A LONGITUDINAL CASE STUDY TRACING GROWTH IN MATHEMATICAL UNDERSTANDING THROUGH THE LENS OF THE PIRIE-KIEREN THEORY

By

KARA TEEHAN

A dissertation submitted to the School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of Doctor of Philosophy Graduate Program in Education

Written under the direction of

____

And approved by

____

____

____

New Brunswick, New Jersey

May 2019

ABSTRACT OF THE DISSERTATION

A LONGITUDINAL CASE STUDY TRACING GROWTH IN MATHEMATICAL UNDERSTANDING THROUGH THE LENSE OF THE PIRIE-KIEREN THEORY

by KARA TEEHAN

Dissertation Director: Carolyn A. Maher

How mathematical ideas and ways of reasoning are built, over time, is an important aspect of the concept development for a student in his or her learning process. Using a qualitative, phenomenological approach that is backed by newly constructed video narratives (VMCAnalytics) to illustrate Stephanie's growth in understanding over time, this study analyzes archived data from a ten-year longitudinal study to trace the growth of mathematical understanding of a participant in the longitudinal study from the lens of the Pirie-Kieren model for studying growth in mathematical understanding. Using archived video data, published VMCAnalytics, transcripts, student work, and publications, the study traces growth in mathematical understanding of one student, Stephanie, as she engages in non-routine mathematics problems in formal and informal learning environments. A learning progression was created, attentive to Stephanie's movement in mathematical understanding through various layers of the Pirie-Kieren Model, starting from primitive knowing to formalizing, structuring, and inventising. Attention was given to following Stephanie's folding back in tracking her growth in understanding, particularly as she makes connections and recognizes the structural relationships between and among task solutions. The VMCAnalytics created to trace Stephanie's growth illustrate how she revisits inner layers of understanding to rebuild and extend that understanding. This study contributes to addressing gaps in the literature by focusing on Stephanie's reasoning as she works with a partner, small group, and in whole class settings. It extends the Pirie-Kieren work by attending to Stephanie's growth in collaborative settings. Also, analyses of response to researcher moves and interactions with others, and the influence of these on Stephanie's growth extend earlier work using the Pirie-Kieren framework. This study demonstrates growth in and among layers of understanding though video narratives, with a learning progression showing visual evidence of mathematical growth in understanding. A major finding includes a proposal for an addendum to the Pirie-Kieren model for studying growth in mathematical understanding that encompasses collaboration's effect on individual learners' growth. Another finding highlights the significance of folding back on growth in mathematical understanding. A third finding indicates that interaction with researchers were instrumental in advancing Stephanie's growth in understanding and development of mathematical ideas.

Table of Contents

Chapter 1: Introduction

1.1 Statement of the Problem

The nature of mathematical understanding has become an area of great interest for mathematics education researchers. Cobb, Yackel, and Wood (1992) and Skemp (1976) investigated the characteristics of mathematical understanding. Defining "understanding" is an evolving process. Growth is defined as a dynamic, nonlinear, and recursive process. Pirie and Kieren (1991) developed a theory about studying the way a student grows in mathematical understanding, and provided an accompanying model for the layers of understanding through which a learner traverses. This theory aligns with constructivist views, and focuses on the notion that understanding occurs in action and interactions. Pirie and Kieren's model provides a framework for a study of growth in understanding over time by defining layers of understanding through which a learner traverses.

This research focuses on the process of growing in mathematical understanding of one learner, Stephanie, as she engages in non-routine math tasks. Using a phenomenological approach, video data were analyzed from longitudinal studies of Stephanie as she engaged in problem-solving sessions, clinical interviews, and semi-structured interviews from early elementary: grade two through grade eight. There is a documented need in the literature for progression models that trace the growth in mathematical understanding over time for individual students. This research produced deliverables in the form of a learning progression focusing on one student, Stephanie, with visual evidence for each level of her learning progression in the form of video narratives.

Work has been done on folding back and the role of prior knowledge in understanding, but there is a lack of specific research on the influence that primitive knowing has on the way a student moves through the layers of understanding. Further, the study is unique in that instances of Stephanie's mathematical behavior are identified and analyzed, such as when Stephanie folds back to earlier ideas. These movements were illustrated by events of video narratives (VMCAnalytics), providing visual and auditory evidence of occurrences of folding back throughout Stephanie's progression of growth in mathematical understanding. This research is important because in-depth, qualitative case studies of a single student provide a model for researchers to create individualized studies of other students. Few students are followed longitudinally over years; this study will offer researchers and educators a window into the progression of learning for a single student's growth in learning. Although the findings are not generalizable to all students, they provide important insights into the application of the Pirie-Kieren theory for studying student longitudinal growth in mathematical understanding. This research offers practicing and pre-service teachers a lens of how to be attentive to and accommodate individual learners' needs in building mathematical understanding.

From this research, video narratives (VMCAnalytics) were created to demonstrate Stephanie's growth in understanding over time and to provide educators with a visual model along with text. These VMCAnalytics can also serve to supplement teacher professional development. Together, they show the learning progression of a student, which has the potential to develop insight into their own students' growth in mathematical understanding in similar content domains.

1.2 Background of the Longitudinal Study

Stephanie, a longitudinal study participant, was part of the group of students in the study originating from the partnership between Rutgers University and the Kenilworth School District. The longitudinal study was partially funded by NSF grants.[1]

The study began in 1989 with a class of first graders at a public school in the working-class district of Kenilworth, NJ, and has been referred to as the Rutgers-Kenilworth Study. The research continued following students doing mathematics from elementary grades to high school and beyond beginning with the first-grade students who were grouped together until grade 4. It continued with a focus group of students, along with a few new students who joined in later grades (Martino, 1992; Maher & Martino, 1996a; Tarlow, 2004). The main goal of the longitudinal study research was to explore and analyze how students build mathematical ideas and ways of reasoning. Strands of well-defined, open-ended tasks were developed to understand how students built mathematical ideas, under certain conditions that encouraged exploration and collaboration. During the first few years, researchers met with the students in two- to three-day blocks for about three hours, about four times per year. The content strands

---

[1] NSF grant MDR 9053597 directed by Robert B. Davis and Carolyn Maher, REC9814846 directed by Carolyn Maher, 93-992022-8001 from the N.J. Department of Higher Education.

were number operations, algebra, counting/combinatorics, probability, and precalculus/calculus. Students were asked to work together as partners or in groups to develop meaningful solutions to the tasks and provide verbal and written justifications. In addition to the collaborative problem-solving sessions, students were interviewed by researchers informally and in semi-structured interview settings.

The data for this research come from videotaped sessions from the Rutgers-Kenilworth longitudinal study that took place between 1992 and 2009, although not all years of the longitudinal study were used for this study. These video data are stored at Rutgers Robert B. Davis Institute for Learning (RBDIL). The recently digitized versions of the 4,500+ hours of data are available for study and for ingesting into the Video Mosaic Collaborative repository that currently stores about 400 hours of video data. Along with these video data, records of the student work, questionnaires, researcher field notes, and transcripts are available for analysis. To better understand the development of mathematical understanding of Stephanie over time, her movement through the layers of the Pirie Kieren model for studying mathematical understanding was traced. To address a relevant and apparent need in the literature for investigation of the folding back between layers of understanding, as well as the role of foundational primitive knowledge in understanding, this study supplements and extends a larger body of work considering the role of social interaction, teacher moves, student argumentation, and teacher/student questioning in a student's mathematical growth. To analyze Stephanie's mathematical growth, the transcribed video data were coded to identify segments that identify an appropriate layer of the Pirie-Kieren model in Stephanie's learning. Both auditory and visual data were coded, focusing specifically on the interaction between Stephanie and other students, and the interaction of Stephanie with a researcher. Argumentation produced by Stephanie was given special attention, specifically segments in her argumentation that demonstrated evidence of Stephanie moving between and among layers of understanding.

1.21 Coding. The images and representations that Stephanie created in her progression through the problem-solving sessions were identified through the coding process. This coding involved deductive and inductive methods of analysis (Hatch, 2002). A deductive coding model from Mueller (2007) was used to identify specific codes evidencing that Stephanie's mathematical understanding is progressing, and specifically in what layer. For example, Stephanie's statements about "guessing and checking" and "trying out all options" are coded as "trial and error" and grouped together as a theme about Stephanie's understanding at the level of primitive knowing. When Stephanie builds unique towers from the unifix cubes available, the behavior is coded as image making, providing evidence that she is now in this layer of understanding. In instances when Stephanie no longer uses the blocks but discusses solving a similar problem based on previous experience interacting with the physical blocks, this is coded as "image having" providing evidence of Stephanie's problem solving being at the "image having" place in her learning progression. Specifically, the statements, actions, and interactions of Stephanie with other participants were coded to highlight instances of Stephanie's exploration that provided evidence of being in, or progressing through, levels of mathematical understanding. Thus, through this study, a learning progression was developed showing Stephanie's. growth in understanding of counting tasks. After the initial coding, the data were also coded inductively to develop codes not yet established.

This allowed for additional and more comprehensive themes to emerge. One or more independent researchers reviewed and coded the data as well to establish reliability of the coding scheme, which needed to be slightly adjusted throughout the process. Additional documents, including student work, were analyzed as well. This afforded the researcher the opportunity to verify and corroborate the written and verbal statements, arguments, and solutions proposed by Stephanie.

1.22 Limitations. This is a case study that provides a model for studying students' learning growth. The results show how a learner accesses and extends previously developed mathematical images as the ideas are expressed in more formal representations showing growth in problem solving, reasoning, and justification of solutions. The resulting learning progression that has emerged is unique as it describes Stephanie's journey, over time. While the progression provides insight into one student's learning, it is not offered as a theoretical progression; rather, it is the result of a detailed case study, not to be generalized to other progressions.

1.3 Research Questions

Several questions guided this study:

(1) What are the characteristics of events where Stephanie "folds back" to inner layers of understanding to continue to build her mathematical ideas?

(2) What does Stephanie's learning progression through the layers of understanding look like, and what features of understanding can be defined at each layer and in each event?

(3) What evidence is there of her understanding of the structure of the solution, and solution process?

An outcome of this research was to develop a learning progression for Stephanie’s growth in mathematical understanding over time by analyzing her actions, interactions, verbalizations, and use of tools from the video data of the problem-solving sessions over a period of years. This study traces Stephanie’s engagement with problem-solving tasks, and maps her actions and words to defined layers of understanding.

More specifically, the following questions guided the research:

1. How does Stephanie traverse through layers of understanding over time as she engages in math problem solving tasks?
2. What influence does “folding back” have on Stephanie’s development and growth of mathematical understanding and her movement among layers of understanding?
3. How do Stephanie’s words, written work, actions, and interactions evidence movement among layers of understanding?
4. What evidence is there that Stephanie has traversed to a different layer?
5. In what ways do the roles of social interaction, teacher moves, student argumentation, and teacher/student questioning contribute, enhance, hinder, or otherwise affect a student's mathematical growth?

Chapter 2: Review of the Literature

Mathematical ideas are built up in students through developing meaning, creating representations, and by connecting context to procedures, processes, formulas, and symbolic representations. Relating constructed images to prior knowledge, real world context, and newly-learned mathematical ideas is key. Students construct mathematical ideas and create meaning for those ideas as they build their knowledge base. They explore to understand how different pieces of math connect and support each other. There are multiple theoretical perspectives that describe and represent the creation of mathematical ideas from different lenses with distinctive perceptions. In each of these theoretical perspectives, mathematical learning that occurs is viewed through the lens of the chosen theoretical learning perspective. Students may encounter difficulties in any learning environment supported by any theoretical perspective. We will define and explore the theoretical perspective of "constructivism" since the data collected was from a constructive perspective. This view is concerned with learning in which students create meaning for the mathematical ideas that are investigated.

2.1 Constructivist Approach to Learning

Constructivism is a theory of learning and can be used to explain how particular mathematical ideas are built by the learner. Noddings (1973) refers to constructivism as a "cognitive position" and a "methodological perspective." Constructivism is a cognitive position maintaining that all knowledge is constructed, or built by bringing together conceptual elements. Constructivism is a methodological perspective under the assumption that human behavior is not without purpose, and humans have the keen ability to organize knowledge. A constructivist perspective assumes that humans are knowing, and that they behave with purpose and arrange and categorize knowledge (Magoon, 1997). Neisser (1967) claims that all mental processes, even passive ones, are constructive, and that there is no clearly defined line between cognition and perception. Under this claim, even seemingly routine learning would involve constructing knowledge.

In mathematics education specifically, constructivism entails a belief about teaching that recognizes learners as active knowledge seekers; Davis and Maher (1990) focus on the child's perspective and the "power of a student's own mental representations and the logic of the student's own thought processes" (p. 90). Davis and Maher propose that is essential for teachers to be aware of a student's thinking about a mathematics problem, and for the teachers to try to make sense of what students are doing and why, to gain insight into the way students' representations are being developed (p. 89). Noddings (1990) asserts that even those who embrace pedagogical constructivism and its methods of learning may not necessarily accept constructivism as a basis. While constructivist views differ in some respects, there are general agreements by constructivists about this theoretical perspective on learning. One is that all knowledge is constructed, and that mathematical knowledge is constructed all or in part through "reflective abstraction." Reflective abstraction is "the main mechanism for the mental constructions in the development of thought and mental mechanism by which all mathematical structures are developed in the mind of the individual" (Piaget 1975, p. 143). Piaget describes reflective abstraction as reflection encompassing awareness and contemplation of learning. Constructivists see cognitive structures that are continuously adapting and developing as an integral part of the cognitive activity that is learning.

Constructivism is a theory of learning and knowing that draws from philosophy, psychology, and science (Walker & Lambert, 1995). One general understanding among social constructivists is that learning occurs when there is interaction with others to develop meaning as they learn. Hurst, Wallace, and Nixon (2013) studied the impact of social interaction on learning, and looked at students' perceptions of the value of the classroom social interaction on their learning. Their findings showed that students perceived social interaction having a positive effect on their learning, and this supports Dewey's (1963) philosophy that learning is foundationally a social activity. Hurst (1998) aligns with this philosophy stating that the person doing the work is the one doing the learning. This falls under social constructivism, a theory based on the belief that knowledge is actively constructed by the learner, and that knowing and understanding is an active process where information must be mentally acted upon by the learner to develop meaning (Piaget, 1979).

Pedagogical constructivism proposes ways of teaching that agree with cognitive constructivism. Davis, Maher, and Noddings (1990) write about constructivist views on the teaching and learning of mathematics, highlighting the implications of constructivism in teaching. Noddings (1990) writes that researchers must investigate learners' perceptions, purposes, premises, interaction with environment, and problem solving process to understand behavior (p. 15). Noddings also writes that students will construct, but these constructions should be purposefully guided by math and need for mathematical justification, not simply to figure out what answer the teacher wants, or where the teacher is heading with the instruction (p. 16).

As there are implications of constructivism in teaching, constructivist learning environments can be facilitated and guided by the teacher, with the teacher setting up the conditions for learning under a constructivist perspective based on constructivist learning theory. The teacher will first attempt to understand where students are coming from in terms of background, establish their prior knowledge and beliefs about a mathematical concept, and then create situations that cause students to question their prior knowledge or ideas they have about a subject or concept. Since students may take different approaches to work through problems, with prior knowledge and beliefs influencing their approaches, the teacher can monitor their problem solving and invite students to provide justification and reasoning. The teacher may also stipulate constraints about the problem, and ask for evidence for statements along the path to a solution. The teacher can introduce correct mathematical language and symbols as the students need them in their work. The teacher can also guide the students toward sense-making of these newly-built mathematical ideas. The teacher can provide tools, or direct students toward what they need so that they can either individually or collaboratively work to solve the problems with a new set of mathematical ideas and resources.

2.11 Collaboration in constructivist environments. Constructivist techniques will have individual and collaborative components. Each learner may construct individual, personal, representations and images, and use the learning approaches he or she sees works best. These learners may discuss and cooperate with peers to resolve issues, question each other, and work through providing evidence and justification.

Francisco (2013) offered insight into how learning progresses in social, collaborative settings, discussing how students build from their peers' ideas to augment and extend their own understanding of mathematics. As part of the larger study about the development of mathematical ideas in problem-solving settings, he longitudinally studied high school students collaborating on challenging probability problems in an after-school setting. His findings showed that collaboration was key in fostering mathematical understanding since collaborative activities offer opportunities for students to build on each other's ideas, evaluate the claims they make based on facts, and develop sophisticated justifications.

Martin & Towers (2016) build on Pirie and Kieren's work studying how students learn and construct understanding. They note that the work of Pirie (1996) is still extremely relevant, and they build upon this work, presenting findings about describing and theorizing the growth of mathematical understanding at the level of the individual, as well as of the collection. Martin and Towers look at the growth in collective mathematical understanding and consider the role that the teacher plays. Martin and Towers use Pirie-Kieren theory to analyze collaborative math learning, and noted that it was impossible at times to "develop clear mappings of the growth of understanding for individual learners" (p. 286). Although growth was occurring, it was not located in the actions of individual learners, but was a "property of group interactions that could be observed in the way ideas were picked up, worked with, elaborated, and shared collectively" (p.286). Martin and Towers pursued a new framework to describe collective mathematical understanding and to augment their existing perspectives on individual mathematical understanding. They define Collective Image Making as the process where a group of students work together and no one learner develops an image for a mathematical concept, but individuals offer fragments of ideas which are elaborated on and developed by other members in the group. The group members create a cohesive, collective representation for the mathematical concept collaboratively. Martin and Towers proposed an elaboration and modification of Pirie-Kieren Theory that drew on other theories, extending the definitions of Image Making, Image Having, Property Noticing, and Folding Back to include and account for collective learning and collaborative development of representation. Martin, Towers, and Pirie (2006) discuss kinds of learning and understanding that occurs when learners collaborate on mathematical problems, that can be defined by collective understanding. They characterize collective growth in mathematical understanding as “emergent and improvisational” (p. 149) and discuss implications for teaching practice.

Maher (1990) discusses constructivism as a theory of learning, and not teaching. Davis, Maher, and Noddings (1990) define characteristics of what may be labeled as “constructivist teaching,” but maintain that constructivism regards learning. One such characteristic would be teachers encouraging purposeful use of manipulatives, and to work to promote interactive student exploration in a whole class situation, possibly using models proposed by Davis (1984) or Schoenfeld (1985). Characteristics of constructivist teaching include teachers modeling and eliciting by asking questions, following leads, and encouraging conjectures instead of providing students with clear, immediate answers and procedures. The researchers write that constructivism does not offer pedagogical directions, and point out that telling the students the “correct way” to solve a problem will not suffice. Student learning requires serious consideration of abstracting reality during learning, which may require a change in approach to teacher education (Davis, Maher, and Noddings, pp. 188-189). It is suggested that mathematical activity and reflection on an activity for both students and teachers is important in a constructivist learning environment (p. 170). Some benefits of constructivist classrooms include: student autonomy and being able to encourage independent, yet guided problem solving, using data and manipulatives and interacting with the physical world; driving lessons and delivery of content based on student responses while learning; engaging students in dialogue; and helping students to elaborate their justifications and mathematical discussions.

In summation, constructivism is a theory of learning emphasizing learners' active creation of knowledge through social communication and active involvement in building new knowledge that is created experientially by relating new experiences to prior knowledge. Conversation and social interaction are key components to social constructivism. Reflection and relation to real world contexts solidify the newly constructed ideas. According to Fosnot (1996), manipulatives, tools, and symbols enhance learners' problem solving. Dienes (1963, 1969) proposed that learners be exposed to multiple situations to develop an idea, meaning that tools can help learners create meaning. Piaget (1951) suggested that children between seven and ten years old work mainly in concrete ways, and abstract mathematical ideas can be made accessible to them through practical resources. Based on this notion, Dienes (1963) states that interaction with manipulatives can lead to recognition of regularities and patterns, and "when a rule is found, …children formulate a rule structure and get closure which ties up the loose ends of past experience" (p. 23). Also building on Piaget's idea, Gattegno and Cuisenaire (1954) developed "Cuisenaire rods," which are physical manipulatives students engage with to explore mathematics in a hands-on way, specifically exploring arithmetic operations, fractions, and counting principals. Moyer (2001) writes that manipulatives are objects that are created with the intent to represent abstract math ideas in a concrete way, through their visual and tactile allure (p. 176).

Mathematical ideas are built as the learner relates prior knowledge to new experiences, often with manipulatives or other tools available, and an environment that fosters collaboration and conversations with peers, and opportunities to reflect on the process.

2.2 How Learning Occurs in a Constructivist Environment

A learning environment that embodies a constructivist learning approach will promote experiential learning. There is a necessary connection between the new learning and prior experience and knowledge (Dewey, 1938). Each student creates his or her own learning and discovers individual truths. The constructivist view does not view learning in which students "absorb" knowledge through direct instruction, application of rules without meaning, and memorization of facts and formulae without understanding. Rather, a constructivist approach views the learner as active and engaged in the meaningful pursuit of knowledge, building on prior knowledge, through exploration and active engagement. Students' learning may be expressed by the representations they create. Learners may, for example, begin with building a model using manipulatives and physical models through initial exploration. This provides them with an opportunity to build a concrete representation of their ideas and contextualize by providing a familiar model. One example would be using multibase arithmetic blocks as a model to explore and build up mathematical ideas about arithmetic (Carpenter et al. 1999). Students can then explore new ideas by building new models to represent the mathematical situation they are asked to explore. They can build new mathematical ideas by first creating a representation of the situation, connecting the situation to their solving process, and then relating that solution and method to the context for future attempts at problem solving. In this way, the students can build up ideas and mental representations so that the symbols and definitions that they are eventually given will connect meaningfully to the images they have created.

Since learning is concerned with building meaning, the solutions that emerge from students come from a constructivist process that is grounded in mathematical meaning that students build for themselves. Students are developing their knowledge as they make sense of the models in several contexts in relation to certain concepts. As students build mathematical ideas, interactions and collaboration between students may be a key component in solution development and communication of learning, both verbal and written. As students cooperate with each other, verbalize their reasoning, and question each other, their conceptual building of mathematical ideas can be facilitated as they build on each other's developing knowledge. Learning can be effective when promoted in a collaborative learning environment, suggesting that contact with learning peers is important. Thus, mathematical ideas are built as students develop their mental representations and connect these images to meaning in their own worlds. Thus, a social constructivist environment can offer students an opportunity to build mathematical ideas dynamically.

2.3 Defining Understanding

"Understanding" is a liberally used tem in education. Skemp (1976) makes a distinction between knowledge and understanding and further categorized mathematical understanding as relational understanding or instrumental understanding. From these categorizations came even more characterizations, and eventually a more general view that understanding is the "development of connections between ideas, facts, and procedures" (Davis, 1984), or a process of connecting representations to a structured network. In this research, this definition of understanding will serve as a basis for viewing growth in mathematical understanding, enabling the study of growth using different models and theories as a foundation, guide, and lens.

2.4 The Pirie-Kieren Model

Pirie and Kieren (1994) presented a model for studying the growth of mathematical understanding. It contains eight potential layers, which emerged from considering mathematical understanding as a leveled, non-linear, and recursive phenomenon through which a learner can move. These layers, from the most basic level to the most intricate, include: primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventising. Students can begin in any layer as they grow in their mathematical understanding, but the more foundational layers are still encompassed in the subsequent, outer layers. Students can retrace their steps, revisit layers, and move along the layers in a non-linear fashion. Figure 1 gives the Pirie Kieren model for studying growth in mathematical understanding.

Figure 1. Pirie-Kieren Model for studying growth in understanding taken from Pirie & Kieren, 1994, p. 167.

2.41 Primitive Knowing. At the core of mathematical understanding is *primitive knowing*, a place from which to start, or one's prior knowledge (Saxe, 1988).

2.42 Image Making. The second layer in the model is called image making, and it is the layer in which a student can make distinctions in a problem based on prior knowledge. For example, a student can create images of different aspects of an idea that may or may not yet be connected as a representation. The student may suggest mental representations by building physical models, for example. Students, then, may begin to connect symbolic representations to their actions.

2.43 Image Having. Image having is the third layer, in which a student takes his or her isolated images from the previous layer, and connects them to form a mental representation of the whole process. Pirie and Kieren (1992) describe this layer as the point where learners create a mental image allowing them to continue without reliance on physical objects, or actions with concrete representations, to help them solve a problem. Learners in this layer can visualize and imagine without the physical representations or the processes that provoked the initial creation of the images.

2.44 Property Noticing. Property noticing occurs as learners can observe their own mental image and recognize characteristics and properties of the image. At this point in learning, the learners make connections between and among different mental images, noticing properties of and patterns between individual images. These connections highlight the most significant difference between the image having layer and the property noticing layer. Pirie and Kieren (1992) describe the validation of a connection as a key distinction between the two layers. They suggest that the connections come about from the earlier explorations of the ideas by the learner. Consequently, it is in this layer that learners may notice connections and use those connections between images to formulate definitions.

2.45 Formalizing. Formalizing, the fifth layer, occurs when learners can group together images according to their properties and common traits, and can organize the images into “classes” based on the properties they may have noticed in the fourth layer of the model, describing the classifications of these images results in full formal or informal definitions. Also, within this layer, learners can develop an appropriate mathematical definition that demonstrates understanding of the idea, and begin to connect similar ideas. Physical action involving manipulatives is no longer needed to solve problems as learners begin to develop an understanding of a general rule.

2.46 Observing. In the sixth layer, *observing*, learners can organize thoughts and processes, gaining “an overall conceptualization of the subject matter and its development” (Michener, 1978, p. 376). Learner now make use of definitions, examples, and theorems to connect ideas and move between and among them. This is more complex than property noticing, because learners’ observations become formal.

2.47 Structuring. The seventh layer of the model is *structuring*. A learner at this level can organize previously made observations, and can work to see if these formal observations are valid. At this point, the learner can explain connections between observations. The learner, in this layer, can relate underlying ideas and axioms to the new idea, and can analyze incorrect ideas about the concept. The learner can now view concepts in a logical, *proof-like* manner.

2.48 Inventising. The eighth and outermost layer is *inventising*, which was formerly called “inventing” but was changed to distinguish this layer from actions sometimes taken at the lower levels of mathematical understanding (Pirie & Kieren, 1994). At this layer, a student breaks free of structured mathematical knowledge. A learner in this layer has a rich understanding of the concept and can pose questions that allow the learner to delve deeper and to build new concepts. The learner’s mathematical understanding is not at all restricted, allowing a student to wonder about what results would be if certain axioms were changes, or the problem situation was altered. The learner is investigating beyond the bounds of the problem and concept.

2.5 Conceptualizing the model

The dynamic nature of the model is a noteworthy aspect. A learner’s movement through the layers is recursive, and growth in the outer levels implies embedded inner levels as well (Pirie & Kieren, 1994). A significant piece of the dynamic nature of the model is evident in a process that learners engage in called *folding back*. A student working through a problem may need to fold back to, or revisit, an inner layer to develop and extend a current insufficient understanding. The student will examine understandings at the revisited inner layer motivated by the need to understand at the inner layer in a different way than previously understood. Thus, growth in understanding cannot be identified just from actions in one layer, but must take into consideration inner layer knowledge as well (Pirie & Kieren, 1992).

2.6 The Role of Constructivism

Constructivism plays an important role in this framework. It would be interesting to explore movement through the layers when students are engaged in a constructivist environment, looking at understanding through this lens as a continuous, unending, reflective progression of systematizing knowledge structures. Pirie and Kieren (1992) describe constructivist learning environments as those where learners use their experience and prior knowledge to perceive, act, and organize during their learning, in contrast to specific activities that define constructivism (p. 506). The Pirie-Kieren theory for studying the growth of mathematical understanding aims to help describe the process of understanding growth in mathematics and to observe learning differences between students (Pirie & Kieren, 1994a).

2.7 Mathematical Understanding and Reasoning

Powell, Francisco, & Maher (2003) propose an analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. The model was based on the Rutgers-Kenilworth longitudinal study which involved investigating the mathematical work and growth of students engaged in mathematical inquiry. This study provides an example of what it means to study growth of mathematical understanding and ways of reasoning. This study used longitudinal data across sixteen years with a focus group of students through videotape data to study learning experiences in environments where sense making is encouraged as a norm. The Rutgers-Kenilworth longitudinal study was based on certain goals for tracing the development of mathematical ideas in students over many years. It provided individual and small group case studies on how students build proof-like justifications for solutions to problems, and to connect and understand how previous ideas contributed to these justifications. Another interest was to track how representations come into play for student reasoning and proof-like justifications. The analytical model provided a model for addressing a gap in the literature and studying growth of mathematical understanding.

Martin and Towers (2016) and Martin and Pirie (1998) used the Pirie-Kieren model to study how prior knowledge of a learner plays into building connected mathematical understandings. Additionally, they expanded upon the concept of folding back and explored how teachers can facilitate folding back in students and get them to build their mathematical understandings from their previously constructed knowledge (Martin & Pirie, 1998, Martin, 1999; Martin & Towers, 2016). Martin (1999) studied the nature of folding back within the Pirie-Kieren theory for the growth of mathematical understanding, and this study will build on and contribute this research. Martin and Towers (2016) studied what it means to build and study the growth of mathematical understanding in students and identified folding back as a key aspect, which is specifically motivated and supported by Pirie-Kieren theory. Pirie, Martin, and Kieren (1996) looked at the role of folding back in collecting.

Maher and Yankelewitz (2017) studied fourth grade children's reasoning while building foundational fraction ideas in a cross-sectional study at an elementary school in Colts Neck, NJ. They found that deductive and inductive types of reasoning were used by the fourth graders in supporting their solutions to fraction problems. Similar findings were declared by Maher and Martino (1996a, 1996b, 2000), Yankelewitz (2009), and Maher and Yankelewitz (2017), from the early years of the Rutgers-Kenilworth study with young children, where they found the children used generic reasoning, reasoning by cases, recursive reasoning, and justification by contradiction. They also showed students' justification, argumentation, conjectures, and extending previously developed concepts. These studies used video data to analyze students working together as they built their mathematical reasoning. The students engaged in their fraction tasks by building physical models using Cuisenaire rods as tools, and created numeric and symbolic representations. They explored abstract ideas regarding fractions, such as the density of the rational numbers.

Throughout these studies, conceptual movement was the theme, specifically in encouraging "mathematical behavior" (p. 201). Open exploration was the initial approach in the tasks, where the students and the adults discussed and reasoned through questions together. Later the students' reasoning was explored and documented through various tasks in which they engaged in this collaborative environment. Analyses of students' argumentation as they offered models to support claims are supported by video narratives (VMCAnalytics) available open source on Rutgers Video Mosaic Collaborative: www.videomosaic.org (Maher & Yankelewitz, 2017).

The VMCAnalytics provide the visual evidence and are intended to supplement the analysis and to illustrate with video events, the Pirie-Kieren (1994) model for studying the growth in mathematical understanding. The goal is to illustrate Stephanie's movement along the layers, "fold back", and revisit layers to employ and strengthen her understanding at more inner layers as her understanding is charted moving through the layers of the model. The trajectory described by Pirie and Kieren is exemplified in the video narratives and described in the writing.

There are other approaches to studying the growth of mathematical understanding in learners. Some approaches involve studying students as they work through tasks in different settings like individual clinical interview settings, small group problem solving sessions, and whole class sessions.

2.8 Individual Clinical Interview

One approach to studying the growth of mathematical understanding is in the context of individual clinical interviews. Aboelnaga (2011) conducted a case study of eight-grader Stephanie tracing the development of her algebraic reasoning. The goal of her research was to analyze the mathematical growth and development of Stephanie as she engaged in a teaching experiment of Stephanie's understanding of the binomial theorem and her recognition of its isomorphism to Pascal's triangle. The study examined Stephanie's representations, explorations, justifications, and ways of dealing with obstacles to understanding. As part of the teaching experiment, Stephanie participated in eight individual task-based interviews and from these, the data for the Aboelnaga study materialized. The research questions that guided her study were:

1. "What representations does Stephanie use to construct, develop, and present her responses to the tasks, problems, and/or questions posed?",
2. "What explanations and justifications does she give for her solutions and/or the representations that she constructs?",
3. "What, if any, obstacles to understanding does she encounter?", and
4. "How, if at all, does she overcome these obstacles?"

   In this work, Aboelnaga analyzed Stephanie's performance on the algebra tasks given to her and reported on how Stephanie responded and engaged in explaining her reasoning as she was given time to explore ideas under flexible conditions, having extended periods of time to pursue her mathematical thinking (Maher, Davis & Alston, 1991).

There are some theoretical assumptions that supported the data collection methods applied to the Aboelnaga study. Constructivism, as a perspective for learning, supported the assumption that mathematical knowledge can be built through creation of mental representations. This assumption supports the types of interventions for Stephanie's algebra learning. The study cites Davis and Maher (1990) as advocates for algebra reform, as they emphasized teaching algebra for understanding and not memorization of procedures. They provided examples of contexts that offered opportunities for discussion, exploration, and discovery of important ideas as students worked to build up mental representations of algebra concepts. The environment was designed to facilitate justification and argumentation in active learning and inventing methods to solve problems. Davis and Maher (1990) showed that children learning mathematics can create personal representations to use in problem solving, like the way mathematicians work, an assumption consistent with conditions established in the Stephanie task based interviews.

Another assumption that supported the data collection methods in the algebra study concerns the learning environment for this type of constructivist learning and building of mathematical ideas to transpire. The researcher cites Maher and Martino's (1996b) description of such an environment, and say that there must be opportunities for students to work in multiple social settings, a flexible curriculum that allows additional time for exploration on current or new mathematical ideas, and student thinking that is teacher guided, not teacher directed. In such an environment, a learner can build representations, search for prior knowledge about a content area being studied, and then apply the representation they built to try to solve the problem. The students can simultaneously and subsequently discuss, justify, and argue with their peers about their representations. These assumptions supported the recording of the student while she engaged in tasks, as well as the recording of the interview process to consider the way the student engaged in the learning process and to get an in-depth view and listen in on of her take on each task and its constraints.

The methods of analysis used in the longitudinal study were qualitative. Video involving Stephanie during an early algebra strand while she was in eighth grade were captured. Data of the videotaped sessions were transcribed. The video data were then digitized for the eight task-based interviews. These data included different camera views - a work view and a people view, from which the researcher could capture both Stephanie's written work and the way she interacted with the interviewer. The interview data included questions, answers, and a space to talk about the math she engaged in during school hours. Aboelnaga (2011) notes that this study used Maher and Speiser's (1997) interview structure, which includes engaging the child in exploration, and later student initiated discourse, and then the researchers question the child to push them to form connections and construct explanations. In addressing the research questions, Aboelnaga described how Stephanie explored and built algebraic ideas and meaning while engaging in problem solving. She transcribed and analyzed each of the eight interview sessions individually and chronologically, to gain insight into Stephanie's growth in mathematical understanding by examining her representations, conjectures, and ideas. Claims made were that Stephanie used a variety of heuristics, or methods of investigation used in attempt to learn something, during her problem-solving processes. One finding was that Stephanie used the heuristic of reviewing and writing down a process when she encountered and obstacle to understanding, that she built meaning as a heuristic to overcoming an obstacle to understanding. Another finding was that Stephanie considered a simpler problem during times where she was unsure of where to begin and then connected the solution to the simpler problem to the one proposed. Another heuristic was substituting in numbers to expressions to either test out her simplification of a representation, or to disprove a conjecture. For each finding, the researcher provided one or more illustrative examples from the data, complete with the date of occurrence and detailed description of proceedings. Multiple examples of how a heuristic was used by Stephanie in her attempt to overcome obstacles to her mathematical understanding were offered. In this case study, Aboelnaga (2011) warns that the findings are results of a case analysis and no claims should be made for generalizability with the methods used.

2.9 Small Group Problem-Solving Sessions

Another approach to studying the growth of mathematical understanding is studying learners in the context of small group problem-solving sessions. As part of the longitudinal study of the development of math ideas in students, Maher and Martino (1996a) studied a small group of fourth-grade students engaging in a problem-solving task about combinatorics where they shared their justifications with each other. They examined how the group built up their mathematical ideas under conditions that promoted thoughtfulness. The students were Jeff, Michelle, Milin and Stephanie and the session came to known as, “The Gang of Four”. The students shared their ideas, discussed, and eventually provided a justification for their solution indicating two forms of arguments: cases and induction. Their arguments were on previously built mathematical ideas and prior knowledge, which was recorded and studied by the researchers. Eight sessions are reported, five of which were in the fourth grade, two of which were in the third grade, and one that took place in second grade. Four of the sessions were interviews, and not all sessions focused on all four students. The last session, where the students produced their proof arguments, was the small group problem-solving session where all four students, discussed, and produced their justifications. The data are stored in the VMC Repository allowing access to researchers for tracing details of a learner’s thinking over many years.

The context for the study was established with the view that mathematical ideas are built over time through exploration, social interaction, collaboration, and questioning. The social component of constructivism is an important theoretical assumption in this study, as the collaborative reasoning and proofs emerged as students questioned each other, pushing for clarification, and sharing ideas. The research was based on the idea that sharing personal representations for a problem solution and building on existing ideas as feedback and input from other students is received and as ideas are challenged or supported, learning occurs (Maher & Martino, 1992a). The constructivist-based assumption that supported the study design is that student interactions can lead to students rejecting, modifying, or strengthening their primary ideas and arguments.

Another constructivist-based assumption supporting the study design is that the development of a new idea comes from the process of the student retrieving existing mental representations, reorganizing and constructing new representations, and then extending current knowledge (Davis & Maher, 1990). These assumptions support the structure of the learning environments in which the researchers studied the Gang of Four as the children shared solutions to an earlier problem. This study attended to the forms of student reasoning, examining how solutions become refined, challenged and accepted by the classroom community to form the basis for justifications (Maher & Martino, 1996a). The approach is consistent with Balacheff's (1988) distinction of justification as discussion with a goal of convincing others a statement is true, proof as community-accepted explanation, and mathematical proof as proof that is created and accepted by mathematicians. The data source for this study included use of the library of videotapes, transcripts, and student work data from the longitudinal study to trace the development of students' reasoning over time. The videotaped sessions are dated, named, and described when reported on in this study. The researcher emphasized one videotape "The Development of Fourth-graders' Ideas About Mathematical Proof" as a main component of the study. The researcher additionally used many other videos that lead up to this session to trace the foundation of these ideas and how they developed over time. The researchers illustrated how children, in a natural way, invented justifications for solutions that were "proof like". The process of analyzing tapes, transcripts, and student work of the students as they built up mathematical ideas over time creates a perfect narrative for a reader to follow. The researchers provided visual evidence of the claims through the student work, and reference live data of the sessions to justify those claims with transcripts of the audio included in the report. The authors make no claims for or against generalizability of this study with the methods used.

2.10 Whole Class Sessions

A third approach to studying the growth of mathematical understanding is studying learners in the context of a whole class session. Van Ness (2017) studied students' argumentation about the density of the set of rational numbers. Data for this study came from the longitudinal study and from the videos that were ingested in the VMC. Van Ness created an accompanying analytic from which the narrative descriptions of the sessions come, illustrating the reasoning of a class of fourth-grade students as they attempt to understand how to place fractions on a line segment and explored the density of the fractions. The research question guiding her study was: "How do students reason abstractly about the density of fractions, and how do students concurrently engage in argumentation?"

The theoretical assumptions that supported the data collection methods applied to the study are rooted in constructivism. The view that students build mathematical ideas through active learning, collaboration with peers, discussion and questioning is at the core of this study. One such assumption is that reasoning is important for students to engage in during the process of proof creation. Another assumption is that student engagement in argumentation is important so that claims can be supported or questioned. There is an inherent theoretical assumption that students create arguments naturally when attempting to reason through and justify a solution, and social interaction with peers enhances this process. The method of studying students in a class session where argumentation is prompted and promoted requires support of a constructivist perspective. Another theoretical assumption is that students revisit earlier understandings and ideas, as “folding back” to prior images recursively. This folding back allows them to build new ideas from earlier ideas (Pirie & Kieren, 1994).

A method applied in this study included creating a video narrative (VMCAnalytic) to view and display the data. Each event was accompanied by a video clip, narrative, often with transcripts, to demonstrate the argumentation and discourse transpiring in the problem-solving session. Van Ness shows in her study what student engagement in argumentation and reasoning abstractly look like by creating video narratives. She identified events showing the forms of arguments displayed by the children. Elements included claims, counterclaims, and warrants for justifications. A link is provided to demonstrate the argumentation expressed in each event in the accompanying analytic. Further, her inclusion of clear classroom images of the students, transcripts for their articulations, and visual evidence of their diagrams, provide the necessary justification for her claims. Thus, her method of narrating the process of argumentation through visual evidence satisfies the necessary defense for the claims made. The researcher makes no claims for generalizability with the methods used.

2.11 Longitudinal Studies

One approach to studying the growth of mathematical understanding is studying learners longitudinally as they engage in mathematics problem-solving tasks across many years. Ahluwalia (2011) studied one student, Robert, longitudinally over sixteen years, and analyzed how external representations created by him helped him in building his mathematical understanding over this period. The study shows how Robert built counting techniques through tracing his problem-solving strategies, justifications, and representations, as well as at how he connects his learning to prior experience in problem solving. It examines how Robert makes connections to earlier problem solving, and investigates Robert's ideas about Pascal's Triangle and Pascal's Pyramid. The researcher reports how Robert used Pascal's Pyramid to identify representations of earlier tasks. Brookes (2015) identified student roles in collaborative mathematics groups, and analyzed the ways these roles impact learning of students involved in collaborative mathematics groups. He followed Jeff, a student participant in the longitudinal study, from grade two through grade twelve.

Steffero (2010) analyzed data from a student in this study, Romina, over seventeen years, as she engaged in problem solving tasks. Studying one student over time allows for an in-depth analysis of how he or she grows in mathematical understanding over time, showing how the student engages with the tasks, as well as how the student interacts with peers during problem solving. The overarching goal of Steffero's research was to analyze the connections between the mathematical beliefs of a student and the behaviors she engages in during problem solving. The data for this study came from the longitudinal study at Rutgers, and from the videos ingested into the VMC. Research questions guiding this involved how Romina's representations and justifications developed when she engaged in problem solving, and how interaction with others influenced her ideas. The researcher also revisited Romina's early mathematical behavior during an adult interview with her about her learning views on learning environments and the learning process, emphasizing how Romina's knowing and sense-making was important in her building knowledge. Thus, the researcher aimed to find how math ideas grew as this student engaged in problem solving over time.

Data were collected for the longitudinal study via videotaping. The data that were collected reflected the problem-solving situations that the student was put in, and the way she interacted with the tasks she engaged in. The theoretical assumptions that supported the data collection methods in this study have deep constructivist roots. The notion that mathematical ideas are built up over time through active learning and interaction with peers stems directly from constructivism. An important part of this is the theoretical assumption that "doing mathematics" is "building a collection of individual mental representations that can be applied, revisited, and modified as new experiences are encountered" (Davis, 1984). The theoretical assumption, again based in constructivism, that new ideas come from old ideas informs the data collection methods of this study and the task-based problem-solving situations in which the student was studied. The assumption that conditions listed by Maher and Martino (2000) including time for exploration, student choice in becoming cognitively involved in a task, and teachers who support reinvention of mathematics, are necessary in creating a “culture of sense-making”.

The methods applied in the Steffero study were qualitative, using the videotaped data from the longitudinal study, transcripts, student work, and field notes from the researcher. The researcher analyzed interviews with the student she studied, along with problem solving task sessions. She used an analytical model that used transcription, coding, and narrative to analyze the student interviews. She viewed the videos, transcribed and verified the interviews, identified significant statements, clustered the information by themes, and wrote a descriptive narrative. This whole process was used to describe the student’s behavior. The significant statements summarized the general idea of Romina’s responses, the clustering allowed the researcher to group similar or corresponding statements thematically, and the narrative was Important to describe the way Romina behaved throughout the length of time studied. Steffero (2010) makes the claim that certain instructional interventions support the development of students’ mathematical ideas over time (p.361). She states that the analysis of Romina’s understanding supports this claim, and the researcher analyzed and illustrated Romina’s interactions with mathematical tasks over many years. Thus, her methods justify this claim. The researcher also states that Romina builds ideas through “association” and “relationship”, noting the models she built for her mathematical ideas, and the associations she made between concepts. The researcher provides evidence of this by referencing the videotaped sessions of Romina engaging in problem solving and constructing mathematical ideas, and describes these sessions in detail with provision of transcripts and student work. Again, the researcher’s methods justify her claims providing evidence from video data, noting that is a case study, and cites Stake (1995) who cautions against generalization research where results develop from a case study. The researcher notes that this and most case studies do not allow for generalization or modification, since this is a study of one student and not representative of all students. The researcher notes that while this study was not meant to generalize all mathematics students based on Romina, one may "infer some knowledge of other cases, but the emphasis must be on 'understanding the case itself'" (Steffero, 2010, p.406). Thus, no claims were made by the researcher for generalizability with the methods used, and the author argues to the contrary.

A goal of the longitudinal studies was to gain a deeper understanding of how students develop mathematical ideas under certain conditions where students are given the opportunity to express the way they think about math by building mathematical ideas, creating models, inventing notations, and justifying and extending their ideas (Maher, 2002).

2.12 Polka's Approach

Polka's framework for studying Stephanie's problem solving approaches was used in this study as an additional method for classifying Stephanie's growth in mathematical understanding. Polka's four-step approach to problem solving includes preparation, thinking time, insight, and verification. The preparation step is for the learner to understand the problem by learning the underlying math concepts necessary for the problem, and developing the needed terminology and notation. The next step is thinking time, where the learner will devise a plan and implement strategies such as drawing pictures, using variables, guessing and checking, looking for patterns, and making a list.

The next step is “insight”, where the learner carries out the plan and uses a problem-solving approach to problem solve. Sometimes the learner must start over, try a new approach, or return to an approach later. The fourth step is “verification”, where the learner looks back to see if the solution work and is reasonable (Polya, 1962).

Chapter 3: Methodology

3.1 Design of the Proposed Study

This research took a qualitative, phenomenological approach to studying one student longitudinally from grades two through eight, using video data of the problem-solving sessions and semi-structured interviews, transcripts, and student work from the data base of the Rutgers longitudinal study. The design addressed two methodological issues: studying Stephanie's growth in mathematical understanding as she engaged in problem solving tasks throughout the seven years, and studying how she progressed through those layers of understanding.

To acquire and establish complete descriptions for experiential reflection and subsequent analysis, one must understand the underlying framework for the phenomenon of human behavior. Through interpretation of the original situation, empirical phenomenology provided the foundation for the researcher to understand the phenomenon in the original setting and develop meaning and description of the subject's experience (Moustakas, 1994). Giorgi (1985) supports qualitative, phenomenological research for being an avenue for discovering significance in types of studies such as this one.

3.2 Data Source

This study makes use of video data stored at Robert B. Davis Institute for Learning (RBDIL) from longitudinal studies, partially funded by NSF grants MDR 9053597, directed by Robert B. Davis and Carolyn Maher, and REC9814846, directed by Carolyn Maher, as well as by grant 93-992022-8001 from the N.J. Department of Higher Education, spanning seven years of data of Stephanie and other students doing mathematics. The videos include individual semi-structured interviews, small-group problem solving sessions, and whole class sessions with Stephanie as a participant. Metadata include transcripts of video data, Stephanie's work and the set of problems she worked on over time. The interviews ranged between 30 and 120 minutes. Video data used is stored online in the Video Mosaic Collaborative (VMC). The videos in this database were also used to create the VMC Analytics that demonstrated visual evidence of the growth in understanding of Stephanie (see www.videomosaic.org).

3.21 Setting. Classroom sessions were videotaped with 2-3 cameras, a videographer and sound person. Views focused on the students and on their work. All data have been digitized and are stored at RBDIL, Rutgers Graduate School of Education. There are over 4,500 hours of digitized video consisting of whole class, small group, individual, and interview sessions, and a large subset of those data are from the longitudinal study. Data was retrieved and analyzed that included sessions where Stephanie is engaged in problem solving and when she is being interviewed by researchers. A subset of the data has been ingested for permanent storage on the Video Mosaic Collaborative (VMC) repository. New data was identified and prepared for ingestion from the RBDIL digitized collection. In preparation for ingestion, the full videos of sessions that were analyzed will be summarized and prepared for ingestion as per Rutgers Library standards to enable a search of the contents. Included is student work and relevant transcripts.

3.3 Analysis of Data

Powell, Francisco, and Maher (2003) provide an analytical model through which the interview analysis will be based that includes the necessary transcription, coding and narrative. The interview data and the problem-solving sessions were analyzed using a similar approach. The research used this approach for both the interviews and problem solving sessions. This model was built from the foundation provided by Moustakas's (1994) and Giorgi's (1985) data analysis methods for phenomenological qualitative research. The video data of the interviews were viewed with the intention of understanding the interview structure, and then will be repeatedly watched for confirmation. For any interviews and problem solving sessions in the digitized archive that do not have accompanying transcripts, these transcripts were created and subsequently verified. The researcher then analyzed the interviews and identify significant statements which were clustered by theme into categories. The researcher then developed a table to organize the observations made with the significant statements, summary, and researcher interpretation, as suggested by Francisco (2004) and Moustakas (1994). These steps were taken for the interview analysis and are described in detail. They include: viewing video, transcribing and verifying interviews, determining significant statements, clustering into themes, and writing descriptive narrative with coding scheme.

The researcher observed the videos and listen to the audio of the videos multiple times to learn the structure of the events and interviews, and to understand the progression of student learning in the videos. She followed the Powell, et al (2003) analyses to "watch and listen attentively while making a conscious effort not to view the videos through a specific, predetermined lens" (Powell, Francisco, & Maher, 2003). According to these three researchers, transcribing the data allows the researcher to code a dynamic, constantly progressing problem-solving session and to be able to analyze discourse and communication. Visual and auditory is not always sufficient to get a comprehensive idea of the entire scope of events. Transcripts for video data without current accompanying transcripts were created and checked for accuracy by at least one other person. The transcripts include line numbers, time and speaker codes, and word for word record of what the subjects verbalize.

After the creation of transcripts, analysis of the verbalizations of the subject, and using the model created by Francisco (2004), significant statements were identified in the transcripts based on the statements' ability to summarize the perceived relevance of the statement to the phenomenon being studied, in this case the growth and progression of Stephanie's mathematical understanding over time. The identified significant statements were clustered into categories by "clustering" (Francisco 2004). This allowed themes to be created based on the clustered statements that have certain similarities. Lastly, a narrative describing the structure of the subject's statements was written. In this narrative, the progression of Stephanie's learning and her growth in mathematical understanding as she traverses through the layers of the Pirie-Kieren model for studying growth in mathematical understanding are explained to identify and interpret the phenomenon being studied, which is how a learner grows in mathematical understanding over time and how primitive knowing and prior layers assist in this growth. An accompanying coding scheme was created that encompassed the major themes that emerged during the analysis of significant statements. These schemes were created after codes were identified, and these codes were verified with a group of other research students to establish reliability. Researchers coded with approximately 80% reliability, calculated by comparing aligning codes to total codes identified for both research students when coding a data set. In the initial stages, the research students would code small sets of data individually, identify differences in coding, and discuss how to refine the code definitions to make the coding consistent. The research students subsequently coded a subset of two different datasets together to establish reliability and discuss inconsistencies as they arose. After this, the research students coded a full data set, using video B61 where Stephanie works in a one-on-one interview in fourth grade with Researcher 1, exploring her work with the five-tall towers problem from the class before when she worked with a partner. This individual coding data set was used to calculate the reliability of the coding scheme.

For the problem-solving sessions, in addition to the above process, critical events were identified. These are events that demonstrated significant or obvious change from previous understanding, or a jump from what was earlier understood (Powell et al. 2003, p. 416). The identification of critical events has been widely used in video data analyses (Kick, 2000; Maher, 2002; Maher & Martino, 1996a; Steencken, 2001). As with the interview analysis process, codes were identified, themes emerged, and a coding scheme was applied to develop a narrative. This narrative describes how Stephanie grew in mathematical understanding over time. It identifies key instances and events that demonstrated her arrival at a certain layer of understanding, and provides evidence of folding back or revisiting a previous layer of understanding to strengthen her understanding as she moved to more outer layers.

Chapter 4: Results

4.1 Introduction

The results are organized into two main sections, with the first section presenting a detailed description of the coding processes used to analyze Stephanie's problem-solving sessions. The second section is broken into subsections based on problem solving sessions that Stephanie engaged in, including whole class sessions, one-on-one interviews with a researcher, and small group/partner sessions.

4.11 Coding. The coding processes for all interview, whole class, and small group sessions are detailed in this section, including an explanation of the coding scheme, each code, and how reliability was established. The codes for the Pirie-Kieren layers of understanding are shown in Table 1.

Table 1 Definitions of layers for Pirie-Kieren Model.

Code	Definition/Key Phrases
Primitive Knowing	Background knowledge, a starting place for a student to begin learning at, prior knowledge, foundational knowledge
Image Making	A student can make distinctions in a problem based on prior knowledge, student can create images of different aspects of an idea that may or may not yet be connected as a representation. The student may suggest mental representations by building physical models, for example, may connect symbolic representations to their actions. Drawing a picture, building a model
Image Having	Student takes his or her isolated images from the previous layer, and connects them to form a mental representation of the whole process; learner creates a mental image allowing him or her to continue without reliance on physical objects, or actions with concrete representations, to help them solve a problem; student, now, can visualize and imagine without the physical representations or the processes that provoked the initial creation of the images Student solves parts of problem without reliance on manipulatives/references former process using drawings or manipulatives and draws conclusions without building or redrawing.
Property Noticing	Student can observe his or her own mental image and recognize characteristics and properties of the image; student makes connections between and among different mental images, noticing properties of and patterns between individual images; student may notice connections and use those connections between images to formulate definitions.
Formalizing	A learner can group together images according to their properties and common traits, and can organize the images into "classes" based on the properties they may have noticed in the fourth layer of the model; student can develop an appropriate mathematical definition that demonstrates understanding of the idea. Learner is beginning to connect similar ideas. Physical action involving manipulatives is no longer needed to solve problems as the learner has begun to develop an understanding of a general rule.
Observing	Connection of ideas based on definitions, examples, and learned theorems

Structuring	Proof-like learning with logical organization of observations about problem
Inventising	Non-restricted math learning, beyond bounds of the problem and concept, wondering about differing results if the axioms of the problem were changed

In addition, researcher moves, student collaboration, and problem solving techniques and heuristics are identified in the videos and transcripts.

4.12 Coding Scheme. Establishing reliability. To ensure validity of codes and to minimize researcher bias, a second coder, a Rutgers Graduate Student in the Math Education Master's program, coded multiple documents that the researcher also coded simultaneously, independent from each other. The codes were compared and the coding scheme and definitions were refined multiple times. In addition, one other coder individually coded two sessions independently, and then his codes were compared to the other two coders and reliability was established with about 80% reliability and compatibility. The researcher and the first coder for reliability met multiple times to code small parts of the data together to discuss definition adjustments and identify areas needing refinement and clearer definitions.

4.2 Session Results

Each problem-solving session was viewed multiple times alongside the transcript. The sessions were coded and checked using the process explicated above. Below is a description of the growth in mathematical understanding through the lens of the Pirie-Kieren model, along with identification of problem-solving strategies, researcher moves, and heuristics that affected Stephanie's growth in mathematical understanding

4.20 Session 0: Shirts and Pants Second Grade. Second grade students Stephanie, Dana, and Michael are working in a group during to explore the Shirts and Pants problem. Specifically, the problem is, "Stephen has a white shirt, a blue shirt and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?" Stephanie verbally articulates problem to her group mates. Michael immediately says, "You can only make two outfits." Stephanie argues against Michael's solution candidate, proposing that "a whole lot of different outfits" can be made. Dana claims that "he can make all three of these shirts with it", and Stephanie says, "shh" and then argues that you can make different outfits and she give an example, saying, "Yeah, but shh. You can make it different ways too just like look white and white, that's one by doing W and W. Two could be blue, blue jeans and a white shirt."

Stephanie makes a pictorial representation and labels her shirt colors with letters, saying, "I'm going to make a shirt and I'm going to put W for white in it." The students each begin to create a representation of their problem solving by drawing their own picture. Dana proposes that yellow and white cannot be an outfit since they don't match, demonstrating her interpretation of the problem's constraints, saying, "The yellow shirt can go with the white." Stephanie proposes her interpretation of the constraints of the problem, claiming that if a shirt and pants pair are matched together, they count. She says to Dana, "But how many outfits can it make it doesn't matter if it doesn't match. As long as it can make outfits. It doesn't have to go with each other Dana."

Stephanie lists out different outfit options, saying, "Number four it could be blue shirt and blue pants. Number five can be a white shirt and wait it can be a blue." Stephanie claims that there are five combinations and uses her diagram as justification. She is using the strategy of guess and check to randomly create outfits by drawing the shirts and pants and labeling with a letter to represent the color, and listing out "1, 2, 3…" with "WW" for white pants, white shirt, "BW for blue shirt white pants, and so on. She uses her diagram representation as justification, saying, “It’s a blue shirt with white pants. A yellow shirt, did I do yellow and blue? A yellow shirt and blue pants. You know there’s five combinations. There’s only five combinations because look.”

Stephanie comes up with five possible outfits, and verbalizes the different outfit matches that she has, saying, “You can do this, listen Michael. You can do five combinations. You can do number one – White and white, number two blue and white, number there, yellow and white, number four blue and blue and number five yellow and blue.” Stephanie voices that she is “sure of” her answer. She says, “You can do [five] combinations Michael. I’m sure of it.” Stephanie is learning in the primitive knowing layer of the Pirie-Kieren model. She is working to develop a solution using what she knows about matching outfits, and using a guess-and-check heuristic to make a prediction about the number of outfits.

4.21 Session I: Shirts and Pants, Third Grade. Third graders Stephanie and Dana are engaged in an open-ended task called “Shirts and Pants.” Specifically, the problem is, “Stephen has a white shirt, a blue shirt and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?” The video, transcripts, and student work were analyzed, with particular attention being paid to Stephanie’s learning through the lens of the Pirie-Kieren model: how her words, written work, actions, and interactions evidence movement among layers of understanding. In addition, the ways that the roles of social interaction, teacher moves, student argumentation, and teacher/student questioning contribute, enhance, hinder, or otherwise affect a student's mathematical growth were observed and analyzed.

During this interview session, Stephanie traverses among layers of the Pirie-Kieren Model. Researcher 2 poses the problem to students and emphasizes the importance of the problem-solving process, not simply the students getting to a correct answer. Researcher 2 prompts the students to provide reasoning for their solutions, and promotes argumentation by asking the partners to come to a single agreed-upon solution. She says,

> "...what you have to do is solve the problem, but it is like last year guys I'm really interested in how you solved the problem. I want you to be able to explain that to me on your paper. Okay, so say your answer was twenty-four or something like that. That's fine and good, but I want to know how you got that twenty-four okay? And you can do that in any way that you'd like. You can write, you can draw, whatever. Explain that to me but whatever you do on here, I want you and your partner to decide what you're going to put on the large paper."

Stephanie works with a partner, Dana, creating pictures to represent white, blue, and yellow shirts and blue and white pairs of pants using letters to represent the colors of clothing pieces. They keep track of their outfit combinations by drawing lines from shirts to pants. Stephanie says, "Well why don't we draw a picture?" and proceeds to use markers to recreate the problem in visual form. Here they are moving from primitive knowing, where their prior knowledge includes that shirts and pants together make an outfit combination, to image making, where the students are using physical drawings of the situation to represent the problem and work out their solution. The two students each create their own drawing, and Dana comes up with six outfits while Stephanie only comes up with four initially. Stephanie asks, “What are your other combinations? I have white and blue, I got white and white, I’ve got blue and white, I’ve got yellow and white. What were your two other combinations?” Dana recognizes that Stephanie missed the yellow and blue, and the blue and white combination so they both have the same number of combinations. Dana then states, “Amy we’re done”, and Stephanie counters, “No Dana we don’t know if we can make any more combinations or not”, which shows evidence of Stephanie wanting to find reasoning beyond the picture for why there are no more than six outfit combinations. Researcher 2’s teacher move further prompts their argumentation, saying, “Make sure okay? Talk about it and make sure.”

When Stephanie and Dana insist there are six outfits, Researcher 2 prompts the students more, saying, “Are you both convinced of that? Can you explain it to me?” Stephanie and Dana alternate to provide their reasoning, describing their process of drawing the shirts and pants, drawing the lines to make the combinations, and then counting the number of lines to find the total number of combinations. Stephanie justifies why these lines create all possible outfits, saying, “So we could make sure that we were, that we did not do that again and say seven, eight, nine, ten. We drew lines so that we could count our lines and say oh we can’t do that again, we can’t do that again.” This is her justification in her argument.

As an extension to this problem, Researcher 2 poses the problem “How many outfits can Stephen make if he has an additional pair of black jeans?.” At first, Dana proposes twelve outfits can be made, after counting by threes. She immediately responds, “It would be twelve. Like everything goes with black. Cause six plus six is twelve.” Stephanie proceeds to draw the additional pair of pants on her paper, and connect lines from each shirt to this new black pair of pants. She keeps track of the pants and shirts, verbalizing her work saying, “White, black, wait blue, and yellow. And then we have white, blue and black. Okay let’s see. White, (inaudible) and nine, Dana it’s nine!.”

Stephanie counts the total number of outfits created by the connected lines and realizes there are nine possible outfits, instead of Dana’s proposed twelve. Stephanie comes to her solution by drawing out the shirts and pants in the conditions of the problem, and counting up the number of outfits. She is still learning in the image-making layer of the Pirie Kieren model, because she is creating the outfits and keeping track of them by making drawings and working out the problem on paper. As a method of problem solving, Stephanie’s solution strategy includes checking her work by having Dana and her each work to their own solutions individually and then compare, stating, “No Dana first I want you to figure it out, we may get different answers. Look, see you got white, white, and white.” Dana subsequently comes to the same solution after creating and counting up the number of outfits. Stephanie decides that the second candidate solution, nine outfits, is correct, saying, “So Dana your calculations were wrong. The answer is nine.” The researcher comes over and prompts the students to explain how they got their solution of nine outfits, and Stephanie responds, “See we drew shirts, and since each one of them can go to three pairs of jeans, three, six, nine. Because there are three pairs and shirts and three pairs of pants. If each shirt could go to three pairs it would be three, six, nine. Three on top, three on the bottom.”

Researcher 2 promotes argumentation by further questioning the students, “What do you think if you had four pairs of jeans? What would happen? Think about it, you don’t have to do that one.” She proposes that they use their created images to solve this new problem without drawing out all the outfit combinations. This strategy by the researcher not only prompts argumentation, but this researcher move ventures to help Stephanie move from the image making layer of the Pirie-Kieren model to the image having layer.

4.22 Session II: Stephanie explores Towers problem. In this session, Stephanie is introduced to the Towers problem by Researcher 2. Stephanie works with her partner Dana to build unique towers of height 4, selecting from two different colors of cubes, and to figure out exactly how many of these unique towers they could make. Researcher 2 poses the task to the students, “How many different towers four blocks tall can you build when selecting from two colors?.” Stephanie and Dana’s first approach is to work separately to build towers. They do not have enough cubes, so they collaborate and work to develop a solution together by comparing their towers, and eliminating repeated towers.

During the session, certain researcher moves provoke the students to use multiple problem-solving strategies and Stephanie re-represents created images via drawing. These moves help the students to engage in the problem using approaches such as guess and check and collaboration. The actions taken by the researcher helped the students to move from the primitive knowing layer of learning from the Pirie-Kieren Model, to the image-making layer. Evidence of this is apparent throughout the video and transcript of this session.

When introducing the problem, Researcher 2 asks the students to convince their partners that all towers have been found, once they come up with a solution. In doing so, the researcher prompts the students to justify response and provide reasoning for their solution, stating, “I want you to talk about it with your partner and again it’s like the shirts and pants, you have to convince that you found them all.” In this statement, Researcher 2 is also comparing the requirements of this problem to those of the Shirts and Pants problem, which the students engaged in previously. This is a researcher move that attempts to get the students to access prior knowledge of a problem’s requirements.

One problem solving strategy that Stephanie uses is collaboration, at the indirect suggestion of Researcher 2 when she says, “Are you two working together?” Stephanie and Dana decide to compare the towers that they have each made individually to see if they have some of the same types of towers. Stephanie says, “If we worked together, we would have more blocks and more combinations. Let’s see what we can eliminate.” She proceeds to compare her towers and Dana’s and eliminate towers that are the same, keeping those that are unique. This compare and eliminate method is a problem-solving strategy that they chose as they worked through the problem. Stephanie offers a solution candidate, saying, “I think we’re only going to be able to make seventeen [towers].” Stephanie cites their collaboration strategy, explaining, “Dana built them, I checked them.” When the researcher asks the students if there is a way to be sure that they have all towers without making any twice, Stephanie suggests taking each tower and comparing it to every other one, and them pushing back the ones that don’t match.

Throughout this task, Stephanie uses much background knowledge and prior knowing. Evidence of Stephanie learning in the primitive knowing layer of the Pirie-Kieren model is demonstrated when Stephanie is trying out different patterns for the towers, which is what she can do initially with the task. She says, “I’m gonna make this kind of pattern. And then I could make red on the top and blue on the bottom. That’s a different tower. Then I could make all red, all blue… I know a different combination. Red one, and one, two, three, one…” Stephanie shows that she knows how to build towers with different orders of the cube colors, that she can differentiate between seemingly unique towers, and create towers in a random fashion.

Stephanie moves from primitive knowing to image making during her engagement with this task. Stephanie is dependent on the physical towers that she builds to figure out the number of towers created and their uniqueness. Her place from which to start was her knowledge of towers, what different towers look like, and how to build towers. Stephanie makes distinctions between repeated towers that needed to be eliminated when physically comparing towers. Her ideas are not yet connected as a full representation, showing that Stephanie is in the process of image making. She is “tied to the action and doing", which is a characteristic of image making. She is doing the physical action of creating towers and double checking to get an idea of the concept, evidencing learning in the image-making layer, with physical towers serving as a representation for her ideas. Stephanie says, "…we had to build them straight so she got the idea of taking them and making patterns" and proposes, instead of just checking all the towers against each other, to "take this one and check it and put it back in its spot" as another way to check for towers that are the same. Stephanie depends on her physical towers to explore the constraints of the problem.

Stephanie decides to draw a picture of the towers as another representation of her ideas, stating, "I'm gonna write a picture for mine" as she double-checks the towers she has built. Stephanie is representing her image using additional media, which is not validated by the Pirie-Kieren model. Her picture serves as a heuristic to keep track of her work and try to come up with additional patterns. These multiple representations form a foundation for folding back that Stephanie later uses to find evidence to support a claim that she makes.

Stephanie recognizes characteristics of her towers, naming towers that are the same, just oriented differently sitting on the table, as "cousins." For example, she says that two towers are cousins because, "this one has blue on the bottom and this one has blue on the top, turn one around and they're the same." She uses this to find towers that are not unique in her collection, further evidencing her learning in the image-making layer, depending on the physical towers to explore the problem. She represents her ideas further by classifying certain duplicates as "cousins", recognizing patterns within this strategy of recognizing duplicate towers.

When another student says that there are seventeen total towers, Stephanie states, "Then you must have something that matches 'cause we got sixteen. Double check…." Stephanie says that she is not sure how she knows that she got all of them with sixteen towers, and decides to try to make one more tower. Soon, she says, "I don't think we can make another one. I really and truly don't." She says that double-checking is how she was convinced that she had all towers, again demonstrating dependence on physical model and evidence of her movement to, and learning in, the image-making layer. She is trying to build reasoning for her conclusion of sixteen four-tall unique towers that could be made from 2 colors of cubes.

4.23 Session III: Grade 3 Towers Additional Problem. After the class has engaged with the four-tall towers problem and discussed as a class to decide on sixteen towers as the final answer, Researcher 3 poses a question to the students about what would happen if she changed the condition of the problem to require three-tall towers instead of four, while still selecting from two colors for the cubes. She says,

> "Okay, I'm going to ask all of you to think for a minute. I want you to think really hard and see if you can imagine. Suppose instead of towers that had four cubes, you could only have three cubes in each tower. Do you think there would be more towers or do you think there would be fewer towers? What do you think? This means, if you have only three box in each tower, you think there would be more towers than sixteen or do you think there would be fewer towers than sixteen? What do you think?"

Researcher 3 prompts the students to consider whether there would be more or fewer towers for the three-tall case than the four-tall case. One student says, "There would be more towers", and the researcher asks him how many he thinks there would be. He says, "Nineteen", to which the researcher asks for him to explain. The researcher calls on a student, Brian, to see what he thinks and he responds that there would be more towers because there are a fewer numbers of blocks." There is some discussion around the class, with one student stating that with a shorter tower height, there are more blocks available to make towers out of, and thus a greater number of towers can be made for three-tall towers, reasoning which is repeated and summarized by the researcher, saying "oh, you mean you would have more blocks than towers out of… but because there are more blocks you think there might be more towers." There is more discussion, and mixed answers for whether more or fewer towers can be made for the three-tall.

The researcher invites students to share their ideas about the patterns for towers that emerge for three tall. One student says that because towers with a height of three requires fewer cubes than a tower with a height of four, there would be similar patterns but fewer numbers. The researcher asks the student why he thinks there would be "the same patterns." The researcher asks Stephanie and Dana to find a tower different from ones produced by another student. She then asks the students to explore the problem, saying "I will let you think for a little bit. I want you are your partner to come up with a good guess."

During a subsequent class discussion, Stephanie shares that she and Dana think that the same number of towers, sixteen, can be produced for three-tall towers as four-tall. She says, "Well, because you are just taking one away from here it's not like it's going to change the whole thing." The researcher clarifies, saying, "Okay, What I hear Stephanie and Dana are saying is, if you can take one away it doesn't change. That's what they are saying. So that's your argument for there being sixteen." One student suggests taking one block off from each pattern and counting the number of towers. The researcher says to the students, "Remember that each one has to be different." Stephanie begins counting towers and observing them, realizing that some towers are the same. She says, "so we can have one that looks like this. Red-red-blue and red-blue-blue. Yup we do. And, let's see. If we had something like blue-red-blue…." They try out a few more patterns of towers, seeing if there are one or more of these towers. Stephanie says, "There are two red-blue-blue. Oh, so that would be less than sixteen. So we take this one away and throw it in the trash," and reasons that taking away a cube sometimes gives two of the same tower. She continues to compare towers to find the duplicates, saying, "So we are taking away one part from the tower. And when you take them apart, they can be different", asking for a tower's "match" for duplicate towers. Stephanie concludes that there are eight towers possible, and the other towers were matches for those eight. In this session, Stephanie folds back to her image making to try to find evidence to support her claim that there will be the same number of towers for three-tall as four tall, and while folding back to her images created during the four-tall tower problem, she recognizes duplicate towers and states that there are less than sixteen towers for the three-tall condition. She has rebuilt her understanding, and goes back to comparing towers to eliminate those duplicates, eventually coming to the answer of eight towers for three-tall.

4.24 Session IV: Grade 3 Towers interview (4-tall, 3-tall). After engaging in the class task of trying to find out how many unique four-tall towers could be made when selecting from two colors, Stephanie sits with the researcher in a one-on-one interview session. The researcher asks Stephanie how she was sure that she had all towers, because she "seemed very definite that 16 was all, and some people were saying 17 and 18." Stephanie responds, "We had to check it a couple times and we tried to make some different ones but we were checking and checking and they all came out the same." Stephanie is providing an explanation of her answer and a reason for how she knew her answer was right. She is describing her comparison method for making sure each tower did not match up with any other that she had already created.

During the classroom session, the researcher had posed a follow up question about what would happen if they had to make towers of three tall, while still selecting from two different colors of cubes. Stephanie acknowledges a misconception about thinking that more towers can be made from the additional blocks coming off the four-tall towers, saying, "…you might think there would be more because there's less blocks and there's more combinations you can make. There's less because once you take one block off, [the towers can be] the same." She says that there would be less towers for the three-tall than the four-tall tower condition of the problem, specifically eight versus sixteen respectively. The researcher asks how Stephanie figured that out. Stephanie says, "we pulled the blocks off and then we matched them up. So it was a matching game. One block off could mean a big difference", and then goes on to describe an example where taking one cube off a tower creates a tower that is the exact same as another tower of height three. In this interview, Stephanie's description of her problem-solving process shows evidence of learning in the image-making layer of the Pirie-Kieren model, as she depends on the physical building and reconstructing of towers to develop her solution to the number of towers that could be made.

4.25 Session V: Grade 4 Five-Tall Towers Problem Classroom Session. In fourth grade, Stephanie investigates the towers problem for towers five cubes high. This is a classroom session taking place on February 5th, 1992. After the problem is presented, Researcher 1 emphasizes that the students must convince each other and the researchers that they have found the exact number of possible towers for five-tall towers, selecting from two colors. She says, "…and you have to be able to convince us that you have found all possibilities-that there are no more or no less. Got the problem? Have fun!" This is a researcher move that prompts the students to justify their answer instead of simply providing an answer for the number of towers. Another researcher move is the last sentence of her directions, where she tells the students to "have fun." This encourages students to enjoy the task and gives a positive connotation to the task and the mathematics. This sets the tone for the tasks.

Stephanie and Dana begin creating five-tall towers. Stephanie says, "Okay, we'll start out with the easiest one. One, two, three, four, five reds and five yellows." Dana says that she has five towers and Stephanie says that she has four, and suggests standing up the towers to compare. She says, "Stand them up straight so we know what we have." This is a strategy for checking the towers to make sure the towers are unique and none are missing. Another strategy that Stephanie uses is creating opposites. She is working with a group of students who are building towers and their opposites, and pairing them together. When Dana says that she has another idea, Stephanie says, "Well, tell me it so that I can do the opposite." This is a strategy for working together to create types of towers in pairs. Stephanie creates all the towers with a red cube at the top, holding that one cube fixed and then changing the other cubes in the towers. She comes up with a strategy when she is controlling for variables, with the red cube at the top being the constant variable. Stephanie folds back to when she was building towers to find all the possibilities when she first created images during the four-tall towers problem. She is now rebuilding her understanding with an organized, variable controlled process. She says, “I’ll do the red- and I’ll do it with red at the top.”

Stephanie folds back in her understanding again to when she classified towers as “cousins” when they were the exact same tower, but oriented differently where because of the way they were laying. The top and bottom cube lost generality, and she recognized that these were different towers during the previous session. In this session, Stephanie does not call tower pairs “cousins”, but she does recognize towers being the same if you just turned one around. She says, “No, it doesn’t [make a pair] because if you turn it around, it’s the same, so that doesn’t go with that one.”

4.26 Session VI: Grade 4 Stephanie Revisits Five-Tall Towers Problem Interview. In the day following the class session where Stephanie and her classmates worked on the five-tall towers problem, Stephanie sits down one-on-one with Researcher 1 to discuss the problem-solving strategies that she employed. Stephanie participated in this interview in the fourth grade, and discussed for about 48 minutes how she and her partner Dana had engaged with the towers problem on the day previous, attempting to figure out exactly how many unique towers of height five could be built when selecting from two colors of Unifix cubes.

At the start of the session, Researcher 1 prompts Stephanie to describe how she and her partner Dana worked on the problem. She says, “So how did you work together?”, which was a researcher move prompting Stephanie to explain her collaboration with her partner. Stephanie responded that she would build a tower, Dana would build its "match" (the tower with opposite coloring), or they would switch roles. Stephanie states that they made 32 towers, and the researcher asks, "Do you believe that's the answer?", which is a researcher move prompting student evaluation of the answer, provoking reasoning and explanation. Researcher 1 asks Stephanie to create an argument that could convince someone that her proposition of 32 possible towers unique towers was correct. Prompting justification as another researcher move, she says, "So if they said to you, 'I think there were thirty'. What would you say to that person?" Stephanie responds, "Well, I would say like what we did yesterday, when we were up at the board with the one block yellow, and then the two [blue blocks]", and then goes on to explain her process.

Stephanie begins building towers with a staircase pattern, with the first having three yellow and two blue cubes consecutively, the next having two yellow three blue all consecutively, the next having one yellow followed by four blue cubes, and the last having all blue cubes. Stephanie is using previously created images that she constructed while physically building towers. Her use of this created image is evident when she says, "And then there's five right there and then you build it backwards." She does not actually build this second set, but recognizes its properties, thus evidencing her learning in the image having layer of the Pirie-Karen model. Stephanie's "building backwards" is a strategy that she uses to create different types of towers. Stephanie goes on to explain more, and asks, "Alright, should I draw the patterns on the paper?", suggesting creating another representation of her work, which is a heuristic for showing her reasoning for being able to create 32 towers. She names her set "One to five."

Stephanie then classifies the next type of tower group, with two blue cubes together that start at one level and one spot with the creating of each new tower. She names this "two blues together." To demonstrate these towers, Stephanie draws on her paper, labeling "B" for blue and "Y" for yellow. This is a representation of her problem-solving process that does not involve building the towers. Stephanie has created an image based on her experience building the towers, and now is re-representing this image to reconstruct her problem-solving process.

Stephanie states that she would replace each of the blue cubes so that a yellow cube takes its spot. She says specifically, "And then we reverse it…and then it would be two yellow together." When doing this, Researcher 1 pointed out some towers that were duplicates between the two groups, and Stephanie discusses how in the previous day, she recognized some of her towers were duplicates, yielding more than 32. Stephanie says, "We ended up counting a lot over. We had thirty-four so we subtracted I think three groups, because we were down to twenty-eight then we added two groups." Stephanie provides reasoning for why she ended up with duplicate towers. Specifically, she says that she didn't cross check for duplicates across pattern classes. She says, "We kept finding different patterns, but we didn't check it with the other patterns."

Researcher 1 prompts Stephanie to assess solution and provide justification in a researcher move, asking, "Are you convinced?", following up by asking why she is convinced. Stephanie provides reasoning, saying, "Well, because we did these groups with the orange and the blues- the yellow and the blues. So, you know that this group is over, so you can't make another group like this." She explains a trial and error approach to making sure a tower is not a duplicate tower. She also says that if a person were to create another tower, it is either a duplicate, or a tower of a taller height would need to be created. She specifically says, "You can only build it five high you'd have to have it so it would be seven high, not six high to build another one." This is evidence of Stephanie learning in the image having layer. Stephanie does not have to build towers to realize that another tower of this type would not fit in the class, and would create. Based on the required conditions of the problem, one would need taller towers to accommodate another tower in one of her sets. Carolyn asks if putting an extra cube on the bottom would suffice, and Stephanie provides reasoning through informal proof by contradiction, saying, "Then you would be making it over. If you put it here (starts to redraw column three by writing B and B in the first two rows) you would be making what is here. (points to column three)", referring to towers she had already made.

To describe how she came up with the rest of the towers, Stephanie says, "All right. Well, we just went and built patterns. Another pattern is this one. One blue, one orange, one blue, one orange, and one blue. And then you can make the opposite, which is orange, blue, orange, blue, orange. That's the opposite one. She refers to creating "opposites", which is a strategy and image that she created previously. After Stephanie creates multiple types of towers, including "three in the middle," "one in the middle" and the reverse of those towers, Researcher 1 says, "Oh, now this is interesting, you have either a yellow in the middle or a blue in the middle and all the rest are the other color. And here you have three in the middle, you don't have that any place else and in the end you have the other color, interesting. I haven't seen anybody else do this. I am glad I had a chance to talk with you, this is different." It is evident that Stephanie has developed a unique solution during her solving process.

Stephanie demonstrates evidence of image having as she talks about how she came to 28 towers. She does not have to build the towers, yet she can discuss different cases for each instance. She says, "That's twenty-six, and then we reversed it so that - we reversed it and then - we went down to the second one. And same thing here. And that's two and that's twenty-eight." In particular, this is evidence of Stephanie generalizing as she describes doing the "same thing here", without actually going through and building the tower. As a researcher move, Researcher 1 asks Stephanie to compare towers to look for similarities without pointing out duplicate towers, saying, "Okay, let's look at these for a minute then (pause) Are any of these alike in any way?." Stephanie is able to describes towers in her sets, evident when she says, "Oh right. Well, these two ((points to Y, B, B, B, B and Y, Y, Y, B, Y)) are somewhat alike because they both have one but at a different place. We just moved [it-]."

As Stephanie begins talking about the "first spot" and "second spot" of the tower, she clarifies that she is working from the top floor. She describes moving one yellow cube among the spots in the tower, starting from the "top floor" and moving it down one position each time. As she did this, she checked to make sure she eliminated a tower if it repeated the pattern of a tower that she had already made. Researcher 1 asks Stephanie to think about and try to develop an "absolute" way of being certain that she has found all of the towers, with no duplicates for the next time they met.

In a researcher move, Researcher 1 says to Stephanie, "Let me ask you another question. Suppose I was building towers of four, what do you think?." Stephanie says, "I think you would get less because with five you have 32, so you are subtracting now, you get less. If you were adding one, you might have gotten more." Stephanie provides a counterexample, and evidences learning in the image having layer of the Pirie-Kieren model. She uses no physical towers to make her prediction, and provides a counterexample as her reasoning, building on her previously created images from when she worked to build towers of height five. In another researcher move, Researcher 1 proposes writing on paper for additional means of representation, taking individual think time, and keeping track of the towers by numbering them. She says, “Do you have any idea how that would work? Think about that for a minute. (pause) get a piece of paper. Also number this one. We lost track of our numbers here.”

Stephanie responds by making predictions. She proposes candidate ideas like having ten towers in a set as opposed to eight, when going from towers of height five to four respectively. Stephanie says, “This one is four (pause) this one is five (numbering pages). Hmmm. Mum, hmm. It would probably work same way we worked with five, like (pause) one blue two blue three blue four blue only you would go to four. So instead of having ten on this you would have eight.” This prediction again shows Stephanie learning in the image having layer, as she is noticing some features of her sets and using created images to make predictions.

In another researcher move, Researcher 1 says, “Okay, write that down. You can go through your thinking. You can look at anything you have done so far. Yeah, write the eight. Try to keep record of what you are thinking”, proposing keeping a record and thinking through the process after. As Stephanie works through building the towers, she realizes that some tower types would be impossible to build. She says, “No, you can’t do one in the middle like this, cause you can only do two in the middle because four is even number, so you could go like this but there is no possible way you can get one (pause) in the middle (pause) one in the middle of four." Stephanie recognizes that four is an even-numbered tower height, and that this is going to result in fewer unique towers that can be created. Researcher 1 prompts Stephanie to try to come up with a "nice" way to find the patterns to make it easier without creating all of the towers, where she will have no doubt in her mind that she has all towers and is not repeating or missing towers.

Researcher 1 prompts Stephanie to try to come up with a way to find the patterns without building them all, that will allow her to know for certain that she has all and has not duplicated any towers. Researcher 1 says, "Is there a way you can do that faster without building all of them; think about them in your head."

After this, Researcher 1 asks Stephanie if this problem reminded her of anything they had worked on before. Stephanie described a problem she and Dana had worked on previously, called the "Shirts and Pants Problem", saying "Mmmm, it sort of remind me of the shirts and shorts, the way Dana and I were fixing them yesterday, we were putting them into pairs." Stephanie and Dana were tasked to find all possible outfits given three different colored shirts and two different pants options. She describes their process for solving the shirts and pants problems, and works to make a connection between that problem and the tower problem she had been working on during the previous day and revisiting today. She says, "Well, what it is the (pause) the shirts and shorts, you have (pause) with the towers, you have two colors. You have two colors of building blocks and you are trying to make a tower of five with these colors, how many towers can you make? And well you (pause) What we do is we reverse the towers we make. We would say I have a building block of blue and four orange building blocks. A building block of orange and four blue building blocks." Stephanie has not made a full connection between the characteristics of the two tasks, but she is working toward noticing properties between the two tasks and trying to connect the conditions and constraints.

4.27 Session VII Grade 4 Six-Tall Towers Problem. This session about building towers of height 6, selecting from two colors, took place between Stephanie and Researcher 1 in the fourth grade on February 21, 1992 as a one-on-one interview. Stephanie had worked at home on the six-tall towers problem, and came to the session with some of her work on paper. She explored the question, "How many different six-tall towers can be made selecting from two different colored Unifix cubes?." Stephanie describes the work that she did at home, which includes a group of towers labeled with a name, how many towers in the group, and how many towers in the group of "opposites", where the opposite tower consists of the same pattern of cubes, with the opposite color for each cube. In her drawings, she uses "B" for blue cubes and "O" for orange cubes, and set up her towers in the drawings as tables with the letter name for the cube representing the cube of blue or orange color.

Stephanie shows her work to Researcher 1, pointing out a set called "one at a time" where the blue cube moves through each position on the tower and the rest of the cubes are orange. Stephanie was in the image-making layer when she worked on this problem at home, creating a representation of the towers in her drawing to develop her solution. She provides direct reasoning based on the conditions of the problem, justifying why it is not possible to create another unique tower meeting the requirements of the problem while not breaking the conditions of the problem. She states, "because you can only make six-blocks high towers…and if you go any further you have to add another block on." For these towers, the label was "1 at a time=6 Double [opposite] total=12."

This was describing the opposite-colored towers that could be created if each blue and orange cube switched color. In recognizing these opposite towers, Stephanie shows evidence of learning in the image having layer of the Pirie-Kieren model. Her drawings were an additional representation of the images she had created, and her recognition of six additional towers that should be in this group, without actually drawing or building them, shows that she used her image to draw additional conclusions. She immediately follows her answer of six towers with, "and we add the opposite and the total is 12."

Stephanie introduces her next group of towers, saying, "Okay then this is the two at a time." She says that there are five of these towers plus their opposites, for a total of ten towers. Her paper showed this, as it said, "double [opposite] total = 10."

She says that they were created by beginning with two blue cubes at the top of the tower and then, "cross[ing] over one [position down]." Stephanie is saying that each next tower with two blue cubes together is one position down from the previous tower. For example, a tower beginning with two blues at the top with precede the tower with a yellow at the top, two blues below, and the rest tallow. Along the same lines, Stephanie's third group has four towers and this is named "3 at a time", with their four opposites as well for eight total towers.

At this point, Stephanie counts thirty total towers. When the researcher asks if these thirty are all different, Stephanie says "Yeah because this one is choosing three blocks, this one is choosing two blocks…" describing how her sets have different characteristics. Stephanie begins to see that some of her "doubles" look the same as other towers. She starts to identify which tower types match up with duplicates, saying "One two three four, that's also four at a time. That two at a time is also four at a time." She begins to realize patterns with the duplicates, an example being when she states, “I think it might have a duplicate. Like the blue blue orange orange orange. If you choose the blue blue orange orange, it's upside down.” Stephanie is creating images of duplicates that she can use later to identify duplicate towers and avoid counting them.

Stephanie realizes that two at a time is also four at a time with the opposite colors, and “five at a time…this is the one at a time.” After this realization, she is able to adjust her total amount of four and five consecutive blue cubes “at a time” towers. She thought that there were six “four at a time” towers and four “five at a time” towers. After her realization about the duplicates, she changes this to two “four at a time” and zero “five at a time” unique towers.

Stephanie uses this same reasoning to work through the different “at a time” towers.” She eventually comes to 34 towers for six-tall. Other names for patterns of towers that she created include “patchworks”, or two towers with consecutive cubes alternating in color. She says she calls them this, “Because you know we've got blue, orange, blue, orange, blue, orange”, or alternating colors. Another type of tower was the “2 separated” which were the 3 unique towers that had a blue cube fixed at the top of the tower and the second blue cube moving up one position from the bottom.

After this, the discussion moves to towers with “two blues separated” in ways not restricted to the top and bottom of the towers being the blue cubes. Stephanie struggles to find a unique “two blues separated” tower. Stephanie tries to explore an argument like the one she had to the one together, two together, three together, etc. Stephanie works in the image making layer, building towers with two whites separated by one red, and three red cubes underneath. She then holds the top white cube fixed and moves the second white cube to each position below. This gave her three unique towers. Then she moved the one white cube down one spot and varied the second white cube, making sure that the white cube was always at least one red cube apart from the fixed white cube. When she got to the white in the fifth position, or the bottom of the tower, she realized that this contradicted the condition of the white cubes being at least one red cube apart. Stephanie discusses her methods for classifying the towers, and Researcher 1 suggests that she uses this new process to solve the four-tall and five-tall tower problems similarly, with already knowing the number of unique towers that can be produced.

4.28 Session VIII: Grade 4 Stephanie Explores Four-tall Towers Interview. This session was a one-on-one interview between Stephanie and Researcher 1 where Stephanie explores methods for solving the four-tall tower problem, selecting from two colors of cubes. The interview took place on March 6th, 1992 about a month after Stephanie's classroom session and interview about the five-tall towers problem, where she made predictions about building towers of height four. We will focus on specific excerpts of this interview.

Stephanie begins by showing Researcher 1 a new method for building unique towers of four. She predicts "around twenty" total unique towers are possible. Researcher 1 prompts Stephanie to justify why there are no more towers with just one white cube. The researcher says, "I'm supposed to be – Stephen, is it? -- and say to you I think there are more with one white and three black. (points to her towers)", in a move to elicit from Stephanie an argument to support her claim. Stephanie responds, saying:

> Once you get down to the last one, at the bottom, you can't move the white back up, because then you'll just be repeating these things. But if you move the white down one you'll be missing a space… and if you can only use four blocks, you can't have another one… if you move the white on the next position on top it'll be

> like this. And you need another -- you'd need another block here, but you can't do that (see Appendix I, lines 56-62).

The researcher asks Stephanie why not, and Stephanie responds by pointing out that she is not permitted to violate a condition of the problem, "because there's only four blocks. [There can't be five] because we have to use four." Here, we see Stephanie offering an argument by contradiction. The "given" was that towers were to be four tall; the "contradiction to the given" was Stephanie claiming to have five-tall towers, which was inconsistent with the condition of the problem.

After about ten minutes, Stephanie shows three whites cubes, glued together, two white glued together, and then removes one to separate the "stuck together" cubes, stating, "now then we go back to the two whites stuck together and we make it… apart." This is one of her strategies for making new towers. She then shares another strategy, taking the "reverse" of a tower, which will create additional towers. Her classification for a tower being the "reverse" is different from what she had previously defined. She says, "when you show it upside down it's reversed." The twenty towers that she counted resulted in part from a "double" rule, where Stephanie created new towers and then counted that number of towers twice, since she could theoretically create the reverse. Stephanie shows use of an image that she created to solve this problem without building a model with physical manipulatives. Although she ended up with additional towers, she employed a new strategy that she developed while learning in the image having layer. By eventually returning to image making, she can reconstruct her idea about "reverse towers."

Researcher 1 asks, "what was your reasoning for getting the double?" Stephanie responds, "This isn't the only way you can get two apart. You can make two apart by taking two black and separating them." She continues to describe her argument, saying "I think about them in… wise… well, if I'm doing a pattern where I'm using whites apart then I'm not going to use a pattern where I use blacks apart with it but when I'm taking this and going and doubling the pattern I just turn it upside down and that's how I get my other pattern."

Stephanie then describes pairs and other patterns., Researcher 1 says:

> Let's talk about the strategy where you did double and pairs. In that strategy, you said these are exactly two whites. Now again I'm going to pretend I'm Stephen, and I'm saying you've convinced me there only two together… there's three. Now I'm asking you to convince me when the two whites apart that there are only three when you have exactly two white and exactly two black when the two whites are apart They're exactly three. How would you convince me of that? Is this the way you wanted to put them? Is this the order you put it (see Appendix J, line 80)?

The scenario challenges Stephanie to create an argument for her "double" method.

Stephanie and Researcher 1 go back and forth through different methods of moving around the placement of black and white cubes, and redefining "opposites." Eventually, Researcher 1 says:

> Now this is the big question. When you started you thought there would be twenty because you found ten by going through a certain plan and you said because of your opposites that's the word you were using but now you went through this plan and you convinced me there are no more when there are no whites than this. I believe it, right? There are no more when there's exactly one white. You went through all plans where there's exactly two whites, three whites, four whites. How could you solve your problem when you only paying attention to the whites now? Not even worrying about the opposites. Come up with sixteen and convince me there are no more. And when you did opposites you ended up with twenty. Do you think that's possible? Do you see how somebody might get confused about that? What do you think (see Appendix J, line 183)?

Researcher 1 pushes Stephanie to be clear in which argument she is using, as she uses words like "opposites. Stephanie is eventually able to systematically talk through each of her groups. She says:

> So for the opposite of this and this group in the same group. Because it just cause um cause you're using the same amount of blocks… let's say, ok, here you are using two blocks in the middle and two blocks separated. Here you are using two

> blocks in the middle, two blocks separated. And like here even if you didn't notice it until the end you're using three blocks and one block and down here you're using three blocks and one block (see Appendix J, line 204).

Stephanie is classifying different types of towers. Stephanie has used the "opposite strategy." Stephanie says that "the opposite strategy can work, but it is better to … double check."

Researcher 1 acknowledges that Stephanie has convinced her that there are no more than sixteen towers, and asks where she thinks the number of twenty towers came from. As a researcher move, Researcher 1 asks Stephanie to think about how she would explain her argument to someone.

Stephanie and Researcher 1 build towers two high using black and white cubes. Stephanie creates classes of all white, no white, and one white. Stephanie creates four total unique towers. After some discussion, Researcher 1 introduces Stephanie into the image having layer of learning in the Pirie-Kieren Model, asking for towers of two, if she thought there would be four. Stephanie says, "You know if has to be less than eight", and the researcher says, "[What about for] towers of one? Don't make them, guess." Stephanie first responds that there is one way to make towers of one. Researcher Maher prompts, "Okay, let's make towers of one using no whites." Stephanie says, "Oh!" She responds that there will be two ways. Stephanie goes through and shows that for towers of one that are two, for towers of two there are four possible, for towers of 3 there are eight possible towers, and for towers of 4 there are sixteen. Stephanie makes a connection and observation, stating, "That's weird look. Two times two is 4 and 4 times two is 8 and 8 times two is 16. It goes like in a pattern. Two times two is 4. And 4 times 4 is 8 and 8 times 8 is 16." Stephanie writes these numbers down on her papers. She notices more patterns saying, “It also turns out that every number is even.” Researcher 1 asks Stephanie, “Now if this is pattern what would you guess would be towers of 5?”, to which she responds, “Based on this pattern I would guess 32.” Researcher 1 says, if this works, you’ll be able to do this pretty fast. Is there a reason why this could work?” Stephanie recalls when she worked on the towers of five problem in class, and says that she got 32 towers during that session also. Researcher 1 asks why she would expect, for example, four towers for towers of two.

Stephanie mentions that the number of towers doubles as the tower heights increase. Stephanie works through having white on the “top floor” or “bottom floor.” With white on the bottom floor, she has two options for colors for the cube above: white or black. Stephanie shows the same thing with black on the bottom floor. Stephanie works to show that she gets “twice as many” for towers of three. Stephanie says, “with white on the bottom, you have four, and with black on the bottom, you have four.” Researcher 1 challenges Stephanie also to think about what happens for the cubes in the middle that are not on the bottom or top floor. She asks Stephanie to see if she can connect these problems to other problems that she has seen before. Stephanie says, “Usually these problems remind me of the Shirts and Shorts, but this one doesn’t, because most of the time you have to pair up.” At each step, Stephanie provides her reasoning and makes connections to previous tower heights, showing learning in the property-noticing layer. She says, “Because we doubled it.” Stephanie recognizes this property of the problem and the answers she was getting for the numbers of towers, and she works to figure out a reason behind this. Researcher 1 acknowledges Stephanie’s recognition of this property, and prompts her to develop this idea and explore why this property emerges and to predict if this pattern continues.

Stephanie connects the towers problem to the Shirts and [Pants] Shorts problem. Researcher 1 asks, “Do these problems remind you of any other problems?” Stephanie says that these problems usually remind her of the Shirts and [Pants] Shorts because usually you must “pair off.” Stephanie explores whether the three-tall tower problem could be connected to the Shirts and Pants problem. In an instance of folding back, Stephanie begins building this idea from the two-tall tower problem with her towers in front of her. She revisits the ideas about the Shirts and Pants problem that she had worked on previously, and builds on her understanding about what the different outfit articles represent, and the meaning behind the black or white options for the articles of clothing. Researcher 1 suggests showing her idea with a picture, so Stephanie creates a drawing representation, including one pair each of white and black pants, and one white and one black shirt. She draws lines to connect the shirts and pants to each other to create outfits. Stephanie then extends this problem to the one-tall with one pair of pants and one shirt. Researcher 1 acknowledges that in context, the problem is difficult to connect because there is just one article of clothing.

Stephanie then creates a situation and new drawing where she adds in both a white hat and a black hat to see how many combinations emerge. Stephanie then draws lines between hats, shirts, and pants. Researcher 1 mentions that the drawing is hard to read, so Stephanie creates another representation: a list where “WH” represents white hat, “BP” represents black pants, and so on. Stephanie lists out all the different outfit combinations that include on hat, one shirt, and one pair of pants. She divides them up using lines to differentiate between the outfits, and then counts them up and got seven outfits. She compares the six that she found from her drawing, and the seven that she created in her list. She draws the conclusion, “I think I messed up some place.” Researcher 1 and Stephanie talk through what could have happened. Researcher 1 suggests that Stephanie refer to the towers and think of the different outfit combinations including just shirts and pants as the two-tall towers, and then put “a black or white hat on top of each outfit” because “that will make it really easy to see. What would they look like?” Stephanie begins creating a third representation of the outfit problem using the towers and creates three-tall towers by adding the black hats to the four two-tall towers, then creating the same two-tall towers, and adding a “white hat”, or white cube, to the top. Stephanie finds eight towers and states, “This makes sense now.”

Stephanie is learning in the image-making layer. She works to create outfits that include another article of clothing, and draws out the shirts, pants, and hats, using connecting lines to create outfit combinations. She then lists out the outfit combinations, still building her ideas. In comparing the two representations, Stephanie is exploring why her two representations do not agree, and this helps. As she folds back to the towers problem, she is able to rebuild her understanding of matching the outfits to create unique outfits. Connecting her multiple representations helped her to make these connections. She tried out the third approach, adding a black or white cube on to each two-tall tower, which she previously connected to the Shirts and Pants Problem, and built a new image connecting the outfit problems to the tower problem representation.

Polya’s problem solving techniques are evident here. Stephanie prepared to solve the problem by talking through the conditions in order to understand what is being asked.

She then devises her plan to solve. This is during her "thinking time" where she is drawing pictures, making a list, and looking for patterns. Stephanie then carries out her plan. She confronts an obstacle when her drawing and list did not result in the same number of outfits. To work through this issue, she started over with a new representation where she used the towers and built them three tall, building from the two-tall towers. Stephanie then verified her solution, checking the towers and talking through the solution. She verbalizes finding her answer reasonable, saying, "that makes sense."

Researcher 1 extends the earlier problem and challenges Stephanie further, "What about if you had to different colored feathers?" Stephanie responds verbally without physically building the towers, demonstrating her learning in the image-having layer. She predicts the effect of adding an additional article of clothing (the feather) to the already three-part outfit. She describes what effect this would have if building-up to each three-tall tower, and how the number of towers with the white feather in the cap would be the same number of outfits with a black feather in the hat. Stephanie finds sixteen, which is what she predicted based on a pattern that she observed. Researcher 1 says, "How about if I added a flower [to the outfit]? A black or white flower; without doing it, tell me how many combinations of outfits would you get?" Stephanie predicts, "I would say, probably, 20." Researcher 1 reiterates the question, and asks her to just tell her what will happen, not do it. Stephanie talks through adding a black cube to the top of each of the 16 four-tall towers, and then adding a white to another set of 16 of the four-tall towers. Stephanie concludes, "So there would be 32. It is like the Shirts and Shorts problem." Stephanie is learning in the image-having layer when she predicts 32 outfits based on the pattern that she has observed as she kept adding an article of clothing, and the number of possible outfits doubled.

Stephanie moves to the property-noticing layer of the Pirie-Kieren model when she connects the Shirts and Pants problem to the towers problem and recognizes that each article of clothing that is added to the outfit is synonymous with adding a cube to the height of the tower. "That's what we got [in the tower problem] when we did the problem the last time." Stephanie draws the conclusion that to get the number of possible towers for any height, or to find the number of outfits suing any article of clothing, "All you have to do is multiply the last number of [outfits/towers] by two." Researcher 1 asks Stephanie is she is sure of this, and she responds affirmatively.

Researcher 1 asks how many towers could be made for six-high towers, and Stephanie says, "Okay, so if towers of five are 32, then [there will be] 64. You take the previous number and multiply by two." Researcher 1 pushes Stephanie to describe her method and strategy to find this, not just the process of multiplying by two. Stephanie says, "You can use the Shirts and Shorts problem. [Or], you can go line by line and [create cases] and there will l be four in the first case." She describes building cases where for towers of six she would have to go up to six cases. Researcher 1 asks Stephanie to "pull together all of her ideas in a story" in a researcher move where she guides Stephanie to connect her ideas and restate them cohesively, along with a description for how to find how many towers could be built ten high. Stephanie shows multiplying out 64 by two, then 128 by two, then 256 by 2, and then 512 by two to get 1024 ten-tall towers.

This part of the interview including the exploration of the solution of the towers problems that connected to the solutions of the corresponding Shirts and Pants problems demonstrates Stephanie learning in the property-noticing layer of the Pirie-Kieren model. Stephanie both uses and connects multiple problem representations. While Stephanie is building images about the outfit problem by building towers and exploring constraints of that problem, she connects properties of her images about both problems throughout her exploration. Stephanie reaches the formalizing layer of the Pirie-Kieren model in her learning. Formalizing is defined as the level where a method, rule, or property is generalized from the identified properties. Stephanie recognizes a pattern and property of the outfit problem that she also connects back to the towers problem. She mentions a "doubling rule", where to find the number of possible towers of a certain height, she multiplies the number of possible towers by the previous amount of towers. This was the same for the outfits. For each additional article of clothing that could be selected from two different colors, to get the possible number of outfits, she doubled the previous number of outfits. This is evidence of Stephanie learning in the property-noticing layer because she has developed a rule based on a pattern that she recognized. She has a recursive rule to find the next number of outfits or towers that can be made. She demonstrates how to use this rule by finding how many towers ten-high could be made. She starts with the 64 towers six-tall that can be made, and multiplies by two four more times to get 1,024 towers.

4.29 Session IX: Grade 4 Gang of Four. This session takes place in fourth grade on March 10th, 1992, in a small-group interview session facilitated by Researcher 1, where Stephanie is working with classmates Jeff, Michelle, and Milin. This interview occurs after the students have worked in a classroom session figuring out how many unique five-tall towers could be made selecting from two colors of Unifix cubes. Researcher 1 begins the group interview by prompting the students to discuss what they remember about their problem solving. She says, “And do you remember what you did with those towers of 5? …tell me about it. What was the problem?” This is a move by the researcher that encourages students to respond and reiterate the problem. After some discussion and back and forth conversation about how they worked through figuring out the different tower possibilities, Researcher 1 says, “Okay, let’s, let’s get a piece of paper and write down what you’re saying and see if you all agree” in a move that encourages written representation ideas to organize verbal ideas that the students could not agree on. Additionally, she prompts the students to assess their peers’ ideas, evident in her questioning, saying, “Do you agree with that? Do you know what she’s talking about?.” The students work during this session initially discussing the unknown, the question, and the conditions. They work through each student’s suggestion taking care not to contradict the conditions of the tower problem or to provide reasoning or examples that is insufficient, such as duplicate towers. The students engage in problem solving, showing evidence of problem solving by Polya’s second principal: devise a plan. The students are looking for a pattern, using a model, and drawing a picture.

The students discuss patterns they noticed during the classroom session exploring the problem and its constraints. Referencing Polya’s problem solving techniques, the students are engaging in Polya’s first principle” understand the problem. Evidence of engagement in this problem-solving technique comes from Researcher 1 as well as from the students. She researcher asks the students if they “agree” and “know what she’s talking about." She suggests using a diagram, or writing down what they are doing.

The students are discussing why, if there are 2 possible towers for one-tall, and 4 possible towers for two-tall, there are eight and not 6 possible towers for three-tall towers. Alluding to collaborative efforts and pattern recognition, Stephanie says, "that's not the pattern that we're working on. The pattern we saw was this: for 1 block at a time we found two." There is much discussion amongst the students about building the towers and about how the eight unique towers come about with the three tall towers. Eventually, the researcher prompts the students to convince one student, Jeff, why there are eight towers. She calls on different students, and when she acknowledges Stephanie, Stephanie's response is, "I found it like this. I drew my lines. And then I went red-red-red, blue-blue-blue, blue-red-blue, red-blue-blue, blue-blue- red, red-red-blue, red-blue-red, blue-red-red." She uses a drawing as her representation to create different tower types. She has an all-red tower, a no-red tower, a one-red type of tower, and a two-red type of tower. Milin asks what would happen if she could make more towers. Stephanie verbally recognizes need to justify to "convince" her classmate that there are exactly eight possible towers for height three. The researcher restates problem for what needs to be explained as a researcher move, saying, "Yeah, but you haven't, he's proved to me from the 4, you could only make 8. You could get two from this one, and two from this one, and two from this one, and two from this one."

As Stephanie goes through explaining her process, Milin proposes that she is missing towers in her groups. In a student move that shows confidence in her understanding and approach, Stephanie says, "Yeah, but that's not what I'm doing. I'm doing it so that they're stuck together." Stephanie is sure of her process and recognizes that hers and Milin's approaches are different, and that mixing the two approaches is not in her method. Another classmate in the session, Jeff, insists multiple times that "there should be one with red", referring to an additional tower that Stephanie seemingly missed. Stephanie responds, "Well, that's not how I do it", again demonstrating certainty in her method. In a researcher move that allows for contributions from multiple students and to show that there is room for different ideas and ways of solving, Researcher 1 says, "Let's hear how, how Steph...we'll hear, we'll hear that other way. That's interesting. Okay, now, so what you've done so far…." Stephanie is carrying out her plan, which is the third principle by Polya for techniques in problem solving.

When Milin makes a claim that Stephanie does not have all the "two-blue" case, Stephanie hands her paper to Milin and says, "Okay, show me another two-blue [tower]. With them stuck together, because that's [how] I'm doing it." This is a verbal assertion by Stephanie that her cases meet certain requirements, and that she recognizes there are other towers with two-blue, but that she has currently identified all the "two-blue" that meet her specified requirements. She asks for some contradiction that another exists in the "two-blue" group. Michele contributes to the Milin's proposed argument that there are more "two-blue" cases, saying "But, but if you just had 2 blues and they weren't stuck together, you could...." Stephanie overlaps her statement, responding, "But that's not what I'm doing, I'm doing the blues stuck together." Stephanie's statements show evidence that she is certain of her cases. This is the beginning stages of Stephanie developing a proof for her work, as she is giving evidence of her reasoning using a system she developed.

Some argumentation transpires, as Milin and Michelle claim that Stephanie should have included more towers in her group. Stephanie responds, “Well, you’re following your pattern. But my pattern goes no red, one red. This was not meant to be like that. That’s not. It’s in the category of one blue. That. I could stick that some place in another category. But I want this to be in the category of one blue. Not in the category of the opposite of this one.” Stephanie insists that the method she used to group the towers worked, and that her peers can work using a different pattern but it would not change her groups. She follows with a description of the tower she put in the category, saying, “And then I have this one, the red-red-blue. So to you, that, you might put that way at the end of the line. But I put it right here.” This insistence demonstrates confidence in methods and reasoning. Stephanie developed her approach and is reviewing it step by step to her peers. Evidence of Stephanie working in Polya’s fourth principal of problem solving, “looking back”, is shown as Stephanie examines her solution step-by step, explaining her process to her peers and checking her argument.

Jeff asks the group, “Do you have to make a pattern?... why is everybody going by a pattern?.” Stephanie reasons that it easier than taking a random approach, and acknowledges that others’ patterns may be different. She states, “Okay, but what I’m saying is that it’s, that it’s just easier to work with a pattern It’s harder to check [for duplicates]. It’s harder to check just having them like come up from out of the blue.” Stephanie has worked with the towers problem multiple times under different conditions. Over this time, she has constructed images and explored the problem’s constraints, possibly developing a notion about the necessity for working with patterns. This is evidence of image having, since Stephanie is using images made during her engagement with the problems to figure out methods that work best for solving the problem.

Jeff questions Stephanie about how she knows that no more towers can be made in the group with "three together" with one blue cube. Stephanie responds, "Look. Okay. Start here. Start here. Okay? You have the 3 together? The one 1 blue. You have the 1 blue. How could I build another one blue?" Stephanie goes on to state that if she could not make another of this type without adding a cube to the tower height. Jeff says, "You can go r-r-b or you can go b-r-r", and Stephanie tells him that these towers are the same and she has this tower. She concludes her argument that there are no more one blue, saying "Look, but I have those three. Look. B-r-r, r-b-r, r-r-b, but then how am I supposed to make another one once that blue got down to my last block?"

Stephanie continues an informal proof by cases, moving on to another category of towers, the two-blue case. She says, "Two blue. Here's one, right? 2 blue. We have one, b-b-r, then we have r-b-b. How am I supposed to make another one?", requesting a contradiction to her argument. Jeff suggests a tower consisting of blue-red-blue. Stephanie responds that Milin had proposed the same argument, but that the blue cubes need to be together in this category. She then moves on to subsequent cases, saying, "Two split apart, which you can only make 1 of. And then you could make, you could find the opposites right in the same group. All right, so then I've convinced you that there's only 8?" Stephanie responds with excitement when Jeff agrees that she has convinced him that there are eight towers that can be built three tall, selecting from two colors or Unifix cubes. She says, "Yes!", which is a student response demonstrating investment and excitement in the task.

The researcher asks, "How many if you're making towers of four?", to which Stephanie immediately gives the answer of sixteen towers. This is a researcher move used to prompt the student to make predictions based on previously developed images. Stephanie's ability to immediately decide how many towers can be made for height four evidences image having, since she does not physically build or draw towers to create categories. She moves to the property-noticing layer of the Pirie-Kieren model for studying growth in mathematical understanding, evidenced by her discussion of a pattern that she noticed emerging as she kept increasing the towers by one cube, and the number doubled. Further evidence of this is shown when the researcher asks how many towers can be built for towers of height five, and Milin, Michelle, and Stephanie all respond that there are thirty-two, and that this works the same way for towers of all heights. Stephanie explains, "The hard part is making patterns. Like, you, from now, we know how to just oh you could give us a problem, like how many in 10 and we'd know. I know the answer. I figured it out. It's 1,024." This is evidence of property noticing because Stephanie has observed her own mental image and recognized properties of the image. She has made connections between images she created for the one-tall, two-tall, and three-tall problems. She noticed patterns between the different individual images for how many towers could be built as she increased the height by one cube, and the number of towers for the previous height was multiplied by two to obtain the number of towers for the increased height. There is evidence of validation of a connection between the towers of different heights. This connection possibly came about from the earlier explorations of the ideas by the learner, as suggested by Pirie and Kieren (1992).

Stephanie assesses her own ideas and propositions. She recognizes a trial during her explorations that did was not valid for towers of all heights. She reflects and it is evident that there is a need for Stephanie to fold back to rebuild the image based on the properties that she discusses noticing. Stephanie says, “You’re timesing, no you don’t times it. It’s the same thing I did. I counted ahead. I just counted ahead 5 or 6, and I said oh, I could just multiply it by that and that’ll give me the same answer, but it didn’t work. You have to figure out what’s in between that”, describing needing to figure out how many towers could be built for five, six, seven, eight, and-tall in order to find how many are able to be built for towers ten-tall, for example. She says, “Do you want me to figure out 10, right? But, in order to figure out 10, I was only up to 5. So what I had to do was I had to go and I had to say, well, what’s 6, what’s 7, what’s 8, and what’s 9, and times that times the last number I had.” Stephanie discusses a trial she engaged with in order to test her conjecture about patterns that she recognized for towers of shorter heights. She describes different attempts she made at discovering the pattern, saying, “So I multiplied, I tried, first of all, I tried multiplying it times 8 because I figured well, all I have to do was 6+4 times 2 that’s 8 so 64 times 8. First, I thought, well, I don’t want to go ahead, and I don’t want to have to multiply 7, 8, 9, and 10. 7, 8, 9 before I get 10. So I figured 6+4 equals 10. And since I’m timesing times 2, I’ll multiply 4 times 2 to get 8 and then just multiply 64 times 8.” As Stephanie discusses this trial, she describes how she figured out that she should try out something else as this did not work the same for all numbers to get the result that she anticipated. Michelle says, “But she was wrong.” Stephanie agrees, and responds that she tried multiplying the previous number by two to get her answer.

Stephanie is again engaging in the fourth principal of Polya’s problem-solving techniques. She is trying to use her result and method to solve another problem and is examining her solution each time to assess her argument. In a researcher move, Researcher 1 says, “So in other words, could this have worked, that’s my question. Now, when would this work? Why didn’t the 8 work?.” Stephanie proposes an idea, saying, “Ahh, I just thought of something. I’m wondering if this will work. This 8 is #8, okay? This is #8, right? This is the answer to #8.” Researcher 1 restates, saying, “Okay. So what you’re suggesting is multiplying by 8 didn’t work. It gave you 512, which was….” The students begin suggesting ideas and Researcher 1 says, “But, you know what I’m thinking, I’m thinking maybe what we should do is I want you to, I don’t want to throw away Stephanie’s idea here, okay, because what Stephanie has here in her idea, once she got to towers of 9, right, she said there were 512. That’s by each time multiplying it by 2. Why didn’t multiplying by eight work when she had towers of six?.” The researcher then poses a challenge, in a move prompting the students to reason beyond what they had already figured out. She says, “Okay, but why, how could she be sure? In other words if 8 didn’t work, do you understand my, my challenge to you? All you mathematicians here. My challenge to you is I don’t want to throw out this idea because, you know, because if Stephanie has something here, she’ll save you a lot of work in the future, right? If she has a good idea here? … maybe you could invent another way. If I said towers of 80. Now, and I said I’ll give you a calculator, but you have to know what to do with your calculator, right?” Stephanie, referring to the process she developed for finding how many towers are possible based on the number of towers possible for the previous tower height, says, “There’s a problem because you have to go all the way from 10 to 80.” Researcher 1’s researcher move is to break the problem into something more accessible. She says, “Well, my question is let’s not worry about that big problem for a moment. Let’s try to do it with a simple problem. Suppose you didn’t know towers of 6 were 64 and towers of 7 were, what did you say that was? What do you have there?" She says that this is a challenging question, allowing students space to struggle, and says, "Suppose you didn't know that. How could you jump from towers of 6 to towers of 10 without going through all those steps and why?." She leaves this problem for them to think about and come back to next time.

Evidence of Stephanie learning in the property noticing layer emerges again after the researcher asks the students to go back and solve this problem again for towers of ten, instead of height eighty. She asks the students to pretend that they only know the answer to towers of six, and to figure out another way than just building up from the number of towers they figured out for towers of six. Stephanie says, "You want us to try and figure it out the way I tried to figure it out the first time", and researcher 1 responds, "Right with only multiplying by 1 number. And convince me that that number makes sense to multiply by. Does that make sense? Do you understand?", checking in to make sure that the students know what she is expecting. After some discussion by the students, the researcher guides them, saying, "…when you had to build towers of, 2 of 2, how many times, of 3 high, how many times did you multiply by 2? When you had to build towers of 3, how many times did you need to multiply by 2? Okay, 2 times 2. That's one time you multiply it by 2. You got 4. Then you multiply by 2 again...." After some responses from the students, Researcher 1 agrees, "And that gave you 8. So how many times did you multiply by 2?", to which Stephanie answers, "You multiplied the amount of times you...." Researcher 1 gives the students guidance for their work after this session, which they plan on coming back to discuss later. She says, "Well, twice. You multiplied it once. This is 2 times 2 once, right? And then you multiplied it by 2 again, right? 2 times 2, let me write this. 2 times 2 gave you 4. That was one time. Then you multiplied it by 2 again another time and you got 8. So you multiplied it twice to build towers of 3, is that right?." They agree to think about this and come back together to discuss again at another time.

4.210 Session X Interview 5 Pascal's Triangle, Part 1. Stephanie is in eighth grade working in a one-on-one interview session with Researcher 1. She had previously been exploring combinatorial notation. Researcher 1 discusses with Stephanie how to connect finding the total number of towers for towers of height four when selecting from red and yellow Unifix cubes. Researcher 1 asks, "So, if I wanted to know the total number where you could have no reds, exactly one, exactly two, exactly three, exactly four. What does it turn out to be?." Stephanie responds that it would be sixteen. She demonstrates evidence of learning in the property noticing layer of the Pirie-Kieren model when she says, "It's the same thing, like with just the towers." Stephanie makes connections between the number of towers created selecting from two colors and the combinatorics notation.

Stephanie looks through five cases: one case each for a height of n=4 and r=0, 1, 2, 3, 4 where r is the number of red cubes. Stephanie figures out how many towers are in each case, and then adds those all together to get the total amount. Stephanie observes that the number of towers that are possible doubles as she adds one cube to the height, or increases n by one.

Stephanie begins to rebuild her understanding of building towers to develop the tower "family." Stephanie justifies her solution by creating a tree diagram where the eight towers for height three are created by adding on towers to the two-high tower cases. Stephanie makes predictions about how many towers can be created based on patterns she observed, saying "[For four high there would be] sixteen." She predicts that this doubling pattern will continue, based on her observations. She says, "Cause, I guess, there's always going to be two combinations with whatever you have on the bottom- 'cause if you're building it from here, it's got to have three reds on the bottom, and there's only two other things 'cause you only have two colors. So you can only do two other things with that. You can either put a red on top or a yellow", justifying her answer with an explanation. Some questions asked of Stephanie in this session that elicit a numerical answer include, "Based on your combinatorics notation, what is the total number of Unifix Towers that are four-cubes tall?" and "What are the total numbers for towers that are: one-cube tall, two-cubes tall and three-cubes tall?" Other questions asked by the researcher during the session that ask Stephanie to explain and predict were, "What would you predict as the total for towers 5-cubes tall?", "If there are exactly two towers one-cube tall, how can you generate towers that are 2-cubes tall? Three cubes tall?" Another question asked of Stephanie during the session that prompts her to explain why and then predict was, "Why does this pattern work? Do you think that it would continue for towers 4-cubes tall and taller?"

4.211 Session XI Interview 5 Pascal's Triangle, Part 2. Stephanie continues to work on the five cases for towers of height four. Researcher 1 asks Stephanie if they should add sixteen additional towers, since if they look at the same types of cases but use yellow as the cube changing position in the tower, there are sixteen towers for that two. She says, "For these four high, you can imagine these sixteen there. And, of these sixteen, I could say, of these sixteen, there'll be no reds and there's going to be one of those. And there's going to be exactly one red…what about yellows? Don't we have to do the same thing for yellows? So wouldn't that give us 32? and if I said, let's now find out how many exactly no yellows, let's find out exactly one yellow out of the four, exactly two yellows out of the four, three yellows out of the four, don't you agree that you'd get another sixteen?" This is a researcher move where the researcher summarizes and follows up with questions to prompt Stephanie's thinking.

Researcher 1 confirms what they did and discusses focusing on the red as the changing position-uses this to extend thinking to what about the yellow cubes. She says, "But then 16 and 16 gives you 32, not 16." She is questioning and asks for either counterexample or reasoning. Stephanie responds, "But wouldn't it be the same thing? Like, only the opposite way? 'Cause, look, if there's two red, then there's two yellow. [*writing*] And if there's three red, then there's one yellow. And if there's one red, then there's three yellow, so isn't it the same thing?." Stephanie is in property noticing layer, demonstrated by her connection made, based on previously created images from building towers and their "opposites."

Stephanie recognizes that these are the same towers, or just the "opposites." She connects this to her developed ideas about opposites and images of "opposite" towers that she created over the years. Thus, there are just sixteen and not thirty-two possible towers of height four.

Researcher 1 says, "…and that's why if you think about that as a strategy, if you've already figured out exactly one, do you know exactly three?." The researcher proposes using this as a strategy to solve problems later on, pushing Stephanie along property noticing. The main question posed to Stephanie is, "Based on the number of towers in each case for n equal height 4 and r equal to the number of red cubes from zero to four, will there not be an equal number of towers for n equal height 4 and r equal to the number of yellow cubes from zero to four, resulting in a final total of 32 towers? Justify your answer."

4.212 Session XII Interview 5 Pascal's Triangle, Part 3. Stephanie works on using combinatorics notation to write out the number of towers for height of one. Stephanie makes sense of the number of one high towers that can be made, saying "there's one" since there are no reds. Researcher 1 uses questioning to help Stephanie develop the meaning, saying, "But you look at this notation and say, "What does this mean?" But see, this will help you think of selections. Ok, so if we were to think about this, um, if we're thinking of for towers for n = 1, that's one high towers, right?."

Researcher 1's questioning challenges Stephanie as she asks, "So I thought we'd do something else that might... now two. Right? So if we're doing two now, again, what do you want to think of red or yellow? Does it matter? You told me it doesn't matter." Stephanie responds, "Well, because there's always going to- if there's- you can't do none of one, and there's another color, it's obviously going to be all the other color."

She then develops this idea for towers of height two and three, and she begins writing this in rows that resemble the pattern for Pascal's triangle. Stephanie makes sense of two-tall towers. There is evidence of her learning in the image-having layer, based on Stephanie not needing to build the towers, and being able to make conjectures and reason through without using the physical manipulatives, but basing her answer on previously built images. Stephanie proposes multiple answers to a question about what happens when n=3, and in a researcher move, Researcher 1 asks "Want to think about that?", allowing Stephanie time to explore the problem on her own. Building on her questions about what would happen when selecting one from four, Researcher 1 asks, "What would you think it would be if I could select one from *n*?" to which Stephanie responds, "n?." In a Researcher move, the researcher challenges Stephanie to generalize and asks, "can you imagine that?" Stephanie proposes that selecting exactly one from 4 would be 4 options, and five would be five, and n would be n. Stephanie can use a visual image to make her conjecture, without building. This is evidence of image having, and is the beginning of property noticing as she is starting to connect the images.

Stephanie begins to explore characteristics and patterns of the images that she has and is connecting, saying, "Oh, is it, cause like, the 1 and 2- 1 and 1 are 2, 1 and 2 are 3, 1 and 2 are 3, 1 and 3 are 4, 1 and 3 are 4, 3 and 3 is 6?." This is evidence of property noticing, as she has connected previous images to notice characteristics and make a connection.

Researcher 1 says, "But remember you told me, like, if I took a number to the zero power, that doesn't make any sense? Well, this is almost like that. It doesn't make any sense, but if you want this picture to be so nice and symmetry and all, and if you want it to turn out to be that way, what would you want it to be?" Stephanie responds that it would have to equal one. Researcher 1 says, "Is there another reason to make that one? I don't know of any. Do you? Taking no things from nothing?" The researcher is prompting Stephanie to make a prediction based on what they are developing. Researcher 1 asks Stephanie to predict what the numbers would be in the row for five high. Stephanie responds, "…it would be 1 + 3, oh, 5. And then it would be 10, 10, 5, 1." This is evidence of property noticing. Stephanie makes a prediction based on connected images and connection of numbers in Pascal's triangle to the tower heights and how many towers can be produced.

As Stephanie is writing out these numbers, she recognizes the addition property of Pascal's triangle. Based on this observed pattern, Stephanie predicts the fourth row's numbers, and then the fifth row. She recognizes that the numbers in each row correspond to the number of possible towers for each case, and that each row represents a tower height, with the second row being for towers of height two, the third row for towers of height three, and so on if the first row is considered row zero. Questions included in this session include:

- "Does the number of towers in each case for n equal 1 to 3-cubes tall and r equal to the number of red cubes from zero to n, when selecting from two colors, match the corresponding entry in each of the first 3 rows of Pascal's triangle?"
- "How can this representation be used to predict rows 4 and 5 in the triangle and the corresponding number of towers 4 and 5-cubes tall?"

4.213 Session XIII Interview 5 Pascal's Triangle, Part 4. Stephanie continues to use the addition pattern that she has recognized in the triangle showing the cases for towers of different heights. In a Researcher move, Researcher 1 asks Stephanie to state what she visualizes "in her mind" and Stephanie verbally confirms what she is seeing. Researcher 1 says, "Okay, so you're going to make this, oh ok. So the five would be one from five, you're saying? And you believe that? You can see that in your mind? What are you seeing? I'm curious." Stephanie demonstrates learning in the property noticing layer of the Pirie-Kieren model. Stephanie recognizes a pattern, states that she would expect certain numbers on the triangle, and predicts the next number would be four. She makes prediction based on connected images, which is a characteristic of learning in the property-noticing layer. This is evident when Stephanie says, "And, I don't know, I know how many combinations I get for each row." Stephanie then predicts the next row based on the patterns that she observed, and she connects her observation of the number of combinations for towers.

She uses this pattern to represent and predict the numbers that will be in the next rows: row five and six. Researcher 1 then names this triangle as Pascal's triangle, after they have developed the concept. Researcher 1 says, "And this thing is called Pascal's triangle. And so, I don't think you realize, when you read this paper now, and see how hard you worked, you were really working pieces of Pascal's triangle." This is a researcher move that challenges Stephanie to develop a mathematical idea before formalizing it with a name.

Stephanie checks and confirms that the total for numbers in each row is consistent with the number of towers possible for towers of the height for each row. Researcher 1 says, "You know, if you hadn't done all that hard work all those years this would make no sense to you now. I mean, I don't know. But it's hard to visualize and see 'cause they only deal with the numbers. They just learned this rule that you add these numbers you get this and you add these numbers, you get this." Researcher 1 attests to the process of building up the concepts and developing the mathematical ideas that allowed Stephanie to come to these conclusions. Questions posed to Stephanie by Researcher 1 in this session include, "According to the addition pattern and the values that you have determined for rows 1 through 4, what would be the values in row 5 of the triangle?", "What is the total of the values in this row?", and the task, "Predict from the addition pattern the values for row 6 and check to determine that the total is consistent with your knowledge of the number of towers 6 cubes tall." These questions promote exploration and prediction, demonstrating Researcher 1 guiding Stephanie to build up the mathematical concepts and ideas.

Chapter 5: Findings and Conclusions

5.1 Introduction

The purpose of this study was to trace the growth in mathematical understanding of one student over a period of seven years, and to map this growth to the layers of the Pirie-Kieren model for studying the growth in mathematical understanding. A goal of this study was to develop a learning progression for Stephanie's growth in mathematical understanding over time by analyzing her actions, interactions, verbalizations, and use of tools from the video data of the problem-solving sessions over a period of years. This study traced Stephanie's engagement with problem solving tasks, and mapped her actions and words to defined layers of understanding. The questions that guided this study were:

1. How does Stephanie traverse through layers of understanding over time as she engages in math problem solving tasks?
2. What influence does "folding back" have on Stephanie's development and growth of mathematical understanding and her movement among layers of understanding?
3. How do Stephanie's words, written work, actions, and interactions evidence movement between and among layers of understanding?
4. What evidence is there that Stephanie has traversed to a different layer?
5. In what ways do the roles of social interaction, teacher moves, student argumentation, and teacher/student questioning contribute, enhance, hinder, or otherwise affect a student's mathematical growth?

Special attention was paid to: (a) what Stephanie's progression of mathematical knowledge and ways of reasoning through the layers of understanding looked like; (b) what features of understanding defined each layer in each event, and (c) the evidence of Stephanie's understanding in the structure of her solution of mathematical tasks and her solution process.

The creation and publication of VMCAnalytics assisted in answering the first, second, third, fourth, and fifth research questions. All the above research questions were answered through analysis of the video data through the lens of Pirie-Kieren theory. This chapter summarizes findings for the questions answered through the video analysis. Key themes in the findings outside the Pirie-Kieren model are that certain teacher/researcher moves, student collaborations, and peer/facilitator questioning strategies help explain student growth in mathematical understanding.

The finding section is partitioned into subsections based on the research questions posed. The first, third, and fourth research questions are addressed in detail by the following five VMCAnalytics: "Tracing Stephanie's Growth in Primitive Knowing through the Pirie-Kieren lens", "Tracing Stephanie's Growth in Image Making through the Pirie-Kieren lens", "Tracing Stephanie's Growth in Image Having through the Pirie-Kieren lens", "Tracing Stephanie's Growth in Property Noticing through the Pirie-Kieren lens", "Tracing Stephanie's Growth in Mathematical Understanding: A learning progression through the Pirie-Kieren lens", and "Tracing Stephanie's Growth in Mathematical Understanding through Researcher Moves."

5.2 Folding Back

Folding back is a feature of building mathematical understanding where the student reconsiders his or her current understanding of a concept, revisits the idea, and reconstructs that knowledge in a meaningful way. Folding back occurs when a learner remains on a level of understanding, and revisits earlier, inner layers of understanding to rebuild or reconstruct previous understanding in a different way. Stephanie folds back during her construction of new mathematical ideas. Instances of Stephanie folding back are recorded in this research. Folding back influenced Stephanie's development, growth of mathematical understanding, and movement between and among layers of understanding. In Session II, when Stephanie is initially exploring the Towers problem in third grade, Stephanie works with the blocks to produce towers, explains her work, and draws an illustration of the towers. These multiple representations of her ideas help her form a foundation, and such a foundation is helpful for a learner to fold back and advance in learning. In order for Stephanie to fold back to a more inner layer of understanding to reconstruct her knowledge, she must have an understanding of the problem and its constraints and be able to represent and explore the problem in different, often deeper, ways.

In Session III, which takes place during the same day as Session II, additional towers problems are introduced. Instead of building four-tall towers, Stephanie is challenged to make predictions about how many towers can be made while building three-tall towers, and whether there would be more or fewer than the sixteen towers created for the four-tall towers problem. Stephanie folds back to the images she created during the four-tall towers problem. There are new constraints to the new problem, and Stephanie revisits ideas about duplicate towers. She recognizes that as she removes a cube from the top of the towers, some towers end up being the same, since the towers only differed by that one cube. Stephanie's initial conjecture was that the number of possible towers would be the same for four-tall and three-tall towers. While looking for evidence to support her claim, folding back helps Stephanie to reconstruct understanding about unique towers and how the duplicate towers emerge. This is an example of how folding back contributes to building mathematical understanding. Sometimes a learner cannot progress in understanding, because some aspect of their understanding must be revisited and/or reconstructed so that the problem can be addressed through a different lens.

In Session V, Stephanie is working on building five-tall towers with Dana in the fourth grade. In a previous session, Stephanie had classified towers as “cousins” when two towers were the same, but laying in opposite vertical directions so they appeared to be different. This was a time when Stephanie identified that a tower was same as one that she had already made. In this session, Stephanie recalls towers that are repeated. She recognizes a characteristic of towers to identify when the towers are the same. She recalls how to address this, and turns one around to show that they are matching towers. She clarifies that these are repeated towers, and rebuilds her understanding of orientation of towers and recreates her image of unique towers, so that she can use this to develop patterns and classes for towers later on that do not include duplicate towers.

In Session VIII, when Stephanie engages in a one-on-one interview with Researcher 1, Stephanie connects the Towers problem to the Shirts and Pants problem, and uses folding back as a tool to reconstruct meaning behind the different variables of the problem. She chose to work on a simpler problem, specifically the Shirts and Pants problem where she has one black and one white shirt, along with one black and one white pair of pants, and must find the number of possible outfits that can be made. She connects the black and white clothing articles to the black and white cubes, and the tower height of two tall with two different articles to choose from. She finds that the number of possible outfits is equivalent to the number of possible towers in this case. Folding back, with researcher guidance, allows Stephanie to take this newly constructed idea and extend this to the three-tall towers problem, where a black or white hat option is added to the outfit, and a cube to the tower height. She continues this process and can make predictions about what it means to add a black or white feather to the hat as an option for the outfit, and how this connects to the aspects of the towers problem. In this situation, folding back was essential for Stephanie to be able to make predictions. Stephanie, in revisiting a simpler problem, identifies variables, and make connections between the Towers and the Shirts and Pants problem on a more foundational level. After this process, Stephanie made conjectures when looking at the more challenging three-tall and four-tall towers problem, and the outfit problem involving more options for additional articles of clothing.

5.3 Addendum to the Pirie-Kieren Model

To answer the fifth research question about collaboration's role in a student's learning, a focus on studying student growth in mathematical understanding using a model that incorporates collaboration can provide more insight into learning. This researcher acknowledges Pirie and Kieren's extensive work on studying the growth in mathematical understanding, and acknowledges Martin, Towers, and Pirie's work on studying collective understanding using the Pirie-Kieren model as a framework (2009). Martin, Towers, and Pirie (2009) explored the phenomenon of mathematical understanding and discussed the nature of collective understanding, characterizing the growth of collective mathematical understanding as "a creative and emergent improvisational process" (p. 149). They show how the consideration of collective understanding can explain growth in mathematical understanding in more detail. They also look at implications for practice and how this consideration affect context and creation of mathematical tasks in a classroom.

As such, the researcher proposes an addition to the model that incorporates collaboration's effect on helping individual learners move along layers of the Pirie-Kieren model for studying growth in mathematical understanding. This differs from the work of Martin, Towers, & Pirie (2006) and the work of Martin & Towers (2016) in that instead of studying collective growth as collaboration occurs, the model will be applied to study designs that initially set up conditions for collaboration.

5.31 Individual instances of collaboration's effect on growth

In Session I, when Stephanie is in third grade engaging with the Shirts and Pants problem, her collaboration with Dana is crucial to their collective movement from primitive knowing to image making. Instead of guessing and checking, Stephanie proposes that they "draw a picture" and they both represent the problem visually using paper and markers to sketch the shirts and pants. Although Stephanie is the one who proposes creating a representation of the problem by drawing, the girls individually create their own pictures to work out the solution. Each student originally comes up with a different number of possible outfits for the three different colored shirts and two different color options for pairs of pants. Stephanie comes up with four, and Dana with six. Individually working, Stephanie may not have questioned her work. If she did question her work, without a partner she has no one's work to compare her own to. Because of the collaborative engagement, she asks Dana, "What are your other combinations? I have white and blue, I got white and white, I've got blue and white, I've got yellow and white. What were your two other combinations?" Dana is able to guide Stephanie to see that she has missed two unique possible outfit combinations. In order for Stephanie to create a concrete image that she can use to solve problems later on, she must have a comprehensive idea about the constraints of the problem and processes by which to explore the components of the task. Stephanie may not have been able to make adjustments to her image and develop her ideas without input from Dana.

This demonstrates a need for an addition to the Pirie-Kieren model that accounts for collaboration's effect on learners' movement between and among layers. Stephanie contributes to furthering this collaborative learning when Dana declares to the researcher that they have finished the problem, and Stephanie argues that they need to know for sure that they cannot make any combinations, emphasizing the importance of reasoning. This helps in the image making learning process because answers without justification and sound reasoning could lead to obstacles and misconceptions as the learner tries to apply the image later on. Stephanie may not have emphasized the need for reasoning, had Dana not pointed out Stephanie's missing outfits, and had Dana not moved to finalize her answer without reasoning for why they had found all outfits. The need for justification follows Stephanie working on later problems as she recalls the need for backing up her answers. Stephanie's image making is affected once again during this session when she and Dana are exploring an extension problem, where there is an additional pair of black jeans introduced. Dana proposes an answer, and Stephanie requests that she and Dana work out the problem separately and compare their work. She uses the same tools as before, drawing out the shirts and pants and labeling the clothing with letters representing the colors of the clothing articles, and then connecting lines from shirts to pants to create outfits. Stephanie is learning in the image making layer still, and as a part of constructing this image, she works with Dana to compare and check her work. Stephanie declares that the correct answer is nine after that, demonstrating confidence in her work and satisfaction with the way that she checked her work.

In Session V, Stephanie, now a fourth grader, is working on the five-tall towers problem in a classroom session with her partner Dana. When Dana says that she has another idea to try, Stephanie says, "Well, tell me it so that I can do the opposite." By working with Dana and requesting to hear her idea, Stephanie is exposed to an additional strategy for building the towers, and she builds onto Dana's idea by proposing that she will build the "opposites." Collaboration facilitated this growth in understanding and helped Stephanie develop a problem-solving process.

In session IX, Stephanie is working with Michelle, Milin, and Jeff about patterns that they found for the five-tall towers problem that emerged for towers of other heights. In the beginning of the session, the students collaboratively discuss the problem that they were working on and describe, in turn, to Researcher 1, alternating in speaking. Researcher 1 says that they will write down their work to "see if they all agree." As the students share their ideas, they talk about a pattern they recognized where the number of possible towers seemed to be numbers hat counted by two, until the three-tall tower session. In this session, Stephanie recognizes some towers being the same after reorientation. She says, "No, it doesn't [make a pair] because if you turn it around, it's the same, so that doesn't go with that one."

5.32 Characterization

In these instances, Stephanie is growing in mathematical understanding and moving among layers of learning. Some of that is fostered and facilitated by questioning by her classmates, guidance from the researcher, and collaborative strategies used by Stephanie and her classmates. When students produced answers with which Stephanie disagreed, the collaborative environment allowed her to verbally question the proposed solution, and to provide a different solution, and develop a reason for her proposal. When Stephanie and a partner's answers disagreed, she was able to reconstruct images, diagrams, towers, and ideas and discuss the characteristics of the problem with a classmate to develop an answer that they both supported, and an answer that they could back up with mathematical reasoning. Stephanie demonstrated learning in the property noticing layer during the "Gang of Four" session, and this was facilitated by collaboration with three other classmates. Stephanie was provided with the opportunity to talk through her ideas, to create diagrams, to assess her classmates' work, and to have her classmates assess and question her solutions and process. In this case specifically, Stephanie is adamant that she has a process for classifying the towers, and when a student claims that she is missing a tower, she explains her way of approaching the problem and says, "That is what you are doing. But that is not the way I am doing it."

The students recognized a doubling pattern, and collaboration provided a space for the students to explore this doubling pattern and to propose ideas for how to formalize this to find the number of possible towers for towers of any height. It is possible that a student learns from another student's ideas, based on the verbal evidence that the student built on and extended from what another student said or did, making adjustments to the process and thoughts offered by others.

Verbalizing math reasoning is a complex task, and a collaborative group setting for math tasks provides an opportunity and space for the learners to demonstrate their ideas verbally, through drawings, and in writing. The expectation of students in the studied session for this research operated under the expectation that if an answer is proposed, a justification and descriptive reasoning must follow. In the sessions where Stephanie works in a group, Stephanie verbally provides reasoning for the way she classified towers, how she knew that she had created all towers, and patterns that she noticed and predicted would continue in the problems. In a group setting, she demonstrated her ideas publically to her small group.

When students talk about their problem-solving process toward a solution, they get a chance to explore their ideas in a different way, trying to make sense of what they have discovered and gain ownership over their ideas. Engle and Conant (2002) write that productive engagement in educational content occurs in learning environments where students have authority to dialogue with others, hold each other accountable, compare their solution and work with others, and help other students to utilize the right context, manipulatives, or diagrams. Applebee et al. (2003), Nystrand (2006), and Soter et al. (2008) state that when teachers ask students to talk about their process, discuss other students' processes, and agree or disagree with others based on reasoning, the teacher is able to focus more on student comprehension. Evidence of this occurs in the sessions where Stephanie collaborates with her classmates. The researcher poses probing questions to individual students and then invites them to explore together. The collaborative environments encouraged the students on multiple occasions to claim that their classmate's solution was wrong, and to provide justification for those statements and sometimes even offer a counter solution. Stephanie and her collaboration partners offered ideas like creating a diagram, rebuilding towers, or making a list as tools to assist in their problem-solving process. When Stephanie compared her work with a partner, she used her partner's input as a method to confirm her work, to assess her partner's process, or as an indicator that she needed to reconstruct her understanding because she had missed something. Stephanie used these opportunities to question her classmates and redefine her problem-solving process in a way that worked for her. She also used these collaborative opportunities to try out ideas offered by her partner, or test out a proposition offered by another classmate. This learning can be characterized as productive engagement since learning in layers of understanding, and moving among layers of understanding was made possible through peer assistance, offering tools, questioning data and solutions, self-assessment with peer input, comparing work, and verbal argumentation and discussion.

The researcher proposes that elements of collaboration be added in to the descriptions of layers, and evidence of a student moving to another layer of understanding include instances where a partner's suggestion moves the learner, or where collective understanding develops from collaborative engagement, verbal argumentation, and individual growth in understanding is demonstrated in work, verbal statements, or justifications of the student's own work, or his or her partner's work. Additionally, in the expanded model, movement among layers can be characterized by a learner adjusting another student's work, suggesting a correction to a partner's work, or providing reasoning for someone else's solution based on the learner's own understanding of the other student's process.

5.4 Researcher Moves

Researcher moves and interaction were essential in advancing Stephanie's thinking and assisting in progressing her growth in mathematical understanding. This section highlights specific researcher moves and the effect on Stephanie's development and learning process. Questioning strategies, provision of ideas, prompts to explain and reason, providing a place from which to start, and specific mathematical guidance all contributed to Stephanie's mathematical growth in understanding. For the purposes of this study, "researcher" is synonymous with "facilitator/researcher" when deducing possible implications of the study. All participants referred to were researchers in this study, and their role could be extended to a facilitator, whose approach to learning respects collaborative learning and building deep understanding of mathematical ideas, for purposes of inference and educational implications beyond this study.

In Session I, Researcher 2 assists in facilitating and extending Stephanie's learning from the image making layer to the image having layer of learning when she questions the students and elicits reasoning from them. She uses phrases such as, "What do you think if you had four pairs of jeans? What would happen? Think about it, you don't have to do that one", and she proposes that they attempt to solve the problem without making physical drawings, but to think about what they already constructed.

In Session II, Researcher 2 references a previous problem and makes a comparison to the problem, saying that the students should "talk about it with a partner …like the shirts and pants problem, you have to convince that you found them all." This offers to students suggestions for how to approach the problem. The researcher also encourages collaboration when she asks the students to talk with a partner, and when she challenges the students to convince their partners that all possible towers have been found. These researcher moves sets expectations for collaborative work, and for answers backed by reasoning and justification.

In Session V, Stephanie is in grade four and working on the five-tall towers problem in a classroom setting. Researcher 1 asks the students to "convince them" that they had found all possible towers. This posing of the instructions for the task leaves the question open-ended, and does not just call for an answer, but a reason as to why no more towers can be made. Researcher 1 also ends her instructions telling the students to "have fun" with the task, which sets a positive, non-threatening tone for the task.

In Session VI, Stephanie is sitting in a one-on-one interview with Researcher 1 to discuss the five-tall towers classroom session from session V. Researcher 1 asks how she and Dana worked together, which highlights the collaborative effort between the two students, and encourages Stephanie to verbalize and summarize their process. When Stephanie says that they made exactly thirty-two five-tall towers, Researcher 1 asks if Stephanie believes this is the answer. Instead of providing Stephanie with a summative and immediate evaluation of her answer being correct or incorrect, Researcher 1 offers Stephanie the opportunity to reflect on her learning. She further prompts a justification from Stephanie by providing an alternate scenario, and requesting how she might respond. Researcher 1 asks Stephanie what she would say to a student who proposed that thirty towers could be made instead of the thirty-two that Stephanie proposed. Later in that interview, Researcher 1 prompts further self-assessment by asking Stephanie, "Are you convinced?" and following up asking why she is convinced. Later in the interview when Stephanie is describing how she came up with different sets of towers, Researcher 1 asks Stephanie to compare towers and to look for similarities, without explicitly telling her which towers are similar or what characteristics to look for. This move elicits description of sets of towers by Stephanie. As Stephanie is working later in the interview, Researcher 1 proposes that Stephanie write her work on paper for an additional representation method of her work. Researcher 1 proposes taking some individual thinking time and keeping track of towers using numbers. This move encourages Stephanie to be organized and to represent her work differently to possibly elicit different understanding.

In Session VIII, Researcher 1 makes a move of asking Stephanie about a specific strategy where she did "double and pair." Researcher 1 summarizes the strategy and asks Stephanie to convince her that she has all the towers in this specific case. This move specifies to Stephanie what she needs to justify. Researcher 1 then prompts Stephanie to think about how she would explain her argument to someone, again highlighting the importance of being able to provide backing for a given answer.

In Session IX where a small group of students are working on the three-tall towers problem, the researcher restates the problem and clarifies what needs to be justified. She says, "He has proved to me that from the four, you could only make eight" and encourages Stephanie to provide complete reasoning. Students are taking different approaches in their work, and Stephanie is insisting that she is using a different method (cases) and does not seem to want input from other students about using different tower groupings. Researcher 1 encourages multiple representations of solutions and allows for contributions from all students when she asks Stephanie to explain her method, and assigns value to Stephanie's input when she calls her method "interesting."

Later in this small-group interview, Researcher 1 makes the move of asking the students to call on previous work to make a prediction, asking how many towers are possible for towers of height four. Later the students are trying to generalize a rule for finding how many towers can be made with different heights, and Researcher 1 asks questions about the different methods that the students propose. She asks questions like, "Could this have worked?" and "When would this work?" and "Why didn't the eight work?" As the students work to try to generalize a rule, Researcher 1 makes moves to break the problem into something more accessible. She says, "Let's not worry about that big problem for a moment. Let's try to do it with a simple problem." This not only introduces a new heuristic in problem solving, but makes the current problem easier to approach, making it accessible to build up from where students seemed to be in their problem solving.

In Session 12, Researcher 1 uses questioning strategies in a one-on-one interview with Stephanie that challenge her to advance her thinking. After Researcher 1 poses a question, she allows Stephanie time to verbally conjecture, and then asks, "Want to think about that?" which allows Stephanie time for individual exploration. Researcher 1 asks Stephanie to generalize throughout the interview, and provides Stephanie with notation needed to generalize. She says, "What would you think it would be if I could select one from n?" and asks Stephanie, "Can you imagine that?"

In Session XIII, Researcher 1 asks Stephanie to verbalize what she is visualizing about the problem in order to get Stephanie to talk through her ideas. Soon, Researcher 1 works with Stephanie to develop a pattern and connects this pattern to the numbers of combinations of towers possible for each row number. They develop the pattern and concept, and only then does Researcher 1 provide Stephanie with the formal name for the pattern: Pascal's triangle. This move allows Stephanie to develop a new concept and make predictions, in contrast to approaches in which students are passively given a formal term, definition, and properties.

In summation, these researcher moves highlight the importance of questioning strategies that provide space for open-ended thinking, as well as phrasing questions and problems in ways that require justification and reasoning to support the solution. The researcher moves serve to promote collaborative problem-solving efforts, and to encourage all students' ideas to be shared, examined, and compared. The researcher moves encourage posing multiple representations for solutions and for justification of those solutions. Researcher moves sometimes suggested that students break down problems, recall similar problems, or introduce formal notation and properties after a concept is developed.

Chapter 6: Discussion

This study contributes to the field of research by illustrating Stephanie's mathematics learning progressions over seven years, her longitudinal growth in mathematical understanding and reasoning, and effects of her collaboration with other classmates in her growth of mathematical understanding. The details of Stephanie's learning over the years provide insights into the complexity of learning as one traces her growth in understanding through the layers of the Pirie-Kieren model. Products of the research include accompanying video narratives in the form of six VMCAnalytics published on the Video Mosaic Collaborative. These video stories provide potential tools for following Stephanie's mathematical learning trajectory over the years and are available worldwide for pre-and in-service teacher education. Stephanie's story can be shared at a variety of grade levels where she began her task investigations in the lower elementary grades, revisited and extended her work in the middle-school years, and provided thoughtful insights and connections during her high-school explorations. The narrative of Stephanie's mathematical learning, captured in the VMCAnalytics are recommended for use in professional development opportunities for teachers. These stories demonstrate what learning looks like in the different layers of understanding, and how moving across layers exemplifies the learning progression for one individual student. The video narratives offer a visual demonstration of how concepts are built over time. They show how Stephanie draws large, meaningful conclusions based on her earlier explorations of mathematical ideas. Also, they illustrate how her investigations provided a solid foundation of multiple representations of concepts. Finally, they demonstrate

Stephanie's revisiting and extending of her knowledge, as she makes connections between and among ideas over the years of her mathematical investigations.

This study has been accomplished by analyzing years of video data archived from longitudinal studies that occurred in both informal and classroom settings, mapping one individual student's growth in mathematical understanding as she engaged in open-ended mathematical tasks from grades two through eight. By examining researcher questioning and pedagogical moves in classroom settings and individual, task-based interviews, insight can be gained to creating opportunities for student learning, based on earlier performance and ongoing assessments of student understandings and potential to build deeper knowledge. By attending to the researcher moves, that included opportunities to revisit tasks, collaborate with other students, and share multiple representations of ideas, teachers studying the work may find implications of findings of this study to their practice.

The longitudinal nature of the study and the conditions of the research that supported student collaboration enabled an extension of the theoretical model proposed by Pirie and Kieren for studying growth in mathematical understanding. Data were available to study Stephanie's problem solving in a variety of contexts and over many years – working with a partner, a small group, with researchers in a teaching experiment settings, for example. An instance of Stephanie working with a partner is seen in the grade two Shirts and Pants problem data in Session 0, as well as the grade three Shirts and Pants Problem in Session I when Stephanie worked with a partner Dana, and grade three initial towers exploration in grade three, Session II, when Stephanie again worked with Dana to explore building towers four-high. She again works with Dana in Session V when they explore the five-tall towers problem in a classroom session. Stephanie works with a small group in Session IX, nicknamed the “Gang of Four”, where Stephanie works with Milin, Michelle, and Jeff to discuss patterns for building towers three tall, and explore the doubling pattern that they found as they increased the tower height by one cube each time. Evidence of Stephanie working with researchers in interview settings can be seen in Session VI when Stephanie revisits the five-tall tower problem that she had worked on in the previous class session in a one-on-one interview with Researcher 1. Stephanie again works in an interview setting with Researcher 1 in the four-tall towers interview in grade four, in Session XIII, when Stephanie makes important connections between different problems she had engaged with. In Sessions X-XIII, Stephanie works with Researcher 1 in an interview setting to explore the connection of the towers problem to Pascal’s triangle in the eighth grade.

6.1 Discussion on Findings

This study endeavored to take on the challenge of studying and documenting the development of one student’s growth in mathematical understanding and constructing of mathematical ideas and ways of reasoning in a variety of settings from elementary through secondary school, and create video narratives to support the creation of a student’s learning trajectory. Mathematics education researchers aim to prescribe a detailed idea about how human’s think and reason about solutions to mathematics problems so that learners can overcome learning obstacles (Davis, 1984). Contributing to this mission, this researcher aimed to focus on the learning of on one individual, over time, and conduct a detailed, qualitative study using text and video data, tracing her growth in understanding and illustrate her learning progression journey. The complexity of learning in teaching experiments and classroom settings involves attending to, researcher moves that may have influenced her growth in understanding. Also, her collaboration with peers as her ideas were challenged, revisited, became further refined and developed were relevant in studying her movement between and among layers of understanding.

With increased emphasis on personalized learning in mathematics education, and interest on developing learning progressions for tracing growth in mathematics, this research offers several lenses for attending to the complexity of student growth and for studying individual students' progressions of mathematical learning. The importance of attending to the benefits of personalized, differentiated instruction, with access to open-ended tasks that can be revisited in greater complexity and with different levels of abstraction over time, with opportunities for collaboration, suggests future studies and the production of new learning progressions.

Stephanie's approach to problem solving is consistent with that proposed by Polya (1962). Stephanie works through the steps of understanding what is being asked, devising a plan to solve, taking individual thought time to employ strategies such as making lists, drawing pictures, working with manipulatives, and looking for patterns, and then carries out her plan. Stephanie encounters many obstacles in her problem-solving processes, and worked past these obstacles by starting over to create a new representation, discussing with a partner, or going back over her written work. Stephanie consistently worked to verify her solution and make sure that it made sense. She verbally recalled solutions to previous tasks to discuss the validity and reasonableness of her proposed solution. This is in keeping with Polya's systematic process used to reach a solution to a problem.

6.2 Implications

Results from an individual case study can offer detailed insights; however, these results should not be generalized, as cautioned by Stake (1995). Nevertheless, some results have implications for practice and for teacher education. The application and extension of the theoretical model proposed by Pirie and Kieran may be useful for other researchers wanting to engage in similar research. This study was not meant in any way to generalize Stephanie's learning progression to all or any mathematics leaners. However, one may make observations based on this study of the importance of the individual case being considered and the data offered over time. Ahluwalia (2011) conducted a longitudinal study using a student, Robert, as her subject and analyzed how external representations created by Robert helped him in building mathematical understanding over a sixteen-year period. Steffero (2010) also studied one of the original participants from the longitudinal study, Romina, over a period of seventeen years. Steffero analyzed the relationship between Romina's beliefs and mathematical behaviors. Brookes (2015) studied another student, Jeff, from the original longitudinal study from grades two to twelve. Brookes studied student roles in collaborative mathematics groups and analyzed how these groups appeared to impact learning

As indicated previously, Stephanie's story has implications about learning mathematics for classroom teachers. This study adds to existing work that, taken together with the previously cited longitudinal studies, may offer generalizations about conditions for student learning. The researcher moves identified in this study might serve to inform classroom teachers about effective questioning strategies, interventions, task design, the promotion of argumentation as well as other behaviors that have potential to not only engage students but shift responsibility for gaining knowledge to the student. Instructor moves that tend to elicit justification and argumentation, and model building, verbal or written reasoning, and the notion of revisiting strands of tasks over time that invite collaborative, explorative work and enable building more abstract and general ideas. As the researchers encouraged students to “convince their partner”, or provide scenarios where an older student, “a fifth grader” provides an alternate solution and the student must either disprove that solution or provide backing for why the “fifth grader” provided a convincing argument. Researchers sometimes suggested that the students consider a simpler problem so that the opportunity to revisit and rebuild his or her understanding enabled to taking on a more complex question. The researchers also would introduce students to similar problems, providing a context for making connections and/or noticing differences between and among the structure of other problems and mathematical ideas. Researchers, refraining from serving as the “authority” for the correctness of a solution, did not tell students whether they were “correct” or “incorrect”, but rather would challenge the student to provide a justification for why the student thinks a certain answer is correct, or why another answer was wrong. Prompting further exploration and individual assessment of ideas assists the students in learning about a problem’s characteristics and constraints. “Researcher moves” in this study can serve as a guide for “teacher moves” to model their questioning techniques and pedagogical moves.

Implications for classroom practice can be observed in each classroom session and small group learning session. The structure of the strand of open-ended mathematical tasks, as well as the option and encouragement of multiple representations of solutions offer a model for instruction across all grade levels. When Stephanie and her classmates explored the different problems, they were encouraged to build physical models, represent their ideas with a drawing, verbalize their reasoning, discuss ideas with classmates, and convince themselves and others of the validity of their solution. The encouragement for collaboration serves as a model across grade levels as well. Evidenced by Stephanie and her classmate's engagement in collaborative tasks throughout grade levels, group settings where partner discussions become a norm serves the students well as they grow in their mathematical understanding. A classroom practice implication also comes from the expectation that solutions make sense and are backed by reasoning. In the earlier grade sessions, researchers would follow up task introduction with reminders about providing reasoning for solutions, and continuous questioning of "Why do you think that is true?" and "Are you convinced?" Later, researchers need less and less to encourage students to provide reasoning with their answers, as this is a classroom norm and a known expectation of these tasks, and spontaneously taken on by the students, themselves.

Implications for researchers also emerge from this study. There are other students who participated in the Rutgers Longitudinal Study, and these students would be very interesting subjects to follow longitudinally using the same framework. Specifically, a comparison of the studies for studying students longitudinally over a period of years as they grow in mathematical understanding using the Pirie-Kieren model would produce useful and interesting results. The potential studies provide a context for research that can be done with longitudinal study data that are preserved and available to researchers. Thus, for researchers considering conducting a longitudinal study and wondering about the benefits of future research, this study shows an example of work that can result from data collected longitudinally.

6.3 Future Work

Research that analyzes collaborative contexts where students develop a concept over a period of time conceptually, and connect representations and ideas from different isomorphic problems is recommended for further study. Specifically, Stephanie engages in an eleventh-grade problem-solving session where she works with a small group on a "pizza problem", which is a counting problem. By extending this study to incorporate Stephanie's development of isomorphic relationships between and among the pizza problem, the towers problems, and the shirts and pants problems that she engaged in previously, the depth of her understanding can be captured. Also examining how formal notation and concept introduction connects with conceptual development of Stephanie's mathematical ideas would provide important knowledge about how a transition from personal to formal knowledge is built. Using the Pirie-Kierenlensin a context where student collaboration is a norm would be another area of research in growth of mathematical understanding, especially in the context of a longitudinal study such as this one. Conducting similar studies on a smaller scale, over a shorter period of time, with high-school students working on algebra-based tasks is another area where research is needed, especially in a setting where collaboration and justification of solutions are encouraged. Researcher/teacher moves, student moves, and task structure also monitored and studied in the context of observing growth of mathematical understanding of the individual students being studied are other lenses for future work.

Expanding on the data explored in session VIII of the current study will enable one to unpack the connections Stephanie made between the Towers and Shirts and Pants problems. Learning by productive engagement through a longitudinal case study lens, inspired by the research in this dissertation, is another area recommended for study. Student verbal argumentation is another important lens to be explored in further research.

Future work is suggested to contribute to this and the three other longitudinal studies following participating students. There is a need, also, to develop VMCAnalytics to accompany the written text of these other studies. Considering the study of Stephanie's growth in understanding, together with the longitudinal studies of Robert, Romina, and Jeff, possible generalizations about conditions for student learning can emerge.

References

Aboelnaga, Eman Y. Rutgers The State University of New Jersey - New Brunswick, ProQuest
Dissertations Publishing, 2011. 3464710.

Anderson, D. S. & Piazza, J. A. (1996). Changing beliefs: teaching and learning Mathematics in
constructivist preservice classrooms. Action in Teacher Education, 18, 51-62.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa Fuentes, S., Trigueros, M. and Weller, K.
(2014). APOS Theory. A framework for Research and Curriculum Development in
Mathematics education. Springer.

Ahluwalia, Anoop. (2011) Tracing the building of Robert's connections in mathematical problem
solving. Retrieved from https://doi.org/doi:10.7282/T3NP243N

Applebee, Arthur N., Judith A. Langer, Martin Nystrand, and Adam Gamoran. 2003.
"Discussion-Based Approaches to Developing Understanding: Classroom Instruction and
Student Performance in Middle and High School English." *American Educational Research Journal* 40, no. 3 (Autumn): 685–730.

APOS Theory: A Framework for Research and Curriculum Development in Mathematics Education, Springer, ISBN: 978-1-4614-7965-9 (Print) 978-1-4614-7966-6 (Online)

Asiala, M., et al. (1996), A framework for research and curriculum development in undergraduate mathematics education, *Research in Collegiate Mathematics Education II*,

CBMS Issues in Mathematics Education, 6, 1-32.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.) *Mathematics, teachers and children,* (pp. 216-235). (Trans. By D. Pimm).

London: Hodder & Stoughton.

Brookes, Elijah A. (2015) Student roles in collaborative math groups. Retrieved from https://doi.org/doi:10.7282/T3Z89F44

Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E. & Weisbeck, L. (1993). Models of

problem solving: A study of kindergarten children's problem-solving processes. *Journal*

for Research in Mathematics Education, 24(5), 427-440.

Carpenter, T. P., Fennema, E., Franke, M., Levi, L. & Empson, S. B. (1999). *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Fennema, E., Franke, M., Levi, L. & Empson, S. B. (2000). *Cognitively Guided*

Instruction: A Research-Based Teacher Professional Development Program for Elementary Mathematics. Research Report 003. Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Carpenter, T.P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's* *mathematics: Cognitively guided instruction* (2nd ed.), Portsmouth, NH:

Heinemann.

Carpenter, T. P., Fennema, E., Franke, M. (1996). *Cognitively Guided Instruction: A*

Knowledge

Base for Reform in Primary Mathematics Instruction. The Elementary School

Journal,

97(1) 3-20.

Davis, R.B. (1984) *Learning mathematics: The cognitive science approach to*

mathematics

education. Norwood, New Jersey: Greenwood Publishing Group.

Davis, R. B. and Maher, C.A. (1990). What do we do when we “do mathematics”?

Constructivist

views of the teaching and learning of mathematics (Monograph no. 4, pp. 65–78).

Reston,

VA: National Council of Teachers of Mathematics.

Davis, R. (1990). Constructivist Views on the Teaching and Learning of

Mathematics. Journal for Research in Mathematics Education: Monograph No. 4.

Journal

for Research in Mathematics Education Monograph.

Dewey, John. (1938) *Experience and education,* New York, Macmillan.

Dienes, Z. (1969) Building up mathematics. London: Hutchinson Education

Dubinsky, E. (1991). The constructive aspects of reflective abstraction in advanced

mathematics,

in (L. P. Steffe, ed.) *Epistemological Foundations of Mathematical Experiences*, New York: Springer-Verlag.

Dubinsky E., Leron, U., Dautermann, J., and Zazkis, R.: 1994, ‘On learning fundamental concepts of group theory’, *Educational Studies in Mathematics* **27**, pp. 267–305.

Dubinsky, E. & McDonald, M. (2001). APOS: A Constructivist Theory of Learning in Undergraduate Mathematics Education Research, In D. Holton et. (Eds.), *The Teaching and Learning of Mathematics at University Level*: An IResearcher 1I Study, Kluwer Academic Publishers, 273-280.

Engle, Randi A., Faith R. Conant. 2002. “Guiding Principles for Fostering Productive Disciplinary Engagement: Explaining an Emergent Argument in a Community of Learners Classroom.” *Cognition and Instruction* 20 (4): 399–483.

Erlwanger S. H. (1973) Benny’s conception of rules and answers in IPI mathematics. Journal of Children’s Mathematical Behavior, 1(2) 7–26.

Fosnot, C.T.: Constructivism: A Psychological Theory of Learning. In Fosnot, C.T. (ed.) Constructivism: Theory, Perspectives and Practice. pp. 8-33 Teachers College Press, New York (1996).

Francisco, John. (2013). Learning in collaborative settings: Students building on each other's ideas to promote their mathematical understanding. Educational Studies in Mathematics.

82. 10.1007/s10649-012-9437-3.

Gattegno, C. & Cuisenaire, G. (1954) Numbers in Colour. London: Heinneman Langley, D.

(2013) Division with Dienes. Primary Mathematics 17(2) p 13 – 15. Leicester: Mathematical Association

George, L. G. (2017). *Children's learning of the partitive quotient fraction sub-construct and the elaboration of the don't need boundary feature of the Pirie-Kieren theory* (Ph.D. Dissertation), University of Southampton.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258-288. doi:10.2307/30034868

Kamii, C. & Rummelsburg, (2008). Arithmetic for first graders lacking number concepts. *Teaching Children Mathematics 14*(7), 389-394.

Kieren, T.E. (1994a) Beyond metaphor: Formalising in mathematical understanding within constructivist environments. *For the Learning of Mathematics,* 14 (1), 39-44.

Magoon A. J. (1977) Constructivist approaches in educational research. Review of Educational Research, 47(4) 651–693.

Maher, C. A. & Yankelewitz, D. (Eds.) (2017). *Children's reasoning while building fraction ideas*. Heidelberg/Dordrcht/Rotterdam: Sense Publishers.

Maher, Carolyn & Powell, Arthur & Uptegrove, Elizabeth. (2011). *Combinatorics and reasoning. Representing, justifying and building isomorphisms*. 10.1007/978-94-007-0615-6.

Maher, C. A. (2002). How students structure their own investigations and educate us: What we have learned from a fourteen-year study. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the Twenty-sixth Annual Meeting of the International Group for the Psychology of Mathematics Education (PME26)* (Vol. 1, pp. 31–46). Norwich, England: School of Education and Professional Development, University of East Anglia.

Maher, C. A., Davis, R.B., and Alston, A.S. (1991). Implementing a "Thinking curriculum" in mathematics. *Journal of Mathematical Behavior* 10, 219–224.

Maher, C. A., and Martino, A.M. (1996a). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education* 27, 194–214.

Maher, C. A., and Martino, A.M. (1996b). Young children invent methods of proof: The gang of four. In L.P. Steffe, P. Nesher, P. Cobb, G,A, Goldin, & B. Greer (Eds.), *Theories of*

Mathematical Learning (pp. 41-90). Norwood, NJ: Ablex.

Maher, C. A., & Martino, A. M. (2000). From patterns to theories: Conditions for conceptual

change. *Journal for Mathematical Behavior, 19*(2), 247-271.

Maher, C. A., and Speiser, R. (1997). How far can you go with block towers? *Journal of Mathematical Behavior* 16(2): 125–132.

Maher, C. A., & Yankelewitz, D. (2017). *Children's reasoning while building fraction ideas*.

Martin, L. C. (1999). The nature of the folding back phenomenon within the Pirie-Kieren theory

for the growth of mathematical understanding and the associated implications for teachers and learners of mathematics. Unpublished doctoral dissertation, Oxford University.

Martin, L. C., & Towers, J. (2016). Folding back, thickening and mathematical met-befores. *The*

Journal of Mathematical Behavior, *43*, 89-97.

Martin, L., & Towers, J. (2016b). Folding back and growing mathematical understanding: A

longitudinal study of learning. *International Journal for Lesson and Learning Studies, 5*(4), 281-294.

doi:http://dx.doi.org.proxy.libraries.rutgers.edu/10.1108/IJLLS-

04-2016-0010

Martin, L.C., Towers, J., & Pirie, S. (2006) Collective Mathematical Understanding as Improvisation, Mathematical Thinking and Learning, 8:2, 149-183, DOI: 10.1207/s15327833mtl0802_3

Meel, D.E. (2003). Models and theories of mathematical understanding: Comparing Pirie and Kieren's model of the growth of mathematical understanding and APOS theory, *Research in Collegiate Mathematics Education V, (pp. 132-181)*. Providence RI: American Mathematical Society.

Michener, Edwina Rissland (1978). *Understanding Understanding Mathematics*. Cognitive Science, 2 (4):361-383.

Moyer, P. (2001) Are we having fun yet? How teachers use manipulatives to teach mathematics. Educational Studies in Mathematics 47: 175-197. Netherlands: Kluwer.

Neisser U. (1967) Cognitive psychology. New York: Appleton-Century-Crofts.

Noddings N. (1973) Constructivism as a base for a theory of teaching. Unpublished doctoral dissertation, Stanford University.

Noddings N. (1990) Constructivism in mathematics education. In: Davis R. B., Maher C. A. & Noddings N. (eds.) Constructivist views on the teaching and learning of mathematics.

National Council of Teachers of Mathematics, Reston VA: 7–18. Available at http://cepa.info/2961

Nystrand, Martin. 2006). “Research on the Role of Classroom Discourse as It Affects Reading

Comprehension.” *Research in the Teaching of English* 40, no. 4 (May): 392–412.

Piaget, J. (1952) The Child's Conception of Number. New York: Humanities Press.

Pirie, S. E. B. and Kieren, T. E. (1991). ‘Folding back: Dynamics in the growth of mathematical understanding’, in F. Furinghetti (ed.), *Proceedings Fifteenth Psychology*

of Mathematics Education Conference, Assisi.

Pirie, S., & Kieren, T. (1992). Creating constructivist environments and constructing creative mathematics. *Educational Studies in Mathematics, 23,* 505-528.Pirie, S., &

Pirie, S., & Kieren, T. (1994). Growth in Mathematical Understanding: How Can We Characterise It and How Can We Represent It? *Educational Studies in Mathematics,*

26(2/3), 165-190.

Pirie, S. E. B., Martin, L. C. & Kieren, T. E. (1996). Folding back to collect: Knowing you know

what you need to know. In L. Puig & A. Gutierrez (Eds.), Proceedings of the Twentieth Annual Meeting of the International Group for the Psychology of Mathematics Education, Vol. 4 (pp. 147-154). Valencia, Spain: PME.

Polya, G. (1962). Mathematical discovery: On understanding, learning, and teaching problem

solving. New York: John Wiley.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the

development of learners' mathematical ideas and reasoning using videotape data. *The*

Journal of Mathematical Behavior, 22(4), 405-43

Saxe, G.B. (1988). Studying working intelligence. In B. Rogoff & J. Lave (Eds.), *Everyday*

cognition (pp. 9-40). Cambridge, MA: Harvard University Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (Special Issue Editor). (2000). Examining the complexity of teaching. Special

issue of the Journal of Mathematical Behavior, 18(3).

Skemp, R. R. (1976). *Relational understanding and instrumental understanding. Mathematics*

Teaching, 77, 20-26

Soter, Anna O., Ian A. Wilkinson, P. Karen Murphy, Lucila Rudge, Kristin Reninger, and

Margaret Edwards, M. 2008. "What the Discourse Tells Us: Talk and Indicators of High-

Level Comprehension." *International Journal of Educational Research* 47 (6):

372–91.

Stake, R. (1995). *The art of case study research*. Thousand Oaks, CA: Sage Publications.

Steffero, Maria (2010). *Tracing beliefs and behaviors of a participant in a longitudinal study for*

the development of mathematical ideas and reasoning: A case study.

Van Ness, Cheryl (2017). *Creating and using VMCAnalytics for preservice teachers' studying of*

argumentation.

Vista, A. (2013). The role of reading comprehension in math achievement growth: Investigating

the magnitude and mechanism of the mediating effect on math achievement in Australian

classrooms. *International Journal of Educational Research, 62*, 21-35. doi:10.1016/j.ijer.2013.06.009

Walker, D., & Lambert, L. (1995). Learning and leading theory: A century in the making. In L Lambert, D. Walker, D. P. Zimmerman, J. E. Cooper, M. D. Lambert, M. E. Gardner, & P. J. Ford Slack, *The constructivist leader* (pp. 1-27). New York, NY: Teachers College Press, Columbia University.

Webb, N. M., Franke, M. L., Ing, M., Chan, A., De, T., Freund, D., & Battey, D. (2008). The

role of teacher instructional practices in student collaboration. *Contemporary Educational*

Psychology, 33, 360-381.

Windschitl, M. (1999). A vision educators can put into practice: portraying the constructivist classroom as a cultural system. School Science and Mathematics, 99 (4), 189-196.

Yankelewitz, D. (2009). *The development of mathematical reasoning in elementary school students' exploration of fraction ideas* (Unpublished doctoral dissertation). Rutgers, The State University of New Jersey, New Brunswick, NJ.

Appendix A: Researcher Identification

Researcher 1: Researcher Carolyn Maher

Researcher 2: Researcher Amy Martino

Researcher 3: Researcher Alice Alston

Appendix B: Transcript Session 0-Shirts and Pants Second Grade

Stephanie engages in the Shirts and Pants problem in second grade with Dana and Michael at Harding Elementary School under researcher Amy Martino on May 30th, 1990.

Description: Clip 8 of 8: Shirts and pants with Stephanie, Dana and Michael Content: Harding Elementary School Research: Amy Martino Tape: Non-routine counting problems Date: 1990-05-30	Authors: Madeline Yedman Verified: Dasom Lee Date: 2013-11-25 Page: 1 of 4

Line	Time	Speaker	Transcript
1	0:26	**Stephanie**	I'm going to make a shirt and I'm going to put W for white in it
2		**Michael**	White shirt, white pants
3		**Stephanie**	Blue and then Yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?
4		**Michael**	He can only make two outfits
5		**Stephanie**	No how many different outfits. He can make a whole lot of different outfits look. He can make white and white
6		**Dana**	He can make all three of these shirts with it
7		**Stephanie**	Yeah, but shh. You can make it different ways too just like look white and white, that's one by doing W and W. Two could be blue, blue jeans and a white shirt
8		**Dana**	Yeah, we'll just put with with..

9		**Stephanie**	Shhhh. Okay yellow shirt and number three could be a yellow shirt and
10		**Dana**	The yellow shirt can go with the white
11		**Michael**	I'm doing blue pants and white shirt and then I'm doing blue pant blue shirt.
12		**Stephanie**	But how many outfits can it make it doesn't matter if it doesn't match. As long as it can make outfits. It doesn't have to go with each other Dana. It can make more if you put them mixed up, just watch. I'm on my third one right here. Number four it could be blue shirt and blue pants. Number five can be a white shirt and wait it can be a blue

Appendix C: Transcript Session I-Shirts and Pants Third Grade

Stephanie engages in the Shirts and Pants problem in third grade with Dana under researcher Amy Martino on October 11th, 1990.

Description: Clip 1 of 2: Introducing and working on the problem Content: Shirts and Pants with Stephanie and Dana Researcher: Amy Martino Date: 1990-10-11	Authors: Madeline Yedman Verifier: Dasom Lee Date: 2014-05-05 Page: 1 of 6

Line	Time	Speaker	Transcript
1	1:00	**Researcher 1**	Okay we are going to do two problems today. And I think that you're going to find that they are challenging, but they are fun. Okay, now I am going to give you your very first problem and you're each getting this paper with the first problem on it. And what you have to do is solve the problem, but it is like last year guys I'm really interested in how you solved the problem. I want you to be able to explain that to me on your paper. Okay, so say your answer was twenty-four or something like that. That's fine and good, but I want to know how you got that twenty-four okay? And you can do that in any way that you'd like. You can write, you can draw, whatever. Explain that to me but whatever you do on here, I want you and your partner to decide what you're gunna put on the large paper. Remember when we did this last year? Okay so you're going to decide what your group answer is, and how you did it and put it on a large paper for me.
2		**Stephanie**	I guess we're gunna use these (holding up marker)
3		**Researcher 1**	You can use the markers if you want, just be careful

4		**Stephanie**	I'll use the light blue marker, and then if I draw a picture ill use the dark one.
5		**Dana**	Yeah, I'm gunna use them both
6		**Stephanie**	The little one doesn't write that big, so I'll have more room. Reading the problem: Stephanie. Wait Stephen. Stephan has a white shirt, a blue shirt, and a yellow shirt. Want me to read it out loud?
7		**Dana**	No I'll do it. He has a pair of blue jeans and a pair of white

Description: Clip 2 of 2: Extending the problem with additional pairs of jeans Content: Shirts and Pants with Stephanie and Dana Researcher: Amy Martino Date: 1990-10-11	Authors: Madeline Yedman Verifier: Dasom Lee Date: 2014-05-05 Page: 1 of 3

Line	Time	Speaker	Transcript
1		**Stephanie**	We're done with this
2		**Researcher 1**	You're all done? Okay, people are still recording these, so you still hold those okay? Can I ask you a question?
3		**Dana**	Yeah
4		**Researcher 1**	You know what I'd like you to try on the back while we're waiting? What if I now gave you all the same clothes, but I also gave you another pair of jeans. A black pair of jeans. See if I can find out how many outfits with three different color shirts…
5		**Dana**	It would be twelve.
6		**Researcher 1**	Well lets see.

7		**Dana**	Like everything goes with black. Cause six plus six is twelve.
8		**Researcher 1**	Well do it and see okay? Remember so you got the same three shirts and the same two pairs of pants but now you got a new pair of pants for Christmas. You got a black pair.
9		**Stephanie**	Now what were the colors of the shirt? White, black, yellow.
10		**Dana**	It's gunna be twelve. It's gunna be twelve.
11		**Stephanie**	White, black, wait blue, and yellow. And then we have white, blue and black. Okay lets see. White, (inaudible) and nine, Dana it's nine!
12		**Dana**	Okay.
13		**Stephanie**	No Dana first I want you to figure it out, we may get different answers. Look, see you got white, white, and white.

Appendix D: Transcript- Session II Stephanie explores Towers Problem

Stephanie engages in the Towers problem in third grade with Dana at Harding Elementary School under Researcher 1 on October 11th, 1990.

Description: Towers with Stephanie and Dana, Clip 5 of 5: Recording their solution Content: Harding Elementary School Researcher: Researcher 1 Tape: Towers with Stephanie and Dana Date: 10/11/90	Authors: Madeline Yedmen Verified: Robert Sigley Date: 12/07/13 Page: 1 of 2

Line	Time	Speaker	Transcript
1		**Researcher 1**	Alright we are going to do something really different today. We're going to build towers with the Unifix cubes. Is that okay? There are certain rules though that you use to do that. Okay first of all, everyone should know what a tower is. What do you think a tower is with Unifix cubes? Billy.
2		**Billy**	When you put things together, like straight up.
3		**Researcher 1**	Like this? Would that be a tower if I were standing them up like this?
4		**Billy**	It's too small
5		**Researcher 1**	Does that look like a tower?
6		**Billy**	No

7		**Researcher 1**	It's little; it's a little tower. Okay, what we're going to do today is okay. We're going to present that we have some pretty teeny tiny people that we're building towers for. We're going to build towers today that have four stories to them. Okay so four blocks to them. Okay so every tower we make today is going to have four, okay? Alright, you're going to get two colors of Unifix cubes, red and blue everybody is going to get. Your job is to find out how many different looking towers you can make that are four high. Okay they all have to be four high, but I want to see as many different ways as you can do that possible and I want you to talk about it with your partner and again it's like the shirts and pants, you have to convince that you found them all. Okay so I'm going to pass out the problems and you can read this.
8		**Stephanie**	One, two, three, four. One, two, three, four.
9		**Dana**	No, how many different looking towers.
10		**Stephanie**	Different looks towers that we can make that are four stories high?
11		**Dana**	Four squares
12		**Stephanie**	Four squares. This one has four squares and my tower is flat.
13		**Researcher 1**	Okay
14		**Stephanie**	One of my towers are flat

15		**Researcher 1**	You're challenged to find out how many of these you can make and there is the problem in writing. You might want to take a look at that.
16		**Stephanie**	How many you can make using different kinds of things?
17		**Researcher 1**	Each one has to be different, but they all have to be four high. (Girls begin to create towers flat on the table) No, towers have to go this way.
18		**Dana**	Aw
19		**Researcher 1**	Yes they do. Because this is the point on the tower.
20		**Stephanie**	Great how many towers can we make? Here Dana I'm gunna read this out okay?
21		**Dana**	Ohhh, we can make them different colors.
22		**Stephanie**	Yeah, and Dana listen to this. (Reading directions) Your group has two different color Unifix cubes... You're right Dana! You're a genius; I'm gunna make this kind of pattern. Two, two. And then I could make red on the top and blue on the bottom. That's a different tower. Then I could make all
			red, all blue.

23		**Dana**	Look at this!
24		**Stephanie**	And I could make one, one, one, one. There, these things are easy. Alright so I'd have to put red on the bottom, and this on the top and red here and this here. Dana look these are my combinations. Okay? Ah ha, I know a different combination. Red one, and one, two, three, one. This is simple. Then two, wait wait wait. Two, one.
25		**Dana**	One of each color.
26		**Stephanie**	Oh nuts I ran out of them. I just have blue. I'm gunna have to use some of your blues Dana. I don't have enough of them. I'm gunna have to use some of your blues, and I'll give you a couple of my reds. Cause I need some of your blue. Um, one red. Then the others blue. God these can be put in many different ways.
27		**Researcher 1**	They sure can! Are you two working together?
28		**Dana**	After this we are..
29		**Stephanie**	Dana that would be a better idea. If we worked together then we would have more blocks and more combinations
30		**Researcher 1**	You really should. So why don't you compare and see which one of those you could eliminate.
31		**Stephanie**	Lets see what we can eliminate. We can eliminate…eliminate that one. And we could use these blocks for something. We can (comparing towers) I know I have this some place. I think I have it. We don't eliminate this one, we can put this at the end of the line.

32		**Dana**	It has to be the same one as that red one.
33		**Stephanie**	I know, I know. You can eliminate this one. Which we could use that one.
34		**Dana**	Look you have
35		**Stephanie**	Yeah, eliminate that one. No, no, no, nope, no, no, we can keep this one over here. The all blue eliminate it.
36		**Dana**	You have all blue?
37		**Stephanie**	Yes!
38		**Dana**	You have all red
39		**Stephanie**	Eliminate those. I don't have all blue at the bottom… I don't have all red at the bottom and blue at the top. So far none of this, no, no, no, no, nope, yup eliminate it. And this one I think I have actually, I'm not sure. Yup, eliminate it. A keep. Now look how many more blocks we have to use.
40		**Dana**	You're right.

Clip 2

Line	Time	Speaker	Transcript
1		**Dana**	Lets…
2		**Stephanie**	I'm gunna make, I'm gunna make, I'm gunna make, this! One red, blue, blue, wait… oh it's three red. Wait one blue, this one, this one and this one. Dana what are you doing? (Dana hold up a tower) I think I have that one.
3		**Dana**	No you don't

4		**Stephanie**	Hold on let me check. I think, I just think. Oh no I have it the other way blue at the top. Everything we make we have to check. Put that in line. Everything we make, lets make a deal-everything we make we have the check.
5		**Dana**	Alright, I'll always make it you'll always check it.
6		**Stephanie**	Alright, you make it, and I'll check it. (Dana hands a tower and Stephanie checks it) It's good.
7		**Dana**	You have that one right there.
8		**Stephanie**	Okay, eliminate it. (Counting towers they have so far) Sixteen. No Dana just make them I'll check them. Dana why don't you try this?
9		**Dana**	Did you keep that one I just made?
10		**Stephanie**	Blue, blue. No I eliminate it. Red, blue.
11		**Dana**	Then I'll do red, red, blue…
12		**Stephanie**	Uh no, I have this one. Well maybe if I did this! Nope, I have that one too Dana. I don't think we get anymore but, one, two, three, four, five…
13		**Dana**	Here
14		**Stephanie**	Okay. That's a good idea, got it. But why don't you try blues at the bottom and two reds in the middle? I think we're making a pretty good business here. We're making a lot of buildings.
15		**Researcher 1**	You're making a lot of buildings?
16		**Stephanie**	Yup. Got it. Dana I meant like this. Oh wait didn't you just make, see it doesn't match many of them. That doesn't match any of them.

17		**Dana**	I think that's the only one we're gunna get.
18		**Stephanie**	Hang on Dana we can always try more. We have to be almost positive. This is tricky here. What if I went like this? I think I may have this here. Make another one. I got it, why don't we raise the blue one? See if it works. It might work is we raise the blue just one. Okay go back to the beginning and check it again. Wait, we have to raise the blue another one, now at the way top. Again, stumped! Dana I think we have this one, yup we do. Aw nuts we can't make anything. I'm almost positive.
19		**Dana**	Here. You have it, I see it, I see it already.
20		**Stephanie**	Where? You're right we do have it.
21		**Dana**	Alright
22		**Stephanie**	I think we're only going to be able to make seventeen.
23		**Researcher 1**	Seventeen?
24		**Stephanie**	Yeah I think we're only going to be able to make…
25		**Researcher 1**	Lets see, count them up again.

26		**Stephanie**	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen.
27		**Researcher 1**	Okay
28		**Stephanie**	So should I write down seventeen?
29		**Researcher 1**	First off, is every one of these different?
30		**Dana/ Stephanie**	Yes
31		**Researcher 1**	Are you certain?
32		**Dana**	Yeah we build them, and then check them like this…
33		**Stephanie**	Cause Dana built them, and I checked them
34		**Researcher 1**	How can you be sure that you haven't made any of them twice, or that you have got them all? Is there a way you could be sure?
35		**Stephanie**	Well there is a way. We can take one. Like say we could take this one, this red with the blue on bottom. And we could go and we could compare it to every one. And the ones that don't match push back.
36		**Dana**	And then we eliminate.

37		**Researcher 1**	Oh I see.
38		**Stephanie**	And that is a way to figure out so.
39		**Researcher 1**	Could you double check for me to make sure?
40		**Dana**	(writing) so we got seventeen.
41		**Stephanie**	(checking) so we have to push these all back. Ah my towers are falling.
42		**Dana**	We have seventeen
43		**Stephanie**	Seventeen I double-checked every one. I double-checked every single one. I'm gunna write a picture for mine. (writing) We got seventeen making patterns.

Clip 3

Line	Time	Speaker	Transcript
1		**R2**	How many do you have?
2		**Stephanie**	Seventeen.
3		**R2**	And you see any of them the same as each other?
4		**Stephanie**	Nope, we double-checked every one by going like this. (moving one tower throughout entire set of them) And the ones that matched we pushed back, and then we pushed them back forward. Dana…

5		**Dana**	What?
6		**Stephanie**	I only checked one…
7		**Dana**	We have sixteen?
8		**Stephanie**	Sixteen. Let me check another one.
9		**R2**	How did you do them?
10		**Stephanie**	What?
11		**R2**	How did you do them? Did you just do them?
12		**Stephanie**	Well we checked by going across and across and across.
13		**R2**	Oh I understand how you checked, but how did you build them?
14		**Stephanie**	Well Dana got the idea because we had to build them all straight so she got the idea of taking them and making patterns.
15		**R2**	Oh, show the patterns you were making.
16		**Dana**	Like this one, it's the pattern with the different colors.
17		**Stephanie**	Or it could be all different
18		**R2**	So pretty, do you think they're anymore?
19		**Stephanie**	I don't know if there are anymore that match
20		**Dana**	No because we used every single block and we had a lot of them too. And the ones that we had double we would take one and if we had double we would take away and eliminate it.
21		**R2**	You eliminated it. So how could you be sure you're done?
22		**Dana**	Because we did every block that we had and these are the ones we are left with.
23		**Stephanie**	You know what happened when we had to start eliminating some, we were running out of blocks we didn't have enough.
24		**R2**	Yeah but you have some left
25		**Stephanie**	Yeah, because these are the ones we eliminated. Amy came over and Amy said why don't you check these you might

			have more. So we went and we checked these and there were a lot that we had to eliminate.
26		**R2**	Yeah, you think you've eliminated them all? How can you be sure?
27		**Stephanie**	Well I think I'm gunna check them. I'm just gunna make a pattern.
28		**R2**	Yeah I'm sure a pattern is really important.
29		**Stephanie**	Oh I know how we could check! This is how we do it. I take this one and I check then I take it and put it at the end of the line. Check check put it at the end. Check.
30		**R2**	But your line is never gunna end.
31		**Stephanie**	Maybe if we put them back
32		**R2**	Otherwise you'd just be checking and checking and checking and checking.
33		**Stephanie**	Well I know what we could do like this. We take the first one and we check and we put it back in its spot. Until we get down to the blue red red blue. We could do that. Dana wanna do that?
34		**Dana**	But do we have a red blue blue red?
35		**Stephanie**	A red blue blue red. I think we do some place.
36		**Dana**	It's right here.
37		**Stephanie**	Yeah red blue blue red. Okay I'm gunna check. Now these one see- multi colored. Cousins.
38		**R2**	You think they're cousins, why are the cousins?
39		**Stephanie**	Well this one has blue on the bottom and this one has blue on the top, turn one around and they're the same.
40		**R2**	I thought that you might say that this one and this one might be cousins.
41		**Stephanie**	Oh yeah.
42		**R2**	Why?
43		**Stephanie**	Because they have sort of the same pattern. Red one at the top and blue one on the bottom. And blue one at the top and red one at the bottom. (Back to checking) This one can go back to the beginning of the line. Okay this one can go back to the beginning of the line there is no trouble.
44		**Dana**	We're done Steph okay?

45		**Stephanie**	I'm just checking Dana.
46		**R2**	Don't mess up your towers because you're going to save them to share. But can you each fill out and see if you understand and agree with these questions.
47		**Stephanie**	(checking) hey, I just figured out something too. Okay they're all nice. Now I have to fill out these forms. How many towers did your group find? Sixteen. We made patterns. Dana why don't we see if we got the same answers now. Are you done? I'm done.
48		**Dana**	Yeah the sentences may be a little different how we explained them.
49		**Stephanie**	For the first one, sixteen?
50		**Dana**	Yeah
51		**Stephanie**	Second one, yeah?
52		**Dana**	Yeah.
53		**Stephanie**	Okay now you do the third one and read your answer and I'll do the third one and read my answer.
54		**Dana**	Third, we used all of our block and we eliminated the ones that matched.
55		**Stephanie**	I put because we kept checking the towers.
56		**Dana**	Okay, that's enough. We used all of our blocks and we had matches and the ones that matched we eliminated.
57		**Stephanie**	I put, we made patterns and eliminated the ones that matched.
58		**Dana**	Well we said the same.
59		**Stephanie**	Lets give ourselves an "A"
60		**Dana**	No they want it!
61		**Stephanie**	I'm just giving myself an "A+"

Clip 4

Line	Time	Speaker	Transcript

1		**Dana**	What do you have Michael? Michael what do you have?
2		**Michael**	Sixteen
3		**Dana**	So do we.
4		**Other**	We have seventeen
5		**Stephanie**	Then you must have something that matches cause we got sixteen. Double-check your young man.
6		**Other**	We already did.
7		**Researcher 1**	Stephanie what makes you so sure that you got everything?
8		**Stephanie**	I don't know.
9		**Dana**	Well we just checked it. Cause we used all of our blocks and then we had matches and the ones that matched we took one of them that matched and we eliminated them.
10		**Researcher 1**	Did you miss one?
11		**Dana**	No.
12		**Researcher 1**	How come?
13		**Dana**	Cause we double-checked about four times.
14		**Stephanie**	Okay Dana I'm gunna try to make one more.
15		**Dana**	Fine. See if we match.
16		**Stephanie**	So we are, we're very smart too. Maybe a blue, a red, a blue. Wait how about a blue…
17		**Dana**	Blue, red, blue, red.
18		**Stephanie**	No. Blue, blue, red, red. That would be good.
19		**Dana**	Blue, blue, red, red. Right here. Red, red, blue, blue right there.
20		**Stephanie**	I don't think we can make another one. I really and truly don't.
21		**Other**	We have sixteen.
22		**Dana**	We have sixteen too.

23		**Stephanie**	I told you guys.
24		**Dana**	Alright straighten those up more; I want ours to be the best!
25		**Stephanie**	Doesn't it feel like just a big patchwork? With all different patterns on it.
26		**Researcher 1**	Okay girls, so you got sixteen?
27		**Dana/ Stephanie**	Yeah
28		**Researcher 1**	Okay so lets leave these here cause they're really nice and we want to share these during group sharing.
29		**Stephanie**	Okay.
30		**Researcher 1**	What I want you to do now is, I'm giving you a recording sheet. Put your name on, you've got pens. You're going to make these towers for me now that you've made record them on here.
31		**Stephanie**	Oh so you mean I take this one…and put blue red red blue
32		**Researcher 1**	Color it in. Right. Now listen very carefully. I want you to do it in such a way so that when we share it shows how you knew you had all of them. Okay? Organize them in a certain way, I want to see the way you knew that you had them all. Oaky? And then when you're all done with that girls when
			you're all done you can share doing this group one okay you fill them in the same way.
33		**Stephanie**	What do you mean in a way everyone knows that?
34		**Researcher 1**	Well what convinced you that you had them all?
35		**Stephanie**	We double-checked.
36		**Researcher 1**	You double-checked okay. And so you had them lined up like this? Okay well then record them in that way.
37		**Stephanie**	So I record…
38		**Researcher 1**	Yeah the way that you have them there just keep going. Okay? There may be extra spaces here but don't worry about that.
39		**Stephanie**	Red, red, red, blue. I'm on red, blue, red, blue.
40		**Dana**	Red, blue, red, blue?
41		**Stephanie**	Red, blue, red, blue.
42		**Dana**	I'm only on the second one from over here.

43		**Stephanie**	Dana all you have to do, it doesn't have to be perfectly colored. Make like this Dana, like this.
44		**Dana**	I'm on blue, blue, red, blue
45		**Stephanie**	Well I'm going this way. I'm almost done Dana, I just need the last one. Blue, blank, blue, blue. I'm on my last one then I'm done with it. I'm gunna call Amy over and tell her that I'm done.

Clip 5

Line	Time	Speaker	Transcript
1		**Stephanie**	Well I finished mine, but Dana's still doing hers. I just scribbled in it.
2		**Researcher 1**	That's fine. Do you want to work on the group sheet?
3		**Stephanie**	Okay, same thing as the recording sheet?
4		**Researcher 1**	That's the same thing
5		**Stephanie**	Okay. (saying different tower patterns aloud) I'm going to do all read first then come to the blue. Red, red, red. Dana I'm doing an easier way on the big sheet see what I'm doing? First I'm just taking care of red, then I move on to the blue that I missed. Okay then red, red. There's red at the top and red at the bottom. Oh well it can be fixed; I'll just go over it with a little bit of blue. Now you see how fast that was Dana, how faster it is. Red, red at the bottom. Then it goes two reds at the top.
6		**Dana**	Oh no Steph!
7		**Stephanie**	Then it goes all red. Then I skip the next one cause the next ones all blue. And I go down to here, which would be red, red, red. Then comes red, blank, red.
8		**Dana**	I'm done, I'll do the blue with you.

9		**Stephanie**	Wait hold on, red in the middle. Every blank spot that's not colored in is blue. Even this one. There's one of them that I put all blank. Wait where was it. Now I don't know where I was. Wait red at the top, then one in the second one. Red in the middle. I'm gunna put stop where you stop. See that's where you stop. At this one. There should be one that's not colored in at all. There should be one that's all red. This one shouldn't have been colored in. Nuts! Nuts, nuts, nuts. Hang
			on, all red and then that should have been all blue. I'm gunna cross this out, and I'm gunna put this one right here. Now this ones all blue, okay Dana? Okay now you do the blue cause I did all the red okay?
10		**Researcher 1**	Okay everyone, we only have a few minutes left I'd like you to finish your recording sheet coloring those in and don't take your towers apart leave them because we're going to talk about them tomorrow. So leave them in the nice neat rows that you have them in okay?
11		**Dana**	We're going to be like this all day.
12		**Stephanie**	Okay I'll get a blue pen and I'll help you out.

13		**Researcher 1**	Okay if you have your towers in a special order leave them that way okay? So we'll organize them in the way that you want to save them
14		**Dana/ Stephanie**	(Finish coloring up to the sixteenth tower)

Appendix E: Session III: Transcript-Grade 3 Towers Additional Problem

Stephanie engages in the Towers problem in third grade at Harding Elementary School under Researcher 1.

Description: Towers Group Sharing Clip 2, 3, 5, 6

Guessing how many towers can be built three cubes high

Content: Harding Elementary School Researcher: Researcher 1

Tape: Towers Group Sharing

Date: 10/11/90

Authors: Madeline Yedmen

Verified: Robert Sigley Date: 12/07/13

Page: 156 of 5

Clip 2

Line	Time	Speaker	Transcript
1	00:00	**T**	Do you remember what we talked about?
2	00:09	**Dana**	If they would tell them how much people can fit in like three of these.
3	00:25	**T**	Okay, if we were going to make towers that were just three, but you still had two colors. But there were just three, what was (inaudible)?
4	00:36	**Dana**	It was like how many people can fit inside.
5	00:42	**T**	(Inaudible: Student s is yelling out “I know.”)

6	00:48	**Student 1**	I know! The question was if you have________, if there would be more, same, or less.
7	00:59	**T**	Okay, the question was, if there is a tower, instead of four blocks there are three blocks in it, (mumbles) how many towers are there?
8	1:18	**Student 2**	There would be sixteen.
9	1:20	**T**	There would be sixteen if there were four. Every single person seemed to think that there were sixteen, is that what you got? Okay. And so, the question was, supposed that there are only three blocks in each tower. Would there be more than sixteen, or would there be fewer than sixteen, or would there be sixteen? And so we had Jamie and Michael and they first said that there would be fewer than sixteen and they changed their minds and that there would be the same. Then, we have Mike and Paul and they said there would be more, probably twenty. And then we had Michael and Giardo(?) ad they thought there would be the same. And we had Brian and Jeff and they thought it would be the same.

			Who haven't I heard from? Dana and Stephanie?
10	2:14	**Stephanie**	We think it's the same.
11	2:18	**T**	Why?
12	2:12	**Stephanie**	Well, because you are just taking one away from here it's not like it's going to change the whole thing. It's gonna be one less.
13	2:33	**T**	Okay, What I hear Stephanie and Dana are saying is, if you can take one away it doesn't change. That's what they are saying. So that's your argument for there being sixteen. Okay, what about the (inaudible)?
14	2:56	**Student 3**	Five.
15	3:00	**T**	And you got one? What do you think?
16	3:03	**Student 4**	Same.
17	3:05	**T**	So you agree that (inaudible). So, what about you?
18	3:12	**Student 5**	We think it's more.
19	3:14	**Student 6**	We think it's the same.
20	3:14	**T**	(inaudible)?
21	3:18	**Student 6**	I know but I think it's two hundred.
22	3:22	**T**	Oh, so it's a lot more.
23	3:23	**Student 5**	I think it's the same.
24	3:27	**T**	Okay. What about you two, Steven and what's your name? Michelle? What do you think?
25	3:47	**Michelle**	Less.

26	3:49	**T**	You think it's going to be less? If you had to guess, what would you guess? How many do you think you'll have?
27	3:58	**Student 7**	Fourteen.
28	3:59	**Steven**	Oh, fourteen. Is that what you think? What do you think?
29	4:08	**Michelle**	About twenty.
30	4:11	**T**	Okay, now, who did I not get? I didn't get this group over here all together. Hello, what's your name.
31	4:16	**Michelle**	Michelle.
32	4:18	**T**	And your name is?
33	4:19	**Erin**	Erin.
34	4:20	**T**	Michelle and Erin. Michelle was working with other Michelle when Erin was gone. What do you think Michelle?
35	4:26	**Michelle**	More.
36	4:28	**T**	Okay. Michelle and Erin, do you agree?
37	4:35	**Erin**	Yeah.
38	4:37	**T**	And you decided there would be more. Why? What I heard from Michael and Geran(?), Michael said earlier, no, it was you. Tell me your name again.
39	4:58	**Matthew**	Mathew.
40	5:02	**T**	Mathew.
41	5:09	**Matthew?**	We said not the same.
42	5:10	**T**	Did you change your mind from the same?
43	5:14	**Matthew**	Yeah.
44	5:15	**T**	And down here, Michelle, Michelle and Erin think there is going to be more. (inaudible).

45	5:21	**Michelle**	(Voice is very soft so it's inaudible).
46	5:24	**T**	You said that there is about eight? (Inaudible). How do we figure that out?
47	5:43	**Matthew**	…each pattern.
48	5:45	**T**	And then?
49	5:46	**Matthew**	And then count up how many you have.
50	5:49	**T**	Do you think we have time?
51	5:50	**T2**	We should be.
52	5:54	**T**	Everybody (inaudible) their partners and see if you (inaudible because people start talking). Remember each one has to be different.

Clip 3

Line	Time	Speaker	Transcript
1	00:00	**T**	Do you remember what we talked about?
2	00:09	**Dana**	If they would tell them how much people can fit in like three of these.
3	00:25	**T**	Okay, if we were going to make towers that were just three, but you still had two colors. But there were just three, what was (inaudible)?
4	00:36	**Dana**	It was like how many people can fit inside.
5	00:42	**T**	(Inaudible: Student s is yelling out "I know.")
6	00:48	**Student 1**	I know! The question was if you have________, if there would be more, same, or less.
7	00:59	**T**	Okay, the question was, if there is a tower, instead of four blocks there are three blocks in it, (mumbles) how many towers are there?
8	1:18	**Student 2**	There would be sixteen.

9	1:20	**T**	There would be sixteen if there were four. Every single person seemed to think that there were sixteen, is that what you got? Okay. And so, the question was, supposed that there are only three blocks in each tower. Would there be more than sixteen, or would there be fewer than sixteen, or would there be sixteen? And so we had Jamie and Michael and they first said that there would be fewer than sixteen and they changed their minds and that there would be the same. Then, we have Mike and Paul and they said there would be more, probably twenty. And then we had Michael and Giardo(?) ad they thought there would be the same. And we had Brian and Jeff and they thought it would be the same.

Clip 5

Line	Time	Speaker	Transcript
1	00:00	**T**	Okay, I'm going to ask all of you to think for a minute. I want you to think really hard and see if you can imagine. Suppose instead of towers that had four cubes, you could only have three cubes in each tower. Do you think there would be more towers or do you think there would be fewer towers? What do you think? This means, if you have only three box in each tower, you think there would be more towers than sixteen or do you think there would be fewer towers than sixteen? What do you think?
2	00:49	**Student (boy)**	There would be more towers.
3	00:50	**T**	You think there would be more towers than sixteen with just three cubes? How many do you think there would be?
4	00:59	**Student (boy)**	Nineteen.
5	00:59	**T**	Nineteen? Why do you think there would be more?
6	1:06	**Student (boy)**	(inaudible)
7	1:09	**T**	That would be more? What do you think Brian?

8	1:10	**Brian**	Because there are fewer numbers of blocks.
9	1:16	**T**	So you think there would be more than sixteen or fewer than sixteen?
10	1:21	**Brian**	More.
11	1:22	**T**	You think there would be more, also. What do you think?
12	1:26	**Student 2 (girl)**	I think it would be less towers.

Clip 5

Line	Time	Speaker	Transcript
1	00:00	**T**	What were we trying to figure out? After you figured out?
2	0:08	**Matthew**	Take one block and there would be two. One take one block away from each pattern.
3	00:14	**T**	And then?
4	00:15	**Matthew**	And then count up how many you have.
5	00:19	**T**	Do you think we have enough time?
6	00:20	**T2**	We should be.
7	00:21	**T**	Say everybody, with your partners, see if you can figure out…(everyone talking at the same time.) Remember that each one has to be different. And all the others are different.
8	1:00	**Stephanie**	One, two, three, four, five, six, seven, eight, nine, ten, eleven. Okay, twelve, thirteen, fourteen, fifteen, and sixteen. And then we can probably go one, two, so we can have one that looks like this. Red-red-blue and red-blue-blue. Yup we do. And, let's see. If we had something like blue-red-blue…
9	2:05	**Dana**	How about, try red-blue-red? I mean, red-blue-blue?
10	2:08	**Stephanie**	I doubt it. Let's try these ones, okay? How about red-blue-red?

11	2:17	**Dana**	We have one.
12	2:24	**Stephanie**	There are two red-blue-blue. Oh, so that would be less than sixteen. So we take this one away and throw it in the trash.
13	2:43	**Dana**	Amy, we think there is less.
14	2:44	**Amy**	Oh, why?

Clip 6

Line	Time	Speaker	Transcript
1	00:00	**T**	Okay, can everybody look up
2	00:04	**Jeff**	Six, seven, eight, we got eight.
3	00:06	**T**	Okay, now, before I say anything, does anybody want to change their minds?
4	00:15	**Students**	Yes!
5	00:21	**T**	Does anybody want to change their minds from what they've said here? Everybody wants to change their minds? Well, okay. Now, this was Dana and Stephanie (inaudible) over here. You said a while ago there was the same: that would have been sixteen. And then now you want to change your mind? What did you get now?
6	00:42	**Stephanie**	It's less, there is only eight.
7	00:45	**T**	How come?
8	00:46	**Stephanie**	Well, because once you take these apart, you start to see that …
9	1:00	**Dana**	The match.
10	1:03	**Stephanie**	Because one took it off and made a whole difference.
11	1:06	**T**	All of them, taking one off? Taking one off changed the answer to eight. How many of them were a match?

Authors: Madeline Yedmen
Verified: Robert Sigley

12	1:17	**Dana**	Eight.
13	1:18	**T**	Eight of them, then. Gosh, so you changed your mind and you said eight. Okay, what about Michelle, Michelle, and Erin?

Appendix F: Session IV: Transcript-Grade 3 Towers Interview (4-tall, 3-tall)

Stephanie engages in the Towers problem in third grade at Harding Elementary School under Researcher 1.

Description: Stephanie Grade 3 Towers interview excerpts **Location: Harding Elementary School** **Researcher: Amy Martino**	

Line	Time	Speaker	Transcript
1		**Researcher 1**	First of all, what do you think you learned from what you did?
2		**Stephanie**	Well, we learned that, well with the Unifix cubes we learned that even though there might be less, there might be, um, less, you might think there would be more because there's less blocks and there's more combinations you can make> There's less because once you take one block off. Say you have red, red, red, red and you have red, red, red, blue once you take one red away and one blue away they are the same.
3		**Researcher 1**	Ohh, you're right. Okay, alright. So you won't have more you would have…
4		**Stephanie**	Less
5		**Researcher 1**	How are you sure that you had them all because you two seemed very definite that 16 was all and some people were saying 17 and 18 but you seemed to be sure it was 16.
6		**Stephanie**	Well, we had to check it a couple times and we tried to make some different ones but we were checking and checking and they all came out the same.
7		**Researcher 1**	When you made the three cube towers were there more of less than

8		**Stephanie**	Less
9		**Researcher 1**	There was less, do you remember how many you got?
10		**Stephanie**	We got eight
11		**Researcher 1**	Eight okay. And how did you do that? Like how, explain to me what you were doing there. You were pulling blocks.

12		**Stephanie**	Well, we pulled the blocks off and then we matched them up. So it was like a matching game.
13		**Researcher 1**	So it was like a matching game and then what happened? What did you notice happened after you pulled one block off?
14		**Stephanie**	One block off could mean a whole big difference. Say again, you have blue, red, blue, blue and you have blue, blue, red, red. Wait yea, no hold on yea. What did I just say before?
15		**Researcher 1**	I forgot, start again.
16		**Stephanie**	Say you have blue, red, blue, blue and you have blue, red, blue, red. If you take off that red, if you take off that other blue you have blue, red, blue. Blue, red, blue.
17		**Stephanie**	You always have to think there’s more, because you can’t go, you never know if there’s gonna be, you can’t say I found two that’s enough because you always have to think there’s more. Because you never know if it’s enough or not – you know what I mean. Until you find out the answer

Appendix G: Session V: Transcript-Grade 4 Five-Tall Towers Problem Classroom Session

Stephanie engages in the Five-tall Towers problem in fourth grade at Harding Elementary School under Researcher 1.

Description: PUP Math - Towers **Location: Harding School – Kenilworth, NJ** **Researcher: Researcher 1**	**Transcriber(s): Private Universe Project** **Verifier(s): Sigley, Robert, Sran, Kiranjeet** **Date Transcribed: Spring 2000** **Page: 1 of 11**

Line	**Time**	**Speaker**	**Transcript**
54.		**Narrator**	16 months later, in the fourth grade, the Kenilworth students investigated towers five cubes tall.
55.		**Researcher 1**	...and you have to be able to convince us that you have found all possibilities - that there are no more or no less. Got the problem? Have fun!
56.		**Stephanie**	Okay, we'll start out with the easiest one. One, two, three, four, five reds and five yellows.
57.		**Dana**	One, two, three, four, five.
58.		**Stephanie**	I only have four. Okay, well, stand them up straight so we know what we have.

59.		**Researcher 1**	In towers five tall, to make a convincing argument that you found them all is harder except for when you have all of a color or one of a color.
60.		**Shelly**	Now we take one of these, one of these.
61.		**Narrator**	Building towers five tall offered a richer, more
			complex challenge for the students to investigate. Students spontaneously invented strategies, such as making a tower and then building its "opposite."
62.		**Brian**	...this one matches with this.
63.		**Romina**	Put the pairs.
64.		**Brian**	Like the opposites.
65.		**Dana**	And then I got another idea.
66.		**Stephanie**	Well, tell me it so I can do the opposite.
67.		**Dana**	I'm going to do the red - this, that-
68.		**Stephanie**	Show me. Oh, okay, and I'll do the red - and I'll do it with the red at the top.
69.		**Researcher 1**	They were holding one variable fixed, constant, and then varying the other. It was exciting that these children at a very early age were showing evidence of controlling for variables. It's lovely. And they were being exhaustive.
70.		**Brian**	I have to do the opposite. I'll do this-
71.		**Stephanie**	We made a pair!
72.		**Dana**	No, look. Look, that's fine. That goes with this one.
73.		**Stephanie**	No it doesn't because if you turn it around, it's the same, so that doesn't go with that one.
74.		**Dana**	That one goes with that one.
75.		**Stephanie**	Wait, let me check. Let me make sure...No that doesn't because...

76.		**Romina**	I think we have them all.
77.		**Researcher 1**	Do you think it's possible to have an odd number?
78.		**Student**	No.
79.		**Researcher 1**	They have an odd number - 35.
80.		**Mike**	You're not supp- You can't because when you have a number, you could have the opposite. And if you would have one of this, you have
			another one because it's the opposite. And if you have 10 of these, you have another one that's opposite, so that makes 20.
81.		**Shelly**	We found 32.
82.		**Researcher 1**	You found 32? How did you do that?
83.		**Jeff**	Easy. You just go this way and then-
84.		**Researcher 1**	You're tired, Jeff. Jeff, how do I know that you don't have duplicates?
85.		**Jeff**	You can check: all you want.
86.		**Researcher 1**	Because you checked it. How? ... That's how you checked it, you compared? How do you know you there're not 34?
87.		**Jeff**	I can't make any more. My brain is tired!
88.		**Researcher 1**	Your brain is tired?
89.		**Researcher 1**	So you might ask us - Why did we ask them to convince us? Why do we ask them to justify? Well, we do that because beginning when they start, they solve their problem randomly. It's sort of guess and they try something. When you don't know what to do, you try something, so you'll build something. And maybe you'll notice certain kinds of patterns in your building; maybe you won't. You might just do trial and error, trial and error, trial and error. We want students to get past trial and error.
90.		**Researcher 1**	Okay, let's take another set and try to convince me the same way.
91.		**Jeff**	We'll show you the other set.
92.		**Researcher 1**	Okay. I believe this one, too - you can have one red, right? And you have the other possibilities. I buy that.

93.		**Jeff**	There's only two kinds of these because there are alternates.
94.		**Researcher 1**	Okay, I buy that. All right. You're convincing me. That's great.
95.		**Jeff**	This, we just ... How are we going to convince her about this one?
96.		**Researcher 1**	You've got to convince me about this one. Why don't you think about this? I'm convinced about these that there are no other possibilities when you have one of a color - either one yellow or one red. Okay? I'm not so sure I'm convinced if there's two reds or if there's two yellows, so why don't you work on convincing me of that? You think about it and I'll be back, and you can call me.
97.		**Researcher 1**	But they're thinking was still very, very exhaustive and it was very organized - when they had to justify their solutions. What it does, then, is it enables them to look at what they have, that they did just by hard work and drive, which we skip in school; we skip that piece of it. How awful - because we don't have time. You know, we skip that drudgery of that going through this hard, hard work we might not see the point of. We don't look enough. Because as they're going through this real intense, hard work, they're noticing things about the structure of the problem - maybe not seeing it overall, but they begin to notice relationships, they begin to notice sub- patterns, and they invent names for these. They really get to know the task well. This is what we expect mathematicians to do in their work.

Appendix H: Session VI: Transcript-Stephanie Revisits Five-Tall Towers Problem

Interview

Stephanie engages in the Five-tall Towers problem in an interview with Researcher 1.

Line #	**Speaker**	**Time Stamp**	**Utterance**
1.	**1**		(Inaudible)
2.	**Stephanie**	00:00:12	Did you do (pause) do another math problem?
3.	**1**	00:00:14	Oh, that's an idea but we didn't. It's a good idea. Now we were interested in telling us about how you and Dana did yesterday. How did that work you did with Dana?
4.	**Stephanie**	00:00:21	Pretty good. We were (pause) We worked pretty good.
5.	**1**	00:00:26	So how did you work together?
6.	**Stephanie**	00:00:29	Well, what we did was we would, we would take the block and we would build a match but I would use the yellow block. So, If I built the match yellow, red, yellow, red, yellow. Dana would build the match red, yellow, red, yellow, red. And then we'd put them together. And then we'd make another match.
7.	**1**	00:00:48	What did you decide in terms of the numbers of towers that can be built?
8.	**Stephanie**	00:00:54	We made thirty-two towers
9.	**1**	00:00:56	Do you believe that's the answer?

10.	**Stephanie**	00:00:59	Mmmh, well, it wasn't (pause) I guess
11.	**1**	00:01:08	Suppose that there was something at stake here that someone came in, a fifth grader, and said I don't think there were thirty-two.Now you built thirty-two. You were convinced they were all different. So if they said to you, "I think there were thirty." What would you say to that person?
12.	**Stephanie**	00:01:20	Well, I would say like what we did yesterday, when we were up at the board with the one block yellow, and then the two [blue blocks.-]
13.	**1**	00:01:27	[Show me then.] That's it. Why do we use blue and yellow? Is that alright? [Blue] and yellow instead of red?
14.	**Stephanie**	00:01:31	[Okay]
15.	**1**	00:01:32	Yeah, try to convince this fifth grader-
16.	**Stephanie**	00:01:37	Like this, we went like this ((Stephanie builds tower yellow, yellow, yelllow, yellow, blue)). And then we went-
17.	**1**	00:01:46	And you could even, you know, write it out, or tell me about it-, or you can build it. Whatever you would like.
18.	**Stephanie**	00:01:52	-Yea. ((Stephanie builds tower yellow, yellow, yellow, blue, blue)) And like that. ((Stephanie builds tower yellow, yellow, blue, blue, blue)) And like that. ((Stephanie builds tower yellow, blue, blue, blue, blue)) And like that. ((Stephanie builds tower blue, blue, blue, blue, blue)) And then like that. And then there's five right there and then you build it backwards so that it's-
19.	**1**	00:02:25	Okay, well you don't have to do that if you don't want to, you can say (pause) Why don't you describe...describe this set. How would you describe this set?

20.	**Stephanie**	00:02:37	Well, I would describe it like as a ((counting)) one, two, three, four, five patterns.
21.	**Researcher 1**		Okay.
	Stephanie		Like you go.
	Researcher 1		That's interesting.
22.	**Stephanie**		Alright, should I draw the patterns on the paper?
	Researcher 1		You can, sure.
23.	**Stephanie**	00:02:57	And this one would be blue ((pointing at the one shaded box in drawing)) and then you have the next one. But then these two would be blue.((pointing at the two shaded boxes in drawing)) The next one. And then the last one would be all blue.
24.	**Researcher 1**	00:03:36	Okay, (inaudible). How am I going to remember that that's yellow and that's blue?
25.	**Stephanie**	00:03:38	All right, well.
26.	**Researcher 1**		Oh, okay. Okay, here.
27.	**Stephanie**		Yeah
28.	**Researcher 1**	00:04:02	Okay now, mmh, suppose (pause). Why don't you write how many we have here so that we can keep a record as we are explaining (to make sense[of what we are doing?])
	Stephanie	00:04:11	[Ok, well we have five.]
	Researcher 1		[So you have five], and why don't you give a name to this pattern, what about (pause) how would you? What name would you give it? If you [had to call it]

29.	**Stephanie**	00:04:18	[One to five.]
30.	**Researcher 1**		Ok, One to five pattern. One to five blue pattern?
	Stephanie		Yes.
	Researcher 1		One to five blue.
31.	**Stephanie**	00:04:27	And then, we can do the same thing?
32.	**Researcher 1**	00:04:27	So, write it down,you have to describe (just write?)- and so you have five one-to-five blues and then you're going to have ((pause)) one-to-five one-to-five yes.So, so far you have ten,why don't you write the number ten there. That's good (pause) mmmmh, okay, so what else did you do?
33.	**Stephanie**	00:04:50	And then we had the pattern (pause:3s) actually it was-
34.	**Researcher 1**	00:04:52	Alright, why are you taking this from here?
35.	**Stephanie**	00:04:53	Because (pause) well because this one we had the pattern the two and the two blocks up, and then the two blocks up.
	Researcher 1	00:05:04	Yes. Okay.
36.	**Stephanie**		So, I was going to use that and then- so yeah. ((creates towers using blocks))
	Researcher 1	00:05:24	You could also draw it too.
37.	**Stephanie**	00:05:56	Yeah, alright I'll draw it instead. (long pause:30s) ((Drawing)) Now, you could- (pause) this one was the two blues, up at a time.Actually you would have to go up one more. ((Draws a fifth row in drawing)). ((draws the letter B in column two row three and row four)) And then we had the two in the middle ,((draws the letter B

			in column three row two and row three)) and then we had the two here, ((draws the letter B in column four row one and row two))and the two here.
38.	**Researcher 1**	00:06:24	Okay, so that was the other pattern you had.
39.	**Stephanie**	00:06:27	mmh, so that equals ((writes five on a paper)) five.
40.	**Researcher 1**	00:06:31	How did you get five? Show me.
41.	**Stephanie**	00:06:33	Well, oh no, that's not five. (I am saying, I forgot something?) that's four, so- then we have a four pattern.
42.	**Researcher 1**	00:06:45	How would you describe this four? Four of- How would you describe this pattern? What did you call these two blue?
43.	**Stephanie**	00:06:51	Mmmh, Two blues?
44.	**Researcher 1**	00:06:52	Two blue.
45.	**Stephanie**	00:06:55	Mmmh, I don't know (pause)
46.	**Researcher 1**	00:06:56	Give it a name maybe, think about what name you might [use?]
47.	**Stephanie**	00:06:58	[Two] at a time
48.	**Researcher 1**	00:06:59	Two what at a time?
49.	**Stephanie**	00:07:01	Two blues at a time
50.	**Researcher 1**	00:07:03	Okay so, all these have two blues?
51.	**Stephanie**	00:07:05	Yeah.

52.	**Researcher 1**	00:07:06	What do you mean by at a time?
53.	**Stephanie**	00:07:07	Well you have the two blue and the rest are yellow. And that is one tower.
54.	**Researcher 1**	00:07:10	Okay
55.	**Stephanie**	00:07:13	And then the two blue and there and that's another yellow. (pause) ...So you are not using more than two blues at a time
56.	**Researcher 1**	00:07:23	At a- I'm not too sure by what you mean by ''at a time'' I could [imagi-]
57.	**Stephanie**	00:07:26	[For one tower]
58.	**Researcher 1**	00:07:28	But I could imagine using two blues - (pause:10s) Something like this ((Create tower Y,B,Y,Y,B)), this is two blues.
59.	**Stephanie**	00:07:43	mmh, two blues together
60.	**Researcher 1**	00:07:45	Okay,so why don't you write that down? Two blues together, so we can know what kinds you mean exactly.
61.	**Stephanie**	00:07:56	Okay, together. And then we had- we reverse it.
62.	**Researcher 1**	00:08:00	Okay, so what would you call those? It wouldn't be two blues together.
63.	**Stephanie**	00:08:03	And then it would be two (.) yellow (pause) together.
64.	**Researcher 1**	00:08:12	How many [of those?]
65.	**Stephanie**	00:08:13	[So that's] eight.

66.	**Researcher 1**	00:08:14	I am confused though, how did you know that some of these ((pointing at "two blues together" and "two yellow together")) aren't these? ((pointing to "Blue1-5 Yellow"))
67.	**Stephanie**	00:08:20	Oh that's right, this one is this one. ((Point at yellow,yellow, yellow, blue, blue in top diagram and yellow,yellow, yellow, blue, blue in bottom diagram)) This one-(.) This one is not eight it's- (pause) This one isn't four, this is a three, and this is six.
68.	**Researcher 1**	00:08:36	Okay, I see, how did you deal with that yesterday? Did you end up counting things?
69.	**Stephanie**	00:08:41	We ended up counting a lot over. We had thirty-four and we had (inaudible) so we subtracted I think three groups, because we were down to twenty-eight then we added two groups
70.	**Researcher 1**	00:08:52	So (pause) so you think that's what was happening yesterday?
71.	**Stephanie**	00:08:56	Yeah
72.	**Researcher 1**	00:08:57	Okay
73.	**Stephanie**	00:09:00	We kept finding different patterns but we didn't check it with the other patterns.
74.	**Researcher 1**	00:09:02	Ahaa (pause) okay. Are you convinced now, however, that there are only exactly three that have (pause) mmmh two blue together that are different from what you have up here. You are absolutely convinced of that? If someone said that five hundred dollars prize depended on your getting it right or wrong? I mean are you- which way would you- which way would you bet?
75.	**Stephanie**	00:09:27	Am I convinced that there is only one group, with the two blues and the two orange, yeah!

78.	**Researcher 1**	00:09:34	Okay. Why are you convinced about that?
79.	**Stephanie**	00:09:35	Well, because we did these groups with the orange and the blues- the yellow and the blues. So, you know that this group is over, so you can't make another group like this.
80.	**Researcher 1**	00:09:44	Okay, okay
81.	**Stephanie**	00:09:47	So-
83.	**Researcher 1**	00:09:48	How do you know there aren't any more of these? Suppose I- Suppose this fifth grader said to you (pause) mmm okay, you found four and I agree with you, that one of them you found earlier but I think there is still another one. What would you say to them about that?
84.	**Stephanie**	00:10:05	You can only build it five high you'd have to have it so it would be seven high, not six high in order to build another one.
85.	**Researcher 1**	00:10:12	What about low though? You can put it on the bottom.
86.	**Stephanie**	00:10:15	Then you would be making it over. If you put it here (pause) ((Starts to redraw column three by writing B and B in the first two rows)) you would be making what is here. ((points to column three))
87.	**Researcher 1**	00:10:25	Okay. So you are convinced of these so far. Ok, so that was another pattern you made. Can you tell me- So, so far how many did you find?
88.	**Stephanie**	00:10:33	Mmm (pause) we found sixteen
89.	**Researcher 1**	0:10:36	How many more do you have to find?
90.	**Stephanie**	00:10:44	(pause: 7s) Oooh twelve, wait, yeah, (pause:3s) wait a second (pause: 7s) sixteen.

91.	**Researcher 1**	00:11:04	You have sixteen more to find. Okay let's find those other sixteen. Show me how you found those
92.	**Stephanie**	00:11:10	All right. Well, we just went and built patterns. Another pattern is this one. One blue, one orange, one blue, one orange, and one blue. And then you can make the opposite, (pause: 3s) which is orange, blue, orange, blue, orange. That's the opposite one.
93.	**Researcher 1**	00:11:50	So in this case, mmmh Let me see- (pause) you made this one first, and that's three blues, exactly three blues separated by a orange. Isn't there another way you can do that?
94.	**Stephanie**	00:12:06	Mmm (pause) Well three blues separated.
95.	**Researcher 1**	00:12:10	By an orange?
96.	**Stephanie**	00:12:13	Is there another way to separate three blues by an orange? (pause) No.
97.	**Researcher 1**	00:12:22	How would you convince this fifth grader?
98.	**Stephanie**	00:12:24	Because, mmm we have three blues and two oranges, but we have five blocks. (Pause) So, you can only- there is only- there is three blues. So you can't- you could put it blue, blue, orange, blue orange or blue, orange, blue, or blue, blue orange blue orange, or orange, blue, orange, blue, blue,or blue, blue, orange, blue, orange. But you can't put it so that these two as separate.
99.	**Researcher 1**	00:13:13	Why not?
100.	**Stephanie**	00:13:14	Because there is only five blocks
101.	**Researcher 1**	00:13:17	(inaudible) So, you found how many more?
102.	**Stephanie**	00:13:20	So there is two more.

103.	**Researcher 1**	00:13:21	Write up here, how would you describe that particular block?
104.	**Stephanie**	00:13:26	Oooh (pause) blue- mmmh which-a-ma-call-it mmmh blue orange
105.	**Researcher 1**	00:13:35	Okay
106.	**Stephanie**	00:13:40	There we are, and blue yellow. That's how you do it
107.	**Researcher 1**	00:13:51	So, what are we up to now?
108.	**Stephanie**	00:13:54	Well now we have (pause) we had sixteen, right? Eighteen.
109.	**Researcher 1**	00:14:02	Okay
110.	**Stephanie**	00:14:03	And I saw (pause), I think another pattern we found was…
111.	**Researcher 1**	00:14:12	You can use that piece of paper. Why don't you put number two on here. Number this paper number two.
112.	**Stephanie**	00:14:13	Okay
113.	**Researcher 1**	00:14:20	Are you sure you didn't find that anyplace else?
114.	**Stephanie**	00:14:24	Mmmh [Another one-]
115.	**Researcher 1**	00:14:25	[What convinced] you that you couldn't (make any more?)
116.	**Stephanie**	00:14:28	Well, over here we did the patterns, ((points to column of four yellows)) yellow, yellow, yellow, yellow,(pause) ((points to column of three yellows)) yellow, yellow yellow (pause) ((points to column of two yellows)) yellow yellow(pause) ((points to column with one yellow)) yellow, but so that had nothing to do with- with the mmh- blue yellow, blue yellow and down here we did blue, blue, yellow, yellow, yellow

			and yellow, blue, blue it had nothing to do separating blues and yellows
117.	**Researcher 1**	00:14:50	Oh ok, ok, gotcha, alright. So, this is the number three.
118.	**Stephanie**	00:14:56	And we did one like this, we did one blue, blue, yellow, blue, blue
119.	**Researcher 1**	00:15:17	Oh, okay, so tell me about this one
120.	**Stephanie**	00:15:22	Well, it had the color in the middle, one color in the middle
121.	**Researcher 1**	00:15:25	One color in the middle, that's no place else
122.	**Stephanie**	00:15:28	Mmh, and the other way too
123.	**Researcher 1**	00:15:41	So you want to write how many of those you have?
124.	**Stephanie**	00:15:43	Two more- so that's- Now we have mmmh-
125.	**Researcher 1**	00:15:47	Let's give this a name
126.	**Stephanie**	00:15:50	Okay, one in the middle
127.	**Researcher 1**	00:15:51	One in the middle, is fine. Good name. Okay, so where are we?
128.	**Stephanie**	00:16:01	Twenty.
	Researcher 1		Twenty one (pause) Do you remember?
			So we need ten more.

129.	**Researcher 1**	00:16:02	A lot of remembering?
130.	**Stephanie**	00:16:07	Mmmh, yeah, I think we did one like this, yeah we did. We went ((creates diagram Blue, Yellow, Yellow, Yellow, Blue)) with the two there.
	Researcher 1		What's- Tell me-
	Stephanie		On the top and the bottom.
131.	**Researcher 1**	00:16:38	Oh, now this is interesting, you have either a yellow in the middle or a blue in the middle and all the rest are the other color. And here you have three in the middle, you don't have that any place else and in the end you have the other color, ahaaa, interesting. I haven't seen anybody else do this.. I am glad I had a chance to talk with you, this is different. (Inaudible)
132.	**Stephanie**	00:17:11	mmm
133.	**Researcher 1**	00:17:12	Three in the middle, so how many of these did you find?
134.	**Stephanie**	00:17:14	Two
135.	**Researcher 1**	00:17:16	And are you convinced they are not any place else here?
136.	**Stephanie**	00:17:20	Mhmm.
137.	**Researcher 1**	00:17:21	Okay, let's go to the next page. Make that number four so that we don't lose track
138.	**Stephanie**	00:17:32	We did one, we did the one, oh one where we went like this (pause) oh, I think we did this, wait we did one in the middle, with the - Oh, just the one on top.
139.	**Researcher 1**	00:17:59	Yeah, I thought you did something with ones?
140.	**Stephanie**	00:18:00	Yeah, it was- ((draws Y,B,B,B,B)) and the opposite.

141.	**Researcher 1**	00:18:13	Okay, that was with one yellow.
142.	**Stephanie**	00:18:14	That's two more, that's twenty four. and we did… oh we did one, the same thing, we reversed it.
143.	**Researcher 1**	00:18:25	Oh
144.	**Stephanie**	00:18:43	We reversed it, it's twenty four
145.	**Researcher 1**	00:18:48	Okay that's interesting
146.	**Stephanie**	00:18:50	And we did (pause) I think we did one where we (pause) put, Oh, one where we went (pause) no that's the same thing we did before, we did the blue, yellow, blue yellow, blue yellow. There was one where we put one in the- Oh, one in the middle, like blue, yellow, blue, blue, blue, and then yellow, blue, yellow, yellow, yellow
147.	**Researcher 1**	00:19:46	Mmh
148.	**Stephanie**	00:19:47	That's twenty six, and then we reversed it so that - we reversed it and then - we went down to the second one . And same thing here. And that's two and that's twenty eight. And that's what we had originally come to, we wanted to stop and then we looked over and we (inaudible)
149.	**Researcher 1**	00:20:37	Okay, let's look at these for a minute then (pause) Are any of these alike in any way?
150.	**Stephanie**	00:20:44	Well they all have to do with- mmh one or two at some point
151.	**Researcher 1**	00:20:52	Is it one or two?
152.	**Stephanie**	00:20:53	Well, these two have the one at [one point]

153.	**Researcher 1**	00:20:56	[Let's] look at the first column. Ok. Oh, I see (pause) because you sometimes did it differently. Okay, I'm a little confused here.
154.	**Stephanie**	00:21:04	Oh right. Well,these two ((points to Y, B,B,B,B and Y,Y,Y,B,Y)) are somewhat alike because they both have one but at a different place. We just moved [it-]
155.	**Researcher 1**	00:21:11	[Okay.] They both have one but at a different place, okay
156.	**Stephanie**	00:21:14	This has the yellow in the first spot and this one has the yellow in the second spot.
157.	**Researcher 1**	00:21:17	Okay, so what about this one?
158.	**Stephanie**	00:21:20	This one (pause) has (pause) the yellow-
159.	**Researcher 1**	00:21:24	Okay, so let's write this down, yellow in the first spot, okay. (pause: 10s) ((Stephanie is writing out of view)) Okay, then you said what [about]
160.	**Stephanie**	00:21:37	[yellow] in the second spot.
161.	**Researcher 1**	00:21:38	Write that down. Yellow in the second spot. By second, you mean top floor or bottom floor of this building?
162.	**Stephanie**	00:21:49	Top floor
163.	**Researcher 1**	00:21:58	Top floor, okay. (pause:8s) the top floor is which floor?
164.	**Stephanie**	00:22:01	This one
165.	**Researcher 1**	00:22:03	Could you give a number name to that floor?
166.	**Stephanie**	00:22:10	Okay. (pause)
	Researcher 1		The top one would be which floor if it were a building?

	Stephanie		Oh, no the top one would be (pause) four.
167.	**Researcher 1**	00:22:13	The fourth one?
168.	**Stephanie**	00:22:15	mmmh
169.	**Researcher 1**	00:22:16	How tall is this building?
170.	**Stephanie**	00:22:17	Five
171.	**Researcher 1**	00:22:21	Okay. Alright. So you have yellow in the top. Right? And yellow in the next to the top. So you are calling second this one, right?
172.	**Stephanie**	00:22:30	Yeah
173.	**Researcher 1**	00:22:34	Okay, what are the other possibilities?
174.	**Stephanie**	00:22:37	The other possibilities could be yellow here, (pause) yellow in the fourth (pause) and we already did the yellow in the fifth.
175.	**Researcher 1**	00:22:57	Ok, did you do the yellow in the fourth place here- Do I see yellow in the fourth any place here? Okay, why don't you draw that one so we will have it.
176.	**Stephanie**	00:23:20	Wait- yellow in the fourth place
177.	**Researcher 1**	00:23:25	Okay. Is that all possibilities?
178.	**Stephanie**	00:23:29	We could have yellow in the fifth place, but we already did that.
179.	**Researcher 1**	00:23:37	Let's write that down. Okay, anything else possible?
180.	**Stephanie**	00:23:47	No. Because there are only five places

181.	**Researcher 1**	00:23:48	Only five places. Ok so, how did you find here?
182.	**Stephanie**	00:23:52	We found five and there is the other one.
183.	**Researcher 1**	00:23:55	Oh, okay. So how many do you have?
184.	**Stephanie**	00:23:58	So there’s ten
185.	**Researcher 1**	00:23:59	Okay. You should write that down. How are we doing? So you are convinced there is only that and you told me why (pause) what do we have here?
186.	**Stephanie**	00:24:12	Okay, before we started this pattern we had sixteen and ten, that’s twenty six
187.	**Researcher 1**	00:24:21	Let’s do this again. Let’s keep a running total (pause) First one there we had what?
188.	**Stephanie**	00:24:23	Uh, ten
189.	**Researcher 1**	00:24:24	Write ten, that on page. Under there we got how many more?
190.	**Stephanie**	00:24:29	We got six.
191.	**Researcher 1**	00:24:33	Okay, here we found how many?
192.	**Stephanie**	00:24:36	On the second page we found two
193.	**Researcher 1**	00:24:38	Mmmh
194.	**Stephanie**	00:24:41	Okay
195.	**Researcher 1**	00:24:42	Third page?

196.	**Stephanie**	00:24:43	On the third page we found two and another two. And the fourth page we found ten. Oh right so that's ten, twenty, twenty-two, twenty-four, twenty-six, twenty-six and six is thirty two.
197.	**Researcher 1**	00:25:08	How did you get thirty-two?
198.	**Stephanie**	00:25:10	Okay, ten and ten are twenty. Six and two is eight and two is ten and two is twelve and twelve and ten- wait- okay ten.
	Researcher 1		Why don't you write the numbers down.
	Stephanie		Yeah. Ten and ten is twenty. I mean no- yeah ten and ten is twenty. Okay, then six and two is eight plus two is ten plus two is twelve plus twenty is thirty two.
199.	**Researcher 1**	00:26:00	I am confused (pause) Did you find these yesterday?
200.	**Stephanie**	00:26:02	Yeah
201.	**Researcher 1**	00:26:04	Did you count those any place?
202.	**Stephanie**	00:26:05	Yeah. Here. When we had the pattern yellow, yellow, yellow, yellow, yellow, yellow, yellow, yellow, yellow, yellow
203.	**Researcher 1**	00:26:14	Where is the all yellows there?
204.	**Stephanie**	00:26:16	Just scribbled it in so it's all yellows.
	Researcher 1		That last one.
	Stephanie		We did these all blues first.

205.	**Researcher 1**	00:26:24	So do you think you would be able to persuade this fifth grader that was nothing else possible?
206.	**Stephanie**	00:26:29	Mmmh
207.	**Researcher 1**	00:26:34	You are absolutely convinced they will believe this? Let me ask you another question. Suppose I was building towers of four, what do you think?
208.	**Stephanie**	00:26:47	Four using (inaudible). I think you would get less.
211.	**Researcher 1**	00:26:55	Okay, why do you think that?
212.	**Stephanie**	00:26:56	Because with five you have thirty two
213.	**Researcher 1**	00:26:58	Mmmh
214.	**Stephanie**	00:26:59	So you are subtracting now, you get less. If you were adding one, you might have gotten more
215.	**Researcher 1**	00:27:06	Okay
216.	**Stephanie**	00:27:07	You might have gotten more. But you're subtracting. So you probably will get less
217.	**Researcher 1**	00:27:22	Do you have any idea how that would work? Think about hat for a minute. (pause) get a piece of paper. Also number this one. We lost track of our numbers here.

218.	**Stephanie**	00:27:24	Okay. This one is four (pause) this one is five (numbering pages). Hmmm. Ummm, hmm. It would probably work the same way we worked with five, llike one lue two blue three blue four blue four blue only you would go to four. So instead of having ten on this you would have eight.
219.	**Researcher 1**	00:27:53	Okay, write that down. You can go through your thinking. You can look at anything you have done so far. Yeah, write the eight. Try to keep record of what you are thinking.
220.	**Stephanie**	00:28:1	And then the same way you would do the next one (pause) next one (pause) then instead of getting siz here because we get two together.
221.	**Researcher 1**	00:28:19	Mmmmh
222.	**Stephanie**	00:28:20	You'd get two together, two together, two together. You'd get- stil you would get six because-
223.	**Researcher 1**	00:28:30	How is that possible that you get six with four and six with-
224.	**Stephanie**	00:28:33	Five
225.	**Researcher 1**	00:28:35	Five? How is that possible? That sounds weird.
226.	**Stephanie**	00:28:40	Okay, two together. Two together (pause) no you'd only get three so the total is six. And then you'd have two together (pause) So you would get six.
227.	**Researcher 1**	00:29:02	I am really confused now because you're telling me it will be the same.
	Stephanie		Mmmh
	Researcher 1		And here you didn't get six at first, and here you have eight at first.
228.	**Stephanie**	00:29:10	Yeah. But we found that we had patterns from up top.

229.	**Researcher 1**	00:29: 25	Oh, Can that happen when you make towers of four?
230.	**Stephanie**	00:29: 18	Yes. So, we made this and (pause) the second one would be one of the towers
231.	**Researcher 1**	00:29: 27	Oh, okay
232.	**Stephanie**	00:29: 32	The first one is one on the towers. The first one is one on the towers.
233.	**Researcher 1**	00:29: 35	Why is that?
234.	**Stephanie**	00:29: 36	Because the first one is the second tower
235.	**Researcher 1**	00:29: 45	Okay, so how many do you have now?
236.	**Stephanie**	00:29: 48	So, we only have, okay, so we subtract ((counting)) one, two three, one two, three. We only have four. And then, each ((shuffling through papers)) page two, we had the-one like this. ((builds tower blue, yellow, blue, yellow)) So that's four, and four (pause) two groups. So you can make two of these.
237.	**Researcher 1**	00:30: 27	Same as the four?
238.	**Stephanie**	00:30: 28	Yeah
239.	**Researcher 1**	00:30: 29	Is that possible? It is gonna be the same when you have a four as the towers of five before, that makes sense to you?
240.	**Stephanie**	00:30: 33	Well, yeah. You can make two of the towers
241.	**Researcher**	00:30: 37	Mmm, and you haven't made these before?
242.	**Stephanie**	00:30: 39	((nodding the head)) No

243.	**Researcher 1**	00:30:40	How do you know that?
244.	**Stephanie**	00:30:41	Well, because (pause) coz you made towers, you made- the one (pause) two together, so these aren't together at all and the other ones are one, two, three, four, five.
245.	**Researcher 1**	00:30:58	Okay (pause) Alright.
246.	**Stephanie**	00:31:05	And then (pause) one in the middle
247.	**Researcher 1**	00:31:19	One in the middle?
248.	**Stephanie**	00:31:20	No, you can't do one in the middle like this
249.	**Researcher 1**	00:31:22	Why
250.	**Stephanie**	00:31:23	Cause you can only do two in the middle because four is even number, so you could go like this but there is no possible way you can get one (pause) in the middle (pause) one in the middle of four
251.	**Researcher 1**	00:31:36	Have you gotten that one before?
252.	**Stephanie**	00:31:39	Yes,(Pause) so that one can't be number three. (pause) Okay this is the one we had all the stories. So this one we will take a while
253.	**Researcher 1**	00:32:02	Is there a way you can do that faster without building all of them, (pause) think about them in your head
254.	**Stephanie**	00:32:07	Well, you could (pause) yeah, you could just-
255.	**Researcher 1**	00:32:09	Imagine in your head
256.	**Stephanie**	00:32:14	Okay, yeah, you could, you could do this, the second one in the middle, so that's another two. And-

257.	**Researcher 1**	00:32:32	I'd like you to tell me about all those together…stop here. What can you tell me about this page ((points to page of Stephanie's representation of all possible towers drawn)) together?
258.	**Stephanie**	00:32:39	All right this page? (pause) All right, well, it's just, it's really the same pattern in different places
259.	**Researcher 1**	00:32:49	Right
260.	**Stephanie**	00:32:50	It's taking one- building on one pattern (pause) it's okay. So it started with the pattern at the top
261.	**Researcher 1**	00:32:59	Mmmh
262.	**Stephanie**	00:33:00	You are taking that pattern, and then moving it down one. And then moving it down another, and another until you have all five patterns
263.	**Researcher 1**	00:33:06	There are five of them?Same as here?
264.	**Stephanie**	00:33:07	Wait (pause) no (pause) I am not sure how many patterns we have of these. But this should have five (pause) with the four you would have (pause) wait a second. With the four, doing that would be just like this
265.	**Researcher 1**	00:33:38	Mmmh
266.	**Stephanie**	00:33:40	Doing (pause) So you can't do this. This one part you can't do because, doing this one((points to tower B,Y,Y,Y)) would be just like doing this over((creates tower B,Y,Y,Y)). The one at the bottom- (long pause)
267.	**Researcher 1**	00:34:10	So you are saying that's the same as that, but that is a five, so I am a little confused. I'd rather you tell me about the one that's the four
268.	**Stephanie**	00:34:15	Well, okay. Well this is the four, right here ((points to representation of four towers. Each tower with blue in a different position)).

269.	**Researcher 1**	00:34:18	Mmmh
270.	**Stephanie**	00:34:19	Like here. And we are using the four (pause) and the four, the first one is the same as that
271.	**Researcher 1**	00:34:24	Okay. Did we count this twice before, maybe?
272.	**Stephanie**	00:34:27	Mmmh, I don't think so
273.	**Researcher 1**	00:34:30	Mmmh, What about this? ((points to B,B,B,B,Y and Y,Y,Y,Y,B))
274.	**Stephanie**	00:34:39	We did (pause) so the answer is thirty
275.	**Researcher 1**	00:34:44	Interesting
276.	**Stephanie**	00:34:45	Well this one you can't do (pause) with the one.
277.	**Researcher 1**	00:34:49	Okay
278.	**Stephanie**	00:34:50	So this eliminates one two. So there is only eight. one two
279.	**Researcher 1**	00:34:58	So how many can you do then?
280.	**Stephanie**	00:35:00	You can do the others
281.	**Researcher 1**	00:35:01	How many 'others' are there? With the four
282.	**Stephanie**	00:35:07	You can do the other four, I think
283.	**Researcher 1**	00:35:09	There are other four?
284.	**Stephanie**	00:35:10	See. Four others from the five, you can probably do with

285.	**Researcher 1**	00:35:15	Mmm
286.	**Stephanie**	00:35:16	Mmm
287.	**Researcher 1**	00:35:17	Well there are the four others, why don't you make them for me.
288.	**Stephanie**	00:35:22	Oh, this is one, so we make that (pause) and we can do that (pause) that's two
289.	**Researcher 1**	00:35:34	Make them all before you write them down, I want to see them all together ((Dr. 1 is asking Stephanie to build all of the possible towers five tall))
290.	**Stephanie**	00:35:38	Mmmm (pause) we can do the one (pause) with the blue (pause) that. And we can do the one (pause) the blue
291.	**Researcher 1**	00:36:09	That's only three you told me you can make four more
292.	**Stephanie**	00:36:10	And the other one is- (pause) you can make three more because there's only three (pause) there's only four blocks
293.	**Researcher 1**	00:36:19	Aaaha, okay.
294.	**Stephanie**	00:36:21	So that (pause) so that's about it. That's all the patterns we made from the five
295.	**Researcher 1**	00:36:37	That's it?
296.	**Stephanie**	00:36:42	Twenty
297.	**Researcher 1**	00:36:44	Okay, twenty, I guess i'm a little confused about how you, in the first problem, how you counted that one twice. I guess I am a little confused about (pause) how I know whether or not you counted some others twice, you know where in your mind you're checking that?

298.	**Stephanie**	00:37:14	You can check it on the paper by first flipping over the paper
299.	**Researcher 1**	00:37:17	Mmm (pause) but you can miss it, can’t you. Can you find an easy to do it when you make them?
300.	**Stephanie**	00:37:21	Mmm
301.	**Researcher 1**	00:37:22	I wonder if there is another way of sort of- You made these for instance- (pause) how do you make these? Remember you counted the blue on the first bottom of this position and you pulled it out.And you remember to do it this time.
302.	**Stephanie**	00:37:43	You would have to-. I only remember, I guess what you did.
303.	**Researcher 1**	00:37:47	That’s cool. I have a lot of trouble remembering all of that. It’s hard for me. I get mixed up.
304.	**Stephanie**	00:37:50	I can’t remember it either
305.	**Researcher 1**	00:37:55	Just wondering if we can come up with a way that can make it easier to remember. Because it’s a nice way of trying to find them like your patterns but I worry about (pause) mmm, if we’re missing some or counting some twice. That’s tricky.You might want to think about that way of trying tocome up with a way to do that. Getting tired?
306.	**Stephanie**		Mmmm
307.	**Researcher 1**		So, you might want to think about these towers of four, when you leave, maybe draw a picture of something and show it to us next time and come up with a way of really checking yourself, for a way of recording- and you developing a nice way of recording your steps. Okay you have all these kinds together. But let me ask you another question. What about towers of five? You said- are sure now? is it thirty two? is it thirty?, is it (pause) What do you think is happening here?

308.	**Stephanie**	00:39:05	Well, it's (pause) it could be thirty, it could be (pause) thirty because we did one like we found in the four. We did one, so it could be pretty much thirty.
309.	**Researcher 1**	00:39:27	Suppose I said to you maybe you missed some too? Maybe you didn't count something twice, but maybe you missed something else?
310.	**Stephanie**	00:39:33	Yeah, how could you be absolutely positively-
311.	**Researcher 1**	00:39:37	Yeah, why?
312.	**Stephanie**	00:39:41	I guess, mmm a very lucky guess.
313.	**Researcher 1**	00:39:45	But math isn't a guess, math you should be able to figure it out, be convinced and- I mean I don't think anybody would be able to convince you that there- you said yesterday to me that if I were only making towers with only one blue, lets say, exactly one blue.(inaudible) Is anything else possible, other than exactly one blue?
314.	**Stephanie**	00:40:17	No
315.	**Researcher 1**	00:40:18	You're convinced of that.
316.	**Stephanie**	00:40:20	Yeah
317.	**Researcher 1**	00:40:21	This question is you may have counted it in different pattern categories, right?
318.	**Stephanie**	00:40:23	Yeah
319.	**Researcher 1**	00:40:24	But you are convinced that nothing else is possible. What was your argument you gave me yesterday? Why you are actually convinced they are only four when you build towers of four, and if I build towers of five, how many are there?

320.	**Stephanie**	00:40:35	Five
321.	**Researcher 1**	00:40:37	And towers of six?
322.	**Stephanie**	00:40:38	Six
323.	**Researcher 1**	00:40:40	So if I were saying exactly one of a color
324.	**Stephanie**	00:40:43	Mmm, you can only build that many
325.	**Researcher 1**	00:40:46	If it's blue (pause) four, yellow four. So, you (pause) so you don't have any doubt in your mind about that
326.	**Stephanie**	00:40:48	Yeah
327.	**Researcher 1**	00:40:50	You don't have any doubt in your mind about that. Okay, now a lot of the ways you were thinking about building these- you should have again no doubt in your mind about that when you are finished you should be absolutely convinced and certain and ready to fight the world for big prize money to be convinced and I don't think you are quite convinced yet? Am I right? With the other cases?
328.	**Stephanie**	00:41:10	Yeah
329.	**Researcher 1**	00:41:12	You sort of have a nice way of finding this, but you are not really convinced to have them all. Or might have counted some more then once.
330.	**Stephanie**	00:41:19	You can't really be convinced because there is no absolute way, you can go and say, 'I'm right'
331.	**Researcher 1**	00:41:26	Well, this is an absolute way
332.	**Stephanie**	00:41:27	Yeah, this is one of the absolute ways

333.	**Researcher 1**	00:41:29	I think, I think that you can (pause) this absolute way is when you look at only one blue
334.	**Stephanie**	00:41:34	Yeah
335.	**Researcher 1**	00:41:36	And I wonder if you could find other absolute ways of looking at maybe just two blues or just three blues or just four blues
336.	**Stephanie**	00:41:41	You can
337.	**Researcher 1**	00:41:42	Is an absolute way. How many ways can you do exactly four blues?
338.	**Stephanie**	00:41:45	Once
339.	**Researcher 1**	00:41:46	You are convinced oh that, right?
340.	**Stephanie**	00:41:48	With the four towers.
341.	**Researcher 1**	00:41:49	Yeah (pause) You're convinced of that right? No one can persuade you otherwise, right? Well, you are convinced of this, you are convinced to that, can you figure out ways of getting convinced to those middle cases, exactly two, or exactly three
342.	**Stephanie**	00:42:00	Yeah
343.	**Researcher 1**	00:42:01	Yeah, what do you think?
344.	**Stephanie**	00:42:03	It is possible to have a certain number and get it right by figuring out the number of two's you could have in the middle, three you know, fours, ones, depending on the number of blocks you have.

345.	**Researcher 1**	00:42:21	Okay. You should be sure that our answer at some point. It shouldn't be- why don't- do you want- will you think about the problem when you go home and maybe we can come back next week or something, to talk to you more about it, maybe we'll let you take some blocks home with you. We'll let you take some. You'll take some of our blocks home. And then we come back next week, and after you have really had a chance to think about it- you can think about it in the case of four and the case of five. if you want we'll make copies of this, for all the hard work you have done. Because I really think you are very good persuasive (pause) person, and we can pretend there is a million dollars at stake. You got to convince us or not convince us. Fair enough?
346.	**Stephanie**	00:43:03	Yes
347.	**Researcher 1**	00:43:04	Okay. That would be great fun. Have you done this before any other time?
348.	**Stephanie**	00:43:08	Mmmm, not of yesterday, we have done different problems with the cubes (pause) but nothing of this sort, not really.
349.	**Researcher 1**	00:43:23	Does it remind you of anything?
350.	**Stephanie**	00:43:25	Mmmm, it sort of remind me of the shirts and shorts, the way Dana and I were fixing them yesterday, we were putting them into pairs.
351.	**Researcher 1**	00:43:39	Oh, okay
352.	**Stephanie**	00:43:41	Like-
353.	**Researcher 1**	00:43:42	Tell me about that
354.	**Stephanie**	00:43:43	We would go ((takes tower Y,Y,Y,Y,B)), and go((creates tower B,B,B,B,Y)). Okay this is five. And this goes on the side. I'll reverse it. And there's the shirt and shirts you would go

355.	**Researcher 1**	00:44:08	I mean you will wear the pants at the bottom and the shirt at the top, I don't understand that.
356.	**Stephanie**	00:44:13	No, (laughing), like we go, we have like blue pants, white pants (pause)) and we have a- and we have- (pause) we are going to say we have-
357.	**Researcher 1**	00:44:27	Tell me about the problem, I don't know the problem you are talking about. The shirts and pants problem we are talking about.
358.	**Stephanie**	00:44:30	Oh right. We made like- or I made like a shirt and shorts problem in my head. We had a blue and white pair of pants and a shirt, a white shirt. You had to find out how many you can make. So, like you could go and put it (pause) with the blue pants, or you could go and put it with the white pants, and that's what we could do, we could put the blue with the five, with the four yellows (pause) but we could go and put orange with the blue.
359.	**Researcher 1**	00:45:04	Okay, so which is the pants, and which is the shirts?
360.	**Stephanie**	00:45:09	Well, here you could have two pairs of this shirt and two pairs of pants
361.	**Researcher 1**	00:45:14	And you told me you only had one shirt, two pairs of pants or do you have two of each?
362.	**Stephanie**	00:45:18	Two of each will be like ((drawing on the paper))
363.	**Researcher 1**	00:45:19	Mmm
364.	**Stephanie**	00:45:23	And then we have this (pause) you could put this shirt with this shirt, that shirt with that shirt (pause) but you can only do it once because you couldn't do it twice. Dana saw- Dana saw a duplicate of one of the group and she went and said, ‘’oh they are in different groups’’ and me and Dana said, ‘’oh they are in different groups we can still use them,” And returned and we were checking and we looked and we said, ‘’what is this doing here?’’

365.	**Researcher 1**	00:45:50	I see, I see, so how many outfits did you get here?
366.	**Stephanie**	00:45:54	Well, you could have (pause) like I said you have blue pants-
367.	**Researcher 1**	00:45:59	What are you making a tower from that is like the two shirts and two pants? Maybe a tower problem that's like this one.
368.	**Stephanie**	00:46:09	Okay, written or (inaudible) just like a-
369.	**Researcher 1**	00:46:12	What will the towers look like if you were making towers like shirts and pants?
370.	**Stephanie**	00:46:20	Okay, this is the blue one (inaudible) Okay we are going to see that
371.	**Researcher 1**	00:46:25	Orange and blue shirts, make it like the towers?
372.	**Stephanie**	00:46:34	((Draws tower O,O,O,O,B and O,O,B,B,O))Here we go, and then we ((writing on the paper)) so that the tower-
373.	**Researcher 1**	00:46:51	But here we only have four outfits
374.	**Stephanie**	00:46:53	Mmmh
375.	**Researcher 1**	00:46:55	Right, I don't understand, I didn't know-
376.	**Stephanie**	00:47:00	Well, what it is is the (pause) the shirts and shorts, you have (pause) with the towers, you have two colors. You have two colors of building blocks and you are trying to make a tower of five with these colors, how many towers can you make? And well you (pause) What we do is we reverse the towers we make. We would say I have a building block of blue and four orange building blocks. A building block of orange and four blue building blocks

377.	**Researcher 1**	00:47: 40	Mmmh, Okay, do you have any sense of- mmm how many towers there should be in that first problem? Maybe you guess now. You got thirty two and then you thought you counted something twice then, how sure are you then?
378.	**Stephanie**	00:47: 55	Well we counted something twice, so I think it is in the league of thirty something between thirty and thirty (pause) then thirty
379.	**Researcher 1**	00:48: 03	Somewhere, it is an estimate now
380.	**Stephanie**	00:48: 04	Yeah?
381.	**Researcher 1**	00:48: 05	Okay, now you will come- we will come next week, Amy? And do the interview
382.	**Amy**	00:48: 09	Sure
383.	**Researcher 1**	00:48: 10	And then you promise you are going to work on this?
	Stephanie	00:48: 15	Yeah.
	Researcher 1		It's going to be great. I can't wait to talk with you about this some more. So, imagine the four towers and imagine the five towers and if you have time. And that is it.
384.	**Stephanie**	00:48: 23	Alright, and I'll work on a way to be definite-
385.	**Researcher 1**	00:48: 25	Yea that would be exciting, that would be real fun. Terrific. Well, thank you so much, you have a good weekend. I enjoyed this very much.
386.	**Stephanie**	00:48: 33	Thank you

Appendix I: Session VII: Transcript-Grade 4 Six-Tall Towers Problem

Stephanie engages in the Six-Tall Towers problem in fourth grade under Researcher 1.

Line	Speaker	Time stamp	Utterance
1.	**Researcher 1**		Yeah so tell me tell me tell me about these six forms…
2.	**Stephanie**	00:00:27	So I got a lot of the sixes but I didn't have-I didn't get a lot of time because I was sick a lot. [...]
3.	**Stephanie**		Yeah, ok. Alright then. Ah I did just like this I made the pages like this and I made these pages. Okay, and this is the first page and what's this, this is th six cubes and this is the first page and this is the one at time.
4.	**Researcher 1**	00:00:51	Please explain to me what- what
5.	**Stephanie**		Ok it's one at a time, ok, so that means, ah, here is the al we have the blue, blue. Ok we have the blue orange orange orange orange orange. Orange blue orange orang orange orange. Orange orange blue orange orange orang Orange orange orange blue orange orange. Orange oran orange orange blue orange. Orange orange orange orang orange blue.

6.	**Researcher 1**	00:01:25	And you're saying they are, how many of these altogether?
7.	**Stephanie**	00:01:26	There are six.
8.	**Researcher 1**		And how come - are you sure there are (no?) more?
9.	**Stephanie**		Yeah
10.	**Researcher 1**		Why?
11.	**Stephanie**		Because, because you can only make six towers… you can only make six…blocks high towers.
12.	**Researcher 1**	00:01:38	Ok, I see.
13.	**Stephanie**		And if you go any further you have to add another block on.
14.	**Researcher 1**		I see. Ok, so I believe you got there…
15.	**Stephanie**	00:01:45	And then we did the (?) messed up on the writing and w had the opposite and the total is 12.
16.	**Researcher 1**	00:01:50-4	Ok. Did you work with anyone on this?

17.	**Stephanie**	00:01:52-1	No, we didn’t get a chance to.
18.	**Researcher 1**	00:01:53-7	Did you talk to Michelle at all? Not even on the telephone?
19.	**Stephanie**	00:01:55	No [...]
20.	**Researcher 1**	00:02:07	Okay, well
21.	**Stephanie**		Okay then this is the two at a time
22.	**Researcher 1**	00:02:10	Two at a time, okay.
23.	**Stephanie**	00:02:10	And there you have, the two blue [pointing to the tower with blue, blue at the top and continuing to point at eac block as she talks]orange, orange, orange, orange. Then the orange blue, blue, orange, orange, orange, and so fo [pointing now to the two blue together in the third and fourth position from the top]. You just go- well, what happened is [points back to the first tower with the 2 blu at the top] we go blue blue, then you cross over one [referring to the next tower the two blues are now in the second and third position, skipping the first position] bl

			blue; cross over one blue blue ; cross over one blue, blu cross over one blue blue and then you have your last on
24.	**Researcher 1**	00:02:34-8	Ok, you gave these a name
25.	**Stephanie**	00:02:36-1	Yeah. Two at a time
26.	**Researcher 1**	00:02:37-6	Two at a time and you say they are five?
27.	**Stephanie**	00:02:39-7	Five and the double opposites they [are ten.
28.	**Researcher 1**	00:02:42-3	Oh ok the double opposites. So these are two at a time. are these the only two at a time that we have? Let's keep record here of what you wrote down. For the first one where you're writing[
29.	**Stephanie**	00:02:51-7	One at a time.

30.	**Researcher 1**	00:02:52-6	And you have six and why don't you write six here and keep track, looking at the work you had done.
31.	**Stephanie**	00:02:57-4	((a faint 'and' is mentioned first)) and we have another for that because we did the opposite
32.	**Researcher 1**	00:03:03-0	Ok and here we have
33.	**Stephanie**	00:03:04-3	Five (inaudible)
34.	**Researcher 1**	00:03:07-6	Okay. And what did you do?
35.	**Stephanie**	00:03:10-1	Okay. Then I (???) we have three at a time.
36.	**Researcher 1**	00:03:12-6	Okay
37.	**Stephanie**		And then to three

38.	**Researcher 1**	00:03:14-9	Okay you have three look back are these the only two a a time you can have?
39.	**Stephanie**	00:03:17	Ah, together
40.	**Researcher 1**	00:03:20-5	Oh these are together. Why don't you write together her I see it's sort of like they can never be separated like the are glued or something?
41.	**Stephanie**	00:03:27-8	Yeah. Researcher 1: Okay.
42.	**Stephanie**		Yeah [because of the two at a time.
43.	**Researcher 1**	00:03:30-1	Okay. I see that. And now these are three?
44.	**Stephanie**	00:03:33-6	At a time.
45.	**Researcher 1**		Together?
46.	**Stephanie**		Together.

47.	**Researcher 1**		Okay we write together so I don't get stuck ((quite a lon pause as Stephanie writes)). And you're convinced that there no more ones, on these?
48.	**Stephanie**	00:03:45	On these kinds, yeah.
49.	**Researcher 1**		Why?
50.	**Stephanie** **Researcher 1** **Stephanie**	00:03:54	Now like, now like, ah, the two together and three together but you can have two at a time. Ah. There are different ways you can have the two at a time
51.	**Researcher 1**	00:03:56	But when they're together these are the only ones
52.	**Stephanie**	00:03:58	Yeah.
53.	**Researcher 1**		And how can you...
54.	**Stephanie**	00:04:02	Prove it? [Stephanie quickly suggests and finishes her question]
55.	**Researcher 1**	00:04:03	Prove it.

56.	**Stephanie**		Well, okay. You have, you, once you get down to the las two blocks,
57.	**Researcher 1**	00:04:07	Yes
58.	**Stephanie**		You've used all the six and you're on your last two blocks,
59.	**Researcher 1**		Yes.
60.	**Stephanie**		You can't go down here, blue blue.
61.	**Researcher 1**		Yes
62.	**Stephanie**	00:04:16	Because you are missing a block, I mean you would nee another block.
63.	**Researcher 1**	00:04:18	Ok.
64.	**Stephanie**		You need another block
65.	**Researcher 1**	00:04:20	Ok. You might not have six anymore. Ok I see that. Anc here, same argument? 00
66.	**Stephanie**	00:04:25	Three, four. And it was four and four [pauses in thought Five? 00

67.	**Researcher 1**	00:04:31	Now, when you said four and four I was forced to imagi in my head that there could have been three orange here and three blue, right when[you do your partners. Is that what you mean by that? 00
68.	**Stephanie**	00:04:42	Mmmhh
69.	**Researcher 1**		What was the opposite of this one?
70.	**Stephanie**	00:04:46	This one?
71.	**Researcher 1**		The second one.
72.	**Stephanie**		The opposite of this would be Blue Orange Orange Orange Blue Blue 00
73.	**Researcher 1**	00:0453	Okay and so forth and those are the partners. And, ok, s you have and how many do you have so far? Can you ju do a total of that so far? 00
74.	**Stephanie**	00:05:25	Thirty
75.	**Researcher 1**		Okay. So far you have 30
76.	**Stephanie**		Mmmmhhh

77.	**Researcher 1**		Okay, and um, and these thirty are all different?
78.	**Stephanie**	00:05:16	Mmmhhh mmmhhh
79.	**Researcher 1**		And you are sure they are all different?
80.	**Stephanie**	00:05:20	Yeah because this one is choosing three blocks
81.	**Researcher 1**	00:05:22	Mmmhhh
82.	**Stephanie**		This one is choosing two blocks and that one…
83.	**Researcher 1**	00:05:24	Choosing one block, even when you switch them aroun
84.	**Stephanie**	00:05:27	Mmmhhh
85.	**Researcher 1**		Ok. What did you do then?
86.	**Stephanie**	00:05:30	And this one is four at a time
87.	**Researcher 1**	00:05:32	Oh four at time?ok.
88.	**Stephanie**		And then this is the same argument as up there and it's just adding an extra one

89.		00:05:39	And.. ok. So, here you had four at a time, and you ende up with six because they are double. But I am confused
90.	**Stephanie**	00:05:48	You double them. You'd go... okay, look [she points at h work]. The opposite of this one will be orange orange orange orange blue blue.
91.	**Researcher 1**	00:06:00	Ok. And so, you… you are counting this as one of your doubles and you have never used that before?
92.	**Stephanie**	00:06:06	Mmmhhh. Yeah. We haven't used it because we use the ones, we use the twos, we use three and then we use fou
93.	**Researcher 1**	00:06:14	Look here...
94.	**Stephanie**		Mmmh. So we did use that. So then that wouldn't be thr anymore. That would be...
95.	**Researcher 1**	00:06:27	I wonder why that's happening! Does that happen any place else? When you turn, when you switch these roun right?
96.	**Stephanie**	00:06:34	Mmmhhhmm

97.	**Researcher 1**		You tell me this is what you get and that happens to be here. What about that?
98.	**Stephanie**	00:06:40	That one? Mmmhhh [Stephanie is in deep thought]. I... No. I don't think anyone is gonna duplicate that so far.... because, look; We did, so far we've been doing them all altogether
99.	**Researcher 1**	00:06:59	Yes
100.	**Stephanie**		So, these two oranges are separated
101.	**Researcher 1**	00:07:04	Why do you suppose this happened? Why do you suppo this duplicate[pauses] occurred?
102.	**Stephanie**	00:07:09	Mmmmhhh. Well, I don't know! [sounds emotional-affect]
103.	**Researcher 1**	00:07:12	You see, why were they here?
104.	**Stephanie**	00:07:14	Here, they were ah,the two at a time.
105.	**Researcher 1**	00:07:20	How can they be same as this is two at a time and this is four at a time?

106.	**Stephanie**	00:07:23	Oh...
107.	**Researcher 1**		How can that be? I am confused
108.	**Stephanie**	00:07:27	One two three four, thats' also four at a time. That two a time is also four at a time
109.	**Researcher 1**	00:07:32	Why is that?
110.	**Stephanie**		Because look; one two three four, there's four oranges together and two blues at time too
111.	**Researcher 1**	00:07:41	Ohhhh. So when you think of the two blue at a time, yo are thinking of four oranges at a time...
112.	**Stephanie**	00:07:47	Mmmmhhh
113.	**Researcher 1**		Or when you are thinking of the two oranges at a time, you are thinking of the four blue at a time and that's hov you get them?
114.	**Stephanie**	00:07:54	Mmmmhhh

115.	**Researcher 1**		Oh. Did you check for that?
116.	**Stephanie**	00:07:57	Aha
117.	**Researcher 1**		Did you check for those duplicates then, and you then g duplicates that way?
118.	**Stephanie**	00:08:01	Aha
119.	**Researcher 1**		Okay we will have to remember that and go back. Ok. That's neat. What about this one?
120.	**Stephanie**	00:08:06	This one?
121.	**Researcher 1**		Yeah. That's kind of neat. I like what you are doing; one two three four at a time, that's neat.
122.	**Stephanie**	00:08:11	Yes. I think it might have a duplicate. Like the blue blue orange orange orange. If you choose the blue blue orang orange, it's upside down.
123.	**Researcher 1**	00:08:21	Yeah. I see! [there's a pause]

124.	**Stephanie**		It will be orange orange blue blue blue blue which is... o it will,oh that's all like, if you turn it upside down, that will be orange orange [orange...
125.	**Researcher 1**	00:08:35	Why do you suppose this one didn't come up and the others did? Why isn't that…
126.	**Stephanie**	00:08:40	Because this is the only one where the two of them aren stuck together.
127.	**Researcher 1**	00:08:45	Ohhhh. That's so interesting. Isn't that amazing, that's right. Yeah.
128.	**Stephanie**	00:08:50	The two of them are stuck together.
129.	**Researcher 1**	00:08:51	Ok, ok, so in these, they were still stuck together. Ok, h many new ones did you this way?
130.	**Stephanie**	00:08:55	Ah, two [points at her work]
131.	**Researcher 1**		Ok. Let's record that. This is very interesting. Now we will all refer to it. Ok.
132.	**Stephanie**	00:09:04	Mmmhh. And now here we are. And then it's just one at time which is... I have just thought of something.

133.	**Researcher 1**	00:09:13	What?
134.	**Stephanie**		This is the one a time.
135.	**Researcher 1**	00:09:15	Huh.. Oh well, let's see. Let's find it at once? Ok, oh tha neat.
136.	**Stephanie**	00:09:19	You see, if you turn... [words not clear] here the orange..[ok. If you turn this one around which is like; blue blue blue blue blue.
137.	**Researcher 1**		Ok.
138.	**Stephanie**		There you go!
139.	**Researcher 1**	00:09:34	There it is! yeah.
140.	**Stephanie**	00:09:35	And then the same with this one.
141.	**Researcher 1**	00:09:37	Yeah, Yeah. So when you do one at a time, you are automatically doing...
142.	**Stephanie**	00:09:42	Five at a time
143.	**Researcher 1**	00:09:43	Five at a time. That's it. So you have no new ones on thi but then that’s neat? Ok, great, let's put this one aside.

			And you gotta write a note or something to show that w already in some place, so we so, ‘cause I get mixed up. These are the same as what? One at a time?????
144.	**Stephanie**	00:09:59	We have these.

Appendix J: Session VIII: Transcript-Grade 4 Stephanie Explores Four-tall Towers Interview

Stephanie engages in the Four-tall Towers problem under Researcher 1.

Line #	Time stamp	Speaker	utterance
1.	00:38	Researcher 1	Anyway let's see, where were we the last time?
2.		S	Well, when I came here you wanted me to do the towers of four.
3.		Researcher 1	Oh right, right, and you were going to [mumbling] your new method to show me, and how many towers of four did you tell there were going to be?
4.		S	I think I said around twenty.
5.		Researcher 1	You said around twenty.
6.		S	…and I did and when I did it I got twenty.
7.		Researcher 1	You did. And you were using?
8.		S	My method. The one. Alright.

9.		Researcher 1	Can you tell me about this? Tell me why you need to do this?
10.			(Talking others) Can you wait just a minute. Hold on just a second. Make a big difference. How are you
11.	1:27	S	Okay. First we have the towers with one white block, and the white block is on the top, and then it's there and then (making towers with exactly one white block in an elevator pattern from top to bottom) it's there -- and then, it's -- there, and then it's… there. And there's our first group of towers.
12.		Researcher 1	Okay, now remember, I'm supposed to be – Stephen, is it? -- and say to you I think there are more with one white and three black. (points to her towers)
13.		S	With one white – no, there's no more with one white…

14.		Researcher 1	But I'm Stephen, and I say I think there are…
15.	2:35	S	Well there's – well, okay. You have – okay. Once you get down to the last one, [mumbling] at the bottom, you can't move the white back up, because then you'll just be repeating these things. But if you move the white down one you'll be missing a space… and if you can only use four blocks, you can't have another one.
16.		Researcher 1	Why can't you move the white on the next position on top?
17.		S	Because if you move the white on the next position on top it'll be like this. And you need another -- you'd need another block here, but you can't do that.
18.		Researcher 1	Why can't you?
19.		S	Because there's only four blocks.

20.		Researcher 1	Why can't there be five?
21.		S	Because well... the assignment said we have to use four.
22.		Researcher 1	Oh ok. So these are. Should we keep a record while we're doing it of how many?
23.		S	Ok that's four. Should I draw them?
24.		Researcher 1	Well you can. Any way you want to record it. You can write a statement to describe it since you have them made already.
25.	4:10	S	Alright. Okay and here's the one moving down. Cause it starts here and it goes down here. Alright. Then we have the two. And, we go like this. That's one… and that's two, and that's three.
26.		Researcher 1	Okay and how do you describe this group?

27.		S	Well, um, two moving down, two together moving together down?
28.		Researcher 1	Okay, so when you wrote moving down, what do you mean? Okay, why don't you write that there's exactly one white… and exactly… how else can we say there's exactly one white?
29.		S	Um…
30.		Researcher 1	Want to say something about the black?
31.		S	With three black.
32.		Researcher 1	Okay…
33.	5:50	S	Okay there. And so this one we can go two… two white moving down, down with two black. And our total there is three.
34.		Researcher 1	Okay do you want to say anything about those two? They're the only ways you can have two white moving down?

35.		S	No they're stuck.
36.		Researcher 1	Stuck.
37.		S	Okay there, that way you can't get it mixed up with the ones that were apart. Then you have your three white.
38.		Researcher 1	Now hold on. This is the only way to do two white?
39.	7:12	S	No, those are not the only way you can do two white but these are the only way you can do two white together.
40.		Researcher 1	Okay so these are exactly one white and these are exactly two white. Want to jump to three white before you finish the two whites?
41.		S	When I did the pattern I did one white stuck together two white stuck together, four white stuck together. then went back and went two white apart.

42.		Researcher 1	I see.
43.		S	And then there’s the one.
44.		Researcher 1	I’m sorry to go back but I’m kind of thinking about Stephen again. Suppose he says, “How do you know have all of them?”
45.		S	Because if you take the last one you can’t move them down another one because you’re only using four blocks. Okay, here's the first one… and here’s the second one.
46.		Researcher 1	Okay.
47.		S	That’s two, three, two together. Alright there's your next one because you have these two and you have the three at top, two at the bottom, one on top.
48.	9:18	Researcher 1	And no more because?

49.		S	Because if you take this one and you wanted to move the white down another one you'd need another block to put in here. Now four… [drops a block]
50.		Researcher 1	Don't worry about it.
51.		S	Okay, four…
52.		Researcher 1	Let me just see what you have here. You have three together whites, three glued together. Glued together. Two white glued together. Three together, okay.
53.		S	Okay. And then we have one, wait… okay and that's the last one. And… now then we go back to the two whites stuck together and we make it… apart.
54.		Researcher 1	Here let's look it here next to these twos.
55.		S	And we can do it like… this… and that’s it. Wait… hmm,

			let's see. Um… oh I forgot. You can also go like this. That's the reverse.
56.		Researcher 1	Well let's see.
57.		S	That's the reverse of that one.
58.	12:18	Researcher 1	In what way reversed, Stephanie?
59.		S	Ok well when you show it upside down it's reversed.
60.		Researcher 1	But you're not, you're standing it up... that's the tower.
61.		S	Well, what I'm saying is I would go and when I'm finished, I'd go I have four here, three here, two here, one here … and I would go double that and I'd go I have two here, four here, six here, eight here... and I'd add that, but if I go like this…
62.		Researcher 1	Ok that's… thats how you got your twenty. what was your reasoning for getting the double.

63.		S	What it is is this isn't the only way you can get two apart. You can make two apart by taking two black and separating them.
64.		Researcher 1	Ok but. That's true. That's true. So when you compare these two, how do you think about them when you use that argument?
65.		S	Well, when I use the argument for I double this? I think about them in… wise… well, if I'm doing a pattern where I'm using whites apart then I'm not going to use a pattern where I use blacks apart with it but when I'm taking this and going and doubling the pattern I just turn it upside down and that's how I get my x my other pattern.
66.		Researcher 1	And you had a name for that the last time, didn't you?
67.		S	This kind of pattern?

68.		Researcher 1	Yeah you had a name for the other one that went with it
69.		S	Yeah pair.. I forget. I forget I think it was um… hm…
70.		Researcher 1	You use the word pair now. Which means the last time you had this you automatically found the other one. What did that look like. Just tell me, you don't have to .
71.		S	It looked like black on top and white…
72.	14:38	Researcher 1	Right. That’s how you got 20 by that strategy. Just now you didn’t do that here. When you just found the ten how many more two more just now,
73.		S	Two more
74.		Researcher 1	You said these were also two whites is that correct? Exactly two whites. Different than the two whites here? In what way? Just you can tell me.

75.		S	These are apart and these are together.
76.		Researcher 1	But these two are different apart in a certain way. What's the difference between in what way these are apart.
77.		S	This one has the, uh, black top. This Is the black at the top and this is the white at the top
78.		Researcher 1	Ok
79.		S	But it’s also this pattern because if you look here it’s black white black white. which is one of these patterns because the whites are still separated.
80.		Researcher 1	Let me stop here then. Here we have… let's talk about the strategy where you did double and pairs. In that strategy you said these are exactly two whites. Now again I’m going to pretend I’m Stephen, and I’m saying you've convinced

			me there only two together… there's three. Now I'm asking you to convince me when the two whites apart that there are only three when you have exactly two white and exactly two black when the two whites are apart They're exactly three. How would you convince me of that? Is this the way you wanted to put them? Is this the order you put it?
81.		S	It doesn't matter.
82.		Researcher 1	That's not how I think you had it I think you had this one first.
83.		S	This one had white at the top. And that's what I started off with had the white at the top then the blue – the black, then the white, then the black.
84.	16:59	Researcher 1	This one white on the top then we had a separation then the next white here… okay, this one.

85.		S	Then, I made a white at the top.
86.		Researcher 1	Again white at the top. Okay.
87.		S	I had a white at the top. With black in the middle, then a white.
88.		Researcher 1	Now these two here... are they alike in any way?
89.		S	Yes, they are; they both have white separated.
90.		Researcher 1	Is there any other way that they are alike, or maybe another way of saying it? These all have white separated, are these different from these in any way?
91.		S	This one is different from this one, because this one has the two black glued together.
92.		Researcher 1	That's true… that's… but I'm asking is there any way that one of these… these two are different.

93.		S	No not both of them. Well I guess yes… this has the black at the top but these don't.
94.	18:13	Researcher 1	If I asked you these two are different than these… these two had whites at the top and then you had to skip, now the only way you skip and this starts here and you had to skip. How do I know there's no more? [Stephanie yawns] You seem tired, Steph…
95.		S	Yeah, I woke up at six. I couldn’t sleep.
96.	18:58	Researcher 1	So now you need a nap… if you want to stop, you decide.
97.		S	No, that’s okay.
98.		Researcher 1	I’m Stephen and I want to be sure that when you have exactly two whites separated, that there can't be any others than the ones you have here.
99.		S	Okay... okay… the only problem is when you go to make

			your doubles, you can't make the doubles with this because there's already made the doubles…
100.		Researcher 1	Let's just try not to worry about the made the doubles… let’s try to prove…
101.		S	It's always harder with the separates, um… oh, okay…
102.		Researcher 1	I wish Stephen were here. Because I think you'd come up with a way of convincing him.
103.		S	Okay. You have the two whites separated by two blacks… you have the white, black, white, black…
104.		Researcher 1	I’m not convinced… that you're just telling me what you have, that doesn't convince me.
105.		S	But then you have the black, white, black, white, and if you want to separate again you'd have to have another piece. because the white on the bottom.

106.		Researcher 1	I don't really see, see it I'm not sure seems to me that if I'm Stephen you're just trying to make a case of each one of these three why they're there and I'm not really sure that I'm quite convinced that you did all of that
107.		S	Ok. Well let me put it this way, we know we can't make another one of these because you can't fit any more than two in between these so you know that this one can't be made again
108.		Researcher 1	So When you have white at the top and white at the bottom
109.		S	Yeah you can't fit any more than two in the middle. You know that ones finished… but then these two you have white black white black okay and then you move the white down one
110.		Researcher 1	Mmhmm….

111.		S	Ok and if I move the white down again if I were to say let me put the white in the black space and the black... hm… and... if I were to put the white here with the two blacks here i'd be getting.. this. If I were to put the white here and the two blacks here and the white here, I'd be getting that. If I were to put the white here and a black here and a white there and a black there you'd be getting this. And if were to put I shifted it here no I didn't , shifted it here, no I didnt shift it there, I would've put this this this and that the other one you'd be getting the same thing. So I shifted to every single level and you can't make another one.
112.		Researcher 1	I think I followed everything you said here so there are possible exactly two white you've added how many more.

113.		S	Three more.
114.		Researcher 1	So why don't we get it exactly two white next to it.
115.		S	Ok.
116.		Researcher 1	Let's put it next to it here so we have all of this row to be exactly two white next to it. You see this box here, why don't we put it right alongside it. This is not exactly two white. These are the three whites… its this box here these were the two glued together whites and these are the two whites that are separated. Okay. But you didn't tell me how many whites had to be here. How many whites exactly are there here. So you've convinced me that you four of exactly one whites, you've convinced me that there are exactly six of exactly two whites. And are there two of exactly three whites?
117.		S	Maybe. maybe. These are separate whites… let's say we'll

			start at the top. No, we can’t start it at the top. Okay we’ll start it right here. Okay there's your first one.
118.		Researcher 1	Oh, now we're talking about exactly three whites but some separation.
119.		S	Mmhmm…
120.		Researcher 1	Okay…
121.	25:17	S	There’s your second one because you moved the black down one. And that’s it. You can't make a third one
122.		Researcher 1	Why?
123.		S	Because if you move your black another one and if you move your black down to the last space you would have this.
124.		Researcher 1	Ok I see that but you don’t have a black on top here
125.		S	If you don't have a black on top, then you would have this.

126.		Researcher 1	So you have how many extras exactly three whites by having separations?
127.		S	Two.
128.		Researcher 1	Ok so why don’t you do that on alongside… so you have exactly three whites… And then this one was exactly four whites... we lost it somehow.
129.		S	Oh well. and then you can only make one.
130.		Researcher 1	Wait a minute aren't we doing whites? You can only make whites.
131.		S	Oh…
132.		Researcher 1	How many whites are there here exactly now? So why don't you tell me how exactly no whites? Why don't you put it down here. How don’t you tell me exactly how many. Keep focusing on whites… how many did you get with exactly no whites. Why don't you tell me how

			many exactly no whites? Keep a record.
133.		S	Exactly.
134.		Researcher 1	Zero whites? No whites?
135.		S	Ok I got one.
136.		Researcher 1	What do we have here?
137.		S	So far? [sigh] okay…
138.		Researcher 1	No whites? How many?
139.		S	One?
140.		Researcher 1	One white.
141.		S	Four.
142.		Researcher 1	Exactly four whites.
143.		S	Five.
144.		Researcher 1	Ok. Exactly two whites.
145.		S	Two… three… that's six… that's eleven.

146.		Researcher 1	Exactly three whites.
147.		S	That’s four… eleven… that's fifteen.
148.		Researcher 1	Exactly four whites.
149.		S	Sixteen.
150.		Researcher 1	You told me there was going to be twenty. you just convinced me there can’t be any more; I’m very confused.
151.		S	There are eleven.
152.		Researcher 1	What's the opposite of this? Put it back… let's see if all the opposite of these are okay… let's just leave here what you have… ‘cause you see this. Did you see any of these? This opposite happens to be here. Isn’t that interesting? This is exactly one white
153.		S	But you know why.
154.		Researcher 1	Why that's very confusing

155.		S	This is also three white. This is also a three white, so these are all going to be the opposite.
156.	28:39	Researcher 1	What you do you mean, these are all going to be the opposite? What opposite?
157.		S	This is the opposite of this.
158.		Researcher 1	Why does that happen?
159.		S	Because you're using three whites, this one was the three white column but you're using three whites, this one was one white column but you're using three blacks, so it's the opposite
160.		Researcher 1	Oh, so is that always going to work? So you're telling me when you're using exactly one white. All of the opposites of the exactly one white turn up where?
161.		S	Here…
162.		Researcher 1	In the exactly three whites, and that's because these are these

			opposites. Okay, so I understand how you got these opposites… the no whites with the four whites, and then one white with the three whites. All the opposites in here. for the exactly two writes.
163.		S	I thought I saw these ones.
164.		Researcher 1	All these are, these are exactly two w. hitesDo you see these opposites any place here? I you do.
165.		S	Hm
166.		Researcher 1	I do I see it here. So that's that opposite. That's the opposite of its very interesting. That's exactly two whites. That's very interesting. That's the opposite.
167.		S	I wonder. I guess…
168.	30:07	Researcher 1	You did convince me that everything has an opposite and the opposite of this one turned out here, the opposite of these turned out…
169.		S	These turned out here.

170.		Researcher 1	…of exactly one white turned out to be with exactly three whites. And the opposites of the exactly two whites came out with the exactly two whites separated.
171.		S	Why? It did not come out…
172.		Researcher 1	Oh… two whites separated…
173.		Researcher 1	Look this is the opposite of these here, somewhere in the exactly two whites you found the opposite of exactly two whites, the exactly two blacks. Why would you find those here when you had to find these here and these here. I wonder why that would be…
174.		S	Well that was the only one left.
175.		Researcher 1	That's a reason. That's not a convincing one; I don't think Stephen would buy that.
176.		S	Um…

177.		Researcher 1	But is there any reason other than that that's true that we have to find them for because we used them all up is there another reason why it would make sense that they would have to be here and couldn't be somewhere else
178.		S	Hm I guess. Like um. They have to be here they couldn't be someplace else. They have to be here they couldn't be somewhere else. I have a reason for that. Because if you're using the two, you can put the two with the three.
179.		Researcher 1	So when you're using exactly two whites, how many blacks are you using?
180.		S	Two.
181.		Researcher 1	Why can't you use more than two blacks when you're using two blacks?

182.		S	Because the assignment said to use four and you had to use another one.
183.		Researcher 1	Now this is the big question. When you started you thought there would be twenty because you found ten by going through a certain plan and you said because of your opposites that's the word you were using but now you went through this plan and you convinced me there are no more when there are no whites than this. I believe it, right? There are no more when there's exactly one white. You went through all plans where there's exactly two whites, three whites, four whites. How could you solve your problem when you only paying attention to the whites now. Not even worrying about that opposites. Come up with sixteen and convince me there are no more. And when

			you did opposites you ended up with twenty. Do you think that's possible? Do you see how somebody might get confused about that? What do you think?
184.		S	How could I explain? All right. You have your two, right, so there's your opposites. you can't look for any more opposites, 'cause these are solid color. We're gonna say these are solid colors. These have the all black and all white. There's no more of these. Then you have this… now I convinced you that there is no more of these kind. And you look for the opposites.
185.		Researcher 1	But you didn't do these by finding opposites remember. You didn't convince me by opposite. Cause if you're gonna use an opposite argument with me, I would say how do you know there keep more and more and more of them

			you've convinced me because you said there's exactly three whites.
186.		S	I convinced you that there were exactly three whites.
187.		Researcher 1	Just turned out that there was exactly three whites… that's not how you convinced me
188.		S	Hm I convinced you that there were no more of these but then when you put them together you find out that they are opposites and you know that you can't get any more of these.
189.		Researcher 1	Ok.
190.		S	Now that you can't get anymore. Then, you have, the two. And the two you have this one and this one
191.		Researcher 1	And how did you convince me of that exactly

192.		S	I convinced you that there exactly, I convinced you for the twos.
193.		Researcher 1	Right
194.	34:22	S	Now, these and these are opposites so you know you can't get any more of this kind.
195.		Researcher 1	Yeh but that's not how you convinced me of more of these oh I see you want me to accept that you already convinced me. okay why don't you go through that again so you know that Mrs. O'Brien just came in and she made one here, how you convinced me that there are no more than the exactly two whites.
196.		S	I convinced you because when I took the, I took it I would go and I would go and I compared it for this part of it I compared it. And

			I can make it this way this way or this way or this way.
197.		Researcher 1	And how about this part of it.
198.		S	This part of it. Well you know you can't make any more because when you have this down the bottom wait hold on, yeah this down the bottom, (mutters) if you have this down the bottom. if you move the two down the bottom then you need another one in the middle. Ok. and you find the opposite of this one, so you know you can't make any more of that kind. Yeah have the opposite of this one and the opposite of this one,
199.		Researcher 1	And more Mrs. O’Brien again, could you tell her why the opposites were exactly two whites you would find the opposites for exactly two blacks for that group?

200.		S	This group? Because of what happens is well first of all there's no groups left. And if you um go and you turn this upside down, you have your pair which means if you turn it over again, you have the opposites
201.		Researcher 1	But you went from the exactly one white to find an opposite to find three whites, why didn't you go to a different group?
202.		S	Well…
203.		Researcher 1	Because you started here and you realized there were no groups left. Why didn't you if I said we didn't know this let's say there was someone doing this for the first time, and they had looked for the opposites of the exactly two whites, they might not go to this same group they might look for a different group. Why does it make sense if it does to find the opposite in this group?

204.		S	So for the opposite of this and this group in the same group. Because it just cause um cause you're using the same amount of blocks… lets say, ok, here you are using two blocks in the middle and two blocks separated. Here you are using two blocks in the middle, two blocks separated. And like here even if you didn't notice it until the end you're using three blocks and one block and down here you're using three blocks and one block.
205.		Researcher 1	Ok and now you said sixteen and you convinced me there are sixteen. How do you think… where do you think the twenty come from? Do you think there's sixteen or do you think they're twenty?
206.	37:37	S	I think there's sixteen. Cause I've went over it and we checked and we made sure there’s no duplicates…

207.		Researcher 1	We did that with twenties and See I made these and they have all their opposites.
208.		S	Well I think when I did the twenty I must've not checked for the um extras.
209.		Researcher 1	Could we have done this problem, instead of looking at exactly no whites and one white and two white and three white and four white, could we have done it by looking at exactly no blacks… and one black, and two blacks...
210.		S	Mmhmm that would just be doing the opposite way! You could do that!
211.		Researcher 1	Does it work?
212.		S	Yep. you don't have to use the two whites together, you could go two blacks, but it’s gonna be the same thing as using two whites.

213.		Researcher 1	How would… sort of… could you… sort of talk about it… how you would explain to someone. Suppose you said to someone, Mrs. Barnes said to Stephen, “I want you to do the same things Stephanie did, but I don't want you to go through the argument with exactly no whites, one white, two whites, three whites, but I want you to do it with blacks.” What would you advise Stephen to do?
214.		S	Do it the same way that I did it with whites because it's going to turn out the same way.
215.		Researcher 1	So I could how would you start if I said exactly no blacks? What would you expect that Stephen would show you?
216.		S	Exactly no blacks? He’d show me this.

217.		Researcher 1	I see, I see… and exactly one black which group would he go to do you think?
218.		S	He would show me one black, he would show me this.
219.		Researcher 1	Which was what group for you?
220.		S	This was three group.
221.		Researcher 1	Exactly three whites… what about exactly two blacks? Which group would he be doing?
222.		S	He would go to this group. This was my white group; my two white at the bottom.
223.		Researcher 1	So was it the same group?
224.		S	Actually he was just doing the opposite as me. I would put it in the same group.
225.		Researcher 1	Ok that's very, very interesting. if you had to do another problem, you had two strategies, you had an opposite strategy that

			gave you twenty, you would expect that you missed something. You were really convinced of the sixteen. Cause you've really gone through all those cases. Are there any other possible cases? One white, two white, three white, four white?
226.		S	You can use the black system, but other than that no.
227.		Researcher 1	That's a way of doing it but the opposite strategy as me but its that a good way of doing it. But the opposite strategy is that a good way of doing it?
228.		S	The opposite strategy can work, but I think it's better to go back and make the opposites.
229.		Researcher 1	Oh so it's a good thing to double check by finding that you have opposites. Once you've done this you go back… okay, that's very interesting… okay that's very

			interesting. So you have sixteen… why don't we put these all here for a minute the way we had them in the groups… I'm gonna move these here so we can get a picture of them. Now we had exactly no whites, how was this? …and where's the exactly one white that was this, exactly two whites here, exactly three whites here and exactly no blacks?
230.		S	Exactly no blacks.
231.	41:14	Researcher 1	Okay, You really think you feel comfortable with this argument? …and if I asked you -- which I may ask you to do because you've done it so nicely here even though you're tired -- I might ask you to share this with your class… would you be uncomfortable doing that?
232.		S	No.

233.		Researcher 1	And explaining it? So I… What I'm interested in now… you feel comfortable sharing that. I might ask you to come to my graduate class…. I might ask you to do that Stephanie. Do you think this argument would work in the same way, if you had to build towers of five now? I don't want you to do it I want you to talk about what kinds of cases you would have… here you have case one: no whites, case two: exactly one white. Case three: which you did three-a, two whites together… two whites separated by at least one black then you did case four: three whites, might have a separation.. In a sense these cases is a very powerful method of proof… do you know in very advanced math people prove things by cases. And it's a good proof if you consider all cases.

234.		S	With this you could probably make an estimate of how many towers you make get with 5
235.		Researcher 1	How would that work?
236.		S	Well okay You know you've done this and you've checked it and you've found that you've got four, with groups of two you've got 3. With groups of well 6. And with groups of um, but when you started out you got 3.
237.		Researcher 1	That's interesting…
238.		S	And that's how it would work
239.		Researcher 1	i just noticed something , I just noticed something you've got one 1 4 6 4 1. Isn't that interesting.
240.		S	Hm interesting it turns out to kind of like pattern
241.		Researcher 1	A different kind of pattern huh

242.		S	When you started out you had had one wait you had uh four, you had um three, had two and you had one. And with that you could go and you could say well I’m gonna estimate if this is how many you have but four with five I might go and say you can go and you can say you had one with the four. With five you had five. Well you'll probably get five. That's what you get with the six that's the same thing. And with the five here I might get one more than this.
243.		Researcher 1	Why do you think one more?
244.		S	One more, because here you're gonna get one more… you're gonna get one more because when we did the pattern with six you got one more than when you did the pattern with five

245.		Researcher 1	What do you think you'd get with exactly one white what do you think you'd get with exactly three white?
246.		S	Not counting the opposites. Probably one white… probably another one.
247.		Researcher 1	Probably? You're guessing one more.
248.		S	Think this would be five, four, three, two. No… five, four, three, um, one… instead of four, three, two, one.
249.		Researcher 1	That would be your guess before you start.
250.		S	Yea that would be my guess.
251.		Researcher 1	That's interesting… now let's go back here you think in this group when you're building towers of five you would get one more when you were building towers of 5. [talks to someone else on the side… third person comes in and says "I'm not

			sure I’m not sure what's in common here.”)
252.		S	Oh wow I must've separated the cases wrong. Because these two are glued together -- the whites are glued together.
253.	45:52	Researcher 1	You want this one in there too.
254.		S	No that one goes with that. Oh this one doesn't belong .
255.		Researcher 1	Those are interesting guesses and let’s try not to lose those guesses. But before we do that i'd like us to try something different. I'd like us to think about something maybe that's not more. But I'd like to think about towers of 3.
256.		S	Towers of three, okay…
257.		Researcher 1	And i'd like to think of it if you were using this strategy of towers of three would it work?

258.		S	Yeah you might you'd probably get something like this you probably get something with 3.
259.		Researcher 1	Ok these you'd probably, towers of 3
260.		S	You'd probably get something like three two one because you'd get less than tower 4 you know that so you get probably you're gonna get three here. And then on the bottom when you go for towers of three You're gonna get one here. Which means you're probably gonna get two in the middle.
261.		Researcher 1	So how many total
262.		S	3 two one um 6. And then when you do your Opposites
263.		Researcher 1	But Wait a minute. We didn't get extras by opposites before.
264.		S	Yeah that's right
265.		Researcher 1	we got extras by cases

266.		S	Just without the opposites you'd get 6.
267.		Researcher 1	You think 6. okay why don't you show me.
268.		S	Okay
269.		Researcher 1	Let's not lose these. Why don't we move these over here. aNd why don't you Show me What would happen with towers of 3.
270.		S	Ok
271.		Researcher 1	I'd like you to set up a Parallel argument if you don't mind. thats why I'm thinking now.
272.		S	Ok We're gonna start with one w
273.		Researcher 1	no You're gonna start with no w.
274.		S	Oh no w. okay so that's three b. Then
275.		Researcher 1	Any other ways of doing this. (laughs) no.
276.		S	No. (laughs) then, oh ok. I'm doing the wrong pattern…. (short

			pause) I have to get the, I have to get it through my head.
277.		Researcher 1	How many will there be, don't make them, tell me before you make them.
278.		S	3
279.		Researcher 1	You can make them now. Just wanted to help you…
280.		S	Okay (pause) right there
281.		Researcher 1	You thought 6 do you still think 6.
282.		S	No I think 7. I think it's gonna..
283.		Researcher 1	Why do you think 7?
284.		S	Because I didn't include the no w I forgot to include the no w
285.		Researcher 1	You said in class one time there cant be an odd number. I-i- That would mess up the whole opposite thinking. Still think 7
286.		S	Yeah could be 7

287.		Researcher 1	So you think there doesn't always have to have an opposite
288.			Uh yeah You don't always have to have an opposite
289.		S	So we did exactly one w now What were we doing after exactly one w
290.		Researcher 1	Exactly two w 49:21
291.		S	Exactly two w,
292.		Researcher 1	i would go.
293.		S	Glued together ones first
294.		Researcher 1	Then you have like that and there was my estimate of two
295.		S	Glued together
296.		Researcher 1	and then you have three so far you have one two three 4 5 6 7. There's 7
297.		S	But do the w always have to be, you do you do

298.		Researcher 1	Yeah you don't have to have the w glued together you can go like this
299.		S	Is there any other way of not having the w glued together
300.		Researcher 1	M m
301.		S	So how many did you find
302.		Researcher 1	3… okay three and three is 6. 8
303.		S	Oh. What do you think
304.		Researcher 1	I think well... um I think. Well- I think that. That's all you could make with that. That is might... I think that hm... I think that I found the opposite. I think that it's going to turn out the same way that it turned out here without the opposites.
305.		S	What's that.
306.		Researcher 1	Well you know how we found the opposites from here to here

307.		S	Yeh from the it's exactly one w
308.		Researcher 1	Oh exactly one w
309.		S	The opposites turns out to be I wonder why that is you think that's accident
310.		Researcher 1	I don't know
311.		S	It's very interesting
312.		Researcher 1	Yeah. And then these two opposites so there you go from sixteen to 8.
313.		S	No you went from 6 to
314.		Researcher 1	To 7 to 8 yeah
315.		S	What made you (inaudible)
316.		Researcher 1	(inaudible)
317.		S	What made you change your mind from 6 to 7 to 8 gain
318.		Researcher 1	I forgot to count from w

319.		S	What made you change your mind from the 7 to the 8
320.		Researcher 1	I saw that I forgot about this one the one with the two separate
321.		S	Ok Lets put these back from the way you found them
322.		Researcher 1	Ok um well they all go down here so it doesn't matter
323.		S	No because if it's exactly one w.
324.		Researcher 1	No
325.		S	Oh exactly no w
326.		Researcher 1	Make sure you have that so mrs austin and msr o'brien doesn't say there's not more and they can look and be convinced
327.		S	There is exactly one b here
328.		Researcher 1	You sure you have them all
329.		S	M hm
330.		Researcher 1	And here you are sure you have them all.

331.		S	Mhm
332.		Researcher 1	And you did them together and you said you did them separated as we chose to do okay now exactly 3. Oh what the heck let's do exactly... towers of 2. I'm gonna drive mrs demi crazy here with the camera but we'll do that anyway. Exactly towers of 2. Same same same way of argument. Exactly no proofs. Exactly now. How many? would you have guessed they would all be 4 without doing it?
333.		S	I would have guessed that not exactly 4 but probably around it.
334.		Researcher 1	Ok
335.		S	Bc you know it has to be less than 8.
336.		Researcher 1	Towers of one don't make them, guess
337.		S	Towers of 1, one,

338.		Researcher 1	You think there's one way of doing towers of one okay lets do towers of one exactly no w You think there's one way to do towers of 1. okay what do you think
339.		S	There's ok. theres 2.
340.		Researcher 1	Ok let's see then we have exactly no w. Towers of 1. Would be a good idea to write some of this down. okay so we have towers of 1234
341.		S	1 2
342.		Researcher 1	Leave some room cause we're gonna go to 5.
343.		S	34
344.		Researcher 1	Its alright you found how many total?
345.		S	2
346.		Researcher 1	And here?
347.		S	4
348.		Researcher 1	ok

349.		S	8
350.		Researcher 1	Nd here
351.		S	16
352.		Researcher 1	Interesting
353.		S	That's weird look two times two is 4 and 4 times two is 8 and 8 times two is 16.
354.		Researcher 1	Oh so what
355.		S	It goes like in pattern. two times two is 4. Nd 4 times 4 is 8 and 8 times 8 is 16.
356.		Researcher 1	Oh I wonder why
357.		S	Well it also turns out that every number is even.
358.		Researcher 1	Well if this is pattern what would you guess would be rowers of 5
359.			If I had to guess? 32.
360.		S	You would guess 32

361.		Researcher 1	I would guess 32
362.		S	Gee, if this works you'd be able to do this pretty fast. Is there a reason why that would work
363.		Researcher 1	I remember when we did the towers of class in class.
364.		S	56:00 (inaudible) I wonder why this works. I wonder why this works. Let's look at what we have here. Let's kind of think about what happens when we build towers of one and towers of two. Why do we have twice as many.
365.		Researcher 1	Because we doubled it
366.		S	What would mrs austin will suggest let's pretend these are the towers of one now suppose I start with this and now build towers of 2. Why would I expect 4 rathian 1. why? Now I'm building starting with my tower of one which is w on

			the bottom floor. Why might it be now right? suppose I start with w on the bottom floor
367.		Researcher 1	With1
368.		S	With 2
369.		Researcher 1	We know there's only one I could build with w on the bottom floor
370.		S	W w on the bottom floor I could do this right
371.		Researcher 1	what kind of towers can I build with w on the bottom floor. Right
372.		S	You can build 2. You can build the solid w. With w on the bottom floor you could build this and you could build this
373.		Researcher 1	I could either put w on the bottom floor or b on the bottom floor. now if we doesn't have to be on the bottom floor

374.		S	It could be on the top floor. It could be on the top floor.
375.		Researcher 1	Right and you told me you could put w on the bottom floor and b on the top of it or another w on top of it but I have to only have b on the bottom floor
376.		S	No you could build it and There's 2
377.		Researcher 1	How could ibuild w on the bottom floor
378.		S	I could put w on the top floor
379.		Researcher 1	And this way
380.		S	and that's 4. well that helps me see how I could go from towers two to towers 4. What if I'm going from the top floor we h have twice as many how could we go twice as many if we start with these towers? we have these four now

381.		Researcher 1	Ok now we;re going from here to here. ok. With w on the bottom then you have this one and you have 1, two ,3 4,. And these all have w on the bottom. And with black on the bottom you have 4.
382.		S	How do I know. Paying attention. Here I have.
383.		Researcher 1	You know what I saying these two w with on the bottom floor you have 2. Now the total here is 4. And with w on the bottom floor. You get 4.
384.		S	Yeah but I could have said in these here you could either have w on the bottom floor or b or w on the top floor or black. It made sense to think of w on the top floor or b on the bottom floor because started with these two I think.
385.		Researcher 1	Yeah.

386.		S	do these problems remind you of any other kinds of problems you've ever seen
387.		Researcher 1	Um these kinds if problems. hm. well only the one that we did when last time when you came with the um 5.
388.		S	The 5
389.		Researcher 1	Um. usually the problems remind me of the shirt and shorts
390.		S	Right
391.		Researcher 1	Usually they remind me but this one doesn't not really usually this one reminds me of the shirts and shorts but this one doesn't.
392.		S	What way does it remind me of shirts and shorts
393.		Researcher 1	Well usually you have to pair up
394.		S	Can't imagine if these could be shirts and pants of some sort an outfit or combination.

395.		Researcher 1	You might me able to it depends on if you might make it into a problem. You could say Jim has a pair of w and b pants or b and w top. how many outfits could four 4 he could make with the 4
396.		S	How would you make 4.
397.		Researcher 1	W with b and b with the w the w.
398.		S	Is there any way you could show me with a picture.
399.		Researcher 1	Mhm.
400.		S	W with the b b with the w. B w the b w with the w.
401.		Researcher 1	Ok in a sense this is like the shirt with the pants. Could you imagine this with shirt and pants.
402.		S	Yeah you could go jim has a b pair of pants and a w top. How many outfits can he make. Um
403.		Researcher 1	Well if he doesn't have shirts and pants here he just has.

404.		S	You could say well, rr you could go like this. Jim has a b pair of pants a w pair of pants, and oh wow it doesn't matter. And a w shirt and then it would make two but
405.		Researcher 1	Yeah here he has two articles of clothing right a shirt and a pants here he only has one article of clothing
406.		S	Yeah It's a little hard to see Yeah it's a little harder to work with two pieces of clothing
407.		Researcher 1	He just has the pants here
408.		S	Yeah
409.		Researcher 1	Maybe it's summer he walks around without a shirt well look it seems like. This really does seem like a outfit problem of 2. Is there any way we could think of the outfits problem with another piece of clothing let's go back to this. Two

			shirts two pants. Now suppose we had two hats. B hat and a w hat.
410.		S	Ok
411.		Researcher 1	Maybe I should have asked you to do it all on another piece of paper.
412.		S	We already did this we have two hats two shirts two pants.
413.		Researcher 1	Alright
414.		S	B hat w hat. Learn How many combinations do we have. one 04 23 (long pause)
415.		Researcher 1	Alright so you have w white black b then you could go b b b , w w w, b w b, that's three b w w, thats four, b b w, thats five, w w b, thats six and then you could go do that (w b b) do that. alright makes six.
416.		S	Want to write out the letters you've lost me kind of show me as you're doing it

417.		Researcher 1	W hat w pants. Oh perfect I put w blue.
418.		S	Remember you already have these outfits here. So you think when you have the w outfits here
419.		Researcher 1	You get two more outfits
420.		S	B pants black shirt. Black pants black hat b h w s. B pants w shirt w hat. B pants w shirt b hat. Wait I did that one. Then you could move on to the what w shiurt b pants. W shirt b hat b pants. No that's the same made that. B pants. W pants b shirt. W hat. W pants b shirt b hat. And that one the same one that one must be my favorite I keep making it up. Am I sure I did this? I didn't do that one already? No. hm.
421.		Researcher 1	And you're not satisfied with that? What bothers you about it

422.		S	Here I got six but here I got 7 and I'm not sure I think I made one over but I didn't.
423.		Researcher 1	Is it possible to have 7?
424.		S	No.
425.		Researcher 1	Why
426.		S	Because you'd be walking around with a hat and a shirt or a hat and a pants or pants and a shirt.
427.		Researcher 1	Ok so what do you think
428.		S	I think I messed up someplace.
429.		Researcher 1	Let's think here, here are your outfits.
430.			Ok
431.		S	You made them and you showed me. Here there were four. You were sure of that. And it's like the towers. You believed it. Now here I am. now Imagine you've

			come home with two really smashing hats. A black one and a w 1. What could you do with these four outfits when you have these two hats.
432.		Researcher 1	What can I do with these outfits when you only have four hats
433.		S	Cause now you have three articles of clothing
434.		Researcher 1	Combinations oh I think you'd still have 6.
435.		S	Ok your b hat.
436.		Researcher 1	My b hat could go here. Could go here. Thats 8. no thats 4.
437.		S	Ok keep thinking
438.		Researcher 1	Ok then k.
439.		S	You've got 4 more with your b hat. But You came home with two hats.
440.		Researcher 1	That’s it you'd have w hat w hat w hat w hat w hat.
441.		S	Well now its not 7

442.		Researcher 1	Its 8
443.		S	Here's some towers and now you're making outfits with three articles of clothing. is there any similarity to this?
444.		Researcher 1	Is there any similarity from these and these?
445.		S	Now you've told me now with three articles of clothing. b shirt w pants. You could now make 8 outfits. And you're absolutely convinced. Cause you started with the four combinations. And you told me you could put a w hat on top and a b hat on top. Now if I asked you to make them to produce the 8 I bet you could make them really easily but you could kind of imagine what they look like.
446.		Researcher 1	They would look like this.
447.		S	Right.

448.		Researcher 1	And they would look like this one. And they would look like this one.
449.		S	But wait a minute what color hat are you using
450.		Researcher 1	Oh we're using b hat. This one and this one. This one. And …
451.		S	W on the bottom floor. B on the middle.
452.		Researcher 1	Yeah this one.
453.		S	You could also give me the other outfits
454.		Researcher 1	This one. This one. This one does go here. This one. This one. Where'd my last one go.
455.		S	2 w on the bottom.
456.		Researcher 1	That one must go there
457.		S	First two floors. Now isn't that interesting
458.		Researcher 1	Yeah.

459.		S	Now if I gave you two feathers a b feather and a w feather.
460.		Researcher 1	Ok. then. we go to the fourth blocks
461.		S	Why
462.		Researcher 1	Because we're adding another piece of clothing.
463.		S	lets we talk about it.
464.		Researcher 1	Oops b feathers on the very top
465.		S	You don't have to do it you could just tell me what you would do.
466.		Researcher 1	Ok well. Ok.
467.		S	now we have all of these. The w hats on the top of all those outfits. Right. Ok.
468.		Researcher 1	Now we have b hat on top.
469.		S	On top of what where would you be putting the b hats on top of.

470.		Researcher 1	B hats on top of these?
471.		S	Mhm
472.		Researcher 1	So all of these would now have a b feather on top thats what youre telling me
473.		S	Not the right order. I was wondering why I couldn't find the rest.
474.		Researcher 1	I was wondering why I could find it.
475.		S	Started to worry right
476.		Researcher 1	yeh.
477.		S	You lose it?
478.		Researcher 1	I’m not sure? Oh.
479.		S	Ok so how many did we get when you put the b on these very outfits.
480.		Researcher 1	You have 8

481.		S	Now we put w on the very top
482.		Researcher 1	Thats.
483.		S	Ok you have a w on the very top
484.		Researcher 1	So.
485.		S	Thats 16.
486.		Researcher 1	Does that make sense
487.		S	Wait. We have 9.
488.		Researcher 1	Ok. These are the ones with the w on the very top. Lets flatten them out because (inaudible).
489.		S	They all they're not the same because does that make you feel better. I was getting worried there too stephnie. Now I don't want you to do this. I know you're getting tired and I want you to go back. These are your shirts pants hats feathers. And now we're gonna have

			a flower. B and w flower to go on top or some kind of design (laughs) Without doing it tell me how many combinations do you get.
490.		Researcher 1	I would say okay if you get ..probably.. 20.
491.		S	Now you have 16, sixteen outfits. Now you're going to put on top of this either a b feather or w flower. Or decoration.
492.		Researcher 1	Instead of making the towers.
493.		S	I'm not even asking you to make them suppose you put a b.
494.		Researcher 1	Ontop .
495.		S	Ok
496.		Researcher 1	just tell me
497.		S	Oh yeah you could
498.		Researcher 1	What's gonna happen

499.		S	Ok youre gonna get b on these
500.		Researcher 1	Yeah ok.
501.			Do you have b on .
502.		S	Ok so You're going to get 32. So it is sort of like the shirts and shorts problem. You could figure out. That's what we got when we multiplied and when we did the problem the last time.
503.		Researcher 1	What did you have
504.		S	The method
505.		Researcher 1	What's the method
506.		S	All you have to do is take the last number you had and multiply by 2.
507.		Researcher 1	You're convinced it's always going to work? What I would like you to do is you've done extraordinary work. Nd you have

			really today You've come up with another method. You have one method you made from last time. Now so if I were to say to you if you were building towers of 6 what would you say to me.
508.		S	Ok towers of 6 and towers of 5. 32. Towers of 6 that's um. 64
509.		Researcher 1	64 right. And You also have a method of finding also 64 what was it
510.		S	Multiply the last one by 2.
511.		Researcher 1	That was the answer but how would you show me What was your strategy.
512.		S	Shirt and shorts
513.		Researcher 1	That's another way but How would remember when you did towers of 4
514.		S	Um You would go one by 1.
515.		Researcher 1	Well you said there were first first case. First cases

516.		S	I think there were 4 in the first case. four
517.		Researcher 1	One in the first case
518.		S	Youre talking about the b 1.
519.		Researcher 1	You used all those cases one w two w up to how many?
520.		S	With towers of 6 you could go up to 5 4 6 to 64
521.		Researcher 1	You'd go to 6
522.		S	Exactly 6 w. What id want you to do and You've been wonderful and you've had so little sleep. I just wish you could write me sort of a story about if you were gonna be the teacher or you were gonna explain to a friend. Hey look if were doing these problems Amy might come in again. She may say okay everybody were going to be doing Towers of 6. You'd have all this information and it sort of isn't

			fair. I'd love if you could do that for me and we'd have to plan some time so you could share it. You just were wonderful. I'd like you to talk to michelle Ask her say to her tell me michelle how you did this. I'd like you to come in and tell me how many towers you could build with exactly 6 high that are exactly two colors. How many towers could you build that are exactly 10 high. With exactly two colors. You think you'd know how to do that.
523.		Researcher 1	Yeah
524.		S	Yeah I think you could too.
525.		Researcher 1	All you'd have to do then is we know we have towers of 6 times it 6 7 8 9 actually 6 7 8 9 10 that's four. By four not by four by 8 because what happens is you multiply by 2.

526.		S	What would multiply by two what would multiplying by two give you
527.		Researcher 1	It would give you. Or you could just go like this 64 times two equals um. Then you could just go that would be the 7 towers. sixteen one 4 5 two eight that's the eight towers. 9 towers. 8 towers. Thats 12 one 10 eleven one four five five five 12 is the 9 towers eighteen nineteen. That's the ten towers. one thousand. Wait huh.
528.		S	Check this
529.		Researcher 1	Thats 4 that's two that's 10 one goes on top
530.		S	You wrote 9 I don't know why
531.		Researcher 1	Yeh but thats 10.
532.		S	Oh thats why, okay so there is 1024

533.		Researcher 1	Would that be same as you were gonna do multiplying by 8
534.		S	No that's 51 512. Gave me 8 towers.
535.		Researcher 1	Wonder why it didn't work multiplying towers by 8. Wonder why that was. The 9 towers . why do you think so
536.		S	I think I it was actually easier going like this trying to figure out
537.		Researcher 1	I think you had a good idea
538.		S	You had to be sure
539.		Researcher 1	What i'd like you to think about I think the idea was very good. You multiplied it by 8. But notice what you did here. You multiplied it by 8. You didn't multiply it by 8. You multiplied it two times two times two times 2.
540.		S	Thats 12. 16.

541.		Researcher 1	So if you had multiplied by sixteen it would've worked.
542.		S	You gotta think about it why the idea didn't work but almost work. Its nice to have shortcuts if you're not sure they're gonna work then wow. Did you think you learned anything today
543.		Researcher 1	Methods
544.		S	People would love to have those methods. Well again I say thank you I would like to come back and have your essay.
545.		Researcher 1	Ok
546.		S	I would love to see if you go back to those towers of 6 and you could prove all the cases there should be how many
547.		Researcher 1	64

548.		S	And convince your classmates that you have not left ny out because left ny out duplicates if you catch them you're okay but if you don't catch them. That's wonderful.
549.			One at a time so that means here's the we have the blue blue okay we have the blue orange orange orange orange orange, orange blue orange orange orange, orange orange blue orange orange, orange orange orange blue orange, orange orange orange orange blue.
550.			If you go any farther you're going to have to have another block
551.			This is the two at a time blue blue orange orange orange, orange orange blue blue orange, and so forth, but what happens is you go blue blue and crossover one.
552.			You gave these a name?

553.			Double opposite. Two at a time.
554.			Lets keep a record.
555.			One at a time.
556.			And you have six choices.
557.			(inaudible)
558.			And then what did you do.
559.			Then we have three at a time.
560.			Are these the only two at a time you have?
561.			Yeah.
562.			What did you call them
563.			Together
564.			And now these are three at a time?
565.			Together, yeah
566.			They can never be separated like they’re glued or something
567.			You're convinced there are no more
568.			Yeah not like, on the three together on the two together.

569.			When they are together these are the only ones how can you
570.			Prove it? You use all the six and you're at your last block you can't go down here blue blue, because you're missing a block.
571.			Okay I see that.And here, same argument?
572.			Three four and then four and four.
573.			Now when you see four and four I’m supposed to imagine in my head that it could have been three orange here. Well what would the opposite of this be?
574.			The opposite of this one would be orange orange orange.
575.			How many do you have so far? Total so far?
576.			So far you have thirty?
577.			These thirty are all different?

578.			Yeah because the section is all different? Even when you switch it around? Why did you do that?
579.			(inaudible)
580.			Okay
581.			You double them, the opposite of this one would be orange orange orange orange orange orange blue blue.
582.			You're counting this as one of your doubles?
583.			Three and one.
584.			I wonder why that’s happening. When you switch these around, you tell me thats what they get and that happens to be here.
585.			THat one? Hm.. No I dont think anyones going to duplicate that one because look so far we've been doing half of these are separated
586.			Why do you supposed this duplicate occurred?

587.			Here they were two at a time
588.			How can they be the same if these were two at a time and those were four at a time?
589.			That two at a time is also four at a time. Because look, one two three four theree’s for oranges together.
590.			Oh so when you think of the two blue at a time you're thinking of the four orangeat a time. Did you check for that? Duplicated?
591.			What about this one?
592.			Yes I think it might have a duplicate. Orage orange orange. And there's nothing left, if you turn it upside down.
593.			Why Do you suppose this one didn't come up?
594.			Because it's the only one that this one isn't stuck together.
595.			So how many new ones did you find this one.

596.			And then it's just give at a time which is I just thought of something. This is one at a time, let's find one, oh that's neat.
597.			If you turn this one around, which is um blue blue blue blue, blue then the same one (inaudible)
598.			So when you do one at a time you automatically do five at a time
599.			Yeah
600.			Um I don't think, no I don't think so because we have the two, we might have and the one with the two already been separated by the two. Yeah that one. That one inaudible) okay second one is the blue, orange orange orange orange. Oh I started wirth the four in between,
601.			Okay well in all of these I noticed blue is in the top floor, in any case blue is in the top floor?

602.			Yes
603.			Interesting
604.			You could've done it the other way.
605.			Okay let's get all these papers.
606.			Okay first we started with blue blue blue orange orange orange orange blue. Instead of that.
607.			And what would the third one look like?
608.			Third one would look like this.
609.			Okay instead of having the blue moving up to the top you move it
610.			Move from top to bottom. Okay let me try to understand this better. When you moved this how many more did you find?
611.			Four.
612.			And all these have exactly two blues. Tell me about these

613.			Alright, they have two blues, and they're separated.
614.			Can you tell em about heyre separation?
615.			They’re separated by the orange.
616.			If you were telling me over the telephone I wonder what I would write down.
617.			Well they’re separated from the bottom one up. You start from the bottom to the top and the bottom would move up one.
618.			Then two, then three then four.
619.			Yeah
620.			Why wouldn't you have them move up five.
621.			Because then we would have that case
622.			Where it's not separated
623.			So we start moving up from the bottom separated by four three

			or two. So you made four of these . How many more do we have now.
624.			Okay we made four. Then you turn it around and we have 8.
625.			Why don't you write that down here.
626.			Then we can make 8 more by doing it from the bottom up.
627.			Are you sure there are any duplicated, you have to remember not only these but how can we be sure of that?
628.			This one is one down right, oh wait. Oh yeah.
629.			You want to fix that up? Let me sure I understand this.
630.			I was looking up here and I’m looking at ten but there's for there and why am I getting three but it's right,
631.			So youre saying you're getting three here and these are three that you haven't used before?

632.			Well let's see we know we're using this one already. But we haven't done the two separated already. So we know, two together, three together , four together one,
633.			Well how do you know these are the same ones separated.
634.			Does it come out
635.			You sure?
636.			You don't get any duplicates.
637.			So how many more edo we have then.
638.			12
639.			So far?
640.			Probably could do the three apart.
641.			We're pulling them apart?
642.			Two apart, three apart, top moving down.
643.			Okay, top blue moving down. As I look at this I see rows. You're always separating the blues

			in here but one of the blue doesn't move. And here is that happening?
644.			Bottom one does not move. That's the difference
645.			Between them, why don't we write that down the bottom doesn't move. What Else do you think is possible?
646.			Three aparts.
647.			What about with two blues?
648.			I don't think you could do it any more than that. We already did two together. And all the two aparts.
649.			How do you know you did all the two aparts. I see blues on bottom and top row. Is that the only thing?
650.			Well look you only have 6 spaces.
651.			You had the blues on the top and the bottom
652.			Yeah sure theres other fours here here here but we've used those.

653.			Why couldn't this blue be here?
654.			Because they might be together.
655.			Why couldn't this blue be here?
656.			Because he’s already been here.
657.			Why can't I move all my bottom floor blues to the next bottom floor.
658.			Because it will be next to this blue.
659.			Here?
660.			It’s already been, then okay because if you move this one here.
661.			They're not glued together. What about here?
662.			You might be able to do that.
663.			24:40 ish
664.			You cant you have different floor so of all blue
665.			You could do that probably.

666.			Top floor
667.			Depends on the pattern right here and the blue here and other blue here.
668.			What I worry about is maybe there are other patterns that we arent finding.
669.			I figured out that you can always do one pattern. But you can't but you're always going to come up with.
670.			What about my pattern where we change the floors.
671.			Okay.
672.			What floor are you making now.
673.			I was going to say if you bring them up to here, oh no. I was saying we could do that but it was already done.
674.			What if we move this floor to, the bottom to the middle
675.			Well, the pictures help.

676.			Do you want to go through and change the floor and see what other you can find? I’m getting confused
677.			Me too. You're bound to come across someone who says I don't believe you. I know you can come up with by going orange orange orange orange. But after you get out of the three and four and five, checked all that, you can convince, but when you start separating them.
678.			3144
679.			Suppose you had an argument, developed and argument, what if you could think of a way to keep track easily so when you go back you can separate. We’re talking about six right? Now if we start to separate them. Think of all ways we can separate them when they’re not together.

680.			We get to separating like this.
681.			Okay this would be when in a sense you have the white on top and you've gone down one. You just jump down one. You want to keep white on top for all these?
682.			Yeah
683.			Okay so now if you have go down one what can we do next?
684.			Go down ten
685.			Lets try that. Whites on top, I want to go down two. White on top and you went down two.
686.			Okay let me see how do you survive this these towers. All of them huh. Right and what else. To separate it. Only have two write sand separate it.
687.			This one is separated white on top separated by five. Because you need seven high tower.

688.			Why can't you do white on top separated by none
689.			Because it would be the two pattern
690.			Are you absolutely convinced this is the only way you could do separated on top. I want to be sure.
691.			We can try it with the two. Then we can separate it by one. We can start by separating it by one.
692.			Thats white separated on the first floor by one okay.
693.			Then we have that. You need more? Why not? Same argument you used with the one right? Okay
694.			Samr argument is going to come up when can you make another one when you have a six tower and you want to make it with five. Same argument on the map.
695.			We have white next to the top floor is that all.

696.			We can make white on the third floor.
697.			Okay
698.			Start to see a pattern four three two, right here it would be one. One more pattern with this always being with this coming down.
699.			39
700.			If we build our way back up from the top from the bottom like this we'd be doing the exact same thing. We;re going to say were starting with white at the bottom.
701.			What's your bet you're really sure of this? How would you be able to convince Dr ??
702.			I'd show her we could move it down one, move it down two, move it down three.
703.			Why can't move it down four?

704.			You'd be Moving down the pattern.
705.			What about opposites what kind of world do they play in?
706.			Opposites? They'd be this…
707.			So what does that mean?
708.			That means we can make twenty.
709.			Twenty? I have another question for you. You told me this is exactly two white and they're separated by at least one and these are exactly two red we're gonna go by at least one. You know what confuses me? Suppose I would say these were exactly four white and separated by at least one. Let's look at all of these you made. Those are saying are exactly four red.
710.			Four red separated by at least one. It can be done.
711.			Two white separated by at least one red.

712.			You're using a different amount of red. But it's the same category.
713.			Why?
714.			Well it would sort of be in the same category. THey're being separated.
715.			So do I have to do fours?
716.			Yeah.
717.			Reds have to be somehow separated.
718.			Can I use the lock?
719.			I'd ,like her to know what we have so far. Let's just summarize what we have at this point what we have so far because I get mixed up.
720.			Okay first one is white red white red ….. NExt one is white red white white red white red red… Next one is white red red red white.
721.			So it's this set.
722.			Okay now we have red, white red white red red. Red red

			white red. Red red white red red red white.red red white red white red. Red red white red red white. Last one is red red white red white red. Two separated. Two of white separated by at least one red.
723.			What's the total?
724.			Ten
725.			How many more did we find?
726.			48, 58.
727.			I don't know if I believe some of these/
728.			Yeah
729.			Start with what we really believe.
730.			We believe the one at a time. Total is 5
731.			Do you believe two at a time.
732.			2 together , three together
733.			2 at a time
734.			Ten, twenty…

735.			We have some more things here, exactly one, then two equals together, two separated, some threes. Okay threes go together. We have three glued together we could probably do another one.
736.			Third person: this says ten on it but I don't see it
737.			Girl: opposites are written on it. Total of twenty
738.			The ones who have a total of three, twos separated is twenty. Then threes go together.
739.			
740.			55 min approx
741.			Next girl comes
742.			You have been thinking about…..
743.			I tried with four towers high, then six, when I did four I got less than 32 and when I did 6 I got more.
744.		3:20	Steven I Think there are more

745.			Yeah I think there are
746.			Why
747.			There's no more.
748.			I think there are
749.			Well. Okay one you get down to the last one the one at the bottom you can't move the white back up because you will just be revealing these three. But if you move the white down one you'll be missing one. And if you can only use four blocks. You can only have another one.
750.			Why can't you move the white down
751.			Then if you only move the white it'll be like this. And you need another block here.
752.			Why can't you
753.			Because there's only four blocks
754.			Why can't it be moved

755.			Because the assignment said we only have these four.
756.			Should we keep a record while we're doing it?
757.			Four, should I draw them?
758.			You can, any way you want to record it you can write a statement to describe it, since you have them made already.
759.			
760.			Alright… 4:07 Here's the one moving down. Then we have two. We go like this that's one... and that's two, and that's three.
761.			Ok how do you describe this group?
762.			Well, Two moving down? Two together moving down?
763.			Okay so when you wrote moving down you mean
764.			the one
765.			Okay why don't you write that one white, how else can we say

			this, there's exactly one white. Want to say something about the black?
766.			With three black. Okay there, so this one we can go, two white moving down with two black and our total there is three.
767.			Do you want to say anything about those two? Are these the only ways you can have two white moving down?
768.			No. They're stuck.
769.			Stuck.
770.			Ok that way you cant get it mixed up with the ones moving apart. And then you have your three white
771.			Now hold on, these are the only ways you could do two white?
772.			No these is not the only ways you can do two white, but this is the only way you can do two white together.

773.			Ok. These are exactly one white and these are exactly two white. You want to jump to three white before you finish the two white?
774.			Well, I was gonna, when it did the pattern I did one one white stuck two white stuck together three white stuck together, then I went back and went two white apart. And then, here's the one….
775.			I see. I'm sorry to go back but I'm kind of thinking about steven again, suppose he says how do you know you have all of them?
776.			Because if you take the last one then you cant move it because you only have four boxes.
777.			Okay
778.			Okay and here's the first one. And… the second one. That's two... three, two together. Alright, there's your next one because you

			have these two nd you have the three up top and one on the bottom two on the bottom one up top.
779.			9:25
780.			And no more because?
781.			Because if you take this one and you want to move the white down another one you need another block put in here.
782.			Ok
783.			Now for… *drops something*
784.			That's ok. Now let me just see what you have here three together whites three glued together, two white glued together, three together.
785.			And then we have one white. And that's the last one. Now we go back to the two white stuck together and we make it a par.
786.			Ok.
787.			We can do it like this.

788.			Let's put it next to these twos.
789.			We can do it like this and that’s it. Wait let’s see. Um, you can also go like this. 12:00 That’s the reverse.
790.			Wait lets see
791.			That’s the reverse of that one
792.			In what way reversed stephanie
793.			You have the reversed
794.			But you’re not you're standing them up
795.			When I’m finished I have four here ii here six here and eight here and i'd add that
796.			Okay. So that's how you got your twenty. What was your reasoning for getting a double.
797.			Well what it is is This is the only way you can make two a par you can make two a par by taking two black and separating them

798.			So That's true so when you compare these two, how do you think about them then when you use that argument
799.			Well when you use that argument for when I double this?
800.			Mhm
801.			Well I think about is it wise well when I'm using a pattern then I'm not going to put the black apart with it. But when I'm taking this and I'm going and doubling the pattern I just turn it upside down and that's how I get my x my other pattern
802.			And you had a name for it, the last time
803.			The pattern?
804.			Yeah You had a name for that pattern you had a name for it when you found the other one that went with it the last time

805.			I forget I forget I think it was um,
806.			You use the word pair now
807.			yeah
808.			What did that look like
809.			It black on top
810.			You used that strategy you didn't do that here when you found the ten when you just found two more just now um you said these were also two whites is that correct, exactly wto whites, different from the two whites here you can tell me
811.			These are apart and these are together
812.			What's the difference in the way these are apart
813.			Um this one has the black at the top and what this is the um black at the top and this is white at the top but its alo this pattern because of you look here it's black white black white because the whites…

814.			Let me stop here, let's talk about the strategy when you do the double with the pairs when you said these are exactly two whites now again I’m going to pretend I’m stephen. You've convinced me that when the two are together there's are only three now I’m asking you to convince me when the two whites are apart that there are only three. When you have exactly two whites and exactly two black now you've convinced me. Is this the order
815.			Doesn't matter
816.			That's not how I think you had it you just had this one first
817.			White at the top then the black then the white then the black
818.			White on the top but it had to have separation and you put the next white here
819.			Then I had a white at the top
820.			Again the white at the top

821.			With black in the middle and the white
822.			Ok now these two here are they alike in any

Appendix K: Session IX: Transcript- Grade 4 Gang of Four

Stephanie engages in the Towers problem in fourth grade with classmates Jeff, Michelle, and Milin under Researcher 1.

Description: B41, The Gang of Four (Jeff and Stephanie view), Grade 4, March 10, 1993, raw footage	Authors: Elizabeth Snee Verified: Robert Sigley Date Transcribed: 4/2/2014

Line	Time	Speaker	Transcript
1		**Researcher 1**	Anyway, do you know why you're here?
2		**Jeff**	No.
3		**Michelle**	Yeah. About the towers.
4		**Researcher 1**	About the tower. Milin?
5		**Milin**	Yeah about the towers. We're going to talk about this.
6		**Researcher 1**	Yeah, yeah, yeah. You know the tower problem?
7		**Jeff**	Yes I do.
8		**Researcher 1**	Yeah, the last one in class you did, remember what that was about?
9		**Jeff**	Robin Hood. That was the last one we did.
10		**Stephanie**	The towers of 5.
11		**Stephanie & Jeff**	The towers of 5.
12		**Researcher 1**	The towers of 5. And do you remember what you did with those towers of 5?

13		**Jeff & Stephanie**	Yeah.
14		**Researcher 1**	And tell me about it. What was the problem?
15		**Jeff**	How many [Michelle: You had to figure out…] to make 5.
16		**Michelle**	…how many towers you could make.
17		**Jeff**	Different towers you can have.
18		**Michelle**	Different towers you could make from 5 blocks up.
19		**Researcher 1**	Any 5 blocks?
20		**Stephanie**	No.
21		**Michelle & Milin**	It had to be 2 colors.
22		**Researcher 1**	Okay, and did you figure that out?
23		**All students**	Yeah.
24		**Researcher 1**	And what is it, do you remember?
25		**All students**	32
26		**Researcher 1**	You sure of it?
27		**All students**	Yes.
28		**Researcher 1**	How could you be so sure?
29		**Milin**	Because we all ready checked!
30		**Researcher 1**	How could you be so sure?
31		**Jeff**	Remember we did all those, the, the charts, the thingys for, [Milin: And then remember...] and all those different patterns? Remember I convinced you up in the, the watchamacallit…
32		**Researcher 1**	Yes, in the room.

33		**Jeff**	I don't feel like convincing you again.
34		**Researcher 1**	You don't feel like convincing me again, okay. Okay, but I remember saying to you, Jeff, and I remember saying to you, Michelle, and to you, Stephanie, and Stephanie did try to work on towers of 6. And I asked all of you…

35		**Michelle**	So did I.
36		**Researcher 1**	You did it. If you were building towers of 6, how many would there be.
37		**Jeff**	I didn't do it.
38		**Michelle**	I, I did some, but I didn't…
39		**Researcher 1**	But do you know how many?
40		**Stephanie**	Yeah.
41		**Milin**	Probably 64.
42		**Researcher 1**	Why do you think 64?
43		**Milin**	Well because there was a pattern.
44		**Researcher 1**	What's that?
45		**Milin**	You just times them by 2, and then…
46		**Researcher 1**	Times what by 2?
47		**Milin**	The towers by 2 because 1 is 2 and then you figure out 2 is 2 and then I mean 4. And then…
48		**Michelle**	See, if you only had 1 block...
49		**Jeff**	You're not making much sense.
50		**Researcher 1**	See, Jeff, okay, okay.

51		**Michelle**	If you have 1 block and 2 colors then you would have 2 towers and we figured out that the other day that you keep on adding 2…
52		**Jeff**	That was the opposites.
53		**Michelle**	You, you like 2 times 2 would be 4 and then the 3…

54		**Researcher 1**	So 4 would be for what?
55		**Michelle**	Four for there, there would be 4 towers for 2 high.
56		**Researcher 1**	Okay.
57		**Jeff**	It's always opposite though.
58		**Researcher 1**	Okay, but let me, let me hear what Michelle has to say.
59		**Michelle**	And then for the 3 high then you would have 8 towers. And then for 4 high, you would have 12 towers. And we kept on doing it like that.
60		**Researcher 1**	Do you agree with that?
61		**Jeff**	I don't know what you're talking about.
62		**Stephanie**	Well, what it is, is…
63		**Researcher 1**	Let's, let's stop here.
64		**Michelle**	And then for 5 towers, it'd be 25 then. And then…
65		**Researcher 1**	Okay, let's, let's get a piece of paper and write down what you're saying and see if you all agree. And I think Jeff hasn't been with us for a while, and he doesn't know what we're talking about, but let's take one at a time. And let's just agree as we're moving along. Go, go ahead, Michelle.
66		**Michelle**	If you had one high, saying there's red and blue. Then you would have 2. And then if you had…

67		**Researcher 1**	Okay, write that down. 2. Do, do you agree with that? [Jeff: Yeah] Do you know what she's talking about?
68		**Jeff**	There's one red and there's one blue. So you, there's only one way to do it so it's 2.
69		**Researcher 1**	One way to do it so it's 2?

70		**Jeff**	Yeah, see, if, if you have to make towers of one, and there's only two colors.
71	3:04	**Milin**	He doesn't beyond(?)doing that.
72		**Michelle**	It'd be 2.
73		**Jeff**	It'd be 2. It'd be 2.
74		**Researcher 1**	All right, let's go on.
75		**Michelle**	If you add, two towers high would be 4 because you have…
76		**Jeff**	Yeah, I agree with that. Okay. I agree.
77		**Researcher 1**	Okay, but hold on.
78		**Michelle**	See, but you times that. Two times 2.
79		**Researcher 1**	All right. Write the 4 down. But I don't, can you explain to me [Milin: I know] why from 2 you get to 4. Milin, tell me why.
80		**Milin**	Because you, for each one of them you could add 1, no 2 more or on because there's a black, I mean a blue and red…
81		**Jeff**	Yeah, but what she's doing…
82		**Researcher 1**	Shh. Let her finish. Okay.
83		**Milin**	See for that, you just put one more for red you put a black on top and a red on top, I mean a blue on top instead of a black. And on blue, you put a blue on top and a red on top. You keep on doing that.

84		**Researcher 1**	Do you understand what he's talking about? [Stephanie: Umm-hmm] You all understand what he's talking about? [Jeff: Yeah] All right, so, so we agree 4. What happens if you're building towers 3 high? What did you say it would be?

85		**Milin & Michelle**	It would be 8.
86		**Researcher 1**	Write 8 down. Can, can you give me an argument, you don't have to do it, why we jump from 4 to 8?
87		**Jeff**	That's what I want to know. [Michelle, Milin start talking over each other]
88		**Researcher 1**	That's what Jeff wants to know. So, shh. Go slowly. It's Jeff you're convincing. Not me. Jeff.
89		**Michelle**	See, see, there's, there's, there's 2 blue. There's 2 here.
90		**Jeff**	I know that.
91		**Michelle**	And we went to 4 so it would have to be times 2. And then 4 times 2 would equal 8.
92		**Researcher 1**	That doesn't help Jeff understand. He just knows you're multiplying 2 [Milin: I know. I know. I know.] Okay. One at a time.
93		**Jeff**	If this was like a pattern, it would go 2-4-6 in between.
94		**Researcher 1**	Yeah, that's what he's saying.
95		**Milin**	No, no, no!
96		**Stephanie**	No, but [Researcher 1: Okay, one at a time] that's not the pattern that we're working on.
97		**Researcher 1**	Okay, go ahead, Stephanie.
98		**Stephanie**	The pattern we saw was this: for 1 block at a time we found 2.

99		**Jeff**	We all ready got 2 and 4 though.
100		**Stephanie**	I know.

101		**Milin**	2-4-6.
102		**Stephanie**	And then 8. 8, right? Two-4-then 8.
103		**Researcher 1**	But why 8? Jeff wants to know. [Milin: I know!] Go ahead, let, let Milin persuade Jeff.
104		**Milin**	If you do that, you just have to add for each one of those, you have to add one…
105		**Researcher 1**	Which one of what? These 4?
106		**Milin**	Yeah. You have to add one more color per…
107		**Researcher 1**	Which way are you adding it? Where are you putting that one more color, Milin?
108		**Milin**	Two more colors for each one. See.
109		**Researcher 1**	So this one with red on the bottom and blue on the top…
110		**Milin**	You could put another blue or another red.
111	5:21	**Researcher 1**	Do you agree with that you could put a blue or red on top and that would be…
112		**Milin**	So that will be 2 and then on this you could do, put another read or a blue on top. That would be 4.
113		**Jeff**	That’s the same right there.
114		**Researcher 1**	No, this is blue-red.
115		**Jeff**	No, her, look. Look, it’s. Okay, okay I see it.
116		**Milin**	See? Now you see it?
117		**Researcher 1**	Do you find what Milin is saying? [Jeff: Yeah] And down here you could put?

118		**Milin**	A red or a blue and the same thing with here.

119		**Researcher 1**	Do you understand that? [Jeff: Yeah] So, do you see how you get 8? Do you agree with that?
120		**Michelle**	Yeah because, ‘cause there’s 2 here and then 2 more.
121	5:51	**Jeff**	No, but, if you’re [indecipherable] bound to get a different thing. If you use two colors, you’re bound to get a different thing. Do, do another one for blue and she’ll all ready have it. She will. Do it.
122		**Milin**	Do 4. Four that would be 16.
123		**Jeff**	No. Look, she…
124		**Researcher 1**	Let’s get another piece of paper. Would you give me another piece of paper, please? Go ahead Jeff, show me what…
125		**Jeff**	She, she has 8 blocks with still only 2 colors
126		**Researcher 1**	Eight blocks with 2 colors. Okay, what, let’s see this.
127		**Jeff**	She has like this: red and she kept on alternating. Blue-red-blue-red and she blue-red until she got 8. And then, blue. And then she did the same thing up here.
128		**Michelle**	I didn’t do the same thing on top.
129		**Jeff**	Yeah, you kept on alternating.
130		**Researcher 1**	Why don’t you do that? See what happens. It’s what he thought he saw you do, but, that’s interesting, maybe you didn’t know you were doing that, Michelle. But look, blue-red-blue-red-blue-red-blue-red. He saw you alternating them on the bottom.
131		**Jeff**	Now, look, you got red and blue.
132		**Researcher 1**	So, so, you’re saying that all of these are alternating and these are opposite alternating? Look, this is blue-red-blue-

			red-blue-red-blue-red.
133		**Jeff**	You see. This is the same thing up here. Red and blue. Red and blue. See, you have to cross off this one. And now red and blue, red and blue. It's a different way. You have to cross off that one because there's another one all ready there. And then over here, you have that right there.
134		**Milin**	But, Jeff, Jeff, Jeff.
135		**Jeff**	But the thing is that's exactly…
136		**Researcher 1**	Listen to Milin.
137		**Milin**	But that, this is for 3. So you could add 2 for each one of the 3.
138		**Jeff**	You didn't say that. She was only doing…
139		**Michelle**	It was supposed to be 3 high.
140		**Milin**	Yeah, she was doing the bottom ones first. That's why.
141		**Researcher 1**	She wasn't finished. She wasn't crossing them out yet because she hadn't finished the tops. Is that right, Michelle?
142		**Michelle**	It's 3 high, not 2. See. See how you got 8? There would be 3 high, so there would be…
143		**Researcher 1**	Can you tell me a little bit about how you would get the 8 now from here? Mil…I like Milin was, was helping me a minute ago or someone was helping. I even forget who it was.
144		**Milin**	You have to keep on putting 2 for this. Two for this. Two for this. And two for this. And it'll work out.
145		**Researcher 1**	Do you agree with that, Jeff?
146		**Jeff**	Yes.

147		**Researcher 1**	Okay. So.
148		**Milin**	Because you can't. There's only two colors. You can't put anymore on them.
149		**Researcher 1**	Okay, now imagine we have our 8. Where, where do we go from 8? Because I heard, Michelle say 12.
150		**Milin**	Sixteen.
151		**Researcher 1**	Well, what is it? Stephanie?
152		**Jeff**	I, I still want to see you build 3 up and then see if there's any…
153		**Researcher 1**	Okay, Jeff isn't convinced you have the 8 here, so you were going to fast.
154		**Milin**	Using Unifix cubes you could still…
155		**Jeff**	It doesn't matter. It's just easier to draw it. [Michelle draws]
156		**Stephanie**	I know.
157		**Blonde teacher**	Milin, why don't you draw what you think?
158		**Milin**	Okay. [Draws]
159		**Jeff**	You got that and that's the same as that. Blue-blue-red. Blue-blue. [Milin: No, look, look…] No, that's red.
160		**Researcher 1**	Red-blue-red.
161		**Jeff**	Yeah.
162		**Researcher 1**	Any ideas Stephanie how to show from 4 to 8?
163		**Stephanie**	I do. All right.
164	~9:50	**Researcher 1**	How about you, Jeff? Before, Milin said you understood when he got 4.

165		**Jeff**	I understand that, but…
166		**Researcher 1**	Okay, now, if this is what he got for 4 and if you understood what he just talked about, can you write down these 4, and use his idea to see if you could build 8? Why don't you write these down what he has here. These 4. Start with these 4 because that's what Milin said to start with. These 4.
167	10:24	**Jeff**	And then…
168		**Researcher 1**	You're going to run out of space if you're going to build them up.
169		**Jeff**	I'll go down.
170		**Researcher 1**	Okay, that's fine. Does it matter?
171		**Jeff**	No.
172		**Researcher 1**	Okay, now, what was his idea?
173		**Jeff**	I put too many things of red. You've got 2 reds. Two blues and the opposite. Then you're adding onto one to make this blue. And then you have to…
174		**Researcher 1**	But hold on. He said. That's not what Milin said.
175		**Jeff**	What did he say?
176		**Researcher 1**	Let's wait 'til he's finished thinking a minute and ask him because I think that's the key to it to know what Milin said and to see if that makes sense.
177		**Milin**	The thing is that you have to keep on adding 2.
178		**Researcher 1**	Okay, Milin, let's talk about this one. He had this, right?
179		**Milin**	Okay, then you just…
180		**Researcher 1**	Now, he said from here, add a blue. Is that what you said?
181		**Milin**	Yeah, you have to keep on adding to the top.

182		**Jeff**	It doesn't matter.
183		**Researcher 1**	No, he wants to add on the bottom.
184		**Milin**	You see for this, see, you have, have to have the bottom down here because you can't put the top otherwise it'll be different.
185		**Researcher 1**	Okay. But, he put a bottom "B" here.
186		**Jeff**	It would be the same. See, if you put a "b" up there, it'd be the same just as if you put on there. It'd be blue-blue-red. And then if you crossed that off, it'd be, put a "b" up there, it'd be blue-blue-red. Just like before.
187		**Stephanie**	Yeah, but, Jeff, Jeff.
188		**Researcher 1**	Okay, but…Hold on.
189		**Michelle**	If you look at this, you have it…
190		**Researcher 1**	Time out for a minute. I'm getting very confused because all of you are talking and you have all different ideas, and I think it would help me if we got one idea on the table at a time. Now the one idea that we have on the table that I wish we would explore before we hear all the new ideas is this one here. Now, I would like all of you to consider what Milin said here. Do you all see that? Get another piece of paper and write this down. Maybe write in the middle so we could build them both ways, and see if there's a difference. You don't have to cross off what you did. Now, what you have here, in this space is on the bottom red-blue-red-blue. And on the top you have blue-red-red-blue.
191		**Milin**	See, if you put this right…

192		**Researcher 1**	But hold on a minute. Let everyone get this down. You might want to separate them, too. Might be a good idea to do it just the way Milin or Michelle wrote it. I only want to see 4 down there because we're looking at Milin's strategy. That's not what Milin did. You made a chart. Milin didn't do that. I'm interested in [Stephanie: Oh]. Okay. Milin actually drew pictures or Michelle drew these pictures. Right? Did you draw this, Michelle, or Milin? You draw pictures of what these towers are going to look like. And see that's not really quite the same. That's interesting.
193		**Jeff**	But that's what we did. That's what she did. She did…
194		**Milin**	So you could do it anyway but, see, they're just put together.
195		**Researcher 1**	Now, what I think Milin is asking us all to do is, is to imagine in front of us, can you all see in front of you the towers of 2 that are these colors? Can you all imagine that in your mind? Can you see the first one? Red on the bottom? Blue on top? Do you see that in your mind, in the middle of the table there? Can you see it? The other one blue on the bottom, red on the top. I see these 4 towers. Now Milin is calling our attention to this first tower. Right? Red on the bottom and blue on the top. And what is he asking us to do with it?
196		**Milin**	Put another blue and then make another thing exactly…
197		**Researcher 1**	Right, put another blue. Now, can you draw a picture of what that tower will looks like of 3. This is a tower of 3. He's putting another blue. Now, he chooses to put it on the top or bottom, Milin? Next question we'll ask of you.
198		**Milin**	[Draws from top B-R-B tower] See, you put a blue here or you could put a red there and this one, [draws from top R-R-B] you could put this way. You could put a red instead of the blue.

199		**Researcher 1**	Okay, Milin, can you show us in the middle here what you just did with that one tower? Thank you.
200		**Milin**	See, from this towers, right,
201		**Researcher 1**	From, from this one tower?
202		**Milin**	See, so I put the blue here, the red here on top of it so it's like this. And then I added one more. That would be red. But then I did like this: blue. Then I put the red back on top of it. Then I put a blue because there's only 2 colors, and I all ready...
203		**Researcher 1**	So what you're, what you're telling me here if I could, if I could make my picture, if I were doing what Milin asked me to do, where we had a blue and red, what he's telling me to do is he's saying from this tower, I'm going to put a blue on the top.
204	15:06	**Milin**	Or red.
205		**Researcher 1**	Or from this tower, I'm going to put a red on the top. [Milin: Yeah] Is that what you're telling me to do? So from this tower, we get these two? [Milin: Yeah] Is that what…
206		**Milin**	Yeah, and for each one you keep on doing that and for 6 you get 64.
207		**Researcher 1**	Does that make any sense? [Jeff: Yeah]
208		**Milin**	[Camera shows paper squares B-B-R in left column, RBR BR in right column] It followed a pattern to 5, why can't it follow a pattern to 6?
209		**Researcher 1**	I guess what, what I'm confused about, Jeff, is you took this one with blue and red and you only put a blue on the top. And you've only done, you've only made this one. Milin is telling you could also make this one. That you could've put…

210		**Jeff**	But I made that.
211		**Researcher 1**	Where is that? Red on the bottom, blue…
212		**Jeff**	Red.
213		**Researcher 1**	From this one, you could've put two things on the top. You only put one. From this one, you could….
214		**Jeff**	Okay, I understand.
215		**Researcher 1**	Does that make sense what he's talking about? And he's saying so from these 4…
216		**Milin**	You could make 8.
217		**Researcher 1**	So tell me now, convince Jeff why it's going to be 8 – why it's going to be double.
218		**Jeff**	Convinced?
219		**Michelle**	I all ready figured it out with this.
220		**Researcher 1**	And what's different about the way you did it, Michelle?
221		**Michelle**	I just, I just, I didn't do it the way Milin did it. I just made them out and I didn't find any that weren't the same.
222		**Researcher 1**	That's not what Milin did. He did something very different. How about you, Steph?
223		**Stephanie**	I found it like this. I drew my lines. And then I went red-red-red, blue-blue-blue, blue-red-blue, red-blue-blue, blue-blue-red, red-red-blue, red-blue-red, blue-red-red.
224		**Researcher 1**	Is yours different than the way Milin did it?
225		**Milin**	Yes.
226		**Stephanie**	Well, yes
227		**Researcher 1**	In what way?

228		**Stephanie**	He built his towers up like this. He went red, blue, red, blue, red, blue, and so on.
229		**Researcher 1**	I didn't see him do that.
230		**Stephanie**	Michelle did it like that.
231		**Researcher 1**	That's not what he did. He started with red and blue. Right? And from this red…
232		**Milin**	I put a red.
233		**Stephanie**	He put like…
234		**Researcher 1**	He put a red on top.
235		**Milin**	And a blue one that's like.
236		**Researcher 1**	You put a red on top. And a blue on top. So we've got blue, red, red, red. And from the blue…
237		**Milin**	I did the same thing.
238		**Researcher 1**	A red on top. That's how he got his red-blue. And then he put a blue on top and got blue-blue.
239		**Stephanie**	But that's like what he's like, that's what's different from mine. I just like took the things and went like, I just took one and went…
240		**Milin**	And kept on going
241		**Stephanie**	Here's one red-red-red, blue-blue-blue. And then I'd go like red-blue-blue, b-r-b.
242		**Researcher 1**	So what I'm hearing you say is that you're just you [Milin: Guessing] believe this 8, but you say guessing. Why does
			that sound like guessing?
243		**Milin**	Because what if you could make more?

244		**Stephanie**	Okay, this is the 3 high, right? And you're convinced you could make 8. I'm convinced I could make 8.
245		**Researcher 1**	Yeah, but you haven't, he's proved to me from the 4, you could only make 8. You could get two from this one, and two from this one, and two from this one, and two from this one.
246		**Milin**	But could you convince her?
247		**Stephanie**	Michelle? Him?
248		**Milin**	No, her.
249		**Stephanie**	Her?
250		**Milin**	Yeah.
251		**Stephanie**	All right. I've done this before. Okay.
252		**Researcher 1**	Take another piece of paper if you want to because it sounds like your approach is a little different.
253		**Stephanie**	Okay.
254		**Researcher 1**	You gotta convince me there are 8 and only 8. No more or fewer.
255		**Milin**	Whoa. You do draw big.
256		**Researcher 1**	Now, now Jeff, this is, this might be a little different here. Let's see what's going on here.
257		**Stephanie**	Okay, first you have without any blue. With just red. R-R-R.
258		**Researcher 1**	Okay, no blues.
259		**Stephanie**	Then you have with one blue.
260		**Researcher 1**	Okay.
261		**Stephanie**	B-R-R. Or R-B-R. Or R-R-B.

262		**Researcher 1**	Anything else?
263		**Michelle**	And you would do the same pattern for everything else?
264		**Stephanie**	No, not with the blue. Not with one blue.
265		**Michelle**	You would, you would do it with the one red and two blues…
266		**Jeff**	You would alternate like…
267		**Michelle**	You would do it the other way around.
268		**Researcher 1**	That's not what she said. Let her finish. That's what you would do. Let's hear what Stephanie does. Maybe she's not the same.
269		**Stephanie**	Well, there's no more of these because if you had to go down the other one, you'd have to have another [indecipherable]. But okay.
270		**Researcher 1**	You buy that? That's all there is of those? [Yeah] Okay.
271		**Stephanie**	Then you have with 3 blues. Well, no, not with 3 blues. I'll go like that.
272		**Researcher 1**	You have no blues and now you have exactly one blue.
273		**Stephanie**	Okay. Now you have exactly 2 blues. Wait, wait, actually, yeah, that's what I did last time because I did it back with 2 things.
274	20:08	**Researcher 1**	Okay, let's see that.
275		**Milin**	Then with 3 blues!
276		**Stephanie**	Which is…
277		**Milin**	Then you get every single one.
278		**Stephanie**	You could put B-B-R. You could put R-B-B.
279	20:26	**Milin**	You could put B-R-B. You could put…

280		**Stephanie**	Yeah, but that's not what I'm doing. I'm doing it so that they're stuck together.
281		**Jeff**	There should be one with one red. Then you could make it up and then there's one with two reds, and there's one with 3 reds.
282		**Milin**	Ahh, but see you did the same thing, but there's blue…
283		**Jeff**	There's all reds. And then there's 3 reds, 2 reds. There should be one with one red. And then you change it to blue.
284		**Stephanie**	Well, that's not how I do it.
285		**Researcher 1**	Let's hear how, how Steph…we'll hear, we'll hear that other way. That's interesting. Okay, now, so what you've done so far is…
286		**Stephanie**	One blue. Two blue.
287		**Researcher 1**	Okay. No blues.
288		**Stephanie**	One blue. Two blue.
289		**Researcher 1**	One blue and two blues. But Milin just said you don't have all two blues, and you said, that's, why is that?
290		**Stephanie**	[Hands paper to Milin] Okay show me another 2 blue? With them stuck together because that's what I'm doing it.

291		**Milin**	In that case [Hands back paper]
292		**Researcher 1**	Okay, now what are you doing, Stephanie?
293		**Michelle**	But, but if you just had 2 blues and they weren't stuck together, you could…
294		**Stephanie**	But that's what I'm doing, I'm doing the blues stuck together.
295		**Researcher 1**	Okay.

296		**Stephanie**	Then we had 3 blues that you could only make one of. Then you want 2 blues stuck apart, not stuck apart, took apart.
297		**Researcher 1**	Separated?
298		**Stephanie**	Yeah, separated. And you can go blue, red, red - do I have that? No. Blue.
299		**Researcher 1**	Okay, so Milin wanted to stick that in earlier, I thought and Michelle, right? When you were doing 2 blues? You wanted that stuck…
300		**Milin**	Because see, look at this. For 2 reds and one blue. Two reds…
301		**Michelle**	And that's stuck together here for 2 reds.
302		**Milin**	Yeah, so you're following no pattern.
303		**Michelle**	And you have more stuck together here.
304		**Stephanie**	Well, you're following your pattern. But my pattern goes no red, one red. This was not meant to be like that. That's not. It's in the category of one blue. That. I could stick that some place in another category. But I want this to be in the category of one blue. Not in the category of the opposite of this one. And then I have this one, the red-red-blue. So to you, that, you might put that way at the end of the line. But I put it right here.
305		**Jeff**	I have a question. Do you have to make a pattern?
306		**Milin & Stephanie**	No.
307		**Jeff**	Well then why is everybody going by a pattern?
308		**Milin**	Because we liked it.
309		**Stephanie**	Yeah, it's easier.

310		**Michelle**	It's easier…
311		**Stephanie**	Because it's easier than just going ooh-ooh.
312		**Michelle**	Because if you, because if you, because if you just keep on guessing like that, you're not sure if there's going to be more.
313	~23	**Stephanie**	It's easier maybe like Shelly and Milin's pattern was to go put this in a different category…
314		**Jeff**	I know their patterns.
315		**Stephanie**	Okay, but what I'm saying is that it's, that it's just easier to work with a pattern.
316		**Milin**	Oh here's another one! Let's see…
317		**Stephanie**	Yeah, I'll put that in.
318		**Michelle**	Because you might have a duplicate. And, and you may not know.
319		**Stephanie**	It's harder to check. It's harder to check just having them like come up from out of the blue.
320		**Milin**	Then just going like this and getting 2 from…

321		**Jeff**	How do you know there's different things in the pattern?
322		**Milin**	Since, see, look at this. These are all different, right?
323		**Jeff**	I see that. Yeah.
324		**Milin**	Yeah, see? From this, right, you can make two more so because here there's a blue-red and then a blue, red…
325		**Michelle**	Because, because there's only 2 colors more so you know you can't make more. Yeah, so.
326		**Milin**	And then there's red, I mean blue-red-red. And you can't make anymore on this one so you go onto the next one.

327		**Stephanie**	All right, and then...
328		**Jeff**	How do you know you can't make any more from that?
329		**Stephanie**	Because…
330		**Milin**	Because there's not any more color.
331		**Stephanie**	Look. Okay. Start here. Start here. Okay? You have the 3 together? The one 1 blue. You have the 1 blue. How could I build another one blue?
332		**Jeff**	You, you can't.
333		**Stephanie**	All right. So I've convinced you that there's no more 1 blue. All right.
334		**Michelle**	But if you didn't have that pattern, it would be harder to convince you.
335		**Stephanie**	If I went, I'll put this one blue over here. And that blue will it'll be on another piece of paper. However that goes.
336		**Jeff**	Yeah, but you can make a blue different what…if you go like this.
337		**Michelle**	That's if you have 4.
338		**Jeff**	If you go like this. You can go r-r-b or you can go b-r-r. Red…
339		**Stephanie**	That's what I have. No.
340		**Jeff**	No. They're all, they're all different. You can do…
341		**Stephanie**	What I'm saying is this is 1 blue. This is one blue.
342		**Jeff**	Yeah but there's 2 more different with one blue.
343		**Stephanie**	Yeah. There is…
344	25:07	**Milin**	No, but only on the bottom.

345		**Stephanie**	Look, but I have those three. Look. B-r-r, r-b-r, r-r-b, but then how am I supposed to make another one once that blue got down to my last block?
346		**Jeff**	Okay.
347		**Stephanie**	Okay. So I've convinced you there's no more 1 blue? [Jeff: Yeah] All right, now.
348		**Michelle**	Then you have to go to 2 blue.
349		**Stephanie**	Two blue. Here's one, right? 2 blue. We have one, b-b-r, then we have r-b-b. How am I supposed to make another one?
350		**Jeff**	B-r-b.
351		**Stephanie**	No, this is the other. Milin gave me that same argument.
352		**Michelle**	She means, she means together.
353		**Jeff**	But the thing is it doesn't matter…
354		**Stephanie**	I don't…
355		**Michelle**	No, she means stuck together.
356		**Stephanie**	Stuck together, that means, like okay I took…
357		**Jeff**	I know.
358		**Stephanie**	Okay, so can I make any more of that kind?
359		**Michelle**	Then you have to move to three, which you can only make 1.
360		**Stephanie**	Yeah, you can only make 1 and then you could make the 3, without blue, and where there's 3 red.
361		**Michelle**	Then you can make 2 split apart.

362		**Stephanie**	Two split apart, which you can only make 1 of. And then you could make, you could find the opposites right in the same group. [Jeff: Okay] All right, so then I've convinced you that there's only 8?
363		**Jeff**	Yeah. [Stephanie: Yes!]
364		**Researcher 1**	How many if you're making towers of 4?
365		**Michelle, Milin, Stephanie**	16
366		**Researcher 1**	You agree, Jeff?
367		**Jeff**	Yeah.
368		**Michelle**	Because you have…
369		**Researcher 1**	Jeff, why do you agree? Don't let them go by so easily. This could be pressure here.
370		**Michelle**	See, look it's because, say you add a red or a blue, you can add a red or blue here.
371		**Researcher 1**	Make a drawing for your sentence showing it..
372		**Jeff**	I understand because you can only, you could put…
373		**Michelle**	Put 2 colors here, you could put 2 colors there. You can keep on going.
374		**Jeff**	You can keep doing 2 colors for each one, and that's…
375		**Michelle & Jeff**	2-4-6-8-10-12-14-16.
376		**Researcher 1**	And so that's for towers of…
377		**Jeff**	4
378		**Milin**	My guess is 16, but…

379		**Jeff**	We all ready got 16.
380		**Milin**	Why, why did she say in the beginning of the whole thing that 12
381		**Jeff**	This…
382		**Michelle**	It's, it's like, it's like…
383		**Researcher 1**	Why did you say 12, Michelle?
384		**Jeff**	Listen, you could do a red or a blue. You could do either a red or a blue. A red or a blue.
385		**Milin**	Jeff, Jeff, Jeff. I know that [indecipherable] But I want to know why she said 12 before?
386		**Stephanie**	Yeah, Michelle, why did you?
387	27:11	**Jeff**	Because she was guessing, not making patterns.
388		**Researcher 1**	Is that true, Michelle? Poor Michelle, it's okay. You think 12 or 16, Michelle?
389		**Michelle**	16.
390		**Researcher 1**	Michelle thinks 16. Now, now you made towers of 5 in class, and what did you get?
391		**Milin, Michelle, Stephanie**	32
392		**Researcher 1**	Does that work the same way?
393		**Milin, Michelle, Stephanie**	Yeah.
394		**Milin**	If you get towers of 4
395		**Jeff**	They're multiples of 2.

396		**Stephanie**	The hard part is making patterns. Like, you, from now, we know how to just oh you could give us a problem, like how many in 10 and we'd know.
397		**Researcher 1**	Okay, how many in 10? You know the answer?
398		**Stephanie**	I know the answer. I figured it out. It's 1,024.
399		**Researcher 1**	1,024.
400		**R2**	Are you sure?
401		**Stephanie**	Uh-huh.
402		**Jeff**	Don't try to convince me.
403		**Researcher 1**	Try to convince him.
404		**Milin, Michelle**	No! No!
405		**Milin**	Okay, okay, okay

406		**Stephanie**	I think we have 1,000 units.
407		**Researcher 1**	You could do that later. However, you were saying you know the answer, but…
408		**Stephanie**	But the problem is, the hard part is you could just give us a problem and we could go like well, we'll go 22 times 2…
409		**Michelle**	See for how we're doing you keep on adding what you have already. For here, you add 2 more. For here, you add another 4 so for here and for the 16
410		**Jeff**	You sure it's 1000? You sure?
411		**Stephanie**	Yeah.
412		**Jeff**	Because look you have…
413		**Stephanie**	Now, see you're dividing the…
414		**Jeff**	I'm not dividing…

415		**Stephanie**	The problem. You're timesing, no you don't times it. It's the same thing I did. I counted ahead. I just counted ahead 5 or 6, and I said oh, I could just multiply it by that and that'll give me the same answer, but it didn't work.
416		**Jeff**	Okay. It didn't work.
417		**Stephanie**	Okay. You have to figure out what's in between that.
418		**R2**	What did you find then?
419		**Teacher**	What do you mean?
420		**Stephanie**	In between, okay.
421		**R2**	Show me a little bit
422		**Stephanie**	Do you want me to figure out 10, right? But, in order to figure out 10, I was only up to 5. So what I had to do was I had to go and I had to say, well, what's 6, what's 7, what's 8, and what's 9, and times that times the last number I had.
423		**Researcher 1**	Well, let's, let's take a look at what you had here. This, this is what Stephanie had, guys. If you want to do it yourself for a minute. When I asked Stephanie how many for towers of 10, what Stephanie, why don't you say what you did to get 1,024 and then let's talk about this…
424		**Milin**	Yup, she's right.
425	~29:20		[indecipherable]
426		**Stephanie**	I was up to 5 so I took the 6. I was up to 6.
427		**Researcher 1**	64 is…Why don't you write that? Okay. Towers of 6.
428		**Stephanie**	Okay, now I was up to 6.
429		**Researcher 1**	You agree with that?
430		**Jeff**	Yeah

431		**Stephanie**	So I multiplied, I tried, first of all, I tried multiplying it times 8 because I figured well, all I have to do was 6+4 times 2 that's 8 so 64 times 8.
432		**Jeff**	What are you saying?
433		**Milin**	She did it wrong.
434		**Researcher 1**	No, no, no, let's hear what she's saying. Let's hear her thinking.
435		**Stephanie**	First, I thought, well, I don't want to go ahead, and I don't want to have to multiply 7, 8, 9, and 10. 7, 8, 9 before I get 10. So I figured 6 plus 4 equals 10. And since I'm timesing times 2, I'll multiply 4 times 2 to get 8 and then just multiply 64 timese 8.

436		**Michelle**	But she was wrong.
437		**Stephanie**	Yeah.
438		**Michelle**	And then, and then, no, she was right here. She only timesed it by 2 so she was right.
439		**Stephanie**	Then I did…
440		**Milin**	You keep timesing it by 2
441		**Stephanie**	Then I did 128 times 2. 256, 512, and then…
442	30:29	**Milin**	You get your answer.
443		**Researcher 1**	Except that, this is where I'm very, very interested what she did. How come she got something, she got 512…
444		**Jeff**	And you all ready got 512 over there.
445		**Researcher 1**	So Is that so very wrong?
446		**Jeff**	And then you could've timesed this by 2…
447		**Milin**	No, that's the same thing.

448		**Jeff**	But you could've just timesed this by 2 and you would've had it a lot easier than going, times, times, times.
449		**Researcher 1**	So in other words, could this have worked, that's my question. Now, when would this work? Why didn't the 8 work? Why did you have to keep…
450		**Stephanie**	Ahh, I just thought of something. I'm wondering if this will work. This 8 is 8, okay? This is 8, right? This is the answer to 8.
451		**Jeff**	You had it right, you just didn't follow a pattern, you just took a guess. And then if you filled it out exactly.
452		**Researcher 1**	Okay. So what you're suggesting is multiplying by 8 didn't work. It gave you 512, which was…
453		**Jeff**	Which gave you to 8.
454		**Researcher 1**	To 9, to 8? Or to 9?
455		**Jeff**	So, if you…
456		**Michelle**	This pattern works here.
457		**Researcher 1**	If I plugged in 8 or 9…
458		**Milin**	It would've worked, her pattern would've…
459		**Researcher 1**	Let's get another piece of paper and see what happened here because this is just a mess.
460		**Milin**	It would've worked where, but then she has to [Researcher 1: Get me another piece of paper. Let's start again.] to times it by 2 after she gets her number. She has to times it by 2 after she gets her number.

461		**Researcher 1**	But, you know what I'm thinking, I'm thinking maybe what we should do is I want you to, I don't want to throw away Stephanie's idea here, okay, because what Stephanie has here in her idea, once she got to towers of 9, right, she said there were 512. That's by each time multiplying it by 2.
462		**Michelle**	And then you have to move [Researcher 1: But, hold on a minute] [indecipherable] This would work if you multiply it times 2. You still get 1024 like over here.
463		**Researcher 1**	Right, but why, why didn't multiplying by 8 work when she had towers of 6?
464		**Michelle**	Because, because she wasn't so sure about going like this…
465		**Researcher 1**	Okay, but why, how could she be sure? In other words if 8 didn't work, do you understand my, my challenge to you?
			[Yeah] All you mathematicians here. My challenge to you is I don't want to throw out this idea because, you know, because if Stephanie has something here, she'll save you a lot of work in the future, right? If she has a good idea here? Do you understand the problem here? And I think what we'll do, I want to be sure. I don't know if Mrs. Barnes is gone. I want to be sure your teacher understands what's going on here so to sort of push you to think about this so that next time I come, maybe you could invent another way. If I said towers of…
466		**Michelle**	80
467		**Researcher 1**	80. Now, and I said I'll give you a calculator, but you have to know what to do with your calculator, right?
468		**Stephanie**	There's a problem because you have to go all the way from 10 to 80.

469		**Researcher 1**	Well, my question is let's not worry about that big problem for a moment. Let's try to do it with a simple problem. Suppose you didn't know towers of 6 were 64 and towers of 7 were, what did you say that was? What do you have there?
470		**Milin**	Towers of 7…
471		**Researcher 1**	128? Is that what you have, Milin?
472		**Milin**	Yeah, I think.
473		**Researcher 1**	And so. Suppose you didn't know that. How could you jump from towers of 6 to towers of 10 without going through all those steps and why?
474		**Milin**	Get out.
475		**Researcher 1**	But isn't that a nice, challenging question? I have one more question to get to. I'm going to put this one aside for a minute because that's going to take some time. When we come back, then we'll talk about it. You could bring your calculator, Jeff. Fair enough? Okay, now look. You said this was like shirts and pants, and I would like for you to say if you agree it's like shirts and pants…
476		**Jeff**	I agree.
477		**Researcher 1**	But why?
478		**Michelle**	But if you kept on going up, you would have to add…
479		**Researcher 1**	Okay, one at a time. Let's hear Jeff.
480		**Jeff**	You have the same pattern, same pattern
481		**Researcher 1**	In what way?
482		**Jeff**	Because with shirts you have to keep on alternating the shirts with the pants. And keep on alternating pants with the shirts…

483		**Researcher 1**	I'm not so sure I follow what you're saying.
484		**Jeff**	Neither do I.
485		**Researcher 1**	Stephanie is working on the towers of 10.
486		**Stephanie**	I might have it here. I'm thinking if I multiply the last number I got which was 1,024 times 80, that I got, [Michelle: You would get the answer probably] 81,120, but I'm not sure if I'm right or not. You know, I'd have to go through all that…
487		**Michelle**	Nuh-uh.
488		**Milin**	Nuh-uh.
489		**Michelle**	Or maybe you would multiply it by 70 because you all ready go 10.

490		**Milin**	No, but times it by 8 [Jeff: You guys are losing me here] because you have to have 8 more.
491		**Researcher 1**	Me too, I'm lost too.
492		**Stephanie**	You wouldn't times it by 8 because we timesed it by 8 when we were on 8. We times it by 80 when we're on 80.
493		**Jeff**	True.
494		**Researcher 1**	But when, I don't understand. Hold on.
495	35:00	**Milin**	Nuh-uh. Did you times it by 80 when you were on 80?
496		**Stephanie**	I went, I said well, there was …
497		**Milin**	8 times 8. 64. How could that be?
498		**Stephanie**	Actually, you would multiply it by 1,600.
499		**Researcher 1**	Can, can we call time out for a minute?
500		**Jeff**	What are you guys talking about?

501		**Researcher 1**	Yeah, I'm a little lost and Jeff is lost. And I don't know how Michelle is doing here. And you two can continue this when we leave and work this out. However…
502		**Michelle**	Finish your fight.
503		**Researcher 1**	I don't really want you to really solve the problem for towers of 80. I want you to solve the problem of towers of 10.
504		**All**	We did that.
505		**Researcher 1**	But hold on, you have to pretend, you only know the answer for towers of 6.
506		**Michelle**	Just keep on building.

507		**Researcher 1**	That's one way.
508		**Milin**	I all ready did that. I all ready did that.
509		**Stephanie**	You want us to try and figure it out the way I tried to figure it out the first time.
510		**Researcher 1**	Right with only multiplying by 1 number. And convince me that that number makes sense to multiply by. Does that make sense? Do you understand?
511		**Milin**	This [holding up his work]…
512		**Jeff**	But all you did…
513		**Researcher 1**	Okay. Hold on. Time out.
514		**Jeff**	You didn't know times 2 times 2 would help you.
515		**Milin**	I did.

516		**Researcher 1**	Well, you sort of know it. But I want to save all those intermediate steps because if you had to go to, to build towers of 80, let's see, when you had to build towers of, 2 of 2, how many times, of 3 high, how many times did you multiply by 2?
517		**Milin**	She's right, Jeff. You should really multiply by 8(?).
518		**Researcher 1**	When you had to build towers of 3, how many times did you need to multiply by 2?
519		**Milin**	Times 3. 4 times 2.
520		**Researcher 1**	I said by 2.
521		**Milin**	Oh. 4. Same thing.
522		**Researcher 1**	Okay, 2 times 2. That's one time you multiply it by 2. You got 4. Then you multiply by 2 again…

523		**Milin**	8
524		**Researcher 1**	And that gave you 8. So how many times did you multiply by 2?
525		**Stephanie**	You multiplied the amount of times you…
526		**Researcher 1**	Well, twice. You multiplied it once. This is 2 times 2 once, right? And then you multiplied it by 2 again, right? 2 times 2, let me write this. 2 times 2 gave you 4. That was one time. Then you multiplied it by 2 again another time and you got 8. So you multiplied it twice to build towers of 3, is that right?
527		**Jeff**	Yeah.
528		**Milin**	No.
529		**Researcher 1**	No?
530		**Milin**	Because to get towers of 2, then it will be much easier.

531		**Jeff**	Yeah, but the thing is it's right, it's easier…
532		**Researcher 1**	I think we've run out of time.
533		**Jeff**	Yeah, we did.
534		**Researcher 1**	Will you come back?
535		**All**	Yeah, okay.
536		**Researcher 1**	Okay. Would you come back? Would you come back another time? Can we come back another time? [Yeah] Okay, next question is I want to know what this has to do with shirts and pants.
537		**Milin**	Shirts and pants?
538		**Stephanie**	No.

539		**Jeff**	Oh no. I have no idea. I didn't think of it…
540		**Researcher 1**	You can talk about it before and share it.
541		**Stephanie**	Can I tell you what I told you last time I was here about shirts and pants?
542		**Researcher 1**	What, what?
543		**Stephanie**	Because remember, it was the problem with the shirts, the pants. You had to match up Steven's pants with the shirts to make like a tower…
544		**Researcher 1**	Yeah.
545		**Stephanie**	Remember?
546		**Milin**	He has to have at least big hands.
547		**Researcher 1**	Well thank you. This was great. Well, thank you so very much. This was fun. I love coming to talk to you about math.

548		**Milin**	Thank you.
549		**Researcher 1**	My budding mathematicians here.

Appendix L: Session X: Transcript-Interview 5 Pascal's Triangle Part 1

Stephanie engages in an interview investigating the doubling pattern for Unifix towers of increasing heights, exploring the connection to Pascal's triangle with Researcher 1.

Description: Clip 4 of 10, Investigating the "Doubling Pattern" for Unifix Towers of Increasing Heights Parent Tape: Early Algebra Ideas About Binomial Expansion, Stephanie's Interview Five of Seven Date: 1996-03-13 Location: Harding Elementary School Researcher: Researcher 1	Transcriber(s): Aboelnaga, Eman Verifier(s): Yedman, Madeline Date Transcribed: Fall 2010 Page: 366 of 5

Time	Line	Speaker	Transcript
0:00	1	**Researcher 1**	Okay, so we've looked at selecting, right?
	2	**Stephanie**	Mm-hmm.
	3	**Researcher 1**	Well we're going to do a little algebra here. We have four and we're selecting *r* and *r* could go- be zero, one, two, three or four. Isn't that right?
	4	**Stephanie**	Yeah.
	5	**Researcher 1**	When *r* is zero, we have this, and you told me that's one. Right?
	6	**Stephanie**	Ok.
	7	**Researcher 1**	When *r* is one, you told me that was . . . [*writing*]
	8	**Stephanie**	Um, with one red, four.
	9	**Researcher 1**	And this was . . . [*writing*]
	10	**Stephanie**	Six.
	11	**Researcher 1**	And this was . . . [*writing*]
	12	**Stephanie**	Four.
	13	**Researcher 1**	And this was . . . one, two, three. [*writing*]

	14	**Stephanie**	Four out of four, you'd have one.
	15	**Researcher 1**	One. Right?
	16	**Stephanie**	Yeah.
	17	**Researcher 1**	So, if I wanted to know the total number-
	18	**Stephanie**	Mm-hmm.
	19	**Researcher 1**	-where you could have no reds, exactly one, exactly two, exactly three, exactly four. What does it turn out to be?
	20	**Stephanie**	Sixteen.
	21	**Researcher 1**	Does that surprise you?
	22	**Stephanie**	Not really. I-I mean, I wasn't thinking about it like that-
	23	**Researcher 1**	I know.
	24	**Stephanie**	-but I mean, no.
	25	**Researcher 1**	Isn't that interesting?
	26	**Stephanie**	Yeah, it's the same thing.
	27	**Researcher 1**	What do you mean?
	28	**Stephanie**	Like with just the towers-
	29	**Researcher 1**	Mm-hmm.
Time	Line	**Speaker**	Transcript
	30	**Stephanie**	-except that I just did it different.
	31	**Researcher 1**	How did you do it differently with the towers?
	32	**Stephanie**	Well, with the towers, I just didn't have this, to, like, say "All right, now I'm going to try it with three." I just, like, did all these different things until we couldn't do them any more.
	33	**Researcher 1**	Mm-hmm.
	34	**Stephanie**	So, it was like, more just like guessing. You know?
	35	**Researcher 1**	Well, but I noticed in the towers later on you did something different. Um, something I just looked at recently. Um, you didn't start, y-you- in order to figure out how many you can build, let's say, four high-
	36	**Stephanie**	Mm-hmm.
	37	**Researcher 1**	-you started building one high. Like, you said this is one high. You said it could be a red or a yellow-
	38	**Stephanie**	Mm-hmm.
	39	**Researcher 1**	-you did some family thing.
	40	**Stephanie**	Yeah, and we had them, I think, when we first showed it we had them all lined up. Like, and their opposites. We did, like, one red, all red, all yellow. And stuff like that.
	41	**Researcher 1**	Do you remember how you built up the family? This was for one high, right?
	42	**Stephanie**	Oh, okay.

Time	Line	Speaker	Transcript
	43	**Researcher 1**	Then, when you went for two high, right-
	44	**Stephanie**	Mm-hmm.
	45	**Researcher 1**	-you built on top of. You all were talking about a way of doing that. Um, you said that, something like, I remember you starting something like, someone asked you how many can you build one high when they could be red or yellow.
	46	**Stephanie**	Mm-hmm. And, there could be two.
	47	**Researcher 1**	There could be red.
	48	**Researcher 1/Steph**	Or yellow.
	49	**Researcher 1**	And then you built those.
	50	**Stephanie**	Yes.
	51	**Researcher 1**	And you see them standing in front of the camera. Beautiful shots of red or yellow.
	52	**Stephanie**	Yeah.
	53	**Researcher 1**	And then, you talked about, "Ok, now I want to move from one to two high."
	54	**Stephanie**	Mm-hmm.
	55	**Researcher 1**	So you said, "Ok, if I start with the red, what could I do to make two high?"
	56	**Stephanie**	Well, I could have um, red-red.
	57	**Researcher 1**	You did something like this, right? [draws a tree diagram showing how the towers build by adding a red and yellow to each previous tower.]
	58	**Stephanie**	Yeah. Or I could have yellow-yellow. Oh if you want to use the red, you can have red-yellow.
	59	**Researcher 1**	If you start with red on the bottom?
	60	**Stephanie**	Well, yellow-red.
	61	**Researcher 1**	Is that right?
	62	**Stephanie**	Yeah.
	63	**Researcher 1**	Millan did something like this. Do you remember that?
	64	**Stephanie**	Mm-hmm.
	65	**Researcher 1**	So you got two, the family grew.
	66	**Stephanie**	Yeah.
	67	**Researcher 1**	You did something like that. Do you remember that?
	68	**Stephanie**	Yes.
	69	**Researcher 1**	And then you used the same argument here.
	70	**Stephanie**	That'd be yellow-yellow and red-yellow.

	71	**Researcher 1**	And you could put, ok, you could put yellow on the top or you could put red on the top of that yellow.
	72	**Stephanie**	Mm-hmm.
	73	**Researcher 1**	And so, two high you ended up–for one high you ended up with a total of two, and for two high, you ended up with a total of-
	74	**Stephanie**	Four.
	75	**Researcher 1**	And then you predicted for three high, there'd be how many?
	76	**Stephanie**	Um, Eight.
Time	Line	**Speaker**	Transcript
	77	**Researcher 1**	And then you predicted for four high, there'd be
	78	**Stephanie**	Sixteen.
	79	**Researcher 1**	Sixteen, and?
	80	**Stephanie**	Thirty-two.
	81	**Researcher 1**	And so, yeah, but how did you get the eight from these four?
	82	**Stephanie**	Um, well, you could do red-red-red or you could do red-yellow-red or red-red-yellow.
	83	**Researcher 1**	I'm having trouble following you if you're making a family.
	84	**Stephanie**	Oh ok, if you're doing- ok. You could do it. And I have to have two red on the bottom?
	85	**Researcher 1**	Well, I don't know, you tell me, I don't…
	86	**Stephanie**	Well, here, I have to have-I can have [writing] red-red-red or I can have red-red-yellow or I can have . . .
	87	**Researcher 1**	That goes from that one?
	88	**Stephanie**	Yeah, that goes from the red-red. Or I can have, [writing] like, red-yellow-red. Or I can have – whoops – red-yellow-yellow. You can't see that. Or I can have, um, yellow-yellow-yellow. Or I can have yellow-yellow-red. Or I can have, um, yellow-red- yellow. Or I can have yellow-red-red. Yeah.
	89	**Researcher 1**	So where did the eight come from, from the four?
	90	**Stephanie**	From the four? Well, like, red-red-red or yellow-red-red.
	91	**Researcher 1**	How did that happen that you got two from that one? Did you always get two from the one?
	92	**Stephanie**	Um…
	93	**Researcher 1**	As you build up from one, you got two here, didn't you?
	94	**Stephanie**	Mm-hmm.
	95	**Researcher 1**	From this one, you got two here, right?
	96	**Stephanie**	Yeah, probably. Yeah.

	97	**Researcher 1**	Why?
	98	**Stephanie**	‘Cause, I guess, there’s always going to be two combinations with whatever you have on the bottom-
	99	**Researcher 1**	Mm-hmm.
	100	**Stephanie**	-like, ‘cause if you’re building it from here, it’s got to have three
Time	Line	**Speaker**	Transcript
			reds on the bottom, and there’s only two other things ‘cause you only have two colors. So you can only do two other things with that. You can either put a red on top or a yellow.
	101	**Researcher 1**	So, so that means four high, you would get?
	102	**Stephanie**	You would get sixteen.
6:51	103	**Researcher 1**	You would get sixteen, so, in this, I’m not gonna ask you to do that, you just told me what it would look like and I can follow what you’re saying. So you do get sixteen four-high.

Appendix M: Session XI: Transcript-Interview 5 Pascal's Triangle Part 2

Stephanie engages in an interview investigating symmetry for the two colors across cases when building towers four high, exploring the connection to Pascal's Triangle with Researcher 1.

Description: Clip 5 of 10: Recognizing the Symmetry for the two Colors across the Cases when building Unifix Towers 4-cubes tall Parent Tape: Early Algebra Ideas About Binomial Expansion, Stephanie's Interview Five of Seven Date: 1996-03-13 Location: Harding Elementary School Researcher: Researcher 1	Transcriber(s): Aboelnaga, Eman Verifier(s): Yedman, Madeline Date Transcribed: Fall 2010 Page: 371 of 3

Time	Line	Speaker	Transcript
0:00	1	**Researcher 1**	So you do get sixteen four-high.
	2	**Stephanie**	Mm-hmm.
	3	**Researcher 1**	Right?
	4	**Stephanie**	Yes.
	5	**Researcher 1**	And, um, in all of these, I focused on red. Talked about the positions for red, right?
	6	**Stephanie**	Mm-hmm.
	7	**Researcher 1**	For these four high, you can imagine these sixteen there. And, of these sixteen, I could say, of these sixteen, there'll be no reds and there's going to be one of those. And there's going to be exactly one red-
	8	**Stephanie**	And there'd be four of those.
	9	**Researcher 1**	And so forth, right? Um, what about yellows? Don't we have to do the same thing for yellows? So

			wouldn't that give us 32?
	10	**Stephanie**	Yeah.
	11	**Researcher 1**	But this thing only produces sixteen. If I were to do the same thing here for yellow, right-
	12	**Stephanie**	Mm-hmm.
	13	**Researcher 1**	-and if I said, let's now find out how many exactly no yellows, let's find out exactly one yellow out of the four, exactly two yellows out of the four, three yellows out of the four, don't you agree that you'd get another sixteen?
	14	**Stephanie**	Yeah.
	15	**Researcher 1**	But then 16 and 16 gives you 32, not 16.
	16	**Stephanie**	But wouldn't it be the same thing? Like, only the opposite way? 'Cause, look, if there's two red, then there's two yellow. [*writing*] And if there's three red, then there's one yellow. And if there's one red, then there's three yellow, so isn't it the same thing?
	17	**Researcher 1**	Is it?
	18	**Stephanie**	Yeah.
	19	**Researcher 1**	Ok, you're sure of that?
Time	Line	**Speaker**	Transcript
	20	**Stephanie**	Yeah.
	21	**Researcher 1**	And-and that's why if you think about that as a strategy, if you've already figured out exactly one, do you know exactly three?
	22	**Stephanie**	Um?
	23	**Researcher 1**	See this was the exactly one here, right?
	24	**Stephanie**	Mm-hmm.
	25	**Researcher 1**	Right?
	26	**Stephanie**	Yes.
	27	**Researcher 1**	That was exactly one red. And when you did exactly three red, I asked you to move one, you also got four.
	28	**Stephanie**	Yeah, well, I guess it's just the opposite.
	29	**Researcher 1**	Isn't that interesting?
	30	**Stephanie**	Yeah.
	31	**Researcher 1**	So, it saves you some work.
	32	**Stephanie**	Yeah.
	33	**Researcher 1**	And that's kind of important to realize. If you know exactly none, right, do you know exactly all?
	34	**Stephanie**	Yeah, but I mean, I wouldn't have thought of that. Like-
	35	**Researcher 1**	Yeah, well, that kind of pulls some of the ideas

			together.
	36	**Stephanie**	Yeah.
	37	**Researcher 1**	I think also if you think about that, it might help you. So if we went, to towers five, it might be interesting to look at some of this, now that you're looking at it from another point of view – combinations or selections – which, by the way, um, is a field of math that's called counting, and counting, um, is a field of math that you study as sort of a prelude to studying things like probability
	38	**Stephanie**	Mm-hmm.
	39	**Researcher 1**	and statistics. So it's a very important field, and, um, if you start to pick up a book at the college level or advanced high school, and you see all these formulas and you see all this notation, and with the notation, there's formulas.
Time	Line	**Speaker**	Transcript
25:00 – 29:59	40	**Stephanie**	Mm-hmm.
	41	**Researcher 1**	There are students who work with this and have no sense of what it means. See, the advantage you're going to have when you get to work with this is if you could think about what this means, you say "Oh, selection, towers."
	42	**Stephanie**	Yeah.
	43	**Researcher 1**	You know what I'm saying?
	44	**Stephanie**	Yeah.
	45	**Researcher 1**	That's like exactly one out of the four being this. See what helps is if you can, all the work- all the hard work you've done for years, if you can, in your mind, try to say, "This is like this" or "This is almost like this", then you can build on these ideas and then when you get the formulas, you know, they don't always apply directly. It's like, sort of, the problem you had yesterday with the factoring.
	46	**Stephanie**	Yeah.
3:59	47	**Researcher 1**	It really was the same problem. You know, sort of tricky, wasn't it? Once you saw it a certain way, you realized it was the same problem. Well that's part of what you have to do. You have to be able to see it, you know, to be able to visualize it, which is part of the strength.

Appendix N: Session XII: Transcript-Interview 5 Pascal's Triangle Part 3

Stephanie engages in an interview investigating the combinatorics notation for towers choices when selecting from two colors to the first five rows of Pascal's triangle, exploring the connection to Pascal's triangle with Researcher 1.

Description: Clip 6 of 10: Connecting the Combinatorics Notation for Tower Choices when Selecting from Two Colors to the First 5 Rows of Pascal's Triangle Parent Tape: Early Algebra Ideas About Binomial Expansion, Stephanie's Interview Five of Seven Date: 1996-03-13 Location: Harding Elementary School Researcher: Researcher 1	Transcriber(s): Aboelnaga, Eman Verifier(s): Yedman, Madeline Date Transcribed: Fall 2010 Page: 374 of 5

Time	Line	Speaker	Transcript
0:00	1	**Researcher 1**	Let's do this. If I picked none, exactly none, out of one.
	2	**Stephanie**	Out of one?
	3	**Researcher 1**	Does that make any sense? Okay, I have one high. I have this one high, if I have no red. I still have my yellow-
	4	**Stephanie**	But- oh- but you have the yellow though.
	5	**Researcher 1**	See notice that it didn't make any sense, but once you started thinking about-
	6	**Stephanie**	Oh, well then there's one.
	7	**Researcher 1**	Oh, isn't that right? And if I said to you, "Exactly one out of one." See this is no reds. You said there's one, right?

	8	**Stephanie**	Yeah.
	9	**Researcher 1**	Exactly one red.
	10	**Stephanie**	That would be one.
	11	**Researcher 1**	That would be one. See, now it has meaning.
	12	**Stephanie**	Yeah.
	13	**Researcher 1**	But you look at this notation and say, "What does this mean?" But see, this will help you think of selections. Ok, so if we were to think about this, um, if we're thinking of for towers for n = 1, that's one high towers, right?
	14	**Stephanie**	Mm-hmm.
	15	**Researcher 1**	So, we can think about this as [*writing*] this and this, right? Or we can think about this as one and one. Isn't that cool?
	16	**Stephanie**	Mm-hmm.
	17	**Researcher 1**	So I thought we'd do something else that might. .... no w two. Right? So if we're doing two now, again, what do you want to think of red or yellow? Does it matter? You told me it doesn't matter.
	18	**Stephanie**	Yeah, it would be one.
	19	**Researcher 1**	There's one way. You saw that right away. What made you see that right away?
	20	**Stephanie**	Well, because there's always going to- if there's- you can't do
Time	Line	**Speaker**	Transcript
			none of one, and there's another color, it's obviously going to be all the other color.
	21	**Researcher 1**	Good, that's great. Ok, so now, if we're gonna do – I'm going to pick one out of two.
	22	**Stephanie**	Um, two ways, I guess. One on top or one on bottom.
	23	**Researcher 1**	Mm-hmm. Can you see that?
	24	**Stephanie**	Yes.
	25	**Researcher 1**	And if it's two out of two?
	26	**Stephanie**	It would be one.
	27	**Researcher 1**	Okay. So, when I have n = 2, here I had one, right, that's no reds or one, that was one red, which was one high. Now, if I'm talking two high, I could have one red, I could have two reds, or I could have one red. No reds. One red or two reds. So this one is this piece, this

			one is this piece, this one is . . . let me just put the numbers in now.
	28	**Stephanie**	Okay.
	29	**Researcher 1**	See if you notice what's happening here. n = 3.
	30	**Stephanie**	Ok, so, for, like, there's one.
	31	**Researcher 1**	Okay.
	32	**Stephanie**	Um, I don't know, maybe there's two?
	33	**Researcher 1**	Want to think about that? (inaudible) yeah-
	34	**Stephanie**	Yeah, I think there's more than I don't know.
	35	**Researcher 1**	Think about it.
	36	**Stephanie**	Um, I need a few…
	37	**Researcher 1**	Yeah, that's fair enough. It's always good to take your time to think about it.
	38	**Stephanie**	There's one choice, I'm gonna do them, like, as towers this time. When there's three it could be, um, you have red and yellow, it could be red-yellow-yellow and there's gonna be three. It could be red and it could be like that. There's three.
	39	**Researcher 1**	You absolutely sure of that? What was-um, what was-combinations were you selecting one from?
Time	Line	**Speaker**	Transcript
	40	**Stephanie**	Two.
	41	**Researcher 1**	Ok. Um, what do you think it would be when selecting one from four? Exactly one from four?
	42	**Stephanie**	Four?
	43	**Researcher 1**	What would you think it would be if I could select one from n?
	44	**Stephanie**	n?
	45	**Researcher 1**	See that? Can you imagine that?
	46	**Stephanie**	Yes.
	47	**Researcher 1**	If it's five, can you see them all up there? If it's six, can you see them? You can make it as tall as you want, you can just see them exactly-
	48	**Stephanie**	Yes.
	49	**Researcher 1**	Isn't that helpful?
	50	**Stephanie**	Yeah.
	51	**Researcher 1**	To have that visual kind of thing?
	52	**Stephanie**	Yes.
	53	**Researcher 1**	You didn't even have any Unifix cubes, that's great. Okay, so-
	54	**Stephanie**	So, there would be three-
	55	**Researcher 1**	You know that, do you know exactly two? Do you know that?

			Do you have to think a lot?
	56	**Stephanie**	I don't know. There's- oh- wouldn't it be the same thing?
	57	**Researcher 1**	Why?
	58	**Stephanie**	Because it's just the opposite, right?
	59	**Researcher 1**	Isn't that right?
	60	**Stephanie**	So that would be three. And then, three, three, is one.
	61	**Researcher 1**	Right?
	62	**Stephanie**	Yeah.
	63	**Researcher 1**	See how fast you got those?
	64	**Stephanie**	Yeah.
	65	**Researcher 1**	Now, I'm going to write for n equals three here, look, put a one, three, three, one. Now do you notice something happening here. I have a one-one, for these two. I have a one-two-one, a one-
Time	Line	**Speaker**	Transcript
			two-one for none, one and two. I have a one-three-three-one, one-three-three-one for the case of three. Do you want to predict what it's going to be like for four?
	66	**Stephanie**	It's going to be, like, one-four and then there's another number. And then, four-one.
	67	**Researcher 1**	Okay, now that's the interesting. . . .
	68	**Stephanie**	Well, I know that that one's six though.
	69	**Researcher 1**	Oh, but notice something, no?
	70	**Stephanie**	Oh, is it, cause like, the 1 and 2- 1 and 1 are 2, 1 and 2 are 3, 1 and 2 are 3, 1 and 3 are 4, 1 and 3 are 4, 3 and 3 is 6?
	71	**Researcher 1**	Isn't that exciting? Now, I'd like to have this case in here [writes].
	72	**Stephanie**	Okay.
	73	**Researcher 1**	It looks pretty, doesn't it? So, what would that be? Gosh. This was $n = 1$.
	74	**Stephanie**	Mm-hmm.
	75	**Researcher 1**	This would have to be $n = 0$. Right? Right?
	76	**Stephanie**	Mm-hmm.
	77	**Researcher 1**	So, what would you have to make selecting none from none, by definition, to make this all look pretty?
	78	**Stephanie**	Selecting none from none?
	79	**Researcher 1**	See it makes almost no sense to think about.
	80	**Stephanie**	Yeah, cause like . . .
	81	**Researcher 1**	But remember you told me, like, if I took a number to the zero power, that doesn't make any sense?

	82	**Stephanie**	Yeah.
	83	**Researcher 1**	Remember we had that conversation in the car?
	84	**Stephanie**	Yes.
	85	**Researcher 1**	Well, this is almost like that. It doesn't make any sense, but if you want this picture to be so nice and symmetry and all, and if you want it to turn out to be that way, what would you want it to be?
Time	Line	**Speaker**	Transcript
	86	**Stephanie**	I guess it would have to equal one.
	87	**Researcher 1**	Yeah. So people find it convenient to make that one. That's how definitions sometimes arise. There's- motivated by some symmetry or beauty. Is there another reason to make that one? I don't know of any. Do you? Taking no things from nothing? One way? [to researchers]
	88	**R2**	Well, (inaudible)
	89	**Researcher 1**	See, it just works out nicely. Can you guess five high, what these numbers would be?
	90	**Stephanie**	All right. It would be 1. Um, and then it would be 1 + 3, oh, 5. And then it would be 10, 10, 5, 1.
6:44	91	**Researcher 1**	I put the one there.

Appendix O: Session XIII: Transcript-Interview 5 Pascal's Triangle Part 4

Stephanie engages in an interview investigating Pascal's Triangle with Researcher 1.

Description: Clip 7 of 10: Continuing Investigation of Pascal's Triangle: Generating Rows 5 and 6 and calculating the totals for each row Parent Tape: Early Algebra Ideas About Binomial Expansion, Stephanie's Interview Five of Seven Date: 1996-03-13 Location: Harding Elementary School Researcher: Researcher 1	Transcriber(s): Aboelnaga, Eman Verifier(s): Yedman, Madeline Date Transcribed: Fall 2010 Page: 379 of 4

Time	Line	Speaker	Transcript
	24	**Stephanie**	Um, two from five. And that equals two.
	25	**Researcher 1**	And that's ten cases. You wouldn't want to write those out. You kinda wish this is gonna be true, don't you?
	26	**Stephanie**	Yeah.
	27	**Researcher 1**	Actually, you did write that out when you were in the fourth grade.
	28	**Stephanie**	Oh yeah.
	29	**Researcher 1**	Right, you really did. We have a video to show it. Ok, and this ten, would that surprise you that it would be-if this is two, this would be three?
	30	**Stephanie**	No. I mean-
	31	**Researcher 1**	You would expect that wouldn't you?
	32	**Stephanie**	Yeah.
	33	**Researcher 1**	Because if you've done one, you've done half your work.
	34	**Stephanie**	Mm-hmm.

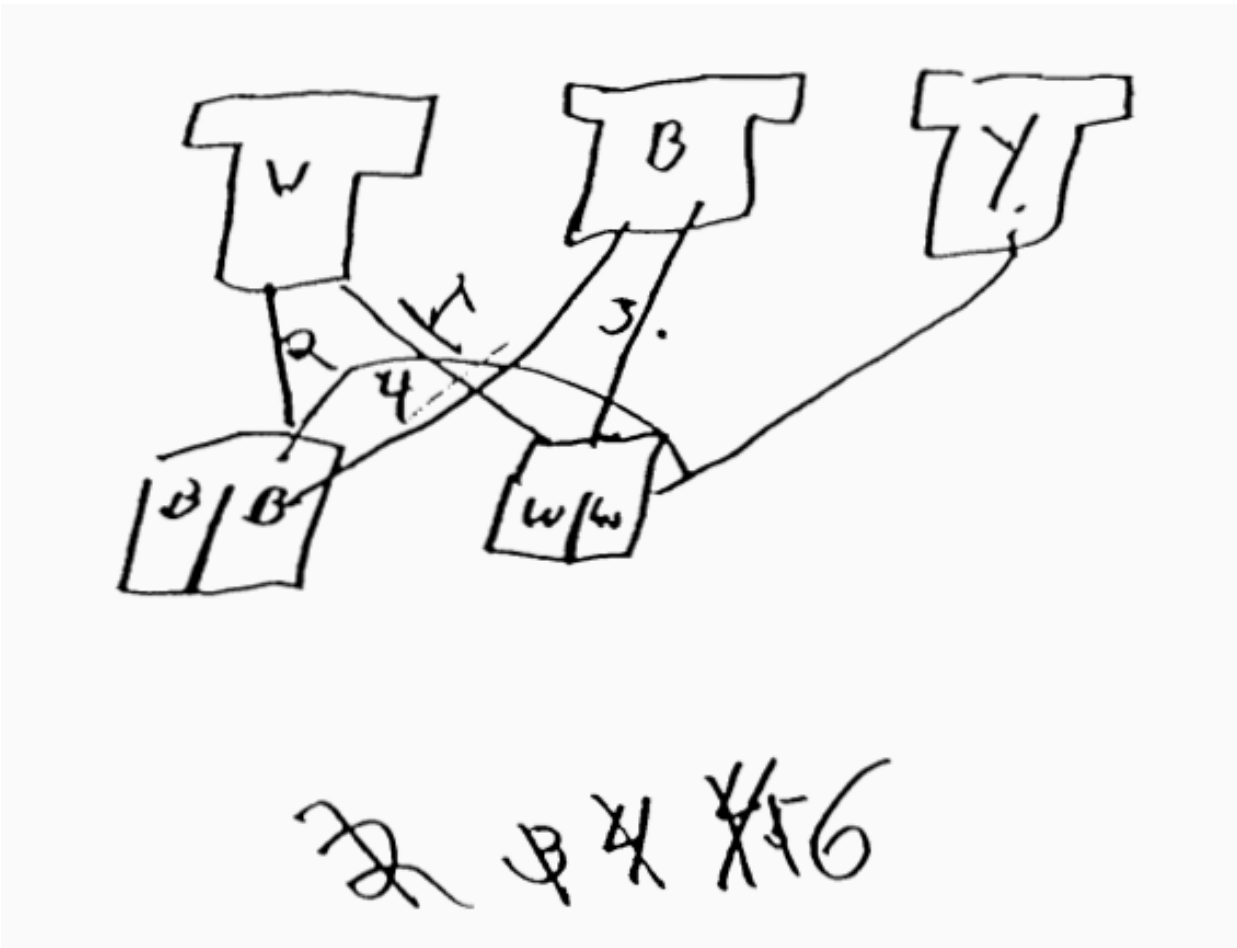
	35	**Researcher 1**	See this nice symmetry here. And the next one will be . . .
	36	**Stephanie**	Four.
	37	**Researcher 1**	And that doesn't surprise you, does it? That that's like this?
	38	**Stephanie**	Nope and the last one will be five. One.
	39	**Researcher 1**	So if I asked you, I'm now building these six, could you tell me how many that are exactly no red-
	40	**Stephanie**	Yeah. Yes.
	41	**Researcher 1**	-exactly one, exactly two, exactly three, exactly four? Now, you expect this should all add up to what if it's five high? If you total them, you should get a total of?
	42	**Stephanie**	Um, 32?
	43	**Researcher 1**	And does it? 6? 11? 21? Wait a minute, something's wrong here. Oh, I shouldn't be adding the 5- 6, 16, 26, 31, 32. So if this thing works, what should it add- what should this next row add up to?
	44	**Stephanie**	Um, 64?
	45	**Researcher 1**	Let's try it. Let's predict what this is going to be.
Time	Line	**Speaker**	Transcript
	46	**Stephanie**	It's going to be 1, 6, 15, 20, 15, 6, 1.
	47	**Researcher 1**	And does that add up to 64?
	48	**Stephanie**	Um, 30, 50, um, 12, Yeah.
	49	**Researcher 1**	You like that?
	50	**Stephanie**	Yes.
	51	**Researcher 1**	So not only do you know how many towers you're going to get by adding, what else do you know?
	52	**Stephanie**	I know the next row.
	53	**Researcher 1**	You know the next row.
	54	**Stephanie**	And, I don't know, I know how many combinations I get for each row.
	55	**Researcher 1**	Mh-hmm.
	56	**Stephanie**	Um.
	57	**Researcher 1**	Wasn't it clever, the person who found this out? Do you know who that was, would you like to know?
	58	**Stephanie**	Yes.
	59	**Researcher**	I don't know the guy's first name, but the last name is

		1	Pascal. Does anybody know his first name?
	60	**R3**	Blaise. B-l-a-i-s-e.
	61	**Researcher 1**	B-l-a-i-s-e. How do you say that? “Blaze” Pascal?
	62	**R3**	(inaudible) I’m not French.
	63	**Researcher 1**	And this thing is called Pascal’s Triangle. And so, I don’t think you realize, when you read this paper now, and see how hard you worked, you were really working pieces of Pascal’s Triangle.
	64	**Stephanie**	Hmm. It makes it easier.
	65	**Researcher 1**	It makes it easier?
	66	**Stephanie**	A lot easier.
	67	**Researcher 1**	You know something, Stephanie? I hate to get preachy, ‘cause my son will tell me “Ma, you’re getting preachy”, but if you hadn’t done all that hard work all those years
	68	**Stephanie**	Yeah.
Time	Line	**Speaker**	Transcript
	69	**Researcher 1**	this would make no sense to you now, I don’t think. Because I taught college and Mrs. Muter teaches college and Mrs. Steencken teaches college and the students work with this and they don’t see it. You know what I mean by see it?
	70	**Stephanie**	Yeah.
	71	**Researcher 1**	You see those cubes. You worked so hard at those.
	72	**Stephanie**	Yeah.
	73	**Researcher 1**	You know what I’m saying?
	74	**Stephanie**	Mh-hmm.
	75	**Researcher 1**	I mean, I don’t know. But it’s hard to visualize and see ‘cause they only deal with the numbers. They just learned this rule that you add these numbers you get this and you add these numbers, you get this.
	76	**Stephanie**	Mm-hmm.
	77	**Researcher 1**	And if someone asks me what is the combinations of selecting exactly one of a color from five. You know, they’ll give you the answer to that, but they have no picture of what they are giving you the answer to. They just are picking it out as a formula.
	78	**Stephanie**	Yeah.

	79	**Researcher 1**	You see that difference?
4:30	80	**Stephanie**	Yeah.

Appendix P: Session I: Student Work

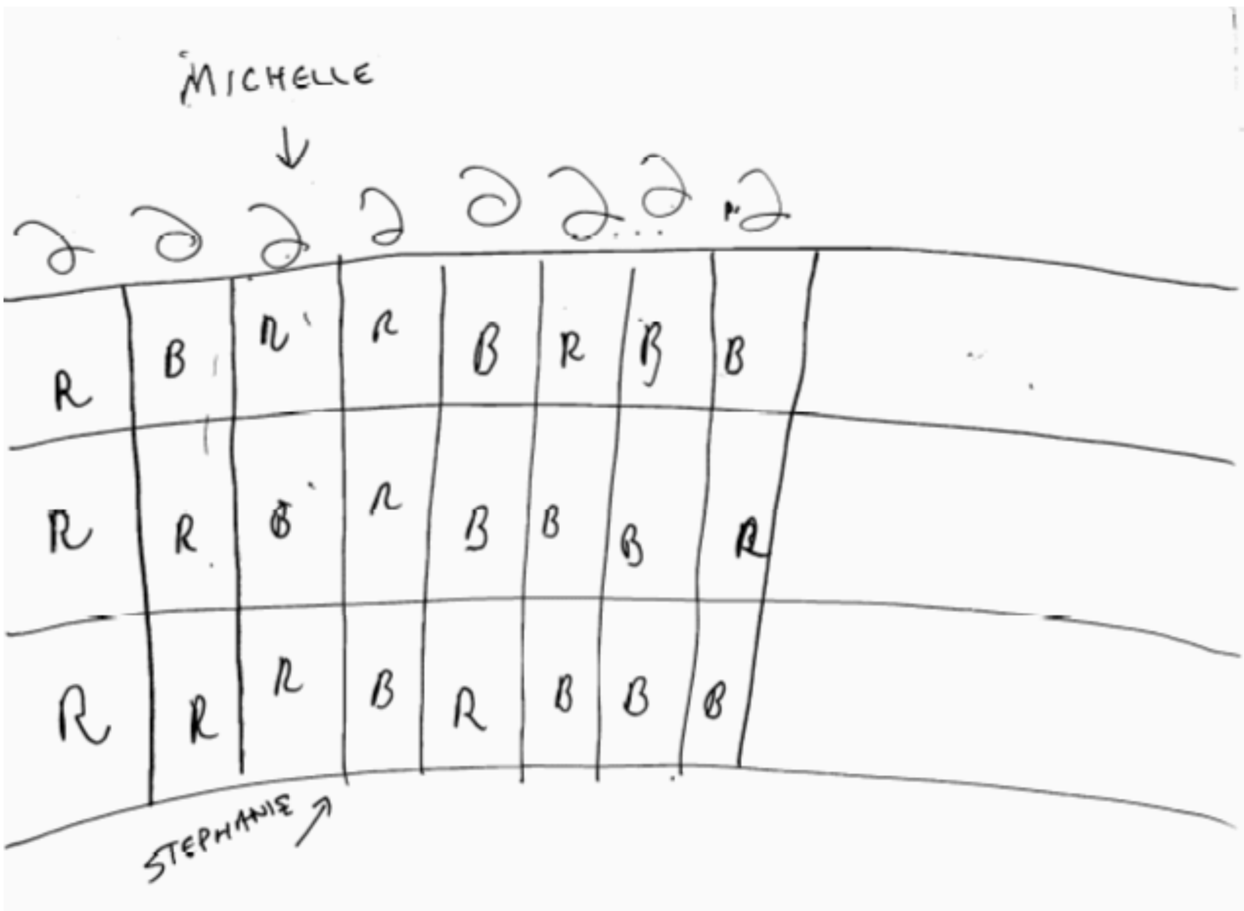
Stephanie's Grade 3 work for the Shirts and Pant's activity



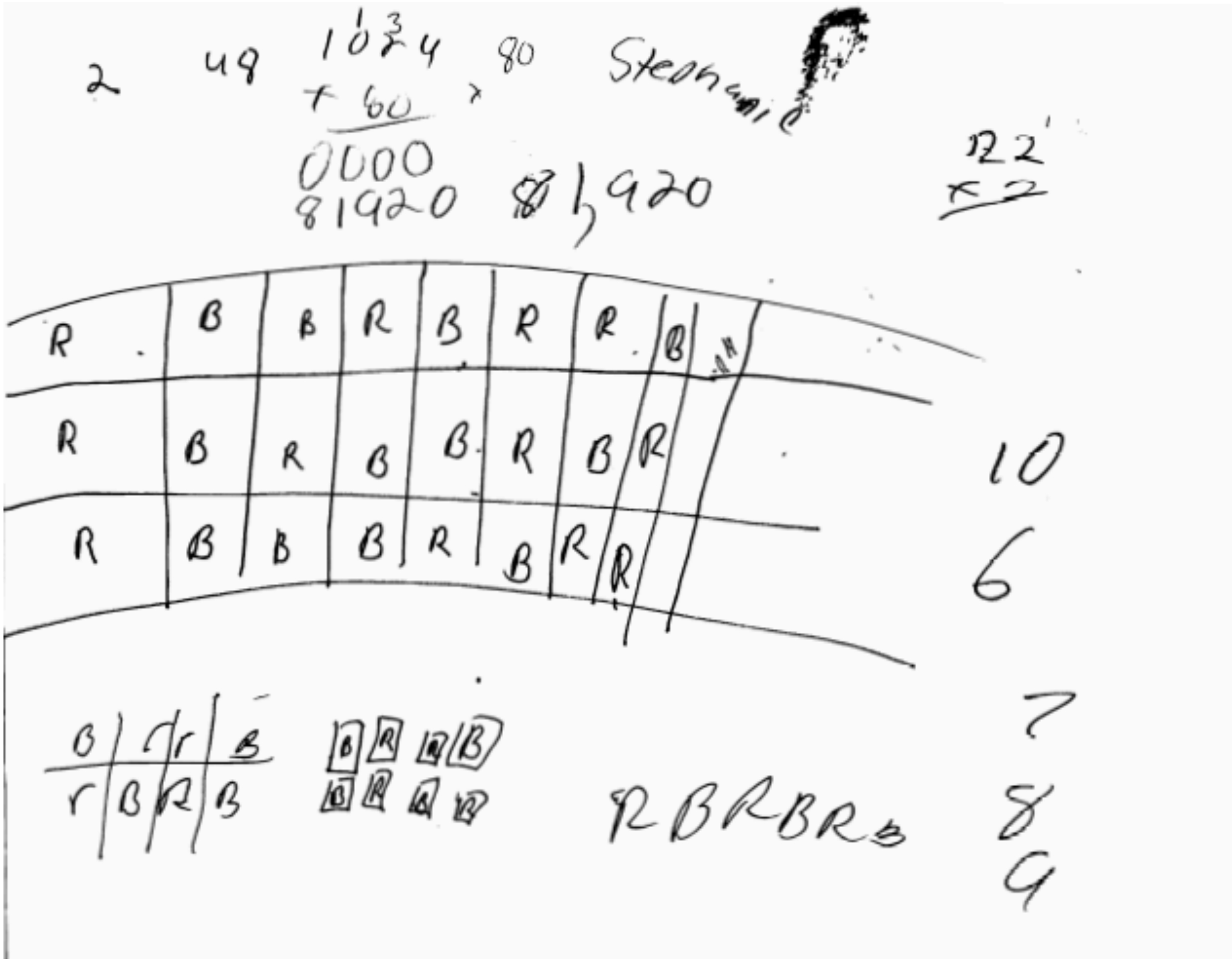
Appendix Q: Session IX: Student Work

Stephanie, Milin, Michelle, and Jeff's work during the Gang of Four Session on building towers of height five, selecting from two colors of Unifix cubes

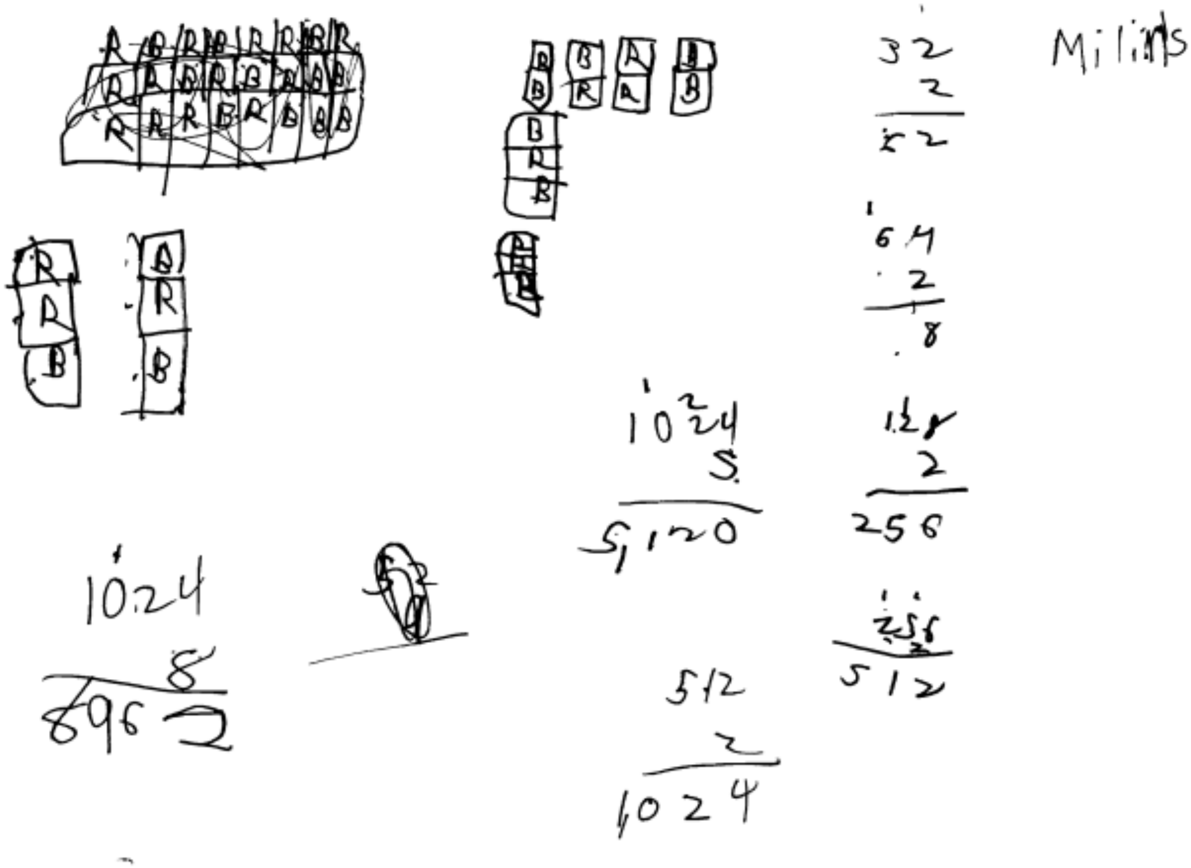
I. Stephanie and Michelle's student work for five-tall towers problem from Session IX



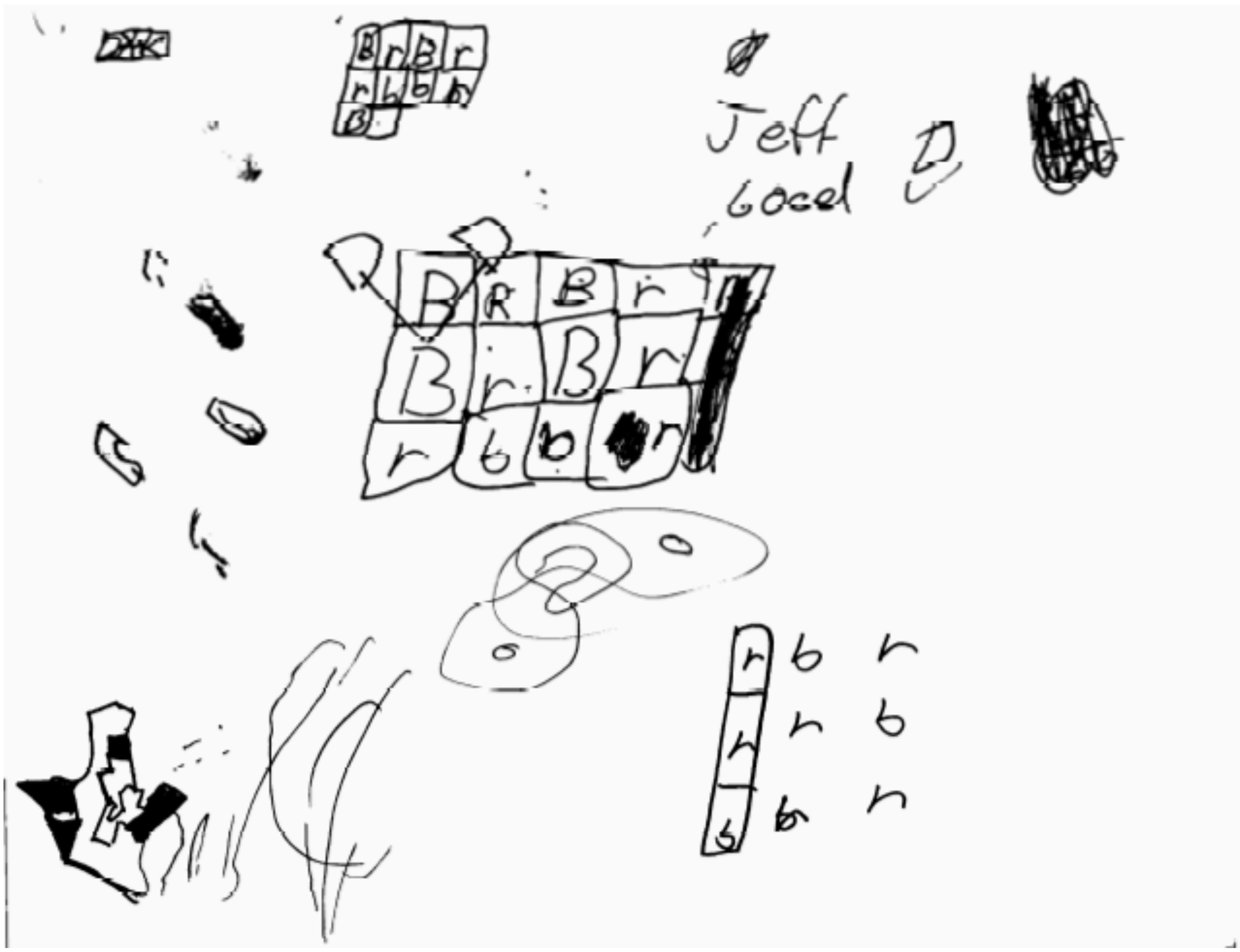
II. Stephanie's student work for five-tall towers problem from Session IX



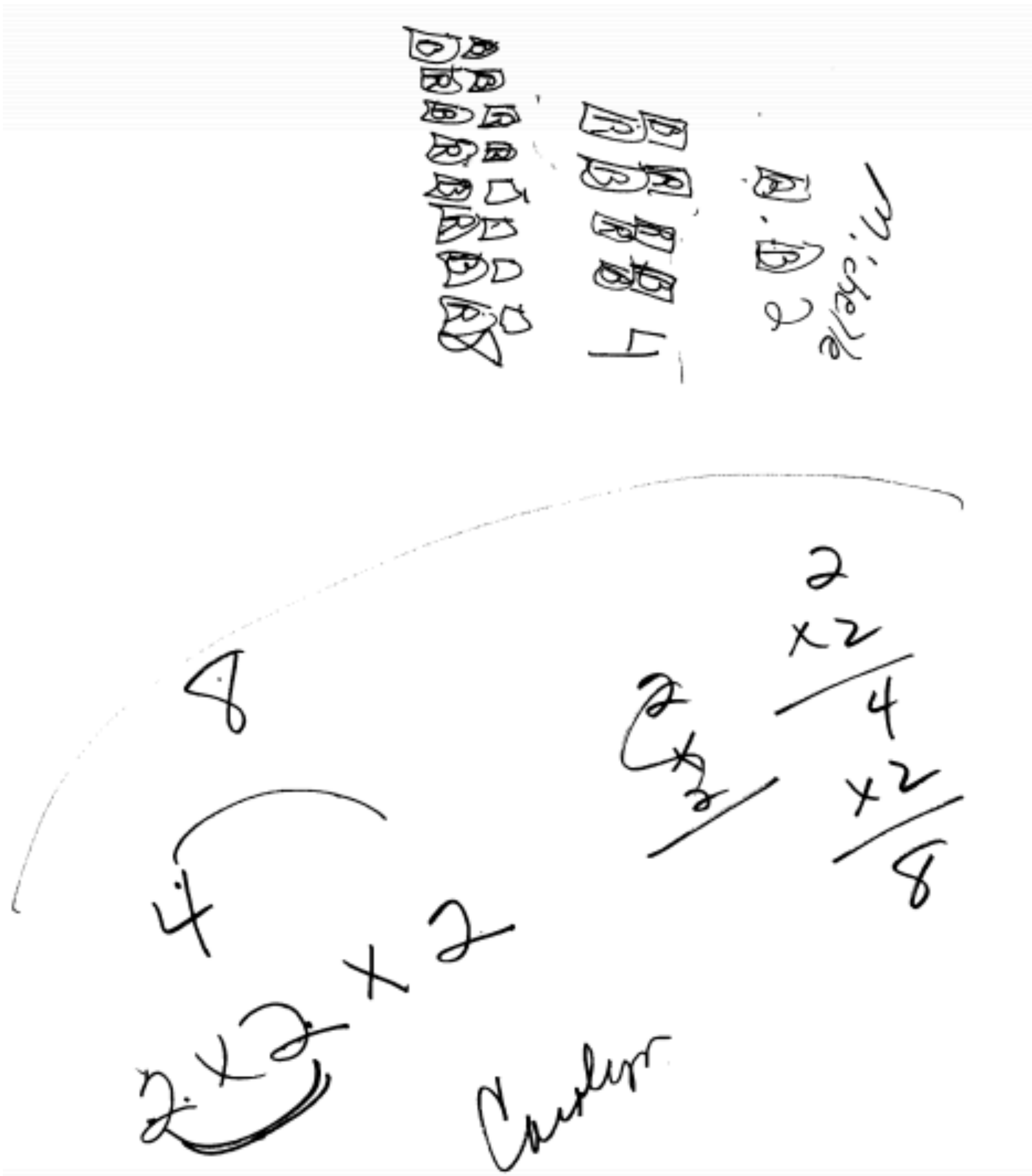
III. Milin's student work for five-tall towers problem from Session IX



IV. Jeff's student work for five-tall towers problem from Session IX



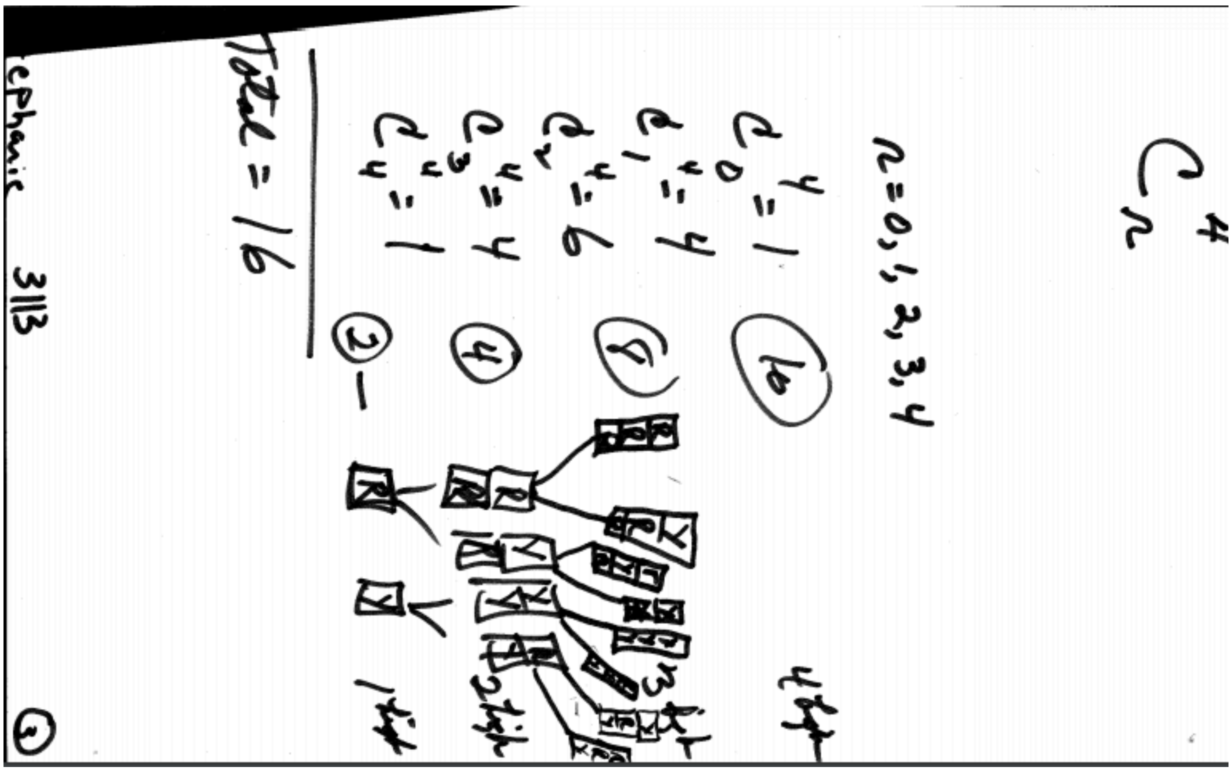
V. Michelle's student work for five-tall towers problem from Session IX



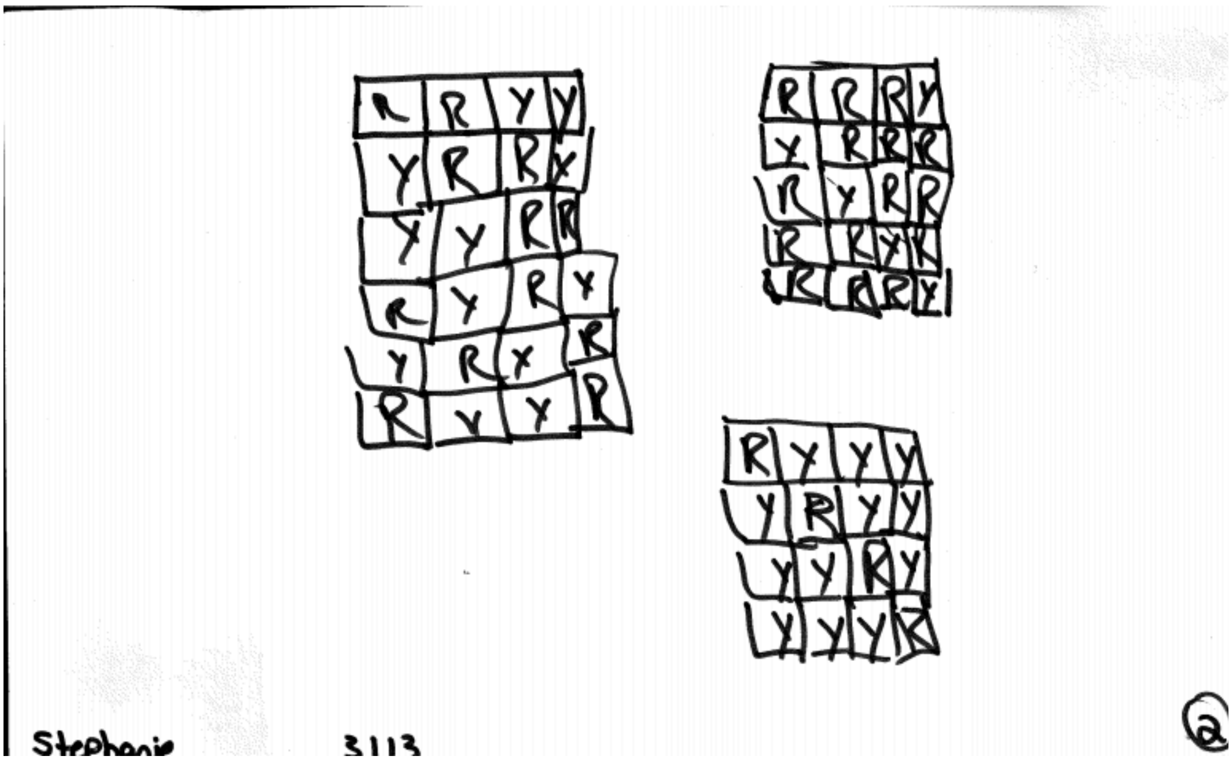
VI. Michelle's student work for five-tall towers problem from Session IX (continued)

Appendix R: Sessions X-XII: Student Work

I. Stephanie's student work for Sessions X to XIII



II. Stephanie's student work for Sessions X to XIII (continued)


Stephanie
3113