mathcal{H}}(x^{\omega,k}), \mathcal{T}_{\mathcal{H}}(x^{\alpha,k})\right] \text{ for } k = 1, \ldots, p. \end{aligned}$$

We claim that these intervals cover all values in the interval $[m(x), \mathcal{T}_{\mathcal{H}}(x) - 1]$, as stated in the lemma. To see this claim, we show the following inequalities:

$$\begin{aligned} \mathcal{T}_{\mathcal{H}}(x^{\alpha,k}) &\geq \mathcal{T}_{\mathcal{H}}(x^{\omega,k-1}) - 1 \quad \text{for } 1 \leq k \leq p, \text{ and} \\ \mathcal{T}_{\mathcal{H}}(x^{\omega,p}) &\leq m(x). \end{aligned}$$

The first group of inequalities follow by Corollary 1. For the second inequality observe that by our choice the set $H' = H_p$ intersects every hyperedge that does not contain $j \in V$. Thus, the only possible moves from $x^{\omega,p}$ are H-moves for hyperedges $H \in \mathcal{H}$ that contain element j. Since $x_j = m(x)$, the total number of such moves is limited by $m(x)$, as stated. □

Lemma 60 *If a hypergraph $\mathcal{H} \subseteq 2^V$ satisfies properties (A1) and (D1), then it also satisfies property (C2), that is, for every position $x \in \mathbb{Z}^V_{\geq}$ and integer $1 \leq \eta < y_{\mathcal{H}}(x)$ there exists a move $x \to x'$ such that $m(x') = m(x)$ and $y_{\mathcal{H}}(x') = \eta$.*

Proof: Let us fix a position $x \in \mathbb{Z}^V_{\geq}$ and assume that $x_j = m(x)$. Since $y_{\mathcal{H}}(x) + 1 = \mathcal{T}_{\mathcal{H}}(x - m(x)e)$ is a Tetris value, there exists a hyperedge $H \in \mathcal{H}$ such that $y_{\mathcal{H}}(x^{s(H)}) = y_{\mathcal{H}}(x) - 1$. By property (A1) there exists a hyperedge $H' \in \mathcal{H}$ that intersects all hyperedges of $\mathcal{H}$ that do not contain element $j \in V$. Then by property (D1) there exists a chain $\mathcal{C} = \{H_0, \ldots, H_p\}$ such that $H_0 = H$ and $H_p = H'$.

Then let $x^{\alpha,0} = x^{s(H_0)}$, and define $x^{\omega,0}$ by

$$x_i^{\omega,0} = \begin{cases} m(x) & \text{for } i \in H_0 \\ x_i & \text{otherwise,} \end{cases}$$

and positions $x^{\alpha,k}$ and $x^{\omega,k}$ for $k = 1, \ldots, p$ by

$$x_i^{\alpha,k} = \begin{cases} m(x) & \text{if } i \in H_{k-1} \cap H_k \\ x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \\ x_i & \text{if } i \notin H_k, \end{cases} \quad \text{and} \quad x_i^{\omega,k} = \begin{cases} m(x) & \text{if } i \in H_k \\ x_i & \text{if } i \notin H_k. \end{cases}$$

Let us observe next that the sets of $y_{\mathcal{H}}(x')$ values for the ranges $x^{\alpha,k} \geq x' \geq x^{\omega,k}$, $k = 0, 1, \ldots, p$ form intervals by Lemma 3. Namely, we have

$$\left\{ y_{\mathcal{H}}(x') \mid x^{\alpha,k} \geq x' \geq x^{\omega,k} \right\} = \left[y_{\mathcal{H}}(x^{\omega,k}), y_{\mathcal{H}}(x^{\alpha,k}) \right] \text{ for } k = 0, 1, \ldots, p.$$

We claim that these intervals cover all values in the interval $[1, y_{\mathcal{H}}(x) - 1]$. To see this claim, we show the following relations:

$$\begin{array}{rcll} y_{\mathcal{H}}(x^{\omega,p}) & = & 1, & \\ y_{\mathcal{H}}(x^{\alpha,k}) & \geq & y_{\mathcal{H}}(x^{\omega,k-1}) - 1 & \text{for } 1 \leq k \leq p, \text{ and} \\ y_{\mathcal{H}}(x^{\alpha,0}) & = & y_{\mathcal{H}}(x) - 1. & \end{array}$$

The second group of inequalities follow by Corollary 1, since these $y_{\mathcal{H}}$-values are essentially Tetris values by their definition (24b). The first equality is true, since $H_p = H'$ intersects all hyperedges that avoids element $j \in V$, and thus, we have $\mathcal{T}_{\mathcal{H}}(x^{\omega,p} - m(x)e) = 0$. The last equality $y_{\mathcal{H}}(x^{\alpha,0}) = y_{\mathcal{H}}(x^{s(H)}) = y_{\mathcal{H}}(x) - 1$ follows by our choice of the set H. □

Lemma 61 *If a hypergraph $\mathcal{H} \subseteq 2^V$ satisfies properties (A1) and (D1), then it also satisfies property (C3), that is, for every position $x \in \mathbb{Z}_{\geq}^V$ and integers $0 \leq \mu < m(x)$ and $m(x) - \mu \leq \eta \leq y_{\mathcal{H}}(x)$ there exists a move $x \to x'$ such that $m(x') = \mu$ and $y_{\mathcal{H}}(x') = \eta$.*

Proof: Let us fix a position $x \in \mathbb{Z}_{\geq}^V$ and assume that $x_j = m(x)$. Let us further fix an integer $0 \leq \mu < m(x)$.

Let us first choose an arbitrary hyperedge $H \in \mathcal{H}$ such that $j \in H$. By property (A1) there exists another hyperedge $H' \in \mathcal{H}$ that intersects all hyperedges of $\mathcal{H}$ that do not contain element $j \in V$. Then by property (D1) there exists a chain $\mathcal{C} = \{H_0, \ldots, H_p\}$ such that $H_0 = H$ and $H_p = H'$.

Let us then define $x^{\alpha,0}$ and $x^{\omega,0}$ by

$$x_i^{\alpha,0} = \begin{cases} \mu & \text{if } i = j \\ x_i - 1 & \text{if } i \in H_0 \setminus \{j\} \\ x_i & \text{otherwise,} \end{cases} \quad \text{and} \quad x_i^{\omega,0} = \begin{cases} \mu & \text{for } i \in H_0 \\ x_i & \text{otherwise,} \end{cases}$$

and define positions $x^{\alpha,k}$ and $x^{\omega,k}$ for $k = 1, \ldots, p$ by

$$x_i^{\alpha,k} = \begin{cases} \mu & \text{if } i \in H_{k-1} \cap H_k \\ x_i - 1 & \text{if } i \in H_k \setminus H_{k-1} \\ x_i & \text{if } i \notin H_k, \end{cases} \quad \text{and} \quad x_i^{\omega,k} = \begin{cases} \mu & \text{if } i \in H_k \\ x_i & \text{if } i \notin H_k. \end{cases}$$

Let us observe next that the sets of $y_{\mathcal{H}}(x')$ values for the ranges $x^{\alpha,k} \geq x' \geq x^{\omega,k}$, $k = 0, 1, \ldots, p$ form intervals by Lemma 3. Namely, we have

$$\left\{ y_{\mathcal{H}}(x') \mid x^{\alpha,k} \geq x' \geq x^{\omega,k} \right\} = \left[y_{\mathcal{H}}(x^{\omega,k}), y_{\mathcal{H}}(x^{\alpha,k}) \right] \text{ for } k = 0, 1, \ldots, p.$$

We claim that these intervals cover all values in the interval $[1, y_{\mathcal{H}}(x) - 1]$, as stated in the lemma.

To see this claim, we prove the following relations.

$$\begin{array}{lll} y_{\mathcal{H}}(x^{\omega,p}) & = 1, & \\ y_{\mathcal{H}}(x^{\alpha,k}) & \geq \; y_{\mathcal{H}}(x^{\omega,k-1}) - 1 & \text{for } 1 \leq k \leq p, \text{ and} \\ y_{\mathcal{H}}(x^{\alpha,0}) & \geq y_{\mathcal{H}}(x). & \end{array}$$

The second group of inequalities follow by Corollary 1, since these $y_{\mathcal{H}}$-values are essentially Tetris values by their definition (24b). The first equality holds since $H_p = H'$ intersects all hyperedges that avoids element $j \in V$, and thus we have $\mathcal{T}_{\mathcal{H}}(x^{\omega,p} - \mu e) = 0$. The last inequality holds since $x^{\alpha,0} - \mu e \geq x - m(x)e$. $\square$

Proof of Theorem 8: Clearly, property (A1) is stronger than property (A0). Lemmas 59, 60 and 61 imply that properties (B1), (C2), and (C3) hold. Thus, the statement follows by Corollary 15. $\square$

5.4 Matroid hypergraphs

Let us call $\mathcal{H} \subseteq 2^V$ a *matroid hypergraph* if the following exchange property holds for all pairs of hyperedges $H, H' \in \mathcal{H}$:

$$\forall i \in H \setminus H' \;\; \exists j \in H' \setminus H : \;\; (H \setminus \{i\}) \cup \{j\} \in \mathcal{H}. \tag{M}$$

In other words, $\mathcal{H}$ is a matroid hypergraph if it is the set of bases of a matroid (see [35, 36]).

Lemma 62 *Matroid hypergraphs satisfy (D1) and (D2).*

Proof: For property (D1), let us fix two arbitrary hyperedges $H, H' \in \mathcal{H}$, and let us consider a chain $\mathcal{C} = \{H_0, \ldots, H_p\} \subseteq \mathcal{H}$ such that $H_0 = H$ and $d(\mathcal{C}) = |H' \setminus H_p|$ is as small as possible. Since there are only finitely many different chains in $\mathcal{H}$, the quantity $d(\mathcal{C})$ is well defined. We claim that $d(\mathcal{C}) = 0$, which implies property (D1), since this applies to any two hyperedges. To see this claim, assume indirectly that $d(\mathcal{C}) > 0$, and apply the exchange axiom for sets H_p and H'. Since $d(\mathcal{C}) > 0$ we have an element $i \in H_p \setminus H'$, and by axiom (M) there exists an element $j \in H' \setminus H_p$ such that $H_{p+1} = (H_p \setminus \{i\}) \cup \{j\} \in \mathcal{H}$. Then for $\mathcal{C}' = \{H_0, \ldots, H_p, H_{p+1}\}$ it follows that $d(\mathcal{C}') = d(\mathcal{C}) - 1$, contradicting the fact that $d(\mathcal{C})$ is as small as possible. This contradiction proves our claim.

For property (D2), let us consider an arbitrary subfamily $\mathcal{F} \subseteq \mathcal{H}$ such that $V(\mathcal{F}) \neq V$. Let us choose two distinct sets $S \in \mathcal{H}$ and $F \in \mathcal{F}$ with minimum $|(S \setminus F) \cap V(\mathcal{F})|$. We claim that $|(S \setminus F) \cap V(\mathcal{F})| = 0$, and hence these sets S and F show property (D2).

Assume a contrary that there exists an element i in $(S \setminus F) \cap V(\mathcal{F})$. By the exchange axiom there exists a $j \in F \setminus S$, such that $S' = (S \setminus \{i\}) \cup \{j\} \in \mathcal{H}$. Then we have $|(S' \setminus F) \cap V(\mathcal{F})| < |(S \setminus F) \cap V(\mathcal{F})|$, which contradicts our assumption. $\square$

Theorem 9 *Let $\mathcal{H}$ be a matroid hypergraph. Then $\mathcal{H}$ is a JM hypergraph if and only if it is minimal transversal-free.*

Proof: It follows from Theorem 8 and Lemmas 51 and 62. $\square$

Corollary 16 *Self-dual matroid hypergraphs are JM.*

Proof: We apply Theorem 9, and show that self-dual matroid hypergraphs are minimal transversal-free.

Let $\mathcal{H}$ be a self-dual matroid. Since for every hyperedge in $\mathcal{H}$, its complement is also a hyperedge, no $H \in \mathcal{H}$ is a transversal of $\mathcal{H}$. Furthermore, by self-duality of $\mathcal{H}$, any hyperedge $H \in \mathcal{H}$ has size $k = n/2$. In any proper induced subhypergraph on at most $2k - 1$ elements any two hyperedges of size k must intersect. Therefore $\mathcal{H}$ is minimal transversal-free. $\square$

Recall that any matroid hypergraph is k-uniform for some k. If $k > n/2$, then it is not transversal-free, and hence not JM. If $k = n/2$, then we can see that a matroid hypergraph is

JM if and only if it is self-dual. For $k < n/2$, we remark that no matroid hypergraph is self-dual. However, the following discussion shows that there are a number of JM matroid hypergraphs.

Let $V = \{1, \ldots, 7\}$, and define $\mathcal{H} \subseteq 2^V$ by $\mathcal{H} = \binom{V}{3} \setminus \mathcal{F}$, where $\mathcal{F}$ denotes Fano plane, i.e.,

$$\mathcal{F} = \{\{1,2,3\}, \{1,4,5\}, \{2,4,6\}, \{1,6,7\}, \{2,5,7\}, \{3,4,7\}, \{3,5,6\}\}.$$

Then we can see that $\mathcal{H}$ is a matroid hypergraph and minimal transversal-free.

We extend the above example and show that there exists a large family of matroid hypergraphs with $n = 2k + \delta$ for $\delta > 0$, which are minimal transversal-free and hence JM.

Let δ and k be integers such that $0 < \delta \leq k - 2$, and assume that $V = \{1, \ldots, n\}$ where $n = 2k + \delta$. Let us consider a $(k + \delta - 1)$-uniform hypergraph $\mathcal{K} \subseteq 2^V$ satisfying the following three conditions.

(K1) $|K \cap K'| \geq \delta$ for all hyperedges $K, K' \in \mathcal{K}$.

(K2) $|K \cap K'| \leq k - 2$ for all distinct hyperedges $K, K' \in \mathcal{K}$.

(K3) No singleton is a transversal of $\mathcal{K}$.

Define

$$\mathcal{H} = \binom{V}{k} \setminus \left\{ \binom{K}{k} \mid K \in \mathcal{K} \right\}. \tag{28}$$

Lemma 63 *Let $\mathcal{K}$ be a $(k + \delta - 1)$-uniform hypergraph that satisfies (K2), $S \subseteq V$ be a set of size $|S| = k - 1$, and $W \subseteq V \setminus S$. Assume that for any $v \in W$, $S \cup \{v\}$ is not contained in $\mathcal{H}$. Then we have $|W| \leq \delta$.*

Proof: Since $S \cup \{v\}$ is not contained in $\mathcal{H}$ for every $v \in W$, there exists K_v in $\mathcal{K}$ such that $K_v \supseteq S \cup \{v\}$. Note that the union of the sets K_v is of size at least $k - 1 + |W|$, and thus if $|W| \geq \delta + 1$, we have two elements v and v' in W such that $K_v \neq K_{v'}$. However, since $K_v \cap K_{v'} \supseteq S$, this contradicts (K2). □

Lemma 64 *If a $(k + \delta - 1)$-uniform hypergraph $\mathcal{K}$ satisfies (K1), (K2), and (K3), then $\mathcal{H}$ is minimal transversal-free.*

Proof: We first show that $\mathcal{H}$ is transversal-free, i.e., any hyperedge $H \in \mathcal{H}$ has a hyperedge $H' \in \mathcal{H}$ such that $H \cap H' = \emptyset$. Let $S \subseteq V \setminus H$ with $|S| = k - 1$, and $W = V \setminus (H \cup S)$. Since $|W| > \delta$, by Lemma 63 we must have at least one $v \in W$ such that $H' = S \cup \{v\}$ belongs to $\mathcal{H}$.

We next show that for any nonempty set $R \subseteq V$, the induced subhypergraph $\mathcal{H}_{V \setminus R}$ is either empty or it has a transversal $T \in \mathcal{H}_{V \setminus R}$.

If $|R| \geq \delta + 1$, then any two hyperedges in $\mathcal{H}_{V \setminus R}$ intersect. Thus it remains to consider the case of $|R| \leq \delta$. Let $K \in \mathcal{K}$ be a hyperedge that maximizes $|R \setminus K|$. Then by (K3) we have $R \not\subseteq K$, and thus $|V \setminus (R \cup K)| \leq k$ is implied.

If $|V \setminus (R \cup K)| < k$, let S be a set such that $V \setminus (R \cup K) \subseteq S \subseteq V \setminus R$ and $|S| = k - 1$, and let $W = K \setminus (R \cup S)$. Since $|W| = k + \delta + 1 - |R| > \delta$, by Lemma 63 we must have at least one $v \in W$ such that $H' = S \cup \{v\}$ belongs to $\mathcal{H}$. We claim that H' is a transversal of $\mathcal{H}_{V \setminus R}$. Indeed, $V \setminus (H' \cup R)$ is a subset of K, and thus no subset of it (of size k) is contained in $\mathcal{H}$.

On the other hand, if $|V \setminus (R \cup K)| = k$, then $H' = V \setminus (R \cup K)$ is a transversal hyperedge in $\mathcal{H}_{V \setminus R}$. Indeed, $H' \in \mathcal{H}_{V \setminus R}$, since otherwise there exists a $K' \in \mathcal{K}$ such that $H' \subseteq K'$. This implies $|K \cap K'| < \delta$, contradicting (K1). Furthermore, $\mathcal{H}_{V \setminus R}$ contains no hyperedge disjoint from H', since $V \setminus (R \cup H')$ is a subset of $K \in \mathcal{K}$. □

Lemma 65 *If a $(k+\delta-1)$-uniform hypergraph $\mathcal{K}$ satisfies (K2), then $\mathcal{H}$ is a matroid hypergraph.*

Proof: Consider two distinct hyperedges $H, H' \in \mathcal{H}$. We assume that (M) does not holds for H and H', and derive a contradiction.

If $|H \cap H'| = k - 1$ then (M) clearly holds, and hence we have $|H \cap H'| \leq k - 2$. By our assumption, there exists an element $i \in H \setminus H'$ such that any $j \in H' \setminus H$ satisfies $(H \setminus \{i\}) \cup \{j\} \notin \mathcal{H}$. This means that for any $j \in H' \setminus H$, we have $K_j \in \mathcal{K}$ such that $K_j \supseteq (H \setminus \{i\}) \cup \{j\}$. We note that $H' \setminus H$ contains two elements j, ℓ that satisfy $K_j \neq K_\ell$, since otherwise K_j contains

H', a contradiction on its own. Since for these indices j and ℓ, we have $|K_j \cap K_\ell| \geq k-1$, we get a contradiction with (K2). $\square$

Theorem 10 *If a $(k+\delta-1)$-uniform hypergraph $\mathcal{K}$ satisfies (K1), (K2), and (K3), then $\mathcal{H}$, defined by* (28)*, is a JM and matroid hypergraph.*

Proof: Follows from Lemmas 64 and 65. $\square$

Now let us construct a hypergraph $\mathcal{K}$ with the desired properties.

Let δ and k be integers such that $0 < \delta \leq k-2$. Define V by $V = W \cup U$, where $W = \{1, \dots, k+\delta-1\}$, and $U = \{1', \dots, (k+1)'\}$. Note that $|V| = (k+\delta-1)+(k+1) = 2k+\delta$. Let

$$\mathcal{K} = \{W, \{1, \dots, \delta\} \cup \{1', \dots, (k-1)'\}, \{\delta+1, \dots, 2\delta\} \cup \{3', \dots, (k+1)'\}\}.$$

It is not difficult to see that $\mathcal{K} \subseteq 2^V$ is a $(k+\delta-1)$-uniform hypergraph satisfying (K1), (K2), and (K3).

5.5 JM hypergraphs arising from graphs

Given an integer $k < |E|$, let us define

$$\mathcal{F}_{e,c}(G,k) = \{F \subseteq E \mid |F| = k, (U(F), F) \text{ is connected}\} \tag{29}$$

$$\mathcal{F}_{v,c}(G,k) = \left\{ U(F) \subseteq U \;\middle|\; \begin{array}{c} F \subseteq E, |F| = k, \\ (U(F), F) \text{ is connected} \end{array} \right\} \tag{30}$$

$$\mathcal{F}_{e,t}(G,k) = \{F \subseteq E \mid |F| = k, (U(F), F) \text{ is a tree}\} \tag{31}$$

$$\mathcal{F}_{v,t}(G,k) = \left\{ U(F) \subseteq U \;\middle|\; \begin{array}{c} F \subseteq E, |F| = k, \\ (U(F), F) \text{ is a tree} \end{array} \right\}. \tag{32}$$

Lemma 66 *If $G = (U, E)$ is a connected graph and $k < |E|$, then the hypergraphs $\mathcal{F}_{e,c}(G,k)$, $\mathcal{F}_{e,t}(G,k)$ for $k \geq 2$ and $\mathcal{F}_{v,c}(G,k)$, $\mathcal{F}_{v,t}(G,k)$ for $k \geq 1$ satisfy property (D1).*

Proof: We are going to prove the statement for $\mathcal{F}_{e,c}(G,k)$. For the others similar proofs work. Let $A, B \in \mathcal{F}_{e,c}(G,k)$ and define

$$d(A,B) = -\mu(A,B) + \rho(A,B),$$

where $\rho(A,B)$ denotes the length of a shortest path between $U(A)$ and $U(B)$ in G, and $\mu(A,B)$ is the size of a maximum connected component in $A \cap B$. We claim that if $A \neq B$ then there exists a $D \in \mathcal{F}_{e,c}(G,k)$ such that $A \cap D \neq \emptyset, |D \setminus A| = 1, d(A,B) > d(D,B)$. By repeatedly applying this claim, we can construct a chain from A to B.

Case 1: $U(A) \cap U(B) = \emptyset$. Let $P \subseteq E$ be a shortest path connecting $U(A)$ with $U(B)$, and $e \in P$ be the first edge incident with $U(A)$. There exists an edge $f \in A$ such that $D = (A \cup \{e\}) \setminus \{f\}$ is connected: If A contains a cycle then f could be any edge of this cycle, otherwise $A \cup \{e\}$ is a tree, and therefore it must have a leaf edge $f \neq e$.

Case 2: $U(A) \cap U(B) \neq \emptyset$. Choose a maximum connected component $K \subseteq A \cap B$. Choose $e \in B \setminus A$ incident with K. Such an edge exists since B is connected and $B \neq A$. Then there exists $f \in A \setminus K$ such that $D = (A \cup \{e\} \setminus \{f\}$ is connected. To see this contract edges $K \cup \{e\}$. If $A \setminus K$ contains a cycle after this contraction, then any edge of this cycle can be chosen. Otherwise we choose a leaf edge $f \in A \setminus K$.

Thus, the set D satisfies the claim in both cases. $\square$

Lemma 67 *If $G = (U, E)$ is a connected graph and $1 \leq k < |E|$, then the hypergraphs $\mathcal{F}_{e,c}(G,k)$, $\mathcal{F}_{e,t}(G,k)$, $\mathcal{F}_{v,c}(G,k)$, and $\mathcal{F}_{v,t}(G,k)$ satisfy property (D2).*

Proof: We are going to prove the statement for $\mathcal{F}_{e,c}(G,k)$. For the others similar proofs work.

Assume that $\mathcal{F} \subseteq \mathcal{F}_{e,c}(G,k)$ is a subfamily such that $\bigcup_{F \in \mathcal{F}} F \neq E$. Let us denote by W the set of vertices incident to some of the sets in $\mathcal{F}$, and let e be an edge of G incident with W that does not belong to any of the sets in $\mathcal{F}$. Such an edge e exists by our assumption. Let us denote by $w \in W$ the vertex with which e is incident (note that e maybe incident with two vertices of

W). Since $w \in W$, there is a set $F \in \mathcal{F}$ such that $w \in U(F)$. We claim that there exists $f \in F$ such that $S = (F \setminus \{f\}) \cup \{e\} \in \mathcal{F}_{e,c}(G,k)$. Namely, if $F \cup \{e\}$ contains a cycle then any $f \neq e$ of this cycle can be chosen, otherwise $F \cup \{e\}$ is a tree, and therefore it must have a leaf edge $f \neq e$. Then the pair of sets F and S proves that property (D2) holds. $\square$

Theorem 11 *Let $G = (U,E)$ be a connected graph. If $\mathcal{F}_{e,*}(G,k)$ for $k \geq 2$ is minimal transversal-free, where $* \in \{c,t\}$, then it is JM. Similarly, if $\mathcal{F}_{v,*}(G,k)$ for $k \geq 1$ is minimal transversal-free, where $* \in \{c,t\}$, then it is JM.*

Proof: Property (A1) follows by our assumption, while properties (D1) and (D2) follow by Lemmas 66 and 67. Thus, the statement is implied by Theorem 8. $\square$

There are several infinite families of graphs for which we can apply Theorem 11. We use standard graph theoretical notation, see e.g., [21]. We denote by C_n the simple cycle on n vertices, $K_{a,b}$ the complete bipartite graph with a and b vertices in the two classes, etc.

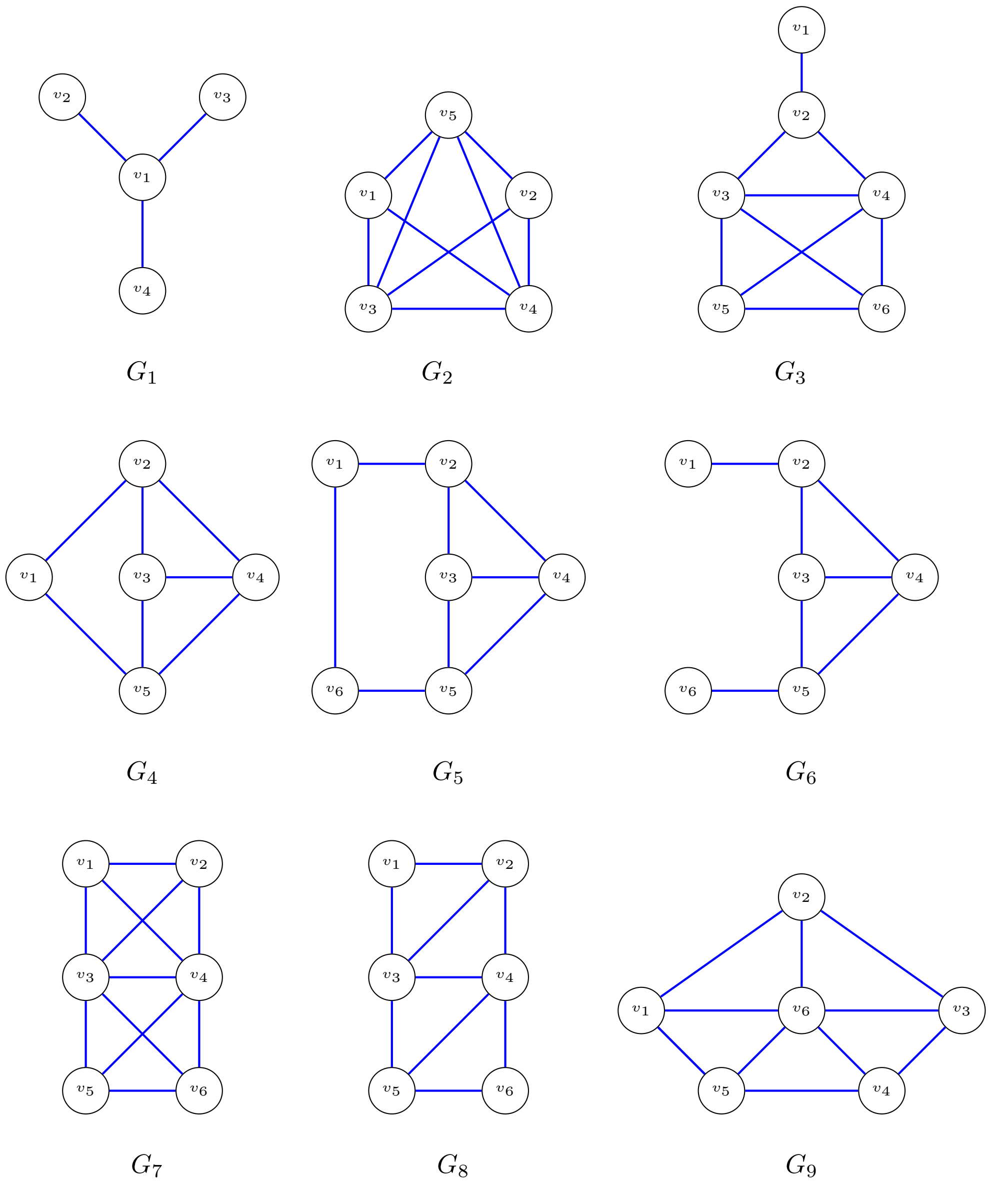

Figure 7: The nine forbidden induced subgraphs characterizing line graphs, see [2].

Circulant hypergraphs: For any given value of $k \geq 2$, it is easy to see that the graphs C_{2k}

and C_{2k+1} yield JM hypergraphs with all four definitions. In fact, they are isomorphic families for both C_{2k} and C_{2k+1}.

Additional self-dual matroids: Graphs $K_{2,k}$ are also good example for $k \geq 2$ with both definitions (29) and (31). Interestingly, both hypergraphs are self-dual matroids, and they are not isomorphic with one another for $k \geq 4$.

Trees: For any $k \geq 2$ the following subfamily of trees on $k^2 + k$ edges provide good examples with both definitions involving edge subsets. Let T_i, $i = 1, \ldots, k+1$ be an arbitrary trees of k edges each on distinct sets of vertices, and let v_i be a leaf vertex of T_i for all $i = 1, \ldots, k+1$. Then we can get a tree T by identifying these leaf vertices. T has $k^2 + k$ edges, and the family $\mathcal{F}_{e,c}(T, k)$ is minimal transversal-free. (Note that definitions (29) and (31) yield isomorphic hypergraphs in this case.)

Star of cliques: Another example is a graph G formed by $k+1$ cliques on k vertices each, joined by one-one edges to a common root vertex. In this case only definition (31) yields a hypergraph that is, $\mathcal{F}_{e,t}(G, k)$ minimal transversal-free and has $(k+1)(\binom{k}{2} + 1)$ vertices.

Petersen: Finally, a singular example is provided by the Petersen graph P for which the family $\mathcal{F}_{e,c}(P, 7)$ is minimal transversal-free.

In Section 5.8, we show that the number of JM hypergraphs defined by (29), (30), (31), and (32) is bounded by a function of k.

5.6 JM graphs

In this subsection we provide a complete characterization of JM *graphs*, which are 2-uniform JM hypergraphs. Note that

$$E = \mathcal{F}_{v,c}(G, 1) = \mathcal{F}_{v,t}(G, 1) \tag{33}$$

holds for any graph $G = (V, E)$, which implies the following result.

Theorem 12 *A graph G is JM if and only if it is connected and minimal transversal-free.*

Proof: The necessity follows from Lemmas 50 and 51, and the sufficiency follows from Theorem 11 and (33). □

In the sequel we characterize all connected minimal transversal-free graphs.

Lemma 68 *If a graph $G = (V, E)$ is connected and minimal transversal-free, then it is the line graph of a simple graph.*

Proof: Let us indirectly assume that G is not a line graph. Then G must contain one of the 9 forbidden induced subgraphs shown in [2], see Figure 7. We claim that none of these 9 graphs can be an induced subgraph of G, since we assumed that G is minimal transversal-free. For this, note that in each of the graphs G_i, $i = 2, \ldots, 9$ contains as a proper induced subgraph at least one of C_4, C_5, K_4, or $2K_2$. In all cases we do not have an edge that would intersect all others.

We claim that the claw G_1 cannot be an induced subgraph of G. Since G_1 is not transversal-free, we assume that G_1 is a proper induced subgraph of G, and derive a contradiction. Remove from G a leaf of the claw, say v_2. Then $G \setminus v_2$ has an intersecting edge e, i.e., an edge e in $G \setminus v_2$ intersects all edges in $G \setminus v_2$. We note that e is incident with the center of the claw v_1.

Suppose that e is an edge of the claw, say $e = (v_1, v_3)$. Then v_4 is a vertex of degree 1 in G. Let us denote by e' the intersecting edge in $G \setminus v_4$. The edge e' must be incident with the center v_1, and hence e' is an intersecting edge of G, which contradicts that G is transversal-free.

Suppose now that e is not contained in the claw, say $e = (v_1, v_5)$ where v_5 is not in G_1. If v_3 or v_4 is of degree 1 then by the same argument G has an intersecting edge. Otherwise G contains edge $e_1 = (v_3, v_5)$. Since G is transversal-free, there must exist an edge e_2 disjoint from e through v_2, say $e_2 = (v_2, v_6)$. Consider again $G \setminus v_4$. By our assumption it has an intersecting edge e_3. As we know, e_3 is incident with v_1. However, no such edge can intersect both e_1 and e_2. This contradicts that e_3 is an intersecting edge in $G \setminus v_4$. □

The following statement is straightforward from the definition.

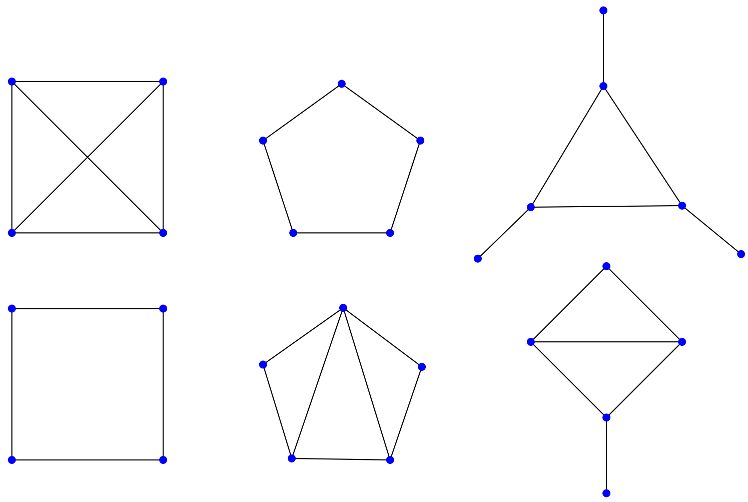

Figure 8: The six JM graphs

Lemma 69 *For every simple graph G we have that $\mathcal{F}_{e,c}(G,2)$ is the edge set of the line graph of G.* $\square$

Lemma 70 *If $G = (V, E)$ is a JM graph then we have $4 \leq |V| \leq 6$.*

Proof: It follows from Theorem 12 that G has at least two disjoint edges. Hence we have $|V| \geq 4$. By Theorem 12 and Lemmas 68 and 69, there exists a graph $G^* = (V^*, E^*)$ such that $E = \mathcal{F}_{e,c}(G^*, 2)$. By Theorem 12 and Lemma 75 we have $|E^*| \leq 2^2 + 2 = 6$, where Lemma 75 can be found in the Section 5.8. Since G is the line graph of G^*, we have $E^* = V$, implying $|V| \leq 6$. $\square$

Theorem 13 *Among all graphs, only six graphs in Figure 8 are JM.*

Proof: It is easy to see that all six graphs are connected and minimal transversal-free. Since graphs correspond to $\mathcal{F}_{v,c}(G, 1)$, by Theorem 11 they are JM graphs.

To show that no other JM graph exists, it is sufficient to check all graphs $G = (V, E)$ with $4 \leq |V| \leq 6$ by Lemma 70; see e.g., [21] for a complete list of graphs with up to 6 vertices. $\square$

Before concluding this subsection, we remark that Lemma 70 provides a tighter bound than the one in (iii) of Lemma 75, when $k = 1$.

5.7 Symmetric hypergraphs

Consider an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \leq n$ and an associated hypergraph

$$\mathcal{H}(\lambda) = \bigcup_{j=1}^{k} \binom{V}{\lambda_j}. \tag{34}$$

The sequence λ is called the *size sequence* of the symmetric hypergraph $\mathcal{H}(\lambda)$. All symmetric hypergraphs have a size sequence and arise in this way (recall that a hypergraph considered is not $\emptyset$ and does not contain $\emptyset$).

The following two theorems summarize our main results.

Theorem 14 *A symmetric hypergraph is JM if and only if it is minimal transversal-free and $n \geq 3$.*

Let us remark that for symmetric hypergraphs $\mathcal{T}_{\mathcal{H}}(x) = \mathcal{T}_{\binom{n}{\lambda_1}}(x)$ by Lemma 11, which can be calculated in polynomial time, for every position x.

Theorem 15 *A symmetric hypergraph $\mathcal{H}(\lambda)$ defined by (34) is minimal transversal-free if and only if its size sequence satisfies the following relations:*

(i) $\lambda_{i+1} - \lambda_i \leq \lambda_1$ *for all* $i \in [k-1]$.

(ii) $\lambda_1 + \lambda_k = n$.

Theorems 14 and 15 extend two previous results stating that $\bigcup_{j=1}^{n-1} \binom{V}{k}$ and $\binom{V}{n/2}$ are JM hypergraphs [9, 23].

Let us also add that by Theorem 7 we can derive numerous non-symmetric JM hypergraphs from the above family of symmetric ones.

We finally remark that it is not easy to describe the closed form of the SG function of symmetric hypergraph NIM games, in general. For example, the case $\binom{[4]}{1} \cup \binom{[4]}{2}$ seems to be difficult. In this case at least the losing positions are known, due to [25]. For the case of $\binom{V}{2}$ for $n = 5$ we are not even aware of a useful characterization of the set of losing positions. In contrast, $\binom{V}{2} \cup \binom{V}{3}$ is a symmetric JM hypergraph by our theorems above.

5.7.1 Proof of Theorem 15

Let us assume first that $\mathcal{H} = \mathcal{H}(\lambda)$ is minimal transversal-free.

If there is an index i such that $\lambda_{i+1} - \lambda_i > \lambda_1$, then let us consider a proper subset $S \subseteq V$ of size $|S| = \lambda_{i+1} - 1 < n$. The induced subhypergraph $\mathcal{H}_S$ has no transversal hyperedge. This is because the largest hyperedge in $\mathcal{H}_S$ is of size λ_i, and for any such edge its complement in S still contains a hyperedge of size λ_1. This contradicts our assumption. Hence, $\lambda_{i+1} - \lambda_i \leq \lambda_1$ must hold for all $i = 1, \ldots, k-1$.

If $\lambda_1 + \lambda_k < n$, then consider a subset $S \subseteq V$ of size $|S| = n - 1$. By a similar argument as above, we can see that $\mathcal{H}_S$ has no transversal hyperedge, leading to a contradiction.

Finally, if $\lambda_1 + \lambda_k > n$, then any hyperedge of size λ_k is a transversal hyperedge of $\mathcal{H}$, contradicting our assumption. Thus, we must have $\lambda_1 + \lambda_k = n$.

Assume next that conditions (i) and (ii) of Theorem 15 hold.

In this case condition (ii) implies that $\mathcal{H}$ has no transversal hyperedge. Let us consider an arbitrary proper subset $\emptyset \neq S \subsetneq V$. If $|S| < \lambda_1$, then $\mathcal{H}_S$ is an empty hypergraph. If $\lambda_i \leq |S| < \lambda_{i+1}$ for some index $1 \leq i \leq k$ (assuming $\lambda_{k+1} = n$), then any hyperedge of size λ_i inside S is a transversal of $\mathcal{H}_S$. Therefore conditions (i) and (ii) imply that $\mathcal{H} = \mathcal{H}(\lambda)$ is minimal transversal-free.

□

5.7.2 Proof of Theorem 14

It is easy to verify that if $n \leq 2$, then there exist no JM hypergraphs. Thus, we can assume in what follows that $n \geq 3$.

Observe next that by Lemma 51 a JM hypergraph must be minimal transversal-free.

For the reverse direction we consider a symmetric minimal transversal-free hypergraph $\mathcal{H}(\lambda)$. If $\lambda_1 = 1$ then by Theorem 15 we have $\lambda = (1, 2, \ldots, n-1)$. Consequently the hypergraph $\mathcal{H}(\lambda)$ coincides with the one considered in [23], and thus their result implies our claim.

For the remaining cases, when $\lambda_1 > 1$, we show that the sufficient conditions of Lemma 56 hold. We break this proof into three technical lemmas, and start with the simplest one.

For positions $a, b \in \mathbb{Z}_+^V$, $a \leq b$ we define $[a, b] = \{x \in \mathbb{Z}_+^V \mid a \leq x \leq b\}$.

Lemma 71 *If $\mathcal{H} = \mathcal{H}(\lambda)$ is a symmetric hypergraph such that its size sequence λ satisfies conditions (i) and (ii) of Theorem 15, then condition (C2) holds.*

Proof: Let us consider a position $x \in \mathbb{Z}_+^V$ such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_{\mathcal{H}}(x) > 1$. Let us define $\ell = \max\{j \mid x_{\lambda_j} > m(x)\}$. By the assumption $y_{\mathcal{H}}(x) > 1$ it is well-defined.

Next we define $\lambda_0 = 0$ and positions a^i, b^i for $i = 1, \ldots, \ell$ as follows.

$$a_j^i = \begin{cases} m(x) & \text{if } j \leq \lambda_{i-1}, \\ x_j - 1 & \text{if } \lambda_{i-1} < j \leq \lambda_i, \\ x_j & \text{otherwise,} \end{cases} \qquad b_j^i = \begin{cases} m(x) & \text{if } j \leq \lambda_i, \\ x_j & \text{otherwise.} \end{cases}$$

Note first that for all indices $i = 1, \ldots, \ell$ we have $a^i \geq b^i$, and there exists a hyperedge $H^i \in \mathcal{H}$ such that both $x \to a^i$ and $x \to b^i$ are H^i-moves.

Let us note next that due to the definition of $y_{\mathcal{H}}$, condition (i) of Theorem 15, and Lemma 8 we have $y_{\mathcal{H}}(a^{i+1}) \geq y_{\mathcal{H}}(b^i) - 1$ for $i = 1, \ldots, \ell - 1$. Furthermore $y_{\mathcal{H}}(a^1) = y_{\mathcal{H}}(x) - 1$ by Lemma 8 and $y_{\mathcal{H}}(b^\ell) = 1$.

Let us then note that we have $m(x') = m(x)$ for all $x' \in \bigcup_{i=1}^{\ell}[a^i, b^i]$. Furthermore we can apply Lemma 2 to the pairs (a^i, b^i), $i = 1, \ldots, \ell$ and obtain

$$\bigcup_{i=1}^{\ell} \{y_{\mathcal{H}}(x') \mid x' \in [a^i, b^i]\} \;=\; [1, y_{\mathcal{H}}(x) - 1].$$

$\square$

Lemma 72 *If $\mathcal{H} = \mathcal{H}(\lambda)$ is a symmetric hypergraph such that its size sequence λ satisfies conditions (i) and (ii) of Theorem 15 and $\lambda_1 > 1$, then condition (C3) holds.*

Proof: Let us consider a position $x \in \mathbb{Z}_+^V$ such that $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_{\mathcal{H}}(x) > 1$. Let us also consider an integer μ that satisfies $0 \leq \mu < m(x)$ and $m(x) - \mu + 1 \leq \eta \leq y_{\mathcal{H}}(x)$, as in (C3).

Next we define positions a^i, b^i for $i = 0, \ldots, k$ as follows.

$$a_j^0 = \begin{cases} x_j - 1 & \text{if } j < \lambda_1, \\ \mu & \text{if } j = n, \\ x_j & \text{otherwise,} \end{cases} \qquad b_j^0 = \begin{cases} \mu & \text{if } j < \lambda_1 \text{ or } j = n, \\ x_j & \text{otherwise.} \end{cases}$$

$$a_j^1 = \begin{cases} \mu & \text{if } j < \lambda_1, \\ x_j - 1 & \text{if } j = \lambda_1, \\ x_j & \text{otherwise,} \end{cases} \qquad b_j^1 = \begin{cases} \mu & \text{if } j \leq \lambda_1, \\ x_j & \text{otherwise.} \end{cases}$$

Note that $m(a^1) = \mu$ because we assumed $\lambda_1 > 1$.

For $i = 2, \ldots, k$ we set

$$a_j^i = \begin{cases} \mu & \text{if } j \leq \lambda_{i-1}, \\ x_j - 1 & \text{if } \lambda_{i-1} < j \leq \lambda_i, \\ x_j & \text{otherwise,} \end{cases} \qquad b_j^i = \begin{cases} \mu & \text{if } j \leq \lambda_i, \\ x_j & \text{otherwise.} \end{cases}$$

Note first that for all indices $i = 0, \ldots, k$ we have $a^i \geq b^i$, and there exists a hyperedge $H^i \in \mathcal{H}$ such that both $x \to a^i$ and $x \to b^i$ are H^i-moves.

Let us note next that due to the definition of $y_{\mathcal{H}}$, condition (i) of Theorem 15, and Lemma 8 we have $y_{\mathcal{H}}(a^{i+1}) \geq y_{\mathcal{H}}(b^i) - 1$ for $i = 1, \ldots, k - 1$. Furthermore $y_{\mathcal{H}}(a^0) \geq y_{\mathcal{H}}(x)$ and $y_{\mathcal{H}}(b^k) = m(x) - \mu + 1$ because of condition (ii) of Theorem 15.

Let us then note that we have $m(x') = \mu$ for all $x' \in \bigcup_{i=0}^{k}[a^i, b^i]$ because we assumed $\lambda_1 > 1$. Furthermore we can apply Lemma 2 to the pairs (a^i, b^i), $i = 0, \ldots, k$ and obtain

$$\bigcup_{i=0}^{k} \{y_{\mathcal{H}}(x') \mid x' \in [a^i, b^i]\} \supseteq [m(x) - \mu + 1, y_{\mathcal{H}}(x)].$$

$\square$

Lemma 73 *If $\mathcal{H} = \mathcal{H}(\lambda)$ is a symmetric hypergraph such that its size sequence λ satisfies conditions (i) and (ii) of Theorem 15 and $\lambda_1 > 1$, then condition (B1) holds.*

Proof: Let us consider a long position $x \in \mathbb{Z}_+^V$ such that $x_1 \geq x_2 \geq \cdots \geq x_n$. Next we define positions a^i, b^i for $i = 0, \ldots, k$ and c^0 as follows.

$$a_j^0 = \begin{cases} x_j - 1 & \text{if } j < \lambda_1 \text{ or } j = n, \\ x_j & \text{otherwise,} \end{cases} \qquad b_j^0 = \begin{cases} 0 & \text{if } j < \lambda_1 \text{ or } j = n, \\ x_j & \text{otherwise.} \end{cases}$$

$$c_j^0 = \begin{cases} x_j - 1 & \text{if } j < \lambda_1, \\ 0 & \text{if } j = n, \\ x_j & \text{otherwise,} \end{cases}$$

$$a_j^1 = \begin{cases} 0 & \text{if } j < \lambda_1, \\ x_j - 1 & \text{if } j = \lambda_1, \\ x_j & \text{otherwise,} \end{cases} \qquad b_j^1 = \begin{cases} 0 & \text{if } j \le \lambda_1, \\ x_j & \text{otherwise.} \end{cases}$$

For $i = 2, \ldots, k$ we set

$$a_j^i = \begin{cases} 0 & \text{if } j \le \lambda_{i-1}, \\ x_j - 1 & \text{if } \lambda_{i-1} < j \le \lambda_i, \\ x_j & \text{otherwise,} \end{cases} \qquad b_j^i = \begin{cases} 0 & \text{if } j \le \lambda_i, \\ x_j & \text{otherwise.} \end{cases}$$

Note first that we have $a^0 \ge c^0 \ge b^0$ and $a^i \ge b^i$ for all indices $i = 1, \ldots, k$. There exists a hyperedge H^0 such that $x \to a^0$, $x \to c^0$, and $x \to b^0$ are all H^0-moves. For $i = 1, \ldots, k$ there exists a hyperedge $H^i \in \mathcal{H}$ such that both $x \to a^i$ and $x \to b^i$ are H^i-moves. Furthermore, we have $m(a^0) = m(x) - 1$, $m(c^0) = m(b^0) = 0$, and $m(a^i) = m(b^i) = 0$ for all $i = 1, \ldots, k$. It follows that all positions in the set

$$X = [c^0, a^0] \cup [b^0, c^0] \cup \bigcup_{i=1}^{k} [b^i, a^i]$$

are long, since x was assumed to be long, and they are all reachable from x by a single move.

Let us note next that due to the definition of $\mathcal{T}_{\mathcal{H}}$, condition (i) of Theorem 15, and Lemma 8 we have $\mathcal{T}_{\mathcal{H}}(a^{i+1}) \ge \mathcal{T}_{\mathcal{H}}(b^i) - 1$ for $i = 1, \ldots, k-1$. Furthermore $\mathcal{T}_{\mathcal{H}}(a^0) = \mathcal{T}_{\mathcal{H}}(x) - 1$ by Lemma 8 and $\mathcal{T}_{\mathcal{H}}(b^k) = m(x)$ because of condition (ii) of Theorem 15.

Therefore we can apply Lemma 2 to the pairs (a^0, c^0), (c^0, b^0), and (a^i, b^i), $i = 1, \ldots, k$ and obtain

$$\{\mathcal{T}_{\mathcal{H}}(x') \mid x' \in X\} \;=\; [m(x), \mathcal{T}_{\mathcal{H}}(x) - 1].$$

$\square$

5.8 Size of k-uniform JM hypergraphs

In this subsection we study the bound for the size of k-uniform JM hypergraphs. We first provide upper bound for the size of k-uniform minimal transversal-free hypergraphs, implying upper bound for the size of k-uniform JM hypergraphs.

Lemma 74 ([4]) *Assume that $\mathcal{H} \subseteq 2^V$ is a k-uniform minimal transversal-free hypergraph. Then, we have*

$$|V| \;\le\; k\binom{2k}{k}.$$

Proof: Since $\mathcal{H}$ is a minimal transversal-free, for every $i \in V$ we have a hyperedge $H_i \in \mathcal{H}$ such that $H_i \cap H' \neq \emptyset$ for all $H' \in \mathcal{H}$ with $i \notin H'$. Let us denote by $\mathcal{H}' = \{H_i \mid i \in V\}$ the family of these hyperedges. By the transversal-freeness we also have for every hyperedge $H \in \mathcal{H}'$ a disjoint hyperedge $B(H) \in \mathcal{H}$, $H \cap B(H) = \emptyset$. Let us now choose a minimal subhypergraph $\mathcal{B} \subseteq \mathcal{H}$ such that

$$\forall H \in \mathcal{H}' \;\; \exists B \in \mathcal{B} : \quad H \cap B = \emptyset. \tag{35}$$

Let us note first that such a $\mathcal{B}$ must form a cover of V, i.e., $V = \bigcup_{B \in \mathcal{B}} B$. This is because for all $H_i \in \mathcal{H}'$ we have a $B \in \mathcal{B}$ such that $H_i \cap B = \emptyset$, and consequently, $i \in B$. Let us observe next that for all $B \in \mathcal{B}$ we have at least one $A(B) \in \mathcal{H}'$ such that $A(B) \cap B = \emptyset$ and $A(B) \cap B' \neq \emptyset$ for all $B' \in \mathcal{B} \setminus \{B\}$. This is because we choose $\mathcal{B}$ to be a minimal family with respect to (35). Let us now define $\mathcal{A} = \{A(B) \in \mathcal{H}' \mid B \in \mathcal{B}\}$. The pair $\mathcal{A}$, $\mathcal{B}$ of hypergraphs now satisfies the conditions of a classical theorem of Bollobás [5], which then implies that

$$|\mathcal{A}| = |\mathcal{B}| \leq \binom{2k}{k}.$$

Since $\mathcal{B}$ is a k-uniform hypergraph that covers V, our claim follows. $\square$

This clearly implies that $|\mathcal{H}| \leq 2^{k\binom{2k}{k}}$.

An example, provided by D. Pálvölgyi [29], almost matches the upper bound above on the size of V, and we recall it here for completeness.

Let $V = U \cup W$, where $|U| = 2k-2$, $|W| = \frac{1}{2}\binom{2k-2}{k-1}$, and $U \cap W = \emptyset$.

Consider all $(k-1)$ subsets of U, labeled as A_i and B_i such that $A_i \cap B_i = \emptyset$ for $i = 0, \ldots, r-1$, where $r = \frac{1}{2}\binom{2k-2}{k-1}$. Assume further that $W = \{w_0, w_1, \ldots, w_{r-1}\}$, and define

$$\mathcal{H} = \{B_i \cup \{w_i\}, A_{i+1} \cup \{w_i\} \mid i = 0, \ldots, r-1\},$$

where indices are taken modulo r. The hypergraph $\mathcal{H}$ is k-uniform.

Easy to see that it is a minimal transversal-free hypergraph. Namely, if we delete some points from U then all remaining hyperedges are intersecting already in U. If we delete some points from W but not U then consider an index i such that we deleted w_i and not w_{i+1}. Then $B_{i+1} \cup \{w_{i+1}\}$ intersects all remaining hyperedges.

We next consider the size of JM hypergraphs discussed in Section 5.3. As mentioned in the introduction, self-dual matroid hypergraphs $\mathcal{H}$ are k-uniform for $k = n/2$, and satisfy

$$2^k \leq |\mathcal{H}| \leq \binom{2k}{k}.$$

Since we characterize JM graphs in Section 5.6, in the rest of this subsection, we provide an upper bound for the size of JM hypergraphs defined by (29), (30), (31), and (32). For this, we prove that the size of a graph, for which definitions (29), (30), (31), and (32) yield minimal transversal-free hypergraphs, is bounded by a function of k.

Lemma 75 *Let $G = (U, E)$ be a connected graph.*

(i) If $\mathcal{F}_{e,c}(G,k)$ is minimal transversal-free, then $|E| \leq k^2 + k$.

(ii) If G is simple and $\mathcal{F}_{e,t}(G,k)$ is minimal transversal-free, then $|E| \leq \frac{k^3}{2} + \frac{k}{2} + 1$.

(iii) If $\mathcal{F}_{v,c}(G,k)$ or $\mathcal{F}_{v,t}(G,k)$ is minimal transversal-free, then $|U| \leq 2k^3 + 4k^2 + k + 2$.

Proof: We prove first (i). Let us choose an edge e, such that the deletion of e does not disconnect the graph G. Such an edge always exists, since we can pick an edge on a cycle or a leaf edge. Then, by the minimal transversal-freeness, after the deletion of e we must have a connected subgraph T of k edges such that no disjoint connected subgraph of k edges exists. This means that if we delete in addition the edges of T, then the graph decomposes into connected subgraphs, each having at most $k-1$ edges. Since these connected subgraphs intersect the vertex set of T in disjoint sets, and since T has at most $k+1$ vertices, we cannot have more than $(k-1)(k+1)+k+1 = k^2+k$ edges in G.

For (ii) let us repeat the same argument and note that in each connected component now we cannot have a tree of k edges. This means that each connected component has at most k vertices, that is, at most $\binom{k}{2}$ edges, since G is assumed to be simple. Thus, in total we get that

$$|E| \leq 1 + k + (k+1)\binom{k}{2} = \frac{k^3}{2} + \frac{k}{2} + 1.$$

For (iii) we provide a proof for $\mathcal{F}_{v,c}(G,k)$. The same proof works for $\mathcal{F}_{v,t}(G,k)$, as well.

Let v be a vertex in G such that $G - v$ is connected. If $G - v$ contain no connected k-edge subgraph, then we have $|U| \leq k+1$. Otherwise, since $\mathcal{F}_{v,c}(G,k)$ is minimal transversal-free, there exists a hyperedge $F_v \in \mathcal{F}_{v,c}(G,k)$ that intersects all $F \in \mathcal{F}_{v,c}(G,k)$ with $v \notin F$. Let C_i $(i = 1, \ldots, p)$ be connected components of $G - (\{v\} \cup F_v)$. Then we have $|V(C_i)| \leq k$, since C_i

contains at most $k-1$ edges by the definition of F_v and the hypergraph $\mathcal{F}_{v,c}(G,k)$. Furthermore, we have $N_G(C_i) \subseteq \{v\} \cup H_v$ for all i, where $N_G(C_i)$ denotes the set of neighbors of C_i in G.

For any component C_i, let w be a vertex in C_i. We first claim that a hyperedge F_w satisfies

$$N_G(C_i) \not\subseteq F_w, \tag{36}$$

where we recall that F_w is a hyperedge in $\mathcal{F}_{v,c}(G,k)$ that intersects all $F \in \mathcal{F}_{v,c}(G,k)$ with $w \notin F$. Since $\mathcal{F}_{v,c}(G,k)$ is minimal transversal-free, $\mathcal{F}_{v,c}(G,k)$ contains a hyperedge that is disjoint from F_w and contains w. This implies the claim.

Let u be a vertex in $N_G(C_i) \setminus F_w$, Then it holds that

$$|N_G(u) \cap (\bigcup_j C_j)| \leq 2k, \tag{37}$$

since otherwise, $|N_G(u) \setminus (\{w\} \cup F_w)| \geq k$, implying that $\mathcal{F}_{v,c}(G,k)$ contains a hyperedge F disjoint from $\{w\} \cup F_w$, a contradiction.

By (36) and (37), the number of connected components C_i is bounded by $2k|\{v\} \cup F_v| = 4k^2 + 2k$. Thus, the number of vertices of G is bounded by

$$(4k^2 + 2k)k + k + 2 = 4k^3 + 2k^2 + k + 2.$$

□

The above bounds imply that for any given k we have only finitely many different such JM hypergraphs, with all four definitions. The examples derived from trees and stars of cliques show that bounds (i) and (ii) are sharp.

5.9 Further examples and concluding remarks

Let us first show a small example for which property (A1) holds, but both properties (D1) and (D3) fail. This example is not JM, showing that not all minimal transversal-free hypergraphs are JM. Unfortunately, we cannot prove the necessity of properties (D1) or (D3), though we conjecture that property (D3) may be necessary for a hypergraph to be JM.

Our example is $\mathcal{H}_{cube}$ formed by the facets of the 3-dimensional unit cube. The vertex set is $V = \{0,1\}^3$, and the six hyperedges of $\mathcal{H}_{cube} \subseteq 2^V$ are the subsets $H_{i,\alpha} = \{\sigma \in V \mid \sigma_i = \alpha\}$ for $i = 1,2,3$ and $\alpha = 0,1$. This is a 4-uniform hypergraph on 8 vertices, and it is clearly minimal transversal-free. On the other hand it does not satisfy any of three properties (D1), (D2), (D3). To see that it is not a JM hypergraph, assume that $m = \binom{3p+1}{2}$ and $q = m + p$ for some positive integer p, and consider the position $x \in \mathbb{Z}^V_{\geq}$ defined as $x_{000} = m$, $x_{100} = x_{010} = x_{001} = q$, $x_{110} = x_{101} = x_{001} = 2q$ and $x_{111} = 3q$. It is easy to see that for this position we have $m(x) = m$, $y(x) = 3p+1$ and $T(x) = 3q$, consequently this is a long position. Furthermore, every Tetris move $x \to x'$ is an $H_{i,1}$-move for some $i = 1,2,3$ and we must have $m(x') = m(x)$ and $y(x') < y(x)$. Consequently, x' is always a short position. Hence, the necessary property (B1) with $z = T(x) - 1$ is violated, and thus, $\mathcal{H}_{cube}$ cannot be JM. We show a smallest such position with $p = 1$ in Figure 9.

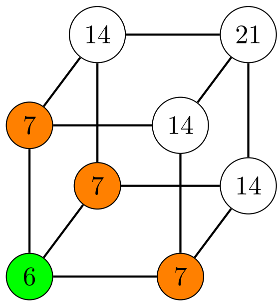

Figure 9: A long position of $\mathcal{H}_{cube}$ that shows that it is not JM.

Let us remark that by its definition, a JM hypergraph $\mathcal{H} \subseteq 2^V$ is minimal non-Tetris, in the sense that for any proper induced subhypergaph $\mathcal{H}_S$, $S \subset V$ the SG function of $\mathrm{NIM}_{\mathcal{H}_S}$ is the Tetris function of $\mathcal{T}_{\mathcal{H}_S}$.

6 Conclusion and open ends

Hypergraph NIM is a very broad generalization of the game NIM as it includes many already proposed NIM generalizations. However, hypergraph NIM is in turn much more complex. To summarize our results, we have only solved it for 2 families of hypergraphs. The simplest are the SG-decreasing hypergraphs which act like a single pile NIM and the optimal play is trivial (the game is 1 move away from ending at all times). We have described some ways to recognize when we are dealing with such a hypergraph, we do not know however how to classify them when dim ≥ 4. The second type of hypergraph are the JM hypergraphs which are can be viewed as just one tier of complexity above the SG-decreasing hypergraphs since removing any vertex makes them a SG-decreasing hypergraph. We have described many families of hypergraphs that are JM, however all are yet to be discovered. Our research also pointed towards certain types of $\mathcal{H}$-combinations of JM hypergraphs lead to another JM hypergraph. However, in most cases we came short and this are mostly open ends. For example, is a conjunctive compound (an $\mathcal{H}$-combination for $\mathcal{H} = \{\{1, 2, ..., n\}\}$) of JM hypergraphs JM? Another open question is whether the is a selective compound (an $\mathcal{H}$-combination for $\mathcal{H} = \{\{1\}, \{2\}, \{1, 2\}\}$) of a JM hypergraph with a singleton also JM, or not. As mentioned, we tackled only the simplest types of hypergraphs. We have still no idea how to play for example on slightly more complex hypergraphs, for example $\binom{[5]}{2}$.

My next step would be to figure out this details as well as try to find the next simplest tier of hypergraphs. Various hypergraph $\mathcal{H}$-combinations also show a lot of promise. We also never tried to figure out how to play the minimal transversal free hypergraphs that are not themselves JM. Given the simplicity of these hypergraphs, they may yield more positive results. At one point in time I did want to make a video game based on hypergraph NIM. One may even use such a game to gather info on how these are played, if they become popular enough. Of course one may also use a self learning AI like AlphaGo to study these games further. Lastly, I would like to say that understanding simpler games does ever so slightly give insight on how to play the bigger games, and someday I would like to see hypegraph NIM become a popular minigame or solve something bigger.

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