INFLUENCE OF STUDENTS' CONVERSATIONS IN CONSTRUCTING MATHEMATICAL KNOWLEDGE

By

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ABSTRACT OF THE DISSERTATION

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The purpose of this study is to investigate how students' verbal conversation facilitates the exchange of mathematics ideas and the construct of mathematical knowledge when students work in a learning activity. Students often involve artifacts or mathematical models to represent their mathematics ideas and use conversational language to exchange and explain those ideas. In line with these premises, the following questions guided the study: (1) How does student use of semiotic models facilitate the constructing of mathematical ideas? (2) How does the use of conversational language facilitate the communication and appropriation of those ideas? These questions are important as they provide the guidelines to investigate how the use of mathematical models and conversational language by students interplay in their construct of mathematical knowledge.

Video data archive files from NSF-funded research stored in the Video Mosaic Collaborative and RBDIL video collections are used to investigate the influence of students' conversation in constructing mathematical knowledge. This study consists of a fine-grained analysis of a single out-of-school session, lasting 1 hour and1 minute, where a group of five high school junior students, one girl and 4 boys, work in a collaborative activity. The students are invited to solve combinatorics problems related to binomial expansion situations. The video data illustrates how the students involve mathematical models to create meanings to the

combinatorics situations they confront, and how they explain their ideas during their mathematics conversations.

In approaching the issue of how students' mathematics conversations influence their construct of mathematical knowledge, I created the concepts of functional linguistic structures (FLS) and functional semiotic models (FSM) as forms of mathematics expressions and representation. Three functional categories of linguistic structures and three functional categories of semiotic models are introduced in this study to investigate how the use of conversational language and mathematical models interact and influence each other in the students' learning of mathematics.

The method of grouping the data in large clusters by functional categories facilitates the coding of the transcripts and data analysis. This unified approach renders visibility to investigate, in a generalized way, how the students construct new mathematics ideas and how they, through verbal conversations, interchange their ideas to construct further mathematical knowledge. The study identifies students' preferring styles of communicating mathematics ideas and presents forms of mathematics expressions and representation that may be used to improve the teaching and learning of mathematics.

This study presents the importance of taking conversational language as one functional resource that interacts with the mathematical models in the learning of mathematics, and demonstrates from data how the FSM and the FLS interact and influence each other in promoting the semantic expansion and growth of mathematical knowledge. The study brings important implications to the learning of mathematics by considering a multi-modal approach taking conversational language as one resource that interacts with mathematical models to create

meanings in mathematical discourse. The multi-modal approach facilitates the creation of mathematics ideas and provides alternatives for preferring styles of mathematics communication, which may be used to improve the learning of mathematics in classrooms.

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Table of Contents

List of Figures

List of Tables

CHAPTER 1 – INTRODUCTION

Students' performance in mathematics is an issue of concern by researchers and educators. As the increase in diverse student population brings distinct educational expectations and styles, instructional strategies are constantly reviewed to improve the teaching of mathematics. Some researchers emphasize the importance of language in school to help students acquire knowledge and improve their understanding in academic disciplines such as mathematics and sciences (Bailey, Blackstock-Bernstein, & Heritage, 2015; DiCerbo, Anstrom, Bakere, & Rivera, 2014). Other research studies have demonstrated that students can adopt their own methods of learning when they work in peer groups. They can learn mathematics when they work individually by using artifacts or models in a learning environment, or when they interact socially in peer learning groups or in class through conversations (Carroll, Fuson, & Diamond, 2000; Davis, 1992; Maher, Powell, & Uptegrove, 2010; Uptegrove, 2005).

1.1 Nature of the Problem

In order to improve the teaching and learning of mathematics, it is important to understand the nature of mathematics as an academic discipline. This study shares the philosophical view among many that mathematics exists, that it can't be created but it can be discovered. This stance does not purpose to deemphasize other differing perspectives but admit the complexity of this issue. Mathematics ideas, nevertheless, can be created and expressed in many forms. This train of thoughts is, to some extent, in line with following Albert Einstein's quote about mathematics (O'Connor & Robertson, 2007): "One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts." If mathematics in its own discourse is viewed as a natural (science) language that one can use to represent and express ideas, how does vernacular language influence the process of teaching and learning mathematics in school? This question about language and mathematics inspired the development of this study to investigate the interplay between the use of mathematical representations and the use of conversational language in constructing mathematics ideas.

1.2 Research Approach

In approaching the issue of how students' conversations influence their construct of mathematical knowledge, I recognized the necessity of creating a suitable way to study systematically how students' use of verbal language and mathematical models interact and influence each other during the conversation. On this attempt, I introduce the concepts of functional linguistic structures (FLS) and functional semiotic models (FSM) as forms of mathematics expressions and representation. The FLS (referred to in this study as the linguistic structures) seeks to identify and classify the mathematics language students use to describe the mathematics situations they confront. In same respect, the FSM (referred to in this study as the semiotic models) seeks to identify and classify the mathematical models students use to represent the mathematics situations they confront. This study takes the premise that the linguistic structures and the semiotic models interact simultaneously and influence each other in the students' construct of mathematical knowledge when they solve problems in collaboration.

This study takes into account two forms of interactions when students work on a learning activity task in collaboration. One is when the students interact with the learning environment of the task, and the other is when the students interact socially with other students. The form of interaction in which the students learn from the learning environment of the task is referred to in this study as a dialectical discourse. When the students interact socially to explain and negotiate those ideas through verbal conversations with other students, the interaction becomes a dialogical one. Students make use of artifacts or mathematical models of their own construction when solving non-routine mathematics problems. They use them to represent and alter the mathematical configuration of the problem in distinct forms, some of which may be interpreted as new mathematics ideas, and use conversational language to explain and exchange those ideas when working in groups.

To investigate how the students' use of mathematical models and conversational language interact and influence each other in their learning of mathematics, the following research questions guided the study: (1) How does student use of semiotic models facilitate the constructing of mathematical ideas? (2) How does the use of conversational language facilitate the communication and appropriation of those ideas?

Video data and transcripts retrieved from the Rutgers' Video Mosaic Collaborative (VMC) repository were used to conduct the research. The selected video data present a group of five high school junior students, one girl and 4 boys, solving combinatorics problems in discussion with each other as they work on a learning activity task for 1 hour and 1 minute. The participating students do not make use of actual physical artifacts or manipulative materials to construct tangible models to represent the mathematics situations they confront while working on the problems of the learning activity task. Instead, they illustrate on the classroom board, on paper, or by other means, these forms of representation recreated from manipulative experiences they gained through previous learning activities (see Section 3.1). Hence, this study uses the functional category of *visual manipulative* when referring to those forms of (alterable) illustrations made by the students to model the mathematics situations they confront while solving the problems of the task.

The research project was conducted in five phases to watch the video data to observe how the students work on the learning activity task, identify the video data segments where mathematics conversations develop, prepare the transcripts for coding and analysis, identify patterns of engagement involving the use of mathematical models in the mathematics conversations, and evaluate findings to draw conclusions. The research questions are important as they provide the guidelines for the coding of the transcripts of the students' conversations. Specific forms of mathematical models and conversational language are considered in this study. Findings demonstrate that the students' verbal conversations facilitate the transmission and exchange of their mathematics ideas and promote their development of semiotic models conducive to their construct of mathematical knowledge.

1.3 Background Theories

The lenses of cognitive perspectives on peer learning (Golbeck & DeLisi, 1999; Hogan & Tudge, 1999) give basis to examine how the forms of interactions that happen when the students work on the learning activity task influence their cognitive development of mathematics ideas. The socio-cognitive theory takes the interplay of observation and imitation as important role in the individual learning (Bandura, 2005). This form of learning in which the students interact socially is commonly observed from the video data. It is evident from the transcripts that every student, at one moment or another, imitates or repeats what others do or say during the learning activity. Contrary to the socio-cognitive perspective that takes the individual learning as a natural process that happens in social interaction (Bandura, 1976), the socio-cultural perspective, founded by the psychologist Lev S. Vygotsky (1987/1934), claims that students can learn when they use mediation artifacts in the learning environment of the task. By using the artifacts, or the mathematical models, the students can make changes to the learning environment of the task and investigate how it responds to those changes (Cobb, 1985). This form of learning in which the students interact with the environment, physically or mentally, is also observed in this study. Both perspectives of learning help in analyzing the students' learning of mathematics by using multi-semiotic models and conversational language.

1.4 Related Concepts

This study builds upon two concepts related to the teaching and learning of mathematics. They are the mathematics register and the construct of representation. The concept of the mathematics register presents a multi-modal approach of teaching and learning mathematics by using a variety of forms of mathematics utility, such as graphs, symbols, notations, models, etc., that facilitate the expression of mathematics ideas (O'Halloran, 2015; Sigley & Wilkinson, 2015; Uptegrove, 2015). The multi-modal approach takes language as one resource that interacts with those forms of expression to create meanings in mathematical discourse (O'Halloran, 2015).

The construct of representation comprises a process of configuring real-life situations conducive to the formation of complex structures to create mental imaging in the individual (Goldin, 2002). Goldin describes two forms of representation. They are the external (physical) representation and the internal (mental) representation. The concept of the external representation finds common ground with the mathematics register as they include the same utility resources, as described in this study, in constructing the mathematics ideas. The internal representation is not considered for the development of this study as it comprises the configuring (or reconfiguring) of more complex structures constructed mentally. The study considers the external representation only, and language as one separate resource that interacts with it. The external resources are essential for the development of this study as they display observable patterns of student behavior during the students' mathematics conversations.

Three types of meanings surround the multi-modal approach (Halliday, 1978). They are experiential meaning (happenings in the world), logical meaning (logical relations between those happenings), and interpersonal meaning (enacting social relations). The first two, i.e., the experiential and logical meanings, are dependent upon the implicit semantic alignment of mathematical results obtained from previous mathematical discourse, and the process of conceptualization develops in constructing a hierarchy of mathematical concepts upon existing knowledge and previous results (O'Halloran, 2015). In this respect, this study takes the process of conceptualization as one defined by a vertical ascent of meanings.

The multi-modal formulations of mathematical knowledge involve the grammatical integration of language, mathematics symbolism and images leading to the formation of complex linguistic structures found in mathematical discourse (O'Halloran, 2015). This study recognizes the difficulties associated with the grammar of mathematical language, which make the explaining of mathematics difficult to the students when they are unfamiliar with specialized grammatical features. In the process of explaining, the interplay between the resources of language and the symbolic notations and images promotes the semantic expansions and the migration of meanings by re-representing the element of one resource into another resource. According to O'Halloran (2015), meaning may be expanded when the re-represented element is contextualized in conformance with the grammatical structure of the other resource without altering its semiotic content.

During mathematics discourse, conversational language is commonly used by the students to contextualize the mathematical concepts learned previously or at the moment. In contrast to written mathematics, which is characterized for its density in multi-semiotic resources and rigorous grammar, the conversational mathematics is expeditious and fluid, and practical for classroom and group discussion (O'Halloran, 2015), which is observable in this study. The grammatical complexities associated with the mathematical symbolism and images in mathematics conversations become a major source of learning difficulties when the students are unfamiliar with those concepts. The linguistic, symbolic, and visual components embedded in mathematics discussions, take the explaining of mathematics ideas to a new level. This study demonstrates that the symbolic resources have the capacity to pack and unpack information in economic and unambiguous ways, while the visual resources have different grammatical structures that facilitate the making of meaning. O'Halloran (2015) emphasizes the importance of the multi-modal approach to the mathematics register to promote the inter-semiotic, metaphorical expansions of meaning. For this reason, this study analyzes the students' verbal conversations in constructing mathematics ideas from a multi-modal perspective.

The mathematics register components, in alignment with the external representation, include objects, artifacts, models, graphs, diagrams, measuring systems, equations, symbols, and notations and other forms of representation. When these multi-modal components inter-act in the learning of mathematics, the students increase their potential to understand and engage with mathematics in multiple and productive ways (O'Halloran, 2015). The specialized (or mathematics) language, contrary to the so-called everyday or informal language, is characterized for being rich and dense in disciplinary content (Sigley & Wilkinson, 2015). Hence, this study considers the multi-modal approach to investigate the students' capacity for learning mathematics by using specific mathematical models and conversational language. The selected group of students, working on non-routine mathematics (combinatorics) problems, was observed to investigate how they use these particular forms of mathematics representations to construct new mathematics ideas, and how their conversations facilitate the negotiation of those ideas.

CHAPTER 2 – THEORETICAL FRAMEWORK

This study considers the constructivist perspective on learning to study how the group of students working in collaboration finds solutions to mathematics problems as they work on the learning activity task by themselves. Origins of the constructivist theory were introduced by the developmental psychologist Jean Piaget (1985), who emphasized that students can construct their own understanding and knowledge when they are engaged in a learning activity. The constructivist view presents the teacher as a facilitator who provides the activity task, a learning environment, and tools with which students construct their own knowledge. The theory of constructivism is proven to be effective in the learning of mathematics (Cobb, 1994; Davis, 1992; Davis, Maher, & Noddings, 1990).

This study investigates the students' interaction in the learning of mathematics from two ends, i.e., (1) the use of (multi-modal) verbal conversations and (2) the use of (non-linguistic) semiotic models. In this respect, some aspects about the theories of socio-cognitive development and socio-cultural perspective of learning play out an important role in this study. The socio-cognitive theory takes the individual learning as a natural process that happens in social interaction (Bandura, 1976). This theory addresses the interplay between *observation and imitation* in the students' learning of mathematic. This form of interaction is clearly observable during the students' conversation. It is evident from the transcripts that every student, at one moment or another, imitates or repeats what others do or say during the learning activity.

In similar fashion, the socio-cultural perspective, founded by the psychologist Lev S. Vygotsky (1987/1934), claims that the students can learn when they use models to represent mathematics ideas in context with a problem situation. The students can arrange a model configuration and investigate how the configuration of the problem responds to changes provoked by the students (Cobb, 1985). In this case, the interplay between *changes and responses* originated by the students in the configuration of the problem are also observable patterns of behavior during the students’ conversation.

Goldin (1998) suggests the importance of developing a *unified model* for the development of a theoretical framework in mathematics education, and that a selected model must consider adequate integration and expansion of earlier ideas and representational systems resulting from the evolution of diverse theoretical perspectives on mathematical learning. According to Goldin the evolutionary trajectory of these perspectives began with the behavioral theorists who consider learners to be conditioned by external stimuli and rules, and advanced to the cognitivist and constructivist perspectives (among others) in which learners are seen as problem solvers with internal reasoning processes. This shift of perspectives suggests the importance of different kinds of representational systems in the constructs of cognitive structures and schemata (Goldin, 1998).

Pursuant to this view, this study introduces and implements three functional categories of semiotic models to investigate how the students construct and use the models, and to classify what type of models they use to represent their mathematics ideas while solving the problems. The three semiotic models are (1) the visual manipulative, (2) the quantitative systems, and (3) the symbolic notations. The students’ conversational language is regarded as a (multi-modal) form of mathematics communication the students use to negotiate and exchange the new mathematics ideas they built by involving the semiotic models.

The study is conducted with the following thematic question: How can the students' verbal conversations facilitate the creation or development of the functional models conducive to the construct of new mathematics ideas and mathematical knowledge when they work on non-routine mathematics problems in group learning activities?

2.1 Literature Review

This literature review presents some studies that describe the student capacity for learning mathematics from two perspectives: (1) the use of semiotic models to construct mathematics ideas or (2) the use of conversational language to explain those ideas. The former investigates how students adopt, create, or develop semiotic forms of representing mathematics ideas when they work in problem situations, and the latter evaluates how the students' use of language plays out in the process of learning mathematics when they work in groups.

2.1.1 The Use of Artifacts to Construct Mathematics Ideas

Research studies describe how the learning environment and its artifacts, when used as mediation objects, stimulate students' learning by individual experiences (Hung, 2001; McGowen & Tall, 2010; Wong, Marton, Wong, & Lam, 2002). Their findings demonstrate that one's ability to construct mental images to represent mathematical ideas is a reflection of how one thinks and processes the information acquired from the learning environment, and that individual experiences play an important role in the development of one's mathematical thinking and ability to construct those mental images. This method of learning that relies on the utilization of objects (or models) in a learning environment is rooted in the socio-cognitive theory of constructivism (Golbeck & DeLisi, 1999).

Researchers have demonstrated that when students work in context with real life problem situations (Carroll et al., 2000; Hung, 2001; Walkington, Sherman, & Petrosino, 2012), they are able to understand mathematics ideas and solve problems. This particular form of learning, rooted in situated cognition (Hung, 2001), helps students to improve their capacity to construct mental images from previous experiences. Situated cognition claims that meanings are perceived in relation to the interaction that take place between students and their learning environments, a process referred to as relational learning. According to Hung (2001) "meanings are perceived as inseparable from interpretation, and knowledge is linked to the relations of which it is a product."

In essence, situated learning is a process of reconciliation between the student and its learning environment, which involves relational transactions that interact between them in an effort to resolve a problem situation. In relational terms, it is believed that learning evolves as the student advances from using (situated) concrete artifacts to constructing (relational) abstract images in an effort to (mentally) represent a problem situation and solution, and it culminates when the student's cognitive skills shift from "contextualization" to "conceptualization" (Iannone & Nardi, 2005; Walkington et al., 2012).

Problem-posing experiences enhance students' abilities to solve story word problems and contextualized mathematical situations, and the approach of telling stories during class discussion proves to be an effective instructional strategy as learners take an active role in their learning by own sensory experiences (Carroll et al., 2000). According to McGowen and Tall (2010), students' early experiences have an impact in their ability to create abstract models and construct new mathematical knowledge later on. Evidence presented by McGowan and Tall (2010) demonstrates how students' knowledge built from previous experiences influences their abilities to understand mathematics. They use the term "met-before" to illustrate the phenomenon of experiential learning as a process that has stronger effects on students' learning than the traditionally used strategy of representing mathematics ideas with metaphors. This is explained in the following quote:

> The notion of metaphor is used widely in mathematics education to denote the way in which we think about mathematics in terms of physical and mental actions, and the general idea of metaphor, especially when used to perform an intellectual analysis of how concepts are conceived, does not necessarily give a complete view of how students learn (McGowen & Tall, 2010, p. 169).

In this respect, experiential learning, seen as a process of contextualization, plays a stronger role in the student process of learning mathematics than the traditional practice of using metaphors to convey mathematical ideas. Basically, the term "met-before", i.e., the implicit meaning of experiential learning, takes precedence over the common instructional practice of using metaphors (words without the building of experiential meanings) to facilitate the conceptualization of mathematical ideas. The building of experiential meanings is contingent upon the implicit semantic alignment of mathematical results obtained from previous mathematical discourse. In this regard, the process of conceptualization (driven by a vertical ascent of meanings) develops in constructing a hierarchy of mathematical concepts upon existing knowledge and previous results (O'Halloran, 2015).

The theoretical implication of "met-before", contrary to the term "metaphor", focuses on the way students interpret ideas based on previous experiences, a topic that has been investigated by researchers (Presmeg, 1998; Weinberg, Wiesner, & Fukawa-Connelly, 2014; Winter, 2017; Wong et al., 2002). In some respect, this form of learning by experience, rooted in constructivism, is a natural of way of acquiring knowledge and building up ideas (cognitive structures) when students interact with the environment by using mediation artifacts. "It is even more important to take into account the particular mental structures available to the individual that have been built from experience, that the individual has "met-before" (McGowen & Tall, 2010). The notion of "met-before" can be used to analyze and explain how students construct and reconstruct mathematical ideas in context with "met-before" experiences. The contextual implication of "met-before" and its declaring importance for the cognitive development of students are enormous.

The meaningful idea behind "met-before" presents the process of learning mathematics as one in which students construct ideas in context with previous relational and situational experiences. Seen from the constructivist lens, "met-before", in a sense, embraces a process of acquisition of knowledge and development of mental structures that results from those experiences. This study strongly builds on these findings to examine how the semiotic forms of representing mathematics ideas facilitate the exercise of this method of learning mathematics from "met-before" experiences, and how students' conversational language plays out in this process of learning.

2.1.2 The Use of Language to Expressing Mathematics Ideas

The research findings presented in the previous section indicate that the semiotic forms of mathematics representation (models) facilitate the assimilation of perceptible information as part of the process of constructing mathematics ideas. Other studies have demonstrated that vernacular language is an essential communication conduit (or cognitive tool) the students use to express and negotiate those ideas, to transmit and appropriate mathematical knowledge when they work in group learning activities (Ju & Kwon, 2007; Krummheuer, 2007; Presmeg, 1998;

Shepherd & Van de Sande, 2014; Towers, Martin, & Heater, 2013). In line with these perspectives, this study centralizes in how the three generalized forms of mathematics representation develop as primary mediation tools in the process of constructing mathematics ideas and knowledge, and how verbal language facilitates the negotiation and appropriation of the built ideas when the students work in collaboration.

When students are provided with the opportunity to negotiate and appropriate knowledge, to position themselves as participants in doing mathematics in group learning activities, they can elaborate and present better arguments to communicate their mathematics ideas verbally (Blanton & Stylianou, 2014; Ju & Kwon, 2007; Krummheuer, 2007). This form of interaction in which the students are provided with the opportunity to discuss (compare, contrast, explain, justify, clarify, and elaborate) mathematics ideas in groups promotes the students' development of transactive reasoning. According to Blanton and Styliano (2014), transactive discussion (reasoning) is a "significant source of productive metacognitive activity because these discussions lead to the public scrutiny of ideas among peers."

Students can also co-construct and communicate mathematics ideas by writing from remote locations via computer (language) technology (Hung, 2001). A case presented by Hung illustrates how two students co-construct mathematical ideas through a game he calls "*playing with ideas*". In this case, "conjectured ideas act as mediational artifacts between the social and individual levels of cognition" (Hung, 2001, p. 247). The mathematical expressions, condensed and packed in semiotic forms of symbols and notations, are the artifacts (models) both students use to explain their mathematical ideas. Under these circumstances, the process of learning mathematics takes place as they communicate and negotiate ideas expressed in condensed forms (symbols and notations) via the computer language. This study recognizes that these compact and expedient forms of expressing and representing mathematics ideas play a similar role as the semiotic models (algebraic language) commonly used for the teaching and learning of mathematics in schoolrooms.

2.1.3 Learning Mathematics in Classroom Discourse

When students are provided with the opportunity to share their past experiences in class, they learn mathematics in meaningful and mindful ways. This form of learning is situated and involves a dialectical discourse between the students and their learning environment. It improves when the students advance from using situated (concrete) artifacts to constructing relational (abstract) images in an effort to (mentally) represent a problem situation and solution (Carroll et al., 2000; Hung, 2001; Walkington et al., 2012). While contextualization is viewed as a process of constructing meaningful mathematical ideas from students' past experiences, the linguistic forms of reading, writing, speaking, and listening are still essential forms of communication to expressing those constructed ideas in classroom discourse (Drageset, 2015; McGowen & Tall, 2010; Reid, 2002; Walkington et al., 2012; Weinberg et al., 2013).

When students conceive mathematics instruction as a passive routine, they see learning as a transmission of procedures and formulas from the teacher. But when they are provided with the opportunity to conceptualize mathematical ideas based on individual experiences by the mediation of artifacts (or models) in a learning environment, they will be able to represent and express meaningful ideas (Cifarelli, 1998). Cifarelli presents the importance of using artifacts (or models) to construct and represent mathematical ideas, and reinforces that the process of creating abstract representation models appears to be essential for the development of conceptual knowledge and problem solving skills. When teachers understand their students' improving capacity to create abstract representation from experience, they can help them to construct further knowledge and develop problem-solving skills in class.

2.2 Study Outlooks and Implications

Research findings indicate that students improve their mathematics skills as they advance from concrete to abstract, from experiential to expressional, from contextual to conceptual representations (Carroll et al., 2000; Iannone & Nardi, 2005; Shepherd & Van de Sande, 2014). This cognitive development shift implies that the use of semiotic models takes a lead over the use of vernacular language in the process of constructing new mathematics ideas. When students are provided with the opportunity to build up their own framework of understanding from own experiences, their possibilities to succeed in further mathematics activities increase (Swanson, 2014; Weinberg et al., 2014; Wong et al., 2002; Reid, 2002).

In addition to the theoretical perspective of constructivism, this study considers various linguistics theories and perspectives about the use of language that are significant for the interpretation of the students' mathematics conversations. Some of these perspectives take the use of language as a semiotic system and as a functional system of signification. The concepts of denotative semiotics and connotative semiotics present, in textual analysis, a distinction between a meaning that is intended to be transmitted and a meaning that is interpreted by the receiver. Ferdinand de Saussure introduces the concept of a sign system and describes the process of transmission of information in terms of two parts: (1) signifier and (2) signified (Davidse, 1987). The signifier represents the information entity that people perceive with their senses, and the signified represents the mental image of an entity that the person conceives rather than the entity itself. In line with this perspective, Louis Hjelmslev defines the function of signification as a single process of interpretation comprised of the two functions, the denotation and connotation (Haas, 1987).

Expanding to this approach, in which the information comes from external sources only, the concept of metacognition provides a theoretical framework upon which multiple abstract meanings can be created intra-mentally and expressed by means of linguistics structures in situational contexts. The syntagmatic dimension of language, in similar respect, approaches language as a communication tool that is subordinate to the external and inflexible restrictions of grammatical structures. This perspective reinforces the use of syntactic rules as framework and takes the structural organization of the grammar as primary property of the texts in their making.

Contrary to this perspective, John Rupert Firth introduces the idea of considering language as a communication system of (internal, flexible) choices that is subordinate to the linguistics structure (Davidse, 1987). Michael Alexander Kirkwood Halliday adopts this systemic approach from his teacher (John Rupert Firth) and modifies it with a functional dimension; and he calls it systemic functional linguistics (SFL) (Webster, 2015; Hasan, 2015). Although he emphasizes the functionality of language over its structural aspects in the making of meaning, he indicates that the paradigmatic process of the SFL (i.e., a system of choices) always culminates in a (realized) text characterized for its concrete syntagmatic structure (Davidse, 1987, p. 46).

The SFL perspectives are relevant for the development of this study. To investigate how the students' conversations facilitate the exchange of mathematics ideas and transmission of meanings, this study takes language as a functional system that involves choices along a paradigmatic axis. This functional approach helps analyze the students' mathematics conversations from a functional standpoint to understand what they intend to transmit, and not to focus on the syntagmatic structure they use when expressing the mathematics ideas.

In line with this perspective, this study introduces the concept of the functional linguistic structures (or linguistic structures) to investigate how the students' use of language influences the construct of mathematics ideas during their conversation. The linguistic structures call for the presence of (1) mathematical nouns and definitions, (2) mathematical verbs and operations, or (3) logic and conditional statements in the students' use of language during their mathematics conversation. The utilization of these linguistic structures takes the coding of students' conversations to a different level to investigate how the students' use of language influences the making and transmission of meanings and lines up with the students' involvement of the semiotic models in their construct of mathematics ideas.

Halliday defines three kinds of semantic components (metafunctions) that comprise his SFL perspective. They are the ideational function, the interpersonal function, and the textual function. This study centralizes in the ideational function, which is, as Halliday describes, the departure to the process of making meanings, the one in which clauses are used to represent patterns of experiences and to construe the world and the logical relations in the world. He emphasizes three forms of representational entities that mediate the making of meanings and logical relations in this functional (ideational) domain. They are the *processes*, *participants*, and *circumstances*. With this in perspective, this study delineates a coding scheme to incorporate the three specific linguistics structures in the ideational making of mathematical meanings. By considering the function of signification (denotative semiotics and connotative semiotics) and the use of language as a functional system this study establishes a coding protocol for the coding of a selected set of video data and transcripts that shows a group of students working in a collaborative learning activity.

Also, Halliday describes the clause as the entry point to the lexicogrammar in a language. Hence, this study evaluates the video data and transcripts from this perspective by identifying moments of departure where the students initiate the formation of the linguistics structures that transmit (wrong or right) mathematics meanings. In respect to Halliday, this moment in time in which the students initiate any expression of mathematics ideas in context with the students' conversation is referred to in this study as a moment of instantiation.

In concert with the literature review and in line with the theoretical perspective of this study, the following preliminary themes are considered. (1) When an activity demands situated cognition as part of a learning practice in which ideas and knowledge are acquired through relational and meaningful negotiation in class, the activity ought to be preceded by an activity that involves situational transactions between student and environment through dialectical discourse by means of mediation artifacts or models. (2) When conversational language is used in class to externalize and negotiate gained ideas through dialogical discourse, i.e., student to student, any dialectical discords among the different semiotic forms of mathematics representation ought to be resolved at the moment. In this process, (3) learning happens when student appropriates, restores, repairs or replaces mathematics knowledge in an effort to resolve any cognitive disequilibrium (O'Donell, 1999) provoked during class discussion.

2.3 Concepts and Definition

For purposes of this study, the following concepts and words are defined:

1. Representation models – concrete or abstract forms of representation drafted or crafted by the students to model (simulate) a mathematics situation.
2. Generalized forms of representation – representation models grouped by three functional categories (below)
3. *Functional semiotic models* – functional form of representation with assigned meaning with visual, quantitative, or symbolic characterization
    a. Visual manipulative – alterable representation that is referred to from earlier models built with manipulatives or illustrations of models
    b. Quantitative systems – form of representation used for counting, measuring, building patterns to determine the number of configuration options to an existing problem situation
    c. Symbolic notations – form of representation used to formulate mathematics operations to interpolate or extrapolate solutions to the problem
4. Mathematics conversation – verbal language in consort with mathematics symbolism and images (Halliday & Matthiessen, 2004)
5. Multi-semiotic models – multi-modal form of expression by integrating language, symbolism, and images (O'Halloran, 2015).
6. *Functional linguistic structures* – language forming to describe nominal, procedural, or logical connections in mathematics conversations
    a. Mathematical entity – phrase, clause, or sentence forming to describe mathematical nouns or definition (nominalization)

b. Mathematical activity – phrase, clause, or sentence forming to describe mathematical procedures (operation)
c. Mathematical logic – phrase, clause, or sentence forming to describe mathematical conditionals

7. Learning environment of a problem – physical or mental domain where student can gain and reflect on own experiences and understanding of an existing mathematics situation

2.3.1 Assumptions

The following assumptions and baselines are drawn in this study.

1. The generalized forms of representation (multi-semiotic models) are mediation tools the students use (physically or mentally) to construct mathematics ideas.
2. Conversation (verbal language) is a form of communication the students use to negotiate and appropriate the built ideas during social interaction.
3. The Night Session video data from a previously conducted longitudinal research are used to provide adequate task-based content for the development of this study.
4. Data acquisition is limited to observable responses of students from recorded video episodes and transcripts in context with the objective of this study.
5. Patterns of non-verbal or behavioral responses by the students are not considered for evaluation.
6. Gestures and utterances by the students are considered only when those expedient forms of expressions support specific axial coding (Creswell, 2013).
7. Inferences and conclusions will be drawn only from coded descriptions or narrations of the recorded episodes and transcripts.

2.4 Research Questions

The following research questions were used as guide in the research to capture emergent moments of observable (recorded) responses that are relevant for the study:

1. Question (Q1): How might the type of mathematics conversation that the students adopt during the activity facilitate their development of the semiotic models?
   a. How do the students initiate the mathematics conversation?
   b. How does the type of linguistic structures used by the students influence the development or use of the semiotic models?
2. Question (Q2): How does the use of the semiotic models by the students promote the mathematics conversation?
   a. How do the students use the semiotic models?
   b. How do the semiotic models used by the students facilitate the formation of the linguistic structures?

CHAPTER 3 - METHODOLOGY

The video data and transcripts retrieved from the Rutgers' Video Mosaic Collaborative (VMC) repository was used to conduct the research of this study. Specific forms of mathematics representation and verbal expressions are considered for the development of this study to investigate how students' verbal conversations influence the students' construct of mathematical knowledge. Focusing on the transmission and exchange of mathematics ideas among students, the VMC's digital video data are used to watch a group of students working on non-routine mathematics problems in a collaborative learning activity.

The research was conducted in five phases to (1) watch the video data to observe how the students represent and express mathematics ideas while working on the non-routine mathematics problems, (2) identify the video data segments in which mathematics discussions are involved, (3) download and prepare the transcripts for coding and analysis, (4) identify patterns involving the use of the semiotic models and the linguistic structures, and (5) evaluate findings to draw themes and conclusions (Creswell, 2013).

The process of coding was executed in two phases to identify (1) how the students use conversational language and (2) how they use mathematical models. To facilitate the coding of the transcripts and to promote a uniform data collection process, a coding protocol was developed in context with the scope of the study. The process of coding was periodically evaluated to refine or redefine, if necessary, the implemented protocol.

3.1 Virtual Environment and Setting

The VMC platform creates a virtual academic environment that provides the opportunity to browse the video to find and analyze moments where the students use (physically or mentally) forms of mathematical representation (concrete or abstract) and verbal expressions while working as a group. The retrievable transcripts are coded to track those moments during the students' conversation. The study uses the retrievable transcripts as primary source of data to code and track the students' conversations as they construct mathematics ideas while working on the mathematics problems. The video data present a problem-solving session of a longitudinal research project directed by Carolyn Maher (2010) and colleagues, which is accessible through the VMC.

The longitudinal study (extended for more than 25 years) traces the mathematics performance of a small, volunteer group of students from Kenilworth, New Jersey. The longitudinal research team video-recorded the students doing mathematics throughout primary and secondary school. During their last three years of high school, groups of students from among the original participants in the longitudinal study met about four times a year for one to three hours to work in collaborative learning activities after school. During this period of time, the students were presented with open-ended challenging, mostly combinatorics, problems. The purpose of the meeting sessions was not to teach the students but to provide a learning environment and an activity task for them to work on and solve the task problems themselves. During each session, the students were not presented with standard notations of combinatorics. They created their own forms of representations to make sense of the combinatorics problems and to explain their problem solutions.

The Night Session video corpus (A28, Night Session, Pascal's Identity, May 12, 1999) is selected for this study as it shows five high school junior students, 1 girl and 4 boys, working after school in a collaborative learning activity conducted at David Brearley High School in Kenilworth. By that time, the students are already familiar with the numbers of Pascal's Triangle and how to generate them using the towers (of Unifix cubes) and pizza models. The researchers participating in this session are Carolyn A. Maher and Regina Kiczek. The students in the session are: Jeff, Romina, Michael, Ankur (who comes in 9 minutes after the session started), and Brian (who comes in 44 minutes late to join the group).

The Night Session presents a single-session activity where the students are invited to work as a group on combinatorics problems related to binomial expansion situations. The activity lasts 1 hour and 1 minute. The students work on the task in a voluntary evening session. The classroom, small-group session is videotaped. The students work in collaboration to express the Addition rule for Pascal's Triangle, using their understanding of the meaning and express their result in symbolic notation. They use a heuristic approach to build meaning of their solution as they converse with each other and build on each other's ideas. This study shows how the students' mathematical activity during the "Night Session" illustrates the use of semiotic models to represent mathematical ideas and how they explain and justify their reasoning while conversing with each other.

3.2 Description of the Task

The Night Session activity and tasks, obtained from the VMC, are described below.

> Description: In this full-session, raw footage video, students have come to school in the evening for a night session. The group, made up of Jeff, Michael and Romina begin discussing the coefficients of the binomial expansion, specifically

> (a+b) to the 10th power. In attempting to explain why 45 is the coefficient of the third term in this expansion, the students refer to counting how many 10-tall towers have exactly two cubes of a specific color. As they are joined by another member, Ankur, they discuss the formula for "n choose x" using factorial notation and what the factorial symbol means. When asked to explain "why you multiply," Ankur responds by making use of an analogy of counting the number of ways to arrange three different colors. They then investigate the reason for dividing n! by (n-x)! and x! when calculating "n choose x." In explaining the specific example of "5 choose 2," they use the analogies: of arranging five people on a line when you are concerned about the positions of only two of the people and counting the number of 5-tall towers having exactly two cubes of one color. They then discuss the notation for combinations, which they called "choose" notation, and how it relates to Pascal's Identity (the addition rule for Pascal's Triangle). Michael describes how the third row of Pascal's Triangle can be written in "choose notation" (1 3 3 1 becomes "3 choose 0, "3 choose 1," "3 choose 2," and "3 choose 3"). He and the other students explain how the answers to the 3-topping pizza problem are related to row 3 of Pascal's Triangle, and how a specific instance of Pascal's Identity can be understood as generating specific 4-topping pizzas from 3-topping pizzas. The researcher then asks the students to write Pascal's Triangle in this notation, including a general row (row n). The students then explain to Brian, a late-comer, the meaning of Pascal's Identity (the addition rule for Pascal's Triangle) in terms of operations on the pizzas that are represented by specific entries in Pascal's Triangle. They write Pascal's Identity in general form using standard notation.

The learning activity of the Night Session promotes the involvement of various semiotic models by the group of students to represent problem situations in distinct ways, and the communication of mathematics ideas through verbal conversation. The settings and the tasks described in the Night Session video data are adequate for the development of this study.

3.3 Data Collection

The video data and transcripts of the Night Session are used to study the mathematics conversation sustained by the students while working on the task problems. The video data are watched to investigate those episodes where the students create or use mathematical models (concrete or abstract) that facilitate the exchange of mathematics ideas through verbal conversations, and episodes where the students initiate or sustain mathematics conversations that promote the creation of mathematical models. The transcripts were downloaded from the VMC repository and stored in text format in a flash drive.

This study centralizes particularly in the involvement of mathematical models and the use of conversational language by the students to represent mathematics ideas and to explain those ideas while working in collaboration. These forms of expressions and representation are grouped by functional categories as described in the coding schemes of Table 1 and Table 2. These tables describe three specific forms of verbal expressions, referred to as the linguistic structures, and three specific forms of mathematics representation, referred to as the semiotic models. They are both considered for the coding of the transcripts (the students' conversation) to investigate how the students represent and express mathematics ideas.

To answer the research questions, the coding schemes described in these tables are used to investigate how the students use the semiotic models and the linguistic structures to construct mathematics ideas. The data acquisition process centralizes in capturing moments, or instances, in the conversation when the students use these specific forms of mathematics representation and verbal expressions. The number of instances is discretely counted and time-recorded to identify patterns of behavior, commonality and frequency, in the use of the linguistic structures and the semiotic models throughout the students' mathematics conversation.

Table 1
Coding scheme for the use of the linguistic structures.

Code	Description
A1 Entity	The student uses the linguistic structures to describe (or identify) a problem situation by means of words that represent mathematics entities such as numbers, variables, terms, or concepts. When the student uses a phrase or clause to describe or identify a mathematics entity, the entire sentence is coded A1. The use of noun(s) in the making of meaning through a mathematics entity is essential for the coding of A1. Leading particulars: (1) The student relates a problem situation to a mathematical entity and uses a noun (phrase) to refer to it by a name or definition. (2) The student describes a concept and uses a (specific) noun to refer to it by a term or concept specific to mathematics. (3) The student classifies a concept according to underlying properties and uses a (specialized) noun to refer to it by a technical concept. Examples: Ten, forty-five, n, r, x, 5x, one-two-one, one-three-three-one, like terms, binomial, combinations, repeats, five-choose-two, n-choose-r, factorial, Pascal's Triangle, etc.
A2 Activity	The student uses the linguistic structures to describe (or identify) a problem situation by means of words that represent the performing of mathematics activities such as the selection of groups, the exchange of positions, the making of combinations, the formation of patterns (or arrays) and the execution of mathematics operations. When the student uses a phrase or clause to describe or identify a mathematics activity, the entire sentence is coded A2. The use of verb(s) in the making of meaning through a mathematics activity is essential for the coding of A2. Leading Particulars: (1) The student describes an exchange (or selection) of entities, a formation of patterns or an execution of mathematics operations, and uses a noun (group) to identify the activity as a process. (2) The student describes a mathematical interaction among concepts and processes and uses a noun (group structure) to identify it as a new process. Examples: Pick three things out of those ten, let a be red and b blue, a plus b, ten percent of a hundred, ten a to the ninth and b to the first, divide by 2 factorial, etc.

Table 1
Coding scheme for the use of the linguistic structures (*continued*)

Code	Description
A3 Logic	The student uses logical statements to describe chains of logical relations among mathematics entities and activities by means of logical connectors. The student makes use of logical connectors to establish such a relationship. When the student uses two or more phrases or clauses to describe or identify a logic relation among mathematics entities or activities or both, the entire sentence is coded A3. The use of logical connectives is essential for the coding of A3. Leading particulars: The student describes chains of logical relations among mathematics entities and activities and makes use of logical connectors to establish such a relationship as statements. Examples: *and, or, if, not, nor, then, so,* or other *logical* conjunctions or connectives.

Note. Leading question: What type of linguistic structures do the students use to explain a problem situation during the mathematics conversations?

Table 2
Coding scheme for the involvement of the semiotic models.

Code	Description
B1 Visual	Coding for B1: The student makes use of visual manipulative (as models) to represent (simulate) mathematics situations to a problem. The student may craft or draft the visual manipulative models and use them to examine the configuration and extent of the problem situation by (re)grouping, (re)collecting, or (re)connecting them as a whole or in parts. The phrase, clause, or sentence where this moment is recorded in video time is coded B1. Leading particulars: The student creates (crafts or drafts) forms of mathematical representation (concrete or abstract) to investigate (physically or mentally) the configuration and extent of a problem situation. Examples: Unifix cubes, blocks, towers, pizza pies, group of people, group of objects, choices of parts, (ordinal or cardinal) numbers, switch of places, people, or objects, “put it there and there”, “move around different spots”, chairs and desks, and others.

Table 2
Coding scheme for the use of the semiotic models (*continued*)

Code	Description
B2 Quantitative	Coding for B2: The student makes use of counting and measurement systems (as models) to quantify specific mathematics situations to a problem. The student may apply existing or create own systems and use them to exhaust all possible configuration options or to identify potential resolutions to the problem by counting, measuring, or forming patterns (in arrangement). The phrase, clause, or sentence where this moment is recorded in video time is coded B2. Leading particulars: The student uses forms of mathematics representation for counting, measuring, graphing, or (con)-forming patterns in arrangement to deplete all possible (con)-figuration options. Examples: Pascal's Triangle, family trees, block diagrams, making arrangements, making patterns or repeats, counting people, counting objects, computation, numerical values (quantities), 6!, "5 choose 3", basic arithmetic, "add", "take away", "multiply", "divide", "eliminate", "get rid of", etc.
B3 Symbolic	Coding for B3: The student uses formulas, symbols, and notations (as models) to express particular mathematics solutions to the problem. The student may apply existing or create new mathematics expressions and use them to interpolate or extrapolate solution possibilities by composing, processing, or performing mathematics operations. The phrase, clause, or sentence where this moment is recorded in video time is coded B3. Leading particulars: The student uses forms of mathematics representation to formulate mathematics operations to interpolate or extrapolate solutions to the problem. Examples: Algebraic expressions and equations, polynomial expansions, mathematical operators and formulas, factorial notation, combination formula, permutation formula, n!, "n choose m", "pick n out of m", etc.

Note. Leading question: What type of semiotic models do the students use to represent a problem situation during the mathematics conversation?

3.3.1 Protocol for the Coding of the Linguistic Structures

The purpose of this protocol is to establish the guidelines for the coding of the video data and transcripts of the Night Session – Pascal's Identity, Session of May, 1999. The students' verbal conversation is studied following the coding scheme of Table 1.

The coding of the transcripts was conducted in the following sequence:

1. Coding for A3 (logic): The use of logical connectives is essential for the coding of A3.
2. Coding for A2 (activity): The use of mathematics verb(s) in the making of meaning through a mathematics activity is essential for the coding of A2.
3. Coding for A1 (entity): The use of noun(s) in the making of meaning through a mathematics entity is essential for the coding of A1.

The following ground rules apply:

1. Phrases, clauses, and sentences recorded in the transcripts are coded only when they transmit mathematical meaning.
2. Right or wrong explanations are equally considered as they represent the iterative nature of building mathematics ideas.
3. A sentence containing a coded phrase or clause demarks a moment of instantiation in the transcripts and therefore the sentence is assigned only one code.
4. Only lexical phrases, clauses, and sentences are considered for coding and not indexical expressions and pronouns.

5. When a student is interrupted, each one of the phrases, clauses, or sentences recorded during the discourse is coded accordingly.
6. When the verb *to be* is used (in any of its conjugated forms) to indicate the presence or existence of an entity, its phrase, clause, and sentence are coded A1 and not A2.
7. When the verb *to have* is used (in any of its conjugated forms) to imply the possession, giving, or receiving of an entity, its phrase, clause, and sentence are coded A1 and not A2.
8. The performing of mathematics activities (A2) are coded only if they are in context with any form of interaction that takes place among mathematics entities only.
9. Explanations of real problem situations in which students (people) are taken as (mathematics) entities are coded accordingly.
10. When conjunctions (*and, then, so,* and others) are used to concatenate procedural (mathematics) steps without logic content, the sentence is coded A2 (activity) and not A3 (logic).

The following conditions are not considered for coding:

1. Discussions lacking mathematical meaning and expressions of confirmation, opposition, verification, or validation without the describing of mathematical entities, activities, or logics
2. Explanations about how to use technology, calculator
3. The telling of what other students did in the past

The following are special cases of ascents in the coding for the linguistic structures:

1. When an A1-structured phrase, clause, or sentence describes an entity in anticipation to the (immediate) construct of an A2 structure by the student or another student, it is promoted to A2 for its contribution to the making of mathematical meaning.
2. When an A2-structured phrase, clause, or sentence describes an activity in anticipation to the (immediate) construct of an A3 structure by the student or another student, it is promoted to A3 for its contribution to the making of mathematical meaning.
3. When an A1-structured phrase, clause, or sentence describes an entity in anticipation to the (immediate) construct of an A3 structure by the student, it is promoted to A3 for its contribution to the making of mathematical meaning.
4. When an A1-structured phrase, clause, or sentence describes an entity in anticipation to the (imminent) construct of an A3 structure by another student, it is promoted to A2 for its contribution to the making of mathematical meaning.
5. When a clause, phrase, or sentence describes the extension or restriction of an entity (i.e., its circumstances), it is coded A1.

The following are special cases of descents in the coding for the linguistic structures:

1. When a sentence contains a prospect A3 structure that doesn't transmit a conclusive mathematical meaning in context with the conversation, it is demoted to A2.
2. When a sentence contains a prospect A2 structure that doesn't transmit a conclusive mathematical meaning in context with the conversation, it is demoted to A1.
3. When a sentence contains a prospect A1 structure that doesn't transmit a conclusive mathematical meaning in context with the conversation, it is un-coded (left blank).

4. Indexical expressions and pronouns are coded accordingly only when they transmit mathematical meaning in context with the conversation.

The socio-cognitive view of *imitation-by-observation* plays a role in the students' mathematics conversations. It is evident from the transcripts that every single student, at one moment or another, repeats what others do or say during the learning activity. Below are special cases of descents in the coding for the repeats of the linguistic structures.

1. When an A3-structured phrase, clause, or sentence repeats the same mathematical meaning transmitted (described) by another student in context with the conversation, the repeat is demoted to A2.
2. When an A2-structured phrase, clause, or sentence repeats the same mathematical meaning transmitted (described) by another student in context with the conversation, the repeat is demoted to A1.
3. When an A1-structured phrase, clause, or sentence repeats the same mathematical meaning transmitted (described) by another student in context with the conversation, the repeat is coded A1.

3.3.2 Protocol for the Coding of the Semiotic Models

The purpose of this protocol is to establish the guidelines for the coding of the video data and transcripts of the Night Session – Pascal's Identity, Session of May, 1999. The students' use of the semiotic models is studied following the coding scheme of Table 2. The purpose of coding the transcripts is to track moments in the conversation in which the students involve the semiotic models during the mathematics conversation.

The coding of the transcripts was conducted in the following sequence:

1. Coding for B1 (visual image): The use of visual manipulative (concrete or abstract) to represent (model) a mathematics situation to a problem is essential for the coding of B1.
2. Coding for B2 (quantitative systems): The use of counting and measurement systems (as models) to quantify (estimate) specific mathematics situations to a problem is essential for the coding of B2.
3. Coding for B3 (symbolic notations): The use of formulas, symbols, and notations to estimate particular mathematics solutions to a problem is essential for the coding of B3.

The following ground rules apply:

1. All models, effective or ineffective, are equally considered as they represent the iterative nature of building mathematics ideas.
2. A phrase, clause, or sentence making reference to, or involving the use of, a semiotic model in context with the mathematics conversation demarks a moment of instantiation and therefore it is assigned a code.
3. Indexical phrases and clauses referring to any of the semiotic models are coded accordingly.
4. Formulas or notations displayed on the keypad of a calculator are coded only if the students use them as a reference during an explanation. Calculations made with the calculator are not considered for coding.

5. When a phrase, clause, or sentence makes use of (mathematical) words like addition, subtraction, multiplication, division, simplification, notation, probability, and combination without involving any type of formulas, symbols, or notations, it is coded B2.

The socio-cultural view of *responses-to-changes* plays an important role in the students' learning of mathematics when they alter the configuration of the problems. Below are special cases of ascents in the coding for the semiotic models.

1. When a phrase, clause, or sentence describes the construct of a B3 model by involving a B2 platform, it is coded B3.
2. When a phrase, clause, or sentence describes the construct of a B3 model by involving a B1 platform, it is coded B3.
3. When a phrase, clause, or sentence describes the construct of a B2 model by involving a B1 platform, it is coded B2.

The following are special cases of descents in the coding for the semiotic models:

1. When a phrase, clause, or sentence describes the deconstruction of a B3 model by involving a B2 platform, it is coded B2.
2. When a phrase, clause, or sentence describes the deconstruction of a B3 model by involving a B1 platform, it is coded B1.
3. When a phrase, clause, or sentence describes the deconstruction of a B2 model by involving a B1 platform, it is coded B1.

3.4 Description of the Transcripts of Appendices A and B

The video data and transcripts were used to track the forms of verbal explanations that lead to the creation or utilization of the semiotic models, and to track the forms of mathematical models that lead to the initiation or continuation of constructive mathematics conversations. To answer the research questions Q1 and Q2, the students' verbal conversations were coded from the transcripts and tabulated in time sequence with the video to track the students' use of the three semiotic models and the three linguistic structures. Appendix A contains the table of the coded transcripts that corresponds to the use of the linguistic structures, and Appendix B contains the table of the coded transcripts that corresponds to the use of the semiotic models. Each table includes the conversation line number, the time each conversation line was recorded, the name of the student who talked, the statement(s) being coded, and the type of code(s) assigned to each coded statement.

3.5 Data Analysis

The transcripts were analyzed in context with the research questions following the coding schemes of Table 1 and Table 2. The data were used to investigate how the linguistic structures and the multi-semiotic models facilitate the exchange and construct of mathematics ideas by the students. The transcripts were coded by hand and stored in table format as shown in Appendix A and Appendix B. The table facilitates the tracking of the episodes that embark the initiation or continuation of verbal conversations, and the creation or utilization of mathematical models by the students conducive to the construction of new mathematics ideas.

The coded data were analyzed to identify patterns of engagement between the three semiotic models and the three linguistic structures as the students' verbal conversation progresses. The coded data, clustered in groups of three by functional categories, are displayed in graphs and charts (Chapter 4) to illustrate the tracing and engagement of the multi-semiotic models and the linguistic structures in the construct and exchange of mathematics ideas by the students throughout the activity. The display of the data facilitates the identification of emergent and consistent patterns of behavior among the codes.

The conversational language was investigated to identify shifts and patterns of behavior among the three linguistic structures. The transcripts were coded in a way that for each moment in the conversation that a linguistic structure was used by a student, it was assigned and designated as a coded instance. Likewise, the data for the mathematical models were analyzed to identify shifts and patterns of behavior among the three semiotic models during the students' conversation. For each time a semiotic model was involved during the students' conversation, it was also assigned and designated as a coded instance.

3.6 Inter-rater Reliability

The students' verbal conversation was coded for the usage of the linguistic structures and the involvement of the semiotic models. The students' usage of the linguistic structures was coded by two independent coders. The inter-rater reliability analysis is prepared to quantify the level of consistency in the coding of the transcripts by the two coders. The computational method of Cohen's Kappa, taking the ratings of the three linguistic structures (i.e., mathematical entity, mathematical activity, and mathematical logic) as nominal values, provides a way of estimating the degree of agreement among the ratings provided by the two coders. The explicit and referential meanings of the three semiotic models involved in the students' mathematics conversations made the coding for these forms of mathematics representation straight forward and therefore were not considered for inter-rater reliability analysis.

During the coding for the three linguistic structures, two pairs of codes were found to be very susceptible to inter-rater disagreement. They are (1) the coding for mathematical logic versus mathematical activity, and (2) the coding for mathematical activity versus mathematical entity. All disagreements and the circumstances that caused those disagreements were investigated. It was found that the utilization of conjunctions in the students' mathematics conversations made the coding for mathematical logic versus mathematical activity susceptible to discrepant interpretations by the coders. Disagreements on the coding for mathematical activity versus mathematical entity were mainly attributed to how the verbs were used by the students when describing salient mathematics situations in context with the problem of the learning activity task. The students' usage of these two grammatical components, i.e., conjunctions and verbs, were investigated and considered for the development of a protocol for the coding of mathematical logic, mathematical activity, and mathematical entity. The coding protocol provides guidelines to prevent misinterpretation of the coded lines and to promote inter-rater agreement during the coding for the three linguistic structures. Since the coding for mathematical entity versus mathematical logic were never subject to inter-rater disagreement, the inter-rating of these two linguistic structures are considered to be exclusive events for purposes of reliability in this study.

The coding of the transcripts of Appendix A was executed in various rounds. After each round the data were evaluated for agreement and disagreement by the coders. For the first rounds of coding (before establishing the protocol), approximately 3 out of 4 (i.e., 75%) of the lines coded were in agreement. For the final rounds of coding (after following the protocol) the rate of agreement became closer to 90%. Table 3 provides data related to moments in the students' conversations that were found to be susceptible to inter-rater disagreement on the ratings.

Table 3
Coded lines with high likelihood of disagreement.

E-A disagreement descriptor	Transcripts line item locator
Coded lines in which verbs are used to describe the presence, existence, or possession of mathematical entities – including nominalization of mathematics expressions – and not to execute mathematical activities.	54, 57, 59, 67, 309, 310, 312, 313, 314, 348, 369, 489, 525, 527, 556, 568, 569, 570, 571, 621, 623, 624, 668, 670, 671, 677, 679, 690, 692, 716, 717, 718, 719, 720, 721, 723, 751, 851, 852, 943, 1020, 1195, 1203, 1205, 1210, 1217, 1306, 1314 Total number of lines – 48 Percent of coded lines – 5.2%
A-L disagreement descriptor	**Transcripts line item locator**
Coded lines in which conjunctions are used to concatenate sequences of mathematical activities and not to connect conditionals and logic statements.	36, 90, 162, 196, 206, 259, 455, 459, 466, 498, 507, 548, 560, 577, 684, 686, 724, 736, 893, 1020, 1054, 1146 Total number of lines – 22 Percent of coded lines – 2.4%

Note: E-A stands for Entity-Activity and A-L stands for Activity-Logic.

The total count of lines coded from the transcripts equals 429 for mathematical entity. For mathematical activity and mathematical logic the total counts are 329 and 174 respectively. Figure 1, Figure 2 and Figure 3 show modified figures of inter-rating data normally distributed between Coder 1 and Coder 2 in terms of (a) the number of lines coded for the linguistic structures, and (b) the probability of expected inter-rater agreement in percent. These data are hypothetically used to calculate the corresponding inter-rater reliability using the equation of Cohen's Kappa

$$k = \frac{p(o) - p(c)}{1 - p(c)}$$

In this equation, k represents the Cohen's Kappa index, $p(o)$ the observed percentage of agreement and $p(c)$ the probability of agreement by chance.

Rating of 75% agreement
Coder 1

Coder 2	E	A	L	
E	321	54		375
A	54	245	42	341
L		42	174	216
	375	341	216	932

(a) Count of coded lines

Rating of 75% agreement
Coder 1

Coder 2	E	A	L	
E	34.4	5.8		40.2
A	5.8	26.3	4.5	36.6
L		4.5	18.7	23.2
	40.2	36.6	23.2	100

(b) Probability matrix for the coding

Figure 1. Coding estimates using the rating of 75% agreement before the protocol.

The probability of obtaining observed percentage of agreement based on the data of Figure 1 equals the sum of the diagonal values of the probability matrix divided by 100.

$$\frac{34.4 + 26.3 + 18.7}{100} = .794$$

The total probability of obtaining chance agreement on mathematical entity, mathematical activity, and mathematical logic equals the sum of the squares of their respective marginal values divided by 10,000.

$$\frac{40.2^2 + 36.6^2 + 23.2^2}{10{,}000} = .349$$

Using these estimates, Kappa is computed using the equation

$$k = \frac{p(o) - p(c)}{1 - p(c)} = \frac{.794 - .349}{1 - .349} = .682$$

Rating of 90% agreement
Coder 1

Coder 2	E	A	L	
E	386	21		407
A	21	296	17	334
L		17	174	191
	407	334	191	932

(a) Count of coded lines

Rating of 90% agreement
Coder 1

Coder 2	E	A	L	
E	41.4	2.3		43.7
A	2.3	31.8	1.8	35.9
L		1.8	18.6	20.4
	43.7	35.9	20.4	100

(b) Probability matrix for the coding

Figure 2. Coding estimates using the rating of 90% agreement after the protocol.

Likewise, the probability of obtaining observed percentage of agreement based on the data of Figure 2 equals the sum of the diagonal values of the probability matrix divided by 100.

$$\frac{41.4 + 31.8 + 18.6}{100} = .918$$

The total probability of obtaining chance agreement on mathematical entity, mathematical activity, and mathematical logic equals the sum of the squares of their respective marginal values divided by 10,000.

$$\frac{43.7^2 + 35.9^2 + 20.4^2}{10{,}000} = .361$$

Using these estimates, Kappa turns out to be

$$k = \frac{p(o) - p(c)}{1 - p(c)} = \frac{.918 - .361}{1 - .361} = .872$$

High disagreement likelihood
Coder 1

Coder 2	E	A	L	
E	381	24		405
A	24	307	11	342
L		11	174	185
	405	342	191	932

(a) Count of coded lines

High disagreement likelihood
Coder 1

Coder 2	E	A	L	
E	40.9	2.6		43.5
A	2.6	32.9	1.2	36.7
L		1.2	18.6	19.8
	43.5	36.7	19.8	100

(b) Probability matrix for the coding

Figure 3. Coding estimates using the parameters of high likelihood of disagreement.

If the disagreement descriptors of Table 3 are assumed to be the only source of inter-rater discrepancy, then the modified count of coded lines and the probability matrix for the coding are estimated to be normally distributed as shown in Figure 3. After using these parameters, the value of k approximates to .881. Hence, it is estimated that the value of the Cohen's Kappa index for the coding of the linguistic structures, thus the inter-rater reliability of this study, fluctuates between .68 and .88.

CHAPTER 4 – RESULTS

The coded transcripts of Appendix A and Appendix B shows moments in the conversation where the students use the linguistic structures and involve the semiotic models respectively. A phrase, clause, or sentence coded for a linguistic structure or a semiotic model defines a moment of instantiation, or a coded instance. The coded instance and the recorded time when this instance occur in the video comprise a set of data. The cumulative count of coded instances and the recorded time where the accumulation sums up comprises a co-ordinate point of the data. The coordinates are used to generate graphs to illustrate the trends in the usage of the linguistic structures and the semiotic models by the students throughout their conversations.

*4.1 C*umulative Counts of Coded Conversation Instances (Appendix C)

The cumulative counts of coded conversation instances involving the linguistic structures and the semiotic models are plotted with respect to the recorded time of the students' conversation in the semi-logarithmic graphs of Appendix C. The semi-logarithmic plots accentuate the display of curve trends and behavior, which can be used to locate patterns of engagement and inter-semiotic shifts among the linguistic structures and the three semiotic models in the students' conversation. These graphs show how frequent the linguistic structures and the semiotic models are used by the students throughout the learning activity.

To investigate how the engagement among the linguistic structures and the semiotic models plays out in the students' mathematics conversation, the data are analyzed from two ends: (1) Comparing the usage of the linguistic structures throughout the students' conversation, and (2) comparing the involvement of the semiotic models throughout the students' conversation. The cumulative counts of the coded conversation instances involving these forms of verbal expressions and mathematics representations are plotted following the coding schemes of Table 1 and Table 2.

4.2 Engagement of the Linguistic Structures in the Students' Verbal Conversation

Figures C1 through C4, in Appendix C, shows the graphs of the data that correspond to the use of the linguistic structures by the students throughout the conversation. The data are grouped into the three functional categories. They are (1) the mathematical entity, (2) the mathematical activity, and (3) the mathematical logic. The cumulative counts of coded conversation instances involving the linguistic structures are plotted with respect to the recorded time of the students' conversation in a semi-logarithmic graph. This graphing format accentuates the display of curve trends and behaviors that can be used to infer how the three linguistic structures engage in the students' conversation throughout the students' conversation. The graphs also help identify how often the linguistic structures are used by the students as the learning activity progresses. Each coded instance, or moment of instantiation, assigns only one linguistic structure coded under one of the following categories: Mathematical entity, mathematical activity, or mathematical logic. At the termination of the conversation, the total accumulation of coded instances is 932.

Figure 4 illustrates the progression of the usage of the linguistic structures throughout the students' conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the linguistic structures divided by the total count of coded instances accumulated as of to that moment in the conversation. After the duration of the conversation the total count of coded instances is distributed as follows: Entity - 429; Activity - 329; and Logic - 174.

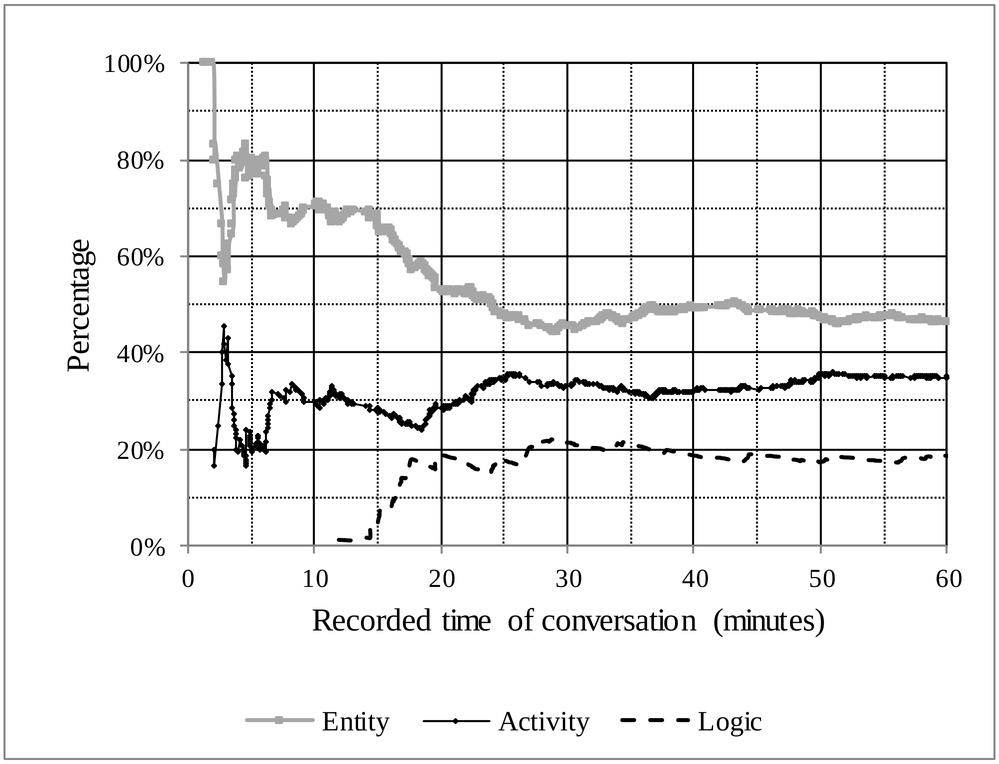

Figure 4. Progression of the linguistic structures usage throughout the students' conversation.

As shown in the graph, the mathematical entity is the linguistic structure most frequently used throughout the conversation followed by the mathematical activity. During the first 10 minutes of conversation, the data that correspond to the linguistic structures of mathematical entity and mathematical activity are the only ones shown in the graph. The linguistic structure of mathematical logic emerges after the passing of 10 minutes of conversation. This lagging factor displays a form of retardation in the formation or utilization of this linguistic structure by the students. Factors of retardation like this one are exhibited in other graphs.

After the emergence of the mathematical logic, the curve that corresponds to the linguistic structure of mathematical entity fairly decreases between the time marks of 10 and 30 minutes in the graph. This is an indication that the students start forming conditional statements to describe and explain the mathematical operations of the problem. During this period of time the curves that correspond to the mathematical activity and the mathematical logic display sporadic (ex)-changes in rate. During this exchange the students describe the mathematics operations and conditions under which these operations take place, and adopt the use of mathematical activity and mathematical logic as preferring forms of linguistic structures in the conversation. They use less nominal phrases in their mathematics conversation.

Curve behavior of lessening rate is evident in the time interval of 12-18 minutes for the linguistic structure of mathematical activity while the curve that corresponds to mathematical logic ramps up. After the 30-minute mark, the three curves sustain a consistent pace, with an average usage rate of 48%, 32%, and 20% among the linguistic structures of mathematical entity, mathematical activity, and mathematical logic respectively.

Figure 5 shows the progression of the linguistic structures usage when the students involve visual manipulative in their conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the linguistic structures divided by the total count of coded instances accumulated as of to that moment in the conversation. At the termination of the conversation the total count of coded instances is distributed as follows: Entity - 71; Activity - 69; and Logic - 57.

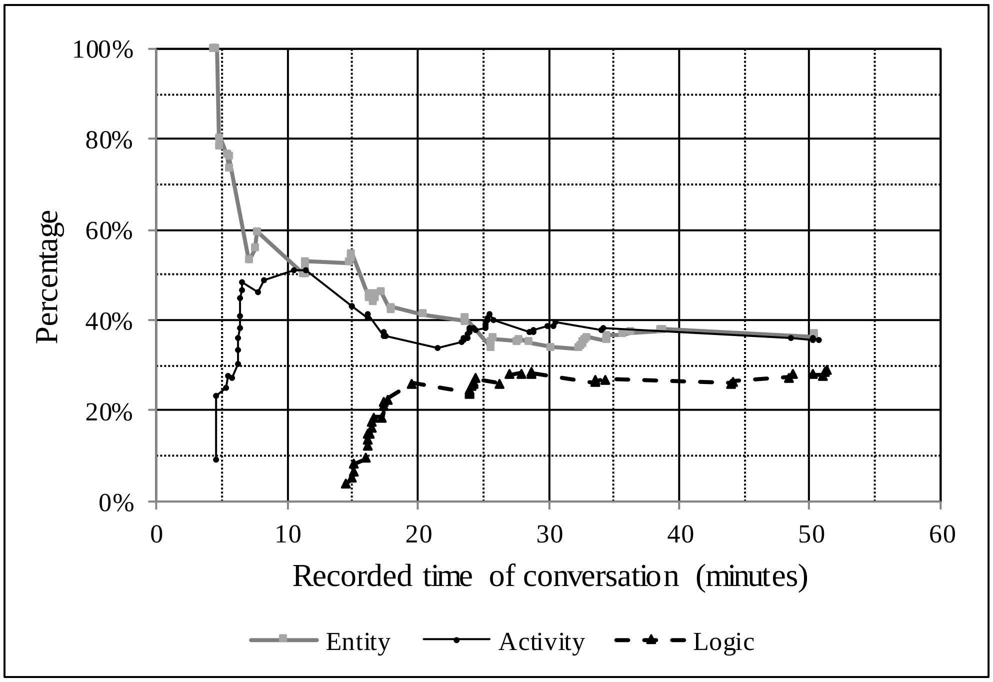

Figure 5. Progression of the linguistic structures usage when the students involve the visual manipulative in their conversation.

The curves of Figure 5 show how the involvement of the visual manipulative influences the progression in the usage of the linguistic structures during the students' conversation. At the beginning of the conversation, the decline of the curve that corresponds to the linguistic structure of mathematical entity is an indication that the students reduce the usage of nominal phrases in their explaining. In exchange, the students start using language to describe mathematical activities.

After the 15-minute mark, the linguistic structure of mathematical logic emerges. The horizontal shift of its curve characterizes a lagging in the formation or utilization of this linguistic structure by the students. While the curve for the mathematical logic ramps up, the curves for the mathematical entity and the mathematical activity gradually descend. This curves behavior reflect an inter-linguistic shift leading to the formation and utilization of logic and conditional statements in the students' conversation.

It is inferred from the graphs that mathematical entity and mathematical activity are preferring forms of linguistic structures when the students involve visual manipulative. Two reasons might be attributed to this condition and they are described from two ends. (1) When the students involve visual manipulative in their mathematics conversation, the linguistic structures of mathematical entity and mathematical activity are simple ways of describing the mathematical components of a problem situation, and of explaining the mathematics interaction that take place among those components. (2) When the students use the linguistic structures of mathematical entity and mathematical activity to communicate their mathematics ideas, the involvement of visual manipulative helps them contextualize the mathematics situation of the problem.

It is important to underscore that the process of contextualization, characterized by a vertical trajectory of descent, helps the students to perceive or conceive new mathematics ideas, to find or create new forms of representation during a mathematics discourse. And when the students attempt to explain their new mathematics ideas, they involve more sophisticated forms of representation and use more effective forms of verbal expressions, like mathematical logic. It is inferred that in the process of building up mathematics ideas, i.e., conceptualization, the students create chains of logical thoughts built upon those visual manipulative experiences. This form of learning is rooted in constructivism in which the students use previous experiences as a platform to build up further mathematical knowledge. This process of conceptualization is characterized by a vertical trajectory of ascent.

Figure 6 shows the progression of the linguistic structures usage when the students involve quantitative systems in their conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the linguistic structures divided by the total count of coded instances accumulated as of to that moment in the conversation. At the termination of the conversation the total count of coded instances is distributed as follows: Entity - 210; Activity - 171; and Logic - 87.

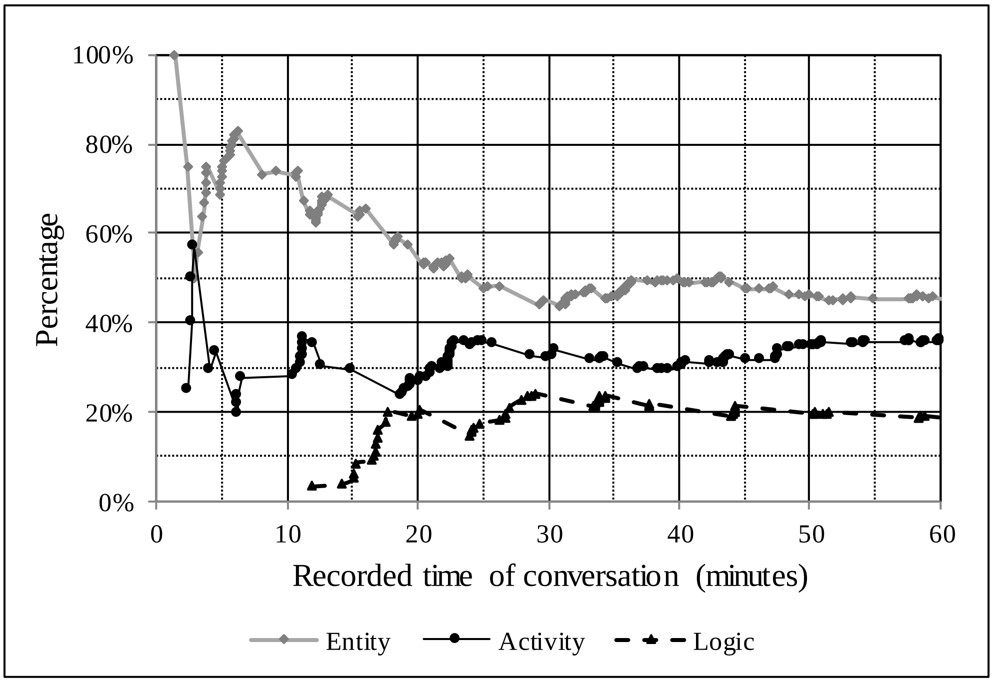

Figure 6. Progression of the linguistic structures usage when the students involve the quantitative systems in their conversation.

The curves of Figure 6 show how the involvement of the quantitative systems influences the usage of the linguistic structures during the students' conversation. The behavior of these curves is similar to the behavior of the curves of Figure 5. The curves of the mathematical entity and the mathematical activity display sporadic (ex)-changes in rate. During the exchange, it is inferred that the students investigate the mathematics operations and the conditions under which these operations take place. They use the linguistic structures of mathematical entity and mathematical activity when exchanging their mathematics ideas.

After the first 12 minutes of conversation, the linguistic structure of mathematical logic surfaces up in the students' conversation. Again, this time shift displays a factor of retardation in the formation or utilization of mathematical logic. This factor of retardation is also evident when the students involve visual manipulative. The profiles of the three curves (Entity, Activity, and Logic) of Figure 6 are similar to those shown in Figure 5. A difference between them is the proximity between Entity and Activity. When the students involve the visual manipulative in their conversation, these two linguistic structures are used at a same rate after the 35-minute mark in the graph. This is not the case when the students involve the quantitative systems in their conversation.

When the students confront a mathematics problem they use the linguistic structure of mathematical entity in alignment with the visual manipulative to define the mathematics configuration of the problem. They use the linguistic structure of mathematical activity in alignment with the quantitative systems to estimate solution possibilities to the problem. In this respect, it is inferred that the interaction or engagement between these two forms of semiotic models and these two forms of linguistic structures comprises an iterative process executed during the students' verbal conversations.

It is seen from the graphs that the mathematical entity and the mathematical activity are predominant forms of linguistic structures when the students involve the visual manipulative and the quantitative systems in their verbal conversations. It is inferred that when the mathematical entity and the mathematical activity are constantly used by the students, the involvement of the quantitative systems fuels the formation and utilization of chains of logical thoughts built upon their previous manipulative experiences.

Figure 7 shows the progression of the linguistic structures usage when the students involve the symbolic notations in their conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the linguistic structures divided by the total count of coded instances accumulated as of to that moment in the conversation. At the termination of the conversation the total count is distributed as follows: Entity - 148; Activity - 89; and Logic - 30.

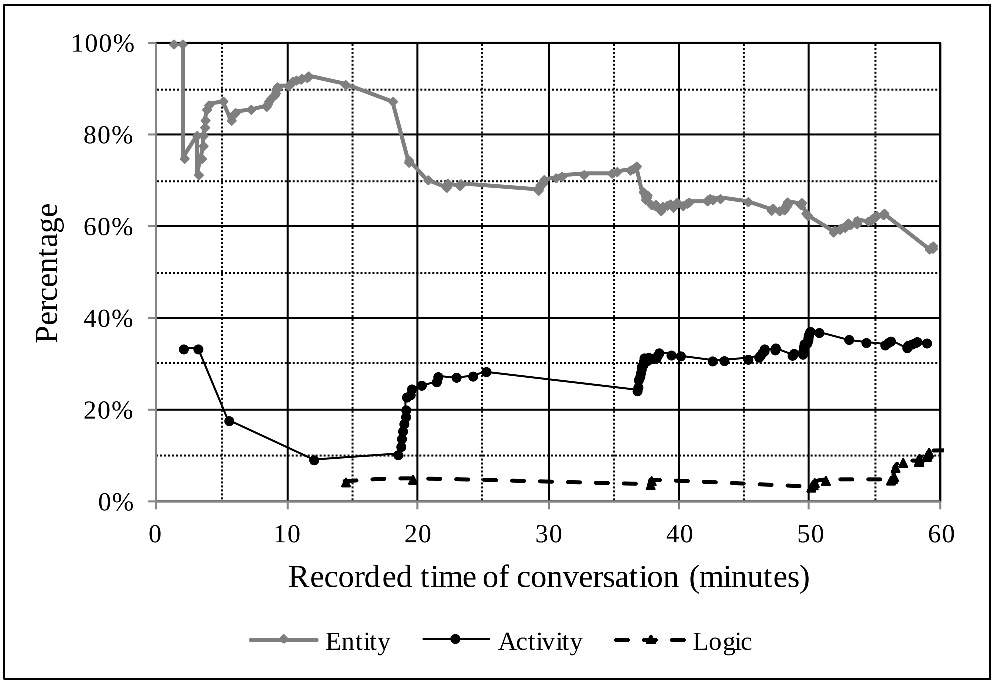

Figure 7. Progression of the linguistic structures usage when the students involve the symbolic notations in their conversation.

The graph shows how the involvement of the symbolic notations lineup with the usage of the three linguistic structures in the students' verbal conversations. Although the graph shows drastic changes in the distribution of coded instances among the three linguistic structures (Entity, Activity, and Logic), the usage of the mathematical entity is always at a higher rate while the mathematical logic is always at a lower rate throughout the conversation when the students involve the symbolic notations. The presence of the lagging factor in the formation or utilization of logic expressions during the students' verbal conversations is also observable.

The students use the linguistic structures of mathematical entity and mathematical activity to contextualize the mathematics situations of the problem. When they involve the visual manipulative and the quantitative systems, the language they use become rich and thick with narration. In a sense, they use language to contextualize the mathematics situations involving these two semiotic models. Their language turns narrative describing the mathematical activities that take place during their dialectical discourse.

Since the symbolic notations have the capacity to pack and unpack information in economic and unambiguous ways, the students' use of language turns less dense. The symbolic notations are functional semiotic models that facilitate the transmission of denotative meanings in a very effective and compact way. It is inferred, then, that the involvement of the symbolic notations promotes a multimodal expansion of meaning by embedding metaphorical expressions of mathematics language (like algebra) in the student's verbal conversations with less descriptive narratives. This is reflected in the Logic plot of the graph with less counts of coded instances.

4.3 Engagement of the Semiotic Models in the Students' Verbal Conversation

Figures C5 through C8, in Appendix C, illustrates the graph of the data that correspond to the use of the semiotic models by the students during their mathematics conversations. The data are grouped into three functional categories. They are (1) visual manipulative, (2) quantitative systems, and (3) symbolic notations. Each datum plotted in the graph defines a moment in the conversation where a semiotic model interacts with one and only one linguistic structure as coded in the transcripts. The cumulative counts of coded conversation instances involving the semiotic models are plotted with respect to the recorded time of the students' conversation in a semi-logarithmic graph. This graphing format accentuates the display of curve trends and behaviors that can be used to infer how the three semiotic models engage in the students' conversation throughout the learning activity. The graphs also help identify how often the semiotic models are used by the students as the learning activity progresses. Each coded instance, or moment of instantiation, assigns only one semiotic model coded under one of the following categories: Visual manipulative, quantitative systems, or symbolic notations. At the termination of the conversation, the total accumulation of coded instances is 932.

Figure 8 illustrates the progression of the involvement of the semiotic models throughout the students' conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the semiotic models divided by the total count of coded instances accumulated as of to that moment in the conversation. After the duration of the conversation the total count of coded instances is distributed as follows: Visual - 197; Quantitative - 468; and Symbolic – 267.

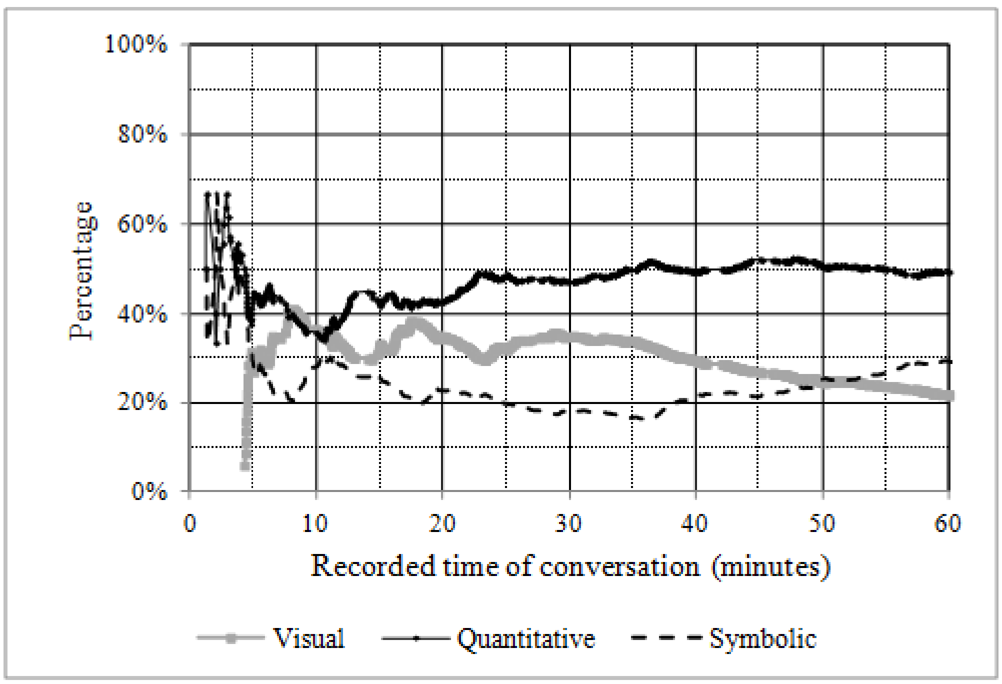

Figure 8. Progression of the semiotic models involvement throughout the students' conversation.

As shown in the graph, the semiotic model most frequently used throughout the conversation is the quantitative systems. During the first 4 minutes of conversation, the curves that correspond to the symbolic notation and the quantitative systems take turns in the conversation. This curve behavior describes multi-semiotic shifts between these two semiotic models in the students' conversation. After 4 minutes of conversation, the students start involving the visual manipulative. This delay presents a form of retardation in the involvement of this semiotic model by the students. The curves that correspond to the quantitative systems and the symbolic notations exhibit a decrease in their rate of involvement in the conversation. It is inferred that during this period of time the students involve the visual manipulative to define the configuration of the problem.

For the time interval of 10-35 minutes, the curves that correspond to the visual manipulative and the quantitative systems exchange turns during the conversation, while the curve that correspond to the symbolic notations sustain a gradual decrease in rate. After that, during the time interval of 35-60 minutes, the inter-semiotic shift from visual to symbolic representation is evident. And at the end of the conversation, the symbolic notations become more attractive than the visual manipulative.

Figure 9 shows the progression of the semiotic models involvement when the students use the mathematical entity in their conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the semiotic models divided by the total count of coded instances accumulated as of to that moment in the conversation. At the termination of the conversation the total count of coded instances is distributed as follows: Visual - 71; Quantitative - 210; and Symbolic – 148.

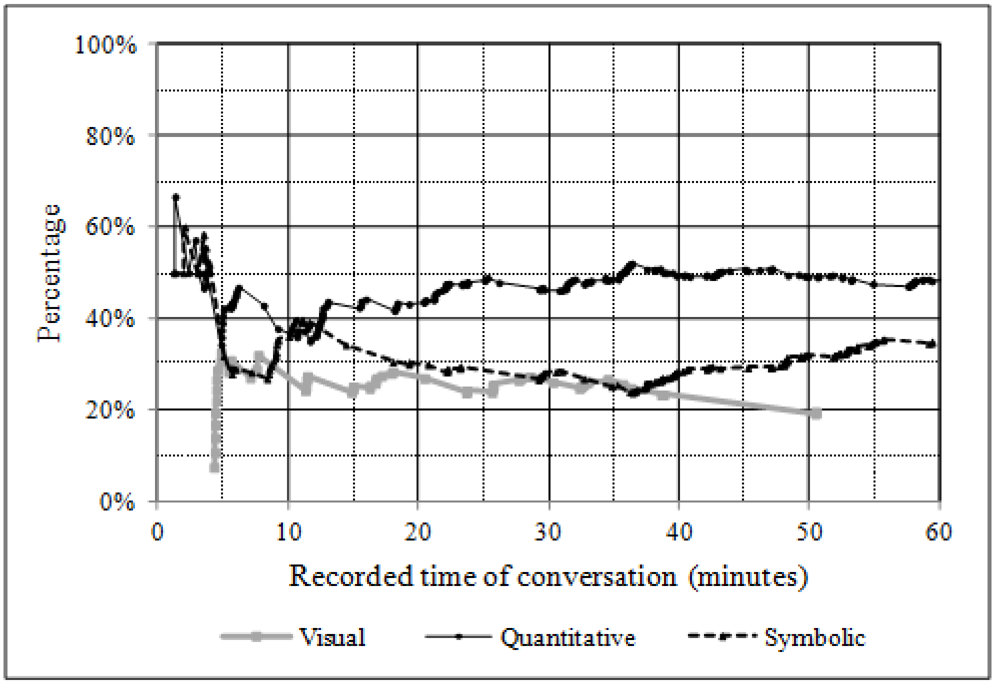

Figure 9. Progression of the semiotic models involvement when the students use the mathematical entity in their conversation.

The curves in the graph can be used to infer how the involvement of the semiotic models influences the progression in the usage of the mathematical entity, or how the mathematical entity lineups with the three semiotic models during the conversation. The students starts off the conversation involving the quantitative systems and the symbolic notations, as primary forms of representation, to describe and define the problem of the task. These two semiotic models take turns during the initiation of the conversation.

After the 4-minute mark, the curve that corresponds to the visual manipulative ramps up for a short period of time. This is an indication that the students shift to the utilization of visual manipulative to model the configuration of the problem situation. They are descending, as part of a process of contextualization, to the core configuration of the problem to identify its mathematical components. In this respect, the students' use of language (i.e., mathematical entity) lines up with the involvement of the semiotic model of visual manipulative in their conversation. The involvement of the visual manipulative is relatively sustained at an average rate of 25% in the time interval of 10-38 minutes, and it drops to 20% with one last coded instance at the 52-minute mark of the graph.

At the beginning of the conversation, it can be inferred that during the inter-semiotic exchange between the quantitative systems and the symbolic notations, the students communicate mathematics ideas expressed in the form of mathematical entities in an attempt to define the mathematical components of the problem. During the exchange, the students involve the visual manipulative to illustrate their ideas in a simple form. This shift demarks a process of descent or analysis in describing the mathematics situation of the problem, i.e., a process of contextualization.

During the duration of the conversation, the involvement rate of the symbolic notations slowly surpasses the involvement rate of the visual manipulative by the students in their mathematics conversation. During the last 20 minutes of the conversation, the use of quantitative systems and symbolic notations are predominant forms of representation when the linguistic structure of mathematical entity is used. The involvement of visual manipulative is no longer a preferring form of mathematics representation when the students' verbal conversation emphasizes mathematical entities. It can be inferred that since the students' use of words conveys denotative meanings, they no longer require the use of visual manipulative to support their explanations. The linguistic structure of mathematical entity is now used by the students to exchange the new mathematics ideas they built at the beginning of the learning activity through visual experiences.

Figure 10 shows the progression of the semiotic models involvement when the students use mathematical activity in their conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the semiotic models divided by the total count of coded instances accumulated as of to that moment in the conversation. At the termination of the conversation the total count of coded instances is distributed as follows: Visual - 69; Quantiitative - 171; and Symbolic - 89.

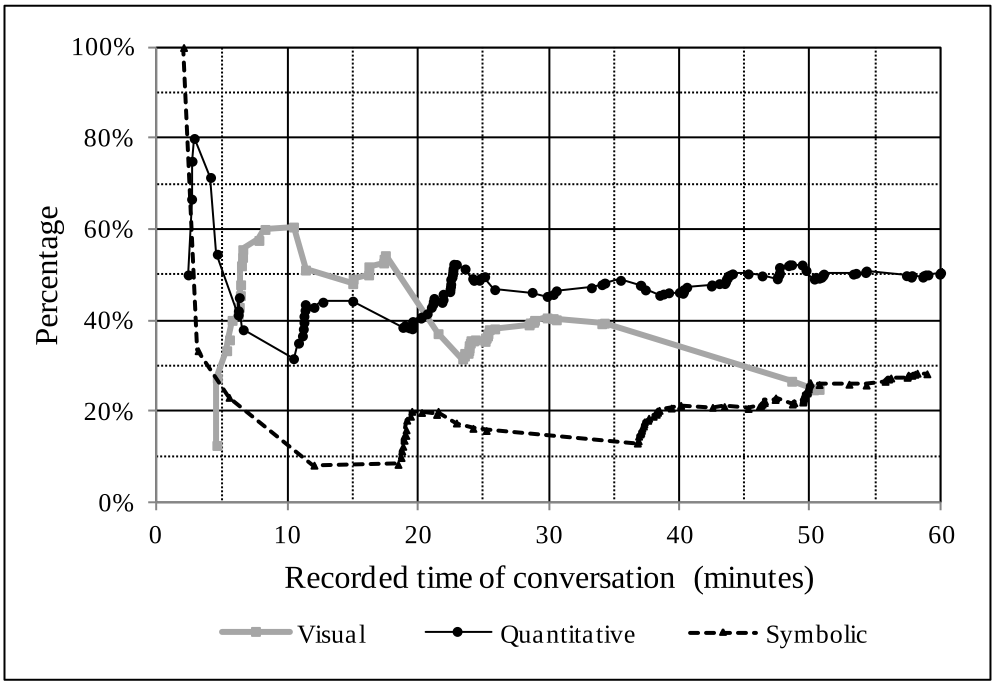

Figure 10. Progression of the semiotic models involvement when the students use the mathematical activity in their conversation.

The curves in the graph can be used to describe how the involvement of the semiotic models influences the progression in the usage of the mathematical activity during the students' conversation, and to illustrate how the mathematical activity lineups with the three semiotic models during the progression of the conversation. The students starts off the mathematics conversation using the linguistic structure of mathematical activity to explain the problem of the task while involving quantitative systems and symbolic notations as primary forms of representation. These two semiotic models take turns during the initiation of the conversation. This is a similar situation as for the mathematical entity, Figure 9.

As shown in the graph, the curves for the three semiotic models take turns. The involvement of the visual manipulative ramps up between the time marks of 4 and 7 minutes of conversation. The curves for the symbolic notations and the quantitative systems do not cross each other. For the first 10 minutes of conversation, the involvement rate for the symbolic notations drops abruptly from 100% to 10%, the quantitative system drops from 80% to 32%, and the visual manipulative rise from 5% to 60%. This change in rates demarks an inter-semiotic shift from quantitative-symbolic to visual manipulative.

During the following 25 minutes after the 10-minute mark of conversation, the graph illustrates how the semiotic models of visual manipulative and quantitative systems line up with the linguistic structure of mathematical activity. During this period of time these two semiotic models take turns in the conversation. And the last two coded instances for the involvement of the visual manipulative takes place at the 35-minute and the 50-minute marks. During this same period of time, the involvement rate for the symbolic notations starts to increase.

It is inferred that during this period of time, the rate increase for the symbolic notations demarks the initiation of a process of vertical ascent (or conceptualization) in the learning of mathematics during the conversation. This preferring form of representing mathematics ideas, i.e., by means of symbolic notations, gets stronger when the students' verbal expressions emphasize mathematical activities. The behavior of its curve reflects a semiotic shift in which the use of the symbolic notations becomes a preferring option over the use of the visual manipulative in the construct (or synthesis) of mathematics ideas when the students converse verbally in terms of mathematical activities. This study describes this type of shift as one of vertical growth (ascent) in the learning of mathematics, one in which the visual manipulative and quantitative systems stimulate the creation and utilization of symbols, notations, and equations in the solving of mathematics problems.

Figure 11 shows the progression of the semiotic models involvement when the students use mathematical logic in their conversation. The percentage shown on the vertical axis is calculated by counting the number of coded instances for each one of the semiotic models divided by the total count of coded instances accumulated as of to that moment in the conversation. At the termination of the conversation the total count of coded instances is distributed as follows: Visual - 57; Quantitative - 87; and Symbolic - 30.

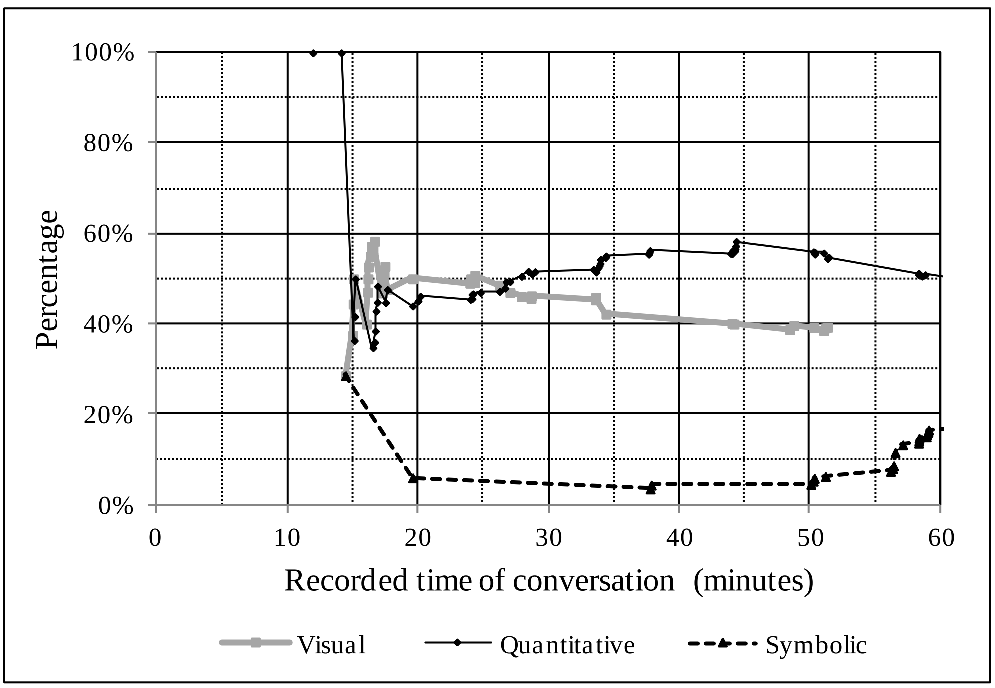

Figure 11. Progression of the semiotic models involvement when the students use the mathematical logic in their conversation.

Contrary to the other two forms of linguistic structures, i.e., the mathematical entity and the mathematical activity, the mathematical logic does engage in the conversation late. This time delay is illustrated in the graphs with a horizontal shift of 12 minutes. The time delay represents a lagging in the formation or utilization of the linguistic structure of mathematical logic. It is inferred that this form of cognitive engagement is time dependent and arises in response to the quality of the initiation of the mathematics conversation.

After the 15-minute time mark, the mathematical logic comes up involving, mainly, the visual manipulative and the quantitative systems as preferring forms of semiotic modeling, while the symbolic notations are the least preferring ones. At the end of the conversation, the use of symbolic notations ramps up sharply in the time intervals of 50-52 and 56-59 minutes of the conversation. During these intervals, the involvement rate for the symbolic notations increases when the students use mathematical logic. This increase demarks a moment of growth characterized by a vertical movement of its curve. This is an indication that the use of symbolic notations becomes a preferring option over the visual manipulative for modeling in the construct (or synthesis) of mathematics ideas when the students' verbal conversation emphasizes mathematical logic. This form of engagement (or disengagement) is similar to the one exhibited in Figure 10. The alignment of this semiotic model with the linguistic structures of mathematical activities and mathematical logic facilitates the development of most sophisticated and expeditious forms of packing and unpacking mathematics ideas by involving mathematical symbols, notations, and equations with denotative meanings.

The data presented in Section 4.2 and Section 4.3 are used to draw the following noteworthy findings. When two semiotic models are consistently used to represent a particular mathematics situation for an extended period of time, the verbal conversation turns out to be one that promotes the exchange of mathematics ideas by engaging both semiotic models in a dialectical discourse while the other remains in remission. It can be inferred that this form of engagement describes a bi-modal inter-semiotic shift that might take place when the students challenge each other during the exchange. This inter-semiotic shift is reflected in the graph as an exchange between the increase and the decrease in the rates of the (two) corresponding curves. In contrast, when the use of one semiotic model ramps up significantly more than the use of the other two semiotic models, it can be inferred (from the graph) that this same model might be in use by the students to deconstruct (analyze) or construct (synthesize) mathematics ideas, to confirm or create denotative meanings during their verbal conversation. Under this circumstance, the linguistic structures play an important role in the students' explaining of mathematics ideas during this hierarchical trajectory.

In general, the data show that when the students start working on a problem of the learning activity task, they often involve the visual manipulative and the quantitative systems to represent the mathematics situations of the problem. Patterns of engagement involving these semiotic models promote the utilization of the linguistic structures of mathematical entity and mathematical activity in the students' verbal conversations. In similar respect, when the quantitative systems and the symbolic notations are engaged in the mathematics conversations, they propitiate the formation of the linguistic structures of mathematical logic conducive to the construct of mathematical knowledge expressed in terms of symbols, notations, and equations.

4.4 Patterns of Behavior and Shifts during the Students' Verbal Conversation

This study considers two forms of interactions that take place during the students' verbal conversation as they work on the learning activity task. One is when the students interact with the learning environment of the task through a dialectical discourse, and the other one is when the students interact socially during a dialogical discourse. The coded transcripts (Appendix A and Appendix B) identify those moments in the learning activity when the students explain and exchange their mathematics ideas as they converse verbally.

The coded data for the linguistic structures and the semiotic models are used to create bar charts to help identify patterns of behavior and multi-modal shifts in the use of these forms of mathematics expressions and representation throughout the students' conversation. The number of coded instances are conglomerated independently into four 15-minute time intervals (or quartiles) of the (approximate) 60-mintute conversation. The total count of coded instances for each one of the intervals is used to create the bar charts of Figures 12 through 19. The charts illustrate the usage distribution among the three linguistic structures and the three semiotic models throughout the entire conversation. The percentage shown on the vertical axis of each chart is calculated by counting the total number of coded instances for each one of the linguistic structures, or the semiotic models, divided by the total count of coded instances conglomerated into the corresponding time interval of the recorded conversation. The chart results are explained in subsequent tables.

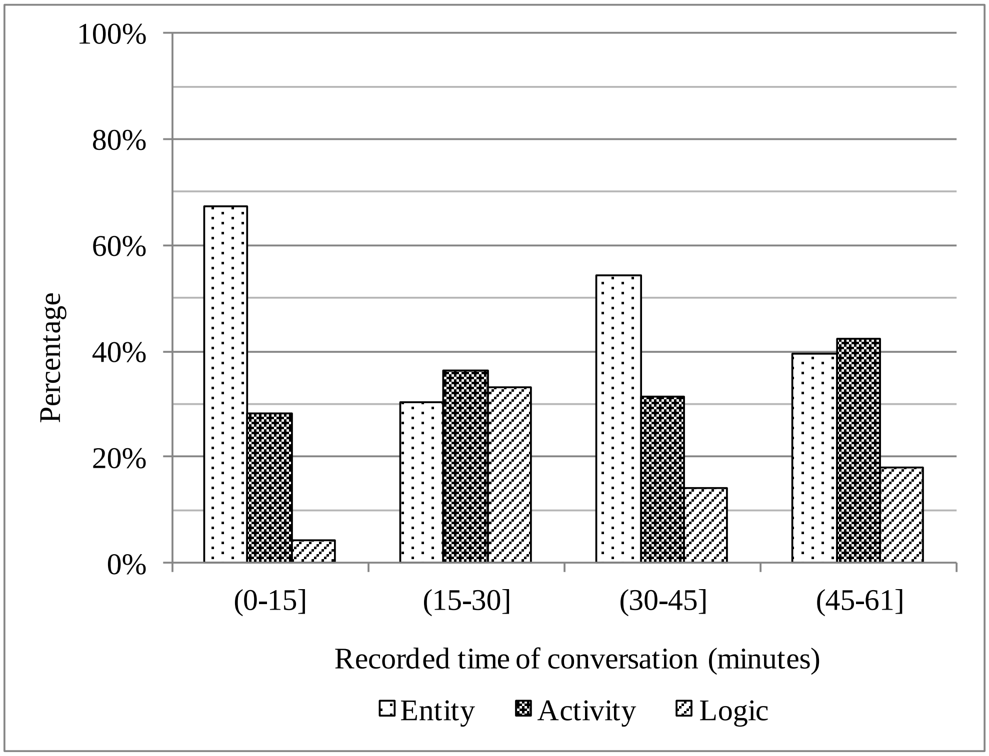

Figure 12. Distribution shift in the usage of the linguistic structures during the students' conversation.

Table 4
Inferences from the bar chart of the linguistic structures.

Code	Description and inferences (I)
A1	The usage of the mathematical entity is high in the first quartile of the conversation with a rate of 67%, and it reduces to 30% in the second quartile. The third quartile reflects a rate of 55% and the final quartile reflects a rate of 55% in the usage of the mathematical entity.
A2	The progression in the usage of the mathematical activity shows a reverse trajectory to the one exhibited by the mathematical activity. The usage rate of the mathematical activity in the first quartile of the conversation is 28%; followed by 37%, 31%, and 42% for the last three corresponding quartiles.
A3	The progression in the usage of the mathematical logic shows a parallel trajectory to the one exhibited by the mathematical activity. The usage rate of the mathematical logic is low at the beginning of the conversation with a value of 4%, and it increases to 33%; followed by a 14% and 18% for the last two quartiles respectively.
I	It is inferred that an inter-linguistic shift takes place between Entity and Activity, and between Activity and Logic, within those corresponding time intervals of the students' verbal conversation, that promote the formation of Logic statements by the students conducive to the construct of mathematical knowledge. This characterizes a process of conceptualization.

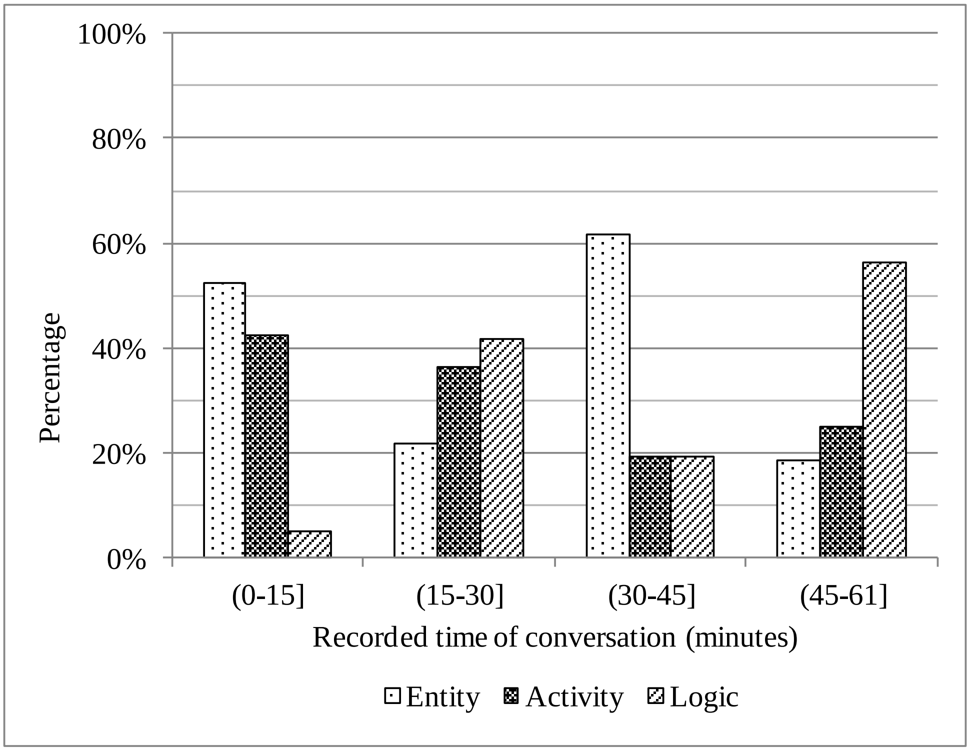

Figure 13. Distribution shift in the usage of the linguistic structures when the students involve the visual manipulative in their conversation.

Table 5
Inferences from the bar chart of the linguistic structures when the students involve the visual manipulative.

Code	Description and inferences (I)
A1	The usage rate of the mathematical entity decreases drastically from a first quartile value of 52% to a second quartile value of 22%. A similar trend is reflected for the third quartile (the highest) and the last quartile with values of 61% and 19% respectively.
A2	The usage of the mathematical activity decreases gradually from the first, to the second, and to third quartiles with rate values of 42%, 36%, and 20% respectively. The last quartile value is 25%.
A3	The distribution in the usage of the mathematical logic shows a reverse trend to the one exhibited by the mathematical entity. The usage rate of the mathematical logic for the first quartile of the conversation is 5%; followed by 42%, 20%, and 56% for the last three corresponding quartiles.
I	It is observed that an inter-linguistic shift takes place between Entity and Logic within those corresponding time intervals of the students' verbal conversation. This may imply that the students' use of language transitions from narrative to deductive when their verbal conversation involves the visual manipulative. This characterizes a process of conceptualization.

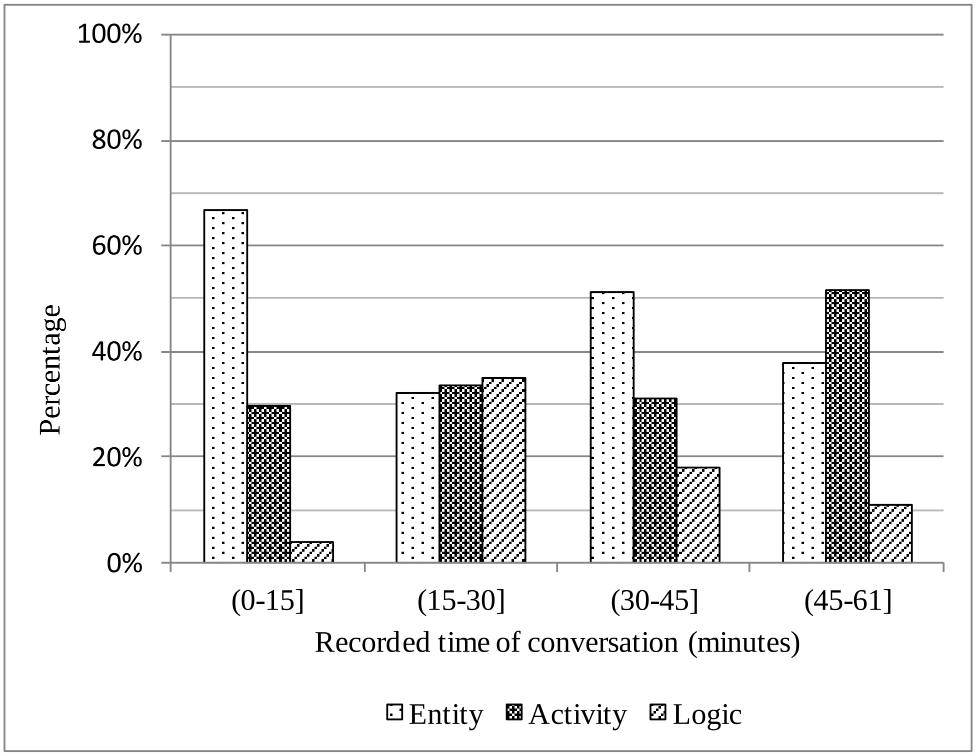

Figure 14. Distribution shift in the use of the linguistic structures when the students involve the quantitative systems in their conversation.

Table 6
Inferences from the bar chart of the linguistic structures when the students involve the quantitative systems.

Code	Description and inferences (I)
A1	The usage rate of the mathematical entity decreases from a first quartile value of 52% to a second quartile value of 22%. A similar trend is reflected for the third quartile and the last quartile with values of 61% and 19% respectively.
A2	The usage rate of the mathematical activity sustains an average value of 31% for the first three quartiles. For the last quartile, its rate increases to 51%.
A3	The distribution in the usage of the mathematical logic shows a reverse trend to the one exhibited by the mathematical entity. The rate of usage of the mathematical logic in the first quartile of the conversation is 5%; followed by 41%, 20%, and 56% for the last three corresponding quartiles.
I	It is inferred that a major inter-linguistic shift takes place between Entity and Logic during the students' conversation. This may imply that the students' use of language transitions from narrative to deductive when their verbal conversation involves the quantitative systems.

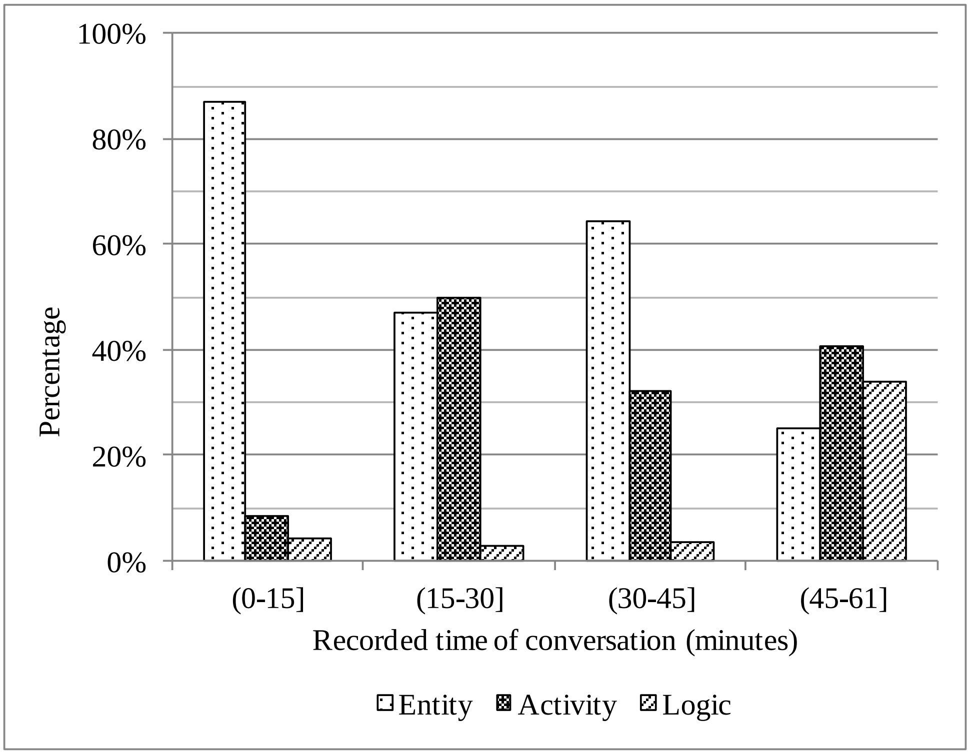

Figure 15. Distribution shift in the use of the linguistic structures when the students involve the symbolic notations in their conversation.

Table 7
Inferences from the bar chart of the linguistic structures when the students involve the symbolic notations.

Code	Description and inferences (I)
A1	The usage rate of the mathematical entity decreases drastically from a first quartile value of 87%, to a second quartile value of 47%. A similar trend is reflected for the third quartile and the last quartile with values of 65% and 25% respectively.
A2	The distribution in the usage of the mathematical activity shows a reverse trend to the one exhibited by the mathematical entity. The rate of usage of the mathematical activity in the first quartile of the conversation is 9%; followed by 50%, 65%, and 25% for the last three corresponding quartiles.
A3	The usage rate of the mathematical logic sustains an averaged low value of 4% for the first three quartiles, and it drastically increases to 34% in the last quartile.
I	It is inferred that a major inter-linguistic shift takes place between Entity and Logic during the students' conversation. This may imply that the students' use of language transitions from narrative to deductive when their verbal conversation involves the symbolic notations.

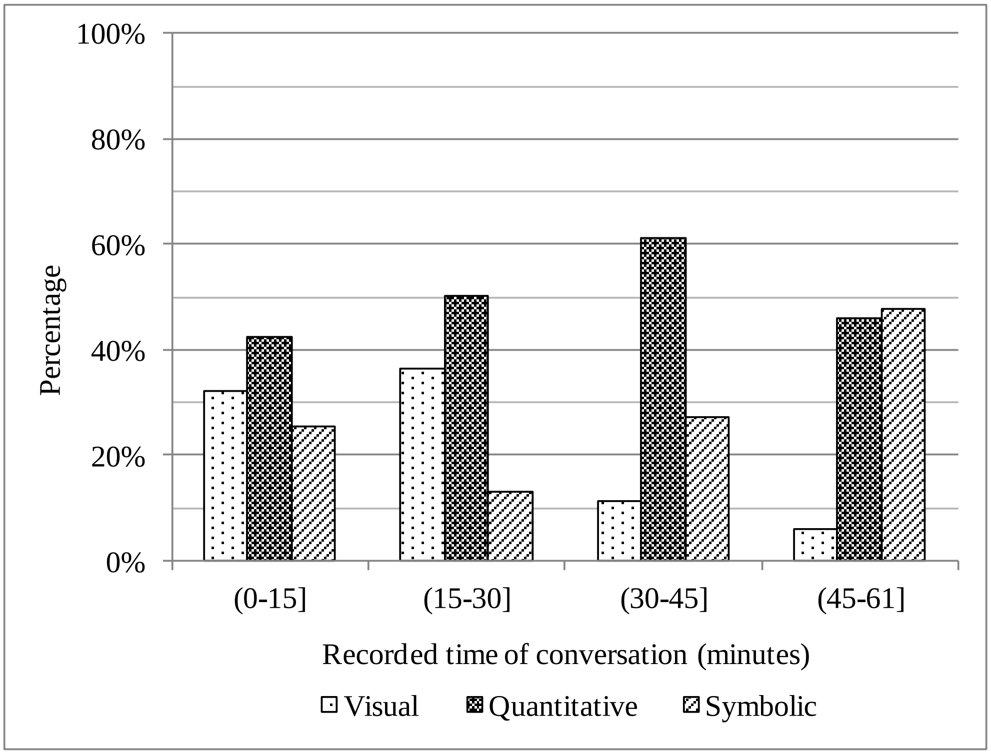

Figure 16. Distribution shift in the involvement of the semiotic models during the students' conversation.

Table 8
Inferences from the bar chart of the semiotic models.

Code	Description and inferences (I)
B1	The involvement rate for the visual manipulative is higher in the first two quartiles than in the last two quartiles, with averaged rates of 35% and 8% respectively.
B2	The involvement rate for the quantitative systems gradually increases from the first to the second and to the third quartiles. Their corresponding rate values are 42%, 50%, 61%. The involvement rate decreases from the third to the fourth quartile value of 46%.
B3	The involvement rate for the symbolic notations decreases in the first two quartiles, from 25% to 14%, and increases in the last two quartiles, from 26% to 48%.
I	A reverse trend of behavior between visual-quantitative and symbolic is shown during the last two quartiles. It is inferred that the students involve the symbolic notations to unpack prior knowledge during the initiation of the conversation, and to pack new knowledge during the termination of the conversation. This depicts a vertical trajectory of descent (analysis) followed a vertical trajectory of ascent (synthesis) in the students' conversation.

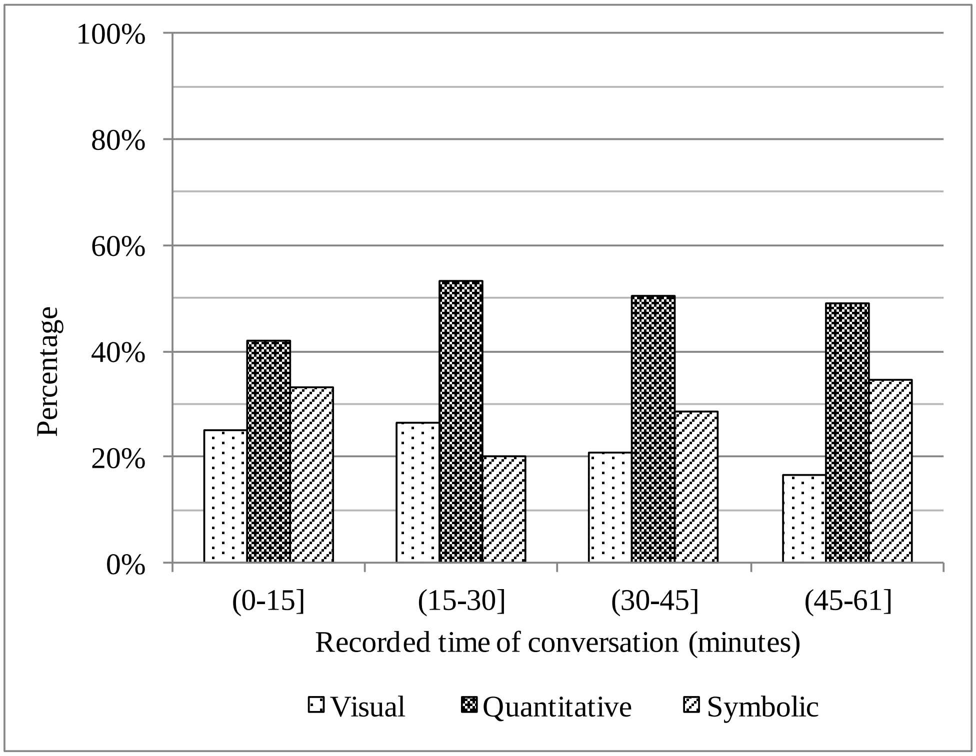

Figure 17. Distribution shift in the involvement of the semiotic models when the students use the mathematical entity in their conversation.

Table 9
Inferences from the bar chart of the semiotic models when the students use the mathematical entity.

Code	Description and inferences (I)
B1	The involvement rate for the visual manipulative increases a bit from 25% to 27% during the first two quartiles, and gradually decreases from 21% to 17% during the last two quartiles.
B2	The involvement rate for the quantitative systems is the highest one in all the quartiles. It gradually increases from a first-quartile value of 42% to a second-quartile value of 53%, and decreases from a third-quartile value of 51% to a fourth-quartile value of 49%.
B3	The involvement of the symbolic notations shows a decrease in its rate from 33% to 20% during the first two quartiles and an increase from 28% to 35% during the last two quartiles. This shows a reverse shift to the one exhibited by the visual and the quantitative models.
I	It is inferred that a minor inter-semiotic shift takes place between visual-quantitative and symbolic when the students use mathematical entity during their conversation. This depicts a vertical trajectory of descent followed by a vertical trajectory of ascent in the students' conversation.

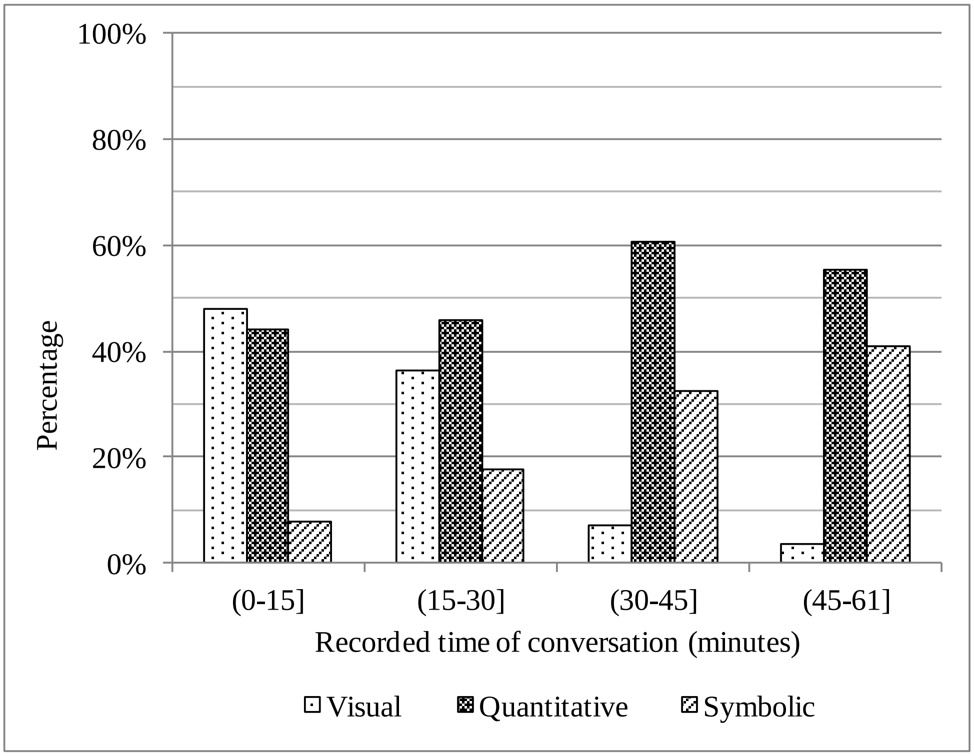

Figure 18. Distribution shift in the involvement of the semiotic models when the students use the mathematical activity in their conversation.

Table 10
Inferences from the bar chart of the semiotic models when the students use the mathematical activity.

Code	Description and inferences (I)
B1	The involvement rate for the visual manipulative decreases from 46% to 37% during the first two quartiles and from 7% to 4% during the last two quartiles. This shows a significant reduction in the involvement of visual manipulative during the progression of the students' conversation.
B2	The involvement rate for the quantitative systems takes the values of 44% and 46% for the first two quartiles, and 61and 55% for the last two quartiles. The involvement rate for the quantitative systems increases by an average of 10% when the students use the mathematical activity.
B3	The involvement rate for the symbolic notations gradually increases from 7%, to, 17%, to 32%, and to 41% for the first, second, third, and fourth quartiles when the students use the mathematical activity in their conversations. The progression in the involvement of the symbolic notations shows a reverse trajectory to the one exhibited by the visual manipulative.
I	It is inferred that an inter-semiotic shift takes place between the visual manipulative and the symbolic notations when the students use mathematical activity in their conversations. This depicts a vertical trajectory of descent followed a vertical trajectory of ascent in the students' conversation.

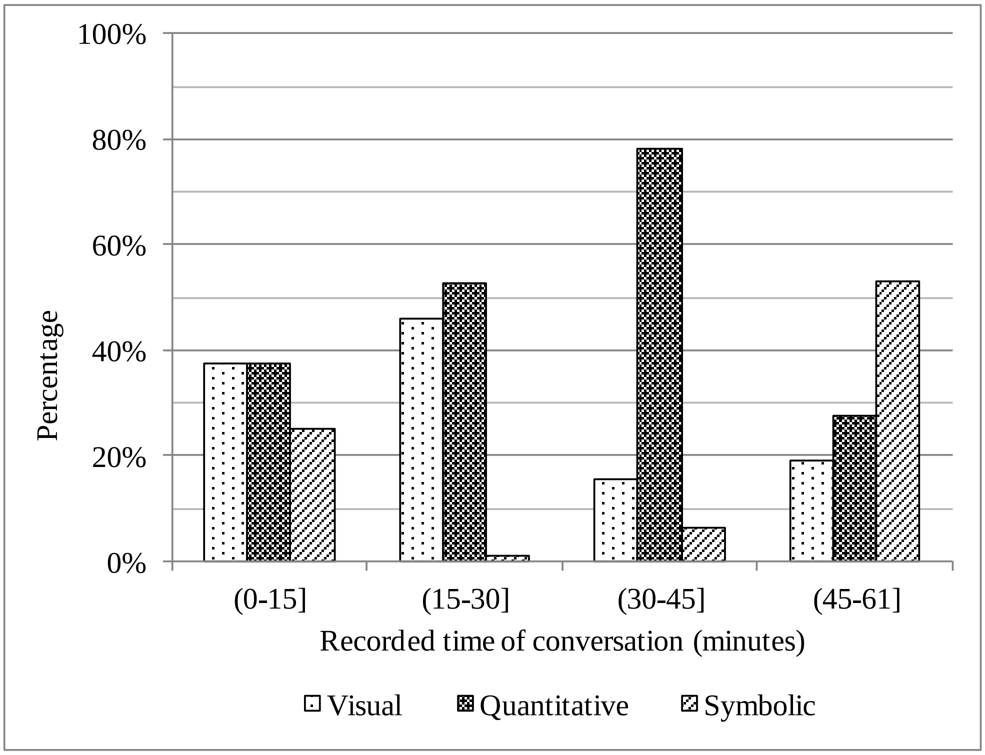

Figure 19. Distribution shift in the involvement of the semiotic models when the students use the mathematical logic in their conversation.

Table 11
Inferences from the bar chart of the semiotic models when the students use the mathematical logic.

Code	Description and inferences (I)
B1	The involvement rate for the visual manipulative decreases from 42% (average between the values of the first and second quartiles) to 16% (average between the values of the third and fourth quartiles).
B2	The involvement rate for the quantitative systems increases from 37%, to 52%, and to 78% for the first, second, and third quartiles respectively. This almost-linear trajectory comes to an end when its rate reaches the value of 27% in the last quartile.
B3	The involvement of the symbolic notations is relatively high at the beginning of the conversation with rate of 25%, and (higher) at the end of the conversation with a rate of 53%. Its rate is very low (1% and 6%) in the second and third quartiles.
I	It is inferred that the students involve the symbolic notations using mathematical logic to analyze the configuration of the problem at the beginning of their conversation, and to synthesize in symbolic forms their solutions expressed in terms of logic and conditional statements at the end of their conversation.

4.5 Examples of Mathematics Conversation Episodes from the Transcripts

This section presents specific examples from the recorded conversation transcripts of how the students use the linguistic structures and the semiotic models either to describe the mathematics situation of a problem or to explain the mathematics ideas they create to solve the problem as they work on the learning activity task. The recorded episodes show how the linguistic structures and the semiotic models help the students get engaged in their mathematics conversations. Table 12 summarizes the legend of codes used in the transcripts. Tables 13 through 15 include three compendia of particular examples from the coded transcripts accompanied with comments, typed in *italicized fonts*, on the linguistic structures and the semiotic models the students use while working on the task during their verbal conversations.

Table 12
Legend of codes.

Code	Legend	
A1		Mathematical entity
A2		Mathematical activity
A3		Mathematical logic
B1	B1	Visual manipulative
B2	B2	Quantitative systems
B3	B3	Symbolic notations

The episodes of Table 13, line items 689-705, show the students conversing about "the choose thing" of the combinatorics problem involving the mathematics operation of factorials, the Pascal's Triangle and the 3-topping pizza problem situation. Researcher R1 (Carolyn) starts a dialog with Michael asking him to elaborate on the addition rule for Pascal's Triangle. In response, the students use, primarily, the linguistic structure of mathematical entity to describe the situation of the problem. They involve Pascal's Triangle as counting utility and the 3-topping pizza problem as a visual manipulative when explaining the addition rule. These episodes illustrate a momentary inter-semiotic shift (descent) in the involvement of semiotic models from a quantitative system (Pascal's Triangle) to a visual manipulative (the 3-topping pizza). In contrast, the use of the linguistic structures by the students exhibits a gradual inter-linguistic shift of ascent from mathematical entity to mathematical activity culminating in the formation of mathematical logic statements as shown at the end of this transcripts excerpt.

Table 13
Compendium of transcripts excerpts 689-734. Comments are included in *italicized font* for purposes of analysis in this study.

Line	Time	Talker	Transcripts
689	00:30:59	R1:	And now you're telling me you can write them as the choose way,[B3] you've called that. So can you take, let's say another row or two and show me the addition rule and what it looks like with your new notation.[B2]
		This episode illustrates a moment of descent in the use of the semiotic models, an inter-semiotic shift from symbolic notation to quantitative system.	
690	00:31:17	Michael:	You're talking about the addition rule[B2] when you
691	00:31:18	R1:	For a particular, for a particular row.[B2]
692	00:31:20	Michael:	Add this and this and go like that?[B2]
693	00:31:21	R1:	Sure, or three and three, six.[B2] Show me what that looks like with that new notation.[B2] Do you understand my question?
694	00:31:29	Michael:	Uh, I don't really.
695	00:31:29	Romina:	I don't understand.

696	00:31:29	Ankur:	Instead of writing three you write-[B2]
697	00:31:31	R1:	Write your next row, Michael.[B2] Now some time ago you, you had a reason. You explained to me-

These episodes (689-697) show persistency in the use of the linguistic structure of mathematical entity and the involvement of Pascal's Triangle as a semiotic model, quantitative system.

698	00:31:45	Michael:	Why you add.[B2]
699	00:31:46	R1:	Why you add.[B2]
700	00:31:47	Michael:	Yeah.
701	00:31:48	R1:	You remember that? You, might, might be useful for folks who haven't heard it to hear it whatever way you want to explain it.
702	00:31:53	Michael:	I don't think I can explain it too good. Um.
703	00:31:55	R1:	Um, you know, however you want to explain it. You've had it a few ways.
704	00:32:00	Michael:	Um, I can't, I can't remember too well. I know why you add,[B2] if I explain it, I don't think anyone will understand.

These episodes (698-704) illustrate how Pascal's Triangle is used as a platform to describe the problem situation. The use of the linguistic structure of mathematical entity by Researcher R1 (Carolyn) is neutral and invites the student(s) to conduct further analysis on the problem situation.

705	00:32:13	R1:	Try.
706	00:32:15	Michael:	I didn't. Didn't I tell you guys like last time I came here?
707	00:32:18	Jeff:	Well, go for it, dude, just-
708	00:32:20	Romina:	You could try.
709	00:32:20	Michael:	You don't have that paper, do you? You can just hand them, hand that out.
710	00:32:21	Romina:	You started talking about toppings.[B1] I think something-

The mention of the word "toppings" demarks a moment of descents in the use of the semiotic models, an inter-semiotic shift from quantitative systems to visual manipulative.

711 00:32:24 Michael: Hand that out instead.

712 00:32:25 Jeff: Just-

713 00:32:27 Michael: Um, all right. If, all right, let's go to, let's go to this one. This would be like three different places I guess.[B1] And um-

714 00:32:37 Jeff: Which one are we looking at?[B1]

715 00:32:38 Michael: That one right there.[B1] You have three-[B1]

The involvement of the pizza toppings model (visual manipulative) helps the students identify the mathematical components of Pascal's Triangle.

716 00:32:41 Jeff: That would be a plus b to the third.[B3]

This episode illustrates a moment of ascent, an inter-semiotic shift with the use of the symbolic notations of $(a+b)^3$ by Jeff. The student relates this binomial expansion to the third row of Pascal's Triangle.

717 00:32:42 Michael: All right, let's say you have like, here's a number, all right? Zero means no toppings.[B2] One would, this, one would be-[B2]

718 00:32:51 Romina: It would be, one's a topping.[B2]

719 00:32:51 Michael: One would be a topping.[B2] So first category is everything with no toppings.[B1] And that's, you can't make, that's, that's your number for that one.[B2]

720 00:33:01 Michael: Next would be- There's all the, the ones that have one topping.[B1]

721 00:33:12 Jeff: Right, you got to make that zero at the end.[B2] You messed up.

722 00:33:14 Michael: What?

723	00:33:14	Jeff:	Last one should be a hundred, not a hundred and one.[B2]

These episodes (717-723) show a back-and-forth shifting in the students' use of the semiotic models between the manipulative object (represented by the 3-topping pizza problem) and the quantitative system (represented by Pascal's Triangle). This back-and-forth shift in the use of the semiotic models is a reflection of the iterative nature of the mathematics solving by the students.

724	00:33:15	Michael:	I knew that. There's your, um, your three choose one.[B2] And there's three different combinations you could put that.[B2] Um, I can go on forever doing this.

This episode shows how Michael starts making connection between the "choose thing" (i.e., the notation for combinations), the 3-topping pizza problem (i.e., the visual manipulative), and Pascal's Triangle (i.e., the quantitative system). The use of the linguistic structure is precise in the form of mathematical activity describing the interaction (the happenings) among the mathematical components of the "choose thing" operation. It is inferred that Pascal's Triangle (the quantitative system) is Michael's preferring form of semiotic modeling at this moment of the conversation.

725	00:33:25	Michael:	But, um, when you have a new, when you add another place, another topping-[B2]
726	00:33:34	Jeff:	That could be one or the other, one or the other, one or the other[B1]
727	00:33:36	Michael:	So it could be one or the other.[B1]
728	00:33:37	Michael:	It could be a zero or a one, a zero or a one, a zero or a one.[B2]
729	00:33:38	Jeff:	Yeah. All right.
730	00:33:39	Michael:	So all these threes would either move up a step onto the next category and, uh, have two toppings.[B2]
731	00:33:47	Michael:	Or they might stay behind and still only have one if they have the zero.[B2]
732	00:33:52	Michael:	So three, three will get a topping, and go to this one.[B2]

733	00:33:56	Michael:	And three won't, will stay.[B2]
734	00:33:58	Michael:	And obviously this guy's going to get a topping.[B2] That's why you add this one[B2]

These episodes (725-734) show Michael explaining how the "choose thing" operation is altered by the addition of another topping in the 3-topping pizza situation (i.e., the visual manipulative), and how this alteration is reflected in Pascal's Triangle (i.e., the quantitative system). Michael elaborates mathematical logic statements in his explanation. His use of language is characterized by the presence of conjunctions conducive to the formation of conditional statements and propositions. These episodes demonstrate how the involvement of the two semiotic models plays out in the construct of the mathematics ideas in the form of mathematical logic.

The previous episodes illustrate how the students elaborate on the identity relations between the "choose notation", Pascal's Triangle (i.e., the quantitative system), and the pizza problem (i.e., the visual manipulative) for the specific instances of the third and fourth row in the Triangle and the 3- and 4-topping pizzas. The following episodes illustrate how the students, after being asked to write the Triangle using the "choose notation" for a general row "N", establish the general addition rules for Pascal's Triangle involving the symbolic notation for "choosing". Jeff starts the conversation writing on the board the "choose notation" in terms of "N", "0", and "X" as shown below.

$$\binom{N}{0} \quad \cdots \quad \binom{N}{X} \quad \cdots \quad \binom{N}{N}$$

Jeff continues on explaining the general addition rule expanding the Triangle in terms of "N-1", "X-1", "X-2", etc. The conversation takes a momentary turn when the students start explaining to Brian, who came in late to join the group, the meaning of the addition rule for Pascal's Triangle. The following episodes (in Table 14) illustrate how the students (primarily) involve the symbolic notations and Pascal's Triangle (the quantitative system) as preferring forms of semiotic modeling when explaining their mathematics ideas associated with the addition rule as their verbal conversation progresses.

Table 14
Compendium of transcripts excerpts 1020-1082. Comments are included in *italicized font f*or purposes of analysis in this study.

Line	Time	Talker	Transcripts
1020	00:45:15	Jeff:	All right. Say we have this row right here.[B2] We got um, N choose 0.[B3] And over here we have N choose X.[B3] And then over here we have N choose N.[B3] All right? Then this right here would be- Oh, we're explaining the general addition, the addition rule using this type of, to fill out the triangle.[B2] Using chooses to fill out the triangle and this here would be N choose X plus one and then N, N choose X plus two and so on to whatever N equals.[B3] Right there'd be dot dot-[B2] I didn't, I didn't leave enough room. And this here would be X minus one and then-[B3]
			Jeff starts the conversation describing the mathematical entities and the mathematical activities of the problem involving Pascal's Triangle as a quantitative system and the "choosing" operation in terms of the symbolic notations of "N" and "X". This episode illustrates an inter-linguistic shift in the use of the linguistic structures. Jeff uses mathematical entity when describing the mathematical components involved in the problem then uses mathematical activity when explaining the interactions that happen among those mathematical components.
1021	00:46:02	Ankur:	You did that one man.
1022	00:46:03	Jeff:	What?
1023	00:46:04	Ankur:	Nothing.
1024	00:46:05	Jeff:	That'd be X minus two and so on each way.[B3] Right? So it'd be that.

1025	00:46:10	Ankur:	Can I see the row above that?[B2]
1026	00:46:12	Jeff:	And the row above this would be N minus one, right?[B3] Yeah.
1027	00:46:17	Michael:	Mm hm.
1028	00:46:19	Jeff:	Um, choose zero.[B2] This again would be N, N minus one choose X and then-[B3]
1029	00:46:29	Michael:	N minus one.[B3]
1030	00:46:30	Jeff:	N minus one, N minus one.[B3] That's a one. Um, how do you want me to, to- Where do you want me to go from here?

These episodes (1024-1030) show the students describing the general addition rule as Jeff is writing on the board the Triangle in terms of "N", "X", "N-1", "X-1", "X-2", etc. The use of mathematical activity prevails in the students' verbal conversation as preferring form of linguistic structure.

1031	00:46:40	R1:	Well, you know, um, Brian wasn't here, so you might want to give him some background to what you've been doing.
1032	00:46:46	Jeff:	Start at the beginning? We did, we worked for an hour and a half getting to this point. Explaining this, doing this. All right, um.
1033	00:46:54	R1:	But Brian's a quick study.
1034	00:46:54	Brian:	That's what I am.
1035	00:46:56	Jeff:	All right. We did, uh, this is Pascal's Triangle using- [B2]

This episode demarks the moment Jeff starts explaining to Brian, who came in late to join the group, the meaning of the addition rule for Pascal's Triangle. From this moment on the linguistic structure of mathematical entity is used by the students to describe the "choosing" operation. This shift in the use of mathematics language depicts a form of engagement in which the student (Jeff) descents to the core of the problem to identify and describe the mathematics *participants involved in the mathematics operation of the addition rule.*

1036	00:47:02	Brian:	The whole choose thing.[B3]
1037	00:47:03	Jeff:	-the choose situation.[B3] That's what this is.
1038	00:47:04	Michael:	You know how choose works, like one, three, three, one.[B2]
1039	00:47:06	Brian:	Yeah.
1040	00:47:07	Jeff:	Yeah.
1041	00:47:07	Michael:	Three choose zero, three choose one-[B2]
1042	00:47:08	Brian:	One, four, six-[B2]
1043	00:47:09	Michael:	Yeah. It's all like chooses of something.[B3]

In these episodes mathematical entity is the prevailing form of linguistic structure in the students' verbal conversation. It is inferred that this form of linguistic structure is used by the students because they want to help Brian understand the problem situation by analyzing the configuration of the Triangle first. It is believed that the use of the linguistic structure and the involvement of the corresponding semiotic models are in alignment with the intent of the explanation.

1044	00:47:11	Jeff:	All right. So, um, I don't- Um, how would you like to, uh, how do you want to do this? How do you want to-
1045	00:47:19	Michael:	We're just-[B3]
1046	00:47:20	Jeff:	Well, tell him what we did.
1047	00:47:21	Michael:	-replacing the three in the chooses by N's and X's.[B3]
1048	00:47:24	Jeff:	Yeah, exactly. And rather doing, like, uh, rather- Say this is the, uh-
1049	00:47:29	Michael:	If N was three.[B2]
1050	00:47:30	Jeff:	Yeah, say if N was the third row, it would be three choose zero.[B2] That would give you one.[B2]

1051 00:47:36 Ankur: Like, you know how it's one, three, three, one.[B2] Three choose zero gives you one.[B2]

1052 00:47:38 Jeff: Three choose one.[B2]

1053 00:47:39 Michael: That'd be three.[B2]

These episodes (1045-1053) illustrate the students explaining the general addition rule in terms of "N", "X", "N-1", "X-1", "X-2", etc. Mathematical activity is the prevailing form of linguistic structure used by the students in their verbal conversation. It is inferred that this inter-linguistic shift in the use of verbal language is a form of engagement conducive to the development of mathematics ideas by the students, which is believed to be part of a process referred to in this study as mathematics synthesis.

1054 00:47:39 Jeff: That would give you the three.[B2] The three choose two.[B2] That would give you the other three.[B2] That's equal to three and then three choose three.[B2] That equals the other one.[B2] And like that's filling out this part of the triangle and so on.[B2] And that's what, that's what we're doing now. We went, other stuff we did we did the whole, we found that equation to find out choose.[B3]

This episode demarks a moment of shift (vertical descent) from mathematical activity to mathematical entity in the use of the linguistic structures, and a shift (vertical ascent) from quantitative systems to symbolic notations in the involvement of the semiotic models. It is inferred that these shifts are in response to the students' need of refining (or redefining) the mathematics operation of "choosing", of imparting denotative meanings to the word "choosing" as part of a process referred to in this study as nominalization . The following episodes illustrate how the students impart denotative meaning, as part of this nominalization process, to the word "choosing" during their verbal conversations.

1055 00:48:01 Michael: What choose means.[B3]

1056 00:48:02 Jeff: Yeah, we did all that.

1057 00:48:03 Romina: And choose.[B3]

1058	00:48:04	Jeff:	But you missed out on all that. That's the choose equation.[B3]
1059	00:48:05	Romina:	That's the choose equals.[B3]
1060	00:48:08	Jeff:	And we spent time explaining. That's what we spent the bulk, bulk of the thing, trying to figure out how to explain that. And-
1061	00:48:14	Brian:	What's that little exclamation point?[B3]
1062	00:48:15	Michael:	//Factorial.[B3]
1063	00:48:16	Romina:	//Factorial.[B3]
1064	00:48:16	Ankur:	//Factorial.[B3]
1065	00:48:16	Jeff:	Factorial.[B3]

These episodes (1055-1065) illustrate the students using the mathematical entity to define or refine (i.e., analyze) the mathematics operation of "choosing" involving symbolic notations.

1066	00:48:17	Brian:	That's what it is?
1067	00:48:17	Romina:	Yeah.
1068	00:48:17	Jeff:	Yeah.
1069	00:48:18	Brian:	All right.
1070	00:48:18	Jeff:	It was really excited, like N! [Michael laughs]
1071	00:48:20	Romina:	You want to know what this is? That's all the combinations.[B2] [Romina points to her paper.] That's minusing.[B2] You know how like they're saying-[B2]
1072	00:48:26	Brian:	Yeah.
1073	00:48:26	Romina:	-three choose two.[B2]
1074	00:48:27	Brian:	Yeah.
1075	00:48:27	Romina:	We don't care about the three, so that's like when the threes are switching, not the twos.[B1] And that's when

the twos are like in the first place and the third place, and they just switch and nothing else moves.[B1]

These episodes (1071-1075) illustrate a moment of ascent in the use of the linguistic structures and a moment of descent in the involvement of the semiotic models by Romina when explaining to Brian the mathematics operation of "choosing". Rather than reading the written symbolic notations of the combination formula, Romina refers to the mathematical components of the "choosing" operation in terms of visual manipulative. Her use of language in line 1075 is characterized by the presence of conjunctions conducive to the formation of conditional statements. It is inferred that the involvement of the visual manipulative helps Brian understand her explaining in the form of mathematical logic.

1076 00:48:35 Brian: So this-

1077 00:48:35 Romina: It's basically the same thing.[B1]

1078 00:48:35 Brian: Is this, is that this over this?[B2]

1079 00:48:37 Michael: Yeah.

1080 00:48:38 Romina: It's N, N factorial over N minus X factorial times X factorial.[B3]

1081 00:48:45 Michael: And that equals N choose X.[B3]

These episodes illustrate a gradual inter-semiotic shift of ascent from manipulative to quantitative and from quantitative to symbolic forms of mathematics representations. The symbolic notations written in terms of "N" and "X" bring denotative meanings to the conversation.

1082 00:48:46 Romina: Like this is when the- the things we don't- No, I'm just saying these are the things that we don't care about when they- they switch and this is when the things we do care about, just switch in the same place and everything stays the same.[B1]

These episodes illustrate a moment of ascent in the use of the linguistic structures as Romina explains the mathematics operation of "choosing" (again) in terms of visual manipulative. Rather than *reading the written symbolic notations, Romina involves the visual manipulative to explain the "choosing" operation. Her use of language is characterized by the presence of conjunctions conducive to the formation of conditional statements. And her mathematical logic statement is supported by the involvement of a simple and perceptual form of mathematics representation.*

The previous episodes illustrate how the students describe the addition rules for Pascal's Triangle involving the "choose notation" in terms of "N", "0", and "X", "N-1", "X-1", "X-2", etc. The students involve Pascal's Triangle and the 3-topping problem to explain the meaning of the addition rule. They further develop the general addition rule using the "N-choose-X" notation and then express it as an equation in terms of the factorial notations themselves (shown below).

$$\binom{N}{X} + \binom{N}{X+1} = \binom{N+1}{X+1}$$

$$\frac{N!}{(N-X)!\,X!} + \frac{N!}{\big(N-(X+1)\big)!\,(X+1)!} = \frac{(N+1)!}{\big((N+1)-(X+1)\big)!\,(X+1)!}$$

In sequence with this development, to conclude, the episodes (in Table 15) illustrate how the students and Researcher R1 (Carolyn) exchange mathematics ideas associated with the addition rule to simplify the addition statement written in factorial notations. The students proof addition rule identity involving quantitative operations of factorial and culminate the conversation with the formation of mathematical logic statements.

Table 15
Compendium of transcripts excerpts 1342-1375. Comments are included in *italicized font* for purposes of analysis in this study.

Line	Time	Talker	Transcripts
1342	00:59:24	R1:	Would you like to know how to do that? [B3] Would you like to know how to do the algebra of factorials? [B3] I bet you know how to do a little bit already. I'll just show you one thing that I know you know and I'll leave you to think about this because everyone is getting tired, but let's just take something like this, right? [B3] Six choose two, right? [B2] And you know, you, you told me you could write that how? As-
		Researcher R1 (Carolyn) uses the linguistic structure of mathematical entity to inquire about some of the symbolic notations (previously) used by Michael. The questioning invites the student to conduct further investigation about the new mathematics representation.	
1343	00:59:55	Michael:	Um, six factorial over- [B2]
		Michael starts responding in terms of mathematical activity, involving quantitative values. This episode illustrates the student's use of language advancing from mathematical entity to mathematics activity, which is an indication of his mathematical understanding of the problem situation.	
1344	00:59:57	R1:	Six factorial. [B2]
1345	00:59:59	Michael:	Three fact, four factorial times two factorial. [B2]
1346	01:00:03	R1:	Times two factorial, right? [B2]
		Researcher R1 uses the same linguistic structure and the same semiotic model when probing. Michael continues building up mathematics ideas using the same linguistic structure and the same semiotic model.	
1347	01:00:05	Romina:	Mm hm.
1348	01:00:06	R1:	And you know what six factorial is, right? [B2] Six times five. [B2]

The Researcher's questioning invites the student to evaluate and validate his approach by rewriting 6 factorial as 6 times 5 factorial. Researcher R1 maintains a neutral stance in the use of mathematics language and representation. These episodes illustrate the iterative nature of the learning of mathematics.

1349 01:00:11 Michael: Times one-twenty. B2

1350 01:00:12 Jeff: Thirty. B2 Yeah.

Michael executes the proper mathematics operation and computes the correct factorial value. Jeff voiced the wrong answer. He takes the values of six and five as two mathematical entities in the operation of six-times-five. This episode illustrates a lack of attendance (by Jeff) to considering the nominalization of the word "factorial" as a mathematical (activity) operation. Hence, this episode illustrates an inter-linguistic shift (descent) in the student's interpretation of language by Jeff.

1351 01:00:13 R1: I'm not going to do that though. I don't like to. I don't like to do multiplication. B2 I'm very lazy. I'm just going to write six times five times four factorial. B2 Is that okay?

1352 01:00:21 Jeff: That's, that's simplifying is great, then you can- B2 [Students all talk at once.]

The Researcher promotes further investigation by altering the configuration of the factorial structure. Jeff understands how to simplify the mathematical expression with the new restructuring of the factorial expression, and gets ready to explore other ways of solving the problem. These episodes illustrate how the modified factorial representation helps the student situate himself in the (new) problem configuration and solve the problem from a different perspective.

1353 01:00:24 R1: But can I do that?

1354 01:00:26 Romina: Yeah.

1355 01:00:26 Michael: And then you could cross out the four factorials and- B2

Michael resumes the conversation using the linguistic structure of mathematical activity describing how the factorial operation *(quantitative system) can be simplified. Michael keeps using the same linguistic structure and the same semiotic model.*

1356 01:00:27 Romina: Oh.

1357 01:00:28 R1: Oh, then I can cross out the four factorials. [B2]

The Researcher reinforces Michael's approach. This episode illustrates a moment of "repeat-to-reinforce", a demotion in the use of mathematics language for rigging purposes.

1358 01:00:28 Jeff: Oh, all right, that makes sense.

1359 01:00:29 R1: Right?

1360 01:00:31 Jeff: So you just get thirty divided by, you get thirty divided by two. [B2]

Jeff explains how the factorial operation can be simplified. This episode illustrates a moment of ascent in the use of mathematics language, a shift from mathematical entity to mathematical activity describing factorial as a quantitative operation.

1361 01:00:33 R1: Yeah. Look at all the time that will save you in an SAT question.

1362 01:00:35 Jeff: That'd be big.

1363 01:00:37 R1: But, but if you think about this-

1364 01:00:39 Jeff: She broke, she broke it down farther. [B2]

1365 01:00:40 Romina: Oh yeah she just-

1366 01:00:42 Jeff: Like rather than say you have six factorial- [B2]

Jeff keeps using the same linguistic structure and the same semiotic model describing further factorization, and presents an argument about the mathematical activity conducive to the formation of logical relations (later on).

1367 01:00:43 Ankur: Mm hm.

1368 01:00:43 Jeff: She broke it down until she got a number that she got that she wanted.[B2]

Jeff describes how the mathematical expression can be simplified by altering the structure of the factorial. There is consistency in the use of the linguistic structures and the semiotic models. Also, the use of the linguistic structure of mathematics activity is in alignment with the development of the solution.

1369 01:00:45 Romina: She had two numbers. [B2]

Romina uses the linguistic structure of mathematical entity to identify the mathematical components involved in the activity described by Jeff. This episode illustrates a moment of inter-linguistic shift as she descends to the core of the factorial structure to identify the mathematical components involved in this mathematics operation.

1370 01:00:47 Jeff: That matched the number on the bottom. [B2]

Jeff sticks to the use of mathematical activity and gets ready to explain a sequence of mathematics events that take place in the simplification (or mathematics synthesis) process.

1371 01:00:48 Ankur: All right. Yeah.

1372 01:00:50 Jeff: Then you end up like with the two factorial and then cross out and that's thirty over the two factorial and that's two. [B2] So it's just fifteen. [B2]

Jeff explains a sequence of mathematical conditions and activities taking place in the simplification (synthesis) process of the problem. His use of language is characterized by the presence of conjunctions conducive to the formation of conditional statements and propositions.

1373 01:00:51 Michael: But then it would probably be even longer than that. [B2] Cause if N is a big number- [B3]

Michael collaborates in the mathematics conversation involving the symbols and notations to extrapolate further solutions. This episode illustrates a moment of inter-semiotic shift (ascent) in the involvement of semiotic models from quantitative systems to symbolic notations for purposes of extrapolation.

1374 01:00:55 R1: Does it matter?

1375	01:00:59	Michael:	-you'd have to write, you would have to write N times N minus one times N minus [B3]

Michael finishes his explanation about the factorial operation. These episodes (1372-1375) illustrate a sudden increase in the students' usage of mathematical conditions and operation sequences conducive to the formation of mathematical logic statements. These episodes demonstrate the vertical ascent (or hierarchical trajectory) in the learning of mathematics in which the proper use of language and models signify the construct of new mathematics ideas conducive to the semantic expansion of mathematical knowledge, a process this study refers to as the synthesis in the learning of mathematics.

CHAPTER 5 – CONCLUSIONS

This study demonstrates the importance of the multi-modal approach in the learning of mathematics when students work in collaborative learning activities. In alignment with the concepts of the mathematics register and the external representation, the functional linguistic structures (FLS) and the functional semiotic models (FSM) prove to be effective in directing the investigation on how the students' verbal conversation facilitates their construct of mathematics ideas when working in groups. These concepts claim necessary the use of a variety of forms of mathematics utility, such as artifacts, graphs, symbols, notations, models, etc., for the creating and expressing of mathematics ideas. The taking of language as one functional resource that interacts with those functional forms of representation is found to be an effective approach to investigate how the students create mathematics ideas when working in the collaborative learning activity. The functional linguistic structures are forms of expressions that are manifested in dialogical (social) discourse, and the functional semiotic models are forms of representation that develop in dialectical (environmental) discourse. This study demonstrates that these two mathematics discourses interact simultaneously and influence each other in the construct of mathematical knowledge by the students as they converse verbally.

5.1 Summary

This study underscores two important aspects about the nature of the students' mathematics conversations as they work on the learning activity task. First, the students use the linguistic structures either (1) to describe the mathematics situation of a problem, or (2) to explain the mathematics solutions to the problem. Secondly, the involvement of the semiotic models by the students has an impact in their ability (1) to identify and define the leading particulars of the problem situation, and (2) to fabricate a framework upon which they can build new mathematics ideas. The alignment between the usage of the linguistic structures and the semiotic models in the students' mathematics conversations reveals patterns of engagement that help explain how these forms of mathematics expressions and representation interact and influence each other in the construct of mathematical knowledge by the students while working in the collaborative learning activity.

This study recognizes the importance of the utilization of the linguistic structures in mathematics conversations. According to the data, the mathematical entity is the linguistic structure most commonly used throughout the students' verbal conversation followed by the mathematical activity (Refer to Figure C1 in the Appendix). During the initiation of the conversation, the mathematical entity is mainly used by the students to describe the mathematics situation of the problem, to identify and define its components and circumstances. During this process, the students strongly make use of the visual manipulative and the quantitative systems. It is inferred that the students involve the visual manipulative at the beginning of the learning activity to descend to the core configuration of the problem, and the quantitative systems to estimate the extent of the problem. In a sense, this process resembles one describing the breakdown of a configured system into its functional components, i.e., a method of analysis. It is believed that these semiotic models are used by the students to examine the mathematics configuration of the problem, and to investigate how its configuration responds to changes provoked by the students through dialectical discourses.

This study also recognizes how important the utilization of the semiotic models is for the building of mathematics ideas when students work on learning activities. According to the data, the semiotic model most commonly involved in the students' verbal conversation is the quantitative systems followed by the visual manipulative (Refer to Figure C5 in the Appendix). During the initiation of the conversation, the students involve the symbolic notations to unpack mathematics information they acquired from previous experiences or sources related to the mathematics situation of the actual problem. In alignment with the mathematical entity, the students' use of the symbolic notations serves as a point of departure to descend to the core of the problem. In this context, it is inferred that in this process of analysis, referred to in this study as a vertical trajectory of descent, the students unpack denotative mathematical meanings represented by the symbols and notations as they investigate and describe the mathematics configuration of the problem. It turns out that the students' use of the symbols and notations plays an important role for the initiation of their mathematics conversation.

5.2 Findings

The socio-cognitive perspective on learning is evident when the students explain their ideas during their mathematics conversations. The interplay between observation and imitation is commonly observed from the video data and transcripts throughout the students' conversation. It is evident from the transcripts that every single student, at one moment or another, repeats what others do or say during the learning activity. Also, the students are able to observe how the semiotic models can be used to represent the configuration of a problem situation, and how the configuration responds to changes made by the students to those models.

The following findings associated with the usage of the mathematical logic and the symbolic notations are worth to underline. Although the mathematical logic is the linguistic structure least used throughout the students' verbal conversation, their curves exhibit unique patterns of behavior characterized by the presence of lagging factors and moments of abrupt increases in their usage during the conversation (Refer to the Figures of Appendix C). This linguistic structure is characterized by the use of conjunctions and the formation of conditional statements, and when used in alignment with symbols and notations takes the students' conversation to a different level. The involvement of the symbolic notations, when provided with denotative meaning, expedites the formation, or re-formation, of the students' trains of thoughts conducive to their construct of mathematical knowledge during their mathematics discourse.

The results of this study show that the involvement of the symbolic notations is strong at the beginning of the conversation and weakens thereafter when the students use mathematical logic. It strengthens again at the end of the conversation helping in the formation of the students' conclusive logical statements. It is evident from the data that the students use the mathematical logic in their verbal conversations to describe sequential and conditional interactions that take place among the mathematical components represented in terms of symbols and notations. It is inferred that the usage of the mathematical logic and the symbolic notations interact and influence each other in the process of synthesis (ascent) in constructing mathematics ideas during the students' verbal conversation. The involvement of symbolic notations facilitates the initiation of directed verbal conversations leading to productive and richer dialogical and dialectical discourses that culminate with the formation of conclusive conditional statements expressed in the form of logical statements involving symbols, notations, and equations.

It was found that when the students use the linguistic structures to describe a problem situation, they initiate the process of analysis by first and briefly involving the symbolic notations. The students' trajectory of descent progresses gradually by shifting from the utilization of the symbolic notations to the utilization of quantitative systems, and from the utilization of the quantitative systems to the utilization of visual manipulative in their effort to define the problem of the task. As the mathematics conversation progresses, the students repeatedly make use of the visual manipulative to represent the configuration, or re-configuration, of the problem, and the quantitative systems to estimate the solution to the problem as part of an interactive and iterative process.

Also, when the students use the linguistic structures to explain the new mathematics ideas they formulate symbolic notations to represent the mathematical knowledge they acquired during the learning activity in a compact and concise form. As shown in the data, the students' conversation takes a trajectory of ascent by gradually shifting from the utilization of visual manipulative to the utilization of quantitative systems (in the interval of 15-45 minutes), and from the utilization of quantitative systems to the utilization of symbolic notations (in the interval of 30-60 minutes). This shift resembles a buildup or synthesis process in which the students consolidate their new mathematics idea as they re-configure and re-solve the problem.

It was also observed that when patterns of behavior in the students' use of language reveal the promotion, or demotion, of inter-semiotic shifts between quantitative systems and visual manipulative, one of two forms of cognitive mechanisms actuates in dialectical discourse (1) to analyze the mathematics situations of the problem by gradually shifting from the utilization of quantitative systems to the utilization of visual manipulative, or (2) to synthesize new mathematics ideas by gradually shifting from the utilization of visual manipulative to the utilization of quantitative systems during the conversation. It is inferred that at this moment the linguistic structures are used by the students either to describe the mathematics situation of the problem (i.e., analysis), or to explain the development of the new mathematics ideas (i.e., synthesis). The data of this study confirm that the usage of these two linguistic structures and these two semiotic models by the students pursues this purpose.

Also, when the students encounter a mathematics problem, they often make use of the visual manipulative to define the mathematical components of the problem. They involve quantitative systems when arranging, or re-arranging, those mathematical components in distinct and differing ways to count and find the solution to the problem. To describe those mathematical components and how they interact among each other, the students use the linguistic structures of mathematical entity and mathematical activity. It is inferred that during this process of analysis, the students make use of the visual manipulative to investigate the core configuration of the problem.

At the end of the conversation, in contrast, when the students use the linguistic structures to explain the new mathematics ideas they acquired during the learning activity, they involve the symbolic notations more than the visual manipulative. It is inferred that the students involve the symbolic notations to consolidate or pack the new mathematical knowledge they learned through dialectical experiences in a compact and concise form. During this process of synthesis (ascent or buildup), the students start forming the linguistic structure of mathematical logic to describe the connectivity and interactivity among those mathematical components represented in terms of symbols, notations, and equations.

The results of this study clearly demonstrate the importance of the multi-modal approach in the learning of mathematics. The utilization of the three linguistic structures in concert with the three semiotic models proves to be effective for the development of this study as they help infer how the students create mathematics ideas when working in the collaborative learning activity. The results of this study validate the multi-modal approach as they show forms of engagement that take place among the linguistic structures and the semiotic models that promote the students' growth of mathematical knowledge as they converse verbally.

5.3 Implications

This study investigates the importance of taking conversational language as one functional resource that interacts with the mathematical models in the learning of mathematics, and demonstrates that the functional semiotic models and the functional linguistic structures interact simultaneously and influence each other in mathematics discourse conducive to the construct of mathematical knowledge. In this respect, the multi-modal approach ought to be considered in the learning of mathematics by taking conversational language as one functional resource that interacts simultaneously with the mathematical models when students participate in collaborative learning activities or in mathematics class discussions.

To conclude, the method of grouping the data in large clusters by functional categories, i.e., the three linguistic structures and the three semiotic models, was found to be effective for the development of this study. If properly implemented in mathematics curricula, this method of classification may help infer how students construct their mathematics ideas, how they learn mathematics, and how they, through verbal conversations, exchange mathematics ideas and construct further mathematical knowledge when working in collaborative learning activities.

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Appendix A: Coding of the transcripts for the linguistic structures.

Legend

Mathematical entity | Mathematical activity | Mathematical logic

Line	Time	Talker	Transcripts
1	00:00:00	__	[Romina, Jeff, and Michael are sitting at a table talking with Researcher 1. At the start, the display says 02:01:01.27. No sounds can be heard. Researcher 2 walks across the room, opens the door, and goes out. The camera shows the rest of the room. Researcher 2 returns.]
2	00:01:05	R1:	-mentioned some of what went on. I have, I don't have a clue. Can you sort of tell me about it and how some of you suggested it's connected to other things you had done? I'm really curious. Feel free to use the board and show me and tell me.
3	00:01:16	Jeff:	Well.
4	00:01:17	R1:	But I don't know what went on.
5	00:01:17	Romina:	When, when we came up with that thing that almost was like the Pascal's Triangle. What was it with e? What were we doing?
6	00:01:21	Jeff:	It was, um-
7	00:01:22	Romina:	Ten, ten percent of a hundred.
8	00:01:25	Jeff:	Is that what it was?
9	00:01:26	Romina:	Yeah.
10	00:01:28	Jeff:	Um, I'm not sure. I don't know.
11	00:01:31	Romina:	Can we have a calculator? Are we allowed to have one?
12	00:01:33	R1:	Sure. Hope you know where they are.
13	00:01:36	R4:	Yeah.
14	00:01:37	R1:	You may want to have them around anyway.

15	00:01:43	Jeff:	This could be our first time ever using calculators.
16	00:01:45	Romina:	Yeah.
17	00:01:46	R1:	Wow, first time.
18	00:01:46	Jeff:	We usually-
19	00:01:46	R1:	You never use them?
20	00:01:47	Romina:	Thank you.
21	00:01:48	R3:	You're welcome.
22	00:01:52	Michael:	Are there any games on this?
23	00:01:58	Jeff:	We said one times or one minus.
24	00:01:59	Romina:	What, what am I doing?
25	00:02:00	Jeff:	Oh that's good right there.
26	00:02:01	Romina:	Yeah. [Romina laughs.]
27	00:02:02	Jeff:	One. Was it one minus one over a hundred?
28	00:02:02	Romina:	Hm.
29	00:02:06	Jeff:	Hundred raised to-
30	00:02:11	Romina:	OK, why am I- Oh, okay. I didn't know what I did wrong.
31	00:02:14	Jeff:	Oh.
32	00:02:17	Romina:	Wasn't it like, weren't we doing this?
33	00:02:18	Jeff:	Yeah, that's what it was.
34	00:02:20	Romina:	Yeah.
35	00:02:21	Michael:	That's this- [unintelligible; chair is moving].
36	00:02:23	Romina:	OK, this is scary. Look when we were- cause we were discussing like percentages. And, uh, like an increase and we did a hundred and we took ten percent of it and that's one two one, that's one three three one, and, you know, that is-
37	00:02:38	Jeff:	Yeah, we kept going it-

38	00:02:39	Romina:	It doesn't come out yeah. After a while it goes.
39	00:02:41	Jeff:	It kind of makes you think. After a while it stops, but we were, uh-
40	00:02:44	Romina:	We really thought that was it, look.
41	00:02:45	Jeff:	We were into it.
42	00:02:48	R1:	Oh. So what does it mean?
43	00:02:50	Jeff:	Uh, we didn't, we didn't know.
44	00:02:51	Romina:	We didn't know because then it stops, though.
45	00:02:53	Jeff:	Yeah, but it was interesting for, for a while.
46	00:02:55	Romina:	While it was going on it was very-
47	00:02:57	Jeff:	We were kind of, uh-
48	00:02:59	Michael:	Are we going to [Inaudible.]?
49	00:03:00	Jeff:	But um, what was the question? What were you-
50	00:03:02	Romina:	We wanted to know what we did in class today.
51	00:03:04	Jeff:	Um, we were looking a lot at, at working at e and, and the equation for it.
52	00:03:09	Michael:	And how it, how it, how it connects with ln and-
53	00:03:11	Jeff:	Yeah, um-
54	00:03:11	Romina:	And we were also trying to find like, you know how we had when we had a plus b to the n? We want to know what- And we had like numbers before it when we got to big numbers, we want to know, you figure out what the numbers were, like in front of the a, you know, cubed.
55	00:03:24	Jeff:	You know, that's, that's like, um-
56	00:03:25	R1:	You could use the board too.
57	00:03:26	Jeff:	Uh, we just- Like if you were looking, if we were looking for like a plus b-

58	00:03:33	Romina:	To the tenth.
59	00:03:33	Jeff:	To the tenth say, um, obviously it- Was the first one ten? Was it //one a to the tenth and //then ten-
60	00:03:39	Michael:	//No it's one, yeah.
61	00:03:40	Romina:	//b. Oh no, you're right. Sorry.
62	00:03:42	Jeff:	Ten a to the ninth b to the first, right?
63	00:03:45	Romina:	Mm hm.
64	00:03:45	Jeff:	And then how to find out //this number.
65	00:03:47	Romina:	//What the next one was.
66	00:03:48	Michael:	It's forty-five.
67	00:03:49	Jeff:	And it was forty-five but we were working on how to figure it out when we were doing it. We knew it was the choose thing, whatever that means. The- You do a forty- What was it? Ten choose two?
68	00:03:58	Michael:	Yeah.
69	00:03:58	Romina:	Uh-huh.
70	00:03:58	Jeff:	You know what I'm talking about? Like, uh, was it N-C-R- actually that's supposed to be lower case. Two- is that how you do it? Right?
71	00:04:05	Michael:	Yeah, it's one of these things like that.
72	00:04:06	Jeff:	And that equals forty-five and that's the answer. You know. I'm not, we're not really sure how all this works but it's like, what is that, if-
73	00:04:13	Romina:	We, we learned that, we learned that with her.
74	00:04:15	Jeff:	Yeah. Yeah the- Yeah, we, we went, we went over that, remember that? With the total-
75	00:04:19	Romina:	We tried to go over that. [Romina laughs.]
76	00:04:20	Jeff:	If you have ten different, what was it? Ten different things.

77	00:04:24	Michael:	You have-
78	00:04:25	Romina:	//Ten high. //Ten high.
79	00:04:26	Jeff:	//Ten high. How many-
80	00:04:26	Romina:	//How many would have two reds, only two reds.
81	00:04:27	Jeff:	//How many would have two, two reds.
82	00:04:29	R1:	One more time.
83	00:04:31	Jeff:	If you had towers// of ten high.
84	00:04:32	Michael:	//If you had like towers.
85	00:04:32	R1:	Towers.
86	00:04:35	Jeff:	If you have towers with ten high //and two colors.
87	00:04:35	Michael:	//How many different places can you put two reds in there?
88	00:04:36	Jeff:	Yeah.
89	00:04:37	Romina:	Yeah.
90	00:04:37	Jeff:	And like a would be one color and b would be blue, um, b would be the other color. Then how many would you have, a being two in the whole thing? And that would be forty-five and that's, that's what this number would be.
91	00:04:50	R1:	And these towers are how tall?
92	00:04:52	Jeff:	Ten tall.
93	00:04:53	Romina:	Ten.
94	00:04:54	Jeff:	That'd be the ten there.
95	00:04:54	Romina:	Mm hm.
96	00:04:54	Jeff:	The two would be the two colors and then, right?
97	00:04:58	Michael:	No.
98	00:04:58	Romina:	No, two of one color.

99	00:04:59	Jeff:	No, ten would be the two of the one color and the two is implied that there's two, only two colors? Or-
100	00:05:04	Michael:	The two is the-
101	00:05:04	Romina:	It's only a plus b.
102	00:05:06	Jeff:	Yeah but in the, when you write this, I mean is it implied that there's only two colors?
103	00:05:10	Romina:	I believe it is but-
104	00:05:12	Jeff:	Is that, is it implied?
105	00:05:14	Romina:	I, I'll go with the yeah. I don't know. [Romina laughs.]
106	00:05:16	Michael:	Uh, You talking about this?
107	00:05:17	Jeff:	Yeah.
108	00:05:18	Michael:	//No, It, it,
109	00:05:18	Romina:	//Is that like-
110	00:05:19	Jeff:	Is that one, the only one works for-
111	00:05:20	Michael:	It's just like you have ten things where, how many different places can you put these two? That's all.
112	00:05:25	Jeff:	Yeah, I know but-
113	00:05:25	Michael:	You know what I'm saying?
114	00:05:25	Jeff:	But if there's, oh, yeah, two. All right, I see what you're saying.
115	00:05:25	Michael:	That's all.
116	00:05:28	Jeff:	There could be a hundred colors but it would still-
117	00:05:31	Michael:	Yeah you pick two things out of those ten.
118	00:05:32	Jeff:	Yeah.
119	00:05:33	Michael:	How many different places can you put them?
120	00:05:34	Jeff:	Put them. All right. All right.
121	00:05:35	Michael:	Forty-five, I think,

122	00:05:37	R1:	So, so you're saying that's forty-five and what if I wanted eight red? Eight red ones or eight a's?
123	00:05:41	Jeff:	Then it would be ten-
124	00:05:41	Michael:	Um.
125	00:05:42	Romina:	Ten choose eight.
126	00:05:43	Jeff:	Choose eight, yeah.
127	00:05:44	Michael:	A smaller number.
128	00:05:45	Jeff:	Because //that would be how many different spots can you move those eight of one color in the tower of ten.
129	00:05:47	Romina:	//Now how do you-
130	00:05:50	Michael:	It's forty-five also.
131	00:05:51	R1:	Why?
132	00:05:52	Romina:	Like how do //you, how do you, how do you do that on a calculator?
133	00:05:53	R1:	//How'd you do that so fast Michael?
134	00:05:53	Jeff:	Um.
135	00:05:54	Michael:	No, I just like did it all in my head, that's all.
136	00:05:54	Jeff:	You go to, uh, math.
137	00:05:56	R1:	Tell us how you did it.
138	00:05:57	Michael:	Um.
139	00:05:57	Jeff:	Probability.
140	00:05:58	Michael:	There's a button that-
141	00:06:00	Jeff:	N-C-R.
142	00:06:00	Michael:	Take ten, that button then eight.
143	00:06:02	Romina:	Then math.
144	00:06:03	Michael:	And it comes out forty-five.

145	00:06:05	Jeff:	Why is that the case?
146	00:06:07	Romina:	Hm.
147	00:06:09	Michael:	Well if you take like on the-
148	00:06:10	Romina:	Well because-
149	00:06:12	Michael:	You know how on Pascal's Triangle.
150	00:06:13	Romina:	That's like the two.//You have eight left over.
151	00:06:14	Jeff:	//Oh, cause you could switch them all around. Is that, is that, I guess you're counting. //You got, you got, yeah
152	00:06:17	R1:	I don't know. Tell me.
153	00:06:17	Michael:	Cause then you would have-
154	00:06:18	Romina:	//Is that the same thing as that because, like, the eight left over to get to the ten, right?
155	00:06:18	Michael:	//It'll be- It would be the same thing.
156	00:06:22	Jeff:	Exactly.
157	00:06:23	Romina:	It's like almost switching colors.
158	00:06:23	Jeff:	Yeah.
159	00:06:24	Romina:	It'd be like two of the other color.
160	00:06:25	Jeff:	And then, and then, yeah, exactly.
161	00:06:26	R1:	Say that one more time, Romina.
162	00:06:28	Romina:	It'd be two of the other color instead of, like say you started with red for this two. That was for the reds and then when you//make red eight.
163	00:06:33	Michael:	//That would be the other eight.
164	00:06:34	Romina:	The, like, say the blues have two.
165	00:06:36	Jeff:	And it's seven. And then obviously three should be the same as that.

166	00:06:39	Romina:	Yeah. Yeah.
167	00:06:47	R1:	So, you're pressing the calculator, you have a new command that gets you those numbers.
168	00:06:53	Romina:	We know how to do it, I mean it's not-
169	00:06:54	R1:	But if you didn't have the calculator?
170	00:06:57	Romina:	We'd write them out.
171	00:06:57	Jeff:	You'd have to write them all out.
172	00:06:58	Michael:	Well, Bob-
173	00:07:00	R1:	Because Alex wants to know how you do that without a calculator.
174	00:07:03	Jeff:	Well, I obviously if the calculator-
175	00:07:04	R1:	Can you, can you help him understand that?
176	00:07:06	Jeff:	Well we would make a, say, tower of ten.
177	00:07:10	Michael:	Can I say something? //All right, um-
178	00:07:10	Romina:	//I don't know.
179	00:07:11	Jeff:	Bob. Yeah go for it.
180	00:07:12	Michael:	No, I'm talking, I just wanted to say that Bob Sidley had like an actual formula to write the equals and-
181	00:07:17	Jeff:	Do we know, do we know what it is? Or-
182	00:07:19	Michael:	I think so. I don't know if I can remember it.
183	00:07:19	R1:	Why don't I leave you a few minutes and think about explaining this to us.
184	00:07:24	Michael:	It depends on-
185	00:07:24	Jeff:	Well it's not, it's not that hard to explain.
186	00:07:25	Michael:	You remember it? I forgot it.
187	00:07:26	R1:	OK, I'll stay then.

188	00:07:28	Alex:	Well, actually no.
189	00:07:29	Romina:	It's not right.
190	00:07:30	Michael:	All right.
191	00:07:31	R1:	He has a bad memory.
192	00:07:32	Michael:	I got to like do trial and error to see if I can figure it out what it was.
193	00:07:35	R1:	OK.
194	00:07:35	Jeff:	All right, say that's ten? Then, um, you would just have to find- Say you had, uh, one was one color and two was the other color.
195	00:07:43	Romina:	Why, why don't you show her how to do it for like three. Show them how we can get all of them. That you have to draw the tower. [Romina laughs.]
196	00:07:48	Jeff:	You have two colors. And out of this tower of three you'd have to find out all the different places you could put those two colors in. So you could put it there and there. Or you could put it, uh, there and there. Or, am I missing any? Yes, I am.
197	00:08:04	R1:	I understand.
198	00:08:05	Romina:	You could just do like-
199	00:08:06	Jeff:	Yeah.
200	00:08:06	Romina:	Do you want to go for another one?
201	00:08:07	Jeff:	No, go for it.
202	00:08:08	Romina:	No, you could just do, you could do like our blue, blue, blue.
203	00:08:12	Jeff:	You gonna write every one?
204	00:08:14	Romina:	Well, there wasn't that many. No I'm just like giving you an example.
205	00:08:15	Jeff:	Yeah.

206	00:08:16	Romina:	And then you just kind of move it through. And that's how we figure them out when we have to write them out.
207	00:08:23	R1:	So you're saying there's a way of getting these without the calculator.
208	00:08:30	Jeff:	Yeah. And there's a, there's a formula that somebody-
209	00:08:34	Romina:	Not too-
210	00:08:34	Jeff:	-had come up with but I don't know, I don't know how it, how it goes. I'm really not sure.
211	00:08:38	Romina:	I've seen it.
212	00:08:39	Jeff:	I don't remember it.
213	00:08:40	Romina:	Yeah, there's some-
214	00:08:41	Michael:	Yeah.
215	00:08:41	Romina:	Something to that effect.
216	00:08:42	Michael:	It was this guy.
217	00:08:43	Romina:	That's it?
218	00:08:43	Michael:	Yeah.
219	00:08:44	Jeff:	It's this right here?
220	00:08:45	Michael:	Yeah.
221	00:08:46	R1:	Why don't you show us up here, Michael.
222	00:08:48	Michael:	Oh, man. I, I didn't come up with this, so don't ask me why [unintelligible, chair moving]
223	00:08:49	R1:	It doesn't matter that you came up with it.
224	00:08:52	Michael:	If you would have like n choose x.
225	00:09:02	Romina:	That's on, that's on the division, n to the x, or is that just like your-
226	00:09:05	Jeff:	That n to the x?

227	00:09:07	Michael:	No, that's, that's choose to the, that's how you write it I think. I think that's how you write it.
228	00:09:08	Romina:	That's just, that's what it is?
229	00:09:11	R1:	Do you want an equals sign there?
230	00:09:13	Michael:	No. That's, that's not in- Yeah. Yeah, I could do that. Times x. That, that would be the number.
231	00:09:24	R1:	OK. Hi, Ankur. Come on in.
232	00:09:25	Ankur:	Hi. Sorry I'm late.
233	00:09:27	R1:	We're glad you're here.
234	00:09:29	Jeff:	Didn't you go with them?
235	00:09:30	Ankur:	No, I didn't go with them. I went with Steve.
236	00:09:34	Jeff:	That's dirty.
237	00:09:34	R1:	Hi, did you eat?
238	00:09:36	Ankur:	No.
239	00:09:37	R1:	Are you hungry? Yes.
240	00:09:39	Ankur:	Yeah I guess so. But it's all right. It's all right.
241	00:09:39	Romina:	You can, uh-
242	00:09:39	R1:	I'll tell you what. I, I-.
243	00:09:41	Michael:	I hate stopping-
244	00:09:58	Jeff:	All right, what are we going to do?
245	00:09:59	Michael:	Oh. Oh yeah, um-
246	00:10:01	Romina:	What, what does that get? That gets you, like-
247	00:10:03	Michael:	That gives you that choose thing.
248	00:10:04	Jeff:	That gives you-
249	00:10:05	Michael:	I don't, I don't know what it means.

250	00:10:08	Romina:	I was working with him one day when he brought that up but he lost me.
251	00:10:11	Jeff:	That was the day that me and, and my table were doing the, uh, finding the square roots without a calculator.
252	00:10:16	Michael:	Yeah, but he did that before like-
253	00:10:18	Romina:	Not with me.
254	00:10:19	Michael:	-in, in class when he was talking about choosing.
255	00:10:19	Jeff:	Who?
256	00:10:20	Romina:	I was in your group.
257	00:10:21	Jeff:	Oh, gee, you got an eyelash.
258	00:10:24	Michael:	No, in class when he was talking about choosing. He figured it out. And
259	00:10:27	Jeff:	All right, um, well you figure, say you do, uh, say you're doing three, right? So that would be three times two times one. That would be each space, I imagine.
260	00:10:40	Romina:	//And x is how many, how many you want of the color?
261	00:10:41	Michael:	//Yeah, I guess that would be how many combinations.
262	00:10:43	Jeff:	Yeah, that would, because that would give you the total number of-
263	00:10:45	Romina:	Yeah. Yeah that'd be it, yeah.
264	00:10:46	Jeff:	Total number of combinations?
265	00:10:48	Michael:	Oh I guess, yeah.
266	00:10:50	Jeff:	All right. So then that would be, say, six factorial. Divided by. That would be- Why would you- //why, where does that work?
267	00:10:58	Michael:	//No wait. //This is, that would be-
268	00:11:00	Romina:	//Come on, x is the number we want and x is like the number we want to get, like the choose number.
269	00:11:05	Michael:	That's the, yeah, choosing number.

270	00:11:07	Jeff:	And-
271	00:11:08	Romina:	So I guess this would-
272	00:11:10	Michael:	He was telling me like this was-
273	00:11:12	Romina:	That would like take away all the, all the ones we would choose.
274	00:11:15	Michael:	He said something about repeats. One would take away //the repeats.
275	00:11:15	Jeff:	//Yeah, this would this would take away the repeats, right?
276	00:11:17	Michael:	I guess.
277	00:11:19	Romina:	And will this, and this will give you-
278	00:11:20	Michael:	And this will, this will take away the-
279	00:11:21	Jeff:	This will take away all the other ones?
280	00:11:22	Michael:	The other ones that, that you don't care where they are.
281	00:11:22	Romina:	Like the ones that are higher than-
282	00:11:24	Jeff:	Yeah. Yeah.
283	00:11:25	Michael:	You only care about, you only care about the two that are moving. Not the other-
284	00:11:26	Jeff:	Yeah, exactly.
285	00:11:28	Michael:	Not the other, uh, four. It's just-
286	00:11:29	Jeff:	And then, and X-
287	00:11:30	Romina:	That makes sense.
288	00:11:32	Jeff:	Yeah. That make, let's see if it works. So say-
289	00:11:35	Romina:	Where's factorial on this?
290	00:11:36	Jeff:	Where's the exclamation point?
291	00:11:37	Michael:	Math.
292	00:11:39	Jeff:	Ah.

293	00:11:39	Michael:	Probability 4.
294	00:11:41	Romina:	What are we doing, three or six?
295	00:11:41	Michael:	Just hit four.
296	00:11:42	Jeff:	All right.
297	00:11:43	Michael:	I mean would you-
298	00:11:45	Jeff:	I don't even know why I did that. That was stupid. Uh, quit, all right. Six divide- Mm. Six divided by-
299	00:11:53	Romina:	Where is the little, where is that? I don't know.
300	00:11:56	Jeff:	Uh.
301	00:11:57	Michael:	Are you doing. why are you doing the six only? Oh, because three factorial is six, right?
302	00:12:01	Jeff:	You got it. That's not six factorial. Um, divided by three minus, what's X? We don't know?
303	00:12:09	Michael:	Do two.
304	00:12:10	Jeff:	Minus two.
305	00:12:14	Michael:	Times-
306	00:12:16	Romina:	You're a lot farther on that than I am.
307	00:12:20	Michael:	You have to put a parenthesis around that whole thing, too. Later on that.
308	00:12:24	Jeff:	Then times.
309	00:12:24	Michael:	No, you got to, at the beginning of that and the end of this thing.
310	00:12:31	Michael:	Get rid of that one there.
311	00:12:35	Jeff:	Don't we have to close that in, though?
312	00:12:36	Michael:	No, you don't have to close that.
313	00:12:37	Jeff:	Oh. All right. So do I have to delete that other one? No.
314	00:12:39	Michael:	No, leave that like that. Two.

315	00:12:42	Jeff:	So divided by two factorial. Um. Let's see.
316	00:12:50	Michael:	Let's see. And do it the other way and it should come out.
317	00:12:53	Jeff:	What way, what was the other way?
318	00:12:55	Romina:	Did, did it work? Four.
319	00:13:00	Michael:	No, it's three.
320	00:13:01	Romina:	Oh.
321	00:13:01	Jeff:	Oh, that's all right.
322	00:13:03	Michael:	Yeah, it works.
323	00:13:04	Jeff:	All right. Yeah that's the case then. All right, so when Ankur's done eating, you can explain all this. And we can explain it to Ankur? Cause you know that's coming. Ankur missed the beginning, so explain it to him.
324	00:13:15	Michael:	There wasn't really a beginning, though.
325	00:13:16	Jeff:	It doesn't matter. We're still going to have to. I didn't know. I forgot that we were, uh, we have to explain things.
326	00:13:23	Michael:	I thought it was just recess or something. [Romina laughs.]
327	00:13:25	Jeff:	Yeah, I know. I'm like that. And, uh-
328	00:13:34	Ankur:	I'm not normally in this class and everyone else is.
329	00:13:37	Alex:	You can go ahead and sit anywhere you want. //That's probably a good spot.
330	00:13:37	Romina:	//I saw you out of the corner of my eye. You confused me. I'm like what is he doing?
331	00:13:44	Romina:	I drove him.
332	00:13:45	Jeff:	Ah.
333	00:13:46	Romina:	I drove him here. I drove Jeff here.
334	00:13:48	Ankur:	By yourself?
335	00:13:48	Jeff:	Yeah righ.

336	00:13:49	Romina:	Well, my dad. [Inaudible]
337	00:13:50	Jeff:	All right, um. Do you want to hear our explanation of, of this and why this works?
338	00:14:00	R1:	That would be really great, but it would help us enormously if you would use the board.
339	00:14:02	Michael:	I wrote that, so you don't have to.
340	00:14:04	R1:	Would you mind, Jeff?
341	00:14:05	Jeff:	All right, I need help because-
342	00:14:06	R1:	They'll help you.
343	00:14:07	Jeff:	I don't want to get stuck. I got stuck up there last time by myself and I was looking like an idiot. All right, the reason why this works- We have no chalk. Right there in the side. All right, the reason why this works, first you get all the total number of this will cover all the total possibilities of your tower.
344	00:14:26	Romina:	//And factorial. [Inaudible]
345	00:14:27	Jeff:	//Say in terms of a tower. The factorial right here. So say you're doing towers of three. You have the three factorial and that'll cover all of the different combinations that you could put three in with two colors. All right? [There is a pause. Jeff waves his hands.] Yes?
346	00:14:45	Ankur:	Sounds good.
347	00:14:48	R1:	Why don't you go through and when you're all done I'll ask my question. Just go.
348	00:14:51	Michael:	Like, you should use the explanation like she used. Like the people on the line. That's better because you have like the first one. Then you have-
349	00:14:57	Romina:	Two spaces.
350	00:14:58	Jeff:	All right, I'll do people on the line.
351	00:14:59	Michael:	Two spaces. You have two people left so that's times two.
352	00:15:00	Jeff:	All right, say we're doing us three right here.

353	00:15:01	Michael:	Yeah, on the line.
354	00:15:02	Jeff:	This, us three, um-
355	00:15:04	Romina:	There's three different people to fill in the first spot.
356	00:15:07	Jeff:	Yeah. Then there's, after, then once one goes there, there's only two people left to fill in this spot. And then-
357	00:15:10	Romina:	So you multiply three and two.
358	00:15:12	Jeff:	Three times two and then once, once someone goes in the other, there's only one person left. And they get the last spot, so that's times the one.
359	00:15:18	Romina:	And that's everyone.
360	00:15:19	Jeff:	That make more sense?
361	00:15:20	R1:	Well I'm, I didn't mind your other example here.
362	00:15:23	Jeff:	Yeah, I, I just like the okay through the way so I could move- You know, steady progress.
363	00:15:27	R1:	Um. But I guess, so why are you multiplying?
364	00:15:30	Romina:	We don't like that question.
365	00:15:30	Jeff:	Ah.
366	00:15:31	R1:	You don't like that question.
367	00:15:32	Romina:	No. That, that one gets us all the time.
368	00:15:34	R1:	Why aren't you adding?
369	00:15:35	Jeff:	Uh, because you don't add. It's just, you don't do it. [Romina laughs]. There's no adding going on it any where anymore. That's like out of style. [Romina laughs.]
370	00:15:41	R1:	That's not the answer.
371	00:15:42	Jeff:	I know that doesn't, that doesn't work. Um, you do it because, uh-
372	00:15:52	Michael:	I can't help you on this one.

373	00:15:53	Jeff:	Yeah, I know.
374	00:15:54	Romina:	Yeah, we're-
375	00:15:56	Michael:	That's a good question.
376	00:15:57	R1:	OK, I'll leave you to tell me.
377	00:15:58	Michael:	Why do you multiply?
378	00:15:59	R1:	You'll figure that out.
379	00:16:00	Romina:	We never know this one.
380	00:16:02	Jeff:	Yeah it's like the //eternal question.
381	00:16:03	Ankur:	//Yeah it's cause, if, if you have three things, there's three things you put here, right?
382	00:16:03	Romina:	Mm hm.
383	00:16:09	Ankur:	There's red, white and blue. And then there's only-
384	00:16:09	Romina:	Uh, are we [Inaudible.].
385	00:16:10	Ankur:	-two things.
386	00:16:12	Michael:	//And if there's two more-
387	00:16:12	Ankur:	//Out of that two-
388	00:16:13	Romina:	//We're doing just two colors. We're doing two colors.
389	00:16:14	Jeff:	Yeah, just do- No, we're- Yeah.
390	00:16:16	Michael:	//If you have like three things, right
391	00:16:17	Romina:	//To explain it, maybe you want to do three different colors?
392	00:16:18	Jeff:	No. Yeah, all right, maybe we can do that. All right, how you saying this?
393	00:16:22	Ankur:	There's red, white and blue, right?
394	00:16:25	Romina:	OK.

395	00:16:26	Ankur:	You take, if red goes over here, that means you only have, with red there could go either go white and blue.
396	00:16:32	Romina:	Mm hm.
397	00:16:33	Ankur:	Like it's each one of those three goes with two more. You know what I mean? There's three things-
398	00:16:40	Michael:	You could see how you got this.
399	00:16:41	Ankur:	-here and then there's two things here.
400	00:16:39	Michael:	You can say you have-
401	00:16:40	Jeff:	All right, yeah.
402	00:16:41	Ankur:	Each one of those, those three goes with //two other.
403	00:16:42	Jeff:	//Those three things go with-
404	00:16:43	Romina:	//Oh OK, like with our line thing.
405	00:16:44	Ankur:	//So it's three times two.
406	00:16:45	Jeff:	All right.
407	00:16:45	Romina:	Like our line thing.
408	00:16:47	Michael:	Or you could say like you have two more colors to add on. So you could do, you could make these into two different combinations.
409	00:16:52	Ankur:	Yeah.
410	00:16:53	Michael:	So that's two.
411	00:16:53	Jeff:	Yeah. That's- Yeah, that's why. All right.
412	00:16:54	Michael:	That's like times. That's why you multiply.
413	00:16:55	Ankur:	That's how you-
414	00:16:56	Michael:	That's just why. All right? Don't ask us anymore.
415	00:16:59	Jeff:	All right, so then, all right. Uh, //Researcher 1.
416	00:17:00	Romina:	//Researcher 1. [Romina laughs.]

417	00:17:03	Jeff:	All right, I think we're good with this.
418	00:17:06	R1:	I'll stay here. Explain it to me on the board.
419	00:17:07	Jeff:	All right, the reason- here, Ankur.
420	00:17:10	Ankur:	Just do it; you're right there. You’re standing.
421	00:17:11	Romina:	You could just say it.
422	00:17:13	Jeff:	Um, just do it with three colors?
423	00:17:15	Ankur:	Yeah.
424	00:17:16	Jeff:	All right, say you have three colors, red, white and blue. Uh, here you do it.
425	00:17:21	Ankur:	Yeah, one of those colors goes in the first.
426	00:17:22	Jeff:	All right.
427	00:17:23	Ankur:	One of those colors goes in the first spot.
428	00:17:24	Jeff:	So, say you have your three spots. Say red goes in the first one, all right? Then you could do-
429	00:17:28	Ankur:	Either one of them-
430	00:17:29	Romina:	Draw the line to the white and the blue.
431	00:17:31	Ankur:	One, one color goes in the first spot, so there's two colors left. So there's three different colors that can go in the first spot and each of those colors can go with two other colors.
432	00:17:39	Jeff:	Two other ones. So this is either going to be a white and blue or a blue and a white. Right? And then- Or the white could to the first thing and this is going to be one of the two other colors or the blue's going to go here and it's going to be, the other two are going to be the combination either way of the other one. So that's why you multiply.
433	00:17:57	Romina:	Make that a B.
434	00:17:57	Jeff:	I used to have a Band-Aid on and now I can get the chalk to stick to my finger.

435	00:18:00	Michael:	It is impressive, huh? [Romina laughs.]
436	00:18:03	R1:	What does this have to do with the towers and what you were showing me about a plus b to the n?
437	00:18:11	Michael:	Well, you want- you asked us why you multiply.
438	00:18:12	R1:	And why, why they would be-
439	00:18:13	Ankur:	We just answered why we multiply.
440	00:18:14	Michael:	Yeah.
441	00:18:15	Jeff:	Yeah. We're not there yet.
442	00:18:16	R1:	OK.
443	00:18:17	Romina:	We're still working on that.
444	00:18:19	Jeff:	Yeah, all right.
445	00:18:20	Michael:	All right. So that's why you multiply.
446	00:18:22	Jeff:	All right. Moving on. So that's why that's three factorial. So that's, all right, that's good.
447	00:18:24	Romina:	That's all your combinations right there.
448	00:18:27	Jeff:	Yeah. All right. All right, now we're going to put that number over, um, n minus x, um-
449	00:18:41	Romina:	Factorial.
450	00:18:42	Michael:	Explain that part.
451	00:18:44	Jeff:	All right. This, the n would be the number you were-
452	00:18:49	Michael:	You want to, you know you're choosing from. Like, let's say two. This is three choose two. You want to know how many different places you could put those three.
453	00:18:55	Jeff:	Yeah. So that's where the n comes in.
454	00:18:57	Michael:	So you-

455	00:18:58	Jeff:	So you're going to take a number so that would be, that would be three and the same reason the three's up there, it's coming down here.
456	00:19:03	Michael:	Minus, minus the x.
457	00:19:04	Jeff:	Minus-
458	00:19:06	Michael:	Then it'll give you one.
459	00:19:07	Jeff:	Wait, the reason you subtract, that's why you're raising the, how come the x is there? Because you're raising it to two, um. That's it. Right?
460	00:19:17	Michael:	Right.
461	00:19:17	Jeff:	That's why it's there.
462	00:19:18	Romina:	And then-
463	00:19:18	Jeff:	The x.
464	00:19:18	Romina:	Multiply-
465	00:19:19	Jeff:	And then that subtracted will give you, will give you-
466	00:19:24	Michael:	If this was, if this was a higher number like five choose two, you, that n minus x would be like a three and-
467	00:19:30	Jeff:	And the factorial-
468	00:19:32	Michael:	Those-
469	00:19:32	Jeff:	-will eliminate all the other ones-
470	00:19:34	Michael:	Yes, and those-
471	00:19:34	Jeff:	-that you don't want.
472	00:19:35	Michael:	Those three, it doesn't, you don't want to know where, It doesn't matter where they are. That's why you want, you want, you know, eliminate them. Because you only, you're only worrying about the two. How many different combinations that you could put those two in. Cause it's five choose two. You're only worried about like the, the two. Like I said, the people on the

			line. Five people on the line, you want to know how many different places you could put those, those two people.
473	00:19:59	Jeff:	All the other ones where-
474	00:20:00	Michael:	Now there's going to be, there's going to be a lot of, a lot of repeats because you're also going to count by those other three people where they're going to be and you're not worried about those other three people.
475	00:20:08	Jeff:	So that only makes-
476	00:20:10	Michael:	So that's, that's why you would divide, to get rid of the, to get rid of them.
477	00:20:12	Jeff:	To subtract them.
478	00:20:15	Michael:	No, that's divide. Why divide that, n minus x?
479	00:20:18	Jeff:	Oh, that's the way. All right. All right. OK, all right.
480	00:20:22	Michael:	And so, you say the next part. I don't know. [Inaudible.]
481	00:20:24	Jeff:	All right. And, why, why do we want-
482	00:20:26	Michael:	I don't know. I don't, I, no it's times, huh.
483	00:20:28	Romina:	Times.
484	00:20:29	Jeff:	Actually, that was supposed to be another one in there. Why is, why-
485	00:20:37	Romina:	Didn't, didn't you guys say something about repeats?
486	00:20:40	Michael:	Yeah, that's what Bob said. I don't know. I don't trust that kid.
487	00:20:41	Romina:	That gets like the repeats out.
488	00:20:44	Michael:	But it worked. It works. That's all.
489	00:20:45	Jeff:	All right, we don't know where the, the final x comes from.
490	00:20:47	R1:	Why don't you, um-
491	00:20:47	Michael:	Why don't we think about it?
492	00:20:48	Jeff:	Work on it?

493	00:20:49	R1:	You need to work out a piece of the problem and see if you can tell me. Ankur's not convinced. He's looking at me, not being convinced.
494	00:20:55	Jeff:	What, what um-
495	00:20:56	Michael:	You're not convinced, Ankur?
496	00:20:57	Jeff:	Like I, I mean how-
497	00:20:59	R1:	Are you convinced, Ankur, about this?
498	00:21:00	Jeff:	Yeah but say, all right, say we're doing five choose two, right, with this. Then we go five factorial. Which is what?
499	00:21:07	Michael:	That'll give you all the combinations they can put everybody in.
500	00:21:09	Jeff:	Uh, twenty times three.
501	00:21:11	Ankur:	OK. Sixty.
502	00:21:12	Jeff:	Would be sixty times two.
503	00:21:14	Ankur:	One-twenty.
504	00:21:14	Jeff:	One-twenty? That would be; it's one-twenty, right, Romina?
505	00:21:18	Romina:	Yeah.
506	00:21:20	Jeff:	We're faster than the calculator, around here. [Romina laughs.] We're good like that. So that'd be one-twenty.
507	00:21:24	Michael:	And, and if you're doing choose two, obviously there's going to be a lot of times where those two are going to be in the same spot as the other three are going to be-
508	00:21:30	Romina:	What are you doing, five choose two?
509	00:21:31	Michael:	-you know, I guess moving around different spots.
510	00:21:31	Jeff:	Yeah.
511	00:21:31	Michael:	That's why you want to get rid of the, the n minus x thing.
512	00:21:35	Jeff:	Yeah, we got, that makes sense.
513	00:21:36	Michael:	Yeah, that, that makes sense to you?

514	00:21:37	Jeff:	That, that part right here, is this all good? Up to this point? Do you understand why this is all happening?
515	00:21:44	R1:	I'm waiting for the whole thing.
516	00:21:47	Michael:	Whole thing? Oh we're not done with that yet.
517	00:21:49	Jeff:	Then, um, then you multiply. Well, at this point here you have three.
518	00:21:54	Romina:	That's six.
519	00:21:54	Jeff:	Yeah, it's six. So you have one-twenty over six times five factorial.
520	00:22:03	Romina:	No isn't it-
521	00:22:03	Michael:	Oh I think its the repeats-
522	00:22:04	Jeff:	Or-
523	00:22:04	Michael:	Would, would be like-
524	00:22:05	Romina:	Isn't it three factorial, two factorial?
525	00:22:07	Jeff:	Three factorial. Oh two, oh, it's act-, all right, yeah. Two.
526	00:22:10	Michael:	Yeah, I guess the, the x-
527	00:22:12	Jeff:	That's the number you were raising-
528	00:22:14	Michael:	That x.
529	00:22:15	Jeff:	-and, and five choose x, say and there was-
530	00:22:15	Michael:	That's what. Since you- Mm hm.
531	00:22:16	Jeff:	And this was-
532	00:22:18	Ankur:	I get it. I get it. I get it. I get it. [Romina laughs.]
533	00:22:21	Michael:	I, I got it now.
534	00:22:23	Jeff:	Like that.
535	00:22:25	Michael:	All right, then the last number would be-
536	00:22:26	Jeff:	Because this just gives you the number.

537	00:22:28	Michael:	You have- Yeah.
538	00:22:29	Jeff:	You're going to multiply by the number.
539	00:22:29	Michael:	Those, those, you want to get rid of those. The, all the combinations that the three are moved around and those, those two aren't.
540	00:22:33	Jeff:	Yeah, they-
541	00:22:35	Michael:	But then those two themselves will be repeat-
542	00:22:36	Jeff:	Yeah-
543	00:22:37	Michael:	You will be mixed up.
544	00:22:38	Jeff:	Be repeating that's what you- that's why you
545	00:22:39	Michael:	That's why you want to get rid of that, too.
546	00:22:40	Jeff:	Exactly. And then, so that would be just two.
547	00:22:42	Michael:	Yeah.
548	00:22:43	Jeff:	So it would be one-twenty divided by twelve and you get ten. Is that what it is?
549	00:22:55	Michael:	Yeah it is. Do you get like why we divide by the n minus x and the, the x? You know, you, you get that?
550	00:23:07	R3:	I don't get that. Could you [Inaudible.]?
551	00:23:07	Michael:	You don't get that?
552	00:23:08	R1:	Ankur, did you have that?
553	00:23:09	Jeff:	What, what part don't, don't..
554	00:23:10	R1:	I wonder if Ankur has that? I wonder if Ankur could explain.
555	00:23:11	Romina:	I don't think the x [Inaudible.].
556	00:23:15	Michael:	All right. The top thing, the n to the, the n to the, uh, factorial was going to give you how many?
557	00:23:21	Romina:	That's all the combinations.

558	00:23:22	Michael:	That's every single combination.
559	00:23:23	Romina:	I got that. That I got.
560	00:23:24	Michael:	Right? Now you're, you're only worried about them, those two people in that line. So there's going to be some instances where those two people are going to be in the same place and those three-
561	00:23:32	Jeff:	Are the ones changing.
562	00:23:33	Michael:	Will be, you know, will be switch, you know, changing.
563	00:23:34	Jeff:	And that's-
564	00:23:35	Michael:	So that's, that would be the, the three factorial. You want to, you want to get rid of that. You want to get rid of them.
565	00:23:40	Ankur:	Wait, say that again.
566	00:23:41	Romina:	Hold on. Well, we-
567	00:23:41	Michael:	Don't worry about that three, we're doing like five.
568	00:23:43	Romina:	No, we're doing this one so the two-
569	00:23:43	Ankur:	All right, so you have the five minus two, is that what you're explaining on there?
570	00:23:46	Romina:	Five minus two, that's-
571	00:23:46	Michael:	So you have the hundred and twenty different combinations.
572	00:23:46	Ankur:	Yeah.
573	00:23:47	Jeff:	Total.
574	00:23:49	Michael:	All right. But you don't think like when those two people are going to be in these two spots-
575	00:23:52	Jeff:	And everyone else is changing.
576	00:23:54	Michael:	-not those other three.
577	00:23:54	Jeff:	And those are, those are, those make no difference because all we're worried about are where those two people are.

578	00:23:56	Romina:	Oh like when, oh, oh, okay, okay, okay.
579	00:23:58	Michael:	All right, those two people are going to be moving around and it- you know, they're like-
580	00:23:59	Jeff:	These people are going to stay the same and every, all the three people, they're just going-
581	00:24:00	Michael:	-the two people staying in the same place. So that's why you get rid of that.
582	00:24:02	Jeff:	You know, going nuts.
583	00:24:02	Michael:	But then those two people themselves could switch places too.
584	00:24:06	Ankur:	Yeah. [Ankur nods.]
585	00:24:07	Michael:	You know what I'm saying?
586	00:24:07	Ankur:	Um-huh.
587	00:24:08	Michael:	Or if-
588	00:24:08	Ankur:	So then you got to get rid of those, too.
589	00:24:08	Michael:	-there were three that could go on.
590	00:24:10	Jeff:	So that's why you get rid of the three.
591	00:24:11	Ankur:	That's why you do the x factorial
592	00:24:12	Michael:	Then you get rid of the, you know-
593	00:24:14	Jeff:	The other one.
594	00:24:15	Ankur:	Yeah, so you get rid of those.
595	00:24:16	Romina:	OK.
596	00:24:17	Jeff:	And then, then-
597	00:24:17	Romina:	Oh, there you go. That makes sense.
598	00:24:19	Michael:	Because you're not worried about every, each person.
599	00:24:20	Romina:	Just the two.
600	00:24:21	Michael:	Just worry about two, right.

601	00:24:22	Jeff:	Just those two. Exactly.
602	00:24:23	Romina:	Yeah, we all have, I got it. I'm good.
603	00:24:24	Michael:	Extension?
604	00:24:26	R1:	Ankur? Can you explain this because poor Researcher 3 is trying to understand this, and she's not following Michael.
605	00:24:36	Ankur:	Something like, I understood it but-
606	00:24:39	Jeff:	Just go through it dude.
607	00:24:40	Ankur:	All right. The top number is five factorial, that's the total number of possibilities for, for five, for five people.
608	00:24:45	Michael:	One twenty
609	00:24:46	Ankur:	And then the five minus two comes, comes in where you're not worried about everyone, you're just worried about two people at a time. So we need to subtract the five minus two. Those get, that gives you and you do factorial, that gives you all the possibilities of just two people, right?
610	00:25:05	Michael:	No, that gives you
611	00:25:05	Romina:	Three people.
612	00:25:06	Jeff:	No, three extras.
613	00:25:07	Michael:	The three that you don't, you're not worried about.
614	00:25:08	Jeff:	That's going to eliminate everyone except the two people you're worried about.
615	00:25:12	Ankur:	OK. Everyone except the two people you're worried about. And then the x factorial eliminates, except the-
616	00:25:18	Michael:	When the two people-
617	00:25:19	Romina:	Two people, yeah.
618	00:25:20	Ankur:	Yeah. When the two people are switched back and forth when you have the same ones over again. [Romina laughs].
619	00:25:25	Jeff:	OK, [Inaudible.].

620	00:25:26	R3:	It's, it's getting better. It's getting better. So they switch back and forth you're saying and with your fingers. I think I'm getting switch back- So could you give me an example?
621	00:25:38	Ankur:	Like when you have, when you have like person A and, over here.
622	00:25:41	Michael:	You want to stand up and show them?
623	00:25:41	Ankur:	And person B over here. And then you have person B and person A.
624	00:25:42	Michael:	You want to be in a line and we'll show them?
625	00:25:43	R1:	Michael, start from the beginning very slow.
626	00:25:45	Michael:	All right. You have five people.
627	00:25:46	R1:	Stand up and show us.
628	00:25:47	R3:	Stand up and show us.
629	00:25:50	Jeff:	All right, I'm going to sit in your seat cause I can't see.
630	00:25:51	Michael:	I'm going to write it nice and clear so you all can see. All right. You got five people, in a line. You agree with me that's how many different combinations you can put those five people.
631	00:26:10	R3:	That part I understand.
632	00:26:11	Michael:	All right.
633	00:26:11	R3:	I understood the multiplication that you showed.
634	00:26:14	Michael:	Now in, you're only worried, you want to know how many different places you can put those two people. All right? So, in all the combinations you're going to have, they're going to be repeated a lot. A lot. When you have like, the two people in a certain place and you know, those three. If the three are, are like this. And then one of them switches, that's another combination. And you get a lot of repeats like that.
635	00:26:37	R3:	Oh, I see. OK.
636	00:26:39	Michael:	So by eliminating that, you eliminate the combinations that repeat by the three people moving around.

637	00:26:45	R3:	Uh-hum.
638	00:26:45	Michael:	Then let's say you just have those two people in, in any given combination. If, if one, if this guy switches the place with this guy it's the, they're different combinations, but in this we're not worried about where they are. We just, you understand?
639	00:27:00	R3:	Mm hm.
640	00:27:02	Michael:	That's why we get rid of the, the two factorial to, to, uh, eliminate the amount like as many times as you could, as many combinations as you could put those two people. Right? Like the three would, would be to eliminate the combinations you could put those three people that you're not worried about. Then the two, they would repeat because those people too, they move around. They, they could, they move around in the, in the line also. And then when, when, when you're done with all that, you just get, um, you get how many places you can just put that two. Like you're not worried if, like you don't care who they are. You don't care like if this guy has a switch with this guy. You understand like why you would eliminate, how that eliminates.
641	00:27:41	R1:	OK. I don't want to think of people. I want to think of the tower now. Isn't that what Jeff said? And now I'm thinking of towers that are five tall?
642	00:27:50	Jeff:	Yeah. You can, we just-
643	00:27:51	R1:	And we're talking of those that have two reds?
644	00:27:54	Jeff:	Yeah. Well. [Inaudible.]
645	00:27:54	R1:	Explain it to me with that.
646	00:27:55	Jeff:	All right. Say, say we're doing, we're doing towers that were, were five tall. Towers of five tall with two different colors in it. Then that's the total amount of possibilities is the five factorial that you could have. All right, in, with, with five high with the combinations. So that's where, that's the five factorial on top. Then the three factorial on the bottom would be five different, five different spots minus the two spots that you're concerned about, leaving you with the three other spots-

647	00:28:26	Romina:	You could say-
648	00:28:26	Jeff:	-that you don't care about. That's going to eliminate all of them.
649	00:28:29	Romina:	That's like, if you say like the reds. Let's say reds are our two colors that they stay in the same place, and like-
650	00:28:34	Jeff:	Reds.
651	00:28:34	Romina:	They're. Like yeah, the two stay in the same place and then the other three are just switching while they're in staying in the same place.
652	00:28:39	Jeff:	Yeah, they're staying in the same spot.
653	00:28:40	Romina:	But we're not concerned with them.
654	00:28:41	Jeff:	That's why you're not concerned with those.
655	00:28:43	Michael:	It's going to repeat like six times.
656	00:28:44	Jeff:	Yeah. So that's where the three factorial comes from, and you're multiplying that by the two factorial. Those are what you're-
657	00:28:50	Romina:	That's to say like the first place and the third place and then they just switch.
658	00:28:51	Michael:	Yeah, like- this way
659	00:28:52	Jeff:	Exactly.
660	00:28:54	Michael:	They just don't have a name on them so the, they're the same thing.
661	00:28:56	Romina:	Yeah.
662	00:28:57	Jeff:	And then that's where the bottom number comes from and then you divide them by each other and that gives you what we're looking for.
663	00:29:04	R1:	OK, so I think I follow what you said. But why were we doing this?
664	00:29:09	Jeff:	Uh, you, we don't-
665	00:29:11	Michael:	We were talking about-

666	00:29:12	Romina:	We want, you wanted us to explain choose.
667	00:29:13	Michael:	The choose that we, all right, whoa-
668	00:29:15	Romina:	Which goes back to Pascal's Triangle and see where a plus b-
669	00:29:19	Michael:	Yeah.
670	00:29:19	Romina:	-to the n. And we could figure out the beginning number.
671	00:29:20	Michael:	All right. Over here, you wanted, the a plus b to the n thing, you wanted to know how we got the choose thing. What does that mean?
672	00:29:24	Jeff:	Yeah, how we got the third number.
673	00:29:25	Romina:	Yeah.
674	00:29:26	Jeff:	And that's how we got off to, to here.
675	00:29:30	R1:	OK, so what did that have to do with what you did in class today?
676	00:29:32	Romina:	That's how we would get the number.
677	00:29:33	Jeff:	We were looking at, we were doing this in class today. That's what we were doing. We were looking at a plus b-
678	00:29:37	Romina:	We're going to be-
679	00:29:38	Michael:	It was like in Pascal's Triangle things go like, by that. Like this choose this. Like, um, if you go to the one, three, three, one part of it, it would be, um-
680	00:29:47	R1:	Show me on the board, Michael.
681	00:29:49	Jeff:	Go get 'em, Mike.
682	00:29:52	Michael:	This would be like, all right, this would be like three choose one. How many different places you put that one, that one guy. There's only one place. There's only, oh, I'm wrong. What am I doing?
683	00:30:18	Romina:	That's when you only have like, it's all one color.
684	00:30:20	Michael:	No, there, there's a way it has something to do with- I think that would be three choose zero, I guess. No. All right, and then the

			next one would be three choose three. Obviously three different places.
685	00:30:32	R1:	Three choose what? What was the next one?
686	00:30:34	Michael:	Three choose one. The next would be three choose two, which we just figured that out. There's three. And last one is three choose three. You can only put those three people in those three places. You can't, you know, no more places to put them.
687	00:30:48	R1:	OK so that's really interesting. That's really very interesting. So you've put something else together. I have another question. You could write more rows of that triangle.
688	00:30:58	Michael:	Yeah.
689	00:30:59	R1:	And now you're telling me you can write them as the choose way, you've called that. So can you take, let's say another row or two and show me the addition rule and what it looks like with your new notation.
690	00:31:17	Michael:	You're talking about the addition rule when you
691	00:31:18	R1:	For a particular, for a particular row.
692	00:31:20	Michael:	Add this and this and go like that?
693	00:31:21	R1:	Sure, or three and three, six. Show me what that looks like with that new notation. Do you understand my question?
694	00:31:29	Michael:	Uh, I don't really.
695	00:31:29	Romina:	I don't understand.
696	00:31:29	Ankur:	Instead of writing three you write-
697	00:31:31	R1:	Write your next row, Michael. Now some time ago you, you had a reason. You explained to me-
698	00:31:45	Michael:	Why you add.
699	00:31:46	R1:	Why you add.
700	00:31:47	Michael:	Yeah.

701	00:31:48	R1:	You remember that? You, might, might be useful for folks who haven't heard it to hear it whatever way you want to explain it.
702	00:31:53	Michael:	I don't think I can explain it too good. Um.
703	00:31:55	R1:	Um, you know, however you want to explain it. You've had it a few ways.
704	00:32:00	Michael:	Um, I can't, I can't remember too well. I know why you add, if I explain it, I don't think anyone will understand.
705	00:32:13	R1:	Try.
706	00:32:15	Michael:	I didn't. Didn't I tell you guys like last time I came here?
707	00:32:18	Jeff:	Well, go for it, dude, just-
708	00:32:20	Romina:	You could try.
709	00:32:20	Michael:	You don't have that paper, do you? You can just hand them, hand that out.
710	00:32:21	Romina:	You started talking about toppings. I think something-
711	00:32:24	Michael:	Hand that out instead.
712	00:32:25	Jeff:	Just-
713	00:32:27	Michael:	Um, all right. If, all right, let's go to, let's go to this one. This would be like three different places I guess. And um-
714	00:32:37	Jeff:	Which one are we looking at?
715	00:32:38	Michael:	That one right there. You have three-
716	00:32:41	Jeff:	That would be a plus b to the third.
717	00:32:42	Michael:	All right, let's say you have like, here's a number, all right? Zero means no toppings. One would, this, one would be-
718	00:32:51	Romina:	It would be, one's a topping.
719	00:32:51	Michael:	One would be a topping. So first category is everything with no toppings. And that's, you can't make, that's, that's your number for that one.
720	00:33:01	Michael:	Next would be- There's all the, the ones that have one topping.

721	00:33:12	Jeff:	Right, you got to make that zero at the end. You messed up.
722	00:33:14	Michael:	What?
723	00:33:14	Jeff:	Last one should be a hundred, not a hundred and one.
724	00:33:15	Michael:	I knew that. There's your, um, your three choose one. And there's three different combinations you could put that. Um, I can go on forever doing this.
725	00:33:25	Michael:	But, um, when you have a new, when you add another place, another topping-
726	00:33:34	Jeff:	That could be one or the other, one or the other, one or the other
727	00:33:36	Michael:	So it could be one or the other.
728	00:33:37	Michael:	It could be a zero or a one, a zero or a one, a zero or a one.
729	00:33:38	Jeff:	Yeah. All right.
730	00:33:39	Michael:	So all these threes would either move up a step onto the next category and, uh, have two toppings.
731	00:33:47	Michael:	Or they might stay behind and still only have one if they have the zero.
732	00:33:52	Michael:	So three, three will get a topping, and go to this one.
733	00:33:56	Michael:	And three won't, will stay.
734	00:33:58	Michael:	And obviously this guy's going to get a topping. That's why you add this one.
735	00:34:03	Jeff:	Uh-huh.
736	00:34:03	Michael:	So now this guy's going to have, without toppings. You're going to add a topping onto him. That's going to be one topping. These three with one topping won't get one so, you know-
737	00:34:14	Jeff:	That's their four.
738	00:34:15	Michael:	You put, you can put them in the same category as this one.
739	00:34:17	Jeff:	Yeah.
740	00:34:17	Michael:	That's four.

741	00:34:17	Jeff:	Those are your four.
742	00:34:18	Michael:	And you know-
743	00:34:19	Ankur:	Three.
744	00:34:19	Jeff:	Those three.
745	00:34:20	Michael:	The three that had two toppings won't get any.
746	00:34:23	Jeff:	Yeah. So they'll go to [Inaudible.].
747	00:34:23	Michael:	And you could put them in together with the ones that did get something. That's why you would add. Keep on adding.
748	00:34:28	R3:	What do you mean by toppings?
749	00:34:29	Michael:	Pizza toppings.
750	00:34:30	R3:	Um, for example-
751	00:34:31	Michael:	Like here you would have a choice of three different ones. Here you would have a choice of five and like the ones would be like the mushrooms, the peppers, the whatever, just by going like- The one would indicate you have it or not.
752	00:34:46	R1:	OK. OK. I remember.
753	00:34:48	Michael:	You remember.
754	00:34:49	R1:	I remember this. But now I don't want to think of the numbers in that triangle, I want to think of those as chooses. So for example, let's just take this row. One, three, three, one.
755	00:35:11	Michael:	Mm hm.
756	00:35:13	R1:	All right. If I wrote these as chooses the way you're writing them-
757	00:35:19	Michael:	Three choose zero, three choose one.
758	00:35:20	R1:	This is three choose zero.
759	00:35:21	Michael:	Yeah.
760	00:35:22	R1:	This is three choose one.

761	00:35:23	Jeff:	Choose one. Same thing.
762	00:35:24	R1:	Three choose-
763	00:35:25	Michael:	Two and three choose, then three choose, three choose three.
764	00:35:28	R1:	Right.
765	00:35:29	Jeff:	So that's how you get it. It's like the same thing, cause like three and zero is like three and three, right? And then three two.
766	00:35:32	R1:	OK, so-
767	00:35:34	Michael:	You want us to write the triangle looking like that?
768	00:35:36	R1:	I would, I would, I would like you to do that and then tell me what the general rule is.
769	00:35:41	Jeff:	All right.
770	00:35:42	R1:	With this notation. Do you understand my question? I'll leave you to work on that. So, so I'd like you to write out some of the rows with the triangle, and then I'd like-
771	00:35:51	Michael:	So to use it like, like that. Like the next one would be, uh, four choose zero.
772	00:35:55	Jeff:	Yeah and-
773	00:35:56	Romina:	Four choose -
774	00:35:56	Michael:	The four choose zero then //four choose one, four choose two-
775	00:35:57	Jeff:	//Four choose one, four choose two.
776	00:35:58	Ankur:	Four choose three.
777	00:36:00	Michael:	We're in a bad place.
778	00:36:02	R1:	Right. You probably want to use this.
779	00:36:03	Michael:	Yeah.
780	00:36:03	R1:	So that people can read it.
781	00:36:04	Michael:	Um.

782	00:36:05	Alex:	Ask them your question one more time.
783	00:36:06	R1:	OK, so I'd like you to rewrite your triangle if you like.
784	00:36:09	Michael:	From top to bottom?
785	00:36:10	R1:	Top to bottom.
786	00:36:11	Romina:	Do you want the ones and like-
787	00:36:13	Jeff:	All right. So what-
788	00:36:14	R1:	I want everything-
789	00:36:14	Jeff:	What would-
790	00:36:14	R1:	I want everything written in this form. Do you understand?
791	00:36:16	Ankur:	Uh-huh. [Ankur nods.]
792	00:36:17	Michael:	That's, that's easy.
793	00:36:18	R1:	And then I would like the general row.
794	00:36:19	Jeff:	Is that one?
795	00:36:19	R1:	What would the general row look like? Where you have towers?
796	00:36:24	Romina:	That's a zero, no that's zero choose zero
797	00:36:27	Ankur:	X high.
798	00:36:28	R1:	Something like that.
799	00:36:29	Jeff:	All right, well that's [Inaudible]
800	00:36:30	R1:	Ankur understands. So he can tell you.
801	00:36:37	Romina:	See, like that?
802	00:36:38	Michael:	So it would be, um, like N over, not two over.
803	00:36:42	Ankur:	Well, it would be-
804	00:36:43	Michael:	N choose-
805	00:36:44	Ankur:	It would be-

806	00:36:46	Romina:	Well, and N, make N like your height or something.
807	00:36:49	Jeff:	All right, so say
808	00:36:50	Romina:	N equals height.
809	00:36:52	Jeff:	Well that would-
810	00:36:52	Ankur:	Well, write the X. Write a plus b to the whatever it is next to it.
811	00:36:57	Jeff:	Yeah.
812	00:36:58	Ankur:	You know what I mean?
813	00:36:59	Jeff:	Yeah. So right. That would be a plus b to the-
814	00:37:00	Michael:	This would be nothing, you know, it would be adding.
815	00:37:02	Jeff:	Yeah, zero, one, two. So a plus b to the second.
816	00:37:05	Romina:	Well, it'd be like N over N minus, but what?
817	00:37:07	Jeff:	Yeah, well, a plus b to the second, so it would be if, or a plus b to the nth.
818	00:37:13	Romina:	To the-
819	00:37:14	Ankur:	No, all you need is like-
820	00:37:14	Romina:	n is factorial.
821	00:37:14	Jeff:	It'd be n, n over-
822	00:37:16	Michael:	n, fa-
823	00:37:18	Jeff:	n mi-
824	00:37:18	Romina:	No, that's just like- No, it's not right. I'm just saying like-
825	00:37:21	Jeff:	It would be-
826	00:37:23	Romina:	You would have to multiply it.
827	00:37:24	Jeff:	n over-
828	00:37:28	Michael:	Well, if you had an n, it would be, uh-
829	00:37:30	Ankur:	To the height of the tower which is n, right?

830	00:37:32	Michael:	You'd have a bunch of n's.
831	00:37:33	Jeff:	Yeah, and it'd be over, just z-
832	00:37:34	Michael:	There'd be n plus one n's going this way.
833	00:37:37	Jeff:	Yeah. If-
834	00:37:38	Michael:	All right?
835	00:37:38	Jeff:	it would be n over 0.
836	00:37:39	Michael:	So if n was three, you'd have four n's going this way.
837	00:37:42	Jeff:	Yeah.
838	00:37:42	Michael:	And the bottom numbers would be just going from 0 to
839	00:37:44	Jeff:	Just-
840	00:37:45	Michael:	To-
841	00:37:45	Jeff:	Yeah. Well, yeah.
842	00:37:46	Michael:	0 to n.
843	00:37:50	Jeff:	Exactly.
844	00:37:51	Michael:	To n.
845	00:37:51	Jeff:	To n. Whatever n equals.
846	00:37:53	Romina:	Is there a way to write that, you know how to write over times [Inaudible.]?
847	00:37:58	Ankur:	I guess.
848	00:37:59	Jeff:	Yeah, so how do you, yeah, wait, now that makes sense but, so it would be n over 0 to the nth. And whatever-
849	00:38:08	Michael:	Zero, what are you talking about?
850	00:38:09	Jeff:	Wherever you're looking for.
851	00:38:09	Ankur:	What are you talking about, 0 to the n?
852	00:38:11	Michael:	0 minus n? That would be negative.

853	00:38:13	Jeff:	No, not minus, like that's to whatever n is. n over 0, n over 1.
854	00:38:18	Romina:	1
855	00:38:19	Jeff:	Not divided by like n, 1, n, uh, 2, n, 3.
856	00:38:25	Michael:	That was-
857	00:38:26	Jeff:	All the way until n could be over n. You know what I'm saying?
858	00:38:28	Michael:	Yeah.
859	00:38:29	Jeff:	Not, not divided by. I was using bad, uh, bad looking things there. But-
860	00:38:34	Michael:	Each of those would be a number-
861	00:38:35	Jeff:	Yeah, it's what, 0 to n.
862	00:38:37	Ankur:	And n represents the height of the tower?
863	00:38:39	Romina:	The height of the tower, yup.
864	00:38:42	Michael:	Yeah, n, n represents-
865	00:38:43	R1:	Do you want that divided sign here?
866	00:38:45	Michael:	No.
867	00:38:45	R1:	On that one?
868	00:38:46	Jeff:	No.
869	00:38:46	Ankur:	No. Cross that off.
870	00:38:46	Romina:	No.
871	00:38:46	Jeff:	I was using it to separate, and that was, that's a habit of mine, it looks bad.
872	00:38:49	Michael:	Oh, sorry about that. It would be, uh, as many, it's like height of the tower with two colors. You have two numbers.
873	00:38:59	Jeff:	Yeah. How do you, how are you, can you write that to get this?

874	00:39:04	Romina:	Like that's what I meant. Like I didn't mean factorial. I meant like when we used four first and like three first. I don't know how to write that, though.
875	00:39:10	R1:	So you go 0, 1, 2, 3, dot, dot, dot, up to n.
876	00:39:16	Jeff:	Yeah.
877	00:39:16	Michael:	Mm hm.
878	00:39:17	R1:	Can we get one in the middle there, like n choose r?
879	00:39:22	Jeff:	Like how would you just go right to n choose 3? Or n choose r? Like what- [Researcher 1 nods.]
880	00:39:29	Michael:	What are you talking about?
881	00:39:30	Romina:	Like instead of using 0, 1, 2, 3.
882	00:39:31	Jeff:	r being any number on the bottom.
883	00:39:35	R1:	Because you said n choose x up there.
884	00:39:37	Jeff:	Yeah.
885	00:39:37	R1:	//I just picked what I wanted-
886	00:39:38	Michael:	//Oh, you want uh, you want to do that.
887	00:39:39	Jeff:	Yeah, so, so it would be-
888	00:39:40	Michael:	Um-
889	00:39:40	Ankur:	n choose-
890	00:39:44	Michael:	It would be n.
891	00:39:49	Jeff:	Wouldn't that just be n choose r for whatever r you wanted? Whatever number you wanted up to, as long as it didn't exceed n?
892	00:39:59	Michael:	This, this is different than that. Isn't it? Like this, these are just like a list of numbers. That's, that's just giving you one of these numbers.
893	00:40:05	Jeff:	Uh, you know all that, but I'm saying, if you wanted to write n choose to get a certain number, wouldn't it just be n choose r?

			Like that? And then as long as r doesn't exceed n or it's less than 0 like r-
894	00:40:15	Ankur:	Wouldn't that equal that?
895	00:40:16	Romina:	Yeah, wouldn't it?
896	00:40:16	Michael:	I guess you could write one of those.
897	00:40:18	Romina:	Yeah. Isn't it supposed to equal that?
898	00:40:18	Michael:	Right there.
899	00:40:19	Ankur:	That's- that is.
900	00:40:19	Romina:	It's the same thing.
901	00:40:21	Ankur:	That does.
902	00:40:24	Michael:	You could do that. It's a lot of-
903	00:40:26	R1:	OK, so you've written out three rows and then you wrote out the nth row.
904	00:40:33	Michael:	The reason why, 0, 1, 2, 3 is that number is always going to be that number. It's not, it's never going to change.
905	00:40:35	R1:	[Researcher 1 walks to the board.] OK. I'll buy that. But something in here could be an n choose r. Right? Something in here could be an n choose r.
906	00:40:41	Romina:	Mm hm.
907	00:40:42	R1:	That's what I heard you say, Jeff?
908	00:40:43	Jeff:	Yes.
909	00:40:43	R1:	Sort of a general one in here, n choose x.
910	00:40:46	Jeff:	That's what-
911	00:40:47	R1:	Whatever you choose to use.
912	00:40:47	Jeff:	Yeah, that's what that is. So, yeah.
913	00:40:49	R1:	OK. OK, so this is my question to you. You've written out two rows and you have the third one there.

914	00:40:55	Jeff:	Mm hm.
915	00:40:56	R1:	Maybe somebody will come up here and write these up nicely.
916	00:40:59	Jeff:	Is that what you want?
917	00:41:01	R1:	Yes. Because then I want to ask, I want; after you do that I have a question to ask you. Thanks.
918	00:41:06	Michael:	You want to erase those?
919	00:41:17	Jeff:	You want to make that the line so bad. I know.
920	00:41:19	Michael:	No, don't do that.
921	00:41:30	Ankur:	How far do you want him to go?
922	00:41:34	Michael:	One more.
923	00:41:34	Jeff:	I want to, uh. You want one more for good measure?
924	00:42:02	Michael:	No. Don't worry about it.
925	00:42:03	R1:	Go to the nth one, then.
926	00:42:06	Jeff:	Wouldn't that just be-
927	00:42:07	R1:	Dot, dot, dot.
928	00:42:08	Jeff:	N zero
929	00:42:10	Michael:	Dot, dot, dot, N to the N.
930	00:42:20	R1:	And the last one, Jeff. Is the last one N N?
931	00:42:24	Michael:	Yeah.
932	00:42:25	Romina:	Mm hm.
933	00:42:25	Jeff:	Yeah.
934	00:42:26	R1:	Do you want to put it at the end?
935	00:42:28	Michael:	Yeah, put it at the end, make it nice.
936	00:42:30	R1:	What's the middle one there? What would you, how would you show the middle one?

937	00:42:31	Jeff:	Uh, actually, you could put N, X.
938	00:42:33	R1:	OK. N choose X, N choose N.
939	00:42:40	Jeff:	Those are dots because you can't really make a dot. Now you can.
940	00:42:44	R1:	OK, now, now, show me, show me, while you're up there, Jeff, just show me, uh, an addition rule of Pascal's Triangle. Let's say from, give me an example from the third, fourth row to the fifth row.
941	00:42:55	Jeff:	Fourth row to this?
942	00:42:57	R1:	Fourth row to the fifth.
943	00:42:59	Michael:	The three to the four.
944	00:43:00	Jeff:	Oh, fourth row. All right. Um.
945	00:43:02	R1:	Show me that three plus three is six. Which ones would it be?
946	00:43:07	Jeff:	That would, like you're saying from here [3 choose 1] to here [3 choose 2] going to there [4 choose 2]?
947	00:43:10	Michael:	Uh-huh.
948	00:43:10	R1:	OK, show me. How would you draw your little arrow to shows that?
949	00:43:15	Michael:	This one and that one.
950	00:43:16	Jeff:	Yeah, is that it? Is that all, so that's all you want?
951	00:43:18	Michael:	Yeah.
952	00:43:18	R1:	Is that true? Do you believe that?
953	00:43:20	Jeff:	Yeah.
954	00:43:20	Michael:	Yeah, I believe so.
955	00:43:21	R1:	You all believe that?
956	00:43:22	Romina:	Yeah.
957	00:43:22	Michael:	Uh-huh.

958	00:43:22	R1:	No one could persuade you otherwise?
959	00:43:23	Ankur:	No.
960	00:43:23	Michael:	No.
961	00:43:25	R1:	OK, so you're saying three choose one, plus //three choose two equals four choose two. Right?
962	00:43:27	Jeff:	//Three choose two should equal four choose two.
963	00:43:30	Romina:	Look at all the numbers are added up.
964	00:43:32	R1:	OK. So what's four choose two plus four choose three?
965	00:43:35	Jeff:	Four choose two plus four choose three? That would be, [Michael laughs.] that would be five-
966	00:43:40	Michael:	Oh, five-
967	00:43:41	Ankur:	Five choose-
968	00:43:43	Michael:	Five choose three.
969	00:43:44	Ankur:	Yeah.
970	00:43:46	Michael:	Right?
971	00:43:47	Ankur:	Yeah.
972	00:43:48	Jeff:	Yeah.
973	00:43:48	R1:	I don't know if Romina's convinced.
974	00:43:50	Jeff:	Why is it five choose three?
975	00:43:52	R1:	Yeah, I don't think Jeff is either.
976	00:43:52	Jeff:	Is this here-
977	00:43:53	Romina:	Yeah, I don't really-
978	00:43:53	Ankur:	Because it's, it's always the one on the right.
979	00:43:55	Michael:	Because, see, this guy gets another topping, I guess, so he turns, he would be a two.
980	00:44:01	Jeff:	Uh huh.

981	00:44:02	Michael:	Whatever it is in here. And this guy doesn't, so it stays two.
982	00:44:03	Jeff:	Ah, it doesn't, so that's two.
983	00:44:04	Michael:	So-
984	00:44:05	Jeff:	It wasn't that.
985	00:44:06	Michael:	Because he's moving up, this bottom number's going to change.
986	00:44:09	Jeff:	Oh, all right.
987	00:44:09	R1:	Explain that one more time, Michael, please.
988	00:44:10	Jeff:	Here.
989	00:44:11	Michael:	Um, wherever this guy goes, wherever this guy goes he's going to get another topping because he's moving this way.
990	00:44:15	Romina:	Um-hm.
991	00:44:15	Jeff:	So that turns it into a two.
992	00:44:16	Michael:	So this bottom number's going to change to two.
993	00:44:19	Michael:	This guy's not going anywhere. Cause the bottom number stays the same.
994	00:44:21	Michael:	So it's going to be five. Because you know the next one's going to be five and it, it has to be a two because- You understand why you add? All right. Good.
995	00:44:33	Romina:	I'm with you.
996	00:44:34	R1:	OK, so that's really very interesting. Let me ask you to explain that to Brian for a minute, but we'll let him eat first. Did you eat, Brian?
997	00:44:40	Brian:	No.
998	00:44:40	R1:	Just help yourself. You can watch us.
999	00:44:43	Jeff:	We don't get another break?
1000	00:44:45	R1:	All right, Brian, just eat. You can.
1001	00:44:46	Brian:	I don't think you want to know what I went through.

1002	00:44:48	Ankur:	Well at least you got a tux.
1003	00:44:49	R1:	We're glad you're here.
1004	00:44:50	Brian:	Neither did I. I didn't.
1005	00:44:52	Ankur:	I didn't either.
1006	00:44:52	Romina:	What happened?
1007	00:44:52	Ankur:	[Inaudible.] what happened to my coat.
1008	00:44:53	Brian:	The coat is like fit for a midget. [Break in tape.]
1009	00:44:57	Alex:	Keep going.
1010	00:44:57	Michael:	All right.
1011	00:44:59	R1:	[Side conversation.] OK, sure, why not.
1012	00:45:00	Alex:	OK. Good.
1013	00:45:01	Jeff:	All right. Well, all right.
1014	00:45:02	Ankur:	[Inaudible.] you remember.
1015	00:45:04	Jeff:	All right, we're looking at, we're looking at this right here. You guys got to pay attention to it.
1016	00:45:08	R1:	Erase it better, Jeff, before you start, because-
1017	00:45:09	Ankur:	Yeah, remember and you didn't pay me back for like three months.
1018	00:45:12	Brian:	I had it the whole time.
1019	00:45:14	Ankur:	Yeah, but it cancelled out.
1020	00:45:15	Jeff:	All right. Say we have this row right here. We got um, N choose 0. And over here we have N choose X. And then over here we have N choose N. All right? Then this right here would be- Oh, we're explaining the general addition, the addition rule using this type of, to fill out the triangle. Using chooses to fill out the triangle and this here would be N choose X plus one and then N, N choose X plus two and so on to whatever N equals.

			Right there'd be dot dot- I didn't, I didn't leave enough room. And this here would be X minus one and then-
1021	00:46:02	Ankur:	You did that one man.
1022	00:46:03	Jeff:	What?
1023	00:46:04	Ankur:	Nothing.
1024	00:46:05	Jeff:	That'd be X minus two and so on each way. Right? So it'd be that.
1025	00:46:10	Ankur:	Can I see the row above that?
1026	00:46:12	Jeff:	And the row above this would be N minus one, right? Yeah.
1027	00:46:17	Michael:	Mm hm.
1028	00:46:19	Jeff:	Um, choose zero. This again would be N, N minus one choose X and then-
1029	00:46:29	Michael:	N minus one.
1030	00:46:30	Jeff:	N minus one, N minus one. That's a one. Um, how do you want me to, to- Where do you want me to go from here?
1031	00:46:40	R1:	Well, you know, um, Brian wasn't here, so you might want to give him some background to what you've been doing.
1032	00:46:46	Jeff:	Start at the beginning? We did, we worked for an hour and a half getting to this point. Explaining this, doing this. All right, um.
1033	00:46:54	R1:	But Brian's a quick study.
1034	00:46:54	Brian:	That's what I am.
1035	00:46:56	Jeff:	All right. We did, uh, this is Pascal's Triangle using-
1036	00:47:02	Brian:	The whole choose thing.
1037	00:47:03	Jeff:	-the choose situation. That's what this is.
1038	00:47:04	Michael:	You know how choose works, like one, three, three, one.
1039	00:47:06	Brian:	Yeah.

1040	00:47:07	Jeff:	Yeah.
1041	00:47:07	Michael:	Three choose zero, three choose one-
1042	00:47:08	Brian:	One, four, six-
1043	00:47:09	Michael:	Yeah. It's all like chooses of something.
1044	00:47:11	Jeff:	All right. So, um, I don't- Um, how would you like to, uh, how do you want to do this? How do you want to-
1045	00:47:19	Michael:	We're just-
1046	00:47:20	Jeff:	Well, tell him what we did.
1047	00:47:21	Michael:	-replacing the three in the chooses by N's and X's.
1048	00:47:24	Jeff:	Yeah, exactly. And rather doing, like, uh, rather- Say this is the, uh-
1049	00:47:29	Michael:	If N was three.
1050	00:47:30	Jeff:	Yeah, say if N was the third row, it would be three choose zero. That would give you one.
1051	00:47:36	Ankur:	Like, you know how it's one, three, three, one. Three choose zero gives you one.
1052	00:47:38	Jeff:	Three choose one.
1053	00:47:39	Michael:	That'd be three.
1054	00:47:39	Jeff:	That would give you the three. The three choose two. That would give you the other three. That's equal to three and then three choose three. That equals the other one. And like that's filling out this part of the triangle and so on. And that's what, that's what we're doing now. We went, other stuff we did we did the whole, we found that equation to find out choose.
1055	00:48:01	Michael:	What choose means.
1056	00:48:02	Jeff:	Yeah, we did all that.
1057	00:48:03	Romina:	And choose.
1058	00:48:04	Jeff:	But you missed out on all that. That's the choose equation.

1059	00:48:05	Romina:	That's the choose equals.
1060	00:48:08	Jeff:	And we spent time explaining. That's what we spent the bulk, bulk of the thing, trying to figure out how to explain that. And-
1061	00:48:14	Brian:	What's that little exclamation point?
1062	00:48:15	Michael:	//Factorial.
1063	00:48:16	Romina:	//Factorial.
1064	00:48:16	Ankur:	//Factorial.
1065	00:48:16	Jeff:	Factorial.
1066	00:48:17	Brian:	That's what it is?
1067	00:48:17	Romina:	Yeah.
1068	00:48:17	Jeff:	Yeah.
1069	00:48:18	Brian:	All right.
1070	00:48:18	Jeff:	It was really excited, like N! [Michael laughs]
1071	00:48:20	Romina:	You want to know what this is? That's all the combinations. [Romina points to her paper; refer to Figure J18.] That's minusing. You know how like they're saying-
1072	00:48:26	Brian:	Yeah.
1073	00:48:26	Romina:	-three choose two.
1074	00:48:27	Brian:	Yeah.
1075	00:48:27	Romina:	We don't care about the three, so that's like when the threes are switching, not the twos. And that's when the twos are like in the first place and the third place, and they just switch and nothing else moves.
1076	00:48:35	Brian:	So this-
1077	00:48:35	Romina:	It's basically the same thing.
1078	00:48:35	Brian:	Is this, is that this over this?
1079	00:48:37	Michael:	Yeah.

1080	00:48:38	Romina:	It's N, N factorial over N minus X factorial times X factorial.
1081	00:48:45	Michael:	And that equals N choose X.
1082	00:48:46	Romina:	Like this is when the- the things we don't- No, I'm just saying these are the things that we don't care about when they- they switch and this is when the things we do care about, just switch in the same place and everything stays the same.
1083	00:48:57	Brian:	All right.
1084	00:48:58	Romina:	And that's all of them. [Romina laughs.]
1085	00:49:00	Ankur:	The Reader's Digest version.
1086	00:49:01	Romina:	Yeah.
1087	00:49:01	R1:	What was that, Ankur?
1088	00:49:02	Ankur:	No, I just said like the Reader's Digest version or something. [Romina laughs.]
1089	00:49:05	R1:	The Reader's Digest version?
1090	00:49:07	Jeff:	Yeah. So where, where do you want to go with, with this?
1091	00:49:10	R1:	Well, I want you to show me how the addition rule works in general.
1092	00:49:14	Jeff:	All right. Well that's not much of a problem-
1093	00:49:16	R1:	So you showed me what N minus one choose X-
1094	00:49:17	Michael:	Go from, go from, go from N X and N X plus one.
1095	00:49:19	Jeff:	Wait, this is, this is //[Inaudible]
1096	00:49:21	Ankur:	Yeah, add that in terms of X. Like below it, you know what I mean?
1097	00:49:23	Michael:	Add these two. What are these two going to equal?
1098	00:49:26	Jeff:	All right, well that's gonna be-
1099	00:49:27	Michael:	We want the next-
1100	00:49:28	Jeff:	//N plus one over-

1101	00:49:30	Michael:	//N plus one over-
1102	00:49:30	Ankur:	X plus one.
1103	00:49:33	Jeff:	X plus one?
1104	00:49:33	Michael:	N.
1105	00:49:34	Ankur:	Yeah. I think. Uh-huh.
1106	00:49:37	Jeff:	That's what these two are going to come into?
1107	00:49:39	Ankur:	Mm hm.
1108	00:49:40	Jeff:	Right?
1109	00:49:41	Michael:	Yeah.
1110	00:49:41	Ankur:	Yeah.
1111	00:49:40	Jeff:	And that's cause-
1112	00:49:41	R1:	Can you write it, can you write it as an equation? Just like you wrote three plus three equals six.
1113	00:49:46	Jeff:	Um, that would-
1114	00:49:48	Ankur:	N plus, just that plus that.
1115	00:49:50	R1:	Why don't you do it on the side?
1116	00:49:51	Jeff:	Just N. Oh, would it be-
1117	00:49:51	Michael:	Oh, N choose X.
1118	00:49:52	Jeff:	N choose X, um, plus-
1119	00:49:53	Ankur:	Plus.
1120	00:49:54	Jeff:	-N choose X plus one.
1121	00:49:57	Michael:	Equals that.
1122	00:50:00	Jeff:	Plus one, equals that right there.
1123	00:50:02	R1:	//[Inaudible]

1124	00:50:04	Jeff:	Then, well, that's, that's because this would be gaining an X and going into the X plus one.
1125	00:50:14	Michael:	Yeah.
1126	00:50:15	Jeff:	And this would be losing an X.
1127	00:50:16	Michael:	No, no, not losing, not getting anything.
1128	00:50:16	Ankur:	Staying the same.
1129	00:50:17	Romina:	No.
1130	00:50:18	Ankur:	It's not getting anything.
1131	00:50:18	Jeff:	That would be staying the same and that's-
1132	00:50:19	Ankur:	That's, yeah, the plus that.
1133	00:50:20	Jeff:	-is the X plus one.
1134	00:50:22	Michael:	And the top numbers have changed because you have more.
1135	00:50:24	Jeff:	Because you're adding more things.
1136	00:50:25	Ankur:	One more.
1137	00:50:25	Jeff:	One more-
1138	00:50:27	Michael:	Topping or-
1139	00:50:27	Jeff:	Place
1140	00:50:28	R1:	Say it so Brian can follow it because he wasn't here for the earlier pizza discussion.
1141	00:50:31	Michael:	He follows, you can follow it?
1142	00:50:32	Brian:	I can just sit in the back and watch.
1143	00:50:33	R1:	Go ahead, Brian. Don't be easy on them, Brian, make them work.
1144	00:50:35	Jeff:	What, what we're doing is the next line of the triangle- Remember how today in class you know how the other triangle was one, two-

1145	00:50:40	Brian:	Yeah.
1146	00:50:41	Jeff:	-three, that whole row there? Well, that's the increase in N, and then the X plus one. If you added another topping onto your whole. Say we're doing pizzas.
1147	00:50:50	Brian:	All right.
1148	00:50:51	Jeff:	If you add another topping onto it?
1149	00:50:53	Romina:	You know how we get the triangle and how we go one two one and add those two together.
1150	00:50:56	Brian:	Yeah.
1151	00:50:56	Jeff:	Yeah.
1152	00:50:57	Romina:	That's what we're doing right there.
1153	00:50:57	Jeff:	Yeah. Well, that's what we're doing.
1154	00:50:58	Ankur:	We're just adding it.
1155	00:50:58	Michael:	You know why, do you know why we add, though?
1156	00:50:58	Brian:	That's all you're all doing?
1157	00:50:59	Romina:	That's all we're doing.
1158	00:51:02	Jeff:	We, we were explaining why you add.
1159	00:51:03	Brian:	All right, keep going.
1160	00:51:03	Jeff:	And why you do it, is it cause when you add another topping like onto it, this one- Say the toppings were one and zero.
1161	00:51:10	Brian:	Uh huh.
1162	00:51:11	Jeff:	If it gets a topping, that's why it goes up to the X plus one. And since it doesn't get anything, it'll stay the same. And in this one, it's staying the same, right?
1163	00:51:20	Michael:	Yeah.
1164	00:51:21	Jeff:	And that's why it's going there. Like saying that's the zero.
1165	00:51:25	Brian:	OK.

1166	00:51:26	Jeff:	And going to there. Make sense?
1167	00:51:28	Brian:	Yes. It actually does.
1168	00:51:30	Jeff:	So, so that would be the general addition rule in this case? That's it?
1169	00:51:34	R1:	Are you impressed?
1170	00:51:35	Jeff:	Impressed?
1171	00:51:37	R1:	Mm hm.
1172	00:51:37	Michael:	Not really.
1173	00:51:37	Jeff:	Not really. I don't think we did anything that spectacular.
1174	00:51:42	Michael:	Yeah, that's all.
1175	00:51:43	R1:	Well, you might be.
1176	00:51:44	Ankur:	Nothing more than we ever did before.
1177	00:51:45	R1:	You might pick up a probability book in-
1178	00:51:46	Jeff:	Is this all in-
1179	00:51:47	R1:	-freshman college and see if you recognize this.
1180	00:51:51	Jeff:	I mean, I don't know. It just, just seems like-
1181	00:51:52	Romina:	We just talked
1182	00:51:53	R1:	If someone said to you, why does this work and this is a rule and you've shown me things with factorials, you can probably write those in factorial notations. I bet you could. In fact, I wish someone would do it on the board on the right there. Write that addition statement using factorial notations.
1183	00:52:11	Jeff:	All right. Um, you want to do that? Want to do it?
1184	00:52:14	Michael:	Just that thing real quick?
1185	00:52:15	Jeff:	We're writing this right here?
1186	00:52:16	R1:	Sure.

1187	00:52:16	Jeff:	The addition rule in factorial notation?
1188	00:52:19	R1:	That's another form, isn't it?
1189	00:52:20	Jeff:	Yeah.
1190	00:52:22	R1:	Brian would like to know that, I know he would.
1191	00:52:25	Romina:	Bless you. [Someone says Thanks]
1192	00:52:27	Brian:	Right.
1193	00:52:27	Jeff:	I'm thrilled
1194	00:52:27	Ankur:	Oh, yeah.
1195	00:52:28	Michael:	That whole thing plus-
1196	00:52:31	Ankur:	Plus.
1197	00:52:35	Michael:	Aw this is gonna be a pain.
1198	00:52:39	Michael:	No.
1199	00:52:40	Ankur:	No, it's just N.
1200	00:52:41	Jeff:	Yeah, N factorial.
1201	00:52:42	Michael:	I just, I just saw that. Um.
1202	00:52:48	Ankur:	Over, just do everything it is.
1203	00:52:50	Michael:	N minus X.
1204	00:52:53	Ankur:	X, parenthesis.
1205	00:52:54	Michael:	Plus one.
1206	00:52:58	Ankur:	Yeah. And then add and do the X factorial. Put that all in parentheses.
1207	00:53:04	Jeff:	It's not an X, it's not X. Yeah, there you go. There you go.
1208	00:53:10	Ankur:	No, it's not the top.
1209	00:53:12	Michael:	Yeah, the whole thing.
1210	00:53:13	Ankur:	Plus one? Do you have that plus one on the bottom?

1211	00:53:18	Michael:	Yeah. Equals. Um. [Michael laughs.] Um, this whole thing on the bottom, um.
1212	00:53:30	Ankur:	It's the same, it's the same thing. Just copy it.
1213	00:53:33	Jeff:	Yeah.
1214	00:53:34	Ankur:	N.
1215	00:53:35	Jeff:	N.
1216	00:53:35	Ankur:	Minus X.
1217	00:53:36	Jeff:	Minus X plus, exactly. You know how like intimidating this equation must be, like if you just pick up a book and look at that? There you go. Yeah.
1218	00:53:57	Michael:	There you go. That's what you want, I think.
1219	00:54:03	R1:	Do you all agree?
1220	00:54:04	Jeff:	Yeah. I got chalk all over my pants like Dr. Zabrower.
1221	00:54:11	Michael:	That means like-
1222	00:54:12	Jeff:	That's-
1223	00:54:13	Michael:	It's too confusing?
1224	00:54:14	R3:	Is that the same thing?
1225	00:54:15	Michael:	Yeah.
1226	00:54:15	Ankur:	It is the same thing.
1227	00:54:17	R3:	It is?
1228	00:54:17	Michael:	Yeah. N.
1229	00:54:17	Ankur:	As that. Yeah.
1230	00:54:18	Michael:	This thing, all right, you see how that is that?
1231	00:54:20	R1:	Mm hm.
1232	00:54:22	Michael:	You know how- I'll go up there again.

1233	00:54:27	Jeff:	We just wrote out the, yeah, exactly, we wrote out the equation, how to find N choose, exactly.
1234	00:54:33	Michael:	That's, that's, I guess that's what you want.
1235	00:54:37	Jeff:	Yeah. It's exactly- We just wrote, we instead of writing-
1236	00:54:39	Michael:	You agree with this? Right? So we just wrote, we wrote that-
1237	00:54:45	Jeff:	We wrote it in the, in the form.
1238	00:54:45	Ankur:	In that form.
1239	00:54:45	Michael:	It still doesn't look, it doesn't look too good.
1240	00:54:47	Jeff:	Yeah. It looks kind of mean.
1241	00:54:49	Michael:	We wrote that like that.
1242	00:54:53	R1:	Did you all very carefully check that arithmetic?
1243	00:54:55	Michael:	You think we're wrong?
1244	00:54:57	Ankur:	What, you found an error?
1245	00:54:58	Jeff:	All right. Well what's, what, go to the, uh, write the regular equation down.
1246	00:55:02	Romina:	Here's a-
1247	00:55:02	Ankur:	There it is, right there.
1248	00:55:02	R1:	Why don't you get a piece of paper and-
1249	00:55:04	Jeff:	Where is it?
1250	00:55:05	Ankur:	It's right above N over X.
1251	00:55:05	Michael:	Oh, yeah. Never mind.
1252	00:55:06	Jeff:	All right.
1253	00:55:06	Romina:	You found it?
1254	00:55:06	Jeff:	Yeah.
1255	00:55:06	Ankur:	The first one.

1256	00:55:14	Michael:	There you go.
1257	00:55:17	Jeff:	Yeah, all right.
1258	00:55:19	R1:	You sure?
1259	00:55:21	Michael:	Yeah, I'm sure. You got anything else? Yeah, I guess.
1260	00:55:24	R1:	Did you check it?
1261	00:55:26	Michael:	What do you mean? Is it wrong?
1262	00:55:29	R1:	Now that, that's really, really very frightening.
1263	00:55:32	Michael:	Yeah.
1264	00:55:32	R1:	What do you think? Is that foreboding?
1265	00:55:35	Jeff:	I guess.
1266	00:55:36	R1:	I wonder if there's a way of simplifying it.
1267	00:55:39	Jeff:	Of what?
1268	00:55:39	Michael:	Simplifying it. Hey!
1269	00:55:40	Ankur:	Yeah, you could [Inaudible.]; that's simplifying.
1270	00:55:42	Jeff:	Yeah that's, that's pretty-
1271	00:55:44	R1:	That's a way to simplify it. But you know I see N plus one parenthesis minus parenthesis X plus one. That looks like that could be a little simpler. See that N plus one parenthesis that Michael just put there.
1272	00:55:59	Michael:	Yeah.
1273	00:56:00	R1:	Minus the expression X plus one. Suppose you distributed that minus one.
1274	00:56:07	Jeff:	So you want, all right, so- All right.
1275	00:56:10	Michael:	Why would you want to do that?
1276	00:56:10	Jeff:	So distributing, say over there, right? You'd have, you'd have N plus one minus X minus one factorial?

1277	00:56:19	Romina:	Mm hm.
1278	00:56:20	Jeff:	Um, that would be in, in parenthesis.
1279	00:56:24	Michael:	Oh yeah, yeah, there you go.
1280	00:56:24	Jeff:	And then, well, that-
1281	00:56:27	Romina:	Why don't you get another piece of paper?
1282	00:56:31	Jeff:	So, all right, so it'd be N plus one factorial divided by, um, N plus one in parentheses minus X minus one factorial. All right? And then, well, that's, that's pretty much all you can do there. Then X plus one factorial, so you could actually can, you can cancel out? Can you cancel that out? The X, minus X minus one and the X plus one? Or-
1283	00:57:04	R1:	That's what I'm asking you to think about. Not right, not now necessarily, but, um-
1284	00:57:06	Jeff:	Yeah, can you, I mean, can you cross out factorials or is that the first factorial on the bottom of the one all the way to the right? Does that affect, that's affecting the N plus one too, so can you, are you allowed to cross out like that? Cross these both out?
1285	00:57:20	R1:	What that's a good question. What do you all think?
1286	00:57:22	Jeff:	Well, can we throw in numbers and see?
1287	00:57:25	Romina:	Would we be able to cross out the N plus ones?
1288	00:57:27	Jeff:	Well then what are you left with?
1289	00:57:29	Romina:	Yeah. Yeah. It doesn't-
1290	00:57:30	Jeff:	Factorial divided by factorial?
1291	00:57:33	Michael:	Now wouldn't that just be, uh-
1292	00:57:35	Jeff:	Now I'm saying you could.
1293	00:57:36	Michael:	But now you're talking about simplifying, wouldn't that just be, uh-
1294	00:57:38	Jeff:	Yeah.

1295	00:57:39	Romina:	I don't, would that, this whole thing be-
1296	00:57:41	Jeff:	Yeah then it would be nothing, right?
1297	00:57:42	Ankur:	Plus one.
1298	00:57:43	Romina:	Yeah.
1299	00:57:44	Jeff:	Then that would cross out and that would cross out.
1300	00:57:45	Romina:	You get two factorials.
1301	00:57:47	Ankur:	You can't do that.
1302	00:57:47	Michael:	You know that
1303	00:57:48	Jeff:	Yeah.
1304	00:57:47	Michael:	She's talking about simplifying, and you just like, you know, put that negative in there and it would be just N minus X?
1305	00:57:56	Jeff:	Where? Where's this at?
1306	00:57:57	Michael:	Right at N minus. minus, that one right there.
1307	00:57:59	Romina:	The one all the way to the side.
1308	00:58:01	Jeff:	Oh yeah, and then the, all right, so you, so you do that, N minus X factorial.
1309	00:58:01	Michael:	That. That could be-
1310	00:58:01	Jeff:	N minus. Yeah exactly.
1311	00:58:04	Michael:	Uh, I'm not too good with my uh-
1312	00:58:07	Jeff:	Simplification.
1313	00:58:08	Michael:	Yeah.
1314	00:58:08	Jeff:	Yeah, because that, it would be- You got the plus one.
1315	00:58:11	Michael:	I'm just wondering. Wouldn't you, wouldn't that equal N plus one minus X minus one?
1316	00:58:19	Jeff:	Yes, then the plus one and the minus one-
1317	00:58:19	Michael:	Are gone.

1318	00:58:19	Jeff:	So it would be N minus X factorial.
1319	00:58:20	Michael:	N minus X so-
1320	00:58:21	Jeff:	It'd be N minus X factorial, um, times X plus one factorial? Right? Yeah.
1321	00:58:34	Michael:	A little simpler. I still don't like it though.
1322	00:58:37	Jeff:	Then, but then you could cross out, OK, could you cross out?
1323	00:58:39	Michael:	Which are you talking about?
1324	00:58:40	Jeff:	Up, no, the bottom and the top.
1325	00:58:42	Romina:	The top.
1326	00:58:42	Jeff:	Oh, that's plus one. All right, my bad, I wasn't even paying attention.
1327	00:58:45	Michael:	Anything else to simplify?
1328	00:58:49	Jeff:	Well, if X equals negative one, just-
1329	00:58:51	Ankur:	And can't you do that on the other side too?
1330	00:58:51	Michael:	Um.
1331	00:58:51	Romina::	Um.
1332	00:58:54	Jeff:	That would be, um-
1333	00:58:56	Ankur:	It would be N minus one.
1334	00:58:56	Jeff:	N minus X minus one factorial. No.
1335	00:59:01	Michael:	No, it'll still be the same number.
1336	00:59:02	Jeff:	Yeah. And it'll be X plus one.
1337	00:59:03	Michael:	You want us to do that, do that too? Or don't even bother.
1338	00:59:05	Jeff:	Factorial.
1339	00:59:08	R1:	I'm, I'm impressed that twenty of ten you're doing this arithmetic. Um, you know, of course the next thing to do is to

			learn how to do the algebra of factorials so that you indeed could do the addition.
1340	00:59:23	Michael:	[Inaudible.].
1341	00:59:23	Jeff:	[Inaudible.] the factorial.
1342	00:59:24	R1:	Would you like to know how to do that? Would you like to know how to do the algebra of factorials? I bet you know how to do a little bit already. I'll just show you one thing that I know you know and I'll leave you to think about this because everyone is getting tired, but let's just take something like this, right? Six choose two, right? And you know, you, you told me you could write that how? As-
1343	00:59:55	Michael:	Um, six factorial over-
1344	00:59:57	R1:	Six factorial.
1345	00:59:59	Michael:	Three fact, four factorial times two factorial.
1346	01:00:03	R1:	Times two factorial, right?
1347	01:00:05	Romina:	Mm hm.
1348	01:00:06	R1:	And you know what six factorial is, right? Six times five.
1349	01:00:11	Michael:	Times one-twenty.
1350	01:00:12	Jeff:	Thirty. Yeah.
1351	01:00:13	R1:	I'm not going to do that though. I don't like to. I don't like to do multiplication. I'm very lazy. I'm just going to write six times five times four factorial. Is that okay?
1352	01:00:21	Jeff:	That's, that's simplifying is great, then you can- [Students all talk at once.]
1353	01:00:24	R1:	But can I do that?
1354	01:00:26	Romina:	Yeah.
1355	01:00:26	Michael:	And then you could cross out the four factorials and-
1356	01:00:27	Romina:	Oh.

1357	01:00:28	R1:	Oh, then I can cross out the four factorials.
1358	01:00:28	Jeff:	Oh, all right, that makes sense.
1359	01:00:29	R1:	Right?
1360	01:00:31	Jeff:	So you just get thirty divided by, you get thirty divided by two.
1361	01:00:33	R1:	Yeah. Look at all the time that will save you in an SAT question.
1362	01:00:35	Jeff:	That'd be big.
1363	01:00:37	R1:	But, but if you think about this-
1364	01:00:39	Jeff:	She broke, she broke it down farther.
1365	01:00:40	Romina:	Oh yeah she just-
1366	01:00:42	Jeff:	Like rather than say you have six factorial-
1367	01:00:43	Ankur:	Mm hm.
1368	01:00:43	Jeff:	She broke it down until she got a number that she got that she wanted.
1369	01:00:45	Romina:	She had two numbers.
1370	01:00:47	Jeff:	That matched the number on the bottom.
1371	01:00:48	Ankur:	All right. Yeah.
1372	01:00:50	Jeff:	Then you end up like with the two factorial and then cross out and that's thirty over the two factorial and that's two. So it's just fifteen..
1373	01:00:51	Michael:	But then it would probably be even longer than that. Cause if N is a big number-
1374	01:00:55	R1:	Does it matter?
1375	01:00:59	Michael:	-you'd have to write, you would have to write N times N minus one times N minus

Appendix B: Coding of the transcripts for the semiotic models.

Legend

$\boxed{}$[B1] Visual manipulative $\boxed{}$[B2] Quantitative systems $\boxed{}$[B3] Symbolic notations

Line	Time	Talker	Transcripts
1	00:00:00	__	[Romina, Jeff, and Michael are sitting at a table talking with Researcher 1. At the start, the display says 02:01:01.27. No sounds can be heard. Researcher 2 walks across the room, opens the door, and goes out. The camera shows the rest of the room. Researcher 2 returns.]
2	00:01:05	R1:	-mentioned some of what went on. I have, I don't have a clue. Can you sort of tell me about it and how some of you suggested it's connected to other things you had done? I'm really curious. Feel free to use the board and show me and tell me.
3	00:01:16	Jeff:	Well.
4	00:01:17	R1:	But I don't know what went on.
5	00:01:17	Romina:	When, when we came up with that thing that almost was like the $\boxed{\text{Pascal's Triangle}}$[B2] What was it with $\boxed{e}$[B3]? What were we doing?
6	00:01:21	Jeff:	It was, um-
7	00:01:22	Romina:	Ten, $\boxed{\text{ten percent of a hundred}}$[B2].
8	00:01:25	Jeff:	Is that what it was?
9	00:01:26	Romina:	Yeah.
10	00:01:28	Jeff:	Um, I'm not sure. I don't know.
11	00:01:31	Romina:	Can we have a calculator? Are we allowed to have one?
12	00:01:33	R1:	Sure. Hope you know where they are.
13	00:01:36	R4:	Yeah.

14	00:01:37	R1:	You may want to have them around anyway.
15	00:01:43	Jeff:	This could be our first time ever using calculators.
16	00:01:45	Romina:	Yeah.
17	00:01:46	R1:	Wow, first time.
18	00:01:46	Jeff:	We usually-
19	00:01:46	R1:	You never use them?
20	00:01:47	Romina:	Thank you.
21	00:01:48	R3:	You're welcome.
22	00:01:52	Michael:	Are there any games on this?
23	00:01:58	Jeff:	We said one times or one minus[B3].
24	00:01:59	Romina:	What, what am I doing?
25	00:02:00	Jeff:	Oh that's good right there.
26	00:02:01	Romina:	Yeah. [Romina laughs.]
27	00:02:02	Jeff:	One. Was it one minus one over a hundred[B3]?
28	00:02:02	Romina:	Hm.
29	00:02:06	Jeff:	Hundred raised to[B3]-
30	00:02:11	Romina:	OK, why am I- Oh, okay. I didn't know what I did wrong.
31	00:02:14	Jeff:	Oh.
32	00:02:17	Romina:	Wasn't it like, weren't we doing this?
33	00:02:18	Jeff:	Yeah, that's what it was.
34	00:02:20	Romina:	Yeah.
35	00:02:21	Michael:	That's this- [unintelligible; chair is moving].
36	00:02:23	Romina:	OK, this is scary. Look when we were- cause we were discussing like percentages[B2]. And, uh, like an increase and we

			did a hundred and we took ten percent of it[B2] and that's one two one, that's one three three one[B2], and, you know, that is-
37	00:02:38	Jeff:	Yeah, we kept going it-
38	00:02:39	Romina:	It doesn't come out yeah. After a while it goes[B2].
39	00:02:41	Jeff:	It kind of makes you think. After a while it stops[B2], but we were, uh-
40	00:02:44	Romina:	We really thought that was it, look.
41	00:02:45	Jeff:	We were into it.
42	00:02:48	R1:	Oh. So what does it mean?
43	00:02:50	Jeff:	Uh, we didn't, we didn't know.
44	00:02:51	Romina:	We didn't know because then it stops[B2], though.
45	00:02:53	Jeff:	Yeah, but it was interesting for, for a while.
46	00:02:55	Romina:	While it was going on it was very[B2]-
47	00:02:57	Jeff:	We were kind of, uh-
48	00:02:59	Michael:	Are we going to [Inaudible.]?
49	00:03:00	Jeff:	But um, what was the question? What were you-
50	00:03:02	Romina:	We wanted to know what we did in class today.
51	00:03:04	Jeff:	Um, we were looking a lot at, at working at e and, and the equation for it[B3].
52	00:03:09	Michael:	And how it, how it, how it connects with ln[B3] and-
53	00:03:11	Jeff:	Yeah, um-
54	00:03:11	Romina:	And we were also trying to find like, you know how we had when we had a plus b to the n[B3]? We want to know what- And we had like numbers before it when we got to big numbers, we want to know, you figure out what the numbers were, like in front of the a, you know, cubed[B2].

55	00:03:24	Jeff:	You know, that's, that's like, um-
56	00:03:25	R1:	You could use the board too.
57	00:03:26	Jeff:	Uh, we just- Like if you were looking, if we were looking for like a plus b[B3]-
58	00:03:33	Romina:	To the tenth[B3].
59	00:03:33	Jeff:	To the tenth[B3] say, um, obviously it- Was the first one ten[B2]? Was it //one a to the tenth and //then ten[B2]-
60	00:03:39	Michael:	//No it's one[B2], yeah.
61	00:03:40	Romina:	//b[B3]. Oh no, you're right. Sorry.
62	00:03:42	Jeff:	Ten a to the ninth b to the first[B3], right?
63	00:03:45	Romina:	Mm hm.
64	00:03:45	Jeff:	And then how to find out //this number[B2].
65	00:03:47	Romina:	//What the next one was[B2].
66	00:03:48	Michael:	It's forty-five[B2].
67	00:03:49	Jeff:	And it was forty-five[B2] but we were working on how to figure it out when we were doing it. We knew it was the choose thing[B3], whatever that means. The- You do a forty- What was it? Ten choose two[B3]?
68	00:03:58	Michael:	Yeah.
69	00:03:58	Romina:	Uh-huh.
70	00:03:58	Jeff:	You know what I'm talking about? Like, uh, was it N-C-R-[B3] actually that's supposed to be lower case. Two- is that how you do it? Right?
71	00:04:05	Michael:	Yeah, it's one of these things like that.

72	00:04:06	Jeff:	And that equals forty-five[B2] and that's the answer. You know. I'm not, we're not really sure how all this works but it's like, what is that, if-
73	00:04:13	Romina:	We, we learned that, we learned that with her.
74	00:04:15	Jeff:	Yeah. Yeah the- Yeah, we, we went, we went over that, remember that? With the total-
75	00:04:19	Romina:	We tried to go over that. [Romina laughs.]
76	00:04:20	Jeff:	If you have ten different [B1], what was it? Ten different things[B1].
77	00:04:24	Michael:	You have-
78	00:04:25	Romina:	//Ten high. //Ten high[B1].
79	00:04:26	Jeff:	//Ten high[B1]. How many-
80	00:04:26	Romina:	//How many would have two reds, only two reds[B1].
81	00:04:27	Jeff:	//How many would have two, two reds[B1].
82	00:04:29	R1:	One more time.
83	00:04:31	Jeff:	If you had towers// of ten high[B1].
84	00:04:32	Michael:	//If you had like towers[B1].
85	00:04:32	R1:	Towers[B1].
86	00:04:35	Jeff:	If you have towers with ten high //and two colors[B1].
87	00:04:35	Michael:	//How many different places can you put two reds in there[B1]?
88	00:04:36	Jeff:	Yeah.
89	00:04:37	Romina:	Yeah.
90	00:04:37	Jeff:	And like a would be one color and b would be blue, um, b would be the other color[B1]. Then how many would you have, a

			being two in the whole thing[B1]? And that would be forty-five and that's, that's what this number would be[B2].
91	00:04:50	R1:	And these towers are how tall[B1]?
92	00:04:52	Jeff:	Ten tall[B1].
93	00:04:53	Romina:	Ten[B2].
94	00:04:54	Jeff:	That'd be the ten there[B2].
95	00:04:54	Romina:	Mm hm.
96	00:04:54	Jeff:	The two would be the two colors[B2] and then, right?
97	00:04:58	Michael:	No.
98	00:04:58	Romina:	No, two of one color[B2].
99	00:04:59	Jeff:	No, ten would be the two of the one color and the two is implied that there's two, only two colors[B2]? Or-
100	00:05:04	Michael:	The two is the[B2]-
101	00:05:04	Romina:	It's only a plus b[B3].
102	00:05:06	Jeff:	Yeah but in the, when you write this, I mean is it implied that there's only two colors[B2]?
103	00:05:10	Romina:	I believe it is but-
104	00:05:12	Jeff:	Is that, is it implied?
105	00:05:14	Romina:	I, I'll go with the yeah. I don't know. [Romina laughs.]
106	00:05:16	Michael:	Uh, You talking about this?
107	00:05:17	Jeff:	Yeah.
108	00:05:18	Michael:	//No, It, it,
109	00:05:18	Romina:	//Is that like-
110	00:05:19	Jeff:	Is that one, the only one works for-

111	00:05:20	Michael:	It's just like you have ten things[B1] where, how many different places can you put these two[B1]? That's all.
112	00:05:25	Jeff:	Yeah, I know but-
113	00:05:25	Michael:	You know what I'm saying?
114	00:05:25	Jeff:	But if there's, oh, yeah, two[B1]. All right, I see what you're saying.
115	00:05:25	Michael:	That's all.
116	00:05:28	Jeff:	There could be a hundred colors[B2] but it would still-
117	00:05:31	Michael:	Yeah you pick two things out of those ten[B3].
118	00:05:32	Jeff:	Yeah.
119	00:05:33	Michael:	How many different places can you put them[B1]?
120	00:05:34	Jeff:	Put them[B1]. All right. All right.
121	00:05:35	Michael:	Forty-five[B2], I think,
122	00:05:37	R1:	So, so you're saying that's forty-five[B2] and what if I wanted eight red[B1]? Eight red ones or eight a's[B1]?
123	00:05:41	Jeff:	Then it would be ten[B2]-
124	00:05:41	Michael:	Um.
125	00:05:42	Romina:	Ten choose eight[B3].
126	00:05:43	Jeff:	Choose eight[B3], yeah.
127	00:05:44	Michael:	A smaller number[B2].
128	00:05:45	Jeff:	Because //that would be how many different spots can you move those eight of one color in the tower of ten[B1].
129	00:05:47	Romina:	//Now how do you-
130	00:05:50	Michael:	It's forty-five also[B2].

131	00:05:51	R1:	Why?
132	00:05:52	Romina:	Like how do //you, how do you, how do you do that on a calculator?
133	00:05:53	R1:	//How'd you do that so fast Michael?
134	00:05:53	Jeff:	Um.
135	00:05:54	Michael:	No, I just like did it all in my head, that's all.
136	00:05:54	Jeff:	You go to, uh, math.
137	00:05:56	R1:	Tell us how you did it.
138	00:05:57	Michael:	Um.
139	00:05:57	Jeff:	Probability[B2].
140	00:05:58	Michael:	There's a button that-
141	00:06:00	Jeff:	N-C-R[B3].
142	00:06:00	Michael:	Take ten, that button then eight[B2].
143	00:06:02	Romina:	Then math.
144	00:06:03	Michael:	And it comes out forty-five[B2].
145	00:06:05	Jeff:	Why is that the case?
146	00:06:07	Romina:	Hm.
147	00:06:09	Michael:	Well if you take like on the-
148	00:06:10	Romina:	Well because-
149	00:06:12	Michael:	You know how on Pascal's Triangle[B2].
150	00:06:13	Romina:	That's like the two[B1].//You have eight left over[B2].
151	00:06:14	Jeff:	//Oh, cause you could switch them all around[B1]. Is that, is that, I guess you're counting[B2]. //You got, you got, yeah
152	00:06:17	R1:	I don't know. Tell me.

153	00:06:17	Michael:	Cause then you would have-
154	00:06:18	Romina:	//Is that the same thing as that because, like, the eight left over to get to the ten[B2], right?
155	00:06:18	Michael:	//It'll be- It would be the same thing[B1].
156	00:06:22	Jeff:	Exactly.
157	00:06:23	Romina:	It's like almost switching colors[B1].
158	00:06:23	Jeff:	Yeah.
159	00:06:24	Romina:	It'd be like two of the other color[B1].
160	00:06:25	Jeff:	And then, and then, yeah, exactly.
161	00:06:26	R1:	Say that one more time, Romina.
162	00:06:28	Romina:	It'd be two of the other color instead of, like say you started with red for this two[B1]. That was for the reds and then when you//make red eight[B1].
163	00:06:33	Michael:	//That would be the other eight[B1].
164	00:06:34	Romina:	The, like, say the blues have two[B1].
165	00:06:36	Jeff:	And it's seven[B2]. And then obviously three should be the same as that[B2].
166	00:06:39	Romina:	Yeah. Yeah.
167	00:06:47	R1:	So, you're pressing the calculator, you have a new command that gets you those numbers.
168	00:06:53	Romina:	We know how to do it, I mean it's not-
169	00:06:54	R1:	But if you didn't have the calculator?
170	00:06:57	Romina:	We'd write them out.
171	00:06:57	Jeff:	You'd have to write them all out.
172	00:06:58	Michael:	Well, Bob-

173	00:07:00	R1:	Because Alex wants to know how you do that without a calculator.
174	00:07:03	Jeff:	Well, I obviously if the calculator-
175	00:07:04	R1:	Can you, can you help him understand that?
176	00:07:06	Jeff:	Well we would make a, say, tower of ten[B1].
177	00:07:10	Michael:	Can I say something? //All right, um-
178	00:07:10	Romina:	//I don't know.
179	00:07:11	Jeff:	Bob. Yeah go for it.
180	00:07:12	Michael:	No, I'm talking, I just wanted to say that Bob Sidley had like an actual formula to write the equals[B3] and-
181	00:07:17	Jeff:	Do we know, do we know what it is? Or-
182	00:07:19	Michael:	I think so. I don't know if I can remember it.
183	00:07:19	R1:	Why don't I leave you a few minutes and think about explaining this to us.
184	00:07:24	Michael:	It depends on-
185	00:07:24	Jeff:	Well it's not, it's not that hard to explain.
186	00:07:25	Michael:	You remember it? I forgot it.
187	00:07:26	R1:	OK, I'll stay then.
188	00:07:28	Alex:	Well, actually no.
189	00:07:29	Romina:	It's not right.
190	00:07:30	Michael:	All right.
191	00:07:31	R1:	He has a bad memory.
192	00:07:32	Michael:	I got to like do trial and error to see if I can figure it out what it was.
193	00:07:35	R1:	OK.

194	00:07:35	Jeff:	All right, say that's ten[B1]? Then, um, you would just have to find- Say you had, uh, one was one color and two was the other color[B1].
195	00:07:43	Romina:	Why, why don't you show her how to do it for like three[B1]. Show them how we can get all of them[B1]. That you have to draw the tower[B1]. [Romina laughs.]
196	00:07:48	Jeff:	You have two colors[B1]. And out of this tower of three you'd have to find out all the different places you could put those two colors in[B1]. So you could put it there and there[B1]. Or you could put it, uh, there and there[B1]. Or, am I missing any? Yes, I am.
197	00:08:04	R1:	I understand.
198	00:08:05	Romina:	You could just do like-
199	00:08:06	Jeff:	Yeah.
200	00:08:06	Romina:	Do you want to go for another one?
201	00:08:07	Jeff:	No, go for it.
202	00:08:08	Romina:	No, you could just do, you could do like our blue, blue, blue[B2].
203	00:08:12	Jeff:	You gonna write every one?
204	00:08:14	Romina:	Well, there wasn't that many. No I'm just like giving you an example.
205	00:08:15	Jeff:	Yeah.
206	00:08:16	Romina:	And then you just kind of move it through[B1]. And that's how we figure them out when we have to write them out[B1].
207	00:08:23	R1:	So you're saying there's a way of getting these without the calculator[B3].
208	00:08:30	Jeff:	Yeah. And there's a, there's a formula that somebody[B3]-
209	00:08:34	Romina:	Not too-

210	00:08:34	Jeff:	-had come up with[B3] but I don't know, I don't know how it, how it goes. I'm really not sure.
211	00:08:38	Romina:	I've seen it.
212	00:08:39	Jeff:	I don't remember it.
213	00:08:40	Romina:	Yeah, there's some-
214	00:08:41	Michael:	Yeah.
215	00:08:41	Romina:	Something to that effect.
216	00:08:42	Michael:	It was this guy[B3].
217	00:08:43	Romina:	That's it?
218	00:08:43	Michael:	Yeah.
219	00:08:44	Jeff:	It's this right here?
220	00:08:45	Michael:	Yeah.
221	00:08:46	R1:	Why don't you show us up here, Michael.
222	00:08:48	Michael:	Oh, man. I, I didn't come up with this, so don't ask me why [unintelligible, chair moving]
223	00:08:49	R1:	It doesn't matter that you came up with it.
224	00:08:52	Michael:	If you would have like n choose x[B3].
225	00:09:02	Romina:	That's on, that's on the division, n to the x[B3], or is that just like your-
226	00:09:05	Jeff:	That n to the x[B3]?
227	00:09:07	Michael:	No, that's, that's choose to the, that's how you write it I think[B3]. I think that's how you write it.
228	00:09:08	Romina:	That's just, that's what it is?
229	00:09:11	R1:	Do you want an equals sign there[B3]?

230	00:09:13	Michael:	No. That's, that's not in- Yeah. Yeah, I could do that. Times x[B3]. That, that would be the number[B2].
231	00:09:24	R1:	OK. Hi, Ankur. Come on in.
232	00:09:25	Ankur:	Hi. Sorry I’m late.
233	00:09:27	R1:	We’re glad you’re here.
234	00:09:29	Jeff:	Didn’t you go with them?
235	00:09:30	Ankur:	No, I didn’t go with them. I went with Steve.
236	00:09:34	Jeff:	That’s dirty.
237	00:09:34	R1:	Hi, did you eat?
238	00:09:36	Ankur:	No.
239	00:09:37	R1:	Are you hungry? Yes.
240	00:09:39	Ankur:	Yeah I guess so. But it’s all right. It’s all right.
241	00:09:39	Romina:	You can, uh-
242	00:09:39	R1:	I’ll tell you what. I, I-.
243	00:09:41	Michael:	I hate stopping-
244	00:09:58	Jeff:	All right, what are we going to do?
245	00:09:59	Michael:	Oh. Oh yeah, um-
246	00:10:01	Romina:	What, what does that get? That gets you, like-
247	00:10:03	Michael:	That gives you that choose thing[B3].
248	00:10:04	Jeff:	That gives you-
249	00:10:05	Michael:	I don't, I don't know what it means.
250	00:10:08	Romina:	I was working with him one day when he brought that up but he lost me.
251	00:10:11	Jeff:	That was the day that me and, and my table were doing the, uh, finding the square roots without a calculator[B3].

252	00:10:16	Michael:	Yeah, but he did that[B3] before like-
253	00:10:18	Romina:	Not with me.
254	00:10:19	Michael:	-in, in class when he was talking about choosing[B3].
255	00:10:19	Jeff:	Who?
256	00:10:20	Romina:	I was in your group.
257	00:10:21	Jeff:	Oh, gee, you got an eyelash.
258	00:10:24	Michael:	No, in class when he was talking about choosing[B3]. He figured it out. And
259	00:10:27	Jeff:	All right, um, well you figure, say you do, uh, say you're doing three[B1], right? So that would be three times two times one[B2]. That would be each space[B1], I imagine.
260	00:10:40	Romina:	//And x is how many, how many you want of the color[B3]?
261	00:10:41	Michael:	//Yeah, I guess that would be how many combinations[B2].
262	00:10:43	Jeff:	Yeah, that would, because that would give you the total number of[B2]-
263	00:10:45	Romina:	Yeah. Yeah that'd be it, yeah.
264	00:10:46	Jeff:	Total number of combinations[B2]?
265	00:10:48	Michael:	Oh I guess, yeah.
266	00:10:50	Jeff:	All right. So then that would be, say, six factorial[B2]. Divided by[B2]. That would be- Why would you- //why, where does that work?
267	00:10:58	Michael:	//No wait. //This is, that would be-
268	00:11:00	Romina:	//Come on, x is the number we want and x is like the number we want to get, like the choose number[B3].
269	00:11:05	Michael:	That's the, yeah, choosing number[B3].

270	00:11:07	Jeff:	And-
271	00:11:08	Romina:	So I guess this would[B2]-
272	00:11:10	Michael:	He was telling me like this was-
273	00:11:12	Romina:	That would like take away all the, all the ones we would choose[B2].
274	00:11:15	Michael:	He said something about repeats[B2]. One would take away //the repeats[B2].
275	00:11:15	Jeff:	//Yeah, this would this would take away the repeats[B2], right?
276	00:11:17	Michael:	I guess.
277	00:11:19	Romina:	And will this, and this will give you[B1]-
278	00:11:20	Michael:	And this will, this will take away the[B2]-
279	00:11:21	Jeff:	This will take away all the other ones[B2]?
280	00:11:22	Michael:	The other ones that, that you don't care where they are[B1].
281	00:11:22	Romina:	Like the ones that are higher than[B1]-
282	00:11:24	Jeff:	Yeah. Yeah.
283	00:11:25	Michael:	You only care about, you only care about the two that are moving[B1]. Not the other[B1]-
284	00:11:26	Jeff:	Yeah, exactly.
285	00:11:28	Michael:	Not the other, uh, four[B1]. It's just-
286	00:11:29	Jeff:	And then, and X[B3]-
287	00:11:30	Romina:	That makes sense.
288	00:11:32	Jeff:	Yeah. That make, let's see if it works. So say-
289	00:11:35	Romina:	Where's factorial on this[B3]?

290	00:11:36	Jeff:	Where's the exclamation point[B3]?
291	00:11:37	Michael:	Math.
292	00:11:39	Jeff:	Ah.
293	00:11:39	Michael:	Probability 4.
294	00:11:41	Romina:	What are we doing, three or six[B2]?
295	00:11:41	Michael:	Just hit four.
296	00:11:42	Jeff:	All right.
297	00:11:43	Michael:	I mean would you-
298	00:11:45	Jeff:	I don't even know why I did that. That was stupid. Uh, quit, all right. Six divide- Mm. Six divided by[B2]-
299	00:11:53	Romina:	Where is the little, where is that? I don't know.
300	00:11:56	Jeff:	Uh.
301	00:11:57	Michael:	Are you doing. why are you doing the six only[B2]? Oh, because three factorial is six, right[B2]?
302	00:12:01	Jeff:	You got it. That's not six factorial[B2]. Um, divided by three minus, what's X[B3]? We don't know?
303	00:12:09	Michael:	Do two[B2].
304	00:12:10	Jeff:	Minus two[B2].
305	00:12:14	Michael:	Times[B2]-
306	00:12:16	Romina:	You're a lot farther on that than I am.
307	00:12:20	Michael:	You have to put a parenthesis around that whole thing[B2], too. Later on that.
308	00:12:24	Jeff:	Then times[B2].
309	00:12:24	Michael:	No, you got to, at the beginning of that and the end of this thing[B2].

310	00:12:31	Michael:	Get rid of that one there[B2].
311	00:12:35	Jeff:	Don't we have to close that in[B2], though?
312	00:12:36	Michael:	No, you don't have to close that[B2].
313	00:12:37	Jeff:	Oh. All right. So do I have to delete that other one[B2]? No.
314	00:12:39	Michael:	No, leave that like that[B2]. Two[B2].
315	00:12:42	Jeff:	So divided by two factorial[B2]. Um. Let's see.
316	00:12:50	Michael:	Let's see. And do it the other way and it should come out.
317	00:12:53	Jeff:	What way, what was the other way?
318	00:12:55	Romina:	Did, did it work? Four[B2].
319	00:13:00	Michael:	No, it's three[B2].
320	00:13:01	Romina:	Oh.
321	00:13:01	Jeff:	Oh, that's all right.
322	00:13:03	Michael:	Yeah, it works[B2].
323	00:13:04	Jeff:	All right. Yeah that's the case then. All right, so when Ankur's done eating, you can explain all this. And we can explain it to Ankur? Cause you know that's coming. Ankur missed the beginning, so explain it to him.
324	00:13:15	Michael:	There wasn't really a beginning, though.
325	00:13:16	Jeff:	It doesn't matter. We're still going to have to. I didn't know. I forgot that we were, uh, we have to explain things.
326	00:13:23	Michael:	I thought it was just recess or something. [Romina laughs.]
327	00:13:25	Jeff:	Yeah, I know. I'm like that. And, uh-
328	00:13:34	Ankur:	I'm not normally in this class and everyone else is.
329	00:13:37	Alex:	You can go ahead and sit anywhere you want. //That's probably a good spot.

330	00:13:37	Romina:	//I saw you out of the corner of my eye. You confused me. I'm like what is he doing?
331	00:13:44	Romina:	I drove him.
332	00:13:45	Jeff:	Ah.
333	00:13:46	Romina:	I drove him here. I drove Jeff here.
334	00:13:48	Ankur:	By yourself?
335	00:13:48	Jeff:	Yeah righ.
336	00:13:49	Romina:	Well, my dad. [Inaudible]
337	00:13:50	Jeff:	All right, um. Do you want to hear our explanation of, of this and why this works?
338	00:14:00	R1:	That would be really great, but it would help us enormously if you would use the board.
339	00:14:02	Michael:	I wrote that, so you don't have to.
340	00:14:04	R1:	Would you mind, Jeff?
341	00:14:05	Jeff:	All right, I need help because-
342	00:14:06	R1:	They'll help you.
343	00:14:07	Jeff:	I don't want to get stuck. I got stuck up there last time by myself and I was looking like an idiot. All right, the reason why this works- We have no chalk. Right there in the side. All right, the reason why this works, first you get all the total number of this will cover all the total possibilities of your tower[B2].
344	00:14:26	Romina:	//And factorial[B3]. [Inaudible]
345	00:14:27	Jeff:	//Say in terms of a tower[B1]. The factorial right here[B3]. So say you're doing towers of three[B1]. You have the three factorial and that'll cover all of the different combinations that you could put three in with two colors[B3]. All right? [There is a pause. Jeff waves his hands.] Yes?
346	00:14:45	Ankur:	Sounds good.

347	00:14:48	R1:	Why don't you go through and when you're all done I'll ask my question. Just go.
348	00:14:51	Michael:	Like, you should use the explanation like she used. Like the people on the line[B1]. That's better because you have like the first one[B1]. Then you have-
349	00:14:57	Romina:	Two spaces[B1].
350	00:14:58	Jeff:	All right, I'll do people on the line[B1].
351	00:14:59	Michael:	Two spaces[B1]. You have two people left so that's times two[B2].
352	00:15:00	Jeff:	All right, say we're doing us three right here[B1].
353	00:15:01	Michael:	Yeah, on the line[B1].
354	00:15:02	Jeff:	This, us three[B1], um-
355	00:15:04	Romina:	There's three different people to fill in the first spot[B1].
356	00:15:07	Jeff:	Yeah. Then there's, after, then once one goes there, there's only two people left to fill in this spot[B2]. And then-
357	00:15:10	Romina:	So you multiply three and two[B2].
358	00:15:12	Jeff:	Three times two and then once, once someone goes in the other, there's only one person left[B2]. And they get the last spot, so that's times the one[B2].
359	00:15:18	Romina:	And that's everyone.
360	00:15:19	Jeff:	That make more sense?
361	00:15:20	R1:	Well I'm, I didn't mind your other example here.
362	00:15:23	Jeff:	Yeah, I, I just like the okay through the way so I could move- You know, steady progress.
363	00:15:27	R1:	Um. But I guess, so why are you multiplying[B2]?
364	00:15:30	Romina:	We don't like that question.

365	00:15:30	Jeff:	Ah.
366	00:15:31	R1:	You don't like that question.
367	00:15:32	Romina:	No. That, that one gets us all the time.
368	00:15:34	R1:	Why aren't you adding[B2]?
369	00:15:35	Jeff:	Uh, because you don't add[B2]. It's just, you don't do it. [Romina laughs]. There's no adding going on it any where anymore[B2]. That's like out of style. [Romina laughs.]
370	00:15:41	R1:	That's not the answer.
371	00:15:42	Jeff:	I know that doesn't, that doesn't work. Um, you do it because, uh-
372	00:15:52	Michael:	I can't help you on this one.
373	00:15:53	Jeff:	Yeah, I know.
374	00:15:54	Romina:	Yeah, we're-
375	00:15:56	Michael:	That's a good question.
376	00:15:57	R1:	OK, I'll leave you to tell me.
377	00:15:58	Michael:	Why do you multiply[B2]?
378	00:15:59	R1:	You'll figure that out.
379	00:16:00	Romina:	We never know this one.
380	00:16:02	Jeff:	Yeah it's like the //eternal question.
381	00:16:03	Ankur:	//Yeah it's cause, if, if you have three things, there's three things you put here, right[B1]?
382	00:16:03	Romina:	Mm hm.
383	00:16:09	Ankur:	There's red, white and blue[B1]. And then there's only[B1]-
384	00:16:09	Romina:	Uh, are we [Inaudible.].
385	00:16:10	Ankur:	-two things[B1].

386	00:16:12	Michael:	//And if there's two more[B1]-
387	00:16:12	Ankur:	//Out of that two[B1]-
388	00:16:13	Romina:	//We're doing just two colors[B1]. We're doing two colors[B1].
389	00:16:14	Jeff:	Yeah, just do- No, we're- Yeah.
390	00:16:16	Michael:	//If you have like three things[B1], right
391	00:16:17	Romina:	//To explain it, maybe you want to do three different colors[B1]?
392	00:16:18	Jeff:	No. Yeah, all right, maybe we can do that. All right, how you saying this?
393	00:16:22	Ankur:	There's red, white and blue[B1], right?
394	00:16:25	Romina:	OK.
395	00:16:26	Ankur:	You take, if red goes over here, that means you only have, with red there could go either go white and blue[B1].
396	00:16:32	Romina:	Mm hm.
397	00:16:33	Ankur:	Like it's each one of those three goes with two more[B2]. You know what I mean? There's three things[B1]-
398	00:16:40	Michael:	You could see how you got this.
399	00:16:41	Ankur:	-here and then there's two things here[B1].
400	00:16:39	Michael:	You can say you have-
401	00:16:40	Jeff:	All right, yeah.
402	00:16:41	Ankur:	Each one of those, those three goes with //two other[B2].
403	00:16:42	Jeff:	//Those three things go with[B1]-
404	00:16:43	Romina:	//Oh OK, like with our line thing[B1].
405	00:16:44	Ankur:	//So it's three times two[B2].

406	00:16:45	Jeff:	All right.
407	00:16:45	Romina:	Like our line thing[B1].
408	00:16:47	Michael:	Or you could say like you have two more colors to add on[B2]. So you could do, you could make these into two different combinations[B2].
409	00:16:52	Ankur:	Yeah.
410	00:16:53	Michael:	So that's two[B2].
411	00:16:53	Jeff:	Yeah. That's- Yeah, that's why. All right.
412	00:16:54	Michael:	That's like times[B2]. That's why you multiply[B2].
413	00:16:55	Ankur:	That's how you-
414	00:16:56	Michael:	That's just why. All right? Don't ask us anymore.
415	00:16:59	Jeff:	All right, so then, all right. Uh, //Researcher 1.
416	00:17:00	Romina:	//Researcher 1. [Romina laughs.]
417	00:17:03	Jeff:	All right, I think we're good with this.
418	00:17:06	R1:	I'll stay here. Explain it to me on the board.
419	00:17:07	Jeff:	All right, the reason- here, Ankur.
420	00:17:10	Ankur:	Just do it; you're right there. You're standing.
421	00:17:11	Romina:	You could just say it.
422	00:17:13	Jeff:	Um, just do it with three colors[B1]?
423	00:17:15	Ankur:	Yeah.
424	00:17:16	Jeff:	All right, say you have three colors, red, white and blue[B1]. Uh, here you do it.
425	00:17:21	Ankur:	Yeah, one of those colors goes in the first[B1].
426	00:17:22	Jeff:	All right.

427	00:17:23	Ankur:	One of those colors goes in the first spot[B1].
428	00:17:24	Jeff:	So, say you have your three spots[B1]. Say red goes in the first one[B1], all right? Then you could do[B1]-
429	00:17:28	Ankur:	Either one of them[B1]-
430	00:17:29	Romina:	Draw the line to the white and the blue[B1].
431	00:17:31	Ankur:	One, one color goes in the first spot, so there's two colors left[B2]. So there's three different colors that can go in the first spot and each of those colors can go with two other colors[B2].
432	00:17:39	Jeff:	Two other ones[B2]. So this is either going to be a white and blue or a blue and a white[B1]. Right? And then- Or the white could to the first thing and this is going to be one of the two other colors or the blue's going to go here and it's going to be, the other two are going to be the combination either way of the other one[B2]. So that's why you multiply[B2].
433	00:17:57	Romina:	Make that a B[B1].
434	00:17:57	Jeff:	I used to have a Band-Aid on and now I can get the chalk to stick to my finger.
435	00:18:00	Michael:	It is impressive, huh? [Romina laughs.]
436	00:18:03	R1:	What does this have to do with the towers[B1] and what you were showing me about a plus b to the n[B3]?
437	00:18:11	Michael:	Well, you want- you asked us why you multiply[B2].
438	00:18:12	R1:	And why, why they would be-
439	00:18:13	Ankur:	We just answered why we multiply[B2].
440	00:18:14	Michael:	Yeah.
441	00:18:15	Jeff:	Yeah. We're not there yet.
442	00:18:16	R1:	OK.

443	00:18:17	Romina:	We're still working on that.
444	00:18:19	Jeff:	Yeah, all right.
445	00:18:20	Michael:	All right. So that's why you multiply[B2].
446	00:18:22	Jeff:	All right. Moving on. So that's why that's three factorial[B2]. So that's, all right, that's good.
447	00:18:24	Romina:	That's all your combinations right there[B2].
448	00:18:27	Jeff:	Yeah. All right. All right, now we're going to put that number over, um, n minus x[B3], um-
449	00:18:41	Romina:	Factorial[B3].
450	00:18:42	Michael:	Explain that part.
451	00:18:44	Jeff:	All right. This, the n would be the number you were[B3]-
452	00:18:49	Michael:	You want to, you know you're choosing from[B3]. Like, let's say two. This is three choose two[B2]. You want to know how many different places you could put those three[B2].
453	00:18:55	Jeff:	Yeah. So that's where the n comes in[B3].
454	00:18:57	Michael:	So you-
455	00:18:58	Jeff:	So you're going to take a number so that would be, that would be three and the same reason the three's up there, it's coming down here[B2].
456	00:19:03	Michael:	Minus, minus the x[B3].
457	00:19:04	Jeff:	Minus[B3]-
458	00:19:06	Michael:	Then it'll give you one[B2].
459	00:19:07	Jeff:	Wait, the reason you subtract, that's why you're raising the, how come the x is there[B3]? Because you're raising it to two[B3], um. That's it. Right?

460	00:19:17	Michael:	Right.
461	00:19:17	Jeff:	That's why it's there[B3].
462	00:19:18	Romina:	And then-
463	00:19:18	Jeff:	The x[B3].
464	00:19:18	Romina:	Multiply[B2]-
465	00:19:19	Jeff:	And then that subtracted will give you[B2], will give you-
466	00:19:24	Michael:	If this was, if this was a higher number like five choose two, you, that n minus x would be like a three and[B3]-
467	00:19:30	Jeff:	And the factorial[B3]-
468	00:19:32	Michael:	Those-
469	00:19:32	Jeff:	-will eliminate all the other ones[B2]-
470	00:19:34	Michael:	Yes, and those[B2]-
471	00:19:34	Jeff:	-that you don't want[B2].
472	00:19:35	Michael:	Those three, it doesn't, you don't want to know where, It doesn't matter where they are[B1]. That's why you want, you want, you know, eliminate them[B2]. Because you only, you're only worrying about the two[B1]. How many different combinations that you could put those two in[B2]. Cause it's five choose two[B3]. You're only worried about like the, the two[B1]. Like I said, the people on the line[B1]. Five people on the line, you want to know how many different places you could put those, those two people[B1].
473	00:19:59	Jeff:	All the other ones where-
474	00:20:00	Michael:	Now there's going to be, there's going to be a lot of, a lot of repeats because you're also going to count by those other three

			people where they're going to be and you're not worried about those other three people[B2].
475	00:20:08	Jeff:	So that only makes-
476	00:20:10	Michael:	So that's, that's why you would divide, to get rid of the, to get rid of them[B2].
477	00:20:12	Jeff:	To subtract them[B2].
478	00:20:15	Michael:	No, that's divide[B2]. Why divide that, n minus x[B3]?
479	00:20:18	Jeff:	Oh, that's the way. All right. All right. OK, all right.
480	00:20:22	Michael:	And so, you say the next part. I don't know. [Inaudible.]
481	00:20:24	Jeff:	All right. And, why, why do we want-
482	00:20:26	Michael:	I don't know. I don't, I, no it's times[B2], huh.
483	00:20:28	Romina:	Times[B2].
484	00:20:29	Jeff:	Actually, that was supposed to be another one in there[B1]. Why is, why-
485	00:20:37	Romina:	Didn't, didn't you guys say something about repeats[B2]?
486	00:20:40	Michael:	Yeah, that's what Bob said. I don't know. I don't trust that kid.
487	00:20:41	Romina:	That gets like the repeats out[B2].
488	00:20:44	Michael:	But it worked. It works. That's all.
489	00:20:45	Jeff:	All right, we don't know where the, the final x comes from[B3].
490	00:20:47	R1:	Why don't you, um-
491	00:20:47	Michael:	Why don't we think about it?
492	00:20:48	Jeff:	Work on it?
493	00:20:49	R1:	You need to work out a piece of the problem and see if you can tell me. Ankur's not convinced. He's looking at me, not being convinced.

494	00:20:55	Jeff:	What, what um-
495	00:20:56	Michael:	You're not convinced, Ankur?
496	00:20:57	Jeff:	Like I, I mean how-
497	00:20:59	R1:	Are you convinced, Ankur, about this?
498	00:21:00	Jeff:	Yeah but say, all right, say we're doing five choose two, right, with this[B2]. Then we go five factorial[B2]. Which is what?
499	00:21:07	Michael:	That'll give you all the combinations they can put everybody in[B2].
500	00:21:09	Jeff:	Uh, twenty times three[B2].
501	00:21:11	Ankur:	OK. Sixty[B2].
502	00:21:12	Jeff:	Would be sixty times two[B2].
503	00:21:14	Ankur:	One-twenty[B2].
504	00:21:14	Jeff:	One-twenty[B2]? That would be; it's one-twenty[B2], right, Romina?
505	00:21:18	Romina:	Yeah.
506	00:21:20	Jeff:	We're faster than the calculator, around here. [Romina laughs.] We're good like that. So that'd be one-twenty[B2].
507	00:21:24	Michael:	And, and if you're doing choose two, obviously there's going to be a lot of times where those two are going to be in the same spot as the other three are going to be[B3]-
508	00:21:30	Romina:	What are you doing, five choose two[B2]?
509	00:21:31	Michael:	-you know, I guess moving around different spots[B1].
510	00:21:31	Jeff:	Yeah.
511	00:21:31	Michael:	That's why you want to get rid of the, the n minus x thing[B3].
512	00:21:35	Jeff:	Yeah, we got, that makes sense.

513	00:21:36	Michael:	Yeah, that, that makes sense to you?
514	00:21:37	Jeff:	That, that part right here, is this all good? Up to this point? Do you understand why this is all happening?
515	00:21:44	R1:	I'm waiting for the whole thing.
516	00:21:47	Michael:	Whole thing? Oh we're not done with that yet.
517	00:21:49	Jeff:	Then, um, then you multiply[B2]. Well, at this point here you have three[B2].
518	00:21:54	Romina:	That's six[B2].
519	00:21:54	Jeff:	Yeah, it's six[B2]. So you have one-twenty over six times five factorial[B2].
520	00:22:03	Romina:	No isn't it-
521	00:22:03	Michael:	Oh I think its the repeats[B2]-
522	00:22:04	Jeff:	Or-
523	00:22:04	Michael:	Would, would be like-
524	00:22:05	Romina:	Isn't it three factorial, two factorial[B2]?
525	00:22:07	Jeff:	Three factorial[B2]. Oh two, oh, it's act-, all right, yeah. Two[B2].
526	00:22:10	Michael:	Yeah, I guess the, the x[B3]-
527	00:22:12	Jeff:	That's the number you were raising[B2]-
528	00:22:14	Michael:	That x[B3].
529	00:22:15	Jeff:	-and, and five choose x, say and there was[B3]-
530	00:22:15	Michael:	That's what. Since you- Mm hm.
531	00:22:16	Jeff:	And this was[B2]-
532	00:22:18	Ankur:	I get it. I get it. I get it. I get it. [Romina laughs.]
533	00:22:21	Michael:	I, I got it now.

534	00:22:23	Jeff:	Like that[B2].
535	00:22:25	Michael:	All right, then the last number would be[B2]-
536	00:22:26	Jeff:	Because this just gives you the number[B2].
537	00:22:28	Michael:	You have[B2]- Yeah.
538	00:22:29	Jeff:	You're going to multiply by the number[B2].
539	00:22:29	Michael:	Those, those, you want to get rid of those[B2]. The, all the combinations that the three are moved around and those, those two aren't[B2].
540	00:22:33	Jeff:	Yeah, they-
541	00:22:35	Michael:	But then those two themselves will be repeat[B2]-
542	00:22:36	Jeff:	Yeah-
543	00:22:37	Michael:	You will be mixed up[B2].
544	00:22:38	Jeff:	Be repeating that's what you- that's why you[B2]
545	00:22:39	Michael:	That's why you want to get rid of that, too[B2].
546	00:22:40	Jeff:	Exactly. And then, so that would be just two[B2].
547	00:22:42	Michael:	Yeah.
548	00:22:43	Jeff:	So it would be one-twenty divided by twelve and you get ten[B2]. Is that what it is[B2]?
549	00:22:55	Michael:	Yeah it is[B2]. Do you get like why we divide by the n minus x and the, the x[B3]? You know, you, you get that?
550	00:23:07	R3:	I don't get that. Could you [Inaudible.]?
551	00:23:07	Michael:	You don't get that?
552	00:23:08	R1:	Ankur, did you have that?
553	00:23:09	Jeff:	What, what part don't, don't.

554	00:23:10	R1:	I wonder if Ankur has that? I wonder if Ankur could explain.
555	00:23:11	Romina:	I don't think the x[B3] [Inaudible.].
556	00:23:15	Michael:	All right. The top thing, the n to the, the n to the, uh, factorial was going to give you how many[B3]?
557	00:23:21	Romina:	That's all the combinations[B2].
558	00:23:22	Michael:	That's every single combination[B2].
559	00:23:23	Romina:	I got that. That I got.
560	00:23:24	Michael:	Right? Now you're, you're only worried about them, those two people in that line[B1]. So there's going to be some instances where those two people are going to be in the same place and those three[B1]-
561	00:23:32	Jeff:	Are the ones changing[B1].
562	00:23:33	Michael:	Will be, you know, will be switch, you know, changing[B1].
563	00:23:34	Jeff:	And that's-
564	00:23:35	Michael:	So that's, that would be the, the three factorial[B2]. You want to, you want to get rid of that. You want to get rid of them[B2].
565	00:23:40	Ankur:	Wait, say that again.
566	00:23:41	Romina:	Hold on. Well, we-
567	00:23:41	Michael:	Don't worry about that three, we're doing like five[B1].
568	00:23:43	Romina:	No, we're doing this one so the two[B1]-
569	00:23:43	Ankur:	All right, so you have the five minus two, is that what you're explaining on there[B2]?
570	00:23:46	Romina:	Five minus two[B2], that's-
571	00:23:46	Michael:	So you have the hundred and twenty different combinations[B2].

572	00:23:46	Ankur:	Yeah.
573	00:23:47	Jeff:	Total.
574	00:23:49	Michael:	All right. But you don't think like when those two people are going to be in these two spots[B1]-
575	00:23:52	Jeff:	And everyone else is changing[B1].
576	00:23:54	Michael:	-not those other three[B1].
577	00:23:54	Jeff:	And those are, those are, those make no difference because all we're worried about are where those two people are[B1].
578	00:23:56	Romina:	Oh like when, oh, oh, okay, okay, okay.
579	00:23:58	Michael:	All right, those two people are going to be moving around and it- you know, they're like[B1]-
580	00:23:59	Jeff:	These people are going to stay the same and every, all the three people, they're just going[B1]-
581	00:24:00	Michael:	-the two people staying in the same place[B1]. So that's why you get rid of that[B2].
582	00:24:02	Jeff:	You know, going nuts[B1].
583	00:24:02	Michael:	But then those two people themselves could switch places too[B1].
584	00:24:06	Ankur:	Yeah. [Ankur nods.]
585	00:24:07	Michael:	You know what I'm saying?
586	00:24:07	Ankur:	Um-huh.
587	00:24:08	Michael:	Or if[B2]-
588	00:24:08	Ankur:	So then you got to get rid of those, too[B2].
589	00:24:08	Michael:	-there were three that could go on[B1].

590	00:24:10	Jeff:	So that's why you get rid of the three[B2].
591	00:24:11	Ankur:	That's why you do the x factorial[B3]
592	00:24:12	Michael:	Then you get rid of the[B2], you know-
593	00:24:14	Jeff:	The other one[B1].
594	00:24:15	Ankur:	Yeah, so you get rid of those[B2].
595	00:24:16	Romina:	OK.
596	00:24:17	Jeff:	And then, then-
597	00:24:17	Romina:	Oh, there you go. That makes sense.
598	00:24:19	Michael:	Because you're not worried about every, each person[B1].
599	00:24:20	Romina:	Just the two[B1].
600	00:24:21	Michael:	Just worry about two[B1], right.
601	00:24:22	Jeff:	Just those two[B1]. Exactly.
602	00:24:23	Romina:	Yeah, we all have, I got it. I'm good.
603	00:24:24	Michael:	Extension?
604	00:24:26	R1:	Ankur? Can you explain this because poor Researcher 3 is trying to understand this, and she's not following Michael.
605	00:24:36	Ankur:	Something like, I understood it but-
606	00:24:39	Jeff:	Just go through it dude.
607	00:24:40	Ankur:	All right. The top number is five factorial, that's the total number of possibilities for, for five, for five people[B2].
608	00:24:45	Michael:	One twenty[B2]
609	00:24:46	Ankur:	And then the five minus two comes, comes in where you're not worried about everyone, you're just worried about two people at a time[B2]. So we need to subtract the five minus two[B2].

			Those get, that gives you and you do factorial, that gives you all the possibilities of just two people, right[B2]?
610	00:25:05	Michael:	No, that gives you[B2]
611	00:25:05	Romina:	Three people[B2].
612	00:25:06	Jeff:	No, three extras[B2].
613	00:25:07	Michael:	The three that you don't, you're not worried about[B1].
614	00:25:08	Jeff:	That's going to eliminate everyone except the two people you're worried about[B1].
615	00:25:12	Ankur:	OK. Everyone except the two people you're worried about[B1]. And then the x factorial eliminates, except the[B3]-
616	00:25:18	Michael:	When the two people[B1]-
617	00:25:19	Romina:	Two people, yeah[B2].
618	00:25:20	Ankur:	Yeah. When the two people are switched back and forth when you have the same ones over again[B1]. [Romina laughs].
619	00:25:25	Jeff:	OK, [Inaudible.].
620	00:25:26	R3:	It's, it's getting better. It's getting better. So they switch back and forth you're saying and with your fingers[B1]. I think I'm getting switch back[B1]- So could you give me an example?
621	00:25:38	Ankur:	Like when you have, when you have like person A and, over here[B1].
622	00:25:41	Michael:	You want to stand up and show them?
623	00:25:41	Ankur:	And person B over here[B1]. And then you have person B and person A[B1].
624	00:25:42	Michael:	You want to be in a line and we'll show them[B1]?
625	00:25:43	R1:	Michael, start from the beginning very slow.

626	00:25:45	Michael:	All right. You have five people[B1].
627	00:25:46	R1:	Stand up and show us.
628	00:25:47	R3:	Stand up and show us.
629	00:25:50	Jeff:	All right, I'm going to sit in your seat cause I can't see.
630	00:25:51	Michael:	I'm going to write it nice and clear so you all can see. All right. You got five people, in a line[B1]. You agree with me that's how many different combinations you can put those five people[B2].
631	00:26:10	R3:	That part I understand.
632	00:26:11	Michael:	All right.
633	00:26:11	R3:	I understood the multiplication that you showed[B2].
634	00:26:14	Michael:	Now in, you're only worried, you want to know how many different places you can put those two people[B1]. All right? So, in all the combinations you're going to have, they're going to be repeated a lot. A lot[B2]. When you have like, the two people in a certain place and you know, those three[B1]. If the three are, are like this[B1]. And then one of them switches, that's another combination[B2]. And you get a lot of repeats like that[B2].
635	00:26:37	R3:	Oh, I see. OK.
636	00:26:39	Michael:	So by eliminating that, you eliminate the combinations that repeat by the three people moving around[B2].
637	00:26:45	R3:	Uh-hum.
638	00:26:45	Michael:	Then let's say you just have those two people in, in any given combination[B2]. If, if one, if this guy switches the place with this guy it's the, they're different combinations, but in this we're not worried about where they are[B2]. We just, you understand?
639	00:27:00	R3:	Mm hm.

640	00:27:02	Michael:	That's why we get rid of the, the two factorial to, to, uh, eliminate the amount like as many times as you could, as many combinations as you could put those two people[B2]. Right? Like the three would, would be to eliminate the combinations you could put those three people that you're not worried about[B2]. Then the two, they would repeat because those people too, they move around[B2]. They, they could, they move around in the, in the line also[B1]. And then when, when, when you're done with all that, you just get, um, you get how many places you can just put that two[B1]. Like you're not worried if, like you don't care who they are[B1]. You don't care like if this guy has a switch with this guy[B1]. You understand like why you would eliminate, how that eliminates[B2].
641	00:27:41	R1:	OK. I don't want to think of people[B1]. I want to think of the tower now[B1]. Isn't that what Jeff said? And now I'm thinking of towers that are five tall[B1]?
642	00:27:50	Jeff:	Yeah. You can, we just-
643	00:27:51	R1:	And we're talking of those that have two reds[B1]?
644	00:27:54	Jeff:	Yeah. Well. [Inaudible.]
645	00:27:54	R1:	Explain it to me with that.
646	00:27:55	Jeff:	All right. Say, say we're doing, we're doing towers that were, were five tall[B1]. Towers of five tall with two different colors in it[B1]. Then that's the total amount of possibilities is the five factorial that you could have[B2]. All right, in, with, with five high with the combinations[B2]. So that's where, that's the five factorial on top[B2]. Then the three factorial on the bottom would be five different, five different spots minus the two spots that you're concerned about, leaving you with the three other spots[B2]-
647	00:28:26	Romina:	You could say-

648	00:28:26	Jeff:	-that you don't care about[B2]. That's going to eliminate all of them[B2].
649	00:28:29	Romina:	That's like, if you say like the reds[B1]. Let's say reds are our two colors that they stay in the same place, and like[B1]-
650	00:28:34	Jeff:	Reds[B1].
651	00:28:34	Romina:	They're. Like yeah, the two stay in the same place and then the other three are just switching while they're in staying in the same place[B1].
652	00:28:39	Jeff:	Yeah, they're staying in the same spot[B1].
653	00:28:40	Romina:	But we're not concerned with them.
654	00:28:41	Jeff:	That's why you're not concerned with those[B1].
655	00:28:43	Michael:	It's going to repeat like six times[B2].
656	00:28:44	Jeff:	Yeah. So that's where the three factorial comes from, and you're multiplying that by the two factorial[B2]. Those are what you're-
657	00:28:50	Romina:	That's to say like the first place and the third place and then they just switch[B1].
658	00:28:51	Michael:	Yeah, like- this way
659	00:28:52	Jeff:	Exactly.
660	00:28:54	Michael:	They just don't have a name on them so the, they're the same thing[B1].
661	00:28:56	Romina:	Yeah.
662	00:28:57	Jeff:	And then that's where the bottom number comes from and then you divide them by each other and that gives you what we're looking for[B2].

663	00:29:04	R1:	OK, so I think I follow what you said. But why were we doing this?
664	00:29:09	Jeff:	Uh, you, we don't-
665	00:29:11	Michael:	We were talking about-
666	00:29:12	Romina:	We want, you wanted us to explain choose[B3].
667	00:29:13	Michael:	The choose that we[B3], all right, whoa-
668	00:29:15	Romina:	Which goes back to Pascal's Triangle[B2] and see where a plus b[B3]-
669	00:29:19	Michael:	Yeah.
670	00:29:19	Romina:	-to the n[B3]. And we could figure out the beginning number[B2].
671	00:29:20	Michael:	All right. Over here, you wanted, the a plus b to the n thing, you wanted to know how we got the choose thing[B3]. What does that mean?
672	00:29:24	Jeff:	Yeah, how we got the third number[B2].
673	00:29:25	Romina:	Yeah.
674	00:29:26	Jeff:	And that's how we got off to, to here.
675	00:29:30	R1:	OK, so what did that have to do with what you did in class today?
676	00:29:32	Romina:	That's how we would get the number[B2].
677	00:29:33	Jeff:	We were looking at, we were doing this in class today. That's what we were doing. We were looking at a plus b[B3]-
678	00:29:37	Romina:	We're going to be-
679	00:29:38	Michael:	It was like in Pascal's Triangle things go like, by that[B2]. Like this choose this[B3]. Like, um, if you go to the one, three, three, one part of it[B2], it would be, um-
680	00:29:47	R1:	Show me on the board, Michael.

681	00:29:49	Jeff:	Go get 'em, Mike.
682	00:29:52	Michael:	This would be like, all right, this would be like three choose one[B2]. How many different places you put that one, that one guy[B1]. There's only one place[B1]. There's only, oh, I'm wrong. What am I doing?
683	00:30:18	Romina:	That's when you only have like, it's all one color[B1].
684	00:30:20	Michael:	No, there, there's a way it has something to do with- I think that would be three choose zero, I guess[B2]. No. All right, and then the next one would be three choose three[B2]. Obviously three different places[B1].
685	00:30:32	R1:	Three choose what[B3]? What was the next one?
686	00:30:34	Michael:	Three choose one[B2]. The next would be three choose two[B2], which we just figured that out. There's three[B2]. And last one is three choose three[B2]. You can only put those three people in those three places[B1]. You can't, you know, no more places to put them[B1].
687	00:30:48	R1:	OK so that's really interesting. That's really very interesting. So you've put something else together. I have another question. You could write more rows of that triangle[B2].
688	00:30:58	Michael:	Yeah.
689	00:30:59	R1:	And now you're telling me you can write them as the choose way[B3], you've called that. So can you take, let's say another row or two and show me the addition rule and what it looks like with your new notation[B2].
690	00:31:17	Michael:	You're talking about the addition rule[B2] when you
691	00:31:18	R1:	For a particular, for a particular row[B2].
692	00:31:20	Michael:	Add this and this and go like that[B2]?

693	00:31:21	R1:	Sure, or three and three, six[B2]. Show me what that looks like with that new notation[B2]. Do you understand my question?
694	00:31:29	Michael:	Uh, I don't really.
695	00:31:29	Romina:	I don't understand.
696	00:31:29	Ankur:	Instead of writing three you write[B2]-
697	00:31:31	R1:	Write your next row[B2], Michael. Now some time ago you, you had a reason. You explained to me-
698	00:31:45	Michael:	Why you add[B2].
699	00:31:46	R1:	Why you add[B2].
700	00:31:47	Michael:	Yeah.
701	00:31:48	R1:	You remember that? You, might, might be useful for folks who haven't heard it to hear it whatever way you want to explain it.
702	00:31:53	Michael:	I don't think I can explain it too good. Um.
703	00:31:55	R1:	Um, you know, however you want to explain it. You've had it a few ways.
704	00:32:00	Michael:	Um, I can't, I can't remember too well. I know why you add[B2], if I explain it, I don't think anyone will understand.
705	00:32:13	R1:	Try.
706	00:32:15	Michael:	I didn't. Didn't I tell you guys like last time I came here?
707	00:32:18	Jeff:	Well, go for it, dude, just-
708	00:32:20	Romina:	You could try.
709	00:32:20	Michael:	You don't have that paper, do you? You can just hand them, hand that out.
710	00:32:21	Romina:	You started talking about toppings[B1]. I think something-
711	00:32:24	Michael:	Hand that out instead.
712	00:32:25	Jeff:	Just-

713	00:32:27	Michael:	Um, all right. If, all right, let's go to, let's go to this one. This would be like three different places[B1] I guess. And um-
714	00:32:37	Jeff:	Which one are we looking at[B1]?
715	00:32:38	Michael:	That one right there[B1]. You have three[B1]-
716	00:32:41	Jeff:	That would be a plus b to the third[B3].
717	00:32:42	Michael:	All right, let's say you have like, here's a number, all right? Zero means no toppings[B2]. One would, this, one would be[B2]-
718	00:32:51	Romina:	It would be, one's a topping[B2].
719	00:32:51	Michael:	One would be a topping[B2]. So first category is everything with no toppings[B1]. And that's, you can't make, that's, that's your number for that one[B2].
720	00:33:01	Michael:	Next would be- There's all the, the ones that have one topping[B1].
721	00:33:12	Jeff:	Right, you got to make that zero at the end[B2]. You messed up.
722	00:33:14	Michael:	What?
723	00:33:14	Jeff:	Last one should be a hundred, not a hundred and one[B2].
724	00:33:15	Michael:	I knew that. There's your, um, your three choose one[B2]. And there's three different combinations you could put that[B2]. Um, I can go on forever doing this.
725	00:33:25	Michael:	But, um, when you have a new, when you add another place, another topping[B2]-
726	00:33:34	Jeff:	That could be one or the other, one or the other, one or the other[B1]
727	00:33:36	Michael:	So it could be one or the other[B1].
728	00:33:37	Michael:	It could be a zero or a one, a zero or a one, a zero or a one[B2].

729	00:33:38	Jeff:	Yeah. All right.
730	00:33:39	Michael:	So all these threes would either move up a step onto the next category and, uh, have two toppings[B2].
731	00:33:47	Michael:	Or they might stay behind and still only have one if they have the zero[B2].
732	00:33:52	Michael:	So three, three will get a topping, and go to this one[B2].
733	00:33:56	Michael:	And three won't, will stay[B2].
734	00:33:58	Michael:	And obviously this guy's going to get a topping[B2]. That's why you add this one[B2].
735	00:34:03	Jeff:	Uh-huh.
736	00:34:03	Michael:	So now this guy's going to have, without toppings[B1]. You're going to add a topping onto him[B2]. That's going to be one topping[B2]. These three with one topping won't get one so[B2], you know-
737	00:34:14	Jeff:	That's their four[B2].
738	00:34:15	Michael:	You put, you can put them in the same category as this one[B1].
739	00:34:17	Jeff:	Yeah.
740	00:34:17	Michael:	That's four[B2].
741	00:34:17	Jeff:	Those are your four[B2].
742	00:34:18	Michael:	And you know-
743	00:34:19	Ankur:	Three[B2].
744	00:34:19	Jeff:	Those three[B2].
745	00:34:20	Michael:	The three that had two toppings won't get any[B2].
746	00:34:23	Jeff:	Yeah. So they'll go to [Inaudible.].

747	00:34:23	Michael:	And you could put them in together with the ones that did get something[B1]. That's why you would add[B2]. Keep on adding[B2].
748	00:34:28	R3:	What do you mean by toppings[B1]?
749	00:34:29	Michael:	Pizza toppings[B1].
750	00:34:30	R3:	Um, for example-
751	00:34:31	Michael:	Like here you would have a choice of three different ones[B1]. Here you would have a choice of five and like the ones would be like the mushrooms, the peppers, the whatever, just by going like[B1]- The one would indicate you have it or not[B2].
752	00:34:46	R1:	OK. OK. I remember.
753	00:34:48	Michael:	You remember.
754	00:34:49	R1:	I remember this. But now I don't want to think of the numbers in that triangle[B2], I want to think of those as chooses[B3]. So for example, let's just take this row. One, three, three, one[B2].
755	00:35:11	Michael:	Mm hm.
756	00:35:13	R1:	All right. If I wrote these as chooses the way you're writing them[B3]-
757	00:35:19	Michael:	Three choose zero, three choose one[B2].
758	00:35:20	R1:	This is three choose zero[B2].
759	00:35:21	Michael:	Yeah.
760	00:35:22	R1:	This is three choose one[B2].
761	00:35:23	Jeff:	Choose one[B2]. Same thing.
762	00:35:24	R1:	Three choose[B2]-
763	00:35:25	Michael:	Two and three choose, then three choose, three choose three[B2].
764	00:35:28	R1:	Right.

765	00:35:29	Jeff:	So that's how you get it. It's like the same thing, cause like three and zero is like three and three[B2], right? And then three two[B2].
766	00:35:32	R1:	OK, so-
767	00:35:34	Michael:	You want us to write the triangle looking like that[B2]?
768	00:35:36	R1:	I would, I would, I would like you to do that and then tell me what the general rule is[B2].
769	00:35:41	Jeff:	All right.
770	00:35:42	R1:	With this notation[B3]. Do you understand my question? I'll leave you to work on that. So, so I'd like you to write out some of the rows with the triangle[B2], and then I'd like-
771	00:35:51	Michael:	So to use it like, like that. Like the next one would be, uh, four choose zero[B2].
772	00:35:55	Jeff:	Yeah and-
773	00:35:56	Romina:	Four choose[B2] -
774	00:35:56	Michael:	The four choose zero then //four choose one, four choose two[B2]-
775	00:35:57	Jeff:	//Four choose one, four choose two[B2].
776	00:35:58	Ankur:	Four choose three[B2].
777	00:36:00	Michael:	We're in a bad place.
778	00:36:02	R1:	Right. You probably want to use this.
779	00:36:03	Michael:	Yeah.
780	00:36:03	R1:	So that people can read it.
781	00:36:04	Michael:	Um.
782	00:36:05	Alex:	Ask them your question one more time.

783	00:36:06	R1:	OK, so I'd like you to rewrite your triangle[B2] if you like.
784	00:36:09	Michael:	From top to bottom[B2]?
785	00:36:10	R1:	Top to bottom[B2].
786	00:36:11	Romina:	Do you want the ones and like[B2]-
787	00:36:13	Jeff:	All right. So what-
788	00:36:14	R1:	I want everything-
789	00:36:14	Jeff:	What would-
790	00:36:14	R1:	I want everything written in this form[B3]. Do you understand?
791	00:36:16	Ankur:	Uh-huh. [Ankur nods.]
792	00:36:17	Michael:	That's, that's easy.
793	00:36:18	R1:	And then I would like the general row[B2].
794	00:36:19	Jeff:	Is that one[B2]?
795	00:36:19	R1:	What would the general row look like[B2]? Where you have towers[B1]?
796	00:36:24	Romina:	That's a zero, no that's zero choose zero[B2]
797	00:36:27	Ankur:	X high[B3].
798	00:36:28	R1:	Something like that.
799	00:36:29	Jeff:	All right, well that's [Inaudible]
800	00:36:30	R1:	Ankur understands. So he can tell you.
801	00:36:37	Romina:	See, like that?
802	00:36:38	Michael:	So it would be, um, like N over, not two over[B3].
803	00:36:42	Ankur:	Well, it would be-
804	00:36:43	Michael:	N choose[B3]-

805	00:36:44	Ankur:	It would be-
806	00:36:46	Romina:	Well, and N, make N like your height or something[B3].
807	00:36:49	Jeff:	All right, so say
808	00:36:50	Romina:	N equals height[B3].
809	00:36:52	Jeff:	Well that would-
810	00:36:52	Ankur:	Well, write the X[B3]. Write a plus b to the whatever it is next to it[B3].
811	00:36:57	Jeff:	Yeah.
812	00:36:58	Ankur:	You know what I mean?
813	00:36:59	Jeff:	Yeah. So right. That would be a plus b to the[B3]-
814	00:37:00	Michael:	This would be nothing, you know, it would be adding[B2].
815	00:37:02	Jeff:	Yeah, zero, one, two[B2]. So a plus b to the second[B3].
816	00:37:05	Romina:	Well, it'd be like N over N minus, but what[B3]?
817	00:37:07	Jeff:	Yeah, well, a plus b to the second, so it would be if, or a plus b to the nth[B3].
818	00:37:13	Romina:	To the-
819	00:37:14	Ankur:	No, all you need is like-
820	00:37:14	Romina:	n is factorial[B3].
821	00:37:14	Jeff:	It'd be n, n over[B3]-
822	00:37:16	Michael:	n, fa[B3]-
823	00:37:18	Jeff:	n mi[B3]-
824	00:37:18	Romina:	No, that's just like- No, it's not right. I'm just saying like-
825	00:37:21	Jeff:	It would be-

826	00:37:23	Romina:	You would have to multiply it[B2].
827	00:37:24	Jeff:	n over[B3]-
828	00:37:28	Michael:	Well, if you had an n, it would be[B3], uh-
829	00:37:30	Ankur:	To the height of the tower which is n[B3], right?
830	00:37:32	Michael:	You'd have a bunch of n's[B3].
831	00:37:33	Jeff:	Yeah, and it'd be over, just z[B2]-
832	00:37:34	Michael:	There'd be n plus one n's going this way[B3].
833	00:37:37	Jeff:	Yeah. If-
834	00:37:38	Michael:	All right?
835	00:37:38	Jeff:	it would be n over 0[B3].
836	00:37:39	Michael:	So if n was three, you'd have four n's going this way[B2].
837	00:37:42	Jeff:	Yeah.
838	00:37:42	Michael:	And the bottom numbers would be just going from 0 to[B2]
839	00:37:44	Jeff:	Just-
840	00:37:45	Michael:	To[B2]-
841	00:37:45	Jeff:	Yeah. Well, yeah.
842	00:37:46	Michael:	0 to n[B3].
843	00:37:50	Jeff:	Exactly.
844	00:37:51	Michael:	To n[B3].
845	00:37:51	Jeff:	To n. Whatever n equals[B3].
846	00:37:53	Romina:	Is there a way to write that, you know how to write over times [Inaudible.]?
847	00:37:58	Ankur:	I guess.

848	00:37:59	Jeff:	Yeah, so how do you, yeah, wait, now that makes sense but, so it would be n over 0 to the nth[B3]. And whatever-
849	00:38:08	Michael:	Zero, what are you talking about[B2]?
850	00:38:09	Jeff:	Wherever you're looking for.
851	00:38:09	Ankur:	What are you talking about, 0 to the n[B3]?
852	00:38:11	Michael:	0 minus n[B3]? That would be negative[B2].
853	00:38:13	Jeff:	No, not minus, like that's to whatever n is. n over 0, n over 1[B3].
854	00:38:18	Romina:	1[B2]
855	00:38:19	Jeff:	Not divided by like n, 1, n, uh, 2, n, 3[B3].
856	00:38:25	Michael:	That was-
857	00:38:26	Jeff:	All the way until n could be over n[B3]. You know what I'm saying?
858	00:38:28	Michael:	Yeah.
859	00:38:29	Jeff:	Not, not divided by[B2]. I was using bad, uh, bad looking things there. But-
860	00:38:34	Michael:	Each of those would be a number[B2]-
861	00:38:35	Jeff:	Yeah, it's what, 0 to n[B3].
862	00:38:37	Ankur:	And n represents the height of the tower[B3]?
863	00:38:39	Romina:	The height of the tower[B1], yup.
864	00:38:42	Michael:	Yeah, n, n represents[B3]-
865	00:38:43	R1:	Do you want that divided sign here[B3]?
866	00:38:45	Michael:	No.
867	00:38:45	R1:	On that one?
868	00:38:46	Jeff:	No.

869	00:38:46	Ankur:	No. Cross that off.
870	00:38:46	Romina:	No.
871	00:38:46	Jeff:	I was using it to separate[B2], and that was, that's a habit of mine, it looks bad.
872	00:38:49	Michael:	Oh, sorry about that. It would be, uh, as many, it's like height of the tower with two colors[B1]. You have two numbers[B2].
873	00:38:59	Jeff:	Yeah. How do you, how are you, can you write that to get this?
874	00:39:04	Romina:	Like that's what I meant. Like I didn't mean factorial[B3]. I meant like when we used four first and like three first[B2]. I don't know how to write that, though.
875	00:39:10	R1:	So you go 0, 1, 2, 3, dot, dot, dot, up to n[B2].
876	00:39:16	Jeff:	Yeah.
877	00:39:16	Michael:	Mm hm.
878	00:39:17	R1:	Can we get one in the middle there, like n choose r[B3]?
879	00:39:22	Jeff:	Like how would you just go right to n choose 3[B3]? Or n choose r[B3]? Like what- [Researcher 1 nods.]
880	00:39:29	Michael:	What are you talking about?
881	00:39:30	Romina:	Like instead of using 0, 1, 2, 3[B2].
882	00:39:31	Jeff:	r being any number on the bottom[B3].
883	00:39:35	R1:	Because you said n choose x up there[B3].
884	00:39:37	Jeff:	Yeah.
885	00:39:37	R1:	//I just picked what I wanted-
886	00:39:38	Michael:	//Oh, you want uh, you want to do that.
887	00:39:39	Jeff:	Yeah, so, so it would be-
888	00:39:40	Michael:	Um-

889	00:39:40	Ankur:	n choose[B3]-
890	00:39:44	Michael:	It would be n[B3].
891	00:39:49	Jeff:	Wouldn't that just be n choose r for whatever r you wanted[B3]? Whatever number you wanted up to, as long as it didn't exceed n[B2]?
892	00:39:59	Michael:	This, this is different than that. Isn't it? Like this, these are just like a list of numbers[B2]. That's, that's just giving you one of these numbers[B2].
893	00:40:05	Jeff:	Uh, you know all that, but I'm saying, if you wanted to write n choose to get a certain number, wouldn't it just be n choose r[B3]? Like that? And then as long as r doesn't exceed n or it's less than 0 like r[B3]-
894	00:40:15	Ankur:	Wouldn't that equal that[B2]?
895	00:40:16	Romina:	Yeah, wouldn't it?
896	00:40:16	Michael:	I guess you could write one of those[B3].
897	00:40:18	Romina:	Yeah. Isn't it supposed to equal that[B2]?
898	00:40:18	Michael:	Right there.
899	00:40:19	Ankur:	That's- that is[B2].
900	00:40:19	Romina:	It's the same thing[B2].
901	00:40:21	Ankur:	That does[B2].
902	00:40:24	Michael:	You could do that. It's a lot of-
903	00:40:26	R1:	OK, so you've written out three rows and then you wrote out the nth row[B2].
904	00:40:33	Michael:	The reason why, 0, 1, 2, 3 is that number is always going to be that number[B2]. It's not, it's never going to change[B2].

905	00:40:35	R1:	[Researcher 1 walks to the board.] OK. I'll buy that. But something in here could be an n choose r[B3]. Right? Something in here could be an n choose r[B3].
906	00:40:41	Romina:	Mm hm.
907	00:40:42	R1:	That's what I heard you say, Jeff?
908	00:40:43	Jeff:	Yes.
909	00:40:43	R1:	Sort of a general one in here, n choose x[B3].
910	00:40:46	Jeff:	That's what-
911	00:40:47	R1:	Whatever you choose to use.
912	00:40:47	Jeff:	Yeah, that's what that is. So, yeah.
913	00:40:49	R1:	OK. OK, so this is my question to you. You've written out two rows and you have the third one there[B2].
914	00:40:55	Jeff:	Mm hm.
915	00:40:56	R1:	Maybe somebody will come up here and write these up nicely.
916	00:40:59	Jeff:	Is that what you want?
917	00:41:01	R1:	Yes. Because then I want to ask, I want; after you do that I have a question to ask you. Thanks.
918	00:41:06	Michael:	You want to erase those?
919	00:41:17	Jeff:	You want to make that the line so bad. I know.
920	00:41:19	Michael:	No, don't do that.
921	00:41:30	Ankur:	How far do you want him to go?
922	00:41:34	Michael:	One more.
923	00:41:34	Jeff:	I want to, uh. You want one more for good measure?
924	00:42:02	Michael:	No. Don't worry about it.
925	00:42:03	R1:	Go to the nth one[B2], then.

926	00:42:06	Jeff:	Wouldn't that just be-
927	00:42:07	R1:	Dot, dot, dot[B2].
928	00:42:08	Jeff:	N zero[B3]
929	00:42:10	Michael:	Dot, dot, dot, N to the N[B3].
930	00:42:20	R1:	And the last one, Jeff. Is the last one N N[B3]?
931	00:42:24	Michael:	Yeah.
932	00:42:25	Romina:	Mm hm.
933	00:42:25	Jeff:	Yeah.
934	00:42:26	R1:	Do you want to put it at the end[B2]?
935	00:42:28	Michael:	Yeah, put it at the end[B2], make it nice.
936	00:42:30	R1:	What's the middle one there? What would you, how would you show the middle one[B2]?
937	00:42:31	Jeff:	Uh, actually, you could put N, X[B3].
938	00:42:33	R1:	OK. N choose X, N choose N[B3].
939	00:42:40	Jeff:	Those are dots because you can't really make a dot[B2]. Now you can.
940	00:42:44	R1:	OK, now, now, show me, show me, while you're up there, Jeff, just show me, uh, an addition rule of Pascal's Triangle[B2]. Let's say from, give me an example from the third, fourth row to the fifth row[B2].
941	00:42:55	Jeff:	Fourth row to this[B2]?
942	00:42:57	R1:	Fourth row to the fifth[B2].
943	00:42:59	Michael:	The three to the four[B2].
944	00:43:00	Jeff:	Oh, fourth row[B2]. All right. Um.

945	00:43:02	R1:	Show me that three plus three is six[B2]. Which ones would it be[B2]?
946	00:43:07	Jeff:	That would, like you're saying from here [3 choose 1] to here [3 choose 2] going to there [4 choose 2][B3]?
947	00:43:10	Michael:	Uh-huh.
948	00:43:10	R1:	OK, show me. How would you draw your little arrow to shows that[B2]?
949	00:43:15	Michael:	This one and that one[B2].
950	00:43:16	Jeff:	Yeah, is that it? Is that all, so that's all you want?
951	00:43:18	Michael:	Yeah.
952	00:43:18	R1:	Is that true? Do you believe that?
953	00:43:20	Jeff:	Yeah.
954	00:43:20	Michael:	Yeah, I believe so.
955	00:43:21	R1:	You all believe that?
956	00:43:22	Romina:	Yeah.
957	00:43:22	Michael:	Uh-huh.
958	00:43:22	R1:	No one could persuade you otherwise?
959	00:43:23	Ankur:	No.
960	00:43:23	Michael:	No.
961	00:43:25	R1:	OK, so you're saying three choose one, plus //three choose two equals four choose two[B3]. Right?
962	00:43:27	Jeff:	//Three choose two should equal four choose two[B2].
963	00:43:30	Romina:	Look at all the numbers are added up[B2].
964	00:43:32	R1:	OK. So what's four choose two plus four choose three[B2]?

965	00:43:35	Jeff:	Four choose two plus four choose three[B2]? That would be, [Michael laughs.] that would be five[B2]-
966	00:43:40	Michael:	Oh, five[B2]-
967	00:43:41	Ankur:	Five choose[B2]-
968	00:43:43	Michael:	Five choose three[B2].
969	00:43:44	Ankur:	Yeah.
970	00:43:46	Michael:	Right?
971	00:43:47	Ankur:	Yeah.
972	00:43:48	Jeff:	Yeah.
973	00:43:48	R1:	I don't know if Romina's convinced.
974	00:43:50	Jeff:	Why is it five choose three[B2]?
975	00:43:52	R1:	Yeah, I don't think Jeff is either.
976	00:43:52	Jeff:	Is this here-
977	00:43:53	Romina:	Yeah, I don't really-
978	00:43:53	Ankur:	Because it's, it's always the one on the right[B2].
979	00:43:55	Michael:	Because, see, this guy gets another topping, I guess, so he turns, he would be a two[B2].
980	00:44:01	Jeff:	Uh huh.
981	00:44:02	Michael:	Whatever it is in here[B1]. And this guy doesn't, so it stays two[B2].
982	00:44:03	Jeff:	Ah, it doesn't, so that's two[B2].
983	00:44:04	Michael:	So[B2]-
984	00:44:05	Jeff:	It wasn't that.

985	00:44:06	Michael:	Because he's moving up, this bottom number's going to change[B2].
986	00:44:09	Jeff:	Oh, all right.
987	00:44:09	R1:	Explain that one more time, Michael, please.
988	00:44:10	Jeff:	Here.
989	00:44:11	Michael:	Um, wherever this guy goes, wherever this guy goes he's going to get another topping because he's moving this way[B1].
990	00:44:15	Romina:	Um-hm.
991	00:44:15	Jeff:	So that turns it into a two[B2].
992	00:44:16	Michael:	So this bottom number's going to change to two[B2].
993	00:44:19	Michael:	This guy's not going anywhere[B2]. Cause the bottom number stays the same[B2].
994	00:44:21	Michael:	So it's going to be five[B2]. Because you know the next one's going to be five and it, it has to be a two[B2] because- You understand why you add[B2]? All right. Good.
995	00:44:33	Romina:	I'm with you.
996	00:44:34	R1:	OK, so that's really very interesting. Let me ask you to explain that to Brian for a minute, but we'll let him eat first. Did you eat, Brian?
997	00:44:40	Brian:	No.
998	00:44:40	R1:	Just help yourself. You can watch us.
999	00:44:43	Jeff:	We don't get another break?
1000	00:44:45	R1:	All right, Brian, just eat. You can.
1001	00:44:46	Brian:	I don't think you want to know what I went through.
1002	00:44:48	Ankur:	Well at least you got a tux.
1003	00:44:49	R1:	We're glad you're here.

1004	00:44:50	Brian:	Neither did I. I didn't.
1005	00:44:52	Ankur:	I didn't either.
1006	00:44:52	Romina:	What happened?
1007	00:44:52	Ankur:	[Inaudible.] what happened to my coat.
1008	00:44:53	Brian:	The coat is like fit for a midget. [Break in tape.]
1009	00:44:57	Alex:	Keep going.
1010	00:44:57	Michael:	All right.
1011	00:44:59	R1:	[Side conversation.] OK, sure, why not.
1012	00:45:00	Alex:	OK. Good.
1013	00:45:01	Jeff:	All right. Well, all right.
1014	00:45:02	Ankur:	[Inaudible.] you remember.
1015	00:45:04	Jeff:	All right, we're looking at, we're looking at this right here[B2]. You guys got to pay attention to it[B2].
1016	00:45:08	R1:	Erase it better, Jeff, before you start, because-
1017	00:45:09	Ankur:	Yeah, remember and you didn't pay me back for like three months.
1018	00:45:12	Brian:	I had it the whole time.
1019	00:45:14	Ankur:	Yeah, but it cancelled out.
1020	00:45:15	Jeff:	All right. Say we have this row right here[B2]. We got um, N choose 0[B3]. And over here we have N choose X[B3]. And then over here we have N choose N[B3]. All right? Then this right here would be- Oh, we're explaining the general addition, the addition rule using this type of, to fill out the triangle[B2]. Using chooses to fill out the triangle and this here would be N choose X plus one and then N, N choose X plus two and so on to whatever N equals[B3]. Right there'd be dot dot[B2]- I didn't, I

			didn't leave enough room. And this here would be X minus one[B3] and then-
1021	00:46:02	Ankur:	You did that one man.
1022	00:46:03	Jeff:	What?
1023	00:46:04	Ankur:	Nothing.
1024	00:46:05	Jeff:	That'd be X minus two and so on each way[B3]. Right? So it'd be that.
1025	00:46:10	Ankur:	Can I see the row above that[B2]?
1026	00:46:12	Jeff:	And the row above this would be N minus one[B3], right? Yeah.
1027	00:46:17	Michael:	Mm hm.
1028	00:46:19	Jeff:	Um, choose zero[B2]. This again would be N, N minus one choose X[B3] and then-
1029	00:46:29	Michael:	N minus one[B3].
1030	00:46:30	Jeff:	N minus one, N minus one[B3]. That's a one. Um, how do you want me to, to- Where do you want me to go from here?
1031	00:46:40	R1:	Well, you know, um, Brian wasn't here, so you might want to give him some background to what you've been doing.
1032	00:46:46	Jeff:	Start at the beginning? We did, we worked for an hour and a half getting to this point. Explaining this, doing this. All right, um.
1033	00:46:54	R1:	But Brian's a quick study.
1034	00:46:54	Brian:	That's what I am.
1035	00:46:56	Jeff:	All right. We did, uh, this is Pascal's Triangle[B2] using-
1036	00:47:02	Brian:	The whole choose thing[B3].
1037	00:47:03	Jeff:	-the choose situation[B3]. That's what this is.
1038	00:47:04	Michael:	You know how choose works, like one, three, three, one[B2].

1039	00:47:06	Brian:	Yeah.
1040	00:47:07	Jeff:	Yeah.
1041	00:47:07	Michael:	Three choose zero, three choose one[B2]-
1042	00:47:08	Brian:	One, four, six[B2]-
1043	00:47:09	Michael:	Yeah. It's all like chooses of something[B3].
1044	00:47:11	Jeff:	All right. So, um, I don't- Um, how would you like to, uh, how do you want to do this? How do you want to-
1045	00:47:19	Michael:	We're just[B3]-
1046	00:47:20	Jeff:	Well, tell him what we did.
1047	00:47:21	Michael:	-replacing the three in the chooses by N's and X's[B3].
1048	00:47:24	Jeff:	Yeah, exactly. And rather doing, like, uh, rather- Say this is the, uh-
1049	00:47:29	Michael:	If N was three[B2].
1050	00:47:30	Jeff:	Yeah, say if N was the third row, it would be three choose zero[B2]. That would give you one[B2].
1051	00:47:36	Ankur:	Like, you know how it's one, three, three, one[B2]. Three choose zero gives you one[B2].
1052	00:47:38	Jeff:	Three choose one[B2].
1053	00:47:39	Michael:	That'd be three[B2].
1054	00:47:39	Jeff:	That would give you the three[B2]. The three choose two[B2]. That would give you the other three[B2]. That's equal to three and then three choose three[B2]. That equals the other one[B2]. And like that's filling out this part of the triangle and so on[B2]. And that's what, that's what we're doing now. We went, other stuff we did we did the whole, we found that equation to find out choose[B3].

1055	00:48:01	Michael:	What choose means[B3].
1056	00:48:02	Jeff:	Yeah, we did all that.
1057	00:48:03	Romina:	And choose[B3].
1058	00:48:04	Jeff:	But you missed out on all that. That's the choose equation[B3].
1059	00:48:05	Romina:	That's the choose equals[B3].
1060	00:48:08	Jeff:	And we spent time explaining. That's what we spent the bulk, bulk of the thing, trying to figure out how to explain that. And-
1061	00:48:14	Brian:	What's that little exclamation point[B3]?
1062	00:48:15	Michael:	//Factorial[B3].
1063	00:48:16	Romina:	//Factorial[B3].
1064	00:48:16	Ankur:	//Factorial[B3].
1065	00:48:16	Jeff:	Factorial[B3].
1066	00:48:17	Brian:	That's what it is?
1067	00:48:17	Romina:	Yeah.
1068	00:48:17	Jeff:	Yeah.
1069	00:48:18	Brian:	All right.
1070	00:48:18	Jeff:	It was really excited, like N! [Michael laughs]
1071	00:48:20	Romina:	You want to know what this is? That's all the combinations[B2]. [Romina points to her paper; refer to Figure J18.] That's minusing[B2]. You know how like they're saying[B2]-
1072	00:48:26	Brian:	Yeah.
1073	00:48:26	Romina:	-three choose two[B2].
1074	00:48:27	Brian:	Yeah.

1075	00:48:27	Romina:	We don't care about the three, so that's like when the threes are switching, not the twos[B1]. And that's when the twos are like in the first place and the third place, and they just switch and nothing else moves[B1].
1076	00:48:35	Brian:	So this-
1077	00:48:35	Romina:	It's basically the same thing[B1].
1078	00:48:35	Brian:	Is this, is that this over this[B2]?
1079	00:48:37	Michael:	Yeah.
1080	00:48:38	Romina:	It's N, N factorial over N minus X factorial times X factorial[B3].
1081	00:48:45	Michael:	And that equals N choose X[B3].
1082	00:48:46	Romina:	Like this is when the- the things we don't- No, I'm just saying these are the things that we don't care about[B1] when they- they switch and this is when the things we do care about, just switch in the same place[B1] and everything stays the same.
1083	00:48:57	Brian:	All right.
1084	00:48:58	Romina:	And that's all of them. [Romina laughs.]
1085	00:49:00	Ankur:	The Reader's Digest version.
1086	00:49:01	Romina:	Yeah.
1087	00:49:01	R1:	What was that, Ankur?
1088	00:49:02	Ankur:	No, I just said like the Reader's Digest version or something. [Romina laughs.]
1089	00:49:05	R1:	The Reader's Digest version?
1090	00:49:07	Jeff:	Yeah. So where, where do you want to go with, with this?
1091	00:49:10	R1:	Well, I want you to show me how the addition rule works in general[B2].
1092	00:49:14	Jeff:	All right. Well that's not much of a problem-

1093	00:49:16	R1:	So you showed me what N minus one choose X[B3]-
1094	00:49:17	Michael:	Go from, go from, go from N X and N X plus one[B3].
1095	00:49:19	Jeff:	Wait, this is, this is //[Inaudible]
1096	00:49:21	Ankur:	Yeah, add that in terms of X[B3]. Like below it, you know what I mean?
1097	00:49:23	Michael:	Add these two[B2]. What are these two going to equal[B2]?
1098	00:49:26	Jeff:	All right, well that's gonna be[B3]-
1099	00:49:27	Michael:	We want the next[B3]-
1100	00:49:28	Jeff:	//N plus one over[B3]-
1101	00:49:30	Michael:	//N plus one over[B3]-
1102	00:49:30	Ankur:	X plus one[B3].
1103	00:49:33	Jeff:	X plus one[B3]?
1104	00:49:33	Michael:	N[B3].
1105	00:49:34	Ankur:	Yeah. I think. Uh-huh.
1106	00:49:37	Jeff:	That's what these two are going to come into[B2]?
1107	00:49:39	Ankur:	Mm hm.
1108	00:49:40	Jeff:	Right?
1109	00:49:41	Michael:	Yeah.
1110	00:49:41	Ankur:	Yeah.
1111	00:49:40	Jeff:	And that's cause-
1112	00:49:41	R1:	Can you write it, can you write it as an equation[B3]? Just like you wrote three plus three equals six[B2].
1113	00:49:46	Jeff:	Um, that would[B3]-

1114	00:49:48	Ankur:	N plus, just that plus that[B3].
1115	00:49:50	R1:	Why don't you do it on the side?
1116	00:49:51	Jeff:	Just N[B3]. Oh, would it be[B3]-
1117	00:49:51	Michael:	Oh, N choose X[B3].
1118	00:49:52	Jeff:	N choose X, um, plus[B3]-
1119	00:49:53	Ankur:	Plus[B2].
1120	00:49:54	Jeff:	-N choose X plus one[B3].
1121	00:49:57	Michael:	Equals that[B3].
1122	00:50:00	Jeff:	Plus one, equals that right there[B3].
1123	00:50:02	R1:	//[Inaudible]
1124	00:50:04	Jeff:	Then, well, that's, that's because this would be gaining an X and going into the X plus one[B3].
1125	00:50:14	Michael:	Yeah.
1126	00:50:15	Jeff:	And this would be losing an X[B3].
1127	00:50:16	Michael:	No, no, not losing, not getting anything[B2].
1128	00:50:16	Ankur:	Staying the same[B1].
1129	00:50:17	Romina:	No.
1130	00:50:18	Ankur:	It's not getting anything[B1].
1131	00:50:18	Jeff:	That would be staying the same[B1] and that's-
1132	00:50:19	Ankur:	That's, yeah, the plus that[B2].
1133	00:50:20	Jeff:	-is the X plus one[B3].
1134	00:50:22	Michael:	And the top numbers have changed because you have more[B2].

1135	00:50:24	Jeff:	Because you're adding more things[B2].
1136	00:50:25	Ankur:	One more[B2].
1137	00:50:25	Jeff:	One more[B2]-
1138	00:50:27	Michael:	Topping[B1] or-
1139	00:50:27	Jeff:	Place[B1]
1140	00:50:28	R1:	Say it so Brian can follow it because he wasn't here for the earlier pizza discussion[B1].
1141	00:50:31	Michael:	He follows, you can follow it?
1142	00:50:32	Brian:	I can just sit in the back and watch.
1143	00:50:33	R1:	Go ahead, Brian. Don't be easy on them, Brian, make them work.
1144	00:50:35	Jeff:	What, what we're doing is the next line of the triangle[B2]- Remember how today in class you know how the other triangle was one, two[B2]-
1145	00:50:40	Brian:	Yeah.
1146	00:50:41	Jeff:	-three, that whole row there[B2]? Well, that's the increase in N, and then the X plus one[B3]. If you added another topping onto your whole[B2]. Say we're doing pizzas[B1].
1147	00:50:50	Brian:	All right.
1148	00:50:51	Jeff:	If you add another topping onto it[B2]?
1149	00:50:53	Romina:	You know how we get the triangle and how we go one two one and add those two together[B2].
1150	00:50:56	Brian:	Yeah.
1151	00:50:56	Jeff:	Yeah.
1152	00:50:57	Romina:	That's what we're doing right there.

1153	00:50:57	Jeff:	Yeah. Well, that's what we're doing.
1154	00:50:58	Ankur:	We're just adding it[B2].
1155	00:50:58	Michael:	You know why, do you know why we add[B2], though?
1156	00:50:58	Brian:	That's all you're all doing?
1157	00:50:59	Romina:	That's all we're doing.
1158	00:51:02	Jeff:	We, we were explaining why you add[B2].
1159	00:51:03	Brian:	All right, keep going.
1160	00:51:03	Jeff:	And why you do it, is it cause when you add another topping like onto it[B2], this one- Say the toppings were one and zero[B1].
1161	00:51:10	Brian:	Uh huh.
1162	00:51:11	Jeff:	If it gets a topping, that's why it goes up to the X plus one[B3]. And since it doesn't get anything, it'll stay the same[B1]. And in this one, it's staying the same[B1], right?
1163	00:51:20	Michael:	Yeah.
1164	00:51:21	Jeff:	And that's why it's going there[B2]. Like saying that's the zero[B1].
1165	00:51:25	Brian:	OK.
1166	00:51:26	Jeff:	And going to there[B2]. Make sense?
1167	00:51:28	Brian:	Yes. It actually does.
1168	00:51:30	Jeff:	So, so that would be the general addition rule in this case[B2]? That's it?
1169	00:51:34	R1:	Are you impressed?
1170	00:51:35	Jeff:	Impressed?
1171	00:51:37	R1:	Mm hm.
1172	00:51:37	Michael:	Not really.
1173	00:51:37	Jeff:	Not really. I don't think we did anything that spectacular.

1174	00:51:42	Michael:	Yeah, that's all.
1175	00:51:43	R1:	Well, you might be.
1176	00:51:44	Ankur:	Nothing more than we ever did before.
1177	00:51:45	R1:	You might pick up a probability[B2] book in-
1178	00:51:46	Jeff:	Is this all in-
1179	00:51:47	R1:	-freshman college and see if you recognize this[B3].
1180	00:51:51	Jeff:	I mean, I don't know. It just, just seems like-
1181	00:51:52	Romina:	We just talked
1182	00:51:53	R1:	If someone said to you, why does this work and this is a rule and you've shown me things with factorials, you can probably write those in factorial notations[B3]. I bet you could. In fact, I wish someone would do it on the board on the right there. Write that addition statement using factorial notations[B3].
1183	00:52:11	Jeff:	All right. Um, you want to do that? Want to do it?
1184	00:52:14	Michael:	Just that thing real quick?
1185	00:52:15	Jeff:	We're writing this right here?
1186	00:52:16	R1:	Sure.
1187	00:52:16	Jeff:	The addition rule in factorial notation[B3]?
1188	00:52:19	R1:	That's another form, isn't it[B3]?
1189	00:52:20	Jeff:	Yeah.
1190	00:52:22	R1:	Brian would like to know that, I know he would.
1191	00:52:25	Romina:	Bless you. [Someone says Thanks]
1192	00:52:27	Brian:	Right.
1193	00:52:27	Jeff:	I'm thrilled
1194	00:52:27	Ankur:	Oh, yeah.

1195	00:52:28	Michael:	That whole thing plus[B2]-
1196	00:52:31	Ankur:	Plus[B2].
1197	00:52:35	Michael:	Aw this is gonna be a pain.
1198	00:52:39	Michael:	No.
1199	00:52:40	Ankur:	No, it's just N[B3].
1200	00:52:41	Jeff:	Yeah, N factorial[B3].
1201	00:52:42	Michael:	I just, I just saw that. Um.
1202	00:52:48	Ankur:	Over, just do everything it is[B3].
1203	00:52:50	Michael:	N minus X[B3].
1204	00:52:53	Ankur:	X, parenthesis[B3].
1205	00:52:54	Michael:	Plus one[B3].
1206	00:52:58	Ankur:	Yeah. And then add and do the X factorial[B3]. Put that all in parentheses[B3].
1207	00:53:04	Jeff:	It's not an X, it's not X[B3]. Yeah, there you go. There you go.
1208	00:53:10	Ankur:	No, it's not the top[B2].
1209	00:53:12	Michael:	Yeah, the whole thing[B2].
1210	00:53:13	Ankur:	Plus one[B2]? Do you have that plus one on the bottom[B2]?
1211	00:53:18	Michael:	Yeah. Equals[B2]. Um. [Michael laughs.] Um, this whole thing on the bottom[B2], um.
1212	00:53:30	Ankur:	It's the same, it's the same thing[B2]. Just copy it.
1213	00:53:33	Jeff:	Yeah.
1214	00:53:34	Ankur:	N[B3].
1215	00:53:35	Jeff:	N[B3].

1216	00:53:35	Ankur:	Minus X[B3].
1217	00:53:36	Jeff:	Minus X plus, exactly[B3]. You know how like intimidating this equation must be[B3], like if you just pick up a book and look at that? There you go. Yeah.
1218	00:53:57	Michael:	There you go. That's what you want, I think.
1219	00:54:03	R1:	Do you all agree?
1220	00:54:04	Jeff:	Yeah. I got chalk all over my pants like Dr. Zabrower.
1221	00:54:11	Michael:	That means like-
1222	00:54:12	Jeff:	That's-
1223	00:54:13	Michael:	It's too confusing?
1224	00:54:14	R3:	Is that the same thing[B2]?
1225	00:54:15	Michael:	Yeah.
1226	00:54:15	Ankur:	It is the same thing[B2].
1227	00:54:17	R3:	It is?
1228	00:54:17	Michael:	Yeah. N[B3].
1229	00:54:17	Ankur:	As that. Yeah[B2].
1230	00:54:18	Michael:	This thing, all right, you see how that is that[B2]?
1231	00:54:20	R1:	Mm hm.
1232	00:54:22	Michael:	You know how- I'll go up there again.
1233	00:54:27	Jeff:	We just wrote out the, yeah, exactly, we wrote out the equation, how to find N choose[B3], exactly.
1234	00:54:33	Michael:	That's, that's, I guess that's what you want.
1235	00:54:37	Jeff:	Yeah. It's exactly- We just wrote, we instead of writing-
1236	00:54:39	Michael:	You agree with this? Right? So we just wrote, we wrote that[B3]-

1237	00:54:45	Jeff:	We wrote it in the, in the form[B3].
1238	00:54:45	Ankur:	In that form[B3].
1239	00:54:45	Michael:	It still doesn’t look, it doesn’t look too good.
1240	00:54:47	Jeff:	Yeah. It looks kind of mean.
1241	00:54:49	Michael:	We wrote that like that[B3].
1242	00:54:53	R1:	Did you all very carefully check that arithmetic[B2]?
1243	00:54:55	Michael:	You think we’re wrong?
1244	00:54:57	Ankur:	What, you found an error?
1245	00:54:58	Jeff:	All right. Well what’s, what, go to the, uh, write the regular equation down[B3].
1246	00:55:02	Romina:	Here’s a-
1247	00:55:02	Ankur:	There it is, right there[B3].
1248	00:55:02	R1:	Why don’t you get a piece of paper and-
1249	00:55:04	Jeff:	Where is it?
1250	00:55:05	Ankur:	It’s right above N over X[B3].
1251	00:55:05	Michael:	Oh, yeah. Never mind.
1252	00:55:06	Jeff:	All right.
1253	00:55:06	Romina:	You found it?
1254	00:55:06	Jeff:	Yeah.
1255	00:55:06	Ankur:	The first one.
1256	00:55:14	Michael:	There you go.
1257	00:55:17	Jeff:	Yeah, all right.
1258	00:55:19	R1:	You sure?
1259	00:55:21	Michael:	Yeah, I’m sure. You got anything else? Yeah, I guess.

1260	00:55:24	R1:	Did you check it?
1261	00:55:26	Michael:	What do you mean? Is it wrong?
1262	00:55:29	R1:	Now that, that's really, really very frightening.
1263	00:55:32	Michael:	Yeah.
1264	00:55:32	R1:	What do you think? Is that foreboding?
1265	00:55:35	Jeff:	I guess.
1266	00:55:36	R1:	I wonder if there's a way of simplifying it[B3].
1267	00:55:39	Jeff:	Of what?
1268	00:55:39	Michael:	Simplifying it[B3]. Hey!
1269	00:55:40	Ankur:	Yeah, you could [Inaudible.]; that's simplifying[B3].
1270	00:55:42	Jeff:	Yeah that's, that's pretty-
1271	00:55:44	R1:	That's a way to simplify it[B3]. But you know I see N plus one parenthesis minus parenthesis X plus one[B3]. That looks like that could be a little simpler[B3]. See that N plus one parenthesis that Michael just put there[B3].
1272	00:55:59	Michael:	Yeah.
1273	00:56:00	R1:	Minus the expression X plus one[B3]. Suppose you distributed that minus one[B3].
1274	00:56:07	Jeff:	So you want, all right, so- All right.
1275	00:56:10	Michael:	Why would you want to do that[B3]?
1276	00:56:10	Jeff:	So distributing, say over there[B3], right? You'd have, you'd have N plus one minus X minus one factorial[B3]?
1277	00:56:19	Romina:	Mm hm.
1278	00:56:20	Jeff:	Um, that would be in, in parenthesis[B3].
1279	00:56:24	Michael:	Oh yeah, yeah, there you go.

1280	00:56:24	Jeff:	And then, well, that[B3]-
1281	00:56:27	Romina:	Why don't you get another piece of paper?
1282	00:56:31	Jeff:	So, all right, so it'd be N plus one factorial divided by, um, N plus one in parentheses minus X minus one factorial[B3]. All right? And then, well, that's, that's pretty much all you can do there[B3]. Then X plus one factorial, so you could actually can, you can cancel out[B3]? Can you cancel that out[B3]? The X, minus X minus one and the X plus one[B3]? Or-
1283	00:57:04	R1:	That's what I'm asking you to think about. Not right, not now necessarily, but, um-
1284	00:57:06	Jeff:	Yeah, can you, I mean, can you cross out factorials or is that the first factorial on the bottom of the one all the way to the right[B3]? Does that affect, that's affecting the N plus one too, so can you, are you allowed to cross out like that[B3]? Cross these both out[B3]?
1285	00:57:20	R1:	What that's a good question. What do you all think?
1286	00:57:22	Jeff:	Well, can we throw in numbers and see[B2]?
1287	00:57:25	Romina:	Would we be able to cross out the N plus ones[B3]?
1288	00:57:27	Jeff:	Well then what are you left with[B3]?
1289	00:57:29	Romina:	Yeah. Yeah. It doesn't-
1290	00:57:30	Jeff:	Factorial divided by factorial[B3]?
1291	00:57:33	Michael:	Now wouldn't that just be, uh-
1292	00:57:35	Jeff:	Now I'm saying you could.
1293	00:57:36	Michael:	But now you're talking about simplifying[B2], wouldn't that just be, uh-
1294	00:57:38	Jeff:	Yeah.

1295	00:57:39	Romina:	I don't, would that, this whole thing be[B2]-
1296	00:57:41	Jeff:	Yeah then it would be nothing[B2], right?
1297	00:57:42	Ankur:	Plus one[B2].
1298	00:57:43	Romina:	Yeah.
1299	00:57:44	Jeff:	Then that would cross out and that would cross out[B2].
1300	00:57:45	Romina:	You get two factorials[B2].
1301	00:57:47	Ankur:	You can't do that[B2].
1302	00:57:47	Michael:	You know that
1303	00:57:48	Jeff:	Yeah.
1304	00:57:47	Michael:	She's talking about simplifying, and you just like, you know, put that negative in there and it would be just N minus X[B3]?
1305	00:57:56	Jeff:	Where? Where's this at[B2]?
1306	00:57:57	Michael:	Right at N minus. minus, that one right there[B2].
1307	00:57:59	Romina:	The one all the way to the side[B2].
1308	00:58:01	Jeff:	Oh yeah, and then the, all right, so you, so you do that, N minus X factorial[B3].
1309	00:58:01	Michael:	That. That could be[B2]-
1310	00:58:01	Jeff:	N minus. Yeah exactly[B2].
1311	00:58:04	Michael:	Uh, I'm not too good with my uh-
1312	00:58:07	Jeff:	Simplification[B2].
1313	00:58:08	Michael:	Yeah.
1314	00:58:08	Jeff:	Yeah, because that, it would be- You got the plus one[B2].

1315	00:58:11	Michael:	I'm just wondering. Wouldn't you, wouldn't that equal N plus one minus X minus one[B3]?
1316	00:58:19	Jeff:	Yes, then the plus one and the minus one[B2]-
1317	00:58:19	Michael:	Are gone[B2].
1318	00:58:19	Jeff:	So it would be N minus X factorial[B3].
1319	00:58:20	Michael:	N minus X so[B3]-
1320	00:58:21	Jeff:	It'd be N minus X factorial, um, times X plus one factorial[B3]? Right? Yeah.
1321	00:58:34	Michael:	A little simpler[B2]. I still don't like it though.
1322	00:58:37	Jeff:	Then, but then you could cross out, OK, could you cross out[B2]?
1323	00:58:39	Michael:	Which are you talking about[B2]?
1324	00:58:40	Jeff:	Up, no, the bottom and the top[B2].
1325	00:58:42	Romina:	The top[B2].
1326	00:58:42	Jeff:	Oh, that's plus one[B2]. All right, my bad, I wasn't even paying attention.
1327	00:58:45	Michael:	Anything else to simplify?
1328	00:58:49	Jeff:	Well, if X equals negative one, just[B2]-
1329	00:58:51	Ankur:	And can't you do that on the other side too[B2]?
1330	00:58:51	Michael:	Um.
1331	00:58:51	Romina::	Um.
1332	00:58:54	Jeff:	That would be, um[B3]-
1333	00:58:56	Ankur:	It would be N minus one[B3].
1334	00:58:56	Jeff:	N minus X minus one factorial[B3]. No.

1335	00:59:01	Michael:	No, it'll still be the same number[B2].
1336	00:59:02	Jeff:	Yeah. And it'll be X plus one[B3].
1337	00:59:03	Michael:	You want us to do that, do that too? Or don't even bother.
1338	00:59:05	Jeff:	Factorial[B3].
1339	00:59:08	R1:	I'm, I'm impressed that twenty of ten you're doing this arithmetic[B2]. Um, you know, of course the next thing to do is to learn how to do the algebra of factorials so that you indeed could do the addition[B3].
1340	00:59:23	Michael:	[Inaudible.].
1341	00:59:23	Jeff:	[Inaudible.] the factorial[B3].
1342	00:59:24	R1:	Would you like to know how to do that[B3]? Would you like to know how to do the algebra of factorials[B3]? I bet you know how to do a little bit already. I'll just show you one thing that I know you know and I'll leave you to think about this because everyone is getting tired, but let's just take something like this[B3], right? Six choose two[B2], right? And you know, you, you told me you could write that how? As-
1343	00:59:55	Michael:	Um, six factorial over[B2]-
1344	00:59:57	R1:	Six factorial[B2].
1345	00:59:59	Michael:	Three fact, four factorial times two factorial[B2].
1346	01:00:03	R1:	Times two factorial, right[B2]?
1347	01:00:05	Romina:	Mm hm.
1348	01:00:06	R1:	And you know what six factorial is[B2], right? Six times five[B2].
1349	01:00:11	Michael:	Times one-twenty[B2].
1350	01:00:12	Jeff:	Thirty[B2]. Yeah.

1351	01:00:13	R1:	I'm not going to do that though. I don't like to. I don't like to do multiplication[B2]. I'm very lazy. I'm just going to write six times five times four factorial[B2]. Is that okay?
1352	01:00:21	Jeff:	That's, that's simplifying is great, then you can[B2]- [Students all talk at once.]
1353	01:00:24	R1:	But can I do that?
1354	01:00:26	Romina:	Yeah.
1355	01:00:26	Michael:	And then you could cross out the four factorials[B2] and-
1356	01:00:27	Romina:	Oh.
1357	01:00:28	R1:	Oh, then I can cross out the four factorials[B2].
1358	01:00:28	Jeff:	Oh, all right, that makes sense.
1359	01:00:29	R1:	Right?
1360	01:00:31	Jeff:	So you just get thirty divided by, you get thirty divided by two[B2].
1361	01:00:33	R1:	Yeah. Look at all the time that will save you in an SAT question.
1362	01:00:35	Jeff:	That'd be big.
1363	01:00:37	R1:	But, but if you think about this-
1364	01:00:39	Jeff:	She broke, she broke it down farther[B2].
1365	01:00:40	Romina:	Oh yeah she just-
1366	01:00:42	Jeff:	Like rather than say you have six factorial[B2]-
1367	01:00:43	Ankur:	Mm hm.
1368	01:00:43	Jeff:	She broke it down until she got a number that she got that she wanted[B2].
1369	01:00:45	Romina:	She had two numbers[B2].

1370	01:00:47	Jeff:	That matched the number on the bottom[B2].
1371	01:00:48	Ankur:	All right. Yeah.
1372	01:00:50	Jeff:	Then you end up like with the two factorial and then cross out and that's thirty over the two factorial and that's two[B2]. So it's just fifteen[B2].
1373	01:00:51	Michael:	But then it would probably be even longer than that[B2]. Cause if N is a big number[B3]-
1374	01:00:55	R1:	Does it matter?
1375	01:00:59	Michael:	-you'd have to write, you would have to write N times N minus one times N minus[B3]

Appendix C: Semi-logarithmic plots of cumulative count of coded conversation instances.

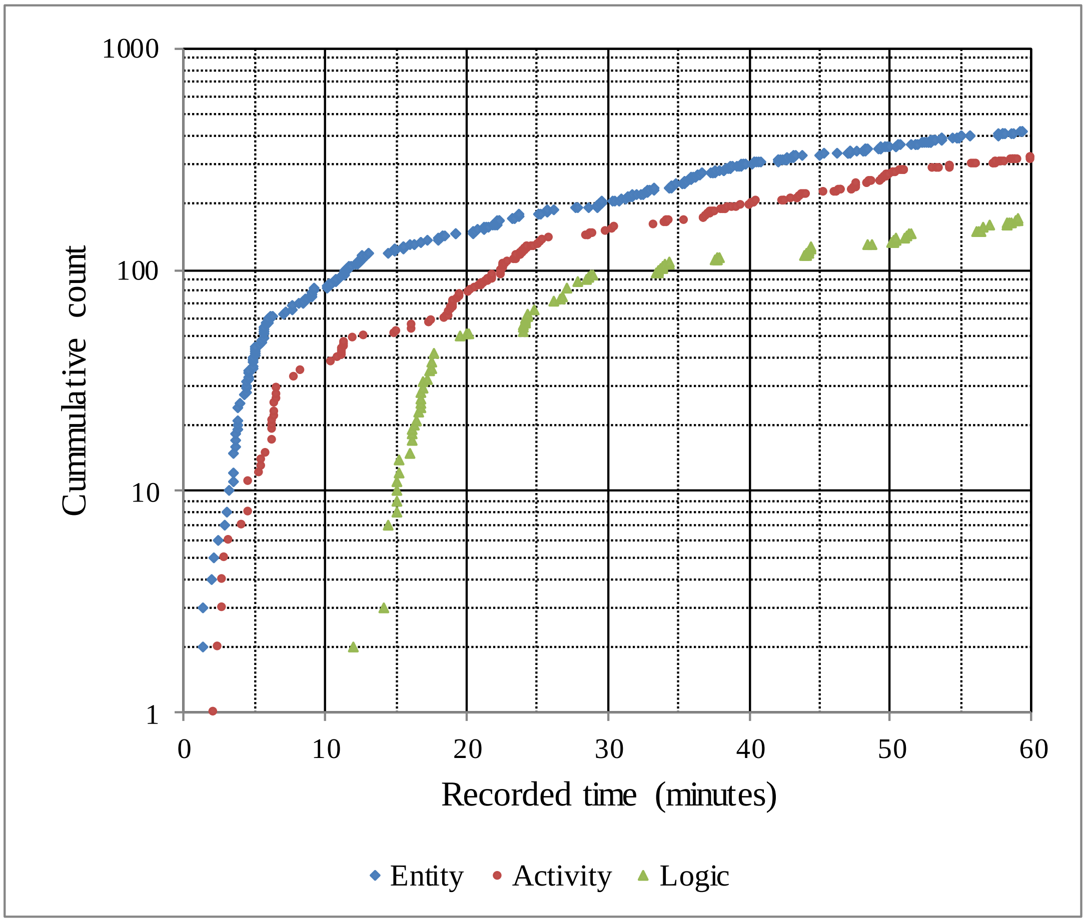

Figure C1. Count of conversation instances coded for the linguistic structures. The total number of coded instances is distributed as follows: Entity – 429, Activity – 329, and Logic – 174.

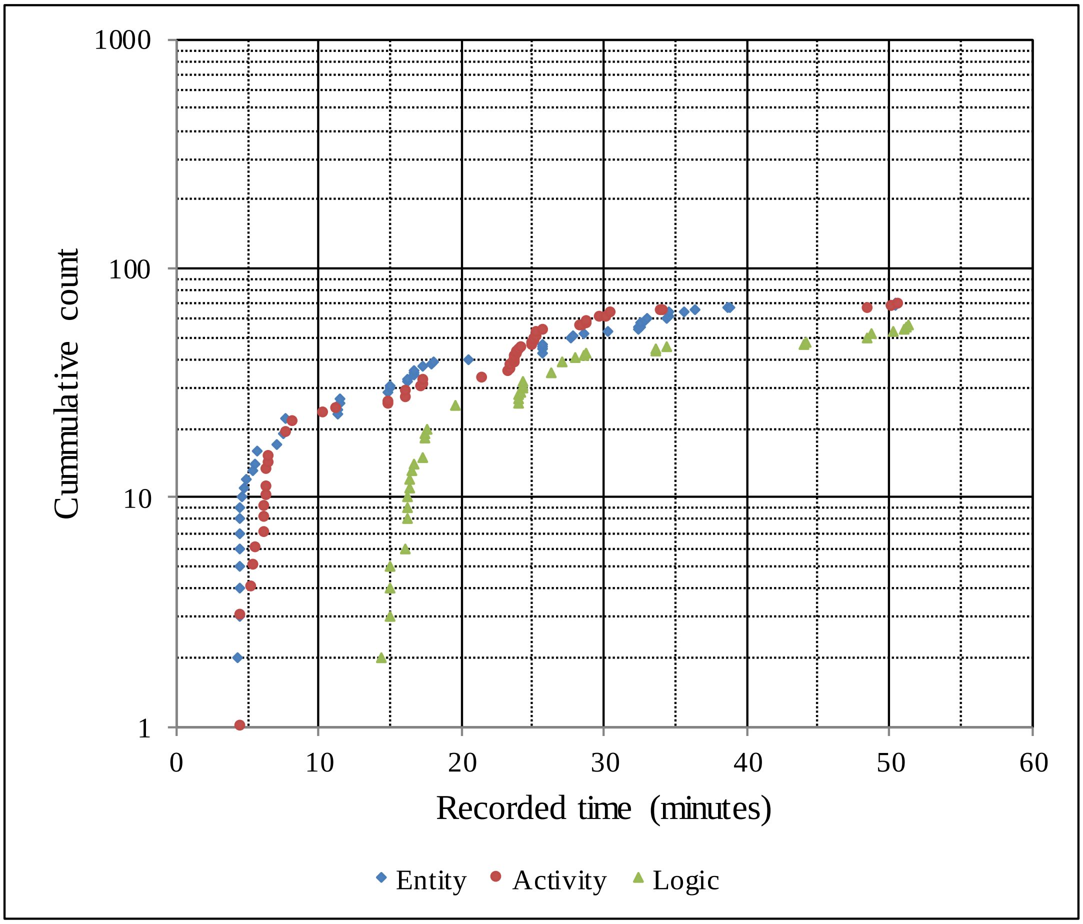

Figure C2. Count of conversation instances coded for the linguistic structures when the students involve the visual manipulative. The total number of coded instances is distributed as follows: Entity – 71, Activity – 69, and Symbolic – 57.

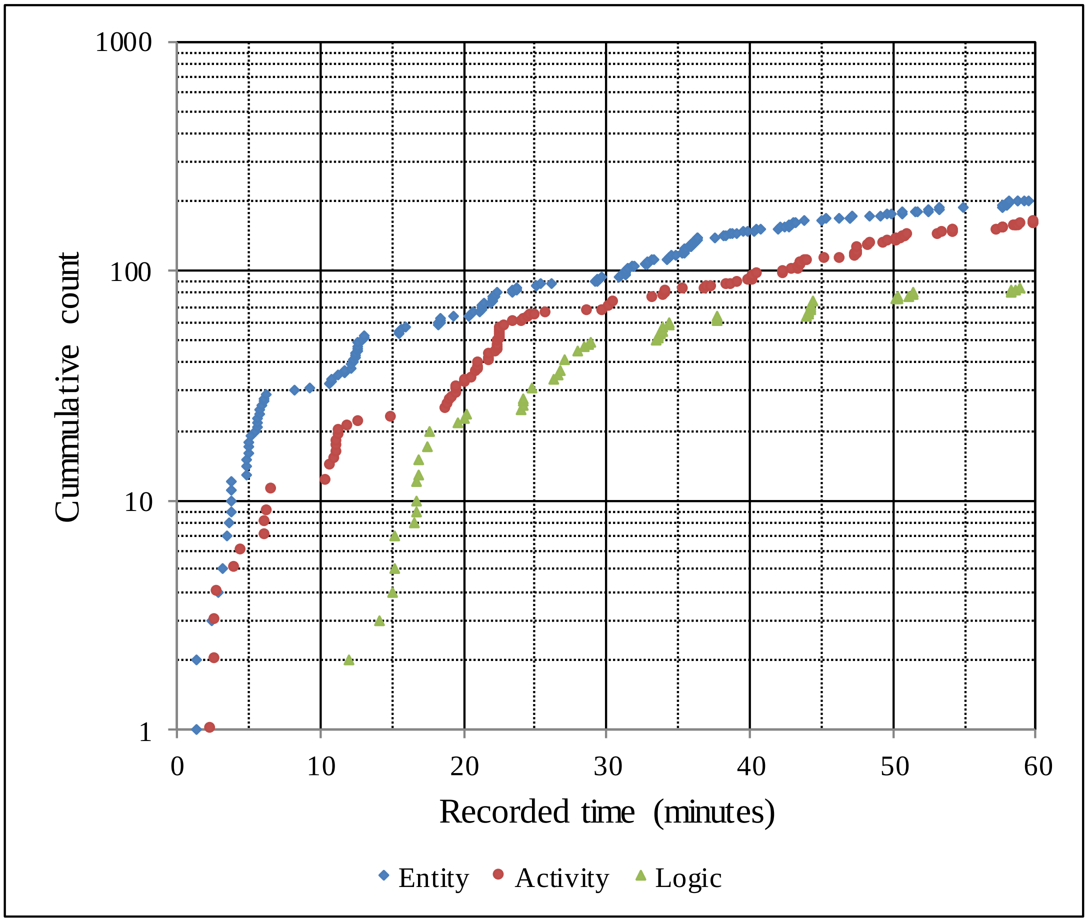

Figure C3. Count of conversation instances coded for the linguistic structures when the students involve the quantitative systems. The total number of coded instances is distributed as follows: Entity – 210, Activity – 171, and Symbolic – 87.

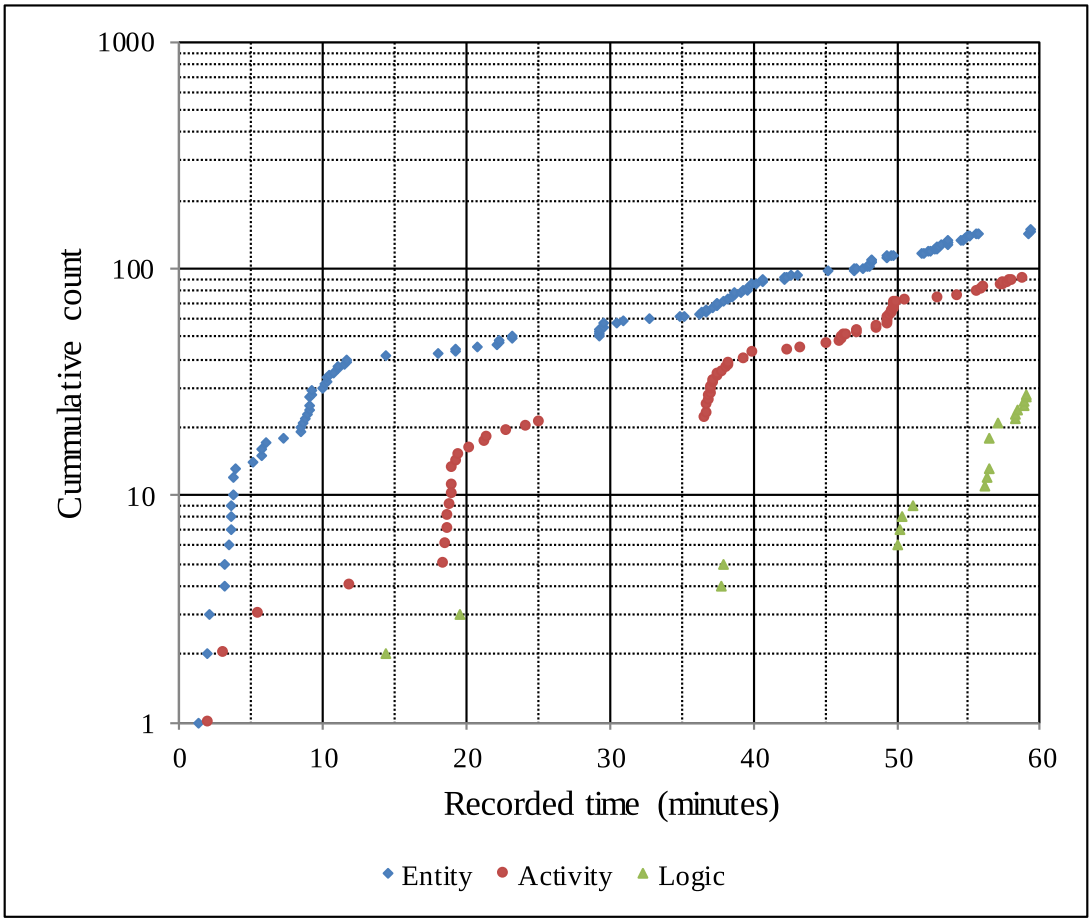

Figure C4. Count of conversation instances coded for the linguistic structures when the students involve the symbolic notations. The total number of coded instances is distributed as follows: Entity – 148, Activity – 89, and Symbolic – 30.

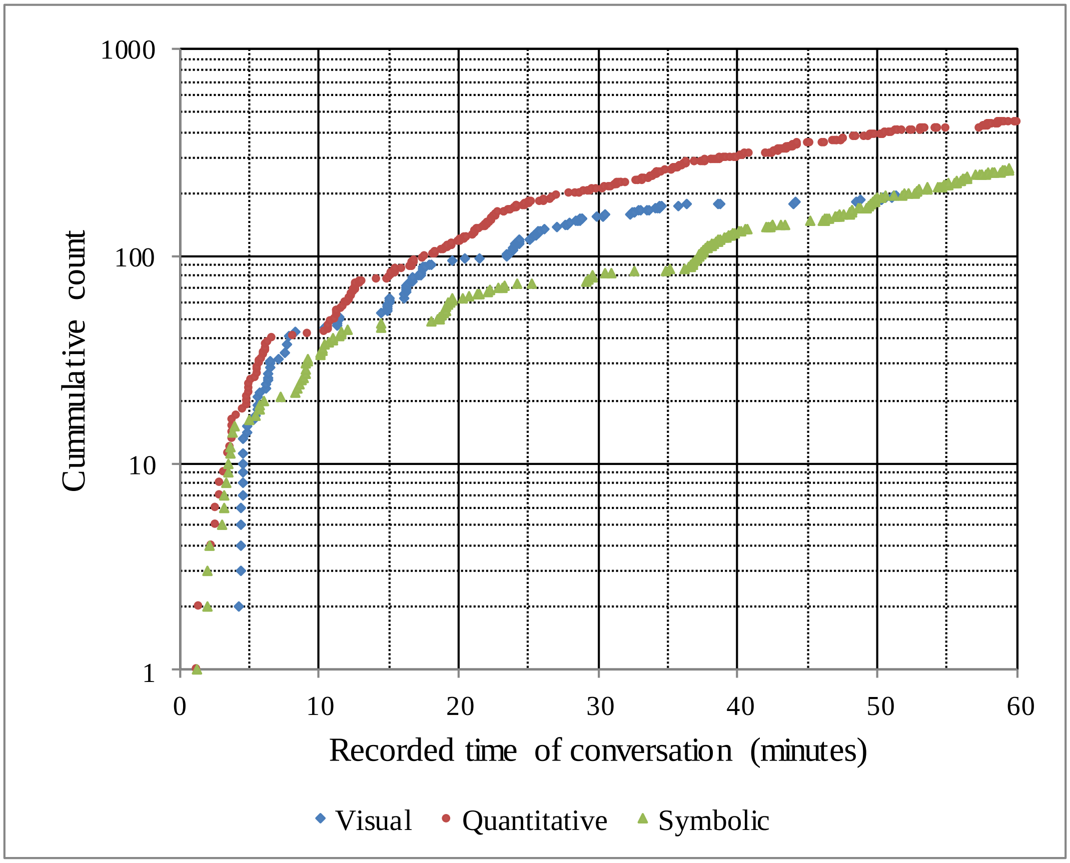

Figure C5. Count of conversation instances coded for the semiotic models. The total number of coded instances is distributed as follows: Visual – 197, Quantitative – 468, and Symbolic – 67.

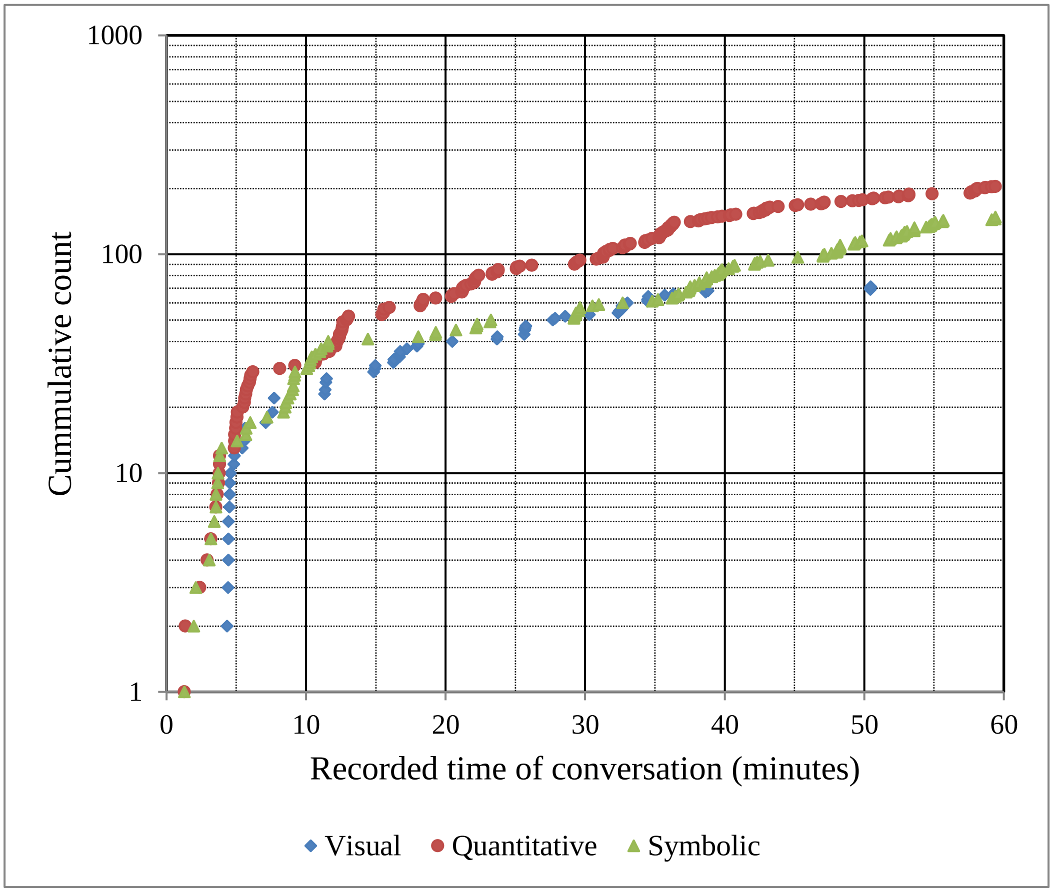

Figure C6. Count of conversation instances coded for the semiotic models when the students use the mathematical entity. The total number of coded instances is distributed as follows: Visual – 71, Quantitative – 210, and Symbolic – 148.

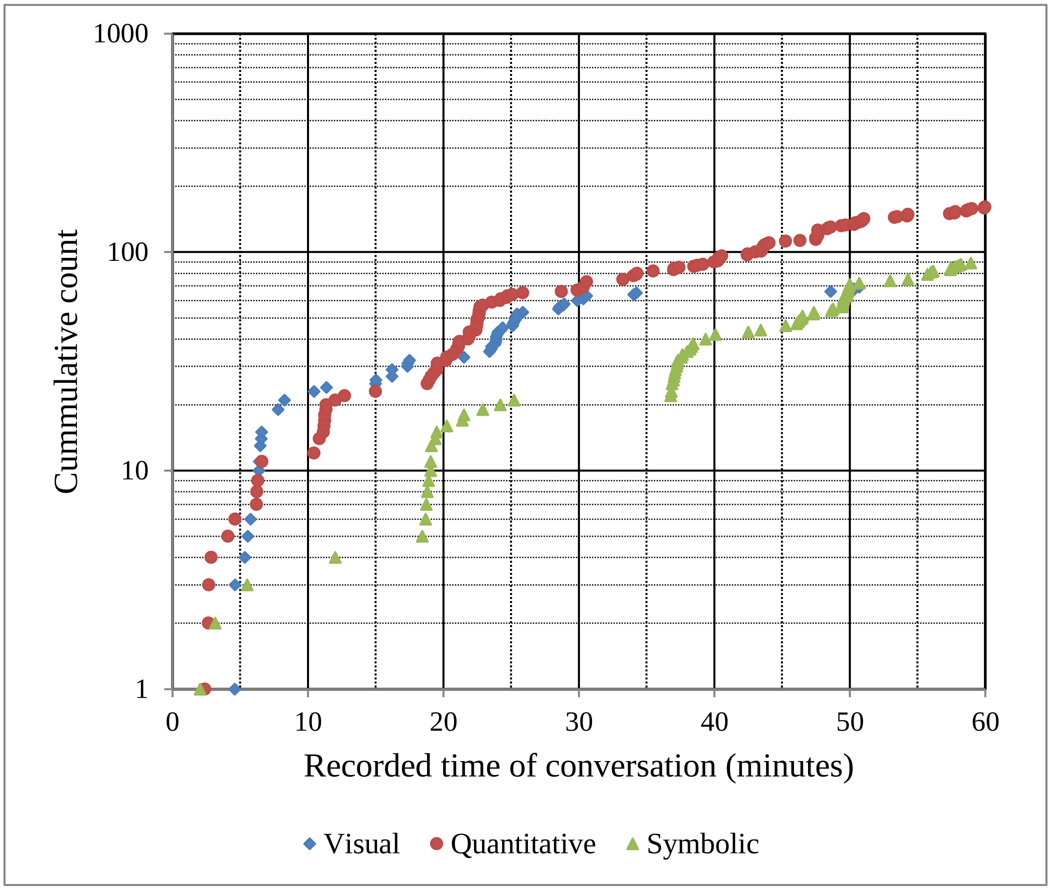

Figure C7. Count of conversation instances coded for the semiotic models when the students use the mathematical activity. The total number of coded instances is distributed as follows: Visual – 69, Quantitative – 171, and Symbolic – 89.

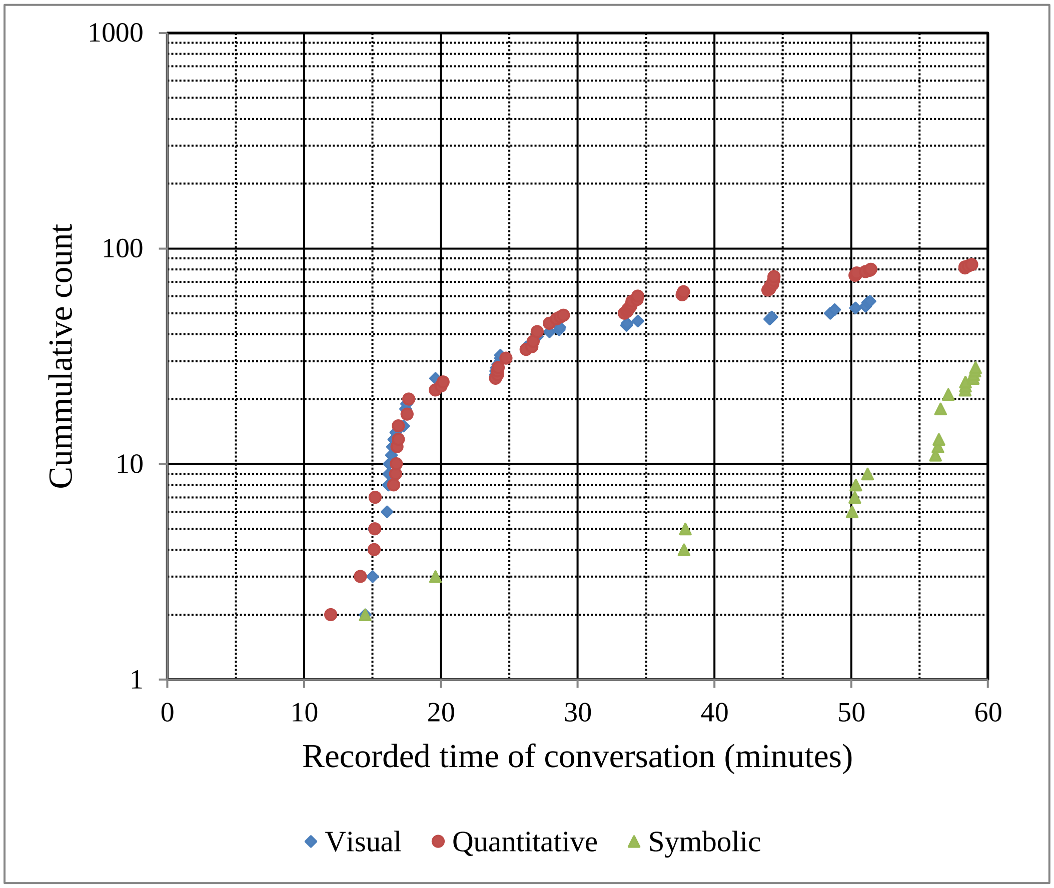

Figure C8. Count of conversation instances coded for the semiotic models when the students use the mathematical logic. The total number of coded instances is distributed as follows: Visual – 57, Quantitative – 87, and Symbolic – 30.