SUBJECTIVE RATIONALISM IN LIBERAL ARTS MATHEMATICS

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ABSTRACT OF THE DISSERTATION

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The majority of American college students pursue a degree outside one of the STEM fields (US News & World Report, 2015), yet these same students must still complete a minimum level of coursework in mathematics for graduation. The intent of liberal arts math courses is to help students who are majoring in one of the liberal arts successfully meet these mathematics proficiency requirements. Colleges offer liberal arts math courses to students who are unlikely to need specialized math skills, instead aiming to improve functional mathematical literacy, provide a sample of practical applications and problem-solving techniques, and to develop an overall appreciation of mathematics. Liberal arts mathematics, and the subset of quantitative literacy courses, are presently characterized by a diversity of curricular offerings with an absence of consistently defined core concepts (Dingman & Madison, 2010). Furthermore, many of these programs fail to support the overarching mission of a liberal arts education – to literally liberate the mind and prepare individuals for fully-informed and active citizenship, across multiple disciplines (King, Brown, Lindsay, & VanHecke, 2007; Stanton, 1987). The first part of this study explores three prototypical textbooks for liberal arts mathematics. The textbooks are examined for their commonalities, strengths and weaknesses, and the extent to which they facilitate the objectives of a liberal arts education. The second part of this study summarizes a progressive new curriculum for liberal arts mathematics, founded upon the notion of threshold concepts (Meyer & Land, 2003), and developed in the context of social justice. The practice of deductive reasoning, while attending to matters of probability and personal preference, collectively outline a new theory of subjective rationalism in this paper. The third part of this study presents a retrospective analysis on the creation and evolution of the new curriculum. This research explored the idea that the optimal set of math skills is neither computational nor algorithmic, rather, it lies within the realm of mathematical reasoning – the essence of which is founded upon key principles in logic and probability. The identification and cultivation of threshold concepts in mathematical reasoning offers to bring clarity and consistency into the field of liberal arts mathematics. This research is significant because a liberal arts math program may be a student's only postsecondary math course and the last opportunity to develop a useful set of math tools. Research data in this retrospective analysis spans five years across secondary and postsecondary implementations of the evolving curriculum; data include multiple iterations of the curriculum and the research practitioner's field notes reflecting on instructional interventions and classroom discourse, as well as reflections on students' performances with written assessments.

TABLE OF CONTENTS


| | | |
|---|---|---|
| ABSTRACT | | ii |
| TABLE OF CONTENTS | | iv |
| LIST OF TABLES | | vi |
| LIST OF FIGURES | . | vii |
| CHAPTER | | |
| 1 | INTRODUCTION | 1 |
| 2 | ANALYSIS OF TEXTBOOKS FOR LIBERAL ARTS MATH PROGRAMS | |
| | 2.1 Literature Review | 7 |
| | 2.2 Methods of Analysis | 15 |
| | 2.3 Analysis of Textbooks | 19 |
| | 2.4 Proposal for a New Liberal Arts Math Curriculum | 48 |
| 3 | A NEW CURRICULUM FOR LIBERAL ARTS MATHEMATICS | |
| | 3.1 Critical Social Theory and a Non-Neutral Curriculum | 52 |
| | 3.2 Subjective Rationalism and Threshold Concepts | 53 |
| | 3.3 Overview of a New Liberal Arts Mathematics Curriculum | 54 |
| | 3.4 Rationale for a New Curriculum | 56 |
| | 3.5 Unit One: Social Choice Mathematics | 59 |
| | 3.6 Unit Two: Logic | 62 |
| | 3.7 Unit Three: Probability | 73 |
| | 3.8Argumentation in Mathematical Reasoning | 79 |
| | 3.9 Intra-Curricular Connections | 82 |

3.10 Supplemental Units of Study for the Curriculum 85

3.11 Remarks: Threshold Concepts and a Curriculum for Social Justice 91

4 A RETROSPECTIVE ANALYSIS OF CURRICULUM DEVELOPMENT

4.1 Motivation for Research 94

4.2 Methodology 100

4.3 Results: Retrospection on Three Phases of Curriculum Development 114

5 DISCUSSION 143

WORKS CITED 148

APPENDICES

Appendix A 152

Appendix B 155

Appendix C 156

Appendix D 157

Appendix E 158

Appendix F 160

Appendix G 161

Appendix H 163

Appendix I 164

Appendix J 170

LIST OF TABLES


| | | |
|---|---|---|
| 1 | Representatives for Three Types of Liberal Arts Mathematics Textbooks | 15 |
| 2 | Liberal Arts Math Textbook Categories and Areas of Analysis | 19 |
| 3 | Summary of Broad Survey Textbook | 27 |
| 4 | Summary of Application-Based Textbook | 36 |
| 5 | Summary of Textbook with Emphasis on Reasoning | 42 |
| 6 | Traffic Stops by Ethnicity, Ferguson Police Department | 77 |
| 7 | Perceived Socioeconomic Status and Class Mobility | 77 |
| 8 | Educational Attainment by Socioeconomic Status | 78 |
| 9 | Game Theory Payoff Matrix | 89 |
| 10 | Game Theory Cardinal Payoffs | 89 |
| 11 | Traffic Stops by Ethnicity, Ferguson Police Department (repeat) | 136 |
| 12 | Perceived Socioeconomic Status and Class Mobility (repeat) | 136 |
| 13 | Correctional Education and Recidivism | 138 |
| 14 | Conditional Probability Assessment | 152 |
| 15 | Correctional Education and Recidivism (repeat) | 154 |
| 16 | Ratings and Rankings of Stressful Cities | 156 |
| 17 | UC-Berkeley Admissions Records, 1973 | 157 |
| 18 | U.S. Median Weekly Earnings and CPI | 159 |
| 19 | Data Set: School-to-Prison Pipeline | 162 |
| 20 | Incarceration Rates by Country | 163 |

# LIST OF FIGURES


| | | |
|---|---|---|
| 1 | Toulim's Model of an Argument | 80 |
| 2 | Wenger's Model of Subjective Rationalism and Argumentation | 80 |
| 3 | Framework of Conjecture Mapping | 101 |
| 4 | Conjecture Mapping, Summary of Three Phases | 102 |
| 5 | Phase One Conjecture Mapping | 121 |
| 6 | Direct Argument, Simplified | 125 |
| 7 | Direct Argument, Tautology | 126 |
| 8 | Phase Two Conjecture Mapping | 132 |
| 9 | Phase Three Conjecture Mapping | 141 |

# CHAPTER ONE

# INTRODUCTION

At the postsecondary level, the majority of American college students enroll in a liberal arts mathematics course to satisfy a degree requirement (US News & World Report, 2015). Colleges offer these liberal arts math courses to students who are unlikely to need specialized math skills, instead aiming to improve functional mathematical literacy, provide a sample of practical applications and problem-solving techniques, and develop an overall appreciation of mathematics. These courses and their accompanying textbooks vary widely in both their content and context; the academic field has neither established definitive guidelines nor explicit learning objectives for liberal arts math programs (Dingman & Madison, 2010; Karaali et al., 2016; Steen, 2001). If the overarching mission of a liberal arts education is to liberate the mind and prepare individuals for fully-informed and active citizenship across multiple disciplines (King, Brown, Lindsay, & VanHecke, 2007; Stanton, 1987), then the mathematics component of a liberal arts education should provide a platform to develop critical thinking skills in support of this mission.

Tools of arithmetic, algebra, probability, and logic, can aid students in constructing a more sophisticated lens through which to view the world and engage in thoughtful reflection. A course in liberal arts mathematics is an opportunity to build a useful set of mathematical reasoning skills that serve a student along a variety of academic and professional careers. However, traditional postsecondary math curricula often present decontextualized algorithmic processes and algebraic abstractions that are ineffective and uninspiring to students outside the STEM fields – resulting in high failure rates, math anxiety, disengagement, and reduced capacity to effectively apply math tools for situations arising in life and work (Hacker, 2016; Steen, 2001). In the absence of historical, cultural, and political context, students are less likely to make connections between the mathematics content and its applications (Lesser & Blake, 2007). Liberal arts math programs generally claim to develop tools for informed citizenship – yet they typically do so with politically neutral data sets. Critical mathematics challenges the notion that we are citizens in an equitable society. A pedagogy of critical mathematics argues in favor of increasing student awareness of social injustices and inspiring students to become agents of change. Liberal arts mathematics and critical mathematics essentially share the same mission of informed citizenship; the former provides content for a program, and the latter offers context. Since there is a dearth of literature exploring the intersection of these two fields, the academic community would benefit from research that uses student feedback to gain insight into the development and efficacy of a liberal arts math curriculum imbued with themes of social justice.

My research investigates current offerings in the field of liberal arts mathematics from the perspective of a critical pedagogy and with an interest to identify threshold concepts in mathematical reasoning. Threshold concepts (Meyer & Land, 2003) are fundamental building blocks within an academic discipline, and when learned, they open up new pathways for understanding. I argue that it is essential to cultivate them within a liberal arts math curriculum if the underlying intention is to build a set of mathematical reasoning skills that are useful across multiple disciplines. This analysis focuses on specific content areas that can help shape the academic field, and examines the impact of framing the content within sociopolitical contexts. As a research practitioner, I explored theories of critical math pedagogy, instructional interventions, the influence of student feedback, and an evolution of ideas leading to a new curriculum for liberal arts mathematics. I have written three chapters that (A) analyze three prototypical liberal arts math textbooks, (B) offer a new curriculum for liberal arts mathematics founded upon social justice and threshold concepts, and (C) present a retrospective analysis of the development of this new curriculum. My research connects liberal arts mathematics with critical mathematics, and seeks to identify threshold concepts fundamental to mathematical reasoning. This paper is written with the mindset that a progressive math education can increase students' awareness of the world's inequities and encourage them to participate in positive social change.

In chapter two, the field of liberal arts mathematics and accompanying textbooks are distilled into three categories based upon their overarching goal of whether they aim to develop an appreciation of general math concepts, a set of practical applications, or reasoning processes. The vastness of mathematics naturally allows for a variety of approaches to the subject. A liberal arts math curriculum is presented with the task of serving a diverse population while maintaining relevance to individual paths. To succeed in this endeavor, educators are challenged to identify meaningful content and context – a math program that lacks either threatens to perpetuate the disconnect that many non-STEM students feel towards the subject. My research demonstrates the diversity within the field, the absence of core concepts, and the conservative nature of context found throughout these programs.

The third chapter outlines a progressive liberal arts math curriculum, designed to aid students in constructing a more sophisticated lens through which to view the world. The curriculum is presented as an alternative to the current offerings described in the second chapter. My interest in critical math pedagogy motivated me to integrate sociopolitical themes into the curriculum. Much of the content, assignments, and assessments are situated in the context of social justice in order to increase student engagement and bring relevancy to the material. In addition, my realization that validity and reliability are threshold concepts in mathematical reasoning resulted in the curriculum's careful attention to logic and probability.

The approach to liberal arts mathematics that I offer in the third chapter is designed to develop *subjective rationalism*. I define subjective rationalism as the practice of deductive reasoning, while attending to matters of probability and personal preference. The quest for understanding is analogous to the act of argumentation, and it is at the crux of Plato's classic theory of knowledge as a justified true belief. The act of justification – providing reasons that support a claim, is inherently a mathematical endeavor. Subjective rationalism is my theory that an individual may reach an optimal level of understanding by operating within a logically valid framework while being mindful of the probabilities necessarily attached to the premises of an argument. The assumption is that the argument is of personal interest to the individual, as such, pure objectivism fails. This theory emerged through my experience of curriculum development, the details of which are explained in my retrospective analysis in chapter four. A curriculum designed for subjective rationalism necessarily integrates threshold concepts from logic and probability, while addressing topics from social choice mathematics in order to cultivate one's ability to connect qualitative and quantitative values. My emphasis on threshold concepts and social justice fits neatly within the current discourse on equity in education and the expanding field of liberal arts math programs. The curriculum outlined in the third chapter now exists as a full-length textbook for postsecondary liberal arts mathematics.

In chapter four, I present a retrospective analysis that explored three distinct phases of my curriculum development and examined my time as a research practitioner. Practitioner research, also known as action research, is a cyclical process of acting-observing-reflecting-changing-planning, and then acting again (Skovsmose & Borba, 2004). As an extension of action research, I implemented a design of conjecture mapping (Sandoval, 2014), in which I tested a series of high-level conjectures that drove the learning experience. The evolution of the curriculum was a continuous state of theoretical refinement and practical improvement, in which I sought to understand the ideal content and context for a liberal arts mathematics course.

Throughout the span of five years, I engaged in curricular and pedagogical changes in order to create a transformative learning experience for my students. Six years ago, I set out to bring more relevance into an application-based math program for high school students concurrently enrolled in the local community college. My primary intention was to increase student engagement by offering a more meaningful course of study. The high school phase was the birth of the curriculum; I launched the study by changing the content and context of a traditional liberal arts math program. The second phase began when I started teaching in correctional facilities, and this phase was characterized by a shift in what I perceived to be important for my students. During this time, I continued to develop my curricular ideas and was challenged by a demanding student body armed with unfiltered feedback. The end of the second phase is when I discovered the notion of a threshold concept. In the third phase, I explicitly targeted the concepts of validity and reliability by attending to logic and probability as essential domains within mathematical reasoning. I grew to understand that the intersection of these seemingly disparate worlds of certainty and uncertainty are fundamental to the principles of argumentation and one's search for knowledge. My personal realization about the central roles played by validity and reliability informed a multitude of changes and brought the curriculum to its current form.

The retrospective analysis adds to our existing knowledge of liberal arts mathematics by analyzing the identification and development of threshold concepts within mathematical reasoning, as well as addressing a gap in academic literature about the role of liberal arts mathematics in correctional education. Ideally, the findings in this research study will reach a wide audience and help to shape the fields of both liberal arts mathematics and critical mathematics, by exploring their intersection.

# CHAPTER TWO

# ANALYSIS OF TEXTBOOKS FOR LIBERAL ARTS MATH PROGRAMS

## 2.1 Literature Review

### The Prevalence of Liberal Arts Mathematics Courses

The majority of American college students enroll in a liberal arts math course to satisfy a degree requirement (US News & World Report, 2015). These students represent a diverse audience with interests outside the field of mathematics, and many of them may never take another math class in college. The question is, and has been for a long time – what do we do with these students?

The goal of a liberal arts education is to develop critical thinking, to literally liberate the mind and free it to reach its fullest potential (King, Brown, Lindsay, & VanHecke, 2007; Stanton, 1987). Effective reasoning, writing, and clear communication are central to its mission; accordingly, it has a social purpose. Liberal arts courses prepare students for full participation in society and promote broad learning in multiple disciplines.

Courses labeled "Math for Liberal Arts," or "Liberal Arts Math," can be found on nearly every college campus in America and provide an opportunity for students to satisfy their school's graduation requirements. College students that pursue a major outside one of the STEM fields must successfully complete a minimum level of coursework in mathematics, increasingly called a "quantitative reasoning" requirement. This is no small matter, given that the failure to pass mandatory math courses is cited as the primary academic reason for students not graduating college (Hacker, 2016).

Liberal arts math is commonly presented as a survey course containing a variety of math topics from different branches, such as algebra, geometry, and set theory. However, the manner in which colleges present their liberal arts math courses – including content, context, assessment, and pedagogy, often varies widely from one campus to the next. Given the diverse nature of its audience, the field of liberal arts math textbooks has not established a set of core content standards. Additionally, the growth of quantitative literacy courses, as a subset of liberal arts math courses, also reveals inconsistent practices among math educators (Dingman & Madison, 2010). This chapter explores a sample of these offerings, and seeks to identify fundamental principles that can unify the field of liberal arts math for a diverse audience – an audience that just happens to be the majority of American college students.

**Dismal Irrelevance**

Non-STEM students, when mandated to enroll in a college level math course, often bring with them low motivation, anxiety, and misconceptions about the irrelevance of math (Hacker, 2016). Instructors for these courses are presented with the challenge of positively impacting students who carry around negative perspectives on mathematics. Hacker (2016) sympathizes with these students, and argues that much of the traditional postsecondary math curricula is both irrelevant and unrealistic in its claim to prepare students with career-readiness.

Traditional math classes are loaded with abstractions that are unlikely to be encountered in real life. These classes, Hacker writes, do not serve the interests of the students. In many cases, the mandatory math curriculum is deeply entrenched, and serves as a “gatekeeper” preventing students from advancing with their academic and professional careers (Hacker, 2016). The consequence of such a widely ineffective curriculum is that many Americans suffer from *innumeracy* – a word popularized by best-selling author and math professor John Allen Paulos. Innumeracy is an inability to work with numbers or failing to effectively engage in mathematical reasoning. Paulos (1998) posits that an ideal liberal arts math curriculum would contain topics such as combinatorics, game theory, and especially probability. He draws a direct link between the innumeracy he observes among the general public, and "the poor mathematical education received by so many people" (p. 108). Dismal performances on standardized international math exams (Cavanaugh, 2012) and a widely innumerate citizenship, collectively point to a curriculum in need of reform. Progressive math educators agree that the traditional American curriculum has been overburdened with algebra, is generally ineffective in training students how to reason with numbers, and basically fails to develop a student's ability to apply mathematical thinking to everyday situations (Packer, 2003; Steen, 2001). At the turn of the century, Lyn Steen led the *National Forum on Quantitative Literacy* and the subsequent publication of its proceedings. His work has inspired a national network, an online peer-review journal, and countless new course offerings across the country. Steen's work gave birth to the quantitative literacy movement, which has since become a growing subset of liberal arts math courses, yet it too remains undefined.

**Quantitative Literacy and Quantitative Reasoning**

Liberal arts math and quantitative literacy courses intend to serve the majority of American college students on their path towards informed citizenship. Both call for sophisticated reasoning with elementary arithmetic, and both use tools of basic algebra.

These courses essentially share the same mission, and many textbooks are written with both audiences in mind. Yet, while there is overlap between them, there is also tremendous variety within them. Neither liberal arts math nor quantitative literacy is clearly defined as an academic field, and the question of how to best align mathematics education with relevant, everyday needs remains unanswered (Dingman & Madison, 2010).

Driven by a civic rationale and a sense of pragmatism, Lyn Steen (2001) argued that the mathematics typically taught in schools bore little relationship to the actual mathematics needed for citizenship. Steen's dissatisfaction with the mainstream mathematics curriculum led him to collaborate with others on the seminal publication *Mathematics and Democracy*, in which they extolled the importance of a quantitatively literate citizenry. His team was in agreement on "a sense of something missing, some important preparation for life that was ignored by this traditional mathematics education" (Steen, 2001, p. 109). He was not afraid to admit that "although almost everyone believes quantitative literacy to be important, there is little agreement on just what it is" (p. 4).

Colleges cite the need to offer a more relevant math option for their liberal arts students (Madison, 2014; Todd & Wagaman, 2015; Tunstall et al., 2016). Many schools have developed programs to help their non-STEM students satisfy the mathematics graduation requirement, to provide more meaningful educational experiences by connecting the classroom to the lives of the students, and to prepare them to be better at their jobs and become more informed citizens. However, quantitative literacy, was and is still, not clearly defined (Karaali, Villafane Hernandez, & Taylor, 2016). In 2001, Steen presented a description as "an aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem-solving skills that people need in order to engage effectively in quantitative situations arising in life and work" (p.7). However, this set of skills is not only called different names, but has a variety of interpretations too. Generally speaking, the terms quantitative literacy, quantitative reasoning, numeracy, mathematical literacy, and mathematical reasoning, refer to a collection of problem solving skills, data analysis tools, habits of mind, and communication capabilities.

In the first 2016 issue of *Numeracy*, Karaali et al., published an article entitled "What's in a Name? A Critical Review of Definitions of Quantitative Literacy, Numeracy, and Quantitative Reasoning." The authors write that in the fifteen years since the publication of *Mathematics and Democracy*, the quantitative literacy movement still lacks definitive guidelines; the mathematics content best suited for equipping citizens with the skills needed to process quantitative data in their everyday personal and professional lives has not been agreed upon (Karaali et al., 2016). There is even less consensus among the collection of liberal arts math courses.

**Content and Context**

Textbooks for liberal arts math programs characteristically fall into three categories based upon their overarching goal of whether they aim to develop (A) an appreciation of general math concepts, (B) a set of practical applications, or (C) reasoning processes. I have distilled the field into these three categories through discussions with my colleagues and peers at national math education conferences, and through informal research during my fourteen years of professional teaching experience at the secondary and postsecondary levels. The first type of textbook contains a broad survey of common math facts people should know, including a variety of algebraic, geometric, and elementary statistical concepts. Books in this category often incorporate historical anecdotes, references to interesting patterns in nature, and exercises with number theory. These books have a tendency to teach the accomplishments of math, and they typically present developmental exercises in a conservative context.

The second category of liberal arts math textbooks claims to offer highly practical and relevant applications of math. Content in these books usually emphasize consumer math and personal finance, linear and exponential modeling, matters of probability, ratios and proportions, and an introduction to data analysis. A smaller percentage of liberal arts math books fit into a third category that gives its attention to mathematical thinking. These books frame their content around an exploration into various types of mathematical reasoning – from deductive to inductive, while developing a list of heuristics for problem solving.

The vastness of mathematics naturally allows for a variety of approaches to the subject. A liberal arts math curriculum is presented with the task of serving a diverse population while maintaining relevance to individual paths. To succeed in this endeavor, educators are challenged to identify meaningful content and context – a math program that lacks either threatens to perpetuate the disconnect that many non-STEM students feel towards the subject.

Courses in liberal arts mathematics illustrate the use of math as a tool for understanding and making sense of everyday situations, in all aspects of one's life – public, private, personal, and professional. Accordingly, the quality of a math education may be measured by its capacity to transform aspects of students' lives as current and future citizens. Whether the goal is to develop productive workers, democratic citizens, or free-thinking individuals, it is worth reflecting on what may be the most relevant contexts in which to situate the content in a math program. To this end, a growing number of progressive educators argue that sociopolitical themes provide effective and engaging context for their students (Gutstein, 2006; Lesser, 2007).

Lesser and Blake (2007) discuss math courses void of historical, cultural, and political context, and they argue that when a course lacks relevant context students fail to make connections between the content and its applications. Furthermore, if teachers use data sets relating to issues of social justice and contextualize course content with provocative themes, then student motivation and engagement can increase (Lesser & Blake, 2007). Research practitioner Rico Gutstein (2005), presents many examples of math exercises with sociopolitical contexts, demonstrating that students can engage in sophisticated mathematical reasoning to investigate the unequal distribution of wealth, wasteful government spending, racial profiling, the correlation between a family's income and the child's academic achievement, misinterpretations of medical diagnostic testing, and capital punishment. Fundamental to the argument made by Lesser and Blake (2007) is the notion that controversial topics capture student interest, with the benefits of more retention and mastery of the math content, as well as an increased ability to generalize and apply mathematical concepts across more cases.

**Critical Pedagogy, Democracy, and the Goals of Liberal Arts Mathematics**

A liberal arts college education purports to develop critical thinking and prepare individuals for full participation in a democratic society (King, Brown, Lindsay, &

VanHecke, 2007; Stanton, 1987). An illiterate or innumerate public may make bad decisions in self-governance, as such, the mathematics component of a liberal arts program offers a platform for students to develop a useful set of reasoning skills and reflect on social issues through a mathematical lens. Steen (2001) argues that an individual can learn to be a more constructive, concerned, and reflective citizen, through a quantitative literacy course that develops one's capacity for logical thinking and reasoning with data. Ideally, liberal arts math programs for non-STEM students, including the subset of quantitative literacy courses, will improve students' abilities to lead informed lives and enrich their understanding of other subjects. However, educators continue to debate content and context in these programs, and this disagreement is reflected in the variety of textbooks used to support these courses.

A course in liberal arts mathematics is an opportunity to build a more sophisticated lens through which to see the world, using tools in arithmetic, algebra, probability, and logic. A pedagogy of critical mathematics is fundamentally about increasing student awareness of sociopolitical issues. One can argue that liberal arts mathematics and critical mathematics share the same mission of increasing a student's capacity for thoughtful reflection; and while one outlines the content, the other provides the context. Liberal arts math programs generally claim to develop tools for informed citizenship – yet they often do so with politically neutral data sets. Critical mathematics challenges the notion that we are citizens in a just society. Lesser and Blake (2007) argue that we should extend the "math needed for informed citizenship" into the realm of critical inquiry as it relates to awareness of social injustices. Henry Giroux (2011) adds, "educators must assume the responsibility for connecting their work to larger social issues" and to help students "learn the tools of democracy and how to make a difference in one's life as a social agent" (p. 171).

This chapter explores three prototypical liberal arts math textbooks from the perspectives of critical pedagogy and progressive curriculum reform. Attention is given to the structure and sequence of each book, the authors' rationale for the selection of content and context, and an analysis of whether the books succeed in the mission to prepare students for fully-informed, active citizenship. The critique investigates the extent to which each textbook provides a platform for a transformative learning experience.

**2.2 Methods of Analysis**

Three books are included in this analysis, and each book was carefully chosen to exemplify a category of liberal arts math textbooks: (1) broad survey books, (2) books emphasizing applications, and (3) books that primarily attend to the development of mathematical thought processes. These classifications are consistent with the three general approaches to liberal arts math courses. My research identified these three particular books as popular choices, frequently adopted by postescondary math departments across the country (see Table 1). Accordingly, these books are appropriate selections as representatives for the three main categories of liberal arts math textbooks.

Table 1
*Representatives for Three Types of Liberal Arts Mathematics Textbooks*

| | |
|---|---|
| Type 1:<br>Broad Survey | *A Survey of Mathematics with Applications* (8th edition).<br>Authors: Allen Angel, Christine Abbott, and Dennis Runde<br><br>Adopted at over 300 state universities, private colleges, and two-year community colleges, including, Kean, Rowan, Arkansas State, Delaware State, Trinity, Hawaii, Iowa State, Wichita State, Towson, |

| | |
|---|---|
| | UMass, Michigan, Minnesota State, UNC, Penn State, Texas A&M, Houston, Vermont, and West Virginia |
| Type 2: Emphasis on Applications | *For All Practical Purposes: Mathematical Literacy in Today's World* (8th edition). Project Director: Solomon Garfunkel<br><br>Currently used by more than 80 colleges and universities, including Auburn, Columbia, FDU, Indiana, LSU, Manhattan College, Miami, Montclair State, NYU, Rice, The College of New Jersey, Georgia, Rhode Island, and Villanova |
| Type 3: Emphasis on Reasoning | *The Heart of Mathematics: An Invitation to Effective Thinking* (3rd edition). Authors: Edward B. Burger and Michael Starbird<br><br>Consistently used at the following state universities: Texas, Tennessee, Connecticut, Wisconsin, as well as these private colleges: Creighton, Bradley, Loyola, and Greensboro |

***Analysis Procedure***

The analysis progressed in three layers. First, the books were read in their entirety, from preface to conclusion, to gain a holistic perspective. The holistic perspective was to assess the book's capacity to provide a platform for a transformative learning experience. This overall reading was followed by a narrowed focus on specific chapters that highlight the book's successes and/or shortcomings in meeting the objectives of a liberal arts math education. A holistic reading approach allows the reader to get a feeling for general themes, contextual patterns, and the sequence of development for key ideas. The focused reading positions the reader to identify particular content areas valued by the authors. The third layer of my analysis was to carefully examine individual exercises put forth by the authors. This chapter presents some exercises from each text; the examples are intended to promote a representative view, avoiding an unfair selection of any examples that could be construed as particularly weak or strong. Examples were chosen based upon their ability to effectively demonstrate the authors' approach to developmental exercises.

Each of these textbooks was studied for the presence of intra-connections within the content to see if key concepts are consistently reinforced by the authors. The content was analyzed for its capacity to impact the development of mathematical reasoning, and the context was critiqued according to its potential to cultivate critical reflection. Special attention was given to authors' introductions and their stated objectives. The books were examined to determine whether the authors created a text that achieves their goals, and whether the book allows an instructor to incorporate local and sociopolitical themes into the curriculum.

In summary, four key areas are addressed in this analysis. First, the books were considered in terms of their scope – what is the overall objective? Second, the books were reviewed for the authors' deliberate selection and sequence of content – do the books tell a story, do they have a cumulative effect, are they self-referential, and do they develop skills that are transferrable across other fields in the liberal arts? Third, how are the books presented contextually – are they conservative, do they explore any sociopolitical themes that offer implications for ethical citizenship? Finally, to what extent does each book attend to threshold concepts in mathematical reasoning? This fourth factor – the attention to threshold concepts, is examined with particular import.

Meyer and Land (2003) advance the notion of a threshold concept as a fundamental building block within an academic discipline, yet one that typically represents troublesome knowledge. They characterize threshold concepts as transformative, integrative, and irreversible – and understanding them naturally engenders more sophisticated ways of thinking. In chapter four of this research study, the retrospective analysis identifies the concepts of a logically valid framework, and the reliability of a premise viewed through the lens of compound probability, as central components in mathematical reasoning, and therefore essential in a liberal arts mathematics experience. The search for knowledge is analogous to the act of argumentation – an individual needs to operate within a logically valid framework while being mindful of probabilities attached to the premises of an argument. The conclusion of a sound argument, i.e. an argument that is both valid and reliable, effectively constitutes new knowledge. When learned, key principles in formal logic and probability switch from gatekeepers to gateways, opening up pathways to expand one's knowledge across disciplines. A liberal arts math course is an opportunity to learn valuable skills in mathematical reasoning – at the heart of which, lie the threshold concepts of validity and reliability. Accordingly, the three textbooks analyzed in this chapter are assessed for their treatment of inductive and deductive reasoning, as well as their development of probabilistic thinking.

***Imitative and Creative Reasoning***

Although reasoning is at the heart of mathematics education research, there is a lack of detail surrounding the aspects of mathematical reasoning that determine its quality (Lithner, 2006). Lithner categorizes two types of reasoning – imitative and creative, while building upon Schoenfeld's (1985) definition of what constitutes a math problem as opposed to a math exercise. It is important to distinguish between computational exercises in which an individual already has access to a solution schema, and problems in which the individual is at an intellectual impasse. Real problems require creative thinking. Sophisticated mathematical reasoning goes beyond the recall of basic facts and procedures; however, students spend the bulk of their time solving computational exercises (Lithner, 2006). It is no surprise, Lithner adds, “Mathematics is often seen as a meaningless, fragmentary collection of algorithms to be memorized.” (p.24, 2006). Memorized reasoning and algorithmic reasoning constitute the two types of imitative reasoning. Exercises that merely require the recall of facts or procedures not only persist in traditional mathematics curricula, they dominate the nature of student tasks in modern textbooks (Lithner, 2003, 2004).

In contrast, Lithner (2006) depicts creative reasoning to possess traits of novelty, flexibility, fluency, and plausibility. When engaged in creative reasoning, students employ both predictive and verificative argumentation to support the selection and efficacy of a particular strategy. The importance of providing reasons to justify one’s endeavors with creative reasoning is consistent with the significance attached to logical frameworks. Mathematical problems that require creative reasoning are also problems whose solution is the product of a sound argument, founded upon threshold concepts of reliability and validity. Thus, while focusing on the four key areas mentioned in the preceding section, the analysis performed in this chapter will also classify the distribution of tasks in the three textbooks, according to the type of reasoning required to produce a solution.

**2.3 Analysis of Textbooks**

The analysis progresses by each text. Table 2 summarizes the three genres of liberal arts math textbooks and the four areas of analysis in this paper.

Table 2
*Liberal Arts Math Textbook Categories and Areas of Analysis*

| Genre | Scope | Content | Context | Threshold Concepts |
| --- | --- | --- | --- | --- |
| Type 1<br><br>Broad Survey | Expose students to an array of traditional topics. Present a review of basic math facts everyone should know. | Elementary algebra and geometry, number systems, and set theory. Books often include a basic introduction to data analysis | Exercises are mostly algorithmic and purely mathematical, devoid of context. Context that is used is typically conservative, such as sports, entertainment, shopping, nature, and the physical sciences. | Topics in logic and probability are included, but are neither tied to each other nor to the principles of argumentation. The explicit lack of connection between these concepts limits the reader's ability to effectively develop more profound reasoning skills. |
| Type 2<br><br>Emphasis on Applications | Develop proficiency in applying formulas to real life scenarios. Students follow prescribed procedures and learn to use mathematical tools already derived. | Consumer math, linear models, exponential growth and decay functions, voting systems, apportionment, rates and percentages, probability and descriptive statistics, cryptography. | Patterns in the natural world, sports and entertainment, retail sales and personal finance, health and medicine, agriculture, automotive and travel scenarios. | Little attention to formal logic, and probability is studied playfully. The concepts are not associated with each other in the text; validity and reliability are not addressed in a manner that would aid the reader in constructing a lens for a more critical world view. |
| Type 3<br><br>Emphasis on Reasoning | Develop problem-solving strategies and highlight various forms of mathematical thinking. Engage students in pattern recognition and promote the communication of ideas. | Informal approaches to inductive and deductive reasoning practices, excluding formal propositional logic. Explorations into non-Euclidean geometries, graph theory, perceptions of infinity, and combinatorics. How to deal with uncertainties and interpret data. | Many games and puzzles, including fictitious stories and riddles. Sequences and patterns found in nature. Pure math contexts, such as number theory and geometrical objects, provide the setting for many reasoning exercises that engage in notions of infinity. | Deductive reasoning is developed for the purpose of problem solving, not for an understanding of validity in argumentation. Probabilities are discussed informally, and they are not associated with the concept of reliability for an argument's premise. The informal approach to both reduces the likelihood of building a transferrable set of sophisticated math reasoning skills. |

**Distribution of Reasoning Types Employed in Textbook Tasks**

Applying Lithner's definitions for imitative and creative reasoning, I analyzed tasks presented at the end of chapters, including review exercises, tests, and extension projects. I focused this part of my analysis on the chapters that address principles in logic and probability because of my assertion about the central roles they play in the development of an individual's mathematical reasoning skills. In the third text, only the probability chapter was examined for its reasoning types because the authors do not have a section that explicitly attends to formal logic, and they present a high number of tasks in the probability chapter. Although math educators value creative problem solving, I found that the tasks in these books overwhelmingly elicit procedural, rather than creative thinking. The overemphasis on imitative reasoning – consisting of factual recall and algorithmic thinking, is revealed in the statistics below.

In the "broad survey" book, 97% of the 193 tasks assigned to the student require only imitative reasoning to reach a solution, and there are no creative reasoning tasks found in the textbook's chapter assessment. In the "applications" book, 95% of the 175 tasks are exercises that prompt the recall of facts and procedures, the remaining problems require creative thinking beyond pre-established solution schema. In the third text, representing "reasoning" books, 94% of 169 probability tasks do not compel the student to use more than imitative reasoning; this is not a significant improvement over the other genres, given its overarching commitment to the development of a reader's mathematical thinking skills. Note, this chapter's critique of the three textbooks proceeds without additional reference to imitative and creative reasoning processes, hereafter focusing on the four previously mentioned areas of analysis.

**Analysis: A Survey of Mathematics with Applications**

***Scope***

*A Survey of Mathematics with Applications* (Angel, Abbott, & Runde, 2009) is a widely used textbook for liberal arts math courses at colleges across America; it has undergone ten editions, and it features contributions from over seventy reviewers from around the country. The three primary authors – Allen Angel, Christine Abbott, and Dennis Runde, are community college mathematics professors with over sixty combined years of teaching experience. The authors state their objectives in the opening passages of the book:

> The text is intended for students who require a broad-based general overview of mathematics, especially those majoring in the liberal arts…. It is particularly suitable for those courses that satisfy the minimum competency requirement in mathematics for graduation or transfer. (p. XII)

In their note to the student, they write "The primary purpose is to provide material that you can read, understand, and enjoy," and "We hope to teach you some practical mathematics that you can use every day and that will prepare you for further mathematics courses" (p. XI). It is important to note that the authors do not make a commitment towards the cultivation of critical thinking skills, rather, they aim to develop a reader's appreciation for the "beauty" of math. As is typical within the first category of liberal arts math textbooks, these authors present what has already been accomplished by other people (i.e. mathematicians throughout history), and then attempt to demonstrate some uses of these accomplishments.

***Content***

This book features an algorithmic approach with its topics and promotes procedural operations. The topic of functions is important in mathematics because of its usefulness in modeling real-life situations. The derivation of a function is an opportunity to develop valuable reasoning skills; students could be prompted to create algebraic expressions and functions by incorporating their understanding of fixed and variable costs, and exponential growth rates, etc. However, these authors omit valuable discussions surrounding the creation of a model, and instead ask students to simply perform a lower-level skill of evaluating these expressions and functions. Consider the following two examples:

> The cost of operating a taxi, C, is given by the function C(M) = 52 + .23M, where M is the number of miles driven per week. What is the weekly cost if he drives 200 miles in a week? (p.392)
>
> Marta Rivera is a part owner of a newly opened bagel company. Marta's yearly profit, in dollars, is given by the function P(X) = .3X – 4000, where X is the number of bagels sold per year. If Marta sells 150,000 bagels a year, determine her yearly profit. (p.403)

Exercises such as these do not develop reasoning skills, they develop memorization skills. Mathematical reasoning is an ability to apply mathematical thinking, not mathematical procedures, to new scenarios. Content in this book, especially that which is contained in the middle chapters on algebra, may serve to support students in further math courses as the authors claim in their introduction, yet they do not develop critical thinking skills that could aid students in their respective liberal arts studies. A liberal arts math course may support students' work in other fields by attending to the underlying nature of mathematical thinking – a process that incorporates both inductive and deductive reasoning. Algebraic expressions and functions are traditional topics in a math class; however, their evaluation is a mechanical procedure, while their creation promotes sophisticated reasoning skills. This textbook gives its attention to the former, rather than the latter.

The book presents a sequence of topics that do not build upon each other cumulatively. The drawback with broad survey books is that while they are diverse in their collection of topics, they are too diverse to build connections across all the content areas, unavoidably resulting in a fragmented arrangement of text material. The authors travel from sets and logic to number theory and number systems, then from algebra to the metric system, from geometry to groups and modular arithmetic, then from consumer math to probability and statistics. Only the final two chapters offer a connected order of topics, because probability establishes a strong foundation for statistics. The authors provide no rationale – neither to the instructor nor the student, why they have chosen to present the content in the given sequence. The textbook is prototypical of a survey course in that its table of contents lacks a connected flow of ideas.

In broad survey books, authors include basic math topics they believe everyone should know. While it is helpful to know the mathematics underlying the metric system, as well as consumer mathematics such as interest calculations on savings and loans, the authors insert them as entire chapters that add to an already voluminous collection of fragmented content. Neither of these topics, which combined constitute over 13% of the text, develop sophisticated reasoning skills that expand pathways to knowledge in other liberal arts fields. Their inclusion reinforces the perspective of the text as a broad survey, rather than as platform for a transformative learning experience.

***Context***

In this textbook, there is no central story for the reader, nor are there any dominant themes – the index of applications at the end of the book lists over nine hundred different contexts for its exercises, ranging from applying lawn fertilizer to weight restrictions on a road in France. Students are consistently guided to follow a sequence of clearly defined steps for each topic and then apply them in conservative contexts. The following example presents itself as a real-life scenario, yet is merely a mechanical application to practice the skill of evaluating an algebraic expression through the established order of operations.

> The rate of growth of grass, in inches per week, depends on a number of factors, including rainfall and temperature. For a certain area, this rate can be approximated by the expression $0.2\ R^2 + 0.003RT + 0.0001T^2$, where R is the weekly rainfall, in inches, and T is the average weekly temperature, in degrees Fahrenheit. Find the amount of growth of grass for a week in which the rainfall is 2 in, and the average temperature is 70˚F. (p.316)

The decision to use conservative context also appears in the book's chapter on logic. A formal study of logic enables one to better understand the principles of argumentation – to be able to construct a valid framework in support of one's own view, and to deconstruct the articulated position of another person. The content in this chapter is comprehensive in introducing and explaining the inner workings of both categorical and propositional logic. However, it does not use the tools to probe relevant or provocative issues. Consider this chapter test exercise:

> Translate the following argument into symbolic form. Determine whether the argument is valid or invalid by comparing the argument to a recognized form or by using a truth table. "If the soccer team wins the game, then Sue played fullback. If Sue played fullback, then the team is in second place. Therefore, if the soccer team wins the game, then the team is in second place." (p.179)

This culminating task, found at the end of the logic chapter, follows countless other exercises that are either decontextualized or situated in a context that does not aid in developing informed citizenry. Rather than examining social themes in the dominant discourse, including the role of the media in promulgating invalid messages, these authors put forth examples that are unlikely to engage students, and they do not demonstrate how the tools of logic transfer into other liberal arts fields. Instead, for example, the authors choose to refer to the habits of consumers, "If the sale is on Tuesday and I have money, then I will go to the sale" (p.178).

The authors boldly state their goal with the book's opening chapter on critical thinking. "The goal of this chapter is to help you master the skills of reasoning, estimating, and problem solving" (p.1). They then allocate less than four pages to exemplify inductive and deductive reasoning, and they do not revisit the topic of logic until one hundred pages later in the textbook. For the authors, critical thinking primarily includes the topics of pattern recognition, estimation and problem solving, while avoiding any notion of critical reflection. Inductive reasoning leads to generalizations, and is the cause of social stereotypes. This particular topic opens the door to relate mathematical thought processes to a variety of social issues, yet that opportunity is missed in chapter one, and is subsequently ignored throughout the entire book. If the goal of a liberal arts program is to prepare individuals for informed citizenship, then the context should be relevant to social issues. Instead, the authors target the topics of estimation and problem solving in the contexts of restaurant bills and cooking recipes, as examples.

***Threshold Concepts***

The topics of inductive and deductive reasoning are briefly defined within the first five pages of the textbook, yet these phrases are not referenced again throughout the rest of the book. Geometry is presented in chapter nine, yet it is not used to build upon the process of deductive reasoning in the first chapter, nor does the text surrounding numerical sequences in chapter five make a connection to pattern-recognition and the process of inductive reasoning from the first chapter. Geometry offers an ideal platform to engage in the "if-then" thought process emblematic of deductive reasoning. However, these authors treat Euclidean geometry as a basic kind of math knowledge everyone should know, and present algorithmic applications of formulas in conservative contexts such as landscaping or sports and entertainment, and continue to emphasize following procedures rather than building a foundation for understanding logic. The following example illustrates this approach: "A National Basketball Association basketball court is a rectangle that is 94 ft long and 50 ft wide. If you were to walk around the outside edge of a basketball court, how far would you walk?" (p.548).

*A Survey of Mathematics with Applications* misses the chance to connect its chapters on sets, logic, and probability, through their commonality with Venn diagrams and the connectors of *and* and *or*. The authors write, "The words *and* and *or* are very important in many areas of mathematics. We use these words in several chapters in this book, including the probability chapter" (p.64). While they highlight the use of these words, they do not convey the strength of the relationship between logic and probability, nor do they emphasize how the words are depicted in Venn diagrams and truth tables. A deeper discussion around the words *and* and *or* would demonstrate that the tools of logic and probability are inextricably linked, and it would reveal how their interaction is foundational to mathematical reasoning. Both of these words play pivotal roles in establishing logically valid frameworks and in determining the value of compound probabilities; comprehending these ideas is necessary to fully internalize an understanding of the threshold concepts of validity and reliability. Also in the logic chapter, the authors give more attention to electrical circuitry than to an explicit explanation of the significance of the conditional statement. Conditional statements in propositional logic are the analog for conditional probability. In the long run, a reader would be better served to see the fundamental connection between logic and probability as it relates to conditional relationships, rather than seeing seven pages of text about switching electrical circuits.

Table 3
*Summary of Broad Survey Textbook*

| Book Type 1 "Broad Survey" | Example #1 (p.403) | Example #2 (p.548) | Example #3 (p.178) |
|---|---|---|---|
| Content | Linear functions | Perimeter of a rectangle | Truth tables and logical equivalencies |
| Context | Bagel Sales | Basketball court | Shopping during a retail sales event |
| Reasoning | Low-level evaluation of functions rather than the derivation of mathematical models | Application of a pre-established formula, rather than engaging in a complex problem that could require multiple steps of deductive reasoning | Merely recognizing the forms of conditional, converse, inverse, and contrapositive statements, without discussing the implications of their interpretations |

**Analysis: For All Practical Purposes: Mathematical Literacy in Today's World**

***Scope***

The pioneering text, *For All Practical Purposes* (Garfunkel, 2009), was originally published in 1987. The book is generally regarded as one of the first, and still one of the best textbooks for liberal arts math programs, and it is currently on its 10th edition. This book is produced by the Consortium for Mathematics and its Applications (COMAP). COMAP is a non-profit organization that works with teachers, students, social scientists, and members of the business community to develop learning experiences which explore how math is used in real world scenarios. In contrast to traditional postsecondary math programs, *For All Practical Purposes* emphasizes applications to the social sciences rather than the physical sciences.

*For All Practical Purposes* is unique in its structure; it consists of twenty-three chapters grouped into seven parts, and it is the result of a partnership among highly competent and talented mathematicians, educators, and publishers. The book is an extensive collection of topics, yet each chapter is written independently of the others by leading experts in their field. The book's great variety of topics includes business management and consumer finance, manufacturing and distribution, technology and information science, probability and statistics, voting systems, apportionment, game theory, and patterns in the natural world. A diversity of content is delivered through a collection of authors, and this format simultaneously creates the book's strength and weakness. The advantage gained from having twelve expert authors and wide-ranging content unfortunately precludes the book from having a single voice; there is no central message, nor is there a unifying theme across the content. Individual authors are independently committed to presenting their separate fields of knowledge, as a result, the text does not cooperatively reinforce its own topics to the reader.

This book contains three short paragraphs in the preface, and no conclusion. The lack of a single, consistent author's voice throughout the book is apparent from beginning to end. Without a central theme and without a clearly identified mission, the reader of *For All Practical Purposes* is left to wonder about the overarching goal of the book. The absence of a predetermined unifying message also gives freedom to a course instructor to use this book for a variety of purposes, without any guarantee that it will be used to satisfy the primary objectives of a liberal arts mathematics education. The book lacks a self-referential quality; there is neither a purposeful sequence to the content, nor an emphasis on building connections across the chapters. As a result, the presentation of material does not contribute towards the natural, cumulative process of learning mathematics because the concepts do not intentionally build upon each other, nor does it deliberately develop students' habits of critical reflection. If the goal of a liberal arts math class is to show students an abundance of applications, then this book facilitates that goal. If, however, the goal is to develop the critical thinking skills of citizens, then this book has its limitations. *For All Practical Purposes* offers demonstration, not transformation, for its readers, and this is foreshadowed in the preface:

> (The) goal is to convey the power of mathematics by showing you the great variety of problems that can be modeled and solved by quantitative means… For All Practical Purposes offers you the tools to succeed in the course and apply your new knowledge to daily life experiences. (p.xii)

The key words, "showing… tools… apply" aptly describe the authors' general approach to the design and presentation of content.

***Content***

This book consists of twenty-three chapters, totaling over one hundred separate sections of content. The amount of content is vast, and chapters are presented in isolation without reference to one another. As a result, instructors who use this book for a course may extract the concepts in different sequences, to create a variety of learning experiences. In this regard, *For All Practical Purposes* positions itself as a reference encyclopedia of useful math applications, and is the prototypical textbook for the genre of application-based liberal arts math books.

*For All Practical Purposes* offers chapters on social choice mathematics, including voting systems, fair division, apportionment, and game theory. These topics naturally open doorways to discussions of positivism and post-structuralism, while disrupting the common misconception that math must always produce a single, definite answer. While the authors in this section effectively demonstrate the notion that different outcomes are equally justifiable, they do not pursue the opportunity to engage students more deeply in a debate about the subjectivity of knowledge.

> "This search for good voting systems, as we shall see, is plagued by a variety of counterintuitive results and disturbing outcomes. In fact, it turns out that one can prove (mathematically) that no one will ever find a completely satisfactory voting system for three or more candidates." (p.285)

Later in the same section, the authors add "Our task of finding a reasonable procedure is impossible" (p.303).

Having empathy and respect for diverse perspectives are components of critical awareness. *For All Practical Purposes* features a comprehensive exploration into more than fifteen voting systems, strategies, and power indices, ten approaches to fair division, and four different apportionment methods. A liberal arts student is unlikely to need or absorb the extensive depth provided in this section of the book, and the expertly-written content risks getting lost in too much detail during the Shapley-Shubik Power Index for weighted voting systems, the Selfridge-Conway procedure for envy-free arrangements, and the Webster method for apportionment. One could argue that the larger purpose of studying social choice mathematics is to develop an awareness of how different parties can support different conclusions based on the same data, and that full, consistent agreement is impossible. It is important that students learn how to effectively transform individual preferences into a single collective preference, but students must also be compelled to reflect on the ramifications of assigning quantitative values to qualitative feelings. The following problems exemplify the authors' attention to computation, rather than interpretation, of the human dynamics at play in social choice.

> A nine-member committee has a chairperson and eight ordinary members. A motion can pass if and only if it has the support of the chairperson and at least two other members, or if it has the support of all eight ordinary members. (A) Find an equivalent weighted voting system. (B) Determine the Banzhaf power index. (C) Determine the Shapley-Shubik power index. (D) Compare the results of parts (B) and (C): Do the power indices agree on how power is shared in this committee? (p.365)
>
> John and Mary inherit their parents' old house and classic car. John bids $28,225 on the car and $55,900 on the house. Mary bids $32,100 on the car and $59,100 on the house. How should they arrive at a fair division (using the Knaster inheritance procedure? (p.428)

This content creates a space in which to develop students' capacity for critical reflection, and the inclusion of sociopolitical contexts could better prepare students with a more sophisticated perspective of social cohesion. Consider the following example which illustrates how easily topics within social choice mathematics lend themselves to thoughtful reflections: "If you and another person are using divide-and-choose to divide something between you, would you rather be the divider or the chooser? Assume that neither of you knows anything about the preferences of the other" (p.429).

The final chapter on social choice mathematics in this book is a thorough introduction to game theory – a topic which presents an ideal platform to cultivate critical reflection for greater social awareness and informed citizenship. Game theory uses mathematical tools to study situations of conflict and cooperation among two or more parties, it is engaging and useful, and it is especially relevant in the fields of economics and politics. Classic scenarios, including the prisoners' dilemma and the game of chicken, are studied in this section. Given that many social transactions have an element of the prisoner's dilemma, this topic appeals to students with a variety of liberal arts interests, and is fittingly included in the textbook.

***Context***

The contexts used for explanations and exercises in the book are conservative, as to be expected. For example, the authors include probability problems about decks of cards and automobile colors (see Table 4). However, such content lends itself to explore more provocative themes, for example, conditional probabilities surrounding educational attainment by socioeconomic status and traffic stops by ethnicity. While the book provides clarity and pragmatism with its applications, the authors' decisions about context steer the reader away from opportunities to develop these tools for purposes of critical reflection and fully-informed citizenship. The following example illustrates this point. In designing a liberal arts math experience, instructors must ultimately decide whether to include these traditional types of exercises, or to deliberately incorporate themes of social justice that will expand students' awareness of important issues.

> Choose a new car or light truck at random and note its color. Here are the probabilities of the most popular colors:
>
> | Color | Silver | White | Black | Gray | Blue | Brown |
> |---|---|---|---|---|---|---|
> | Probability | .201 | .184 | .116 | .115 | .088 | .085 |
>
> What is the probability that the car you choose has any color other than the six listed? What is the probability that a randomly chosen car is either silver or white? (p.276)

***Threshold Concepts***

This book clearly explains and exemplifies a multitude of applications, yet there is a dearth of formal logic, and it offers little discussion around fundamental reasoning processes. The book's exposition of myriad topics extensively shows students how to act like a mathematician, but it neglects to teach students how to think like a mathematician – a practice founded upon inductive and deductive reasoning. *For All Practical Purposes* omits chapters on inductive and deductive reasoning, a surprising revelation, given that so many other liberal arts math textbooks attend to them. Rules of logic constitute the grammar for the language of mathematics. Along with probabilistic thinking, the study of logically valid frameworks and the process of deductive reasoning are central to the development of mathematical reasoning skills that support student growth in other academic fields. A website for the book states that additional chapters are available through custom publishing, including logic, set theory, and geometry, yet the implication is that these topics are secondary in importance to the collection of math applications already included in the text. There is only one section that addresses truth tables, presented at the end of the chapter on information science. The author provides a terse introduction to truth tables, but only explains the *and* and *or* connectors. Moreover, he does not reference the fact that the same connectors are used in the language of compound probability – students are left to discover that critically important link on their own. In this chapter, the author's brief treatment of mathematical logic serves to prepare his reader for a discussion on message routing in computers and web searches. The conditional *if-then* connector is only found in exercises at the end of the chapter; and is referred to as the "implication connective." Consider the following exercise:

> Using the implication connective and other connectives, variables, and truth tables, determine whether the statement "If it snows, there will be no school," is

> logically equivalent to the statement "It is not the case that it snows and there is school." (p.566)

The text does not mention the direct, inverse, converse, or contrapositive forms of conditional statements, nor does the author describe the concept of validity or link these aspects of deductive reasoning to the principles of argumentation.

Logically valid frameworks form the backbone of mathematical theorems; this concept is central to mathematical thought processes. This book misses a pivotal opportunity to steer the reader in the direction of a threshold concept. Instead, the author writes about coding and cryptography because those topics fit neatly within the theme of his individual chapter. Consider this example, in the same section: "Suppose we code a four-symbol genetic set {A, C, T, G} into binary form as follows: A→0, C→10, T→110, G→111. Convert the sequence ACAAGTAAC into binary code" (p.565). This exercise is a treatment of binary notation and data compression, yet a reader may translate and complete the above task using the "if-then" language inherent in deductive reasoning processes. Content in the information science chapter opens pathways into a formal study of deductive reasoning, however, the flaw of this book reveals itself when separate contributors do not share a common mission. Each author carefully addresses his or her own niche, yet collectively they do not craft a larger, cohesive message to the reader – a message that would need to be expressed through mutually agreed upon threshold concepts.

Given that we live in a world of uncertainty, an understanding of the principles of probability is essential for the construction of a sophisticated lens through which to view and interpret the world. The author of the probability chapter overlooks an important connection between the fields of logic and probability. The more students are able to build connections across content, the more their understanding will grow (Davis, 1992). It is important to note that the same connectors *and*, *or*, and *if-then*, play significant roles in two fields that are fundamental in mathematical reasoning. The same words used to join together multiple events in compound probability, are also used to join together multiple statements in propositional logic. *For All Practical Purposes* misses an opportunity to facilitate students in building these valuable intra-connections because separate authors are used for separate chapters in the book. The author of the probability chapter is more intent on providing the reader with a background for statistics than he is with making connections to logic, and the previously mentioned author of the information science chapter makes no effort to connect his content to matters of probability. The absence of these connections highlights the book's inability to cultivate threshold concepts in mathematical reasoning. While the book provides a collection of useful applications, the core concepts within mathematical reasoning are ignored, thereby reducing the likelihood that a student will develop a transferrable set of reasoning skills.

Authors in this book follow the format of define, explain, and exemplify, then they provide exercises to showcase the applications and assess the reader's comprehension. The selected content is engaging, but the book's emphasis is on the application of the content, not the derivation of its truth. The process of deriving mathematical truths epitomizes the process of mathematical reasoning – and more attention to this matter would serve to be useful in developing the critical thinking skills of liberal arts students.

Table 4
*Summary of Application-Based Textbook*

| Book Type 2 “Applications” | Example #1 (p.566) | Example #2 (p.276) | Example #3 (p.365) |
|---|---|---|---|
| Content | Conditional statements | Simple and compound probabilities | Voting systems |
| Context | Snow storm | Automobile colors | Fictitious committee |
| Reasoning | Reader is asked to determine logical equivalency only, without prompting further exploration into logically valid frameworks. | Identifying correct values to use from a table, and joining them together with addition. Does not engage reader in higher-order thinking. | Algorithmic calculation of power indices. No discussion about social dynamics of decision-making processes or implications for strategic voting. |

**Analysis: The Heart of Mathematics: An Invitation to Effective Thinking**

***Scope***

Since its original publication in 1999*, The Heart of Mathematics* (Burger & Starbird, 2009) became “the most widely adopted new textbook in liberal arts mathematics in over ten years” (p.IE-4). This book optimally represents the third category of liberal arts math textbooks because of its devotion to mathematical thought processes; the book excels in its ability to use expository prose for modeling types of mathematical thinking, and the authors integrate entertaining examples to engage students. The authors’ goals are explicitly stated in a lengthy introduction – they aim to develop reasoning skills that will aid a diversity of students in their development of how to think mathematically, and “for students to appreciate and enjoy mathematics” (p.IE-3).

> Students can learn great ideas and practical methods of effective thinking that can change their lives and how they view and understand the world around them…. Students can learn methods for developing insight such as: first formulating questions arising from observations; then looking for patterns, analogies, generalizations, examples, and beauty; and, finally, making and verifying conjectures. (p.IE-2)

> Our goal is to inspire students to be actively engaged in mathematical thought… (to) learn various techniques of thought through repeated exposure… to help them learn innovative modes of thought to empower them to approach and conquer all types of issues within and outside mathematics. (p.IE-4)

***Content***

In the introductory remarks, the authors promote a pedagogy of discovery-based learning; they state the importance of building connections through repetition across the content areas, and they concede that "this book contains many more topics than can be treated in one semester (or even two semesters)" (p.IE-5). The content is extensive; the authors present the book as "a network of intriguing ideas – not a dry, formal list of techniques" (p.xi). The "entertaining and stimulating" (p.IE-4) ideas are intended to develop key thought processes, and each section concludes with open-ended questions to add a writing component to the course. Topics includes elementary number theory and ideas surrounding infinity, geometry and patterns in nature, fractals and chaos, probability, statistics, and social choice mathematics. It is interesting to note, however, that despite the authors' attention to the development of thought processes, they neglect to include a chapter on formal logic. Furthermore, the principles of inductive and deductive reasoning are alluded to throughout the book, yet never directly addressed.

***Context***

The first chapter uses puzzles to introduce students to critical thinking. The problem-solving techniques for these puzzles foreshadow heuristics that reappear throughout the book. In the second chapter, entitled "Number Contemplation," the authors begin to guide students towards an intuitive understanding of inductive and deductive processes. This chapter addresses sequences – numerical and natural, including the Fibonacci sequence. Here the text also offers a proof for the prime factorization theorem, and it discusses Goldbach's Conjecture, Fermat's Last Theorem, modular arithmetic, and the density of the number line. The topics raised in this treatment of elementary number theory allow one to informally explore inductive processes and make logical deductions, yet they do so in the context of pure mathematics. As one example, Burger and Starbird lead the reader through a proof for the irrationality of the square root of two. The authors present their proof without mentioning the logic of the contrapositive. Instead, they call it a "counterintuitive revelation" and an "elegant line of reasoning" (p.134). Through this example, the authors aim to develop an intuitive notion about the fragility of assumptions, making "legal deductions" (p.133) along the way and eventually arriving at a contradiction.

Explorations into inductive reasoning and logical frameworks create space for more sophisticated mathematical reasoning in a variety of contexts, unfortunately, the authors limit their discussions to a purely mathematical domain. They do, however, hint at the usefulness of their informal approach to logic: "An effective strategy for analyzing life is to make an assumption and see what consequences follow logically. If a logical consequence is a contradiction, then the assumption must be wrong" (p.138). This chapter epitomizes the nature of the text and the authors' approach to mathematical reasoning. They give significant attention to thought processes, yet they do not promote critical reflection within a sociopolitical context. Note, this is not a broken promise; at no point in the text do the authors claim to incorporate socially progressive themes into their book. However, in the spirit of a liberal arts education aiming to prepare students for informed citizenship, one could argue that students would be well served to develop mathematical reasoning skills in the context of relevant social issues. The exercises provided in this chapter relate to number theory; and while they encourage students to make connections to real life situations, the authors do not illustrate how to make these connections. Bridging this gap, from content in the text to situations in real life, is an important piece of the puzzle, and the students are asked to do it alone. Consider the following two tasks presented as writing prompts at the end of the section: "Write an imaginative story that involves or evokes the ideas of this section (p.77), and "Provide several real-life issues – ideally, from your own experience – that some of the strategies of thought presented in this section would effectively approach and resolve" (p.77). Identical writing prompts are presented to the reader at the end of several sections in this chapter. This book, while attending to thought processes, limits the likelihood that students will transfer their newly developed mathematical thinking skills into other fields because the authors do not model that process for them.

*The Heart of Mathematics* allots four chapters to investigate ideas about infinity and space, including fractals, topology, graph theory, non-Euclidean geometry, and the fourth dimension. Each topic has the opportunity to stimulate the reader with a variety of exercises that demand thoughtful reasoning. These chapters navigate between aesthetics and analytics; the authors guide the reader to consider simpler versions of complex problems, to seek patterns, and to look for relationships that reveal deep structures. Through these chapters, the authors help students learn how to think, yet the authors are consistent in their approach to contextualize this developmental process within the domain of pure mathematics (see Table 5). The following examples are prototypical in that they cultivate careful thinking within a confined context:

> At stage 0, the Sierpinksi triangle consists of a single, filled-in triangle. At stage 1, there are three smaller, filled-in triangles. How many filled-in triangles are

> there at stage 2? How many at stage 3? What's the pattern? How many triangles are there at stage 4? How many will be there be at stage n? (p.487)

> Take a Golden Rectangle and draw the largest circle inside it that touches three sides. The circle will touch two opposite sides of the rectangle. If we connect those two points with a line and then cut the rectangle into two pieces along that line, will either of the two smaller rectangles be a Golden Rectangle? Explain your reasoning. (p.272)

The beauty of fractals reveals that simple, repeated processes can lead to surprisingly interesting outcomes – a notion that foreshadows the counterintuitive nature of compound probability. The process of arriving at generalizations through inductive reasoning and pattern recognition are fundamental in the skill set of mathematical reasoning tools, and they are useful across the liberal arts. However, the authors limit coverage of these thought processes to purely mathematical scenarios.

Principles of probability, elementary data analysis, and social choice mathematics comprise the final three chapters of the book. Each of these are ideal avenues to explore themes of social justice. Instead, the authors present their readers with Yahtzee and Monty Hall, distributions of donut production, and cake-cutting exercises, among others. The following exercise typifies the authors' use of traditional and conservative contexts, with their added touch of humor to entertain the reader.

> You have a small bag of candy-coated chocolates that melt in your mouth; three are red, four are yellow, two are green, and five are blue. If you take a piece out of the bag at random, what is the probability it is green? What is the probability it is blue? What is the probability that you will eat it? (p.599)

Such exercises offer clear opportunities to apply the basic principles of probability, but more sophisticated contexts could challenge the reader to apply these reasoning tools for informed citizenship, and not just for games. These final three chapters in *The Heart of Mathematics* have the potential to challenge the reader to expand his or her awareness of social issues. Burger and Starbird write, “mathematical thinking can contribute substantial insight to this most human of problems (re: the challenge of fair division)” (p.869). The authors address conditional probability, statistical inference, and the fair division of assets, yet they do not engage students in discussions of equality versus equity, or disproportionate distributions such as traffic stops by ethnicity or educational attainment by socioeconomic status. Cake-cutting provides a model to develop thinking strategies for dividing resources, yet the authors do not illustrate a real application towards profound societal problems, instead resorting to the familiar writing prompt asking readers to think of their own examples.

***Threshold Concepts***

While the authors engage the readers in countless thinking exercises, they refrain from specifically naming logically valid structures, and they treat deductive reasoning as a general guideline for problem-solving strategies in pure mathematics. Geometry offers an ideal platform through which to develop powerful deductive procedures, and the authors pose a variety of interesting questions, including the aforementioned Golden Rectangle task. However, they abide by their commentary in the book’s preface by omitting a formal exposition of deductive reasoning – taking the position that it would not benefit readers in the long run. “Students will quickly forget technical terminology, notation, and details of proofs” (p.IE-2). A discussion of logically valid frameworks is a natural extension of geometry exercises. Given that deductive reasoning is foundational in mathematical reasoning, the reader’s understanding of deductive procedures could be enhanced by exploring variations of *if-then* conditional statements and truth tables to determine logical equivalencies among direct, inverse, converse, and contrapositive statements. However, the authors choose not to follow this path and withhold those vocabulary terms. Similarly, the authors do not offer a connection between principles in probability and the reliability of premises in an argument. Such a connection would help the reader to internalize the integrative nature of a threshold concept. Thus, while the authors facilitate student comprehension of key ideas in logic and probability, they present them in isolation of each other, obscuring the notion that they could work together to facilitate new understandings across academic fields.

The authors conclude with a summative farewell, expressing their hopes that readers have "expanded (their) repertoire of strategies and modes of thought" and that the book's lessons helped them to "strengthen (their) confidence to face challenging life issues and conquer them" (p.886). It is unclear how the writers expect their audience to do this, given that this important process was not directly modeled in the book.

Table 5
*Summary of Textbook with Emphasis on Reasoning*

| Book Type 2 "Reasoning" | Example #1 (p.487) | Example #2 (p.272) | Example #3 (p.599) |
|---|---|---|---|
| Content | Fractals and patterns | Aesthetically pleasing math facts | Simple probability |
| Context | Sierpinski Triangle (Geometry) | Golden Rectangle (Geometry) | Candy consumption |
| Reasoning | Asks reader to make diagrams, identify the pattern, and generalize the relationship with a formula. | Asks reader to draw diagram, study the figure and deduce a conclusion about the ratio of side lengths. | Low-level thinking, requires reader to insert numbers into a formula, does not challenge reader with applications in more sophisticated context. |

**Discussion of Analysis**

Liberal arts math programs may be designed to help prepare citizenry for informed self-governance. Given that these programs often represent the last time non-STEM students formally study math at the postsecondary level, they are a valuable opportunity to purposefully cultivate mathematical reasoning skills that could be useful across a variety of academic and professional paths. To help achieve this, a liberal arts math textbook may devote attention to critical thinking, critical reflection, and situate its content within socially responsible contexts. However, the majority of books in this genre do not adhere to this design. Critics argue that traditional mainstream mathematics fails to emphasize the applications of its tools in real context and are often ineffective in developing a student's ability to apply mathematical thinking to everyday situations (Packer, 2003; Steen, 2001).

This chapter critiques examples from three categories of liberal arts math textbooks: broad survey books, application-based books, and books that emphasize mathematical thinking processes. Although the authors bring different approaches to the field of liberal arts mathematics, their objectives should be the same – to provide students with a positive learning experience that develops and equips them with a useful set of mathematical reasoning skills. Each textbook offers value to its readers through varied content and thoughtful exercises that facilitate individual growth in mathematics. From general principles that everyone should know, to common applications, to clever puzzles that demand logical thinking – the three categories align with typical expectations for a liberal arts mathematics experience. Unfortunately, the representative books in this critique also exhibit shortcomings in both content and context, indicative of the field at large. For example, consider a common textbook exercise which asks the reader to use to the formula for the area of a trapezoid to determine the amount of fertilizer required for an irregularly shaped lawn. Such a problem contains elements of mathematical thinking, yet it does not develop sophisticated reasoning skills that would broadly serve a student in the liberal arts. Note, the textbooks in this critique and mainstream mathematics outside the realm of critical pedagogy neither seek nor claim to cultivate a sophisticated lens for sociopolitical awareness, nor do they identify and deliberately develop threshold concepts in the experience. To clarify, this critique intends to initiate conversations about potential gains from pursuing new avenues in liberal arts math programs.

Upon examining the current field of liberal arts math textbooks, some common traits emerge – fragmented content, arbitrary sequencing of topics, and an absence of core concepts. Furthermore, given that the typical course offering is not aligned with a critical pedagogy, the pervasive use of conservative contexts deprives students of chances to explore themes of social justice that could prepare them for more informed and ethical citizenship. The following discussion addresses each of these aforementioned ideas.

Topics within social choice are addressed by two of the books – *For All Practical Purposes* and *The Heart of Mathematics*, yet in each case, the authors do not capitalize on opportunities to discuss the subjective nature of knowledge. The tasks of fairly dividing resources and coalescing individual preferences into a single collective decision inherently require open-mindedness, ethics, and empathy. The books present their readers with procedural tools, yet they do not apply them to important social problems, nor do they engage them in reflections about the deeper implications of equality versus equity, or individual versus institutional struggles for power and control.

Structurally, mathematical reasoning takes the shape of an argument. It begins with premises – statements assumed to be true, and proceeds forward in a logically valid framework. Accordingly, it would benefit a liberal arts math student if his or her textbook gave attention to the principles of argumentation. A formal study of mathematical logic, supported with the use of truth tables, is essential for developing an appreciation of valid structures and for determining the strength of a conclusion. Only the first book examined in this critique, *A Survey of Mathematics with Applications*, directly addresses propositional logic and truth tables, yet it is insufficient with the translation component. If students are to actually apply the tools of formal mathematical logic in other settings, they first need training in how to model real life situations with propositional logic. If a person struggles to translate a situation into mathematical terms, then that same person is unlikely to use mathematical tools to better understand that situation. Two of these textbooks ignored formal logic, and the third book addressed it ineffectually, by neglecting to demonstrate the range of its applications.

A common weakness of these textbooks is the fragmented arrangement of content; the authors' sequencing is ostensibly arbitrary, and they are remiss in their efforts to explicitly highlight key intra-connections within the books. Chapters in these books can stand alone as independent units of study, and the chapters can be studied in any order. Without a single voice crafting a story with consistent themes, the reader is deprived of the transformative impact that accompanies a cumulative learning experience. To build deep understanding, a student needs to make many connections across the content – this is difficult to achieve, especially if a book does not facilitate the process.  This shortcoming is epitomized by the under-emphasized connection between logic and probability in *A Survey of Mathematics with Applications*.

In critiquing this survey book, from chapter to chapter, the sequence of content reveals itself to be fragmented and disconnected.  Herein lies a typical flaw of books in this genre: when topics are presented in isolation, the reader is deprived of opportunities to make meaningful and lasting connections within the curriculum, thereby reducing the likelihood of long-term retention.  Student learning is built upon previous understandings, and new information is processed through an assimilation paradigm in which an individual compares it to something he or she already knows (Davis, 1992).  A textbook and/or curriculum that is self-referential and highlights intra-connections in a cumulative fashion, would serve to facilitate student growth.  However, authors in this genre typically do not incorporate that approach in their textbooks.

Moreover, *A Survey of Mathematics with Applications,* is typical in the broad survey category in that it contains two very traditional chapters on algebra.  Such chapters often consist of linear and nonlinear equations, graphs of functions, quadratics, systems of linear equations, and other content typically found in a high school algebra curriculum.  It should be noted that liberal arts math courses, as well as courses in the domain of quantitative reasoning, are credit-bearing courses at the college level.  The algebra content in these books are often treated as prerequisite knowledge for such courses, with the notion of building upon this knowledge – not using the accompanying textbook to teach it again.  Algebra, by nature, is an algorithmic subject that does not inherently teach students to think critically.  Not surprisingly, the algebra problems in such books are either completely decontextualized, or situated in conservative contexts such as determining the amount of fertilizer needed for a lawn. For these reasons, the algebra chapters are consistent with the book's scope as a survey course, but do not support the overarching goal of a liberal arts math program to liberate one's mind with sophisticated critical thinking skills. Similarly, the pages within *For All Practical Purposes* consistently meet expectations for an application-based liberal arts math textbook, but they are not aligned with the broader mission of cultivating a transferrable set of reasoning skills.

There is significant overlap across the fields of logic and probability as they relate to conclusions and premises with inductive and deductive reasoning processes. Inductive reasoning does not produce conclusions with certainty; arguments with deductive reasoning are built upon premises, and premises are inherently probabilistic. Additionally, it is important to consider how a book's use of context may be framed in relation to the interests and experiences of the reader. That is to say, logic and probability can effectively be studied in conjunction with discussions about the subjective nature of knowledge, and all three can intersect in a liberal arts math experience. New information and new experiences are filtered through one's understanding of logic and probability, and the development of critical thinking may be viewed as a fundamental objective of a liberal arts education. With this perspective, a mathematics education can enable a student to construct a lens through which to view the world with greater sophistication. To achieve this, the liberal arts math textbook would need to be purposely designed to attend to threshold concepts within deductive reasoning and probabilistic thinking, in particular, the concepts of validity and reliability. Authors in each book do not link together these essential concepts, in part because the community of liberal arts math educators has neither identified nor established the core concepts of mathematical reasoning. Liberal arts math programs, including quantitative reasoning courses, continue to lack definitive guidelines and remain without consensus on learning goals for these courses (Steen, 2001; Madison & Dingman, 2010; Karaali et al., 2016). If one accepts the pedagogical stance that curricula should be locally generated to appeal to the needs and interests of the local population, then it is reasonable to expect diversity in the content and the context. However, for the purposes of developing an informed citizenry, careful consideration should be given to sociopolitical contexts and selection of the most useful content.

The books in this critique choose not to address any themes of social justice. They include examples and exercises relating to real life situations, but they are conservative in nature. Unless aligned with the principles of a critical pedagogy, a liberal arts math textbook is unlikely to be the basis of a transformative experience for its readers. Giroux (2011) argues that one's education should connect to larger social issues, and that it should help students "learn the tools of democracy and how to make a difference in one's life as a social agent" (p. 171). Education, therefore, becomes more than preparation for citizenship, it becomes a form of political intervention that creates possibilities for social transformation. A liberal arts mathematics curriculum has the opportunity to develop citizens who will engage in critical reflection and be willing to act in a socially responsible way. In the realm of critical pedagogy, mathematical proficiencies are not enough, students must also be equipped with the ability to critique social issues, develop their own sense of agency, and challenge inequities in society (Frankenstein, 1983; Frankenstein 2001; Gutstein, 2006). Accordingly, textbooks and curricula may be designed to support this endeavor. For example, the study of inductive reasoning creates space to either address and challenge social stereotypes, or perpetuate them. Deductive reasoning provides tools to deconstruct hidden messages in the dominant discourse. And, the principles of probability offer a profound perspective through which to view social issues in America, as they relate to ethnicity and economic class. If the goals of a liberal arts math program are to develop a prosocial identity, sociopolitical awareness, agency, activism, and ultimately to empower a student to build a sophisticated lens for viewing the world, then core concepts in logic and probability would need to be explicitly addressed in meaningful contexts.

**2.4 Proposal for a New Liberal Arts Math Curriculum**

This critique suggests expanding discussions with students about the subjective nature of knowledge, while highlighting the importance of mathematical logic and probability. Liberal arts mathematics is ready for a new curriculum to address general weaknesses in the academic field, typified by the textbooks in this critique. Liberal arts math programs would benefit from a textbook that deliberately and repeatedly cultivates a learner’s understanding of threshold concepts within mathematical reasoning. Such a curriculum would have carefully sequenced content, contextualized with themes of social justice, and be composed with a consistent voice that is committed to providing students with a transformative learning experience. These features, combined with the identification and implementation of threshold concepts, would enable a student to construct a more sophisticated lens through which he or she can explore sociopolitical issues and become a more informed citizen.

Deductive reasoning and probabilistic thinking contain threshold concepts, they are fundamental in the experience of learning to successfully engage in mathematical reasoning and in developing critical thinking skills that are useful outside the math classroom. Key principles in logic and probability intersect to provide a foundation for understanding argumentation; the ability to present a sound argument and justify one's position is valuable across all academic disciplines. The threshold concept of validity is central in mathematical logic, it establishes a proper framework for thinking and supports confidence in conclusions. The threshold concept of reliability is a matter of probability – stemming from inductive reasoning, then maturing through the mechanics of compounded events. Validity and reliability are natural components within a mathematics curriculum. Together, they create strong reasoning skills, and when fully understood, they produce an irreversible, integrative, and transformative effect on an individual's mindset.

The word "mathematics" originates from the ancient Greek *mathema* – which translates into "that which can be learned." That which can be learned, reciprocally, is that which can be taught. At its core, mathematics is a way of thinking; students can be taught how to think logically and how to work with quantitative calculations in a meaningful way. Through the careful cultivation of threshold concepts, students can learn how to build upon basic axioms and first principles, follow rules of deduction and inference, and ultimately arrive at sound conclusions. Mathematical reasoning is comparable to the act of argumentation – they both lead to the advancement of one's knowledge, but knowledge is only useful when it solves problems.

Citizens who participate in a self-governing democracy and citizens who possess awareness of society's relevant issues, must also be citizens who are aware of the injustices in the world. Math can help build a sociopolitical lens and empower an individual to more effectively critique the world, challenge the discourse, understand society and one's place in it. A curriculum can promote student agency, and the right liberal arts mathematics textbook has an opportunity to engage individuals with identity development at a profound level to spark social change. This kind of textbook can challenge students' misconceptions of mathematics, develop their capacity to think within a logical framework, carefully scaffold explanations of probability, and recognize the subjective nature of knowledge. Overall, training in deductive reasoning processes and understanding the validity of certain kinds of logical frameworks has the potential to affect change in society if this generation of students can learn how to bridge the gap between academic achievement and social responsibility. Liberal arts math programs are well suited for a textbook to support these bold ambitions. Accordingly, this paper ends with my proposal for a new liberal arts math curriculum founded upon subjective rationalism.

A curriculum designed for subjective rationalism necessarily integrates threshold concepts from logic and probability, while addressing concepts from social choice mathematics. Ideally, the curriculum is presented with a student-centered, critical pedagogy and provides students with an opportunity for a transformative learning experience. Through the curriculum, students understand how to attach quantitative values to qualitative values, and engage with content that is situated in the context of social justice. Themes of social justice pervade the curriculum in order to escalate student engagement, increase their awareness of important issues, and to inspire students towards active citizenship.

# CHAPTER THREE

# A NEW CURRICULUM FOR LIBERAL ARTS MATHEMATICS

## 3.1 Critical Social Theory and a Non-Neutral Curriculum

This chapter is an affirmation of a point of view – a point of view that rejects the idea of a neutral curriculum. The curriculum outlined here is essentially a political act that promotes criticism as one of the defining aspects of a quality education. As the research practitioner, I designed instructional interventions that were intended to build a language of critique and cultivate a classroom discourse that would advance the notion of a liberatory education. Proponents of critical social theory recognize the "power to change the pedagogical process from one of knowledge transmission to knowledge transformation" (Leonardo, 2004, p. 11). My intention was to provide a transformative learning experience for my students – founded upon a dialogue of critique that implies possibility and hope for the improvement of social realities. In this realm, criticism is not an act of pessimism, it is an emancipatory path that cultivates individual agency necessary for a functional democracy.

This curriculum contains contextualized developmental exercises, as well as instructional examples and written assessments that highlight inequitable relationships between social systems and people. Student feedback, and my observations of their engagement and performance levels, led to multiple iterations of the curriculum. I deliberately integrated themes of social justice into the curriculum to expand student awareness and inspire activism to remedy the inequities of the world. Critical thinking and critical reflection are fundamental habits of an informed citizenry, as such, critical social theory puts criticism at the center of its knowledge production (Leonardo, 2004). The non-neutral curriculum presented here does not view critique as an exercise in rejection, but as an engagement in sophisticated argumentation. This curriculum evolved to emphasize the construction and deconstruction of arguments as not only central to mathematical thinking, but also as an indispensable feature of a liberal arts math course purporting to develop a set of reasoning skills useful across multiple disciplines.

**3.2 Subjective Rationalism and Threshold Concepts**

This liberal arts mathematics curriculum is designed to develop subjective rationalism. Subjective rationalism is the practice of deductive reasoning, while attending to matters of probability and personal preference. An individual's search for understanding is analogous to the act of argumentation and is at the crux of Plato's classic theory of knowledge as a justified true belief. The act of justification – providing reasons that support a claim, is inherently a mathematical endeavor. Subjective rationalism is my theory that an individual may reach an optimal level of understanding by operating within a logically valid framework while being mindful of the probabilities necessarily attached to the premises of an argument. The assumption is that the argument is of personal interest to the individual, as such, pure objectivism fails. Every person has a unique lens through which he or she views the world. Individuals are influenced by past experiences, emotions, and opinions. Therefore, it is essential to recognize the element of subjectivity and the role played by individual perspectives in the pursuit of new knowledge.

Meyer and Land (2003) advance the notion of threshold concepts to identify building blocks of an academic discipline. Threshold concepts are typically troublesome for learners, and may initially be counterintuitive. However, once fully understood, the effects are transformative, integrative, and irreversible. Learners often labor in the liminal space while transitioning into understanding, accordingly, the development of these concepts requires ample time and multiple opportunities to build meaningful connections. The curriculum presented here supports this development through recurring themes and by highlighting intra-connections throughout the course of study.

When learned, key principles in formal logic and probability switch from gatekeepers to gateways, enabling new perspectives and linkages across disciplines. Deductive and inductive reasoning are fundamental human thought processes; this curriculum pairs these processes with a careful development of probabilistic thinking to build an understanding of validity and reliability. Logic and probability deservedly receive significant attention in this curriculum; taken together, their study facilitates the comprehension of two threshold concepts in mathematical reasoning – validity and reliability. When an individual understands these two concepts, it opens up pathways to expand his or her knowledge in all fields.

**3.3 Overview of a New Liberal Arts Mathematics Curriculum**

The mathematics needed for everyday life should be reflected in the mathematics taught in classrooms. The concepts in this curriculum are developed to purposefully aid in the construction of a sophisticated lens through which to view the world and explore today’s relevant issues. The curriculum offers a highly pragmatic and philosophical approach to mathematical reasoning, and wherever possible, situates the math content in sociopolitical contexts.

There are three ways in which we filter the reality of the world around us – subjectively, objectively, and with uncertainty. Accordingly, the curriculum begins with a study of social choice math and the variables used in decision-making. Next, it examines formal logic and an objective search for truth. This second unit includes inductive and deductive reasoning, and the principles of argumentation. The third unit contains a comprehensive introduction to probability – including simple, compound, conditional, and binomial probabilities. A deep understanding and self-awareness within these three domains empowers an individual to develop a more critical worldview.

The fundamentals of mathematical reasoning are the same as the principles of argumentation, and they both constitute a means for advancing knowledge. As such, this course of study gives considerable attention to the process of constructing and deconstructing arguments, and assessing the soundness of their conclusions. Understanding argumentation is important for informed citizenship – it is a necessary skill for dissecting the dominant discourse and for effectively critiquing contemporary society.

As supplements to these core concepts, the curriculum offers an investigation into the logic of Euclidean geometry to foreshadow the formal study of deductive reasoning, and it explores modular arithmetic and operations in other number bases as a means to challenge students' preconceptions about the rigidity of mathematics. An introduction to game theory is presented at the end of the curriculum; it combines essential elements from the three main units and it provides models for studying social phenomena. Through game theory, students engage in discussions about individual and collective motivations, while examining situations of conflict and cooperation between intelligent and rational people. Thus, the curriculum concludes with students learning to reconcile decisions that serve their own best interests with the potentially larger social implications of their actions.

This paper includes representative examples from the curriculum and highlights some of the recurring sociopolitical themes that are explored through mathematical reasoning. Each of the three main units is summarized, and special attention is given to sample exercises in the second and third curricular units which demonstrate the central roles played by the threshold concepts of validity and reliability. The entire curriculum is detailed in a 675-page textbook, *Definite Possibilities* (Wenger, 2018), which fully develops a progressive liberal arts mathematics curriculum founded upon the principles of a critical pedagogy.

### 3.4 Rationale for a New Curriculum

#### Liberal Arts Mathematics, Justice Studies, and Correctional Education

This curriculum was created through student feedback at the secondary and postsecondary levels, and is intended for use with a critical pedagogy. Learning should be an emancipatory experience, one that liberates the individual to a sociopolitical awareness of inequities in the world (Freire, 1972). Such awareness leads to activism, so that individuals transformed with a social justice education will work to remedy society's problems and restore balance in the world. Agency – a person's belief in his or her own capacity to act and make a difference, is strongly promoted within this curriculum. The curriculum allows an instructor of the course to use math as a social lens, to talk about equity in society, and to create awareness of important issues by placing the course content in the context of relevant sociopolitical themes.

Social reality is the product of human action, yet for too many people this social reality is not fair. We do not live in a world with equal access to health care, educational or economic opportunities. The instructor for this course should be aware of the tensions, fears, and doubts of marginalized populations. A lack of opportunities, a neglected youth, the absence of positive role models – each of these are ingredients in a recipe for self-destruction. The challenge is to use students' lived experiences and prior knowledge as a process for dialogue. A critical pedagogy of dialogue can validate student identities through their participation in the classroom, and this liberating dialogue can be carried out at whatever stage of development the students are in. Instructors for this class are likely to see themselves as transformative intellectuals committed to social reconstruction. Education cannot be politically neutral because it either perpetuates or disrupts the status quo – both of which are political actions. The instructor for the course cannot be ambivalent, he or she needs political clarity.

The job of an academic is to help produce informed, ethical, and empathetic citizens. Math is a tool, but even the best tools are worthless if a person does not know how to use them. This curriculum is about teaching people how to use the tools of math to better understand their world. Each student carries a unique lens – a filter through which new information is processed. Every individual lens has been constructed because of prior experiences. The instructor for the course needs to pay attention to perceptions and misconceptions, and utilize the course as a platform to develop students' lenses and all forms of their reasoning processes.

A different approach to education is required when students are adults and not children. Adults demand immediate relevance and application of the content, yet they are also far more capable of working with abstract ideas and engaging in sociopolitical debates. This curriculum includes discussions of individual agency and brings attention to the dominant discourses in society. Furthermore, the curriculum is explicit with its intention to guide students in a liberatory education, to give a sense of self-worth, and to empower students with the responsibility for their own self-improvement. A social justice education is committed to reflection and action upon the world in order to transform it in the name of equity.

People in prison can go backwards, stay the same, or move forward. Education is essential to moving forward. Every day, the incarcerated pay their penitence as they reconcile their past with their present. The struggle for a new life begins with the recognition that the last one was destroyed, and efforts towards self-improvement are founded upon one's hope for a better future. In prison, people lose their identity; through education, they reclaim it. Incarceration is punitive, but it must also be rehabilitative. Correctional education programs help inmates understand their potential while building pathways for a better society. At least 95 percent of people incarcerated in state prisons will be released back to their communities at some point (Hughes & Wilson, 2003). Thus, prisons are encouraged to remember their reformatory roots and provide meaningful programs to inmates that prepare them for full participation in a democratic society.

Each of us interprets the world based on our prior experiences. Our past forms a conceptual framework for our perception of reality. New information is processed through an assimilation paradigm in which we compare it to something we already know, thereby filtering new experiences through a predetermined bias. However, this can be a destructive cycle for those with a jaded past. The Bureau of Justice Statistics reports that approximately 75% of released convicts eventually return to prison (Durose, Cooper, & Snyder, 2014). Given that education can be a transformative experience, correctional education programs become a necessary act of intervention. Recidivism intoxicates a community; there are hidden social costs of broken families and the despair of individual lives recycled through our criminal justice system is immeasurable. High prison rates need to be viewed as economic and moral failures, and the time is right for more progressive educational programs to play a role in the reconstruction of society. Mass-incarceration is not the solution to our social problems.

The goals of education are multi-faceted. We want students to gain critical thinking skills that will allow them to be economically self-sustaining, cooperative, and productive workers. We also want students to be empathetic and ethical citizens. To this end, the curriculum outlined in this paper provides opportunities to study social choice mathematics, and to logically explore issues of equity and the subjective nature of knowledge in a world filled with uncertainty. The commitment to social justice distinguishes this curriculum from most courses presently offered in liberal arts mathematics.

This chapter summarizes an innovative curriculum for a liberal arts math course and it is based on extensive research, years of experience, and valuable student feedback. The curriculum aims to provide students with useful skills to help them become informed and participating citizens, and it fits neatly within the parameters of a typical university course for "Math for Liberal Arts." The underlying motivation for this curriculum is the belief that we can break the cycles of inequity by cultivating more student awareness of the injustices in the world and by encouraging them to take action to remedy these problems.

### 3.5 Unit One: Social Choice Mathematics

#### Subjectivism and Mathematics

The first part of the curriculum is an introduction to social choice mathematics – a field of study that develops mathematical reasoning in the framework of decision-making. Social choice math acknowledges the presence of personal feelings in the world of mathematics and illustrates the significant role played by emotions and how they interact with numbers. The language and tools of math may coexist with subjective interpretations of reality and help us understand the world in which we live. The beginning of the curriculum investigates how math can be used to make decisions and investigate issues of equity in society. Students explore the subjective nature of knowledge and overcome the common misconception that math must always produce a single definite answer. In particular, the curriculum explores the mathematics behind voting systems, ratings and rankings, compensation arrangements and fair division of assets.

In social choice mathematics, we admit that emotions play an important role in the decision-making process. As a premise, we accept that feelings are not just relevant but also essential to the process, as such, we work to deliberately attach quantitative values to them. For example, we can assign weights to the various factors that influence our decision in the weighted sum method for ratings and rankings. Additionally, we can assign monetary values to objects that must be shared among multiple parties so that compensation arrangements can be made to those who do not physically receive the objects to which they have entitlement. In voting systems, we can numerically rank our full range of preferences and consider all available choices rather than selecting only one option. Social choice math makes it acceptable to combine personal feelings with mathematical concepts to reach conclusions. A key takeaway from the curriculum's first unit is that numbers do not speak for themselves, people must interpret them – yet each person will do so through a unique lens.

The mathematics underlying voting systems is simple arithmetic, however, students are challenged and enlightened by the notion that the same votes can be used to produce different results. Students learn that in an election, it is not a matter of who you vote for, but rather how your vote is counted – and understanding this is important for fully-informed citizenship. The value in this section lies with the surprising realization that math can be used to manipulate outcomes, and contrasting conclusions are equally justifiable.

Through exercises in compensation arrangements and the fair division of assets, students learn how to bridge the gap between quantitative and qualitative values. Given that groups and individuals have different value systems and must navigate social situations replete with contrasting interests and motivations of others, the algebra-based tools of fair division provide students with highly useful skills to dissect and direct their decisions in a variety of settings. This section also includes problems about shared inheritances, divorce arrangements, and disputes between roommates. Algebraic tools offer valid frameworks with which to operate, but the subjective nature of each context highlights the variability of premises that individuals may use. Students reflect on how their satisfaction with an outcome ultimately depends upon the reliability of the quantitative values that they attach to the related qualitative values.

**The Importance of Studying Social Choice Mathematics**

Being able to make thoughtful decisions is a valuable skill, and it is one of the central tenets for this curriculum of mathematical reasoning. This curriculum advances a theory of subjective rationalism by recognizing the role that emotions and opinions play in our decisions and by combining subjectivity with the tools of mathematics. The principles of social choice math may be used to gain deeper insight into the sociopolitical issues of our times and/or personal situations in life. Students are asked to consider (A) how a vote is counted is just as important as for whom the vote is cast, (B) ranking systems are easily manipulated by the selection of categories and the weights that are assigned to them, (C) the allocation of resources and distribution of assets can be determined in many ways, and (D) there is a significant difference between sameness and fairness (i.e. equality vs. equity).

The mathematics of social choice – voting schemes, public rankings, apportionment practices, and compensation arrangements are all talking points in the media and in local communities. To effectively use the tools mathematical reasoning, one must first learn to recognize their presence in the public discourse and opportunities for their applications in everyday situations. By doing so, students can become more informed citizens and assume more active roles in shaping the world around them.

**3.6 Unit Two: Logic**

**Understanding Validity: A Threshold Concept**

Unit Two develops the threshold concept of a logically valid framework. Logic is the grammar for the language of mathematics; and the concept of validity informs a learner on how to engage in proper thought processes. As such, the concept of logical validity is fundamental to mathematical reasoning, and its inclusion is essential in a liberal arts math course because it is useful across all disciplines. A valid framework is a logical structure that guarantees the truth of a conclusion if the premises are true. Deductive reasoning is the process of reasoning from one or more premises to a conclusion. The manner in which one does this is critically important because it determines the strength of the conclusion.

Mathematical theorems are logically derived conclusions built upon accepted truths. The strength of the conclusion rests upon the reliability of the initial facts and the structure of the argument. Understanding the reliability of premises is carefully examined in the third unit of probability, and is also explored here in a discussion about inductive reasoning. The dynamics of deductive reasoning, developed through categorical and propositional logic, provide the platform to examine various forms of argumentation. Valid forms, such as the direct, contrapositive, disjunctive, and transitive arguments are studied, and contrasted with invalid forms, including the inverse, converse, and non-sequitur false chains.

The process of argumentation is an act of justification, and it is central to the notion that knowledge is a justified true belief. An argument is an expression of knowledge, and when a student fully understands the concept of logical validity, the internalized logical framework dramatically affects how he or she perceives and expresses new ideas. One’s thought processes become irreversibly transformed, and the new perspective is applied in all settings – these are the defining characteristics of a threshold concept. Upon learning the concept, a student also develops the ability to identify and deconstruct an invalid argument, which facilitates sophisticated dialogue and critical insight into the dominant discourse.

### *Inductive Reasoning and a Foundation for Statistical Inference*

The first unit in the curriculum illustrates how to assign quantitative values to subjective qualitative values. Self-awareness of individual preferences significantly influences future actions, and mathematical tools can be used to navigate one's direction within the realm of social choice. The second unit in the curriculum begins with more consideration for the impact of individual lived experiences – cultural upbringing, relationships with friends and family, and profound events that have shaped individual perspectives on life. The cumulative effect of these experiences is a filter – a psychological lens through which one sees the world.

Inductive reasoning is the act of drawing a conclusion from experiments or observations. A defining feature of inductive reasoning is that it is based on incomplete information. During this process, we naturally fill in gaps, make predictions, and reach conclusions based on our expectations. But, our expectations are a result of our experiences. We do not label inductive conclusions as right or wrong, instead, we classify them as strong or weak. Most of what we think we know about the world – including scientific understandings, is the result of inductive reasoning. Consistent observations and repeated results of experiments point us towards generalized understandings. Inductive reasoning takes us beyond the evidence and guides us towards a suggestion of what is probably true.

A formal study of inductive reasoning provides a strong foundation for statistical reasoning; accordingly, it is given significant attention in this curriculum. An increasing number of liberal arts programs are requiring students to develop proficiency with the tools of data analysis, as such, postsecondary enrollment in statistics courses has steadily risen throughout the country (Cobb, 2005; Lutzer, Maxwell, & Rodi, 2000). Inductive reasoning provides students with skills to interpret findings, to recognize the limitations of statistical inference, and to deepen their intuitive understanding of probability.

***Deductive Reasoning and the Principles of Argumentation***

Deductive reasoning is a process of reasoning that uses known facts and applies them to specific cases; it is a process that takes us from initial premises to a conclusion. A premise is a statement used as the basis of an argument; it is an assumption that something is true. The certainty of a conclusion from deductive reasoning is based upon the reliability of the premises, and the manner in which the premises are combined. Through deductive reasoning, students learn how to build a framework for constructing proper arguments. The curriculum explores logic as the internal structure of mathematics; understanding valid logical frameworks is fundamental to one's ability to work with mathematical tools, and it is a threshold concept. A successful learning experience in a mathematical reasoning course will develop the habits of mind that facilitate one's ability to use logic in a variety of contexts. These habits of mind, including the internalization of a valid framework, are essential for a person to be able to process new information in a way that allows for a trustworthy conclusion. The curriculum thoroughly explores deductive reasoning, with attention to both categorical and propositional logic. Students learn to distinguish between valid and invalid structures, and they are provided with multiple opportunities to identify logical fallacies and to construct sound arguments.

***Logic and the Axiomatic Method***

Mathematics is generally considered to be the most absolute of all academic disciplines because it is carefully founded upon a collection of self-evident truths called axioms. The axioms of mathematics are premises which are so obviously true that we cannot imagine living in a world where they are not true. Students are asked to briefly revisit the algebraic axioms of equality which allow for the substitutions, simplifications, and balancing of equations. The inclusion of axioms here builds upon earlier discussions in the curriculum surrounding the postulates in Euclidean geometry.

The axiomatic method is referenced throughout the curriculum. By frequently reminding and explicitly highlighting these connections, students gradually grow to appreciate the internal structure of mathematical logic and the necessity of reliable premises within a valid framework. When probabilities are attached to premises, as they often are in real life, they see the interplay between logic and probability as critical components for constructing a more sophisticated worldview.

The curriculum offers a carefully scaffolded introduction to propositional logic, including extensive work on translations, truth tables, and structures of arguments – most of which is situated in the context of justice studies. Propositional logic studies relationships among statements joined together using the words *and*, *or*, and *if-then*. These same logical connectors reappear later in the curriculum during the study of compound and conditional probability. The curriculum guides students to discover the connection between these two ostensibly disparate domains – the world of certainty and the world of uncertainty, through the use of common language and the supporting visual aid of Venn diagrams. Students achieve a profound insight when they understand how statements in logic and events in probability are connected by means of using the same language.

***A Formal Study of Arguments***

An argument is defined as a set of reasons given with the aim of persuading others that an idea is right or wrong. Classic theory states that knowledge must be a justified true belief. To claim knowledge of something, one must believe it, it must be true, and most importantly – one must provide justification. This last component is the essence of a mathematical proof and argumentation. Justification is what allows knowledge to grow and spread to others. Careful training in mathematical reasoning can equip students with this important skill. The quest for knowledge is an innate drive within each of us; we want to know things to satisfy our natural curiosity of the world, to become more productive, and to reach our individual potential.

The basic principle of argumentation is that it must be impossible for a false conclusion to result from true premises. If we accept the "if P, then Q" structure ($P \rightarrow Q$), as a true causal relationship for a given situation, then this represents an understanding of the world and serves as a premise for future arguments. An argument must have at least two premises. A valid structure ensures that the true premises lead to a true conclusion. In contrast, an invalid structure for an argument allows for the possibility of a false conclusion to arise from a given set of true premises. The curriculum explores various forms of argumentation, including:

| | | |
|---|---|---|
| Direct (modus ponens) | $[\,(P \rightarrow Q) \wedge P\,] \rightarrow Q$ | valid |
| Contrapositive (modus tollens) | $[\,(P \rightarrow Q) \wedge {\sim}Q\,] \rightarrow {\sim}P$ | valid |

| | | |
|---|---|---|
| Transitive | $[ (P \rightarrow Q) \wedge (Q \rightarrow R) ] \rightarrow (P \rightarrow R)$ | valid |
| Disjunctive | $[ (P \vee Q) \wedge \sim P ] \rightarrow Q$ | valid |
| Inverse | $[ (P \rightarrow Q) \wedge \sim P ] \rightarrow \sim Q$ | invalid |
| Converse | $[ (P \rightarrow Q) \wedge Q ] \rightarrow P$ | invalid |

This section of the curriculum is developed in the context of criminal justice. The examples provided are intended to be more relevant and engaging than traditionally conservative contexts used in other textbooks. Consider the following scenario and the given statements for P and Q:

> Scenario: On October 22nd, a 7-Eleven was robbed in Cincinnati. The local police subsequently make an arrest.
>
> Fact: If a suspect did indeed rob the 7-Eleven, then that suspect must have been in Cincinnati on October 22nd. (We use this true fact as a premise in the argument).
>
> P: The defendant robbed the 7-Eleven.
> Q: The defendant was in Cincinnati on October 22nd.
>
> If we learn that the defendant was in Cincinnati on October 22nd, can we safely conclude that he must be guilty of this crime? $[ (P \rightarrow Q) \wedge Q ] \rightarrow P$
>
> If we learn that the defendant was not in Cincinnati on October $22^{nd}$, can we safely conclude that he did not commit the robbery? $[ (P \rightarrow Q) \wedge \sim Q ] \rightarrow \sim P$

The curriculum provides extensive training with deductive reasoning and numerous exercises in identifying the structures of arguments. The scaffolded approach begins with translating written text into symbolic expressions and building truth tables based on the properties of logical connectors. The approach is designed to equip students with the skills to recognize whether conclusions are valid or invalid.

> ***Sample Exercise: Education and Socioeconomic Status***
>
> Many studies have researched the relationship between educational attainment and socioeconomic status in America. A recent report from the National Center of Education Statistics (NCES) revealed that the majority of adults who have not completed their high school diplomas end up in the lower class. Identify the two conclusions that are consistent with the findings from the NCES:
>
> a) If a person is in the lower class, then it is likely that he or she does not have a high school diploma.

b) If a person has a high school diploma, then he or she is likely to be above the lower class.
c) If a person does not have a high school diploma, then he or she is likely to be in the lower class.
d) If a person is not in the lower class, then he or she is likely to have more than a high school diploma.

The relationship between educational attainment and socioeconomic status is revisited later in the curriculum during the section on conditional probabilities. Several themes such as this one purposefully reappear throughout the curriculum. Students grow to learn that a variety of mathematical reasoning tools can be applied to the same story, and collectively these tools work together to support more profound insight. This section of the curriculum culminates with assignments that ask students to analyze arguments within sophisticated discourse. In particular, students address arguments made by Martin Luther King, Jr. in his “Letter from Birmingham Jail,” as well as issues of poverty, education, crime, and recidivism. Through mathematical reasoning, the curriculum teaches students how to deconstruct public discourse and how to optimally construct their own arguments in support of their positions on important issues. Students spend a lot amount of time working on these assignments and discussing them with their peers. The context of these problems is extremely relevant to their personal experiences, and as a result, they are highly motivated to apply their new understanding of logic to navigate the discourse surrounding these social issues.

***Sample Exercise: Poverty, Education, and Crime***

An individual that has a college education is likely to have more job opportunities than someone who does not have a college education. More job opportunities mean more financial security for those highly educated people. Altogether, a community with many educated residents is less likely to have poverty, and less poverty means less crime. Society should invest in the education of all its citizens, including the incarcerated, because everyone benefits from living in a community of educated individuals. However, some people argue that increases

> in crime rates are simply the result of implementing stricter laws. What if there was no poverty and what if the laws were not strict? Are crime rates an economic or legal issue? Provide your interpretation of the relationship between education, poverty, strict laws, and the incidence of crime within a community. Identify appropriate statements for P, Q, R, and S, and express an argument using symbolic logic.

> ***Sample Exercise: Correctional Education and Recidivism***

> The United States Bureau of Justice Statistics reports national recidivism rates of approximately 67% during the first three years following a person's release from incarceration. One state in particular is progressive with its efforts to offer college education to inmates while serving their sentences, and, this same state reports recidivism rates of approximately 33% during the first three years of one's release. Recent data shows recidivism to be 5% among the inmates who directly participate in that college program. However, despite the positive results of correctional education programs, many American citizens oppose the use of taxpayer money to subsidize college courses for inmates. Some taxpayers say that the money should be used to provide tuition assistance for law-abiding citizens instead. Depending on the number of courses taken, the cost of enrolling an inmate in a state-sponsored college program is approximately $5,000 per year. Meanwhile, state governments across America are spending an average of $30,000 per year per inmate, to incarcerate a population of more than 1.3 million people in their state facilities. Perform some calculations and then use symbolic logic to express a valid argument in favor of government-subsidized correctional education.

In this exercise, students are deliberately led in a direction to conclude that correctional education is worthy of public funding. Typically, in critical pedagogy, a teacher is not to impose his or her political views upon the students, rather, the students are to be presented with the tools and data to reach and support their own conclusions (Giroux, 1983). However, in the experience of creating this curriculum, it was both beneficial and efficient to have students working towards a shared understanding so that the attention was on the formulation of arguments and not on a debate surrounding contrasting views. Given that all of the students assigned this task were in a correctional education program, it was a unanimous point of view that correctional education programs should receive taxpayer funding anyway.

The curriculum outlined in this paper contains exercises that sometimes lead students with a particular agenda. That is, the manner in which exercises are framed admittedly contains a biased point a view. For example, the context of the preceding exercise presents data which is easily used to justify funding for higher education in correctional facilities. At this point in the curriculum, the students have been trained in constructing arguments using propositional logic, and are now asked to build an argument that supports public funding for correctional education. This particular exercise presents students with a straightforward opportunity to create a mathematical argument in support of an issue that has direct relevance to their lives. Note, this topic was frequently raised by students who communicated their need and appreciation for higher education, while pleading their cases for financial support. Their studies in other courses reinforced their intuition that an enlightened mind is less likely to engage in criminal behavior, and the data provided in this exercise validated their beliefs in the power of higher education to reduce recidivism rates.

The statistics included here, along with proper training in propositional logic, collectively arm students with tools to advocate for themselves, both academically and as citizens in an unjust society. The pervading sentiment among inmates is that they have not enjoyed the same economic freedoms nor privilege to pursue higher education as their more affluent peers. The combination of information – in the form of real, verifiable data, and the tools of mathematical reasoning that enable them to apply logic to construct an argument, together enable students to construct an argument on their own behalf – an empowering experience that benefits them intellectually and socially. Theoretically, the consequence of which cultivates a sense of agency and inspires them to share their sophisticated understanding of the world with others. Although the preceding exercise about correctional education suggests a foregone conclusion to the students, the curriculum provides scaffolding from forced-choice exercises to support their efforts with more open-ended questions which allow them to effectively decide their own conclusion. In particular, exercises with compound and conditional probability that appear later in the curriculum provide substantial opportunities for students to navigate ambiguity, choose among a multitude of paths, and arrive at a variety of possible conclusions.

***Sample Assessment Questions***

1. When charged for a crime, you are either guilty or innocent of the charge against you. Use the given P and Q to create a truth table for the expression below and interpret the results.

   P: You are guilty.
   Q: You are innocent.

   $[(P \vee Q) \wedge \sim P] \rightarrow Q$

2. A courtroom trial follows the form of an argument, in which two opposing sides present reasons to justify their positions. This process aims to result in a valid conclusion. The most important premise in the American criminal justice system is the presumption of innocence. When someone is accused of a crime, the proceedings of our system are based on the assumption that the individual is "innocent until proven guilty."

   P: The defendant is innocent.
   Q: Proof of innocence exists.
   R: Absolute proof of guilt exists.
   S: The defendant is set free.

   Translate each of the following symbolic expressions into complete sentences.

   $(P \wedge Q) \rightarrow S$ $\qquad$ $\sim R \rightarrow S$

   $P \rightarrow (Q \vee \sim R)$ $\qquad$ $(\sim Q \vee R) \rightarrow \sim S$

3. Create a truth table for the argument: $[(P \wedge Q) \vee \sim R\,] \rightarrow S$

> Is this a sound argument? Comment on the reliability of the premises, and create a truth table with 16 rows to analyze the validity of the conclusion. Explain the implications of your answer.

Students who successfully complete these assessment questions discover that the argument breaks down in the fourth row. Our criminal justice system is founded upon the premises that a defendant will be set free if there is either proof of innocence or if absolute proof of guilt beyond a reasonable doubt does not exist. Students recognize the implied power of these two premises: $[(P \wedge Q) \rightarrow S]$ and $[\sim R \rightarrow S]$. Along with their interpretations of the table, students provide thoughtful reflections on their experiences within the criminal justice system. At this point in the curriculum, students are equipped with a potent grasp of inductive and deductive reasoning processes, and many of them naturally apply the tools of mathematical logic to their own legal cases – seeking greater insight and a clearer understanding of their individual situations.

**3.7 Unit Three: Probability**

**Understanding Reliability: A Threshold Concept**

This curriculum explores mathematical reasoning to clarify our thought processes, to understand the nature of knowledge, and to develop the means to arrive at reliable and valid conclusions. When we reflect on our ways of thinking we become more self-aware with how we perceive the world around us, and with what constitutes actual knowledge. In doing this, we are naturally drawn to the act of justification – and justification is central to doing mathematics. An argument is built upon premises; premises are evaluated based on their reliability, and these levels of reliability are essentially matters of probability.

As a threshold concept, reliability is reinforced throughout the curriculum. The reliability of a premise is first introduced as a subjective matter in social choice mathematics, and then reappears in the study of inductive reasoning. Students recall how it is human nature to make predictions and generalizations based on limited observations and pattern recognition. Conclusions reached through inductive reasoning are labeled either strong or weak, depending on the reliability of their premises. These same conclusions then become the premises used in deductive reasoning. Therefore, premises used in deductive reasoning should have probabilities attached to them. In propositional logic, deductive premises are treated in binary fashion, assigned either a true or false value. Yet, if in reality they are matters of probability, then the conclusion of deductive reasoning must also be viewed from a perspective of probability – in addition to be being labeled as either valid or invalid. The reasoning process is valid if the truth of the premises guarantees the truth of the conclusion. The truth of the premises is a matter of reliability, and herein lies the need to study probability.

Understanding the concepts inherent in reasoning with probabilities opens up new ways of thinking about something. A student can neither progress with a meaningful study of statistics, nor fully appreciate the tools of data analysis without knowing how to manage uncertainties from a quantitative perspective. Probability lays the groundwork for statistical reasoning; accordingly, this curriculum serves as an ideal precursor to a statistics course. Logic provides the fundamental framework for thinking, but probability enables one to effectively cope with ambiguity and the unpredictability of life's random events. Together, threshold concepts in logic and probability fundamentally transforms one's capacity to process new information and construct arguments, in an integrative and irreversible way.

***Probability: An Essential Component of Mathematical Reasoning***

The topic of probability is both relevant and engaging to students. Given that we live in a world of certainty, an understanding of basic probability principles is highly useful for making sense of everyday occurrences. Probability has a deserving place in the field of liberal arts math courses, however, the research performed in chapter two of this paper reveals that probability often does not receive sufficient attention. The practice of probabilistic thinking captures the interest of students, yet it is often counterintuitive and takes time to develop. When probability topics are situated within relevant contexts, students discover powerful tools with which they may investigate social issues. Probability is the formalization of common sense; it is based on experience and stems from an understanding of both inductive and deductive reasoning. Students learn that probability concepts are not only born out of logic, they also form a foundation for logical arguments.

The search for knowledge must occur within logical frameworks; this curriculum demonstrates that this quest is founded upon probabilities. Our thoughts and actions are based upon premises, and the reliability of each premise is a matter of probability. Throughout the curriculum, students recognize that individual emotions inevitably play a role in decision-making, inductive reasoning is a natural human inclination which generates premises for future arguments, premises are probabilities, and a logically valid structure produces a trustworthy conclusion – but only if the initial premises themselves are reliable. Thus, one's understanding of the world is essentially based upon careful combinations of feelings about probabilities.

The curriculum presents a scaffolded approach to the subject of probability, beginning with the simple probability of a single event, followed by the compounded probability of multiple independent events, then conditional probabilities with dependent events, and concluding with a study of binomial probability and an examination of binomial distributions. The content in the first section on simple probability includes a discussion of expected value, permutations, odds and fair payouts, the law of large numbers, and probability distributions. Next, students explore the compound probability of multiple independent events joined together using the words "and" and "or." An important connection is made between logic and probability because the same logical connectors used with deductive reasoning and truth tables are also used in compound probability. The overlap between logic and probability is illustrated through frequent use of Venn diagrams in this section. The curriculum next examines conditional probabilities among dependent events. The language of conditional probability utilizes an "if-then" construction which is the third fundamental connector in propositional logic. The probability unit concludes by combining the principles of compound probability with combinatorics, to develop an understanding of binomial probabilities. This section reveals a surprising array of applications for Pascal's triangle, and ultimately demonstrates how the binomial probability distribution serves as a model for the normal distribution, thereby developing a strong foundation for future studies in the field of inferential statistics.

***Sample Exercise: Compound Probability***

> The American criminal justice system uses the phrase "beyond a reasonable doubt." But, to say "beyond a reasonable doubt" is to imply an element of uncertainty in the process. If we accept that nothing is certain and that everything has a probability attached to it, then we must decide what is an acceptable comfort level in this process. For this exercise, let's say we are comfortable if we are 99.9% sure. In the last 40 years, there have been 1,429 executions completed as the result of death penalty verdicts in the American criminal justice system. If each verdict had a 99.9% chance of being correct, what is the probability that there was at least one mistake made, that is, at least one innocent person was wrongfully executed?

The topic of compound probability is introduced with a brief discussion of "The People vs. Collins" (1964), a landmark case in which the principles of compound probability were misapplied in the courtroom. Students are presented with the question: What is the probability of seeing a black man with facial hair and a white woman with blonde hair in a ponytail, driving together in a yellow car? Local demographics are used to establish probabilities for each variable, and students learn how to compound these probabilities to reach extremely high odds against seeing such an event (which led the prosecutor to argue that the suspected couple must have been guilty of the charge against them). In this famous case, a guilty verdict was later overturned on the notion that defendants should not have their guilt determined by the odds. In a U.S. court of law, an expert is not allowed to calculate probabilities based on estimates, because probabilities are not reliable. Students are reminded of what they learned in the previous unit – that a reliable premise must be consistently true, and that probabilities do not carry guarantees.

***Sample Exercises: Conditional Probability***

> Write about the use of conditional probability as a tool to explore the three social issues expressed below. In what ways can mathematics help build a lens to view the world from a different perspective?
>
> In 2015, *The New York Times* published an article entitled "The Disproportionate Risks of Driving While Black." The author wrote, "A year of turmoil over the deaths of unarmed blacks after encounters with the police in Ferguson, Mo., in

Baltimore and elsewhere has sparked a national debate over how much racial bias skews law enforcement behavior, even subconsciously." His examination of traffic stops in North Carolina uncovered wide racial differences in police conduct; and he quotes one government official as saying, "Racial profiling is a very real phenomenon." In late 2014, the residents in Ferguson, Missouri were outraged that an unarmed black man was shot and killed by a white police officer. Many sociologists attribute a brooding anger to systemic discrimination against the African-American community, as evidenced by the disproportionate number of vehicle stops by ethnicity. Use conditional probability to investigate the story of the public's angst and the cause for civil unrest. Create and answer your own questions.

Table 6
*Traffic Stops by Ethnicity, Ferguson Police Department*

| *Ferguson, MO (2013)* | Stops | Searches | Arrests | No Incidents | Total Registered Drivers |
|---|---|---|---|---|---|
| White | 686 | 47 | 36 | 4,569 | 5,338 |
| Black | 4,632 | 562 | 483 | 4,317 | 9,994 |
| Hispanic | 22 | 1 | 1 | 150 | 174 |
| Asian | 12 | 0 | 0 | 83 | 95 |
| American Indian | 8 | 1 | 1 | 49 | 59 |
| Other | 24 | 0 | 0 | 181 | 205 |
| Total | 5,384 | 611 | 521 | 9,349 | 15,865 |

A meritocracy is a social system in which people's success in life depends primarily on their talents, abilities, and effort. Proponents of the idea that America is a meritocracy argue that the realities of socioeconomic inequalities are simply the result of unequal talents and not the result of societal prejudices or institutional discriminations. In 2012, a national research center gathered data about class mobility. The following table represents their findings (expressed as percentages) regarding Americans' self-perceptions about their socioeconomic status compared to the social class of their own parents. Does this table reinforce or contradict the claim that America is a meritocracy? Begin by thinking about your initial position in this controversial debate, and then search the data for meaningful insight. More than seventy different conditional probabilities can be expressed using the numbers in this table, but some are more revealing than others. Explore the possibilities and investigate the likelihood of an adult ending up in a social class different than that of his or her parents.

Table 7
*Perceived Socioeconomic Status and Class Mobility*

| *Meritocracy* | Parents' Status / Family Background |
|---|---|

| | | Lower | Middle | Upper | Total |
|---|---|---|---|---|---|
| Children as Adults | Upper | 4 | 5 | 6 | 15 |
| | Middle | 15 | 25 | 7 | 47 |
| | Lower | 21 | 13 | 4 | 38 |
| | Total | 40 | 43 | 17 | 100 |

The following table shows the levels of educational attainment by members of each socioeconomic class. Identify and answer four conditional probability questions that can provide insight into the relationship between personal wealth and educational achievement. The data corresponds to the highest level of education achieved, and the numbers are expressed in terms of a representative sample of 1,000 Americans. (This data was drawn from a national center for education statistics).

Table 8
*Educational Attainment by Socioeconomic Status*

| *Educational Attainment* | Less than HS Diploma | High School Diploma | Some Postsecondary | Associate's Degree | Bachelor's or Higher | Total |
|---|---|---|---|---|---|---|
| Upper Class | 2 | 6 | 58 | 14 | 120 | 200 |
| Middle Class | 15 | 65 | 225 | 50 | 145 | 500 |
| Lower Class | 21 | 66 | 147 | 24 | 42 | 300 |
| Total | 38 | 137 | 430 | 88 | 307 | 1,000 |

## 3.8 Argumentation in Mathematical Reasoning

### The Intersection of Logic and Probability

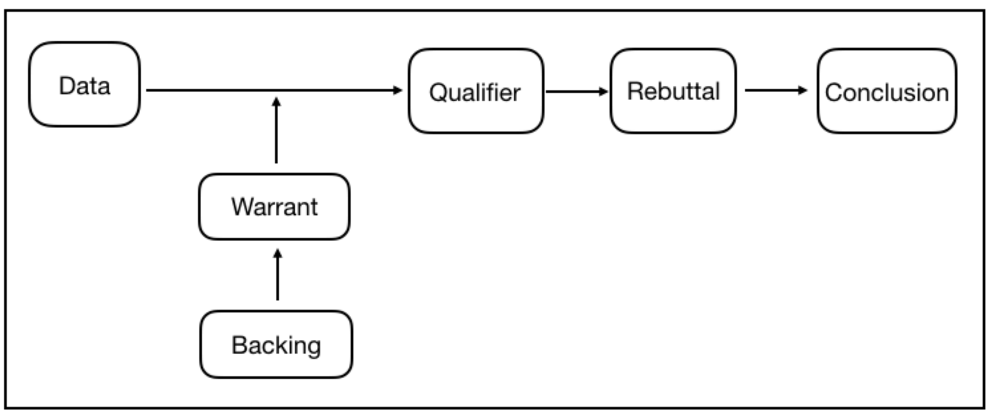
The threads of logic and probability are woven throughout the fabric of our minds. Despite the objectivity of logic, probability is inescapable, and our conclusions can never be certain. Logic and probability are linked through the interplay of validity and reliability as threshold concepts, and they are explicitly connected by their shared use of the key words: *and*, *or*, *if-then*. The same connectors used in the construction of arguments and the study of logically valid frameworks are central in matters of compound and conditional probability. Logic's use of universal and existential quantifiers may also be viewed in quantitative terms, further solidifying its intersection with probability. The principles of logic and probability overlap like two non-mutually exclusive circles in a Venn diagram. Logically valid frameworks offer a definite certainty, and principles of probability offer a way to manage the uncertainties of this world. The interaction of the two is a powerful relationship, the understanding of which facilitates an individual's acquisition of new knowledge and the construction of a more sophisticated lens.

Argumentation is both a search for, and an expression of, knowledge. It begins with premises that stem from an understanding of the world – an understanding that traces back to inductive reasoning and prior experiences. Such premises may or may not be reliable. We use these premises within a logical framework and apply the principles of deductive reasoning to ensure that we arrive at a valid conclusion. The study of probability is critical because it helps us clarify basic assumptions used as premises within logical frameworks. The strength of the conclusion invariably traces back to the strength of the initial premises. This curriculum shows students how to manage multiple uncertainties with compound and conditional probability as they construct arguments and seek knowledge. Together, reliability and validity are the two ingredients of a sound argument. A sound argument is analogous to a sharply-focused lens, one which enables insight into our lives and the sociopolitical issues of our time. At the end of this section of the curriculum, two reflection assignments are given to assess a student's growth within the intersection of logic and probability, in the context of justice studies (see Appendix A and Appendix B).

**Frameworks for an Argument**

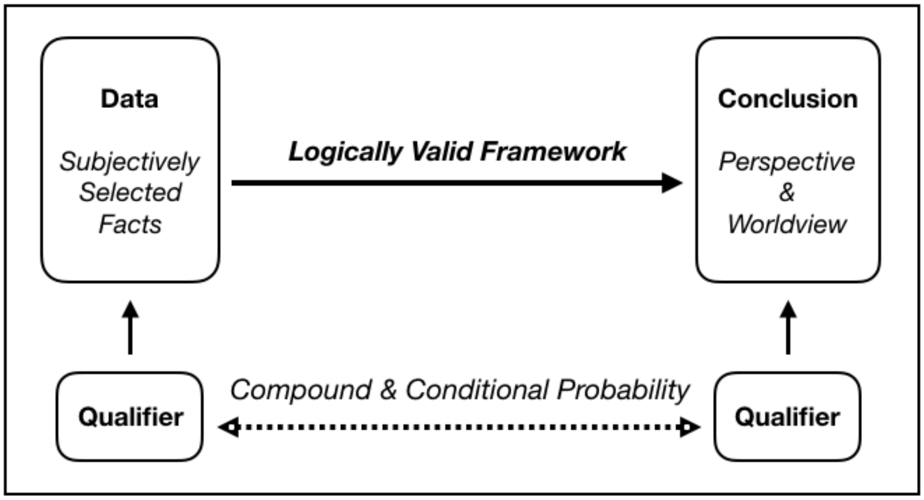
Mathematical reasoning is less about the manipulation of numbers, and more about the construction of arguments. Lithner (2006) highlights two types of argumentation – predictive and verificative, employed by students when they engage in creative reasoning processes for mathematical problem solving. As an act of justification, the argument is central to both mathematical thinking and one's search for new knowledge. Toulmin's (2003) widely-known model of an argument outlines six necessary components to its structure, including the data, warrant and backing, rebuttal, quantifier, and conclusion.

Figure 1
*Toulim's Model of an Argument*




In contrast, this curriculum offers an alternative model that emerges through subjective rationalism. Figure 2 presents a pedagogical and curricular approach to the development of mathematical reasoning, built around the threshold concepts of validity and reliability.

Figure 2
*Wenger's Model of Subjective Rationalism and Argumentation*




Subjective rationalism recognizes the role played by an individual's emotions and personal preferences surrounding any given topic. The data or initial grounds upon which an argument is based, is inherently affected by one's deliberate selection of particular facts. These facts naturally have descriptive statistics attached to them, which become probabilities for the assumed truth of the premises. Pedagogically, this provides an opportunity for the instructor to engage students in discussions that relate matters of probability to the language of existential and universal quantifiers used in formal logic. The curriculum exemplifies this with the following syllogism, found in the section on categorical logic.

| | |
|---|---|
| Premise #1: | All inmates have tattoos. |
| Premise #2: | Joe is an inmate. |
| Conclusion: | (Fill in the blank). |

Students learn to understand that the strength of the conclusion is directly related to the strength of the initial premises. If descriptive statistics reveal that only 70% of inmates actually have a tattoo, then the conclusion for Joe to have a tattoo has only a 70% chance of being correct. The interplay between probability and logic here demonstrate that the quantifier attached to the premise must also be used as the quantifier for the conclusion.

This curriculum also emphasizes the strength of established mathematical theorems by highlighting our use of axioms – universally accepted truths. When we build an argument that stems from axioms, then we can be confident in the strength of the conclusion. However, outside the absolute world of pure mathematics, axioms are rare. Accordingly, students benefit from this liberal arts math curriculum that (A) cultivates an understanding of probability and (B) guides them with their usage inside logically valid frameworks. The combination of which, assists students in developing a set of reasoning skills that are practical across multiple disciplines.

This new model of an argument emerged through the experience of co-constructing a curriculum based on student feedback and the research practitioner's reflections on students' competencies in expressing their points of view on matters that were personally important to them. In response to the tasks presented throughout the curriculum, students were not expected to reach an irrefutable conclusion and convince others, rather to gain proficiency in building an argument, explicitly recognize the role of uncertainty, and effectively share their perspective with others from a more sophisticated point of view.

### 3.9 Intra-Curricular Connections

Key math concepts have been linked together throughout this curriculum – not only to help tell a story with consistent themes, but also because the concepts themselves are inherently related to one another. The connections occur naturally, and taken collectively, they weave an intricate web of ideas. We process new information by comparing it to our prior understanding of related ideas. The more connections we can make, the more sophisticated our interpretative lens can become.

Consider a Venn diagram with three overlapping circles. The image represents three kinds of ways through which we filter the reality of the world around us – subjectively, objectively, and with uncertainty. Interesting patterns and quantifiable relationships are all around us, and to see them we need a sophisticated lens – a lens that integrates elements from all three units of the curriculum.

**The Connection between Unit 1 & Unit 2: Subjectivity and Objectivity**

The academic field of mathematics is typically regarded as an objective discipline where everything is defined, and answers are either right or wrong. But there is a subjective nature of knowledge, and within it, a place for mathematics. Often in life, we situate our subjective values within a logical framework. The "if-then" connector ($P \rightarrow Q$) is central to a direct argument and foundational to the principle of argumentation. Valid conclusions are reached because of the argument's logical structure, but they are commonly founded upon premises influenced by personal preferences. If we assign particular weights to categories, then we reach a particular conclusion with a set of rankings. If we use a particular voting system, then a particular candidate will win an election. Based on reported values, an arbitrator can mathematically resolve a conflict and achieve an equitable distribution of assets for all parties involved. We can use logic to justify a conclusion as valid, but that outcome can easily be manipulated by an individual's subjective influence.

Subjectivity creeps into objectivity when we decide which premises to use in an argument. Consider the expression: "You are entitled to your own opinions, but you are not entitled to your own facts." There is some truth to this. If the temperature inside a classroom is 56˚ F, then that is an indisputable fact that exists apart from you. While

56˚F may be a comfortable temperature for your neighbor, it may feel cold to you. Opinions about the temperature are subjective and will vary amongst people. Conflicts naturally arise because there are always multiple facts surrounding any given situation, and there will always be multiple interpretations of these same facts.

When we build an argument to prove a point or discover new knowledge, we are likely to select facts which will support our position. And, our argument is sure to use premises that reflect what is important to us. After all, why would we construct or deconstruct an argument if we did not care about the issue? Thus, the objectivity of a logically valid framework can be imbued with the subjectivity of personal values. Imagine constructing an argument, either for or against, the idea of an American meritocracy. There are more than seventy different conditional probabilities that can be extracted from the raw data presented in the section on conditional probability (Table 7), but they do not all point in the same direction. The calculations are subject to interpretation. Consider these three conditional probabilities:

$$P\text{ (Adult Child in Lower Class} \mid \text{Parents in Lower Class)} = .5250$$
$$P\text{ (Adult Child in Middle Class} \mid \text{Parents in Middle Class)} = .5814$$
$$P\text{ (Adult Child in Upper Class} \mid \text{Parents in Upper Class)} = .3529$$

Do we argue that the most likely outcome for lower or middle-class children is for them to stay in the same class, or do we argue that it is unlikely for upper class children to remain the same status as their parents? We can justify a conclusion in each of these arguments, but we choose which facts to use along the way.

An individual may manipulate the interpretation of a situation merely through a biased selection of facts. Subjectivity and objectivity initially seem to be separate worlds, yet upon closer inspection we see how easily they infiltrate and influence one another. Subjectivity and objectivity are intertwined.

**The Connection between Unit 1 & Unit 3: Subjectivity and Uncertainty**

In everyday life situations, we attach personal values to particular outcomes. In social choice mathematics, students learn to bridge this gap between qualitative and quantitative value systems. When we combine the value of an outcome with its probability, we create expected values that can aid us with important decision-making processes. Feelings may lead us to believe that certain probabilities will occur. But we must also reflect on how probabilities influence our feelings. The value system we develop as individuals has roots in the probabilities we have observed and internalized throughout our lives. For example, consider your positions on meritocracy and racial profiling. A lifetime of observations and experiences with these matters leads you to develop intuitive feelings about proportions and probabilities (i.e. what you would expect to see in a given situation). These subjective feelings connect with other values you have in life, and collectively shape your identity, your outlook, and your interpretation of social realities.

**The Connection between Unit 2 & Unit 3: Objectivity and Uncertainty**

If nothing is certain, are we truly prohibited from building new knowledge? No, of course not, but that means we must accept the idea that the premises we use in our arguments are inherently unreliable because they are attached to probabilities. We naturally engage in logical thought processes that produce both reliable and unreliable conclusions. The premises of an argument are often formed through inductive reasoning, and the reliability of the conclusion depends on the probability that the premises are reliable. The entire argument – from the premises to the conclusion, is built upon probabilities. It is therefore necessary to understand the principles of probability, in order to clarify our own assumptions and develop a more sophisticated awareness within this uncertain world. This curriculum highlights the interaction between logic and probability – and the corresponding threshold concepts of validity and reliability, as essential elements within mathematical reasoning. When students are able to successfully internalize an understanding of these concepts, they develop powerful reasoning skills that are transferrable across multiple disciplines.

### 3.10 Supplemental Units of Study for the Curriculum

Additional chapters are offered with this curriculum to set up, support, and extend the threshold concepts central to the theory of subjective rationalism. These topics include the logic of Euclidean geometry, modular arithmetic and number bases, the principles of apportionment, and an introduction to game theory. A one-semester course would require substantially more classroom time to engage in these additional topics, necessarily producing a four-credit version of the course to accommodate the added workload within this curriculum for a liberal arts mathematics program.

#### Euclidean Geometry

Geometry may be included in the beginning of the curriculum to plant the seed for more formal deductive reasoning later. The study of Euclidean geometry provides opportunities for students to strengthen their algebraic proficiencies, as well as explore practical applications of area, perimeter, and volume problems.

#### Modular Arithmetic and Number Bases

Modular arithmetic and number bases are presented in the curriculum because they challenge students to reflect on their basic assumptions about the nature of mathematics. By examining other frameworks, students learn that mathematical systems are man-made, and as such, quantitative expressions are open to interpretation and manipulation. Understanding the various outcomes that naturally arise through different number systems sets the stage for the element of subjectivity promoted in the following unit of social choice mathematics. Students are often entertained by the notion of other number systems, and many develop an appreciation for base 2 binary and base 12 dozenal systems.

**Apportionment**

The section on apportionment serves as an informative extension of social choice mathematics. Students learn that there are multiple, contrasting, and equally justifiable ways to proportionally distribute resources among groups. This section naturally references state representation in Congress, yet it also presents students with applications in a variety of contexts – including the allocation of a police force according to regional crime rates. Regular rounding rules lead to controversial decisions, and students become intrigued and surprised that our own government has used many different apportionment methods, dating back to 1790.

**Game Theory and the Prisoner's Dilemma**

The final section of an extended, four-credit curriculum culminates with the principles of strategic decision making. Game theory is the study of mathematical models of conflict and cooperation between intelligent and rational people. The academic field was originally developed to study economic problems, but it has surprising applications in political science, sociology, psychology, finance, and warfare. This topic provides a mathematical foundation for studying social phenomena, and enables the instructor and students to engage in discussions about individual and collective motivations. In this section, students learn to reconcile decisions that serve their own best interests with the potentially larger social implications of their actions. Fittingly, game theory combines essential components from the three main units of study in the curriculum.

Knowledge of social choice mathematics teaches students that people may disagree on perceived values of things. Understanding the inherent role of subjectivity, and how participants may attach numbers to personal values, helps students to effectively navigate the conflict between individual and collective decision-making forces. Game theory also requires pattern recognition, logical thinking, and the powers of deductive reasoning – all developed in Unit Two. And of course, games of strategy naturally contain elements of uncertainty, so it is necessary to draw upon key ideas from probability developed in Unit Three. Game theory fundamentally requires a combination of deductive reasoning and probabilistic thinking, and it demands an understanding of the threshold concepts emphasized throughout the course. The curriculum culminates with a familiar question to the students, “What are the associated levels for the reliability of your premises, and how can you string them together in a logically valid way to produce the strongest possible conclusion?” This section on game theory includes the following content: cooperative vs. total conflict games, symmetric vs. asymmetric power, sequential vs. simultaneous play, zero-sum vs. non-zero-sum games, perfect and complete information, payoff matrices, ordinal and cardinal payoffs, the minimax theorem, and the Nash equilibrium.

| | Duey Confesses | Duey is Silent |
|---|---|---|
| Huey Confesses | Both receive a five-year sentence | Duey is set free, and Huey gets fifteen years |

Consider a scenario in which two individuals are arrested as suspects in a major crime. They are interrogated in separate rooms, and each person has the choice of remaining silent or confessing and implicating his partner in the crime. If both individuals remain silent, then they will each be convicted of a lesser crime and receive a short sentence. If one confesses and the other does not, then the confessor will be set free as a reward for his cooperation, while the other will be given a long sentence. If both confess, then they each receive sentences of medium length. For example, Huey and Duey have been arrested for burglary and are being held in separate interrogation rooms. The police detective pressures each one to confess because there is not enough evidence to convict either person. If Huey and Duey both remain silent, then they will be convicted on a lesser crime for possession of stolen goods, and each will be sentenced to one year in prison. If one confesses and the other does not, then the confessor will be set free as a reward for his cooperation, while the other will be sent to prison for fifteen years. If Huey and Duey both confess, then they will each be sentenced to five years. The following table is a payoff matrix showing each of the four possible outcomes.

Table 9
*Game Theory Payoff Matrix*

| | | |
|---|---|---|
| Huey is Silent | Huey is set free, and Duey gets fifteen years | Both receive a one-year sentence |

Cardinal payoffs are numbers that represent the outcomes of a game, and may be expressed in terms of money, time, or some other unit of measurement. The condensed payoff matrix below shows cardinal payoffs for the prison sentences, measured in years.

Table 10
*Game Theory Cardinal Payoffs*

| (H, D) | Duey Confesses | Duey is Silent |
|---|---|---|
| Huey Confesses | (5, 5) | (0, 15) |
| Huey is Silent | (15, 0) | (1, 1) |

The average time served for the two detainees may be 5 years, 7 ½ years, 7 ½ years, or 1 year. Their best decision is to act with a collective mentality – for both to remain silent and serve one year each. However, if one recognizes this and expects the other to remain silent, then it is in the first person's best interest to confess and be set free. If both prisoners realize this and act accordingly, then the scenario quickly settles on a double confession.

If Huey confesses, then Duey's best move is to confess (reducing his sentence from fifteen to five years). If Huey remains silent, then Duey's best move is to confess (reducing his sentence from one year to no time at all). Either way, Duey's best individual move is to confess, and the same is true for Huey. This way of thinking leads them both to confess and receive five-year sentences, even though it is in their mutual best interest to remain silent. The Nash equilibrium tells us that despite all the cyclical reasoning that each prisoner may engage in, the most likely outcome is that they will both confess. The motivation to initially act in one's own self-interest does not actually serve one's best interest in the long run. The mutual confession produces a worse outcome for the pair of prisoners.

Games structured like the prisoner's dilemma compel an individual to weigh the potential reward of another player's selfless decision, alongside the risk of relying on that same player who may make a selfish decision. Collective success depends on both participants being intelligent human beings making rational choices and selecting the cooperative option. Many scholars have studied an iterated version of the prisoner's dilemma. They have found that when the scenario is repeated many times with the same two players, the players learn the collective benefit and move towards the cooperative option (Fogel, 1993). On a larger scale, we must hope that members in our community can learn to reconcile the conflict between selfish and selfless actions. Our society ultimately succeeds when individualist inclinations yield to a cooperative cause. Shifting our focus to the greater communal good, albeit personally risky, has a reward that justifies the risk. Game theory presents a platform for the instructor and students to debate relationships among individuals and society, and to engage in a dialectic that demands sophisticated mathematical reasoning skills.

**"The Individual and Society" vs. "The Individual or Society"**

The connectors "and" & "or" are introduced in the unit on formal logic, and are used again in the study of compound probability. The curriculum's section on game theory provides a final stage to explore the profound implications of these two simple words and the struggle between "me" and "we." Conflicts exist between individuals, between groups, and between individuals and groups. Decisions that are good for individuals may be bad for the larger group, and vice versa. As a player in a game, or more importantly – as a member of society, it is necessary to balance opposing views and to make tradeoffs between individual and collective interests.

The opening remarks in the curriculum initiate a discussion on the work of Paulo Freire and the idea of *conscientizacao* (Freire, 1972). Critical consciousness demands that we understand and reconcile the difference between individualist and cooperative actions. Almost all social transactions have an element of the prisoner's dilemma, for example, consider the implications surrounding decisions in the context of our environment. Selfish inclinations explain why, despite knowledge of negative ramifications for planet earth and all its inhabitants, humans continue to destroy rainforests, overfish the seas, and emit too much carbon into the atmosphere. Benefits experienced individually or locally often interfere with a willingness to act ethically from a wider perspective. We will always have personal dilemmas that require careful thinking and decision making, the outcome of which will affect others. This curriculum offers students an opportunity to see that the character of a community is reflected by the tendencies of its members and whether they generally act with selfish or selfless motives.

### 3.11 Remarks: Threshold Concepts and a Curriculum for Social Justice

Content in this curriculum has been purposefully selected and developed to aid students in building a more sophisticated lens through which to view the world. The tools of math enable a profound perspective, and can provide an individual with astonishingly deep insight into a variety of issues. The curriculum explores many themes of social justice, including:

- the school to prison pipeline
- relationships among poverty, education, and crime
- racial profiling
- the myth of meritocracy
- capital punishment and the likelihood of false convictions
- the difference between equality and equity
- influence of the media and the dominant discourse
- the relationship between socioeconomic status and educational attainment
- the need to fund correctional education and reduce recidivism
- cultural bias in standardized intelligence tests
- stereotypes in society, including biological determinism
- misuse of probabilities in the courtroom

We cannot be content to live in a world of inequity; education should be a liberatory experience – one that creates awareness of the world's problems and subsequently inspires actions to remedy those problems. Knowledge is only useful when it helps to solve human problems. A liberal arts math program is not meant to be taught in isolation; it does not exist to merely satisfy a graduation requirement. This liberal arts math curriculum is designed to support student learning in other academic fields, and it treats mathematical reasoning as a set of transferrable skills.

The curriculum outlined in this paper emphasizes the construction of arguments and the interpretation of numbers; it is a search for knowledge using mathematical tools. Throughout the curriculum, significant attention is given to the reasoning processes that develop the following important habits of mind: (A) asking probing questions, (B) identifying underlying assumptions, and (C) reaching and supporting a conclusion. This curriculum succeeds when students become more mindful of the role played by their emotions while making decisions, when they habitually assess the reliability of their premises, and when they know to work within a logically valid framework. Individual lenses are already constructed prior to experiencing this curriculum, but the contextualized content in the course assists students in developing a more sophisticated lens – each topic sharpening their focus and helping them see with more depth and greater clarity.

The curriculum summarized in this paper presents a platform for students to develop an understanding of reliability and validity, through a formal study of mathematical logic and the principles of probability. It is a progressive effort to improve liberal arts mathematics and is imbued with relevant sociopolitical themes. Education is elemental in one's journey towards realizing his or her potential and becoming a productive member of society. By situating the content in the context of social justice, this liberal arts math curriculum contributes to the development of ethical, empathetic, and enlightened citizens guided by good reasoning. When students understand and internalize the central roles played by reliability and validity, the concepts become transformative, integrative, and irreversible – the defining features of a threshold concept. In the search for knowledge, an individual engages in deductive and inductive reasoning, while effectively operating in a world of uncertainty. The reliability of a premise is a matter of probability, the certainty of a conclusion is a matter of logic. Together, premises and structure comprise an argument, and the construction and/or deconstruction of an argument is the essence of mathematical reasoning. The curriculum also includes space for the subjective nature of knowledge and teaches students how to effectively incorporate personal values into their mathematical analyses. The curriculum is optimistically designed to provide a transformative experience – in students' perspectives, attitudes, and identities.

# CHAPTER FOUR

# A RETROSPECTIVE ANALYSIS OF CURRICULUM DEVELOPMENT

## 4.1 Motivation for Research

If the intended purpose of a college education is to develop critical thinking skills and prepare individuals for full participation in a democratic society (King, Brown, Lindsay, & VanHecke, 2007; Stanton, 1987), then the mathematics component of this process must provide a platform to develop these skills. Liberal arts math programs, including the subset of quantitative literacy courses, are not clearly defined as an academic field (Dingman & Madison, 2010), and the question of how to best align mathematics education with relevant, everyday needs remains unanswered (Steen, 2001). Meyer and Land (2003) advance the notion of a threshold concept as a fundamental building block within an academic field, one which opens paths to new ways of thinking. Given that educators in the field of liberal arts mathematics continue to debate the content and context of such courses, the education community would benefit from research that identifies threshold concepts for mathematical reasoning. An agreement about threshold concepts would aid in the rational development of curricula in rapidly expanding arenas where there is a strong tendency to overload the curriculum. In particular, these questions are worthy of attention:

- What are the central constructs of a social justice liberal arts mathematics curriculum, and how does a curriculum evolve in response to those constructs?

- How does focusing on threshold concepts of reliability and validity transform a social justice liberal arts mathematics curriculum?

Liberal arts math programs illustrate the use of mathematics as a tool for understanding and making sense of everyday situations in all aspects of one's life. Teaching for social justice has particular import for the field of liberal arts math because it looks at real life, yet is predicated on the idea of education as an emancipatory experience. Critical mathematics, as it is also called, implements a problem-posing pedagogy of critique, and situates the math content within sociopolitical themes. The two fields of liberal arts math and social justice are not mutually exclusive, yet there is a dearth of literature exploring their intersection. Research that pairs critical pedagogy with the cultivation of threshold concepts in mathematical reasoning may bring new perspective to the field of liberal arts mathematics.

**Mathematics Education for an Informed Citizenry**

Hacker (2016) and Steen (2001) argue that the traditional secondary and postsecondary math curricula essentially demand irrelevant training in decontextualized algorithmic processes and algebraic abstractions, rendering the content ineffective. The results of misguided and uninspiring curricula include high failure rates, math anxiety, student disengagement, and reduced capacity to effectively apply math tools for situations arising in life and work (Hacker, 2016; Steen, 2001). Steen writes that the mathematics taught in the classrooms should resemble the mathematics needed for everyday life, and that this should primarily consist of sophisticated reasoning with arithmetic. Paulos (1988) adds to the argument by popularizing the word *innumeracy* and defining it as "an inability to deal comfortably with the fundamental notions of number and chance" (p. 3). Paulos pushes for more training in probability, to prepare students in dealing with uncertainties in life. Hacker (2016) sees the need for greater emphasis on logic, writing that "one of the great goals of math is to be able to understand and construct proofs" (p.94). Logical thinking, evaluating evidence, and searching for truth, deserve increased academic attention, and he notes that "the quest for justice has much in common with mathematics" (p. 94). Paulos (1988) sums it up best when he writes, "Probability, like logic, is not just for mathematicians anymore. It permeates our lives" (p.178). Liberal arts students constitute the majority of U.S. college students, by extension, they represent the majority of U.S. college graduates. Accordingly, liberal arts math programs present an ideal opportunity to cultivate students' reasoning skills and develop a more informed American citizenry. The next two sections address both of these objectives – reasoning skills and informed citizenship, exploring the notion that these objectives may be supported by a curriculum that attends to threshold concepts and themes of social justice.

### *Critical Pedagogy and Democracy*

Advocates of critical pedagogy contend that education is fundamental to democracy; it is more than preparation for citizenship, it is also a form of political intervention that creates possibilities for social transformation (Giroux, 2011). "Educators must assume the responsibility for connecting their work to larger social issues" and to help students "learn the tools of democracy and how to make a difference in one's life as a social agent" (Giroux, 2011, p. 171). Lesser and Blake (2007) write that we should extend the "math needed for informed citizenship" into the realm of critical inquiry and pursue the math needed for citizenship as it relates to awareness of social injustices. Mathematics and democracy are a perfect pair, in that a critical math education can produce more "constructive, concerned, and reflective citizens" (Steen, 2001), but at a level even beyond what Steen envisioned.

There are countless opportunities to embed themes of social justice into a liberal arts math literacy curriculum; some well-known examples have been put forward by research practitioner, Rico Gutstein. He writes of students who engage in mathematical reasoning to investigate topics such as the unequal distribution of wealth, wasteful government spending, racial profiling, misinterpretations of medical diagnostic testing, the correlation between a family's level of income and the child's academic achievement, and capital punishment (Gutstein, 2005).

A liberal arts math course is an opportunity to cultivate a useful set of reasoning skills, in particular, probability and logic skills that aid in constructing a math lens through which to view the world with a more sophisticated perspective. A pedagogy of critical mathematics is fundamentally about increasing student awareness of sociopolitical issues. Liberal arts mathematics and critical mathematics share the same mission; and while one outlines the content, the other provides the context. Liberal arts math programs, although nationally inconsistent, generally claim to develop tools for informed citizenship – yet they typically do so with politically neutral data sets. Critical mathematics takes this a step further by challenging the notion that we are citizens in a just society. Critical mathematics asks how we can remedy societal inequities through educating citizens to be more empathetic and ethical, and by encouraging them to participate in positive social change.

There is a relationship between education and social change, and a critical pedagogy is necessary to bridge the gap between the two. Critical pedagogy stems from Freire's *Pedagogy of the Oppressed* (1972), and begins with a teacher accepting the subjectivity of knowledge. Frankenstein (1983) writes, "Freire's theory compels mathematics teachers to probe the nonpositivist meaning of mathematical knowledge, the importance of quantitative reasoning in the development of critical consciousness," (p. 9) and how "critical mathematics education can develop critical understanding and lead to critical action" (p. 11). If a progressive educator adheres to the principles of critical pedagogy, then he or she operates with the assumption that knowledge and power are subject to debate, and engages students with a politically charged pedagogy because it highlights the struggle for a more socially just world and directs students towards action. Such a pedagogy challenges the status quo, but the absence of this pedagogy and a curriculum void of social justice context serves to perpetuate the status quo and the current power structures, which indirectly also represents a political action. Thus, political intervention is unavoidable in education. Developing a sophisticated understanding of the injustices and harsh realities of the world is fundamental to reforming society, and the notion that awareness leads to activism is one of the central tenets of Freire's teachings. Frankenstein (2001) argues that we can use math "as a tool to interpret and challenge inequities in society" (p. 1). Through critical mathematics, students can deepen their understanding of real issues and be inspired to advocate for social justice.

***A New Perspective for Liberal Arts Mathematics***

Rather than acting as gatekeepers impeding students on their academic and career paths, secondary and postsecondary math courses can effectively serve as gateways to higher learning. However, the field of liberal arts mathematics is broad and inconsistent, and the mathematics content best suited for equipping citizens with the reasoning skills needed in their everyday personal and professional lives has not been agreed upon (Karaali et al., 2016). As such, it is important to identify and cultivate foundational concepts within the curriculum if the underlying intention is to build a useful set of mathematical reasoning skills. This retrospective analysis focuses on pursuing the notion of a threshold concept and developing specific content areas that can be used to shape the academic field.

The search for knowledge is analogous to the act of argumentation; individuals need to understand how to work with reliable premises in a logically valid framework. Logic is the backbone of all mathematical thinking; deductive and inductive reasoning, along with principles of probability, are central to mathematics and foundational for inferential statistics and learning to work with quantitative data – all of which are increasingly emphasized in other liberal arts fields (Cobb, 2005). Accordingly, this research examines the idea that mathematical logic and probability consist of particular threshold concepts that can define a new liberal arts math experience. A postsecondary math course assumes certain computational proficiencies. These competencies are necessary, but more important is an individual's ability with the written and oral expression of ideas – it is this capacity for communication that facilitates full participation in citizenship.

This research study investigates the development of threshold concepts in mathematical reasoning with a critical mathematics pedagogy. The retrospective analysis examines the practitioner's interventions with an experimental math program and the evolution of the curriculum. The research practitioner's field notes attend to the ways in which students experience and demonstrate their knowledge of logic and probability on written assessments and in the classroom, and the fields notes coincide with reflections on the efficacy of the curriculum. By contextualizing the content with themes of social justice and seeking to identify essential elements in a liberal arts math experience, this research offers insight into the fields of liberal arts math and critical mathematics, as well as addressing the dearth of literature on their intersection.

## 4.2 Methodology

### Research Design

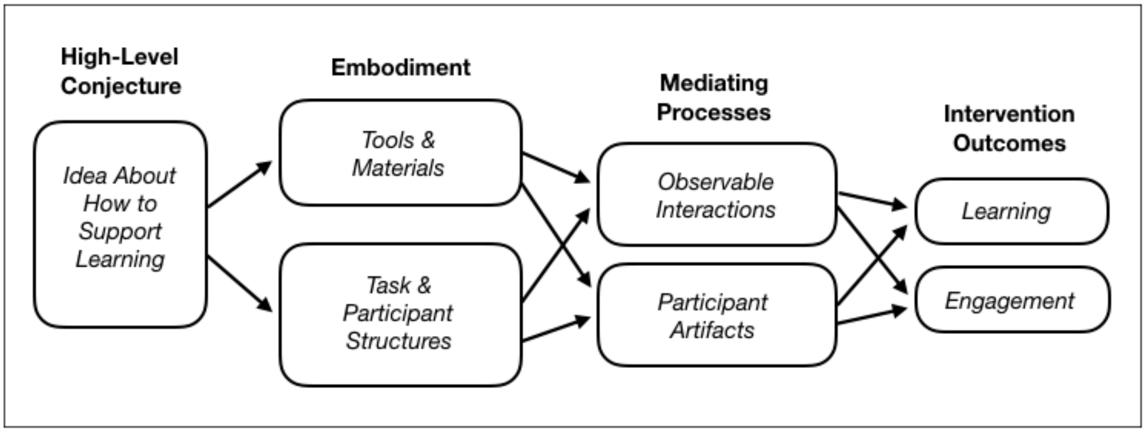
This research studies the development of a liberal arts math curriculum, founded upon the notion of threshold concepts, and imbued with themes of social justice. Data for this study had already been collected for non-research purposes, including multiple iterations of the curriculum, extensive notes on instructional interventions and observations in the classroom, and my reflections on student performances. I made a post hoc decision to engage in a retrospective record review, because I believed that my experience was worth analyzing and sharing with the academic community.

My research design is that of conjecture mapping – it is a methodological approach that describes how curricular and pedagogical experimentation can lead to theoretical refinement. As a research practitioner, I analyzed the efficacy of contextualized content as I continually redesigned the curriculum. Practitioner research, also known as action research, is a cyclical process of acting-observing-reflecting-changing-planning, and then acting again (Skovsmose & Borba, 2004). The defining feature of action research is the occurrence of change, i.e. an action, between the researcher and that which is being researched (Hitchcock & Hughes, 1995). In such cases, the research practitioner is interested in making improvements to a given situation through an active intervention. The practitioner is faced with the challenge of creating the change necessary to bridge the gap between the current situation and the imagined situation. As the practitioner, I imagined a curriculum that fostered the development of a student's mathematical reasoning skills and aided in the construction of a more sophisticated lens through which to view sociopolitical issues.

The method of conjecture mapping offers a guide to systematically test particular conjectures in the joint pursuit of practical improvement and theoretical refinement (Sandoval, 2014). Figure 1 illustrates the four stages of the cyclical process. It is important to clarify what constitutes a successful learning experience, and to seek connections that link observable processes with observable outcomes of instructional interventions. This research design is focused on the learners' growth in relation to the support provided by the practitioner and through the curriculum. I utilized this method for my retrospective analysis, to organize my empirical research and to support my claims about learning mechanisms within the curriculum.

Figure 3

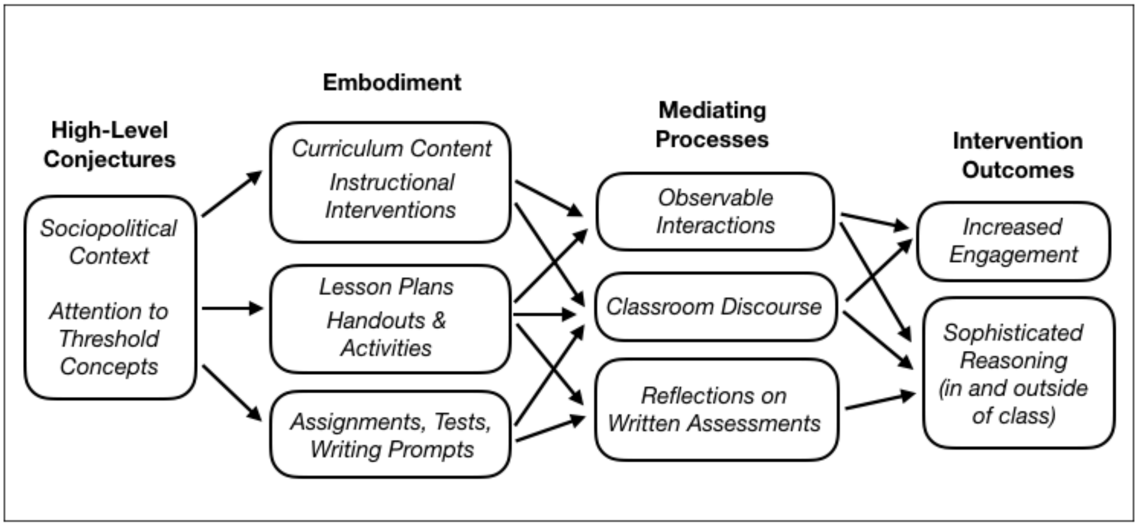
*Framework of Conjecture Mapping*




Conjecture mapping distinguishes between conjectures about how a design functions and how those functions produce learning. The process begins with a high-level conjecture that is not specific, rather, it is an overarching idea about how to support the kind of learning desired for the students (Sandoval, 2014). The initial high-level conjecture drives the design of the learning experience and is embodied by the lesson plans and assignments presented to the students. Students engage in mediating processes as they interact with the curriculum; these processes are revealed through observable interactions or the analysis of artifacts produced from the learning activities. My field journal contains my classroom observations and notes on classroom discourse, as well as my interpretations of students' performances on written assessments. The intention of conjecture mapping is to identify the outcomes of an instructional intervention and to be able to connect it to these mediating processes. My conjecture maps are explained in the three phases of my retrospective analysis, beginning on page 114. The figure below offers a generalized overview, summarizing the components of my three conjecture maps.

Figure 4

*Conjecture Mapping, Summary of Three Phases*




My field notes are aligned with chronicled changes in the curriculum, the combination of which enabled me to identify key moments in the development. This liberal arts math curriculum evolved through three professional phases, hence my research design consists of three conjecture maps. The map in Figure 4 consolidates all three professional phases; detailed maps for each phase are shown in my results section. The retrospective analysis spans the development of the curriculum over five years with approximately two hundred students. The confluence of student feedback, my personal reflections as both a practitioner and a graduate student, contributed to this long process of development. The practice of conjecture mapping promotes the notion that students can participate in the co-construction of a curriculum through the implicit feedback of their engagement levels and performances on tasks, as well as their explicit contributions towards locally generating themes for the mathematics content.

The interpretation of data is inherently subjective, and a research practitioner must be mindful of a natural susceptibility to bias. The curriculum in this study was presented to learners at both the high school and college levels. I knew that a retrospective analysis would need a narrative of my interaction with students in the classroom, so I carefully noted my observations and discussions with students. I understand that confirmation bias is a natural tendency that appears when researchers test hypotheses and filter information. Accordingly, I have gathered data from multiple sources, including my field notes from within the classroom, personal reflections about my instructional interventions, and multiple versions of the curriculum as it evolved over the years. To minimize confirmation bias in my study, I took great care in coding the various data sources and continually reevaluated assumptions I had about the data.

**Context of Study**

There are fundamental contradictions between a typical school's politically correct atmosphere inside the classroom, and the reality of life outside the classroom for many of its students. This research explored the idea that if we wish to prepare students to become fully-informed members of society, then we should consider going beyond the conservative nature of traditional curricula and address sociopolitical themes. The research was motivated by my personal interest in creating a progressive liberal arts mathematics curriculum. I was frustrated with a decade's worth of teaching experience in traditional mathematics curricula that correlate with high levels of student disengagement, and I was disheartened by the inequities facing large segments of my student population. This study began at the time I reflected on the need for curricular change and became determined to create learning opportunities that would equip my students with more powerful reasoning skills, improve their quantitative literacy, and inspire them to become agents of social change.

My experience with curriculum development coincided with the start of a new concurrent enrollment program for our high school students with the local community college. We offered a liberal arts math course titled Quantitative Literacy, aimed at students whose college interests lay outside the STEM fields. For me, the program presented an opportunity to challenge my students' misconceptions of mathematics. The first three years of this retrospective analysis address my curriculum development at the high school level. In this stage, I introduced some new content and focused on framing it in more relevant context to appeal to my classroom demographics – primarily students of working-class Hispanic families. The following two years, I implemented the evolving curriculum in a college-level liberal arts program within correctional facilities. For many of those students, it represented their last, and only, postsecondary mathematics course. Students in that program were grateful for the educational opportunity to improve the quality of their lives. As a result, they typically displayed high levels of engagement, motivation, and work ethic. Taken together, their readiness to learn and my desire to create an impactful course, created the ideal setting in which to experiment with a progressive curriculum. Rather than reviewing a traditional set of algorithmic math skills, I carried out a series of instructional interventions to target the development of mathematical reasoning. As I worked with my students during this span of five years, I attended to their feedback, their demonstrations of understanding, and their ability to use mathematical tools to gain deeper insight into sociopolitical issues highlighted in the course. My retrospective analysis examined changes in the curriculum over that time; I sought to discover why my initial emphasis on social justice context later evolved into an effort to find and develop the threshold concepts necessary for mathematical reasoning.

My high school students were juniors and seniors, and my students in the college program were men and women, ranging in age from 22 to 60 years old, and all ethnicities were represented. At the high school, I had 180 days to develop and implement the curriculum. In the college program, students were enrolled in the course for a full fifteen-week semester, and attended classes that met once per week, for two-hour sessions. The average class size in high school was eighteen students, and thirteen for the college program. None of the students were paid for their participation in the study, nor did they incur any cost. The retrospective analysis did not target students as subjects, rather it examined the research practitioner's reflections on the efficacy of instructional interventions and the continual development of the curriculum.

**Development of the Curriculum**

The retrospective analysis spans five years and is characterized by three professional phases. Each phase corresponds with a new cycle of the four components in conjecture mapping. My efforts to reform the curriculum began when I was working as a public high school math teacher in Central New Jersey while pursuing my doctoral studies. My original intentions were to increase student engagement and to provide a more relevant mathematics education to my students. Curricular changes emerged during the first phase when I learned about the social justice math movement in my graduate studies, integrated discussions of equity into the curriculum, and began to reflect on my classroom observations and interpretations of student performances.

My curriculum development during the first phase was guided by the teachings of Paulo Freire and Henry Giroux, and was strongly influenced by the work of well-known social justice educators, including Rico Gutstein and Marilyn Frankenstein. My participation at the conferences *Creating Balance in an Unjust World* and *Mathematics Education and Society*, led me to further pursue the idea of situating mathematics content in the context of sociopolitical themes. Given that I was working in a primarily Hispanic community whose members sustained a low socioeconomic status, I focused on topics such as income inequality and economic hardships experienced by minority populations. My intention was to help students construct a lens using mathematical tools that would help them gain deeper insight into issues that affected their lives. To achieve this, I realized that I would need to reform the existing curriculum – in both content and context. This initial phase is characterized by my quest to find useful applications of math tools, and experimentation with a variety of contextual settings and data sets that I thought would capture my students' interest. While my students' engagement increased, there were many indicators that the high school students were not comprehending matters on a sophisticated level. During this time, the first iteration of my curriculum was created, along with an opportunity to join the faculty of a progressive liberal arts college program.

When I brought my evolving curriculum into the college program, I intensified the contexts used – including racial profiling, the school to prison pipeline, recidivism, and the criminal justice system. During this second phase of curriculum development I realized again that my students were lacking fundamental reasoning skills. Although many were computationally proficient, they struggled to analyze sociopolitical issues because they did not understand basic principles of argumentation. Taking a position on a subject, expressing one's self, and challenging the assertions of others, all incorporate elements of an argument. In an argument, one must clearly state premises and justify a position with a line of reasoning. The reliability of the initial premises is more deeply understood through the lens of probability, and the act of justification is a matter of working within a logically valid framework. The realization that core concepts in mathematical reasoning are integral to argumentation came to me as the research practitioner, in the middle of a classroom lecture when I heard myself summarizing the semester course out loud.

I was affiliated with a college program that encouraged themes of social justice, and I knew that Andragogical theory asserts the importance of an immediately relevant curriculum (Knowles and Swanson, 2001). Together, these two forces motivated the second phase of my curriculum development. My students had limited access to educational resources, accordingly, I felt compelled to create and provide them with significant amounts of engaging content and enrichment material. As a research practitioner, I noted the classroom discourse and indicators of student growth. This phase of curriculum development underwent deepened context, a restructuring of content, and rigorous assessments that challenged and frustrated my students. The end of this phase was marked by the last day of the fall 2016 semester when I began to understand the critical role of threshold concepts.

The third phase of my curriculum development was a commitment to the concepts of validity and reliability – which I identified as threshold concepts within mathematical reasoning. I deliberately targeted key principles in logic and probability, while continually linking them to argumentation and the search for knowledge. A liberal arts math course can be more than superficial applications of mathematical tools, even in the context of social justice. It is an opportunity to construct a set of sophisticated reasoning skills that are useful across disciplines. To produce an informed citizenry, a liberal arts math course can be designed to help students search for knowledge as they navigate a world where certainty and uncertainty constantly intersect. My curriculum was remodeled during this third phase – most of the curriculum was preserved, but the presentation of the content and context was reframed to deliberately cultivate the threshold concepts. Threshold concepts take time to develop, as such, I worked to reinforce them throughout the curriculum. By explicitly highlighting the role of validity and reliability in mathematical thought processes, I aimed to build a more sophisticated set of reasoning skills for students – skills that are integrative, transformative, and irreversible. The timeline for this retrospective analysis ended with my reflections surrounding the curriculum used in the 2017 spring semester.

During this evolutionary process, my original vision of a more engaging and relevant liberal arts math curriculum changed because I realized that students lacked an understanding of fundamental threshold concepts. I shifted my focus from context to content, and sought to identify and emphasize the threshold concepts of mathematical reasoning. This retrospective analysis is significant because it provides insight into the process of developing a progressive curriculum that aids students in constructing a more critical worldview.

## Data Sources and Data Analysis

### *Data Sources*

For the first three years of this study, my retrospective data consists of extensive field notes and early renditions of a new curriculum from when I taught at a public high school. During this phase, I took advantage of the 180-day length school year to experiment with lesson plans, content and context, and student assignments. This was a time of personal growth as a practitioner; I learned about social justice and had the professional freedom to explore new academic ideas in my classroom. My field notes exist as personal reflections, handwritten directly on top of my lesson plans and assessments. At this stage, I was primarily focused on increasing student engagement through more relevant context. Consequently, my field notes recorded students' comments, described their interest levels in the lessons and assignments, and listed my thoughts on items that could be improved or removed. My lesson plans were typed, and my reflection notes were written either during or immediately after the class. After each topic, I contemplated changes and typed new lessons plans for the future. As a result, I can refer back to those three years and see the evolution of particular pieces in the curriculum.

Data for the second and third phases of this retrospective analysis correspond with my time in a progressive college program initiative within correctional facilities. I continued to maintain an extensive collection of field notes comprised of my classroom observations and informal conversations with students, as well as my personal reflections on the curriculum development. I documented student engagement, students' competence in applying critical perspectives, indications of their confusion with the content, and my thoughts on using different examples to clarify my lesson plans over three semesters, from January 2016 through May 2017. During this time, I worked with approximately one hundred thirty students; many of whom were minorities, and some were women. For this study, I examined particular assignments that gave insight into the evolution of the curriculum by reflecting on students' interaction with the content in a sociopolitical context and their overall development of mathematical reasoning skills. The data offered evidence towards my understanding of students' capabilities with the threshold concepts of validity and reliability, and in navigating themes of social justice. Special attention was given to assignments that I thought would be impactful and were not, as well as assignments in which students demonstrated their understanding of threshold concepts. In particular, I analyzed my reflections on these student assessments:

- a unit test on mathematical logic, entirely situated within the context of the American criminal justice system
- a cumulative exam that required students to synthesize their understanding of conditional probability with deductive reasoning in order to build an argument surrounding the relationship between correctional education and recidivism
- assignments in propositional logic that asked students to translate and analyze sociopolitical arguments propounded in the dominant discourse of mass media
- open-ended tasks with conditional probability, exploring issues such as racial profiling, the myth of meritocracy, and the relationship between one's level of educational attainment and his or her socioeconomic status
- a pair of pre- and post-tests that bracketed a fifteen-week semester, measuring students' comprehension of key concepts – including compound and conditional probability, expected values, logical validity, inductive reasoning, the subjectivity of knowledge, and the relationship between logic and probability

Overall, my data for this retrospective analysis includes multiple iterations of the curriculum, a detailed record of all instructional interventions, reflections on student performances, observations of classroom discourse, and my field notes about students' mediating processes as they engaged with the embodiments of the various high-level conjectures. This data informed the research questions by providing insight into how and why my development of the curriculum evolved through the process of conjecture mapping.

***Data Analysis***

I personally designed the curriculum, classroom lectures, handouts, assignments, and assessments for all students. My research was a retrospective analysis on the evolution and efficacy of this curriculum through continual practitioner experimentation, instructional interventions, and personal reflections. I searched for indicators that the curriculum was or was not achieving its intended goals and targeted my reflections surrounding student performances that exemplified the impact and/or absence of attending to threshold concepts. In addition, I studied my notes about my students' reasoning skills before and after key learning experiences within the curriculum, and I examined evidence that chronicled changes in my approach to the development of the curriculum.

This analysis utilized a coding scheme that was motivated by the intention of the curriculum, that is, to develop students' mathematical reasoning skills for the purpose of building a more sophisticated lens through which to view the world. In accordance with this overarching objective, my research sought to identify key moments of practitioner reflection, highlighting realizations and the factors that led to significant changes in the curriculum. My coding scheme was designed to categorize my field notes relating to students' computational proficiencies, as well as their growth of higher-order thinking skills by identifying pieces of the curriculum that enabled them to engage in acts of analysis and evaluation. My research design efficiently paired with my use of the coding scheme by facilitating my assortment of the data into the different phases of the conjecture map. This organizational approach allowed me to more effectively check the degree to which the conjecture map fit the data. Specifically, in my coding scheme, I searched through my field notes for reflections on instructional interventions that promoted procedural proficiency with the mathematics content (P), assignments and classroom discourse which facilitated and revealed that students were making connections across curricular ideas (C), and notes surrounding students' capacity to work with the structure of an argument (A), that is, the ability to use mathematical reasoning tools to take and support a position. Furthermore, given my objective of developing an informed citizenry with a critical worldview, I included two codes pertaining to social justice. I used the code (SE) when I noted that students effectively applied mathematical tools to demonstrate more sophisticated awareness of a sociopolitical issue. And, I used the code (SI) when student performances revealed an inability, or an ineffectiveness, in using the content for insight into a sociopolitical issue. In summary, I explored my data pieces and attached the following codes to categorize my various types of reflections.

LP the curriculum piece and/or instructional intervention corresponded with demonstrations of low proficiency with threshold concepts; student performances exemplified a lack of procedural knowledge

P proficiency with computational procedures, relating to threshold concepts

SI students were ineffective in utilizing math content for sociopolitical insight

SE student work showed an effective application of mathematical reasoning tools to demonstrate more sophisticated insight into sociopolitical issues

C connections across content and context; students demonstrated awareness of inter-relationships within the curriculum

A the curriculum piece and/or instructional intervention provided students with the opportunity to make a valid argument; students took a position on an issue and used mathematical reasoning tools to provide justification

As the practitioner, I was already familiar with my students' interactions with the curriculum; as the researcher, I categorized my field notes and reflections by their codes and analyzed the development of key curricular ideas. I reviewed the analysis for internal consistency by revisiting it multiple times over the span of several months, each time ignoring student identities and reading the data from different chronological orders. I searched for the collective meaning in all of my notes and reflections. Additionally, my retrospective analysis accounted for external academic influences, such as my graduate studies and my participation at math education conferences, by revisiting my notes from those sources and considering their impact on related curriculum developments. Overall, I reflected on the confluence of pedagogical intentions and my interpretations of student growth, as I sought to understand the fundamental elements of a liberal arts math curriculum. Finally, I went back through the data and looked for disconfirming evidence to further refine those themes.

The research design of conjecture mapping facilitated my retrospective analysis. Initially, my attention was on the design conjectures, for practical purposes as the practitioner. When I shifted roles to a researcher's perspective, I focused more on the theoretical conjectures and analyzed my notes about their mediating processes. My study looked at performance levels, that is, their ability to construct sophisticated arguments in a social justice context, as intervention outcomes. Guided by my research questions, I focused on evidence that relate to the identification of central constructs in a social justice liberal arts mathematics curriculum, and the impact of threshold concepts in transforming the curriculum.

### 4.3 Results: Retrospection on Three Phases of Curriculum Development

The results of my retrospection are explicated with a separate section for each phase. Each section explains a motivating rationale for the high-level conjecture, and includes the corresponding embodiment, mediating processes, and intervention outcomes that unfolded through the experience. A visual representation of the conjecture map for the phase is presented at the end of each section. The significance of social justice context and the targeting of fundamental reasoning skills highlight the overarching themes that cut across this analysis.

#### Phase One

The process of curriculum development effectively began when I started the concurrent enrollment program for our high school students with the local community college. Up until then, their traditional algebra-laden high school math experience had fostered disengagement, frequently prompting students to ask the familiar question: "When will we ever use this stuff in real life?" In contrast, the new liberal arts course featured problem-solving techniques, inductive and deductive reasoning, ratings and rankings, indexes, mathematical modeling, and basic concepts of probability and statistics. Quantitative literacy courses are generally designed to develop one's proficiency in working with numerical data; such courses are founded upon the notion that being able to read, write, and reason with quantitative information is an integral part of informed citizenship. At this stage in my teaching career, I viewed quantitative literacy as important, but characteristically vague. My goal was to provide students with a meaningful and relevant math education experience, and I embraced the opportunity to share a new curriculum with students. Driven by a sense of civic rationale, I recorded a question in my journal that would serve as a motivating force for professional growth: "How can we better understand the world and its people, through quantitative literacy?"

#### *High-Level Conjecture*

At the same time that I was building our school's concurrent enrollment program, I was also immersed in my graduate studies and discovered the social justice mathematics movement. The work of Freire, Giroux, Frankenstein, and Gutstein inspired me to integrate sociopolitical themes into my classroom and encourage students to become agents of social change. The confluence of forces surrounding my early implementation of the liberal arts math course led to my first conjecture map, built around the following high-order conjecture. Conjecture #1: Situating the mathematics content in the context of themes of social justice will positively impact learning by increasing student engagement and facilitating students' ability to make connections across the curriculum. During this time, my growing familiarity with the course content gave me the confidence to create progressive lesson plans, reform the course material, and work towards designing a more impactful learning experience for my students.

***Embodiment of 1st High-Level Conjecture***

The community college's version of the course contained student tasks that were conservative in nature, deliberately disconnected from real issues. Here are two such examples of the college's original course material.

> You are shopping for Thing-a-ma-Bobs for gifts. To pursue this venture in an efficient manner, you have come up with a table of characteristics of Thing-a-ma-Bobs that you have rated, and weights for each characteristic. This will enable you to use the weighted sum method of arrive at a decision about the ideal brand to purchase.
>
> A small company has been producing widgets and has experienced great success. Here is a table showing widgets profits… This data exhibits exponential growth. Calculate the ratios which verify that the growth shown in the data is exponential. Using this ratio, predict the widgets profit for the year 2010.

Another problem assessed comprehension of conditional probability by describing a distribution of different sizes and colors of gumballs, then asked students to find the probability of a red gumball, given that it was one of the larger sizes. I observed student affect to be characteristically disappointed with these "college-level" math problems. Widgets, Thing-a-ma-Bobs, and gumballs were simply not engaging contexts. What I noted from students' responses led me to write "application of content has great potential, but context needs to be more engaging" in my field journal. It was my conviction that tasks presented in such decontextualized or conservative settings would not contribute towards developing informed citizens with sophisticated reasoning skills, nor would it sustain student engagement. The realization of this led me to revise the curriculum with more provocative context. In response to the college's offerings, I built tasks (see

Appendix C and Appendix D) for the same content yet situated them in themes that would connect the application of mathematical tools to larger social issues.

> The American Psychological Association conducted a national survey to learn what are considered to be the most significant sources of stress for Americans. The most common responses were related to money (including job stability), family relationships, health, and personal safety. Create a spreadsheet to rank the following cities based on the criteria listed below. You will need to do a little research to get current numbers for use with your weighted sum method. Note, this method can be used to make statements about the best of something or the worst of something; keep in mind how much power is invested in the person who selects the categories and assigns the weights to each category. In arriving at a conclusion, the data that you use is just as important as how you use it. Provide reasons for the weights you assign and explain how you establish the rating scale within each category. Rank the outcomes of your weighted sum and be prepared to present your findings to the class.
>
> In 1973, the University of California – Berkeley was sued for bias against women who had applied for admission into its graduate schools. The admission figures for the fall of 1973 showed that men applying were more likely than women to be admitted, and the difference was so large that it was unlikely to be due to chance. Compute the joint and marginal distributions for the table above. Determine the conditional probabilities that would be helpful in building a legal case for gender bias against the university. In a surprising twist, officials from the university brought more detailed data into the courtroom, and revealed the admissions records from the six largest academic departments. (The university has over 60 different academic departments within the graduate schools). Use the real data provided in these tables to reach a strong conclusion. If you were the judge, what would be your verdict? Defend your position with the appropriate mathematical calculations.

Changes in the contexts had the desired effect of increasing engagement – which manifested through lively class discussions. I noted that students verbalized lots of opinions on the social topics and I was easily able to solicit ideas from them about more applications for the mathematics of ratings and rankings, however, they did not elaborate with their written responses.

***Mediating Processes***

Students' written work during this phase was typically very short in length. Reflecting on my field journal, I noted that their submissions demonstrated proficiency with computational aspects, yet they neglected to support their conclusions with reasons or personal commentary. For example, the majority of students correctly calculated numbers with the weighted sum method for ratings and rankings, and they were able to successfully compute conditional probabilities with the provided data sets, yet they did not articulate conclusions which relayed their individual outlook on the matter. That is to say, I did not see students explicitly make connections between the use of mathematics and their own lives. It was clear to me that students had been conditioned in prior math courses to believe that math solutions ended with a simple numerical answer, and that it was unnecessary to elaborate on a response with an interpretation of its meaning. A common response would be, "But that's the final answer. It's right, right?" I understood this shortcoming to be a function of the students not knowing how to elaborate, rather than refusing to do so. This habit had been reinforced by assignments that neither provided context nor prompted meaningful applications of the content. For example, consider the following exam question, produced by the community college's math department: Set up and complete the truth table for: not (p or q) and (if q then p).

I captured my own frustration in my field notes; next to the above exam question I wrote, "What is the point of this?!" At this stage, I adopted the belief that context gives knowledge its meaning; in my mind, the task above exemplified the meaningless manipulation of symbols that perpetuated students' perceptions about the irrelevance of mathematics. Such exercises deter student discussions, thereby inhibiting the mediating processes and negatively impacting instructional outcomes. My observations of students during these early stages of the concurrent enrollment program made me realize that not only did students need more training with propositional logic because it was a new topic for them, but they also needed guidance in connecting and applying the tools of deductive reasoning to real issues. That is, they needed much more experience with contextualized content. I added in my journal, "Need context for propositional logic exercises!" I was growing increasingly aware of the importance of contextualizing the mathematics – especially with content that was unfamiliar to students, was abstract, and had high potential for relevant applications. Upon reflection, through my retrospective analysis, I noted that merely adding context would not suffice, and that I would also need to carefully scaffold the content.

***Intervention Outcomes***

A successful learning experience would have been characterized by engaged students, thoughtfully applying mathematical tools for greater insight into a real-life issue. My high school students showed more interest in the content – they appreciated the opportunity to participate in a concurrent enrollment program and they were glad to be doing mathematics that was different from a traditional algebra course. I was confident in my assessment of their increased interest because of my classroom observations and the improved submission rate of homework assignments – from 60% in prior courses, to approximately 95% in the new program. However, despite my attempts to inspire them with themes of social justice, they were not outwardly moved by the topics I offered for context. I designed tasks that referenced the unequal distribution of wealth, access to health care, and racial profiling. In particular, I presented a conditional probability exercise that referenced the riots in Ferguson, Missouri. In my field journal I wrote, “Students give feedback that yes, these things seem important in the news, but they don’t feel strongly connected to the issues (re: Ferguson).” Despite embedding social justice contexts within problems, the students clearly informed me that they did not feel directly connected to these issues.

I reflected on students’ interest levels – implicit in their classroom discussions and the effort they put into assignments, and I realized that I should solicit their thoughts on themes for the course. I asked them, “If I could take these math tools and use them to study anything in the world, which topics would you like to explore?” Given that they were high school seniors, it was not surprising that their responses were overwhelmingly about college. As a result, I reframed some lesson plans and tasks with themes about college rankings, college affordability, and the long-term financial benefits of getting a college degree. The following excerpts are taken from assignments that incorporate mathematical modeling and fluency with the consumer price index. The full assignments are included as Appendix E and F.

> Listed below are the average costs for one year of college tuition in America during the years 1980, 1994, and 2008. Adjust the costs of tuition to reflect real 2012 dollar values, and calculate how much the real cost of college tuition has changed over the years.
>
> The average salary of a college graduate is $45,000 per year, compared to $30,000 for someone without a college degree. Assume that the high school graduate receives a 2% raise per year, and the college graduate is given a 3% raise each year. Construct a graph to compare the accumulated earnings of a high school versus college graduate during a span of twenty years.

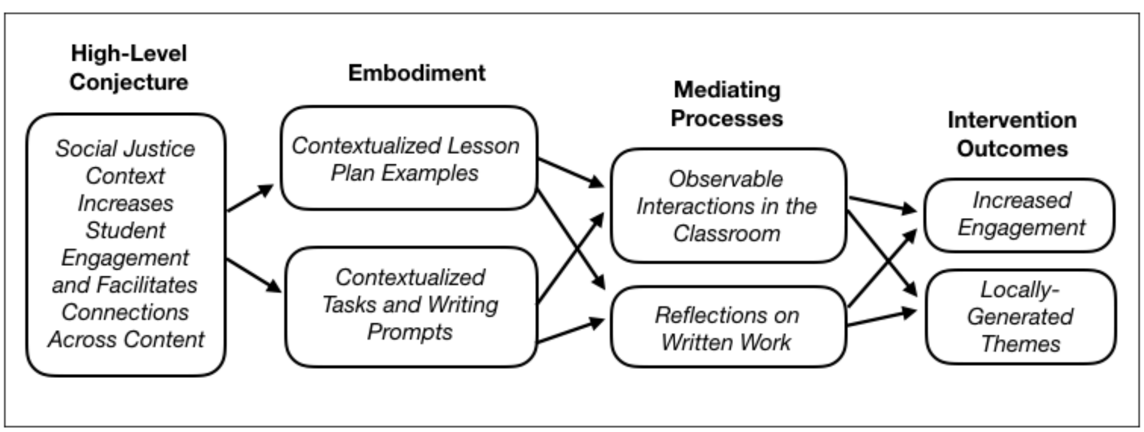
The students’ enthusiasm for exercises like this affirmed my belief in the power of locally-generated curriculum. I reflected in my field journal that the students responded well to the assignments that they inspired; I wrote “Students love college themes for assignments, less interested in politics.” This realization was one of the most significant outcomes from phase one. By collaborating with them on ideas for contextual themes, I opened pathways that would allow them and future students to participate in the co-construction of the curriculum.

***Reflections on 1st Conjecture Map***

During this first phase, students fully participated with the curriculum – in that they completed all the tasks. However, they were not inspired in the beginning because the contexts did not incorporate enough issues of personal importance. While I was able to increase engagement, the students typically gave short responses where longer ones were needed. Students answered questions, but they did not articulate their own position on matters. It was clear to me that any liberal arts mathematics curriculum was going to fall short in its mission to cultivate informed citizens if it did not facilitate students' capabilities in connecting content to areas outside the classroom. Recognition of this led me to build a new conjecture map with a different kind of instructional outcome for my next phase of curriculum development. Figure 5 summarizes the conjecture map for my first phase.

Figure 5

*Phase One Conjecture Mapping*



As I move into the second phase, there are two big things you will see, (A) more attention to the mechanics of propositional logic, and (B) efforts to help students become proficient in translating real-life situations into mathematical terms. I entered phase two with a high-level conjecture about the importance of pairing the tools of formal logic with more engaging contexts, and my emphasis shifted from quantitative literacy to mathematical reasoning. Admittedly, I did not fully understand the nature of mathematical reasoning from a pedagogical standpoint because I had not yet discovered the role of threshold concepts. Nonetheless, I expanded my efforts to not only contextualize the math content, but to also work towards directly teaching students about mathematical thinking.

## Phase Two

Relevant context is important, and it is a hallmark of quantitative literacy courses. However, I learned during the first phase that merely situating content in context does not produce an informed citizenry. My original motivation was evolving from an initially vague ambition to "better understand the world and its people, through quantitative literacy" to now in phase two, wanting my students to be able to engage in sophisticated reasoning about sociopolitical issues. This overarching objective had not been clear to me at the outset of my curriculum project in the beginning of phase one. Accordingly, my process of curriculum development was ready to test a new high-level conjecture for phase two.

### *High-Level Conjecture*

The start of this phase coincided with my new job as an instructor with a progressive college program inside correctional facilities. Given the lived experiences and present environment of my new student population, I intensified the context and explored more sociopolitical issues – including the criminal justice system and themes that characterize life in urban areas. For example, I assigned a task with the weighted sum method to rank a sample of cities, according to the categories that are often implicated in the school-to-prison pipeline (see Appendix G). I knew that I would need to operate in alignment with Andragogical theory, which asserts that adult learners demand immediate relevancy. My new student population, consisted primarily of students of color, and was a population that had struggled with traditional schooling and would likely benefit from an alternative approach. Additionally, due to the restricted setting inside, my students had limited access to academic resources. These factors compelled me to create a lot of new course material for the students, motivated by the following high-level conjecture. Conjecture #2: A formal study of propositional logic, contextualized with themes of social justice, will guide students towards constructing a more critical worldview. I anticipated that student growth in this area, that is, their ability to engage and articulate sophisticated reasoning about sociopolitical issues, would be revealed through classroom discussions and the work they submitted on written assignments. My efforts during this phase emphasized reasoning processes over computational proficiencies.

***Embodiment of 2nd High-Level Conjecture***

At this stage, the curriculum still contained a unit on elementary data analysis. The students worked with basic concepts in descriptive statistics, visual representations of data, and bivariate relationships. The two tasks below exemplify the types of problems

I created for this unit. For the full assignments and data sets, see Appendix G and Appendix H.

> The following data was accessed from the International Centre for Prison Studies, 2016. The table presents the incarceration rates (per 100,000 citizens), for sixty-five nations. Use the "Five Number Summary" of the data to produce a box-and-whisker plot. Perform a test for outliers, and clearly indicate their presence, if any, on the plot. Make a histogram of the data set, using exactly eight class intervals. Identify the class boundaries and make a frequency distribution. Compute the standard deviation for the eight countries listed in Central America.
>
> Utilize the "School-to-Prison Pipeline" data set; draw scatter plots and create linear regression models to analyze the relationship between these pairs of variables:
> - high school suspension rates and violent crime rates
> - high school graduation rates and unemployment rates
> - percent living in poverty and property crime rates

These problems stand in stark contrast to the tasks presented by the community college's math department that situated the same content in the context of cholesterol levels and the fuel efficiencies of faculty members' cars. But they also stand in stark contrast to the contexts I developed early in phase one that were not specific to students' interests. The majority of students demonstrated proficiency with the computational aspects of my assignments, yet they continued to struggle when I asked them to articulate thoughtful conclusions. I began to doubt the value of introducing statistics content into the course. The students' struggles produced a reflective note in my field journal; I asked myself, "What is the purpose of developing these data analysis tools in context now, if in the end, students are not fluent with the reasoning processes needed to reach and express a conclusion?" Despite its inclusion in many quantitative literacy programs, my professional experience as a statistics instructor told me there would be a more suitable time for students to learn that particular content in a future course. Coincidentally, at the same time, I was also teaching an advanced course on data analysis inside the correctional facilities. My students in that class had difficulty making inferences and clearly lacked formal training with inductive and deductive reasoning processes. My awareness of this deficiency reinforced my decision to focus on reasoning processes in my evolving liberal arts mathematics course.

I realized through student interactions that it would be more beneficial to shift attention away from quantitative literacy, and instead towards mathematical reasoning. However, as a research practitioner, I was still trying to understand exactly what that meant. A pivotal moment occurred when the students and I began using the word "argument" while discussing a problem about poverty rates and crime rates. I thought that formal training with truth tables would have been sufficient, but what I saw was the need to approach mathematical reasoning through the framework of propositional logic. The curriculum changed in that moment. I introduced the concept of a valid argument as one in which the truth of a conclusion would be guaranteed by the truth of the premises. Furthermore, I integrated themes from the criminal justice system to illustrate key ideas, as shown in this lesson plan excerpt.

> Assume as a fact, that in a court of law, the strongest evidence is eyewitness testimony. We could therefore make the argument that if an eyewitness saw you commit the crime, then you would be convicted of the charge. The letters P and Q are used to symbolize statements so that we can employ the tools of propositional logic.
>
> P: An eyewitness testifies against you. Q: You are convicted of the charge.
>
> The assumption that P leads to Q, is a conditional statement, and is used a basis for the argument. Thus, $P \rightarrow Q$ is our first premise. The second premise, used as a fact in this argument, is that P has occurred.
>
> If $P \rightarrow Q$ is true, and P is true, then Q must be true. The valid framework of a direct argument guarantees the truth of a conclusion, if the premises are true. Symbolically, the entire argument is represented as: $[(P \rightarrow Q) \wedge P] \rightarrow Q$

The figure below illustrates a simplified truth table, highlighting the instance when both premises are true, and showing that true premise correspond with a true conclusion.

Figure 6

*Direct Argument, Simplified*

| | | Premise #1 | Premise #2 | Conclusion |
|---|---|---|---|---|
| **P** | **Q** | **P → Q** | **P** | **Q** |
| T | T | T | T | T |
| T | *F* | *F* | T | *F* |
| *F* | T | T | *F* | T |
| *F* | *F* | T | *F* | *F* |

The next figure presents a more complex truth table for the argument, showing the correct application of logical connectors. This argument is a tautology, giving students confidence in the conclusion reached through this line of reasoning.

Figure 7

*Direct Argument, Tautology*

| **P** | **Q** | **P→Q** | **(P→Q) ∧ P** | **[(P→Q) ∧ P] → Q** |
|---|---|---|---|---|
| T | T | T | T | T |
| T | *F* | *F* | *F* | T |
| *F* | T | T | *F* | T |
| *F* | *F* | T | *F* | T |

Some students questioned whether to accept P→Q as always true, 100% of the time. Their thoughts surrounding the consistent truth of this first premise here, and with other lesson examples, allowed me to plant seeds for upcoming discussions about the reliability of premises. My handwritten notes after that lesson, state "During probability unit, refer back to students' comments about premises being 100% reliable!" I realized that an important connection could be made between logic and probability, and that an argument exists within the intersection of these two domains.

When I linked together the tools of propositional logic with the principles of argumentation, students gained a new appreciation for the mechanics of truth tables. Students began to demonstrate changes in their perspectives by altering their language in classroom discussions, paying careful attention to their use of logical connectors, and the sequence in which they expressed their thoughts. This was a key development; my thoughts surrounding the potential impact of mathematical reasoning across the liberal arts suddenly changed. I realized that for an individual to have an informed view on a critical social issue, or for that matter – any issue of personal importance, it would benefit that person to understand how to think within the framework of an argument. As a result of this development, I created assignments which I titled "Sophisticated Arguments." Students were asked to read through an issue, take a position, and build a valid argument in support of that position. The following is an example of such a task:

> An individual that has a college education is likely to have more job opportunities than someone who does not have a college education. More job opportunities mean more financial security for those highly educated people. Altogether, a community with many educated residents is less likely to have poverty, and less poverty means less crime. Society should invest in the education of all its citizens, including the incarcerated, because everyone benefits from living in a community of educated individuals. However, some people argue that increases in crime rates are simply the results of implementing stricter laws. What if there was no poverty and what if the laws were not strict? Are crime rates an economic or legal issue? Provide your interpretation of the relationship between education, poverty, strict laws, and the incidence of crime within a community. Identify appropriate statements for P, Q, R, and S, and express an argument using symbolic logic.

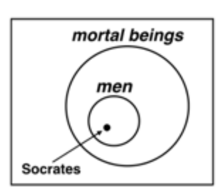
This task differs greatly from all previous logic exercises that I had presented to students because of its deep context and degree of difficulty. In addition, this marks the first time I challenged students with an open-ended task in logic that included several debatable questions. Up to this point, students' prior work had included propositional logic and themes of social justice, but in an algorithmic manner. In contrast, this set of sophisticated arguments explicitly required students to apply mathematical reasoning to take and support a position on a rich topic. I noted in my field journal, as I created this problem set, that I wanted to steer the future path of the curriculum in the direction of sociopolitical themes that required a sophisticated level of thinking.

Unfortunately, students expressed great frustration and had difficulty with these assignments. Many submitted blank pieces of paper, some responses were incoherent, and not a single student correctly utilized propositional logic to demonstrate sophisticated reasoning. The failure of that assignment produced a reflective note in my field journal, speculating that the students could not perform the task because they were not yet proficient at (A) identifying premises for arguments, and (B) recognizing the structure of an argument. My interpretation of this result led to the curriculum changing; as a result, I added layers of scaffolding into the curriculum to build these skills.

The first layer of scaffolding was to practice the act of translation; we looked at proverbs and quotations that could be reframed as logical arguments, such as "Constant dripping will wear away a stone," and "When you fall into a pit, you either die or you get out," and "In the middle of difficulty lies opportunity." Next, we tackled longer translations – including the language used in Proposition 187, on the 1994 California ballot to decide whether or not "illegal aliens" should be entitled to receive social services, such as education and health care. As we got deeper into the practice of translations, and in preparation for one class, students were asked to read Martin Luther King, Jr.'s "Letter from a Birmingham Jail." At the next class meeting, we reflected on the story – to activate engagement and motivate the challenging work of translation. In a guided discussion, we found various types of argument structures employed by MLK, Jr., including direct, disjunctive, contrapositive, and biconditional arguments, as well as his references to false premises used in arguments by those who opposed him. I noted, "Students were highly invested in this assignment." The letter from King, Jr. following the explicit practice of translating written text into propositional terms, solidified the importance of working with meaningful context. Students demonstrated that they were beginning to see conditional relationships implied in everyday language. For example, in the previously-mentioned proverb about dripping water, students were able to interpret it as "If one is persistent, then one can overcome obstacles." Recognizing the pervasive nature of "if-then" conditional relationships brought students to a new level of awareness, and brought a new style of speaking to our class discussions. I observed that students were making more connections between what we learned in the classroom and real-life situations, and were frequently emphasizing the words "if" and "then" in their speech.

***Mediating Processes***

Initially, students struggled with the act of translating because they had difficulty in identifying the statements to use for the building blocks in propositional logic. Furthermore, the students brought a binary perspective into the process because of the truth table structure. The notion that each statement must be either true or false, seemed to confuse and contradict their natural intuition that premises could be subjective. In



response, I drew a connection to a previously studied topic from the first unit on social choice mathematics. I explained, "If *these* weights are assigned to the categories, then the outcome of the weighted sum method produces *these* rankings." And, "If *this* voting system is utilized, then *this* candidate emerges as the winner." The students began to show signs that they understood the ramifications of selecting from competing premises. A pivotal moment occurred in class when I was teaching a lesson on syllogisms. I presented some easy examples, including the classic one about Socrates, and I accompanied it with a visual diagram.

> Premise #1: All men are mortal.
> Premise #2: Socrates is a man.
> Conclusion: Socrates is mortal.

When I wrote these next two premises on the board and asked for a conclusion, students shouted out their responses.

> Premise #1: All inmates have tattoos.
> Premise #2: Joe is an inmate.
> Conclusion: _____________

The mediating processes that occurred in that moment revealed a major breakthrough in students' thought processes. I observed the students taking a big leap forward – suddenly grasping the notion of a premise, its relative uncertainty, how it forms the foundation of an argument, and how it is directly tied to the strength of a conclusion. The lively discussion settled on an unverifiable fact, but we accepted a claim that 70% of the inmates have tattoos, and therefore there was a 70% chance we would be correct if we concluded that Joe has a tattoo. The next few assignments were characterized by a significant increase in the use of "if-then" language by the students, more careful translations, and improvements in clearly stating premises and conclusions for their arguments. This led me to write a reflective note in my field journal, one that would prove to be influential for the third phase of curriculum development. “Key connections are made between logic and probability. Application of logic makes more sense, seems more realistic to students when acknowledge uncertainty of premises.” The evolution of ideas that led to incorporating probability within logical reasoning at first seemed counterintuitive to me, but then through class discussions we collectively agreed that it made the most sense. By that I mean the students voiced their understanding that arguments begin with premises, but premises are assumptions and subject to error. Discovering the important role of probability was essentially an epiphany, and this discovery became a motivating factor for phase three.

### *Intervention Outcomes*

During this second phase of my curriculum, students developed proficiency with translating text into the symbolic notation used in propositional logic, and they improved their ability to select premises and state conclusions. The combination of propositional logic with contextualized themes worked to facilitate forward steps with their reasoning skills. These proficiencies were achieved through scaffolded instruction and classroom discourse with careful attention to word choices.

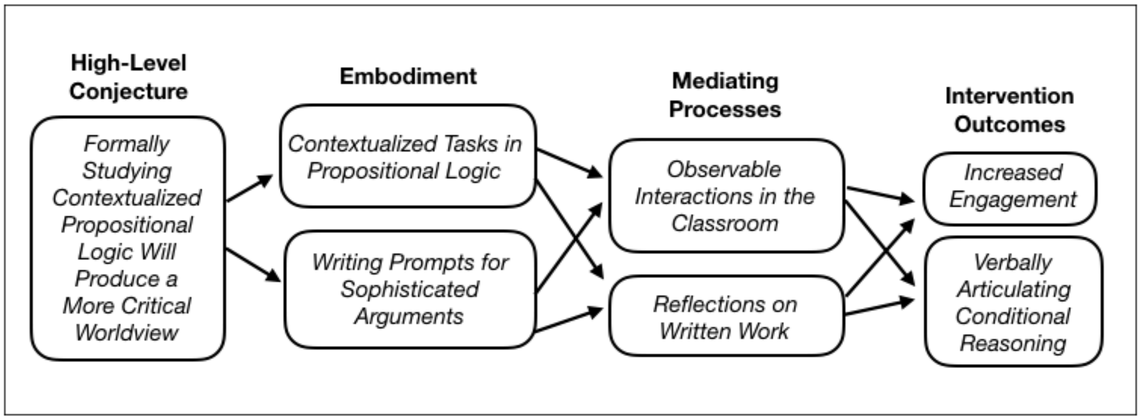
Separately, students demonstrated a weakness in probability exercises, yet they were entertained by topics of compound and conditional probabilities. It was a surprising revelation for students when they understood that premises could have probabilities attached to them, as in the example of the statistical syllogism about tattoos. It was even more surprising when, at the end of the semester, a student observed and pointed out the common language used by both logic and probability. The connectors “and, or, if-then” are used with propositional logic, and are also used in compound and conditional probabilities. Many students admitted their prior misconception that logic and probability were separate topics, yet they now viewed them as inescapably intertwined. Prior to this phase, I thought probability topics would entertain them, and be viewed apart from the formal study of logic. However, what I saw was that the intersection of the two domains intuitively made sense to the students and pushed them to embrace the topic of propositional logic.

### ***Reflections on 2nd Conjecture Map***

Through the course of phase two, I realized that training in propositional logic was, by itself, not sufficient if the goal was to construct a more critical worldview. Additionally, the sociopolitical contexts I used were engaging, but did not directly improve student performance. The students' struggles with the ambiguities of identifying and selecting premises for an argument gave me awareness that they also needed to incorporate subjective influences and the perspective of conditional probability. Figure 8 illustrates my second conjecture map. I started phase two with an emphasis on developing logic and sociopolitical themes, but I realized the curriculum still needed further development.

Figure 8

*Phase Two Conjecture Mapping*




Summarizing the course out loud on the last day of the semester, I heard myself talking about one's search for knowledge and the classic theory of knowledge as a justified true belief. I explained how justification is a matter of a logically valid framework, and truth is a matter of probability – which manifests as the reliability of a premise. My lecture on the last day connected key ideas in the curriculum about attaching quantitative values to qualitative feelings, recognizing the role of emotions and subjectivity in making decisions, and logically combining premises to produce a conclusion and argue a point of view. The realization that students could learn to integrate principles of logic and probability, work with sociopolitical contexts, and operate within the framework of an argument to have a more critical worldview, led me to create my new theory of mathematical reasoning – which I termed *subjective rationalism* (see definition on page 4 in the introduction). I created my theory of subjective rationalism to explain the intersection of multiple forces on individual's mathematical reasoning processes. I aimed to develop a model for sophisticated reasoning that would account for logic, uncertainty, and the influences of lived experiences. Together, my thoughts surrounding subjective rationalism served to motivate my efforts to continue reforming the curriculum in phase three.

During my graduate studies, I discovered research on the notion of a threshold concept (Meyer & Land, 2003). My interests as a research practitioner compelled me to search for the threshold concepts of mathematical reasoning, which I now believed to be the concepts of validity and reliability. I believed the curriculum was on the right track towards developing a set of reasoning skills that would be useful across disciplines. As I move into the third phase, you will see me target the intersection of the two threshold concepts and integrate them throughout the curriculum.

**Phase Three**

Entering phase three, I had the mindset that in order for my students to develop mathematical reasoning skills that would be useful across disciplines, they would need to understand threshold concepts in logic and probability, while being mindful of subjective influences. I was motivated to test my new theory with another high-level conjecture during this third phase of curriculum development. My earlier doubts about the nature of mathematical reasoning were replaced by a strong belief in two particular threshold concepts, and this led me to design a pre/post assessment for my next semester (see Appendix I). I began phase three hoping that I would be able to discern changes in students' perceptions over the course of the semester, as a result of reframing the curriculum to deliberately target students' conceptual understanding of validity and reliability.

***High-Level Conjecture***

My new theory of subjective rationalism drove me to review the existing curriculum it its entirety and examine it from the perspective of threshold concepts. In this phase, I redesigned instructional interventions and student tasks to not only emphasize logic and probability, but to highlight their intersection whenever possible. This time, I was guided by the following high-level conjecture. Conjecture #3: A liberal arts math curriculum that cultivates the threshold concepts of validity and reliability, and situated with recurring themes of social justice, will aid the development of sophisticated reasoning skills and enable students to construct a more critical worldview. The interplay between logic and probability would be woven throughout the semester. My plan was to lead students into thinking within logically valid frameworks, interpret premises as probabilities, and be self-aware that they were viewing social issues through subjective lenses. At the end of this phase, I was prepared to characterize a successful learning experience as one in which students demonstrated an improved ability to apply mathematical reasoning for sophisticated insight into important social issues.

***Embodiment***

I began the first day of the semester by asking questions that would help me understand students' preconceptions of mathematical reasoning. For example, I asked them if correct calculations and valid logical reasoning would always produce conclusions with certainty. I collected their responses on a pretest to learn how they viewed relationships among logic, probability, and mathematics. My new approach to the development of mathematical reasoning is reflected in the following curriculum pieces. The first excerpt includes new language I added to a previous unit test on logic. In the prior semester, I did not ask students to reflect on the reliability of the premises or the implications of their answers on the criminal justice system.

P: You are innocent.
Q: Proof of innocence exists.
R: Absolute proof of guilt exists.
S: You are set free.

> Create a truth table for the argument: $[(P \land Q) \lor \sim R] \rightarrow S$
>
> Is this a sound argument? Identify the premises of the argument. Comment on the reliability of the premises, and use the truth table to analyze the validity of the conclusion. Explain the implications of your answer.

This test question represents my approach to designing tasks in this phase because it exemplifies how all three elements of my high-level conjecture intersect. My analysis of prior phases reveals that earlier tasks on the topic of mathematical logic initially did not have context, then layers sociopolitical context were added, then the topic was framed with the principles of argumentation, and now, finally, the concept of reliability was integrated into the task. My curriculum differs from other liberal arts math programs by not only including these elements, but by including them simultaneously in a single task.

During this semester, I also asked students to reflect on the risk in using a conditional statement $P \rightarrow Q$ as a premise in an argument. At the time it was assigned, students were well-versed with the idea that premises have probabilities attached to them, that premises are often subjectively selected, and that conclusions are unreliable despite the use of valid frameworks. The journal assignment (see Appendix H) prompted them to recall personal experiences with conditional relationships that were used as premises for arguments with other people. I noted in my field journal that student work was increasingly showing connections between the course content and their own lives.

The next two tasks were given to students after they had studied probability. These tasks were included in earlier iterations of the curriculum, but now they were presented as open-ended prompts for students to take and support a position with a valid framework. Students worked on their interpretations outside of class, and the subsequent class meetings were filled with verbal arguments on the issues – but the arguments were noticeably more structured than in previous semesters.

In late 2014, the residents in Ferguson, Missouri were outraged that an unarmed black man was shot and killed by a white police officer. Many sociologists attribute a brooding anger to systemic discrimination against the African-American community, as evidenced by the disproportionate number of vehicle stops by ethnicity. Use conditional probability to investigate the story of the public's angst and the cause for civil unrest. Create and answer your own questions.

Table 11 (repeat)
*Traffic Stops by Ethnicity, Ferguson Police Department*

| Ferguson, MO (2013) | Stops | Searches | Arrests | No Incidents | Total Registered Drivers |
|---|---|---|---|---|---|
| White | 686 | 47 | 36 | 4,569 | 5,338 |
| Black | 4,632 | 562 | 483 | 4,317 | 9,994 |
| Hispanic | 22 | 1 | 1 | 150 | 174 |
| Asian | 12 | 0 | 0 | 83 | 95 |
| American Indian | 8 | 1 | 1 | 49 | 59 |
| Other | 24 | 0 | 0 | 181 | 205 |
| Total | 5,384 | 611 | 521 | 9,349 | 15,865 |

A meritocracy is a social system in which people's success in life depends primarily on their talents, abilities, and effort. Proponents of the idea that America is a meritocracy argue that the realities of socioeconomic inequalities are simply the result of unequal talents and not the result of societal prejudices or institutional discriminations. In 2012, a national research center gathered data about class mobility. The following table represents their findings (expressed as percentages) regarding Americans' self-perceptions about their socioeconomic status compared to the social class of their own parents. Does this table reinforce or contradict the claim that America is a meritocracy?

Table 12 (repeat)

*Perceived Socioeconomic Status and Class Mobility*

| | | Parents' Status / Family Background | | | |
|---|---|---|---|---|---|
| | | Lower | Middle | Upper | Total |
| Children | Upper | 4 | 5 | 6 | 15 |

| | | | | | |
|---|---|---|---|---|---|
| as Adults | Middle | 15 | 25 | 7 | 47 |
| | Lower | 21 | 13 | 4 | 38 |
| | Total | 40 | 43 | 17 | 100 |

> Begin by thinking about your initial position in this controversial debate, and then search the data for meaningful insight. More than seventy different conditional probabilities can be expressed using the numbers in this table, but some are more revealing than others. Explore the possibilities and investigate the likelihood of an adult ending up in a social class different than that of his or her parents.

I designed these problems because (A) students were exploring themes of racial profiling and class mobility in their other courses, and (B) students had directly experienced the impact of these social forces in their own lives. These tasks prompted students to build arguments and I guided our class discussions to incorporate the language of validity and reliability. I noted in my field journal how students' engagement levels reinforced my confidence in my theory of subjective rationalism.

At the end of the semester in phase three, I administered an original assessment investigating the intersection of logic and probability. The following excerpt typifies the manner in which I compelled students to work with threshold concepts while exploring an important issue relevant to their own lives.

> P: An inmate is involved in a correctional education program.
> Q: An ex-convict does not recidivate.
> R: An ex-convict successfully reintegrates back into society.
>
> Analyze the argument: $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$
>
> Provide two reasons why $(P \rightarrow Q)$ could be a flawed premise.
> Provide one reason why $(Q \rightarrow R)$ could be a flawed premise.
>
> Speculate on the relationship between probability and reliable premises, as it relates to the argument on education and recidivism. Insert your own numbers into the blank table; use percentages that you feel would be necessary and sufficient to support the argument in favor of education reducing recidivism.

What do you think the numbers should be, in order for the argument to be widely accepted in the public discourse?

Table 13

*Correctional Education and Recidivism*

| | Recidivate | Do Not Recidivate | Total |
|---|---|---|---|
| Involved with Correctional Education | | | |
| No Correctional Education | | | |
| Total | | | 100% |

Based on the numbers in your table, compute the following probabilities:

P (a randomly selected inmate participates in a correctional education program)
P (an ex-convict does not recidivate)
P (inmate does not participate with correctional education and then recidivates)
P (ex-convict recidivates | inmate was not involved with correctional education)
P (ex-convict does not recidivate | inmate was involved with correctional education)

More experiences will produce more statistical data, and more data allows for greater reliability with the premises of an argument. How much data would be enough data to make a sound argument in favor of correctional education reducing recidivism? If the numbers you provided in the table above are sufficient for use as premises in the argument, how long (how many years) do you think the public would need to see consistent results before they are accepted as reliable premises?

The pedagogical advantage of working with open-ended questions such as these is that students naturally bring their individual perspectives to the problem and must utilize the curriculum content to support their position on an important social issue. The students were extremely engaged with the context of correctional education and recidivism, and I observed that their engagement was synonymous with their motivation to work at a high level and provide thoughtful responses.

***Mediating Processes***

Students scored extremely high on the tasks in this phase, indicating to me that the curriculum content was being effectively scaffolded for comprehension. I would typically introduce an idea in class, provide students the opportunity to familiarize themselves with that content outside of class, and then facilitate a classroom discussion about the topic at our next meeting. Students informed me that they regularly met with each other in between classes and would have in-depth discussions about course material. They shared stories about the amount of time they spent reading and reflecting on the curriculum and in preparing their responses. In my journal, I noted that their written submissions drastically increased in length, and that we engaged in many lively class debates. Not only did they elaborate verbally and in writing, they also articulated their responses with more structure. That is, their style of language changed and the sequence in which they expressed ideas evolved. Students now made a point to clarify their premises to the audience, and then proceeded to make frequent use of the logical connectors "and, or, if-then" in their language. For example, a student would state "If I can do my college inside (while incarcerated), then I'm gonna be in a much better place when I get out. And, if I can continue with my studies at Rutgers, then I might get a decent job." I noted that students were using the content language and working with a sequential flow of ideas in a very natural manner. At this stage, I observed that many students adopted valid frameworks with their thought processes and incorporated them into coursework and classroom vernacular.

It was also clear to me that the students saw connections across content; they demonstrated an awareness of inter-relationships in the curriculum not only with the usage of "and, or, if-then," but with other topics too, including frequent use of the powers of two. Students saw the same numbers used to determine the size of a truth table appear again in probability with the fundamental counting rule and exercises with coin tosses, and then later with Pascal's Triangle to develop key principles in binomial probability. Additionally, I made the effort to use recurring themes of social justice in order to increase their familiarity with the contexts and promote deeper insight into the issues by examining them with a variety of mathematical tools. I would often hear a student say, "Everything seems connected." I reflected in my journal that these connections seemed to reinforce their learning by cultivating multifaceted perspectives.

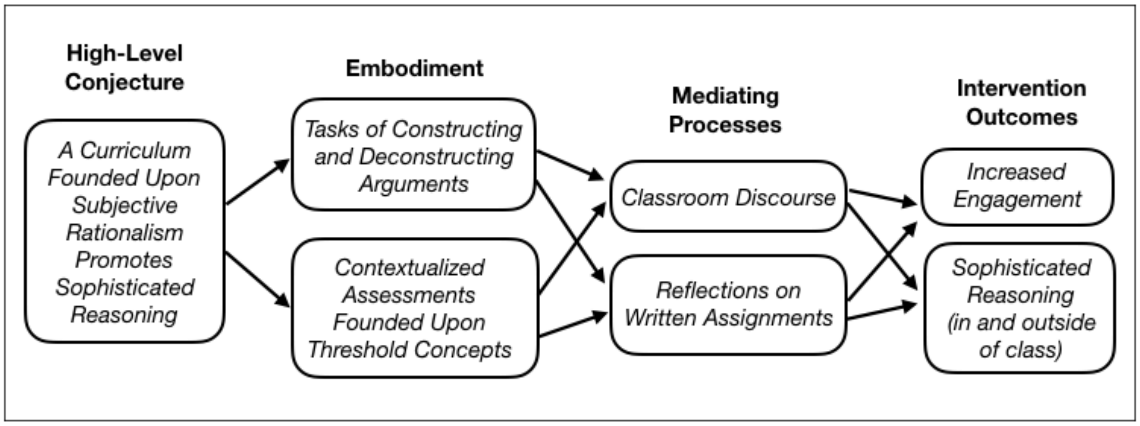
#### *Intervention Outcomes*

My goal with this third conjecture map was to be able to say that if certain mediating processes occurred, then it would lead to my desired learning outcome. The desired learning outcome was to see students demonstrate growth in being able to engage in sophisticated reasoning about sociopolitical issues. The embodiment of my high-level conjecture existed as instructional interventions in which I contextualized my lesson plans and deliberately targeted their comprehension of validity and reliability, as well as student tasks designed to engage and assess them in their development. The mediating processes included thoughtful discussions with their peers, and the mental struggles they endured as they gradually modified their thinking to operate within logically valid frameworks. Analysis of my field observations and my reflections on their written performances reveals that the students embraced the principles of argumentation and became more cognizant of the uncertainty of their premises. Student growth came to fruition at the end of the semester, and the posttest showed significant changes in their perceptions about logic, probability, and what it means to engage in mathematical reasoning.

An insightful moment occurred in class at the end of the semester when a student commented on the “if-then” language used in conditional statements. I recorded this moment in my journal. Paraphrasing, he said the “if-then” construction that is central to deductive reasoning, does not exist without the “if,” and that the “if” is essentially a probability. Therefore, he continued, the uncertainty of a premise was included in the framework of an argument all along! I found this to be a creative interpretation, and one that effectively demonstrated how students’ perspectives fundamentally changed. Many students said they had begun thinking in new ways when talking with friends, while watching television, and in their other college courses. A colleague mentioned that students were outwardly employing their mathematical reasoning skills in his class during a lecture on David Hume’s moral philosophy. They demonstrated sophisticated thinking that perhaps they would not have been able to do previously. The students applied logic, referenced truth tables, and discussed subjectivity in a debate about how moral judgments are matters of perception and not matters of rationality. Through my curriculum, they were more equipped to understand how mistaken moral judgments occur when we try to attach true/false values to our perceptions of good and bad. This curriculum was designed to develop thinking skills that could be useful across disciplines; hearing this kind of feedback from students and colleagues reinforced my conviction in this liberal arts mathematics curriculum. Figure 9 outlines my conjecture map for this phase.

Figure 9

*Phase Three Conjecture Mapping*




### *Reflections on 3rd Conjecture Map*

My third phase targeted principles in probability and identified reliability as a threshold concept. Linking the careful study of probability to the framework of an argument allowed for the emergence of a new model of argumentation. I reflected on my primary intention with the curriculum – to develop students' reasoning skills. The feedback from my colleagues, from students, and my assessment of student work, collectively indicated that the observed outcomes were aligned with my overarching objectives. My curriculum development stopped after phase three, but the field of liberal arts mathematics educators should be encouraged to continue exploring the notion of threshold concepts and the integration of sociopolitical themes into curricula. My theory of subjective rationalism is in its infancy, and additional research surrounding this theory would undoubtedly serve to strengthen the academic field.

# CHAPTER FIVE: DISCUSSION

The tools of mathematical reasoning can aid students in building a more sophisticated lens through which to view the world; my retrospective analysis is about how this works. My experience as a research practitioner reveals the importance of identifying threshold concepts, and building a curriculum around these concepts. My discovery of the critical roles played by validity and reliability, while being cognizant of subjective influence, formed my theory of subjective rationalism. Note, there are inherently too many variables and too many conjectures (Cronbach, 1975; Bereiter, 2002) to be able to definitively state causal relationships among the embodiments, mediating processes, and intervention outcomes of a conjecture map. Thus, it is better if we take the view that conjecture maps can outline process relations, rather than causal mechanisms. There are complex interactions within the four stages of a conjecture map, and ultimately a research practitioner must decide how to determine the value in a learning experience. That is, we ask ourselves the questions "What do I want to see, how do students get there, and what do I provide them with to help them get there?" I decided that, in the end, mathematical reasoning is less about the manipulation of numbers and more about the construction of arguments, and context gives knowledge its meaning. My study offers insight into the process of creating a curriculum that aims to develop a useful set of reasoning skills that can aid individuals in all aspects of life, by attending to key principles in logic and probability, because we are rational beings living in a world of uncertainty.

**Findings in Relation to Existing Literature**

Lyn Steen's (2001) call for an innovative quantitative literacy curriculum provided initial momentum for my work, but the inconsistency and diversity in the academic field meant that I needed to experiment with many ideas. The shortcoming in merely using realistic data sets, as recommended by Steen, is that they often fail to pique student interest. Steen wanted to promote an informed citizenry; he characterized individuals as quantitatively literate in the sense of being able to comprehend messages promulgated by the media. I observed that students needed to be motivated on a deeper level, and that a realistic context was not necessarily a relevant context in the eyes of students. In my experience, we need to provoke our students by examining serious social issues that directly matter to them.

Themes of social justice in a mathematics classroom have the potential to engage students and increase their awareness of important social issues. Gutstein (2003) and Frankenstein (1983) both share practitioner research that demonstrate the impact of contextualizing content with sociopolitical matters. They implement progressive tactics that push Steen's ideas beyond the conservative realm and challenge the apolitical nature of a traditional classroom. My experiences affirm the benefits of this approach; however, I noted that students still struggled to develop a critical consciousness when they were not directly taught about reasoning processes.

Gutstein (2003) is right to promote the use of locally-generated themes to contextualize the mathematics content. Social justice is not limited to stories about racial oppression or the consequences of economic globalization. I learned to solicit input from the students and allow them to co-construct the curriculum by identifying the issues most relevant to their lives. Gutstein is also correct in teaching his students how to analyze and critique social issues using mathematical tools. Educators who engage in critical mathematics are known to implement a pedagogy of questioning – instilling students with the habit of asking questions, critiquing society, and challenging the status quo (Gutstein, 2006). As an extension, my curriculum work adds to this field by exploring the impact of explicitly teaching students how to take and support a position, using mathematical tools. Such tools go beyond the elementary arithmetic that Steen suggested, and encompass propositional logic with key principles in probability. My research has found that liberal arts math programs and the prototypical textbooks that accompany them at the postsecondary level do not center their curriculum around argumentation. Social justice math programs explore issues of equity, but I have not found any existing literature that speaks to using a liberal arts math curriculum for explicit guidance on constructing arguments through mathematical logic. This aspect of the curriculum, along with the theory of subjective rationalism, offer new perspectives for the academic field.

**Implications for Practice and Suggestions for Future Research**

My research proposes several benefits to implementing a curriculum founded upon subjective rationalism. I noted in my retrospective analysis that student engagement increased, and the nature of their language and thought processes evolved to incorporate logically valid structures. The implication here is that educators and liberal arts math programs can serve their students by attending to the act of argumentation, and by contextualizing the content with locally-generated provocative themes of social justice wherever possible.

The current field of liberal arts mathematics is still broad and undefined, yet it can be distilled into three types of programs – broad survey courses, application-based programs, and programs that focus on reasoning processes. Each of these categories can potentially add to a student's learning experience by exploring elements of subjective rationalism. I suggest that more research be performed on (A) the notion of threshold concepts, to continue studying the roles of validity and reliability in mathematical reasoning, (B) the impact of integrating sociopolitical context, to cultivate critical consciousness and develop an informed citizenry, and (C) designing a curriculum that explicitly attends to argumentation, from a mathematical perspective. The classic theory of knowledge – as a justified true belief, and a progressive curriculum founded upon subjective rationalism, together can clarify the diverse field of liberal arts mathematics. The intersection objectivity, uncertainty, and subjectivity exists within the framework of an argument, and all three converge to produce a more sophisticated lens and critical worldview. I would also like to see research that explores how a curriculum of subjective rationalism could facilitate engagement on a societal or interpersonal level, or lead students towards thoughtful reflections on their own reasoning processes, i.e. how they build new knowledge, how they digest knowledge put forth by others, and how they view the world.

The process of curriculum development is cyclical; it demands constant reflection and continual experimentation. It is a slow and gradual process, yet it can change in an instant with critical feedback. The efficiency of my research design compels me to suggest that future research in liberal arts math programs also employ conjecture mapping as a means to clarify core components of the curriculum. This methodology led me to realize that a transformative curriculum needs more than relevant context, it also needs deliberate attention to the threshold concepts of validity and reliability. I am hopeful that the community of liberal arts math educators can ultimately become more consistent with our core content as we continue exploring the development of mathematical reasoning tools that prepare students for more informed citizenship.

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# APPENDIX A

## Test on Logic and Probability

What is the relationship between premises and probabilities? Are the two worlds of logic and probability mutually exclusive, or not? Let's explore this relationship and try to answer these questions. We begin with another wise old saying. "Give a man a fish and you feed him for a day. Teach a man to fish and you feed him for a lifetime."

Part One

Perhaps one's education can play the role of a "fishing pole" – we'll investigate that in Part Two. But first, let's assume that the purpose of going fishing is to catch fish, and that the ultimate goal is to have some fish to eat at your next meal. Whether or not you catch fish depends on several variables, including the type of bait used and the presence of fish in that body of water.

- What type of bait do you use?
- Are there fish in the water?

The following table represents a sample of results from local fishermen last weekend. Does the table affirm or disaffirm the effectiveness of live bait?

Table 14

*Conditional Probability Assessment*

| | Caught Fish | Did Not Catch Fish | Total |
|---|---|---|---|
| Used Live Bait | 21 | 5 | 26 |
| Did Not Use Live Bait | 3 | 6 | 9 |
| Total | 24 | 11 | 35 |

Compute, and comment on, the following probabilities:

P (Catch fish) =
P (Catch fish | do not use live bait) =
P (Use live bait | do not catch fish) =
P (Use live bait | Catch fish) =
P (Catch fish | Use live bait) =

Which of these conditional probabilities provide the most meaningful insight to the relationship between catching fish and the type of bait used?

Veteran fishermen will tell you that live bait is superior to any alternative or artificial bait. Generally speaking, their position on this matter is based on inductive reasoning – a lifetime of observations and experiences which lead them to reach a certain conclusion. The pastime of fishing however, can also be modeled using the tools of deductive reasoning.

P: You use live bait.
Q: You catch fish.
R: You have fish to eat at your next meal.

Premise #1: $P \rightarrow Q$
Premise #2: $Q \rightarrow R$
Conclusion: $P \rightarrow R$

Comprehensive Expression: $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$

For the premises to be reliable as the basis for a conclusion, they must be consistently true. Are these premises reliable? Does the phrase "consistently true" really demand 100% assurance of the event in order for it to be used as a premise in an argument? Is the conclusion valid? Create a truth table for the comprehensive expression, and identify the structure of the argument. Combine your answers to the last two questions – is this a sound argument? Why, or why not? Translate and interpret the validity of $(R \rightarrow P)$.

Part Two

Can fishing serve as an analogy to explore correctional education, as it relates to recidivism? (Note, "recidivism" is defined as a person's relapse into criminal behavior). The purpose of going fishing is to catch fish and have something to eat. Surely, the purpose of correctional education is to reduce recidivism and help ex-convicts succeed on the outside, right? Once again though, the situation is complicated and several factors are involved, including:

- What type of correctional educational programs are implemented?
- Are there opportunities for ex-convicts on the outside?

Compare these questions to their fishing counterparts: is someone using the correct type of bait, and are there fish in the water? The existence of fish in the water and the presence of opportunities on the outside are hidden premises that we must assume to be true, otherwise our actions are futile.

P: An inmate is involved in a correctional education program.
Q: An ex-convict does not recidivate.
R: An ex-convict successfully reintegrates back into society.

The structure of the argument is the same as before: $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$

However, the argument is subject to criticism.
Provide two reasons why $(P \rightarrow Q)$ could be a flawed premise.

Provide one reason why ($Q \rightarrow R$) could be a flawed premise.

Human beings are complicated and diverse, and we live in a world of uncertainty. It is too unrealistic to expect 100% consistency about anything when people are involved. Speculate on the relationship between probability and reliable premises, as it relates to the argument on education and recidivism. Insert your own numbers into the blank table; use percentages that you feel would be necessary and sufficient to support the argument in favor of education reducing recidivism. What do you think the numbers should be, in order for the argument to be widely accepted in the public discourse?

Table 15 (repeat)
*Correctional Education and Recidivism*

| | Recidivate | Do Not Recidivate | Total |
|---|---|---|---|
| Involved with Correctional Education | | | |
| No Correctional Education | | | |
| Total | | | 100% |

Based on the numbers in your table, compute the following probabilities:

P (a randomly selected inmate participates in a correctional education program)
P (an ex-convict does not recidivate)
P (an incarcerated individual does not participate with correctional education and recidivates)
P (an ex-convict recidivates | an inmate was not involved with correctional education)
P (an ex-convict does not recidivate | an inmate was involved with correctional education)

What is the relationship between probability and premises? Are the two worlds of logic and probability mutually exclusive, or not? Explain.

More experiences will produce more statistical data, and more data allows for greater reliability with the premises of an argument. How much data would be enough data to make a sound argument in favor of correctional education reducing recidivism? If the numbers you provided in the table above are sufficient for use as premises in the argument, how long (how many years) do you think the public would need to see consistent results before they are accepted as reliable premises.?

Note, this analogy with fishing and correctional education is presented in a way that shows their identical logical structures. Recall the isomorphisms from binomial probability, and how problems with the same type of structure have the same type of solution. An awareness of isomorphisms and logical equivalencies can provide you with surprising insight into seemingly unrelated issues. Reflecting on isomorphisms can also stretch your creative capabilities by building more connections across a variety of fields.

Part Three
If the recidivism rate in New Jersey is 40%, compute the probability distribution for ten inmates recently released, and determine the probability that exactly four of them recidivate.

## APPENDIX B

### Reflection: Logic, Probability, and a Criminal Investigation

*Ethylenediaminetetraacetic acid* (EDTA) is a chemical used to preserve blood samples held as police evidence in laboratory vials. If a sample of a person's blood is being held as evidence from a prior case, then that person's blood could potentially be used to frame him for a future crime. However, if his blood is in fact planted as evidence at a future crime scene, then the presence of EDTA should be detectable in the blood stain. Consider the following statements:

> P: The blood at the crime scene was extracted from the vial of the preserved blood sample (i.e. the blood stain was planted and did not come from someone actively bleeding at the crime scene).
>
> Q: EDTA is detected in the blood stain at the scene of the crime.

During a high-profile case in Wisconsin, the defense argued that the local police department conspired against him. The defendant's lawyers claimed that statement P was true - the defendant's blood was planted at the scene of the crime.

However, an FBI forensics analyst could not detect any traces of EDTA in the blood stain. This led the prosecution to argue against the conspiracy accusation by asserting that since EDTA was not detected, then the blood stain was not planted.

Is it fair to claim that if the presence of EDTA is not found, then it is not actually there? How does this compare to the informal fallacy of "Appeal to Ignorance" from inductive reasoning? If evidence has not been found, does it mean that evidence does not exist?

Here are excerpts from the testimony of an independent laboratory quality auditor:

> "Just because EDTA is not detected by the laboratory doesn't mean that blood sample came from somebody actively bleeding on that spot."
>
> "It's quite possible that those blood swabs could have come from the defendant's blood tube but simply not been detectable by the laboratory."

The defense used the auditor's testimony to argue that the original premise of ($P \rightarrow Q$) was unreliable, therefore the prosecution's argument was equally unreliable.

The probability of the test's accuracy determines the reliability of the result. The accuracy of the test is used a premise in the prosecutor's argument, therefore the reliability of the test result directly affects the reliability of the conclusion. How can we determine the degree to which the forensic test's conclusion is reliable? Summarize the logic used by the defense and the prosecution in this case. Explain the types of arguments made, and the role that probability plays in these arguments.

# APPENDIX C

## Assignment: Stressful Cities

Upon graduation, you may have to decide among competing job offers in different cities. Where should you live, what factors will influence your decision? Beginning a new job is tough enough, so if you have the choice, pick a city that will not burden you with external pressures. Several publications have created their own ranking systems for the most stressful cities in America, (you can find many online), but they do not fully reveal their methods.

The American Psychological Association conducted a national survey to learn what are considered to be the most significant sources of stress for Americans. The most common responses were related to money (including job stability), family relationships, health, and personal safety. Create a spreadsheet to rank the following cities based on the criteria listed below. You will need to do a little research to get current numbers for use with your weighted sum method.

Note, this method can be used to make statements about the best of something or the worst of something; keep in mind how much power is invested in the person who selects the categories and assigns the weights to each category. In arriving at a conclusion, the data that you use is just as important as how you use it. Provide reasons for the weights you assign and explain how you establish the rating scale within each category. Rank the outcomes of your weighted sum and be prepared to present your findings to the class.

Table 16
*Ratings and Rankings of Stressful Cities*

| | Unemployment | Divorce Rate | Crime Rate | Suicide Rate | Commute Time | Sunny Days | Weighted Sum |
|---|---|---|---|---|---|---|---|
| *(Weight)* | | | | | | | |
| Boston | | | | | | | |
| Cleveland | | | | | | | |
| Dallas | | | | | | | |
| Denver | | | | | | | |
| Detroit | | | | | | | |
| Las Vegas | | | | | | | |
| Los Angeles | | | | | | | |
| Miami | | | | | | | |
| Minneapolis | | | | | | | |
| Philadelphia | | | | | | | |
| Pittsburgh | | | | | | | |
| San Francisco | | | | | | | |
| Santa Fe | | | | | | | |
| Seattle | | | | | | | |
| Tampa | | | | | | | |
| Washington | | | | | | | |

## APPENDIX D

### Assignment: Conditional Probability

"The People vs. UC- Berkeley"
*(A Case of Gender Bias)*

In 1973, the University of California – Berkeley was sued for bias against women who had applied for admission into its graduate schools. The admission figures for the fall of 1973 showed that men applying were more likely than women to be admitted, and the difference was so large that it was unlikely to be due to chance.

| | Admitted | Denied | Total |
|---|---|---|---|
| Men | 3,714 | 4,728 | 8,442 |
| Women | 1,512 | 2,809 | 4,321 |
| Total | 5,226 | 7,537 | 12,763 |

Part I
Compute the joint and marginal distributions for the table above. Determine the conditional probabilities that would be helpful in building a legal case for gender bias against the university.

Part II
In a surprising twist, officials from the university brought more detailed data into the courtroom, and revealed the admissions records from the six largest academic

departments. (The university has over 60 different academic departments within the graduate schools). Use the real data provided in these tables to reach a strong conclusion. If you were the judge, what would be your verdict? Defend your position with the appropriate mathematical calculations.

Table 17
*UC-Berkeley Admissions Records, 1973*

| Dept. | Female Applicants | | |
|---|---|---|---|
| | Admitted | Denied | Total |
| A | 89 | 19 | 108 |
| B | 17 | 8 | 25 |
| C | 202 | 391 | 593 |
| D | 131 | 244 | 375 |
| E | 94 | 299 | 393 |
| F | 24 | 317 | 341 |
| Total | 557 | 1278 | 1835 |

| Dept. | Male Applicants | | |
|---|---|---|---|
| | Admitted | Denied | Total |
| A | 512 | 313 | 825 |
| B | 353 | 207 | 560 |
| C | 120 | 205 | 325 |
| D | 138 | 279 | 417 |
| E | 53 | 138 | 191 |
| F | 16 | 256 | 272 |
| Total | 1192 | 1398 | 2590 |

## APPENDIX E

### Assignment: College Affordability

Listed below are the average costs for one year of college tuition in America during the years 1980, 1994, and 2008. Adjust the costs of tuition to reflect *real* 2012 dollar values, and calculate how much the real cost of college tuition has changed over the years.

1980 average tuition: $2,120 2012 real dollars:

1994 average tuition: $4,450 2012 real dollars:

2008 average tuition: $10,240 2012 real dollars:

1. How many weeks would it take a single parent to pay for one year of a child's college tuition in 1980, if the parent worked full-time and had

    (A) only a high school diploma

    (B) a two-year associate degree

    (C) a four-year college degree

2. How many weeks would it take a single parent to pay for one year of a child's college tuition in 1994, if the parent worked full-time and had

   (A) only a high school diploma

   (B) a two-year associate degree

   (C) a four-year college degree

3. How many weeks would it take a single parent to pay for one year of a child's college tuition in 2008, if the parent worked full-time and had

   (A) only a high school diploma

   (B) a two-year associate degree

   (C) a four-year college degree

How has college affordability changed over the years?

The following table provides data for the median weekly earnings, in dollars, of full-time wage and salary workers, 25 years and older in the US, by educational attainment, from 1980 to 2012.

Table 18
*U.S. Median Weekly Earnings and CPI*

| | | Median Weekly Earnings ($) | | |
|---|---|---|---|---|
| Year | CPI | High School Diploma | College (1-3 yrs) | College (4 or more years) |
| 1980 | 82.400 | 266 | 304 | 376 |
| 1982 | 96.500 | 302 | 351 | 438 |
| 1984 | 103.900 | 323 | 382 | 486 |
| 1986 | 109.600 | 344 | 409 | 525 |
| 1988 | 118.300 | 368 | 430 | 585 |
| 1990 | 130.700 | 386 | 476 | 639 |
| 1992 | 140.300 | 404 | 485 | 697 |
| 1994 | 148.200 | 421 | 499 | 733 |
| 1996 | 156.900 | 443 | 518 | 758 |
| 1998 | 163.000 | 479 | 558 | 821 |
| 2000 | 172.200 | 506 | 598 | 896 |
| 2002 | 179.880 | 535 | 629 | 941 |
| 2004 | 188.900 | 574 | 661 | 986 |
| 2006 | 201.600 | 595 | 692 | 1039 |
| 2008 | 215.303 | 618 | 722 | 1115 |

| | | | | |
|---|---|---|---|---|
| 2010 | 218.056 | 626 | 734 | 1144 |
| 2012 | 229.594 | 652 | 749 | 1165 |

*Source: US Bureau of Labor Statistics*

Questions

1. By what percentage did the CPI increase from 1980 to 2012?

2. By what percentage did the nominal median weekly salary for workers with only a high school diploma increase from 1980 to 2012?

3. By what percentage did the nominal median weekly salary for workers with 1-3 years of college increase from 1980 to 2012?

4. By what percentage did the nominal median weekly salary for workers with four or more years of college increase from 1980 to 2012?

5. Which of the three categories of educational attainment experienced the greatest percentage of "real" earnings growth, when measured using *real* 2012 dollars?

6. When did the CPI experience its most rapid growth, and how did it affect the median weekly earnings for each category?

7. What other conclusions can you make from this data set?

## APPENDIX F

### Assignment: Student Loans and a Comparison of Accumulated Earnings

Part 1

The average salary of a college graduate is $45,000 per year, compared to $30,000 for someone without a college degree. However, college tuition is very expensive and students often need loans to pay for the costs of college. The average student loan debt of today's college graduate is $25,000.

- If the typical student loan has an APR of 4% compounded monthly, and you make payments of $200 at the end of each month, how long will it take to pay off this debt?

- Including the total interest that accumulates, what is the total amount that the student loans actually cost?

Part 2

Assume that the high school graduate receives a 2% raise per year, and the college graduate is given a 3% raise each year. Compute the accumulated earnings for each person during the amount of time that it takes to pay off the student loans. Note, if a person begins working straight out of high school instead of going to college, then he or she could have already earned approximately $120,000 in the four years it takes to graduate from college.

Construct a graph to compare the accumulated earnings of a high school vs. college graduate during a span of twenty years.

## APPENDIX G

### Assignment: Ranking Cities According to the School-to-Prison Pipeline

"In these days, it is doubtful that any child may reasonably be expected to succeed in life if he is denied the opportunities of an education. Such an opportunity, where the state has undertaken to provide it, is a right that must be made available on equal terms."

– Chief Justice Earl Warren, *Brown v. Board of Education* (1954)

Equal access to educational opportunities has been a point of contention for a long time in this country. Civil rights activists argue that underperforming school systems and economic hardships have disproportionately affected African-American and Latino communities, placing incomparable challenges on the minority youth population. The phenomenon known as the "school-to-prison pipeline" is a grave social injustice that has manifested itself in cities across America. Through the "pipeline," students of color are channeled out of the public school system and into the criminal justice system. How does this happen?

Strict school policies may lead to suspensions and expulsions – even for minor offenses. When students are forced out of school for disruptive behavior, they often return home to negative environments and become even further disconnected from their academic studies.

High suspension rates lead to high dropout rates, and impoverished communities with high levels of unemployment offer little opportunity for frustrated youth. Statistics show that strict school policies seem to primarily punish and push out students of color. But do the circumstances actually lead the youth to commit crimes? Investigate this issue and explore how multiple factors come together to create a disaffected youth population.

Utilize the following categories of data to rank twenty-three cities across America, as they relate to the "school-to-prison pipeline." Reflect on the raw data used in conjunction with the weighted sum method, and interpret the results of your ranking system. In your journal reflection, be sure to explain your reasoning behind the weights you assign to each category, and discuss how math can be used to gain additional insight into socioeconomic issues.

**APPENDIX G (CONTINUED)**

Table 19
*Data Set: School-to-Prison Pipeline*

| | Violent Crime Rate (per 1,000 residents) | Property Crime Rate (per 1,000 residents) | Percent Living in Poverty | Unemployment Rate | High School Graduation Rates | High School Suspension Rates |
|---|---|---|---|---|---|---|
| Atlantic City, NJ | 13 | 62 | 37% | 15.4% | 69% | 29.0% |
| Asbury Park, NJ | 14 | 52 | 34% | 5.7% | 68% | 28.4% |
| Camden, NJ | 20 | 37 | 42% | 12.1% | 61% | 20.9% |
| Cherry Hill, NJ | 1 | 26 | 3% | 4.7% | 97% | 4.5% |
| Montclair, NJ | 2 | 19 | 6% | 4.4% | 90% | 8.8% |
| New Brunswick, NJ | 8 | 38 | 27% | 5.6% | 61% | 25.5% |
| Patterson, NJ | 8 | 26 | 22% | 11.2% | 78% | 19.9% |
| Plainfield, NJ | 7 | 24 | 16% | 7.6% | 71% | 25.9% |
| Point Pleasant, NJ | 1 | 12 | 5% | 6.6% | 97% | 3.7% |
| Princeton, NJ | 1 | 19 | 9% | 5.0% | 93% | 0% |
| Watchung, NJ | 1 | 4 | 4% | 4.9% | 98% | 3.9% |
| Westfield, NJ | 1 | 9 | 3% | 3.7% | 100% | 1.8% |
| Iowa City, IA | 3 | 27 | 7% | 2.3% | 91% | 5.8% |
| Baltimore, MD | 14 | 48 | 23% | 7.4% | 74% | 12.4% |
| Boulder, CO | 2 | 30 | 13% | 3.5% | 95% | 4.9% |
| Atlanta, GA | 13 | 62 | 24% | 6.3% | 72% | 28.7% |
| St. Louis, MO | 17 | 63 | 25% | 6.8% | 67% | 31.9% |
| Detroit, MI | 20 | 49 | 26% | 10.2% | 71% | 22.9% |
| Flint, MI | 17 | 41 | 40% | 9.7% | 71% | 33.5% |
| Houston, TX | 10 | 47 | 19% | 3.7% | 78% | 18.2% |
| Pearland, TX | 2 | 20 | 5% | 3.1% | 92% | 3.5% |
| Fresno, CA | 5 | 42 | 25% | 11.1% | 79% | 14.3% |
| Palo Alto, CA | 1 | 20 | 5% | 2.6% | 96% | 2.5% |

Note, this data set is also used for bivariate analysis and linear regression models, comparing:

High School Suspension Rate vs. Violent Crime Rate
High School Graduation Rates vs. Unemployment Rate
Percent Living in Poverty vs. Property Crime Rate

# APPENDIX H

## Assignment: Elementary Data Analysis

The following data was accessed from the International Centre for Prison Studies (ICPS), 2016. The table presents the incarceration rates (per 100,000 citizens), for sixty-five nations.

Table 20
*Incarceration Rates by Country*

| | Country | Rate |
|---|---|---|
| *1* | USA | 693 |
| *2* | El Salvador | 541 |
| *3* | Cuba | 510 |
| *4* | Thailand | 476 |
| *5* | Russia | 450 |
| *6* | Panama | 426 |
| *7* | Belize | 410 |
| *8* | Costa Rica | 352 |
| *9* | Brazil | 307 |
| *10* | Uruguay | 291 |
| *11* | Taiwan | 267 |
| *12* | Mongolia | 266 |
| *13* | Lithuania | 254 |
| *14* | Peru | 251 |
| *15* | Chile | 242 |
| *16* | Colombia | 240 |
| *17* | Kazakhstan | 221 |
| *18* | Singapore | 219 |
| *19* | Czech Republic | 211 |
| *20* | Mexico | 204 |
| *21* | New Zealand | 202 |
| *22* | Honduras | 198 |

| | Country | Rate |
|---|---|---|
| *23* | Albania | 192 |
| *24* | Poland | 187 |
| *25* | Slovakia | 186 |
| *26* | Hungary | 183 |
| *27* | Paraguay | 180 |
| *28* | Fiji | 174 |
| *29* | Malaysia | 172 |
| *30* | Nicaragua | 171 |
| *31* | Ukraine | 168 |
| *32* | Argentina | 160 |
| *33* | Venezuela | 159 |
| *34* | Australia | 152 |
| *35* | Uzbekistan | 150 |
| *36* | Vietnam | 146 |
| *37* | England | 146 |
| *38* | Romania | 142 |
| *39* | Scotland | 141 |
| *40* | Philippines | 140 |
| *41* | Portugal | 137 |
| *42* | Spain | 131 |
| *43* | Bulgaria | 125 |
| *44* | Guatemala | 122 |

| | Country | Rate |
|---|---|---|
| *45* | Bolivia | 122 |
| *46* | Tajikistan | 121 |
| *47* | China | 118 |
| *48* | Cambodia | 116 |
| *49* | Canada | 114 |
| *50* | France | 103 |
| *51* | Belgium | 98 |
| *52* | Austria | 97 |
| *53* | Greece | 91 |
| *54* | Italy | 89 |
| *55* | Switzerland | 84 |
| *56* | Croatia | 81 |
| *57* | Ireland | 79 |
| *58* | Germany | 78 |
| *59* | Norway | 70 |
| *60* | Indonesia | 69 |
| *61* | Netherlands | 69 |
| *62* | Denmark | 61 |
| *63* | Finland | 55 |
| *64* | Sweden | 53 |
| *65* | Iceland | 45 |

A. Use the "Five Number Summary" of the data to produce a box-and-whisker plot. Perform a test for outliers, and clearly indicate their presence, if any, on the plot.

B. Make a histogram of the data set, using exactly eight class intervals. Identify the class boundaries and make a frequency distribution.

C. Compute the standard deviation for the 8 countries listed in

| Country | Rate |
|---|---|
| El Salvador | 541 |
| Panama | 426 |
| Belize | 410 |
| Costa Rica | 352 |
| Mexico | 204 |
| Honduras | 198 |
| Nicaragua | 171 |
| Guatemala | 122 |

Central America.

## APPENDIX I

## Pre/Post Test

On a scale of 1 to 5, indicate how much you agree/disagree with the following statements:

1. Numerical information is very useful in everyday life.
   Strongly disagree 1 2 3 4 5 Strongly agree

2. Numbers are not necessary for most situations.
   Strongly disagree 1 2 3 4 5 Strongly agree

3. Numerical information is vital for accurate decisions.
   Strongly disagree 1 2 3 4 5 Strongly agree

4. Understanding numbers is as important in daily life as reading and writing.
   Strongly disagree 1 2 3 4 5 Strongly agree

5. It is a waste of time to learn information containing a lot of numbers.
   Strongly disagree 1 2 3 4 5 Strongly agree

6. Complete the analogy: "Math is like …"

7. Fill in the blank: 3 x 7 = __________

   On a scale of 1 to 5, indicate your level of confidence with this answer:
   No Confidence 1 2 3 4 5 Very High Confidence
   Comments:

8. True or False? If different people look at the same set of numerical information and do not make any computational errors, then the logic of mathematics will ensure that they arrive at the same conclusion.

9. A. What is your definition of "logic?"
   B. What is the relationship between logic and mathematics?

10. What is the relationship between probability and logic?

11. Explain three different kinds of probability and provide an example of how each one appears in everyday life.

12. Assume that 50% of the residents within a particular town are in favor of stricter gun control laws. If a reporter randomly selects a group of six people from that town, what is the probability that exactly three out of those six people will say that they favor stricter gun control laws?

    a. Less than 33% chance
    b. Between 33% and 67% chance
    c. More than 67% chance
    d. Impossible to determine

    On a scale of 1 to 5, indicate your level of confidence with this answer:
    No Confidence 1 2 3 4 5 Very High Confidence

13. Identify the next term in this sequence: 2, 5, 10, 17, …
    Circle One: 24 26 28 30 32

14. Identify the next term in this sequence: 3, 5, 8, 13, 21, …
    Circle One: 28 30 33 34 36

15. Identify the next term in this sequence: 68, 36, 20, 12, 8, …
    Circle One: 2 3 4 5 6

16. How many different ways can you rearrange the letters of the word C-H-E-S-S ?

17. A gambler must choose between two options:
    - Definitely winning $80.
    - A 40% chance to win $200, and a 60% chance of winning nothing at all.

    State which option is better, and explain why.

18. Imagine that you are a military commander of 600 soldiers and you are being threatened by a superior enemy force. Your intelligence officers report that an ambush is coming, and you must attempt to lead your soldiers to safety by one of two possible routes. If you take the first route, 400 of your soldiers are likely to die. If you take the second route, there is a one-third chance that nobody will die, and a two-thirds chance that all 600 of your soldiers will die. Which route you would take? Use math to support your answer.

19. Chris has a large Yankees bumper sticker on his car. When he got a flat tire on the highway last week, the driver of another vehicle stopped to offer some assistance. Question: Was this other driver more likely to be a Yankees fan, or a Yankees fan and an automobile mechanic?

    Circle One: Yankees fan or Yankees fan and an automobile mechanic

20. In a recent survey, 70% of the Rutgers University faculty reported having brown eyes. In that same survey, 40% of the faculty claimed to be good chess players. Assume that eye color and chess skills are independent of one another, that is, the color of one's eyes have no effect on one's ability to be a good chess player. Determine which of the following would be the most likely outcome, upon meeting a new Rutgers professor:

    a. The professor has brown eyes and is a good chess player.
    b. The professor has brown eyes but is not a good chess player.
    c. The professor does not have brown eyes and is not a good chess player.
    d. The professor is good at chess.

    On a scale of 1 to 5, indicate your level of confidence with this answer:
    No Confidence 1 2 3 4 5 Very High Confidence

21. Define <u>argument</u>:

22. In 2016, a local university studied police records within in a New Jersey town, to investigate the topic of racial profiling as it relates to traffic stops. The lead researcher said, "There is overwhelming data that they are pulling people over based on race." The director of police defended his officers' actions by stating "We are virtually even across the board," and he supported his position by citing the distribution of tickets by ethnicity:

    <u>Facts</u>
    672 tickets to white drivers
    678 tickets to Hispanic drivers
    684 tickets to black drivers

    Do the data support the police director's claim or the researcher's claim? Choose a side and provide a reason to support your answer.

23. Medical doctors agree that aspirin cures a headache. Assume this fact is true: If you have a headache and take aspirin, then your headache will be cured. Which of the following arguments is consistent with this fact?

    (A) If you don't take aspirin, your headache will not be cured.
    (B) If your headache is cured, then you must have taken aspirin.
    (C) If your headache is not cured, then you must have not taken aspirin.

    a. Only (A) is true
    b. Only (B) is true
    c. Only (C) is true

d. Both (A) and (B) are true
e. None of them are true.

24. All inmates have tattoos.
Chris is an inmate.
Conclusion: ___________________________________

On a scale of 1 to 5, indicate your level of confidence with this answer:
No Confidence 1 2 3 4 5 Very High Confidence
Comments:

25. Many studies have researched the relationship between educational attainment and socioeconomic status in America. A recent report from the National Center of Education Statistics (NCES) revealed that 90% of adults who have not completed their high school diplomas end up in the lower class.

Circle the <u>two</u> arguments below that are <u>inconsistent</u> with the findings from the NCES.

a. If you are in the lower class, then it is likely you have less than a HS diploma.

b. If you have more than just your high school diploma, then you are likely to be above the lower class.

c. If you do not have your high school diploma, then you are likely to be in the lower class.

d. If a person is not in the lower class, then he is likely to have more than just a high school diploma.

26. In 2013, U.S. scientists reported the results of a comprehensive study with data spanning hundreds of years. The scientists found that small changes in temperature or rainfall correlated with a rise in violent crimes as well as group conflicts and war. With the current projected levels of climate change, scientists expect the world to become a more violent place. Which of the following arguments listed below is consistent with the position taken by the U.S. scientists? (You may circle more than one).

a. If the world does not see an increase in violent crimes and group conflicts in the future, then we can conclude that the earth's climate will have stabilized and did not change.

b. If the climate does not change, then the world will not become more violent.

c. If the world sees an increase in violent crimes and group conflicts in the future, then we can conclude that the earth's climate must have changed beforehand.

d. Climate change is a result of human pollution caused by greedy economic practices. The system of capitalism has negatively affected our climate and has created an unequal distribution of wealth, which inevitably leads to an increase in crime and group conflicts.

e. If climate change leads to an increase in crime, and a bad economy leads to an increase in crime, then it can be reasonably concluded that a change in the earth's climate will also lead to a bad economy.

27. International experts agree that the primary reason immigrants come to America is the lack of economic opportunities in their home countries. Assume this is true, and identify which of the following arguments is equivalent to the experts' viewpoint.

(A) If an immigrant comes to America, it is likely that he is in search of greater economic opportunities.

(B) If an individual has economic opportunities in his home country, then he is unlikely to immigrate to America.

(C) An immigrant must choose between poverty in his home country and prosperity in America. If there were no jobs in America, then immigrants would not come to America.

a. Only (A)
b. Only (B)
c. Only (C)
d. All three arguments are equivalent to the experts' viewpoint.

The topics of correctional education programs, re-entry, and recidivism, are receiving more attention in today's political discourse. Use the facts below to answer questions #28 and #29. Note, college education programs in prison may be funded by private donors, taxpayer money, or a combination of both.

Fact #1: The U.S. has an incarcerated population of approximately 2.3 million individuals.
Fact #2: The average annual cost to house an inmate in America is $35,000.
Fact #3: The average cost of providing a college degree to an incarcerated student is approximately $5,000 per year.
Fact #4: The national recidivism rate is 67% within the first three years after release.
Fact #5: Recidivism rates decrease by over 40% for inmates that participate in correctional education programs.

28. Based on the above facts, which of the following represents a valid conclusion? (You may circle more than one).

    a. If taxpayer money is used to fund college education for incarcerated individuals, then it will actually save taxpayers' money in the long run.

    b. If a state sees a decrease in its recidivism rates, then it is because the state has offered college education inside its correctional facilities.

    c. If a state offers college education inside its correctional facilities, then it will see an overall reduction in its recidivism rates.

    d. If a state does not offer college education inside its correctional facilities, then it will not see an overall reduction in its recidivism rates.

29. Which of the following represent a <u>flawed</u> conclusion? (You may circle more than one).

    a. These days, more liberal media outlets are presenting a sympathetic side to the causes and effects of mass incarceration in our society, and politicians can gain a significant amount of positive attention by addressing the issue of college education programs in prisons. Thus, politicians should vote in favor of using taxpayer money to fund these programs.

    b. The use of taxpayers' money towards the benefit of individuals that have committed crimes is wrong. Felony convicts pose a threat to society and should therefore not be beneficiaries of public funding.

    c. U.S. citizens must decide between either supporting the use of taxpayer money for college programs in prison, or continuing to see high recidivism rates.

    d. All individuals who participate in college education programs while incarcerated will have successful lives upon re-entry into society.

    e. Celebrity athletes, including many from the National Basketball Association, have spoken out about sociopolitical conditions that have contributed towards mass incarceration. These athletes, given their star power and their lived experiences, should be able to influence legislative changes in our criminal justice system.

30. True or False? Together, correct mathematical computations and valid logical reasoning will always produce a conclusion with certainty. Explain your answer.

*Pre-Test* Expectations for the course:

*Post-Test* Reflections on the course:

## APPENDIX J

### Assignment: The Unavoidable Danger of “if P, then Q” as Premise #1

Conclusions reached through inductive reasoning are labeled either “strong” or “weak” depending on the reliability of their premises. These same conclusions (as we go from confirming local observations to generalized global understandings) become the premises used in deductive reasoning (as we apply global understandings to new specific cases). Accordingly, the premises used in deductive reasoning should have probabilities attached to them. Deductive premises are treated in binary fashion, assigned either a true or false value. Yet, if in reality they are matters of probability, then the conclusion of deductive reasoning must also be viewed from a perspective of probability (in addition to be being labeled as either valid or invalid).

Each of us has our own understanding of the world, and it is captured (both consciously and subconsciously) in a collection of ($P \rightarrow Q$) relationships. For a given topic, ($P \rightarrow Q$) is the personal lens through which we interpret and understand the world. However, there is inherent danger in using ($P \rightarrow Q$) relationships as primary premises for our arguments because they are often based on personal experiences and observations. Our prior knowledge influences how we process new information – it provides a framework for understanding the world in terms of conditional ($P \rightarrow Q$) relationships. But our prior knowledge is essentially based on the generalized results of inductive reasoning, thus it is potentially flawed for each one of us.

We can build an argument that reaches a valid conclusion, but it may not be reliable because of a ($P \rightarrow Q$) premise that we use. We need to constantly reflect on our individual lenses and reevaluate our knowledge. Our understanding of the world is continually evolving because we continue to learn new things every day.

Consider a syllogism about “Whiskey the Cat.” We can use a syllogism to confidently conclude that Whiskey has a tail.

Premise #1: All cats have tails.
Premise #2: Whiskey is a cat.
Conclusion: Whiskey has a tail.

We can reach the same conclusion through a direct argument.

P: The animal is a cat.
Q: The animal has a tail.

Premise #1: $P \rightarrow Q$
Premise #2: P (Whiskey is a cat).
Conclusion: Q (Whiskey has a tail).

The first premise is based on your present knowledge – a lifetime of experiences and observations have given you an understanding that all cats have tails. This is how inductive reasoning sets the stage for deductive reasoning. Using a direct argument (i.e. the tools of mathematical logic) you could reach a conclusion about Whiskey that would be false if your neighbor owned a Manx cat. (Manx cats are a domestic breed of cats that are born without tails, and this piece of information could be absent from your knowledge bank of feline zoology). Deductive reasoning is the process of how we arrive at a conclusion, but inductive reasoning provides us with the premises in the first place. Thus, we are inherently imperfect with the certainty of our conclusions in deductive reasoning when all necessary information is not known, nonetheless, it is still the best approach that we have.

Reflect on your experiences with unreliable ($P \rightarrow Q$) conditional relationships that have been used as premises for arguments with other people. In your reflection, identify at least two examples of ($P \rightarrow Q$) assumptions, and consider how they played a critical role in arguing a particular position.