COHOMOLOGICAL FIELD THEORIES AND FOUR-MANIFOLD INVARIANTS

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ABSTRACT OF THE DISSERTATION

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Four-dimensional cohomological quantum field theories possess topological sectors of correlation functions that can be analyzed non-perturbatively on a general four-manifold. In this thesis, we explore various aspects of these topological models and their implications for smooth structure invariants of four-manifolds.

Cohomological field theories emerge when one considers topological twisting of ordinary quantum field theories with extended ($N = 2$ in the context of this thesis) supersymmetry. The global scalar supersymmetry of these theories allows one to use integrals/sums over their quantum vacua as a tool for their exact analysis. In the case of pure SU(2) $N = 2$ gauge theory this has lead to remarkable success of Witten's field theory formulation of Donaldson invariants and discovery of Seiberg-Witten invariants which are the best presently available tool for distinguishing smooth structures on four-manifolds with fixed topological type. In chapter 3 of this thesis we analyze a new prescription for defining the integral over a Coulomb branch of vacua in Donaldson-Witten theory as well as discuss possible treatment of IR divergences associated with certain BRST-exact operators. Chap-

ter 3 of the thesis is based on the work reported in [20] (arXiv:1901.03540 [hep-th]) and partly has been extracted from that paper.

On general grounds one expects that topological twisting of any $N = 2$ supersymmetric theory defines a smooth structure invariant. However, examples of Lagrangian theories strongly suggest that topological partition functions of Lagrangian theories are expressible through the classical cohomological invariants and Seiberg-Witten invariants. Therefore, the search for new 4-manifold invariants has to be restricted to so-called "non-Lagrangian" $N = 2$ theories. Though full non-Lagrangian theories are, at present, difficult to analyze due to their strongly-coupled nature and the lack of action principle, in chapter 4 we show how one can derive the topological partition function of a simplest non-trivial non-Lagrangian theory discovered by Argyres and Douglas and known as AD3 theory. We obtain a formula for the partition function of topologically twisted version of the AD3 theory on any compact, oriented, simply connected, four-manifolds without boundary and with $b_2^+ > 0$. The result can be, once again, expressed in terms of classical cohomological invariants and Seiberg-Witten invariants. We argue that our results hint at the existence of four-manifolds of new, presently unknown, type as well as narrow the search for new field theory invariants of four-manifolds to Non-Lagrangian superconformal points that admit Higgs branches. Chapter 4 of this thesis is based on the work reported in [40] (arXiv:1711.09257 [hep-th]) and partly has been extracted from that paper.

Finally, in chapter 5 we derive a twisted (0,2) two-dimensional model by putting the abelian low energy theory of single M5 brane described by the PST action on a direct product of a Riemann surface and a four-manifold. The resulting two dimensional topological model can potentially be used as a tool refining the u-plane integral to study topologically twisted $N = 2$ theories of class S.

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Chapter 1

Introduction

In 1988 Witten [1] discovered the mechanism that allows correlation functions of cetrain quantum field theories to be independent of the background metric. The goal was to find field theoretic formulation of Donaldson theory, which describes a set of powerful invariants of smooth four-manifodls allowing one to distinguish different differential structures that can be assigned to a given four-manifold topology by means of intersection theory on instanton moduli spaces. Such formulation was found by Witten to be given in terms of topologically twisted $N = 2$ supersymmetric Yang Mills theory and the general idea behind that relation formed the foundation of what is known now as cohomological quantum field theory.

Models of cohomological field theory have been of immense importance in the last thirty years in both physics and mathematics. Besides the Donaldson theory, their applications involve Gromov-Witten invariants [2], geometric Langlands program [3], evaluation of central charges in superconformal theories [4], etc.

The defining property of cohomological field theory is existence of (at least one) scalar fermionic symmetry (generated by scalar fermionic operator) Q. Given an action S of such theory it must satisfy $Q(S) = 0$. By acting with Q one more time one obtains $Q^2(S) = 0$, so Q^2 has to be among bosonic symmetries of the action.

One of the key observations of [1] was that at the formal level correlation functions of operators in the image of Q (known also as Q-exact operators) all vanish, while Q-invariant (also known as Q-closed) operators are decoupled from the Q-exact ones. That is, correlation functions of the form $\langle Q(a) \cdot b \rangle$ where a is any combination of the fields and b is Q-closed are identically zero. Such property implies, in particular, that correlation functions of Q-invariant operators are invariant under Q-exact deformations of the theory's action. It turns out that among such Q-exact deformations one typically finds variations of the coupling constant as well as deformations of the background metric g with energy momentum tensor

$$\delta_g S = \frac{1}{2} \int_X \sqrt{g}\, \delta g^{\mu\nu}\, T^{\mu\nu}, \qquad\qquad T_{\mu\nu} = Q(\lambda_{\mu\nu}), \tag{1.1}$$

Thus, the Q-symmetry drastically simplifies the theory: it implies the existence of topological (metric independent) sector of observables identified with the Q-cohomology. It also allows one to use several methods to obtain exact results for the sector of Q-invariant correlators such as evaluation via localization to Q-fixed points (the Mathai-Quillen formalism [5]), extension of results obtained at weak coupling, evaluation based on low energy effective description [6,7]. However, the validity of these methods relies on the existence of action as well as on the behaviour near the field space boundary that determines the precise meaning of the Q-closed vs Q-exact decoupling.

Analysis of the decoupling of Q-exact and Q-closed operators

In chapter 3 we study the decoupling of Q-exact and Q-closed operators (by which we mean vanishing of correlation functions of the form $\langle Q(a)\, b \rangle$ for any a and Q-closed b) in the context of low energy effective theory on the Coulomb branch of topologically twisted $N = 2$ sypersymmetric Yang-Mills (SYM) theory with SU(2) gauge group. The corresponding path integral is known [7] to localize to a subtle but finite dimensional integral over the Coulomb branch of vacua known also as u-plane integral.

The Q-closed vs Q-exact decoupling can be obstructed by a contribution from the boundary of the space of vacua. In chapter 3 we show that vacuum expectation values of Q-exact operators can be expressed as u-plane integrals whose integrands can be written as a total derivative after integration over the auxiliary field and the fermionic zero modes. In the case of the pure SU(2) Donaldson-Witten theory there is a relation between the Coulomb branch integral and integrals over a modular fundamental domain (this relation is reviewed in section 3.1), the vacuum expectation value (vev) of any Q-exact operator takes the following form

$$\langle [Q, \mathcal{O}\} \rangle = \int_{\mathcal{F}_\infty} d\tau \wedge d\bar{\tau}\, \partial_{\bar{\tau}} F_{\mathcal{O}}, \tag{1.2}$$

where $\mathcal{F}_\infty = \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is a fundamental domain for the modular group $\mathrm{SL}(2,\mathbb{Z})$ and $F_{\mathcal{O}}$ admits a q-expansion of the form

$$\sum_s y^{-s} \sum_{m,n} c_s(m,n)\, q^m \bar{q}^n, \tag{1.3}$$

where $\sum_s$ is a sum over finite number of s values and $q = e^{2\pi i\tau}$.

Such boundary contributions to correlation functions $\langle Q(a)b \rangle$ can be non-zero. An example of finite value of $\langle Q(a)b \rangle$ is given by the well known wall crossing property of the Coulomb branch contributions.

For some a and b the contribution of the cusp at $\tau \to +i\infty$ can be divergent and requires careful interpretation. In chapter 3 we study possible resolutions of such divergences by regularizing modular integrals of the form

$$L_{m,n,s} = \int_{\mathcal{F}_\infty} d\tau \wedge d\bar{\tau}\, q^m \bar{q}^n y^{-s}, \tag{1.4}$$

where f is a non-holomorphic modular form of weight $(2-s, 2-s)$. The integral (1.4) is finite for $m+n>0$ and for $m+n=0$ with $\mathrm{Re}\, s>1$, but diverges exponentially for $y=\mathrm{Im}\,\tau\to\infty$ when $m+n<0$. For a large class of such (m,n), namely when one of the two numbers is non-negative, the integral can be consistently evaluated using a, by now standard, prescription [8–10] to carry out the integral over $x=\mathrm{Re}\,\tau$ first and the integral over y second.

Our main example of operators in Donaldson-Witten (DW) theory which lead to divergent integrals of the form (1.4) with both m and n negative is

$$\int_S \{Q, \mathrm{Tr}\,\bar{\phi}\chi\}, \tag{1.5}$$

where S is a two-cycle in the underlying four-manifold and $\bar{\phi}, \chi$ are the scalar and self-dual fermion fields of DW theory. This operator was studied previously in the context of the CohFT interpretation of Witten-like indices [18], and more recently for the evaluation of Coulomb branch integrals using indefinite theta functions in [19]. Due to presence of both $m<0$ and $n<0$ in the τ-plane integrand of (1.2) the standard prescription of integrating over x first and over y second does not help to remove the infinity. Therefore we formulate a new prescription for defining such correlation functions [20], which is based on analytic continuation of the incomplete Gamma function. [1].

Topological partition function of AD3 theory.

Given Witten's remarkable application of the Seiberg-Witten solution of pure SU(2) $\mathcal{N}=2$ supersymmetric Yang-Mills theory [21] to the theory of four-manifolds, the natural question arises whether other (topologically twisted) 4d $\mathcal{N}=2$ theories can be sensitive to new (other than Seiberg-Witten) four-manifold invariants. This was the main motivation for

[1]This prescription was considered recently by Bringmann-Diamantis-Ehlen [11] in the context of inner products of weakly holomorphic modular forms (see also [12] and [13]). Such integrals have also been studied in the context of one-loop amplitudes in string theory [14–16], and in mathematics as the (Petersson) inner product for cusp forms [17].

the works such as [22]. The main conclusion of [22] was that, the topological partition functions for twisted Lagrangian 4d $N = 2$ theories, while intricate and interesting, will nevertheless be expressible in terms of the classical cohomological invariants and Seiberg-Witten (SW) invariants of a four-manifold. This narrows the search for new invariants to non-Lagrangian superconformal theories. Again, this was the motivation for [23, 24]. Those papers failed to discover new invariants, but did manage to show that the very existence of superconformal theories is related to nontrivial sum rules on the Seiberg-Witten invariants, now known as the "superconformal simple type condition".

The superconformal theory used in [23, 24] is the simplest nontrivial Argyres-Douglas theory and is denoted here as AD3 (it is sometimes also denoted as the (A_1, A_2) theory). It arises in special points of the Coulomb branch of pure SU(3) SYM [25] and in the Coulomb branch of SU(2) SYM coupled to a single hypermultiplet in the fundamental representation [26]. In chapter 4 we complete the story of [23, 24] by giving an explicit formula (4.14) for the topological partition function of the twisted AD3 theory on compact, oriented, simply-connected ($b_1(X) = 0$), four-manifolds without boundary with $b_2^+(X) > 0$, henceforth denoted by X. For manifolds with $b_2^+(X) > 1$ the general formula (4.14) simplifies to (4.109). This formula, once again, expressed in terms of the SW invariants and does not provide new four-manifold invariants.

A standard four-manifold X with $b_2^+(X) > 1$ is said to be of *Seiberg-Witten simple type* if the Seiberg-Witten invariant associated to a spin-c structure is only nonvanishing when the moduli space of solutions to the Seiberg-Witten equations is of dimension zero (for mathematical discussions see [31, 32]). Strangely enough, all known standard four-manifolds X are of Seiberg-Witten simple type. Assuming that X is of Seiberg-Witten simple type (SWST) the formula (4.109) simplifies dramatically to the following equation

$$\langle e^{p\mathcal{O}+\mathcal{O}(S)}\rangle_X^{AD3} = \mathcal{C}_2 \sum_{\lambda\in \overline{H^2(X,\mathbb{Z})+\frac{1}{2}w_2}} e^{i\pi\lambda\cdot w_2} SW(\lambda)\left[\frac{\mathfrak{B}(\mathfrak{B}-1)}{24}S^2(S\cdot\lambda)^{\mathfrak{B}-2} + (S\cdot\lambda)^{\mathfrak{B}}\right], \quad (1.6)$$

where $\mathfrak{B} = -\frac{-7\chi-11\sigma}{4}$, χ, σ, w_2 are respectively the Euler characteristic, signature and integral lift of the second Stiefel-Witney class of X, $S \subset X$ is a two-cycle associated with the surface observable $\mathcal{O}(S)$ and $\mathcal{C}_2$ is a constant factor. The right hand side of (1.6) is independent of p meaning that the 0-observable on a manifold of SWST always lead to zero correlation function. This property is in sharp contrast with the general formulae (4.14), (4.109) for those hypothetical manifolds that are not of SWST. In the absence of any compelling reason for the 0-observable to be a null-vector, we conjecture that there are in fact standard four-manifolds that are not of SWST.

Our result does not imply that topologically twisted N=2 theories can not lead to new four-manifold invariants, although it restricts the search for new invariants. Whether or not other theories lead to new invariants, the computation of these partition functions is an interesting challenge and we expect that our method to be applicable to other non-Lagrangian superconformal points that appear in theories with one dimensional Coulomb branch.

Chapter 2

Review of the field theory approach to four-manifold invariants

2.1 Twisted $N = 2$ SYM in four dimensions

Consider the Lagrangian of $N = 2$ SYM on a four-manifold X

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Im}\,\mathrm{Tr}\left[\tau_0 \int d^2\theta\, W^\alpha W_\alpha + \int d^4\theta\, \Phi^\dagger e^{2gV}\Phi\right]; \qquad \tau_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \tag{2.1}$$

Its expansion for a particular value of θ-angle can be written in the following form

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Big[&-\frac{1}{4}(F^+, F^+) + \frac{1}{2}\left(\nabla\phi^\dagger, \nabla\phi\right) + \frac{1}{2}D^2 + \frac{1}{2}D[\phi^\dagger, \phi] \\ &- i\overline{\psi}^a\hat{\nabla}\psi^a - \frac{i}{2}\epsilon_{ab}[\psi^a, \psi^b]\phi^\dagger - \frac{i}{2}\epsilon_{ab}[\overline{\psi}^a, \overline{\psi}^b]\phi\Big]\end{aligned} \tag{2.2}$$

where $(\,,\,)$ is invariant pairing on p-forms $(\alpha, \beta) = *(\alpha \wedge *\beta)$, ∇ - covariant derivative and $\hat{\nabla} := \bar{\sigma}^\mu\nabla_\mu$, $F^+ = F + \tilde{F}$. Lets fix our convensions by specifying $\nabla_\mu\psi^a_\alpha = \partial_\mu\psi^a_\alpha - i[A_\mu, \psi^a_\alpha] - i(\omega_\mu)^\beta_\alpha\psi^a_\beta$, where α, β are spinor indices and ω_μ is the spin connection compatible with veilbein e^n_μ. In the following we will be using mostly Lorentz indices $n, m, ... = 1, .., 4$.

Recall that the $N = 2$ supersymmety generators $Q^a_\alpha, \bar{Q}^a_{\dot{\alpha}}$ have the following non-zero commutation relations

$$\{Q^a_\alpha, \bar{Q}^b_{\dot{\beta}}\} = 2\varepsilon^{ab}(\sigma_\mu)_{\alpha\dot{\beta}}P^\mu \tag{2.3}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta}Z^{ab} \tag{2.4}$$

where P^μ is the momentum operator and Z^{ab} are the central charges of the theory [1]. The matrix of central charges Z^{ab} is skew-symmetric, so the only non-zero component is $Z \equiv Z^{12} = -Z^{21}$. We also have the following non-zero commutators

$$\begin{aligned}
[J^{\mu\nu}, Q^a_\alpha] &= -(\sigma^{\mu\nu})_\alpha^{\ \beta} Q^a_\beta \\
[J^{\mu\nu}, \bar{Q}^a_{\dot{\alpha}}] &= -(\sigma^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \bar{Q}^a_{\dot{\beta}} \\
[Q^a_\alpha, R] &= Q^a_\alpha \\
[\bar{Q}^a_{\dot{\alpha}}, R] &= -\bar{Q}^a_{\dot{\alpha}s}
\end{aligned} \tag{2.5}$$

where $J^{\mu\nu}$ denotes the generators of the SO(4) rotations and σ_m are the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and σ_{mn} is given by

$$\sigma_{mn} = \frac{1}{4}(\sigma_m\bar{\sigma}_n - \sigma_n\bar{\sigma}_m), \tag{2.6}$$

with $\bar{\sigma}_m$ the complex conjugate of σ_m.

Besides the gauge symmetry and general covariance the Lagrangian above possesses global bosonic symmetry with generators J^{mn}, R^{ab}, U that represent Lorentz, R-symmetry $su(2)_R$ rotations $\psi^a \to \left(e^{-\frac{i}{2}\sigma^i n_i}\right)^a_{\ b}\psi^b$ and u(1) R-transformations. Writing Poincare algebra

[1] Here $\mu = 1, ..., 4$, α and $\dot{\alpha}$ are SO(4) indices and $a, b = 1, 2$ are $SU(2)_R$ indices.

generated by J^{mn} as $\mathrm{su}(2)_- \oplus \mathrm{su}(2)_+$ fields in this Lagrangian decompose in the following representation of the bosonic subalgebra $\mathrm{su}(2)_- \oplus \mathrm{su}(2)_+ \oplus \mathrm{su}(2)_R$

$$\begin{aligned} &\text{bosons } A_m, \phi, \phi^\dagger: \qquad (2,2,1)^0 \oplus (1,1,1)^2 \oplus (1,1,1)^{-2} \\ &\text{fermions } \psi^a_\alpha, \overline{\psi}^a_{\dot\alpha}: \qquad (2,1,2)^1 \oplus (1,2,2)^{-1} \end{aligned}$$

and auxiliary boson D is a singlet. Supersymmetry generators $Q^a_\alpha, \bar{Q}^a_{\dot\alpha}$ transform in the same way as $\psi^a_\alpha, \overline{\psi}^a_{\dot\alpha}$.

The quantum theory is defined by the formal Feynmann functional integral carried over the orbits of gauge transformation in the field space

$$\langle \mathcal{O} \rangle_e = \int \mathscr{D}[A, \psi, \overline{\psi}, \phi, \phi^\dagger, D] \exp\left(-\frac{1}{g^2} \int_X vol \cdot \mathcal{L} \right) \cdot \mathcal{O}$$

This quantum average a priori depends on the veilbein e.

Due to the presence of spin connection Lorentz symmetry $\mathrm{su}(2)_- \oplus \mathrm{su}(2)_+$ is extended to a gauge symmetry. The twisted version of $N = 2$ SYM was first considered in [1] and is defined by turning on a connection (background gauge field) Ω^{ab} for the $\mathrm{su}(2)_R$ symmetry. This boils down to changing the covariant derivative on fermions (since only fermions transform non-trivially with respect to $\mathrm{su}(2)_R$) as follows

$$\hat{\nabla} \quad \to \quad \sigma^\mu \nabla^{ab}_\mu := \sigma^\mu \left(\delta^{ab} \nabla_\mu - i \Omega^{ab}_\mu \right) \tag{2.7}$$

Besides the new R-symmetry connection, $\hat{\nabla}$ contains the spin connection that consists of two components, $(\omega_\mu)^\beta_\alpha$ and $(\bar\omega_\mu)^{\dot\beta}_{\dot\alpha}$ which correspond respectively to $\mathrm{su}(2)_-$ and $\mathrm{su}(2)_+$ parts of the Lorentz symmetry. We then set the new connection Ω^{ab}_μ to be equal to the right-handed part of the spin connection, namely $\Omega^{ab}_\mu := (\bar\omega_\mu)^a_b$. This effectively means that we have modified the right-handed part of the Lorentz symmetry to be $\mathrm{su}(2)'_+ := \mathrm{su}(2)_{diag} = \mathrm{su}(2)_+ \oplus \mathrm{su}(2)_R$.

Since bosonic fields don't feel the su$(2)_R$ rotations all the twisting operation does is changes coupling of the right-haned fermions to gravity. However, the effect it makes on the $N = 2$ supersymmetry charges is that

$$\begin{array}{lll} \text{left-handed} \quad Q^a_\alpha \in (2,1,2)^1 & \rightarrow & Q_\mu \in (2,2)^1 \\ \text{right-handed} \quad \bar{Q}^a_{\dot\alpha} \in (1,2,2)^{-1} & \rightarrow & (Q_{\mu\nu}, Q); \quad Q_{\mu\nu} \in (1,3)^{-1} \text{ and } Q \in (1,1)^{-1} \end{array}$$

In other words index a that was an R-symmetry index turns into $\dot a$ spinor index with respect to the new Lorentz symmetry su$(2)_-\oplus$su$(2)_{diag}$ and certain combination of the supercharges, namely $Q := \frac{1}{2}\delta^{\dot\alpha}_a \bar{Q}^a_{\dot\alpha}$, becomes scalar with respect to the new Lorentz symmetry. Quite analogously, $\psi^{\dot a}_\alpha$ turns into a vector fermionic field ψ_μ via $\psi^{\dot a}_\alpha = (\bar\sigma^\mu)^\alpha_{\dot a}\psi_\mu$ and $\overline{\psi}^a_{\dot\alpha}$ decomposes into a self-dual 2-tensor fermion $\chi_{\mu\nu}$ and a scalar η via $\overline{\psi}^{\dot a}_{\dot\alpha} = i(\sigma^{\mu\nu})^{\dot a}_{\dot\alpha}\chi_{\mu\nu} + \frac{1}{2}\delta^{\dot a}_{\dot\alpha}\eta$, where $\sigma^{\mu\nu} = \frac{1}{4}[\sigma^\mu, \bar\sigma^\nu]$. One can combine these supercharges into three operators valued in differential forms on X: $Q \in \Omega^0(X)$, $K = K_\mu dx^\mu \in \Omega^1(X)$, and $L = L_{\mu\nu}dx^\mu \wedge dx^\nu \in \Omega^{2,+}(X)$.

Lets write down the Lagrangian for twisted theory. The only terms that change under the twist are $-i\overline{\psi}^a\hat\nabla\psi^a$; $-i\epsilon^{ab}[\psi^a,\psi^b]\phi^\dagger$; $-i\epsilon^{ab}[\overline{\psi}^a,\overline{\psi}^b]\phi$. After integrating out the auxiliary scalar field D and writing it in the new notations the Lagrangian is given by

$$\begin{aligned} \mathcal{L} = \mathrm{Tr}\Big[&\frac{1}{4}(F^+,F^+) + \frac{1}{2}(\nabla\phi,\nabla\phi^\dagger) - i\phi\eta^2 - \frac{1}{8}[\phi,\phi^\dagger]^2 \\ &+ i(\nabla\eta,\psi) + i(\nabla\wedge\psi,\chi) - \frac{i}{4}\phi(\chi,\chi) - i\phi^\dagger(\psi,\psi)\Big] \end{aligned} \tag{2.8}$$

The same redefinition of indices for supercharges leads to $\bar{Q}^a_{\dot\alpha} = \frac{1}{2}\left(\delta^a_{\dot\alpha}Q + (\sigma^{\mu\nu})^a_{\dot\alpha}Q_{\mu\nu}\right)$. Scalar fermionic symmetry $Q := \delta^{\dot\alpha}_a\bar{Q}^a_{\dot\alpha}$ is of special interest for us. Unlike $Q^a_\alpha, \bar{Q}^a_{\dot\alpha}$ in the untwisted $N = 2$ SYM, Q does not transform when we go from one chart of the covering on X to another. Thus it defines invariant differential operator, an odd vector field on the space of fields and one can formally consider integration over the orbit of its action as a part of our path integral. In coordinates this vector field has the following form

$$\delta A = i\varepsilon\psi, \quad \delta\psi = -\varepsilon D\phi, \quad \delta\chi = \varepsilon F^+, \quad \delta\phi = 0, \quad \delta\lambda = 2i\varepsilon\eta, \quad \delta\eta = \frac{1}{2}\varepsilon[\phi,\lambda]$$
$$Q = \int_X vol\,\mathrm{Tr}\left[-\psi(x)\frac{\delta}{\delta A(x)} - iD\phi(x)\frac{\delta}{\delta\psi(x)} + iF^+(x)\frac{\delta}{\delta\chi(x)} - 2\eta(x)\frac{\delta}{\delta\lambda(x)} + \frac{i}{2}[\phi,\lambda](x)\frac{\delta}{\delta\eta(x)}\right] \tag{2.9}$$

In the Hamiltonian formalism Q corresponds to a canonical transformation $\{Q_Y,\ \}$ for certain functional Q_Y that can be found from Noether theorem to be

$$Q_Y = \int_Y *J; \qquad J_\mu = \mathrm{Tr}\left[F^+_{\mu\nu}\psi^\nu - \eta D_\mu\phi - \chi_{\mu\nu}D^\nu\phi - \frac{1}{2}\psi_\mu[\phi^\dagger,\phi]\right] \tag{2.10}$$

where Y is a certain three-cycle incide X.

Using equation of motion for $\chi_{\mu\nu}$ field, namely $2(D\wedge\psi)^+ = [\phi,\chi]$ one finds

$$Q^2 = \int d^4x\, e\mathrm{Tr}\left[iD_\mu\phi(x)\frac{\delta}{\delta A_\mu(x)} + [\psi_\mu,\phi]\frac{\delta}{\delta\psi_\mu(x)} - \frac{i}{2}[\phi,\chi]\frac{\delta}{\delta\chi_{\mu\nu}} - i[\phi,\eta](x)\frac{\delta}{\delta\eta(x)}\right] \tag{2.11}$$

It is not hard to see that Q^2 is a vector field representing gauge transformation parametrized by field ϕ.

The crucial property of the twisted $N = 2$ SYM is that its energy-momentum tensor[2] is Q-exact:

$$\begin{aligned} &T_{\mu\nu} = Q(\lambda_{\mu\nu}), \qquad \text{where} \\ &\lambda_{\mu\nu} = \mathrm{Tr}\Big(F_{(\mu,\sigma}\chi_{\nu)}^{\ \sigma} - \frac{1}{4}g_{\mu\nu}F_{\sigma\tau}\chi^{\sigma\tau} + \psi_{(\mu}D_{\nu)}\phi^\dagger - \frac{1}{2}g_{\mu\nu}\psi_\sigma D^\sigma\phi^\dagger + \frac{1}{4}g_{\mu\nu}\eta[\phi,\phi^\dagger]\Big) \end{aligned} \tag{2.12}$$

Therefore metric variation of correlation functions of Q-closed operators under small deformation of background metric

$$\delta\langle\mathcal{O}_1\ldots\mathcal{O}_k\rangle = -\frac{1}{g^2}\int_X d^4xe\,\delta g^{\mu\nu}\langle Q(\lambda_{\mu\nu})\,\mathcal{O}_1\ldots\mathcal{O}_k\rangle \tag{2.13}$$

[2]Since self-duality equation for field χ is not invariant with respect to metric variation, for the calculation of energy-momentum tensor metric variation has to be accompanied with variation of field χ such that $(\delta_\chi + \delta_g)\chi = (\delta_\chi + \delta_g) * \chi$.

2.2 Topological sectors of twisted $N = 2$ SYM

Correlators local Q-closed operators $\langle \mathcal{O}_1^{(0)}(x_1)\dots\mathcal{O}_n^{(0)}(x_n)\rangle$ are independent on the choices of points x_p. For example, differential of a local operator $\mathcal{O}^{(0)}(x) = \frac{1}{2}\mathrm{Tr}\,\phi^2$ has Q-exact form

$$d\mathcal{O}^{(0)} = \mathrm{Tr}\phi D\phi = iQ(\mathrm{Tr}\phi\psi), \tag{2.14}$$

so that

$$\mathcal{O}^{(0)}(x) - \mathcal{O}^{(0)}(x') = \int_{x'}^{x} d\mathcal{O}^{(0)} = iQ\Big(\int_{x'}^{x}\mathcal{O}^{(1)}\Big) \tag{2.15}$$

with $\mathcal{O}^{(1)} := \mathrm{Tr}\phi\psi$ being a 1-form. In fact, one can proceed further and define $\mathcal{O}^{(i)}$, $i = 1, 2, 3$ starting from $\mathcal{O}^{(0)}$:

$$\begin{aligned} d\mathcal{O}^{(1)} &:= Q(\mathcal{O}^{(2)}) & \qquad \mathcal{O}^{(2)} &:= \mathrm{Tr}\left(\frac{1}{2}\psi\wedge\psi + i\phi\wedge F\right) \\ d\mathcal{O}^{(2)} &= Q(\mathcal{O}^{(3)}) & \qquad \mathcal{O}^{(3)} &:= i\mathrm{Tr}\,(\psi\wedge F) \\ d\mathcal{O}^{(3)} &= Q(\mathcal{O}^{(4)}) & \qquad \mathcal{O}^{(4)} &:= -\frac{1}{2}\mathrm{Tr}F\wedge F \end{aligned} \tag{2.16}$$

$d\mathcal{O}^{(4)} = 0$, so this sequence stops. Every $\mathcal{O}^{(k)}$ for $0 \le k \le 4$ is a k-form with ghost number equal to $4 - k$. For a closed k-cycle γ on X integral $\mathcal{O}(\gamma) = \int_\gamma \mathcal{O}^{(k)}$ is Q-closed since $Q(\mathcal{O}(\gamma)) = -i\int_\gamma d\mathcal{O}^{(k-1)} = 0$. Also $\mathcal{O}(\gamma)$ changes by Q-exact term when we change γ by a boundary $\mathcal{O}(\gamma + \partial\beta) = \mathcal{O}(\gamma) + iQ(\int_\beta \mathcal{O}^{(k-1)})$ and thus each homology k-cycle defines BRST-closed observable $\mathcal{O}(\gamma)$.

An alternative way to define the observables $\mathcal{O}^{(i)}$ is to recall that the supersymmetry generators of the twisted theory belong to $(\mathbf{1},\mathbf{1})^{+1} \oplus (\mathbf{2},\mathbf{2})^{-1} \oplus (\mathbf{1},\mathbf{3})^{+1}$ representation of $\mathrm{SU}(2)'_+\times\mathrm{SU}(2)_-\times\mathrm{U}(1)_\mathrm{R}$ and the second term $(\mathbf{2},\mathbf{2})^{-1}$ corresponds to the one-form operator K that satisfies $\{Q, K\} = d$. Such K provides a canonical solution to the following descent equations

$$Q(\mathcal{O}^{(i+1)}) = d\mathcal{O}^{(i)}, \qquad i = 0, \dots, 3, \tag{2.17}$$

by setting $\mathcal{O}^{(i)} = K^i \mathcal{O}^{(0)}$ [7]. Integration of these operators over i-cycles then gives another representation of topological observables $\mathcal{O}(\gamma)$.

The weak coupling regime

Yet another important property of the twisted $N = 2$ SYM is that the variation of the coupling constant g^2 of the twisted $N = 2$ SYM action is Q-exact:

$$\delta \int_X vol \cdot \mathcal{L} = Q\Psi; \qquad \Psi := \int_X vol \cdot \mathrm{Tr}\left[\frac{1}{4}(F^+, \chi) + \frac{1}{2}\left(\psi, D\phi^\dagger\right) - \frac{1}{4}\eta[\phi, \phi^\dagger]\right] \tag{2.18}$$

Variation of $\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle$ with Q-closed $\mathcal{O}$'s with respect to g^2 vanishes. Therefore, we can extract such correlation functions from the classical/weak coupling limit $g^2 \to +0$ in which the path integral is dominated by minima of the action. Minimisation of the first term $\frac{1}{4}(F^+, F^+)$ reduces us to the (anti-)instanton equation $F_+ = 0$.

If our manifold admits $F_+ = 0$ solutions, the instantons have moduli space $\mathcal{M}_{\mathrm{ASD}} = \left\{[A] \in \text{space of gauge connections/group of all gauge transformations} \,|\, F_+(A) = 0\right\}$. Tangent space $T_A\mathcal{M}_{\mathrm{ASD}}$ corresponds to gauge equivalence classes of deformations $A \to A + \delta A$ that preserve the anti-self-dual equation

$$(1 + *)D\delta A = 0 \tag{2.19}$$

In the case of a generic point (that is not invariant under any gauge transformation) in the instanton moduli space the formal dimension of tangent space at this point is equal

$$d_{\mathcal{M}} := \dim \mathcal{M}_{\mathrm{ASD}} = 8k - 3(b_2^+ - b_1 + 1) \tag{2.20}$$

where $k = \frac{1}{8\pi} \int_X \mathrm{Tr}\, F^{\wedge 2}$ is instanton number (the second Chern class of the principal gauge bundle), b_1 is the first Betti number and b_2^+ is the second self-dual Betti number.

Index theorem implies that the number of ψ zero modes minus number of (η, χ) zero modes in the instanton background is equal $d_{\mathcal{M}}$. Thus the path integral measure has definite but non-zero charge under the $U(1)_R$ R-symmetry

$$\mathscr{D}[A, \psi, \chi, ...] = d^{d_{\mathcal{M}}} a d^{n_\psi} \psi_0 d^{n_\eta} \eta_0 d^{n_\chi} \chi_0 \cdot \mathscr{D}[A', \psi', \chi', ...]; \qquad n_\psi - n_\eta - n_\chi = d_{\mathcal{M}} \quad (2.21)$$

where $a, \psi_0, \eta_0, \chi_0$ represent bosonic and fermionic zero modes while A', ψ', etc represent topologically trivial field configurations (non-zero modes). Only correlation functions with operator insertions of total ghost number (charge under the $U(1)_R$) equal $d_{\mathcal{M}}$ can be non-vanishing, the correlation functions total ghost number other than $d_{\mathcal{M}}$ vanish due to fermionic zero modes. In particular, unless $\mathcal{M}_{\text{ASD}}$ consist of discrete isolated points the partition function is zero. Otherwise, if $d_{\mathcal{M}} = 0$ the partition function provides field theoretic definition of one of the four-manifold invariants originally introduced by Donaldson [31]. Integrating out the non-zero modes the path integral measure $\mathscr{D}[A, \psi, \chi, ...]$ reduces to just $d^{d_{\mathcal{M}}}[a, \psi_0]$. The functional integral is Gaussian in the weak coupling limit and given by the ratio

$$\frac{\text{Pfaff}\,(D_F)}{\sqrt{\det\,\Delta_B}} \quad (2.22)$$

of Pfaffian of fermionic modes (a section of Pfaffian line bundle over the space of gauge equivalence classes of zero modes) and determinant of the bosonic ones. Due to the scalar Q-symmetry between bosons and fermions they cancel each other up to a sign. Donaldson proved orientability of the moduli space and therefore triviality of the Pfaffian line bundle. Thus one can consistently integrate this ratio over $\mathcal{M}_{\text{ASD}}$ and fix the sign ambiguity by picking up +1 for some instanton solution.

Cancelation of non-zero modes reduces the path integral to a finite dimensional integral over $\mathcal{M}_{\text{ASD}}$ that can be viewed as integral of a certain forms on $\mathcal{M}_{\text{ASD}}$.

$$\langle \prod_{i=1}^{r} \int_{\gamma_i} \mathcal{O}^{(k_i)} \rangle = \int_{\Pi T \mathcal{M}_{\text{ASD}}} d^{d_{\mathcal{M}}}[a, \psi_0] \cdot \Phi_{i_1 \ldots i_{d_{\mathcal{M}}}} \psi^{i_1} \ldots \psi^{d_{\mathcal{M}}} = \int_{\mathcal{M}_{\text{ASD}}} \Phi \quad (2.23)$$

Note that χ, $\phi^\dagger$ and η are absent in topological operators $\mathcal{O}^{(k_i)}$, so only ψ zero modes can appear in (2.23). For the same reason there are no Wick-theorem contractions between different factors in $\prod_{i=1}^r \int_{\gamma_i} \mathcal{O}^{(k_i)}$, so at least in perturbation theory, the correlation functions factorize as

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \cdot \langle \mathcal{O}_2 \rangle \tag{2.24}$$

for any Q-closed $\mathcal{O}_1$ and $\mathcal{O}_2$ that are made of $\mathcal{O}^{(i)}$'s. This property is violated by non-perturbative effects as there appear contact terms of the surface observables [7, 37]. Gaussian integration over fixed zero mode configurations leads to the following averages

$$\begin{aligned} &\langle \phi^A(x) \rangle' \overset{g^2\to 0}{\to} -i \int_{\mathcal{M}_{\text{ASD}}} \sqrt{g} \left(\frac{1}{\nabla_\mu \nabla^\mu} \right)^{AB} (x, y) \, [\psi_{0,\alpha}, \psi_0^\alpha]_B \, (y) \\ &\langle \psi \rangle' \overset{g^2\to 0}{\to} \psi_0 \qquad\quad \langle F \rangle' \overset{g^2\to 0}{\to} F_{\text{inst}}(a) \end{aligned} \tag{2.25}$$

This gives field theory version of the Donaldson map $\omega_D : H_k(X) \to H^{4-k}(\mathcal{M}_{\text{ASD}})$ [31] given by

$$\omega_D(\gamma) = \langle \mathcal{O}(\gamma) \rangle_{\text{twisted } N\,=\,2 \text{ SYM}} \tag{2.26}$$

In particular, one has $\gamma = x$; $\omega_D(x) = \frac{1}{2}\text{Tr}\langle\phi(x)\rangle^2$, $\omega_D(\gamma) = \int_\gamma \text{Tr}\langle\phi(x)\rangle \wedge \psi_0$, $\gamma = S$; $\omega_D(S) = \int_S \text{Tr}\left(\frac{1}{2}\psi_0 \wedge \psi_0 + i\langle\phi(P)\rangle \wedge F_{\text{inst}}(a)\right)$.

Given a gauge bundle such that $d_{\mathcal{M}}$ is positive, choose homology cycles $\gamma_1, ..., \gamma_r$ of dimensions $k_1, ..., k_r$ such that $d_{\mathcal{M}} = \sum_{k=1}^r (4 - k_r)$. Then correlation functions of periods of $\mathcal{O}^{(k_r)}$ over these cycles $\langle \prod_{i=1}^r \int_{\gamma_i} \mathcal{O}^{(k_i)} \rangle$ are given by intersection numbers on $\mathcal{M}_{\text{ASD}}$

$$P_D((\gamma_0)^l (\gamma_2)^k) := \int_{\mathcal{M}_{\text{ASD}}} \omega_D(\gamma_0)^l \omega_D(\gamma_2)^k \tag{2.27}$$

Such observables in Donaldson-Witten theory match the mathematically defined Donaldson polynomials [1, 31].

2.3 Low energy effective theory of $\mathcal{N} = 2$ SYM

The approach based on the full UV theory sketched in the previous section is rather formal and is not well suited for explicit calculations. A much more powerful approach based on the low energy effective description became available after the Seiberg-Witten solution [21] of $\mathcal{N} = 2$ gauge theories was discovered. This section reviews several things used in the IR representation of the Donaldson-Witten theory: low energy models of abelian $\mathcal{N} = 2$ vector multiplets and their topologically twisted versions, as well as the Seiberg-Witten geometry [21] describing the vacua of the non-abelian $\mathcal{N} = 2$ SYM with SU(2) gauge group.

Aspects of general low energy models of abelian $\mathcal{N} = 2$ vector multiplets

- An abelian vector multiplet consists of a gauge field A, a pair of (chiral, anti-chiral) spinors $(\psi, \bar{\psi})$, a complex scalar Higgs field ϕ (valued in the complexification of the Lie algebra), and an auxiliary scalar field D_{ij} (symmetric in i and j). possible matter representations. These can be combined into a single $\mathcal{N} = 2$ chiral superfield $\Psi(x, \theta, \tilde{\theta})$

- The supersymmetry algebra of the theory contains a central charge $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ where Γ is the lattice of electric and magnetic charges

$$Z(n_{\mathrm{e}}, n_{\mathrm{m}}) = n_{\mathrm{e}} a + n_{\mathrm{m}} a_D, \tag{2.28}$$

  where $(n_{\mathrm{e}}, n_{\mathrm{m}}) \in \Gamma$ is the pair of electric-magnetic charges, and the pair $(a, a_D) \in \mathbb{C}^2$ are the central charges for a unit electric or magnetic charge. The central charge determines the mass of BPS states, $m_{\mathrm{BPS}} = |Z|$.

- A most general two-derivative action respecting the $N = 2$ supersymmetry of the vector multiplet is given by a single holomorphic function $\mathcal{F}$

$$S = \int_X d^4x \sqrt{g} \int d^2\theta d^2\tilde{\theta}\, \mathcal{F}(\Psi(x, \theta, \tilde{\theta})) \tag{2.29}$$

- Given the function $\mathcal{F}$ the parameters a and a_D are related as

$$a_D = \frac{\partial \mathcal{F}(a)}{\partial a}, \tag{2.30}$$

- The quadratic term in the expansion of the $\mathcal{F}$ function determines the effective coupling constant $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2} \in \mathbb{H}$,

$$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}. \tag{2.31}$$

  where θ is the angle, g the Yang-Mills coupling and $\mathbb{H}$ is the complex upper half-plane.

- In case of multiple abelian vector multiplets $\Psi^i(x, \theta, \tilde{\theta})$, $i = 1, \ldots, r$ the story generalizes as

$$\begin{aligned}
S &= \int_X d^4x \sqrt{g} \int d^2\theta d^2\tilde{\theta}\, \mathcal{F}(\Psi^1(x, \theta, \tilde{\theta}), \ldots, \Psi^1(x, \theta, \tilde{\theta})) \\
Z &= n_{\mathrm{e,i}} a^i + n^i_{\mathrm{m}} a_{D,i} \\
a_{D,i} &= \frac{\partial \mathcal{F}(a)}{\partial a^i} \\
\tau_{ij} &= \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}
\end{aligned} \tag{2.32}$$

Topologically twisted low energy effective theory

The field content of the low energy topological twisted theory is a one-form gauge potential A, a complex scalar a, together with anti-commuting (Grassmann valued) self-dual two-form χ, one-form ψ and zero-form η. The auxiliary fields of the non-twisted theory combine to a self-dual two-form D. The action of the BRST operator Q on these fields is given by

$$
\begin{aligned}
&[Q, A] = \psi, && [Q, a] = 0, && [Q, \bar{a}] = \sqrt{2} i\eta, \\
&[Q, D] = (d\psi)_+, && \{Q, \eta\} = 0, && \{Q, \psi\} = 4\sqrt{2}\, da, \\
&\{Q, \chi\} = i(F_+ - D).
\end{aligned}
\tag{2.33}
$$

For later use, it will be useful to express Q as a derivative in field space,

$$
Q = \int_X \left(\psi \frac{\partial}{\partial A} + (d\psi)_+ \frac{\partial}{\partial D} + 4\sqrt{2} da \frac{\partial}{\partial \psi} + \sqrt{2} i\eta \frac{\partial}{\partial \bar{a}} + i(F_+ - D) \frac{\partial}{\partial \chi} \right). \tag{2.34}
$$

The low energy Lagrangian of the Donaldson-Witten theory is given by [7]

$$
\begin{aligned}
\mathcal{L} =& \frac{i}{16\pi}(\bar{\tau} F_+ \wedge F_+ + \tau F_- \wedge F_-) + \frac{y}{8\pi} da \wedge * d\bar{a} - \frac{y}{8\pi} D \wedge * D \\
&- \frac{1}{16\pi} \tau \psi \wedge * d\eta + \frac{1}{16\pi} \bar{\tau} \eta \wedge d * \psi + \frac{1}{8\pi} \tau \psi \wedge d\chi - \frac{1}{8\pi} \bar{\tau} \chi \wedge d\psi \\
&+ \frac{\sqrt{2} i}{16\pi} \frac{d\bar{\tau}}{d\bar{a}} \eta \chi \wedge (F_+ + D) - \frac{\sqrt{2} i}{2^7 \pi} \frac{d\tau}{da} \psi \wedge \psi \wedge (F_- + D) \\
&+ \frac{i}{3 \cdot 2^{11}} \frac{d^2\tau}{da^2} \psi \wedge \psi \wedge \psi \wedge \psi - \frac{\sqrt{2} i}{3 \cdot 2^5 \pi} \left\{ Q, \chi_{\mu\nu} \chi^{\nu\lambda} \chi_\lambda^{\ \mu} \right\} \sqrt{g}\, d^4 x.
\end{aligned}
\tag{2.35}
$$

Seiberg-Witten solution in terms of modular forms of $\Gamma^0(4)$

The Coulomb branch of vacua $\mathcal{B}$ is parametrized by a single order parameter,

$$
u = \frac{1}{16\pi^2} \left\langle \operatorname{Tr} \phi^2 \right\rangle, \tag{2.36}
$$

where the trace is taken over the adjoint representation of the gauge group. The renormalization group flow relates the Coulomb branch parameter u and the effective coupling constant τ.

$$
\frac{u(\tau)}{\Lambda^2} = \frac{\vartheta_2^4 + \vartheta_3^4}{2\vartheta_2^2 \vartheta_3^2} = \frac{1}{8} q^{-\frac{1}{4}} + \frac{5}{2} q^{\frac{1}{4}} - \frac{31}{4} q^{\frac{3}{4}} + O(q^{\frac{5}{4}}), \tag{2.37}
$$

where $q = e^{2\pi i\tau}$ and $\vartheta_i(\tau) = \vartheta_j(\tau, 0)$, where $\vartheta_j(\tau, v)$ are the standard Jacobi theta functions (see appendix A). At the cusps $\tau \to 0$ (respectively $\tau \to 2$) a monopole (respectively a dyon) becomes massless, and the effective theory breaks down since new additional degrees of freedom need to be taken into account.

The Seiberg-Witten curve of the pure SU(2) $N = 2$ SYM is an elliptic curve for $\Gamma^0(4) \subset \mathrm{SL}(2, \mathbb{Z})$ [21]. It implies that the τ-plane can be identified with a fundamental domain of $\Gamma^0(4)$ in the upper-half plane $\mathbb{H}$, which we choose as the images of the familiar key-hole fundamental domain of $\mathrm{SL}(2, \mathbb{Z})$ under $\tau \mapsto \tau + 1$, $\tau + 2$, $\tau + 3$, $\tau + 4$, $-1/\tau$ and $2 - 1/\tau$ displayed in Figure 2.1.

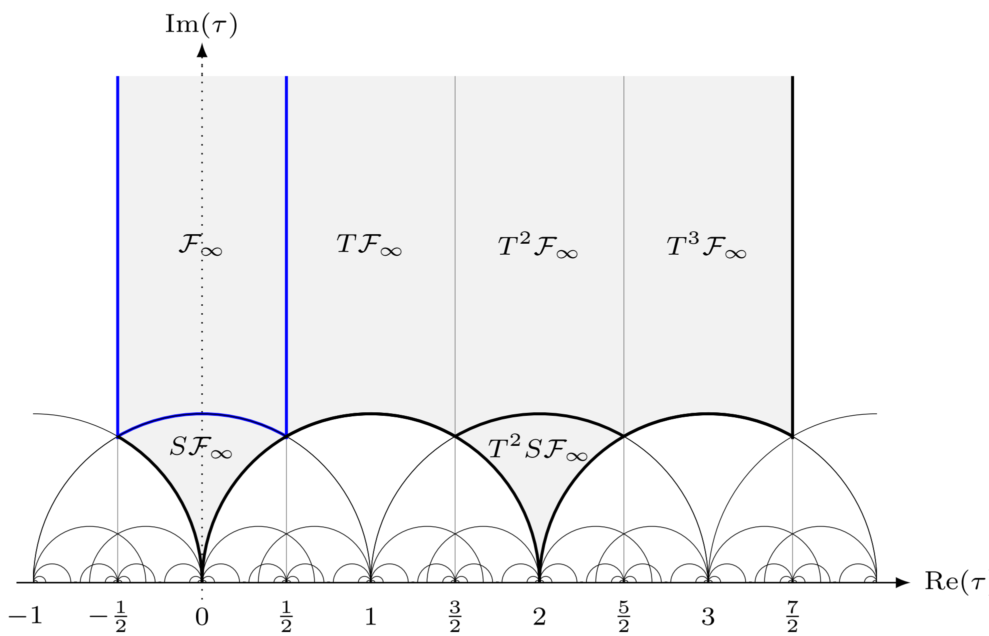

Figure 2.1: The effective coupling constant plane for pure SU(2) $N = 2$ SYM as the fundamental domain of the duality group $\mathcal{D} = \Gamma^0(4)$ (shaded) - it consists of six copies of the fundamental domain for $\mathrm{SL}(2, \mathbb{Z})$ denoted $\mathcal{F}_\infty$ (bounded by blue).

2.4 Basics of four-manifold topology

In dimensions higher than 4, fixing the topology will fix the differential structure up to finitely many choices.

Four-manifolds with $b_2^+ = 1$

Let X be a smooth, simply connected, compact four-manifold without boundary. Its basic topological numbers are its Euler character $\chi(X) = 2-2b_1(X)+b_2(X)$ and signature $\sigma(X) = b_2^+(X) - b_2^-(X)$, where $b_1(X) = \dim(H_1(X,\mathbb{R}))$ and $b_2^\pm(X) = \dim(H_2(X,\mathbb{R})^\pm)$. We will omit the dependence on X unless a confusion may arise. We will restrict in the following to four-manifolds with $b_2^+ = 1$, since the u-plane integral only contributes for this class of four-manifolds. A four-manifold X with $b_2^+ = 1$ admits an almost complex structure, since any simply connected four-manifold with b_2^+ odd does [34]. We denote the canonical class of X by $K_X \in H^2(X,Z)$, which equals the second Stiefel-Whitney class $w_2(X)$ modulo $H^2(X,2\mathbb{Z})$.

The intersection form on the middle cohomology provides a natural bilinear form $(\,,\,) : H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \to \mathbb{R}$ that pairs degree two co-cycles,

$$(\lambda_1,\lambda_2) := \int_X \lambda_1 \wedge \lambda_2, \tag{2.38}$$

and whose restriction to $H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z})$ is an integral bilinear form with signature $(1, b_2 - 1)$. The bilinear form provides the quadratic form $Q(\lambda) := (\lambda,\lambda) \equiv \lambda^2$, which can be brought to a simple standard form [34, Section 1.1.3]. We denote the period point by ω, i.e. the harmonic two-form, satisfying

$$*\,\omega = \omega \in H^2(X,\mathbb{R}), \qquad \omega^2 = 1. \tag{2.39}$$

with $*$ the Hodge $*$-operation. Using the period point, we can decompose elements $\lambda \in H^2(X)$ to its self-dual and anti-self-dual components: $\lambda_+ = \omega(\lambda,\omega)$ and $\lambda_- = \lambda - \lambda_+$. the anti-self-dual part of λ. For later use, we mention that the canonical class is a characteristic vector of $H^2(X,\mathbb{Z})$ and satisfies

$$K_X^2 = \sigma + 8. \tag{2.40}$$

2.5 Coulomb branch contribution

The path integral Z_u in the topologically twisted Seiberg-Witten theory on a simply connected four-manifold[3] reduces to a finite dimensional integral over the zero modes λ, χ, η, a_0, $\bar{a}_0$, b [7], i.e. the fermionic and bosonic non-zero modes cancel due to the scalar Q-symmetry:

$$\begin{aligned} &F = 4\pi\lambda + dA, \qquad D = b + D', \qquad a = a_0 + a', \qquad \bar{a} = \bar{a}_0 + \bar{a}' \\ &\chi = \chi_0 + \chi', \qquad \eta = \eta_0 + \eta' \end{aligned} \tag{2.41}$$

Thus the partition function of such theory boils down to a finite dimensional integral over the Coulomb branch

$$\begin{aligned} Z_u^\omega &= \int_{\mathcal{B}} da d\bar{a} \int d\eta d\chi \int_{\mathbb{R}^{b_2^+}} db\, A(u)^\chi\, B(u)^\sigma\, e^{U+S^2T(a)}\, \Psi^\omega_{\lambda_0,\xi}, \\ \Psi^\omega_{\lambda_0,\xi} &= \sum_{\mathrm{U(1)\ fluxes}\lambda} e^{2i\pi\lambda^i\cdot\xi_i} e^{-S_0+\mathcal{O}(S)} \end{aligned} \tag{2.42}$$

where S_0 and $\mathcal{O}(S)$ are the action (2.35) and two-observable restricted to zero modes:

$$\begin{aligned} S_0 &= i\pi\bar{\tau}_{ij}(\lambda_+^i,\lambda_+^j) + i\pi\tau_{ij}(\lambda_-^i,\lambda_-^j) + \frac{1}{8\pi}\mathrm{Im}\,\tau_{ij}b^ib^j - \frac{i\sqrt{2}}{16}\frac{\partial\bar{\tau}_{ij}}{\partial\bar{a}^k}\eta^i\chi^j(b^k + 4\pi\lambda_+^k) \\ \mathcal{O}(S) &= -iV_i(S,\lambda_-^i) - \frac{i}{4\pi}V_i(S,\omega)b^i \end{aligned} \tag{2.43}$$

[3]Simply connectedness implies that the manifold does not admit zero modes of the ψ field. This assumption significantly simplifies the number of contact terms that one has and, on the other hand, is usually made in the considerations about four-manifolds.

The measure factors

$$A := \alpha \left(\det_{ij} \frac{\partial u^i}{\partial a^j} \right)^{1/2} \qquad B := \beta \Delta^{1/8} \tag{2.44}$$

correspond to the terms in the low energy effective action on the Coulomb branch describing the coupling of the $U(1)$ vectormultiplet to the Euler character χ and the signature σ. Here $\Delta = \prod_s (u - u_s)$ is a holomorphic function with first order zeroes at the discriminant locus $\{u_s\}$ where a hypermultiplet becomes massless. [4] The factors α, β are independent of u but can depend on the theory, the scale Λ, and the masses.

The sum over U(1) fluxes λ goes over the lattice $\Gamma := H^2(X; \Lambda_r)/\mathrm{Tors}$, where Λ_r is the root lattice of the gauge group and Tors is the torsion subgroup of $H^2(X; \Lambda_r)$. The 't Hooft flux and the phase $(-1)^{\lambda_i \cdot \xi^i}$ will be specified in details in the case rank one case below.

The Coulomb branch integral (2.42) vanishes for $b_2^+ > 1$ due to fermionic integrations - there is only one η for any value of b_2^+ that enters the integral only via the last term in S_0.

The Coulomb branch integrand depends on a choice of Riemannian metric on X for $b_2^+ = 1$, but the dependence only enters through the cohomology class of a self-dual two-form $\omega \in H^2(X, \mathbb{R})$, so $\omega = *\omega$. We can normalize it such that $\int_X \omega \wedge \omega = 1$. When $b_2^+ = 1$ there is a Lorentzian signature on the quadratic space $H^2(X, \mathbb{R})$, and we must choose a component of the lightcone, which we can call the "forward light cone", in order to specify ω uniquely. Such ω is sometimes calles a *period point*.

Rank one u-plane integral. It is referred to informally as the "u-plane integral." For additional background and discussion of the u-plane integral see [19, 36, 38].

The original discussion of [7] applied just to SU(2) Yang-Mills coupled to $N_f \leq 4$ fundamental hypermultiplets or one adjoint hypermultiplet, but in fact the measure makes sense for any one-dimensional Coulomb branch.[5] Although the integral is, conceptually, best written as an integral over the u-plane, the path integral derivation leads more naturally

[4] One should, in general, distinguish the "physical discriminant" from the "mathematical discriminant".

[5] What is far less obvious is whether the measure is single-valued on the u-plane and whether the integral over the u-plane is well-defined for other SW families of curves.

to an integral over a special coordinate a so that it becomes

$$
\begin{aligned}
Z_u^{\omega} &= \int da d\bar{a}\, A^{\chi} B^{\sigma}\, e^{2pu+S^2 T(a)} \Psi \\
\Psi &:= \sum_{\lambda \in \lambda_0 + \Gamma} e^{2\pi \mathrm{i} \lambda \cdot \xi} \mathcal{N}_{\lambda} \\
\mathcal{N}_{\lambda} &:= \frac{d\bar{\tau}}{d\bar{a}} \frac{e^{\frac{S_+^2}{8\pi y}\left(\frac{du}{da}\right)^2}}{\sqrt{y}} e^{-i\pi\bar{\tau}\lambda_+^2 - i\pi\tau\lambda_-^2 - i\frac{du}{da}(S,\lambda_-)} \left[\lambda_+ + \frac{i}{4\pi y}\frac{du}{da} S_+\right]
\end{aligned}
\tag{2.45}
$$

Let us explain the main ingredients of (2.45)

The sum Ψ is, essentially, the classical partition function of the $U(1)$ gauge field on the four-manifold. We think of Γ as embedded in the quadratic vector space $H^2(X;\mathbb{R})$. We have introduced a shift λ_0 and a phase ξ. In the case of SU(2), $N_f = 0$ we have $\lambda_0 = \frac{1}{2}\overline{w_2(P)}$ where P is a principal SO(3) bundle, and $\xi = \frac{1}{2}\overline{w_2(X)}$ and the overline denotes an integral lift. When $\tau \to \infty$ as $u \to \infty$, 2ξ must be a characteristic vector on Γ for the measure to be well-defined. In this case we can write $\lambda = v + \lambda_0$, $v \in \Gamma$ and the phase

$$
e^{2\pi \mathrm{i}(\lambda - \lambda_0)\cdot\xi} = (-1)^{v \cdot w_2(X)}
\tag{2.46}
$$

In the case of SU(2) SYM with $N_f > 0$ hypermultiplets we must take $w_2(P) = w_2(X)$ so we should take $2\lambda_0 = 2\xi_0$ to be an integral lift of $w_2(X)$. Note that all anti-holomorphic dependence and all metric dependence of the integrand is subsumed in the expression $\mathcal{N}_{\lambda}$.

p is a fugacity conjugate to the insertion of the 0-observable $\mathcal{O} = u$ in the twisted partition function. $S \in H_2(X;\mathbb{Z})$ is a homology class and determines a canonical 2-observable $\mathcal{O}(S) := \int_S K^2 u$, via the descent formalism [7]. Here K is a one-form supercharge such that $[K, Q] = d$. The expression Z_u^{ω} should be viewed as a formal power series in p, S and it is the contribution of the Coulomb branch to the correlation function $\langle e^{p\mathcal{O}+\mathcal{O}(S)} \rangle$ in the twisted theory on X. a is a special coordinate suitable to a duality frame at $u \to \infty$. It is the period of the Seiberg-Witten differential on a cycle that is invariant (up to a sign) under the path

$u \to e^{2\pi i} u$. Once we choose a B-cycle we have $\tau = x + iy$, decomposed in terms of real and imaginary parts. In the case of SU(2) with $N_f < 4$ this is a frame in which $y = \operatorname{Im}\tau \to +\infty$.

$T(a)$ depends on the choice of duality frame and is known as a "contact term" it is given by

$$T(a) = -\frac{1}{24} E_2(\tau) \left(\frac{du}{da}\right)^2 + H(u) \tag{2.47}$$

It is claimed in [7] that $H(u) = u/3$ for the case of SU(2) SYM with $N_f < 4$. For systematic treatments of such contact terms in twisted 4d $N = 2$ theories see [35–37].

2.6 Wall crossing and Seiberg-Witten contributions

A systematic derivation of the Witten conjecture of four-manifold theory (equation (2.14) of [6]) relating the Donaldson and Seiberg-Witten invariants was presented in [7]. It involves an integral over the Coulomb branch of the SU(2) $N_f = 0$ theory. [6]

The topological partition function of the twisted theory on X

$$Z = \langle e^{2pu + \mathcal{O}(S)} \rangle, \tag{2.48}$$

where $\mathcal{O}(S) = \int_S K^2 u$ is the canonical 2-observable associated to S, is a sum of the u-plane integral with contributions that guarantee that the contribution of the vacua near $u \cong u_s$:

$$Z = Z_u + Z_{SW}, \tag{2.49}$$

where

$$Z_{SW} = \sum_s Z(u_s) \tag{2.50}$$

and the sum over s is a sum over the discriminant locus of the family of Seiberg-Witten curves. When the family of elliptic curves in a neighborhood of u_s is of Kodaira type I_1

[6]Mathematically rigorous proofs of the Witten conjecture have been given in [47] for complex algebraic manifolds and in [48] for all standard four-manifolds of Seiberg-Witten simple type.

(i.e., the discriminant has a first order zero at $u = u_s$ while the Weierstrass invariants g_2, g_3 are nonzero at u_s) the method used in [7] can be applied.

$$Z_{\mathrm{SW},s} = \mathrm{const}\cdot e^{i\phi_s} \sum_{\lambda \in H^2_{\mathbb{Z}}+\xi} e^{2\pi i\lambda\lambda_0} \mathrm{SW}(\lambda) \oint_{a_s=0} \frac{da_s}{a_s} a_s^{-\frac{1}{2}\lambda^2+\frac{1}{8}(2\chi+3\sigma)} C(u)^{\frac{\lambda^2}{2}} P(u)^{\frac{\sigma}{8}} L(u)^{\chi_h} e^{2pu+T_s S^2 - i\frac{du}{da_s}(S,\lambda)} \tag{2.51}$$

The ± 1 jumps of SW invariants for $b_2^+ = 1$ occur at ω such that there exists $\lambda \in H^2_{\mathbb{Z}} + \frac{1}{2}w^X$ satisfying $(\lambda, \omega) = 0$, $\lambda^2 < 0$. These are the same ω for which $Z_u^{\mathrm{disc.}}$ jumps by the following amount

$$\Delta Z_u^{\mathrm{disc}} = e^{i\phi_s} e^{\pi i\lambda w^X} (\pm 1) \oint_{q_s=0} \frac{dq_s}{q_s} q_s^{-\frac{1}{2}\lambda^2+\frac{\sigma}{8}+1} (q_s^{-1}\Delta)^{\frac{\sigma}{8}} \frac{du}{dq_s} \Big(\frac{da_s}{du}\Big)^{\frac{\sigma}{2}-1} e^{2pu+T_s S^2 - i\frac{du}{da_s}(S,\lambda)} \tag{2.52}$$

Therefore, for the functions C, P, L we obtain

$$\begin{aligned} C(u)\, a_s^{-1} &= q_s^{-1} \\ P(u)\cdot a_s &= q_s \Big(\frac{da_s}{du}\Big)^4 (q_s^{-1}\Delta) \\ L(u)\cdot \frac{da_s}{a_s} a_s &= \frac{dq_s}{q_s} q_s \frac{du}{dq_s} \Big(\frac{da_s}{du}\Big)^{-1} \end{aligned} \quad \Rightarrow \quad \begin{aligned} C(u) &= \frac{a_s}{q_s} \\ P(u) &= \frac{q_s}{a_s} \Big(\frac{da_s}{du}\Big)^4 (q_s^{-1}\Delta) \\ L(u) &= \Big(\frac{du}{da_s}\Big)^2 \end{aligned} \tag{2.53}$$

As a result the Seiberg-Witten contribution associated with singularity $u = u_s$ is given by

$$\begin{aligned} Z(u_s) = \alpha^\chi \beta^\sigma\, e^{2\pi \mathrm{i}({\lambda_0}^2 - \xi_s\cdot\lambda_{0,s})}\, e^{i\varphi_s} \sum_{\lambda \in \lambda_{0,s}+\Gamma} e^{2\pi \mathrm{i}\lambda\cdot\xi_s} (-1)^{n(\lambda)} \mathrm{SW}(\lambda)\cdot \\ \left[\left(\frac{a_s}{q_s}\right)^{\chi_h - 1} \frac{du}{dq_s} \left(\frac{\Delta}{q_s}\right)^{\sigma/8} \left(\frac{da_s}{du}\right)^{1-\chi/2} e^{2pu+S^2 T_s(a_s) - \mathrm{i}\lambda\cdot S\,\frac{du}{da_s}}\, q_s^{-n(\lambda)} \right]_{q_s^0} \end{aligned} \tag{2.54}$$

Here $\lambda_{0,s}$ and ξ_s are the theta characteristics resulting from the duality transformation applied to Ψ in the neighborhood of u_s. Similarly, $e^{i\varphi_s}$ is a root of unity arising from the multiplier system in that duality transformation. The expression only makes sense for $\lambda_{0,s} = \frac{1}{2}\overline{w_2(X)}$ so that the sum on λ can be interpreted as a sum over the characteristic class of spin-c structures on X. [7] Then SW(λ) is the corresponding SW invariant associated with the SW moduli space of dimension $2n(\lambda)$ where

$$n(\lambda) = \frac{1}{2}\lambda^2 - \frac{\sigma}{8} - \chi_h \tag{2.55}$$

and $\chi_h := (\chi + \sigma)/4$. (For a complex surface χ_h is the holomorphic Euler characteristic.) The special coordinate a_s vanishes at u_s and the coordinate $q_s = e^{2\pi i \tau_s} \to 0$ as $u \to u_s$. The contact term $T_s(a_s)$ is obtained from $T(a)$ by duality transformation.

[7] (2λ) is the characteristic class of the spin-c structure, modulo torsion.

Chapter 3

Divergent Q-exact operators on the Coulomb branch via modular integrals

The Coulomb branch integral is quite subtle and requires careful definition. The integrand is typically very singular at points $u = u_s$ in the u-plane corresponding to zeroes of the discriminant, as well as at $u = \infty$. The standard method (proposed in [7]) of defining it is to reformulate it as an integral over the values of effective coupling constant $\tau = x + iy$ valued in the fundamental domain $\mathbb{H}/\mathcal{D}$ of the duality group $\mathcal{D}$ of the theory. The resulting integral can be represented as a sum of modular integrals of the form

$$L_{m,n,s} = \int_{\mathcal{F}_\infty} d\tau \wedge d\bar{\tau}\, q^m \bar{q}^n y^{-s}, \tag{3.1}$$

where m, n, s are real numbers such that $m - n \in \mathbb{Z}$. For a large class of such (m, n), namely when one of the two numbers is non-negative, the integral can be defined using a, by now standard, prescription [9, 10, 15] to carry out the integral over $x = \operatorname{Re}\tau$ first and the integral over y second. Under this definition $L_{m,n,s}$ is finite for $m + n > 0$ and $s \in \mathbb{R}$ or $m + n = 0$ with $s > 1$, but diverges exponentially for $y = \operatorname{Im}\tau \to \infty$ when $m + n < 0$.

Another prescription considered recently by Bringmann-Diamantis-Ehlen [11] in the context of inner products of weakly holomorphic modular forms (see also [12] and [13]), employs analytic continuation of the incomplete Gamma function[1]. In this chapter we review and compare the first and second definitions following [38].

3.1 The integrand on the fundamental domain

Making use of Matone's formula [33]

$$\frac{du}{d\tau} = \frac{4\pi}{i}(u^2-1)\left(\frac{da}{du}\right)^2 \tag{3.2}$$

together with identities (2.37) and

$$\frac{da}{du}(\tau) = \frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\Lambda}, \tag{3.3}$$

one arrives at the following expression for the Coulomb branch integral for pure SU(2) $N = 2$ theory:

$$Z^{\omega}_{\lambda_0} = \int_{\mathbb{H}/\Gamma^0(4)} d\tau d\bar{\tau}\, \tilde{\nu}(\tau)\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}_0]\left(\tau,\bar{\tau},\frac{du}{da}\frac{S}{2\pi},0\right). \tag{3.4}$$

The holomorphic "measure term" $\tilde{\nu}(\tau)$ is explicitly given by

$$\tilde{\nu}(\tau) := -8i(u^2-1)\frac{da}{du}\vartheta_4(\tau)^{\sigma}. \tag{3.5}$$

It transforms under the generators $ST^{-1}S : \tau \mapsto \frac{\tau}{\tau+1}$ and $T^4 : \tau \mapsto \tau+4$ of $\Gamma^0(4)$ as follows

$$\begin{aligned} \tilde{\nu}\left(\frac{\tau}{\tau+1}\right) &= (\tau+1)^{2-b_2/2}e^{-\frac{\pi i\sigma}{4}}\tilde{\nu}(\tau), \\ \tilde{\nu}(\tau+4) &= -\tilde{\nu}(\tau). \end{aligned} \tag{3.6}$$

Near the weak coupling cusp $\tau \to i\infty$ one has $\tilde{\nu}(\tau) \sim q^{-\frac{3}{8}}$, and near the monopole cusp $\tau_D = -1/\tau \to i\infty$ $\tilde{\nu}(\tau) \sim q_D^{1+\frac{\sigma}{8}}$.

[1]Such integrals have also been studied in the context of one-loop amplitudes in string theory [9, 14, 15], and in mathematics as the (Petersson) inner product for cusp forms [17].

The sum over U(1) fluxes can be conveniently expressed in terms of a Siegel-Narain theta function $\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}] : \mathbb{H} \to \mathbb{C}$

$$\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}](\tau,\overline{\tau},z,\bar{z}) = \sum_{\lambda\in\Gamma+\lambda_0} \mathcal{K}(\lambda)\,(-1)^{(\lambda,\xi)} q^{-\frac{1}{2}\lambda_-^2} \bar{q}^{\frac{1}{2}\lambda_+^2}\; \exp\left(-2\pi i(z,\lambda_-) - 2\pi i(\bar{z},\lambda_+)\right). \quad (3.7)$$

where the lattice $\Gamma \cong \mathbb{Z}^{b_2}$ is identified with $H^2(X,\mathbb{Z})$, $\lambda_0 = \frac{1}{2}w_2(E) \in \frac{1}{2}\Gamma$ is the t'Hooft flux, $\mathcal{K} : \Gamma \to \mathbb{C}$ is a summation kernel

$$\mathcal{K} = y^{-1/2}(\lambda,\omega) + \frac{i}{4\pi y^{3/2}}\frac{du}{da}(S,\omega). \quad (3.8)$$

z and $\bar{z}$ are elliptic variable related to the surface observable and given by

$$z = \frac{du}{da}\frac{S}{2\pi}, \qquad \bar{z} = 0 \quad (3.9)$$

Modular invariance of the integrand under $\Gamma^0(4)$ transformations is an important requirement for (3.4). The effect of inserting the kernel $\mathcal{K}$ into $\Psi^{\omega}_{\lambda_0,\xi}$ is to increase the weight by $(\frac{1}{2},\frac{3}{2})$ as the factor $1/\sqrt{y}$ contributes $(\frac{1}{2},\frac{1}{2})$ and (λ,ω) contributes $(0,1)$ to the total weight. Then using (3.3), the modular transformation of $\Psi^{\omega}_{\lambda_0,\xi}$ (A.12) as well as the fact that $\xi^2 = w_2(X)^2 = 8+\sigma$ one finds that the integrand of (3.4) is invariant under $\tau \mapsto \frac{\tau}{\tau+1}$ transformation.

For the other generator of $\Gamma^0(4)$ $\tau \to \tau+4$, if $(\lambda_0, w_2(X)) = 0\text{mod}\mathbb{Z}$ $\tau \mapsto \tau+4$ leads to multiplication of the integrand by -1. However, one can show that $\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}_0]$ vanishes in this case, so there is no problem with modular invariance[2]. We conclude therefore that the Coulomb branch integral (3.4) is well defined since the measure $d\tau d\overline{\tau}$ transforms as a mixed modular form of weight $(-2,-2)$ while the product $\tilde{\nu}\,\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}]$ is a mixed modular

[2]The derivative $\frac{da}{du}$ transforms under the generators STS and T^4 of $\Gamma^0(4)$ as

$$\frac{da}{du}(\tau+4) = -\frac{da}{du}(\tau), \qquad \frac{da}{du}\left(\frac{\tau}{\tau+1}\right) = (\tau+1)\frac{da}{du}(\tau). \quad (3.10)$$

Modular form	Mixed weight
$d\tau d\overline{\tau}$	$(-2,-2)$
y	$(-1,-1)$
$\partial_{\overline{\tau}}$	raises $(\ell,0)$ to $(\ell,2)$
$\tilde{\nu}(\tau)$	$(2-b_2/2,0)$
$\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}_0]$	$(b_2/2,2)$

Table 3.1: Modular weights of various ingredients for the u-plane integral. Transformation are in $\mathrm{SL}(2,\mathbb{Z})$ for the first three rows, while in $\Gamma^0(4)$ for the last three rows.

form of weight $(2,2)$ making the integrand modular invariant. We close this section with table 3.1 that collects the weights of the various modular forms that appear in the context of u-plane integrals.

3.2 Regularization of modular integrals

Having transformed the u-plane integral to an integral over τ-domain $\mathbb{H}/\Gamma^0(4)$ the integrand becomes a function admitting Lauren expansion in powers of $q^{1/4}$, $\bar{q}^{1/4}$. In fact, for fixed number of operator insertions the Lauren series has only finitely many negative powers. One can further map the integral over $\mathbb{H}/\Gamma^0(4)$ to an integral over $\mathcal{F}_\infty = \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ with elements 1, S, T, T^2, T^3, T^2S of $\mathrm{SL}(1,\mathbb{Z})$. As a result of this the u-plane integral can be written as a finite sum of formal integrals of the form

$$\mathcal{I}_f = \int_{\mathcal{F}} d\tau \wedge d\overline{\tau}\, y^{-s} f(\tau,\overline{\tau}), \tag{3.11}$$

where f is a non-holomorphic modular form of weight $(2-s,2-s)$, and $\mathcal{F} = \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ is a fundamental domain for the modular group $\mathrm{SL}(2,\mathbb{Z})$. In this section we consider situations when in $\mathcal{I}_f$ is divergent and revisit a regularization of such integrals, which has been developed in the mathematical literature in the context of inner products for weakly holomorphic modular forms [11], i.e. modular forms that are holomorphic on the interior of $\mathbb{H}$ but diverge when $\tau \to i\infty \cup \mathbb{Q}$.

To begin with, consider integral of a single term $q^m\,\bar{q}^n$ in the Fourier expansion of f.[3] To this end, consider the set $\mathcal{T}$ of triples (m, n, s), defined by

$$\mathcal{T} = \{m, n \in \mathbb{R}, s \in \mathbb{Z}/2 \,|\, m - n \in \mathbb{Z}\}\,. \tag{3.12}$$

For $(m, n, s) \in \mathcal{T}$, we define

$$L_{m,n,s} = \int_{\mathcal{F}_\infty} d\tau \wedge d\bar{\tau}\, y^{-s}\, q^m \bar{q}^n, \tag{3.13}$$

where $\mathcal{F}_\infty$ is the common keyhole fundamental domain $\mathcal{F} = \mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ pictured in Figure 2.1. Since $\mathcal{F}_\infty$ is non-compact and the integrand may diverge for $y \to \infty$, this is an improper integral. It should be understood as the limiting value of integrals over compact domains, which approach $\mathcal{F}_\infty$. To this end, we introduce the compact domain $\mathcal{F}_Y$ by restricting $\mathrm{Im}\,\tau \leq Y$ for some $Y \geq 1$.[4] The boundaries of $\mathcal{F}_Y$ are given by the following arcs

$$\begin{aligned}
1: &\quad \tau = \tfrac{1}{2} + iy, && y \in [\tfrac{1}{2}\sqrt{3}, Y],\\
2: &\quad \tau = x + iY, && x \in [-\tfrac{1}{2}, \tfrac{1}{2}],\\
3: &\quad \tau = -\tfrac{1}{2} + iy, && y \in [\tfrac{1}{2}\sqrt{3}, Y],\\
4: &\quad \tau = i\, e^{i\varphi}, && \varphi \in [-\tfrac{\pi}{6}, \tfrac{\pi}{6}].
\end{aligned} \tag{3.14}$$

In the limit, $\lim_{Y\to\infty} \mathcal{F}_Y$ we recover $\mathcal{F}_\infty$. We then define $L_{m,n,s}(Y)$ as

$$L_{m,n,s}(Y) = \int_{\mathcal{F}_Y} d\tau \wedge d\bar{\tau}\, y^{-s}\, q^m \bar{q}^n, \tag{3.15}$$

for $(m, n, s) \in \mathcal{T}$, and define

$$L_{m,n,s} = \lim_{Y\to\infty} L_{m,n,s}(Y), \tag{3.16}$$

---

[3]We will justify in section 3.3 that the Fourier series and the integral can be exchanged.

[4]One may consider a more general upperbound with Y being a function of $\mathrm{Re}\,\tau = x$. This will not affect the final result.

provided the limit exists. To study the dependence on Y, we split the compact domain $\mathcal{F}_Y$ into $\mathcal{F}_1$ plus a rectangle $[-\frac{1}{2}, \frac{1}{2}] \times [1, Y]$ as shown below in Figure 3.1, which yields

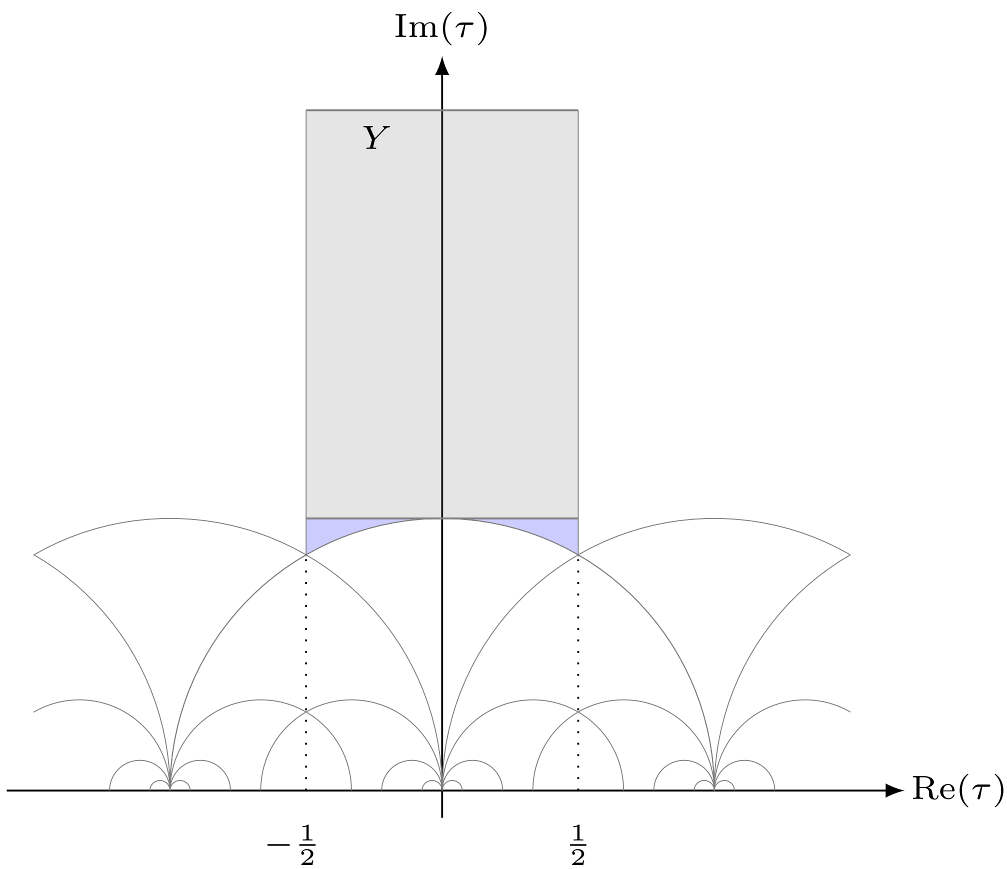

Figure 3.1: Splitting of $\mathcal{F}_Y$ into $\mathcal{F}_1$ (the blue region) and the rectangle $\mathcal{R}_Y$ (gray region).

$$L_{m,n,s}(Y) = \int_{\mathcal{F}_1} d\tau \wedge d\bar{\tau}\, y^{-s}\, q^m \bar{q}^n - 2i \int_{-1/2}^{+1/2} \int_1^Y dx \wedge dy\; y^{-s}\, q^m \bar{q}^n. \tag{3.17}$$

The first term on the right hand side is finite and independent of Y, while integration over $x \in [-\frac{1}{2}, +\frac{1}{2}]$ in the second term yields

$$-\,2i\, \delta_{m,n} \int_1^Y dy\; y^{-s}\, e^{-4\pi m y}. \tag{3.18}$$

Therefore $\lim_{Y\to\infty} L_{m,n,s}(Y)$ converges, except for $m = n < 0$, or $m = n = 0$ with $s \leq 1$. Let us denote this set by $\mathcal{D}$,

$$\mathcal{D} = \{(m, n, s) \in \mathcal{T} \,|\, m = n < 0\} \cup \{(0, 0, s) \in \mathcal{T} \,|\, s \leq 1\}\,. \tag{3.19}$$

As we will see below in section 2.5, there exist Q-exact operators whose contribution on the Coulomb branch contains $(m, n, s) \in \mathcal{D}$, and therefore diverges. This indicates a dangerous IR divergence in presence of such operators.

One way of dealing with the cases $m = n < 0$ is to consider a regularized version $L^{\mathrm{r}}_{m,n,s}$, of $L_{m,n,s}$ for all $(m, n, s) \in \mathcal{T}$. Let us note that the limit of the sum

$$\lim_{Y\to\infty} \left[L_{m,n,s}(Y) + 2i\,\delta_{m,n} \int_1^Y dy\; y^{-s}\, e^{-4\pi m y} \right] = L_{m,n,s}(1) \tag{3.20}$$

is finite. In the definition for $L^{\mathrm{r}}_{m,n,s}$, we will subtract from the two terms in the brackets, a regularized counter part of the second term. To this end, let us introduce the generalized exponential integral $E_\ell(z)$ defined for $\mathrm{Re}\, z > 0$ by

$$E_\ell(z) = \int_1^\infty e^{-z\,t} t^{-\ell} dt. \tag{3.21}$$

$E_\ell(z)$ for ℓ shifted by an integral value can be related to the original $E_\ell(z)$ by partial integration

$$e^{-z} = z\, E_\ell(z) + \ell\, E_{\ell+1}(z). \tag{3.22}$$

We can also express $E_\ell(z)$ in terms of the incomplete Gamma function $\Gamma(k, z)$,

$$\Gamma(k, z) = \int_z^\infty e^{-t}\, t^{k-1} dt = z^k E_{1-k}(z). \tag{3.23}$$

With the analytic continuation of $\Gamma(k,z)$, we can extend the domain of $E_\ell(z)$ to the full complex plane. We define

$$E_\ell(z) = \begin{cases} z^{\ell-1}\displaystyle\int_z^\infty e^{-t}\,t^{-\ell}\,dt, & \text{for } z \in \mathbb{C}^*, \\ \dfrac{1}{\ell-1}, & \text{for } z = 0,\ \ell \neq 1, \\ 0, & \text{for } z = 0,\ \ell = 1, \end{cases} \tag{3.24}$$

where for non-integral ℓ we fix the branch of $t^{-\ell}$ by specifying that the argument of any complex number $\rho \in \mathbb{C}^*$ is in the domain $(-\pi,\pi]$. For $s \in \mathbb{R}^+$, we have $\operatorname{Im} E_\ell(-s) = -\frac{\pi\, s^{\ell-1}}{\Gamma(\ell)}$.

In terms of this function $E_\ell(z)$, we finally define $L^{\mathrm{r}}_{m,n,s}$ for all $(m,n,s) \in \mathcal{T}$ as

$$L^{\mathrm{r}}_{m,n,s} = \int_{\mathcal{F}_1} d\tau \wedge d\overline{\tau}\, y^{-s}\, q^m \bar{q}^n - 2i\,\delta_{m,n}\, E_s(4\pi m), \tag{3.25}$$

This can be used to replace $L_{m,n,s}$ for $(m,n,s) \in \mathcal{D}$.

3.3 Modular invariant integrands

Let us start with the integral of a modular form over the fundamental domain,

$$\mathcal{I}_f = \int_{\mathcal{F}_\infty} d\tau \wedge d\overline{\tau}\, y^{-s}\, f(\tau,\overline{\tau}), \tag{3.26}$$

where $f(\tau,\overline{\tau})$ is a non-holomorphic modular form for $\mathrm{SL}(2,\mathbb{Z})$ of weight $(2-s, 2-s)$, with Fourier expansion

$$f(\tau,\overline{\tau}) = \sum_{m,n \gg -\infty} c(m,n)\, q^m \bar{q}^n, \tag{3.27}$$

given by coefficients $c(m,n)$ that, by the requirement that f is a modular form, are only non-zero when $m - n \in \mathbb{Z}$. We will in fact assume that f is in fact a function on $\mathbb{H} \times \bar{\mathbb{H}}$

satisfying

$$f\left(\frac{a\tau+b}{c\tau+d},\frac{a\sigma+b}{c\sigma+d}\right)=(c\tau+d)^{2-s}(c\sigma+d)^{2-s}f(\tau,\sigma),\tag{3.28}$$

where for $s\in\mathbb{Z}+\frac{1}{2}$, we specify the branch of the square root by requiring that the argument of $c\tau+d$ is in $(-\pi,\pi]$. For a single factor $(c\tau+d)^{2-s}$, consistency of the square root and $\mathrm{SL}(2,\mathbb{Z})$ requires a non-trivial multiplier system. For $f(\tau,\sigma)$, the multiplier systems for τ and σ are complex conjugate and multiply to 1 on the rhs of (3.28).

We allow f to have only finite number of terms with negative $n+m$, i.e. there is an $M\in\mathbb{Z}$ such that $c(m,n)=0$ if $m+n<M$. However, due to the terms with $m+n\leq 0$, the integrand in (3.26) diverges at $y\to+\infty$. If there are no terms with $m=n<0$, the integral is defined using a well-known regularization [9, 10, 15]. For $m=n<0$ we introduce a cut-off Y for $\mathrm{Im}\,\tau$ as in section 3.2, and define $\mathcal{I}_f(Y)$ as an integral of f over the domain $\mathcal{F}_Y$ (3.14),

$$\mathcal{I}_f(Y)=\int_{\mathcal{F}_Y}d\tau\wedge d\overline{\tau}\,y^{-s}\,f(\tau,\overline{\tau}).\tag{3.29}$$

We regularize the divergence of $\mathcal{I}_f(Y)$ by subtracting terms involving the generalized exponential function $E_s(z)$ defined in (3.24). More precisely, we replace $\mathcal{I}_f$ by $\mathcal{I}_f^{\mathrm{r}}$ defined as

$$\mathcal{I}_f^{\mathrm{r}}=\lim_{Y\to\infty}\left[\mathcal{I}_f(Y)-2i\sum_{m\gg-\infty}c(m,m)\,Y^{1-s}E_s(4\pi mY)\right].\tag{3.30}$$

Let us verify that the limit is well-defined. Since the domain $\mathcal{F}_Y$ is compact and the sum over m and n is absolutely convergent on $\mathcal{F}_Y$, we can exchange the double integral and the sum. Thus,

$$\mathcal{I}_f(Y)=\sum_{m,n\gg-\infty}c(m,n)\,L_{m,n,s}(Y),\tag{3.31}$$

with $L_{m,n,s}(Y)$ as in (3.17). We substitute this expression in (3.30). Using

$$\int_1^Y dy\,y^{-s}\,e^{-4\pi my}=E_s(4\pi m)-Y^{1-s}E_s(4\pi mY),$$

we arrive at

$$\mathcal{I}_f^{\mathrm{r}} = \sum_{m,n \gg -\infty} c(m,n)\, L^{\mathrm{r}}_{m,n,s}\,, \tag{3.32}$$

with $L^{\mathrm{r}}_{m,n,s}$ as in (3.25). This is finite since there are at most a finite number of terms with $m = n < 0$, and the sum over other m and n is absolutely convergent by the above made assumptions about f.

3.4 The case of total derivative integrands

If we assume that the integrand can be expressed as a total derivative with respect to $\overline{\tau}$, we can evaluate the integral using Stokes' theorem, and we will find that $\mathcal{I}_f^{\mathrm{r}}$ takes an elegant form in this case. To this end, let us write $y^{-s} f(\tau, \overline{\tau})$ as

$$\partial_{\overline{\tau}} \widehat{h}(\tau, \overline{\tau}) = y^{-s} f(\tau, \overline{\tau}), \tag{3.33}$$

such that the integrand of (3.26) is in fact exact and equal to $-d(d\tau\, \widehat{h})$. Note that this does not imply that $d\overline{\tau}\, \partial_{\overline{\tau}} \widehat{h}$ is exact, since $d\widehat{h} = d\tau\, \partial_\tau \widehat{h} + d\overline{\tau}\, \partial_{\overline{\tau}} \widehat{h}$. For our application to modular integrals, $\widehat{h}(\tau, \overline{\tau})$ transforms as a modular form of weight two. Equation (3.33) can be integrated using $E_\ell(z)$. For $s \neq 1$,

$$\widehat{h}(\tau, \overline{\tau}) = h(\tau) + 2i\, y^{1-s} \sum_{m,n \gg -\infty} c(m,n)\, q^{m-n} E_s(4\pi n y), \tag{3.34}$$

while for $s = 1$, the terms with $n = 0$ in the sum should be replaced by

$$-2i\, \log(y) \sum_{m \gg -\infty} c(m,0)\, q^m.$$

The $c(m,n)$ in (3.34) are the Fourier coefficients of f (3.27), and h is a (weakly) holomorphic function with Fourier expansion

$$h(\tau) = \sum_{\substack{m \gg -\infty \\ m \in \mathbb{Z}}} d(m)\, q^m. \tag{3.35}$$

Since there are no holomorphic modular forms of weight two for $\mathrm{SL}(2,\mathbb{Z})$, $h(\tau)$ is uniquely determined by the coefficients $d(m)$ with $m < 0$. However, since the $d(m)$, $m < 0$, are not determined by the $c(m,n)$, the space of weakly holomorphic modular forms of weight 2 gives an ambiguity in $h(\tau)$. We will discuss below (3.39), that the integral $\mathcal{I}_f^{\mathrm{r}}$ is independent of this ambiguity.

The modular properties of $\widehat{h}(\tau,\overline{\tau})$ imply interesting transformations for $h(\tau)$. Let us consider this for the case that f depends on both τ and $\overline{\tau}$, but is such that the $c(m,n)$ in (3.34) are only non-vanishing for $n > 0$ (or $n \geq 0$ and $s > 1$). We can then express $\widehat{h}$ as

$$\widehat{h}(\tau,\overline{\tau}) = h(\tau) + 2^s \int_{-\overline{\tau}}^{i\infty} \frac{f(\tau,-v)}{(-i(v+\tau))^s} dv. \tag{3.36}$$

Note that the two terms on the right hand side are separately invariant under $\tau \to \tau + 1$, while the transformation of the integral under $\tau \to -1/\tau$ implies for $h(\tau)$,

$$h(-1/\tau) = \tau^2 \left(h(\tau) + 2^s \int_0^{i\infty} \frac{f(\tau,-v)}{(-i(v+\tau))^s}\, dv \right). \tag{3.37}$$

Let us return now to the generic case with $f(\tau,\overline{\tau})$ of the form (3.27) and evaluate $\mathcal{I}_f^{\mathrm{r}}$. The integral over $\mathcal{F}_Y$ can then be carried out using Stokes' theorem, which reduces to a contribution from the interval $[-\frac{1}{2} + iY, \frac{1}{2} + iY]$. We thus find that the integral over $\mathcal{F}_Y$ in (3.30) equals for $s \neq 1$,

$$d(0) + 2i \lim_{Y\to\infty} \sum_{m \gg -\infty} Y^{1-s}\, c(m,m)\, E_s(4\pi m Y), \tag{3.38}$$

using expression (3.34) for $\widehat{h}$. For $s = 1$, we apply the renormalization by analytic continuation in s mentioned below (3.19), which gives the same result.

The last step is to combine (3.38) with the other term in equation (3.30), which yields

$$\mathcal{I}_f^{\mathrm{r}} = d(0). \tag{3.39}$$

As a result the only contribution to the integral arises from the constant term of $h(\tau)$.

Recall that there is an ambiguity in h due to the possibility to add a weakly holomorphic modular form of weight two. To see that the result (3.39) does not depend on this ambiguity, let us note that a basis of weakly holomorphic modular forms of weight 2 is given by derivatives of powers of the modular invariant J-function, $\partial_\tau\big(J(\tau)^\ell\big)$, $\ell \in \mathbb{N}$, which all have vanishing constant terms.

3.5 Q-exact operators as total derivatives

Note that the zero-mode Lagrangian (2.43) can be written as

$$S_0 = 2\pi i\tau(\lambda,\lambda) + Q(W), \qquad \text{with} \qquad W = -\frac{\sqrt{2}}{16\pi}\,\overline{\tau}\chi\,(F_+ + D) \tag{3.40}$$

Let us define $\widetilde{\mathcal{O}} := \mathcal{O}\,e^{-Q(W)}$ and write $Q(\mathcal{O})\,e^{-S_0}$ as $Q(\widetilde{\mathcal{O}})\,q^{-\lambda^2/2}$. Assuming $b_2^+ = 1$ we expand $\widetilde{\mathcal{O}}$ in terms of η and χ yields

$$\widetilde{\mathcal{O}}(S) = \sum_{m=0,1} \widetilde{\mathcal{O}}_{m,0}\,\eta^m + \sum_{m=0,1} \widetilde{\mathcal{O}}_{m,1}\,\eta^m\,(\chi,\omega), \tag{3.41}$$

where $\widetilde{\mathcal{O}}_{m,n}$ are functions of a, $\bar{a}$, λ and b.

$$Q(\widetilde{\mathcal{O}}(S)) = \sqrt{2}i\partial_{\bar{a}}\widetilde{\mathcal{O}}_{0,0}\,\eta + \sqrt{2}i\partial_{\bar{a}}\widetilde{\mathcal{O}}_{0,1}\,\eta\,(\chi,\omega) - i(\lambda_+ - b)\sum_{m=0,1} \widetilde{\mathcal{O}}_{m,1}\,\eta^m. \tag{3.42}$$

Only the term with $\widetilde{\mathcal{O}}_{0,1}$ survives the integration over fermion zero modes,

$$\int d\eta\, d\chi\, Q(\widetilde{\mathcal{O}}(S)) = -\sqrt{2}i\,(S,\omega)\,\partial_{\bar{a}}\widetilde{\mathcal{O}}_{0,1}, \quad \text{with} \quad \widetilde{\mathcal{O}}_{0,1} = \mathcal{O}_{0,1}\, q^{-\lambda_-^2/2}\bar{q}^{\lambda_+^2/2}\, \exp\frac{y}{8\pi}b^2. \tag{3.43}$$

We thus find that the Coulomb branch contribution to any Q-exact insertion can be written as

$$\langle Q(\mathcal{O})\rangle_{\text{C.B.}} = \int d\tau d\bar{\tau}\, \partial_{\bar{\tau}}\left(-i\sqrt{2}(S,\omega)\tilde{\nu} \sum_{\lambda\in\Gamma+\lambda_0} (-1)^{\lambda\cdot\xi} q^{-\lambda_-^2/2}\bar{q}^{\lambda_+^2/2} \int db\, \mathcal{O}_{0,1}(b,\lambda)\, \exp\frac{y}{8\pi}b^2\right) \tag{3.44}$$

for any Q-exact insersion $Q(\mathcal{O})$.

Q-exact operators with finite non-zero contribution. As an example of how the boundary term (3.44) can have sensible non-zero contribution consider ω-derivative of the zero-mode action:

$$\frac{\partial S}{\partial\omega^\alpha} = Q(a_\alpha), \quad \text{where} \quad a_\alpha = \chi\left(iy(\lambda_-)_\alpha + \frac{1}{4\pi}\frac{du}{da}(S_-)_\alpha\right) \tag{3.45}$$

The wall-crossing formula for the Coulomb branch part of $\langle e^{2pu+\mathcal{O}(S)}\rangle$ implies that the ω-derivative of $\langle e^{2pu+\mathcal{O}(S)}\rangle_{\text{C.B.}}$ is given by

$$\frac{\partial}{\partial\omega^\alpha}\langle e^{2pu+\mathcal{O}(S)}\rangle_{\text{C.B.}} = \sum_{\lambda\in\Gamma+\lambda_0}(-1)^{\lambda\cdot\xi}\cdot\left(-\frac{1}{2}\right)\delta(\lambda_+)\left[q^{-\frac{1}{2}\lambda^2}\frac{du}{d\tau}\left(\frac{du}{da}\right)^{\frac{1}{2}b_2}\Delta^{\frac{\sigma}{8}}\, e^{2pu-i\frac{du}{da}(S,\lambda)+TS^2}\right]_{q^0} \tag{3.46}$$

(3.46) on the other hand equals $\langle Q(a_\alpha)\, e^{2pu+\mathcal{O}(S)}\rangle_{\text{C.B.}}$.

3.6 Divergent Q-exact operators

Consider the following operator in the UV description

$$\begin{aligned} I_+(S) &= -\frac{1}{4\pi}\int_S \left\{Q,\left[L,\operatorname{Tr}\bar{\phi}^2\right]\right\} \\ &= -\frac{1}{2\pi}\int_S \{Q,\operatorname{Tr}\bar{\phi}\chi\}, \end{aligned} \tag{3.47}$$

where $S\in H_2(X,\bar{Z})$ is a two-cycle, and L is the twisted supersymmetry generator discussed in section 2.3. The subscript + is to indicate that it involves a self-dual two-form field, and is in a sense a self-dual counterpart of the holomorphic, anti-self dual Donaldson observable $\mathcal{O}(S)$ [19]. Using the action of L, we can determine the image of $I_+(S)$ on the Coulomb branch, denoted by $\widetilde{I}_+(S)$, in terms of the IR fields

$$\begin{aligned} \widetilde{I}_+(S) &= -\frac{1}{4\pi}\int_S \left\{Q,\frac{d\bar{u}}{d\bar{a}}\chi\right\} \\ &= -\frac{i}{\sqrt{2}\pi}\int_S \left(\frac{1}{2}\frac{d^2\bar{u}}{d\bar{a}^2}\eta\chi+\frac{\sqrt{2}}{4}\frac{d\bar{u}}{d\bar{a}}(F_+-D)\right). \end{aligned} \tag{3.48}$$

Integration over b and fermion zero modes yields $\frac{d\bar{\tau}}{d\bar{a}}\mathcal{K}_+(\lambda)$, where $\mathcal{K}_+(\lambda)$ is given by

$$\mathcal{K}_+(\lambda):=\frac{2\,(S,\omega)}{\sqrt{y}}\left(-\frac{1}{2}\frac{d^2\bar{u}}{d\bar{a}\,d\bar{\tau}}+\frac{i}{8\,y}\frac{d\bar{u}}{d\bar{a}}+\frac{\pi i}{2}\frac{d\bar{u}}{d\bar{a}}\,\lambda_+^2\right). \tag{3.49}$$

The sum over the U(1) fluxes can be written in a compact form as a total derivative with respect to $\bar{\tau}$

$$\Psi^\omega_{\lambda_0,\xi}[\mathcal{K}_+](\tau,\bar{\tau})=-\partial_{\bar{\tau}}\left(\frac{(S,\omega)}{\sqrt{y}}\frac{d\bar{u}}{d\bar{a}}\,\Psi^\omega_{\lambda_0,\xi}[1](\tau,\bar{\tau})\right). \tag{3.50}$$

This expression demonstrates that $\Psi^\omega_{\lambda_0,\xi}[\mathcal{K}_+]$ vanishes for $(\lambda_0,K_M)=\frac{1}{2}\mathrm{mod}\mathbb{Z}$, since $\Psi^\omega_{\lambda_0,\xi}[1]$ vanishes in this case. If non-vanishing $\Psi^\omega_{\lambda_0,\xi}[\mathcal{K}_+]$ has the required modular properties: it transforms with modular weight $(\frac{b_2}{2},2)$, and changes sign under $\tau\mapsto\tau+4$.

For the one-point function of $\widetilde{I}_+(S)$, we arrive at the integral

$$\langle \widetilde{I}_+(S)\rangle = -\int_{\mathbb{H}/\Gamma^0(4)} d\tau\wedge d\bar\tau\,\partial_{\bar\tau}\left(\widetilde{\nu}\,\frac{(S,\omega)}{\sqrt{y}}\,\frac{d\bar u}{d\bar a}\,\Psi^{\omega}_{\lambda_0,\xi}[1]\right). \tag{3.51}$$

We can easily evaluate this integral using Stokes' theorem. This reduces to arcs close to the three cusps of $\mathbb{H}/\Gamma^0(4)$, $\tau\to i\infty$, 0 and 2. But here is where the surprise occurs: since $\frac{d\bar u}{d\bar a}$ diverges as $\bar q^{-\frac{1}{8}}$ for $\tau\to\infty$ and $\tilde\nu(\tau)$ as $q^{-\frac{3}{8}}$, we find integrals $L_{m,n,s}$ (3.1) with both m and $n<0$ for the cusp at $i\infty$! The standard prescription of integrating over x first and over y second therefore does not cure the divergence if $\Psi^{\omega}_{\lambda_0,\xi}\sim q^{\frac{1}{4}}$ for $\tau\to i\infty$.

Using the regularization of section 3.2, we can show that the correlation functions of the form $\langle\{Q,\mathcal{O}\}\rangle$ vanish. At this point recall that $\langle\{Q,\mathcal{O}\}\rangle$ can be expressed as

$$\langle\{Q,\mathcal{O}\}\rangle = \int_{\mathcal{F}_\infty} d\tau\wedge d\bar\tau\,\partial_{\bar\tau}F_{\mathcal{O}}, \tag{3.52}$$

with

$$F_{\mathcal{O}}(\tau,\bar\tau) = y^{-s}\sum_{m,n} c(m,n)\,q^m\bar q^n, \tag{3.53}$$

where only a finite number of $c(m,n)\neq 0$ for $m+n<0$. Let us first evaluate (3.52) using section 3.4. Since

$$\partial_{\bar\tau}F_{\mathcal{O}} = -i\,y^{-s}\sum_{m,n} c(m,n)\,(2\pi\,n+\tfrac{1}{2}\,s\,y^{-1})\,q^m\bar q^n, \tag{3.54}$$

we can identify $F_{\mathcal{O}}$ with $\widehat{h}_1+\widehat{h}_2$ following (3.33). Here $\widehat{h}_1$ is of the form (3.34) and $\widehat{h}_2$ as well, but with s replaced by $s+1$. $F_{\mathcal{O}}$ is a (non-holomorphic) modular form of weight 2, and the discussion in section 2.5 did not include a holomorphic function h_1+h_2. Indeed, since $F_{\mathcal{O}}$ is a modular form of weight 2, vanishing of h_1+h_2 is consistent with the modular properties. The sum of constant terms $d_1(0)+d_2(0)$ thus vanishes, which demonstrates that $\langle\{Q,\mathcal{O}\}\rangle$ vanishes.

Alternatively, one may start from (3.30) with $f = \partial_{\bar\tau} F_{\mathcal{O}}$, such that $\langle\{Q, \mathcal{O}\}\rangle$ reads

$$\begin{aligned}\langle\{Q, \mathcal{O}\}\rangle = \lim_{Y\to\infty}\Bigg[\int_{\mathcal{F}_Y} d\tau \wedge d\bar\tau\, \partial_{\bar\tau} F_{\mathcal{O}} \\ -2Y^{-s} \sum_{m\gg-\infty} c(m,m)(2\pi\, m\, Y\, E_s(4\pi m Y) + \frac{s}{2}\, E_{s+1}(4\pi m Y))\Bigg].\end{aligned} \tag{3.55}$$

To evaluate the integral over $\mathcal{F}_Y$, we use Stokes' theorem. Modular invariance of the integrand implies that only the arc at $\mathrm{Im}\,\tau = Y$ contributes. Using (3.22) for the second line, we obtain

$$\langle\{Q, \mathcal{O}\}\rangle = \sum_m c(m,m) \lim_{Y\to\infty}\left[Y^{-s}\, e^{-4\pi Y m} - Y^{-s}\, e^{-4\pi Y m}\right] = 0. \tag{3.56}$$

We have thus demonstrated that the correlation function of a generic Q-exact observable vanishes with the current prescription.

Chapter 4

Topological partition function of Argyres-Douglas theory

Recall that the u-plane integral can be written in a form that applies to any one-dimensional Coulomb branch. In this chapter we will apply it to the SU(2), $N_f = 1$ family (4.1) and the AD3 family (4.4), in which case we will write $Z_u^{SU(2),N_f=1}$ and $Z_u^{ADFamily}$, respectively.

Our basic line of reasoning is the following. From reference [26] we know that in the SU(2) $N_f = 1$ theory at a special value of the quark mass, $m = m_*$, the IR physics near a special vacuum $u = u_*$ is described by the AD3 theory. Moreover there is no noncompact Higgs branch for the SU(2) $N_f = 1$ theory so if we take the $m \to m_*$ limit of the partition function we should be able to extract the AD3 partition function. In section 4.2 below we will make this more precise. In section 4.4 we explain that on manifolds with $b_2^+ = 1$ the topologically twisted AD3 partition function is, in fact, not diffeomorphism invariant, but varies continuously with the metric. In section 4.7 we discuss implications of the main formulae (4.109), (4.111) for four-manifolds.

4.1 SU(2) $N_f = 1$ and AD3

Seiberg-Witten geometry of $N_f = 1$ and AD3 theories

The Seiberg-Witten curve Σ for SU(2) theory coupled to a single hypermultiplet in the fundamental representation was first presented in [59]. The class S presentation of this curve is

$$\lambda^2 = \left(\frac{\Lambda^2}{z} + 3u + 2\Lambda m z + \Lambda^2 z^2\right)\left(\frac{dz}{z}\right)^2 \tag{4.1}$$

where Λ is the dynamically generated scale, m is the mass of the hypermultiplet, u is a coordinate on the Coulomb branch, and $z \in C \cong \mathbb{C}^*$ is a coordinate on the UV curve C. The Seiberg-Witten curve is a subset of T^*C where the restriction of the canonical Liouville one-form on T^*C to Σ is the canonical Seiberg-Witten differential $\lambda = \frac{x}{z}dz$.

As observed in [26] when $m = \frac{3}{2}\omega\Lambda$, with ω a third root of unity, three branch points of the curve collide and the discriminant of the curve has a multiple zero. For definiteness we consider the limiting behavior as $m \to m_* := \frac{3}{2}\Lambda$, so the discriminant has a double zero at $u = u_* := \Lambda^2$ where two roots $u_\pm(m)$ collide. To define a scaling limit we change variables:

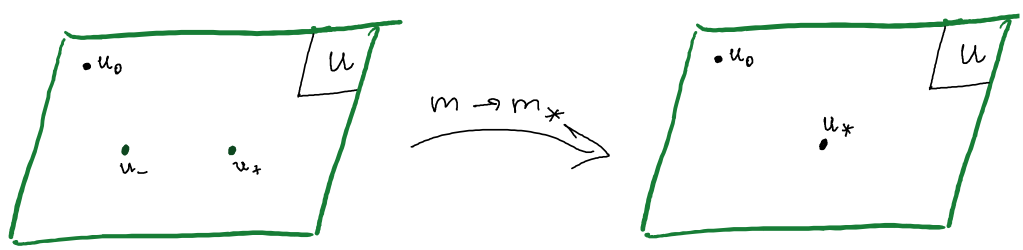

Figure 4.1: Collision of singularities in the Coulomb branch of $N_f = 1$ theory at critical value of the UV quark mass $m = m_*$.

$$m_s = \frac{3}{2}\Lambda_s + \delta m \qquad u = \Lambda_s^2 + \delta u \qquad z = -1 + \tilde{z} \tag{4.2}$$

Define

$$\begin{aligned} \tilde{z} &= -\epsilon z_{AD} \\ 4\Lambda\delta m - 3\delta u &= 3\epsilon^2\Lambda_{AD}^2 \\ 2\Lambda\delta m - 3\delta u &= -\epsilon^3 u_{AD} \\ \lambda &= \epsilon^{5/2}\lambda_{AD} \end{aligned} \tag{4.3}$$

and take $\epsilon \to 0$ holding all quantities with subscript AD fixed. The result (now and in what follows we drop the AD subscript for readability) is the AD3 family of curves as used in [50]:

$$\lambda^2 = (z^3 - 3\Lambda^2 z + u)(dz)^2 \tag{4.4}$$

The AD3 theory is described by the limit $\Lambda \to 0$ at the origin of the Coulomb branch $u = 0$.

A key point made in [26] is that at the points $u_\pm(m)$ of the SU(2), $N_f = 1$ family, the hypermultiplets with mutually nonlocal $U(1)$ charges become massless. Therefore, when $m \to m_*$ and $u \to u_*$ there are massless particles coupled to the $U(1)$ gauge field and its dual and the low energy effective theory cannot be described by a Lagrangian. It is, in fact, the AD3 theory weakly coupled to other degrees of freedom in the SU(2), $N_f = 1$ theory.

We remark that the $AD3$ theory was first discovered at a point in the Coulomb branch of pure SU(3) $N = 2$ SYM [25]. However, in that case the $U(1)$ flavor symmetry associated with the mass parameter is gauged and therefore integrated over. For our purposes it is much better to keep it as a free parameter.

The $U(1)_R$ charge anomaly of AD3

There is a simple conceptual reason for the selection rule $U = 0$ we have found: It is the selection rule enforced by the the $U(1)_R$ symmetry of a superconformal theory. It was pointed out that such $U(1)_R$ selection rules would apply to twisted superconformal collerators in [22] although the background charge for the AD3 theory deduced from the measure of the SU(3) Coulomb branch was incorrectly stated in that paper to be $-\chi/10$. The correct determination of the background charge from the measure, expressed in terms of the conformal anomaly coefficients a and c, was derived in [52]. We briefly recall the derivation here.

The result of [52] is based on the fact that the $U(1)_R$ current is in a superconformal supermultiplet with the energy-momentum tensor. (See [58] for a useful discussion.) The result of [52] can equivalently be summarized in the anomaly eight-form [27]

$$A = (a-c)(c_1(\mathcal{R})p_1(TX) - c_1(\mathcal{R})^3) - (4a-2c)(c_1(\mathcal{R})c_2(E)) \tag{4.5}$$

where a, c are the usual anomaly coefficients of the Weyl anomaly, $\mathcal{R}$ is a $U(1)_R$ symmetry line bundle and E is the $\mathrm{SU}(2)_R$ symmetry bundle.

Applying the usual descent formalism and plugging in the AD3 values $a = 43/120$ and $c = 11/30$ gives indeed the expected value

$$-\frac{1}{20}(7\chi + 11\sigma) \tag{4.6}$$

The $U(1)_R$ charge of the canonical 0-observable is $6/5$ and hence that of the 2-observable $\mathcal{Q}(S)$ is $1/5$. [1] Therefore, dividing the selection rule $U = 0$ by 5 gives the expected $U(1)_R$ symmetry selection rule:

$$\frac{6}{5}\ell + \frac{1}{5}r = \frac{\chi_h - c_1^2}{5} = -\frac{1}{20}(7\chi + 11\sigma) \tag{4.7}$$

Remark: When b_1 is nonzero we can also introduce 1- and 3-observables $\mathcal{Q}(\gamma) = \int_\gamma Ku$ and $\mathcal{Q}(\Sigma) = \int_\Sigma K^3 u$, for $\gamma \in H_1(X;\mathbb{Z})$ and $\Sigma \in H_3(X;\mathbb{Z})$, respectively. The selection rule

[1] To see this note that λ and Z have $U(1)_R$ charge $+1$, the supersymmetry operator K has charge $-\frac{1}{2}$.

now becomes

$$\frac{12}{10}n_0 + \frac{7}{10}n_1 + \frac{2}{10}n_2 - \frac{3}{10}n_3 = \frac{\chi_h - c_1^2}{5} = -\frac{1}{20}(7\chi + 11\sigma) \tag{4.8}$$

The notable feature here is that the expression on the left-hand side can now be negative, and moreover the relative minus sign allows the possibility of infinitely many nontrivial correlation functions, in strong contrast to the simply connected case.

4.2 Topological partition function for AD3

In order to extract the partition function of the AD3 theory from that of the SU(2), $N_f = 1$ theory we use the following principles:

1. The limit of $Z^{SU(2),N_f=1}$ as $m \to m_*$ must exist since there are no noncompact Higgs branches. (Noncompact Higgs branches are the only source of IR divergences given that X is compact and the contribution from $u \to \infty$ is finite.)

2. The resulting path integral must be an integral over all Q-invariant field configurations.

3. According to [26] those Q-invariant configurations include the supersymmetric "states" of the AD theory, perhaps coupled to other degrees of freedom in the theory. However, at $m = m_*$ those couplings should be arbitrarily weak in the scaling region of the u-plane near u_*.

4. We can therefore isolate the AD configurations by focusing on the contribution from an infinitesimally small neighborhood of the colliding singularities $u_\pm(m)$ plus the SW contributions associated with those points.

When $m = m_*$, the family (4.1) has a singularity at $u = u_*$, where two singularities $u_\pm(m)$ have collided, and another singularity u_0 far away from the scaling region. Since the definition of the integral requires a subtle regularization over the noncompact regions it turns out that

$$Z_u^{SU(2),N_f=1} - \int du d\bar{u} \lim_{m\to m_*}\left(\left|\frac{da}{du}\right|^2 A^\chi B^\sigma e^{2pu+S^2T(u)}\Psi\right) \tag{4.9}$$

has a nonzero Laurent expansion in power of ${\lambda_0}^{1/4}$ around $\mu = 0$ where $\mu := (m - m_*)/\Lambda_1$. Here the integral of the $m \to m_*$ limit of the integrand of $Z_u^{SU(2),N_f=1}$ is defined by cutting out disks around u_0 and u_* and taking the limit as the disks shrink. The singular terms in expansion (4.9) will cancel against similar singular terms from Z_{SW}. The constant term (i.e. the coefficient of μ^0) is in general nonzero and does not cancel against the constant term from Z_{SW}.

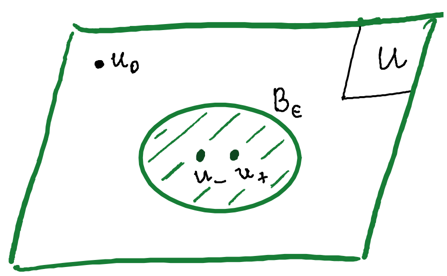

Figure 4.2: The neighborhood $B(\epsilon; u_*)$ for sufficiently small $(m - m_*) << \epsilon$ and $\epsilon << \Lambda$ corresponds to the AD family of curves.

The quantity (4.9) comes from the integration over an infinitesimal region near $u = u_*$. Indeed, for any $\epsilon > 0$ let $B(\epsilon; u_*)$ be a disk around u_* with $|u-u_*| < \epsilon$. When m is sufficiently close to m_* the two colliding singularities $u_\pm(m)$ will be inside this disk. Therefore, for any fixed $\epsilon > 0$:

$$\lim_{m\to m_*}\int_{\mathbb{C}-B(\epsilon;u_*)} du d\bar{u}\left(\left|\frac{da}{du}\right|^2 A^\chi B^\sigma e^{2pu+S^2T(u)}\Psi\right) = \int_{\mathbb{C}-B(\epsilon;u_*)} du d\bar{u} \lim_{m\to m_*}\left(\left|\frac{da}{du}\right|^2 A^\chi B^\sigma e^{2pu+S^2T(u)}\Psi\right) \tag{4.10}$$

Therefore, in view of the limiting behavior reviewed in section 4.1 we should attribute the difference to the contribution of the AD partition function on four-manifolds with $b_2^+ = 1$.

In the SU(2) $N_f = 1$ family when $m \to m_*$ and $u \to u_*$ the AD3 theory is still weakly coupled to other degrees of freedom in the original gauge theory. The detailed considerations of section 4.5 show that we should extract a factor $e^{2p(u_* + \frac{2}{3}\mu) + S^2 T_*}$ to account for these couplings. Here and henceforth we will choose units so that $\Lambda_1 = 1$. The peculiar shift by $2\mu/3$ in the coefficient of p is due to the linear combinations (4.2). Thus we consider the constant term in the Laurant expansion around $\mu = 0$:

$$\left[e^{-2p(u_* + \frac{2}{3}\mu) - S^2 T_*} \left(Z_u^{SU(2), N_f = 1} - \int du d\bar{u} \lim_{m \to m_*} \left(A^\chi B^\sigma e^{2pu + S^2 T(u)} \Psi \right) \right) \right]_{\mu^0} \tag{4.11}$$

Again, the detailed considerations of section 4.5 strongly motivate the following conjectures:

1. The constant term in (4.11) is in fact a polynomial in p and S, in striking contrast to the partition functions of Donaldson-Witten theory. We will denote in by $P_1(p, S)$.

2. Furthermore, if we define a grading of the polynomial $P_1(p, S)$ by "R-charge" with $R[p] = 6$ and $R[S] = 1$ then the highest degree is given by $6\ell + r = \mathfrak{B} := -\frac{1}{4}(7\chi + 11\sigma)$.

3. If one considers the u-plane integral for the AD3 family (4.4) it has a similar expansion in powers of $\Lambda_{AD}^{1/2}$ around $\Lambda_{AD} = 0$ and the constant term $P_{AD}(p, S)$ is also a polynomial in p and S.

4. Finally, defining $P_1^{top}(p, S)$ be the sum of terms with maximal R-charge we have:

$$P_1^{top}(p, S) = N P_{AD}(n_0 p, n_2 S) \tag{4.12}$$

   for suitable constants N, n_0, n_2.[2]

[2] We interpret the terms of lower R-charge in the polynomial $P_1(p, S)$ as effects arising from the coupling of the AD3 theory to other degrees of freedom in the SU(2) $N_f = 1$ theory. It would certainly be useful to understand the physics of the lower order terms better.

The results of sections 4.5 and 4.4 are enough to prove all the above claims for the difference of u-plane integrals for any two choices of metric. Moreover, they establish the above claims absolutely when X has a homotopy type so that the u-plane integral has a vanishing chamber in the sense explained in section 5 of [7].

These considerations motivate our central formula for how to extract the physics of the AD3 theory from the expansion around $\mu = 0$ of the SU(2) $N_f = 1$ partition function[3]:

$$\begin{aligned} \widetilde{Z}_{AD} := & \left[e^{-2p(u_*+\frac{2}{3}\mu)-S^2 T_*} \left(Z_u^{SU(2),N_f=1} - \int du d\bar{u} \lim_{m\to m_*} \left(A^\chi B^\sigma e^{2pu+S^2T(u)} \Psi \right) \right) \right]_{\mu^0}^{TopR-charge} \\ & + \left[e^{-2p(u_*+\frac{2}{3}\mu)-S^2 T_*} \left(Z_{SW}^{SU(2),N_f=1}(u_+(m)) + Z_{SW}^{SU(2),N_f=1}(u_-(m)) \right) \right]_{\mu^0}^{TopR-charge} \end{aligned} \tag{4.13}$$

On the other hand, a very natural way to define the partition function of the AD3 theory is to use directly the family of curves (4.4) and define:

$$Z_{AD} = \lim_{\Lambda_{AD}\to 0} \left[Z_u^{AD family} + Z_{SW}^{AD family} \right] \tag{4.14}$$

We conjecture that, up to an overal constant and renormalization of p and S, we have $\widetilde{Z}_{AD} = Z_{AD}$. Again, a full proof of this statement follows from the considerations of section 4.5, if we consider the difference of partition functions for two metrics, or if we consider a homotopy type X admitting a vanishing chamber. Moreover, if $b_2^+ > 1$ then the statement is an easy consequence of the relationship of the two curves described in section 4.1.

Our main conjecture is that

$$Z_{AD} = \langle e^{p\mathcal{Q}+\mathcal{Q}(S)} \rangle \tag{4.15}$$

for the topologically twisted AD3 theory on four-manifolds X with $b_2^+ > 0$.

[3]Several arguments show that for the AD3 theory $H(u) = 0$.

4.3 The u-plane integrand as total derivative

In this section we will show that if we consider the difference of two u-plane measures at different period points ω and ω_0 then the measure can naturally be written as a total derivative of a well-defined one-form on the u-plane. Our approach here was strongly influenced by [19].

Up to an overall constant the measure on the u-plane can be written as

$$\begin{aligned}
& d\mu^{\omega}_{\text{Coulomb}} = du d\bar{u}\, \hat{\mathcal{H}}\Psi, \quad \text{where} \\
& \Psi = -4i \sum_{\lambda\in\Gamma+\lambda_0} (-1)^{(\lambda-\lambda_0)\xi} \left(\frac{d}{d\bar{u}} E(\rho^{\omega}_{\lambda}, \rho_0)\right) e^{-i\pi\tau\lambda^2 - i\frac{du}{da}(\lambda, S)} \\
& \hat{\mathcal{H}} = \left(\frac{du}{da}\right)^{1-\sigma/2} \Delta^{\sigma/8} e^{2pu+TS^2} E(\rho,\rho_0) := \int_{\rho_0}^{\rho} dt\, e^{-2\pi t^2}
\end{aligned} \tag{4.16}$$

The period point ω is explicitly written in the notation of the measure in (4.16) since the dependence on ω will be important in what follows. The lower bound ρ_0 in the contour integral $E(\rho^{\omega}_{\lambda}, \rho_0)$ is fairly arbitrary. It can depend on λ, u, etc, but not on $\bar{u}$.

Consider the difference of u-plane measures for two different metrics with period points ω and ω_0. Then we can write

$$d\mu^{\omega}_{\text{Coulomb}} - d\mu^{\omega_0}_{\text{Coulomb}} = d\Omega^{\omega,\omega_0} \tag{4.17}$$

where Ω^{ω,ω_0} is a $(1,0)$ form:

$$\begin{aligned}
& \Omega^{\omega,\omega_0} = -du\, \hat{\mathcal{H}} \tilde{\Theta}^{\omega,\omega_0} \\
& \tilde{\Theta}^{\omega,\omega_0} = \tilde{\Theta}^{\omega,\omega_0}(\xi, \lambda_0, \tau, z) := \sum_{\lambda\in\Gamma+\lambda_0} (-1)^{(\lambda-\lambda_0)\xi} E(\rho^{\omega}_{\lambda}(z), \rho^{\omega_0}_{\lambda}(z))\, e^{-i\pi\tau\lambda^2 - 2\pi i(z,\lambda)} \\
& z = \frac{1}{2\pi} S \frac{du}{da}
\end{aligned} \tag{4.18}$$

The indefinite theta series $\tilde{\Theta}^{\omega,\omega_0}$ is both absolutely convergent and satisfies the modular transformation properties

$$\begin{aligned} \tilde{\Theta}^{\omega,\omega_0}(\xi,\lambda_0;\tau+1,z) &= e^{-i\pi\lambda_0^2\tilde{\Theta}^{\omega,\omega_0}}\Big(\xi-\frac{1}{2}(w_2+2\lambda_0),\lambda_0;\tau,z\Big) \\ \tilde{\Theta}^{\omega,\omega_0}(\xi,\lambda_0;-\frac{1}{\tau},\frac{z}{\tau}) &= e^{i\pi/2}(-i\tau)^{b_2/2}e^{-2\pi i\lambda_0\cdot\xi}e^{-i\pi z^2/\tau}\tilde{\Theta}^{\omega,\omega_0}(\lambda_0,-\xi;\tau,z) \end{aligned} \tag{4.19}$$

where b_2 is the rank of Γ. It now follows that the difference of u-plane integrals can be written as

$$\begin{aligned} Z_u^{\omega}-Z_u^{\omega_0} &= \int_{u\text{-plane}} d\Big(du\hat{\mathcal{H}}\Omega^{\omega,\omega_0}(\xi,\lambda_0;\tau,z)\Big) \\ &= \Big(\oint_{|u|=1/\epsilon}-\sum_s\oint_{|u-u_s|=\epsilon}\Big)du\hat{\mathcal{H}}\Omega^{\omega,\omega_0}(\xi,\lambda_0;\tau,z) \end{aligned} \tag{4.20}$$

We now use the contour integral representation (4.19) to show that there is a formal series in p,S expressed as a contour integral, and denoted $G^{\omega}(p,S)$, or just G^{ω}, such that

$$Z_u^{\omega}-Z_u^{\omega_0}=G^{\omega}-G^{\omega_0} \tag{4.21}$$

The point here is that G^{ω} only depends on a single point and is expressed as a contour integral. Before deriving (4.21) let us draw from it some useful consequences.

First, (4.21) implies that $Z_u^{\omega}=G^{\omega}+\omega$-independent power series in p and S. As we will see in the derivation of (4.21) the formula is only valid when ω and ω_0 are in the same component of the light cone in $H^2(X;\mathbb{R})$. On the other hand, Z_u^{ω} is defined for ω in either component and moreover $Z^{-\omega}=-Z^{\omega}$. Therefore we conclude that

$$Z_u^{\omega}=G^{\omega}+C(p,S)\mathrm{sign}(\omega^t) \tag{4.22}$$

where $C(p,S)$ is independent of ω and ω^t is the "time component" of ω.

It remains to prove (4.21). We begin with the contribution of a finite point us and assume that $\mathrm{Im}\,\tau_s\to+\infty$ and assume that da_s/du is finite as $u\to u_s$. Then, provided the metric so that there is no λ with $\lambda_+=0$, the indefinite theta function in (4.18) simplifies and we can replace the difference of error functions by $\frac{1}{\sqrt{8}}(\mathrm{sign}(\lambda \cdot \omega) - \mathrm{sign}(\lambda \cdot \omega_0))$. But now we note that if ω and ω_0 are in the same component of the lightcone then their time components ω^t and ω_0^t have the same sign and hence $\mathrm{sign}(\lambda \cdot \omega) - \mathrm{sign}(\lambda \cdot \omega_0) = 0$ when $\lambda^2 \geq 0$. Therefore, in evaluating the residue integral around $u = u_s$ in (4.20) we can replace $\tilde{\Theta}^{\omega,\omega_0}$ by $F_s^{\omega} - F_s^{\omega_0}$ where

$$F_s^{\omega} := \frac{1}{\sqrt{8}} \sum_{\lambda^2 < 0} \mathrm{sign}(\lambda, \omega)\, e^{-i\pi\tau\lambda^2 - 2\pi i z \cdot \lambda} \cdot (-1)^{(\lambda - \lambda_0)\cdot\xi} \tag{4.23}$$

Note that because of the restriction $\lambda^2 < 0$ this sum converges absolutely. Moreover it is a function purely of ω and not of ω^0. Let G_s^{ω} be the corresponding contour integral

$$G_s^{\omega} := \oint_{u_s} du\, \hat{\mathcal{H}} F_s^{\omega}\,. \tag{4.24}$$

We would like to do something similar to write the integral around $u = \infty$ but there are two cases

1. For SU(2) $N_f < 4$ we have $du/da \to \infty$ as $u \to \infty$.

2. For the conformal theories of interest we have $\tau \to \tau_*$ as $u \to \infty$ and $\mathrm{Im}(\tau)$ does not go to infinity so we cannot replace the error functions by differences of sign functions.

To deal with this complication we note that the u-plane integral really only has meaning as a formal power series in p and S. Therefore, we should use the expansion of the error function

$$E(r + a) = E(r) - 2\pi e^{-2\pi r^2} \sum_{n=1}^{\infty} \frac{(-2\pi a)^n}{n!} H_{n-1}(2\pi r) \tag{4.25}$$

where $H_n(x)$ are the standard Hermite polynomials. Applying (4.25) gives:

$$\begin{aligned} \tilde{\Theta}^{\omega,\omega_0} : = & \sum_{\lambda\in\Gamma+\lambda_0} \Big(E(\sqrt{y}\lambda\cdot\omega) - E(\sqrt{y}\lambda\cdot\omega_0)\Big) e^{-i\pi\tau\lambda^2 - 2\pi i z\cdot\lambda}(-1)^{(\lambda-\lambda_0)\xi} \\ & + \sum_{n=1}^{\infty}(\Theta_n^{\omega} - \Theta_n^{\omega_0}) \end{aligned} \tag{4.26}$$

where the θ_n^{ω} come from the n-th term in the sum in (4.25) and are absolutely convergent as sums in λ. For a fixed monomial $p^{\ell}S^{r}$ only a finite number of such terms will contribute so we do not need to worry about the convergence of the sum on n in $\sum_n \Theta_n$. Now, since we are considering a contour on a circle whose radius goes to infinity, if $y \to +\infty$ we can replace this expression by $F_{\infty}^{\omega} - F_{\infty}^{\omega_0}$ where

$$F_{\infty}^{\omega} := \sum_{\lambda^2<0} \operatorname{sign}(\lambda\cdot\omega) e^{-i\pi\tau\lambda^2 - 2\pi i z\cdot\lambda}(-1)^{(\lambda-\lambda_0)\cdot\xi} + \sum_{n=1}^{\infty}\Theta_n \tag{4.27}$$

is a well defined function of a single period point ω. In the conformal case when $y \to y_*$ has a finite limit as $u \to \infty$ we write

$$E(\sqrt{y}\lambda\cdot\omega) - E(\sqrt{y}\lambda\cdot\omega_0) = E(\sqrt{y}\lambda\cdot\omega;\infty) - E(\sqrt{y}\lambda\cdot\omega_0;\infty) \tag{4.28}$$

Now we can separate terms and obtain a well defined function F_{∞}^{ω}.

Finally, let G_{∞}^{ω} denote the contour integral of $du\hat{\mathcal{H}}F_{\infty}^{\omega}$ around the circle at infinity and let

$$G^{\omega} := G_{\infty}^{\omega} + \sum_{s} G_s^{\omega} \tag{4.29}$$

This completes the proof of (4.21).

4.4 The u-plane contribution to Z_{AD}

We now turn to the u-plane integral $Z_u^{AD3Family}$. We will find that, once again, the coefficient of Λ_{AD}^0 in the expansion around $\Lambda_{AD} \to 0$ is a polynomial with terms satisfying the selaction rule $U = 0$. (In particular, it vanishes for manifolds such that $S^2 \times S^2$ and CP^2, cases where the corresponding intgerals in Donaldson theory are quite interesting.)

We do not know how to give a general contour integral expression for the result of the u-plane integral, but one key feature can be immediately noticed: in the AD3 family the τ-parameter approaches a finite value τ_* as $u \to \infty$. Just as in the case of the SU(2), $N_f = 4$ theory studied in [7] this results in continuous metric dependence: The general arguments for invariance of the topological partition function fail utterly. We expect this to be a generic property of topologically twisted superconformal partition functions on four-manifolds with $b_2^+ = 1$.

Note that for the AD3 family, even when $\Lambda_{AD} \neq 0$ for $u \to \infty$ we have, in any duality frame

$$\begin{aligned} a &\to \kappa u^{5/6} + \dots \\ a_D &\to \kappa \tau_* u^{5/6} + \dots \end{aligned} \tag{4.30}$$

where κ is a nonzero constant and τ_* is in the $PSL(2,\mathbb{Z})$ orbit of $e^{i\pi/3}$. For concreteness, we will choose a frame so that $\tau_* = e^{i\pi/3}$. This means that $da/du \sim u^{-1/6} + \dots$ is not single-valued on the u-plane. It is quite nontrivial, and somewhat remarkable, that the u-plane measure is in fact well-defined at $u \to \infty$. Nevertheless, one can indeed check that it is well defined by directly making the modular transformation of the integrand by $(TS)^{-1}$. From the physical viewpoint it is quite important that the measure be well-defined on the u-plane and not just on some cover.

As explained in section 4.5, it is possible to write the u-plane integral as a sum of contour integrals when we consider the difference of integrals for two period points ω and

ω_0. The continuous metric dependence for the AD3 family comes from the contour at $u \to \infty$ and this difference can be written as $G^{\omega}_{\infty} - G^{\omega_0}_{\infty}$ where G^{ω}_{∞} is a contour integral depending on ω and not both ω, ω_0. Using the expansions in (4.50) et. seq. we can be quite explicit. Up to an overall normalization factor we have:

$$\begin{aligned} G^{\omega}_{\infty} &= -\oint_{\gamma_\infty} \frac{du}{u} u^{-\mathfrak{B}/6} e^{-\frac{\mathbf{w}^2}{24}E_2(\tau_*)} \\ &\left\{\sum_{\lambda}\left(\int_{\sqrt{y_*}\lambda\cdot\omega}^{\infty} e^{-2\pi t^2}dt\right) e^{-i\pi\tau_*\lambda^2 - i\mathbf{w}\cdot\lambda}(-1)^{(\lambda-\lambda_0)\cdot w_2}\right. \\ &\left. +2\pi\sum_{n=1}^{\infty}\frac{\left(\frac{i\mathbf{w}\cdot\omega}{2\sqrt{y_*}}\right)^n}{n!}\sum_{\lambda} H_{n-1}(2\pi\sqrt{y_*}\lambda\cdot\omega) e^{-i\pi\bar{\tau}_*\lambda_+^2 - i\pi\tau_*\lambda_-^2 - i\mathbf{w}\cdot\lambda}(-1)^{(\lambda-\lambda_0)\cdot w_2}\right\} \end{aligned} \tag{4.31}$$

where $\mathbf{w} = \kappa_2 u^{1/6} S$, the constant κ_2 is given in equation (4.54), and H_n are standard Hermite polynomials.

In particular, if $\sigma = -7$ so $\mathfrak{B} = 0$ the we have a nonzero constant:

$$G^{\omega}_{\infty} = -2\pi i \sum_{\lambda}\left(\int_{\sqrt{y_*}\lambda\cdot\omega}^{\infty} e^{-2\pi t^2}dt\right) e^{-i\pi\tau_*\lambda^2 - i\mathbf{w}\cdot\lambda}(-1)^{(\lambda-\lambda_0)\cdot w_2} \tag{4.32}$$

and if $\sigma = -8$ so $\mathfrak{B} = 1$ then we have a linear function of S:

$$\begin{aligned} G^{\omega}_{\infty} = -2\pi\kappa_2 &\left\{\sum_{\lambda}\left(\int_{\sqrt{y_*}\lambda\cdot\omega}^{\infty} e^{-2\pi r^2}dr\right)(S\cdot\lambda) e^{-i\pi\tau_*\lambda^2 - i\mathbf{w}\cdot\lambda}(-1)^{(\lambda-\lambda_0)\cdot w_2}\right. \\ &\left. +\frac{\pi}{\sqrt{y_*}} S\cdot\omega \sum_{\lambda} e^{-i\pi\bar{\tau}_*\lambda_+^2 - i\pi\tau_*\lambda_-^2}(-1)^{(\lambda-\lambda_0)\cdot w_2}\right\} \end{aligned} \tag{4.33}$$

and so on. Clearly, these expressions depend continuously on the metric and do not vanish as ω approaches any boundary of the light cone.

4.5 Details on the relation of SU(2), $N_f = 1$ and $AD3$ Coulomb branch integrals

In this section we prove the claims made between equations (4.9) and (4.15) for the Coulomb branch integrals of SU(2), $N_f = 1$ and $AD3$ theories.

We consider a small disk $B(u_*, \epsilon)$ of radius ϵ around the critical point u_*. Let γ_ϵ be the counterclockwise oriented boundary. Set $\Lambda_1 = 1$ so that $u_* = 1$ and define the deviation from the critical mass by $m = \frac{3}{2} + \mu$. Then we cut out disks of radius δ, with $\delta << \epsilon$ around the colliding points $u_\pm$ in the discriminant locus and let $\gamma_\pm$ be the ccw oriented boundaries of these discs. We are going to prove that

$$P_1(p, S) := \left[e^{-2p(u_* + \frac{2}{3}\mu) - T_* S^2} \left(\oint_{\gamma_\epsilon} \Omega - \oint_{\gamma_+} \Omega - \oint_{\gamma_-} \Omega \right) \right]_{\mu^0} \tag{4.34}$$

is a polynomial in p and S. Here it is understood that we take $\delta \to 0$ then $\epsilon \to 0$. As mentioned above the quantity in square brackets might have divergent terms for $\mu \to 0$. It has a Laurent expansion in $\mu^{1/4}$ around $\mu = 0$. The singular terms will cancel against other terms coming from the Seiberg-Witten contribution to the partition function. In any case, our main focus here is on the constant term, i.e. the coefficient of μ^0.

Moreover, we will compare the polynomial $P_1(p, S)$ to the u-plane contribution for the AD3 theory

$$P_{AD}(p, S) = \left[\left(\oint_{\gamma_\infty} \Omega_{AD} - \oint_{\gamma_+^{AD}} \Omega_{AD} - \oint_{\gamma_-^{AD}} \Omega_{AD} \right) \right]_{\Lambda_{AD}^0} \tag{4.35}$$

where now $\gamma_\pm^{AD}$ are small contours of radius ϵ around the two points in the AD3 discriminant locus $u_\pm = \pm 2\Lambda_{AD}^3$. We will show that $P_{AD}(p, S)$ is also a polynomial in p and S. Furthermore, if we define a grading of the polynomial P_1 by "R charge" with $R[p] = 6$ and $R[S] = 1$ then we will show that the highest degree is given by $6\ell + r = \mathfrak{B} = -\frac{1}{4}(7\chi + 11\sigma)$. Finally, defining $P_1^{\mathrm{top}}(p, S)$ to be the sum of terms with maximal R-charge we will show that

$$P_1^{\rm top}(p,S) = N P_{AD}(n_0 p, n_2 S) \tag{4.36}$$

for suitable constants N, n_0, n_2.

In the proof it is useful to note that for $b_2^+ = 1$ we have $\mathfrak{B} = -7-\sigma$ and $1-\chi/2 = \sigma/2-1$ and we recall that, up to an overall normalization we have (4.17) with

$$\Omega = du \left(\frac{du}{da}\right)^{1-\sigma/2} \Delta^{\sigma/8} e^{2pu+S^2 T} \tilde{\Theta}^{\omega,\omega_0}(\lambda_0,\lambda_0;\tau,z) \tag{4.37}$$

where $\lambda_0 = \frac{1}{2} w_2(X)$. It will be crucial to compare expressions for du/da and u in the relevant expansions in the $N_f = 1$ and AD3 contour integrals.

We begin with the expressions in the $N_f = 1$ theory

$$\left[e^{-2p(u_* + \frac{2}{3}\mu) - T_* S^2} \oint_{\gamma_\epsilon} \Omega \right]_{\mu^0} \tag{4.38}$$

Here we can set $\mu = 0$ in the expressions for Ω so that the two points $u_\pm$ collide at $u = u_*$. In evaluating this integral we expand the integrand in powers of $(u - u_*)$ and perform the contour integral. When $\mu = 0$ we find that $\tau(u)$ approaches $\tau_* = e^{i\pi/3}$ as $u \to u_*$ and indeed

$$\tau = \tau_* + PS((u-u_*)^{1/3}) \tag{4.39}$$

where $PS(x)$ means power series in positive powers of x that vanishes at $x = 0$. Similarly:

$$\frac{du}{da} = \kappa_1 (u-u_*)^{1/6} \Big(1 + PS((u-u_*)^{1/3})\Big) \tag{4.40}$$

with

$$\kappa_1 = \left(-\frac{1}{4} \left(\frac{3}{\rho}\right)^2 \left(-\frac{4}{9}\right)^{1/3} (E_6(\tau_*))^{-1/3} \right)^{1/2} \tag{4.41}$$

Similarly,

$$du\Big(\frac{du}{da}\Big)^{1-\sigma/2}\Delta^{\sigma/8} = \mathcal{N}^1_\infty \frac{d(u-u_*)}{(u-u_*)}(u-u_*)^{-\mathfrak{B}/6}(1+PS\,((u-u_*)^{1/3})) \tag{4.42}$$

with

$$\mathcal{N}^1_\infty = \kappa_1^{1-\sigma/2}(u_*-u_0)^{\sigma/8} \tag{4.43}$$

and finally, $T_* = u_*/3$ and

$$T - T_* = -\frac{\kappa_1^2}{24}E_2(\tau_*)(u-u_*)^{1/3}\Big(1+PS\,((u-u_*)^{1/3})\Big) \tag{4.44}$$

The integral over the phase of $u-u_*$ will kill all terms in the power series except those proportional to

$$\frac{d(u-u_*)}{(u-u_*)}(|(u-u_*)^{1/3}|)^n \tag{4.45}$$

for some integer n, and in our expressions n is always nonnegative. However, since we also take $\epsilon \to 0$ limit, inly the terms with $n = 0$ will contribute. We thus concentrate on the Laurent expansion in $(u-u_*)^{1/3}$ working to zeroth order in the power series expansion in $(\bar{u}-\bar{u}_*)^{1/3}$.

Now, since $(\tau - \tau_*)$ and du/da are expansions in positive powers of $(u-u_*)^{1/3}$ the resulting contour integral is a polynomial in p and S. Moreover, S always multiplies du/da, so by (4.40) if we assign charge +1 to S and +6 to p then the leading powers of $(u-u_*)^{1/3}$ are governed by the natural grading $6\ell + r$. The higher order terms in the expansions in $(u-u_*)^{1/3}$ above will contribute to lower degree terms in the polynomial P_1. So the contribution to P_1^{top} only comes from the leading order terms in the above expansions giving the contribution to the polynomial:

$$P^{\text{top}}_{1,\infty} = \mathcal{N}^1_\infty \oint \frac{d(u-u_*)}{(u-u_*)}(u-u_*)^{\mathfrak{B}/6}e^{2p(u-u_*)}F_\infty(\kappa_1(u-u_*)^{1/6}S) \tag{4.46}$$

where

$$F_\infty(w) = e^{-\frac{\mathbf{w}^2}{24}E_2(\tau_*)} \sum_{\lambda\in\Gamma+\lambda_0} (-1)^{w_2\cdot(\lambda-\lambda_0)} \mathcal{E}(\rho_\lambda^\omega(\mathbf{w});\rho_0) e^{-i\pi\tau_*\lambda^2 - i\mathbf{w}\cdot\lambda} \tag{4.47}$$

and here

$$\rho_\lambda^\omega(\mathbf{w}) := \sqrt{y_*}\lambda_+ - \frac{i}{2\pi\sqrt{y_*}}\mathbf{w}_+ \tag{4.48}$$

Let us compare the above contribution to P_1^{top} with the corresponding expression in the AD3 theory

$$\left[\oint_{\gamma_\infty} \Omega_{AD}\right]_{\Lambda_{AD}^0} \tag{4.49}$$

Since we are after the constant tesm we consider the AD3 family with $\Lambda_{AD} \to 0$. Equation (4.93) can be written as:

$$\frac{(E_4(\tau))^3}{(E_6(\tau))^2} = 4\left(\frac{\Lambda^3}{u}\right)^2 \tag{4.50}$$

and (4.95) can be written as

$$\left(\frac{du}{da}\right)^2 = -\frac{1}{6}\left(\frac{3}{\rho}\right)^2 \frac{E_4(\tau)}{E_6(\tau)}\frac{u}{\Lambda^2} \tag{4.51}$$

Now, as $u \to \infty$,

$$\tau - \tau_* = 2^{2/3}\frac{(E_6(\tau_*))^{2/3}}{E_4'(\tau_*)}\left(\frac{\Lambda^3}{u}\right)^{2/3}\left(1 + PS\left(\left(\frac{\Lambda^3}{u}\right)^{2/3}\right)\right) \tag{4.52}$$

and

$$\left(\frac{du}{da}\right) = \kappa_2 u^{1/6}\left(1 + PS\left(\left(\frac{\Lambda^3}{u}\right)^{2/3}\right)\right) \tag{4.53}$$

$$\kappa_2 = \left(-\frac{1}{12}\left(\frac{3}{\rho}\right)^2 \frac{2^{2/3}}{(E_6(\tau_*))^{1/3}}\right)^{1/2} \tag{4.54}$$

Similarly,

$$du\left(\frac{du}{da}\right)^{1-\sigma/2}\Delta^{\sigma/8} = \mathcal{N}_\infty^{AD}\frac{du}{u}u^{-\mathfrak{B}/6}\left(1 + PS\left(\left(\frac{\Lambda^3}{u}\right)^{2/3}\right)\right) \tag{4.55}$$

with $\mathcal{N}_\infty^{AD} = \kappa_2^{1-\sigma/2}$.

Once again, since we are taking the contour to infinity, we can focus on the holomorphic expansion in $u^{1/6}$. All the higher order terms in the power series have positive powers of Λ_{AD} and hence, again, we need only consider the leading order terms to get the contribution at Λ_{AD}^0. We have

$$P_{AD,\infty}(p,S) = \mathcal{N}_\infty^{AD} \oint_\infty \frac{du}{u} u^{-\mathfrak{B}/6} e^{2pu} F_\infty(\kappa_2 u^{1/6} S) \tag{4.56}$$

with the same function F_∞ defined in (4.47).

Comparing the two expressions we will find an equality of the kind (4.36), for this contribution to the polinomial, provided

$$\mathcal{N}_\infty^1 (2p)^\ell (\kappa_1 S)^r = N\mathcal{N}_\infty^{AD}(2n_0 p^\ell)(n_2\kappa_2 S)^r \tag{4.57}$$

for $r + 6\ell = \mathfrak{B}$. We solve for r in terms of ℓ and $\mathfrak{B}$ and then since different powers of ℓ appear in th polynomial we must have

$$\mathcal{N}_\infty^1 \kappa_1^{\mathfrak{B}} = N\mathcal{N}_\infty^{AD}(n_2\kappa_2)^{\mathfrak{B}} \tag{4.58}$$

$$\left(\frac{\kappa_2}{\kappa_1}\right)^6 = \frac{n_0}{n_2^6} \tag{4.59}$$

Now we consider an analogous computation for the contributions from $\gamma_\pm$. First we consider

$$\left[e^{-2p(u_* + \frac{2}{3}\mu) - T_* S^2} \oint_{\gamma_\pm} \Omega\right]_{\mu^0} \tag{4.60}$$

in the $N_f = 1$ theory. Here we will be writing the integrand as a power series in the local duality frame $q_\pm$.

For small μ the two points in the discriminant locus have an expansion

$$u_+ = 1 + \frac{2}{3}\mu + \left(\frac{2}{3}\right)^{5/2} \mu^{3/2} + \dots \tag{4.61}$$

$$u_- = 1 + \frac{2}{3}\mu - \left(\frac{2}{3}\right)^{5/2} \mu^{3/2} + \dots \tag{4.62}$$

A subtle point is that if we take the limit at $u \to u_*$ with μ held fixed then the expansions for u and du/da involve an infinite series of increasingly divergent terms in μ. The correct scaling limit [4] is to define

$$u = u_\pm + \mu^{3/2} v \tag{4.63}$$

and take the limit $\mu \to 0$ holding v fixed. With this understood we have

$$e^{2pu} = e^{2p(u_* + \frac{2}{3}\mu)} e^{\pm 2p\mu^{3/2}(2/3)^{5/2} E_6/E_4^{3/2}} (1 + \mathcal{O}(\mu^{1/2})) \tag{4.64}$$

where the Eisenstein series are expansions in $q_\pm$ in the standard way. Next we can write

$$\frac{du}{da} = \kappa_2 E_4^{-1/4} \mu^{1/4} \left(1 + PS(\mu^{1/2})\right) \tag{4.65}$$

$$\kappa_1 = \left(\frac{\zeta_s}{2} \left(\frac{3}{\rho} \sqrt{\frac{2}{27}} \right)^2 \right)^{1/2} \tag{4.66}$$

and similarly

$$du \left(\frac{du}{da}\right)^{1-\sigma/2} \Delta^{\sigma/8} = \mathcal{N}_\pm^1 u^{-\mathfrak{B}/4} \frac{dq_\pm}{q_\pm} H(q_\pm) \left(1 + PS(\mu^{1/2})\right) \tag{4.67}$$

where the power series in $\mu^{1/2}$ has coefficients which are themselves power series in $q_\pm$. Here

$$H(q) := \left(q \frac{d}{dq} \Big(\frac{E_6}{E_4^{3/2}} \Big) \right) E_4^{-(\sigma+1)/4} (E_6^2 - E_4^3)^{\sigma/8} \tag{4.68}$$

[4]This is a consequence of the linear combinations we found in equation (4.3) above.

$$\mathcal{N}^1_\pm = \pm\Big(\frac{2}{3}\Big)^{5/2(1+\sigma/4)}\kappa_3^{1-\sigma/2}(u_* - u_0)^{\sigma/8} \tag{4.69}$$

Now the expansion in $p^\ell S^r$ comes with a power $\mu^{(r+6\ell)/4}$ so the μ^0 term satisfies the selection rule and the higher powers in the μ expansion contribute lower order terms. Thus, the contribution to the polynomial from these two singularities is the sum over + and − of

$$P_1^{\text{top}}(p,S) = \eta\mathcal{N}^1_\pm\left[\mu^{-\mathfrak{B}/4}\oint\frac{dq_\pm}{q_\pm}H(q_\pm)e^{\pm 2p\mu^{3/2}(2/3)^{5/2}E_6/E_4^{3/2}}F_\pm(\kappa_3\mu^{1/4}E_4^{-1/4}S)\right]_{\mu^0} \tag{4.70}$$

where

$$F_\pm(\mathbf{w}) = \frac{1}{\sqrt{8}}e^{-\frac{\mathbf{w}^2}{24}E_2(\tau)}\sum_{\lambda\in\Gamma+\mu}(-1)^{w_2\cdot(\lambda-\mu)}(\text{sign}(\lambda,\omega) - E(\rho_\lambda^{\omega_0}(\overline{\tau}_{0,s})))e^{-i\pi\tau\lambda^2 - i\mathbf{w}\cdot\lambda} \tag{4.71}$$

Finally we come to the contributions

$$\left[\oint_{\gamma_+^{AD}}\Omega_{AD}\right]_{\Lambda_{AD}^0} \tag{4.72}$$

in the AD3 family of curves.

In the AD3 family of curves we have the exact formulae for the expansions in $q_\pm$ near $u_\pm$:

$$u = \pm 2\Lambda_{AD}^3\frac{E_6}{E_4^{3/2}} \tag{4.73}$$

$$\frac{du}{da} = \kappa_4 E_4^{-1/4}\Lambda_{AD}^{1/2} \tag{4.74}$$

$$\kappa_4 = \left(-\zeta_s\frac{1}{6}\Big(\frac{3}{\rho}\Big)^2\right)^{1/2} \tag{4.75}$$

and we compute:

$$du\left(\frac{du}{da}\right)^{1-\sigma/2}\Delta^{\sigma/8} = \mathcal{N}_\pm^{AD}\Lambda_{AD}^{\mathfrak{B}/2}\frac{dq_\pm}{q_\pm}H(q_\pm) \tag{4.76}$$

$$\mathcal{N}_\pm^{AD} = \pm 2^{1+\sigma/4}\kappa_4^{1-\sigma/2} \tag{4.77}$$

So these terms contribute to the polynomial

$$P_{AD,\pm} = \eta \mathcal{N}_{\pm}^{AD} \left[\Lambda_{AD}^{-\mathfrak{B}/2} \oint \frac{dq_\pm}{q_\pm} H(q_\pm) e^{\pm 4p\Lambda_{AD}^3 \frac{E_6}{E_4^{3/2}}} F_\pm(\kappa_4 \Lambda_{AD}^{1/2} E_4^{-1/4} S) \right]_{\Lambda_{AD}^0} \tag{4.78}$$

Now to match these using the rescalings (4.36) we have the conditions

$$\mathcal{N}_+^1 \left[2p\Big(\frac{2}{3}\Big)^{5/2} \right)^{\ell} (\kappa_3 S)^r = N \mathcal{N}_+^{AD} (4n_0 p)^{\ell} (\kappa_4 n_2 S)^r \tag{4.79}$$

when $6\ell + r = \mathfrak{B}$. In a way similar to (4.58) and (4.59) we obtain:

$$\mathcal{N}_+^1 \kappa_3^{\mathfrak{B}} = N \mathcal{N}_+^{AD} (n_2 \kappa_4)^{\mathfrak{B}} \tag{4.80}$$

$$\Big(\frac{\kappa_4}{\kappa_3}\Big)^6 = 2\Big(\frac{3}{2}\Big)^{5/2} \frac{n_0}{n_2^6} \tag{4.81}$$

We now ask if there are constants N, n_0, n_2 that allow us to solve the four conditions (4.58), (4.59), (4.80), (4.81). The conditions are not all independent, and in fact, there are such constants iff we have

$$\Big(\frac{\kappa_1 \kappa_4}{\kappa_2 \kappa_3}\Big)^4 = 2^{-3/2} 3^{5/2} \tag{4.82}$$

$$\frac{\mathcal{N}_\infty^1}{\mathcal{N}_+^1} \Big(\frac{\kappa_1}{\kappa_3}\Big)^{\mathfrak{B}} = \frac{\mathcal{N}_\infty^{AD}}{\mathcal{N}_+^{AD}} \Big(\frac{\kappa_2}{\kappa_4}\Big)^{\mathfrak{B}} \tag{4.83}$$

Plugging the above values we can confirm that those conditions are indeed satisfied.

4.6 Seiberg-Witten contribution to Z_{AD}

When X has $b_2^+ > 1$ only Z_{SW} contributes to the partition function. In this section we will evaluate it fairly explicitly for the AD3 family (4.4) in the limit $\Lambda_{AD} \to 0$. Thus we are starting from the definition (2.51).

To begin we put (2.51) in a form which is more suitable for explicit evaluation. In fact our derivation of the result (4.109) below applies to any family of elliptic SW curves with a simple zero of the discriminant at $u = u_s$ such that Weierstrass invariants g_2, g_3 are nonzero at $u = u_s$ (this is Kodaira type I_1). We also assume $\lambda_{0,s} = \xi_s = \frac{1}{2}\overline{w_2(X)}$. Thus holds for the SU(2) $N_f = 1$ family and therefore for the AD3 family. Moreover, the duality transformations needed to transform from the duality frame at $u = \infty$ to u near u_s are all, according to equation (11.17) of [49], conjugate to T. It turns out that the measure of all the u-plane transforms by a character under S and T. Therefore the root of unity η_s is independent of s and we will just denote it by η.

Now we can replace the sum over λ by the average over λ and $-\lambda$. Because $\Lambda_{0,s} = \xi_s = \frac{1}{2}\overline{w_2(X)}$ we have

$$e^{-4\pi i\lambda\cdot\xi_s} = e^{-2\pi i(\nu+\frac{1}{2}w_2)\cdot w_2} = e^{-i\pi w_2^2} = (-1)^{\sigma} \tag{4.84}$$

Moreover it is a standard result of SW theory that

$$SW(-\lambda) = (-1)^{\chi_h} SW(\lambda) \tag{4.85}$$

so in the sum over λ in (2.51) we can freely make the replacement:

$$SW(\lambda)e^{2\pi i\lambda\cdot\lambda_0}e^{-i\lambda\cdot S\frac{du}{da_s}} \to \frac{1}{2}SW(\lambda)e^{2\pi i\lambda\cdot\lambda_0}\left(e^{-i\frac{du}{da_s}S\cdot\lambda} + (-1)^{\chi_h+\sigma}e^{i\frac{du}{da_s}S\cdot\lambda}\right) \tag{4.86}$$

The reason this is useful is that the expansion in $S\cdot\lambda$ only involves powers of $\frac{du}{da_s}$ of a definite parity independent of λ. That will be important since, as we will see below, we can readily determint the q_s-expansion of $\left(\frac{du}{da_s}\right)^2$ near u_s, but taking the square-root could be tricky. Equation (4.86) motivates us to define:

$$\frac{1}{2}\left(e^{-i\frac{du}{da_s}S\cdot\lambda} + (-1)^{\chi_h+\sigma}e^{i\frac{du}{da_s}S\cdot\lambda}\right) := \sum_{n\geq 0}\hat{c}_n^{\chi_h+\sigma}(S)\left(\frac{da_s}{du}\right)^{-n} \tag{4.87}$$

with

$$\hat{c}_n^{\chi_h+\sigma}(S) = \begin{cases} \frac{1}{n!}e^{-i\pi n/2}(S\cdot\lambda)^n & n = (\chi_h+\sigma) \bmod 2 \\ 0 & n \neq (\chi_h+\sigma) \bmod 2 \end{cases} \tag{4.88}$$

Now suppose we have a SW curve presented in the form:

$$y^2 = x^3 + A_2 x^2 + A_4 x + A_6 \tag{4.89}$$

and there is a special coordinate a_s so that $a_s \to 0$ but

$$\frac{da_s}{du} = \frac{\rho}{\pi}\omega_1 \tag{4.90}$$

is nonvanishing as $q_s = e^{2\pi i \tau_s} \to 0$. [5]

In order to evaluate (2.51) we need to know the expansions

$$\begin{aligned} u &= u_s + \mu_1 q_s + \mu_2 q_s^2 + \ldots \\ a &= \kappa_1 q_s + \kappa_2 q_s^2 + \ldots \end{aligned} \tag{4.91}$$

We now show how to extract these expantion - in principle - from the SW curve.

From A_2, A_4, A_6 we can construct the standard Weierstrass invariants g_2, g_3. For SU(2) theories and the AD3 family these will be polynomials in u. In general we have

$$(12)^3 \frac{g_2^3}{g_2^3 - 27 g_3^2} = j(\tau_s) = q_s^{-1} + 744 + 196884 q_s + 21493760 q_s^2 \tag{4.92}$$

Actually, for our purposes, this equation is more usefully written as

$$(27)\frac{g_3^2}{g_2^3} = \frac{E_6^2}{E_4^3} \tag{4.93}$$

[5]Here ρ is a relative normalization between the standard periods ω of the elliptic curve and $\frac{da_s}{du}$. Its value depends on the conventions used to normalize the cental charge. In the conventions of [50] the central charge is $Z(\gamma) = \pi^{-1}\oint_\gamma \lambda$, so $\frac{da}{du} = \pi^{-1}\oint_A \frac{d\lambda}{du}$. Next, for an elliptic curve presented in the form (4.89) the canonically normalized holomorphic differential is $\sqrt{2}\frac{dx}{y}$. Finally, we note that for the AD3 family (4.3) we have $\frac{d\lambda}{du} = \frac{1}{2}\frac{dz}{y}$. We thus conclude that for natural conventions for class S we have $\rho = 1/\sqrt{8}$. However, we leave ρ undetermined above since it is different if one uses other conventions such as those of [49] and [7]. The results for different choices of ρ are simply related by a renormalization of S.

Plugging (4.91) into either version gives a triangular system of equations from which we can extract the coefficients μ_n. Next, if we have chosen a basis so that $\tau = \omega_2/\omega_1$ then the period ω_1 is expressed in terms of coefficients of the elliptic curve τ by

$$\omega_1^2 = 2\left(\frac{\pi}{3}\right)^2 \frac{E_6(\tau)}{E_4(\tau)} \cdot \frac{g_2}{g_3} \tag{4.94}$$

and hence

$$\left(\frac{da_s}{du}\right)^2 = 2\left(\frac{\rho}{3}\right)^2 \frac{E_6(\tau)}{E_4(\tau)} \cdot \frac{g_2}{g_3} \tag{4.95}$$

Now we use the standard expansions of E_4, E_6 interms of q_s and we expand the polynomials g_2, g_3 of u around u_s and use (4.91). This gives κ_1^2 and all the κ_n/κ_1 for $n > 1$.

We also write $\Delta = \mathcal{N}^u_{\text{math}}\Delta^{\text{math}}$ where Δ^{math} is the mathematical discriminant of the elliptic curve,

$$\Delta^{\text{math}} = (e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 = 4(4g_2^4 - 27g_3^2) = 2^{-22}\left(\frac{da_s}{du}\right)^{-12} \eta(\tau_s)^{24} \tag{4.96}$$

where e_i, $i = 1, 2, 3$ are the roots of the cubic. Putting all these things together we can write (2.51) in the form

$$\begin{aligned} Z^s_{SW} = &\left(\sqrt{32}\pi\beta^\sigma\alpha^\chi 2^{-11\sigma/4}(\mathcal{N}^u_{\text{math}})^{\sigma/8}\eta\right) \\ &\sum_{n\geq 0}\sum_{\lambda\in\frac{1}{2}w_2+\Gamma} e^{\pi i\lambda\cdot w_2}(-1)^{n(\lambda)} SW(\lambda)\cdot \hat{c}_n^{\chi_h+\sigma}(S) \\ &\left[\frac{du}{dq_s}\left(\frac{\eta(\tau_s)^{24}}{q_s}\right)^{\sigma/8}\left(\frac{a_s}{q_s}\right)^{\chi_h-1}\left(\frac{da_s}{du}\right)^{1-2\chi_h-\sigma-n} e^{2pu+S^2T_s(q_s)}q_s^{-n(\lambda)}\right]_{q_s^0} \end{aligned} \tag{4.97}$$

This result is a slight generalization of, and improvement upon, equation (11.28) of [7].

We now specialize (4.97) to the AD3 family of curves. For the AD3 family we have $g_2 = 3\Lambda^2$ and $g_3 = -u/2$, Δ is quadratic in u and so there are just two singularities $u_\pm$. [6]

[6] In this section we write Λ instead of Λ_{AD}.

Near each of them we have the expansion in q_s:

$$u = u_s \frac{E_6}{E_4^{3/2}} \tag{4.98}$$

with $u_s = 2\zeta_s \Lambda^3$. Here $\zeta_s = \pm 1$ at the two singularities and E_6, E_4 are power series in q_s beginning at 1. Fractional powers of Eisenstein series are to be interpreted as power series in q_s. Note that

$$\frac{du}{dq} = -\zeta_s (12\Lambda)^3 (q^{-1}\eta^{24}) \cdot E_4^{-5/2} \tag{4.99}$$

From (4.95) we obtain $\kappa_1^2 = -\frac{1}{2}(12\Lambda)^5 \zeta_s$ and

$$\check{E}_1(q) := \frac{a_s/\kappa_1}{q} := 1 + \sum_{n\geq 2}^{\infty} \frac{\kappa_n}{\kappa_1} q^{n-1} \tag{4.100}$$

is independent of s and satisfies the equation:

$$q\frac{d}{dq}(q\check{E}_1(q)) = \eta^{24} E_4^{-9/4} = (12)^{-3}(E_4^3 - E_6^2) E_4^{-9/4} \tag{4.101}$$

from which one may generate its q-series. There does not appear to be any simple expression for $\check{E}_1$ in terms of E_2, E_4, and E_6 and we will, regrettably, take the above as its definition.

Using these formulae and (4.97) we obtain

$$\begin{aligned} Z_{SW}^{ADfamily} &= \mathcal{C}_1 \sum_{r_1\geq 0} \delta_{r_1} \sum_{\lambda} \sum_{\zeta_s=\pm 1} \Lambda^{\frac{1}{2}(r_1-(\chi_h-c_1^2))} e^{i\pi\lambda\cdot w_2} (-1)^{n(\lambda)} SW(\lambda) \frac{(\sqrt{24}S\cdot\lambda)^{r_1}}{(4\rho)^{r_1} r_1!} \\ &\zeta_s^{2\chi_h+\sigma+r_1-\frac{1}{2}(\chi_h+\sigma+r_1)} \left[\check{E}_1^{\chi_h-1} (q^{-1}\eta^{24})^{1+\frac{\sigma}{8}} E_4^{-\frac{1}{4}(2\chi_h+\sigma+9+r_1)} e^{4(\zeta_s\Lambda^3 p)E_6 E_4^{-3/2} + (\zeta_s\Lambda\left(\frac{S}{4\rho}\right)^2)E_2E_4^{-1/2}} \right]_{q^{n(\lambda)}} \end{aligned} \tag{4.102}$$

where δ_{r_1} enforces the constraint $r_1 = (\chi_h + \sigma) \bmod 2$ and

$$\mathcal{C}_1 = \left(\sqrt{32}\pi\beta^{\sigma}\alpha^{\chi}\eta\right)\left(-i2^{11/2}3^{7/2}\rho^{-1}\right)^{\chi_h}\left(i2^{-13/4}3^{1/2}\rho^{-1}\mathcal{N}_{\text{math}}^{u}\right)^{\sigma} \tag{4.103}$$

Next, we expand the terms with p and S^2 in the exponential. We find that the terms proportional to $(S \cdot \lambda)^{r_1}(S^2)^{r_2}p^{\ell}$ come with the power $\Lambda^{U/2}$ where [7]

$$U := r_1 + 2r_2 + 6\ell - \mathfrak{B} \qquad\qquad \mathfrak{B} := \chi_h - c_1^2 = -\frac{7\chi + 11\sigma}{4} \tag{4.104}$$

This is, of course, a reflection of the $\mathrm{U}(1)_R$ symmetry at the superconformal point.

Next, the entire dependence of the expression (4.102) on the two values $s = +$ and $s = -$ is summarized by the power $\zeta_s^{\frac{1}{2}(\sigma + r_1 - \chi_h) + \ell + r_2}$, so the sum over ζ_s imposes the selection rule $U = 0 \bmod 4$. (This selection rule implies that $r_1 = (\chi_h + 2) \bmod 2$ so we can now drop that constraint.) The result of these considerations is that

$$\begin{aligned} Z_{SW}^{AD3family} = 2c_1 \sum_{U \bmod 4} \sum_{\lambda} \Lambda^{\frac{1}{2}U} e^{i\pi\lambda\cdot w_2}(-1)^{n(\lambda)} SW(\lambda) \frac{(\sqrt{24}S \cdot \lambda)^{r_1}}{(4\rho)^{r_1} r_1!} \frac{(S^2)^{r_2}}{(4\rho)^{2r_2} r_2!} \frac{(4p)^{\ell}}{\ell!} \\ \left[\breve{E}_1^{\chi_h - 1}(q^{-1}\eta^{24})^{1+\frac{\sigma}{8}} E_2^{r_2} E_4^{-\frac{1}{4}(9+U-5\chi_h)} E_6^{\ell}\right]_{q^{n(\lambda)}} \end{aligned} \tag{4.105}$$

where the first sum is over all integers $r_1, r_2, \ell \geq 0$ such that $U = 0 \bmod 4$.

Now we wish to take the $\Lambda \to 0$ limit. We can organize the sum by the degree U. Note that there are potentially negative powers of Λ is $\mathfrak{B} > 0$. Nevertheless, the correlators should be finite in the $\Lambda \to 0$ limit. This was the original argument of [23] used to derive sum rules of SW inveriands. However, unlike [23], here we are not assuming that X is of SW simple type. [8]

For any given X there will be a finite number of sum rules, one for each nonnegative integer k such that $k - \mathfrak{B} < 0$ and $k = \mathfrak{B} \bmod 4$. For each such k the sum, for fixed degree $U = k - \mathfrak{B}$ must vanish. To be concrete:

[7] The quantity $\mathfrak{B}$ is very natural in this subject. The quantity $2\mathfrak{B}$ provides a lower bound for the number of SW basic classes of X [23, 24].

[8] "Seiberg-Witten simple type" is often given the acronym SWST below.

1. Suppose $\chi_h - c_1^2 > 0$ and $\chi_h - c_1^2 = 0 \bmod 4$. Then

$$0 = \sum_{\lambda} e^{i\pi\lambda\cdot w_2}(-1)^{n(\lambda)} SW(\lambda) \left[\check{E}_1^{\chi_h - 1}(q^{-1}\eta^{24})^{1+\frac{\sigma}{8}} E_4^{-\frac{1}{4}(9+c_1^2-6\chi_h)} \right]_{q^{n(\lambda)}} \tag{4.106}$$

2. Suppose $\chi_h - c_1^2 > 1$ and $\chi_h - c_1^2 = 1 \bmod 4$. Then the $U = 1 - (\chi_h - c_1^2)$ only gets a contribution from $r_1 = 1$, $r_2 = \ell = 0$ and hence

$$0 = \sum_{\lambda} e^{i\pi\lambda\cdot w_2}(-1)^{n(\lambda)} SW(\lambda)(S\cdot\lambda) \left[\check{E}_1^{\chi_h - 1}(q^{-1}\eta^{24})^{1+\frac{\sigma}{8}} E_4^{-\frac{1}{4}(10+c_1^2-6\chi_h)} \right]_{q^{n(\lambda)}} \tag{4.107}$$

3. Suppose $\chi_h - c_1^2 > 2$ and $\chi_h - c_1^2 = 2 \bmod 4$. Then the $U = 2 - (\chi_h - c_1^2)$ gets a contribution from $r_1 = 2$, $r_2 = \ell = 0$ and $r_1 = 0$, $r_2 = 1$, $\ell = 0$ hence

$$\begin{aligned} 0 = \sum_{\lambda} e^{i\pi\lambda\cdot w_2}(-1)^{n(\lambda)} SW(\lambda) \\ \Big\{ S^2 \left[\check{E}_1^{\chi_h - 1}(q^{-1}\eta^{24})^{1+\frac{\sigma}{8}} E_2 E_4^{-\frac{1}{4}(11+c_1^2-6\chi_h)} \right] \\ -\frac{1}{2}(-24)^{\frac{1}{2}(\chi_h+\sigma+1)}(S\cdot\lambda)^2 \left[\check{E}_1^{\chi_h - 1}(q^{-1}\eta^{24})^{1+\frac{\sigma}{8}} E_4^{-\frac{1}{4}(11+c_1^2-6\chi_h)} \right]_{q^{n(\lambda)}} \Big\} \end{aligned} \tag{4.108}$$

4. And so on: We get rather complicated polynomials in S^2 and $S\cdot\lambda$ which must vanish. If we assume SWST then only the spin-c structures with $n(\lambda) = 0$ contribute and we get the criteria of [23]. In this case the formulae simlify a lot because all the factors of the form $[\check{E}_1^{\chi_h-1}\cdots]_{q^{n(\lambda)}}$ can be put equal to 1.

Now we consider the actual value at $\Lambda = 0$. According to our conjecture above, this should give the partition function of topologically twisted AD3 theory on standard four-manifolds. Technically, we simply keep the terms above with $U = 0$ so our formula is

$$\begin{aligned} \langle e^{p\mathcal{O}+\mathcal{O}(S)}\rangle_X^{AD3} = 2\mathscr{C}_1 \sum_{U=0}\sum_{\lambda} e^{i\pi\lambda\cdot w_2}(-1)^{n(\lambda)} SW(\lambda) \frac{(\sqrt{24}S\cdot\lambda)^{r_1}}{(4\rho)^{r_1} r_1!} \frac{(S^2)^{r_2}}{(4\rho)^{2r_2} r_2!} \frac{(4p)^{\ell}}{\ell!} \\ \left[\check{E}_1^{\chi_h - 1}(q^{-1}\eta^{24})^{1+\frac{\sigma}{8}} E_2^{r_2} E_4^{-\frac{1}{4}(9-5\chi_h)} E_6^{\ell} \right]_{q^{n(\lambda)}} \end{aligned} \tag{4.109}$$

This is the generator of correlation functions of the twisted AD3 theory on four-manifolds X with $b_1 = 0$ and $b_2^+ > 1$. It is only non-vanishing for $\mathfrak{B} = \chi_h - c_1^2 \geq 0$.

We now assume that X has SWST so that only spin-c structures with $n(\lambda) = 0$ contribute. Moreover, we will also assume that X is of superconformal simple type (SCST) with $\mathfrak{B} \geq 4$. According to [23], [24] this means that

$$\sum_{\lambda} e^{i\pi\lambda\cdot w_2} SW(\lambda)(\lambda \cdot S)^k = 0 \qquad 0 \leq k \leq \mathfrak{B} - 4 \tag{4.110}$$

Therefore, given the constraint $U = 0$ the only terms that can contribute are $r_1 = \chi_h - c_1^2 - 2 = \mathfrak{B} - 2$, $r_2 = 1$, $\ell = 0$, and $r_1 = \chi_h - c_1^2 = \mathfrak{B}$, $r_2 = \ell = 0$, and our partition function simplifies to

$$\langle e^{p\mathcal{O}+\mathcal{O}(S)}\rangle_X^{AD3} = \mathcal{C}_2 \sum_{\lambda} e^{i\pi\lambda\cdot w_2} SW(\lambda) \left[\frac{\mathfrak{B}(\mathfrak{B}-1)}{24} S^2 (S\cdot\lambda)^{\mathfrak{B}-2} + (S\cdot\lambda)^{\mathfrak{B}}\right] \tag{4.111}$$

and to get the constant we observe that $\Delta^{\mathrm{math}} = -27(u^2 - (2\Lambda^3)^2)$ so $\mathcal{N}^u_{\mathrm{math}} = -1/27$. After some computation we find:

$$\mathcal{C}_2 = \frac{\sqrt{128}\pi\eta'}{\mathfrak{B}!} \left(\frac{32\beta\rho^{3/2}}{3^{3/8}}\right)^{\sigma} \left(2^{9/4}\alpha\rho^{3/2}\right)^{\chi} \tag{4.112}$$

where η' is an eighth root of unity we have not determined. (One could probably use the fact that SU(2) $N_4 = 1$ theory is time-reversal invariant for Λ real to constrain this phase.)

4.7 Discussion

A striking property of equation (4.111) is that it does not depend on p. This comes about because when the condition $U = 0$ is combined with the SCST condition, the only solutions have $\ell = 0$. This means the 0-observable is a "null vector" in the topological sector. That is, insertions of $\mathcal{O}$ into correlators always vanish for such four-manifolds. Although it is certainly true that $\mathcal{O}_{\text{classical}} = 0$ is not obvious why this should be true in the quantum theory (i.e. why quantum fluctuations can not have non-zero contribution).

It is important to recall that the very existence of the twisted partition function in the limit $m \to m_*$ is nontrivial [23,24] and implies necessary conditions for its finiteness. These are the conditions (4.106), (4.107), (4.108), et. seq. above in the special case of SWST. These conditions are quite complicated so the authors of [23, 24] also formulated the SCST condition, namely, that either $\mathfrak{B} \leq 3$ or (4.110) holds. The SCST condition is, a priori, sufficient but not necessary condition for finiteness of the $\Lambda \to 0$ limit. As far as we know, all known standard four-manifolds satisfy the SCST condition. It was conjectured and checked in multiple examples [23, 24] that in fact all standard four-manifolds are of SCST. The work of [47] gave a different argument that complex algebraic manifolds are of SCST. The more recent work [48] shows - subject to an unproven hypothesis - that for all standard four-manifolds, SWST implies SCST. Therefore, (accepting the work of [48]), all standard four-manifolds of SWST have the property that the topological correlators are given by (4.111), and, in particular, the 0-observable is a "null-vector". That property is not true for the general formula (4.109) corresponding to general four-manifold (not necessarily satisfying the SWST condition). Why this should be so is mysterious at present. There ought to be a good reason why the zero observable acts as a null vector.

The absence of a compelling reason for $\mathcal{O}$ to be a null-vector leads us to take seriously the possibility that there might be standard four-manifolds that are not of Seiberg-Witten simple type since if we drop the SCST condition (4.110) there are many more solutions to $U = 0$, i.e. $r_1 + 2r_2 + 6\ell = \mathfrak{B}$ which will contribute to (4.109). Some will include $\ell \neq 0$, and the necessary conditions (4.106), (4.107), (4.108), et. seq. do not eliminate the p-dependence.

Chapter 5

u-plane integral from single M5 brane

The four dimensional $N = 2$ theories of class S are defined by means of:

1. Putting the hypothetical "non-Lagrangian" six-dimensional (2,0) theory [45, 46] on $C \times X$, where C is a Riemann surface and X is a four-manifold. The bosonic symmetry algebra of the resulting theory on $C \times X$ consists of $so(5)_R$ R-symmetry, two-dimensional Lorentz algebra $so(2)_C$ and the usual four-dimensional Lorentz algebra $su(2)_+ \oplus su(2)_-$.

2. Partial topological twisting on C identifying the diagonal subgroup $so(2)'_C \underset{\text{diag}}{\subset} so(2)_C \oplus so(2)_R$, where $so(2)_R \oplus so(3)_R$ is a maximal subgroup of $so(5)_R$ as the new Lorentz algebra.

3. Compactification on C, which given a direct product metric $g_{C\times X} = t_1^2 g_C \oplus t_2^2 g_X$ corresponds to taking the limit $t_1^2 \to +0$.

As a result of these steps one obtains a broad family of four-dimensional $N = 2$ theories some of which are can be described by Lagrangian and some are believed to have no description in terms of an action principle. Here our aim is to turn on the Donaldson-Witten (DW) twist as well and identify $su(2)'_+ \underset{\text{diag}}{\subset} su(2)_+ \oplus so(3)_R$ as the Lorentz algebra of the twisted 4d theory. This corresponds to considering compactification of the fully twisted (2,0) theory on C (with $so(2)'_C \oplus su(2)'_+ \oplus su(2)_-$ Lorentz algebra).

Assuming the usual metric independence properties of the fully twisted (2,0) theory on $C \times X$ with $g_{C\times X} = t_1^2 g_C \oplus t_2^2 g_X$ the limits $t_1^2 \to +0$ and $t_2^2 \to +0$ must commute. Therefore, DW-twisted class S theory is equivalent to a two dimensional theory on C [42] obtained by compactifying the fully twisted (2,0) theory on X.

The IR limit of the fully twisted (2,0) theory on $C{\times}X$ is described by the effective theory of small perturbations of a single M5 brane theory on $\Sigma \times X$ [43], where $\Sigma \subset T^*C$ plays the role of Seiberg-Witten curve for the class S theory associated compactification on C. In the fully twisted (2,0) on $C \times X$ theory one should topological partition function independent of the overall scale of the metric and thus we expect to have $Z_{UV} = Z_{IR}$. Therefore, the Seiberg-Witten effective field theory of any class S theory should be equivalent to a two-dimensional model obtained by compactifying the single M5 on a branched cover Σ of C [1] with Σ playing the role of the Seiberg-Witten curve for the corresponding class S theory.

In the rest of this chapter we present calculation of the two-dimensional action starting from the PST action [44] for single M5 brane.

5.1 PST action on four-manifold times Riemann suface

Low energy effective action of single M5 brane proposed by Pasti, Sorokin and Tonin [44] is given by

$$S = \int_{M_6} \left[\sqrt{-\det\left(G_{mn} + i\tilde{H}_{mn}\right)} - \frac{1}{2} H \wedge \alpha \wedge \iota_v H + \frac{1}{2} \mathrm{C} \wedge F + \mathbb{X}^* \mathbb{C}_6 \right], \tag{5.1}$$

where

[1] This reasoning is somewhat analogous to the motivation behind the AGT correspondence.

- $\mathbb{X} = (X^{\underline{m}}, \theta^{\underline{\alpha}})$ with $\underline{m} = 0, \ldots, 10$, $\underline{\alpha} = 1, \ldots, 32$ is an embedding $\mathbb{X} : M_6 \to M_{11|32}$ of the M5 brane's worldvolume M_6 (i.e. a bosonic manifold $M_{6|0}$) into the superspace $M_{11|32}$ of eleven-dimensional supergravity.

- $\mathbb{C}_3$, $\mathbb{C}_6$, $\mathbb{G}$ and $\mathbb{E}$ are respectively 11|32 superfield extensions of the 3-form, 6-form, metric and vielbein fields on $M_{11|32}$. Also, $\mathrm{C} = \mathbb{X}^*\mathbb{C}_3$ and $G = \mathbb{X}^*\mathbb{G}$.

- $F = dB$ is field strength associated with two-form field B and $H = F + \mathrm{C}$.

- $\tilde{H} = *_G \iota_{\mathsf{v}} H$ and $\alpha = da$ where a is an auxiliary field known as PST scalar and v is a vector field $\mathsf{v}^m = \frac{\alpha^m}{\sqrt{\alpha_k \alpha^l}}$ (see [44] for details).

- In what follows x^m will be local coordinates on M_6.

Here we consider the regime in which the bosonic embedding X can be decomposed as $X^{\underline{m}} = X_0^{\underline{m}} + Y^{\underline{m}}$, where $X_0^{\underline{m}}(x)$ represents certain geometric configuration of the M5 brane and $Y^{\underline{m}}$ represents small fluctuations. The (six) tangential components of the fluctuations $Y^{\underline{m}}$ can be removed by fixing the reparametrization gauge symmetry, so there are in fact only 5 scalar fields Y^I where I labels normal directions to the background shape given by $X_0^{\underline{m}}$. Expanding in fluctuations and keeping only relavant terms in the Lagrangian density terms we obtain (see appendix (B) for derivation)

$$
\begin{aligned}
S = \int_{M_6} \Big\{ & -\frac{1}{2} F \wedge \alpha \wedge \iota_{\mathsf{v}} (1 - *_g) F + \frac{1}{2} G_{IJ} d^{\nabla} Y^I \wedge *_g d^{\nabla} Y^J \\
& + \frac{1}{8} (1 + *_g)(1 - \alpha \wedge \iota_{\mathsf{v}}) F \wedge \partial_I (*_g) \overline{\theta_+} \Gamma^I \Gamma^{\wedge 3} \theta_+ \\
& + 2 \overline{\theta_+} \Gamma \wedge *_g \Big(d + [\omega]_{\mu^1} + ([\omega_I]_{\mu^{-2}} - \frac{1}{8} *_g^{-1} \partial_J (*_g) \Gamma_I \Gamma^J \Big) d^{\nabla} Y^I) \theta_+ \\
& + \frac{4}{2 \cdot 4!} \overline{\theta_+} \Gamma \wedge *_g \left[\delta_\theta^2 \omega |_{\psi, F=0} \right]_{\mu^1} \theta_+ \Big\}
\end{aligned}
\tag{5.2}
$$

Here $g_{mn} := \eta_{AB} E_M^A(X_0, Y) E_N^A(X_0, Y) \partial_m X_0^M \partial_n X_0^N$ is the metric induced from 11d by a purely bosonic embedding X_0.

The last term in (5.2) is quartic in fermions. It is evaluated in section (B.1) and is given by

$$\int_{M_6} d^6x\,\sqrt{-g}\Big\{\frac{1}{2}R_{IJKL}\Big(\overline{\theta_+}\Gamma^m\Gamma^{IJ}\theta_+\overline{\theta_+}\Gamma_m\Gamma^{KL}\theta_+ + \overline{\theta_+}\Gamma^m\Gamma^{IJT}\theta_+\overline{\theta_+}\Gamma_m\Gamma^{KL}{}_T\theta_+\Big)$$
$$+\,2R_{mInJ}\Big(\overline{\theta_+}\Gamma^m\Gamma^{IK}\theta_+\overline{\theta_+}\Gamma^n\Gamma_K{}^J\theta_+ + \frac{1}{2}\overline{\theta_+}\Gamma^m\Gamma^{IST}\theta_+\overline{\theta_+}\Gamma^n\Gamma_{ST}{}^J\theta_+ - \frac{1}{6}\overline{\theta_+}\Gamma^{mst}\Gamma^I\theta_+\overline{\theta_+}\Gamma^n{}_{st}\Gamma^J\theta_+\Big)\Big\} \tag{5.3}$$

5.2 Decomposition of the action (5.2) on $M_6 = \Sigma \times X$

The 3-form field-strength on $\Sigma \times X$ can be decomposed as $F = F^{1,2} + F^{0,3} + F^{2,1}$ where $F^{1,2}$, $F^{0,3}$, $F^{2,1}$ are elements of the corresponding subspaces:

$$\Omega^3_{M_6} = \underbrace{\Omega^1_\Sigma \otimes \Omega^2_X}_{F^{1,2}\in} \oplus \underbrace{\Omega^0_\Sigma \otimes \Omega^3_X}_{F^{0,3}\in} \oplus \underbrace{\Omega^2_\Sigma \otimes \Omega^1_X}_{F^{2,1}\in}, \tag{5.4}$$

Assuming the direct product metric the six-dimensional Hodge star product factorizes as $*_{M_6} = *_\Sigma *_X$, while the space of self-dual 3-forms on $\Sigma \times X$ can be decomposed as follows

$$(1 + *_{M_6})\Omega^3_{M_6} = (\Omega^{1,+}_\Sigma \otimes \Omega^{2,+}_X) \oplus (\Omega^{1,-}_\Sigma \otimes \Omega^{2,-}_X) \oplus (1 + *_\Sigma *_X)\Omega^2_\Sigma \otimes \Omega^1_X \tag{5.5}$$

Corresponding decomposition of the PST action for chiral two-form:

$$\begin{aligned}
-\frac{1}{2}F \wedge \alpha \wedge \iota_{\mathrm{v}}(1 - *_{M_6})F = &-\frac{1}{2}F^{1,2}_+ \wedge \alpha \wedge \iota_{\mathrm{v}}(1 - *_\Sigma)F^{1,2}_+ \\
&-\frac{1}{2}F^{1,2}_- \wedge \alpha \wedge \iota_{\mathrm{v}}(1 + *_\Sigma)F^{1,2}_- \\
&-\frac{1}{2}F^{0,3} \wedge (F^{2,1} - \mathrm{Vol}_\Sigma \wedge *_X F^{0,3})
\end{aligned} \tag{5.6}$$

Decomposition of the kinetic term for scalars :

$$\begin{aligned}
\frac{1}{2}G_{IJ}\,dY^I \wedge *_{M_6}\,dY^J = &\frac{1}{2}\mathrm{Vol}_X \wedge G_{ij}\,d_\Sigma Y^i \wedge *_\Sigma\, d_\Sigma Y^j + \frac{1}{2}\mathrm{Vol}_\Sigma \wedge G_{ij}\,d_X Y^i \wedge *_X\, d_X Y^j \\
&+ \frac{1}{2}\mathrm{Vol}_X \wedge G_{ab}\,d_\Sigma Y^a \wedge *_\Sigma\, d_\Sigma Y^b + \frac{1}{2}\mathrm{Vol}_\Sigma \wedge G_{ab}\,d_X Y^a \wedge *_X\, d_X Y^b
\end{aligned} \tag{5.7}$$

Our convensions for decomposition of gamma matrices are the following

$$\Gamma^{\mu} = \mathbf{1}_{4\times4} \otimes \mathbf{1}_{2\times2} \otimes \gamma^{\mu}, \quad \Gamma^{a} = \mathbf{1}_{4\times4} \otimes \gamma^{a} \otimes \gamma^{(4)}, \quad \Gamma^{I} = \gamma^{I} \otimes \gamma^{(2)} \otimes \gamma^{(4)} \tag{5.8}$$

where $\gamma^i = \sigma_3 \otimes \sigma^i$ and $\gamma^p = \sigma^p \otimes \mathbf{1}_{2\times2}$.

$$\Gamma^{(6)} = \mathbf{1}_{4\times4} \otimes \gamma^{(2)} \otimes \gamma^{(4)}, \tag{5.9}$$

The charge conjugation matrix

$$C = \Gamma^{10}\Gamma^{8}\Gamma^{6}\Gamma^{4}\Gamma^{2} = i\sigma_1 \otimes \varepsilon \otimes \varepsilon \otimes \mathbf{1}_{2\times2} \otimes \varepsilon, \tag{5.10}$$

$$\begin{aligned} C\Gamma^{i} &= i\sigma_1\sigma_3 \otimes \varepsilon\sigma^{i} \otimes \varepsilon\sigma^{3} \otimes \sigma^{3} \otimes \varepsilon, \\ C\Gamma^{p} &= i\sigma_1\gamma^{p} \otimes \varepsilon \otimes \varepsilon\sigma^{3} \otimes \sigma^{3} \otimes \varepsilon, \\ C\Gamma^{a} &= i\sigma_1 \otimes \varepsilon \otimes \varepsilon\gamma^{a} \otimes \sigma_3 \otimes \varepsilon \end{aligned} \tag{5.11}$$

The right hand sides of (5.10) is symmetric so one has $(C\Gamma^{\underline{m}})^T = C\Gamma^{\underline{m}}$.

Resulting decomposition of the Dirac term:

$$\begin{aligned} 2\overline{\theta_+}\Gamma_m\nabla^m\theta_+ &= 2(\widetilde{D}_{4\times4})_{AB}\Big((\theta_+^A)^T(\gamma^0 \otimes D_{4\times4})(\gamma_a \otimes \gamma^{(4)})\nabla^a\theta_+^B \\ &\qquad + (\theta_+^A)^T(\gamma^0 \otimes D_{4\times4})(\mathbf{1}_{2\times2} \otimes \gamma_\mu)\nabla^\mu\theta_+^B\Big) \\ &\left[\quad \theta_+^A = \frac{1}{2}\begin{pmatrix}\psi^A \\ \tilde{\psi}^A\end{pmatrix}, \qquad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0\end{pmatrix}, \qquad \overline{\psi^A} = (\psi^A)^T\gamma^0 \otimes \varepsilon \quad\right] \\ &= \frac{1}{2}(\widetilde{D}_{4\times4})_{AB}\Big(\overline{\psi^A}\gamma_a \otimes \mathbf{1}_{2\times2}(\nabla^a\psi)^B - \overline{\tilde{\psi}^A}\gamma_a \otimes \mathbf{1}_{2\times2}(\nabla^a\tilde{\psi})^B \\ &\qquad + \overline{\tilde{\psi}^A}\mathbf{1}_{2\times2} \otimes \overline{\sigma}_\mu(\nabla^\mu\psi)^B - \overline{\psi^A}\mathbf{1}_{2\times2} \otimes \sigma_\mu(\nabla^\mu\tilde{\psi})^B\Big) \end{aligned} \tag{5.12}$$

Decomposition of the CF term. The fermionic combination entering the CF term is given by

$$\begin{aligned}\overline{\theta_+}\Gamma_{\mu\nu}\Gamma_a\Gamma^I\theta_+ dx^a &= (\widetilde{D}_{4\times4}\gamma^I)_{AB}(\theta_+^A)^T(\gamma^0\otimes D_{4\times4})(\mathbf{1}_{2\times2}\otimes\gamma_{\mu\nu})(\gamma_a\otimes\gamma^{(4)})\theta_+^B dx^a \\ &= (\widetilde{D}_{4\times4}\gamma^I)_{AB}\begin{pmatrix}\psi^A \\ \tilde{\psi}^A\end{pmatrix}^T \gamma^0\gamma_a\otimes\begin{pmatrix}\varepsilon\sigma_{[\mu}\overline{\sigma}_{\nu]} & 0 \\ 0 & -\varepsilon\overline{\sigma}_{[\mu}\sigma_{\nu]}\end{pmatrix}\begin{pmatrix}\psi^B \\ \tilde{\psi}^B\end{pmatrix}dx^a \\ &= (\widetilde{D}_{4\times4}\gamma^I)_{AB}\Big(\overline{\psi^A}\gamma\otimes\sigma_{[\mu}\overline{\sigma}_{\nu]}\psi^B - \overline{\tilde{\psi}^A}\gamma\otimes\overline{\sigma}_{[\mu}\sigma_{\nu]}\tilde{\psi}^B\Big)\end{aligned} \tag{5.13}$$

where we have used $\overline{\sigma}_{[\mu}\sigma_{\nu]} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\overline{\sigma}^{[\lambda}\sigma^{\rho]}$ and $\sigma_{[\mu}\overline{\sigma}_{\nu]} = -\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\sigma^{[\lambda}\overline{\sigma}^{\rho]}$.

The self-dual current can be written as

$$\begin{aligned}(1+*_{M_6})(1-\alpha\wedge\iota_{\mathrm{v}})F = F^{0,3} &+ \frac{1}{\mathrm{Vol}_\Sigma} *_X (F^{2,1}-\alpha\wedge\iota_{\mathrm{v}}F^{2,1}) \\ &+ F^{2,1} + \mathrm{Vol}_\Sigma\wedge *_X F^{0,3} \\ &+ (1+*_\Sigma)(1-\alpha\wedge\iota_{\mathrm{v}})F_+^{1,2} \\ &+ (1-*_\Sigma)(1-\alpha\wedge\iota_{\mathrm{v}})F_-^{1,2}\end{aligned} \tag{5.14}$$

The resulting CF term is given by

$$\begin{aligned}&\frac{1}{8}\int_{M_6}(1+*_{M_6})(1-\alpha\wedge\iota_{\mathrm{v}})F\wedge\partial_p(*_\Sigma)*_X\overline{\theta_+}\Gamma^{\wedge3}\Gamma^p\theta_+ \\ &= -\frac{1}{8}\int_\Sigma\int_X \mathrm{Vol}_X\wedge(1+*_\Sigma)(1-\alpha\wedge\iota_{\mathrm{v}})F_{+\mu\nu}^{1,2}\wedge\partial_p(*_\Sigma)\overline{\tilde{\psi}^A}\gamma\otimes\overline{\sigma}^{[\mu}\sigma^{\nu]}\tilde{\psi}^B(\widetilde{D}_{4\times4}\gamma^p)_{AB} \\ &\quad -\frac{1}{8}\int_\Sigma\int_X \mathrm{Vol}_X\wedge(1-*_\Sigma)(1-\alpha\wedge\iota_{\mathrm{v}})F_{-\mu\nu}^{1,2}\wedge\partial_p(*_\Sigma)\overline{\psi^A}\gamma\otimes\sigma^{[\mu}\overline{\sigma}^{\nu]}\psi^B(\widetilde{D}_{4\times4}\gamma^p)_{AB}\end{aligned} \tag{5.15}$$

Decomposition of the scalar-fermion interaction. The corresponding combination of fermions can be decomposed as follows

$$\begin{aligned}\overline{\theta_+}\Gamma_a\Gamma^I\Gamma^J\theta_+ dx^a &= (\widetilde{D}_{4\times4}\gamma^I\gamma^J)_{AB}(\theta_+^A)^T(\gamma^0\otimes D_{4\times4})(\gamma\otimes\gamma^{(4)})\theta_+^B \\ &= (\widetilde{D}_{4\times4}\gamma^I\gamma^J)_{AB}\begin{pmatrix}\psi^A\\ \tilde{\psi}^A\end{pmatrix}^T\gamma^0\gamma\otimes\mathbf{1}_{2\times2}\otimes\varepsilon\begin{pmatrix}\psi^B\\ -\tilde{\psi}^B\end{pmatrix} \\ &= (\widetilde{D}_{4\times4}\gamma^I\gamma^J)_{AB}\Big(\overline{\psi^A}\gamma\otimes\mathbf{1}_{2\times2}\psi^B - \overline{\tilde{\psi}^A}\gamma\otimes\mathbf{1}_{2\times2}\tilde{\psi}^B\Big)\end{aligned} \tag{5.16}$$

The resulting $Y\theta^2$ term is given by

$$\begin{aligned}&-\frac{1}{4}\int_{M_6}\nabla Y^I\wedge\partial_p(*)\overline{\theta_+}\Gamma_I\Gamma^p\Gamma\theta_+ = \\ &= -\frac{1}{4}\int_\Sigma\int_X \mathrm{Vol}_X\wedge\nabla Y^q\wedge\partial_p(*_\Sigma)\Big(\overline{\psi^A}\gamma\otimes\mathbf{1}_{2\times2}\psi^B - \overline{\tilde{\psi}^A}\gamma\otimes\mathbf{1}_{2\times2}\tilde{\psi}^B\Big)(\widetilde{D}_{4\times4}\gamma_q\gamma^p)_{AB} \\ &\quad -\frac{1}{4}\int_\Sigma\int_X \mathrm{Vol}_X\wedge\nabla Y^i\wedge\partial_p(*_\Sigma)\Big(\overline{\psi^A}\gamma\otimes\mathbf{1}_{2\times2}\psi^B - \overline{\tilde{\psi}^A}\gamma\otimes\mathbf{1}_{2\times2}\tilde{\psi}^B\Big)(\widetilde{D}_{4\times4}\gamma_i\gamma^p)_{AB}\end{aligned} \tag{5.17}$$

Quartic fermionic couplings:

$$\begin{aligned}\sim R_{pquv}:\quad \frac{1}{2}R_{pquv}\Big(&+\overline{\theta_+}\Gamma^a\Gamma^{pq}\theta_+\overline{\theta_+}\Gamma_a\Gamma^{uv}\theta_+ + \overline{\theta_+}\Gamma^\mu\Gamma^{pq}\theta_+\overline{\theta_+}\Gamma_\mu\Gamma^{uv}\theta_+ \\ &+\overline{\theta_+}\Gamma^a\Gamma^{pq}\Gamma^i\theta_+\overline{\theta_+}\Gamma_a\Gamma^{uv}\Gamma_i\theta_+ + \overline{\theta_+}\Gamma^\mu\Gamma^{pq}\Gamma^i\theta_+\overline{\theta_+}\Gamma_\mu\Gamma^{uv}\Gamma_i\theta_+\Big)\end{aligned} \tag{5.18}$$

$$\begin{aligned}\sim R_{apbq}:\quad 2R_{apbq}\Big(&-\overline{\theta_+}\Gamma^a\Gamma^p\Gamma^i\theta_+\overline{\theta_+}\Gamma^b\Gamma^q\Gamma_i\theta_+ - \overline{\theta_+}\Gamma^a\Gamma^{pu}\theta_+\overline{\theta_+}\Gamma^b\Gamma^q{}_u\theta_+ \\ &+\frac{1}{2}\overline{\theta_+}\Gamma^a\Gamma^p\Gamma^{ij}\theta_+\overline{\theta_+}\Gamma^b\Gamma^q\Gamma_{ij}\theta_+ + \overline{\theta_+}\Gamma^a\Gamma^{pu}\Gamma^i\theta_+\overline{\theta_+}\Gamma^b\Gamma^q{}_u\Gamma_i\theta_+ \\ &-\frac{1}{6}\overline{\theta_+}\Gamma^a\Gamma^{\mu\nu}\Gamma^p\theta_+\overline{\theta_+}\Gamma^b\Gamma_{\mu\nu}\Gamma^q\theta_+ - \frac{1}{3}\overline{\theta_+}\Gamma^{ac}\Gamma^\mu\Gamma^p\theta_+\overline{\theta_+}\Gamma^b{}_c\Gamma_\mu\Gamma^q\theta_+\Big)\end{aligned} \tag{5.19}$$

5.3 Donaldson-Witten twist on X and S-class twist on Σ

Donaldson-Witten twist on X boils down to replacing fields $Y^i, \psi^{r\dot{a}}_{\alpha}, \tilde{\psi}^{r\dot{a}}_{\dot{\alpha}}$ with $Y_+, \psi^r, \eta^r, \chi^r_+$ according to

$$\sigma_i Y^i = -\frac{1}{4}\overline{\sigma}^{[\mu}\sigma^{\nu]}(Y_+)_{\mu\nu}, \qquad \psi^{r\dot{a}}_{\alpha} = (\sigma^{\mu}\varepsilon)_{\alpha\dot{a}}\psi^r_{\mu}, \qquad \tilde{\psi}^{r\dot{a}}_{\dot{\alpha}} = -i\varepsilon_{\dot{\alpha}\dot{a}}\eta^r + (\sigma_k\varepsilon)_{\dot{\alpha}\dot{a}}\chi^r_k \tag{5.20}$$

The resulting bosonic action can be written as:

$$\begin{aligned} S_{\text{bosonic}} = \int_{\Sigma}\int_{X} \Big\{ &-\frac{1}{2}F^{1,2}_{+}\wedge\alpha\wedge\iota_{\mathrm{v}}(1-*_{\Sigma})F^{1,2}_{+} - \frac{1}{2}F^{1,2}_{-}\wedge\alpha\wedge\iota_{\mathrm{v}}(1+*_{\Sigma})F^{1,2}_{-} \\ &+\frac{1}{2}\Big(d_{\Sigma}Y_+\wedge *_{\Sigma}\, d_{\Sigma}Y_+ + \mathrm{Vol}_{\Sigma}\wedge d_X Y_+\wedge *_X\, d_X Y_+ \\ &+\mathrm{Vol}_X\wedge G_{ab}\, d_{\Sigma}Y^a\wedge *_{\Sigma}\, d_{\Sigma}Y^b + \mathrm{Vol}_{\Sigma}\wedge G_{ab}\, d_X Y^a\wedge *_X\, d_X Y^b\Big)\Big\} \end{aligned} \tag{5.21}$$

The resulting fermionic action after the Donaldson-Witten twist:

$$\begin{aligned} S_{\text{fermionic}} = \int_{\Sigma}\int_{X} \mathrm{Vol}_X\wedge\Big\{ &-4\mathrm{Vol}_{\Sigma}\cdot\varepsilon_{rs}\Big(\overline{\psi^r_{\mu}}\gamma_a\nabla^a\psi^{s,\mu} - \overline{\eta}^r\gamma_a\nabla^a\eta^s - \frac{1}{2}\overline{\chi}^r_{\mu\nu}\gamma_a\nabla^a\chi^{s,\mu\nu} \\ &+\overline{\eta}^r\nabla_{\mu}\psi^{s,\mu} + \overline{\chi}^r_{\mu\nu}\nabla^{\mu}\psi^{s,\nu} - \overline{\psi}^r_{\mu}\nabla^{\mu}\eta^s - \overline{\psi}^{r,\mu}\nabla^{\nu}\chi^s_{\mu\nu}\Big) \\ &+\frac{1}{4}\Big(\frac{1}{2}(1+*_{\Sigma})(1-\alpha\wedge\iota_{\mathrm{v}})F^{1,2}_{+} - \nabla Y_+\Big)^{\mu\nu}\wedge\partial_p(*_{\Sigma})(\varepsilon\gamma^p)_{rs}\overline{\eta}^r\gamma\chi^s_{\mu\nu} \\ &+\frac{1}{4}\Big(\frac{1}{2}(1-*_{\Sigma})(1-\alpha\wedge\iota_{\mathrm{v}})F^{1,2}_{-} + \nabla Y_+\Big)^{\mu\nu}\wedge\partial_p(*_{\Sigma})(\varepsilon\gamma^p)_{rs}\overline{\psi}^r_{\mu}\gamma\psi^s_{\nu} \\ &+\frac{1}{2}\nabla Y^q\wedge\partial_p(*_{\Sigma})(\varepsilon\gamma_q\gamma^p)_{rs}\Big(\overline{\psi}^{r,\mu}\gamma\psi^s_{\mu} - \overline{\eta}^r\gamma\eta^s - \frac{1}{2}\overline{\chi}^{r,\mu\nu}\gamma\chi^s_{\mu\nu}\Big) \\ &+\ \text{quartic terms}\Big\} \end{aligned} \tag{5.22}$$

The S-class twist on Σ is implemented using the following decompositions (recall that $\varkappa, r = 9, 10$):

$$\psi_{\varkappa r} = \begin{pmatrix} \psi_{++} & \psi_{+-} \\ 0 & 0 \end{pmatrix}, \quad \tilde{\psi}_{\varkappa r} = \begin{pmatrix} 0 & 0 \\ \tilde{\psi}_{-+} & \tilde{\psi}_{--} \end{pmatrix} \tag{5.23}$$

$$\gamma^{(2)} = i\sigma_2\sigma_1 = \sigma_3, \quad \gamma_\pm = \sigma_\mp, \quad \varepsilon\gamma_- = -P_-, \quad \varepsilon\gamma_+ = P_+$$
$$\gamma_a = e_a^+\gamma_+ + e_a^-\gamma_-, \quad \gamma_p = E_p^+\gamma_+ + E_p^-\gamma_-, \tag{5.24}$$

$$\psi_{\varkappa r} = \Big(\psi_{++}\frac{1+\gamma^{(2)}}{2} + \psi_{+-}\gamma_-\Big)_{\varkappa r},$$
$$\tilde\psi_{\varkappa r} = \Big(\tilde\psi_{-+}\gamma_+ + \tilde\psi_{--}\frac{1-\gamma^{(2)}}{2}\Big)_{\varkappa r} \tag{5.25}$$

The resulting fermionic action after the S-class twist

$$\begin{aligned}
S_{\text{fermionic}} = \int_\Sigma\int_X \mathrm{Vol}_X \wedge \Big\{ & \mathrm{Vol}_\Sigma \Big(e_a^+\psi_{+-\mu}\nabla^a\psi^\mu_{++} - e_a^+\psi_{++\mu}\nabla^a\psi^\mu_{+-} \\
& + e_a^-\eta_{--}\nabla^a\eta_{-+} - e_a^-\eta_{-+}\nabla^a\eta_{--} + \frac{1}{2}e_a^-\chi^{\mu\nu}_{--}\nabla^a\chi_{-+\mu\nu} - \frac{1}{2}e_a^-\chi^{\mu\nu}_{-+}\nabla^a\chi_{--\mu\nu} \\
& + \eta_{-+}\nabla_\mu\psi^\mu_{+-} - \eta_{--}\nabla_\mu\psi^\mu_{++} + \chi^{\mu\nu}_{--}\nabla_\mu\psi_{++\nu} - \chi^{\mu\nu}_{-+}\nabla_\mu\psi_{+-\mu} \\
& + \psi^\mu_{++}\nabla_\mu\eta_{--} - \psi^\mu_{+-}\nabla_\mu\eta_{-+} + \psi_{++\mu}\nabla^\nu\chi^{\mu\nu}_{--} - \psi_{+-\mu}\nabla_\nu\chi^{\mu\nu}_{-+}\Big) \\
+ \frac{1}{16}\Big(\frac{1}{2}(1+*_\Sigma)(1-\alpha\wedge\iota_{\mathrm{v}})F_+^{1,2} - \nabla Y_+\Big)_{\mu\nu} & \wedge \frac{\partial}{\partial Y_p}(*_\Sigma)\, e_a^- dx^a (E_p^-\eta_{--}\chi^{\mu\nu}_{--} - E_p^+\eta_{-+}\chi^{\mu\nu}_{-+}) \\
+ \frac{1}{16}\Big(\frac{1}{2}(1-*_\Sigma)(1-\alpha\wedge\iota_{\mathrm{v}})F_-^{1,2} + \nabla Y_+\Big)_{\mu\nu} & \wedge \frac{\partial}{\partial Y_p}(*_\Sigma)\, e_a^+ dx^a (E_p^+\psi^\mu_{++}\psi^\nu_{++} - E_p^-\psi^\mu_{+-}\psi^\nu_{+-}) \\
& + \frac{1}{4}\nabla Y_p \wedge \frac{\partial}{\partial Y_p}(*_\Sigma)\, dx^a\Big(e_a^+\psi^\mu_{++}\psi_{+-\mu} - e_a^-\eta_{--}\eta_{-+} - \frac{1}{2}e_a^-\chi^{\mu\nu}_{--}\chi_{-+\mu\nu}\Big) \\
& + \text{ quartic terms}\Big\}
\end{aligned} \tag{5.26}$$

Note that Y^a transforms now as a vector under the twisted Lorentz group SO(2)$'$ on Σ.

5.4 The two-dimensional model of single M5 brane compactified on a four-manifold

In the limit $t_1/t_2 \to 0$ we need to keep zero-modes on X only. For simplicity we consider the case of simply-connected X, so terms containing ψ, $F^{0,3}$ and $F^{2,1}$ can be dropped.

Remaining components of the 2-form's field-strength can be written as

$$F_+^{1,2} = d_\Sigma \mathsf{B}^h \omega_h + (F_+^{1,2})', \qquad F_-^{1,2} = d_\Sigma \tilde{\mathsf{B}}^{\tilde{h}} \tilde{\omega}_{\tilde{h}} + (F_-^{1,2})' \tag{5.27}$$

where ω_h, $h = 1, \ldots, b_2^+$ are representatives of a basis of $H^{2,+}(X)$ and $\tilde{\omega}_{\tilde{h}}$, $\tilde{h} = 1, \ldots, b_2^-$ are representatives of a basis of $H^{2,-}(X)$. Thus we obtain 2d scalar fields on Σ B_+ and $\tilde{\mathsf{B}}_-$ valued in $H^{2,+}(X, \mathbb{R})$ and $H^{2,-}(X, \mathbb{R})$.

$$\mathsf{B}_+ = \mathsf{B}^h[\omega_h], \qquad \tilde{\mathsf{B}}_- = \tilde{\mathsf{B}}^{\tilde{h}}[\tilde{\omega}_{\tilde{h}}], \tag{5.28}$$

Analogously, $Y_+, \chi_{-+}, \chi_{--}$ lead to a 2d fields $\mathsf{Y}_+, \chi_{-+}, \chi_{--}$ valued in $H^{2,+}(X, \mathbb{R})$. The resulting two dimensional action is given by

$$\begin{aligned} S = \int_\Sigma \Big\{ & - \frac{1}{2} \langle d\mathsf{B}_+ \wedge \alpha \wedge \iota_{\mathsf{v}} (1 - *_\Sigma) d\mathsf{B}_+ \rangle - \frac{1}{2} \langle d\tilde{\mathsf{B}}_- \wedge \alpha \wedge \iota_{\mathsf{v}} (1 + *_\Sigma) d\tilde{\mathsf{B}}_- \rangle \\ & + \frac{1}{2} \langle d\mathsf{Y}_+ \wedge *_\Sigma \, d\mathsf{Y}_+ \rangle + \frac{1}{2} G_{ab} \, dY^a \wedge *_\Sigma \, dY^b \\ & + \mathrm{Vol}_\Sigma \Big(e_a^{\bar{z}} \eta_{\bar{z}} \nabla^a \eta - e_a^{\bar{z}} \eta \nabla^a \eta_{\bar{z}} + e_a^{\bar{z}} \langle \chi_{\bar{z}} \nabla^a \chi \rangle - e_a^{\bar{z}} \langle \chi \nabla^a \chi_{\bar{z}} \rangle \Big) \\ & + \frac{1}{16} \langle \Big(\frac{1}{2} (1 + *_\Sigma)(1 - \alpha \wedge \iota_{\mathsf{v}}) d_\Sigma \mathsf{B}_+ - d_\Sigma \mathsf{Y}_+ \Big) \wedge \frac{\partial}{\partial Y_p} (*_\Sigma) \, e_a^{\bar{z}} dx^a (E_p^{\bar{z}} \eta_{\bar{z}} \chi_{\bar{z}} - E_p^z \eta \chi) \rangle \\ & - \frac{1}{4} \nabla Y_p \wedge \frac{\partial}{\partial Y_p} (*_\Sigma) \, dx^a \Big(e_a^{\bar{z}} \eta_{\bar{z}} \eta + e_a^{\bar{z}} \langle \chi_{\bar{z}}, \chi \rangle \Big) + \text{ quartic terms} \Big\} \end{aligned} \tag{5.29}$$

where the angular bracket $\langle \cdot , \cdot \rangle$ denotes the pairing $\int_X \cdot \wedge \cdot \, : H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \to \mathbb{R}$.

It is useful to note that $(...)_\pm$ components are just (anti-)holomorphic 1-form on Σ while $(...)_{-+}$ component is a scalar. Therefore, in (5.29) we used replacements

$$\eta_{--} \to \eta_{\bar{z}}, \qquad \eta_{-+} \to \eta, \qquad \chi_{--} \to \chi_{\bar{z}}, \qquad \chi_{-+} \to \chi \tag{5.30}$$

Chapter 6

Conclusions

In this thesis we have explored non-perturbative aspects of four-dimensional cohomological field theories and their relation to four-manifold invariants. We have encountered a remarkable phenomenon that topological correlation functions in a cohomological field theory can have continuous metric dependence due to a contribution from the boundary of the zero modes space.

We have revisited subtleties of defining integrals over non-compact Coulomb branch of vacua and studied IR divergent operators associated with such non-compactness. We have studied the divergent Q-exact operators requiring proper interpretation. We have seen that the divergence comes entirely from zero momentum modes going to the non-compact quasi-classical direction on the Coulomb branch. The meaning of such divergence in the Hamiltonian formalism is the trace divergence over the continuum of vacua near with large vev of $\mathrm{Tr}\,\phi^2$ operator. This is a special case of the phenomenon of non-normalizable zero modes. Analogous example is the divergence of the partition function of non-compact free massless boson in 2d due to non-normalizable zero mode. However in our case the non-normalizability arises only in presence of certain suficiently divergent operators such as the operator $\tilde{I}_+$. This probably indicates existence of IR instability in presence of operator $\tilde{I}_+$ and therefore we conclude that such operators are problematic as physical observables.

Nevertheless, we have described a prescription that removes the infinity associated with inserting $\tilde{I}_+$ as inserting such operators has been proven to be useful in certain applications in the theory of Mock modular forms ant iterated integrals of theta functions.

As the main result, we have proposed a new method for exracting topological partition function of the Argyres-Douglas non-Lagrangian models using its embedding into a Lagrangian theory. The most unexpected outcome is that our procedure implies that the resulting topological partition function of the AD3 theory can still be expressed through the Seiberg-Witten invariants. Moreover, we have observed the mysterious vanishing of the 0-observable on manifolds of superconformal simple type that requires further understanding.

Our results left many unanswered questions and suggest several directions for future research.

1. Perhaps the most interesting and promising direction is to investigate other theories with one-dimensional Coulomb branches. SCFT's appearing in one-dimensional Coulomb branches have been classified by Argyres, Lotito, Lu, Martone [28–30]. The classification is based on Kodaira type of singularity together with how the singularity splits under relevant deformations. It is important that according to [28–30] all singularities admit maximal splitting, i.e. spliting to cusps where a single dyon becomes massless. This makes the analysis of these cases very analogous to the AD3 case studied above.

   However, the main new feature that should arise at more general rank one theories is that in most cases there would be Higgs branches opening up at superconformal points. The contribution of the superconformal vacua will now have to agree with both, the Coulomb branch integral, and the Higgs branch integral in such a way that the total partition function has proper continuity properties. This suggests the next step in narrowing the search for new four-manifold invariants.

2. The extension to manifolds with $b_1 \neq 0$ is of interest for two reasons. First, in this case the 3-form descendent of the 0-observable has negative ghost number and hence the ghost number selection rule admits the possibility that there is an infinite number of nonzero correlation functions, in strong contrast to the simply-connected case. Moreover, non-simply connected manifolds are probably best suited for comparison with the approach to computing topologically twisted $d = 4$ $N = 2$ partition functions suggested in [27].

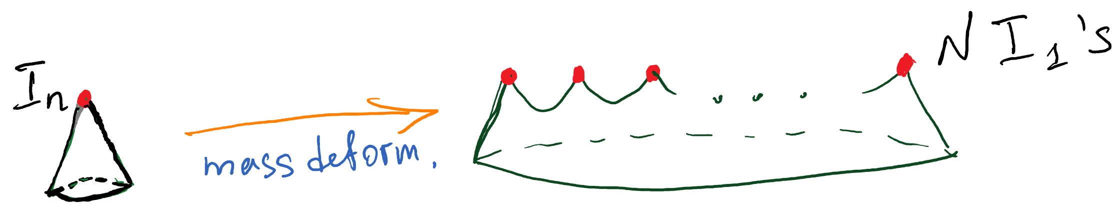

Figure 6.1: Mass deformation of superconformal points splits them into N cusp singularities.

3. Finally, the microscopic interpretation of our partition function in terms of moduli spaces of traditional partial differential equations is an interesting open problem. In principle one should be able to translate our formulae (4.14), (4.109), (4.111) into some subtle aspect of intersection theory on the moduli space of the nonabelian monopole equations.

4. The unknown coefficients α and β entering A and B couplings on the Coulomb branch could depend non-trivially on the dynamically generated scale, masses, and on the conformal manifold in the superconformal case. For example, they can depend on τ_0 in the SU(2), $N_f = 4$ and SU(2), $N = 2^*$ cases. It would be interesting to understand these coefficients in detail, in particular for class S $N = 2$ theories.

5. The (0,2) two-dimensional model derived in chapter 6 can potentially be used as a tool to study topologically twisted $N = 2$ theories of class S that refines the u-plane

integral. It is important to keep in mind that all the non-trivial information about the smooth structure sensitivity is stored in the defects inserted at the punctures on Σ. Understanding this and other aspects of the (0,2) model is an interesting open problem for future research.

Appendix A

Modular forms and theta functions

In this appendix we collect certain aspects of the theory of modular forms and Siegel-Narain theta functions that are used throughout the main text. See for more comprehensive treatments for example [55–57].

The standard Jacobi theta functions $\vartheta_j : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$, $j = 1, \dots, 4$ are defined as

$$\begin{aligned}
\vartheta_1(\tau, v) &= i \sum_{r \in \mathbb{Z}+\frac{1}{2}} (-1)^{r-\frac{1}{2}} q^{r^2/2} e^{2\pi i r v}, \\
\vartheta_2(\tau, v) &= \sum_{r \in \mathbb{Z}+\frac{1}{2}} q^{r^2/2} e^{2\pi i r v}, \\
\vartheta_3(\tau, v) &= \sum_{n \in \mathbb{Z}} q^{n^2/2} e^{2\pi i n v}, \\
\vartheta_4(\tau, v) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2/2} e^{2\pi i n v}.
\end{aligned} \tag{A.1}$$

The function $u(\tau)$ is invariant under transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ given by elements of the congruence subgroup $\Gamma^0(4) \subset \mathrm{SL}(2,\mathbb{Z})$. Recall that the modular group $\mathrm{SL}(2,\mathbb{Z})$ is the group of integer matrices with unit determinant

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z};\ ad - bc = 1 \right\}, \tag{A.2}$$

while the congruence subgroup $\Gamma^0(n)$ is defined as:

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \;\middle|\; b = 0 \bmod n \right\}. \tag{A.3}$$

The functions $\vartheta_j(\tau)$ form direct sum of a one dimensional representation (given by $\vartheta_1(\tau)$) and a three dimensional representation ($\vartheta_i(\tau)$ for $i = 2, 3, 4$) of $\Gamma^0(4)$:

$$\begin{aligned} \vartheta_2(\tau+4) &= -\vartheta_2(\tau), & \vartheta_2\left(\frac{\tau}{\tau+1}\right) &= \sqrt{\tau+1}\,\vartheta_3(\tau), \\ \vartheta_3(\tau+4) &= \vartheta_3(\tau), & \vartheta_3\left(\frac{\tau}{\tau+1}\right) &= \sqrt{\tau+1}\,\vartheta_2(\tau), \\ \vartheta_4(\tau+4) &= \vartheta_4(\tau), & \vartheta_4\left(\frac{\tau}{\tau+1}\right) &= e^{-\frac{\pi i}{4}}\sqrt{\tau+1}\,\vartheta_4(\tau). \end{aligned} \tag{A.4}$$

Eisenstein series and Dedekind eta function The Eisenstein series for even $k \geq 2$ defined as q-series

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \tag{A.5}$$

with $\sigma_k(n) = \sum_{d|n} d^k$ the divisor sum. For $k \geq 4$, E_k is a modular form of $\mathrm{SL}(2,\mathbb{Z})$ of weight k. In other words, it transforms under $\mathrm{SL}(2,\mathbb{Z})$ as

$$E_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k E_k(\tau). \tag{A.6}$$

The space of modular forms of $\mathrm{SL}(2,\mathbb{Z})$ forms a ring that is generated by $E_4(\tau)$ and $E_6(\tau)$, while $E_2(\tau)$ is a quasi-modular form, which means that the $\mathrm{SL}(2,\mathbb{Z})$ transformation of E_2 includes a shift in addition to the weight,

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6i}{\pi}c(c\tau+d) \tag{A.7}$$

The Dedekind eta function $\eta : \mathbb{H} \to \mathbb{C}$ is defined as

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \tag{A.8}$$

It is a modular form of weight $\frac{1}{2}$ under $\mathrm{SL}(2,\mathbb{Z})$ with a non-trivial multiplier system. It transforms under the generators of $\mathrm{SL}(2,\mathbb{Z})$ as

$$\begin{aligned}\eta(-1/\tau) &= \sqrt{-i\tau}\eta(\tau)\\ \eta(\tau+1) &= e^{\frac{\pi i}{12}}\eta(\tau)\end{aligned} \tag{A.9}$$

Siegel-Narain theta functions $\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}] : \mathbb{H} \to \mathbb{C}$ form a large class of theta functions of which the Jacobi theta functions are a special case. For our application, it is sufficient to consider Siegel-Narain theta functions for which the associated lattice Γ is uni-modular and Lorentzian, i.e. it has signature $(1, n-1)$. We denote the bilinear form by (λ_1, λ_2) and the quadratic form $Q(\lambda) \equiv (\lambda, \lambda) \equiv \lambda^2$. Also, let ξ be a characteristic vector of Γ, such that $Q(\lambda) + (\lambda, \xi) \in 2\mathbb{Z}$ for each $\lambda \in \Gamma$. Given an element $\omega \in \Gamma \otimes \mathbb{R}$ with $\omega^2 = +1$, we may decompose the space $\Gamma \otimes \mathbb{R}$ in a positive definite subspace Γ_+ spanned by ω, and a negative definite subspace Γ_-, orthogonal to Γ_+. The projections of a vector $\lambda \in \Gamma$ to Γ_+ and Γ_- are then given by

$$\lambda_+ = \omega(\lambda, \omega), \qquad \lambda_- = \lambda - \lambda_+. \tag{A.10}$$

Then $\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}]$ is defined as follows

$$\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}](\tau, \overline{\tau}, z, \bar{z}) = \sum_{\lambda\in\Gamma+\lambda_0} \mathcal{K}(\lambda)\,(-1)^{(\lambda,\xi)} q^{-\frac{1}{2}\lambda_-^2} \bar{q}^{\frac{1}{2}\lambda_+^2}\, \exp\left(-2\pi i(z, \lambda_-) - 2\pi i(\bar{z}, \lambda_+)\right). \tag{A.11}$$

Modular properties of the Siegel-Narain theta function. The modular properties of $\Psi^{\omega}_{\lambda_0,\xi}[\mathcal{K}]$ depend on $\mathcal{K}$. For $\mathcal{K} = 1$ $\Psi^{\omega}_{\lambda_0,\xi}[1]$ transformations under the generators T and S of $\mathrm{SL}(2,\mathbb{Z})$ as follows

$$\begin{aligned}&\Psi^{\omega}_{\lambda_0+\frac{1}{2}\xi}[1](\tau+1, \overline{\tau}+1, z, \bar{z}) = e^{\pi i({\lambda_0}^2-\frac{1}{4}\xi^2)}\Psi_{\lambda_0+\frac{1}{2}\xi,\xi}[1](\tau, \overline{\tau}, z+\lambda_0, \bar{z}+\lambda_0),\\ &\Psi^{\omega}_{\lambda_0+\frac{1}{2}\xi}[1]\left(\frac{-1}{\tau}, \frac{-1}{\overline{\tau}}, \frac{z}{\tau}, \frac{\bar{z}}{\overline{\tau}}\right) = (-i\tau)^{\frac{n-1}{2}}(i\overline{\tau})^{\frac{1}{2}} e^{\pi i(\frac{-1}{\tau})z^2+\frac{1}{2}\xi^2+\lambda_0\cdot K)}\,\Psi^{\omega}_{\frac{1}{2}\xi,\lambda_0}[1](\tau, \overline{\tau}, z-\lambda_0, \bar{z}-\lambda_0).\end{aligned} \tag{A.12}$$

Using the above $\mathrm{SL}(2,\mathbb{Z})$ transformations and Poisson resummation one verify that $\Psi^{\omega}_{\lambda_0,\xi}[1]$ is a modular form for the congruence subgroup $\Gamma^0(4)$. The transformations under the generators of this group are given by

$$\begin{aligned}
&\Psi^{\omega}_{\lambda_0,\xi}[1]\left(\frac{\tau}{\tau+1},\frac{\overline{\tau}}{\overline{\tau}+1}\right) = (\tau+1)^{\frac{n-1}{2}}(\overline{\tau}+1)^{\frac{1}{2}}\exp\!\left(\tfrac{\pi i}{4}\xi^2\right)\Psi^{\omega}_{\lambda_0,\xi}[1](\tau,\overline{\tau}),\\
&\Psi^{\omega}_{\lambda_0,\xi}[1](\tau+4,\overline{\tau}+4) = e^{2\pi i(\lambda_0,\xi)}\,\Psi^{\omega}_{\lambda_0,\xi}[1](\tau,\overline{\tau}),
\end{aligned} \tag{A.13}$$

where we have set $z=\bar{z}=0$. Transformations for other kernels can be easily determined from these expressions by acting with $\mathcal{K}(\frac{i}{2\pi}\frac{\partial}{\partial z},\frac{i}{2\pi}\frac{\partial}{\partial z})$.

Appendix B

Some formulae for compactification and twisting of PST action

B.1 Superspace gauge completion of $d = 11$ supergravity background

Supersymmetry transformations of $d = 11$ supergravity:

$$
\begin{aligned}
&\delta_\epsilon E^{\underline{a}} = i\bar{\epsilon}\Gamma^{\underline{a}}\psi \\
&\delta_\epsilon C = \bar{\epsilon}\Gamma^{\wedge 2}\wedge\psi \\
&\delta_\epsilon \tilde{C} = \bar{\epsilon}\Gamma^{\wedge 5}\wedge\psi + \delta_\epsilon C\wedge C \\
&\delta_\epsilon \psi_{\underline{m}} = \hat{D}_{\underline{m}}\epsilon + \frac{1}{4!}T_{\underline{m}}{}^{\underline{rstu}}\epsilon\cdot\left(F + \alpha\wedge\iota_{\mathrm{v}}\left(*_{11}\tilde{F} - F\right)\right)_{\underline{rstu}} \\
&\omega_{\underline{m}}{}^{\underline{bc}} = E^{\underline{0}[\underline{b}}E^{\underline{c}]}_{\underline{0},\underline{m}} + E^{\underline{0}[\underline{b}}E^{\underline{c}]\underline{k}}G_{\underline{m0},\underline{k}}
\end{aligned}
\tag{B.1}
$$

Supersymmetry algebra:

$$
\left[\delta_{\epsilon_1},\delta_{\epsilon_2}\right] = \delta_{\epsilon_3} + \delta_{gct}(\xi) + \delta_L(\lambda^{rs}) + \delta_C(\xi_{mn}) \tag{B.2}
$$

We want to represent supersymmetry transformations on superspace. For generic scalar superfield we have

$$\delta_\epsilon \Phi = \hat{\Xi}_\epsilon \Phi, \qquad \hat{\Xi}_\epsilon = \Xi_\epsilon^m \frac{\partial}{\partial X^m} + \Xi_\epsilon^\alpha \frac{\partial}{\partial \theta^\alpha} \tag{B.3}$$

Commutator of two supersymmetry transformations can be expressed through the superspace vector field as

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\Phi = \delta_{\epsilon_1}(\hat{\Xi}_{\epsilon_2}\Phi) - (1 \leftrightarrow 2) = [\hat{\Xi}_{\epsilon_2}, \hat{\Xi}_{\epsilon_1}]\Phi + \delta_{\epsilon_1}(\hat{\Xi}_{\epsilon_2})\Phi - \delta_{\epsilon_2}(\hat{\Xi}_{\epsilon_1})\Phi \tag{B.4}$$

Taking into account the commutation relation for $\delta(e)$ we obtain that in order to represent supersymmetry variation on superspace the vector field $\hat{\Xi}$ must be a solution to the following equation.

$$\begin{aligned} &[\hat{\Xi}_{\epsilon_2}, \hat{\Xi}_{\epsilon_1}] + \delta_{\epsilon_1}(\hat{\Xi}_{\epsilon_2}) - \delta_{\epsilon_2}(\hat{\Xi}_{\epsilon_1}) = 2\overline{\epsilon}_2 \Gamma^m \epsilon_1 \partial_m + \hat{\Xi}_{\epsilon_3} - \frac{1}{4}\lambda_3^{rs}(\Gamma_{rs}\theta)^\alpha \partial_\alpha \\ &\text{for} \quad \epsilon_3 = -2\overline{\epsilon}_2 \Gamma^m \epsilon_1 \psi_m \quad \text{and} \quad \lambda_3^{rs} = -2\overline{\epsilon}_2 \Gamma^m \epsilon_1 \hat{\omega}_m{}^{rs} + \frac{1}{72}\overline{\epsilon}_2 \Big[\Gamma^{rstukl}\hat{F}_{tukl} + 24\Gamma_{tu}\hat{F}^{rstu}\Big]\epsilon_1 \end{aligned} \tag{B.5}$$

This equation can be solved perturbatively in orders of θ

$$\hat{\Xi}_\epsilon = \hat{\Xi}_\epsilon^{(0)} + \hat{\Xi}_\epsilon^{(1)} + \hat{\Xi}_\epsilon^{(2)} + \hat{\Xi}_\epsilon^{(3)} + \dots \tag{B.6}$$

with the following initial condition

$$\hat{\Xi}_\epsilon^{(0)} = \epsilon^\alpha \frac{\partial}{\partial \theta^\alpha} \tag{B.7}$$

Taking zero's order of equation (B.5) we obtain

$$\Xi_{[2}^{(0)\alpha} \partial_\alpha \Xi_{1]}^{(1)m} \partial_m + \Xi_{[2}^{(0)\alpha} \partial_\alpha \Xi_{1]}^{(1)\beta} \partial_\beta = \overline{\epsilon}_2 \Gamma^m \epsilon_1 - \overline{\epsilon}_2 \Gamma^m \epsilon_1 \psi_m^\alpha \partial_\alpha, \tag{B.8}$$

(B.8) implies that the first order Ξ is given by $\Xi_\epsilon^{(1)m} = \bar{\theta}\Gamma^m\epsilon$ and $\Xi_\epsilon^{(1)\beta} = -\bar{\theta}\Gamma^m\epsilon\psi_m^\beta$. General formula for k-th order correction to Ξ and commutator of two Ξ's is the following:

$$\begin{aligned}
\Xi^{(k)} &= \sum_{k=0}^{32}\left(\Xi_\epsilon^{(k)m}\partial_m + \Xi_\epsilon^{(k)\alpha}\partial_\alpha\right) = \sum_{k=0}^{32}\frac{\epsilon^\beta\theta^{\alpha_1}...\theta^{\alpha_k}}{k!}\left(\Xi^m_{\beta|\alpha_1...\alpha_k}\partial_m + \Xi^\alpha_{\beta|\alpha_1...\alpha_k}\partial_\alpha\right)\\
\frac{1}{2}[\hat{\Xi}_2,\hat{\Xi}_1]^{(k)} &= \left(\Xi^\alpha_{[2}\partial_\alpha\Xi^\beta_{1]} + \Xi^m_{[2}\partial_m\Xi^\beta_{1]}\right)^{(k)}\partial_\beta + \left(\Xi^\alpha_{[2}\partial_\alpha\Xi^m_{1]} + \Xi^n_{[2}\partial_n\Xi^m_{1]}\right)^{(k)}\partial_m\\
&= \Big(\epsilon^\alpha_{[2}\partial_\alpha\Xi^{(k+1)\beta}_{1]} + \sum_{i=1}^{k}\Xi^{(i)\alpha}_{[2}\partial_\alpha\Xi^{(k-i+1)\beta}_{1]} + \sum_{i=1}^{k-1}\Xi^{(i)m}_{[2}\partial_m\Xi^{(k-i)\beta}_{1]} + \Xi^{(k)m}_{[2}\partial_m\epsilon^\beta_{1]}\Big)\partial_\beta\\
&\quad + \Big(\epsilon^\alpha_{[2}\partial_\alpha\Xi^{(k+1)m}_{1]} + \sum_{i=1}^{k}\Xi^{(i)\alpha}_{[2}\partial_\alpha\Xi^{(k-i+1)m}_{1]} + \sum_{i=1}^{k-1}\Xi^{(i)n}_{[2}\partial_n\Xi^{(k-i)m}_{1]}\Big)\partial_m
\end{aligned} \tag{B.9}$$

Using (B.9) we can in principle iteratively compute corrections to superfields to any given order. Second order correction to the bosonic components of the super-vielbein is given by

$$\delta^2_\theta\mathbb{E}^{\underline{a}}_M = 2\bar{\theta}\Gamma^{\underline{a}}\Big(-\frac{1}{4}\hat{\omega}_{\underline{m}}{}^{\underline{bc}}\Gamma^{\underline{bc}} + \frac{1}{4!}\hat{F}_{\underline{rstu}}T_M{}^{\underline{rstu}}\Big)\theta = 2\bar{\theta}\Gamma^{\underline{a}}\hat{D}_{\underline{m}}\theta \tag{B.10}$$

This expression equals twice the second θ-order correction $\mathbb{E}^{(2)A}_M$ and this equality agrees with the general formula $\mathbb{E}^{(k)A}_M = \frac{1}{k!}\delta^k_\theta E^A_M$. For M5 brane's action we need also the fourth order term $\mathbb{E}^{(4)A}_M = \frac{1}{4!}\delta^4_\theta E^A_M$ in the background with vanishing gravitino and tensor field strength F. This term has to be proportional to the curvature 2-form, so in order to detect it we can work in the locally flat coordinates:

$$\begin{aligned}
\delta^2_\theta\hat{\omega}_M{}^{BC}\big|_{\psi,F=0} &\sim \frac{1}{2}E^{NB}\bar{\theta}\Gamma^C{}_{EF}\theta\partial_{[M}\omega_{N]}{}^{EF} - \frac{1}{2}E^N_C\bar{\theta}\Gamma^B{}_{EF}\theta\partial_{[M}\omega_{N]}{}^{EF} - \frac{1}{2}E^{LB}E^N_CE^D_M\bar{\theta}\Gamma_{DEF}\theta\partial_L\omega_N{}^{EF}\\
\delta^2_\theta\hat{F}_{RSTU}\big|_{\psi,F=0} &\sim 3!\bar{\theta}\Gamma_{[ST,}\Gamma_{EF}\theta\partial_R\omega_{U]}{}^{EF}
\end{aligned} \tag{B.11}$$

Covariantising (B.11) we obtain the following θ^4 order correction to the vielbein superfield:

$$\mathbb{E}^{(4)\underline{a}}_{\underline{m}} = \frac{1}{4!}\delta^4_\theta \mathbb{E}^{\underline{a}}_{\underline{m}} = \frac{2}{4!}\bar{\theta}\Gamma^{\underline{a}}\delta^2_\theta \hat{D}_{\underline{m}}\theta, \quad \text{where} \quad \delta^2_\theta \hat{D}_{\underline{m}} = \frac{1}{4}\Big[S_{\underline{m}}{}^{\underline{rud}}\bar{\theta}\Gamma_{\underline{dbc}}\theta + T_{\underline{m}}{}^{\underline{rust}}\bar{\theta}\Gamma_{\underline{stbc}}\theta\Big]R_{\underline{ru}}{}^{\underline{bc}}$$
$$S_{\underline{m}}{}^{\underline{rud}} = -\delta^{[\underline{r}}_{\underline{m}}\Gamma^{\underline{u}]\underline{d}} + \frac{1}{2}\Gamma^{\underline{ru}}E^{\underline{d}}_{\underline{m}}, \qquad T_{\underline{m}}{}^{\underline{rstu}} = \frac{1}{12}(\Gamma_{\underline{m}}{}^{\underline{rstu}} - 8\delta^{[\underline{r}}_{\underline{m}}\Gamma^{\underline{stu}]}) \tag{B.12}$$

Vielbein $\mathbb{E}$ on the 11d superspace. bosonic component:

$$\begin{aligned}
\mathbb{E}^{\underline{a}}_{\underline{m}} &= E^{\underline{a}}_{\underline{m}} + \frac{1}{2}\delta^2_\theta E^{\underline{a}}_{\underline{m}}\Big|_{\psi,F=0} + \frac{1}{4!}\delta^4_\theta E^{\underline{a}}_{\underline{m}}\Big|_{\psi,F=0} + \dots \\
\mathbb{E}^{\underline{a}}_{\underline{\alpha}} &= (\bar{\theta}\Gamma^{\underline{a}})_{\underline{\alpha}} \\
\frac{1}{2}\delta^2_\theta E^{\underline{a}}_{\underline{m}}\Big|_{\psi,F=0} &= \bar{\theta}\Gamma^{\underline{a}}\omega_{\underline{m}}\theta \\
\frac{1}{4!}\delta^4_\theta E^{\underline{a}}_{\underline{m}}\Big|_{\psi,F=0} &= \frac{2}{4!}\bar{\theta}\Gamma^{\underline{a}}(\delta^2_\theta D_{\underline{m}}\theta)_{\psi,F=0}
\end{aligned} \tag{B.13}$$

Analogously, the fermionic component $\mathbb{E}^\alpha_M$ is given by

$$\begin{aligned}
\mathbb{E}^{(1)\alpha}_M &= \delta_\theta \mathbb{E}^\alpha_M = (\hat{D}_M\theta)^\alpha \\
\mathbb{E}^{(3)\alpha}_M &= \frac{1}{3!}\delta^3_\theta \mathbb{E}^\alpha_M = \frac{1}{3!}(\delta^2_\theta \hat{D}_M\theta)^\alpha
\end{aligned} \tag{B.14}$$

Second supersymmetry variation of the spin connection:

$$\delta^2_\theta \omega_{\underline{m}}\Big|_{\psi,F=0} = \frac{1}{4}\Big[S_{\underline{m}}{}^{\underline{rud}}\bar{\theta}\Gamma_{\underline{dbc}}\theta + T_{\underline{m}}{}^{\underline{rust}}\bar{\theta}\Gamma_{\underline{stbc}}\theta\Big]R_{\underline{ru}}{}^{\underline{bc}}$$
$$S_{\underline{m}}{}^{\underline{rud}} = -\delta^{[\underline{r}}_{\underline{m}}\Gamma^{\underline{u}]\underline{d}} + \frac{1}{2}\Gamma^{\underline{ru}}E^{\underline{d}}_{\underline{m}}, \qquad T_{\underline{m}}{}^{\underline{rstu}} = \frac{1}{12}(\Gamma_{\underline{m}}{}^{\underline{rstu}} - 8\delta^{[\underline{r}}_{\underline{m}}\Gamma^{\underline{stu}]}) \tag{B.15}$$

Metric $\mathbb{G}$ on the 11d superspace:

$$\begin{aligned}
\mathbb{G} &= dX^{\underline{m}}\mathbb{E}^{\underline{a}}_{\underline{m}}dX^{\underline{0}}\mathbb{E}^{\underline{a}}_{\underline{0}} + 2dX^{\underline{m}}\mathbb{E}^{\underline{a}}_{\underline{m}}d\theta^{\underline{\alpha}}\mathbb{E}^{\underline{a}}_{\underline{\alpha}} + \dots \\
&= G_{\underline{m0}}dX^{\underline{m}}dX^{\underline{0}} + 2E^{\underline{a}}_{\underline{m}}dX^{\underline{m}}\bar{\theta}(d + dX^{\underline{0}}\omega_{\underline{0}})\theta + \frac{4}{4!}E^{\underline{a}}_{\underline{m}}dX^{\underline{m}}\bar{\theta}\Gamma^{\underline{a}}(\delta^2_\theta D_{\underline{0}}\theta)_{\psi,F=0}dX^{\underline{0}}
\end{aligned} \tag{B.16}$$

3-form field $\mathbb{C}$ on the 11d superspace:

$$\begin{aligned} &\delta_\epsilon C_{\underline{klm}} = -3!\bar{\epsilon}\Gamma_{[\underline{kl}}\psi_{\underline{m}]} \\ &\mathbb{C}_{\underline{klm}} = C_{\underline{klm}} + \frac{1}{2}\delta_\theta^2 C_{\underline{klm}}\big|_{\psi,F=0} + \dots \\ &C_{\underline{klm}} = 0 \\ &\frac{1}{2}\delta_\theta^2 C_{\underline{klm}}\big|_{\psi,F=0} = -3\bar{\theta}\Gamma_{[\underline{kl},}\omega_{\underline{m}]}\bar{\theta} \end{aligned} \tag{B.17}$$

6-form field $\tilde{\mathbb{C}}$ on 11d superspace:

$$\begin{aligned} &\tilde{\mathbb{C}}_{\underline{m}_1..\underline{m}_6} = \tilde{C}_{\underline{m}_1..\underline{m}_6} + \frac{1}{2}\delta_\theta^2\tilde{C}_{\underline{m}_1..\underline{m}_6}\big|_{\psi,F=0} + \frac{1}{4!}\delta_\theta^4\tilde{C}_{\underline{m}_1..\underline{m}_6}\big|_{\psi,F=0} + \dots \\ &\tilde{\mathbb{C}}_{\underline{m}_1..\underline{m}_5\underline{\alpha}} = (\bar{\theta}\Gamma_{\underline{m}_1..\underline{m}_5})_{\underline{\alpha}} + \dots \\ &\delta_\epsilon\tilde{C}_{\underline{m}_1..\underline{m}_6} = -12\bar{\epsilon}\Gamma_{[\underline{m}_1..\underline{m}_5}\psi_{\underline{m}_6]} - 3!\bar{\epsilon}\Gamma_{[\underline{m}_1\underline{m}_2}\psi_{\underline{m}_3}\tilde{C}_{\underline{m}_4..\underline{m}_6]} \\ &\tilde{C}_{\underline{m}_1..\underline{m}_6} = 0 \\ &\frac{1}{2}\delta_\theta^2\tilde{C}_{\underline{m}_1..\underline{m}_6}\big|_{\psi,F=0} = -6\bar{\theta}\Gamma_{[\underline{m}_1..\underline{m}_5}\omega_{\underline{m}_6]}\theta \\ &\frac{1}{4!}\delta_\theta^4\tilde{C}_{\underline{m}_1..\underline{m}_6}\big|_{\psi,F=0} = -\frac{12}{4!}\bar{\theta}\Gamma_{[\underline{m}_1..\underline{m}_5}(\delta_\theta^2 D_{\underline{m}_6]}\theta)_{\psi,F=0} + O(\omega^2) \end{aligned} \tag{B.18}$$

B.2 Some details on the derivation of the action (5.2)

Metric induced by the embedding $\mathbb{X}$ of the M5 brane worldvolume M_6 into $M_{11|32}$

$$\begin{aligned} &(\mathbb{X}^*\mathbb{G})_{mn} = (X^*G)_{mn} + 2\bar{\theta}\Gamma_{(m}(\partial_{n)} + \omega_{n)})\theta + \frac{4}{4!}\bar{\theta}\Gamma_{(n}(\delta_\theta^2\omega_{m)}\theta)_{\psi,F=0} \\ &(X^*G)_{mn} = G_{\underline{m}\underline{0}}\partial_m X^{\underline{m}}\partial_n X^{\underline{0}} = \big/\ \partial_m X^{\underline{m}} = \mathsf{T}_m^{\underline{m}} + \mathsf{N}_I^{\underline{m}}\nabla_m Y^I\ \big/ = g_{mn} + G_{IJ}\nabla_m Y^I\nabla_m Y^J, \\ &\text{where}\quad g_{mn} = G_{\underline{m}\underline{0}}\mathsf{T}_m^{\underline{m}}\mathsf{T}_n^{\underline{0}},\quad G_{IJ} = G_{\underline{m}\underline{0}}\mathsf{N}_I^{\underline{m}}\mathsf{N}_J^{\underline{0}}\quad\text{and}\quad G_{\underline{m}\underline{0}}\mathsf{T}_m^{\underline{m}}\mathsf{N}_I^{\underline{0}} = 0 \\ &\mathrm{Vol}_{\mathbb{X}^*\mathbb{G}} = \mathrm{Vol}_g\cdot\Big(1 + \frac{1}{2}g^{mn}(\mathbb{X}^*\mathbb{G} - g)_{mn} + \dots\Big) \end{aligned} \tag{B.19}$$

The 6-form coupling.

$$
\begin{aligned}
\tilde{\mathrm{C}}_{m_1..m_6} &= (\mathbb{X}^*\tilde{\mathbb{C}})_{m_1..m_6} = \partial_{m_1}X^{\underline{m}_1}...\partial_{m_6}X^{\underline{m}_6}\tilde{C}_{\underline{m}_1..\underline{m}_6} + 6\partial_{[m_1}X^{\underline{m}_1}...\partial_{m_5}X^{\underline{m}_5}\partial_{m_6]}\theta^{\underline{\alpha}}\tilde{\mathbb{C}}_{\underline{m}_1..\underline{m}_5\underline{\alpha}} + \dots \\
&= -6\bar{\theta}\Gamma_{[m_1..m_5}\omega_{m_6]}\theta - \frac{12}{4!}\bar{\theta}\Gamma_{[m_1..m_5}(\delta_\theta^2 D_{m_6]}\theta)_{\psi,F=0} \\
\int_X \mathbb{X}^*\mathbb{C}_6 &= \int d^6x\,(-\det X^*G)^{\frac{1}{2}}\cdot\frac{1}{6!}(-\det X^*G)^{-\frac{1}{2}}\epsilon^{m_1..m_6}(\mathbb{X}^*\mathbb{C}_6)_{m_1..m_6} \\
&\quad \Big/\ \frac{1}{5!}(-\det X^*G)^{-\frac{1}{2}}\epsilon^{m_1..m_6}\Gamma_{m_1..m_5} = -\Gamma^{m_6}\Gamma^{(6)}\ \Big/ \\
&= \int d^6x\,\sqrt{-g}\Big(\bar{\theta}\Gamma^m\Gamma^{(6)}(\partial_m+\omega_m)\theta + \frac{2}{4!}\bar{\theta}\Gamma^m\Gamma^{(6)}(\delta_\theta^2\omega_m\theta)_{\psi,F=0}\Big) \\
\mathrm{Vol}_{\mathbb{X}^*\mathbb{G}} + \mathbb{X}^*\mathbb{C}_6 &= \mathrm{Vol}_{X^*G}\Big(1+\bar{\theta}\Gamma_m(1+\Gamma^{(6)})(\partial_n+\omega_n)\theta(X^*G)^{mn} + \frac{2}{4!}\bar{\theta}\Gamma_m(1+\Gamma^{(6)})\delta_\theta^2\omega_n\theta(X^*G)^{mn}\Big)
\end{aligned}
\tag{B.20}
$$

Scalar-fermion interaction.

$$
\begin{aligned}
\bar{\theta}\Gamma^m(1+\Gamma^{(6)})\omega_m\theta &= \bar{\theta}g^{mn}\Gamma_{\underline{m}}(\mathsf{T}^{\underline{m}}_m+\mathsf{N}^{\underline{m}}_I\nabla_m Y^I)\Big(1+\Gamma^{(6)}_0+\frac{1}{5!}(-g)^{-\frac{1}{2}}\epsilon^{r_1..r_6}\Gamma_{r_1..r_5}\Gamma_I\partial_{r_6}Y^I + \dots\Big)(\mathsf{T}^{\underline{0}}_n+\mathsf{N}^{\underline{0}}_I\nabla_n Y^I)\omega_{\underline{0}}\theta \\
&= \bar{\theta}\Gamma_m(1+\Gamma^{(6)}_0)[\omega^m]_{\mu^{-2}}\theta + \bar{\theta}\Gamma_m(1+\Gamma^{(6)}_0)[\omega^m]_{\mu^1}\theta \\
&\quad + \bar{\theta}\Gamma^m(1+\Gamma^{(6)}_0)\nabla_m Y^I[\omega_I]_{\mu^{-2}}\theta + \bar{\theta}\Gamma_I(1+\Gamma^{(6)}_0)\nabla_m Y^I[\omega^m]_{\mu^{-2}}\theta - \bar{\theta}\Gamma_m\Gamma^n\Gamma^{(6)}_0\nabla_n Y^I\Gamma_I[\omega^m]_{\mu^{-2}}\theta
\end{aligned}
\tag{B.21}
$$

Using $[\omega_m]_{\mu^{-2}} = -\frac{1}{4}\Gamma_{\underline{bc}}[\omega_m{}^{\underline{bc}}]_{\mu^{-2}} = -\frac{1}{8}\Gamma^n\Gamma^I\frac{\partial g_{mn}}{\partial Y^I}$ and $[\omega^m]_{\mu^{-2}} = g^{mn}[\omega_n]_{\mu^{-2}} = \frac{1}{8}\Gamma^I\Gamma_n\frac{\partial g^{mn}}{\partial Y^I}$ for the spin connection in the regime of small fluctuations we obtain

$$
\begin{aligned}
&\bar\theta\Gamma_m(1+\Gamma_0^{(6)})[\omega^m]_{\mu^{-2}}\theta = -\frac{1}{4}\overline{\theta_+}\Gamma^I\Gamma_m\Gamma_n\theta_+\frac{\partial g^{mn}}{\partial Y^I} = \overline{\theta_+}\Gamma^I\theta_-\cdot\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial Y^I}\\
&\bar\theta\Gamma^m(1+\Gamma_0^{(6)})\nabla_m Y^I[\omega_I]_{\mu^{-2}}\theta = -\frac{1}{2}\overline{\theta_+}\Gamma^m\Gamma_{JK}\theta_+\cdot\nabla_m Y^I[\omega_I{}^{JK}]_{\mu^{-2}}\\
&\bar\theta\Gamma_I(1+\Gamma_0^{(6)})\nabla_m Y^I[\omega^m]_{\mu^{-2}}\theta = \frac{1}{4}\overline{\theta_-}\Gamma_I\Gamma^J\Gamma_n\theta_-\cdot\nabla_m Y^I\frac{\partial g^{mn}}{\partial Y^J}\\
&\bar\theta\Gamma_m\Gamma^n\Gamma_0^{(6)}\nabla_n Y^I\Gamma_I[\omega^m]_{\mu^{-2}}\theta = -\frac{1}{8}\overline{\theta_+}\Gamma_m\Gamma^n\Gamma_I\Gamma^J\Gamma_k\theta_+\cdot\nabla_n Y^I\frac{\partial g^{mk}}{\partial Y^J} + \frac{1}{8}\overline{\theta_-}\Gamma_m\Gamma^n\Gamma_I\Gamma^J\Gamma_k\theta_-\cdot\nabla_n Y^I\frac{\partial g^{mk}}{\partial Y^J}\\
&\Big/\ \frac{1}{8}\overline{\theta_+}\Gamma_m\Gamma^n\Gamma_k\Gamma_I\Gamma^J\theta_+\cdot\nabla_n Y^I\frac{\partial g^{mk}}{\partial Y^J} = \frac{1}{4}\overline{\theta_+}\Gamma_m\Gamma_I\Gamma^J\theta_+\cdot\nabla_n Y^I\frac{\partial g^{mn}}{\partial Y^J} - \frac{1}{8}\overline{\theta_+}\Gamma_m\Gamma_k\Gamma^n\Gamma_I\Gamma^J\theta_+\cdot\nabla_n Y^I\frac{\partial g^{mk}}{\partial Y^J}\\
&\qquad = \frac{1}{4}\overline{\theta_+}\Gamma_m\Gamma_I\Gamma^J\theta_+\cdot\nabla_n Y^I\frac{\partial g^{mn}}{\partial Y^J} + \frac{1}{8}\overline{\theta_+}\Gamma^n\Gamma_I\Gamma^J\theta_+\cdot\nabla_n Y^I g^{mk}\frac{\partial g_{mk}}{\partial Y^J}\ \Big/\\
&\qquad = -\frac{1}{4}\overline{\theta_+}\Gamma_m\Gamma_I\Gamma^J\theta_+\cdot\nabla_n Y^I\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{mn})}{\partial Y^J} + \frac{1}{4}\overline{\theta_-}\Gamma_m\Gamma_I\Gamma^J\theta_-\cdot\nabla_n Y^I\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{mn})}{\partial Y^J}
\end{aligned}
\tag{B.22}
$$

Adding the above listed contributions and setting $\theta_- = 0$ yields the Dirac+Yukawa terms

$$
\bar\theta\Gamma^m(1+\Gamma^{(6)})(\partial_m+\omega_m)\theta = 2\overline{\theta_+}\Gamma^m\Big(\partial_m+[\omega_m]_{\mu^1}+\nabla_m Y^I[\omega_I]_{\mu^{-2}}\Big)\theta_+ - \frac{1}{4}\overline{\theta_+}\Gamma_m\Gamma_I\Gamma^J\theta_+\nabla_n Y^I\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}g^{mn})}{\partial Y^J}
\tag{B.23}
$$

The 3-form C. Leading order contribution to the pullback of the $\mathbb{C}$-field:

$$
[\mathrm{C}_{pqr}]_{\mu^3} = \bar\theta\Gamma_{[pq}[\omega_{r]}]_{\mu^{-2}}\theta = \frac{1}{8}\bar\theta\Gamma^I\frac{\partial g^{st}}{\partial Y^I}\Gamma_t\Gamma_{[pq}\theta g_{r]s}
\tag{B.24}
$$

$$
\begin{aligned}
(\mathrm{C}_{-*_g})_{m'n'l'} &= \frac{1}{2} g_{m'm} g_{n'n} g_{l'l} \big(g^{mp} g^{nq} g^{lr} - \frac{1}{3!\sqrt{-g}} \epsilon^{mnlpqr} \big) \mathrm{C}_{pqr} \\
&= \frac{1}{8}\frac{1}{2}\Big(\frac{4}{3!^2\sqrt{-g}} g_{m'm} g_{n'n} g_{l'l} \epsilon^{mnlpqr} \partial_I (*_g)_{pqr}{}^{stu} \overline{\theta_+} \Gamma^I \Gamma_{stu} \theta_+ - \frac{4}{3!\sqrt{-g}} \partial_I (\sqrt{-g}) \overline{\theta_-} \Gamma^I \Gamma_{m'n'l'} \theta_- \Big) \\
&= \frac{1}{8}\Big(*_g\, \partial_I (*_g) \overline{\theta_+} \Gamma^I \Gamma^{\wedge 3} \theta_+ - \frac{1}{2\cdot 3!} \overline{\theta_-} \Gamma^I \Gamma^{\wedge 3} \theta_- \partial_I (\log\, g) \Big)_{m'n'l'}
\end{aligned}
$$

Analogously, $\quad (\mathrm{C}_{+*_g})_{m'n'l'} = \frac{1}{8}\Big(*_g\, \partial_I (*_g) \overline{\theta_-} \Gamma^I \Gamma^{\wedge 3} \theta_- - \frac{1}{2\cdot 3!} \overline{\theta_+} \Gamma^I \Gamma^{\wedge 3} \theta_+ \partial_I (\log\, g) \Big)_{m'n'l'}$ (B.25)

A useful expression for C can be obtained by using the following identities

$$
\begin{aligned}
\partial_I g^{ts} \Gamma_t \Gamma_{[pq} g_{r]s} \big(g^{mp} g^{nq} g^{lr} - \frac{1}{3!} (-g)^{-\frac{1}{2}} \epsilon^{pqrmnl} \big) &= \Big/ \ \frac{1}{2} \Gamma_{pq} \epsilon^{pqrmnl} = +(-g)^{\frac{1}{2}} \Gamma^{rmnl} \Gamma^{(6)} \ \Big/ \\
&= \partial_I g^{ts} (\Gamma_t \Gamma^{[mn} \delta^{l]}_s - \frac{1}{3} \Gamma_t \Gamma_s{}^{mnl} \Gamma^{(6)}) \\
\Big/ \ \Gamma^{(t} \Gamma^{s)mnl} = g^{ts} \Gamma^{mnl} - g^{(t,m} \Gamma^{s)nl} - g^{(t,n} \Gamma^{s)ml} - g^{(t,l} \Gamma^{s)mn} &= g^{ts} \Gamma^{mnl} - 3 g^{(t,[m} \Gamma^{nl],s)} \ \Big/ \\
&= \frac{2}{3\sqrt{-g}} \partial_I (\sqrt{-g}) \Gamma^{mnl} \Gamma^{(6)} + \partial_I g^{mm'} g^{nn'} g^{ll'} \Gamma_{m'n'l'} (1 + \Gamma^{(6)}) \\
&= \frac{4}{3!\sqrt{-g}} \partial_I (\sqrt{-g} g^{mm'} g^{nn'} g^{ll'}) \Gamma_{m'n'l'} \frac{1+\Gamma^{(6)}}{2} - \frac{4}{3!\sqrt{-g}} \partial_I (\sqrt{-g}) \Gamma^{mnl} \frac{1-\Gamma^{(6)}}{2} \\
\Big/ \ \sqrt{-g} g^{mm'} g^{nn'} g^{ll'} = \frac{1}{3!} \epsilon^{mnlpqr} (*_g)_{pqr}{}^{m'n'l'} \ \Big/ & \\
&= \frac{4}{3!^2\sqrt{-g}} \epsilon^{mnlpqr} \partial_I (*_g)_{pqr}{}^{m'n'l'} \Gamma_{m'n'l'} \frac{1+\Gamma^{(6)}}{2} - \frac{4}{3!\sqrt{-g}} \partial_I (\sqrt{-g}) \Gamma^{mnl} \frac{1-\Gamma^{(6)}}{2}
\end{aligned}
$$

(B.26)

Kappa symmetry gauge fixing.

$$
\left.
\begin{aligned}
&\theta_- = 0 \\
&X_0^m = X_0^m(x)
\end{aligned}
\right\}
\Rightarrow
\begin{aligned}
&\kappa_- = \text{const (all of non-constant part of } \kappa_- \text{ was used to fix } \theta_- = 0), \\
&\epsilon_- = \text{const (constant part of } \kappa_- \text{ or } \epsilon_- \text{ is used to fix } \theta_- = 0) \\
&\epsilon_+ = 0 \text{ (is needed to preserve gauge fixing condition, otherwise } \delta_{\epsilon_+} X_0^m \neq 0) \\
&\kappa_+ = 0 \text{ (is needed to preserve gauge fixing condition, otherwise } \delta_{\kappa_+} X_0^m \neq 0)
\end{aligned}
$$

(B.27)

The gauge fixing condition $\theta_- = 0$ implies that

$$
\begin{aligned}
&\delta_{\epsilon_-} X_0^m = -\overline{\epsilon_-}\Gamma^m\theta_- \sim 0 && \delta_{\epsilon_-} Y^I = -\overline{\epsilon_-}\Gamma^I\theta_+ \\
&\delta_{\kappa_-} X_0^m = 2\overline{\kappa_-}\Gamma^m\theta_- \sim 0 && \delta_{\kappa_-} Y^I = 2\overline{\kappa_-}\Gamma^I\theta_+ \\
&\delta_{\epsilon_-}\theta_- = \epsilon_- - \overline{\theta_-}\Gamma^m\epsilon_-[\omega_m]_{\mu^{-2}}\theta_+ - \overline{\theta_+}\Gamma^I\epsilon_-[\omega_I]_{\mu^{-2}}\theta_- \sim \epsilon_- && \delta_{\kappa_-}\theta_- = 2\kappa_-
\end{aligned}
\tag{B.28}
$$

There are 16 remaining global supersymmetries $\tilde{\delta}_{\epsilon_-} = \frac{1}{2}\delta_{\kappa_-=\epsilon_-} - \delta_{\epsilon_-}$ preserving the gauge fixing conditions $\theta_- = 0$ and $X_0^m = X_0^m(x)$; i.e. $\tilde{\delta}_{\epsilon_-}\theta_- = 0$.

$$
\begin{aligned}
&\tilde{\delta}_{\epsilon_-} Y^I = \overline{\epsilon_-}\Gamma^I\theta_+ \\
&\tilde{\delta}_{\epsilon_-}\theta_+ = -\frac{1}{2}\left(\Gamma \cdot dY^I\Gamma_I + \frac{1}{3!}(\Gamma^{\wedge 3} \cdot (1-\alpha \wedge \iota_{\mathrm{v}})F)\right)\epsilon_- + \frac{1}{2}\tilde{\delta}_{\epsilon_-} Y^I[\omega_I]_{\mu^{-2}}\theta_+ \\
&\tilde{\delta}_{\epsilon_-} B = \overline{\epsilon_-}\Gamma^{\wedge 2}\theta_+
\end{aligned}
\tag{B.29}
$$

Derivation of the quartic fermionic couplings (5.3)**.**

$$
\begin{aligned}
4[\delta_\theta^2\omega_m\big|_{\psi,F=0}]_{\mu^1} = &\underbrace{-R_{m\underline{ubc}}\Gamma^{\underline{ud}}\overline{\theta_+}\Gamma_{\underline{d}}{}^{\underline{bc}}\theta_+}_{(1)_m} \underbrace{+\frac{1}{2}R_{\underline{rubc}}\Gamma^{\underline{ru}}\overline{\theta_+}\Gamma_m{}^{\underline{bc}}\theta_+}_{(2)_m} \\
&\underbrace{+\frac{1}{12}R_{\underline{rubc}}\Gamma_m{}^{\underline{rstu}}\overline{\theta_+}\Gamma_{\underline{st}}{}^{\underline{bc}}\theta_+}_{(3)_m} \underbrace{-\frac{2}{3}R_{\underline{rubc}}\delta_m^{[\underline{r}}\Gamma^{\underline{stu}]}\overline{\theta_+}\Gamma_{\underline{st}}{}^{\underline{bc}}\theta_+}_{(4)_m}
\end{aligned}
\tag{B.30}
$$

$$(1)_m = -2R_{mInJ}\Gamma^{JK}\overline{\theta_+}\Gamma_K{}^{nJ}\theta_+ \quad \Rightarrow \quad \overline{\theta_+}\Gamma^m(1)_m\theta_+ = 2R_{mInJ}\overline{\theta_+}\Gamma^m\Gamma^{IK}\theta_+\overline{\theta_+}\Gamma^n\Gamma_K{}^J\theta_+$$

$$(2)_m = \frac{1}{2}R_{IJKL}\Gamma^{IJ}\overline{\theta_+}\Gamma_m{}^{KL}\theta_+ + \frac{1}{2}R_{nIkJ}\cdot 2\Gamma^{nI}\overline{\theta_+}\cdot 2\Gamma_m{}^{kJ}\theta_+$$

$$\Rightarrow \quad \overline{\theta_+}\Gamma^m(2)_m\theta_+ = \frac{1}{2}R_{IJKL}\overline{\theta_+}\Gamma^m\Gamma^{IJ}\theta_+\overline{\theta_+}\Gamma_m\Gamma^{KL}\theta_+$$

$$\begin{aligned}
(3)_m &= \frac{1}{12}R_{nIkJ}\cdot 2\Gamma_m{}^{nJ\underline{st}}\overline{\theta_+}\cdot 2\Gamma_{\underline{st}}{}^{kJ}\theta_+ + \frac{1}{12}R_{IJKL}\Gamma_m{}^{IJ\underline{st}}\overline{\theta_+}\Gamma_{\underline{st}}{}^{KL}\theta_+\\
&= \frac{1}{3}R_{nJkJ}\Gamma_m{}^{nIST}\overline{\theta_+}\Gamma^k\Gamma_{ST}{}^J\theta_+ + \frac{1}{3}R_{nIkJ}\Gamma_m{}^{nst}\Gamma^I\overline{\theta_+}\Gamma_{st}{}^k\Gamma^J\theta_+ + \frac{1}{12}R_{IJKL}\cdot 2\Gamma_{mn}\Gamma^{IJT}\overline{\theta_+}\Gamma^n\Gamma_T{}^{KL}\theta_+
\end{aligned}$$

$$\Big/\Gamma^m\Gamma_{mn} = \frac{1}{2}(6\Gamma_n - 2\Gamma_n + 6\Gamma_n) = 5\Gamma_n\Big/$$

$$\begin{aligned}
\overline{\theta_+}\Gamma^m(3)_m\theta_+ &= \frac{5}{3}R_{nJkJ}\overline{\theta_+}\Gamma^n\Gamma^{IST}\theta_+\overline{\theta_+}\Gamma^k\Gamma_{ST}{}^J\theta_+ + \frac{1}{3}R_{nIkJ}\overline{\theta_+}\Gamma^m\Gamma_m{}^{nst}\Gamma^I\theta_+\overline{\theta_+}\Gamma_{st}{}^k\Gamma^J\theta_+\\
&\quad + \frac{5}{6}R_{IJKL}\overline{\theta_+}\Gamma_n\Gamma^{IJT}\theta_+\overline{\theta_+}\Gamma^n\Gamma^{KL}{}_T\theta_+\\
&\quad \Big/\Gamma^m\Gamma_m{}^{nst} = 3\Gamma^{nst}, \qquad \Gamma^m\Gamma_m{}^{st} = 4\Gamma^{st}\Big/\\
&= \frac{5}{3}R_{mJnJ}\overline{\theta_+}\Gamma^m\Gamma^{IST}\theta_+\overline{\theta_+}\Gamma^n\Gamma_{ST}{}^J\theta_+ + R_{mInJ}\overline{\theta_+}\Gamma^{mst}\Gamma^I\theta_+\overline{\theta_+}\Gamma^n{}_{st}\Gamma^J\theta_+\\
&\quad + \frac{5}{6}R_{IJKL}\overline{\theta_+}\Gamma_m\Gamma^{IJT}\theta_+\overline{\theta_+}\Gamma^m\Gamma^{KL}{}_T\theta_+
\end{aligned} \tag{B.31}$$

$$\begin{aligned}
(4)_m &= -\frac{2}{3\cdot 4}R_{\underline{rubc}}(\delta_m^{\underline{r}}\Gamma^{\underline{stu}} - \delta_m^{\underline{s}}\Gamma^{\underline{rtu}} - \delta_m^{\underline{t}}\Gamma^{\underline{sru}} - \delta_m^{\underline{u}}\Gamma^{\underline{str}})\overline{\theta_+}\Gamma_{\underline{st}}{}^{bc}\theta_+\\
&= -\frac{1}{3}R_{m\underline{ubc}}\Gamma^{\underline{ust}}\overline{\theta_+}\Gamma_{\underline{st}}{}^{bc}\theta_+ + \frac{1}{3\cdot 2}R_{\underline{rubc}}\Gamma^{\underline{rtu}}\overline{\theta_+}\Gamma_{m\underline{t}}{}^{bc}\theta_+ + \frac{1}{3\cdot 2}R_{\underline{rubc}}\Gamma^{\underline{sru}}\overline{\theta_+}\Gamma_{\underline{s}m}{}^{bc}\theta_+\\
&= -\frac{2}{3}R_{mInJ}\Gamma^{IST}\overline{\theta_+}\Gamma^n\Gamma_{ST}{}^J\theta_+ - \frac{2}{3}R_{mInJ}\Gamma^{st}\Gamma^I\overline{\theta_+}\Gamma_{st}{}^n\Gamma^J\theta_+\\
&\quad - \frac{1}{3}R_{IJKL}\Gamma^{IJT}\overline{\theta_+}\Gamma_m\Gamma_T{}^{KL}\theta_+ + \frac{4}{3}R_{nIkJ}\Gamma^{nt}\Gamma^I\overline{\theta_+}\Gamma_{mt}{}^k\Gamma^J\theta_+
\end{aligned} \tag{B.32}$$

$$\begin{aligned}
\overline{\theta_+}\Gamma^m(4)_m\theta_+ = &-\frac{2}{3}R_{mInJ}\overline{\theta_+}\Gamma^m\Gamma^{IST}\theta_+\overline{\theta_+}\Gamma^n\Gamma_{ST}{}^J\theta_+ - \frac{2}{3}R_{mInJ}\overline{\theta_+}\Gamma^m\Gamma^{st}\Gamma^I\theta_+\overline{\theta_+}\Gamma_{st}{}^n\Gamma^J\theta_+\\
&+ \frac{4}{3}R_{nIkJ}\overline{\theta_+}\Gamma^m\Gamma^{nt}\Gamma^I\theta_+\overline{\theta_+}\Gamma_{mt}{}^k\Gamma^J\theta_+ - \frac{1}{3}R_{IJKL}\overline{\theta_+}\Gamma^m\Gamma^{IJT}\theta_+\overline{\theta_+}\Gamma_m\Gamma_T{}^{KL}\theta_+
\end{aligned} \tag{B.33}$$

$$\begin{aligned}
&\sim R_{IJKL}: \quad \frac{1}{2}R_{IJKL}(\overline{\theta_+}\Gamma^m\Gamma^{IJ}\theta_+\overline{\theta_+}\Gamma_m\Gamma^{KL}\theta_+ + \overline{\theta_+}\Gamma^m\Gamma^{IJT}\theta_+\overline{\theta_+}\Gamma_m\Gamma^{KL}{}_T\theta_+)\\
&\sim R_{mInJ}: \quad 2R_{mInJ}(\overline{\theta_+}\Gamma^m\Gamma^{IK}\theta_+\overline{\theta_+}\Gamma^n\Gamma_K{}^J\theta_+ + \frac{1}{2}\overline{\theta_+}\Gamma^m\Gamma^{IST}\theta_+\overline{\theta_+}\Gamma^n\Gamma_{ST}{}^J\theta_+ - \frac{1}{6}\overline{\theta_+}\Gamma^{mst}\Gamma^I\theta_+\overline{\theta_+}\Gamma^n{}_{st}\Gamma^J\theta_+)
\end{aligned} \tag{B.34}$$

Appendix C

Seiberg-Witten curves for pure $N = 2$ SYM with compact simple gauge groups

SW curve for classical gauge groups can be written as

$$\frac{F(x)}{\zeta} + \zeta\widetilde{F}(x) = Q(x)\,, \tag{C.1}$$

where Q, F and $\widetilde{F}$ are some polynomials. Note that for pure vector multiplets one always has $F = \widetilde{F}$, so

$$\frac{1}{\zeta} + \zeta = \frac{Q(x)}{F(x)} \qquad \Leftrightarrow \qquad y^2 = 4F(x)^2\Big(\frac{Q(x)^2}{4F(x)^2} - 1\Big), \quad y = \frac{2F(x)}{\zeta} - Q(x) \tag{C.2}$$

The tilded variables below correspond to the variables used in the papers of Konishi et all. Variables without tilde correspond to the variables used in Tachikawa's review.

A_n:

$$\frac{\Lambda^{n+1}}{z} + z\Lambda^{n+1} = x^{n+1} + u_2 x^{n-1} + \ldots + u_n x + u_{n+1}, \qquad \lambda = \frac{x}{z}dz$$
$$P(x) = x^{n+1} + u_2 x^{n-1} + \ldots + u_n x + u_{n+1}, \qquad F(x) = \Lambda^{n+1}, \qquad y = \frac{2F(x)}{z} - P(x) \tag{C.3}$$
$$y^2 = P(x)^2 - 4\Lambda^{2n+2}$$

Change of variables:

$$\begin{aligned} &\tilde{y} = y, \quad \tilde{x} = x, \quad P(x) = \prod_{a=1}^{n+1}(\tilde{x} - \phi_a)^2 \\ &\tilde{y}^2 = \prod_{a=1}^{n+1}(\tilde{x} - \phi_a)^2 - \tilde{\Lambda}^{2n+2}, \qquad \tilde{\Lambda} = 2^{\frac{1}{n+1}}\Lambda \end{aligned} \tag{C.4}$$

$N = 1$ points:

$$\tilde{y}^2 = \tilde{\Lambda}^{2n+2}(T_{n+1}^2(\xi) - 1), \qquad \xi = \frac{\tilde{x}}{2\tilde{\Lambda}} \cdot 2^{\frac{1}{n+1}} e^{-2\pi i k/2(n+1)}, \quad k = 0, 1, ..., n \tag{C.5}$$

This is equivalent to the following condition on $P(x)$:

$$\frac{1}{2}\Lambda^{-n-1}P(x) = T_{n+1}\Big(x \cdot \frac{1}{2\tilde{\Lambda}} \cdot 2^{\frac{1}{n+1}} e^{-2\pi i k/2(n+1)}\Big) \tag{C.6}$$

The maximally degenerate curve in the Hitchin form:

$$\frac{1}{z} + z = 2T_{n+1}\Big(x \cdot \frac{1}{2\Lambda} e^{-2\pi i k/2(n+1)}\Big) \tag{C.7}$$

B_n:

$$x\Big(\frac{\Lambda^{2n-1}}{z^{1/2}} + z^{1/2}\Lambda^{2n-1}\Big) = x^{2n} + u_2 x^{2n-2} + u_4 x^{2n-4} + \ldots + u_{2n}, \qquad \lambda = \frac{x}{z}\,dz$$
$$P(x) = x^{2n} + u_2 x^{2n-2} + u_4 x^{2n-4} + \ldots + u_{2n}, \qquad F(x) = x\Lambda^{2n-1}, \qquad y = \frac{2F(x)}{z^{1/2}} - P(x)$$
$$y^2 = P(x)^2 - 4\Lambda^{4n-2}x^2 \tag{C.8}$$

Change of variables:

$$\begin{aligned} &\tilde{y} = xy \quad \Rightarrow \quad \tilde{y}^2 = x^2 P(x)^2 - 4\Lambda^{4n-2}x^4 \\ &\tilde{x} = x^2 \quad \Rightarrow \quad \tilde{y}^2 = \tilde{x}P(\tilde{x})^2 - 4\Lambda^{4n-2}\tilde{x}^2, \qquad P(\tilde{x}) = \prod_{a=1}^{n}(\tilde{x} - \phi_a^2)^2 \\ &\tilde{y}^2 = \tilde{x}\prod_{a=1}^{n}(\tilde{x} - \phi_a^2)^2 - \tilde{\Lambda}^{4n-2}\tilde{x}^2 \end{aligned} \tag{C.9}$$

$N = 1$ points:

$$\tilde{y}^2 = \tilde{\Lambda}^{4n-2}\tilde{x}^2(T^2_{2n-1}(\xi) - 1), \qquad \xi = \frac{\sqrt{\tilde{x}}}{2\tilde{\Lambda}} \cdot 2^{\frac{1}{2n-1}} e^{-2\pi i k/2(2n-1)}, \quad k = 0, 1, ..., 2n-2 \tag{C.10}$$

This is equivalent to the following condition on $P(x)$:

$$\frac{1}{2}\Lambda^{-2n+1}x^{-1}P(x) = T_{2n-1}\Big(x \cdot \frac{1}{2\tilde{\Lambda}} \cdot 2^{\frac{1}{2n-1}} e^{-2\pi i k/2(2n-1)}\Big), \qquad \tilde{\Lambda} = 2^{\frac{1}{2n-1}}\Lambda \tag{C.11}$$

The maximally degenerate curve in the Hitchin form:

$$\frac{1}{z^{1/2}} + z^{1/2} = 2T_{2n-1}\Big(x \cdot \frac{1}{2\Lambda} e^{-2\pi i k/2(2n-2)}\Big) \tag{C.12}$$

C_n:

$$\frac{\Lambda^{2n+2}}{z^{1/2}} + 2\Lambda^{2n+2} + z^{1/2}\Lambda^{2n+2} = x^2(x^{2n} + u_2 x^{2n-2} + u_4 x^{2n-4} \ldots + u_{2n}), \qquad \lambda = \frac{x}{z}dz$$

$$P(x) = x^{2n} + u_2 x^{2n-2} + u_4 x^{2n-4} \ldots + u_{2n}, \quad F(x) = \Lambda^{2n+2}, \qquad y = \frac{2F(x)}{z^{1/2}} - x^2 P(x) + 2\Lambda^{2n+2}$$

$$y^2 = x^2 P(x)(x^2 P(x) - 4\Lambda^{2n+2}) \tag{C.13}$$

$$\begin{aligned} &\tilde{y} = yx^{-1} \quad \Rightarrow \quad \tilde{y}^2 = P(x)(x^2 P(x) - 4\Lambda^{2n+2}) \\ &\tilde{x} = x^2 \quad \Rightarrow \quad \tilde{y}^2 = P(\tilde{x})(\tilde{x}P(\tilde{x}) - 4\Lambda^{2n+2}), \qquad P(\tilde{x}) = \prod_{a=1}^{n}(\tilde{x} - \phi_a^2) \\ &\tilde{y}^2 = \prod_{a=1}^{n}(\tilde{x} - \phi_a^2)\Big\{\tilde{x}\prod_{a=1}^{n}(\tilde{x} - \phi_a^2) - \tilde{\Lambda}^{2n+2}\Big\} \end{aligned} \tag{C.14}$$

$N = 1$ points (for some reason there are two cases) :

$$\text{even n:} \quad \tilde{y}^2 = 2^{\frac{2n^2}{n+1}}\tilde{\Lambda}^{4n+2}\prod_{a=1}^{n/2}(\xi^2 - (\omega_a^{[n+1]})^2)(T_{n+1}^2(\xi) - 1), \quad \xi = 2^{\frac{1}{n+1}}\frac{\sqrt{\tilde{x}}}{2\tilde{\Lambda}}e^{-2\pi i k/2(n+1)}, \quad k = 0, 1, .., n \tag{C.15}$$

$$\begin{aligned} \text{odd n:} \quad \tilde{y}^2 &= \tilde{\Lambda}^{4n+2}(-1)^{-\frac{1}{n+1}}2^{1-\frac{2}{n+1}}(\xi - 1)U_{\frac{n-1}{2}}^2(\xi)T_{\frac{n+1}{2}}^2(\xi), \qquad \xi = -2^{\frac{2}{n+1}-1}\frac{1}{\tilde{\Lambda}^2}(\tilde{x} - \frac{1}{2}\phi_n^2)e^{-2\pi i k/2(n+1)} \\ &= \tilde{\Lambda}^{4n+2}(-1)^{-\frac{1}{n+1}}2^{1-\frac{2}{n+1}}\frac{1}{4}(\xi - 1)U_n^2(\xi) \\ &= \tilde{\Lambda}^{4n+2}(-1)^{1-\frac{1}{n+1}}2^{-1-\frac{2}{n+1}}\frac{1}{1+\xi}(T_{n+1}^2(\xi) - 1) \end{aligned} \tag{C.16}$$

D_n:

$$\begin{aligned} &x^2\Big(\frac{\Lambda^{2n-2}}{z} + z\Lambda^{2n-2}\Big) = x^{2n} + u_2 x^{2n-2} + \ldots + u_{2n} \\ &P(x) = x^{2n} + u_2 x^{2n-2} + \ldots + u_{2n}, \qquad F(x) = x^2\Lambda^{2n-2}, \qquad y = \frac{2F(x)}{z} - P(x) \\ &y^2 = P(x)^2 - 4\Lambda^{4n-4}x^4 \end{aligned} \tag{C.17}$$

$$
\begin{aligned}
&\tilde{y} = yx \quad \Rightarrow \quad \tilde{y}^2 = x^2 P(x)^2 - 4\Lambda^{4n-4} x^6 \\
&\tilde{x} = x^2 \quad \Rightarrow \quad \tilde{y}^2 = \tilde{x} P(\tilde{x})^2 - 4\Lambda^{4n-4} \tilde{x}^3, \qquad P(\tilde{x}) = \prod_{a=1}^{n} (\tilde{x} - \phi_a^2)^2 \\
&\tilde{y}^2 = \tilde{x} \prod_{a=1}^{n} (\tilde{x} - \phi_a^2)^2 - \tilde{\Lambda}^{4n-4} \tilde{x}^3, \qquad \tilde{\Lambda} = 2^{\frac{1}{2n-2}} \Lambda
\end{aligned}
\tag{C.18}
$$

$N = 1$ points:

$$
\tilde{y}^2 = \tilde{\Lambda}^{4n-4} \tilde{x}^3 (T_{2n-2}^2(\xi) - 1), \quad \xi = 2^{\frac{1}{2n-2}} \frac{\sqrt{\tilde{x}}}{2\tilde{\Lambda}} e^{-2\pi i k/2(2n-2)}, \quad k = 0, 1, ..., 2n-3 \tag{C.19}
$$

This is equivalent to the following condition on $P(x)$:

$$
\frac{1}{2} \Lambda^{-2n+2} x^{-2} P(x) = T_{2n-2}\Big(x \cdot 2^{\frac{1}{2n-2}} \frac{1}{2\tilde{\Lambda}} e^{-2\pi i k/2(2n-2)}\Big) \tag{C.20}
$$

The maximally degenerate curve in the Hitchin form:

$$
\frac{1}{z} + z = 2T_{2n-2}\Big(x \cdot \frac{1}{2\Lambda} e^{-2\pi i k/2(2n-2)}\Big) \tag{C.21}
$$

Exceptional cases

G_2:

The curve is obtained in [71] using the prescription of Martinec-Warner [72]:

$$
\begin{aligned}
&3(z - \frac{\mu}{z})^2 - x^8 + 2ux^6 - [u^2 + 6(z + \frac{\mu}{z})]x^4 + [v + 2u(z + \frac{\mu}{z})]x^2 = 0, \qquad \lambda = \frac{x}{z} dz \\
&\text{where} \quad u = \frac{1}{4} \mathrm{tr}\, \phi^2, \quad v = \frac{1}{6} \mathrm{tr}\, \phi^6 - \frac{1}{96} [\mathrm{tr}\, \phi^2]^3
\end{aligned}
\tag{C.22}
$$

$N = 1$ points:

$$u_k = 2 \cdot 3^{\frac{1}{4}} \Lambda^2 e^{i\pi k/2}, \qquad v_k = -4 \cdot 3^{-\frac{9}{4}} \Lambda^6 e^{3i\pi k/2}, \qquad k = 0, 1, 2, 3$$
$$3(\zeta - \frac{\mu/\Lambda}{\zeta})^2 - \xi^8 + 4 \cdot 3^{\frac{1}{4}} \xi^6 - [4 \cdot 3^{\frac{1}{2}} + 6(\zeta + \frac{\mu/\Lambda}{\zeta})]\xi^4 + [-4 \cdot 3^{-\frac{9}{4}} + 4 \cdot 3^{\frac{1}{4}}(\zeta + \frac{\mu/\Lambda}{\zeta})]\xi^2 = 0,$$
where $\xi = \frac{x}{\Lambda}$, $\zeta = \frac{z}{\Lambda^4}$, $\frac{\mu}{\Lambda}$ is a dimensionless number.

(C.23)

E_6:

The curve was obtained in Lerch-Warner [73] (using the prescription of Lerch and Warner):

$$\frac{1}{2} x^3 \tau^2 - q_1 \tau + q_2 = 0, \qquad \tau = z + \frac{\mu^2}{z} + u_6\,, \tag{C.24}$$

where

$$\begin{aligned} q_1 &= 270x^{15} + 342u_1x^{13} + 162u_1^2x^{11} - 252u_2x^{10} + (26u_1^3 + 18u_3)x^9 - 162u_1u_2x^8 \\ &\quad + (6u_1u_3 - 27u_4)x^7 - (30u_1^2u_2 - 36u_5)x^6 + (27u_2^2 - 9u_1u_4)x^5 - (3u_2u_3 - 6u_1u_5)x^4 \\ &\quad - 3u_1u_2^2x^3 - 3u_2u_5x - u_2^3 \\ q_2 &= \frac{1}{2x^3}(q_1^2 - p_1^2p_2) \\ p_1 &= 78x^{10} + 60u_1x^8 + 14u_1^2x^6 - 33u_2x^5 + 2u_3x^4 - 5u_1u_2x^3 - u_4x^2 - u_5x - u_2^2 \\ p_2 &= 12x^{10} + 12u_1x^8 + 4u_1^2x^6 - 12u_2x^5 + u_3x^4 - 4u_1u_2x^3 - 2u_4x^2 + 4u_5x + u_2^2 \end{aligned}$$

(C.25)

Alternatively we can write the curve as $\tau = z + \frac{\mu^2}{z} + u_6 = \frac{1}{x^3}[q_1 \pm p_1\sqrt{p_2}]$.

$$\begin{aligned}
p_2 &= (\sqrt{12}(x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5))^2 \qquad \Rightarrow \qquad u_2 = u_5 = 0,\ \ u_3 = \frac{1}{2}u_1^3,\ \ u_4 = -\frac{1}{96}u_1^4 \\
&= 12(x^5 + \frac{1}{2}u_1x^3 + \frac{1}{24}u_1^2x)^2 \\
x^{-3}q_1 &= 270x^{12} + 342u_1x^{12} + 162u_1^2x^8 + 35u_1^3x^6 + \frac{3\cdot 29}{32}u_1^4x^4 + \frac{3}{32}u_1^5x^2 \\
x^{-3}p_1\sqrt{p_2} &= 2\sqrt{3}(78x^8 + 60u_1x^6 + 14u_1^2x^4 + u_1^3x^2 + \frac{1}{96}u_1^4)(x^4 + \frac{1}{2}u_1x^2 + \frac{1}{24}u_1^2) \\
\tau &= z + \frac{\mu^2}{z} + u_6 = \ldots
\end{aligned} \tag{C.26}$$

E_7:

This curve was obtained in [74] using the method based on ALE spaces.

$$(\tau + v_{18})^3 + A_2(x)(\tau + v_{18})^2 + A_1(x)(\tau + v_{18}) + A_0(x) = 0\,, \tag{C.27}$$

where

$$\begin{aligned}
A_2 &= \frac{9}{16x^2}(6qp_1 - 3p_2) \\
A_1 &= (\frac{9}{16x^2})^2(9q^2p_1^2 - 6rp_1p_2 - 12qp_1p_3 + 3qp_2^2 + 3p_3^2) \\
A_0 &= -(\frac{9}{16x^2})^3(4r^2p_1^3 + 6qrp_1^2p_2 + 9q^2p_1^2p_3 - 6rp_1p_2p_3 - 6qp_1p_3^2 + 2rp_2^3 + 3qp_2^2p_3 + p_3^3)
\end{aligned} \tag{C.28}$$

Here τ can be identified as $z + \frac{1}{z}$, q and r are certain polynomials in x of degree 10 and 15 respectively, p_1, p_2 and p_3 are polynomials in x of degree 10, 15, 20 (see section [74] for details).

The authors of [74] claim that the curve (C.27) agrees with the prescription of Lerch and Warner [73] applied for the $R = \mathbf{56}$ of E_7. Namely, they claim that (C.27) can be written as

$$\begin{aligned} & P^{\mathbf{56}}_{E_7}(x; v_2, \dots, v_{14}, v_{18} + \tau) = 0\,, \\ & \text{where} \quad P^{\mathbf{56}}_{E_7}(x; v_2, \dots, v_{14}, v_{18}) = -\frac{x^2}{3^6}\,(v_{18}^3 + A_2(x) v_{18}^2 + A_1(x) v_{18} + A_0(x)) \end{aligned} \tag{C.29}$$

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