Coarse Geometry of Out$(A_1 * \ldots * A_n)$

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Abstract

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In this thesis we have examined $\Gamma_n := \mathsf{Out}(G_n)$ from the perspective of geometric group theory, where $G_n = A_1 * ... * A_n$, is a finite free product and each A_i is a finite group. We wanted to inspect hyperbolicity and relative hyperbolicity of such groups. We used the $\mathsf{Out}(G_n)$ action on the Guirardel-Levitt deformation space, [GL07], to find a virtual generating set and prove quasi isometric embedding of a large class of subgroups. To prove non-distortion we used arguments similar to those used in [HM13] and [Ali02]. We used these subgroups to prove that Γ_n is thick for higher complexities. Thickness implies that the groups are non relatively hyperbolic for higher complexities, [BDM09].

To

my family (parents, brother and wife)

Acknowledgements:

In a simpler universe, I would have filled this space by writing a ballad of Lee Mosher's knowledge, kindness and guidance which made this work possible. However, this is a more complex universe and I would be flagrantly ungrateful if I did not acknowledge the contributions of those who positively affected my life and supported my work for the last 7 years. Disclaimer: this is not the most comprehensive list.

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The department of mathematics at Rutgers-Newark has provided me with an excellent environment to learn and teach Mathematics. I would like to take this opportunity to acknowledge the help I received from the administrative staffs-Terry and Harolina (previously Jane and Katie).

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Contents

1 Introduction

Our research has been motivated by trying to answer the following questions:

Problem 1. If each A_i is a finite group, then is $\Gamma_n := \mathsf{Out}(A_1 * A_2 * ... * A_n)$ hyperbolic? If the answer is no, then is it relatively hyperbolic?

Questions similar to these have been answered for $\mathsf{Out}(F_n)$ by Behrstock-Druţu-Mosher[BDM09]. In case of mapping class groups, $\mathsf{MCG}(S)$, They have been independently answered by Karlsson-Noskov [KN04], Bowditch [Bow05]; Anderson-Aramayona-Shackelton[AAS07]; Behrstock-Druţu-Mosher [BDM09]. These are two of the most studied groups in geometric group theory.

1.1 Main theorem

The following theorem answers the original question.

Theorem 1.1. *If each A_i is finite group, and $\Gamma_n := \mathsf{Out}(A_1 * ... * A_n)$, then for*

1. $n \leq 2, \Gamma_n$ is finite.

2. $n = 3, \Gamma_n$ is infinite hyperbolic.

3. $n > 3$, Γ_n is a thick group of order at most one. As a consequence, Γ_n is non relatively hyperbolic when $n > 3$.

Remark 1.2. Hyperbolicity for $n = 3$ was proved by Collins, [Col88]. We will give an independent proof of hyperbolicity in lower complexities ($n \leq 3$) using theorem 1.3 and the topology of the deformation space.

1.2 Methodology

We have employed the following notable tools in our investigation -

1. **Deformation space of G-trees**, is a geodesic metric space on which Γ_n acts by isometries such that the action is properly discontinuous. We follow the work of Guirardel-Levitt [GL07], which is the most general theory of such spaces. Culler-Vogtmann spaces, see [CV86], are examples of Guirardel-Levitt deformation spaces. The outer automorphisms we have used for understanding the action resemble the symmetric outer automorphisms investigated by McCullough-Miller, see [MM96].

2. **Algebraic thickness** of groups introduced by Behrstock-Druţu-Mosher, see [BDM09]. Thickness is sufficient to conclude non-relative-hyperbolicity, see theorem 1.4.

Guirardel-Levitt showed:

Theorem 1.3. *[GL07, Theorem 6.1] Deformation space, $\mathcal{D}(G, \mathcal{H})$, is contractible.*

For $n > 3$, in addition to theorem 1.3 we use our understanding of Γ_n and its action on $\mathcal{D}(G, \mathcal{H})$ to inspect its thickness. Behrstock-Druţu-Mosher showed:

Theorem 1.4. *[BDM09, Corollary 7.9] If G is a finitely generated group which is thick, then G is not relatively hyperbolic.*

$\text{Out}(F_n)$, $\mathcal{MCG}(S)$ and some other classes of geometrically interesting groups are thick for all but finitely many cases and hence non relatively hyperbolic. To prove thickness we have to find suitable *undistorted*, zero thick subgroups of Γ_n. A subgroup is undistorted in Γ_n, if a Cayley graph of the subgroup can be quasi isometrically embedded in a Cayley graph of Γ_n. We use ideas from Handel-Mosher [HM13] to find a coarse Lipschitz retract from the spine of the deformation space to a sub-complex of the spine. Additionally we use ideas from Alibegović [Ali02] to prove non-distortion of another class of subgroups. A full justice to these ideas cannot be done in this short introduction; nonetheless, we would like to mention that one of the most innovative geometric ideas in this work can be found in the definition of coarse Lipschitz-retraction map, see definition 8.10. The author would like to express his gratitude towards Lee Mosher for this idea and most of the other ideas in this work.

Anthony Genevois has communicated that there is a nice argument for proving that Γ_n is NRH for $n > 7$ which depends on [GM].

2 Organization

In this section we will give a brief summary of each section in this exposition.

Section 1 In the introductory section, we have stated the main question, Problem 1. We have then stated our answer to the question in theorem 1.1. We have also briefly discussed the methodologies, subsection 1.2, used to investigate the question.

Section 3 In this section we have discussed some of the basic definitions and results in geometric group theory, which are relevant to our research. The reader can skip this section if the reader feels comfortable about the notions of quasi isometry, Milnor-Švarc lemma, hyperbolicity, relative hyperbolicity, Bass-Serre theory, undistorted subgroups and first barycentric subdivision.

Section 4 In section 4, we have defined the deformation space, definition 4.4 and in section 4.2 we have described the topology and geometry of the deformation space using collapse-expand moves (deformations). The contractibility of the deformation space in this topology is due to the work of Guirardel-Levitt, theorem 1.3. We conclude the section by proving that Γ_n acts geometrically on the spine of the deformation space, $\mathcal{SPD}(G, \mathcal{H})$ (remark 4.26). The homotopy equivalence of the deformation space and its spine follows from lemma 3.24.

Section 5 In section 5.1, we have proved the finiteness of Γ_2 using the triviality of $\mathcal{SPD}(G_2, \mathcal{H})$. An important consequence of this section is the uniqueness (up-to homeomorphism) of $A_i * A_j$-minimal sub-tree, discussed in remark 5.3. This uniqueness has been exploited in various times in sections 6, 8, to prove the ideas circling the most important results. In section 5.2 we have proved that Γ_3 is hyperbolic. This is the only result that uses the full power of the contractibility of the deformation space, theorem 1.3; elsewhere we have used path connectedness of deformation space. Guirardel-Levitt has given credit to Max Forester [For02] for the proof of path connectedness of the deformation space. In section 5.3 we have inspected the orbits of graphs of groups up-to homeomorphism of $\mathcal{SPD}(G_4, \mathcal{H})$ under the action of a finite index subgroup of Γ_4, called Ω_4.

Section 6 In section 6, we have considered a subgroup $\Gamma_n' \leq \Omega_n \leq \Gamma_n$, where Ω_n is the subgroup that fixes conjugacy class of each element in Γ_n and Γ_n' is generated by the outer automorphisms that have a representative automorphism which acts by the identity on at least one of the factors. We showed that Γ_n' is finite index in Γ_n. The idea of the proof is to find a connected sub-complex of $\mathcal{SPD}(G_n, \mathcal{H})$ on which Γ_n' and Γ_n acts where the actions are co-compact and properly discontinuous. The essential part of the proof is to establish path connectedness of the sub-complex, which has been done in corollary 6.25. A concrete example of the idea of this proof can be found for the case of $n = 4$ in claim 6.7 contained inside the proof of the proposition 6.5.

Section 7 Careful inspection of the definition of Γ_n' (definition 6.17), made it clear that there is a substantial collection of subgroups, which are direct products of infinite subgroups, when $n \geq 4$. Once we observed the presence of these infinite subgroups, our motivation was to find a thickly connected network, definition 7.6, of Γ'_4. So, a reader could start from section 7, and *see* that Γ'_n has a thickly connected network, corollary 7.23 . To prove thickness of Γ_n, definition 7.7, we had to prove that Γ'_n is finite index in Γ_n (section 6) and the subgroups in the network are undistorted in Γ'_n (section 8).

Section 8 In section 8, we prove non distortion of certain classes of subgroups, corollaries 8.14, 8.28, 8.37. The idea of the proofs of corollaries 8.14, 8.37 are similar. We found a sub-complex of $\mathcal{SPD}$ which has a Lipschitz retraction from $\mathcal{SPD}$ and are quasi-isometric to these sub-groups. This idea draws inspiration from Handel-Mosher's paper [HM13]. The idea of the proof of corollary 8.28 has been motivated by Alibegović's work [Ali02].

Section 9 In this section we have organized the our conclusions to give a complete overview of the proof of the theorem.

3 Definitions and Preliminaries

In this section, we will define and describe some of the fundamental concepts of geometric group theory. In geometric group theory, often the object of study is a geodesic metric space and a subgroup of its symmetry group. From another point of view the object of study is a group and its action on a geodesic metric space.

3.1 Fundamental observation in geometric group theory

The following fundamental observation in geometric group theory connects a group with the geodesic metric space on which it is acting.

Lemma 3.1 (Milnor[Mil68]-Švarc[Š55] lemma)**.** *For any group G and any proper geodesic metric space X, if there exists a properly discontinuous, co-compact, isometric action $G \curvearrowright X$ then G is finitely generated. Furthermore, for any such action and any point $x \in X$, the orbit map $g \mapsto g \cdot x$ is a quasi-isometry $\mathcal{O} : G \to X$, where G is equipped with the word metric of any finite generating set.*

Definition 3.2 (Geodesic metric space)**.** In a **geodesic metric space** we can define and measure length of any path using a function called **metric**. Additionally, any two points in the space can be connected by a shortest length path called **geodesic**.

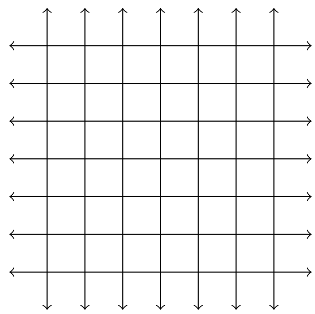
Definition 3.3. A metric space is proper if a closed ball is compact.

Every finitely generated group act on its *Cayley graphs* by isometries. A *geometry* of a finitely generated group means the geometry of a Cayley graph of the group. Equivalently, it means the geometry of any geodesic metric space on which the group acts *geometrically* (properly and co-compactly).

Definition 3.4 (Properly discontinuous action)**.** An action of a finitely generated group G on a geodesic metric space (X, d) is **properly discontinuous** if $\forall x \in X$, there is a neighborhood U_x of x such that the set $\{g \in G | g \cdot U_x \cap U_x \neq \phi\}$ is a finite set.

Definition 3.5 (Co-compact action)**.** An action of a finitely generated group G on a geodesic metric space (X, d) is **co-compact** if the quotient G/X is compact.

Definition 3.6 (Cayley graph)**.** The **Cayley graph** of a group with respect to a finite generating set is a metric space on which the group acts geometrically. Given a finitely generated group G and a finite generating set S, the Cayley graph of G is a graph with vertex set labeled by group elements and two vertices labeled by group elements g_1 and g_2 are connected by an edge directed from the former vertex to the latter vertex if $g_1^{-1}g_2$ is an element of S. If we assign length 1 to each edge and define the distance between any two vertices on the Cayley graph by the minimum number of edges required to connect the two vertices, then the Cayley graph can be realized as a geodesic metric space. The metric on a Cayley graph is a **word metric** on G with respect to the generating set S.



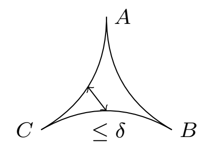
An unlabeled Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$

One of the main objectives in geometric group theory is to classify geodesic metric spaces up-to *quasi isometry*. Quasi isometry captures large scale geometric behaviors of metric spaces.

Definition 3.7 (Quasi Isometry)**.** A geodesic metric space (X, d_X) is said to be (K, C)-quasi isometrically embedded for $k \geq 1, C \geq 0$ in a geodesic metric space (Y, d_Y) if there is a function $f : (X, d_X) \to (Y, d_Y)$ which follows the following inequality $\forall x_1, x_2 \in X$
$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$. f is called a quasi isometric embedding.
Additionally, f is a quasi isometry if there is a $D \geq 0$ such that $\forall y \in Y, \exists x \in X$ with $d_Y(f(x), y) \leq D$. In this case X and Y are said to be quasi isometric.

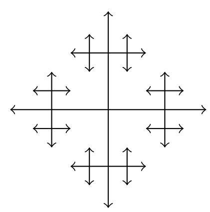
One of the most prominent quasi-isometry invariant is *hyperbolicity*. In other words a non hyperbolic space cannot be quasi isometric to a hyperbolic space.

Definition 3.8 (Hyperbolicity)**.** A geodesic metric space is called **hyperbolic** if all geodesic triangles are δ-thin for some fixed $\delta \geq 0$, i.e., any point on one side is within a distance δ of other two sides. A group is hyperbolic if one of its Cayley graphs is hyperbolic.



A $\delta-$thin geodesic triangle

Example 3.9. A tree with length of each edge 1 is a 0-hyperbolic geodesic metric space.

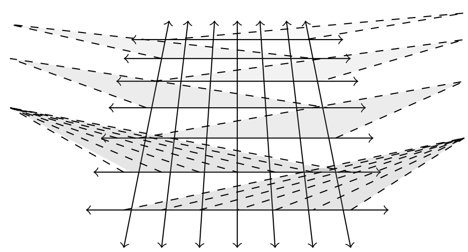


A valence 4 tree

Relative hyperbolicity serves as a quasi-isometry invariant for the groups which fail to be hyperbolic. A group is relatively hyperbolic, if we can construct a hyperbolic space which follows an additional technical condition by converting a collection of infinite diameter regions in a Cayley graph of the group to finite diameter regions using a method called *coning off*.

Definition 3.10 (Relative hyperbolic groups)**.** If G denotes a finitely generated group, $\mathcal{H} = \{H_1, ..., H_n\}$ is a finite family of subgroups of G and $\mathcal{LH}$ denotes the collection of left cosets of $H_1, ..., H_n$ in G. The group G is **weakly hyperbolic relative to** $\mathcal{H}$ if collapsing the left cosets in $\mathcal{LH}$ to finite diameter sets, in a Cayley graph of G, yields a δ-hyperbolic space. The subgroups $H_1, ..., H_n$ are called peripheral subgroups. The group G is (strongly) **hyperbolic relative to** $\mathcal{H}$ if it is weakly hyperbolic relative to $\mathcal{H}$ and if it has the bounded coset penetration property (BCP). BCP property, roughly speaking, requires that in a Cayley graph of G with the sets in $\mathcal{LH}$ collapsed to bounded diameter sets, a pair of quasi-geodesics with the same endpoints travels through the collapsed $\mathcal{LH}$ in approximately the same manner, see [Far98, Osi06, Bow12]. When a group contains no collection of proper subgroups with respect to which it is relatively hyperbolic, we say the group is **non relatively hyperbolic**, (NRH).

Example 3.11. $\mathbb{Z} \oplus \mathbb{Z}$ is weakly hyperbolic relative to $\mathbb{Z}$ but not relatively hyperbolic. In fact $\mathbb{Z} \oplus \mathbb{Z}$ is NRH. If A and B are finitely generated groups, $A * B$ is hyperbolic relative to subgroups A and B.



A coned-off Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$

Remark 3.12.

If G is a finitely generated subgroup and $H \leq G$ is a finite index subgroup, then H is quasi-isometric to G, where the quasi-isometry is given by the inclusion map.

3.2 Undistorted subgroups

The definition of algebraic thickness requires the existence of certain *undistorted* subgroups. In this section we will define an undistorted subgroup of a finitely generated group and then discuss relevant properties of a subgroup to prove non-distortion.

Definition 3.13. A finitely generated subgroup H of a finitely generated group G is said to be **undistorted** if the inclusion map of $H \hookrightarrow G$ induces a quasi-isometric embedding of Cayley graphs.

To prove non-distortion we have to prove only one side of the inequality, lemma 3.16. The relevant side of the inequality can also be stated as a *coarse Lipschitz map*

Definition 3.14 (Coarse Lipschitz map)**.** For constants $K \geq 1, C \geq 0$, a function
$f : (X, d_X) \to (Y, d_Y)$ is $(K, C)-$**coarse Lipschitz** if
$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$ for all $x_1, x_2 \in X$.

The following results gives us a way of proving non-distortion using the action.

Lemma 3.15. *[HM13, Corollary 10] If G is a finitely generated group acting properly discontinuously and co-compactly by isometries on a connected locally finite simplicial complex X , if $H < G$ is a subgroup, and if $Y \subset X$ is a nonempty connected sub-complex which is $H-$invariant and $H-$co-compact, then:*

1. H is finitely generated.

2. H is undistorted in G if and only if the inclusion $Y \hookrightarrow X$ is a quasi-isometric embedding.

3. The following are equivalent:

(a) H is a Lipschitz retract of G.

(b) The $0-$skeleton of Y is a Lipschitz retract of the $0-$skeleton of X.

(c) The $1-$skeleton of Y is a Lipschitz retract of the $1-$skeleton of X.

(d) Y is a coarse Lipschitz retract of X.

Lemma 3.16. *[HM13, Lemma 11] If X is a geodesic metric space and $Y \subset X$ is a rectifiable subspace, and if Y is a coarse Lipschitz retract of X, then the inclusion $Y \to X$ is a quasi-isometric embedding.*

3.3 G-trees and graphs of groups

Our understanding of $\mathsf{Out}(G)$ will be related to our understanding of a space of *G-trees*, called *deformation space* [GL07]. Roughly a deformation space is a space of *G-equivariance* classes of *G-trees*. The trees considered in this exposition are simplicial trees with metrics. It may be convenient at times only to consider the underlying simplicial structure.

Definition 3.17 (G-tree)**.** A group action of G on a metric (resp., simplicial) tree T via isometries (resp., simplicial homeomorphisms) is called **minimal**, if there are no proper G-invariant subtrees of T. A metric (resp., simplicial) tree on which G acts minimally is called a G**-tree**.

Definition 3.18 (G-equivariant isometry)**.** Consider metric G-trees T_1 and T_2. T_1 and T_2 are G**-equivariantly isometric** if there is an isometry $f : T_1 \mapsto T_2$ such that $g \in G \implies f(g.x) = g.f(x), \forall x \in T_1$

A *fundamental domain* for a G-tree gives us much relevant information about the action and the geometry and topology of the tree. In our research, a fundamental domain gives all the necessary information about the G-tree we are interested in.

Definition 3.19 (Fundamental domain of a G-tree)**.** A subtree F of a given G-tree is called a fundamental domain for the action $G \curvearrowright T$, if $G \cdot F \supset T$ and no other proper subtree of F has this property.

This interplay between a G-tree and its fundamental domain is captured by the Bass-Serre theory [Ser80].

Definition 3.20 (Fundamental group of a graph of groups)**.** A **graph of groups** over a graph X is an assignment

1. of a group G_x to each vertex x of X,
2. of a group G_y to each edge y of X, and
3. monomorphisms ϕ_{y_0} and ϕ_{y_1} mapping G_y into the groups assigned to the vertices at its ends.

Denote this graph of groups by **X**. If X is a tree then define the **fundamental group** of **X** is defined as $\Gamma := \langle G_x | x \in vert(X), \phi_{y_0}(e) = \phi_{y_1}(e) \forall e \in edge(X) \rangle$

Theorem 3.21. *[Fundamental theorem of Bass-Serre theory] Let G be a group acting on a tree T without inversions. Let $\mathbf{X}$ be the quotient graph of groups. Then G is isomorphic to the fundamental group of $\mathbf{X}$ and there is an equivariant isomorphism between the tree T and the Bass-Serre covering tree $T_{\mathbf{X}}$ (definition 3.23). That is, there is a group isomorphism $i : G \to \Gamma$ and a graph isomorphism $j : T \to T_{\mathbf{X}}$ such that $\forall g \in G$, $\forall$ vertex $x \in T$ and $\forall$ edge $e \subset T$ we have $j(g \cdot x) = g \cdot j(x)$ and $j(g \cdot e) = g \cdot j(e)$.*

Remark 3.22. In general, Γ is defined for any graph X (not necessarily a tree). However, assuming X to be a tree simplifies the definition and is sufficient for this exposition. This will also simplify the definition of the *Bass-Serre covering tree*.

Definition 3.23 (Bass-Serre covering tree)**.** For a given graph of groups $\mathbf{X}$ with fundamental group Γ and its underlying tree X (we are considering the special case where X is a tree), let G_x represents vertex group of a vertex x of X and G_e represents edge group of an edge e of X. Then define the **Bass-Serre covering tree** of $\mathbf{X}$, $T_{\mathbf{X}}$, as follows:

1. The vertex set of $T_{\mathbf{X}}$ is a disjoint union of points labeled by the cosets: $vert(T_{\mathbf{X}}) := \bigsqcup_{x \in vert(\mathbf{X})} \Gamma / G_x$

2. The edge set of $T_{\mathbf{X}}$ is a disjoint union of edges labeled by the cosets: $edge(T_{\mathbf{X}}) := \bigsqcup_{e \in edge(\mathbf{X})} \Gamma / G_e$

3. An inclusion of groups $G_e \hookrightarrow G_x$ induces a natural surjection map at the level of cosets $\Gamma/G_e \twoheadrightarrow \Gamma/G_x$. An edge is incident on a vertex if the edge label maps to the vertex label under this surjection.

3.4 Simplicial complex

Deformation space has an invariant spine on which $\mathsf{Out}(G)$ acts geometrically. The following theorem implies that the sub-complex spanned by the barycentric coordinates (spine) is 'good enough'substitute if the property of interest is a homotopy invariant.

Lemma 3.24. *Let, S be a connected subset of a finite dimensional simplicial complex, Δ, such that S is the complement of a sub-complex of Δ, then S deformation retracts onto S_B; where S_B is the sub-complex of the 1^{st} barycentric subdivision of Δ consisting of all simplices that lie entirely inside S.*

Proof. Let us denote the 1^{st} barycentric subdivision of Δ by Δ_B. Let us assume that S intersects a k-dimensional simplex of Δ denoted by σ, such that the 0-simplices of σ are denoted by $\{\alpha_0, ..., \alpha_k\}$. Let, $\sigma_B \subset \sigma$ be a k-dimensional simplex of Δ_B whose vertices are denoted by $\{\beta_0, ..., \beta_k\}$ such that $\beta_i = \displaystyle\sum_{j=0}^{i} \frac{\alpha_j}{i+1}$. Without loss of generality, assume that some of the vertices of σ_B are not in S. As S is the complement of a sub-complex of Δ, so $\beta_i \notin S$ for some i implies $\alpha_0 = \beta_0 \notin S$. Additionally, assume that - $\beta_i \notin S,$ when $i \in \{0, ..., l\}$ and $\beta_i \in S,$ when $i \in \{l+1, ..., k\}$.

$\beta_i \notin S \implies \sigma|_{\{\alpha_0,...,\alpha_i\}} \cap S = \phi \implies \sigma_B|_{\{\beta_0,...,\beta_i\}} \cap S = \phi$

Now, we will define a projection map r_{σ_B}

$$r_{\sigma_B} : S \cap \sigma_B \to S_B \cap \sigma_B$$

$$a_0\beta_0 + a_1\beta_1 + ... + a_k\beta_k \mapsto \frac{a_{l+1}}{1 - \sum_{j=0}^{l} a_j}\beta_{l+1} + ... + \frac{a_k}{1 - \sum_{j=0}^{l} a_j}\beta_k$$

If σ'_B is another simplex of Δ_B such that $\sigma_B \cap \sigma'_B \neq \phi$, then we will show that the map $r_{\sigma_B}|_{\sigma_B \cap \sigma'_B} = r_{\sigma'_B}|_{\sigma_B \cap \sigma'_B}$.

Assume that the 0-simplices of $\sigma'_B \cap \sigma_B$ are given by $\{\beta_p, ..., \beta_s\}$, where

$$\beta_i \in S \iff i \in \{r+1, ..., s-1, s\}$$

With these notations,

$$\begin{aligned} & r_{\sigma_B}|_{\sigma_B \cap \sigma'_B}(a_p\beta_p + ... + a_s\beta_s) \\ = & \frac{a_{r+1}}{1 - \sum_{j=p}^{r} a_j}\beta_{r+1} + ... + \frac{a_s}{1 - \sum_{j=p}^{r} a_j}\beta_s \\ & = r_{\sigma'_B}|_{\sigma_B \cap \sigma'_B}(a_p\beta_p + ... + a_s\beta_s) \end{aligned}$$

Hence, r_{σ_B} can be continuously extended to a map $r : S \to S_B$. By, definition $r|_{S_B}$ is the identity map. So, this map is a continuous deformation retract. $\square$

4 $\mathsf{Out}(A_1 * ... * A_n)$ and A Geometric Action

4.1 Ω_n - a finite index subgroup of $\mathsf{Out}(A_1 * ... * A_n)$

We want to understand the coarse geometric structure of $\mathsf{Out}(A_1 * ... * A_n)$. It will be convenient to consider the maximal subgroup of the outer automorphism group that preserves the conjugacy class of every element of each $A_i, i \in \{1, ..., n\}$. In lemma 4.2 we will prove that this subgroup is a finite index subgroup of $\mathsf{Out}(A_1 * ... * A_n)$. As a consequence, $\mathsf{Out}(A_1 * ... * A_n)$ will be quasi isometric to this subgroup.

Definition 4.1. Let, S_n be the symmetry group on first n natural numbers. Consider $\phi \in \Gamma_n$. If $\phi([A_i]) = [A_j],$ where $i, j \in \{1, ..., n\}$, then $s_\phi \in S_n$ is the element such that $s_\phi(i) = j,$ here $i, j \in \{1, ..., n\}$.

Lemma 4.2. *Consider the following map from Γ_n to the symmetry group on first n natural numbers, S_n:*

$$P : \Gamma_n \to S_n$$
$$\phi \mapsto s_\phi$$

P is a homomorphism of groups and the kernel of the map is a subgroup of Γ_n which preserves conjugacy classes of the free factors A_i.

Proof. Let $\phi_1, \phi_2 \in \Gamma_n$ be such that $\phi_1([A_i]) = [A_j]$ and $\phi_2([A_j]) = [A_k]$, then $\phi_2\phi_1([A_i]) = [A_k] \implies P(\phi_2\phi_1) = s_{\phi_2}s_{\phi_1}$. If $\phi([A_i]) = [A_j]$, then $\phi^{-1}([A_j]) = [A_i] \implies P(\phi^{-1}) = s_\phi^{-1}$. Now we will show that the kernel of the map P

is the subgroup of Γ_n which preserves the conjugacy classes of free factors. $\phi \in ker(P) \iff s_\phi = id_{S_n} \iff s_\phi(i) = i, \forall i \iff \phi([A_i]) = [A_i], \forall i$. $\square$

Definition 4.3. $\Omega_n := ker(P)$, is a finite index subgroup of $\mathsf{Out}(A_1 * ... * A_n)$ that preserves the conjugacy classes of the free factors.

4.2 Deformation Space

We will study the algebra and geometry of Ω_n by studying an action of Ω_n on a complete geodesic metric space, which is a subspace of the deformation space. The deformation space is a metric space with the following parameters - a group, (in our case the group is $A_1 * ... * A_n$), and a class of subgroups. As a set, the deformation space is the set of equivalence classes of G-trees with an additional condition on the vertex stabilizers, where two trees are equivalent if they are G-equivariantly isometric. In this section we will define the deformation space.

4.2.1 $\mathcal{D}(G, \mathcal{H})$ as a set of G-trees

Deformation space has been defined in [GL07] more generally. In contrast, we will consider the following definition of deformation space (as a set). This definition will result in a space on which $\mathsf{Out}(A_1 * ... * A_n)$ will act isometrically, and properly discontinuously.

Definition 4.4 (Deformation space as a set of G-trees)**.** Consider a group G, which is a free product of finite number of finitely generated indecomposable subgroups. The deformation space $\mathcal{D}(G, \mathcal{H})$ of G with respect to a collection of finitely generated subgroups $\mathcal{H}$ is the set of equivalence classes of minimal, metric G-trees with the following properties:

1. **Equivalence relation**: Two trees, T_1 and T_2 are equivalent in $\mathcal{D}(G, \mathcal{H})$ if T_1 is G-equivariantly isometric to T_2.

2. $\mathcal{H}$ **is the set of vertex stabilizers** : If T is a G-tree from the equivalence class $[T] \in \mathcal{D}(G, \mathcal{H})$, and $v \in T$ is a vertex of T, then $Stab(v) \in \mathcal{H}$. Conversely, given a $H \in \mathcal{H}, \exists$ a vertex, $v \in T$, with $Stab(v) = H$. Moreover, valence of a vertex with trivial vertex stabilizer must be greater than 2.

3. **Trivial edge stabilizer**: If T is a G-tree from the equivalence class $[T] \in \mathcal{D}(G, \mathcal{H})$, and e is an edge of T; then Stab$(e) = \{id\}$.

Remark 4.5. The deformation space that we have studied in this exposition is a special case of the deformation space discussed in [GL07]. Here, the maximal elliptic subgroups of the group under the group action on a G-tree are conjugates of the free factors of G, which are also vertex stabilizers. Additionally, the edge groups are trivial.

4.2.2 $\mathcal{D}(G, \mathcal{H})$ as a set of graph of groups

The goal of this subsection is to describe the deformation space as a set of equivalence classes of graphs of groups, such that each graph of groups has fundamental group G. Consider a point of $\mathcal{D}(G, \mathcal{H})$, corresponding to a G-tree $T \in \mathcal{D}(G, \mathcal{H})$. This point also corresponds to a graph of groups $\mathbf{X}$, where T is G-equivariantly isometric to the Bass-Serre tree of $\mathbf{X}$.

Consider a G-tree $T \in \mathcal{D}(G, \mathcal{H})$, $X := T/G$ is a finite graph. The finiteness is a result of the minimal action of the finitely generated group G on T. If we choose a fundamental domain for the action of G on T we can associate a graph of groups, $\mathbf{X}$, to T.

Lemma 4.6. *Consider a G-tree T such that the edge stabilizers are trivial and $X := T/G$ is a finite tree, then any fundamental domain of T under the action of G is isometric to X.*

Proof. We will prove that any fundamental domain is isomorphic to the quotient $X := T/G$.

Fix a fundamental domain of T under the action of G and name it Y. As the quotient is a tree, so no two points in Y are in the same orbit. Hence, we can define a unique bijective map f from X to Y.

$$f : X \to Y.$$

$$x \mapsto \text{the unique pre image of } x$$

f is an isometry as G acts on T by isometries. $\square$

Remark 4.7. [GL07, Page 147] If $T_1, T_2 \in \mathcal{D}(G, \mathcal{H})$ are trees in a deformation space then, then the rank of the quotient graphs T_1/G and T_2/G are the same. Hence, the underlying graph of every quotient graph of groups in our case is a tree.

Corollary 4.8. *Consider a G-tree T such that the edge stabilizers are trivial and $X := T/G$ is a finite tree. Fix a fundamental domain Y of T, then G is equal to the internal free product of vertex stabilizer subgroups of vertices in Y.*

Proof. Consider a graph of groups $\mathbf{X}$ whose underlying tree is isometric to X and the vertex group associated to a vertex of $\mathbf{X}$ under this isometry is the vertex stabilizer group of the corresponding vertex of X. Then, the Bass-Serre tree of $\mathbf{X}$ is equivariantly isometric to T. Hence, G is isomorphic to internal the free product of vertex groups of $\mathbf{X}$. □

4.2.3 A dictionary between two points of view of $\mathcal{D}(G, \mathcal{H})$

Consider a tree $T \in \mathcal{D}(G, \mathcal{H})$. We will now describe the graph of groups, $\mathbf{X}$, corresponding to T. The underlying graph of $\mathbf{X}$ is isomorphic to $X = T/G$. If X is a tree then, X is isomorphic to a fundamental domain of T under the action of G. The vertex group associated to a vertex of $\mathbf{X}$ under this homeomorphism is the vertex stabilizer subgroup of the corresponding vertex of X. The valence of a vertex with trivial vertex group is at least 3. The edge groups of $\mathbf{X}$ are trivial as the edge stabilizers of T are trivial.
Hence, a point of the deformation space can be represented by a graph of groups. Two graph of groups $\mathbf{X_1}$ and $\mathbf{X_2}$ represent the same point of $\mathcal{D}(G, \mathcal{H})$ if their Bass-Serre trees are G equivariantly isometric.

Remark 4.9. We will use the following dictionary to change our viewpoint of $\mathcal{D}(G, \mathcal{H})$ from a set of trees to a set of graph of groups and vice versa.

1. Consider a tree $T \in \mathcal{D}(G, \mathcal{H})$, a graph of groups $\mathbf{X}_T \in \mathcal{D}(G, \mathcal{H})$ representing the point T can be constructed once we fix a fundamental domain F of T. $\mathbf{X}_T$ is isometric to F as a graph and the vertex group associated to a vertex of $\mathbf{X}_T$ is the vertex stabilizer group of the corresponding vertex of F.

2. Consider a graph of groups, $\mathbf{X}$, with fundamental group G. The Bass-Serre tree, $T_{\mathbf{X}}$, of $\mathbf{X}$ represents $\mathbf{X} \in \mathcal{D}(G, \mathcal{H})$.

4.3 Geometry of Deformation Space

4.3.1 $\mathbb{O}_T$ - open cone of deformation space

Consider an equivalence class, $[T] \in \mathcal{D}(G, \mathcal{H})$. Let, us choose a tree $T \in [T]$. If T has $k+1$ orbits of edges then every tree in $[T]$ has $k+1$ orbits of edges. In terms of the graph of groups, if $\mathbf{X}$ is a graph of groups corresponding to T; then $\mathbf{X}$ has $k+1$ edges. For the rest of our exposition, we will abuse notation and denote an equivalence class in $\mathcal{D}(G, \mathcal{H})$ by a tree (or a graph of groups) belonging to the equivalence class. Consider the set, $\mathbb{O}_T := \{S \in \mathcal{D}(G, \mathcal{H}) | S$ is G-equivariantly homeomorphic to $T\}.\mathbb{O}_T$ is naturally bijective to the positive orthant of $\mathbb{R}^{k+1}$ which induces a topology on $\mathbb{O}_T$. If the edge lengths of distinct edge orbits of a tree $S \in \mathbb{O}_T$ is given by $e_0, e_1, ..., e_k$, then the bijection is described as follows:

$$\mathbb{O}_T \to \mathbb{R}^{k+1}$$
$$S \mapsto (e_0, e_1, ..., e_k).$$

Hence, $\mathbb{O}_T$ can be realized geometrically as a metric space isometric to the positive orthant of $\mathbb{R}^{k+1}$.

4.3.2 Admissible collapse and expand moves in $\mathcal{D}(G, \mathcal{H})$

A tree, $T \in \mathcal{D}(G, \mathcal{H})$ admits a **collapse move** if collapsing some edge orbits G-equivariantly, produces a tree $T' \in \mathcal{D}(G, \mathcal{H})$. **Admissible collapse move** is a relation (T, T') on $\mathcal{D}(G, \mathcal{H})$. The inverse of an admissible collapse move is an **admissible expand move**.

Admissible expand move can be defined independently. For a given fundamental domain F of T, and the set of vertices $\{v_i | v_i$ is a vertex of $F\}$; T admits an expand move at a vertex v_i, if v_i satisfies one of the following two conditions:

1. $stab(v_i) = id$ with valence of $v_i|_F > 3$.

2. $stab(v_i) \neq id$ with valence of $v_i|_F > 1$.

The following construction describes the tree obtained from T by expanding the vertex v_i. Let F' be a finite tree obtained by attaching the vertices of a tree(F^c) to the connected components of $F \setminus \{v_i\}$ and v_i, at the extremities marked by v_i. Such that, all the vertices of valence 1 and 2 in F^c is attached to at least one of the connected components. F' can be realized as a fundamental domain of a tree T' under the action of G which has been expanded G-equivariantly at the vertex v_i.

If we apply an admissible collapse move on a tree T, then the resulting tree, T', may not be in $\mathbb{O}_T$. In that case T' may be associated to a point on the boundary of the positive orthant, with one or more 0 coordinate, i.e., a point in one of the bounding hyperplanes of $\mathbb{O}_T$.

Remark 4.10. We have to administer our collapse moves cautiously, so that we do not produce a tree whose vertex stabilizer is not in the collection $\mathcal{H}$. Hence, the name *admissible* collapse move. Similarly, we have to exercise caution while applying expand moves to make sure that the resulting tree does not have a vertex of valence ≤ 2 with trivial vertex stabilizer.

4.3.3 Boundary of $\mathbb{O}_T$

After realizing $\mathcal{D}(G, \mathcal{H})$ as a collection of disjoint open orthants, our next goal is to give identification maps to the collection of open orthants.
Let, $\mathbb{O}_{T'}$ be a k' dimensional open simplex and $\mathbb{O}_T$ be a k dimensional open simplex, where $k' \leq k$. $\mathbb{O}_{T'}$ is a boundary of $\mathbb{O}_T$ if and only if T' is isomorphic to a tree obtained by applying collapse move on T.

4.3.4 $\mathcal{PD}(G, \mathcal{H})$ - projectivized deformation space

$\mathbb{R} \setminus \{0\}$ acts on a $k+1$ dimensional open cone and the quotient of the action can be identified with $\sigma_k = \{(e_0, e_1, ..., e_k) | \Sigma_{i=0}^{k} e_i = 1\}$, the k-dimensional open simplex in $\mathbb{R}^{k+1}$.

4.3.5 Topology of $\mathcal{PD}(G, \mathcal{H})$

A set in this space is closed if and only if the intersection of the set with any simplex is a closed subset of the simplex.

4.3.6 $\mathcal{SPD}(G, \mathcal{H})$ - spine of $\mathcal{PD}(G, \mathcal{H})$

The spine of the projectivized deformation space is a subspace of the projectivized deformation space. $\mathcal{SPD}(G, \mathcal{H})$ is the flag complex spanned by the barycenters of $\mathcal{PD}(G, \mathcal{H})$. That is a 0 simplex of $\mathcal{SPD}(G, \mathcal{H})$ is a point of $\mathcal{PD}(G, \mathcal{H})$ having equal value on every coordinate. For example, the 0-simplex corresponding to a $k-$dimensional open simplex of the projectivized deformation space is given by $\{(e_0, e_1, ..., e_k) | e_i = \frac{1}{k}, \forall i\}$. Observe that $\mathcal{PD}(G, \mathcal{H})$ deformation retract onto $\mathcal{SPD}(G, \mathcal{H})$ (lemma 3.24).

Remark 4.11. Contractibility of $\mathcal{D}(G, \mathcal{H}), \mathcal{PD}(G, \mathcal{H}), \mathcal{SPD}(G, \mathcal{H})$ follows from theorem 1.3.

4.4 Action of $\mathsf{Out}(G)$ on $\mathcal{SPD}(G, \mathcal{H})$

The goal of this section is to establish a geometric connection between $\mathsf{Out}(G)$ and $\mathcal{SPD}(G, \mathcal{H})$. We will show that $\mathsf{Out}(G) \curvearrowright \mathcal{SPD}(G, \mathcal{H})$ properly discontinuously and co-compactly.

Remark 4.12. For the rest of our exposition we will inspect the space $\mathcal{D}(G, \mathcal{H})$ for

$$G = G_n = A_1 * ... * A_n, \text{ where each } A_i \text{ is finite, and}$$

$$\mathcal{H} = \{H \leq G | H \text{ is indecomposable, vertex stabilizer subgroup of a } G - \text{tree}\}.$$

We may drop the subscript n from G_n, if it is clear from the context.

4.4.1 Structure of graph of groups in $\mathcal{D}(G_n, \mathcal{H})$

Lemma 4.13. *If a graph of groups has following properties:*

1. *There are exactly n non trivial vertex groups, $n \geq 2$.*

2. *The vertices having trivial vertex groups have valence greater than 2.*

3. *The edge groups are trivial.*

4. *The underlying graph is a finite tree.*

Then $n \leq \mathcal{V} \leq 2(n-1),$ and $,(n-1) \leq \mathcal{E} \leq 2n-3$, where $\mathcal{V}, \mathcal{E}$ represent the number of vertices, edges of the underlying tree, respectively.

Proof. The second set of inequalities follow from the first set of inequalities because in a finite tree the number of vertices is 1 more than the number of edges.
$n \leq \mathcal{V}$ follows from the fact that there are n non trivial vertex groups. Additionally, the lower bound is attained by a tree isometric to $[0, n-1]$ with the integer points of the interval realized as the vertices.
We will prove the second half of the first inequality by induction. Let us inspect a finite tree of groups having two vertex groups. Such a tree has at most 2 vertices of valence 1. The underlying space is homeomorphic to the interval $[0, 1]$, as we do not allow vertices of valence less than 3 for trivial vertex groups. So, the only possible configuration is a tree with 2 vertices and 1 edge. Now, let us assume this statement is true is for $n = k$. That is, a graph of groups with k non trivial vertex groups and satisfying conditions $2, 3,$ and 4 from above has at most $2(k-1)$ vertices; and a tree achieves this

upper bound. Using this tree we will construct a tree with $2k$ vertices having $k+1$ vertices of valence ≤ 2. Take this tree and choose an interior point of an edge. Attach the interval $[0,1]$ to this point by a quotient map where only the point 0 from the interval gets identified to the chosen point. In the quotient space, define the image of 0 and 1 as vertices. This way the quotient space formed can be realized as a tree with exactly $2(k-1)+2 = 2k$ vertices having $k+1$ vertices of valence 1.

Now, if there exists a tree T, with $k+1$ vertices of valence ≤ 2 satisfying conditions $2, 3,$ and, 4 and e is an edge containing a terminal vertex (a vertex of valence 1); then the larger connected component of $T \setminus \{\text{interior of } e\}$ is homeomorphic to a tree with k vertices of valence ≤ 2. From the previous paragraph it follows that such a tree can have at most $2(k-1)$ vertices. So, T can have at most $2k$ vertices. $\square$

4.4.2 $\mathsf{Out}(G)$ action on $\mathcal{D}(\mathcal{G},\mathcal{H})$

We will take the help of the following proposition to define an action of $\phi \in \mathsf{Out}(G)$ on the deformation space.

Definition 4.14. If $\Phi \in \mathsf{Aut}(G)$ is an automorphism and T is a G-tree, then define $\Phi(T)$ to be a G-tree which is isometric to T with a *twisted* action of G on T. The action is denoted by $\cdot_\Phi$ and is defined as-

$$g \cdot_\Phi x := \Phi(g) \cdot x, \forall x \in T, g \in G.$$

here the action($\cdot$) on the right denotes the original action.

Proposition 4.15. *If $\Phi_1, \Phi_2 \in \mathsf{Aut}(G)$ are two automorphisms representing the same outer automorphism class $\phi \in \mathsf{Out}(G)$ and $T \in \mathcal{D}(G, \mathcal{H})$ is a G-tree, then $\Phi_1(T)$ is G-equivariantly isometric to $\Phi_2(T)$.*

Proof. We will prove that if Φ is a non-identity automorphism representing the identity outer automorphism class, then $\Phi(T)$ is G-equivariantly isometric to T.

Let, Φ represent the trivial outer automorphism then $\exists h \in G$, such that $\Phi(g) = hgh^{-1}, \forall g \in G$. This motivates the definition of an isometry, f, between T and $\Phi(T)$

$$\begin{aligned} f : T \to& \Phi(T) \\ x \mapsto& h \cdot x \end{aligned}$$

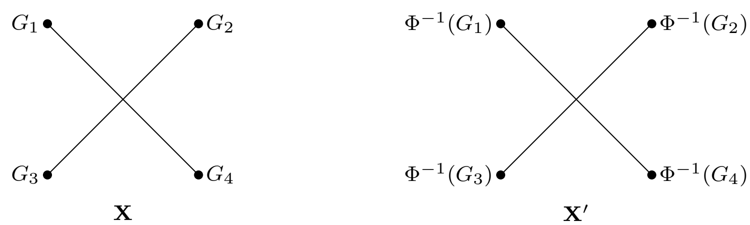
Next, we will verify the G-equivariance of the map f.
In T we have, $f(g \cdot x) = h \cdot (g \cdot x), \forall g \in G$.
In $\Phi(T)$ we have, $g \cdot_\Phi f(x) = hgh^{-1} \cdot (h \cdot x) = (hg) \cdot x, \forall g \in G$.
Hence, f is a G-equivariant isometry. $\square$

Definition 4.16 (Definition of the action)**.** Consider a G-tree $T \in \mathcal{D}(G, \mathcal{H})$ and $\phi \in \mathsf{Out}(G)$. $\phi(T)$ is the equivalence class of G-equivariant trees represented by $\Phi(T)$, where Φ is an automorphism from the class of outer automorphism ϕ.

Remark 4.17. If v is a vertex of T, then $\text{stab}_{\Phi(T)}(v) = \Phi^{-1}(\text{stab}_T(v))$, where $\Phi \in \mathsf{Aut}(G)$. If F is a fundamental domain of T with vertices $v_1, v_2, ..., v_d$ and vertex stabilizers $G_{v_1}, G_{v_2}, ..., G_{v_d}$, respectively; then F is a fundamental domain in $\Phi(T)$ and the vertex stabilizers of the vertices $v_1, v_2, ..., v_d$ are given by $\Phi^{-1}(G_{v_1}), \Phi^{-1}(G_{v_2}), ..., \Phi^{-1}(G_{v_d})$, respectively.

Following remarks 4.9 and 4.17, we can give a simpler description of the action $\mathsf{Out}(F_n) \curvearrowright \mathcal{D}(G, \mathcal{H})$, when the latter is considered as a space of graph of groups.

Definition 4.18. Let $\mathbf{X} \in \mathcal{D}(\mathcal{G}, \mathcal{H})$ be a graph of groups whose underlying graph is denoted by X; and $\Phi \in \mathsf{Out}(G)$ be an automorphism. Define $\Phi(\mathbf{X})$ (denoted by $\mathbf{X}'$) to be a graph of groups whose underlying graph, X', is related to X by an isometry $i : X \to X'$, such that if v is a vertex of $\mathbf{X}$ having G_v as the vertex group; then the vertex group corresponding to $i(v)$ is $\Phi^{-1}(G_v)$.



Proposition 4.19. *Let $\Phi \in \mathsf{Aut}(G)$ and $\mathbf{X} \in \mathcal{D}(G, \mathcal{H})$ be a graph of groups whose Bass-Serre tree is denoted by $T_{\mathbf{X}}$, then the Bass-Serre tree of $\Phi(\mathbf{X})$ is $\Phi(T_{\mathbf{X}})$.*

Proof. Let the vertices and the vertex groups of $\mathbf{X}$ be labeled as $v_1, v_2, ..., v_d$, and $G_{v_1}, G_{v_2}, ..., G_{v_d}$, respectively. We can use the same vertex labeling for the vertices of $\Phi(\mathbf{X})$ and the associated vertex groups are $\Phi^{-1}(G_{v_1}), \Phi^{-1}(G_{v_2}), ...,$ $\Phi^{-1}(G_{v_d})$, respectively.

So, there is a fundamental domain of $T_{\mathbf{X}}$ and $T_{\Phi(\mathbf{X})}$(Bass-Serre tree of $\Phi(\mathbf{X})$) with vertex stabilizers of the vertices given by $\{G_{v_1}, G_{v_2}, ..., G_{v_d}\}$ and $\{\Phi^{-1}(G_{v_1}), \Phi^{-1}(G_{v_2}), ..., \Phi^{-1}(G_{v_d})\}$, respectively. On the other hand, $\Phi(T_{\mathbf{X}})$ has a fundamental domain with vertex stabilizer group given by $\{\Phi^{-1}(G_{v_1}), \Phi^{-1}(G_{v_2}), ..., \Phi^{-1}(G_{v_d})\}$.
From the bijective correspondence between the fundamental domain and the Bass-Serre tree in $\mathcal{D}(G, \mathcal{H})$ we conclude that $\Phi(T_{\mathbf{X}})$ is G-equivariantly isometric to $T_{\Phi(\mathbf{X})}$. □

Corollary 4.20. *If $\Phi_1, \Phi_2 \in \mathsf{Aut}(G)$ are two automorphisms representing the same outer automorphism class $\phi \in \mathsf{Out}(G)$ and $\mathbf{X} \in \mathcal{D}(G, \mathcal{H})$ is a graph of groups, then $\Phi_1(\mathbf{X})$ is G-equivariantly isometric to $\Phi_2(\mathbf{X})$.*

Definition 4.21 ($\mathsf{Out}(G)$ action on a graph of groups)**.** If $\phi \in \mathsf{Out}(G)$ and $\mathbf{X} \in \mathcal{D}(G, \mathcal{H})$, then $\phi \cdot \mathbf{X} := \Phi(\mathbf{X})$, where Φ is an automorphism from the outer automorphism class ϕ.

4.5 Properties of the action

$\mathcal{D}(G, \mathcal{H})$ is locally finite when the the elements of $\mathcal{H}$ are finite subgroups. Later, we will prove proper discontinuity and co-compactness (when restricted to the spine of $\mathcal{PD}(G, \mathcal{H})$) of the action.

Lemma 4.22. *$\mathcal{D}(G, \mathcal{H})$ is a locally finite topological space when $|H| < \infty, \forall H \in \mathcal{H}$.*

Proof. Consider a tree $T \in \mathcal{D}(G, \mathcal{H})$. This point is on the boundary of other open simplices if we can equivariantly expand some edge-orbits of T. The number of edge orbits of T is bounded above by $2n-3$ and the number of vertex orbits are bounded above by $2n-2$. Since, the vertex groups are finite, each vertex has a finite valence. Hence, the number of fundamental domains containing a vertex is bounded above. So, the number of G-equivariant vertex expansions is bounded above for the tree T.

Therefore, the relative open simplex containing T can be a boundary to at most finitely many relative open simplices. As a result $\mathcal{D}(G, \mathcal{H})$ is locally finite. $\square$

Lemma 4.23. *Stabilizer of any point of $\mathcal{D}(G, \mathcal{H})$ under the action of*
$\mathsf{Out}(G)$ *is finite.*

Proof. Consider a tree $T \in \mathcal{D}(G, \mathcal{H})$. ϕ is a stabilizer of the point T, if $\phi(T)$ is G-equivariantly isometric to T. Let us fix a fundamental domain of T and name it F. As $\phi \in stab(T)$, $\phi(T)$ contains a fundamental domain identical to F (isometric and same vertex stabilizers under the action of G).

Now, let us fix a vertex $v \in F$ and choose a representative automorphism Φ from the outer class ϕ such that $stab(v)|_{\Phi(T)} = stab(v)|_T$. So, Φ permutes the fundamental domains isomorphic to F based at $v \in T$. However, there are only finitely many such fundamental domains at a given vertex and finitely many vertices v of F. So, the vertex stabilizer subgroup is finite. $\square$

A corollary of the two previous results is proper discontinuity of the action-

Corollary 4.24. *The action of* $\mathsf{Out}(G)$ *on* $\mathcal{D}(G, \mathcal{H})$ *is properly discontinuous.*

$\mathcal{D}(G, \mathcal{H})$ and $\mathcal{PD}(G, \mathcal{H})$ are not simplicial complexes. The spine of $\mathcal{PD}(G, \mathcal{H})$ is a simplicial complex and is denoted by $\mathcal{SPD}(G, \mathcal{H})$. $\mathcal{PD}(G, \mathcal{H})$ deformation retracts onto $\mathcal{SPD}(G, \mathcal{H})$. The advantage of working with $\mathcal{SPD}(G, \mathcal{H})$ is that the quotient of the action $\mathsf{Out}(G) \curvearrowright \mathcal{SPD}(G, \mathcal{H})$ is compact, which is not true for the action on $\mathcal{PD}(G, \mathcal{H})$.

Proposition 4.25. *The action of* $\mathsf{Out}(G)$ *on* $\mathcal{SPD}(G, \mathcal{H})$ *is co-compact.*

Proof. Consider a graph of groups $\mathbf{X} \in \mathcal{D}(G, \mathcal{H})$. Each vertex group is either trivial or a conjugate of exactly one of the $A_i, i \in \{1, ..., n\}$ such that the fundamental group of the graph of groups is G. Hence, the internal free product of the vertex groups is G and we can define a G automorphism Φ which maps each $A_i, i \in \{1, ..., n\}$ to the vertex group of $\mathbf{X}$ conjugate to A_i. If ϕ is the outer automorphism $[\Phi]$, then $\phi \cdot \mathbf{X}$ is a graph of groups with the set of vertex groups $\{A_1, ..., A_n\}$.
So, under the action of $\mathsf{Out}(G)$ on $\mathcal{D}(G, \mathcal{H})$ every graph of groups is in the orbit of a graph of groups with the set of vertex groups $\{A_1, ..., A_n\}$.
The underlying graph of any graph of groups from $\mathcal{SPD}(G, \mathcal{H})$ is a tree with at most $2n-3$ edges and at least $n-1$ edges. Hence, up-to homeomorphism there are only finitely many graphs of groups with the set of vertex groups $\{A_1, ..., A_n\}$.
$\mathsf{Out}(G)$ acts by isometries on $\mathcal{SPD}(G, \mathcal{H})$, which is a simplicial complex. The quotient is a finite dimensional locally finite simplicial complex such that there are only finitely many vertices. Hence, the quotient is compact. □

Remark 4.26. $\mathsf{Out}(G)$ action on $\mathcal{SPD}(G, \mathcal{H})$ is properly discontinuous and co-compact. Hence, by Milnor-Švarc lemma $\mathsf{Out}(G)$ is quasi isometric to $\mathcal{SPD}(G, \mathcal{H})$. We will exploit this fact to answer the original question in lower complexities and also to find a virtual generating set of $\mathsf{Out}(G)$ in general.

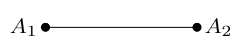
5 Structure of Deformation Space in Lower Complexities

Recall that $\Gamma_n := \mathsf{Out}(A_1 * ... * A_n)$, where each A_i is a finite group. Ω_n is the finite index subgroup of $\mathsf{Out}(A_1 * ... * A_n)$ which preserves conjugacy class of each free factor. In this section we will prove that Ω_2 is finite and Ω_3 is a hyperbolic group (virtually free). We will also inspect a finite index subgroup of Ω_4 and denote it by Γ_4'. This will lay the ground work for a similar inspection for $\Omega_n, (n \geq 5)$.

5.1 Finiteness of Γ_2

Lemma 5.1. *$\mathcal{SPD}(G_2, \mathcal{H})$ is a point.*

Proof. Consider the graph of groups:



The Bass-Serre tree of this graph of groups is a G-tree whose vertex stabilizers are conjugates of $A_i, i \in \{1, 2\}$. There is only one edge orbit. If we collapse an edge equivariantly in this tree, we will get a point. So, no G-equivariant collapses are possible. Contractibility of the deformation space due to theorem 1.3 implies that if there is a different tree non G-equivariantly homeomorphic to the given tree, then they can be connected in the deformation space by a collapse expand path. However, a collapse or expand move is not permissible due to the constraint on the vertex stabilizers. So, we arrive at a contradiction. □

Corollary 5.2. *Ω_2 and Γ_2 are finite.*

Remark 5.3. As a consequence of lemma 5.1, any two G-trees in $\mathcal{D}(G,\mathcal{H})$ have G-equivariantly homeomorphic $(H * K)$-minimal subtrees; where H and K are finite subgroups of G such that $G = H * K * F$ for some $F \leq G$.

5.2 Hyperbolicity of Γ_3

Proposition 5.4. *$\mathcal{SPD}(G_3,\mathcal{H})$ is a 1 dimensional simplicial complex.*

Proof. If $T \in \mathcal{SPD}(G_3,\mathcal{H})$ is a G-tree then the number of edge orbits of T is at most 3 and at least 2. Hence, we can apply only 1-edge orbit collapse move on T. So, $\mathcal{SPD}(G_3,\mathcal{H})$ does not have any 2 dimensional simplex and is a 1 dimensional simplicial complex. □

Corollary 5.5. *Ω_3 and Γ_3 hyperbolic groups.*

Proof. By Guirardel-Levitt's work (theorem 1.3) $\mathcal{SPD}(G_3,\mathcal{H})$ is contractible. Also, $\mathcal{SPD}(G_3,\mathcal{H})$ is a 1 dimensional simplicial complex. So, $\mathcal{SPD}(G,\mathcal{H})$ is a tree.

Ω_3 and Γ_3 act geometrically on $\mathcal{SPD}(G_3,\mathcal{H})$. Using lemma 3.1 we can say that Ω_3 and Γ_3 are hyperbolic. □

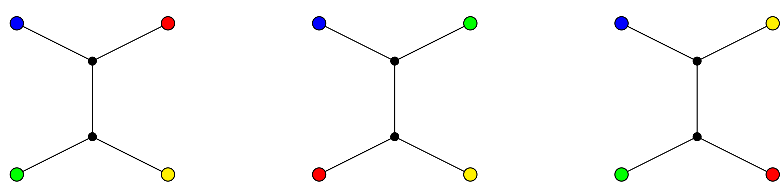
5.3 Orbit of $\mathcal{SPD}(G_4,\mathcal{H})$ under the action of Ω_4

Let us recall the definition of Ω_n (definition 4.3).

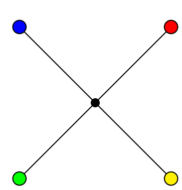
Lemma 5.6. *The following graphs of groups are representatives of a complete list of distinct orbits of graphs of groups (up to homeomorphism of graphs of* groups) in $\mathcal{SPD}(G_4, \mathcal{H})$ under the action of Ω_4. The vertices of the graphs have been color coded to represent vertex groups. The following table associates the colors to the vertex groups-

Vertex Color	Vertex group
Blue	A_1
Red	A_2
Green	A_3
Yellow	A_4

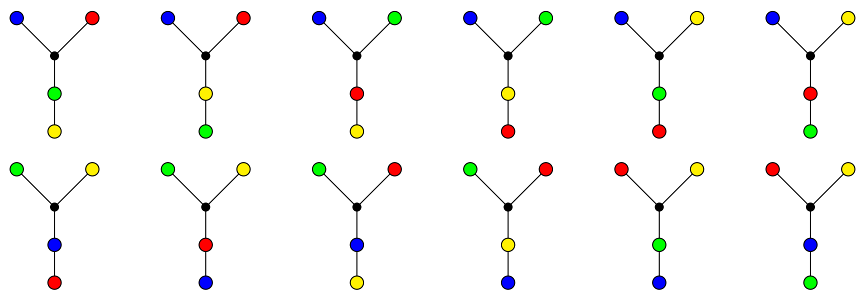
1. Three graphs of groups with 6 vertices and 5 edges. 4 of those vertices have a non trivial vertex group and have valence 1. The other 2 vertices have trivial vertex groups and the valence is 3. The open simplex containing these graphs of groups are 4-dimensional in $\mathcal{PD}(G_4, \mathcal{H})$.



2. One graph of groups with 5 vertices and 4 edges. 4 of those vertices have a non trivial vertex group and have valence 1. 1 vertex has a trivial vertex group and has valence 4. The open simplex containing these graphs of groups are 3-dimensional in $\mathcal{PD}(G_4, \mathcal{H})$.

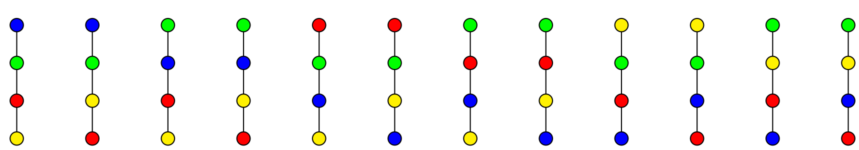


3. Twelve graphs of groups with 5 vertices and 4 edges. 4 of those vertices have non trivial vertex groups. 3 of the vertices having non trivial vertex groups have valence 1; 1 vertex with non trivial vertex group has valence 2. 1 vertex has trivial vertex group and valence 3. Open simplices containing these graphs of groups are 3-dimensional in $\mathcal{PD}(G_4, \mathcal{H})$.



4. Four graphs of groups with 4 vertices and 3 edges. 3 of those vertices have valence 1. 1 vertex has valence 3. Open simplices containing these graphs of groups are 2-dimensional in $\mathcal{PD}(G_4, \mathcal{H})$.

5. Twelve graphs of groups with 4 vertices and 3 edges. 2 of those vertices have valence 2; The other 2 vertices have valence 1. Open simplices containing these graphs of groups are 2-dimensional in $\mathcal{PD}(G_4, \mathcal{H})$.

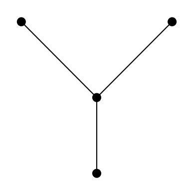


Proof. We will use lemma 4.13 to prove this lemma. Let, $\mathbf{Z} \in \mathcal{D}(G_4, \mathcal{H})$ be a graph of groups. $\mathbf{Z}$ has 4 non trivial vertex groups. So, the underlying graph Z has one of the following structures:

1. 4 **vertices,** 3 **edges** If a finite tree has 3 edges then the sum of all the valences over all vertices is $2 \times 3 = 6$

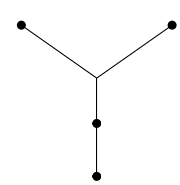
(a) 2 **terminal vertices**: If a finite tree has two terminal vertices then it is homeomorphic to a closed interval. So, the only possible configuration is:

(b) 3 **terminal vertices**: Then the non terminal vertex will have valence 3. So, the possible configuration is:

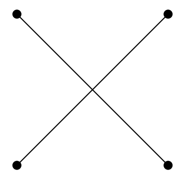


2. 5 **vertices,** 4 **edges** If a finite tree has 4 edges then the sum of all the valences over all vertices is $2 \times 4 = 8$

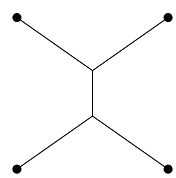
(a) 3 **terminal vertices**: Then the sum of the valences of the non terminal vertices is $8 - 3 = 5$. So, one vertex will have valence 2 and the other one will have valence 3.



(b) 4 **terminal vertices**: Then the sum of the valences of the non terminal vertices is $8 - 4 = 4$. So, the non terminal vertex will have valence 4.



3. 6 **vertices,** 5 **edges**: If a finite tree has 5 edges then the sum of all the valences over all vertices is $2 \times 5 = 10$. In this case the tree cannot have less than 4 terminal vertices. So, 4 terminal vertices will force 2 non terminal vertices to have valence 3 each. The only possible configuration is



$\square$

6 A finite index subgroup of Ω_n

In this section we will investigate a subgroup generated by some elements of Ω_n (definition 4.3) and prove that the subgroup is finite index.

In the first part we will prove this result for Ω_4 and later generalize to all n.

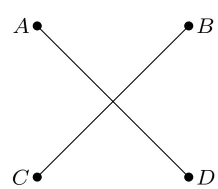
6.1 A finite index subgroup of Ω_4

We will use the following notations to better communicate the ideas.

Notation 6.1. *We will use the following notation in this subsection:*

1. Denote the finite groups by $A, B, C,$ *and* D *instead of* $A_1, A_2, A_3,$ *and* A_4*, respectively.*

2. We will use $\mathbf{X}$ *to represent the the following graph of groups.*



$\mathbf{X}$

3. We will denote the open simplex of $\mathcal{PD}$ *containing the graph of groups* $\mathbf{Y}$ *by* $\sigma_{\mathbf{Y}}$*.* $T_{\mathbf{Y}}$ *will represent the Bass-Serre tree of* $\mathbf{Y}$*.*

Definition 6.2. Consider the subgroup $\Gamma_4' \leq \Omega_4$, generated by outer automorphisms which are represented by automorphisms of the form f_H^w, defined by:

$$f_H^w(u) := \begin{cases} w^{-1}uw & \text{if } u \in H \\ u & \text{else} \end{cases}$$

Here $H \in \{A, B, C, D\}$, $u \in A \cup B \cup C \cup D$, and $w \in A \cup B \cup C \cup D - H$. We will abuse notation to denote the outer automorphism by f_H^w as well.

Lemma 6.3. *If $\mathbf{X}' \in \mathcal{SPD}$ is a graph of groups whose underlying graph is isomorphic to the underlying graph of $\mathbf{X}$, such that the tree produced by an equivariant 1-edge orbit collapse of $T_{\mathbf{X}}$ is G-equivariantly homeomorphic to the tree produced by an equivariant 1-edge orbit collapse of $T_{\mathbf{X}'}$; then $\mathbf{X}'$ can be represented by a graph of groups such that the vertex groups have one of the following forms*

1. $A, a_iBa_i^{-1}, a_jCa_j^{-1}, a_kDa_k^{-1}$, *(where $a_i, a_j, a_k \in A$)*
2. $b_iAb_i^{-1}, B, b_jCb_j^{-1}, b_kDb_k^{-1}$, *(where $b_i, b_j, b_k \in B$)*
3. $c_iAc_i^{-1}, c_jBc_j^{-1}, C, c_kDc_k^{-1}$, *(where $c_i, c_j, c_k \in C$)*
4. $d_iAd_i^{-1}, d_jBd_j^{-1}, d_kCd_k^{-1}, D$, *(where $d_i, d_j, d_k \in D$)*

Proof. Let,

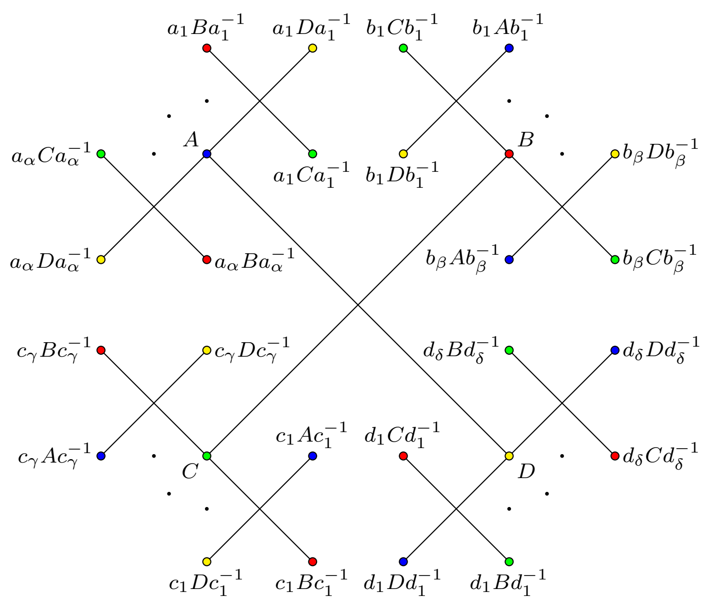
$$A = \{a_0, ..., a_\alpha\},$$

$$B = \{b_0, ..., b_\beta\},$$

$$C = \{c_0, ..., c_\gamma\},$$

$$D = \{d_0, ..., d_\delta\}$$

The local structure of $T_{\mathbf{X}}$ around the vertices having vertex stabilizers $A, B, C,$ and D is given by the following diagram. In the diagram each vertex is labeled by the corresponding vertex stabilizer

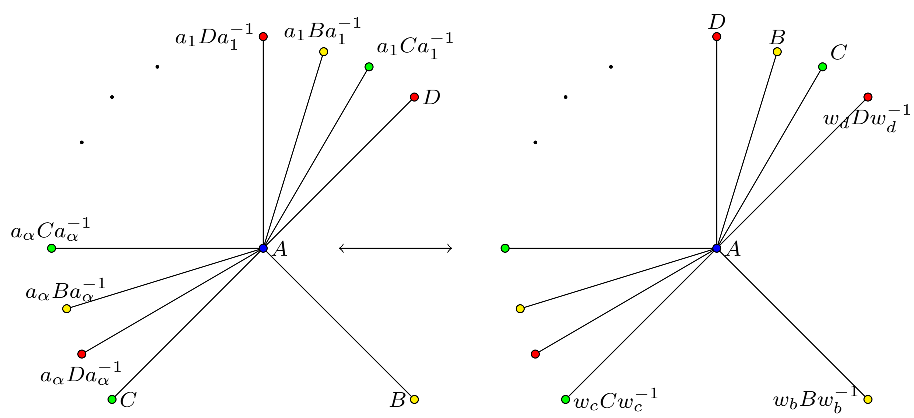


Let us equivariantly collapse a single edge orbit of $T_{\mathbf{X}}$. Without loss of generality, let us collapse the edges attached to the vertices having conjugates of the subgroup A as their stabilizers, i.e., collapse the edges incident on the blue vertices. Let, us call this tree $T^{(A)}$ and the corresponding graph of groups $\mathbf{X}^{(A)}$.

If $\mathbf{X}' \in \mathcal{SPD}$ is a graph of groups whose underlying graph is isomorphic to the underlying graph of $\mathbf{X}$, such that $T^{(A)}$ is G-equivariantly homeomorphic to the tree produced by an equivariant 1-edge orbit collapse of $T_{\mathbf{X}'}$, then $T_{\mathbf{X}'}$ is the result of a single edge orbit expand move applied on the vertices of $T^{(A)}$

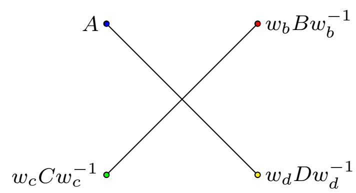
whose stabilizers are conjugates of A. In other words, equivariantly collapsing the edges of $T_{\mathbf{X}'}$ attached to the vertices having conjugates of subgroup A as their stabilizers will produce a tree G-equivariantly homeomorphic to $T^{(A)}$. In $T^{(A)}$ the vertices labeled by conjugates of B (and of C, D) are connected to the vertex labeled by A by a single edge, and the conjugating elements belong to A.

In the following diagram we can see the local picture around the vertex labeled by A of two G-equivariantly homeomorphic trees representing $T^{(A)}$. The tree on the left represents a tree obtained by applying the single edge orbit collapse move on $T_{\mathbf{X}}$, mentioned above. The vertices $\{A,\ w_cCw_c^{-1}$, $w_bBw_b^{-1}$, and $w_dDw_d^{-1}\}$ of the tree on the right represent a choice of fundamental domain for applying the single edge orbit expand move, mentioned above to get $T_{\mathbf{X}'}$.



Two G-equivariantly homeomorphic trees representing $T^{(A)}$

So, there is a representation of $\mathbf{X}'$ in which the non-trivial vertex groups will be given by A, $w_cCw_c^{-1}$, $w_bBw_b^{-1}$, and $w_dDw_d^{-1}$ (where, $w_b, w_c, w_d \in A$). **Note**: if $w_b = w_c = w_d$, then $T_{\mathbf{X}'}$ is G-equivariantly homeomorphic to the original tree $T_{\mathbf{X}}$.



A representation of $\mathbf{X}'$

$\square$

Corollary 6.4. *If*

1. $\mathbf{X}$ *is the graph of groups from the previous lemma (lemma 6.3) and* $\mathbf{X}'$ *is a graph of groups whose underlying graph is isomorphic to the underlying graph of* $\mathbf{X}$.

2. The tree produced by an equivariant single edge orbit collapse of $T_{\mathbf{X}}$ *is* G*-equivariantly homeomorphic to the tree produced by an equivariant single edge collapse of* $T_{\mathbf{X}'}$.

Then $\exists f \in \Gamma_4'$ *such that* $f(\sigma_{\mathbf{X}}) = \sigma_{\mathbf{X}'}$

Proof. Without loss of generality, assume $\mathbf{X}'$ is the same graph of groups that we obtained in the proof of the previous lemma. Let ϕ be the outer automorphism represented by $\Phi \in \mathsf{Aut}(G)$ given by,

$$\Phi(x) := \begin{cases} x & \text{if } x \in A \\ w_b^{-1} x w_b & \text{if } x \in B \\ w_c^{-1} x w_c & \text{if } x \in C \\ w_d^{-1} x w_d & \text{if } x \in D \end{cases}$$

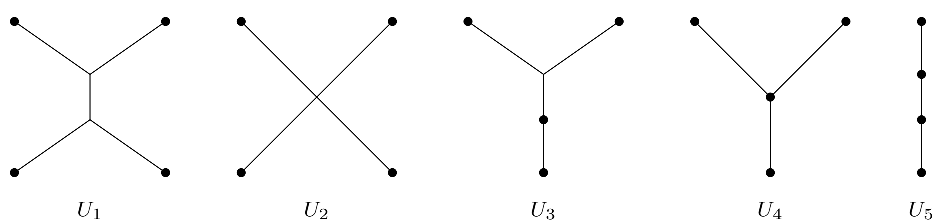
$$f := \phi = f_B^{w_b} f_C^{w_c} f_D^{w_d} \implies f \in \Gamma'_4$$

□

Proposition 6.5. *Under the action $\Gamma'_4 \curvearrowright \mathcal{SPD}(G_4, \mathcal{H})$ there is exactly one orbit of graph of groups whose underlying graph is isomorphic to the underlying graph of* $\mathbf{X}$.

Proof. We will use the following notations to prove this lemma

Notation 6.6. *1. The following picture gives us all the different isomorphism types of graphs occurring as graphs of groups in $\mathcal{SPD}$. These isomorphism classes have been explained in lemma 5.6. From left to right we will denote them by $U_1, U_2, U_3, U_4,$ and U_5, respectively. A tree which is G-equivariantly homeomorphic to the Bass-Serre tree of the corresponding graph of groups will have a similar nomenclature; that is, tree of type U_1, U_2, U_3, U_4, U_5, respectively.*



2. Consider a finite edge path in $\mathcal{SPD}$ starting in $\mathbf{X_{i_0}}$; if the path crosses the vertices $\mathbf{X_{i_0}}, ..., \mathbf{X_{i_k}}$ (in that order) we will code that information by a finite sequence $U_{i_0}, ..., U_{i_k}$, where $U_{i_j} \in \{U_1, ..., U_5\}$, and the underlying graph of $\mathbf{X_{i_j}}$ is isomorphic to graph of type U_{i_j}.

3. For a given tree T and a vertex with vertex stabilizer group H, we will denote the vertex by v_H.

We will use permissible collapse and expand moves on the Bass-Serre tree of the given graphs of groups to navigate through an edge-path in $\mathcal{SPD}$. The following graph gives a graphic representation of the possible collapse and expand moves between different types of trees in $\mathcal{SPD}$ representing different vertices of $\mathcal{SPD}$. The arrows represent possible collapses:

$$\begin{tikzcd}
U_2 \arrow[r] & U_4 & \\
U_1 \arrow[u] \arrow[r] & U_3 \arrow[u] \arrow[r] & U_5
\end{tikzcd}$$

The idea of the proof is to show that the 1 skeleton of the sub-complex of $\mathcal{SPD}(G_4, \mathcal{H})$ consisting of trees of type U_2 and U_4 is connected.

Claim 6.7. *Using these notations, we will show that any two vertices represented by graphs of groups of type U_2 in $\mathcal{SPD}$ can be joined by an edge path consisting only of vertices of type U_2 and U_4.*

Proof of the claim. Guirardel-Levitt showed that $\mathcal{PD}$(and $\mathcal{SPD}$) is contractible, see theorem 1.3. Hence, the 1-skeleton of $\mathcal{SPD}$ is path connected. So, any two vertices represented by graphs of groups of type U_2 can be connected by a collapse-expand edge path. Given any edge path connecting two graphs of groups of type U_2, we will construct an alternative edge path connecting the original vertices consisting of only graphs of groups of type U_2 and U_4.

We will breakdown the construction of the alternative path in three steps.

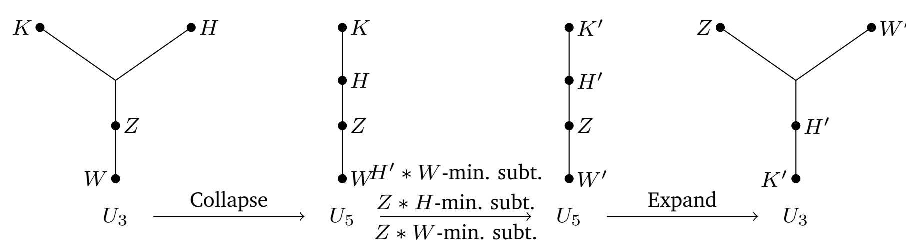
1. We will create an alternative path that does not contain any graph of groups of type U_5.

2. We will take the resulting path from step 1 and create an alternative path without any graph of groups of type U_3 (and U_5).

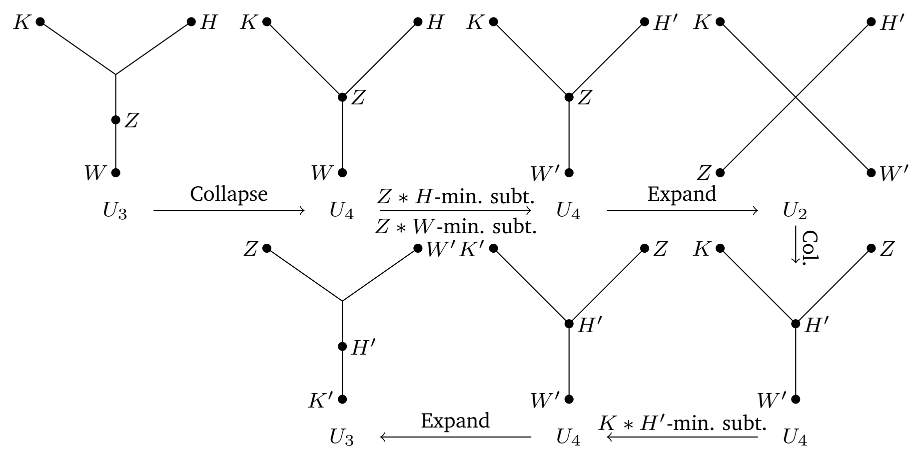
3. We will take the path obtained from step 2 and create an alternative path without any graph of groups of type U_1 (and U_3, U_5).

Collapse-Expand move on graphs of groups: We will be using the collapse-expand terminology for graphs of groups. In practice it means that we are looking at a fundamental domain of the Bass-Serre tree of the graph of groups and collapsing or expanding edges equivariantly in the Bass-Serre tree and observing a conveniently chosen fundamental domain.

Step 1 An edge path beginning and ending in graphs of groups of type U_2 containing a graph of groups of type U_5 must contain a subpath $U_3U_5U_3$ as shown below. In this $\mathcal{SPD}$ edge path we apply a collapse move on a graph of groups of type U_3 and collapse the edge labeled by H to get a graph of groups of type U_5. In the second step we choose a new fundamental domain where the vertices labeled by K, H and W have been replaced by conjugates of respective elements (namely, K', H' and W'). Later, we expand vertex labeled by Z to move from a graph of group of type U_5 to U_3.

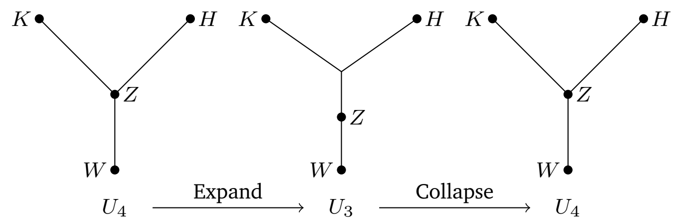


The replacement edge subpath containing graphs of groups of type U_2 and U_4, instead of graphs of groups of type U_5, has been given in the following diagram. The choice of a fundamental domain containing vertices labeled by $\{K', H', Z, W'\}$ was possible for the U_4 because of the uniqueness (remark 5.3) of $Z * H$- minimal subtree, $Z * W$-minimal subtree and $H' * K$-minimal subtree in the trees of the original edge path in $\mathcal{SPD}$ and the replacement edge path.

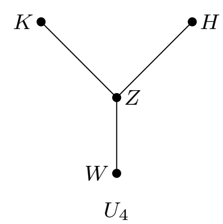


Step 2 A path beginning and ending in graphs of groups of type U_2 containing at least one graph of groups of type U_3; and not containing graphs of groups of type U_5 must have all the type U_3 graphs of groups contained in one following subpaths: $U_4U_3U_4$, $U_1U_3U_1$, $U_1U_3U_4$, and $U_4U_3U_1$.

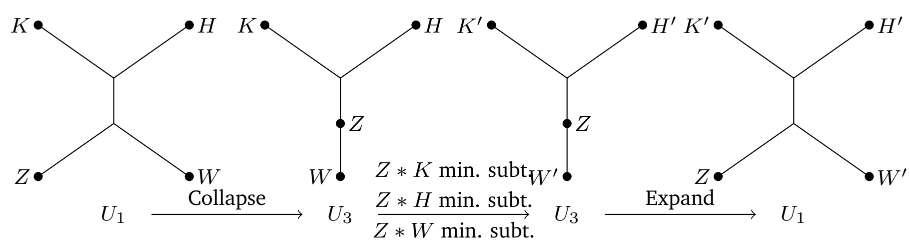
Subpath 1 $U_4U_3U_4$: Without loss of generality let us assume that the expand move expands the edge containing the vertex labeled by Z along the vertex labeled by W. To return to a graph of type U_4 there is only one possible collapse.



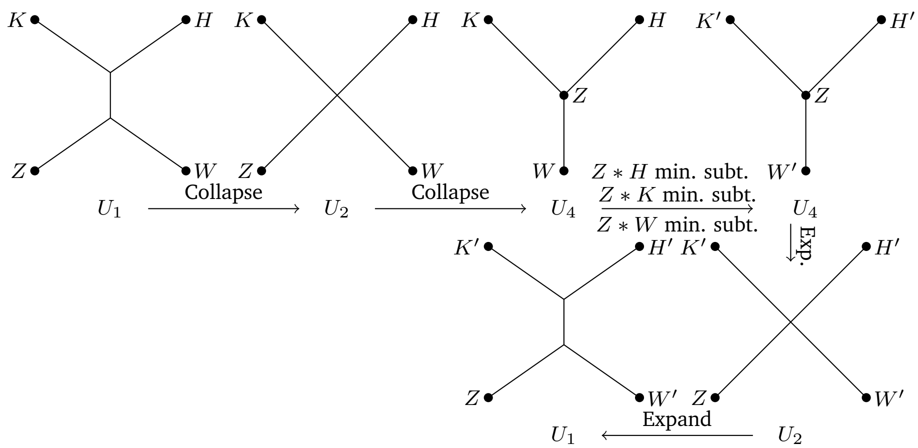
Replacement subpath U_4. This is a trivial subpath.



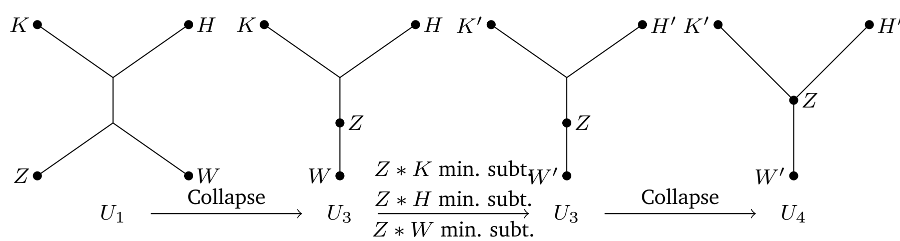
Subpath 2 $U_1U_3U_1$: Without loss of generality, we choose the edge adjacent to the vertex labeled by subgroup Z for collapse. In the second step we choose a new fundamental domain where the vertices labeled by K, H and W have been replaced by conjugates of respective elements (namely, K', H' and W'). In the last step, we expand the vertex labeled by Z to get back a graph of type U_1.



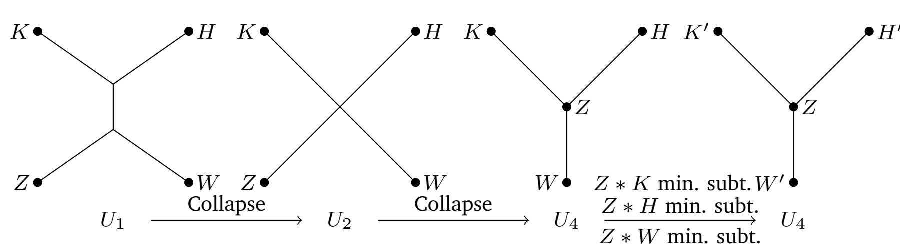
We will replace such paths by a subpath $U_1U_2U_4U_2U_1$. As with the first case, the flow of the diagrams is from top left to top right and then from bottom right to bottom left. The crucial step is choosing a different fundamental domain for the graph U_4.



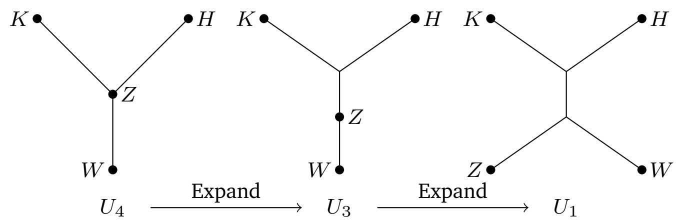
Subpath 3 $U_1U_3U_4$: Without loss of generality, let us assume that the edge adjacent to the vertex labeled by Z is collapsed. The next step is a choice of fundamental domain and replacement of vertices labeled $K, H,$ and W by K', H' and W', respectively.



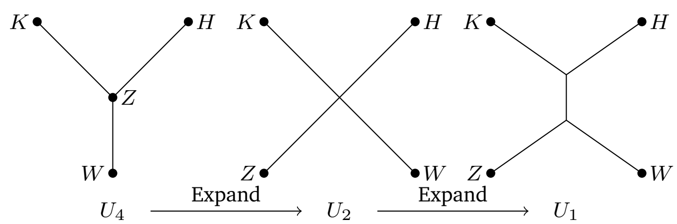
We will replace such paths by subpaths of type $U_1U_2U_4$. The crucial step is the last step, where we choose a different fundamental domain to achieve the necessary graph of groups picture.



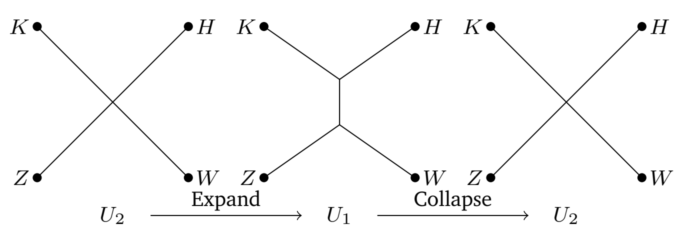
Subpath 4 $U_4U_3U_1$: This is the previous subpath in the opposite direction. However, the vertex stabilizer in the fundamental domain of the final type U_1 graph of groups is forced to come from a fundamental domain of the initial type U_4 graph of groups. (In contrast to the previous case, where the vertex groups in a fundamental domain of the initial U_1 graph might not have been present in the final U_4 graph.)



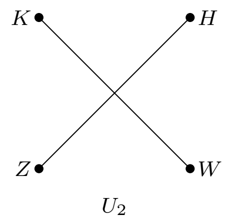
Replacement subpath $U_4U_2U_1$



Step 3 A path beginning and ending in graphs of groups of type U_2 containing at least one graph of groups of type U_1; and not containing graphs of groups of type U_3, and U_5 must have all the type U_1 graphs of groups contained in a subpath of the form $U_2U_1U_2$.



We will replace every such $U_2U_1U_2$ path by U_2. This is again the trivial path.



So, we have proved our claim that any two graphs of groups of type U_2 can be connected in $\mathcal{SPD}$ by a path consisting of graphs of groups of type U_2 and U_4. $\square$

As a consequence of the claim and corollary 6.4, if $\mathbf{X}' \in \mathbf{PD}(G_4, \mathcal{H})$ is a graph of groups whose underlying graph is isomorphic to the underlying graph of $\mathbf{X}$, then there exists an element $g \in \Gamma_4'$ such that $g(\sigma_{\mathbf{X}}) = \sigma_{\mathbf{X}'}$. g depends on the choice of the path from $\mathbf{X}'$ to $\mathbf{X}$ lying inside the $(U_2 - U_4)$-sub-complex of the $\mathcal{SPD}$. Hence, there is exactly one orbit of graphs of groups whose underlying graph is isomorphic to $\mathbf{X}$. $\square$

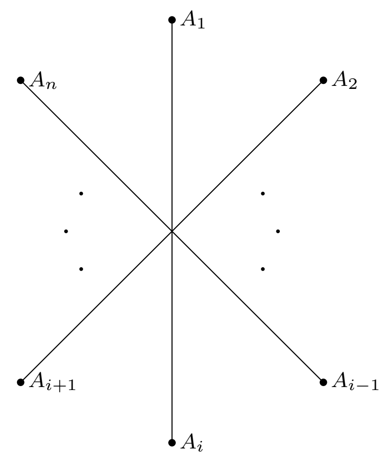
Corollary 6.8. *Γ_4' is finite index in Ω_4*

Proof. The sub-complex of the 1-skeleton of $\mathcal{SPD}(G_4, \mathcal{H})$ consisting of graphs of type U_2 and U_4 is connected and locally finite. The action of Ω_4 and Γ_4' on this sub-complex is co-compact (1 orbit of trees of type U_2 and 4 orbits of trees of type U_4) and properly discontinuous. $\square$

6.2 $\mathcal{D}(G, \mathcal{H})$ and a finite index subgroup of Ω_n

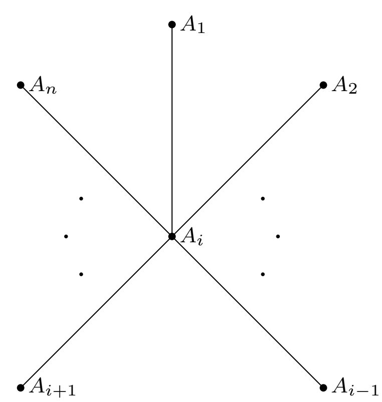
In notation 6.9 we have described some graphs of groups that we will refer frequently in our subsequent discussions.

Notation 6.9. *1. Let,* $\mathbf{X} \in \mathcal{SPD}(G, \mathcal{H})$ *be the vertex of* $\mathcal{SPD}$ *given by the following graph of groups.*



2. Any graph of groups whose underlying graph is isomorphic (simplicially) to the underlying graph of $\mathbf{X}$ *will be called a graph of groups of type* X*. Similarly, any tree* G*-equivariantly homeomorphic to the the Bass-Serre tree of a type* X *graph of groups will be called a tree of type* X*.*

3. Let, $\mathbf{Y_i} \in \mathcal{SPD}(G, \mathcal{H})$ *be the vertex of* $\mathcal{SPD}$ *given by the following graph of groups. The subscript* i *signifies that the vertex associated to the vertex group* A_i *has valence* $n-1$ *and the rest of the vertices have valence* 1*.*



4. Any graph of groups whose underlying graph is isomorphic to the underlying graph of $\mathbf{Y_i}$ will be called a graph of groups of type Y. In other words, a graph of groups with 1 vertex of valence $n-1$ and $n-1$ vertices of valence 1 is a graph of groups of type Y. Similarly, any tree G-equivariantly homeomorphic to the the Bass-Serre tree of a type Y graph of groups will be called a tree of type Y.

6.2.1 Properties of some subgroups of $\mathsf{Out}(G)$

Definition 6.10. Given $w \in \bigsqcup_{j=1}^{n} A_j$ and a fixed integer, $i \in \{1, ..., n\}$, define a map $F^w_{A_i} : \bigsqcup_{j=1}^{n} A_j \to \mathop{\ast}_{j=1}^{n} A_j$ as follows:

$$F^w_{A_i}(a) = \begin{cases} w^{-1}aw & \text{, if } a \in A_i \\ a & \text{, otherwise} \end{cases}$$

By the universal property, this map can uniquely be extended to an automorphism $F^w_{A_i} : \mathop{\ast}_{j=1}^{n} A_j \to \mathop{\ast}_{j=1}^{n} A_j$. In general, for $w \in \mathop{\ast}_{j=1}^{n} A_i$ we define $F^w_{A_i}$ inductively as follows. If $w = uv$, then define $F^w_{A_i} := F^u_{A_i} F^v_{A_i}$.

Definition 6.11. Define f^w_H to be the outer automorphism represented by the automorphism F^w_H, where $H \in \{A_1, ..., A_n\}$ and $w \in \mathop{\ast}_{j=1}^{n} A_j$.

Lemma 6.12. *If $k, m \in \{1, ..., n\}$ are distinct integers, then for any $u, v \in \mathop{\ast}_{\substack{i \neq k,m \\ i=1}}^{n} A_i$, $f^u_{A_k}$ commutes with $f^v_{A_m}$.*

Proof. Definition 6.10 implies $F^u_{A_k} F^v_{A_m} = F^v_{A_m} F^u_{A_k}$, when m and k are distinct integers and $u, v \in \mathop{\ast}_{\substack{i \neq k,m \\ i=1}}^{n} A_i$. So, $f^u_{A_k} f^v_{A_m} = f^v_{A_m} f^u_{A_k}$, when m and k are distinct integers and $u, v \in \mathop{\ast}_{\substack{i\neq k,m\\ i=1}}^{n} A_i$. $\square$

Definition 6.13. For a fixed $i \in \{1, ..., n\}$, define the following subgroups

$$\overline{H_i^j} := \{F_{A_i}^w | w \in A_j\} < \mathsf{Aut}(\mathop{\ast}_{j=1}^{n} A_j)$$

$$H_i^j := \{f_{A_i}^w | w \in A_j\} < \mathsf{Out}(\mathop{\ast}_{j=1}^{n} A_j)$$

Proposition 6.14. *For a fixed $i \in \{1, ..., n\}$, we have the following isomorphisms*

$$\left\langle \overline{H_i^j} | j \in \{1, ..., n\} \setminus \{i\} \right\rangle = \mathop{\ast}_{\substack{j\neq i\\ j=1}}^{n} \overline{H_i^j}$$

$$\left\langle H_i^j | j \in \{1, ..., n\} \setminus \{i\} \right\rangle = \mathop{\ast}_{\substack{j\neq i\\ j=1}}^{n} H_i^j$$

$$\mathop{\ast}_{\substack{j\neq i\\ j=1}} \overline{H_i^j} \cong \mathop{\ast}_{\substack{j\neq i\\ j=1}} H_i^j \cong \mathop{\ast}_{\substack{j\neq i\\ j=1}}^{n} A_j$$

Proof. Let $\Phi \in \left\langle \overline{H_i^j} | j \in \{1, ..., n\} \setminus \{i\} \right\rangle$. Then Φ can be expressed as a composition of $F^{w}_{A_i}$s. That is, $\Phi = F^{u_1}_{A_i} \circ ... \circ F^{u_k}_{A_i}$, where each $u_l \in \bigsqcup_{\substack{j \neq i \\ j=1}}^{n} A_j$. Then

$$\Phi|_{A_t} = \begin{cases} id|_{A_t} & \text{if } t \neq i \\ x_i \mapsto (u_1...u_k)^{-1} x_i (u_1...u_k), \text{if } x_i \in A_i, & \text{and } t = i \end{cases}$$

Hence,

$$\Phi = id \qquad (\text{in } \left\langle \overline{H^j_i} | j \in \{1, ..., n\} \setminus \{i\} \right\rangle) \iff u_1...u_k = id \qquad (\text{in } \mathop{\ast}_{\substack{j \neq i \\ j=1}}^{n} A_j)$$

$$\iff F^{u_1}_{A_i}...F^{u_k}_{A_i} = id \ \ (\text{in } \mathop{\ast}_{\substack{j \neq i \\ j=1}}^{n} \overline{H^j_i})$$

So, the following maps are well defined isomorphisms

$$\begin{array}{ccc} \left\langle \overline{H^j_i} | j \in \{1, ..., n\} \setminus \{i\} \right\rangle \to & \mathop{\ast}_{\substack{j \neq i \\ j=1}}^{n} \overline{H^j_i} \to & \mathop{\ast}_{\substack{j \neq i \\ j=1}}^{n} A_j \\ \Phi \mapsto & F^{u_1}_{A_i}...F^{u_k}_{A_i} \mapsto & u_1...u_k \end{array}$$

Similarly, let $\phi \in \left\langle H^j_i | j \in \{1, ..., n\} \setminus \{i\} \right\rangle$ be an element such that it can be expressed as a product of $f^{w}_{A_i}$s. That is, $\phi = f^{v_1}_{A_i}...f^{v_r}_{A_i}$, where each $v_l \in \bigsqcup_{j=1, j \neq i}^{n} A_j$. Consider the graph of groups $\mathbf{X} \in \mathcal{SPD}(G, \mathcal{H})$ (notation 6.9). Then the underlying graph of $\phi(\mathbf{X})$ is isomorphic to the underlying graph of $\mathbf{X}$ and the corresponding vertex groups are

$\{A_1, ..., A_{i-1}, (v_1...v_r)A_i(v_1...v_r)^{-1}, A_{i+1}, ..., A_n\}$. If $\phi(\mathbf{X})$ is G-equivariantly isometric to $\mathbf{X}$ (denoted by $\phi(\mathbf{X}) \cong_G \mathbf{X}$), then $\Phi|_{A_i} = id, \forall i$.

$$\begin{aligned}
\phi = id(\text{in } \left\langle H_i^j | j \in \{1, ..., n\} \setminus \{i\} \right\rangle) &\iff \phi(\mathbf{X}) \cong_G \mathbf{X} \\
&\iff \phi(T_{\mathbf{X}}) \cong_G T_{\mathbf{X}} \\
&\iff \text{the vertices labeled by} \\
&\qquad A_i, wA_iw^{-1}(w \in A_j, j \neq i) \text{ are} \\
&\qquad \text{adjacent to } A_j(j \in \{1, ..., n\} \setminus \{i\}) \\
&\qquad \text{in } \phi(T_{\mathbf{X}}) \\
&\iff v_1...v_r \in \bigcap_{\substack{j \neq i \\ j=1}}^{n} A_j \\
&\iff v_1...v_r = id \qquad\qquad (\text{in } \mathop{\ast}_{\substack{j \neq i \\ j=1}}^{n} A_j) \\
&\iff f_{A_i}^{v_1}...f_{A_i}^{v_r} = id \qquad\qquad (\text{in } \mathop{\ast}_{\substack{j \neq i \\ j=1}}^{n} H_i^j)
\end{aligned}$$

So, the following maps are well defined isomorphisms

$$\left\langle H_i^j | j \in \{1,...,n\} \setminus \{i\} \right\rangle \to \mathop{\ast}_{\substack{j\neq i\\ j=1}}^{n} H_i^j \to \mathop{\ast}_{\substack{j\neq i\\ j=1}}^{n} A_j$$

$$\phi \mapsto f_{A_i}^{v_1} ... f_{A_i}^{v_r} \mapsto v_1 ... v_r$$

$\square$

Proposition 6.15. *Consider two distinct, fixed integers* $j_1, j_2 \in \{1, ..., n\}$*, then* $\langle H_i^{j_1}, H_i^{j_2} | i \in \{1, ..., n\} \setminus \{j_1, j_2\}\rangle = \bigoplus_{\substack{i\neq j_1,j_2\\ i=1}}^{n} H_i^{j_1} \ast H_i^{j_2} \cong \bigoplus_{\substack{i\neq j_1,j_2\\ i=1}}^{n} A_{j_1} \ast A_{j_2}$

Proof. Let $k, l \in \{1, ..., n\} \setminus \{j_1, j_2\}$ be distinct integers, then $\langle \overline{H_k^{j_1}}, \overline{H_k^{j_2}} \rangle$ commutes with $\langle \overline{H_l^{j_1}}, \overline{H_l^{j_2}} \rangle$. Hence, $\langle H_k^{j_1}, H_k^{j_2} \rangle$ commutes with $\langle H_l^{j_1}, H_l^{j_2} \rangle$.

Consider $\phi \in \langle H_i^{j_1}, H_i^{j_2} | i \in \{1, ..., n\} \setminus \{j_1, j_2\}\rangle$. Due to the commutativity stated previously ϕ can be expressed as a product, $\phi = f_{A_{i_1}}^{w_{i_1}} ... f_{A_{i_s}}^{w_{i_s}}$, where $i_1, ..., i_s \in \{1, ..., n\} \setminus \{j_1, j_2\}$ are distinct integers and $w_{i_1}, ..., w_{i_s} \in A_{j_1} \ast A_{j_2}$. We want to show that ϕ is the identity outer automorphism if and only if $w_1, ..., w_s$ are all identity elements. To prove this we will consider the action of ϕ on the Bass-Serre tree ($T_{\mathbf{X}}$) of the graph of groups $\mathbf{X} \in \mathcal{SPD}(G, \mathcal{H})$ (notation 6.9). The underlying graph of $\mathbf{X}$ has n vertices of valence 1 and 1 vertex of valence n. The non-trivial vertex groups of $\mathbf{X}$ are $\{A_1, ..., A_n\}$ (the groups assigned to the valence 1 vertices). Then the underlying graph of $\phi(\mathbf{X})$ is isomorphic to the underlying graph of $\mathbf{X}$ and the corresponding vertex groups are

$\{w_1 A_1 w_1^{-1}, ..., w_{j_1-1} A_{j_1-1} w_{j_1-1}^{-1}, A_{j_1}, w_{j_1+1} A_{j_1+1} w_{j_1+1}^{-1}, ...,$

$w_{j_2-1}A_{j_2-1}w_{j_2-1}^{-1}, A_{j_2}, w_{j_2+1}A_{j_2+1}w_{j_2+1}^{-1}, ..., w_nA_nw_n^{-1}\}$. Without loss of generality, we have assumed $1 \neq j_1 < j_2 \neq n$.

$$\begin{aligned}
\phi = id &\iff T_{\mathbf{X}} \text{ is } G\text{-equivariantly isometric to } \phi(T_{\mathbf{X}}) \\
&\iff w_i \in A_{j_1} \cap A_{j_2}, \forall i \in \{1, ..., n\} \setminus \{j_1, j_2\} \\
&\iff w_i = id, \forall i \in \{1, ..., n\} \setminus \{j_1, j_2\} \\
&\iff f_{A_i}^{w_i} = id, \forall i \in \{1, ..., n\} \setminus \{j_1, j_2\} \\
&\iff \langle H_k^{j_1}, H_k^{j_2} \rangle \cap \langle H_l^{j_1}, H_l^{j_2} \rangle = \{id\}, \forall k \neq l \in \{1, ..., n\} \setminus \{j_1, j_2\}
\end{aligned}$$

If we combine this with the commutativity of the subgroups $\langle H_k^{j_1}, H_k^{j_2} \rangle$ and $\langle H_l^{j_1}, H_l^{j_2} \rangle$ for distinct $k, l \in \{1, ..., n\} \setminus \{j_1, j_2\}$, then we get decomposition into direct products as follows -

$$\begin{aligned}
\langle H_i^{j_1}, H_i^{j_2} | i \in \{1, ..., n\} \setminus \{j_1, j_2\} \rangle &= \bigoplus_{i \neq j_1, j_2, i=1}^{n} \langle H_i^{j_1}, H_i^{j_2} \rangle \\
(\text{proposition } 6.14) &= \bigoplus_{i \neq j_1, j_2, i=1}^{n} H_i^{j_1} * H_i^{j_2} \\
(\text{proposition } 6.14) &\cong \bigoplus_{i \neq j_1, j_2, i=1}^{n} A_{j_1} * A_{j_2}
\end{aligned}$$

□

Corollary 6.16. *$\mathrm{Out}(\mathop{\ast}_{i=1}^{n} A_i)$ is not hyperbolic, when $n \geq 4$.*

Proof. When $n \geq 4$, the cardinality of the set $\{1, ..., n\} \setminus \{j_1, j_2\}$ is greater than

1. Hence, $\bigoplus_{\substack{i \neq j_1, j_2 \\ i=1}}^{n} H_i^{j_1} * H_i^{j_2}$ is a direct sum of more than one infinite groups, which violates the hyperbolicity of $\bigoplus_{\substack{i \neq j_1, j_2 \\ i=1}}^{n} H_i^{j_1} * H_i^{j_2}$. As a result, Out($\mathop{\ast}_{i=1}^{n} A_i$) is not hyperbolic. $\square$

6.2.2 Γ_n' - a finite index subgroup of Ω_n

Definition 6.17. Consider the subgroup $\Gamma_n' \leq \Omega_n$, generated by outer automorphisms of the form f_H^w (definition 6.11), where $w \in \bigcup_{i=1}^{n} A_i - H$.

Remark 6.18. From definitions 6.13 and 6.17 we get,
$\Gamma_n' = \left\langle H_i^j | i, j \in \{1, ..., n\}, i \neq j \right\rangle$. We will prove that Γ_n' is a finite index subgroup of Ω_n. We will refer to graph of groups $\mathbf{X}, \mathbf{Y_i}$, graph of groups (and G-trees) of type X and type Y from notation 6.9 for our discussion in this section.

Remark 6.19. The strategy to prove that Γ_n' is finite index in Ω_n is described below. Our approach will be a generalization of our approach of the proof $\Gamma_4' \leq \Omega_4$.

1. In lemma 6.20 we will give a relation between the vertex stabilizers of two vertices of a G-tree which are in the same G-orbit and are part of two fundamental domains with non trivial intersection(s).

2. Corollary 6.21 will follow from lemma 6.20. In corollary 6.21 we will establish a relation between the vertex stabilizer subgroups in a selected fundamental domain of two different trees when the trees differ by single edge orbit expansion.

3. In lemma 6.22 we will construct a path between two trees of type X using trees of type X and Y, when the trees have fundamental domains whose vertex stabilizer subgroups share a relation similar to the one described in corollary 6.21.

4. In lemma 6.23 we will prove that any two trees that have the same non trivial vertex stabilizer subgroups in a fundamental domain can be connected.

5. In lemma 6.24 we will connect any two trees of type X by trees of type X and Y under certain restrictions.

6. Corollary 6.25 will follow from lemma 6.24, where we will prove that the sub-complex of $\mathcal{SPD}$ spanned by trees of type X and Y is connected in $\mathcal{SPD}$.

7. In lemma 6.26 we will prove that two trees of type X which are distance 2 apart are in the same Γ'_n orbit.

Lemma 6.20. *Consider $T \in \mathcal{SPD}^0(G, \mathcal{H})$ and two fundamental domains F_1, F_2 of T such that the vertex stabilizer groups of F_1 are given by $W_1, ..., W_n$ and the vertex stabilizer groups of F_2 are given by $V_1, ..., V_n$ where V_k is conjugate to $W_k, \forall k \in \{1, ..., n\}$. Assume that $V_i = W_i$ for a fixed i and for $j \neq i$ the vertices with non trivial vertex stabilizers in the shortest path between W_i, and W_j (excluding W_i and W_j) are labeled as $W_{j_1}, ..., W_{j_{n_j}}$ in increasing order of distance from W_i, then the conjugacy relations are given by -*

$$V_{j_p} = (w_i w_{j_1} ... w_{j_{p-1}}) W_{j_p} (w_i w_{j_1} ... w_{j_{p-1}})^{-1} \tag{6.1}$$

where, $w_r \in W_r$*, for* $r \in \{i, j_1, ..., j_p, .., j_{n_j}\}$

Proof. The choice of vertices in the respective conjugacy classes of subgroups for the fundamental domain based at the vertex W_i is outlined below:

W_{j_1} In the $W_i * W_{j_1}$ minimal subtree of T_1 we choose the vertex labeled by $w_i W_{j_1} w_i^{-1}$ (for the conjugacy class of W_{j_1} in the fundamental domain).

W_{j_2} In the $w_i W_{j_1} w_i^{-1} * w_i W_{j_2} w_i^{-1}$ minimal subtree of T_1 we choose the vertex labeled by $w_i w_{j_1} W_{j_2} w_{j_1}^{-1} w_i^{-1}$ (for the conjugacy class of W_{j_2} in the fundamental domain).

$\vdots$

W_{j_p} In the $(w_i w_{j_1} ... w_{j_{p-2}}) W_{j_{p-1}} (w_i w_{j_1} ... w_{j_{p-2}})^{-1} * (w_i w_{j_1} ... w_{j_{p-2}}) W_{j_p} (w_i w_{j_1} ... w_{j_{p-2}})^{-1}$ minimal subtree of T_1 we choose the vertex labeled by $(w_i w_{j_1} ... w_{j_{p-1}}) W_{j_p} (w_i w_{j_1} ... w_{j_{p-1}})^{-1}$ (in the conjugacy class of W_{j_p} for the fundamental domain).

$\square$

Corollary 6.21. *Consider* $T_1 \in \mathcal{SPD}^0(G, \mathcal{H})$ *and fix a fundamental domain of* T_1 *whose nontrivial vertex stabilizers are given by* $W_1, ..., W_n$*. Let* $T_2 \in \mathcal{SPD}^0(G, \mathcal{H})$ *be obtained from* T_1 *by equivariantly expanding the vertex labeled by* W_i*. Then there exists a fundamental domain of* T_2 *whose non trivial vertex stabilizers are labeled by* $V_1, ..., V_n$*, where each* W_k *is conjugate to* V_k *for* $k \in \{1, ..., n\}$*, and the conjugacy relations are given by-*

1. $V_i = W_i$

2. If $j \neq i$ and the vertices with non trivial vertex stabilizers in the shortest path between W_i, and W_j (excluding W_i and W_j) are labeled as $W_{j_1}, ..., W_{j_{n_j}}$ in increasing order of distance from W_i, then

$$V_{j_p} = (w_i w_{j_1} ... w_{j_{p-1}}) W_{j_p} (w_i w_{j_1} ... w_{j_{p-1}})^{-1} \tag{6.2}$$

where, $w_r \in W_r$, for $r \in \{i, j_1, ..., j_p, .., j_{n_j}\}$

Proof. As the vertex labeled by W_i is expanded, we get $V_i = W_i$. This in turn implies that T_2 has two fundamental domains which satisfies the conditions of lemma 6.20. Hence, we have the result. □

Lemma 6.22. *Consider two trees of type X, T_1 and T_2, satisfying the following properties*

1. $T_1 \in \mathcal{SPD}^0(G, \mathcal{H})$ has a fundamental domain whose non trivial vertex stabilizers are labeled by $W_1, ..., W_n$.

2. $T_2 \in \mathcal{SPD}^0(G, \mathcal{H})$ has a fundamental domain whose non trivial vertex stabilizers are labeled by $V_1, ..., V_n$.

3. For each $k \in \{1, ..., n\}, W_k,$ is related to V_k by equation 6.1 given in corollary 6.21. That is

(a) $V_i = W_i$, for a fixed $i \in \{1, ..., n\}$

(b) If $j \neq i$, then $V_{j_p} = (w_i w_{j_1} ... w_{j_{p-1}}) W_{j_p} (w_i w_{j_1} ... w_{j_{p-1}})^{-1}$ where, $w_r \in W_r$ for $r \in \{i, j_1, ..., j_p, .., j_{n_j}\}$

Then we can connect T_1 and T_2 by a path in $\mathcal{SPD}^1(G, \mathcal{H})$ using trees of type X and Y.

Proof. We will import the notations from corollary 6.21 and construct a collapse-expand path consisting only of trees of type X and Y from T_1 to T_2 leveraging the following conditions

- $V_i = W_i$
- $V_{j_1} = w_i W_{j_1} w_i^{-1}$
- $V_{j_2} = w_i w_{j_1} W_{j_2} w_i^{-1} w_{j_1}^{-1}$

$\vdots$

- $V_{j_p} = (w_i w_{j_1} ... w_{j_{p-1}}) W_{j_p} (w_i w_{j_1} ... w_{j_{p-1}})^{-1}$

The steps of the expand and collapse moves are underlined below:

- On T_1 apply the following moves:

  1. Collapse (equivariantly) the edges of T_1 adjacent to the vertex labeled by W_i, equivariantly. The resulting tree is of type Y.

  2. Choose a fundamental domain replacing each W_r by $w_i W_r w_i^{-1}$, $\forall$ $r \in \{j_1, ..., j_p\}$.

  3. Expand (equivariantly) the vertex labeled by W_i. This tree is of type X.

- On the resulting tree we apply the following moves:

  1. Collapse (equivariantly) the edges adjacent to the vertex labeled by $w_i W_{j_1} w_i^{-1}$, equivariantly. The resulting tree is of type Y.

2. Choose a fundamental domain replacing $w_i W_r w_i^{-1}$ by $w_i w_{j_1} W_r w_{j_1}^{-1} w_i^{-1}$, $\forall\, r \in \{j_2, ..., j_p\}$.

3. Expand (equivariantly) the vertex labeled by $w_i W_{j_1} {w_i}^{-1}$. This tree is of type X.

⋮

- This is the final step:

1. Collapse (equivariantly) the edges adjacent to the vertex labeled by $(w_i w_{j_1} ... w_{j_{p-2}}) W_{j_{p-1}} (w_i w_{j_1} ... w_{j_{p-2}})^{-1}$, equivariantly. The resulting tree is of type Y.

2. Choose a fundamental domain replacing $(w_i w_{j_1} ... w_{j_{p-2}}) W_{j_r} (w_i w_{j_1} ... w_{j_{p-2}})^{-1}$ by $(w_i w_{j_1} ... w_{j_{p-1}}) W_{j_r} (w_i w_{j_1} ... w_{j_{p-1}})^{-1}$, for $r = j_p$.

3. Expand (equivariantly) the vertex labeled by $(w_i w_{j_1} ... w_{j_{p-2}}) W_{j_{p-1}} (w_i w_{j_1} ... w_{j_{p-2}})^{-1}$. This tree is of type X.

□

Lemma 6.23. *Consider $T \in \mathcal{SPD}(G, \mathcal{H})$ with a fundamental domain having non-trivial vertex stabilizer subgroups labeled by $W_1, ..., W_n$. If $T' \in \mathcal{SPD}(G, \mathcal{H})$ has a fundamental domain with the non trivial vertex stabilizer subgroups labeled by $W_1, ..., W_n$, then T and T' are connected by a expand-collapse path in $\mathcal{SPD}(G, \mathcal{H})$ such that every intermediate tree in that path has a fundamental domain with the non trivial vertex stabilizer subgroups labeled by $W_1, ..., W_n$.*

Proof. We will show the existence of such a path in a few steps.

1. T is connected to a tree with maximum number of edge orbits having a fundamental domain such that the non trivial vertex stabilizers are labeled by $W_1, ..., W_n$.

2. Any two trees with maximum number of edge orbits having a fundamental domain with non trivial vertex stabilizer subgroups labels $W_1, ..., W_n$ are connected. This is because both of them are connected to the tree of type X with a fundamental domain labeled by $W_1, ..., W_n$ via collapse moves.

So, T and T' are connected to the same tree of type X. □

Lemma 6.24. *Consider $T, T' \in \mathcal{SPD}^0(G, \mathcal{H})$ such that the distance between them is 1 in $\mathcal{SPD}$. If $S, S' \in \mathcal{SPD}^0(G, \mathcal{H})$ are trees of type X whose nontrivial vertex stabilizer subgroups in a fundamental domain are same as that of in a fundamental domain of T, T', respectively. Then S and S' can be connected by a path in $\mathcal{SPD}$ consisting only of trees of type X and Y.*

Proof. Since T and T' are at a distance of 1. So, without loss of generality let us assume T' is obtained by expanding p edge orbits of T, equivariantly. We can find trees $T = T_0, T_1, ..., T_{p-1}, T_p = T'$ such that T_{i+1} is obtained from T_i by one edge orbit expansion. That is, we find trees so that the p-edge orbit expansions are factored into p singe edge orbit expansions.
For each T_i let S_i denote the tree of type X with a fundamental domain whose non trivial vertex stabilizer subgroups are same as that of a fundamental domain of T_i.
From lemma 6.22 of this subsection we know that S_i and S_{i+1} can be connected by an expand-collapse path consisting only of trees of type X and

Y. □

Corollary 6.25. *The sub-complex of the* 1*-skeleton of* $\mathcal{SPD}(G, \mathcal{H})$ *spanned by vertices corresponding to graph of groups of type* X *and* Y *is connected.*

Proof. If $S, S' \in \mathcal{SPD}^0(G, \mathcal{H})$ are trees of type X. Consider a path of length q connecting them. Starting from S let the trees in this path be given by $T_0 = S, T_1, ..., T_{q-1}, T_q = S'$.

For a given T_i, let S_i represent the tree of type X having a fundamental domain with non trivial vertex stabilizer subgroups identical to that of a fundamental domain of T_i.

Following the previous lemma, lemma 6.24, we see that S_i and S_{i+1} can be connected by a path containing only of trees of type X and Y. So the alternative path would consist of trees $S_0 = S, S_1, ..., S_{q-1}, S_q = S'$ and all the trees between each S_i and S_{i+1}. □

Lemma 6.26. *If* $S, S' \in \mathcal{SPD}(G, \mathcal{H})$ *are two trees of type* X *which are distance* 2 *apart, such that the non trivial vertex stabilizer subgroups in a fundamental domain of the tree* S *are* $A_1, ..., A_n$*. Then there is an outer automorphism* $\phi \in \Gamma'_n$ *such that* $\phi(S) = S'$

Proof. If the distance between S and S' is 2, then there is a tree T such that distance of T from S and S' is 1. We will prove that T must be a tree of type Y. We will show that a path of length 2 S, T, S' must be traveled by collapsing an orbit of edge G-equivariantly of S and then expanding an orbit of vertex G-equivariantly of T.

Complete list of vertex stabilizer subgroups in a fundamental domain uniquely (up to equivariant homeomorphism) determines a tree of type X. So, S and S' do not have a fundamental domain whose non trivial vertex stabilizer subgroup match. To move to a different tree in $\mathcal{SPD}$ from S we must apply either a collapse move or an expand move.

Expand move must be applied to the orbit of vertices with trivial vertex stabilizer subgroup, as the other orbits of vertices have valence 2. The vertex groups of any fundamental domain for a tree of type X is at the extremity of the fundamental domain. Every tree obtained from applying only expand move to a tree of type X must also have a fundamental domain that has all the non trivial vertex groups in the extremities of the fundamental domain.

Any tree with a fundamental domain that has all the vertex groups at the extremities of the fundamental domain does not have edge overlap from two distinct fundamental domains. As a result, one expand move followed by one collapse move on a tree of type X (to get to a tree of type X) does not give rise to a different tree due to inability to choose a different fundamental domain. So, to get to a different tree of type X, we need to apply collapse move first (instead of expand move we have considered in this case) and then an expand move.

Only single edge orbit collapse move is possible. Let the edge adjacent to the vertex group labeled by A_i be collapsed, equivariantly. Let us denote this tree by T_i. Then we have to apply expand move on the vertex labeled by A_i of the tree T_i to get S'.

The choices of vertices for a fundamental domain of T are as follows:

- In the $A_1 * A_i$ minimal subtree we can choose vertex labeled by $a_{i1}A_1a_{i1}^{-1}$.
- In the $A_2 * A_i$ minimal subtree we can choose vertex labeled by $a_{i2}A_2a_{i2}^{-1}$.

$\vdots$

- In the $A_{i-1} * A_i$ minimal subtree we can choose vertex labeled by $a_{ii-1}A_{i-1}a_{ii-1}^{-1}$.
- A_i.
- In the $A_{i+1} * A_i$ minimal subtree we can choose vertex labeled by $a_{ii+1}A_{i+1}a_{ii+1}^{-1}$.

$\vdots$

- In the $A_n * A_i$ minimal subtree we can choose vertex labeled by $a_{in}A_na_{in}^{-1}$

  Here, $a_{ik} \in A_i$, for $k \in \{1, ..., n\}$.

For such a choice, the vertex stabilizer subgroup of a fundamental domain of S' are given by

$a_{i1}A_1a_{i1}^{-1}$, $a_{i2}A_2a_{i2}^{-1}, \dots, a_{ii-1}A_{i-1}a_{ii-1}^{-1}$, A_i, $a_{ii+1}A_{i+1}a_{ii+1}^{-1}, \dots, a_{in}A_na_{in}^{-1}$.

In this case, $\phi := (f_{A_n}^{a_{in}})^{-1} \dots (f_{A_{i+1}}^{a_{ii+1}})^{-1}(f_{A_{i-1}}^{a_{ii-1}})^{-1} \dots (f_{A_1}^{a_{i1}})^{-1}$. □

Corollary 6.27. Γ'_n *is a finite index subgroup of* Ω_n

Proof. Consider the 1-skeleton of the sub-complex of $\mathcal{SPD}(G, \mathcal{H})$ spanned by trees of type X and Y. By corollary 6.25, it is connected. By lemma 6.26, any two trees of type X which are distance 2 apart are in the same Γ'_n orbit. Hence, all trees of type X corresponding to a vertex of $\mathcal{SPD}$ are in the same Γ'_n orbit.

Γ'_n acts co-compactly as there is only 1 orbit of trees of type X (upto G-equivariant homeomorphism). The action is properly discontinuous as $\Gamma'_n \leq \Omega_n$.

So, by Milnor-Švarc lemma, lemma 3.1, Γ'_n is finite index in Ω_n. $\square$

Remark 6.28. The elements considered by McCullough-Miller, see [MM96], were of the form f_H^w, as well. A key difference is we are restricting further by requiring $w \notin H$. So, Γ'_n is a proper subgroup of the symmetric outer automorphisms considered by them.

7 Algebraically Thick Groups

A major tool used in the investigation of the original question (relative hyperbolicity of $\mathsf{Out}(A_1 * ... * A_n)$) for higher complexities is *algebraic thickness*. Theorem 1.4 by Behrstock-Druţu-Mosher underscores the relevance of the study of algebraic thickness. According to theorem 1.4, thickness of a finitely generated group implies non-relative hyperbolicity of the group. Thickness has been developed in full generality by Behrstock-Druțu-Mosher in [BDM09].

In section 7.1, we will briefly describe the terms related to the definition of algebraic thickness. Our exposition closely follow the exposition in [BDM09]. We will start by defining a *non-principal ultrafilter* in definition 7.1. Then, we will define *ultralimit* in definition 7.2. Using the concept of ultralimit of a family of metric spaces we will define the *asymptotic cone* of a metric space (X, dist) in definition 7.4. We will use the concepts of ultrafilter and asymptotic cone to define an *unconstricted metric space* in definition 7.5. Algebraic thickness of a group is an inductive property, where the base case or algebraically thick group of order at most zero are groups which are unconstricted. We will use the notion of *algebraic network of subgroups*, definition 7.6, to define algebraic thickness in higher order, definition 7.7.

7.1 Definition

Definition 7.1. A *non-principal ultrafilter* on the positive integers, denoted by ω, is a non-empty collection of sets of positive integers with the following properties:

1. If $S_1 \in \omega,$ and $S_2 \in \omega$, then $S_1 \cap S_2 \in \omega$.

2. If $S_1 \subset S_2$ and $S_1 \in \omega$, then $S_2 \in \omega$.

3. For each $S \subset \mathbb{N}$ exactly one of the following must occur: $S \in \omega$ or $\mathbb{N} - S \in \omega$.

4. ω does not contain any finite set.

Definition 7.2. For a non-principal ultrafilter ω, a topological space X, and a sequence of points $(x_i)_{i\in\mathbb{N}}$ in X, we define x to be the *ultralimit* of $(x_i)_{i\in\mathbb{N}}$ with respect to ω, and we write $x = \lim_\omega x_i$, if and only if for any neighborhood $\mathcal{N}$ of x in X the set $\{i \in \mathbb{N} : x_i \in \mathcal{N}\}$ is in ω.

Remark 7.3. 1. When X is compact any sequence in X has an ultralimit.

2. If moreover X is Hausdorff then the ultralimit of any sequence is unique.

Fix a non-principal ultrafilter ω and a family of based metric spaces $(X_i, x_i, \text{dist}_i)$. Using the ultrafilter, a pseudo distance on $\prod_{i\in\mathbb{N}} X_i$ is provided by:

$$\text{dist}_\omega((a_i),(b_i)) = \lim_\omega \text{dist}_i(a_i, b_i) \in [0, \infty].$$

One can eliminate the possibility of the previous pseudo-distance taking the value ∞ by restricting to sequences $y = (y_i)$ such that $\text{dist}_\omega(y, x) < \infty$, where $x = (x_i)$. A metric space can be then defined, called the *ultralimit* of

$(X_i, x_i, \text{dist}_i)$, by:

$$\lim_\omega(X_i, x_i, \text{dist}_i) = \left\{ y \in \prod_{i\in\mathbb{N}} X_i : \text{dist}_\omega(y,x) < \infty \right\} / \sim,$$

where two points $y, z \in \prod_{i\in\mathbb{N}} X_i$ we define $y \sim z$ if and only if $\text{dist}_\omega(y,z) = 0$. The pseudo-distance on $\prod_{i\in\mathbb{N}} X_i$ induces a complete metric on $\lim_\omega(X_i, x_i, \text{dist}_i)$.

Definition 7.4. For a metric space (X, dist), consider $x = (x_n)$ a sequence of points in X, called *observation points*, and $d = (d_n)$ a sequence of positive numbers such that $\lim_\omega d_n = \infty$, called scaling constants. The *asymptotic cone of* (X, dist) *relative to the non-principal ultrafilter* ω *and the sequences* x *and* d is given by: $\text{Cone}_\omega(X, x, d) = \lim_\omega \left(X, x_n, \frac{1}{d_n}\text{dist} \right)$.

Definition 7.5. [Definition 3.1 (Unconstricted space/ group)][BDM09] A path connected metric space B is *unconstricted* if the following two properties hold:

1. there exists a non-principal ultrafilter ω and a sequence d such that for every sequence of observation points b, $\text{Cone}_\omega(B, b, d)$ does not have cut-points;

2. for some constant c, every point in B is at distance at most c from a bi-infinite geodesic in B.

An infinite finitely generated *group is unconstricted* if at least one of its asymptotic cones does not have cut-points.

Definition 7.6. [Definition 5.2(Algebraic network of subgroups)][BDM09] Let G be a finitely generated group with a given generating set, let $\mathcal{H}$ be a finite collection of subgroups of G and let $M > 0$. The group G is an $M-$*algebraic network with respect to* $\mathcal{H}$ if:

$\mathbf{AN_0}$ All subgroups in $\mathcal{H}$ are finitely generated and undistorted in G.

$\mathbf{AN_1}$ There is a finite index subgroup G_1 of G such that $G \subset \mathcal{N}_M(G_1)$, such that a finite generating set of G_1 is contained in $\bigcup_{H \in \mathcal{H}} H$.

$\mathbf{AN_2}$ Any two subgroups H, H' in $\mathcal{H}$ can be *thickly connected* in $\mathcal{H}$: there exists a finite sequence $H = H_1, ..., H_n = H'$ of subgroups in $\mathcal{H}$ such that for all $1 \leq i < n, H_i \cap H_{i+1}$ is infinite.

Definition 7.7. [Definition 7.3(Algebraic thickness)][BDM09] Consider a finitely generated group G.

$\mathbf{A_1}$ G is called *algebraically thick of order zero* or 0-*thick* if it is unconstricted.

$\mathbf{A_2}$ G is called M-*algebraically thick of order at most* $n+1$ with respect to $\mathcal{H}$, where $\mathcal{H}$ is a finite collection of subgroups of G and $M > 0$, if:

- G is an M-algebraic network with respect to $\mathcal{H}$;
- all subgroups in $\mathcal{H}$ are algebraically thick of order at most n.

G is said to be *algebraically thick of order at most* $n+1$ *with respect to* $\mathcal{H}$ if there is a $M > 0$, such that G is M-*algebraically thick of order at most* $n+1$ with respect to $\mathcal{H}$. G is said to be *algebraically thick of order* $n+1$ *with respect to* $\mathcal{H}$, when G is algebraically thick of order at most $n+1$ and G is not algebraically thick of order at most n.

Remark 7.8. The algebraic thickness property does not depend on the word metric on G, moreover it holds for any metric quasi-isometric to a word metric.

Remark 7.9. We will show that Γ'_n is algebraically thick of order at most 1 in higher complexities. More specifically, we will find and prove the existence of an algebraic network of undistorted, 0-thick subgroups. Examples inspired by the following classes of unconstricted spaces will be our base 0-thick subgroups.

Example 7.10. [BDM09, Definition 3.4] A cartesian product of two geodesic metric spaces of infinite diameter is an example of an unconstricted space.

Remark 7.11. A Cayley graph of a direct product of groups which have infinite diameter is an example of an unconstricted space.

7.2 In search for thickly connected subgroups

In this section we will work with subgroups generated by carefully selected elements from the set of generators defined in section 6.2. H_i^j from definition 6.13 will serve as the building blocks for a potential thickly connected network of 0-thick subgroups.

7.2.1 Some thickly connected subgroups of Γ'_4

We will consider two separate cases to investigate thickly connected subgroups of Γ_4

Case 7.12. *Each A_i is abelian. We choose to portray this separately as the subgroups used for this case are similar to the subgroups used for $\Gamma'_n, (n > 4)$.*

Case 7.13. *In general we will not assume that A_is are abelian and investigate thickly connected subgroups Ω_4 (Definition 4.3).*

Case 7.12: In this case we will consider H_i^j from definition 6.13, such that $i \neq j$. We will organize the generating subgroups, H_i^j, of Γ'_4 (definition 6.17) into the following table. A subgroup generated by any two subgroups in a row is a direct product of those two subgroups by proposition 6.15. A subgroup generated by any two subgroups in a column is a free product of those two subgroups by proposition 6.14.

	H_2^1	H_3^1	H_4^1
H_1^2		H_3^2	H_4^2
H_1^3	H_2^3		H_4^3
H_1^4	H_2^4	H_3^4	

Lemma 7.14. *If each A_i is an abelian group, then the subgroups in the shaded region of the table generate Γ'_4*

Proof. Fix $a_1 \in A_1$, then

$$f^{a_1}_{A_2} f^{a_1}_{A_3} f^{a_1}_{A_4}(a) = \begin{cases} a_1^{-1} a a_1, \text{when } a \in A_2 \cup A_3 \cup A_4 \\ \\ a, \text{when } a \in A_1 \end{cases}$$

$$A_1 \text{ is abelian } \implies a = a_1^{-1} a a_1, \forall a \in A_1$$

$$\text{Hence, } f^{a_1}_{A_2} f^{a_1}_{A_3} f^{a_1}_{A_4} = id, \text{ (Conjugation by } a_1\text{)}$$

$$\implies f^{a_1}_{A_2} = \left(f^{a_1}_{A_3}\right)^{-1} \left(f^{a_1}_{A_4}\right)^{-1} \implies H^1_2 \subset \left\langle H^1_3, H^1_4 \right\rangle$$

$$\text{Similarly, } H^2_1 \subset \left\langle H^2_3, H^2_4 \right\rangle, H^3_4 \subset \left\langle H^3_1, H^3_2 \right\rangle \text{ and } H^4_3 \subset \left\langle H^4_1, H^4_2 \right\rangle$$

□

Now we will define some 0-thick subgroups of Γ'_4 such that together they can be potential candidates for proving at most 1-thickness of Γ'_4.

Definition 7.15 ($H^3 := \langle N^{12}, N^{34} \rangle$)**.** Fix non identity elements $a_i \in A_i$.

$$g_{12} := f^{a_1}_{A_3} f^{a_2}_{A_3} f^{a_1}_{A_4} f^{a_2}_{A_4} \in \left(H^1_3 * H^2_3\right) \oplus \left(H^1_4 * H^2_4\right)$$

$$g_{34} := f^{a_3}_{A_1} f^{a_4}_{A_1} f^{a_3}_{A_2} f^{a_4}_{A_2} \in \left(H^3_1 * H^4_1\right) \oplus \left(H^3_2 * H^4_2\right)$$

In subsection 8.2 we will define the subgroups N^{ij} more generally. N^{12}, N^{34}

are special cases of that and will be used in this section .

$$N^{12} := \langle g_{12} \rangle \cong \mathbb{Z}$$

$$N^{34} := \langle g_{34} \rangle \cong \mathbb{Z}$$

Define a subgroup H_3 as follows. The last equality will be proved in corollary 8.20

$$H_3 := \langle g_{12}, g_{34} \rangle = \left\langle N^{12}, N^{34} \right\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

Notation 7.16. *The subgroups* H^{12}, H^{34}. H^{12}, H^{34}, H_3 *(notations explained below) are potential candidates for* 0*-thick subgroups of* Γ'_n*. These notations will be generalized and used in sections 8, 8.2 for further discussions.*

1. $H^{12} := (H_3^1 * H_3^2) \oplus (H_4^1 * H_4^2)$
2. $H^{34} := (H_1^3 * H_1^4) \oplus (H_2^3 * H_2^4)$

If we combine all the information from this section. We get

$$\langle H^{12}, H^{34}, H_3 \rangle = \Gamma'_4$$

$$H^{12} \cap H_3 \cong H^{34} \cap H_3 \cong \mathbb{Z}$$

$$H^{12} \cong (A_1 * A_2) \oplus (A_1 * A_2)$$

$$H^{34} \cong (A_3 * A_4) \oplus (A_3 * A_4)$$

$$H_3 \cong \mathbb{Z} \oplus \mathbb{Z}$$

Hence, in the case when each A_i is abelian algebraic thickness of order at most 1 of Γ'_4 will follow, if we can prove that H^{12}, H^{34} and H_3 are undistorted subgroups in Γ'_4. We will do this in sections 8, 8.2.
Case 7.13: Now we will not assume that A_is are abelian. Here, we will investigate a finitely generated subgroup $M_4 \leq \Omega_4$ (definition 4.3) for thickly connected subgroups. The definition will imply $\Gamma'_4 \leq M_4 \leq \Omega_4$. So, M_4 will be a finite index subgroup of $\Gamma_4 (= \mathsf{Out}(\mathsf{A}_1 * \mathsf{A}_2 * \mathsf{A}_3 * \mathsf{A}_4))$.

Definition 7.17. $M_4 := \langle H_i^j | i, j \in \{1, 2, 3, 4\} \rangle$. Recall the definition of H_i^j from definition 6.13.

With the notations described in definition 7.17, subgroups $H_i^j \leq M_4$ can be organized in a table similar to the previous case -

H_1^1	H_2^1	H_3^1	H_4^1
H_1^2	H_2^2	H_3^2	H_4^2
H_1^3	H_2^3	H_3^3	H_4^3
H_1^4	H_2^4	H_3^4	H_4^4

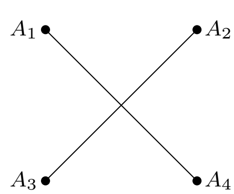
In addition to the subgroups H^{12}, H^{34} (see notation 7.16) considered in the previous case, we have to consider the following subgroups for a thickly connected network of M_4.

Definition 7.18. $M^{12} := \langle H_1^1, H_2^1, H_1^2, H_2^2 \rangle\, ; M^{34} := \langle H_3^3, H_4^3, H_3^4, H_4^4 \rangle$

Lemma 7.19. $\langle M^{12}, M^{34} \rangle = M^{12} \oplus M^{34}$

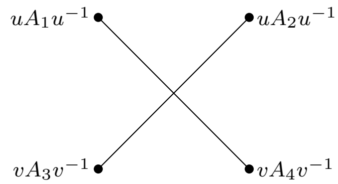
Proof. 1. The generating subgroups of M^{12} commute with the generating subgroups of M^{34}. Hence, $M^{12}, M^{34} \trianglelefteq \langle M^{12}, M^{34} \rangle$

2. Now we will show that $M^{12} \cap M^{34} = \{id\}$. We will show this by considering the action of a generic element of M^{12} and a generic element of M^{34} on the graph of groups $\mathbf{X}$ described below



$\mathbf{X}$

Let, $m_{12} \in M^{12}$ and $m_{34} \in M^{34}$. Then, $m_{12}m_{34}(\mathbf{X}) =$

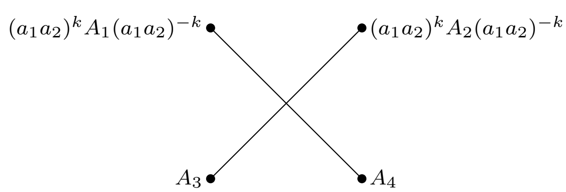


Here, $u \in A_1 * A_2; v \in A_3 * A_4$

$$M^{12} \cap M^{34} \neq \{id\} \implies \exists m_{12}, m_{34}$$
$$\text{such that } m_{12}m_{34}(\mathbf{X}) = \mathbf{X}$$
$$\iff m_{12}m_{34}(T_{\mathbf{X}}) = T_{\mathbf{X}}$$

By uniqueness of $A_1 * A_2$-minimal subtree and $A_3 * A_4$-minimal subtree in every tree of $\mathcal{SPD}$, $m_{12}m_{34}(\mathbf{X}) = \mathbf{X} \implies u = v \implies u = v = id$
Hence, $\langle M^{12}, M^{34} \rangle = M^{12} \oplus M^{34}$. $\square$

Remark 7.20. 1. Fix $a_i \in A_i \setminus \{id_{A_i}\}(i \in \{1, 2, 3, 4\})$, then observe that $(f_{A_1}^{a_1} f_{A_2}^{a_1} f_{A_1}^{a_2} f_{A_2}^{a_2})^k(\mathbf{X}) =$



Here, $a_1 \in A_1; a_2 \in A_2$

Hence, $f_{A_1}^{a_1} f_{A_2}^{a_1} f_{A_1}^{a_2} f_{A_2}^{a_2} \in M^{12}$ is an element of infinite order. Similarly, the order of $f_{A_3}^{a_3} f_{A_4}^{a_3} f_{A_3}^{a_4} f_{A_4}^{a_4} \in M^{34}$ is also infinite.

2. $f_{A_1}^{a_1} f_{A_2}^{a_1} f_{A_3}^{a_1} f_{A_4}^{a_1} = id_{\Gamma_4} \implies f_{A_1}^{a_1} f_{A_2}^{a_1} = \left(f_{A_4}^{a_1}\right)^{-1} \left(f_{A_3}^{a_1}\right)^{-1} \in \langle H_3^1, H_4^1 \rangle$. Similarly,

$f_{A_1}^{a_2} f_{A_2}^{a_2} = \left(f_{A_4}^{a_2}\right)^{-1} \left(f_{A_3}^{a_2}\right)^{-1} \in \langle H_3^2, H_4^2 \rangle \implies f_{A_1}^{a_1} f_{A_2}^{a_1} f_{A_1}^{a_2} f_{A_2}^{a_2} \in \langle H_1^3, H_1^4, H_2^3, H_2^4 \rangle = H^{34}$.

3. Similarly, $f_{A_3}^{a_3} f_{A_4}^{a_3} f_{A_3}^{a_4} f_{A_4}^{a_4} \in \langle H_3^1, H_3^2, H_4^1, H_4^2 \rangle = H^{12}$.

$(M^{12} \oplus M^{34}) \cap H^{12}, (M^{12} \oplus M^{34}) \cap H^{34}$ is infinite. Hence the thickly connected subgroups of M_4 are $H^{12}, H^{34}, (M^{12} \oplus M^{34})$. To prove thickness of M_4 we will show in section 8.3 that all of the above subgroups are undistorted.

7.3 Some thickly connected subgroups of Γ'_n, when $n \geq 5$

In this subsection we will generalize the study of Γ'_4 (done in case 7.12) to $\Gamma'_n, n \geq 5$. A major difference when $n \geq 5$ is that potential algebraic networks can be found in Γ'_n without any assumption on the free factors, A_i (in case 7.12, we assumed each A_i is abelian).

We have organized the subgroups $H_i^j (i \neq j,$ and $i, j \in \{1, ..., n\})$, in the following table. $H_i^j (i \neq j,$ and $i, j \in \{1, ..., n\})$, will be the building blocks for the 0-thick subgroups, which can form algebraic network if the 0-thick subgroups are quasi isometrically embedded in Γ'_n. In contrast to the case 7.13, the diagonal groups, H_i^is, have not been considered.

	H_2^1	$\dots$	H_{n-1}^1	H_n^1
H_1^2		$\dots$	H_{n-1}^2	H_n^2
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
H_1^{n-1}	H_2^{n-1}	$\dots$		H_n^{n-1}
H_1^n	H_2^n	$\dots$	H_{n-1}^n	

The notation for the two different classes of subgroups that we will consider are $H^{ij}(i \neq j \in \{1, ..., n\})$ and $\langle N^{i_1 i_2}, N^{i_3 i_4}\rangle$ $(i_1, i_2, i_3, i_4$ are distinct integers from the set $\{1, ..., n\})$.

Definition 7.21. $H^{ij} := \bigoplus_{\substack{k \neq i,j \\ k=1}}^{k=n} H_k^i * H_k^j$

We observe that, $\Gamma_n' \subset \left\langle \bigcup_{i \neq j}^{n} H^{ij} \right\rangle$. Now we will define an infinite order element of H^{ij}, and call the group generated by that element as N^{ij}

Definition 7.22. Fix distinct integers $i, j \in \{1, ..., n\}$ and $x_i \in A_i \backslash \{id_{A_i}\}, x_j \in A_j \setminus \{id_{A_j}\}$. Define an outer automorphism, $f^{ij} := \prod_{\substack{k \neq i,j \\ k=1}}^{n} \left(f_{A_k}^{x_i} f_{A_k}^{x_j}\right) \in H^{ij}$ and a subgroup of $\Gamma_n' \geq N^{ij} := \langle f^{ij} \rangle$.

In section 8.2 we will prove the following results

1. $N^{ij} \cong \mathbb{Z}$

2. $\langle N^{i_1 i_2}, N^{i_3 i_4}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, where i_1, i_2, i_3, i_4 are all different integers.

3. $\langle N^{i_1 i_2}, N^{i_3 i_4}\rangle$ is undistorted in Γ_n'

The following corollary follows from definition of H^{ij} and N^{ij}

Corollary 7.23. *If* i_1, i_2, i_3, i_4 *are distinct integers from the set* $\{1, ..., n\}$*, then the collection of subgroups of the form* $\{H^{i_1i_2}, \langle N^{i_1i_2}, N^{i_3i_4}\rangle, H^{i_3i_4}\}$ *constitute a thickly connected collection of subgroups of* Γ_n'*, where* $n > 4$.

8 Some Undistorted Subgroups of Γ'_n

In this section we will prove that the subgroups H^{ij}, N^{ij} and $M^{12} \oplus M^{34}$ (discussed in section 7) are quasi isometrically embedded in Γ'_n. The idea of the proof of non-distortion of H^{ij} (and of $M^{12} \oplus M^{34}$) is inspired by work of Handel-Mosher [HM13]. Proof of non-distortion of N^{ij} is inspired by work of Alibegović [Ali02].

8.1 An important class of undistorted subgroups of Γ'_n

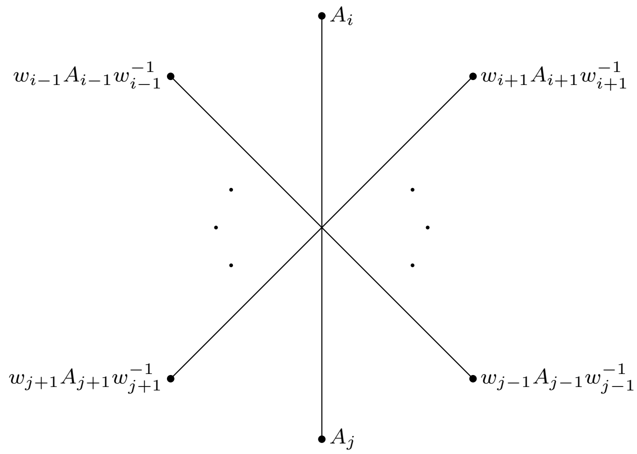
In this section we will prove that H^{ij} is quasi isometrically embedded in Γ'_n. The strategy of the proof is to find a sub-complex, $\mathcal{K}^{ij}$ of $\mathcal{SPD}$, on which H^{ij} acts geometrically and define a Lipschitz retraction map from $\mathcal{SPD}$ to $\mathcal{K}^{ij}$. This will imply quasi isometric embedding of $\mathcal{K}^{ij}$ into $\mathcal{SPD}$.

Definition 8.1. $\mathcal{K}^{ij}$ is the flag sub-complex of $\mathcal{SPD}(G, \mathcal{H})$ spanned by those vertices of $\mathcal{SPD}$ which satisfy the following properties-

1. A tree in ${\mathcal{K}^{ij}}^0$ has a fundamental domain containing vertices stabilized by A_i and A_j.

2. The other vertices in this fundamental domain are stabilized by conjugates of $A_k, k \neq i, j$ and the conjugating elements are from the subgroup $A_i * A_j$.

Example 8.2. A graph of groups representing an element of $\mathcal{K}^{ij}$ from definition 8.1 is given below:



where, $w_k \in A_i * A_j$ for all k. Recall that, a graph of groups whose underlying graph is isomorphic to the underlying graph of this graph of groups is called a graph of groups of type X (in accordance with notation 6.9).

Lemma 8.3. *Let $\mathbf{X} \in \mathcal{K}^{ij^0}$ denote a graph of groups of type X with non trivial vertex groups $A_1, ..., A_n$. Consider a graph of groups $\mathbf{X}' \in \mathcal{K}^{ij}$ of type X, then $\mathbf{X}$ and $\mathbf{X}'$ can be connected by a path in $\mathcal{K}^{ij}$.*

Proof. The proof will be broken down into two parts: In part 1. We will assume that $\mathbf{X}$ and $\mathbf{X}'$ only differ at one vertex (The vertex labeled by the conjugate of the group A_p, for a fixed $p \in \{1, .., n\} \setminus \{i, j\}$). In the second part we will consider more general $\mathbf{X}'$.

1. Consider, graph of groups $\mathbf{X_1}, \mathbf{X_2} \in \mathcal{K}^{ij^0}$ of type X. Assume that $\mathbf{X_1}$ and $\mathbf{X_2}$ are identical except for the vertex corresponding to vertex group congruent to A_p, where $p \in \{1, .., n\} \setminus \{i, j\}$ is an arbitrary fixed integer. The vertex group congruent to A_p in $\mathbf{X_1}$ is A_p; whereas in $\mathbf{X_2}$ the vertex group congruent to A_p is wA_pw^{-1} (where, $w \in A_i * A_j$). In this proof we will show that in such a situation $\mathbf{X_1}$ and $\mathbf{X_2}$ can be connected by a path in $\mathcal{K}^{ij}$.

First, let us assume that $w = vu$ is a word of length 2, such that $u \in A_i$ and $v \in A_j$. So, $\mathbf{X_2} = f^w_{A_p}(\mathbf{X_1})$ (refer to definition 6.11 for $f^w_{A_p} \in \Gamma_n$). In $f^w_{A_p}(\mathbf{X_1})(= \mathbf{X_2})$, the non-trivial vertex group conjugate to A_p is $uvA_pv^{-1}u^{-1}$.

We will give a collapse-expand route from $\mathbf{X_1}$ to $f_p(\mathbf{X_2})$ lying in $\mathcal{K}^{ij}$. Observe that our argument is inductive and we have started with the base case where w is a word of length 2 (instead of 1). However, the description of the collapse-expand path when w has word length 1 is contained in part a of the base case.

(a) i. **Collapse:** Starting from $T_{\mathbf{X_1}}$ we collapse the edges adjacent to the vertex labeled by A_j, equivariantly.

ii. **Expand:** Apply an expand move on the resulting tree to expand vertex A_j after choosing the fundamental domain containing the vertex labeled by vA_pv^{-1} in the $A_j * A_p$ minimal subtree (instead of the vertex labeled by A_p).

(b) Starting from this tree we follow a similar procedure as described above to obtain $f^w_{A_p}(\mathbf{X_1})$.

i. **Collapse:** This time we collapse the edges adjacent to the vertex labeled by A_i, equivariantly,

ii. **Expand:** Apply an expand move on the resulting tree to expand vertex A_i after choosing the fundamental domain containing the vertex labeled by $uvA_pv^{-1}u^{-1}$ in the $A_i * vA_pv^{-1}$ (instead of vA_pv^{-1}). The resulting tree is equivariantly homeomorphic to $f^w_{A_p}(\mathbf{X_1})$.

Notice - w is a word of length 2. More generally, for any word $w \in A_i * A_j$ this proof can be extended by induction on the length of the word w, when

w is expressed as an alternating product of elements of A_i and A_j. So, that concludes the proof of the part 1, where $\mathbf{X}$ and $\mathbf{X}'$ only differ at the vertex labeled by conjugate of A_p.

2. $\mathbf{X}'$ can be expressed as $f(\mathbf{X})$, where $f = \prod_{\substack{w_p \in A_i * A_j \\ p \neq i,j \\ p = p_l}}^{p_1} f_{A_p}^{w_p}$ (definition 6.11), such that $p_i \in \{1, .., n\} \setminus \{i, j\}$. Hence, we can connect

- $\mathbf{X}$ to $f_{A_{p_1}}^{w_{p_1}}(\mathbf{X})$ via a path in $\mathcal{K}^{ij}$.
- $f_{A_{p_1}}^{w_{p_1}}(\mathbf{X})$ to $f_{A_{p_2}}^{w_{p_2}} f_{A_{p_1}}^{w_{p_1}}(\mathbf{X})$ via a path in $\mathcal{K}^{ij}$.

$\vdots$

- $\prod_{\substack{w_p \in A_i * A_j \\ p \neq i,j \\ p = p_{l-1}}}^{p_1} f_{A_p}^{w_p}(\mathbf{X})$ to $\prod_{\substack{w_p \in A_i * A_j \\ p \neq i,j \\ p = p_l}}^{p_1} f_{A_p}^{w_p}(\mathbf{X}) = \mathbf{X}'$ via a path in $\mathcal{K}^{ij}$.

$\square$

Remark 8.4. We will use lemma 6.23 in our following discussion. The lemma states that two different graphs of groups having same vertex groups can be connected by a path consisting of graphs of groups having same vertex groups in $\mathcal{SPD}$.

Corollary 8.5. *$\mathcal{K}^{ij}$ is connected.*

Proof. 1. By lemma 6.23, we can connect any graph of groups in $\mathcal{K}^{ij}$ to a graph of groups of type X via a path contained inside $\mathcal{K}^{ij}$.

2. By lemma 8.3 we can connect any graph of groups of type X via a path contained inside $\mathcal{K}^{ij}$ to a graph of group of type X whose non trivial vertex groups are $A_1, ...,$ and A_n.

$\square$

Remark 8.6. Recall definition 7.21 from section 7, $H^{ij} := \bigoplus_{\substack{k \neq i,j \\ k=1}}^{n} H_k^i * H_k^j$.

Lemma 8.7. *$\mathcal{K}^{ij}$ is invariant under the action of the subgroup H^{ij}.*

Proof. Let, $T \in \mathcal{K}^{ij}$ and $\phi \in H^{ij}$. Assume, that $\Phi \in \mathrm{Aut}(G_n)$ be such that $\phi = [\Phi]$ and

$$\Phi(a) = a, \text{ when } a \in A_i \cup A_j;$$

$$\Phi(a) = u_k^{-1} a u_k (\text{when } a \in A_k, k \neq i,j \text{ and } u_k \in A_i * A_j)$$

There is a fundamental domain of T, such that the non-trivial vertex stabilizers are given by A_i, A_j, and $w_k A_k w_k^{-1}$(where $k \neq i,j$ and $w_k \in A_i * A_j$).

$$\Phi(A_i) = A_i; \Phi(A_j) = A_j; \text{and}$$

$$\Phi(w_k A_k w_k^{-1}) = \Phi(w_k)\Phi(A_k)\Phi(w_k^{-1}) = w_k \Phi(A_k) w_k^{-1} = w_k u_k^{-1} A_k u_k w_k^{-1}$$

So, $\phi(T) \in \mathcal{K}^{ij}$. If e is any edge of length 1 connecting two vertices of $\mathcal{K}^{ij}$, then $\phi(e)$ is also an edge of length 1 as the Γ_n action is isometric. Hence, it is in $\mathcal{K}^{ij}$. $\square$

Lemma 8.8. *$H^{ij} \curvearrowright \mathcal{K}^{ij}$ is properly discontinuous and co-compact.*

Proof. There are only finitely many graphs of groups in $\mathcal{K}^{ij}$ (up-to homeomorphism) such that the non trivial vertex groups are $\{A_1, A_2, ..., A_n\}$. We will prove co-compactness by showing that any other graph of groups in $\mathcal{K}^{ij}$ is in the H^{ij}-orbit of a graph of groups described in the first line.

Let, $T \in \mathcal{K}^{ij}$ be a tree such that the corresponding graph of groups is represented by $\mathbf{X}_T$ and the non trivial vertex groups are given by A_i, A_j and $w_k A_k w_k^{-1}$ (where $k \in \{1, ..., n\} \setminus \{i, j\}$ and $w_k \in A_i * A_j$). Consider $f_{A_k}^{w_k} \in \Gamma'_n$ (definition 6.11), then $\prod\limits_{k \neq i,j} (f_{A_k}^{w_k})(\mathbf{X}_T)$ is the desired graph of groups. So, there are finitely many orbits (up-to homeomorphism) of graphs of groups in $\mathcal{K}^{ij}$ under the action of H^{ij}. $\mathcal{K}^{ij}$ is locally finite. The Bass-Serre tree of any graph of groups have finitely many fundamental domains containing the vertices labeled by A_i and A_j. So, point stabilizer is finite. Hence, the action is properly discontinuous. $\square$

Lemma 8.9. *Fix distinct integers $i, j \in \{1, ..., n\}$ and $w_k \in A_i * A_j$, where $k \in \{1, ..., n\} \setminus \{i, j\}$, then the fundamental group of a graph of groups having non trivial vertex groups $\{A_i, A_j, w_k A_k w_k^{-1} | k \in \{1, ..., n\} \setminus \{i, j\}\}$ is $\mathop{\ast}\limits_{l=1}^{n} A_l$.*

Proof. Consider the map

$$\mathbb{A} : \bigcup_{l=1}^{n} A_l \to \mathop{\ast}_{l=1}^{n} A_l$$

$$a \mapsto \begin{cases} a, & \text{if } a \in A_i \cup A_j \\ w_k^{-1} a w_k, & \text{if } a \in \bigcup\limits_{\substack{l \neq i,j \\ l=1}}^{n} A_l \end{cases}$$

By the universal property of free products, this map can be uniquely extended to a homomorphism denoted by $\mathbb{A} : \mathop{\ast}\limits_{l=1}^{n} A_l \to \mathop{\ast}\limits_{l=1}^{n} A_l$ (abusing notation). The homomorphism defined by the map

$$\mathbb{A}' : \bigcup_{l=1}^{n} A_l \to \mathop{\ast}_{l=1}^{n} A_l$$

$$a \mapsto \begin{cases} a, & \text{if } a \in A_i \cup A_j \\ w_k a w_k^{-1}, & \text{if } a \in \bigcup\limits_{\substack{l \neq i,j \\ l=1}}^{n} A_l \end{cases}$$

can be uniquely extended to a homomorphism denoted by $\mathbb{A}' : \mathop{\ast}_{l=1}^{n} A_l \to \mathop{\ast}_{l=1}^{n} A_l$ (abusing notation) and satisfies $\mathbb{A} \circ \mathbb{A}' = \mathbb{A}' \circ \mathbb{A} = id$. Hence, $\mathbb{A}$ is an automorphism and the fundamental group of a graph of groups having non trivial vertex groups $\{A_i, A_j, w_k A_k w_k^{-1} | k \in \{1, ..., n\} \setminus \{i, j\}\}$ is $\mathop{\ast}_{l=1}^{n} A_l$. $\square$

The goal of our next definition is to assign a tree in $\mathcal{K}^{ij}$ for a given tree in $\mathcal{SPD}(G, \mathcal{H})$, eventually leading to a Lipschitz retraction of $\mathcal{SPD}(G, \mathcal{H})$ onto $\mathcal{K}^{ij}$.

Definition 8.10. Consider a tree $T \in \mathcal{SPD}$ and fix two distinct integers $i, j \in \{1, ..., n\}$. We will build a metric tree, $\overline{T}^{ij}$, using T as follows:

1. Start with the $A_i * A_j$-minimal subtree in T and call it T^{ij}.

2. If the nearest point projection to T^{ij} of the vertex stabilized by $A_k, k \neq i, j$ is contained in the fundamental domain of $A_i * A_j \curvearrowright T^{ij}$ whose extremities are stabilized by the subgroups $w_k A_i w_k^{-1}$ and $w_k A_j w_k^{-1}$, then the nearest point projection of the vertex labeled by the subgroup $w_k^{-1} A_k w_k$ to T^{ij} is contained in the fundamental domain labeled by the subgroups A_i and A_j. If the

nearest point projection of $A_k, k \neq i, j$ is part of more than one fundamental domains of $A_i * A_j \curvearrowright T^{ij}$, then choose the fundamental domain closest to the fundamental domain whose vertices are labeled by A_i and A_j.

3. Construct a graph of groups, such that the geometry of the underlying graph is isometric to the geometry of the smallest subtree of T containing the vertices labeled by the groups from the following set - $\{A_i, A_j, w_k^{-1} A_k w_k | k \in \{1, ..., n\} \setminus \{i, j\}\}$ and the corresponding non trivial vertex groups are $\{A_i, A_j, w_k^{-1} A_k w_k | k \in \{1, ..., n\} \setminus \{i, j\}\}$. $\overline{\mathbf{X}}^{ij}_{\mathcal{PD}}$ is the graph of groups homothetic to the above graph of groups such that the sum of edge lengths is 1. By lemma 8.9 it follows that $\overline{\mathbf{X}}^{ij}_{\mathcal{PD}}$ is an element of $\mathcal{PD}$. Define $\overline{\mathbf{X}}^{ij}$ to be the image of $\overline{\mathbf{X}}^{ij}_{\mathcal{PD}}$ in $\mathcal{SPD}$ under the retraction stated in lemma 3.24 and $\overline{T}^{ij}$ is the Bass-Serre tree of $\overline{\mathbf{X}}^{ij}$.

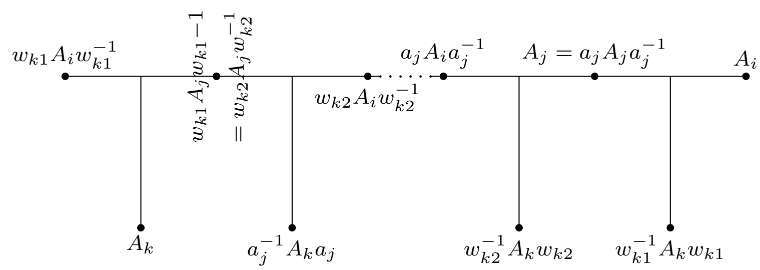
Our next goal is to define a map which can be extended to a Lipschitz retraction.

Definition 8.11. Define a map

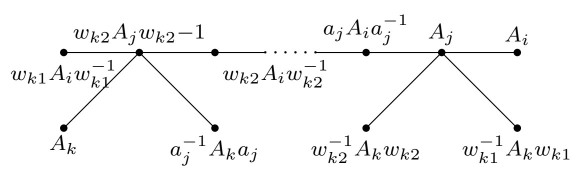
$$L_{ij} : \mathcal{SPD}^0(G, \mathcal{H}) \to \mathcal{K}^{ij}$$
$$T \mapsto \overline{T}^{ij}$$

Lemma 8.12. *If* $T_1, T_2 \in \mathcal{SPD}^0(G, \mathcal{H})$ *satisfies* $d_{\mathcal{SPD}}(T_1, T_2) = 1$*, then* $d_{\mathcal{K}^{ij}}(L_{ij}(T_1), L_{ij}(T_2)) \leq 4$.

Proof. $d_{\mathcal{SPD}}(T_1, T_2) = 1 \implies$ that the two trees are related by a collapse move. Without loss of generality, let us assume that T_2 is obtained by applying collapse moves on edge orbits of T_1. Fix an arbitrary $k \neq i, j$. Let, $w_{k1}, w_{k2} \in A_i * A_j$ be such that the vertex groups conjugate to A_k in a representation of $\overline{\mathbf{X}}_1^{ij}$ and $\overline{\mathbf{X}}_2^{ij}$ are $w_{k1}^{-1}A_k w_{k1}$ and $w_{k2}^{-1}A_k w_{k2}$, respectively. By the uniqueness of $A_i * A_j$-minimal subtrees in T_1 and T_2, we have $w_{k2}^{-1}w_{k1} \in A_i \cup A_j$ (in the example shown below $w_{k2}^{-1}w_{k1} \in A_j$).



T_1



T_2

Hence, the $d_{\mathcal{K}^{ij}}(\overline{T}_1^{ij}, \overline{T}_2^{ij})$ is at most 4. The explanation of distance 4 is as follows -

1. Starting with $\overline{T}_1^{ij}$, we apply maximum number of expand move possible.

2. On the resulting tree we collapse all the edge orbits which are not adjacent to a vertex with non-trivial stabilizer along with the edge orbit adjacent to either the vertex stabilized by A_i or A_j (a tree of type Y, i.e. a tree whose quotient graph of groups has $n-1$ vertices of valence 1 and 1 vertex of valence $n-1$). Now, we choose a different fundamental domain in this tree of type Y, replacing vertex stabilized by $w_{k1}^{-1}A_k w_{k1}$ with vertex stabilized

by $w_{k2}^{-1}A_k w_{k2}$. Such a choice of fundamental domain is possible due to the uniqueness of the $A_i * A_j$ minimal subtree in $T_1, \overline{T}_1^{ij}, T_2$ and $\overline{T}_2^{ij}$.

3. On the tree of type Y from the previous step we apply a maximal expand move such that a collapse move will lead us to $\overline{T}_2^{ij}$.

4. On the resulting tree we apply a collapse move to get $\overline{T}_2^{ij}$.

□

Corollary 8.13. *The map L_{ij} from definition 8.11 can be extended to a continuous Lipschitz retraction $L_{ij} : \mathcal{SPD}^1(G, \mathcal{H}) \to \mathcal{K}^{ij}$*

Proof. We will extend the map linearly on each edge of $\mathcal{SPD}^1$. Lemma 8.12 implies the map is 4-Lipschitz. Definition 8.11 implies the map is a retract.

□

Corollary 8.14. *H^{ij} is an undistorted subgroup of Γ_n*

8.2 A second class of undistorted subgroups of Γ_n'

In this section we will find a class of subgroups N^{ij} (here, $i \neq j \in \{1, ..., n\}$) of Γ_n' which satisfy the following properties:

1. $N^{ij} < H^{ij}$.

2. $N^{ij} \cong \mathbb{Z}$.

3. $\langle N^{i_1 i_2}, N^{i_3 i_4} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, where i_1, i_2, i_3, i_4 are distinct integers from the set $\{1, ..., n\}$.

4. $\langle N^{i_1 i_2}, N^{i_3 i_4}\rangle$ is undistorted in Γ'_n.

We will define N^{ij} next.

Definition 8.15. Fix two distinct integers $i, j \in \{1, ..., n\}$. For $q \in \{i, j\}$ fix $x_q \in A_q \setminus \{id_{A_q}\}$.

$$f^{ij} := \prod_{\substack{p \neq i,j \\ p=1}}^{n} f_{A_p}^{x_i} f_{A_p}^{x_j} = \prod_{\substack{p \neq i,j \\ p=1}}^{n} f_{A_p}^{x_i x_j} \in H^{ij}, N^{ij} := \langle f^{ij}\rangle$$

Lemma 8.16. *Consider distinct integers $i, j \in \{1, ..., n\}$, then $N^{ij} \cong \mathbb{Z}$.*

Proof. We will prove that $(f^{ij})^m = id \implies m = 0$. Consider a graph of groups $\mathbf{X}$ such that the underlying graph has 1 vertex of valence n and n vertices of valence 1; and the non trivial vertex groups are $\{A_1, A_2, ..., A_n\}$.
By definition of f^{ij}, there is a representation of $(f^{ij})^m(\mathbf{X})$, such that the vertex groups are given by $\{A_i, A_j, (x_i x_j)^m A_k (x_i x_j)^{-m} | k \neq i, j\}$.
Let us fix a $k \neq i, j$. In the Bass-Serre tree of $\mathbf{X}$, the distance between the vertex labeled by A_i and A_k is 2. However, in the Bass-Serre tree of $(f^{ij})^m(\mathbf{X})$ the distance between the vertex labeled by A_i and A_k is $4m + 2$. So, $m \neq 0 \implies (f^{ij})^m \neq id$. Hence, $\langle f^{ij}\rangle = N^{ij} = \mathbb{Z}$. $\square$

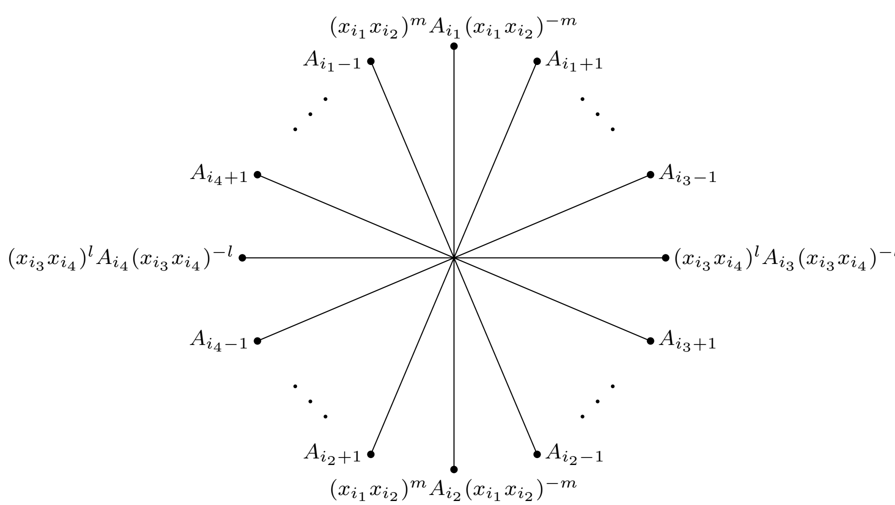
Lemma 8.17. *If $i_1, i_2, i_3, i_4 \in \{1, ..., n\}$ are distinct integers, then $f^{i_1 i_2}$ commutes with $f^{i_3 i_4}$.*

Proof. If an automorphism conjugates every element of the group G_n by a fixed element, then the automorphism represents the outer class of the identity automorphism. So,

$$\begin{aligned}
\prod_{k=1}^{n} f_{A_k}^{x_i} = id_{\Gamma_n} \implies \prod_{\substack{k\neq i,j\\ k=1}}^{n} f_{A_k}^{x_i} &= \left(f_{A_i}^{x_i}\right)^{-1}\left(f_{A_j}^{x_i}\right)^{-1} \\
\implies f^{ij} = \prod_{\substack{k\neq i,j\\ k=1}}^{n} f_{A_k}^{x_i} f_{A_k}^{x_j} &= \left(\prod_{\substack{k\neq i,j\\ k=1}}^{n} f_{A_k}^{x_i}\right)\left(\prod_{\substack{k\neq i,j\\ k=1}}^{n} f_{A_k}^{x_j}\right) \\
&= \left(f_{A_i}^{x_i}\right)^{-1}\left(f_{A_j}^{x_i}\right)^{-1}\left(f_{A_i}^{x_j}\right)^{-1}\left(f_{A_j}^{x_j}\right)^{-1}
\end{aligned}$$

If $i_1, i_2, i_3,$ and i_4 are all distinct numbers, then using an argument similar to the one used in proving lemma 6.12 we see that, $f^{i_1 i_2}$ and $f^{i_3 i_4}$ commute.

Notation 8.18. *Consider the graph of groups,* $\mathbf{X}$*, from notation 6.9, then* $\left(f^{i_1 i_2}\right)^m \left(f^{i_3 i_4}\right)^l (\mathbf{X})$ *can be represented by the following graph of groups:*



□

Lemma 8.19. *If* $i_1, i_2, i_3, i_4 \in \{1, ..., n\}$ *are distinct numbers and* $\left(f^{i_1 i_2}\right)^m = \left(f^{i_3 i_4}\right)^l$*, then* $m = l = 0$*.*

Proof. Recall that we have denoted a graph of groups with underlying graph having 1 vertex of valence n and n vertices of valence 1 as a graph of groups of type X. Let, $\mathbf{X}$ be a graph of groups of type X such that the non trivial vertex groups are $\{A_1, ..., A_n\}$. Since, $i_1, i_2, i_3, i_4 \in \{1, ..., n\}$ are distinct numbers, without loss of generality assume that $1 < i_1 < i_2 < i_3 < i_4 < n$. So, $x_{i_1}, x_{i_2}, x_{i_3}$, and x_{i_4} are all distinct elements of the group G_n. Also, recall the identity $f^{ij} = \left(f^{x_i}_{A_i}\right)^{-1} \left(f^{x_i}_{A_j}\right)^{-1} \left(f^{x_j}_{A_i}\right)^{-1} \left(f^{x_j}_{A_j}\right)^{-1}$.
Then

1. $(f^{i_1 i_2})^m(\mathbf{X})$ is a graph of groups of type X with vertex groups $\{A_1, ..., \ A_{i_1-1}, (x_{i_1}x_{i_2})^{-m}A_{i_1}(x_{i_1}x_{i_2})^m, \ A_{i_1+1}, ..., \ A_{i_2-1},$ $(x_{i_1}x_{i_2})^{-m}A_{i_2}(x_{i_1}x_{i_2})^m, A_{i_2+1}, ..., \ A_n\}$

2. $(f^{i_3 i_4})^l(\mathbf{X})$ is a graph of groups of type X with vertex groups $\{A_1, ..., \ A_{i_3-1}, (x_{i_3}x_{i_4})^{-l}A_{i_3}(x_{i_3}x_{i_4})^l, \ A_{i_3+1}, ..., \ A_{i_4-1},$ $(x_{i_3}x_{i_4})^{-l}A_{i_4}(x_{i_3}x_{i_4})^l, A_{i_4+1}, ..., \ A_n\}$

We will show that $(f^{i_1 i_2})^m \neq (f^{i_3 i_4})^l$ in Γ_n by showing that $(f^{i_1 i_2})^m(T_{\mathbf{X}}) \neq (f^{i_3 i_4})^l(T_{\mathbf{X}})$ in $\mathcal{SPD}$, where $T_{\mathbf{X}}$ is the Bass-Serre tree of $\mathbf{X}$.
The vertex labeled by A_{i_1} is at a distance of 2 from the vertex labeled by A_1 in $(f^{i_3 i_4})^l(T_{\mathbf{X}})$; whereas the vertex labeled by $(x_{i_1}x_{i_2})^{-m}A_{i_1}(x_{i_1}x_{i_2})^m$ is at a distance 2 from the vertex labeled by A_1 in $(f^{i_1 i_2})^m(T_{\mathbf{X}})$. By uniqueness of $A_{i_1} * A_{i_2}$-minimal subtree the vertex labeled by A_{i_1} cannot be at a distance 2 from the vertex labeled by A_1 in $(f^{i_1 i_2})^l(T_{\mathbf{X}})$. Hence, $(f^{i_1 i_2})^m = (f^{i_3 i_4})^l \implies m = 0$. Similarly, The vertex labeled by A_{i_3} is at a distance of 2 from the vertex labeled by A_3 in $(f^{i_1 i_2})^m(T_{\mathbf{X}})$; whereas the vertex labeled

by $(x_{i_3}x_{i_4})^{-l}A_{i_3}(x_{i_3}x_{i_4})^l$ is at a distance 2 from the vertex labeled by A_3 in $(f^{i_3i_4})^l(T_{\mathbf{X}})$. By uniqueness of $A_{i_3} * A_{i_4}$-minimal subtree the vertex labeled by A_{i_3} cannot be at a distance 2 from the vertex labeled by A_3 in $(f^{i_3i_4})^l(T_{\mathbf{X}})$. Hence, $(f^{i_1i_2})^m = (f^{i_3i_4})^l \implies l = 0$ □

Corollary 8.20. $\langle N^{i_1i_2}, N^{i_3i_4}\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$*, where* i_1, i_2, i_3, i_4 *are all different integers.*

Our next goal is to prove that the distance between $\mathbf{X}$ and $\left(f^{i_1i_2}\right)^m \left(f^{i_3i_4}\right)^l (\mathbf{X})$ is at least $2(m+l)$ in $\mathcal{SPD}$.

Consider a non trivial vertex stabilizer subgroup $H \in \mathcal{H}$. We will define a function $g_H^{i_1i_2}$ from the 0-skeleton of $\mathcal{SPD}$ to the real numbers. For a given tree $T \in \mathcal{SPD}$, the function will count the number of vertices labeled by conjugates of A_{i_1} and A_{i_2} on the $x_{i_1}x_{i_2}$-axis between two points on T as described in the following definition.

Definition 8.21. Consider $T \in \mathcal{SPD}$. For $H \in \mathcal{H}$ let $g_H^{i_1i_2}(T)$ be the number of vertices labeled by subgroups of $A_{i_1} * A_{i_2}$ which are conjugates of A_{i_1} and A_{i_2} on the $x_{i_1}x_{i_2}$-axis of T between the following two points.

1. The closest point to the $x_{i_1}x_{i_2}$-axis in T from a vertex labeled by the subgroup H.

2. The vertex labeled by $(x_{i_2}x_{i_1})^m A_{i_2}(x_{i_2}x_{i_1})^{-m}$ on T.

$$g_H^{i_1i_2} : \mathcal{SPD}^0(G, \mathcal{H}) \to \mathbb{R}$$

$$T \mapsto g_H^{i_1i_2}(T)$$

Lemma 8.22. *If* $d_{\mathcal{SPD}}(T_1, T_2) = 1$ *and* $k \neq i_1, i_2$, *then* $|g^{i_1 i_2}_{A_k}(T_1) - g^{i_1 i_2}_{A_k}(T_2)| \leq 1$.

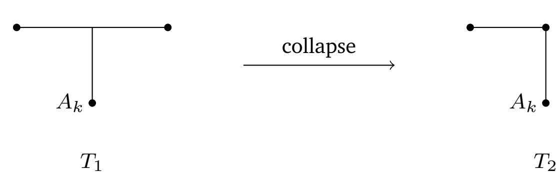
Proof. The idea of the proof is derived from the knowledge of uniqueness (up to $A_{i_1} * A_{i_2}$ equivariant homeomorphism) of $A_{i_1} * A_{i_2}$-minimal subtree inside every tree of $\mathcal{SPD}$ (lemma 5.1).
Without loss of generality, let us assume that

1. T_2 is obtained from T_1 by applying a series of collapse moves on its edge orbits.

2. v_{T_j} is the vertex on the $x_{i_1}x_{i_2}$-axis of T_j closest to the vertex labeled by A_k, where $j \in \{1, 2\}$.

3. The vertex whose stabilizer subgroup is a conjugate of A_{i_j} and is closest to v_{T_1} on the $x_{i_1}x_{i_2}$-axis of T_1 is labeled by $(x_{i_2}x_{i_1})^s A_{i_j} (x_{i_2}x_{i_1})^{-s}$ (or $(x_{i_1}x_{i_2})^s A_{i_j} (x_{i_1}x_{i_2})^{-s}$). Here $j \in \{1, 2\}$.

If v_{T_1} is part of two different fundamental domains of the $x_{i_1}x_{i_2}$-axis, then we choose the fundamental domain closer to $(x_{i_2}x_{i_1})^m A_{i_2} (x_{i_2}x_{i_1})^{-m}$ and its vertex labeling.
Label the vertex whose stabilizer subgroup is a conjugate of A_{i_j} and is closest to v_{T_2} on the $x_{i_1}x_{i_2}$-axis of T_2 by $(x_{i_2}x_{i_1})^r A_{i_j} (x_{i_2}x_{i_1})^{-r}$. Here $j \in \{1, 2\}$.
If we get T_2 by equivariantly collapsing edges of T_1, then $|r - s| \leq 1$



Hence, $|g^{i_1 i_2}_{A_k}(T_1) - g^{i_1 i_2}_{A_k}(T_2)| \leq 1$. ∎

Corollary 8.23. *$g^{i_1 i_2}_{A_k}$ can be continuously extended to a Lipschitz map on all of $\mathcal{SPD}$.*

Proof. This is a result of the definition of a simplicial complex. Any point in a simplicial complex, which is not in the 0-skeleton, is in the interior of a unique simplex. Any point in $\mathcal{SPD}$, which is not in the 0-skeleton can be expressed as a linear combination of the points in the 0-skeleton of the simplex containing them. Hence, we can extend $g^{i_1 i_2}_{A_k}$ linearly, and the resulting extension is Lipschitz. $\square$

Definition 8.24. We will abuse notation to denote the extension of $g^{i_1 i_2}_{H}$ to all of $\mathcal{SPD}$ by $g^{i_1 i_2}_{H} : \mathcal{SPD}(G, \mathcal{H}) \to \mathbb{R}$.

Lemma 8.25. *$g^{i_1 i_2}_{A_k}(T_{\mathbf{X}}) = 2m$ and $g^{i_1 i_2}_{A_k}(\left(f^{i_1 i_2}\right)^m (T_{\mathbf{X}})) = 0$.*

Proof. When $k \notin \{i_1, i_2\}$, the vertices labeled by conjugates of A_{i_1} and A_{i_2} on the $x_{i_1} x_{i_2}$-axis in $T_{\mathbf{X}}$ between the vertex labeled by A_k and $(x_{i_2} x_{i_1})^m A_{i_2} (x_{i_2} x_{i_1})^{-m}$ are listed below in order of increasing distance:

(1) A_2

(2) $x_2 A_1 x_2^{-1} (= (x_2 x_1) A_1 (x_2 x_1)^{-1})$

(3) $(x_2 x_1) A_2 (x_2 x_1)^{-1} (= (x_2 x_1) A_2 (x_2 x_1)^{-1})$

$\vdots$

(2m) $(x_2 x_1)^m A_1 (x_2 x_1)^{-m}$

So, $g^{i_1 i_2}_{A_k}(T_{\mathbf{X}}) = 2m$. When $k \notin \{i_1, i_2\}$, there are no vertices on the tree $(\left(f^{i_1 i_2}\right)^m (T_{\mathbf{X}}))$, with non trivial stabilizer between the vertex labeled by A_k and $(x_{i_2} x_{i_1})^m A_{i_2} (x_{i_2} x_{i_1})^{-m}$. So, $(\left(f^{i_1 i_2}\right)^m (T_{\mathbf{X}})) = 0$. $\square$

Lemma 8.26. $d_{\mathcal{SPD}}(T_{\mathbf{X}}, (f^{i_1 i_2})^m (T_{\mathbf{X}})) \geq 2m$.

Proof. By intermediate value theorem for metric spaces, the image of the path from $T_{\mathbf{X}}$ to $(f^{i_1 i_2})^m (T_{\mathbf{X}})$ under the 1-Lipschitz map $g^{i_1 i_2}_{A_k}$ contains the interval $[0, 2m]$. Hence, $d_{\mathcal{SPD}}(T_{\mathbf{X}}, (f^{i_1 i_2})^m (T_{\mathbf{X}})) \geq 2m$. □

Next we want to prove a similar result about $d_{\mathcal{SPD}}(T_{\mathbf{X}}, (f^{i_1 i_2})^m (f^{i_3 i_4})^l (T_{\mathbf{X}}))$.

Lemma 8.27. $d_{\mathcal{SPD}}(T_{\mathbf{X}}, (f^{i_1 i_2})^m (f^{i_3 i_4})^l (T_{\mathbf{X}})) \geq 2\max(m, l) \geq m + l$.

Proof. Consider $k \notin \{i_1, i_2, i_3, i_4\}$. Then,

$$g^{i_1 i_2}_{A_k}(T_{\mathbf{X}}) = 0$$

$$g^{i_1 i_2}_{A_k}(\left(f^{i_1 i_2}\right)^m \left(f^{i_3 i_4}\right)^l (T_{\mathbf{X}})) = m$$

$$\implies d_{\mathcal{SPD}}(T_{\mathbf{X}}, \left(f^{i_1 i_2}\right)^m \left(f^{i_3 i_4}\right)^l (T_{\mathbf{X}})) \geq 2m.$$

Similarly, we can show that

$$d_{\mathcal{SPD}}(T_{\mathbf{X}}, \left(f^{i_1 i_2}\right)^m \left(f^{i_3 i_4}\right)^l (T_{\mathbf{X}})) \geq 2l.$$

$$d_{\mathcal{SPD}}(T_{\mathbf{X}}, \left(f^{i_1 i_2}\right)^m \left(f^{i_3 i_4}\right)^l (T_{\mathbf{X}})) \geq 2\max(m, l).$$

□

Corollary 8.28. $\langle N^{i_1 i_2}, N^{i_3 i_4} \rangle := \langle f^{i_1 i_2}, f^{i_3 i_4} \rangle$ *is quasi isometrically embedded in* Γ'_n*, when* $i_1, i_2, i_3, i_4 \in \{1, ..., n\}$ *are distinct integers.*

Proof. $\langle N^{i_1 i_2}, N^{i_3 i_4} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, is generated by $f^{i_1 i_2}$, and $f^{i_3 i_4}$.
Consider, $g := (f^{i_1 i_2})^m (f^{i_3 i_4})^l \in \langle N^{i_1 i_2}, N^{i_3 i_4} \rangle$. As $\mathcal{SPD}$ acts geometrically on Γ'_n, we have, $\|g\|_{\Gamma'_n} \approx d_{\mathcal{SPD}}(T_{\mathbf{X}}, g(T_{\mathbf{X}})) \geq m + l = \|g\|_{\langle N^{i_1 i_2}, N^{i_3 i_4} \rangle}$.So, $\langle N^{i_1 i_2}, N^{i_3 i_4} \rangle := \langle f^{i_1 i_2}, f^{i_3 i_4} \rangle$ is quasi isometrically embedded in Γ'_n □

8.3 An undistorted subgroup of $\mathsf{Out}(A_1 * A_2 * A_3 * A_4)$

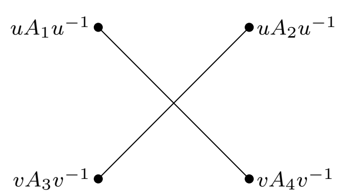
Let us recall the subgroup $M^{12} \oplus M^{34} \leq \Omega_4$ (definitions 7.18, 4.3), where $M^{12} := \langle H_1^1, H_2^1, H_1^2, H_2^2 \rangle$; $M^{34} := \langle H_3^3, H_4^3, H_3^4, H_4^4 \rangle$. To show that $M^{12} \oplus M^{34}$ is undistorted in $\mathsf{Out}(A_1 * A_2 * A_3 * A_4)$ we will

1. define , $\mathcal{M}^4$, an $M^{12} \oplus M^{34}$ invariant, connected sub-complex of $\mathcal{SPD}(G_4, \mathcal{H})$ such $M^{12} \oplus M^{34} \curvearrowright \mathcal{M}^4$ is co-compact; and

2. show that there is a Lipschitz retraction from $\mathcal{SPD}(G_4, \mathcal{H}) \mapsto \mathcal{M}^4$.

Definition 8.29. $\mathcal{M}^4$ is the flag sub-complex of $\mathcal{SPD}(G_4, \mathcal{H})$ spanned by 0-simplices (G_4-trees) of following type- $T \in (\mathcal{M}^4)^0 \iff \exists$ a fundamental domain, F, of T such that the stabilizer of each point of F is either a subgroup of $A_1 * A_2$ or $A_3 * A_4$.

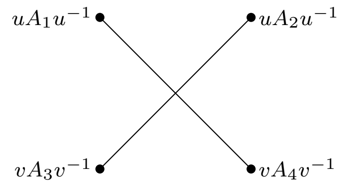
Example 8.30. An example of a graph of groups corresponding to a vertex of $\mathcal{M}^4$ is:



Here, $u \in A_1 * A_2; v \in A_3 * A_4$

Lemma 8.31. *$\mathcal{M}^4$ is an $M^{12} \oplus M^{34}$ invariant sub-complex of $\mathcal{SPD}(G_4, \mathcal{H})$.*

Proof. Let $\mathbf{X}' \in \mathcal{M}^4$ be a graph of groups. Without loss of generality, assume that $\mathbf{X}'$ is given by -



Here, $u \in A_1 * A_2; v \in A_3 * A_4$

So, the vertex group-

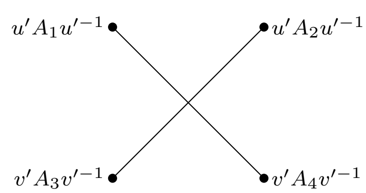
1. conjugate to A_1 is given by uA_1u^{-1}

2. conjugate to A_2 is given by uA_2u^{-1}

3. conjugate to A_3 is given by vA_3v^{-1}

4. conjugate to A_4 is given by vA_4v^{-1}

Here, $u \in A_1 * A_2$ and $v \in A_3 * A_4$. If $f \in M^{12} \oplus M^{34}$, then $f(u) \in A_1 * A_2; f(v) \in A_3 * A_4$. So, f maps the $A_1 * A_2$-minimal subtree of $T_{\mathbf{X}'}$ to the $A_1 * A_2$-minimal subtree of $f(T_{\mathbf{X}'})$. Similarly, f maps the $A_3 * A_4$-minimal subtree of $T_{\mathbf{X}'}$ to the $A_3 * A_4$-minimal subtree of $f(T_{\mathbf{X}'})$. By, uniqueness of $A_1 * A_2$-minimal subtree and $A_3 * A_4$-minimal subtree inside $f(T_{\mathbf{X}'})$, we can represent $f(\mathbf{X}')$ by the following graph of groups -

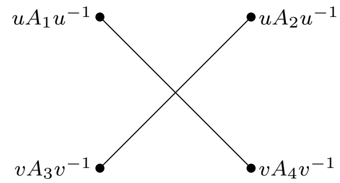


Here, $u' \in A_1 * A_2; v' \in A_3 * A_4$

Hence, $f(\mathbf{X}') \in \mathcal{M}^4$ and $\mathcal{M}^4$ is an $M^{12} \oplus M^{34}$ invariant sub-complex of $\mathcal{SPD}(G_4, \mathcal{H})$. $\square$

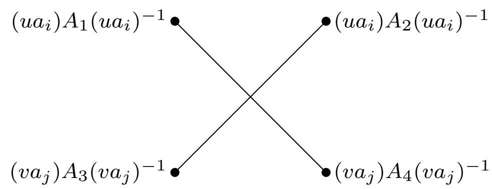
Lemma 8.32. $\mathcal{M}^4$ *is a connected sub-complex of* $\mathcal{SPD}(G_4, \mathcal{H})$.

Proof. We will prove this by induction. Let, $\mathbf{X}' \in \mathcal{M}^4$ be given by -



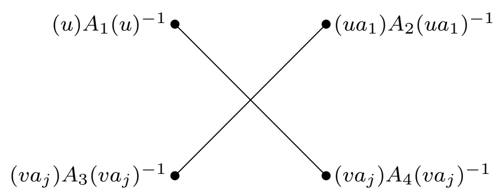
Here, $u \in A_1 * A_2; v \in A_3 * A_4$

We will show that $\mathbf{X}'$ is connected in $\mathcal{M}^4$ to the graph of groups, $\mathbf{X}''$, given by -



Here, $a_i \in A_1 \sqcup A_2; a_j \in A_3 \sqcup A_4$

Without loss of generality assume that $a_i = a_1 \in A_1$. Hence, $\mathbf{X}''$ can be given by -



Here, $a_1 \in A_1 \sqcup A_2; a_j \in A_3 \sqcup A_4$

Now we will give a collapse-expand path from $T_{\mathbf{X}''}$ to $T_{\mathbf{X}'}$ contained inside $\mathcal{M}^4$.

1. **Collapse** the edge adjacent to the vertex labeled by uA_1u^{-1} of $T_{\mathbf{X}''}$, G_4-equivariantly.

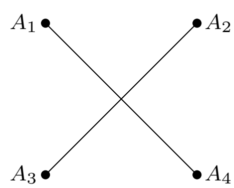
2. In the resulting tree choose the vertex labeled by uA_2u^{-1} (instead of the vertex labeled by $(ua_1)A_2(ua_1)^{-1}$) from the $(u)A_1(u)^{-1} * (ua_1)A_2(ua_1)^{-1}$-

minimal subtree to observe a fundamental domain where the non-trivial vertices are labeled by $\{uA_1u^{-1}, uA_2u^{-1}, (va_j)A_3(va_j)^{-1}, (va_j)A_4(va_j)^{-1}\}$. Observe that this is a tree in $\mathcal{M}^4$.

3. **Expand** the vertex labeled by uA_1u^{-1}, so that in the resulting tree there is a fundamental domain containing the vertices labeled by
$\{uA_1u^{-1}, uA_2u^{-1}, (va_j)A_3(va_j)^{-1}, (va_j)A_4(va_j)^{-1}\}$.

4. Starting from the above tree we will follow similar collapse-expand path (described above) to connect it to a tree containing a fundamental domain in which the vertices are labeled by $\{uA_1u^{-1}, uA_2u^{-1}, vA_3v^{-1}, vA_4v^{-1}\}$. This tree is G_4-equivariantly isometric to $T_{\mathbf{X}'}$.

Now consider the graph of groups $\mathbf{X}$ given by -



Graph of groups: $\mathbf{X}$

By an induction on the word length of $u \in A_1 * A_2$ and $w \in A_3 * A_4$, and repeatedly following the collapse-expand moves described above, we can connect $\mathbf{X}'$ to $\mathbf{X}$ in $\mathcal{M}^4$.

If $\mathbf{Z} \in \mathcal{M}^4$ is a graph of groups, then we can find a collapse-expand path in $\mathcal{M}^4$ from $\mathbf{Z}$ to a graph of groups with same non trivial vertex groups as that of $\mathbf{Z}$, whose underlying graph is isomorphic to the underlying graph of $\mathbf{X}$ (using lemma 6.23). Hence, there is a collapse-expand path in $\mathcal{M}^4$ from $\mathbf{Z}$ to $\mathbf{X}$. $\square$

Lemma 8.33. *The action $M^{12} \oplus M^{34} \curvearrowright \mathcal{M}^4$ is co-compact.*

Proof. We will define an outer automorphism in $M^{12} \oplus M^{34}$, which maps a graph of groups whose non-trivial vertex groups are given by $\{A_1, A_2, A_3, A_4\}$ to any $\mathbf{Z} \in \mathcal{M}^4$, where underlying graph of both graphs of groups are isomorphic.

Assume that the vertex groups of $\mathbf{Z}$ are $\{uA_1u^{-1}, uA_2u^{-1}, vA_3v^{-1}, vA_4v^{-1}\}$, where $u \in A_1 * A_2$ and $v \in A_3 * A_4$ are reduced words given by $u = a_{1i_1}^{\epsilon_1} a_{2i_1} a_{1i_2} a_{2i_2} \ldots a_{1i_k} a_{2i_k}^{\epsilon_2}; v = a_{3i_1}^{\epsilon_3} a_{4i_1} a_{3i_2} a_{4i_2} \ldots a_{3i_l} a_{4i_l}^{\epsilon_4}$. Here, $a_{jp_m} \in A_j$ and $\epsilon_n \in \{0, 1\}$. Then the outer automorphism represented by the following automorphism is the required outer automorphism $(f_{A_1}^{a_{1i_k}})^{\epsilon_1}(f_{A_2}^{a_{2i_k}})\ldots(f_{A_1}^{a_{1i_1}})(f_{A_2}^{a_{2i_1}})^{\epsilon_2}(f_{A_3}^{a_{3i_l}})^{\epsilon_3}(f_{A_4}^{a_{4i_l}})\ldots(f_{A_3}^{a_{3i_1}})(f_{A_4}^{a_{4i_1}})^{\epsilon_4} \in M^{12} \oplus M^{34}$. □

Lemma 8.34. *Let $T \in \mathcal{SPD}$ be a tree and T^{ij} denote the $A_i * A_j$-minimal subtree of T. If $i_1, i_2, i_3, i_4 \in \{1, \ldots, n\}$ are distinct integers, then $T^{i_1i_2} \cap T^{i_3i_4}$ is homeomorphic to a line segment.*

Proof. We will prove this by contradiction. Let, $i_1, i_2, i_3, i_4 \in \{1, \ldots, n\}$ be distinct integers such that $I := T^{i_1i_2} \cap T^{i_3i_4} \neq \phi$. Let, $v \in I$ be a vertex of valence greater than 2.

1. Since, v is a part of $T^{i_1i_2}$, the uniqueness of the minimal subtree $T^{i_1i_2}$ forces the stabilizer subgroup of v to be either a conjugate of A_{i_1} or a conjugate of A_{i_2}, where the conjugating element belongs to the $A_{i_1} * A_{i_2}$.

2. Similarly, v is a part of $T^{i_3i_4}$. So, the stabilizer subgroup of v is either a conjugate of A_{i_3} or a conjugate of A_{i_4}, where the conjugating element belongs to the $A_{i_3} * A_{i_4}$.

We arrive at a contradiction. The intersection, I, cannot have a vertex of valence greater than 2. $\square$

Now we will construct a map from $\mathcal{SPD}(G_4, \mathcal{H})$ to $\mathcal{M}^4$.

Definition 8.35. We will define a map L_4, where $\overline{T}$ is described as follows:

$$L_4 : \mathcal{SPD}^0(G_4, \mathcal{H}) \to \mathcal{M}^4$$
$$T \mapsto \overline{T},$$

1. Consider the smallest subtree of T containing two distinct points whose non-trivial vertex stabilizer subgroups are subgroups of $A_1 * A_2$ and two distinct points whose non-trivial vertex stabilizer subgroups are subgroups of $A_3 * A_4$. In case of an ambiguity, choose the subtree which is closer to the fundamental domain of $A_1 * A_2$-minimal subtree (and $A_3 * A_4$ minimal subtree) whose extremities are labeled by the groups A_1, A_2 (and A_3, A_4).

2. Let $\overline{\mathbf{X}}_{\mathcal{PD}}$ be the graph of groups homothetic to the subtree described above and the non-trivial vertex groups are the subgroups of $A_1 * A_2$ and $A_3 * A_4$ described above, such that the sum of edge length of the underlying graph of $\overline{\mathbf{X}}_{\mathcal{PD}}$ is 1. So, $\overline{\mathbf{X}}_{\mathcal{PD}}$ is an element of $\mathcal{PD}$. Define $\overline{\mathbf{X}}$ to be the image of $\overline{\mathbf{X}}_{\mathcal{PD}}$ in $\mathcal{SPD}$ under the retraction stated in lemma 3.24 and $\overline{T}$ is the Bass-Serre tree of $\overline{\mathbf{X}}$.

Lemma 8.36. *L_4 can be extended to a continuous, Lipschitz map -*

$$L_4 : \mathcal{SPD}(G_4, \mathcal{H}) \to \mathcal{M}^4$$

Proof. The proof is similar to the proof of the lemma 8.12. If $T_1, T_2 \in \mathcal{SPD}^0$ are two trees such that $d_{\mathcal{SPD}}(T_1, T_2) = 1$, then the two trees are related by a collapse move. Without loss of generality, let us assume that T_2 is obtained by applying collapse moves on edge orbits of T_1. If the vertex groups of $\overline{T}_1$ are $\{uA_1u^{-1}, uA_2u^{-1}, vA_3v^{-1}, vA_1v^{-1}\}$ and the vertex groups of $\overline{T}_2$ are $\{u'A_1u'^{-1}, u'A_2u'^{-1}, v'A_3v'^{-1}, v'A_1v'^{-1}\}$, then by the uniqueness of $A_1 * A_2$-minimal subtree we have $u'u^{-1} \in A_1 \cup A_2$. Similarly, by the uniqueness of $A_3 * A_4$-minimal subtree we have $v'v^{-1} \in A_3 \cup A_4$.

Hence, the $d_{\mathcal{M}^4}(\overline{T}_1, \overline{T}_2)$ is at most 6. The explanation of distance 6 is as follows -

1. Starting with $\overline{T}_1$, we apply maximum number of expand move possible.

2. On the resulting tree we collapse all the edge orbits which are not adjacent to a vertex with non-trivial stabilizer along with the edge orbit adjacent to either the vertex stabilized by A_1 or A_2 (a tree of type Y, i.e. a tree whose quotient graph of groups has 3 vertices of valence 1 and 1 vertex of valence 3). Now, we choose a different fundamental domain in this tree of type Y, replacing vertex stabilized by $uA_iu^{-1}, (i \in \{1, 2\})$ with vertex stabilized by $u'A_iu'^{-1}$. Such a choice of fundamental domain is possible due to the uniqueness of the $A_1 * A_2$ minimal subtree in $T_1, \overline{T}_1, T_2$ and $\overline{T}_2$.

3. On the tree of type Y from the previous step we expand one vertex orbit to get a tree with quotient graph of groups having 1 vertex of valence 4 and 4 vertices of valence 1.

4. On the resulting tree we collapse the edge orbit adjacent to either the

vertex stabilized by A_3 or A_4 (a tree of type Y). Now, we choose a different fundamental domain in this tree of type Y, replacing vertex stabilized by $vA_jv^{-1}, (j \in \{3,4\})$ with vertex stabilized by $v'A_iv'^{-1}$. Such a choice of fundamental domain is possible due to the uniqueness of the $A_3 * A_4$ minimal subtree in $T_1, \overline{T}_1, T_2$ and $\overline{T}_2$.

5. On the tree of type Y from the previous step we apply a maximal expand move such that a collapse move will lead us to $\overline{T}_2$.

6. On the resulting tree we apply a collapse move to get $\overline{T}_2$.

Hence, the map L_4 can be linearly extended to $\mathcal{M}^4$ such that

$d_{\mathcal{SPD}}(T_1, T_2) = 1 \implies d_{\mathcal{M}}(L_4(T_1), L_4(T_2)) \leq 6.$ □

Corollary 8.37. *$\mathcal{M}^4$ is a quasi isometrically embedded sub-complex of $\mathcal{SPD}(G_4, \mathcal{H})$. Hence, $M^{12} \oplus M^{34}$ is undistorted in* $\mathsf{Out}(A_1 * A_2 * A_3 * A_4)$.

9 Summary

We will summarize our work together to give a summary of the proof of theorem 1.1 in this section.

Proof of theorem 1.1. 1. Finiteness of Γ_2 follows from corollary 5.2

2. Hyperbolicity of Γ_3 follows from corollary 5.5

3. Γ_4 is thick of order at most 1, when each A_i is finite. The subgroups relevant to our discussion are H^{12}, H^{34} (definition 7.21); $M^{12} \oplus M^{34}$ (definition 7.18) . For a tabular representation refer to table 7.2.1. We will list the reasons whose combination make Γ_4' thick of order at most 1.

(a) $\langle H^{12}, H^{34}, M^{12} \oplus M^{34} \rangle \geq \Gamma_4'$.

(b) H^{12}, H^{34} are undistorted in Γ_4' (corollary 8.14). $M^{12} \oplus M^{34}$ is undistorted in Γ_4 (corollary 8.37).

(c) Proposition 6.15 proves H^{12}, H^{34} is 0-thick. Corollaries 7.19, 8.37 prove $M^{12} \oplus M^{34}$ is 0-thick.

(d) Remark 7.20 proves $H^{12}, H^{34}, M^{12} \oplus M^{34}$ are thickly connected.

4. For $n > 4$, Γ_n is thick of order at most 1, when each A_i is finite. For notations refer to definitions 7.21, 8.15. For a tabular representation refer to table 7.3. We will list the reasons whose combination make $\Gamma_n' \leq \Gamma_n$ thick of order at most 1.

(a) When, $i \neq j$ and $i, j \in \{1, ..., n\}$, then H^{ij} generate Γ_n'

(b) When i_1, i_2, i_3, i_4 are all distinct integers, then $H^{i_1 i_2}$ is undistorted in Γ'_n (corollary 8.14). $\langle N^{i_1 i_2}, N^{i_3 i_4} \rangle$ is undistorted in Γ'_n (corollary 8.28).

(c) When i_1, i_2, i_3, i_4 are all distinct integers, then, proposition 6.15, corollary 8.20 proves $H^{i_1 i_2}, H^{i_3 i_4}, \langle N^{i_1 i_2}, N^{i_3 i_4} \rangle$ are zero thick.

(d) When i_1, i_2, i_3, i_4 are all distinct integers, then corollary 7.23 proves $H^{i_1 i_2}, H^{i_3 i_4}, \langle N^{i_1 i_2}, N^{i_3 i_4} \rangle$ are thickly connected.

□

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