TWO PROBLEMS IN REPRESENTATION THEORY: AFFINE LIE ALGEBRAS AND ALGEBRAIC COMBINATORICS

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ABSTRACT OF THE DISSERTATION

Two Problems in Representation Theory: Affine Lie Algebras and Algebraic Combinatorics

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In this dissertation, we investigate two topics with roots in representation theory. The first topic is about twisted affine Kac-Moody algebras and vector spaces spanned by their characters. Specifically, the space spanned by the characters of twisted affine Lie algebras admit the action of certain congruence subgroups of $SL(2, \mathbb{Z})$. By embedding the characters in the space spanned by theta functions, we study an $SL(2, \mathbb{Z})$-closure of the space of characters. Analogous to the untwisted affine Lie algebra case, we construct a commutative associative algebra (fusion algebra) structure on this space through the use of the Verlinde formula and study important quotients. Unlike the untwisted cases, some of these algebras and their quotients, which relate to the trace of diagram automorphisms on conformal blocks, have *negative* structure constants with respect to the (usual) basis indexed by the dominant integral weights of the Lie algebra. We give positivity conjectures for the new structure constants and prove them in some illuminating cases. We then compute formulas for the action of congruence subgroups on these character spaces and give explicit descriptions of the quotients using the affine Weyl group.

The second topic concerns algebraic combinatorics and symmetric functions. In statistics, zonal polynomials and Schur functions appear when taking integrals over certain compact Lie groups with respect to their associated Haar measures. Recently, a conjecture, related to certain integrals of statistical interest, was proposed by D. Richards and S. Sahi. This conjecture asserts that certain linear combinations of Jack polynomials, a one-parameter family of symmetric polynomials that generalizes the zonal and Schur polynomials, are non-negative when evaluated over a certain cone. In the second part of this dissertation, we investigate these conjectures for Schur polynomials and give a refined version of the conjecture. In addition, we prove some cases and arrive at certain seemingly new combinatorial results. In an important instance, we give an analogous result for Jack polynomials.

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Chapter 1

Introduction

Representation theory is a broad field with many beautiful branches and applications. In this dissertation, we will explore two topics in representation theory that at first glance seem very far apart, but at their core are concerned with representations of Lie algebras. Both address conjectures involving mysterious identities, giving concrete answers in certain important families. Both give rise to even deeper questions about their connections to other fields. Finally, both involve computational and experimental techniques, though ultimately the results are purely conceptual.

This dissertation has two main stand-alone parts, Chapters 2 and 3, each an essay in its own right. In first part, we try to determine what sort of hidden algebraic structures exist among certain classes of representations of *twisted affine Kac-Moody Lie algebras*. More precisely, we try to find an analogue to the Verlinde fusion algebras, which exist for untwisted affine Kac-Moody algebras, for the twisted case. In the second part, we investigate curious conjectures about positivity arising from statistics, but with representation theoretic connections, give partial results confirming them in many cases, and provide a further refinement of the conjectures. The identities are formulated in the generality of Jack polynomials, and, in this work, we focus on the special case of Schur polynomials, where the conjectures are no easier to prove. Many of the results for Schur polynomials generalize easily and directly to Jack polynomials, and the Jack polynomial case will be the object of future study.

Each part has an introduction which will motivate and describe the results in that chapter, but let us a give a broad outline of what is contained there. In the first part, we give a background on Verlinde fusion algebras for untwisted affine Kac-Moody algebras and then discuss what we do for the twisted Kac-Moody algebras. Specifically, we study an embedding of the space spanned by the characters of twisted Kac-Moody algebras into a $SL_2(\mathbb{Z})$-module and then immediately show that, in most cases, this space is isomorphic to a Verlinde fusion algebra. One case stands out among the rest, that is the case of Kac-Moody algebras of type $A_{2\ell}^{(2)}$, where we show that this gives rise to a new fusion algebra. This fusion algebra has special properties that differ from the usual Verlinde fusion algebras. Just like the classical untwisted case, the structure constants of the fusion algebra associated to $A_{2\ell}^{(2)}$, with respect to a natural basis, are integers. Unlike the classical untwisted case, these integers are sometimes *negative*. We proved exactly when they are negative for certain Kac-Moody algebras and give a precise conjecture as to when negative structure constants appear for the remaining ones. This answers questions brought up in [Ho2] and gives evidence that these fusion algebras might not directly compute the dimensions of modules. Further results are given for related fusion algebras defined in [Ho2] and certain identities with respect to important generators of congruence subgroups are proved.

In the second part, we investigate the Richards-Sahi conjectures for Schur polynomials [RS]. These conjectures assert that certain linear combinations of Schur polynomials in d variables $x_1, \dots, x_d$ evaluate to *non-negative* real numbers whenever $1 \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$. As mentioned earlier, a stronger conjecture can be formulated for Jack polynomials with an arbitrary parameter α, and we plan to address Jack case in future work. To give an idea of the types of linear combinations we refer to (see section 3.2 for precise definitions), given a partition μ and and certain integers r^-, r, related to μ,

$$\alpha_\mu^r(x) = \sum_{\nu \in H(\mu)} R_\mu(\nu, r) s_\nu(x) \geq 0$$

where $H(\mu)$ is the set of partitions ν such that $\nu \subseteq \mu$ and $\mu \setminus \nu$ is a horizontal strip, s_ν is the Schur polynomial corresponding to the partition ν, $R_\mu(\nu, r)$ is a certain rational function in r depending on μ and ν, and $1 \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0$. (The coefficients $R_\mu(\nu, r)$ are often negative.) We verify these conjectures for several important families, indexed by partitions, and give combinatorial proofs. Later, we give a stronger, refined conjecture that exploits the combinatorics of partitions. More precisely, we show that in all of the known cases, given $n \in \mathbb{N}$, the set $\mathcal{P}^{\leq n}$ of partitions of integers no greater than n (partially) ordered by inclusion can be partitioned into certain chains $\lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^k$ in such a way so that

$$\sum_{i=0}^{k} R_\mu(\lambda^i, r) s_{\lambda^i}(x) \geq 0.$$

In many cases, proving the positivity of these expressions is easier and uses combinatorial arguments. Surprisingly, in proving special cases of these refined conjectures, we arrive at interesting recursions for important values known as Kostka coefficients. In particular, they give a new recursion for the dimensions of irreducible representations of symmetric groups.

During the course of this research, we wrote software in order to compute many examples and give us insight into these mysteries. These programs, written in `Maple`, are too bulky to include in this text but can be downloaded from the link below.

`http://math.rutgers.edu/~ag930`

Chapter 2

Fusion Algebras, the Verlinde Formula, and Twisted Affine Kac-Moody Algebras

2.1 Introduction

Fusion algebras associated to affine Lie algebras can be traced back to the works of physicists working in conformal field theory in the early 1980's (see for example [BPZ], [KZ], [W]). In particular, they investigated the so-called Wess-Zumino-Witten (WZW) model (sometimes called the WZNW model where N stands for Novikov) which is directly related to the representation theory of affine Kac-Moody Lie algebras. Specifically, they worked with untwisted affine Lie algebras, which are universal central extensions of the the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t]$ where $\mathfrak{g}$ is a finite dimensional simple Lie algebra. Of critical importance in the theory are the so-called *fusion rules* which give a special commutative and associative product structure on the Grothendieck group of a certain category of integrable highest weight representations of an untwisted affine Lie algebra. The resulting ring is called the *fusion ring* or *fusion algebra*. One motivation for such a product structure arises from the fact that the usual tensor product of vector spaces does *not* preserve the action of a distinguished element in the affine Lie algebra: the canonical central element (which acts as a scalar, called the *level*, on these representations). This led to great interest in realizing such a product structure in representation-theoretic terms. Indeed, for untwisted affine Lie algebras, the representation-theoretic product structure was finally constructed using vertex operator algebra theory by Y.-Z. Huang and J. Lepowsky in 1997 (see [HL4]). In this work, they constructed a vertex tensor category structure and, in particular, a braided tensor category structure, that gives rise to the fusion algebra. (For future reference, let $\mathcal{O}_k(\mathfrak{g})$ denote the category of level k integrable highest weight representations of an affine Lie algebra $\mathfrak{g}$.)

Rather early on, the group $SL(2,\mathbb{Z})$ was shown to act on the complex vector space spanned by the characters of certain modules for affine Lie algebras (for example, [FS]). In 1987-8, the endeavor to understand fusion algebras gained further significance after Erik Verlinde conjectured that the modular group transformation S, corresponding to the Möbius transformation $\tau \mapsto -1/\tau$ on $\mathbb{H}$, *diagonalizes* the fusion rules [V]. (Since the left multiplication, with respect to the fusion product, can be represented by commuting symmetric matrices, these matrices can be simultaneously diagonalized.) He presented the *Verlinde formula* which describes how the matrix corresponding to the S action can be used to compute the fusion rules. Since then, many people have presented proofs of the Verlinde formula, most notably G. Faltings [Fal] in 1994 for some affine Lie algebras, and C. Teleman [Te] for affine Lie algebras in general around 1994, with the latest proof, in much greater generality than affine Lie algebras, given by Huang [Hu3] in 2004 based on vertex tensor category theory (referred to above). In fact, the last proof showed that a suitable tensor product structure on $\mathcal{O}_k(\mathfrak{g})$, where $k \in \mathbb{Z}_{>0}$ and $\mathfrak{g}$ is an untwisted affine Lie algebra, (see [HL1], [HL2], [HL3], [Hu1]) gives rise to the same fusion algebra predicted by the Verlinde formula.

The entire story cannot be directly applied to *twisted* affine Lie algebras. Twisted affine Lie algebras can be defined in two equivalent ways: the first via their generalized Cartan matrices and the Chevalley-Serre relations for Kac-Moody algebras, and the second as fixed-point subalgebras of untwisted affine Lie algebras (with respect to so-called diagram automorphisms). We will use the former definition exclusively, but acknowledge the importance of understanding the connection between our results here and diagram automorphisms.

In this paper, we study analogous structures for *twisted* affine Lie algebras and investigate their properties. Let us briefly summarize the contents. To begin with, we use the definition of twisted affine Lie algebras using generalized Cartan matrices, i.e., in terms of generators and relations (the generalized Chevalley-Serre relations). We then proceed to find the most suitable $SL(2,\mathbb{Z})$-module in which to embed the space of characters of twisted affine algebras. This space is then endowed with the structure of a commutative associative algebra, which we refer to as a *Verlinde algebra*, using the action of $S \in SL(2,\mathbb{Z})$ in a fashion analogous to the construction of fusion algebras for untwisted affine algebras. For affine Lie algebras of type $A_{2\ell}^{(2)}$, the space of characters admits the action of $SL(2,\mathbb{Z})$ so we define an algebra structure on it directly. In the remaining cases of twisted affine Lie algebras we consider a strictly larger space on which $SL(2,\mathbb{Z})$ acts.

Using the algebra structure and the affine Weyl group, we construct a natural family of quotients, in which, quite surprisingly, negative structure constants appear. These quotients are isomorphic to the fusion algebras constructed in [Ho2] which encode the trace of diagram automorphisms on conformal blocks. In [Ho2], the author discovers that the structure constants are non-negative in certain families and asks how generally this property holds. Our results show that the non-negativity of the structure constants holds in about 'half' the cases (depending on level), while in the other 'half' we determine when negative structure constants appear. Ultimately, we hope that these Verlinde algebras lead to a greater understanding of the structure of certain categories of modules for twisted affine Lie algebras.

Now we will describe the contents of this paper in more technical detail. The space of characters of an affine Lie algebra of type $X_N^{(r)}$ admits an action of the congruence subgroup $\Gamma_1(r) \subseteq SL(2,\mathbb{Z})$. Untwisted affine algebras correspond to the case when $r = 1$. Therefore, $\Gamma_1(1) = SL(2,\mathbb{Z})$ acts on the space of characters of untwisted affine algebras and, using the action of the element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, one can define a commutative associative algebra structure on this space explicitly using the so-called *Verlinde formula*, which we will define shortly.

The construction of the algebras associated to twisted affine algebras is as follows. First, suppose that V is a finite dimensional $SL(2,\mathbb{Z})$-module with a fixed basis $\{e_\lambda\}_{\lambda \in I}$, where $0 \in I$ is a distinguished index. If the 0th column of the action of $S \in SL(2,\mathbb{Z})$, with respect to this basis, has no zero entries, then one can put a commutative associative algebra structure on V, with e_0 as the identity element, by defining its structure constants as

$$N^{\nu}_{\lambda\mu} = \sum_{\phi\in I} \frac{S_{\phi\lambda}S_{\phi\mu}(S^{-1})_{\nu\phi}}{S_{\phi 0}} \tag{2.1}$$

(see [Ka, ex. 13.34]). In this paper, the above formula will be referred to as the *Verlinde formula*. Notice that any nonzero scalar multiple of S will yield the same structure constants $N^{\nu}_{\lambda\mu}$.

We will now construct such a space associated to the character space of twisted affine algebras. (See section 2.2.1 for definitions.) Consider the space of characters $Ch_k(\mathfrak{g}) = Ch_k$ of integrable highest weight modules with fixed positive integer level k of an affine Lie algebra $\mathfrak{g}$. This complex vector space is spanned by $\chi_\lambda = A_{\lambda+\rho}/A_\rho$, where A_μ are alternating sums of theta functions (which we refer to as *alternants*). The space spanned by $A_{\lambda+\rho}$, for λ ranging over dominant level k weights, is denoted Ch^-_k and is a subspace of $Th_{k+h^\vee}$, the space of theta functions of level $k+h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. In fact, Ch^-_k is a subspace of $Th^-_{k+h^\vee} \subset Th_{k+h^\vee}$, the space of alternating theta functions (alternating with respect to the action of the associated finite Weyl group of $\mathfrak{g}$). The space $Th^-_{k+h^\vee}$ admits an action of $SL(2,\mathbb{Z})$ different from the usual $Mp(2,\mathbb{Z})$ action (see section 2.2.3 for the definition), and the injection above intertwines the action of $SL(2,\mathbb{Z})$ up to scalar. In the untwisted cases and in the case that $\mathfrak{g}$ is of type $A^{(2)}_{2\ell}$, the map $\chi_\lambda \mapsto A_{\lambda+\rho} = \chi_\lambda A_\rho$ is an injection. For the remaining twisted cases, Ch^-_k does not admit an action of $SL(2,\mathbb{Z})$, but its image under S is naturally contained in a certain subspace $V \subseteq Th^-_{k+h^\vee}$. This subspace V is the smallest subspace of $Th^-_{k+h^\vee}$ that contains Ch^-_k, is closed under S, and has a basis $\{A_\mu\}_{\mu\in X}$, where X is a subset of the set of regular dominant weights (not necessarily integral of course). The space V is called the $SL(2,\mathbb{Z})$-*closure* of Ch^-_k. (The condition that V be spanned by A_μ reflects the hope that this space will correspond to integrable highest weight modules for some affine Lie algebra and thus can be interpreted representation-theoretically.) More precisely, we have the following description of V.

Theorem 2.1. *If $\mathfrak{g}$ is a twisted affine Lie algebra, but not of type $A^{(2)}_{2\ell}$, then the $SL(2,\mathbb{Z})$-closure of $Ch^-_k(\mathfrak{g})$ is equal to $Ch^-_k(\mathfrak{g}^t)$, where $\mathfrak{g}^t$ is the transpose affine Lie algebra. In the case where $\mathfrak{g}$ is of type $A^{(2)}_{2\ell}$, $Ch^-_k(\mathfrak{g})$ is an $SL(2,\mathbb{Z})$-module.*

If $\mathfrak{g}$ is untwisted, then the map $\chi_\lambda \mapsto A_{\lambda+\rho}$ is an algebra isomorphism (since the Verlinde formula only depends on the action of S up to scalar). If $\mathfrak{g}$ is of type $A_{2\ell}^{(2)}$, we get a completely new algebra defined on the space of characters. For the remaining twisted cases, we get an algebraic structure on a strictly larger space (the precise increase in size will be discussed in detail in section 2.4). In both cases, the identity element is chosen to be the alternant corresponding to the 'smallest' weight. Our discussion motivates the following unified definition.

Definition 2.2. *Let $\mathfrak{g}$ be an affine Kac-Moody Lie algebra and k be a positive integer. The* Verlinde algebra $V_k(\mathfrak{g})$ *is the algebra whose underlying vector space is the $SL(2,\mathbb{Z})$-closure V with basis $\{A_\lambda\}_{\lambda\in X}$, where X is a subset of regular dominant weights, (see the discussion above Theorem 2.1) and whose structure constants, with respect to the given basis, are given by the Verlinde formula (eq. 2.1).*

This definition has several advantageous features. The first is that the action of $\Gamma_1(r)$ on the space of characters can be exhibited directly by the $SL(2,\mathbb{Z})$ action on the untwisted Verlinde algebra. Indeed, explicit formulas for the action of congruence subgroups on the linear space of characters of twisted affine algebras are given in terms of evaluations of characters of irreducible finite dimensional modules for finite dimensional simple Lie algebras similar to the untwisted case. The second is that the algebra has structure constants that can be computed via the Verlinde formula. Another, as we shall later see, is that the Verlinde algebras can be defined as quotients of the representation ring $\mathcal{R}(\mathring{\mathfrak{g}}^t)$ of the transpose of the underlying simple Lie algebra (the transpose Lie algebra of B_n is C_n, C_n is B_n, and the remaining are self-transpose).

In all cases except for $A_{2\ell}^{(2)}$, these Verlinde algebras turn out to be isomorphic to the Verlinde algebras of untwisted affine algebras at *different levels*. By construction, this isomorphism is compatible with the $SL(2,\mathbb{Z})$ action. More specifically, the following theorem is proved. (Note that in the case of $A_{2\ell}^{(2)}$ this is a tautology since these algebras are isomorphic to their transposes.)

Theorem 2.3. *Let $\mathfrak{g}$ be a twisted affine Lie algebra of type $X_N^{(r)}$, k a positive integer,*

$\mathfrak{g}^t$ *its transpose algebra. Then there is an algebra isomorphism*

$$\varphi_k : V_k(\mathfrak{g}) \to V_{k+h^\vee-h}(\mathfrak{g}^t)$$

that intertwines the action of $SL(2,\mathbb{Z})$ *up to scalar.*

Verlinde algebras have striking features, including the following algebra grading structure whose proof relies on the Kac-Walton algorithm (see section 2.3).

Theorem 2.4. *Let* $\mathfrak{g}$ *be an affine Lie algebra of type* $X_N^{(1)}$, V *be its Verlinde algebra at level* k, *and set* $\mathcal{A} = \mathring{P}/\mathring{Q}$. *The algebra* V *has an* $\mathcal{A}$*-grading*

$$V = \bigoplus_{a\in\mathcal{A}} V_a$$

where $V_a = span\{\chi_\lambda : \lambda \in P_k^+ \text{ and } \overline{\lambda} \in a\}$.

In certain cases, the grading structure highlights the subspace of characters of twisted affine algebras, but not in general. The grading is not immediately evident from studying the S-matrix but can be observed after computing the structure constants. The grading seems to play a significant role in the Verlinde algebras of type $A_{2\ell}^{(2)}$ and in certain quotients of the other types (with multiple root lengths).

In all but the case when the affine Lie algebra is of type $A_{2\ell}^{(2)}$ with even level, the structure constants in these Verlinde algebras are known to be non-negative integers. In the $A_{2\ell}^{(2)}$ even level case, all known examples have some negative integral structure constants. We prove that in the special case $A_2^{(2)}$, at even levels, the Verlinde algebra always has some negative structure constants and the grading helps us determine the signs of all structure constants. The author has not found this phenomenon cited in any of the literature. Based on a large amount of evidence, we conjecture (see 2.5) that 'twisting' the basis by a character on $\mathring{P}/\mathring{Q}$ gives nonnegative integral structure constants in type $A_2^{(2)}$.

Remark. There is some difficulty in working with this definition of the algebra since computing the structure constants is quite tedious if done 'by hand', involving large sums of roots of unity. We overcame this problem by writing and using the Maple package `Affine Lie Algebras`, which can efficiently compute these quantities and helped reveal much of the phenomena studied here. This program helped immensely in testing our conjectures for many families of affine Lie algebras.

The same algebraic structure can be defined in terms of a certain action of the affine Weyl group on the representation ring of an associated finite dimensional simple Lie algebra. While this sidesteps the problem of the aforementioned computations and is more conceptual, working with the algebraic structure in this setting involves finding specific representatives in the infinite orbits of the affine Weyl group, again a difficult task. Nonetheless, in some new cases arising from our methods, we present a recursive method of computing these representatives by finding a natural family of affine Weyl group elements that move the vectors they act on closer to a certain fundamental domain of the action of the group (see section 2.6.2).

2.1.1 Structure of Chapter

In section 2.2, we go over preliminaries, notation, and conventions. Of particular importance is the subsection on theta functions, modular invariance, and characters (section 2.2.3) where we define and discuss the actions of $SL(2,\mathbb{Z})$ that feature throughout this work. In section 2.3, the well-known Verlinde algebras for untwisted affine Lie algebras are reviewed and their structure is discussed. In section 2.4, the definition of Verlinde algebras is investigated for all twisted affine Lie algebras. The grading structure and linear subspace spanned by the twisted affine algebra characters are discussed. In section 2.5, the Verlinde algebras of type $A_{2\ell}^{(2)}$ are more extensively studied. When the level is odd, these algebras are well understood and are isomorphic to an untwisted Verlinde algebra. When the level is even, as mentioned earlier, the structure constants are sometimes negative. Finally, in section 2.6, we show that J. Hong's fusion algebras for affine Kac-Moody Lie algebras are isomorphic to natural quotients of the Verlinde algebras presented here and thereby answer some open questions about non-negative structure constants posed in [Ho2]. The quotient structure of the Verlinde algebra of type $A_{2\ell}^{(2)}$ is then explicitly computed in terms of the affine Weyl group. In the process, we produce an algorithm for bringing weights that lie in $2kA_1$ into kA_1, where A_1 is the fundamental alcove of the affine Weyl group of type C and k is a positive integer. The algorithm is a technical tool can be used to study the negative structure constants but is of interest in its own right. In addition, explicit formulas for the action of congruence subgroups on the space of characters are given, proving certain identities such as the following one (which is very similar to the classical S-matrix action).

$$\sum_{\nu\in \mathring{P}} e^{-2\pi i m_\nu}(S_{k\Lambda_0,\nu})^2 \mathring{\chi}_\lambda\left(e^{-\frac{2\pi i(\nu+\rho)}{k+h^\vee}}\right)\mathring{\chi}_\mu\left(e^{\frac{2\pi i(\nu+\rho)}{k+h^\vee}}\right) = e^{2\pi i(m_\lambda+m_\mu)} S_{k\Lambda_0,\mu}\mathring{\chi}_\lambda\left(e^{-\frac{2\pi i(\mu+\rho)}{k+h^\vee}}\right)$$

The action of the element u_{21}^r of $\Gamma_1(r)$ is computed and shown to be expressible as multiples of evaluations of *summands* of the characters of finite dimensional simple Lie algebras. In the 2.2, we introduce notation and briefly go over the basic representation theory of affine Kac-Moody Lie algebras.

2.2 Preliminaries

2.2.1 Affine Kac-Moody Lie Algebras

This section very briefly reminds the reader about the theory of affine Kac-Moody Lie algebras. For greater detail, the reader is encouraged to consult [Ka]. For the sake of simplicity, affine Kac-Moody Lie algebras will often be referred to as affine Lie algebras or affine algebras.

Let $A = (a_{ij})_{0\le i,j\le \ell}$ be an irreducible affine (generalized) Cartan matrix and $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A, with

$$\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{h}^* \text{ and } \Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset \mathfrak{h}.$$

Let $(\mathring{\mathfrak{h}}, \mathring{\Pi}, \mathring{\Pi}^\vee)$ be a realization of the associated finite root system whose Cartan matrix is $\mathring{A} = (a_{ij})_{1\le i,j\le\ell}$. In particular,

$$\mathring{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\} \subset \mathring{\mathfrak{h}}^* \text{ and } \mathring{\Pi}^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset \mathring{\mathfrak{h}}.$$

The vector spaces $\mathfrak{h}$ and $\mathfrak{h}^*$ have bases $\Pi^\vee \cup \{d\}$ and $\Pi\cup\{\Lambda_0\}$, resp., where $\langle \Lambda_0, \alpha_i^\vee\rangle = \delta_{0i}$ and $\langle \Lambda_0, d\rangle = 0$. The affine Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ is generated by $\mathfrak{h}$ and e_i, f_i, $0 \le i \le \ell$, subject to the Chevalley-Serre relations. The Lie algebra $\mathring{\mathfrak{g}}$ is the Lie subalgebra of $\mathfrak{g}$ generated by $\overset{\circ}{\mathfrak{h}}$ and the generators e_i, f_i, $1 \leq i \leq \ell$, along with their corresponding Chevalley-Serre relations. The algebra $\overset{\circ}{\mathfrak{g}}$ is a finite dimensional simple Lie algebra and is referred to as the *underlying simple Lie algebra* of $\mathfrak{g}$. The transpose A^t has the realization $(\mathfrak{h}, \Pi^\vee, \Pi)$ and has associated affine Kac-Moody algebra $\mathfrak{g}^t := \mathfrak{g}(A^t)$ called the *transpose algebra* of $\mathfrak{g}(A)$.

The irreducible affine Kac-Moody algebras are classified (up to isomorphism) by the affine Dynkin diagrams given in [Ka, Ch. 4]. More precisely, there are three tables of affine algebras: Aff 1, Aff 2, and Aff 3.

WARNING 2.2.1. Unless stated otherwise, the labeled Dynkin diagram of type $A_{2\ell}^{(2)}$ will be assumed to be the transpose of the one appearing in the table *Aff* 2. This is consistent with the construction of twisted affine algebras via non-trivial diagram automorphisms of simply laced, untwisted affine Lie algebras. Moreover, it allows us to uniformly give a definition of Verlinde algebras for twisted affine algebras. Note that in [Ka, pg. 132] there is a related warning.

If an affine algebra is isomorphic to one from table Aff r, it is said to be of *type* $X_N^{(r)}$ for $X \in \{A, B, C, D, E, F, G\}$ and $N \in \mathbb{Z}_{>0}$. The affine algebras in Aff 1 are called *untwisted affine Lie algebras* while those in Aff 2 and Aff 3 are called *twisted affine Lie algebras*. For any affine Lie algebra $\mathfrak{g}$, the Dynkin diagram of its underlying simple Lie algebra $\overset{\circ}{\mathfrak{g}}$ can be obtained from the affine Dynkin diagram of $\mathfrak{g}$ by removing the 0 node (and all edges attached to the 0 node).

For an affine irreducible Cartan matrix A, let $D(A)$ denote its Dynkin diagram. The numerical *labels* of $D(A)$ are $a_0, a_1, \ldots, a_\ell$ and can be found in the aforementioned tables. The *dual labels* $a_0^\vee, a_1^\vee, \ldots, a_\ell^\vee$ of $D(A)$ are the labels of $D(A^t)$. The *Coxeter number* $h = \sum_{i=0}^{\ell} a_i$ and *dual Coxeter number* $h^\vee = \sum_{i=0}^{\ell} a_i^\vee$ play very important roles in the theory.

Using these labels and the Cartan matrix $A = (a_{ij})_{0 \leq i,j \leq \ell}$, define the following symmetric nondegenerate bilinear forms on $\mathfrak{h}$ and $\mathfrak{h}^*$,

$$(\alpha_i^\vee, \alpha_j^\vee) = a_j a_j^{\vee -1} a_{ij},\ (\alpha_k^\vee, d) = a_0 \delta_{0k},\ \text{and } (d, d) = 0\ (i, j, k = 0, \ldots, \ell)$$

$$(\alpha_i, \alpha_j) = a_i^\vee a_i^{-1} a_{ij},\ (\alpha_k, \Lambda_0) = a_0^{-1} \delta_{0k},\ \text{and } (\Lambda_0, \Lambda_0) = 0\ (i, j, k = 0, \ldots, \ell),$$

respectively. The bilinear form on $\mathfrak{h}$ induces an isomorphism

$$\nu : \mathfrak{h} \to \mathfrak{h}^*$$

$$\nu(\alpha_i^\vee) = a_i a_i^{\vee -1} \alpha_i, \;\; \nu(d) = a_0 \Lambda_0$$

which will be used to identify both vector spaces (with little to no warning). We will also be using the map $\alpha \mapsto \alpha^\vee$ where α is a nonisotropic element of $\mathfrak{g}^*$ and $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\nu^{-1}(\alpha)$. Define the important isotropic elements

$$\delta = \sum_{i=0}^{\ell} a_i \alpha_i \in \mathfrak{h}^* \text{ and } K = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee \in \mathfrak{h}.$$

The element K is called the *canonical central element* of $\mathfrak{g}$. Also define the element $\theta = \delta - a_0 \alpha_0$, which is a dominant root of $\mathring{\mathfrak{g}}$. With these new elements define the new bases

$$\{\alpha_1, \ldots, \alpha_\ell, \Lambda_0, \delta\} \subset \mathfrak{h}^*$$

$$\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee, d, K\} \subset \mathfrak{h}$$

and the projection map

$$\pi : \mathfrak{h}^* \to \mathring{\mathfrak{h}}^*$$

$$\alpha \mapsto \overline{\alpha}$$

with respect to the new basis.

The *Weyl group* W of $\mathfrak{g}$ is the group of orthogonal transformations on $\mathfrak{h}^*$ generated by the reflections

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$$

for $i = 0, \ldots, \ell$. Using the isomorphism ν, W also acts by orthogonal transformations on $\mathfrak{h}$. Let $\mathring{W}$ be the subgroup generated by s_i, $1 \le i \le \ell$, then $\mathring{W}$ is the Weyl group of the underlying simple Lie algebra $\mathring{\mathfrak{g}}$. Moreover, the Weyl group W can be decomposed as the semidirect product

$$W = \mathring{W} \ltimes T \cong \mathring{W} \ltimes M$$

where M is the lattice spanned by the set $\mathring{W}(a_0^{-1}\theta)$, T is the abelian group of translations by elements of M (acting on $\mathfrak{h}^*$), and $\mathring{W}$ acts on M in the natural way. The elements of T are written as t_α, $\alpha \in M$. For $\mathfrak{g}$ of type $X_N^{(r)}$, in the case $r = a_0$ the lattice $M = \nu(\mathring{Q}^\vee)$ and in the case $r > a_0$, $M = \mathring{Q}$ (see the next subsection for the definitions of relevant lattices).

2.2.2 Integrable Highest Weight Modules

Throughout this work, we will be dealing with the category of integrable highest weight modules of level k, where k is a positive integer. This category is endowed with the usual direct sum structure. Quite significantly, the usual tensor product does not preserve the level. In order to describe these modules, let us very rapidly recall the various lattices and sets that feature in the representation theory.

Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Recall that the *root lattice* Q is the lattice in $\mathfrak{h}^*$ generated by the simple roots $\alpha_0, \ldots, \alpha_\ell$. The *coroot lattice* $Q^\vee$ is the lattice in $\mathfrak{h}$ generated by $\alpha_0^\vee, \ldots, \alpha_\ell^\vee$. Define the *fundamental weights* to be elements $\Lambda_i \in \mathfrak{h}^*$ such that $\Lambda_i(\alpha_j^\vee) = \delta_{i,j}$, for $i, j = 0, \ldots, \ell$. The *weight lattice* P is the lattice in $\mathfrak{h}^*$ generated by the fundamental weights $\Lambda_0, \ldots, \Lambda_\ell$. The *dominant integral weights* P^+ is the set of $\lambda \in P$ such that $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for $i = 0, \ldots, \ell$. The *regular dominant integral weights* P^{++} are the dominant integral weights such that the above inequality is strict for all $\alpha_i^\vee$. When decorated above by the symbol $\circ$, these lattices are the corresponding lattices for $\mathring{\mathfrak{g}}$.

A very important element of P^{++} is

$$\rho := \Lambda_0 + \Lambda_1 + \cdots + \Lambda_\ell.$$

The level k of an element $\lambda \in \mathfrak{h}^*$ is the quantity $\langle \lambda, K \rangle$. Note that the level of ρ is $h^\vee$. Define the sets

$$P_k = \{\lambda \in P : \langle \lambda, K \rangle = k \text{ and } (\lambda, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in M\},$$

$$P^k = \{\lambda \in P : \langle \lambda, K \rangle = k \text{ and } \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha^\vee \in \mathring{Q}^\vee\},$$

$$P_k^+ = P_k \cap P^+, \; P_k^{++} = P_k \cap P^{++}, \; P^{k+} = P^k \cap P^+, \; P^{k++} = P^k \cap P^{++}.$$

For an affine algebra $\mathfrak{g}$, an *integrable highest weight module of highest weight* λ is a non-trivial $\mathfrak{g}$-module V with decomposition

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu,$$

$$V_\mu = \{v \in V : h \cdot v = \mu(h)v \text{ for all } h \in \mathfrak{h}\},$$

such that $\lambda \in P^+$, there is an element $v_0 \in V_\lambda$ (unique up to scaling) that generates V, and $e_i v_0 = 0$ for all $i = 0, \ldots, \ell$. The irreducible integrable highest weight modules of highest weight $\lambda \in P^+$ are denoted L_λ. The central element K acts by a scalar k, called the *level*, on L_λ. The terminology is consistent since the level of L_λ is the level of λ as an element of $\mathfrak{h}^*$.

2.2.3 Theta Functions, Modular Invariance, and Characters

This subsection discusses characters and theta functions associated with affine Lie algebras. For a full account of the now classical theory, see [KP] or [Ka]. Significantly, at the end of this section, we define an action of $SL(2,\mathbb{Z})$ on the space $Th^-_{k+h^\vee}$ that features prominently in the definition of the Verlinde algebras.

Let $\mathfrak{g}$ be an affine Lie algebra, $k \in \mathbb{Z}_{>0}$, and $\lambda \in P_k$, then the *classical theta function* of level k with characteristic $\overline{\lambda}$ is defined

$$\Theta_\lambda = e^{-\frac{(\lambda,\lambda)}{2k}\delta} \sum_{\alpha \in M} e^{t_\alpha(\lambda)},$$

(recall the notation t_α given at the end of 2.2.1). It can be shown that $\lambda \equiv \mu \pmod{kM + \mathbb{C}\delta}$ implies $\Theta_\lambda = \Theta_\mu$. These functions are defined on the half-space $Y = \{\lambda \in \mathfrak{h} | \operatorname{Re}(\lambda, \delta) > 0\}$. The *space of theta functions* Th_k is a subspace of the vector space of functions on Y certain invariance properties with respect to a Heisenberg group. Due to the following result, which can be found in [Ka,§13], we will not need the formal definition.

Proposition 2.5. *Let $k > 0$, then the set of functions $\{\Theta_\lambda | \lambda \in P_k \pmod{kM + \mathbb{C}\delta}\}$ is a $\mathbb{C}$-basis of Th_k.*

The space Th_k admits a right action of the metaplectic group $Mp(2,\mathbb{Z})$, which is a double cover of

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1 \right\}.$$

The group $SL(2,\mathbb{Z})$ has the generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which play a prominent role. The action of $X \in Mp(2,\mathbb{Z})$ on Θ_λ is denoted $\Theta_\lambda|_X$. The same notation will be used when referring to the action of $SL(2,\mathbb{Z})$ when applicable. For later use, we record the action certain elements of $Mp(2,\mathbb{Z})$ on Th_k,

$$\Theta_\lambda|_{(T,j_1)} = \gamma(T,j_1,\ell)e^{\pi i|\overline{\lambda}|^2/h}\Theta_\lambda$$

$$\Theta_\lambda|_{(S,j_2)} = \gamma(S,j_2,\ell)(-i)^{\ell/2}|M^*/kM|^{-1/2} \sum_{\mu\in P_k, w\in \mathring{W}} \epsilon(w)e^{-2\pi i(w(\overline{\lambda}),\overline{\mu})/k}\Theta_\mu$$

where ℓ is the rank of M, $\lambda \in P_k$, j_1, j_2 are holomorphic functions whose squares are $1, \tau$, resp., and the function γ takes values in $\{1,-1\}$.

For $r = 1,2,3$, the congruence subgroups

$$\Gamma_1(r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{r} \right\},$$

are also important. The group $\Gamma_1(r)$ is generated by the matrices

$$u_{12} = T^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } u_{21}^r = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

which satisfy the relations $(u_{12}u_{21})^3 = -1$, $(u_{12}u_{21}^r)^{2r} = 1$ for $r \in \{2,3\}$, and $S = u_{12}u_{21}u_{12}$ [IS].

If V is a $\mathfrak{g}$-module with weight space decomposition

$$V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$$

such that $\dim V_\mu < \infty$, the *formal character* of V is the formal sum

$$ch(V) = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu.$$

For any $\lambda \in P^{k+}$, define the *normalized character* or simply *character*

$$\chi_\lambda = e^{-m_\lambda \delta} ch(L_\lambda)$$

where m_λ is the so-called *modular anomaly* of λ

$$m_\lambda = \frac{|\lambda + \rho|^2}{2(k + h^\vee)} - \frac{|\rho|^2}{2h^\vee}.$$

The complex vector space spanned by the characters χ_λ of $\mathfrak{g}$ of level k will be denoted $Ch_k := Ch_k(\mathfrak{g})$. The characters can be expressed in terms of classical theta functions. To explain this, first define the *alternants*

$$A_\lambda = \sum_{w \in \mathring{W}} \epsilon(w) \Theta_{w(\lambda)}$$

and let Th_k^- be the space spanned by the A_λ for $\lambda \in P_k^+$. The following result can also be found in [Ka,§12].

Proposition 2.6. *If $k > 0$ and $\lambda \in P^{k+}$, then*

$$\chi_\lambda = A_{\lambda+\rho}/A_\rho.$$

In the case when $r \leq a_0$, the space of characters is invariant under the action of $SL(2,\mathbb{Z})$, i.e., the action of $Mp(2,\mathbb{Z})$ factors through $SL(2,\mathbb{Z})$. In the case $r > a_0$, Ch_k is invariant under $\Gamma_1(r)$. The explicit actions of these subgroups will be discussed in the upcoming sections.

In the case that $r \leq a_0$, the spaces Ch_k and $Th_{k+h^\vee}^-$ are linearly isomorphic. By transport of structure, this defines an action of $SL(2,\mathbb{Z})$ on $Th_{k+h^\vee}^-$ that is different from the (restriction of the) $Mp(2,\mathbb{Z})$ action defined on $Th_{k+h^\vee}$. In the case that $r > a_0$, there is an embedding $Ch_k \hookrightarrow Th_{k+h^\vee}^-$ defined by $\chi_\lambda \mapsto A_{\lambda+\rho}$ and, again by transport of structure, $\Gamma_1(r)$ acts on the image of this embedding. In fact, we can define an $SL(2,\mathbb{Z})$ action on $Th_{k+h^\vee}^-$, for $k \geq 0$, that extends the action of $\Gamma_1(r)$ up to scalar in the following way.

Proposition 2.7. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra, $k \in \mathbb{N}$, and $Th^-_{k+h^\vee}$ be as above. Let $U \subseteq Th^-_{k+h^\vee}$ be the image of Ch_k under the embedding $\chi_\lambda \mapsto \chi_\lambda A_\rho$, equipped with the linear action α_r of $\Gamma_1(r)$ induced by this embedding. Then then there is a linear action α_1 of $SL(2,\mathbb{Z})$ on $Th^-_{k+h^\vee}$ such that for any $F \in U$,*

$$\alpha_1(u_{12}) \cdot F = c_{12}\alpha_r(u_{12}) \cdot F$$

$$\alpha_1(u^r_{21}) \cdot F = c_{21}\alpha_r(u^r_{12}) \cdot F$$

where c_{12} and c_{21} are nonzero scalars.

Proof. If $r \le a_0$, then, by the above discussion, we are done. Suppose that $r > a_0$, then $M = \overset{\circ}{Q}$. Let ℓ be the rank of $\mathfrak{g}$ (or, equivalently, M) and let $\rho^\vee \in P^+_h$ be defined (modulo $\mathbb{C}\delta$) by $(\rho^\vee, \alpha_j) = 1$, for $j = 1, \ldots, \ell$. Note that $A_{\rho^\vee} \neq 0$ and $Th^-_h = \mathbb{C}A_{\rho^\vee}$. The action of $Mp(2,\mathbb{Z})$ on Th^-_h is given by

$$A_{\rho^\vee}|_{(T,j_1)} = \gamma(T, j_1, \ell)e^{\pi i|\overline{\rho^\vee}|^2/h}A_{\rho^\vee}$$

$$A_{\rho^\vee}|_{(S,j_2)} = \gamma(S, j_2, \ell)(-i)^{\ell/2}|M^*/hM|^{-1/2}\sum_{w\in\overset{\circ}{W}}\epsilon(w)e^{-2\pi i(w(\overline{\rho^\vee}),\overline{\rho^\vee})/h}A_{\rho^\vee}$$

where j_1, j_2, γ are defined earlier in this section. Notice that only γ depends on j_1, j_2, and ℓ.

Now, let

$$W_{k+h^\vee} = \bigoplus_{\lambda\in P^+_k}\mathbb{C}A_{\lambda+\rho}/A_{\rho^\vee},$$

then $Th^-_{k+h^\vee} \cong W_{k+h^\vee}$. The natural action of $Mp(2,\mathbb{Z})$ is easily seen (due to the cancellation of the γ terms) to descend to an action of $SL(2,\mathbb{Z})$ on $W_{k+h^\vee}$. Using the isomorphism, we equip $Th^-_{k+h^\vee}$ with that action of $SL(2,\mathbb{Z})$. Denote this action by α_1.

To show that α_1 and α_r differ by scalars, first note that $\alpha_r(T)$ acts diagonally with respect to the basis A_λ, $\lambda \in P^{k++}$. Second, note that $u_{21} = TST$. The $Mp(2,\mathbb{Z})$ action of (u^r_{21}, j), for a holomorphic j such that $j^2 = r\tau+1$, on the numerator of $\chi_\lambda = A_{\lambda+\rho}/A_\rho$ is a scalar multiple of the α_r action of u^r_{21} on $A_{\lambda+\rho}$. Similarly, also by definition, the $Mp(2,\mathbb{Z})$ action of (u^r_{21}, j), as above, on the numerator of $A_{\lambda+\rho}/A_{\rho^\vee}$ is a scalar multiple of the α_1 action of u^r_{21} on $A_{\lambda+\rho}$. The result follows. □

2.3 Verlinde Algebras of Untwisted Affine Algebras

This section introduces the well-known Verlinde algebras of untwisted affine Lie algebras, i.e., affine Kac-Moody Lie algebras of type $X_N^{(1)}$, and discuss their grading structure, including the proof of theorem 2.4.

Let $\mathfrak{g}$ be an untwisted affine Lie algebra and k a positive integer that will serve as the level. Recall that in this case Ch_k is invariant under the action of $SL(2,\mathbb{Z})$. More importantly, the action of the element $S \in SL(2,\mathbb{Z})$ defined earlier 2.2.3 encodes information about a commutative associative algebra structure on Ch_k. The *S-matrix*, which gives the action of S with respect to the basis of characters, encodes the structure constants of this algebra via the Verlinde formula, which we restate for this particular case, $\chi_\lambda \chi_\mu = \sum_{\nu \in P_k^+ \text{ mod } \mathbb{C}\delta} N_{\lambda\mu}^{\nu} \chi_\nu$ where

$$N_{\lambda\mu}^{\nu} = \sum_{\phi \in P_k^+ \text{ mod } \mathbb{C}\delta} \frac{S_{\lambda\phi} S_{\mu\phi} (S^{-1})_{\nu\phi}}{S_{k\Lambda_0,\phi}}.$$

The structure constants $N_{\lambda\mu}^{\nu}$ are known as the *fusion rules*, are well-defined, and correspond to a tensor product structure on an appropriate category of modules for $\mathfrak{g}$ (which will not be discussed here).

First, let us give an explicit description of the $SL(2,\mathbb{Z})$ action on Ch_k, which can be found in [Ka]. Afterwards, it will be apparent that the Verlinde algebras are well-defined. Surprisingly, the S-matrix can be expressed in terms of evaluations of characters of irreducible highest weight $\mathring{\mathfrak{g}}$-modules. In what follows, the function $\mathring{\chi}_{\overline{\lambda}}$ is the character of the irreducible highest weight $\mathring{\mathfrak{g}}$-module with highest weight $\overline{\lambda}$. By [KP], Ch_k is invariant under the action of $SL(2,\mathbb{Z})$. The action, denoted $A : \chi_\lambda \mapsto \chi_\lambda|_A$ where $A \in SL(2,\mathbb{Z})$, is described explicitly in terms of the generators T and S.

Theorem 2.8 (Kac-Peterson)**.** *Let $\mathfrak{g}$ be an affine Lie algebra of type $X_N^{(1)}$ or $A_{2\ell}^{(2)}$ and k be a positive integer. For any $\lambda \in P_k^+$,*

$$\chi_\lambda|_T = e^{2\pi i m_{\overline{\lambda}}} \chi_\lambda$$

$$\chi_\lambda|_S = \sum_{\mu \in P_k^+ \ mod \ \mathbb{C}\delta} S_{\lambda\mu} \chi_\mu$$

where

$$S_{\lambda\mu} = i^{|\overset{\circ}{\Delta}_+|}|M^*/(k+h^\vee)M|^{-1/2}\sum_{w\in\overset{\circ}{W}}\epsilon(w)e^{-\frac{2\pi i(\overline{\lambda}+\overline{\rho},w(\overline{\mu}+\overline{\rho}))}{k+h^\vee}}$$
$$= c_\lambda \overset{\circ}{\chi}_{\overline{\mu}}\left(e^{-\frac{2\pi i(\overline{\lambda}+\overline{\rho})}{k+h^\vee}}\right)$$

and $c_\lambda = S_{k\Lambda_0,\lambda} = |M^*/(k+h^\vee)M|^{-1/2}\prod_{\alpha\in\overset{\circ}{\Delta}_+} 2\sin\frac{\pi(\overline{\mu}+\overline{\rho},\alpha)}{k+h^\vee}$.

Note that the constant $c_\lambda = S_{k\Lambda_0,\lambda}$ in the above theorem is a positive real number. Furthermore, the theorem shows that the S-matrix is symmetric, therefore the product in V_k is commutative. The associativity follows from the fact that $\chi_{k\Lambda_0}$ is decreed to be the identity element and the columns of the S-matrix are assumed to be, up to nonzero scalars, a system of orthogonal idempotents of V_k and, under this assumption, the Verlinde formula simply computes the structure constants of the resulting algebra. It is worth noting that $S^{-1} = \overline{S}^t = \overline{S}$. We now proceed to prove theorem 2.4, i.e., that these Verlinde algebras are graded by $\overset{\circ}{P}/\overset{\circ}{Q}$.

Proof of 2.4. Let us denote by $P(\overline{\mu})$ the set of weights (regarded as a multiset) of the irreducible finite dimensional $\overset{\circ}{\mathfrak{g}}$-module with highest weight $\overline{\mu}$. By the Kac-Walton algorithm [W], the fusion rules for $\lambda, \mu, \nu \in P_k^+$ are of the form

$$N_{\lambda\mu}^{\nu} = \sum_{\phi}\epsilon(w_\phi)N_{\overline{\lambda}\overline{\mu}}^{\overline{\phi}}$$

where ϕ ranges over a finite multiset of integral weights of $\overset{\circ}{\mathfrak{g}}$ and w_ϕ are certain elements in W. Therefore, $N_{\overline{\lambda}\overline{\mu}}^{\overline{\nu}} = 0$ implies that $N_{\lambda\mu}^{\nu} = 0$. Using the Racah-Speiser algorithm [FH,§25.3], the finite fusion rules have the form

$$N_{\overline{\lambda}\overline{\mu}}^{\overline{\nu}} = \sum_{\overline{\phi}}\epsilon(w_{\overline{\phi}})$$

where the sum ranges over all $\overline{\phi} \in P(\overline{\mu})$, such that $w_{\overline{\phi}}(\overline{\lambda}+\overline{\phi}+\overline{\rho}) - \overline{\rho} = \overline{\nu}$ for some $w_{\overline{\phi}} \in \overset{\circ}{W}$. Note that such $w_{\overline{\phi}}$ are unique (when they exist).

In these cases, $M = \overset{\circ}{Q}{}^\vee \subseteq \overset{\circ}{Q} \subseteq M^* = \overset{\circ}{P}$ so let $\mathcal{A} = M^*/\overset{\circ}{Q}$. Notice that $\overset{\circ}{W}$ acts trivially on $\mathcal{A}$. Define the subspaces

$$V_a = \operatorname{span}\{\chi_\lambda | \overline{\lambda} \equiv a \pmod{\overset{\circ}{Q}}\}$$

for $a \in \mathcal{A}$. Recall that whenever $\overline{\phi} \in P(\overline{\mu})$, $\overline{\mu} - \overline{\phi}$ is a sum of positive roots. Therefore, from the above discussion we know that

$$\overline{\nu} = w_{\overline{\phi}}(\overline{\lambda} + \overline{\phi} + \overline{\rho}) - \overline{\rho} \equiv \overline{\lambda} + \overline{\mu} \pmod{\mathring{Q}},$$

proving that $N_{\lambda\mu}^{\nu} = 0$ whenever $\overline{\lambda} + \overline{\mu} \neq \overline{\nu}$ in $\mathcal{A}$. This proves that $V = \bigoplus_{a \in \mathcal{A}} V_a$ is a grading. □

Remark 1. *The theorem above indicates a relationship between the center of the associated simply connected compact Lie group G and the grading structure. There may be an interpretation of this mystery in terms of an action of $Z(G)$ on the Verlinde algebra or on conformal blocks.*

Example 2.9. *Let us observe the grading structure in the case when $\mathfrak{g}$ is of type $A_2^{(1)}$ with level $k = 2$. Consider the ordered basis $\chi_{\lambda_1}, \chi_{\lambda_2}, \ldots, \chi_{\lambda_6}$ where the $\overline{\lambda}_i$ are*

$$0, \overline{\Lambda}_1 + \overline{\Lambda}_2, \overline{\Lambda}_1, 2\overline{\Lambda}_2, 2\overline{\Lambda}_1, \overline{\Lambda}_2,$$

respectively. The weights have been paired off so that they lie in the classes $0, \overline{\Lambda}_1, 2\overline{\Lambda}_1 \in \mathring{P}/\mathring{Q}$ respectively. The matrices for L_i, the operator of left multiplication by χ_{λ_i}, are

$$L_1 = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 0 & & \\ & & 0 & 1 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 1 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} & & & & 0 & 1 \\ & & & & 1 & 1 \\ 1 & 1 & & & & \\ 0 & 1 & & & & \\ & & 1 & 0 & & \\ & & 1 & 1 & & \end{pmatrix} \qquad L_4 = \begin{pmatrix} & & & & 1 & 0 \\ & & & & 0 & 1 \\ 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \end{pmatrix}$$

$$L_5 = \begin{pmatrix} & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \\ 1 & 0 & & & & \\ 0 & 1 & & & & \end{pmatrix} \quad L_6 = \begin{pmatrix} & & 1 & 0 & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \\ & & & & 0 & 1 \\ 0 & 1 & & & & \\ 1 & 1 & & & & \end{pmatrix}.$$

Example 2.10. *Let us observe the grading structure in the case when $\mathfrak{g}$ is of type $B_3^{(1)}$ with level $k = 2$. Consider the ordered basis $\chi_{\lambda_1}, \chi_{\lambda_2}, \ldots, \chi_{\lambda_7}$ where the $\overline{\lambda}_i$ are*

$$0, \overline{\Lambda}_1, 2\overline{\Lambda}_1, \overline{\Lambda}_2, 2\overline{\Lambda}_3,$$

$$\overline{\Lambda}_3, \overline{\Lambda}_1 + \overline{\Lambda}_3$$

respectively. There are only two classes $\{0, \overline{\Lambda}_3\} = \mathring{P}/\mathring{Q}$ and so V_k has a $\mathbb{Z}_2$-grading. (Indeed, this is always the case for type $B_\ell^{(1)}$, with the system of representatives $\{0, \overline{\Lambda}_\ell\}$.) The first row lies in the class of 0 and the second row lies in the class of $\overline{\Lambda}_3$. The matrices for the L_i, the operator of left multiplication by χ_{λ_i}, are $L_1 = id$

$$L_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & & \\ 1 & 0 & 1 & 1 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & 1 & & \\ & & & & & 1 & 1 \\ & & & & & 1 & 1 \end{pmatrix} \quad L_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & 0 & & \\ 1 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 0 & 1 & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}$$

$$L_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & & \\ 0 & 1 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & 0 & & \\ 1 & 0 & 1 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & 0 & & \\ & & & & & 1 & 1 \\ & & & & & 1 & 1 \end{pmatrix}, L_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 1 & 0 & 1 & 0 & & \\ 1 & 1 & 1 & 0 & 0 & & \\ & & & & & 1 & 1 \\ & & & & & 1 & 1 \end{pmatrix}$$

$$L_6 = \begin{pmatrix} &&&&& 1 & 0 \\ &&&&& 1 & 1 \\ &&&&& 0 & 1 \\ &&&&& 1 & 1 \\ &&&&& 1 & 1 \\ 1 & 1 & 0 & 1 & 1 && \\ 0 & 1 & 1 & 1 & 1 && \end{pmatrix} \quad L_7 = \begin{pmatrix} &&&&& 0 & 1 \\ &&&&& 1 & 1 \\ &&&&& 1 & 0 \\ &&&&& 1 & 1 \\ &&&&& 1 & 1 \\ 0 & 1 & 1 & 1 & 1 && \\ 1 & 1 & 0 & 1 & 1 && \end{pmatrix}.$$

This grading structure also applies to the case where $k = \infty$, i.e., when the Verlinde algebra is equal to the representation algebra of $\overset{\circ}{\mathfrak{g}}$, since it depends only on the Racah-Speiser algorithm for computing the tensor product decomposition of finite dimensional simple $\overset{\circ}{\mathfrak{g}}$-modules.

A method of describing this algebra more conceptually is via the representation ring $\mathcal{R}(\overset{\circ}{\mathfrak{g}})$ of the finite dimensional simple Lie algebra $\overset{\circ}{\mathfrak{g}}$. (This interpretation of the Verlinde algebra will be used later on.) There is a map

$$F_k : \mathcal{R}(\overset{\circ}{\mathfrak{g}}) \to Ch_k(\mathfrak{g})$$

$$\overset{\circ}{\chi}_{\overline{\lambda}} \mapsto \begin{cases} \epsilon(w)\chi_{w(\lambda+\rho)-\rho} & \text{if there exists } w \in W,\ w(\lambda+\rho)-\rho \in P_k^+ \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda \in P_k^+$ (note that the w referenced in the definition of F_k is unique). There is also a canonical section

$$S_k : Ch_k(\mathfrak{g}) \to \mathcal{R}(\overset{\circ}{\mathfrak{g}})$$

$$\chi_\lambda \mapsto \overset{\circ}{\chi}_{\overline{\lambda}}.$$

The following proposition was proved in [W].

Theorem 2.11 (Walton)**.** *The Verlinde algebra V_k has the product structure*

$$\chi_\lambda \cdot \chi_\mu = F_k(S_k(\chi_\lambda) \otimes S_k(\chi_\mu))$$

for $\lambda, \mu \in P_k^+$.

Instead of using W in the definition of F_k, we can use the affine Weyl group $W_{aff}^{k+h^\vee} = \overset{\circ}{W} \ltimes (k+h^\vee)\overset{\circ}{Q}^\vee$, which acts on $\overset{\circ}{\mathfrak{h}}^*$. The fundamental alcove $P_{k+h^\vee}^{aff\ +}$ is defined to be the set of $\lambda \in \overset{\circ}{\mathfrak{h}}^*$ such that

$$0 \le (\lambda, \alpha_i) \text{ for } i = 1, \ldots, \ell$$

$$(\lambda, \theta) \le k + h^\vee,$$

this is a fundamental domain for the action of $W_{aff}^{k+h^\vee}$. The set $P_{k+h^\vee}^{aff\ ++}$ consists of all $\lambda \in P_{k+h^\vee}^{aff\ +}$ such that the above inequalities are all strict. For any $\lambda \in \mathfrak{h}^*$ of level k, $\overline{\lambda} + k\Lambda_0 \equiv \lambda \pmod{\mathbb{C}\delta}$. Using $\overline{\lambda} \mapsto k\Lambda_0 + \overline{\lambda}$, the sets $P_k^{aff\ +}$ and $P_k^{aff\ ++}$ can be identified with P_k^+ and P_k^{++}, respectively. Therefore, we can also define the map F_k by the rule

$$\overset{\circ}{\chi}_{\overline{\lambda}} \mapsto \begin{cases} \epsilon(w)\chi_{k\Lambda_0 + w(\overline{\lambda}+\overline{\rho})-\overline{\rho}} & \text{if there exists } w \in W_{aff}^{k+h^\vee},\ w(\overline{\lambda}+\overline{\rho}) \in P_{k+h^\vee}^{aff\ ++} \\ 0 & \text{otherwise} \end{cases}.$$

Later on in 2.6, this definition will prove very useful.

Remark 2. *While the representation ring description of these Verlinde algebras show that the structure constants are integers, these algebras have an interpretation as the Grothendieck ring of a tensor category of modules for untwisted affine Lie algebras and, therefore, the structure constants are all non-negative integers [Hu]. For the twisted cases, the Verlinde algebras will be described as a quotient of the representation ring of a finite dimensional simple Lie algebra and non-negativity of structure constants will not hold in exactly the case* $A_{2\ell}^{(2)}$ *when the level is even.*

2.4 Verlinde Algebras of Twisted Affine Algebras

The affine Kac-Moody Lie algebras of type $X_N^{(r)}$ for $r > 1$ are twisted affine Lie algebras and their spaces of characters behave very differently from those of untwisted affine Lie algebras. In these cases, $M = \overset{\circ}{Q}$ and $P^k \subsetneq P_k$, therefore we cannot use the same techniques used for the algebras of type $X_N^{(1)}$. In particular, there is no action of the S-matrix. Nonetheless, to each twisted affine Lie algebra $\mathfrak{g}$ of this type, we will associate a Verlinde algebra $V_k(\mathfrak{g})$ that contains space of characters of $\mathfrak{g}$ at level k and admits the action of the modular group. In particular, we prove theorems 2.1 and 2.3 in this section by comparing a twisted affine algebra $\mathfrak{g}$ to its transpose $\mathfrak{g}^t$. Since the Verlinde algebras for type $A_{2\ell}^{(2)}$ are fundamentally different from the others, we will more deeply explore them in the next section.

The lattice $\overline{P}_k$ is the dual of the root lattice, therefore it has the basis

$$\frac{a_1}{a_1^\vee}\overline{\Lambda}_1, \frac{a_2}{a_2^\vee}\overline{\Lambda}_2, \ldots, \frac{a_\ell}{a_\ell^\vee}\overline{\Lambda}_\ell.$$

If $\lambda \in P_k^+$ then $(\overline{\lambda}, \theta) = \langle \overline{\lambda}, \theta^\vee \rangle \leq k$. Conversely, any $\overline{\lambda} \in \overline{P}^+$ satisfying the above inequality has unique preimage $\lambda \in P_k^+$ modulo $\mathbb{C}\delta$.

In [Ka], Kac relates each $\mathfrak{g}$ of the type considered here to another affine algebra denoted $\mathfrak{g}'$ (this will be discussed later). For the purpose of investigating Ch_k, it will be helpful to instead relate $\mathfrak{g}$ to an affine Lie algebra of type $X_N^{(1)}$. To each affine Lie algebra with $r > a_0$, associate the transpose algebra $\mathfrak{g}^t = \mathfrak{g}(A^t)$, where A is the Cartan matrix of $\mathfrak{g}$. The following table gives the explicit correspondences.

$\mathfrak{g}$	$\mathfrak{g}^t$	$\mathfrak{g}'$
$A_{2\ell-1}^{(2)}$	$B_\ell^{(1)}$	$D_{\ell+1}^{(2)}$
$D_{\ell+1}^{(2)}$	$C_\ell^{(1)}$	$A_{2\ell-1}^{(2)}$
$E_6^{(2)}$	$F_6^{(1)}$	$E_6^{(2)}$
$D_{2\ell-1}^{(3)}$	$G_\ell^{(1)}$	$D_{2\ell-1}^{(3)}$

The superscript ^{t} will be used when dealing with data associated with $\mathfrak{g}^t$, e.g., the labels of $\mathfrak{g}^t$ are denoted a_i^t, its Cartan subalgebra is denoted $\mathfrak{h}^t$, the invariant bilinear form is $(\cdot, \cdot)^t$, etc. In order to compare the two algebras more readily, define the isometry

$$T : \mathfrak{h}^* \to \mathfrak{h}^t$$

$$\alpha_i \mapsto \alpha_i^{\vee t}$$

$$\Lambda_0 \mapsto d^t$$

which in turn induces an isometry $\mathfrak{h}^* \to (\mathfrak{h}^t)^*$, also denoted T, given by

$$\Lambda_i \mapsto \frac{a_i^\vee}{a_i}\Lambda_i^t \text{ and } \delta \mapsto \delta^t.$$

This map 'transposes' the roots and coroots and is crucial in the construction of the Verlinde algebra for twisted affine algebras. The restriction map $T|_M$ is an isomorphism of lattices $M \cong M^t$ and

$$T(\theta) = \theta^t, T(\theta^\vee) = \theta^{\vee t}.$$

The latter fact shows that $T\left(P^k\right) \hookrightarrow (P^k)^t = (P_k)^t \cong P_k$. Recall that $\{\Theta_\lambda | \lambda \in P_k \pmod{kM + \mathbb{C}\delta}\}$ is a basis of Th_k for $k > 0$. Therefore, T induces a canonical map between the theta functions of $\mathfrak{g}$ and $\mathfrak{g}^t$.

Proposition 2.12. *The isometry T induces an W-invariant isomorphism $Th_k \cong Th_k^t$.*

Proof. A direct check shows that $W^t = \overset{\circ}{W}{}^t \ltimes M^t \cong \overset{\circ}{W} \ltimes M$ and

$$e^{(\lambda,\mu)} = e^{(T(\lambda),T(\mu))^t}.$$

Therefore, the map $\Theta_\lambda \mapsto \Theta^t_{T(\lambda)}$ is an inclusion. Furthermore, since $T(M^*) = (M^t)^*$, proposition 2.5 implies that $T\left(Th_k\right) = Th_k^t$. □

Since the finite Weyl groups of $\mathfrak{g}$ and $\mathfrak{g}^t$ are isomorphic and commute with isometries between their Cartan subalgebras, the space Th_k^-, which is spanned by the alternants

$$A_\lambda = \sum_{w \in \overset{\circ}{W}} \epsilon(w)\Theta_{w(\lambda)}$$

for $\lambda \in P_k$, can be identified with the space $\left(Th_k^t\right)^-$.

At this point, one should notice that $T(\rho) \neq \rho^t$ and so, instead of using the map T between theta functions given above, consider the map

$$\overset{\bullet}{T} : P_k \to P^t_{k+h^\vee-h}$$

$$\lambda \mapsto T(\lambda+\rho) - \rho^t$$

and the map it induces on the spaces of characters, denoted by the same symbol,

$$\overset{\bullet}{T} : Ch_k \to Ch^t_{k+h^\vee-h}$$

$$\chi_\lambda \mapsto \chi^t_{\overset{\bullet}{T}(\lambda)}.$$

Note that this map is a well-defined injection, i.e., $h^\vee - h \geq 0$ and $\overset{\bullet}{T}(\lambda) \in P^+_{k+h^\vee-h}$, and that $\overset{\bullet}{T}(\chi_\lambda) = A_{T(\lambda+\rho)}/A_{\rho^t}$. Furthermore, $\overset{\bullet}{T}$ gives a tight relationship between the S-matrices of the two Lie algebras. Before describing it, recall that for any $\mathfrak{g}$ of the type considered in this section, the *adjacent algebra* $\mathfrak{g}'$ is given by the above table. The S-matrix of $\mathfrak{g}$, as defined in [Ka], does not act on the space of characters but, rather, is a map between different character spaces. Nonetheless, its image can be described in terms of the characters of the adjacent algebra $\mathfrak{g}'$. For more details, please refer to [Ka,§13.9]. In that section, Kac discusses a map $\alpha \mapsto \alpha'$ between the Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{g}'$ that is very similar to the map T in this paper.

Proposition 2.13. *Let $S(\mathfrak{g})$ be the S-matrix acting on the space of theta functions at level k and $S(\mathfrak{g}^t)$ be the analogous S-matrix for $\mathfrak{g}^t$ and level $k + h^\vee - h$. Then*

$$S(\mathfrak{g})_{\lambda,\mu} = \left|\overset{\circ}{Q}{}^\vee/M\right|^{1/2} S(\mathfrak{g}^t)_{\overset{\bullet}{T}(\lambda),T(\mu)+(h^\vee-h)\Lambda_0^t}$$

$\lambda \in P^k$, $\mu \in P_k$.

Proof. The result follows directly from the dictionary between the map given in [Ka] and the construction of the two maps T and $\overset{\bullet}{T}$ in this paper. Since $\overline{\rho}^t = T(\overline{\rho}')$ and $M' = \overset{\circ}{Q}{}^\vee$, comparing the two quantities below

$$S(\mathfrak{g})_{\lambda,\mu} = i^{|\overset{\circ}{\Delta}_+|}|M^*/(k+h^\vee)M|^{-1/2}|M'/M|^{1/2}\sum_{w\in\overset{\circ}{W}}\epsilon(w)e^{-\frac{2\pi i(w(\overline{\lambda}+\overline{\rho}),\overline{\mu}+\overline{\rho}')}{k+h^\vee}}$$

$$S(\mathfrak{g}^t)_{\overset{\bullet}{T}(\lambda),T(\mu)+(h^\vee-h)\Lambda_0^t} = i^{|\overset{\circ}{\Delta}{}^t_+|}|\left(M^t\right)^*/(k+h^\vee)M^t|^{-1/2}\sum_{w\in\overset{\circ}{W}}\epsilon(w)e^{-\frac{2\pi i(w(T(\overline{\lambda}+\overline{\rho})),T(\overline{\mu})+\overline{\rho}^t)}{k+h^\vee}}$$

gives the result. □

Theorem 2.1 asserts that the space V is equal to $Th^{t-}_{k+h^\vee}$. Notice that $Ch^t_{k+h^\vee-h} \cong Th^{t-}_{k+h^\vee}$ via the map $\chi^t_\lambda \mapsto \chi^t_\lambda A_{\rho^t}$. Using this isomorphism, we can study the action of S on $Th^{t-}_{k+h^\vee}$ and prove the theorems 2.1 and 2.3.

Proof of 2.1. The space V is a subspace of $Th^{t-}_{k+h^\vee}$. The action of S on $Th^{t-}_{k+h^\vee}$ differs from its action on $Ch^t_{k+h^\vee-h}$ by a nonzero scalar, see Theorem 2.8. The S-matrix of $Ch^t_{k+h^\vee-h}$ has $S_{k\Lambda_0,,\lambda} = S_{\lambda,k\Lambda_0} \neq 0$ for all $\lambda \in P^{t+}_k$ (see Theorem 2.8) and so the coefficient of A_{ρ^t} in the expansion of $A_{\lambda+\rho}|_S$ (with resect to the alternant basis) is nonzero. It follows that $A_{\rho^t} \in V^-$. Similarly, for any $\mu \in P^{t+}_k$ the coefficient of $A_{\mu+\rho^t}$ in the expansion of $A_{\rho^t}|_S$ is nonzero. It follows that $Th^{t-}_{k+h^\vee} \subseteq V$.

Note that $\widetilde{\rho} = \rho^t$ and that the S matrix has nonzero entries in the column corresponding to $A_{\widetilde{\rho}}$. Therefore, we can use the Verlinde formula to give V the structure of a commutative associative algebra as discussed in section 2.1. □

Theorem 2.3 is a direct consequence of above proof. Using the Verlinde algebra $V^t_{k+h^\vee-h}$ of $\mathfrak{g}^t$ at level $k+h^\vee-h$, one can study the action of the modular group on $Ch_k(\mathfrak{g})$ directly.

Theorem 2.14. *Let $\mathfrak{g}$ be a twisted affine Lie algebra, $\lambda \in P^k$, and $X \in \Gamma_1(r)$ then*

$$T\left(A_{\lambda+\rho}|_X\right) = A^t_{T(\lambda+\rho)}|_X.$$

Furthermore, there is a nonzero scalar $v(k,X)$ such that

$$\overset{\bullet}{T}\left(\chi_\lambda|_X\right) = v(k,X)\overset{\bullet}{T}(\chi_\lambda)|_X.$$

Proof. T is an isometry on the underlying Cartan subalgebras and induces an isomorphism between the spaces of theta functions $Th_{k+h^\vee}$ and $Th^t_{k+h^\vee}$. Therefore, the modular group action (actually, $Mp(2,\mathbb{Z})$ action) commutes with T. The second identity follows from the fact that $A_\rho|_X \in \mathbb{C}A_\rho \setminus \{0\}$ and $A_{\rho^t}|_X \in \mathbb{C}A_{\rho^t} \setminus \{0\}$. □

Let us investigate the structure of V_k relative Ch_k. The grading structure (introduced in 2.4) on $V_k(\mathfrak{g})$, where $\mathfrak{g}$ is a twisted affine Lie algebras different from type $A^{(2)}_{2\ell}$, is easy to describe

$$V_k(A^{(2)}_{2\ell-1}) = V_0 \oplus V_1, \quad V_k(D^{(2)}_{\ell+1}) = V_0 \oplus V_1$$

$$V_k(E^{(2)}_6) = V_0, \quad V_k(D^{(3)}_4) = V_0.$$

Using the grading and comparing it to the map $\overset{\bullet}{T}$, it can be shown that the image of $Ch_k(\mathfrak{g})$ lies in $V_{T(\overline{\rho})-\overline{\rho}^t}$ for $\mathfrak{g}$ of type $A_{2\ell-1}^{(2)}$. The image of Ch_k for type $D_{\ell+1}^{(2)}$ is generally not concentrated in a homogeneous component.

Proposition 2.15. *If* $\mathfrak{g}$ *is of type* $A_{2\ell-1}^{(2)}$, *then* $V_k = V_0 \oplus V_1$ *and* $\overset{\bullet}{T}(Ch_k) = V_1$.

Proof. The first part is clear. For the second part, note that

$$\overset{\circ}{Q}{}^t(A_{2\ell-1}^{(2)}) = \overset{\circ}{Q}(B_\ell^{(1)}) = \langle \Lambda_1^t, \ldots, \Lambda_{\ell-1}^t, 2\Lambda_\ell^t \rangle \text{ and } T(\rho) - \rho^t = \Lambda_\ell^t,$$

where $\langle , \rangle$ means 'spanned by'. Therefore, $V_{T(\overline{\rho})-\overline{\rho}^t} = V_1$. The lattice $\overset{\circ}{Q}{}^t(A_{2\ell-1}^{(2)})$ is self-dual which implies the equality

$$\overset{\bullet}{T}(Ch_k) = V^t_{T(\overline{\rho})-\overline{\rho}^t} = V_1.$$

□

2.5 Verlinde Algebras for $A_{2\ell}^{(2)}$

The case when $\mathfrak{g}$ is of type $A_{2\ell}^{(2)}$ has fascinating features that differ substantially from the other cases. On one hand, at *odd* integer level k the space of characters has the structure of a fusion algebra that is isomorphic to the Verlinde algebra for $C_\ell^{(1)}$ at the level $\frac{k-1}{2}$ [Ho2]. (A proof is included here.) On the other hand, more interesting features reveal themselves when the level is an *even* integer. Among them is the fact that the structure constants are no longer non-negative integers.

It can be verified, see 2.6, that the fusion rules in this case are well-defined and define a commutative associative unital algebra with integral structure constants. Similar to the untwisted Verlinde algebras, one can define an algebraic structure on the representation ring of $\mathfrak{g}$ via a map from the representation ring of $\overset{\circ}{\mathfrak{g}}$. Unlike the untwisted case, the fusion rules can only be shown to be non-negative at odd level (see prop 2.16 below).

Proposition 2.16. *Let* $\mathfrak{g}$ *be of type* $A_{2\ell}^{(2)}$ *and fix the level* $2k+1$, *where* $k \in \mathbb{N}$, *then the Verlinde algebra of* $\mathfrak{g}$ *is isomorphic to the Verlinde algebra of the untwisted affine Lie algebra of type* $C_\ell^{(1)}$ *at level* k.

Proof. It can be checked directly that the S-matrices of both affine Lie algebras are equal, up to conjugation by a permutation matrix. More explicitly, if $\mathfrak{g}'$ is of type $C_\ell^{(1)}$ with Cartan subalgebra $\mathfrak{h}'$,

$$2(\alpha_i^\vee, \alpha_j^\vee) = (\alpha_i^{\vee\prime}, \alpha_j^{\vee\prime})'$$

$$\frac{1}{2}(\overline{\Lambda}_i, \overline{\Lambda}_j) = (\overline{\Lambda}_i', \overline{\Lambda}_j')'$$

for $1 \leq i, j \leq \ell$. Therefore, we can define isometries

$$\varphi : \mathring{\mathfrak{h}}' \to \mathring{\mathfrak{h}}$$

$$\alpha \mapsto \sqrt{2}\alpha$$

and the dual map

$$\varphi^* : \mathring{\mathfrak{h}}'^* \to \mathring{\mathfrak{h}}^*$$

$$\alpha \mapsto \frac{1}{\sqrt{2}}\alpha$$

such that $\langle \varphi(\alpha), \varphi^*(\lambda) \rangle = \langle \alpha, \lambda \rangle$. Note that $\sqrt{2}M = \varphi(M')$ and $\frac{1}{\sqrt{2}}M^* = \varphi^*(M'^*)$, therefore

$$|M'^*/(k+\ell)M'| = |M^*/(2k+2\ell)M|.$$

The S-matrix of $\mathfrak{g}'$ at level k is defined by

$$S_{\lambda\mu} = i^{|\mathring{\Delta}_+|} |M'^*/(k+\ell)M'|^{-1/2} \sum_{w \in \mathring{W}} \epsilon(w) e^{-\frac{2\pi i (w(\overline{\lambda}+\overline{\rho}), \overline{\mu}+\overline{\rho})'}{k+\ell}}$$

while the S-matrix of $\mathfrak{g}$ at level $2k+1$ is defined by

$$S_{\lambda\mu} = i^{|\mathring{\Delta}_+|} |M^*/(2k+2\ell)M|^{-1/2} \sum_{w \in \mathring{W}} \epsilon(w) e^{-\frac{2\pi i (w(\overline{\lambda}+\overline{\rho}), \overline{\mu}+\overline{\rho})}{2k+2\ell}}.$$

Since the structure constants of the Verlinde algebras are computed by the coefficients of the corresponding S-matrices, the algebras are isomorphic. □

In the cases where the level is even, one can observe that the fusion rules are often negative. In fact, it is true in every case we could compute.

Conjecture 2.17. *Let $\mathfrak{g}$ be an affine Lie algebra of type $A_{2\ell}^{(2)}$ and fix the level $2n$ for $n \in \mathbb{Z}_{>0}$. For any $\lambda, \mu \in P_{2n}$,*

$$\chi_\lambda \chi_\mu = \sum_{\nu \in P_{2n}^+ \text{ mod } \mathbb{C}\delta} (-1)^{[\lambda]+[\mu]+[\nu]} |N_{\lambda\mu}^{\nu}| \chi_\nu$$

in $V_{2n}(\mathfrak{g})$, where $[\phi]$ is the class of ϕ in $\mathcal{A} \cong \{0,1\}$. In particular, the structure constants of $V_{2n}(\mathfrak{g})$, with respect to the basis $\{\widetilde{\chi}_\lambda\}_{\lambda \in P_{2n}^+ \text{ mod } \mathbb{C}\delta}$, where $\widetilde{\chi}_\lambda = (-1)^{[\lambda]}\chi_\lambda$, are nonnegative.

We can show exactly when negative structure constants appear in the following case. Let $\mathfrak{g}$ be of type $A_2^{(2)}$ and fix a level $2n$, $n > 0$. The underlying simple Lie algebra is isomorphic to $\mathfrak{sl}_2$. Set $\alpha = \alpha_1$, $\alpha^\vee = \alpha_1^\vee, \overline{\Lambda}_1 = \Lambda$ then

$$\theta = \alpha,\ \theta^\vee = \alpha^\vee,\ \text{and } \overline{\rho} = \alpha^\vee.$$

Note also that $h^\vee = 3$. The of dominant integral weights of level $2n$ project to

$$\overline{P}_+^{2n} = \{m\Lambda\}_{m=0}^n.$$

The set of dominant integral weights P_+ of $\mathfrak{sl}_2$ is indexed by $\mathbb{N}$ and the map between character spaces is

$$F_{2n} : \{\chi_m\}_{m \in P_+} \to \{\chi_\lambda\}_{\lambda \in P_+^{2n}}$$

$$\chi_m \mapsto \begin{cases} \chi_{m'\Lambda} & \text{if } 0 \le m' \le n \text{ and } m \equiv m' \pmod{2n + h^\vee} \\ 0 & \text{if } m \equiv -1 \pmod{2n + h^\vee} \\ -\chi_{m'\Lambda} & \text{if } 0 \le m' \le n \text{ and } m + m' + 2 \equiv 0 \pmod{2n + h^\vee} \end{cases}.$$

The proof of the above map follows from the fact that the affine Weyl group of type $A_{2\ell}^{(2)}$ is $S_2 \ltimes \mathbb{Z}$ and so, for any $m \in \mathbb{N}$, there is a unique m' modulo $2n + h^\vee$ satisfying the conditions in the piecewise definition of F_{2n}. More importantly, there is a canonical section

$$S_{2n} : \{\chi_\lambda\}_{\lambda \in P_+^{2n}} \to \{\chi_m\}_{m \in P_+}$$

$$\chi_{m\Lambda} \mapsto \chi_m$$

and the fusion rules N_{ab}^c are given by the coefficients of the irreducible character χ_c in

$$F_{2n}(S_{2n}(\chi_{a\Lambda}) \otimes S_{2n}(\chi_{b\Lambda})).$$

Th product $S_{2n}(\chi_{a\Lambda}) \otimes S_{2n}(\chi_{b\Lambda})$ is the direct sum of $\chi_{c\Lambda}$ where $|a-b| \leq c \leq a+b$ and $a+b+c \equiv 0 \pmod 2$. The tuples (a,b,c) with the above property are known as *admissible triples.* More explicitly, the tensor product of $\mathfrak{sl}_2$-modules is given by'

$$\chi_{a\Lambda} \otimes \chi_{b\Lambda} = \chi_{|a-b|\Lambda} \oplus \chi_{(|a-b|+2)\Lambda} \oplus \cdots \oplus \chi_{(a+b)\Lambda}.$$

Assume that $a \geq b$. Since $a+b \leq 2n$ we have two cases: $a+b \leq n$ or $n < a+b \leq 2n$. The above discussion proves the following.

Proposition 2.18. *For $\mathfrak{g}$ of type $A_2^{(2)}$ at level $2n$, the product structure of $V_{2n}(\mathfrak{g})$ is explicitly given by*

$$\chi_{a\Lambda} \otimes \chi_{b\Lambda} = \sum_{0 \leq i \leq \frac{n+b-a}{2}} \chi_{(a-b+2i)\Lambda} - \sum_{\frac{n+b-a}{2} < i \leq b} \chi_{(2n+1+b-2i-a)\Lambda}.$$

It follows that $N_{ab}^c < 0$ whenever (a,b,c) is admissible, $a+b > n$, and $c > n$. In particular, $N_{n,n}^1 = -1$ when n is even and $N_{n,n}^2 = -1$ when n is odd. Later on, in section 2.6.2, we will give a more explicit description of the product structure of $V_{2n}(A_{2\ell}^{(2)})$ for any rank ℓ.

The following curious fact shows that when the level is $2n$ and $2n+1$, the underlying character spaces are equal while the Verlinde algebra structures are quite different. Moreover, I observed that, at even level, if the structure constants $N_{\lambda\mu}^{\nu}$ are replaced by their absolute values $|N_{\lambda\mu}^{\nu}|$, then the resulting algebra is *also* associative and commutative.

Lemma 2.19. *Let $k \geq 1$, then $\overline{P}_+^{2n} = \overline{P}_+^{2n+1}$.*

Proof. It is a straightforward check to show that $\overline{P}_+^{2n} \subseteq \overline{P}_+^{2n+1}$. Suppose that $\lambda \in P_+^{2n+1}$ and $\lambda = \sum_{i=0}^{\ell} c_i \Lambda_i$, then $c_i \in \mathbb{N}$ for $0 \leq i \leq \ell$ and

$$\langle \lambda, K \rangle = \left\langle \lambda, \alpha_0^\vee + 2\sum_{i=1}^{\ell} \alpha_i^\vee \right\rangle = 2n+1.$$

This equation implies that $c_0 > 0$ and so we can define $\lambda' \in P_{2n}^+$ by $\lambda' = \lambda - \Lambda_0$. Therefore, the earlier inclusion and $\overline{P}_+^{2n} \supseteq \overline{P}_+^{2n+1}$ implies the lemma. $\square$

Another different feature in this case is that the grading structure (as an algebra) only exists when the level is odd (since it exists in $V_k(C_\ell^{(1)})$). Nevertheless, when the level is even we can still define the same quotient $\mathcal{A} = \overset{\circ}{P}/\overset{\circ}{Q} \cong \mathbb{Z}_2$ and get the decomposition $V_k(\mathfrak{g}) = V_0 \bigoplus V_1$ *as a vector space*. Empirically, we have observed the following phenomenon and conjecture that $V_k(\mathfrak{g})$ can be realized as a commutative associative algebra with nonnegative integral structure constants.

2.6 Applications

2.6.1 Quotients of $V_k(\mathfrak{g})$

The definition of Verlinde algebras for the various types of affine Lie algebras gives an explicit way of computing the structure constants of these algebras. In a recent paper by J. Hong [Ho2], the author attaches a fusion ring to all affine Lie algebras and he expects that the corresponding structure constants are all non-negative. Here we show that Hong's fusion algebras can be realized as *quotients* of the Verlinde algebras presented here. With this realization, we can answer some questions posed in [Ho2]. In the case of $A_{2\ell}^{(2)}$, the Verlinde algebras and Hong's fusion algebras are isomorphic, proving that the structure constants are sometimes negative (specifically, when the level is even they are negative in all known cases). In the remaining cases, computations show that the structure constants are also sometimes negative (depending on the level and type).

First, let us describe the quotients of the Verlinde algebras to be considered. Let k be a positive integer throughout this section. Recall the map

$$F_k : \mathcal{R}(\overset{\circ}{\mathfrak{g}}) \to Ch_k(\mathfrak{g})$$

given in section 2.3 where $\mathfrak{g}$ is an *untwisted* affine Lie algebra of rank ℓ. This map depends on the affine Weyl group

$$\overset{\circ}{W} \ltimes \overset{\circ}{Q}^\vee$$

(where, as usual, $\overset{\circ}{Q}^\vee$ is identified with a lattice in $\mathfrak{h}^*$ using the invariant bilinear form.) Consider, instead, the group $W' := \overset{\circ}{W} \ltimes \overset{\circ}{Q}$. When $\mathfrak{g}$ is simply laced, $W = W'$ and so we may assume that $\mathfrak{g}$ is of type $B_\ell^{(1)}, C_\ell^{(1)}, F_4^{(1)}$, or $G_2^{(1)}$. The group W' is generated by $s_{\alpha'_i}$ where $\alpha'_i = \alpha_i$ for $i = 1 \ldots, \ell$ and $\alpha'_0 = \delta - \theta_s$, where θ_s is the highest short root of $\mathring{\mathfrak{g}}$. The α'_i give a root system (that is equivalent to that of $\mathfrak{g}^{t\prime}$) and there are corresponding notions for fundamental weights $\Lambda'_0, \Lambda'_1, \ldots,$ the element $\rho' = \sum_{i=0}^{\ell} \Lambda'_i$, the weight lattice $P' = \bigoplus_{i=0}^{\ell} \mathbb{Z}\Lambda'_i$, the level k dominant weights $P_k'^{+}$, the space of 'primed' level k characters $Ch'_k(\mathfrak{g}) = \bigoplus_{\lambda \in P_k'^{+} \bmod \mathbb{C}\delta} \mathbb{C}A_{\lambda'+\rho'}/A_{\rho'}$, etc. The corresponding dual Coxeter number $h^{\vee\prime}$ coincides with the Coxeter number h. Note that $\overline{\rho}' = \overline{\rho}$. In a little while, we will see why the 'prime' notation is suitable.

The map $F_{k+h-h^\vee}$ uses the action of $\mathring{W} \ltimes (k+h)\mathring{Q}^\vee$ in its definition. Using the 'primed' version of the affine Weyl group $W_{aff}'^{k+h} = \mathring{W} \ltimes (k+h)\mathring{Q}$, define the map

$$F'_k : \mathcal{R}(\mathring{\mathfrak{g}}) \to Ch'_k(\mathfrak{g})$$

$$\mathring{\chi}_{\overline{\lambda}} \mapsto \begin{cases} \epsilon(w)\chi_{k\Lambda'_0 + w(\overline{\lambda}+\overline{\rho})-\overline{\rho}} & \text{if there exists } w \in W_{aff}'^{k+h},\ w(\overline{\lambda}+\overline{\rho}) \in P_{k+h}'^{aff\ ++} \\ 0 & \text{otherwise} \end{cases}.$$

Since $\mathring{Q}^\vee \subseteq \mathring{Q}$, $W_{aff}^{k+h} \subseteq W_{aff}'^{k+h}$ and F'_k factors through $Ch_{k+h-h^\vee}(\mathfrak{g})$, i.e., there is a map

$$G_k : Ch_{k+h-h^\vee}(\mathfrak{g}) \to Ch'_k(\mathfrak{g}),$$

such that $F'_k = G_k \circ F_{k+h-h^\vee}$. Recall that we may identify $Ch_{k+h-h^\vee}(\mathfrak{g}) \cong V_{k+h-h^\vee}(\mathfrak{g})$ and also, by 2.3, we identify $V_{k+h-h^\vee}(\mathfrak{g}) \cong V_k(\mathfrak{g}^t)$. We use the same notation for the corresponding map

$$G_k : V_k(\mathfrak{g}^t) \to Ch'_k(\mathfrak{g}).$$

Let us now give briefly describe Hong's fusion algebras for *twisted* affine Lie algebras. Let $\mathfrak{g}$ be a twisted affine Kac-Moody Lie algebra that is not of type $A_{2\ell}^{(2)}$ and let k a positive integer, then Hong's fusion algebra $R_k(\mathfrak{g})$ is defined to be the vector space

$$R_k(\mathfrak{g}) = \bigoplus_{\lambda \in P^k \bmod \mathbb{C}\delta} \mathbb{C}\mathring{\chi}_{\overline{\lambda}}$$

with structure constants

$$c_{\lambda\mu}^{\nu} = \sum_{w \in W_k^\dagger} \epsilon(w) N_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}$$

where $W_k^\dagger$ is the set of minimal representations of the left cosets of $\overset{\circ}{W}$ in $W_{aff}^{k+h^\vee}$ and $N_{\overline{\lambda},\overline{\mu}}^{\overline{\nu}}$ is the coefficient of $\overset{\circ}{\chi}_{\overline{\nu}}$ in $\overset{\circ}{\chi}_{\overline{\lambda}}\overset{\circ}{\chi}_{\overline{\mu}}$. When $\mathfrak{g}$ is of type $A_{2\ell}^{(2)}$, we use the same definition of $R_k(\mathfrak{g})$ except that we use the affine Weyl group $\overset{\circ}{W} \ltimes (k+h^\vee)\overset{\circ}{Q}^\vee$. See [Ho2] for more details. The vector spaces $Ch_k(\mathfrak{g})$ and $R_k(\mathfrak{g})$ can be naturally identified using the map $\chi_\lambda \mapsto \overset{\circ}{\chi}_{\overline{\lambda}}$, $\lambda \in P^{k+}$.

Let us now compare our Verlinde algebras with $R_k(\mathfrak{g})$. Let $\mathfrak{g}$ be a twisted affine Kac-Moody algebra. Recall the correspondence between adjacent algebras $\mathfrak{g} \leftrightarrow \mathfrak{g}'$ given in the table in section 2.4. Note that the underlying simple Lie algebra of $\mathfrak{g}'$ is isomorphic to the underlying simple Lie algebra of $\mathfrak{g}^t$. Identify their Cartan subalgebras $\overset{\circ}{\mathfrak{h}}{}' = \overset{\circ}{\mathfrak{h}}{}^t$ and notice that since the θ element of $\mathfrak{g}'$ is the highest short root, $\alpha_0 \in \mathfrak{h}'^*$ and $(\alpha_0^t)' \in \mathfrak{h}^{t*}$ can also be identified. This identification induces a natural bijection $P_k' \cong (P_k^t)'$. Finally, note that $h'^\vee = (h^{\vee t})' = h^t$. All of this shows that there is a natural linear isomorphism $Ch_k(\mathfrak{g}') \cong Ch_k'(\mathfrak{g}^t)$ given by $\chi_\lambda \mapsto \chi_\lambda$.

Now, let $\mathfrak{g}$ be a twisted affine algebra, not of type $A_{2\ell}^{(2)}$, then $\mathfrak{g}^t$ is untwisted. Also, $V_k(\mathfrak{g}) \cong V_{k+h^\vee-h}(\mathfrak{g}^t)$ and $h^\vee = h^t, h = h^{\vee t}$. The maps

$$F_{k+h^\vee-h} : \mathcal{R}(\overset{\circ}{\mathfrak{g}^t}) \to Ch_{k+h^\vee-h}(\mathfrak{g}^t)$$

$$F_k' : \mathcal{R}(\overset{\circ}{\mathfrak{g}^t}) \to Ch_k'(\mathfrak{g}^t) = Ch_k(\mathfrak{g}')$$

$$G_k : Ch_{k+h^\vee-h}(\mathfrak{g}^t) \to Ch_k'(\mathfrak{g}^t),$$

give rise to maps

$$F_{k+h^\vee-h} : \mathcal{R}(\overset{\circ}{\mathfrak{g}^t}) \to V_k(\mathfrak{g})$$

$$F_k' : \mathcal{R}(\overset{\circ}{\mathfrak{g}^t}) \to R_k(\mathfrak{g}')$$

$$G_k : V_k(\mathfrak{g}) \to R_k(\mathfrak{g}'),$$

by abuse of notation, using the identifications discussed earlier. The following theorem appears in [Ho2] and will be used to show that $R_k(\mathfrak{g}')$ is a quotient of $V_k(\mathfrak{g})$. This will give a method of computing the structure constants $c_{\lambda\mu}^\nu$ from the fusion rules $N_{\lambda\mu}^\nu$.

Theorem 2.20. *(Hong) Let $\mathfrak{g}$ be a twisted affine Kac-Moody algebra, then, using the above conventions, F'_k is a surjective homomorphism.*

Theorem 2.21. *Let $\mathfrak{g}$ be a twisted affine Kac-Moody algebra and k a positive integer, then there is a surjective homomorphism $V_k(\mathfrak{g}) \to R_k(\mathfrak{g}')$. In particular, the fusion algebra $R_k(\mathfrak{g}')$ is isomorphic to a quotient of $V_k(\mathfrak{g})$.*

Proof. If $\mathfrak{g}$ is of type $A_{2\ell}^{(2)}$, then $\mathfrak{g} = \mathfrak{g}'$ and the vector spaces $V_k(\mathfrak{g})$ and $R_k(\mathfrak{g})$ can be identified via $A_\lambda \mapsto \overset{\circ}{\chi}_{\overline{\lambda}}$, $\lambda \in P^{k+}$. In [Ho2], the structure constants may be computed by the same Verlinde formula as the one found here and so the algebra structures coincide.

Now, let $\mathfrak{g}$ be a twisted affine algebra not of type $A_{2\ell}^{(2)}$, then $\mathfrak{g}^t$ is untwisted. Since the map

$$F'_k : \mathcal{R}(\overset{\circ}{\mathfrak{g}^t}) \to R_k(\mathfrak{g}')$$

is surjective, so is

$$G_k : V_k(\mathfrak{g}) \to R_k(\mathfrak{g}').$$

For $\chi_\lambda, \chi_\mu \in V_k(\mathfrak{g})$, recall that $\chi_\lambda \chi_\mu = F_{k+h^\vee-h}(\overset{\circ}{\chi}_{\overline{\lambda}} \overset{\circ}{\chi}_{\overline{\mu}})$. Therefore, by 2.20,

$$G_k(\chi_\lambda \chi_\mu) = G_k(F_{k+h-h^\vee}(\overset{\circ}{\chi}_{\overline{\lambda}} \overset{\circ}{\chi}_{\overline{\mu}})) = F'_k(\overset{\circ}{\chi}_{\overline{\lambda}}) F'_k(\overset{\circ}{\chi}_{\overline{\mu}}) = G_k(\chi_\lambda) G_k(\chi_\mu).$$

□

Corollary 2.22. *Let $\mathfrak{g}$ be a twisted affine Lie algebra, then the structure constants $c_{\lambda\mu}^\nu$ of $R_k(\mathfrak{g}')$ are given by*

$$c_{\lambda\mu}^\nu = \sum_{w \in X_\nu} \sum_{\phi \in P'^{k+} \bmod \mathbb{C}\delta} \frac{S_{\lambda\phi} S_{\mu\phi} (S^{-1})_{\nu\phi}}{S_{k\Lambda_0,\phi}},$$

where $X_\nu = \{w \in W_k^\dagger : (w(\overline{\nu} + \overline{\rho}) - \overline{\rho}, \theta^t) \leq k + h^\vee - h\}$.

Example 2.23. *The left multiplication matrices of $G_k(V_k(\mathfrak{g}))$ are given by 'truncating and folding' the left multiplication matrices of $V_k(\mathfrak{g})$. More precisely, one removes all columns corresponding to weights λ such that $\langle \lambda + \rho, \theta^\vee \rangle \geq k + h^\vee$ (here $\theta^\vee$ is the highest* *coroot of* $\mathring{\mathfrak{g}}$*), removes all rows corresponding to* λ *such that* $\langle \lambda + \rho, \theta^\vee \rangle$ *is an integral multiple of* $k + h^\vee$*, and then adds all rows that are in the same orbit under the* dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ *with appropriate sign* $\epsilon(w)$*. In particular, consider the data from example 2.10 where the weights are*

$$0, \overline{\Lambda}_1, 2\overline{\Lambda}_1, \overline{\Lambda}_2, 2\overline{\Lambda}_3,$$

$$\overline{\Lambda}_3, \overline{\Lambda}_1 + \overline{\Lambda}_3.$$

In this case, $\mathfrak{g}$ *is of type* $A_5^{(2)}$ *and* $k = 1$*. The only* $\overline{\lambda} + \overline{\rho}$*, for* $\overline{\lambda}$ *from the list above, satisfying* $\langle \lambda + \rho, \theta^\vee \rangle < k + h^\vee = 1 + 6$ *are* 0 *and* $\overline{\Lambda}_3$*. The weights* $\overline{\Lambda}_1 + \overline{\rho}$*,* $\overline{\Lambda}_2 + \overline{\rho}$*, and* $2\overline{\Lambda}_3$ *satisfy* $\langle \lambda + \rho, \theta^\vee \rangle = 7$ *and so are thrown out. Finally, adding the remaining rows (all zeros) we get*

$$L_1' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } L_2' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2.6.1.1 More Negative Structure Constants

In the remaining twisted cases, negative structure constants also appear in a pattern very similar to the appearance of negative structure constants for the Verlinde algebra of $A_{2\ell}^{(2)}$. Unlike the conjectured state of affairs for $A_{2\ell}^{(2)}$, experimental data shows that it is not possible to 'twist' the basis by a character in order to make the structure constants non-negative. Nonetheless, a grading structure plays a role in determining when the structure constants are non-negative.

Let $\mathfrak{g}$ be a twisted affine Lie algebra that is not of type $A_{2\ell}^{(2)}$. The algebra $R(\mathfrak{g})$ defined in the previous section does not inherit the algebra grading of $V_k(\mathfrak{g}')$. Instead, there is a *vector space grading* indexed by $\mathring{P}/\mathring{Q}$ analogous to the untwisted case, i.e.,

$$R(\mathfrak{g}) = \bigoplus_{a \in \mathcal{A}} R(\mathfrak{g})_a$$

where $R(\mathfrak{g})_a = \text{span}\{\chi_\lambda : \lambda \in P_k^+ \text{ and } \overline{\lambda} \in a\}$. Similar to the $A_{2\ell}^{(2)}$ case, this vector space grading seems to play a role (albeit different) in determining the non-negativity of the structure constants of $R(\mathfrak{g})$. The structure constants $N_{\lambda\mu}^{\nu}$ of $R(\mathfrak{g})$ can be observed to behave according to the following 2/3's rule.

Conjecture 2.24. *Let k be an even positive integer and $\mathfrak{g}$ be a twisted affine Lie algebra that is not of type $A_{2\ell}^{(2)}$. If $\lambda, \mu, \nu \in P_k^+$ and at least 2 out of $\overline{\lambda}, \overline{\mu}, \overline{\nu}$ lie in $\mathring{Q}$, then $N_{\lambda\mu}^{\nu} \geq 0$, where $N_{\lambda\mu}^{\nu}$ are the structure constants of $R(\mathfrak{g})$ with respect to the basis $\{\chi_\lambda\}_{\lambda \in P_k^+}$.*

Example 2.25. *Let us consider the case when $\mathfrak{g}$ is of type $A_5^{(2)}$ with level $k = 2$. In this case there are negative structure constants. Consider the ordered basis $\chi_{\lambda_0}, \chi_{\lambda_1}, \chi_{\lambda_2}, \ldots, \chi_{\lambda_4}$ where the $\overline{\lambda}_i$ are*

$$0, \overline{\Lambda}_1, 2\overline{\Lambda}_1, \overline{\Lambda}_2, \overline{\Lambda}_3,$$

respectively. There are only two classes $\{0, \overline{\Lambda}_3\} = \mathring{P}/\mathring{Q}$. The matrices for the L_i, the operator of left multiplication by χ_{λ_i}, are $L_0 = id$

$$L_1 = \begin{pmatrix} & 1 & & & \\ 1 & & 1 & 1 & \\ & 1 & & & \\ & 1 & & & 1 \\ & & & 1 & \end{pmatrix} \qquad L_2 = \begin{pmatrix} & & 1 & & \\ & 1 & & & \\ 1 & & & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} & & & 1 & \\ & 1 & & & 1 \\ & & & 1 & \\ 1 & & 1 & & \\ & 1 & & & -1 \end{pmatrix} \qquad L_4 = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & & & 1 \\ & 1 & & & -1 \\ 1 & & 1 & -1 & \end{pmatrix}$$

2.6.2 The $A_{2\ell}^{(2)}$ Quotient Structure

In the case where $\mathfrak{g}$ is of type $A_{2\ell}^{(2)}$, the Verlinde algebra $V_k(\mathfrak{g})$ can be realized as a quotient of a Verlinde algebra $V_{k'}(C_\ell^{(1)})$. In this section, we use the affine Weyl groups in question to explicitly describe the quotient $V_k(\mathfrak{g})$. We then explain the relationship between the grading structures of these vector spaces.

Lemma 2.26. *Let $\mathfrak{g}$ be of type $A_{2\ell}^{(2)}$ and fix a positive integer k. Let $\widetilde{\mathfrak{g}}$ be an affine algebra of type $C_\ell^{(1)}$, then there exist a surjective homomorphism*

$$F : V_{k+\ell}(\widetilde{\mathfrak{g}}) \to V_k(\mathfrak{g})$$

given by $F(\chi_\lambda) = F_k \mathring{\chi}_{\overline{\lambda}}$, where F_k is the homomorphism defined at the end of section 2.3.

Proof. The underlying simple Lie algebras of $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ coincide and so we will identify their Cartan subalgebras and their duals. The map F is clearly well-defined as a linear map. To show that it is a homomorphism, it is enough to show that the homomorphism F_k factors through $V_{k+\ell}(\widetilde{\mathfrak{g}})$. Indeed, $V_{k+\ell}(\widetilde{\mathfrak{g}})$ is the quotient of the character ring $\mathcal{R}(\mathring{\mathfrak{g}})$ by the action of $\mathring{W} \ltimes (k+2\ell+1)\widetilde{\nu}(\mathring{Q}^\vee)$ where $\widetilde{\nu}$ is the map $\mathfrak{h} \to \mathfrak{h}^*$ induced by the normalized bilinear form of $\widetilde{\mathfrak{g}}$. If ν is the normalized bilinear form of $\mathfrak{g}$, then note that

$$\widetilde{\nu}(\mathring{Q}^\vee) = 2\nu(\mathring{Q}^\vee) \subset \mathring{Q} \subset \nu(\mathring{Q}^\vee).$$

Since $V_k(\mathfrak{g})$ is the quotient of $\mathcal{R}(\mathring{\mathfrak{g}})$ by the action of $\mathring{W} \ltimes (k+2\ell+1)\nu(\mathring{Q}^\vee)$, it follows that F_k factors through $V_{k+\ell}(\widetilde{\mathfrak{g}})$. □

Remark 3. *Let A_1 be the fundamental alcove of the affine Weyl group. Note that the fundamental domain of $\mathring{W} \ltimes (k+2\ell+1)\nu(\mathring{Q}^\vee)$ is $(k+2\ell+1)A_1$ and the fundamental domain of $\mathring{W} \ltimes (k+2\ell+1)\widetilde{\nu}(\mathring{Q}^\vee)$ is $2(k+2\ell+1)A_1$. We will soon present an algorithm mapping elements from $2(k+2\ell+1)A_1$ into $(k+2\ell+1)A_1$.*

Let $\mathfrak{g}$ be of type $A_{2\ell}^{(2)}$ as before. For the next result, let us use the notation $w_0 := id$ and

$$w_i := s_{i-1}s_{i-2}\cdots s_0$$

for $0 < i \leq \ell$ where $s_i \in W$ are the Coxeter generators of the Weyl group W of $\mathfrak{g}$. For any list of integers $I = (i_1, \ldots, i_m)$, also define

$$w_I = w_{i_m} w_{i_{m-1}} \cdots w_{i_1}.$$

In the proof of this next theorem, we define an explicit recursive algorithm defining the quotient indicated by the previous lemma.

Theorem 2.27. *Let $\mathfrak{g}$ be of type $A_{2\ell}^{(2)}$, fix a positive integer k, and let $\widetilde{\mathfrak{g}}$ be an affine algebra of type $C_\ell^{(1)}$. If λ is a dominant weight of $\widetilde{\mathfrak{g}}$ of level $k+\ell$ such that $\overline{\lambda}+\overline{\rho}$ has trivial stabilizer with respect to the affine Weyl group $\mathring{W} \ltimes \nu(\mathring{Q}^\vee)$, then there is a weight μ of $\mathfrak{g}$ of level k and a sequence of nonnegative integers $I = (i_1, \ldots, i_m)$, where m satisfies $1 \leq m \leq k + 2\ell + 1$, such that*

$$\overline{\mu} + \overline{\rho} = w_I(\overline{\lambda} + \overline{\rho}).$$

In particular, for the homomorphism given in the previous lemma and λ as above,

$$F(\chi_\lambda) = (-1)^{i_1+\cdots+i_m} F_k(\mathring{\chi}_{w_I \cdot \overline{\lambda}}).$$

Proof. Let k be as in the statement and recall that $h^\vee = 2\ell+1$. Also recall that $\theta = 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell$ and $\theta^\vee = \alpha_1^\vee + \cdots + \alpha_\ell^\vee$. We identify $W \cong \mathring{W} \ltimes (k+h^\vee)\nu(\mathring{Q}^\vee)$. The Weyl group W of $\mathfrak{g}$ contains the subgroup $\mathring{W} \ltimes 2(k+h^\vee)\nu(\mathring{Q}^\vee)$ and maps every dominant weight $\overline{\lambda} \in \mathring{P}^+$ to a unique dominant weight $\overline{\mu}$ such that $\langle \overline{\mu}, \theta^\vee \rangle = \frac{1}{2}(\overline{\mu}, \theta) \leq k + h^\vee$. Therefore, for the purpose of describing F, we only need to consider dominant weights $\overline{\mu}$ such that $(\overline{\mu}+\overline{\rho}, \theta) < 2(k+h^\vee)$.

Let $\lambda \in P_{k+\ell}^+(\widetilde{\mathfrak{g}})$ be a *regular* dominant weight, i.e., a weight such that $\overline{\lambda}+\overline{\rho}$ has trivial stabilizer with respect to $\mathring{W} \ltimes (k+h^\vee)\nu(\mathring{Q}^\vee)$, then there is a unique element $w_\lambda \in \mathring{W} \ltimes (k+h^\vee)\nu(\mathring{Q}^\vee)$ such that $w_\lambda(\overline{\lambda}+\overline{\rho}) \in \mathring{P}^+$, and $(w_\lambda(\overline{\lambda}+\overline{\rho}), \theta) < k+h^\vee$. If $(\overline{\lambda}+\overline{\rho}, \theta) < k+h^\vee$, then we can take $w_\lambda = w_0 = id$.

Suppose that $k+h^\vee < (\overline{\lambda}+\overline{\rho}, \theta) < 2(k+h^\vee)$. Write

$$\overline{\lambda} + \overline{\rho} = \sum_{i=1}^{\ell} b_i \alpha_i^\vee = \sum_{i=1}^{\ell} c_i \overline{\Lambda}_i$$

(where $\alpha_i^\vee$ is mapped into $\mathring{\mathfrak{h}}^*$ via ν) such that $b_i > 0, c_i > 0$ for $i = 1, \ldots, \ell$, then $k+h^\vee < 2b_1 < 2(k+h^\vee)$ since $\theta = 2\overline{\Lambda}_1$. It is straightforward to check that $b_1 =$

$c_1 + \cdots + c_\ell$. Applying the affine generator s_0, we get

$$\begin{aligned} s_0(\overline{\lambda}+\overline{\rho}) &= s_\theta(\overline{\lambda}+\overline{\rho}) + (k+h^\vee)\theta^\vee \\ &= \overline{\lambda}+\overline{\rho} - \langle \overline{\lambda}+\overline{\rho}, \theta^\vee\rangle\theta + (k+h^\vee)\theta^\vee \\ &= \overline{\lambda}+\overline{\rho} + (k+h^\vee-2b_1)\theta^\vee \\ &= (k+h^\vee-2b_1+c_1)\overline{\Lambda}_1 + \sum_{i=2}^{\ell} c_i\overline{\Lambda}_i. \end{aligned}$$

It follows that $(s_0(\overline{\lambda}+\overline{\rho}),\theta) = 2(k+h^\vee-b_1) < k+h^\vee$ and, therefore, if $k+h^\vee-2b_1+c_1 > 0$, then $w_\lambda = w_1 = s_0$.

Suppose that $k+h^\vee-2b_1+c_1 < 0$. Compute the following weights

$$\begin{aligned} s_1s_0(\overline{\lambda}+\overline{\rho}) =& -(k+h^\vee-2b_1+c_1)\overline{\Lambda}_1 \\ &+ (k+h^\vee-2b_1+c_1+c_2)\overline{\Lambda}_2 + \cdots + c_\ell\overline{\Lambda}_\ell \\ s_2s_1s_0(\overline{\lambda}+\overline{\rho}) =& c_2\overline{\Lambda}_1 - (k+h^\vee-2b_1+c_1+c_2)\overline{\Lambda}_2 \\ &+ (k+h^\vee-2b_1+c_1+c_2+c_3)\overline{\Lambda}_3 + \cdots \\ s_3s_2s_1s_0(\overline{\lambda}+\overline{\rho}) =& c_2\overline{\Lambda}_1 + c_3\overline{\Lambda}_2 - (k+h^\vee-2b_1+c_1+c_2+c_3)\overline{\Lambda}_3 + \cdots \\ s_{\ell-1}\cdots s_1s_0(\overline{\lambda}+\overline{\rho}) =& c_2\overline{\Lambda}_1 + c_3\overline{\Lambda}_2 + \cdots + c_{\ell-1}\overline{\Lambda}_{\ell-2} \\ &- (k+h^\vee-2b_1+c_1+\cdots+c_{\ell-1})\overline{\Lambda}_{\ell-1} + (k+h^\vee-b_1)\overline{\Lambda}_\ell. \end{aligned}$$

If we set $\overline{\lambda}_i = s_{i-1}s_{i-2}\cdots s_0(\overline{\lambda}+\overline{\rho}) = w_i(\overline{\lambda}+\overline{\rho})$, then it must be the case that $\overline{\lambda}_{i_1} \in \mathring{P}^+$ for some i_1 such that $0 \le i_1 \le \ell$. In fact, $\overline{\lambda}_j$ is a regular dominant weight as soon as $2b_1 - \sum_{i=1}^{j} c_i \le k+h^\vee$ and, by assumption, $b_1 = 2b_1 - \sum_{i=1}^{\ell} c_i < k+h^\vee$. Moreover, for $i>1$, $(\overline{\lambda}_i,\theta) = 2(b_1-c_1) < 2b_1$. If the element $w_{i_1}(\overline{\lambda}+\overline{\rho})$ satisfies $(w_{i_1}(\overline{\lambda}+\overline{\rho}),\theta) < k+h^\vee$ then take $w_\lambda = w_{i_1}$ and we are done. Otherwise, we may continue in this fashion, producing a sequence

$$(\overline{\lambda}+\overline{\rho},\theta) > (w_{i_1}(\overline{\lambda}+\overline{\rho}),\theta) > (w_{i_2}w_{i_1}(\overline{\lambda}+\overline{\rho}),\theta) > \cdots$$

which must eventually lead to $(w_I(\overline{\lambda}+\overline{\rho}),\theta) < k+h^\vee$ in at most $k+h^\vee$ steps for some finite sequence $I = (i_1,\ldots,i_m)$. □

Corollary 2.28. *Let $\mathfrak{g}$, k, $\widetilde{\mathfrak{g}}$, λ, I, m etc. be as in the previous theorem. Recall that $V = V_k(\mathfrak{g})$ has a $\mathbb{Z}_2 \cong \mathring{P}/\mathring{Q}$ grading as a vector space. If k is even, then*

$$F(\chi_\lambda) \in V_{m+[\lambda] \ mod \ 2}$$

if all $i_j > 0$ for $1 \leq j \leq m$, where $[\lambda] \in \mathbb{Z}_2$ is the class of λ in $\mathring{P}/\mathring{Q}$.

Proof. The lattice $\nu(\mathring{Q}^\vee)$ is spanned by $\alpha_1, \ldots, \alpha_{\ell-1}, \frac{1}{2}\alpha_\ell, 2\nu(\mathring{Q}^\vee) \subseteq \mathring{Q}$, and

$$\mathring{W}\left(\frac{1}{2}\alpha_\ell + \mathring{Q}\right) = \frac{1}{2}\alpha_\ell + \mathring{Q}.$$

We can express

$$w_i = s_{i-1} \cdots s_0 = t_{s_{i-1} \cdots s_1 \theta^\vee} s_{i-1} \cdots s_1 s_\theta$$

$$= t_{\alpha_i^\vee + \cdots + \alpha_\ell^\vee} s_{i-1} \cdots s_\theta$$

for any $i > 0$, and so $w_i \overline{\lambda}$ is in class $[\lambda] + 1$. The result follows immediately. □

2.6.3 Congruence Group Action

As a another application, let $\mathfrak{g}$ be a twisted affine Lie algebra of type $X_N^{(r)}$ and k a positive integer, then $Ch_k(\mathfrak{g})$ is a module for the congruence subgroup $\Gamma_1(r)$ (see 2.2.3) [Ka].

Proposition 2.29. *Let $\mathfrak{g}$ be as above and $\lambda \in P^k$, then*

$$\chi_\lambda|_{u_{21}^r} = v_k \sum_{\mu \in P_k \ mod \ \mathbb{C}\delta, \nu \in P^k \ mod \ \mathbb{C}\delta} e^{-2r\pi i m_{\overline{\mu}}} (S_{0\mu}^t)^2 \mathring{\chi}_{\overline{\lambda}}^t \left(e^{-\frac{2\pi i(\overline{\mu}+\overline{\rho})}{k+h^\vee}}\right) \mathring{\chi}_{\overline{\nu}}^t \left(e^{\frac{2\pi i(\overline{\mu}+\overline{\rho})}{k+h^\vee}}\right) \chi_\nu$$

where v_k is a nonzero scalar depending only on k and $\mathfrak{g}$. In particular, when $r = 1$

$$\chi_\lambda|_{u_{21}} = \sum_{\mu \in P_k \ mod \ \mathbb{C}\delta} e^{2\pi i(m_{\overline{\lambda}} + m_{\overline{\mu}})} S_{k\Lambda_0,,\mu} \mathring{\chi}_{\overline{\lambda}} \left(e^{-\frac{2\pi i(\overline{\mu}+\overline{\rho})}{k+h^\vee}}\right) \chi_\mu.$$

Proof. These follow directly from 2.14 and the identities $u_{21}^r = S u_{12}^r S^{-1}$ and $u_{21} = u_{12}^{-1} S u_{12}^{-1}$, respectively. □

Corollary 2.30. *Let $\overset{\circ}{\mathfrak{g}}$ be a finite dimensional simple Lie algebra with weight lattice $\overset{\circ}{P}$ and $\overset{\circ}{\chi}_\lambda$ be the character of the irreducible finite dimensional module of highest weight $\lambda \in \overset{\circ}{P}^+$, then, for any $\lambda, \mu \in \overset{\circ}{P}^+$,*

$$\sum_{\nu \in \overset{\circ}{P}} e^{-2\pi i m_\nu} (S_{k\Lambda_0,\nu})^2 \overset{\circ}{\chi}_\lambda \left(e^{-\frac{2\pi i(\nu+\rho)}{k+h^\vee}} \right) \overset{\circ}{\chi}_\mu \left(e^{\frac{2\pi i(\nu+\rho)}{k+h^\vee}} \right) = e^{2\pi i (m_\lambda + m_\mu)} S_{k\Lambda_0,\mu} \overset{\circ}{\chi}_\lambda \left(e^{-\frac{2\pi i(\mu+\rho)}{k+h^\vee}} \right),$$

where m_ϕ is defined the same way as the modular anomaly is for affine algebras and $S_{k\Lambda_0,\phi} = |M^*/(k+h^\vee)M|^{-1/2} \prod_{\alpha \in \overset{\circ}{\Delta}_+} 2 \sin \frac{\pi(\phi+\rho,\alpha)}{k+h^\vee}$.

The evaluations the characters of $\overset{\circ}{\mathfrak{g}}^t$ in 2.29 can be replaced by multiples of the characters of $\overset{\circ}{\mathfrak{g}}$. It is worth that noting that the action of S on untwisted Verlinde algebras gives the strikingly similar identity

$$\sum_{\nu \in \overset{\circ}{P}} (S_{k\Lambda_0,\nu})^2 \overset{\circ}{\chi}_\lambda \left(e^{-\frac{2\pi i(\nu+\rho)}{k+h^\vee}} \right) \overset{\circ}{\chi}_\mu \left(e^{\frac{2\pi i(\nu+\rho)}{k+h^\vee}} \right) = \delta_{\lambda\mu}.$$

Indeed, the above transformation property is used to define a nondegenerate Hermitian form on the fusion algebras in other works such as [B], [Ho2]. Pursuing this idea further, one can say more about the above identity in 2.29. Let us adopt the notation

$$\left\langle \overset{\circ}{\chi}_\lambda, \overset{\circ}{\chi}_\mu \right\rangle_r = \sum_{\nu \in P_k^+ \text{ mod } \mathbb{C}\delta} e^{-2r\pi i m_{\overline{\nu}}} (S^t_{0\nu})^2 \overset{\circ}{\chi}{}^t_{\overline{\lambda}} \left(e^{-\frac{2\pi i(\overline{\nu}+\overline{\rho})}{k+h^\vee}} \right) \overset{\circ}{\chi}{}^t_{\overline{\mu}} \left(e^{\frac{2\pi i(\overline{\nu}+\overline{\rho})}{k+h^\vee}} \right).$$

Theorem 2.31. *Let $\mathfrak{g}$ be of type $X_N^{(r)}$, k be a positive integer, and $\lambda, \mu \in P^{k+}$, then there exists a $\beta = \beta(\mathfrak{g}, k) \in \overset{\circ}{\mathfrak{h}}^*$ and a nonzero scalar $v = v(\mathfrak{g}, k)$ such that*

$$\chi_\lambda|_{u_{21}^r} = v e^{-\frac{\pi i |\beta|^2}{r(k+h^\vee)}} \sum_{(\mu,w) \in P(\mathfrak{g},k,\lambda) \ mod\ \mathbb{C}\delta} \varepsilon(w) e^{\frac{\pi i}{r(k+h^\vee)} |\overline{\mu}+\overline{\rho}-\overline{w(\lambda+\rho)}|^2} \chi_\mu$$

where the sum is over

$$P(\mathfrak{g}, k, \lambda) = \{(\mu, w) \in P^{k+} \times \overset{\circ}{W} |\ \overline{\mu} + \overline{\rho} - \beta = w(\overline{\lambda} + \alpha) (mod\ (k+h^\vee)M) \text{ where } \alpha \in rM^*\}.$$

In particular, if $\mu + \rho - \beta \in P^{k++}$,

$$\left\langle \overset{\circ}{\chi}_\lambda, \overset{\circ}{\chi}_\mu \right\rangle_r = v e^{\frac{\pi i}{r(k+h^\vee)} (|\overline{\mu}+\overline{\rho}|^2 + |\overline{\lambda}+\overline{\rho}|^2 - |\beta|^2)} \sum_w \varepsilon(w) e^{-\frac{2\pi i}{r(k+h^\vee)} (\overline{\mu}+\overline{\rho}, w(\overline{\lambda}+\overline{\rho}))}$$

where the sum is over $w \in \overset{\circ}{W}$ such that $\overline{\mu} + \overline{\rho} - w(\overline{\lambda} + \overline{\rho}) - \beta \in rM^* = r\overline{P}_k$, and $\left\langle \overset{\circ}{\chi}_\lambda, \overset{\circ}{\chi}_\mu \right\rangle_r = 0$ otherwise.

Proof. By the transformation rules given in [KP,§3], level k theta functions satisfy

$$\Theta_\lambda|_{u_{21}^r} = v \sum_{\alpha\in M^*,\ r\alpha \pmod{kM}} e^{\pi i k^{-1}(\alpha, r\alpha+2\beta)}\Theta_{\lambda+r\alpha+\beta}$$

for some scalar $v \in \mathbb{C}$ such that $|v| = |((k+h^\vee)M + rM^*)/(k+h^\vee)M|^{-1/2}$ and $\beta \in \mathring{\mathfrak{h}}^*$ such that

$$kr|\alpha|^2 \equiv 2(\alpha,\beta) \pmod{2\mathbb{Z}}$$

for all $\alpha \in \frac{1}{r}M \cap \frac{1}{k}M^*$. Applying this to $A_{\lambda+\rho} = \sum_{w\in\mathring{W}} \epsilon(w)\Theta_{w(\lambda+\rho)}$ and finding the coefficient of $\Theta_{\mu+\rho}$ in the expansion of $A_{\lambda+\rho}|_{u_{21}^r}$ for $\mu \in P^{k+}$ gives the first result. Note that for $(\mu,w) \in P(\mathfrak{g},k,\lambda)$, $w(\alpha) = \overline{\mu}+\overline{\rho}-\beta-w(\overline{\lambda}+\overline{\rho}) \in rM^*$ and so

$$(w(\alpha), w(\alpha)+2\beta) = |\overline{\mu}+\overline{\rho}-w(\overline{\lambda}+\overline{\rho})|^2 - |\beta|^2.$$

The second identity in the theorem statement follows immediately from the first and 2.29. □

Notice that when $r=1$, one has that $\beta = 0$ and $\left\langle \mathring{\chi}_\lambda, \mathring{\chi}_\mu \right\rangle_1$ is a scalar multiple of $\mathring{\chi}_{\overline{\lambda}}$, matching proposition 2.30. Indeed, $\left\langle \mathring{\chi}_\lambda, \mathring{\chi}_\mu \right\rangle_r$ is in general a multiple of a *summand* of the character $\mathring{\chi}_{\overline{\lambda}}$ evaluated at $\frac{2\pi i(\overline{\mu}+\overline{\rho})}{r(k+h^\vee)}$. Specifically, the summand is

$$\sum_{\substack{w\in\mathring{W}:\\ \overline{\mu}+\overline{\rho}-w(\overline{\lambda}+\overline{\rho})-\beta\in rM^*}} \epsilon(w)e^{w(\overline{\lambda}+\overline{\rho})}.$$

2.7 Some Data

Below we give the multiplication matrices of certain twisted affine Lie algebras of type $A_{2\ell}^{(2)}$ with the indicated level. The vectors $v_1, v_2, \ldots$ in the row labeled by "Weights" give the ordered basis $A_{\lambda_1}, A_{\lambda_2}, \ldots$ of the corresponding Verlinde algebra such that

$$\overline{\lambda}_j = \sum_{i\geq 1} v_{ji}\overline{\Lambda}_i$$

where $v_j = (v_{j1}, v_{j2}, \ldots)$. The matrices $L_1, L_2, \ldots$ are defined to be

$$L_i = \left(N_{\lambda_i\lambda_j}^{\lambda_k}\right)_{i,j,k},$$

where $N_{\lambda\mu}^{\nu}$ are the fusion rules.

Type: $A_4^{(2)}$

Level: 2

Weights: $(0,0),(0,1),(1,0)$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&-1\\0&-1&1\end{pmatrix},\begin{pmatrix}0&0&1\\0&-1&1\\1&1&-1\end{pmatrix}$$

Type: $A_4^{(2)}$

Level: 3

Weights: $(0,0),(0,1),(1,0)$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$$

Type: $A_4^{(2)}$

Level: 4

Weights: $(0,0),(2,0),(0,1),(0,2),(1,0),(1,1)$

$$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0&0&0\\1&1&1&1&-1&-1\\0&1&1&0&0&-1\\0&1&0&1&0&-1\\0&-1&0&0&1&1\\0&-1&-1&-1&1&2\end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 2 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & -1 & -2 \end{pmatrix}$$

Type: $A_4^{(2)}$

Level: 5

Weights: $(0,0)$, $(2,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Type: $A_6^{(2)}$

Level: 2

Weights: $(0,0,0),(0,1,0),(1,0,0),(0,0,1)$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&1&-1&-1\\0&-1&1&1\\0&-1&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&-1&1&1\\1&1&-1&0\\0&1&0&-1\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&-1&1&0\\0&1&0&-1\\1&0&-1&0\end{pmatrix}$$

Type: $A_6^{(2)}$

Level: 3

Weights: $(0,0,0),(0,1,0),(1,0,0),(0,0,1)$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\0&0&1&1\\1&1&0&0\\0&1&0&0\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

Type: $A_6^{(2)}$

Level: 4

Weights: $(0,0,0),(2,0,0),(0,1,0),(0,2,0),(1,0,1),(0,0,2),(1,0,0),\ (1,1,0),$ $(0,0,1),\ (0,1,1)$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & -1 & -1 & -1 & -1 & 0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & 2
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1
\end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 2 & 0 & 0 & -2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -2 & -1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & 2 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 2 & 0 & 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 2 & 1 & 1 & -1 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -2 & -2 & -1 & 1 & 3 & 1 & 2 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -2 & 0 & 1 & 2 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 1 & 2 & 1 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & -1 & -2 & -1 & 1 & 2 & 1 & 2 \\ 0 & -1 & -1 & -2 & -2 & -1 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 3 & 1 & -1 & -3 & -1 & -2 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 2 & 2 & 1 & 0 & -2 & -1 & -2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\
0 & -1 & -1 & -1 & -1 & 0 & 1 & 2 & 0 & 1 \\
0 & -1 & -1 & -1 & -2 & 0 & 1 & 2 & 1 & 1 \\
0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \\
0 & 1 & 1 & 2 & 2 & 1 & 0 & -2 & -1 & -2 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\
1 & 2 & 1 & 1 & 1 & 0 & -1 & -2 & -1 & 0
\end{pmatrix}$$

Chapter 3

Richards-Sahi Conjectures for Schur Polynomials

3.1 Introduction

Jack polynomials are a family of symmetric polynomials, depending on a parameter usually denoted by α, that generalize Schur and zonal polynomials. They are of great importance in representation theory, combinatorics, and statistics, to name a few areas. For example, they appear in computing integrals of polynomials over the orthogonal and unitary groups. Jack polynomials are further generalized by the important two-parameter family of symmetric polynomials called Macdonald polynomials.

Recently, D. Richards and S. Sahi introduced a conjecture asserting the positivity of certain expressions involving Jack polynomials [RS]. The version of the conjecture, which we call the α_μ^r conjecture, that we investigate in this dissertation can be found in section 3.3. There has been little progress in resolving this conjecture, even in the case of Schur polynomials, i.e., the case when $\alpha = 1$. In this chapter, we develop a stronger version of the conjecture for Schur polynomials and prove it for several important cases. In the process we discover new identities involving Kostka coefficients. One of these identities is a seemingly new recursion for the dimensions of irreducible representations of the symmetric group. As a bonus, we give a proof of the analogue of this identity for Jack polynomials.

The Richards-Sahi conjecture has its roots in statistics. More precisely, in the study of parameter matrices. Nonetheless, the problem can be expressed in purely algebraic and combinatorial terms (the reader is not expected to be familiar with statistics). Let us briefly describe the background. Statisticians often try to determine how "good" certain estimators of a statistic are with respect to some criteria. Suppose that we have an $n \times n$ positive definite random matrix X whose probability distribution is given by an $n \times n$ (symmetric) positive definite parameter matrix Σ (e.g. a covariance matrix). Suppose further that the distribution is invariant with respect to the conjugation action by the orthogonal group $O_n(\mathbb{R})$. Many significant multivariate distributions have this property, such as the Wishart distribution and the multivariate F-distribution.

To estimate Σ, it is natural to use estimators $\widehat{\Sigma}$ that are also invariant under the same action of the orthogonal group. Estimators of this type can be shown to be given by

$$\widehat{\Sigma}(X) = G\Psi(L)G^T$$

where $X = GLG^T$, $G \in O_n(\mathbb{R})$, $L = diag(l_1, \ldots, l_n)$ such that $l_1 > \cdots > l_n > 0$ a.e. and $\Psi(L) = diag(\psi_1(L), \ldots, \psi_n(L))$. Such an estimator is called *order-preserving* if $\psi_1(L) > \cdots > \psi_n(L) > 0$ a.e. In a 2005 paper (see [Sh]), Yo Sheena conjectured that any non-order-preserving estimator $\widehat{\Sigma}$ can be modified (for example, by simply rearranging the ψ_i in decreasing order) into an new estimator $\widehat{\Sigma}'$ so that $\widehat{\Sigma}'$ is a better estimator (with respect to a certain entropy loss function called Stein's loss function). To be more precise, in statistics there is a notion of *admissibility* of estimators. Given some criteria that allows one to compare estimators, an estimator Σ is *inadmissible* if one can find another estimator Σ' that is at least as good as Σ is all instances and strictly better in at least one instance. Sheena's conjecture states that *non-order-preserving estimators are inadmissible.*

In the same paper, Sheena shows that this conjecture can be framed in terms of the non-negativity of certain integrals of linear combinations of zonal polynomials. Zonal polynomials are special cases of Jack polynomials where the Jack parameter $\alpha = 2$. Allowing the parameter α to take the values 1 and $1/2$, respectively, we may restate the same (integral form of the) conjecture for the unitary and symplectic group. Over the three compact groups, integration of polynomials (with respect to their corresponding Haar measures) can be accomplished via combinatorial methods involving Jack polynomials (see [CS]).

In this work, we consider the case when $\alpha = 1$, i.e., the Schur polynomial case, which corresponds to Hermitian matrices (as opposed to the symmetric matrices considered by Sheena). Many of the results and methods generalize directly to the case $\alpha = 2$, i.e., the zonal polynomial case.

3.2 Preliminaries

3.2.1 Partitions

We will first do a quick review of basic combinatorial concepts that we will be using. A *partition* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ of $n \in \mathbb{N}$, is a tuple of non-negative integers such that $\sum_i \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. The entries of a partition λ are referred to as *parts.* For a partition $\lambda = (\lambda_1, ..., \lambda_l)$ of n, the *length* of λ, which we denote by $\ell(\lambda)$, is the total number of strictly positive entries in λ. The *weight* of λ is the sum of its entries and is denoted $|\lambda|$. We write $\lambda \vdash n$ if $|\lambda| = n$. Given a partition λ, we define the *conjugate* λ' to be the partition whose ith row equals the number of boxes in the ith column of the Young diagram of λ.

Sometimes we use the so-called *frequency notation* for partitions. Let λ be the partition

$$\lambda = (\underbrace{\alpha, \ldots, \alpha}_{m_1 \text{ many}}, \underbrace{\beta, \ldots, \beta}_{m_2 \text{ many}}, \ldots)$$

then in frequency notation we write λ as

$$\lambda = (\alpha^{m_1}, \beta^{m_2}, \ldots).$$

In this work, we seldom use frequency notation, but when it is used we will state it explicitly.

Let $\mathcal{P}$ denote the set of all partitions, where were identify λ and μ if $\ell(\lambda) = \ell(\mu)$ and $\lambda_j = \mu_j$ for all $j = 1, \ldots, \ell(\lambda)$. For any positive integer d, we denote by $\mathcal{P}_d$ the subset of $\mathcal{P}$ of all partitions of length *at most* d. Therefore, we naturally identify $\mathcal{P}_d$ with the corresponding subset of $\mathbb{N}^d$.

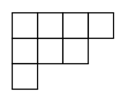
There are two important partial orders defined on the set of partitions that we will be discussing. The first is the *inclusion order*, defined by: $\lambda \subseteq \mu$ if and only if $\lambda_i \leq \mu_i$

for all i. We write $\lambda \subset: \mu$ when $\lambda \subseteq \mu$ and $|\lambda| + 1 = |\mu|$. The second is the *dominance order*, defined by: $\lambda \leq \mu$ if and only if $|\lambda| = |\mu|$ and

$$\sum_{j=1}^{m} \lambda_j \leq \sum_{j=1}^{m} \mu_j$$

for all $m \geq 1$, where we assume that $\lambda_j = 0$, (resp. $\mu_j = 0$) when $j > \ell(\lambda)$ (resp. $j > \ell(\mu)$).

A *Young diagram* of a partition λ is a diagram: a left aligned collection of boxes whose ith row has exactly λ_i many boxes. For example, for $\lambda = (4, 3, 1)$ has the Young diagram



In this way, we think of a partition λ also as a set of boxes

$$\lambda := \{(i, j) : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}.$$

For example, the above figure is the set of boxes

$$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1)\}.$$

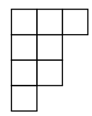
This notion is especially convenient when dealing with the inclusion order. In particular, for any $\lambda, \mu \in P$, $\lambda \subseteq \mu$ is the set of boxes of λ is a subset of the set of boxes of μ. It is also convenient when dealing with the dominance order. In particular, $\lambda <: \mu$ if we can raise a single box in λ to a higher row and produce μ.

Often we use language that alludes to the Young diagram when referring to partitions. For example, 'removing a part of size 2' from $(4, 2, 2, 1)$ yields the partition $(4, 2, 1, 0)$. We will need the following two notions.

Definition 3.1. *Let $\lambda, \mu \in P$ such that $\lambda \subseteq \mu$, then the* skew partition *$\mu \setminus \lambda$ is set of boxes that lie in μ but do not lie in λ, i.e., $\mu \setminus \lambda$ where μ and λ are treated as sets of boxes as above.*

Definition 3.2. *Let $\lambda, \mu \in P$ such that $\lambda \subseteq \mu$. A skew partition $\mu \setminus \lambda$ is called a* horizontal strip *if for every pair of distinct boxes $(a, b), (c, d) \in \mu \setminus \lambda$, $b \neq d$.*

The Young diagram of λ' is simply the transpose of the Young diagram of λ. For example, for $\lambda = (4, 3, 1)$ as above, the conjugate is $\lambda' = (3, 2, 2, 1)$ and its Young diagram is



Let λ be a partition, then a *standard Young tableaux T* of shape λ is a filling of the boxes in the Young diagram of λ with the integers $\{1, \ldots, |\lambda|\}$ so that the integers are strictly increasing along rows (read from left to right) and also strictly increasing along columns (read from top to bottom). Let λ and μ be partitions, then a (column-strict) *semi-standard Young tableaux T* of shape λ and weight μ, is a filling of the boxes in the Young diagram of λ with μ_1 many 1's, μ_2 many 2's, and so on, so that the integers are non-decreasing along rows (read from left to right) and also strictly increasing along columns (read from top to bottom).

3.2.2 Symmetric Polynomials

Let d be a positive integer. Let $\Lambda_{\mathbb{Z},d} := \mathbb{Z}[x_1, \ldots, x_d]$ be the ring of polynomials in d variables with coefficients in $\mathbb{Z}$. For any commutative ring R containing $\mathbb{Z}$, i.e., $\mathbb{Z} \hookrightarrow R$, define $\Lambda_{R,d} := R \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z},d}$. When the ring is understood from context or specified beforehand, we will write Λ_d for simplicity.

The symmetric group S_d acts on Λ_d by permuting the variables, i.e., for any $\sigma \in S_d$, $k \in \mathbb{N}$, and $i_1, \ldots, i_k \in \{1, \ldots, n\}$,

$$\sigma \cdot x_{i_1} \cdots x_{i_k} = x_{\sigma(i_1)} \cdots x_{\sigma(i_k)}.$$

Let us adopt the notation

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} \tag{3.1}$$

where $\alpha \in \mathbb{Z}^d$. Note that S_d acts on $\mathbb{Z}_d$ by

$$\sigma \cdot (\alpha_1, \ldots, \alpha_d) = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(d)}),$$

so that

$$\sigma x^\alpha = x^{\sigma \cdot \alpha}.$$

Assume that we are working over $\mathbb{Q}$, then Λ_d has several important bases indexed by partitions of length at most d. We will quickly introduce just a few. The most natural is the *monomial basis*

$$m_\lambda = \sum_{\mu \in S_d \cdot \lambda} x^\mu$$

where $\lambda \in \mathcal{P}_d$ and $S_d \cdot \lambda$ is the orbit of λ. A second important basis is the *power sum* basis, whose elements are

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_d}$$

where $p_0 = 1$ and, for $r > 0$,

$$p_r := \sum_{i=1}^{d} x_i^r.$$

A third, and very important, basis is comprised of the *Schur polynomials*, defined by

$$s_\lambda = \frac{\det(x_i^{\lambda_j + n - j})_{1 \le i,j \le n}}{\det(x_i^{n-j})_{1 \le i,j \le n}}.$$

Note that the denominator is the Vandermonde determinant and that s_λ is a polynomial over $\mathbb{Z}$. In fact, both the monomial basis and the Schur basis are bases of $\Lambda_{\mathbb{Z},d}$ (but not the power sum basis).

There is a purely combinatorial formula for the Schur polynomials that proves that the entries are non-negative linear combinations in the monomial basis. It is well-known that

$$s_\lambda = \sum_T x^T$$

where the sum is over all semi-standard Young tableaux (see previous subsection) of shape λ and $x^T = x_1^{t_1} x_2^{t_2} \cdots x_d^{t_d}$, such that T has t_j many j's.

3.3 α_μ^r Conjecture

In this section, we will present the Richards-Sahi conjectures and set notation for the remainder of this work. The conjectures are about the positivity of certain linear combinations of Schur polynomials with polynomial coefficients over a prescribed region of $\mathbb{R}^d$.

Let d be a positive integer and $\mathcal{P}_d$ be the set of all partitions of length less than d. For any $\mu \in \mathcal{P}_d$, define

$$G(\mu) = \{1 \leq i \leq d \mid \mu_i > \mu_{i+1}\}$$
$$H(\mu) = \{\nu \in \mathcal{P}_d \mid \mu \setminus \nu \text{ is a horizontal strip}\}.$$

For $\nu \in H(\mu)$ and x a variable. we define the polynomial

$$g_\mu(\nu, x) = \prod_{i \in G(\mu)} (x + \nu_i - i).$$

Next define

$$P(\mu) = \{(i,j) \notin \mu \mid \mu \cup \{(i,j)\} \text{ is a partition}\}$$
$$R(\mu) = \{i - j : (i,j) \in P(\mu)\}.$$

We will refer to elements of these sets as Pieri boxes and Pieri roots of μ, respectively. For $r \in R(\mu)$ and $\nu \in H(\mu)$ we define

$$k_\mu^r(\nu) = \frac{g_\mu(\nu, r)}{\prod_{s \in R(\mu)\setminus\{r\}} (r - s)}.$$

For $a \in \mathbb{R}^d$ define

$$\tilde{s}_\nu(a) = a_1^{|\mu|-|\nu|} s_\nu(a_2, \ldots, a_d),$$

where s_ν is the Schur function in $d-1$ variables. For $r \in R(\mu)$ we define

$$\psi_\mu^r(a) = \sum\nolimits_{\nu \in H(\mu)} k_\mu^r(\nu)\, \tilde{s}_\nu(a)$$
$$\varphi_\mu^r(a) = \frac{1}{(d-r)k_\mu^r(\mu)} \psi_\mu^r(a)$$

The most negative Pieri root is $r^- = -\mu_1$ corresponding to the Pieri box $(1, \mu_1 + 1)$ and we define

$$R^-(\mu) = R(\mu) \setminus \{r^-\}.$$

Finally, define

$$\alpha_\mu^r(a) = \varphi_\mu^{r^-}(a) - \varphi_\mu^r(a),$$

The *Richards-Sahi conjecture* for Schur polynomials is the following.

Conjecture 3.3. *If μ is a partition and $a \in \mathbb{R}^d$ satisfies*

$$a_1 \geq \cdots \geq a_d \geq 0$$

then for all $r \in R^-(\mu)$ we have

$$\alpha_\mu^r(a) \geq 0 \text{ and } \beta_\mu^r(a) \geq 0.$$

If $a_1 = \ldots = a_d = 0$, then the conjecture is true. Assume that $a_d > 0$. Note that by setting $y_i = a_{i+1}$, for $i = 0, \ldots, d-1$, then dividing by $y_0^{|\mu|}$ and defining $x_i = y_i/y_0$, we may replace the polynomials $\tilde{s}_\nu(a)$ by the Schur polynomials $s_\nu(x_1, \ldots, x_{d-1})$. The conjecture then asserts the non-negativity for the corresponding $\alpha_\mu^r, \beta_\mu^r$ on the domain $x_1 \geq \cdots \geq x_{d-1} \geq 0$.

3.4 Special Cases of Richards-Sahi Conjecture

In this section, we prove the conjecture for some important families of partitions. The techniques here do not seem to be directly adaptable for general partitions. These are basically 'brute-force' arguments that take advantage of ad hoc manipulations that are possible in these special cases. Afterward, we give a refined version of the conjecture.

3.4.1 Case $\mu = n^i$

In order to show positivity for α_μ^r we need only show that the

$$\sum_{j=0}^{i} a_{\mu\lambda^j}^r m_{\lambda^j}(1) \leq 0$$

for $\lambda \leq \mu$, $i = 0, \ldots, k$, $k = k(\lambda)$, $\mathcal{C}(\lambda) = \{\lambda^0, \ldots, \lambda^{k(\lambda)}\}$, with equality when $i = k$. Indeed, it suffices to show that $a^r_{\mu\lambda^j} \geq 0$ implies that $a^r_{\mu\lambda^{j+1}} \geq 0$ and

$$\sum_{j=0}^{k} a^r_{\mu\lambda^j} m_{\lambda^j}(1) = 0. \tag{3.2}$$

In the special case $\mu = n^i$, we have that $G_\mu = \{i\}$, $R_\mu = \{-n, i\}$, and every $\nu \subseteq \mu$ such that $\mu \setminus \nu$ is a horizontal strip is of the form $\nu = n^{i-1}, n-j$ for $j = 0, \ldots, n$. The coefficients of the relevant Schur polynomials are

$$\begin{aligned} k^r_{\mu\nu} &= \frac{g(\nu, r^-)}{(d+n)g(\mu, r^-)} - \frac{g(\nu, i)}{(d-i)g(\mu, i)} \\ &= \frac{-n+n-i-j}{(d+n)(-n+n-i)} - \frac{i+n-i-j}{(d-i)(i+n-i)} \\ &= \frac{(dj-in)(n+i)}{in(d-i)(d+n)}. \end{aligned}$$

Let $\mu^j = n^{i-1}, n-j$. Since

$$a^r_{\mu\nu} = K_{\mu^{|\mu|-|\nu|},\nu} k^r_{\mu,\mu^{|\mu|-|\nu|}},$$

for all ν appearing in the monomial expansion of α^r_μ and $k^r_{\mu\mu^j} \geq 0 \implies k^r_{\mu\mu^{j+1}} \geq 0$, we need only to show equation (2). Note that if a particular monomial m_ν appears with nonzero coefficient, then $d > \ell(\nu)$. in particular, for $\lambda \leq \mu$

$$b_1(dj_1 - in) + b_2(dj_2 - in) + \cdots + b_k(dj_k - in) = din - \ell in > 0$$

where $j_m = |\lambda| - |\lambda^m|$, b_m is the associated multiplicity in λ of the removed row, and ℓ is the length of λ.

After clearing denominators, we get that for any $\lambda \leq \mu$ we must show that

$$\sum_{j=0}^{k} m_{\lambda^j}(1) K_{\mu^{|\mu|-|\lambda^j|},\lambda^j} \big(d(|\lambda| - |\lambda^j|) - in\big) = 0.$$

In this case, $K_{\mu^{|\mu|-|\lambda^j|},\lambda^j} = K_{\mu,\lambda}$ for all $j = 0, \ldots, k$ and so this reduces to proving

$$\sum_{j=0}^{k} m_{\lambda^j}(1) \big(d(|\lambda| - |\lambda^j|) - in\big) = 0$$

or the more combinatorially friendly equation

$$\sum_{j=0}^{k} d(|\lambda| - |\lambda^j|) m_{\lambda^j}(1) = in \sum_{j=0}^{k} m_{\lambda^j}(1). \tag{3.3}$$

The $m_{\lambda^j}(1)$ are multinomials and a direct check shows that equation 3.3 is true. Indeed,

$$\sum_{j=0}^{k} d(|\lambda| - |\lambda^j|) \binom{d-1}{b_k, \dots, b_j - 1, \dots b_0} = in \sum_{j=0}^{k} \binom{d-1}{b_k, \dots, b_j - 1, \dots b_0} \tag{3.4}$$

where $b_0, \dots, b_k - 1$ are the multiplicities of the parts of λ and $b_0 + \cdots + b_k = d$. By factoring and clearing denominators we get

$$\sum_{j=0}^{k} d(|\lambda| - |\lambda^j|) b_j = ind \tag{3.5}$$

which is true since, by definition,

$$\lambda = (|\lambda| - |\lambda^k|)^{b_k}, \dots, (|\lambda| - |\lambda^1|)^{b_1}, (|\lambda| - |\lambda^0|)^{b_0}$$

and $in = |n^i| = \sum_{j=0}^{k} (|\lambda| - |\lambda^j|) b_j$.

3.4.2 Case $\mu = m^1 1^n$

We now prove the conjecture for some hooks of the form $\mu = m^1 1^n$. Let's first study these quantities a little more closely.

$$\mu = m^1 1^{n-1} 0^{d-n}.$$

We have $\nu \in H(\mu) = \{k^1 1^{n-2}\}_{k=1}^{m} \cup \{k^1 1^{n-1}\}_{k=1}^{m}$ for $n > 1$, and

$$g_\mu(\nu, -m) = (\nu_1 - 1 - m)(\nu_n - n - m)$$

$$g_\mu(\nu, 0) = (\nu_1 - 1)(\nu_n - n)$$

$$g_\mu(\nu, n) = (\nu_1 - 1 + n)\nu_n.$$

More precisely,

$$g_\mu(k^1 1^{n-2}, -2) = (k - m - 1)(-n - m) = (m - k + 1)(n + m)$$

$$g_\mu(k^1 1^{n-1}, -2) = (k - m - 1)(1 - n - m) = (m - k + 1)(n + m - 1)$$

$$g_\mu(k^1 1^{n-2}, 0) = (\nu_1 - 1)(\nu_n - n) = -(k-1)n$$
$$g_\mu(k^1 1^{n-1}, 0) = (\nu_1 - 1)(\nu_n - n) = -(k-1)(n-1)$$

$$g_\mu(k^1 1^{n-2}, n) = (\nu_1 - 1 + n)\nu_n = 0$$
$$g_\mu(k^1 1^{n-1}, n) = (\nu_1 - 1 + n)\nu_n = n + k - 1$$

Now, we compute $k_\mu^r(\nu)$:

$$k_\mu^{-2}(k^1 1^{n-2}) = \frac{(m-k+1)(n+m)}{-m(-m-n)} = \frac{m-k+1}{m}$$
$$k_\mu^{-2}(k^1 1^{n-1}) = \frac{(m-k+1)(n+m-1)}{-m(-m-n)} = \frac{(m-k+1)(n+m-1)}{m(m+n)}$$

$$k_\mu^{0}(k^1 1^{n-2}) = \frac{-(k-1)n}{-mn} = \frac{k-1}{m}$$
$$k_\mu^{0}(k^1 1^{n-1}) = \frac{-(k-1)(n-1)}{-mn} = \frac{(k-1)(n-1)}{mn}$$

$$k_\mu^{n}(k^1 1^{n-2}) = 0$$
$$k_\mu^{n}(k^1 1^{n-1}) = \frac{n+k-1}{(n+m)n}$$

Putting these expressions together we arrive at the following.

$$\alpha_\mu^0 = \frac{m(m+n)}{(d+m)(n+m-1)} \left(\sum_{k=1}^{m} \frac{(m-k+1)(n+m-1)}{m(m+n)} \tilde{s}_{k^1 1^{n-1}} + \sum_{k=1}^{m} \frac{m-k+1}{m} \tilde{s}_{k^1 1^{n-2}} \right) - \frac{mn}{d(m-1)(n-1)} \left(\sum_{k=1}^{m} \frac{(k-1)(n-1)}{mn} \tilde{s}_{k^1 1^{n-1}} + \sum_{k=1}^{m} \frac{k-1}{m} \tilde{s}_{k^1 1^{n-2}} \right)$$

$$\alpha_\mu^n = \frac{m(m+n)}{(d+m)(n+m-1)} \left(\sum_{k=1}^{m} \frac{(m-k+1)(n+m-1)}{m(m+n)} \tilde{s}_{k^1 1^{n-1}} + \sum_{k=1}^{m} \frac{m-k+1}{m} \tilde{s}_{k^1 1^{n-2}} \right) - \frac{(n+m)n}{(d-n)(m+n-1)} \sum_{k=1}^{m} \frac{n+k-1}{(n+m)n} \tilde{s}_{k^1 1^{n-1}}$$

We rescale $\alpha^r_{m^1 1^1}$ as follows:

$$\widetilde{\alpha^0_\mu} = m(m+n)d(m-1)(n-1)\left(\sum_{k=1}^{m}\frac{(m-k+1)(n+m-1)}{m(m+n)}\tilde{s}_{k^1 1^{n-1}} + \sum_{k=1}^{m}\frac{m-k+1}{m}\tilde{s}_{k^1 1^{n-2}}\right)$$
$$-mn(d+m)(n+m-1)\left(\sum_{k=1}^{m}\frac{(k-1)(n-1)}{mn}\tilde{s}_{k^1 1^{n-1}} + \sum_{k=1}^{m}\frac{k-1}{m}\tilde{s}_{k^1 1^{n-2}}\right)$$
$$= d(m-1)(n-1)\sum_{k=1}^{m}(m-k+1)(n+m-1)\tilde{s}_{k^1 1^{n-1}}$$
$$+(m+n)d(m-1)(n-1)\sum_{k=1}^{m}(m-k+1)\tilde{s}_{k^1 1^{n-2}}$$
$$-(d+m)(n+m-1)\sum_{k=1}^{m}(k-1)(n-1)\tilde{s}_{k^1 1^{n-1}}$$
$$-n(d+m)(n+m-1)\sum_{k=1}^{m}(k-1)\tilde{s}_{k^1 1^{n-2}}$$

$$\widetilde{\alpha^n_\mu} = m(m+n)(d-n)\left(\sum_{k=1}^{m}\frac{(m-k+1)(n+m-1)}{m(m+n)}\tilde{s}_{k^1 1^{n-1}} + \sum_{k=1}^{m}\frac{m-k+1}{m}\tilde{s}_{k^1 1^{n-2}}\right)$$
$$-(n+m)n(d+m)\sum_{k=1}^{m}\frac{n+k-1}{(n+m)n}\tilde{s}_{k^1 1^{n-1}}$$
$$= (d-n)\sum_{k=1}^{m}(m-k+1)(n+m-1)\tilde{s}_{k^1 1^{n-1}}$$
$$+(m+n)(d-n)\sum_{k=1}^{m}(m-k+1)\tilde{s}_{k^1 1^{n-2}}$$
$$-(d+m)\sum_{k=1}^{m}(n+k-1)\tilde{s}_{k^1 1^{n-1}}$$

Simplifying, we get

$$\widetilde{\alpha^0_\mu} = m(n-1)(m+n-1)\sum_{k=1}^{m}(-dk+dm-k+1)\tilde{s}_{k^1 1^{n-1}}$$
$$+(m+n)d(m-1)(n-1)\sum_{k=1}^{m}(m-k+1)\tilde{s}_{k^1 1^{n-2}}$$

$$-n(d+m)(n+m-1)\sum_{k=1}^{m}(k-1)\tilde{s}_{k^1 1^{n-2}}$$

$$\widetilde{\alpha_\mu^n} = (m+n)\sum_{k=1}^{m}((d-n)(m-k)-n-k+1)\tilde{s}_{k^1 1^{n-1}}$$

$$+(m+n)(d-n)\sum_{k=1}^{m}(m-k+1)\tilde{s}_{k^1 1^{n-2}}.$$

Recall that

$$s_\lambda = \sum_{\nu \preceq \lambda} K_{\lambda\nu} m_\nu,$$

so we can set $a_1 = 1$ and rewrite the above quantities in terms of monomials

$$\widetilde{\alpha_\mu^0} = m(n-1)(m+n-1)\sum_{k=1}^{m}(-dk+dm-k+1)\sum_{\nu \preceq k^1 1^{n-1}} K_{k^1 1^{n-1}\nu} m_\nu$$

$$+(m+n)d(m-1)(n-1)\sum_{k=1}^{m}(m-k+1)\sum_{\nu \preceq k^1 1^{n-2}} K_{k^1 1^{n-2}\nu} m_\nu$$

$$-n(d+m)(n+m-1)\sum_{k=1}^{m}(k-1)\sum_{\nu \preceq k^1 1^{n-2}} K_{k^1 1^{n-2}\nu} m_\nu$$

$$\frac{\widetilde{\alpha_\mu^n}}{m+n} = \sum_{k=1}^{m}((d-n)(m-k)-n-k+1)\sum_{\nu \preceq k^1 1^{n-1}} K_{k^1 1^{n-1}\nu} m_\nu$$

$$+(d-n)\sum_{k=1}^{m}(m-k+1)\sum_{\nu \preceq k^1 1^{n-2}} K_{k^1 1^{n-2}\nu} m_\nu.$$

Let's revisit the $m = 2$ case. We have

$$\frac{\widetilde{\alpha_\mu^n}}{2+n} = \sum_{k=1}^{2}((d-n)(2-k)-n-k+1)\sum_{\nu \preceq k^1 1^{n-1}} K_{k^1 1^{n-1}\nu} m_\nu$$

$$+(d-n)\sum_{k=1}^{2}(2-k+1)\sum_{\nu \preceq k^1 1^{n-2}} K_{k^1 1^{n-2}\nu} m_\nu$$

$$= (d-2n)m_{1^n} - (n+1)\sum_{\nu \preceq 2^1 1^{n-1}} K_{2^1 1^{n-1}\nu} m_\nu$$

$$+2(d-n)m_{1^{n-1}} + (d-n)\sum_{\nu \preceq 2^1 1^{n-2}} K_{2^1 1^{n-2}\nu} m_\nu$$

$$= (d-2n)m_{1^n} - (n+1)m_{2^1 1^{n-1}} - n(n+1)m_{1^{n+1}}$$

$$+2(d-n)m_{1^{n-1}} + (d-n)m_{2^1 1^{n-2}} + (n-1)(d-n)m_{1^n}$$

$$= n(d-n-1)m_{1^n} - (n+1)m_{2^1 1^{n-1}} - n(n+1)m_{1^{n+1}}$$

$$+2(d-n)m_{1^{n-1}} + (d-n)m_{2^1 1^{n-2}}$$

Replacing the largest negative monomials by the corresponding inequalities that 'make sense', i.e., that are in the up-set of the existing smaller partitions, we get

$$\frac{\widetilde{\alpha_\mu^n}}{2+n} \geq n(d-n-1)m_{1^n} - t_1(n+1)m_{2^1 1^{n-2}} - t_2 n(n+1)m_{1^n}$$

$$-n(n+1)\frac{d-n-1}{n+1}m_{1^n} + 2(d-n)m_{1^{n-1}} + (d-n)m_{2^1 1^{n-2}}$$

$$= -t_1\frac{(n+1)(d-n)}{n-1}m_{2^1 1^{n-2}} - t_2(n+1)(d-n)m_{1^{n-1}}$$

$$+2(d-n)m_{1^{n-1}} + (d-n)m_{2^1 1^{n-2}}$$

Set $t_1 = \frac{n-1}{n+1}$, $t_2 = \frac{2}{n+1}$ giving

$$\frac{\widetilde{\alpha_\mu^n}}{2+n} \geq -(d-n)m_{2^1 1^{n-2}} - 2(d-n)m_{1^{n-1}} - t_2(n+1)(n-1)m_{1^{n-1}}$$

$$+2(d-n)m_{1^{n-1}} + (d-n)m_{2^1 1^{n-2}}$$

$$= 0.$$

Notice that we chose t_1 so that it canceled the $m_{2^1 1^{n-2}}$ term, i.e., the largest term left.

Continuing in this way, one can prove non-negativity for all small values of m and $r = 0, n$. Later, in section 3.6, we will show a method of organizing the partitions so that non-negativity can be verified for families without resorting to ad-hoc methods as above.

3.5 Another Partial Order on Partitions

Before we discuss the refinement of the Richards-Sahi conjecture, we will introduce a partial order $\prec$ that is of independent interest. Afterward, we define a relation $\prec_H$, in fact a sub-relation of $\preceq$, which features in the refined conjecture. Let us first define the partial order and then show, immediately after, that it is well-defined.

Definition 3.4. *Let $\mathcal{P}$ be the set of partitions. Define a partial order $\preceq$ on $\mathcal{P}$ by $\lambda \preceq \mu$ if and only if there exists a partition ν such that $\lambda \leq \nu$ and $\nu \subseteq \mu$.*

Proposition 3.5. *The relation $\preceq$ is a partial order on the set of partitions.*

Proof. Clearly, $\preceq$ is reflexive. Moreover, if $\lambda \preceq \mu$ and $\mu \preceq \lambda$ then we have ν_1, ν_2 such that

$$\lambda \leq \nu_1 \subseteq \mu$$

$$\mu \leq \nu_2 \subseteq \lambda.$$

If $|\lambda| \neq |\mu|$ then this is not possible. Therefore, $\nu_1 = \mu$ and $\nu_2 = \lambda$ which implies that $\lambda \leq \mu$ and $\mu \leq \lambda$. The antisymmetry of $\leq$ implies the antisymmetry of $\preceq$.

It remains to show transitivity. Suppose that $\lambda \preceq \mu$ and $\mu \preceq \nu$ then we have two intermediate partitions α, β such that

$$\lambda \leq \alpha \subseteq \mu$$

$$\mu \leq \beta \subseteq \nu.$$

If we can find γ such that

$$\alpha \leq \gamma \text{ and } \gamma \subseteq \beta \tag{3.6}$$

then the transitivity of $\leq$ and $\subseteq$ would imply $\lambda \preceq \nu$ as desired.

We proceed by induction on $n = |\nu| - |\lambda|$, where the $n = 0$ case is obviously true (since then the inclusions become equalities in the expressions above). Assume for the moment that $n > 1$ and that it has been proved for all values smaller than n (we will address the $n = 1$ case shortly). If $\mu = \beta$, then $\lambda \leq \alpha \subseteq \mu \subseteq \nu$ and we are done. Suppose then that $\mu < \beta$. We may assume that $\beta = \nu$, otherwise, $|\beta| < |\nu|$ and by the inductive hypothesis there is a γ such that 3.6 holds, giving the result. To summarize, we have reduced it to the case that

$$\lambda \subset \mu < \nu.$$

Since we have assumed that it is true for $n = 1$, we can find α^1 such that

$$\lambda \subseteq \alpha^1 \subset: \mu < \nu$$

and so

$$\alpha^1 \preceq \nu$$

with intermediate γ^1, i.e., such that $\alpha^1 \leq \gamma^1 \subset \nu$. If $|\gamma^1| - |\lambda| < |\nu| - |\lambda|$, then by the induction hypothesis, there is an intermediate λ^1 such that

$$\lambda \leq \lambda^1 \subseteq \gamma^1$$

which in turn implies that $\lambda \leq \lambda^1 \subset \nu$.

It remains to show the case $n = 1$, i.e., that $\lambda \subset: \mu$. Let $\mu^0 = \mu <: \mu^1 <: \cdots <: \mu^m = \nu$. We use a separate inductive argument on m to show that $\lambda \preceq \nu$. The case $m = 0$ was done above. Suppose that $m > 0$ and that we have proved it for all values less than m. If we can find an intermediate partition γ such that

$$\lambda \leq \gamma \subseteq \mu^1,$$

then by the inductive hypothesis, $\gamma \preceq \nu$. This in turn implies that there is a γ^1 such that

$$\lambda \leq \gamma \leq \gamma^1 \subseteq \nu$$

as desired. Therefore, we only need to prove it for $m = 1$.

Assume that $\mu <: \nu$. There are two cases:

1. For some i, $\lambda_i + 1 = \mu_i$ and $\mu_i = \nu_i + 1$.

2. For some i, $\lambda_i + 1 = \mu_i$ and $\mu_i = \nu_i$.

For (1), $\mu_j \leq \nu_j$ for $j \neq i$. Therefore $\lambda_i = \nu_i$ and $\lambda_j \leq \nu_j$ for $j \neq i$. So $\lambda \subseteq \nu$ and we are done.

For (2), let i_0, j_0, k_0 be indices such that

$$\mu_{i_0} = \lambda_{i_0} + 1$$

$$\nu_{j_0} = \mu_{j_0} + 1$$

$$\nu_{k_0} = \mu_{k_0} - 1.$$

Note that $i_0 \neq k_0$, $j_0 < k_0$, $\mu_{k_0+1} < \mu_{k_0}$, and $\mu_{j_0} < \mu_{j_0-1}$. We now have two subcases: $i_0 = j_0 - 1$ and $i_0 \neq j_0 - 1$. When $i_0 \neq j_0 - 1$, define the partition γ by

$$\gamma_j = \lambda_j, \qquad \text{for } j \notin \{j_0, k_0\}$$

$$\gamma_{j_0} = \lambda_{j_0} + 1,$$

$$\gamma_{k_0} = \lambda_{k_0} - 1.$$

This is well defined for two reasons. First, $i_0 \neq k_0$ means that $\lambda_{k_0} = \mu_{k_0}$ and so

$$\gamma_{k_0+1} = \lambda_{k_0+1} \leq \mu_{k_0+1} < \mu_{k_0} = \lambda_{k_0} = \gamma_{k_0} + 1.$$

More to the point, $\gamma_{k_0+1} \leq \gamma_{k_0}$. Second, $\lambda_{j_0-1} = \mu_{j_0-1} = \nu_{j_0-1} > \mu_{j_0}$ implies that $\gamma_{j_0-1} \geq \lambda_{j_0} + 1 = \gamma_{j_0}$ (unless $j_0 = 1$ in which case it is already well defined). Now, $\nu \supset \gamma$ since

$$\gamma_j = \lambda_j \leq \mu_j = \nu_j \text{ for } j \notin \{j_0, k_0\}$$

$$\gamma_{j_0} = \lambda_{j_0} + 1 \leq \mu_{j_0} + 1 = \nu_{k_0}$$

$$\gamma_{k_0} = \lambda_{k_0} - 1 = \mu_{k_0} - 1 = \nu_{k_0}.$$

In the case that $i_0 = j_0 - 1$, define the partition γ by

$$\gamma_j = \lambda_j, \qquad \text{for } j \notin \{i_0, k_0\}$$

$$\gamma_{i_0} = \lambda_{i_0} + 1,$$

$$\gamma_{k_0} = \lambda_{k_0} - 1.$$

To show that this is well-defined, we repeat the same argument for γ_{k_0} and for γ_{i_0} we note that $\lambda_{i_0} + 1 = \mu_{i_0} \leq \lambda_{i_0-1} = \mu_{i_0-1}$ if $i_0 > 1$. We again have that $\nu \supset \gamma$ since

$$\gamma_j = \lambda_j = \mu_j \leq \nu_j \text{ for } j \notin \{i_0, k_0\}$$

$$\gamma_{i_0} = \lambda_{i_0} + 1 = \mu_{i_0} = \nu_{k_0}$$

$$\gamma_{k_0} = \lambda_{k_0} - 1 = \mu_{k_0} - 1 = \nu_{k_0}.$$

It follows that $\lambda \preceq \nu$, completing the proof. $\square$

The partial order $\preceq$ is a natural fusion of the dominance and inclusion order. It has a sub-relation $\preceq_H$, which, despite the unfortunate notation, is not a partial order (it fails transitivity), but is featured in the expansion

$$s_\lambda(x_1, \ldots, x_n, y) = \sum_{\mu \preceq_H \lambda} a_{\lambda\mu} y^{|\lambda|-|\mu|} m_\mu(x_1, \ldots, x_n),$$

were $a_{\lambda\mu}$ are positive integers. Nonetheless, if $\lambda \preceq_H \mu$, then $\lambda \preceq \mu$. First, let us define the relation and then later prove the above expansion.

Definition 3.6. *Let $\mathcal{P}$ be the set of partitions. Define a relation $\preceq_H$ on $\mathcal{P}$ by $\lambda \preceq_H \mu$ if and only if there exists a partition ν such that $\lambda \leq \nu$, $\nu \subseteq \mu$, and $\mu \setminus \nu$ is a horizontal strip.*

Proposition 3.7. *Let $\lambda \in \mathcal{P}$ and $n+1 \geq \ell(\lambda)$, then*

$$s_\lambda(x_1, \ldots, x_n, y) = \sum_{\mu \preceq_H \lambda} a_{\lambda\mu} y^{|\lambda|-|\mu|} m_\mu(x_1, \ldots, x_n),$$

where $a_{\lambda\mu}$ are positive integers.

Proof. Recall that s_λ is the sum of monomials of the form x^T, where T ranges over all semi-standard tableaux of shape λ (where $x_{n+1} = y$), see section 3.2. Now, for any semi-standard tableaux T the $n+1$ entries form a horizontal strip of λ. In the other direction, if we fixed a horizontal strip $\lambda \setminus \mu$ and label those boxes with $n+1$, then we can fill the boxes in μ to form any semi-standard tableaux T' of shape μ, then adding the boxes containing $n+1$ to T' forms a semi-standard tableaux T of shape λ. Therefore, using the semi-standard tableaux expansion of s_λ,

$$\begin{aligned} s_\lambda(x_1, \ldots, x_n, y) &= \sum_{\mu \subseteq \lambda, \lambda \setminus \mu \text{is a hor. strip}} y^{|\lambda|-|\mu|} s_\mu(x_1, \ldots, x_n) \\ &= \sum_{\mu \subseteq \lambda, \lambda \setminus \mu \text{is a hor. strip}} \sum_{\nu \leq \mu} K_{\mu\nu} y^{|\lambda|-|\mu|} m_\nu(x_1, \ldots, x_n) \\ &= \sum_{\mu \preceq_H \lambda} a_{\lambda\mu} y^{|\lambda|-|\mu|} m_\mu(x_1, \ldots, x_n), \end{aligned}$$

where $K_{\mu\nu} \in \mathbb{N}$ are Kostka numbers, which are positive whenever $\nu \leq \mu$, and $a_{\lambda\mu}$ are non-empty sums of Kostka numbers. $\square$

3.6 Certain Chains of Partitions

Given $n \in \mathbb{N}$, there is a way of decomposing the set $\mathcal{P}^{\leq n}$ of partitions of weight at most n into disjoint chains with respect to inclusion. Moreover, each of the chains will have one (and only one, of course) partition of weight n.

Definition 3.8. *Define the following map*

$$\mathcal{C} : \lambda \mapsto \{\lambda = \lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^{k(\lambda)}\}$$

where $k(\lambda)$ is the number of nonzero elements in the set $\{\lambda_1, \ldots, \lambda_{\ell(\lambda)}\}$ and, for $i = 1, \ldots, k(\lambda)$, λ^i is attained from λ by removing its ith smallest part. We will often use the notation $\lambda^* = \lambda^{k(\lambda)}$.

Example 3.9. *If $\lambda = (3, 2, 1)$, then*

$$\mathcal{C}(\lambda) = \{(3, 2, 1), (3, 2, 0), (3, 1, 0), (2, 1, 0)\}.$$

Lemma 3.10. *Let λ be a partition and $n \geq |\lambda|$. If λ^0 is the partition obtained from λ by adding a single row of length $n - |\lambda|$, then $\lambda \in \mathcal{C}(\lambda^0)$.*

Proof. Let $\lambda = (\alpha_1^{m_1}, \alpha_2^{m_2}, \ldots, \alpha_r^{m_r})$ written in frequency notation. Let k be the index such that $\alpha_k \geq n - |\lambda|$ and $\alpha_{k+1} < n - |\lambda|$ where we take $\alpha_0 := \infty$. Then the composition

$$\lambda^0 := (\alpha_1^{m_1}, \ldots, \alpha_k^{m_k}, n - |\lambda|, \alpha_{k+1}^{m_{k+1}}, \ldots, \alpha_r^{m_r})$$

is a partition. Furthermore, $\mathcal{C}(\lambda^0)$ is the sequence:

$$\lambda^0 = (\alpha_1^{m_1}, \ldots, n - |\lambda|, \alpha_{k+1}^{m_{k+1}}, \ldots, \alpha_{r-1}^{m_{r-1}}, \alpha_r^{m_r})$$

$$\lambda^1 = (\alpha_1^{m_1}, \ldots, n - |\lambda|, \alpha_{k+1}^{m_{k+1}}, \ldots, \alpha_{r-1}^{m_{r-1}}, \alpha_r^{m_r - 1})$$

$$\lambda^2 = (\alpha_1^{m_1}, \ldots, n - |\lambda|, \alpha_{k+1}^{m_{k+1}}, \ldots, \alpha_{r-1}^{m_{r-1}-1}, \alpha_r^{m_r})$$

$$\vdots$$

$$\lambda^{r-k} = (\alpha_1^{m_1}, \ldots, n - |\lambda|, \alpha_{k+1}^{m_{k+1}-1}, \ldots, \alpha_{r-1}^{m_{r-1}}, \alpha_r^{m_r})$$

$$\lambda^{r-k+1} = \lambda$$

$$\vdots$$

This completes the proof. □

Proposition 3.11. *Let $n \in \mathbb{N}$, $\mathcal{P}^{\leq n}$ be the set of partitions of weight at most n, and $\mathcal{P}^n \subseteq \mathcal{P}^{\leq n}$ be the subset of partitions of weight n, then the set $\{\mathcal{C}(\lambda)\}_{\lambda \in \mathcal{P}_n}$ is a partition of $\mathcal{P}^{\leq n}$ into $|\mathcal{P}_n|$ many disjoint chains, with respect to the inclusion order.*

Proof. Let $\mu \in \mathcal{P}^{\leq n}$, then by the previous lemma, there is a unique $\lambda \in \mathcal{P}_n$ such that $\mu \in \mathcal{C}(\lambda)$. This proves the disjointness of the chains, the fact that the union of the chains in $Q = \{\mathcal{C}(\lambda)\}_{\lambda \in \mathcal{P}_n}$ equals $\mathcal{P}^{\leq n}$, and that Q contains $|\mathcal{P}_n|$ many chains. □

The sequence $\mathcal{C}(\lambda)$ can be described more easily in terms of frequency notation. Let

$$\lambda = (\alpha_1^{m_1}, \alpha_2^{m_2}, \ldots, \alpha_r^{m_r})$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_r$. The sequence $\mathcal{C}(\lambda)$ is obtained by subtracting 1 from the multiplicities in the following way

$$\lambda^0 = (\alpha_1^{m_1}, \alpha_2^{m_2}, \ldots, \alpha_{r-1}^{m_{r-1}}, \alpha_r^{m_r})$$

$$\lambda^1 = (\alpha_1^{m_1}, \alpha_2^{m_2}, \ldots, \alpha_{r-1}^{m_{r-1}}, \alpha_r^{m_r-1})$$

$$\lambda^2 = (\alpha_1^{m_1}, \alpha_2^{m_2}, \ldots, \alpha_{r-1}^{m_{r-1}-1}, \alpha_r^{m_r})$$

$$\vdots$$

$$\lambda^{r-1} = (\alpha_1^{m_1}, \alpha_2^{m_2-1}, \ldots, \alpha_{r-1}^{m_{r-1}}, \alpha_r^{m_r})$$

$$\lambda^r = (\alpha_1^{m_1-1}, \alpha_2^{m_2}, \ldots, \alpha_{r-1}^{m_{r-1}}, \alpha_r^{m_r}).$$

Proposition 3.12. *Let λ be a partition and $\mathcal{C}(\lambda) = \{\lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^k\}$, then for each $j = 0, \ldots, k-1$, the skew diagram $\lambda^j \setminus \lambda^{j+1}$ is a horizontal strip.*

Proof. The prove that any skew diagram $\lambda \setminus \mu$ is a horizontal strip, it suffices to show: if $\mu_j < \lambda_j$, then $\lambda_j \leq \mu_{j-1}$, for all j. This is so because if $\lambda \setminus \mu$ contains two boxes $(a, j), (b, j)$, $a < b$, then $\mu_a < j \leq \lambda_a$ and $\mu_b < j \leq \lambda_b$, which implies that $\mu_{b-1} < \lambda_b$.

Now, let $j \in \{0, \ldots, k-1\}$ and λ^j (λ^{j+1}, resp.) be attained from λ by removing the i_1th row (i_2th row, resp.) of size n_1 (n_2, resp.). Note that $n_1 < n_2$ and $i_2 < i_1$. Therefore λ^j and λ^{j+1} are equal in rows $i < i_2$ and $i \geq i_1$. Moreover, $\lambda^{j+1}_{i_2} < \lambda^j_{i_2}$ and $\lambda^{j+1}_{i_2} = \lambda^{j+1}_i = \lambda^j_{i+1}$ for $i_2 \leq i < i_1 - 1$. It follows that $\lambda^j \setminus \lambda^{j+1}$ is a horizontal strip. □

The chains have the following curious 'universal' property. If we ever have the situation described by the diagram below, where $\mu^0 \setminus \mu$ is a horizontal strip and λ, λ^0 are as above, then this diagram "commutes", i.e., $\lambda^0 \leq \mu^0$.

$$\begin{array}{ccc} \lambda^0 & --- & \mu^0 \\ \cup & & \cup \ \text{(hor. strip)} \\ \lambda & \leq & \mu \end{array}$$

Proposition 3.13. *Let λ, λ^0 be partitions such that $\lambda \in \mathcal{C}(\lambda^0)$. For any partitions μ, μ^0 such that $\lambda \leq \mu$, $\mu^0 \setminus \mu$ is a horizontal strip, and $|\mu^0| = |\lambda^0|$, then $\lambda^0 \leq \mu^0$.*

Proof. First, let's describe λ^0 more precisely. If k is the index of λ such that $\lambda_k \geq |\lambda^0| - |\lambda|$ and $\lambda_{k+1} < |\lambda^0| - |\lambda|$, then the parts of λ^0 are

$$\lambda^0_j = \begin{cases} \lambda_j & \text{if } j \leq k \\ |\lambda^0| - |\lambda| & \text{if } j = k+1 \\ \lambda_{j-1} & \text{if } j > k+1 \end{cases}.$$

Second, note that $\lambda \leq \mu$ and $\mu \subseteq \mu^0$ implies

$$\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \mu^0_j$$

for all $i \geq 1$.

Now, suppose that $\lambda^0 \not\leq \mu^0$. This means that there is some i_0 so that

$$\sum_{j=1}^{i_0} \mu^0_j - \lambda^0_j < 0.$$

By the inequalities in the second paragraph, it must be the case that $i_0 \geq k+1$. The condition that $\mu^0 \setminus \mu$ is a horizontal strip implies that

$$\sum_{j=i_1}^{i_2} \mu_j^0 - \mu_j \leq \mu_{i_1}^0 - \mu_{i_2}.$$

In particular,

$$\sum_{j \geq i} \mu_j^0 - \mu_j \leq \mu_i^0.$$

Using these facts we see that,

$$\sum_{j=1}^{i_0} \lambda_j^0 = |\lambda^0| - |\lambda| + \sum_{j=1}^{i_0-1} \lambda_j^0 > \sum_{j=1}^{i_0} \mu_j^0$$

which implies

$$|\lambda^0| - |\lambda| - \sum_{j=1}^{i_0-1} \mu_j^0 - \lambda_j = \sum_{j \geq i_0} \mu_j^0 - \lambda_j > \mu_{i_0}^0,$$

a contradiction. The proposition follows. □

3.7 Refinement of Conjecture

In this section, we give a refined version of the conjecture. First, we need to have an understanding of the decomposition of these identities in terms of the monomial basis (see sec. 3.2).

Let P_μ be the set of partitions λ satisfying $\lambda \preceq_H \mu$. It follows that

$$\alpha_\mu^r = \sum_{\lambda \in P_\mu} a_{\mu\lambda}^r m_\lambda$$

for some coefficients $a_{\mu\lambda}^r \in \mathbb{Q}$.

The set $M_\mu \subseteq P_\mu$ of *maximal* partitions is

$$M_\mu = \{\lambda \in P_\mu | \lambda \in \mathcal{C}(\nu) \text{ iff } \lambda = \nu\}.$$

Proposition 3.13 gives us a characterization of the set M_μ. The proof of the following lemma is straightforward using the arguments in the proofs in 3.6.

Lemma 3.14. *The set $\{\mathcal{C}(\lambda)\}_{\lambda \in M_\mu}$ is a partition of the set P_μ.*

Proposition 3.15. *Let μ be a partition, then*

$$M_\mu = \{\lambda : \lambda \leq \mu\}.$$

Proof. If $\lambda \leq \nu$ and $\mu \setminus \nu$ is a horizontal strip, then, using the previous two lemmas, we can find a λ^0 such that $\lambda \in \mathcal{C}(\lambda^0)$ and $\lambda^0 \leq \mu$. □

3.7.1 Positive Functions $\gamma_{\mu\lambda}$

The most important component of the refined conjecture involves functions that are non-negative on the set $1 \geq x_1 \geq x_2 \geq \cdots \geq x_m \geq 0$. Let λ, μ be partitions such that $\mu \supseteq \lambda$. Define $c_{\mu\lambda} := \frac{m_\mu(1)}{m_\lambda(1)}$ and the function

$$\gamma_{\mu\lambda} := c_{\mu\lambda} m_\lambda - m_\mu.$$

Also, consider the set $\mathcal{K} := \{(x_1, \ldots, x_m) : 1 \geq x_1 \geq x_2 \geq \cdots \geq x_m \geq 0\}$.

Proposition 3.16. *If $x \in \mathcal{K}$ and $\mu \supseteq \lambda$, then $\gamma_{\mu\lambda}(x) \geq 0$.*

Proof. If $\lambda = \mu$ then $\gamma_{\mu\lambda} = 0$ and we are done. Suppose that $\lambda \subset \mu$. If $x_1 = \cdots = x_m = 0$, then $\gamma_{\mu\lambda} = 0$. Suppose that $x = (x_1, \ldots, x_m) \in \mathcal{K}$ and $x_m > 0$. The inequality $\gamma_{\mu\lambda} \geq 0$ is equivalent to

$$m_\mu(1) m_\lambda(x) \geq m_\lambda(1) m_\mu(x).$$

Consider the expressions $M_{\nu,\rho}(x, y) = m_\nu(y) m_\rho(x)$ for partitions ν, ρ, then the non-negativity of $\gamma_{\mu\lambda}$ is equivalent to the non-negativity of $M_{\nu,\rho}(x, 1) - M_{\nu,\rho}(1, x)$. If $\nu \subset \rho$ then

$$\begin{aligned} c(M_{\nu,\rho}(x, y) - M_{\nu,\rho}(y, x)) &= \sum_{\sigma,\tau \in S_m} y^{\sigma\cdot\nu} x^{\tau\cdot\rho} - x^{\tau\cdot\nu} y^{\tau\cdot\rho} \\ &= \sum_{\sigma,\tau \in S_m} x^{\tau\cdot\nu} y^{\sigma\cdot\nu} (y^{(\sigma\cdot\rho)\setminus(\sigma\cdot\nu)} - x^{(\tau\cdot\rho)\setminus(\tau\cdot\nu)}) \end{aligned}$$

where c is the product of the sizes of the stabilizers of ν and ρ in S_m. Substituting $y_1 = \cdots = y_m = 1$ and noting that $1 - x^\alpha \geq 0$ for all $\alpha \in \mathbb{N}^m$ gives the result. □

Definition 3.17. *For any partition μ and Pieri root r, we define the values $b^{\mu,r}_{\lambda,i}$ for $\lambda \in M_\mu$ to be the coefficients in the expansion of α^r_μ in $\gamma_{\lambda^{i-1}\lambda^i}$, i.e.,*

$$\alpha^r_\mu = \sum_{\lambda\in M_\mu} \sum_{i=1}^{|\mathcal{C}(\lambda)|-1} b^{\mu,r}_{\lambda,i}\gamma_{\lambda^{i-1}\lambda^i} + \sum_{\lambda\in M_\mu} f_{\mu\lambda^*}m_{\lambda^*}$$

where $\mathcal{C}(\lambda) = \{\lambda = \lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^*\}$.

Conjecture 3.18. *For any partition μ and Pieri root r,*

$$\alpha^r_\mu = \sum_{\lambda\in M_\mu} \sum_{i=1}^{|\mathcal{C}(\lambda)|-1} b^{\mu,r}_{\lambda,i}\gamma_{\lambda^{i-1}\lambda^i}.$$

Furthermore, for all $\lambda \in M_\mu$ and $i \geq 1$, $b^{\mu,r}_{\lambda,i} \geq 0$. In particular, $\alpha^r_\mu(x) \geq 0$ for $x \in \mathcal{K}$.

There is a recurrence relation between the $b^{\mu,r}_{\lambda,i}$ involving the monomial expansion coefficients

$$\alpha^r_\mu = \sum_{\lambda\in P_\mu} a^r_{\mu\lambda}m_\lambda$$

and the weights involved in the definition of the $\gamma_{\mu\lambda}$,

$$\gamma_{\mu\lambda} := c_{\mu\lambda}m_\lambda - m_\mu.$$

Let $b^r_{\mu\lambda^i} := b^{\mu,r}_i$ in the expression of $\Gamma^r_{\mu\lambda}$ for $\lambda \in M_\mu$, then

$$b^r_{\mu\lambda^i} = c_{\lambda^{i-1}\lambda^i}b^r_{\mu\lambda^{i-1}} - a^r_{\mu\lambda^{i-1}} \tag{3.7}$$

Write

$$\lambda = (\alpha_1^{m_k}, \alpha_2^{m_{k-1}}, \ldots, \alpha_k^{m_1}, 0^{m_0}).$$

Another way of writing the above recursion fully in terms of the $a_{\mu\lambda}$ is

$$b^r_{\mu\lambda^i} = -\sum_{j=0}^{i} c_{\lambda^j\lambda^i}a^r_{\mu\lambda^j}.$$

Multiplying out by $m_{\lambda^i}(1)$ we deduce that, $b^r_{\mu\lambda^i} \geq 0$ if and only if

$$\sum_{j=0}^{i} a^r_{\mu\lambda^j}m_{\lambda^j}(1) \leq 0.$$

Now, the coefficients $f_{\mu\lambda^*}$ are explicitly

$$f_{\mu\lambda^*} = a^r_{\mu\lambda^*} - b^r_{\mu\lambda^*} = a^r_{\mu\lambda^*} + \sum_{j=0}^{k-1} c_{\lambda^j\lambda^{k-1}} a^r_{\mu\lambda^j},$$

$k = k(\lambda)$, which has the same sign as

$$m_{\lambda^{k-1}}(1) f_{\mu\lambda^*} = \sum_{j=0}^{k} a^r_{\mu\lambda^j} m_{\lambda^j}(1).$$

In fact, since the value of $m_\lambda(1)$ is a multinomial coefficient, after clearing denominators the right hand side becomes

$$\sum_{j=0}^{k} a^r_{\mu\lambda^j} m_j = \sum_{j=0}^{k} \sum_{\nu:\lambda^j\leq\nu} m_j h^r_{\mu\nu} K_{\nu\lambda^j}$$

where,

$$h^r_{\mu\nu} = \frac{g_\mu(\nu,-\mu_1)}{(d+\mu_1)g_\mu(\mu,-\mu_1)} - \frac{g_\mu(\nu,r)}{(d-r)g_\mu(\mu,r)}. \tag{3.8}$$

The following conjecture is therefore a much stronger version of the assertion in conjecture 3.18 that $f_{\mu\lambda^*} = 0$.

Conjecture 3.19. *For any $r \in R_\mu$,*

$$\sum_{j=0}^{k} \sum_{\nu:\lambda^j\leq\nu} m_j K_{\nu\lambda^j} \frac{g_\mu(\nu,r)}{(d-r)g_\mu(\mu,r)} = K_{\mu\lambda}.$$

This conjecture has been verified empirically. We will show data at the end of this dissertation. Suppose that $a_{\mu\lambda} \geq 0$ for all λ such that $|\lambda| < |\mu|$, then we have the following observation.

Proposition 3.20. *Let μ be a partition of weight n. If $a^r_{\mu\lambda} \geq 0$ for all λ with $|\lambda| < n$ and for all ν, $|\nu| = n$ we have that $f^r_{\mu\nu^*} = 0$, then for all $i \in \{1,\ldots,k(\nu)\}$ $b^r_{\mu\nu^i} \geq 0$.*

$$a_{\mu\lambda^j} = \sum_{\nu:\lambda^j\leq\nu} k^r_{\mu\nu} K_{\nu\lambda^j}$$

where $k^r_{\mu\nu} = 0$ if $\mu \setminus \nu$ is not a horizontal strip.

3.8 Kostka-Content Identities

After some algebraic manipulation, conjecture 3.19 can be put into a polynomial form. This allows for an *inductive* proof for certain families of partitions. This new form does not address the positivity of the $b^{\mu,r}_{\lambda,i}$ coefficients.

Let σ, τ be a partitions and recall that

$$G_\sigma = \{i : \sigma_i > \sigma_{i+1}\}$$

$$g_\sigma(\tau, t) = \prod_{j \in G_\sigma} (t + \tau_j - j).$$

Also recall that $p_\mu(x) = \prod_{r \in R_\mu} x - r$. Enumerate the elements in R_μ and G_μ by

$$R_\mu = \{r_0, r_1, \ldots, r_N\}$$

$$G_\mu = \{g_1, \ldots, g_N\}$$

such that $r_0 = r^-$, $r_0 < r_1 < \cdots < r_N$, and $g_1 < g_2 < \cdots < g_N$. Then we have that

$$\mu_{g_i} = g_{i-1} - r_{i-1}$$

where $g_0 = 0$. Using this notation, we can write the polynomial p_μ as

$$p_\mu(x) = \prod_{i=0}^{N} (x + (g_{i+1} - g_i) + \mu_{g_{i+1}} - g_{i+1})$$

which resembles the polynomials $g_\mu(\lambda, t)$. In fact, using the frequency notation $\mu = (n_1^{m_1} n_2^{m_2} \cdots n_N^{m_N})$, we have that $g_i - g_{i-1} = t_i$. The last factor is

$$x - r_N = x - \ell.$$

Suppose that $N = 1$ then can expand p_μ and put it in the form

$$p_\mu = (x + m_1 + \mu_{g_1} - g_1)(x + \mu_{g_2} - g_2)$$

$$= (x + m_1 + \mu_{g_1} - g_1)(x + \mu_\ell - \ell)$$

$$= (x + \mu_{g_1} - g_1)(x + \mu_\ell - \ell) + m_1(x + \mu_\ell - \ell).$$

It follows that the expansion holds in the case that $\mu = \lambda$ and $N = 1$.

Throughout this section, by abuse of notation, the values $K_{\sigma,\tau}$ refer to the coefficients of the monomial symmetric functions in the expansion of $s_\mu(x_1,\ldots,x_n)$ for a specified n. For the first part, we may assume that $n >> 0$ and so $K_{\sigma,\tau}$ are the Kostka numbers. The following conjecture is equivalent to conjecture 3.19.

Conjecture 3.21 (Kostka-Content Identity)**.** *For any partitions μ, λ such that $\lambda \le \mu$, where $\lambda = (n_1^{m_k}, n_2^{m_{k-1}}, \ldots, n_k^{m_1})$ $n_i > 0$,*

$$K_{\mu\lambda}\prod_{r\in R_\mu} t-r = (t-\ell(\lambda))g_\mu(\mu,t)K_{\mu\lambda} + \sum_{j=1}^{k}\sum_{\nu\in H_\mu:|\nu|=|\lambda^j|} m_j g_\mu(\nu,t)K_{\nu\lambda^j}$$

where $\lambda^j = (n_1^{m_k},\ldots,n_{k-j+1}^{m_j-1},\ldots,n_k^{m_1})$.

The proof of the equivalence of the two conjectures is a straightforward algebraic manipulation that involves clearing denominators and then checking the degree of the polynomial on both sides. This polynomial form gives us the means to more easily check the vanishing part of conjecture 3.18.

3.8.1 An Inductive Argument

In certain cases, we can use induction to prove conjecture 3.21. To do this, note the following relationship between the g and p polynomials:

$$g_{\mu+m}(\mu,t) = p_\mu(t-1) = \text{ and}$$

$$p_{\mu+m}(t) = (t+m)p_\mu(t-1),$$

where $m > \mu_1$. If $m = \mu_1$, then the analogous identities are

$$g_{\mu+\mu_1}(\mu,t) = \frac{p_\mu(t)}{t+\mu_1-1} \text{ and}$$

$$p_{\mu+\mu_1}(t) = \frac{(t+\mu_1)}{(t+\mu_1-1)}p_\mu(t-1).$$

Using these identities we can prove the following.

Proposition 3.22. *Let μ, λ be partitions such that $\lambda \leq \mu$ and conjecture 3.19 holds, then for any $m \geq \mu_1$ conjecture 3.19 holds for $\mu + m, \lambda + m$.*

Proof. Recall that conjecture 3.19 states that, in the same setting as in the proposition statement,

$$K_{\mu\lambda} p_\mu(t) = (t - \ell(\lambda)) g_\mu(\mu, t) K_{\mu\lambda} + \sum_{j=1}^{\ell(\lambda)} \sum_{\nu \in H_\mu : |\nu| = |\lambda| - \lambda_j} g_\mu(\nu, t) K_{\nu, \lambda - \lambda_j}.$$

Let $\hat{\nu} = \nu + m$ for any partition ν. Suppose that $m > \mu_1$. Using the identities above, a straightforward calculation yields the result:

$$(t - \ell(\hat{\lambda})) g_{\hat{\mu}}(\hat{\mu}, t) K_{\hat{\mu}\hat{\lambda}} + \sum_{j=1}^{\ell(\hat{\lambda})} \sum_{\nu \in H_{\hat{\mu}} : |\nu| = |\hat{\lambda}| - \hat{\lambda}_j} g_{\hat{\mu}}(\nu, t) K_{\nu, \hat{\lambda} - \hat{\lambda}_j}$$

$$= (t - \ell(\lambda) - 1) g_{\hat{\mu}}(\hat{\mu}, t) K_{\hat{\mu}\hat{\lambda}} + \sum_{j=1}^{\ell(\lambda)+1} \sum_{\nu \in H_{\hat{\mu}} : |\nu| = |\hat{\lambda}| - \hat{\lambda}_j} g_{\hat{\mu}}(\nu, t) K_{\nu, \hat{\lambda} - \hat{\lambda}_j}$$

$$= (t + m - 1) \left((t - \ell(\lambda) - 1) g_\mu(\mu, t-1) K_{\mu\lambda} + \sum_{1 \leq j \leq \ell(\lambda), \nu \in H_\mu} g_\mu(\nu, t-1) K_{\nu, \lambda - \lambda_j} \right)$$
$$+ g_{\mu+m}(\mu, t) K_{\mu\lambda}$$

$$= (t + m - 1) p_\mu(t-1) K_{\mu\lambda} + g_{\mu+m}(\mu, t) K_{\mu\lambda}$$

$$= (t + m) p_\mu(t-1) K_{\mu\lambda}$$

$$= p_{\mu+m}(t) K_{\hat{\mu}\hat{\lambda}}.$$

In the case that $m = \mu_1$, the last 4 lines do not hold. Instead we have,

$$= m_1 g_{\mu+m}(\mu, t) K_{\mu\lambda} + ((t - \ell(\lambda) - 1) g_\mu(\mu, t-1) K_{\mu\lambda}$$
$$+ \sum_{1 \leq j \leq \ell(\lambda), \nu \in H_\mu} g_\mu(\nu, t-1) K_{\nu, \lambda - \lambda_j} \Bigg)$$
$$= p_\mu(t-1) K_{\mu\lambda} + g_{\mu+m}(\mu, t) K_{\mu\lambda}$$
$$= p_\mu(t-1) K_{\mu\lambda} + \frac{p_\mu(t-1)}{t + \mu_1 - 1} K_{\mu\lambda}$$
$$= (t + \mu_1) \frac{p_\mu(t-1)}{t + \mu_1 - 1} K_{\mu\lambda}$$
$$= p_{\mu+m}(t) K_{\hat{\mu}\hat{\lambda}}.$$

□

Note that the set $H_{\mu+m}$ can be described in terms of H_μ as

$$H_{\mu+m} = \{k + \nu : \nu \in H_\mu, \mu_1 \le k \le m\} = \bigsqcup_{\mu_1 \le k \le m} (k + H_\mu).$$

This leads us to the following corollary to proposition 3.22.

Corollary 3.23. *Let μ, λ be partitions such that $\lambda \le \mu$. Fix i and suppose that $b^{\mu,r}_{\lambda,i}$ is decreasing as a function of r where $r \in R_\mu$, then for any $m \ge \mu_1$, $b^{\mu+m,r}_{\lambda+m,i}$ is decreasing as a function of r where $r \in R_{\mu+m}$.*

3.8.2 Case $\lambda = \mu$

In this subsection, we will prove the Kostka-Content identity for the case when $\lambda = \mu$ using the conventions in 3.21.

Proposition 3.24. *Let μ and λ be partitions, then Kostka-Content identity holds when $\mu = \lambda$.*

Proof. Let $\mu = n_1^{m_k} n_2^{m_{k-1}} \cdots n_k^{m_1}$ and suppose that $\lambda = \mu$ then the right hand side of the Kostka-Content identity is

$$(t - \ell(\mu))g_\mu(\mu, t)K_{\mu,\mu} + \sum_{j=1}^{k} \sum_{\nu \in H_\mu : |\nu| = |\lambda^j|} m_j g_\mu(\nu, t) K_{\nu\mu^j}.$$

Since $\nu \ge \mu^j$ implies that $\nu = \mu^j$ we have that the above polynomial is

$$(t - \ell(\mu))g_\mu(\mu, t) + \sum_{j=1}^{k} m_j g_\mu(\mu^j, t).$$

Furthermore, if we let $R_\mu = \{r_1, \dots, r_k, r_{k+1}\}$ where $-\mu_1 = r_1 < r_2 < \cdots < r_{k+1} = \ell(\mu)$ then one can directly verify that

$$\begin{aligned} g_\mu(\mu^j, t) &= \prod_{p=1}^{k-j} (t - r_p - m_{k-p+1}) \prod_{p=k-j+1}^{k} (t - r_p - m_{k-p+1} + \mu_{i_p} - \mu_{i_{p+1}}) \\ &= \prod_{p=1}^{k-j} (t - r_p - m_{k-p+1}) \prod_{p=k-j+1}^{k} (t - r_{p+1}). \end{aligned}$$

The crucial observation is that

$$(t-\ell(\mu))\prod_{p=1}^{k}(t-r_p-m_{k-p+1})+\sum_{j=1}^{k}m_j\prod_{p=1}^{k-j}(t-r_p-m_{k-p+1})\prod_{p=k-j+1}^{k}(t-r_{p+1})$$

$$= (t-\ell(\mu))(t-r_k)\prod_{p=1}^{k-1}(t-r_p-m_{k-p+1})$$

$$+\sum_{j=2}^{k}m_j\prod_{p=1}^{k-j}(t-r_p-m_{k-p+1})\prod_{p=k-j+1}^{k}(t-r_{p+1})$$

$$= (t-\ell(\mu))(t-r_k)(t-r_{k-1})\prod_{p=1}^{k-2}(t-r_p-m_{k-p+1})$$

$$+\sum_{j=3}^{k}m_j\prod_{p=1}^{k-j}(t-r_p-m_{k-p+1})\prod_{p=k-j+1}^{k}(t-r_{p+1})$$

$$\vdots$$

$$= p_\mu(x).$$

□

3.8.3 Jack Hook-Length Recursion Formula

Conjecture 3.21 for $\mu \vdash n$, $\lambda = 1^n$ can be written as

$$f_\mu \prod_{r\in R_\mu} t-r = (t-n)g_\mu(\mu,t)f_\mu + n\sum_{\nu\subset:\mu} g_\mu(\nu,t)f_\nu$$

where $f_\rho = K_{\rho,1^n}$ for any partition ρ. Note that every ν such that $\nu \subset: \mu$ is of the form $\mu-(i,\mu_i)$, for $i\in G_\mu$. In light of this fact, let μ^i denote the partition $\mu-(i,\mu_i)$. Evaluating both sides at $t=r$ where r is a Pieri root and dividing out by $g_\mu(\mu,r)$ (which is never 0 at Pieri roots) yields

$$0 = (r-n)f_\mu + n\sum_{i\in G_\mu}\frac{r+\mu_i-i-1}{r+\mu_i-i}f_{\mu^i}.$$

Further manipulation yields the more manageable

$$-rf_\mu = n\sum_{i\in G_\mu}\frac{f_{\mu^i}}{i-r-\mu_i}.$$

We will now give a proof of the following version of conjecture 1 for $\lambda = 1^n$. It uses the hook length formula for f_μ crucially. The proof uses the technique (of interpolation polynomials) found in the proof of the hook formula in [Ba].

Proposition 3.25. *For $\mu \vdash n$, $r \in P_\mu$, and $\mu^i = \mu - (i, \mu_i)$ for $i \in G_\mu$,*

$$r f_\mu = n \sum_{i \in G_\mu} \frac{f_{\mu^i}}{r + \mu_i - i}.$$

Proof. Recall that for any partition λ,

$$f_\lambda = \frac{|\lambda|!}{H(\lambda)}$$

where $H(\lambda) = \prod_{s \in \lambda} h_s(\lambda)$ and $h_{(i,j)}(\lambda) = \lambda_i + \lambda_j - i - j + 1$ is the hook length at the box (i, j). Now fix a Pieri root r_0. We will prove the identity for $r = r_0$.

Solving for r_0 on the left hand side of the purported identity, we get

$$r_0 = \sum_{i \in G_\mu} \frac{1}{r_0 + \mu_i - i} \frac{\prod_{s \in \mu} h_s(\mu)}{\prod_{s \in \mu^i} h_s(\mu^i)}.$$

In the quotient $\frac{\prod_{s \in \mu} h_s(\mu)}{\prod_{s \in \mu^i} h_s(\mu^i)}$, there are many cancellations.

Lemma 3.26.

$$\frac{\prod_{s \in \mu} h_s(\mu)}{\prod_{s \in \mu^i} h_s(\mu^i)} = -\frac{\prod_{r \in R_\mu} (c(\mu \setminus \mu^i) + r)}{\prod_{j \in G_\mu, j \neq i} (c(\mu \setminus \mu^i) - c(\mu \setminus \mu^j))}$$

where $c(s)$ is the content of s.

The lemma can be proved directly by carefully keeping track of all of the cancellations. A crucial observation is that $r + \mu_i - i = r + c(\mu \setminus \mu^i)$ and so we must prove that

$$-r_0 = \sum_{i \in G_\mu} \frac{\prod_{r \in R_\mu \setminus \{r_0\}} (c(\mu \setminus \mu^i) + r)}{\prod_{j \in G_\mu, j \neq i} (c(\mu \setminus \mu^i) - c(\mu \setminus \mu^j))}.$$

Now, consider the polynomials

$$P(t) = \sum_{i \in G_\mu} \frac{\prod_{r \in R_\mu \setminus \{r_0\}} (c(\mu \setminus \mu^i) + r)}{\prod_{j \in G_\mu, j \neq i} (c(\mu \setminus \mu^i) - c(\mu \setminus \mu^j))} \prod_{j \in G_\mu, j \neq i} (t - c(\mu \setminus \mu^j)) \text{ and}$$

$$Q(t) = \prod_{r \in R_\mu \setminus \{r_0\}} (t+r)$$

then, by degree considerations and vanishing at $c(\mu \setminus \mu^i)$, we see that

$$Q(t) - P(t) = \prod_{i \in G_\mu} (t - c(\mu \setminus \mu^i)).$$

Solving for $P(t)$ and computing the coefficient of $t^{|G_\mu|-1}$, we get that

$$\sum_{i \in G_\mu} \frac{\prod_{r \in R_\mu \setminus \{r_0\}} (c(\mu \setminus \mu^i) + r)}{\prod_{j \in G_\mu, j \neq i} (c(\mu \setminus \mu^i) - c(\mu \setminus \mu^j))} = \sum_{r \in R_\mu \setminus \{r_0\}} r + \sum_{i \in G_\mu} c(\mu \setminus \mu^i) = -r_0.$$

(This last equality follows from the fact that $\sum_{r \in R_\mu} r + \sum_{i \in G_\mu} c(\mu \setminus \mu^i) = 0$). □

Corollary 3.27. *Let μ be a partition of n for $n \geq 1$, then*

$$f_\mu = \frac{1}{\mu_1} \sum_{(i,j) \in \mu} \frac{f_{\mu-(i,j)}}{\mu_1 - h_{(1,j)}(\mu)}$$

where $f_{\mu-s} = 0$ if $\mu - s$ is not a partition and $h_s(\mu)$ is the hook length of s in μ.

Here is an analogous *Jack version* that uses the fact that the Pieri roots of μ are the negative contents of the *inner corners* of μ. An inner corner of μ is the box (i, μ_{i+1}) where $i \in G_\mu \cup \{0\}$. For this, we will need the concept of interpolation polynomials $P_\lambda^{r\delta}$ defined in [KP].

Proposition 3.28. *Let μ be a partition, x_i be the outer corners of μ, and y_i be the inner corners of μ, and $\mu^i = \mu - x_i$, then*

$$\alpha|\mu| = -\sum_{i \in G_\mu} \frac{\prod_j (c_\alpha(x_i) - c_\alpha(y_j))}{\prod_{j, j \neq i} (c_\alpha(x_i) - c_\alpha(x_j))} = \alpha \sum_{\mu^i} P_{\mu^i}^{r\delta}(\mu + r\delta)$$

where $c_\alpha(i,j) = \alpha j - i$.

Call the values $c_\alpha(s)$, for s a box, the *α-content* of s. Let $x_i = (ai, b_i)$ for $i = 1, \ldots, m$, then

$$y_0 = (0, b_1), y_1 = (a_1, b_2), \ldots, y_n = (a_n, 0).$$

The proof of the above proposition follows along the same lines as the proof of 3.25.

Proof. Let

$$P(t) = -\sum_{\mu^i} \frac{\prod_{j=0}^{m}(c_\alpha(x_i) - c_\alpha(y_j))}{\prod_{1\le j\le m, j\ne i}(c_\alpha(x_i) - c_\alpha(x_j))} \prod_{1\le k\le m; k\ne i}(t - c_\alpha(x_k))$$

$$Q(t) = \prod_{0\le i\le m}(t - c_\alpha(y_i)),$$

then

$$P(t) + Q(t) = (t+\beta)\prod_{1\le i\le m}(t - c_\alpha(x_i))$$

where β is a constant, possibly depending on α. Solving for $P(t)$ we see that the coefficient of t^m is

$$\beta - \sum_{i=1}^{m} c_\alpha(x_i) + \sum_{i=0}^{m} c_\alpha(y_i) = 0$$

and since

$$\sum_{i=1}^{m} c_\alpha(x_i) = (\alpha b_1 - a_1) + (\alpha b_2 - a_2) + \cdots + (\alpha b_n - a_n)$$

$$(\alpha b_1 - 0) + (\alpha b_2 - a_1) + \cdots + (\alpha b_n - a_{n-1}) + (\alpha\cdot 0 - a_n)$$

$$= \sum_{i=0}^{m} c_\alpha(y_i)$$

it follows that $\beta = 0$. Therefore, the coefficient of t^{m-1} of

$$P(t) = -Q(t) + t\prod_{1\le i\le m}(t - c_\alpha(x_i))$$

is

$$\sum_{1\le i<j\le m} c_\alpha(x_i)c_\alpha(x_j) - \sum_{0\le i<j\le m} c_\alpha(y_i)c_\alpha(y_j)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{m} c_\alpha(x_i)\right)^2 - \frac{1}{2}\left(\sum_{i=0}^{m} c_\alpha(y_i)\right)^2 - \frac{1}{2}\sum_{i=0}^{m}(c_\alpha(x_i)^2 - c_\alpha(y_i)^2)$$

$$= -\frac{1}{2}\sum_{i=0}^{m}(c_\alpha(x_i)^2 - c_\alpha(y_i)^2)$$

where we take $x_0 = (0,0) = x_{m+1}$ for convenience. Finally, shifting the sum slightly, note that

$$-\frac{1}{2}\sum_{i=0}^{m}(c_\alpha(x_{i+1}) - c_\alpha(y_i))(c_\alpha(x_{i+1}) + c_\alpha(y_i))$$

$$= \sum_{i=1}^{m} \alpha b_i(a_i - a_{i-1})$$

$$= \alpha|\mu|.$$

To prove that

$$|\mu| = \sum_{\mu^i} P^{r\delta}_{\mu^i}(\mu + r\delta)$$

note that

$$\sum_{|\lambda|=k} P^{r\delta}_{\lambda}(x) = \frac{1}{k!}\left(x_1 + \cdots + x_d - r\binom{n}{2}\right)_k.$$

Therefore,

$$\sum_{|\lambda|=k} P^{r\delta}_{\lambda}(\mu + r\delta) = \binom{|\mu|}{k}.$$

□

Define the upper and lower hook lengths for the box $s = (i, j)$ to be

$$h_\lambda(s;\alpha) = \alpha(\lambda_i - j) + \lambda'_j - i + 1$$

$$h'_\lambda(s;\alpha) = \alpha(\lambda_i - j + 1) + \lambda'_j - i$$

respectively, where λ' is the transpose of λ. Notice the following important fact and its immediate corollary.

Lemma 3.29. *Using the notation above and from 3.28, for $\mu :\supset \mu^i$ and R (resp. C) are the row (resp. column) of the box $x_i = \mu/\mu^i$,*

$$\alpha \prod_{s\in C} \frac{h_\mu(s;\alpha)}{h_{\mu^i}(s;\alpha)} \prod_{s\in R} \frac{h'_\mu(s;\alpha)}{h'_{\mu^i}(s;\alpha)} = -\frac{\prod_j (c_\alpha(x_i) - c_\alpha(y_j))}{\prod_{j,j\neq i}(c_\alpha(x_i) - c_\alpha(x_j))}$$

Define, for any partitions μ, λ such that $\lambda \subset \mu$, the set $Y(\mu/\lambda)$ of boxes $(i, j) \in \mu$ such that $\lambda_i < \mu_i$ and $\lambda'_j = \mu'_j$ and the rational expression

$$\psi_{\mu/\lambda}(\alpha) = \prod_{s\in Y(\mu/\lambda)} \frac{h'_\mu(s;\alpha)/h_\mu(s;\alpha)}{h'_\lambda(s;\alpha)/h_\lambda(s;\alpha)}.$$

Remark 4. *Note that in the 1996 IMRN paper [KS], the authors defined*

$$\psi'_{\mu/\lambda}(\alpha) = \prod_{s \in X(\mu/\lambda)} \frac{h_\mu(s;\alpha)/h'_\mu(s;\alpha)}{h_\lambda(s;\alpha)/h'_\lambda(s;\alpha)}$$

where $X(\mu/\lambda)$ is the set of boxes $(i,j) \in \mu$ such that $\mu_i = \lambda_i$ and $\lambda'_j < \mu'_j$, in order to describe the Pieri rule for Jack polynomials:

$$e_k P_\lambda^{(\alpha)} = \sum_\mu \psi'_{\mu/\lambda}(\alpha) P_\mu^{(\alpha)},$$

where μ runs over partitions such that μ/λ is a vertical k-strip.

Also define the quantity

$$f_\lambda(\alpha) = \frac{|\lambda|!}{\prod_{s\in\lambda} h_\lambda(s;\alpha)},$$

which is the coefficient the monomial m_{1^λ} in the expansion of the Jack polynomial (specifically, the normalized version $P_\lambda^{(\alpha)}$) with parameter α. Notice the striking similarity with the Pieri rule for the case $e_1 P_\lambda^{(\alpha)}$.

Corollary 3.30. *For μ any partition,*

$$f_\mu(\alpha) = \sum_{\lambda \subset: \mu} \psi_{\mu/\lambda}(\alpha) f_\lambda(\alpha).$$

The analogue of 3.25 for the Jack case can be formulated as follows. Let P_μ be the set of inner corners of μ and R_μ be their α-contents. An element of R_μ will be denoted r_α to emphasize the dependence on the parameter α.

Proposition 3.31. *For $\mu \vdash n$, $r \in P_\mu$, and $\mu^i = \mu - (i, \mu_i)$ where $i \in G_\mu$,*

$$r_\alpha f_\mu(\alpha) = n \sum_{i \in G_\mu} \frac{f_{\mu^i}(\alpha)}{r_\alpha + \alpha\mu_i - i}.$$

Equivalently,

$$(n - r_\alpha) f_\mu(\alpha) = n \sum_{i \in G_\mu} \frac{r_\alpha + \alpha\mu_i - i - 1}{r_\alpha + \alpha\mu_i - i} f_{\mu^i}(\alpha).$$

Lemma 3.32. *For any partition μ and $r_\alpha \in R_\mu$, then*

$$r_\alpha = \sum_{i \in G_\mu} \frac{P^{r\delta}_{\mu^i}(\mu + r\delta)}{r_\alpha + \alpha\mu_i - i}.$$

Equivalently,

$$(n - r_\alpha) \prod_{j \geq 1} (r_\alpha - n + \alpha\overline{\mu}_j) = \sum_{\nu \subset: \mu} \prod_{j \geq 1} (r_\alpha - n + \alpha\overline{\nu}_j)) P^{r\delta}_\nu(\overline{\mu}).$$

$$-t \prod_{j \geq 1} (t + \alpha\overline{\mu}_j) = \sum_{\nu \subset: \mu} \prod_{j \geq 1} (t + \alpha\overline{\nu}_j)) P^{r\delta}_\nu(\overline{\mu}).$$

The proofs of these results is almost identical to the proofs in the Schur case, except that the contents c and Pieri roots r are replaced by their Jack versions r_α and c_α respectively. Similar replacements can be made throughout the paper in order to produce Jack versions of different results for Schur polynomials. We hope to address the conjectures in the more general Jack setting at a future date.

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