

MONOPOLES, BPS STATES, AND 'T HOOFT DEFECTS IN 4D $\mathcal{N} = 2$ THEORIES OF CLASS \mathcal{S}

by

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ABSTRACT OF THE DISSERTATION

Monopoles, BPS States, and 't Hooft Defects in 4D $\mathcal{N} = 2$ Theories of Class \mathcal{S}

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Monopoles are a fundamental feature of non-abelian gauge theories. They are relevant to the study of confinement and general non-perturbative quantum effects. In this dissertation we study some aspects of monopoles in supersymmetric non-abelian gauge theories. In particular, we focus primarily on 't Hooft defects (magnetically charged defects) and their interaction with smooth, supersymmetric monopoles. Here we use a semiclassical approximation to study the spectrum of bound states between such monopoles and 't Hooft defects and the phase transitions where this spectrum changes discontinuously. Then, we use string theory and localization techniques to compute the expectation value of 't Hooft defects as operators in the full quantum theory. Using the computed expectation value, we are able to directly study the non-perturbative process called monopole bubbling in which smooth monopoles dissolve into an 't Hooft defect. Then, by combining the results of string theory techniques with localization techniques, we are able to derive general formulas for the full spectrum of monopole bound states in all possible phases of the theory.

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Dedication

To my father Theodore Dennis Brennan Jr.

Table of Contents

Abstract	ii
Acknowledgements	iii
Dedication	iv
1. Introduction	1
1.1. Summary	4
2. Monopoles	7
2.1. Smooth Monopoles	7
2.2. Singular Monopoles	9
2.3. Construction of Monopole Solutions	10
2.3.1. ADHM Construction of Instantons	11
2.3.2. ADHMN Construction of Smooth Monopoles	13
Generalizing to $SU(N)$	15
From Bogomolny to Nahm Data	16
2.3.3. Singular Monopoles and Kronheimer's Correspondence	17
Taub-NUT Spaces	17
Review of Kronheimer's Correspondence	18
2.3.4. Instantons on multi-Taub-NUT and Bows	20
Quick Review of Bows	20
Bow Construction of Instantons	24
2.4. Monopole Moduli Space	27
ADHMN Construction of Monopole Moduli Space	28
2.4.1. Singular Monopole Moduli Space	29
Bow Construction of Singular Monopole Moduli Space	30

Bow Variety Isomorphisms: Hanany-Witten Transitions	30
Singularity Structure: Monopole Bubbling	31
2.4.2. Triholomorphic Killing Vectors and Symmetries	32
2.4.3. Universal Bundle	34
2.5. Rational Map Construction of Monopole Moduli Space	35
2.5.1. Explicit Construction of Flag Data and Rational Maps	39
2.5.2. Physical Interpretation of Rational Map	40
Example: 2 $SU(2)$ Monopoles	41
Example: 2 $SU(3)$ Monopoles	41
2.5.3. Rational Map Formulation of Singular Monopole Moduli Space	43
Example: Singular Monopoles in $PSU(2)$	44
3. Semiclassical BPS States	45
3.1. BPS States	46
3.2. $\mathcal{N} = 2$ Supersymmetric Gauge Theory	48
3.2.1. Low Energy Theory	51
3.3. Vanilla BPS States	53
3.3.1. Collective Coordinate Expansion	56
3.3.2. Universal Hilbert Bundle	59
Collective Coordinate Symmetries	60
3.4. Line Defects	61
3.4.1. Wilson Lines	62
Coadjoint Orbit Quantization	63
SUSY Wilson Lines	66
3.4.2. 't Hooft Defects	66
3.4.3. General Wilson-'t Hooft Defects	69
3.5. Framed BPS States	70
3.5.1. Collective Coordinate Dynamics with Wilson-'t Hooft Defects	71
The Wilson Bundle	73

Hamiltonian Dynamics	74
3.5.2. Quantization	74
Comparison with Low Energy Limit	77
3.5.3. 1-loop Corrections	77
3.5.4. Extended Example	78
3.5.5. Sen Conjecture	79
3.6. BPS States and Hypermultiplets	81
Multiple Hypermultiplets	85
Example	85
3.6.1. Matter Bundle and the Universal Connection	86
3.6.2. Collective Coordinate Dynamics	86
Hamiltonian Formalism	88
3.6.3. Quantization	89
3.6.4. Generalized Sen Conjecture	90
Conserved Charges	91
3.6.5. Summary of Collective Coordinate Analysis	92
3.7. Rational Maps and Hyperholomorphic Vector Bundles	94
3.7.1. Hypermultiplet Matter Bundle	94
Matter Bundle Factor with Vanishing Mass	96
Rational Map Formulation of Matter Bundle	97
Remark on $\mathcal{N} = 2^*$ Theory	101
3.7.2. Wilson Bundle	102
Coadjoint Orbit Quantization	102
3.7.3. Rational Maps and General Bundles for Framed BPS States . . .	106
Matter Bundle for Framed BPS States	106
Wilson Bundle for Framed BPS States	107
4. Wall Crossing of Semiclassical Framed BPS States	108
4.1. Asymptotic Regions Of Moduli Space	108

4.1.1.	Triholomorphic Killing Vectors	112
4.2.	The Asymptotic Dirac Operator	113
4.3.	Framed BPS States and Haloes	117
4.3.1.	Pure 't Hooft Defects	118
4.3.2.	Including Wilson Defects	124
4.3.3.	Wilson-'t Hooft Defects	126
	Pure Wilson Defects	128
5.	BPS States in String Theory	130
5.1.	Reducible 't Hooft Defects in String Theory	131
5.1.1.	SUSY vacua	135
	Low Energy Effective Theory	135
	Vacuum Equations	141
5.1.2.	Monopole Bubbling	144
5.1.3.	Kronheimer's Correspondence and T-Duality	148
	Action of T-Duality on Fields	150
	String Theory Analysis	151
	T-duality and Line Bundles	154
5.2.	Irreducible Monopoles	158
5.2.1.	$SU(2)$ Irreducible 't Hooft Defects	159
5.2.2.	$SU(N)$ Irreducible Monopoles	162
5.2.3.	Physical 't Hooft Charges	164
5.3.	Including Fundamental Hypermultiplets	167
5.3.1.	Fundamental Hypermultiplets and Brane Webs	168
	Brief Review on (p, q) -Brane Webs	168
	Improved Brane Configuration	172
5.4.	The Class \mathcal{S} Construction	173
5.4.1.	The 6D $\mathcal{N} = (2, 0)$ Theory	174
5.4.2.	The Class \mathcal{S} Construction of $SU(N)$ Theories	176

5.4.3.	't Hooft Defects in Theories of Class \mathcal{S}	178
6.	Expectation Value of 't Hooft Defects	181
6.1.	Line Defects in Theories of Class \mathcal{S}	181
6.1.1.	Theories of Class \mathcal{S} and Spectral Networks	183
6.1.2.	Wall Crossing in Spectral Networks	188
6.1.3.	Line Defects and the AGT Correspondence	192
	Complexified Fenchel-Nielsen Coordinates	193
6.2.	Localization for $Z_{mono}(P, \mathbf{v})$	194
6.2.1.	Localization	197
6.2.2.	Bubbling SQMs	199
	Localization	205
	Regularization	208
6.2.3.	Reduction to Contour Integral	210
	Boundary Terms at Infinity	213
	Comparison with Literature	215
6.2.4.	Coulomb and Higgs Branch States	215
	Physical Interpretation of Jeffrey-Kirwan Residues	217
6.2.5.	Ground State Index	219
6.2.6.	Summary	221
6.2.7.	Examples: $SU(2)$ $N_f = 4$ Theory	223
	$Z_{mono}(1, 0)$	223
	$Z_{mono}(2, 1)$	225
6.3.	Proposed Resolution: Coulomb Branch States	228
6.3.1.	Witten Indices with Continuous Spectra	228
	Toy Model on Semi-Infinite Line	228
	Effective Coulomb Branch SQM	233
6.3.2.	Proposal	235
	Relation to Defect Contribution	236

6.3.3. Examples	238
$SU(2)$ $N_f = 4$ Theory	238
$SU(2)$ $N_f = 2$ Theory	240
$SU(2)$ SYM Theory	241
$SU(2)$ $\mathcal{N} = 2^*$ Theory	243
6.3.4. Comment on 4D $\mathcal{N} = 2$ $SU(2)$ Quiver Gauge Theories	244
6.4. Decoupling Flavors	247
6.4.1. Examples	248
$\langle L_{1,0} \rangle$ in the $N_f = 3$ Theory	249
$\langle L_{1,0} \rangle$ in the $N_f = 2$ Theory	250
7. Index Theorems on \mathcal{M} and Characteristic Numbers on Kronheimer-Nakajima Spaces	251
7.1. Fenchel-Nielsen Networks	254
7.1.1. Fenchel-Nielsen Spectral Networks and The Semiclassical Region	256
7.2. Index Theorem and Characteristic Numbers	261
7.2.1. General Theory	261
7.2.2. Example: $SU(2)$ $N_f = 0$ Theory	264
Index Theorem	269
Characteristic Numbers	271
Explicit Example: $\langle L_{2,0} \rangle$ in $SU(2)$ SYM	273
7.2.3. Comments on the $\mathcal{N} = 2^*$ Theory	276
Appendix A. Semiclassical Framed BPS States	280
A.1. Collective Coordinate Calculation for SYM	280
A.1.1. Wilson-'t Hooft Collective Coordinate Supersymmetry	284
A.2. Hypermultiplet Index Calculation	286
A.2.1. Reduction to Boundary	286
Contribution from Defects	289
A.3. Hypermultiplet Collective Coordinates	290

A.4. 1-Loop Mass Contribution from HM	290
A.5. Hypermultiplet Collective Coordinate Supersymmetry	293
A.6. Conventions for Hypermultiplets	295
A.7. $SU(2)_R$ Invariance of Hypermultiplets	296
Appendix B. Kronheimer's Correspondence	300
B.1. Review of Taub-NUT Spaces	300
B.2. Kronheimer's Correspondence for a Single Defect	301
B.3. Generalization to Multiple Defects	306
Appendix C. Wall Crossing	309
C.1. The Two-galaxy Region for Smooth Monopoles	309
C.1.1. Hyperkählerity of the metric	313
C.2. The Dirac Operator in the Two-galaxy Region	314
C.2.1. Vielbein	314
C.2.2. Spin connection	315
C.2.3. Dirac operator	318
C.3. Singular Monopole Moduli Space	320
Appendix D. SQM Localization	323
D.1. $U(1)$ $\mathcal{N} = (0, 4)$ SQM Analysis	323
D.1.1. Matter Ground States	332
D.1.2. Asymptotic States	337
D.1.3. Hermitian Supercharge Operators and Boundary Conditions . . .	341
D.1.4. Extra Contribution to the Witten Index	343
D.1.5. 1D Wilson Lines	345
D.2. $U(1)^3$ $\mathcal{N} = (0, 4)$ SQM Analysis	346
D.2.1. Effective Hamiltonian	356
D.2.2. Ground States	358
Ground States in S_+	360

Ground States in S_-	361
Hermiticity	362
D.2.3. Mixed Branch	363
D.2.4. Contribution to the Witten Index	364
D.3. Behavior of Localized Path Integral at Infinity	365
D.4. A Useful Integral	367
D.5. Bubbling Contribution in the $SU(2) \times SU(2)$ SCFT	368
References	372

Chapter 1

Introduction

Quantum physics governs the laws of the universe at small sizes. Contrary to the philosophy of classical physics, quantum physics is fundamentally probabilistic. At any given point in time, the universe can be in a simultaneous superposition of many different configurations. In quantum theories, only the probability that the universe evolves from one state definite to another is determined.

Our current understanding of quantum physics is primarily based upon the notion of quantizing a classical theory. Generally, the state of a system is described by a quantum probability function. Physical quantities can be computed from this probability function, and are often of the form of the classical quantity plus a series of small quantum corrections that are controlled by a small, dimensionless parameter. In practice, one usually computes these perturbative corrections to the classical result to whatever degree of precision is required for the problem at hand. While this way of approaching quantum physics has been wildly successful, a perturbative understanding of quantum physics leads to an inherently incomplete description of the universe.

There are many physical quantum systems that cannot be adequately described by perturbative quantum physics. Notably, such systems often exhibit the (related)) features where 1.) there is no small expansion parameter or 2.) where there is collective behavior. In the first case, the coupling parameter that would control the size of quantum corrections is large enough that the quantum corrections are an infinite series of terms that are all of the same size (or worse of increasing size). This would require us to compute all of the contributions to a physical quantity, which, by using standard techniques, is technically impossible.

Collective behavior, also goes beyond a perturbative understanding of quantum

physics. It describes the situation when there is some large scale behavior that cannot be understood by looking at the local degrees of freedom. The canonical example of a classical system that has collective behavior the case of a “solitary water wave” which has a profile given by the hyperbolic secant function. Here the water molecules, which are locally swirling around, conspire to support the solitary wave.

In such systems, collective phenomena give rise to an infinite number of terms in the perturbation series which, when truncated to a finite number of corrections, does not correctly describe the behavior of the physical phenomenon. This is analogous to how the Taylor series expansion of a sine function has an infinite number of terms, which, when truncated to any finite polynomial, is infinitely far from approximating the sine function. Or alternatively, we can think of Zeno’s paradox in which after any finite number of steps, a runner who goes half of the remaining distance does not reach the goal – one requires an infinite number of steps to see that the runner does indeed cross the finish line.

One of the fundamental quantum theories of the universe that exhibits both of the problems we have described is the strong force: quantum chromodynamics (QCD). In QCD, the coupling parameter that controls the perturbation series, is not small $g_{QCD} \sim 1$. There are also collective excitations in QCD. Famously, most of the matter in the universe is made out of protons (and neutrons) which are non-perturbative bound states of three quarks, the fundamental constituents of most matter in the Universe. These quarks are held together by non-perturbative “strings” which are collective excitations of the gluon field. Our lack of understanding about this feature of QCD is perhaps most succinctly summarized by the statement that we do not understand why the proton is stable. Said differently, we do not actually know why all of the matter in the universe does not fall apart.

This fundamental lack of understanding about the way the universe works begs for better tools to study quantum physics. One way we can hope to develop a new understanding is by studying collective excitations in theories in which there are other tools that provide more analytic control. One such avenue of research, and the focus of this work, is to study non-local excitations in supersymmetric quantum field theories.

Supersymmetry is a distinguished symmetry of a quantum theory in which every particle/field has a partner “super”-particle/field. This pairing gives rise to extraordinary cancellations and fantastical properties in the full quantum theory. Supersymmetry provides a microscope that can be used to study non-perturbative quantum effects. Using supersymmetry has been wildly successful and has been used in works such as [156, 155, 149, 81, 67] to completely compute quantities in the quantum theory (to all orders in perturbation theory including non-perturbative corrections).

Additionally, supersymmetric theories can often be interpreted as a low energy description of certain string theory configurations. This allows us to use the full power of string theory to study features of certain types of quantum field theories that can be described in this way. This provides us with additional tools to study non-perturbative effects in such theories.

Here, we will mainly focus on the study of monopoles in supersymmetric field theories. Monopoles are a non-local quantum excitations that source magnetic charge. They are one of the fundamental types of non-local excitations. They behave as particles, allowing us to rely on general physical intuition on particles in quantum theories.

Supersymmetry is particularly powerful when used to study monopole configurations. In supersymmetric quantum field theories, monopoles preserve at most half of the supersymmetries of the theory. States of this type are called BPS states. Because of the rigid structure of supersymmetry, these states are very robust: they are protected from decay by supersymmetry. This simplifies the study of such BPS states as compared to non-SUSY states over which we have practically no control.

Here we will primarily focus on studying a special type of BPS state called framed BPS states. These are dyonic states that are bound to a defect operator. This can be thought of as a quantum bound state in a background potential or an infinitely heavy, classical particle. This case distills many of the essential features of general BPS states while simplifying the analysis (since it is generally easier to study $(N - 1)$ -bodies in a fixed background than it is to study general N -body motion).

Due to SUSY, BPS states are generically protected from decay as one changes the

physical parameters of a supersymmetric theory, such as masses and coupling parameters. However, there are special regions in parameter space where the spectrum of BPS states can change. This is the phenomenon known as BPS state wall crossing. As it turns out this BPS state wall crossing occurs in a very controlled way, which has been known for some time [106, 107, 67, 50, 99, 150]. This allows us to infer the BPS spectrum everywhere in parameter space (including strong coupling) from a knowledge of the BPS spectrum in some weakly coupled region where we have analytic control. Thus, studying monopoles in supersymmetric quantum field theories gives us direct insight into quantum physics outside of the perturbative regime. This will be the main topic of this work.

1.1 Summary

In this work we will study framed BPS states in 4D $\mathcal{N} = 2$ theories on $\mathbb{R}^3 \times S^1$. We will primarily focus on using two techniques to study these states: 1.) semiclassical analysis and 2.) string theory constructions. These two tools will give us complimentary, geometric pictures of BPS states.

In Chapter 3 we will study the dynamics of semiclassical BPS states. These have dynamics coming from quantum fluctuations of the fields around the monopole background and dynamics of the monopoles themselves. We will treat this in a Born-Oppenheimer approximation where we integrate out the quantum fluctuations to obtain an effective description of BPS state interactions as a supersymmetric quantum mechanics (SQM) on the space of BPS field configurations (monopole moduli space). This description maps many of the difficult questions in the full QFT to an easily stated quantum mechanics problem. For example, in this formulation, stable BPS states are given by solutions to the Dirac equation on monopole moduli space.

As we will show in Chapter 4, the semiclassical description of (framed) BPS states is particularly useful for studying BPS state wall crossing. In the semiclassical limit, generic BPS states can be thought of as a multi-particle state made up from simple dyonic particles. Here the BPS multi-particle state has a structure similar to that

of a galaxy or a cluster of stars. When BPS wall crossing occurs, this decays into sub-clusters which are no longer bound to each other.

For simplicity, we will study the 2-body decay of such BPS states. Since BPS states are indistinguishable particles, we would expect that there is some kind of universal behavior of multi-body decay. As we will show, this is indeed the case and that the 2-body (or primitive) wall crossing is controlled by the universal behavior of the Dirac equation on the four-dimensional Taub-NUT space.

Then, in Chapter 5 we will go on to give a particularly useful embedding of monopoles and dyonic BPS states in 4D supersymmetric Yang-Mills theory into string theory. In summary, theory can be described as the low energy effective theory on a stack of D3-branes in which smooth monopoles (BPS states) can be realized as D1-branes running between the D3-branes and singular monopoles (magnetically charged line defects) can be described by D1-branes stretched between the D3-branes and transverse NS5-branes.

We will show that this brane configuration gives new insight into the non-perturbative phenomenon of monopole bubbling. Monopole bubbling describes the processes in which smooth monopoles dissolve into a singular monopole. This screening the magnetic charge of the singular monopole and deposits quantum degrees of freedom on its world volume. The brane construction of singular monopoles we present here is especially useful as it allows us to determine the exact content of the SQM that arises from monopole bubbling.

We then go on to study the expectation value of magnetically charged line defects. This is intimately related to the study of color confinement in QCD [163]. This is the mechanism that squeezes the gluon field into QCD strings which hold the nucleons together. Color confinement is exemplified by the curious feature of the universe that quarks do not appear alone. Famously, color confinement is measured by the expectation value of line defects wrapped on a circle [172, 163]. This is because it measures the total energy of a quark-antiquark (or monopole-antimonopole) pair creation and annihilation. This allows one to determine the behavior of the confining potential. Thus, the expectation value of a line defect is of great interest.

In Chapter 6, we discuss the non-perturbative part of the expectation value of an 't Hooft defect that comes from monopole bubbling. We will show that there is an important contribution from the quantum degrees of freedom that are deposited on the world volume of the 't Hooft defect that arise in monopole bubbling.

Upon closer inspection, we find that computing this expectation value is quite tricky. The standard method, for computing supersymmetric operators in a quantum field theory is called localization. This uses the cancellation between super-partners to reduce the path integral of a usual quantum field, which is an integral over an infinite dimensional space, to an integral over a finite dimensional space that can be evaluated by using computational tricks. However, as we will show, the naive application of this machinery to the case of the SQM from monopole bubbling gives the incorrect answer. Because of this, we need to add an additional contribution to the “standard answer.” We will give both a prescription and a physical explanation for this correction of the localization computation.

Then in the final section, we derive some new results in mathematics arising from the comparison of the semiclassical spectrum of framed BPS states bound to an 't Hooft defect with results from the localization computation of expectation value of the same 't Hooft defect. This gives us both an index theorem of Dirac operators on monopole moduli spaces (which encodes the framed BPS indices) and a formula for characteristic numbers of related Kronheimer-Nakajima spaces. These results are novel because there are no known general results for general non-compact spaces.

Chapter 2

Monopoles

Monopoles are non-trivial solutions to the classical equations of motion in Yang-Mills theory that carry a non-trivial magnetic charge. They can be thought of as magnetically charged particles which, in a weak coupling description, are very heavy compared to W-bosons. In a pure Yang-Mills theory monopoles are singular and are called singular (or Dirac) monopoles. However, in the case of non-abelian Yang-Mills theory coupled to a Higgs field, there also exist smooth field configurations that source magnetic charge. These monopoles are dynamical objects in the full quantum theory and what are generally referred when speaking of monopoles. Such monopoles, or smooth monopoles, will be the main focus for this chapter. See [85, 170, 131] for review.

2.1 Smooth Monopoles

Smooth monopoles exist in Yang-Mills theory on $\mathbb{R} \times \mathbb{R}^3$ with non-abelian gauge group G coupled to an adjoint valued Higgs field X . This theory has an action:

$$S = \frac{1}{g^2} \int d^4x \operatorname{Tr} \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu X D^\mu X - V(X) \right\} , \quad (2.1)$$

where $V(X) \geq 0$. The corresponding Hamiltonian can be written as

$$\begin{aligned} H = \frac{1}{g^2} \int d^4x \operatorname{Tr} \{ & E^i E_i + (D_0 X)^2 + |B_i - D_i X|^2 + V(X) \} \\ & + \frac{2}{g^2} \int d^4x \epsilon^{abc} \operatorname{Tr} \{ \partial_a (X F_{bc}) \} . \end{aligned} \quad (2.2)$$

This Chapter is based on material from my papers [24, 28].

Thus, we see that the energy is minimized by solutions to the equation

$$B_i = D_i X , \quad (2.3)$$

subject to the condition $\lim_{r \rightarrow \infty} V(X(r)) = 0$. This equation is the famous Bogomolny equation.

A monopole is a time independent, smooth, solution of these equations with asymptotic magnetic flux and finite energy. Such solutions are specified by γ_m, X_∞ where $\gamma_m \in \Lambda_{cr}$ and $X_\infty \in \mathfrak{t}$. The field configuration corresponding to a monopole with data γ_m, X_∞ has the asymptotic form

$$\begin{aligned} X &= X_\infty - \frac{\gamma_m}{2r} + O(r^{-3/2}) & \text{as } r \rightarrow \infty , \\ F &= \frac{\gamma_m}{2} d\Omega + O(r^{-1/2}) & \text{as } r \rightarrow \infty , \end{aligned} \quad (2.4)$$

where $d\Omega$ is the standard volume form on S^2 .¹ Here γ_m describes the magnetic charge and X_∞ describes the asymptotic Higgs vacuum expectation value which satisfies $V(X_\infty) = 0$.

A smooth gauge field with the behavior (2.4) at infinity necessarily must be defined patch-wise over at least two patches. If we choose two such patches, covering the northern or southern hemispheres, the connection can be written as

$$A = \frac{\gamma_m}{2} (\sigma - \cos(\theta)) d\phi , \quad (2.5)$$

where $\sigma = \pm 1$ in the northern/southern hemisphere. These are related by a gauge transformation $g = e^{i\gamma_m\phi}$ across the equator of the asymptotic 2-sphere, S_∞^2 . Since the gauge transformation above is a map $g : U(1) \rightarrow G$, γ_m is an element of the cocharacter lattice $\gamma_m \in \Lambda_{cochar} \subset \mathfrak{t}$.² Further, since \mathbb{R}^3 is contractible, the gauge transformation must be homotopic to the identity (by contraction to a point) which is equivalent to the requirement that g lifts to an element in the cocharacter lattice of the universal covering group which is isomorphic to the coroot lattice $\gamma_m \in \Lambda_{cochar}(\tilde{G}) \cong \Lambda_{cr}(G)$ [131]. Note that this implies that

$$[\gamma_m, X_\infty] = 0 , \quad (2.6)$$

¹Here we use the convention of a real, anti-hermitian representation of the Lie algebra \mathfrak{g} .

²The cocharacter lattice Λ_{cochar} is defined as: $\Lambda_{cochar} = \{\varphi \in \mathfrak{t} \mid e^{2\pi i\varphi} = \mathbb{1}_G\}$.

and hence the gauge transformation preserves the asymptotic form of the solution X . We will assume that X_∞ is a regular element of the Lie algebra and hence defines a maximal commuting subalgebra \mathfrak{t} spanned by a system of simple roots α_I and co-roots H_I .

A monopole solution is locally (near the monopole center) of the form of the Prasad-Sommerfeld solution of an $SU(2)$ -monopole embedded into the gauge group along a simple coroot [152]. This is a smooth solution that takes the form

$$\begin{aligned} X &= \frac{1}{2}h(r)H_I , \\ A &= \frac{H_I}{2}(\sigma - \cos(\theta))d\phi - \frac{1}{2}f(r)e^{\sigma i\phi}(d\theta - i\sin(\theta)d\phi)E_I^+ \\ &\quad + \frac{1}{2}f(r)e^{-\sigma i\phi}(d\theta - i\sin(\theta)d\phi)E_I^- , \end{aligned} \quad (2.7)$$

where $\sigma = \pm 1$ in the northern/southern hemisphere and

$$h(r) = m_W \coth(m_W r) - \frac{1}{r} , \quad f(r) = \frac{m_W r}{\sinh(m_W r)} , \quad (2.8)$$

where m_W is the mass of the W -boson: $m_W = \langle \alpha_I, X_\infty \rangle$. Here we have taken the convention where $\mathfrak{g}_{\mathfrak{su}(2)} = \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where \mathfrak{h} is the Cartan subalgebra generated by H_I and where \mathfrak{g}_\pm are generated by E_I^\pm which satisfy

$$[E_I^\pm, H_I] = \pm E_I^\pm . \quad (2.9)$$

This can be embedded into any semisimple lie algebra to give a local solution of a smooth monopole in a gauge theory with gauge group G .

2.2 Singular Monopoles

If we lift the requirement that classical solutions to have finite energy, we also can have singular monopoles. These are $U(1)$ Dirac monopoles that have been embedded into the gauge group along a simple coroot and correspond semiclassically to infinitely heavy magnetically charged particles and are used to describe 't Hooft defects. The data of an 't Hooft defect is given by (P_n, \vec{x}_n) where $\vec{x}_n \in \mathbb{R}^3$ specifies the location of the defect and $P_n \in \Lambda_{cochar} \cong \text{Hom}(U(1), T)$, where $T \subset G$ is the maximal torus specified by X_∞ , specifies the magnetic charge. Now the gauge transformation across the equator on the

asymptotic 2-sphere is no longer homotopic to the identity, but rather is homotopic to $e^{i\sum_n P_n\phi}$ and hence the asymptotic magnetic charge takes elements in a torsor of the coroot lattice $\gamma_m \in \Lambda_{cr} + \sum_n P_n \subset \Lambda_{cochar}$.

The defect is then inserted by imposing the following boundary conditions

$$\begin{aligned} X &= -\frac{P_n}{2r_n} + O(r_n^{-3/2}) & \text{as } r_n \rightarrow 0, \\ F &= \frac{P_n}{2}d\Omega_n + O(r_n^{-1/2}) & \text{as } r_n \rightarrow 0, \end{aligned} \tag{2.10}$$

in a local coordinate system centered at \vec{x}_n . In the presence of a collection of singular monopoles with charges $\{P_n\}$, $\gamma_m \in \Lambda_{cr} + \sum_n P_n$ since $e^{i\gamma_m\phi}$ is no longer homotopic to the identity by contracting the infinite sphere.

In our upcoming discussion we will need to distinguish between a reducible and an irreducible singular monopole. An *irreducible singular monopole* is defined as above by the data (\vec{x}, P) . The definition of a reducible singular monopole requires the definition of a *minimal singular monopole* which is simply an irreducible singular monopole with magnetic charge given by a simple (or minimal) coroot h^I . A reducible singular monopole of charge

$$P = \sum_I p_I h^I, \tag{2.11}$$

is then defined as the coincident limit of

$$p = \sum_I p_I, \tag{2.12}$$

minimal singular monopoles such that all of their charges sum to P . Sometimes we will call singular monopoles 't Hooft defects, but here we will generally reserve the term 't Hooft defect to refer to the defect operator in the quantum theory which in our considerations will generally be supersymmetric singular monopoles in a supersymmetric gauge theory.

2.3 Construction of Monopole Solutions

There many known constructions of monopole solutions. In this section we will review some of these constructions. See [130, 115, 167] for more details.

2.3.1 ADHM Construction of Instantons

Before we consider any constructions of monopole solutions, we believe it will be useful to first review the simpler case of the ADHM construction $SU(N)$ instantons with charge k on \mathbb{R}^4 .

Consider two complex vector spaces $V \cong \mathbb{C}^k$ and $W \cong \mathbb{C}^N$ and the set of maps

$$\begin{aligned} B_i &\in Hom(V, V) , \quad i = 1, 2 , \\ I &\in Hom(W, V) , \\ J &\in Hom(V, W) . \end{aligned} \tag{2.13}$$

These can be arranged into a short exact sequence

$$1 \longrightarrow V \xrightarrow{\alpha} V \otimes \mathbb{C}^2 \oplus W \xrightarrow{\beta} V \longrightarrow 1 , \tag{2.14}$$

where \mathbb{C}^2 is identified with the spin bundle of \mathbb{R}^4 [102] and

$$\alpha = \begin{pmatrix} B_1 - z_1 \mathbb{1}_k \\ B_2 - z_2 \mathbb{1}_k \\ J \end{pmatrix} , \quad \beta = \left(-(B_2 - z_2 \mathbb{1}_k) , B_1 - z_1 \mathbb{1}_k , I \right) , \tag{2.15}$$

where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ parametrize the base \mathbb{R}^4 .

Now we can construct the Dirac operator $D^\dagger : V \otimes \mathbb{C}^2 \oplus W \rightarrow V \oplus V$ where

$$D^\dagger = \begin{pmatrix} \alpha \\ \beta^\dagger \end{pmatrix} = \begin{pmatrix} I & B_2 + z_2 & B_1 + z_1 \\ J^\dagger & -B_1^\dagger - \bar{z}_1 & B_2^\dagger + \bar{z}_2 \end{pmatrix} \tag{2.16}$$

which can be more simply be written

$$D^\dagger = \begin{pmatrix} I & B_2 & B_1 \\ J^\dagger & -B_1^\dagger & B_2^\dagger \end{pmatrix} - i \left(\vec{0}^T \quad x_\mu \sigma^\mu \right) , \tag{2.17}$$

where $\vec{0}^T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Now define the subbundle $E \rightarrow \mathbb{R}^4$ of $V \otimes \mathbb{C}^2 \oplus W \rightarrow \mathbb{R}^4$ by

$$E = \text{Ker}[D] . \tag{2.18}$$

E has rank N so that we can construct solutions $\{\psi_i\} : \mathbb{R}^4 \rightarrow \text{Mat}_{2k,N}(\mathbb{C})$. We can then arrange these into a $(2k + N) \times N$ matrix

$$\Psi(x) = \begin{pmatrix} \vdots & \vdots & \vdots \\ \psi_1(x) & \psi_2(x) & \dots & \psi_N(x) \\ \vdots & \vdots & & \vdots \end{pmatrix}, \quad (2.19)$$

which we will normalize as

$$\Psi^\dagger(x)\Psi(x) = \mathbb{1}_N. \quad (2.20)$$

We can then construct the projection operator

$$P = |\Psi(x)\rangle\langle\Psi^\dagger(x)|, \quad (2.21)$$

and the complimentary operator

$$Q = D^\dagger \frac{1}{D^\dagger D} D, \quad (2.22)$$

which together satisfy

$$\mathbb{1} = P + Q. \quad (2.23)$$

Note that

$$D^\dagger D = \begin{pmatrix} \alpha\alpha^\dagger & 0 \\ 0 & \alpha\alpha^\dagger \end{pmatrix} = \begin{pmatrix} \beta^\dagger\beta & 0 \\ 0 & \beta^\dagger\beta \end{pmatrix}, \quad (2.24)$$

is a diagonal $2k \times 2k$ matrix.

The connection can then be constructed from this data as

$$A_\mu = \Psi^\dagger(x)\partial_\mu\Psi(x). \quad (2.25)$$

In this formulation, there is an $SU(k)$ gauge symmetry that act as

$$B_i \mapsto g^{-1}B_i g, \quad I \mapsto g^{-1}I, \quad J \mapsto Jg. \quad (2.26)$$

Now it remains to show that: 1.) A_μ is $\mathfrak{su}(N)$ -valued, 2.) it gives rise to a self dual field strength, and 3.) it has instanton number k .

First, we will show that it is an $SU(N)$ connection. Such matrices obey

$$A^\dagger = -A. \quad (2.27)$$

Computing, we find

$$A_\mu^\dagger = (\partial_\mu \Psi)^\dagger \Psi = -\Psi^\dagger \partial_\mu \Psi , \quad (2.28)$$

where we have used the identity

$$\Psi^\dagger \Psi = 1 \longrightarrow (\partial_\mu \Psi^\dagger) \Psi + \Psi^\dagger \partial_\mu \Psi = 0 . \quad (2.29)$$

Now we can compute the connection

$$\begin{aligned} F_{\mu\nu} &= \partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]} \\ &= \partial_{[\mu} (\Psi^\dagger \partial_{\nu]} \Psi) + (\Psi^\dagger \partial_{[\mu} \Psi) (\Psi^\dagger \partial_{\nu]} \Psi) \\ &= (\partial_{[\mu} \Psi^\dagger) (\partial_{\nu]} \Psi) + (\partial_{[\mu} \Psi^\dagger) (\Psi \Psi^\dagger) (\partial_{\nu]} \Psi) \\ &= (\partial_{[\mu} \Psi^\dagger) Q (\partial_{\nu]} \Psi) \\ &= \Psi^\dagger (\partial_{[\mu} D) \frac{1}{D^\dagger D} (\partial_{\nu]} D^\dagger) \Psi \\ &= -2i \sigma_{\mu\nu} \otimes \frac{1}{D^\dagger D} \end{aligned} \quad (2.30)$$

Now using the fact that $\sigma_{\mu\nu}$ is self-dual, we have that $F_{\mu\nu}$ is self dual.

We can now compute the instanton number. Using the identity [148]

$$*\text{Tr}_N F \wedge F = \frac{1}{2} (\partial_\mu \partial^\mu)^2 \log \det(\alpha \alpha^\dagger) = \frac{1}{2} (\partial_\mu \partial^\mu)^2 \det \log(\alpha \alpha^\dagger) , \quad (2.31)$$

we can compute

$$\int \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr}_N \{F \wedge F\} = \frac{1}{16\pi^2} \int_{\mathbb{R}^4} d^4x (\partial_\mu \partial^\mu)^2 \text{Tr}_N \log(\alpha \alpha^\dagger) . \quad (2.32)$$

Now since the integrand is a total derivative we can take the asymptotic form of $\alpha \alpha^\dagger \rightarrow x^2 \mathbb{1}_k$ such that the integral becomes

$$\int \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr}_N \{F \wedge F\} = \frac{1}{2\pi^2} \int_{S_\infty^3} \hat{r}_\mu d^3\Omega \text{Tr}_k \left\{ \frac{x^\mu}{x^4} \mathbb{1}_k \right\} = \text{Tr}_k \{\mathbb{1}_k\} = k . \quad (2.33)$$

Hence, the connection constructed in (2.25) is indeed a charge k $U(N)$ instanton.

2.3.2 ADHMN Construction of Smooth Monopoles

The ADHM construction of instantons can be generalized to a method to determine monopole field configurations [138]. Here we will review the generalization, called the ADHMN construction for monopoles on \mathbb{R}^3 . See [170, 34, 52] for more details.

First let us consider the case of $SU(2)$ monopoles on \mathbb{R}^3 specified by the asymptotic data

$$\gamma_m = kH_I = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} , \quad X_\infty = vH_I = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} . \quad (2.34)$$

The data for a solution of the Bogomolny equation consists of four \mathbb{C} -valued $k \times k$ matrices over the interval $I = (-\frac{v}{2}, \frac{v}{2})$ which is parametrized by s

$$T_\mu : I \rightarrow M_{k \times k}(\mathbb{C}) , \quad (2.35)$$

such that they solve Nahm's equations

$$\frac{dT_a}{ds} + i[T_0, T_a] + \frac{i}{2}\epsilon_{abc}[T_b, T_c] = 0 \quad , \quad a, b, c = 1, 2, 3 , \quad (2.36)$$

with the boundary conditions

$$\lim_{s \rightarrow \pm \frac{v}{2}} T_a = \frac{\sigma_a}{s \mp v/2} + O(1) . \quad (2.37)$$

Again, construct the Dirac operator:

$$D(s) = \frac{d}{ds} + iT_0(s) \otimes \mathbf{1}_2 - T_a(s) \otimes \sigma_a + x^a \mathbf{1}_k \otimes \sigma_a . \quad (2.38)$$

This operator corresponds to the Dirac operator in the ADHM construction. That is to say that we consider the kernel

$$D^\dagger(s)w_a(s, x) = 0 \quad , \quad \int_{-v/2}^{v/2} ds \, w_a^\dagger(s, r)w_b(s, r) = \delta_{ab} . \quad (2.39)$$

, and construct fields which satisfy the Bogomolny equations:

$$X^{ab} = \int_{-v/2}^{v/2} ds \, s w_a^\dagger w_b \quad , \quad A_\mu^{ab} = -i \int_{-v/2}^{v/2} ds \, w_a^\dagger \partial_\mu w_b , \quad (2.40)$$

where a, b are now $SU(N)$ indices.

Here there is a $SU(k)$ gauge symmetry that acts as

$$T_0 \mapsto g^{-1}T_0g - ig^{-1}\frac{d}{ds}g \quad , \quad T_a \mapsto g^{-1}T_ag . \quad (2.41)$$

As in the case of the ADHM construction, it can be easily shown that the fields X, A_μ constructed this way solve the Bogomolny equation and satisfy the asymptotic boundary conditions (2.34).

Generalizing to $SU(N)$

Now let us generalize this discussion to the case of $G = SU(N)$ monopoles. Consider the asymptotic data

$$\gamma_m = \sum_I n_I H_I \quad , \quad X_\infty = \text{diag}(s_1, \dots, s_N) \quad , \quad s_i < s_{i+1} \quad . \quad (2.42)$$

This choice of X_∞ now gives us partition of the interval $I = [s_1, s_N]$:

$$I = \bigcup_{p=1}^{N-1} \overline{I_p} \quad , \quad I_p = (s_p, s_{p+1}) \quad , \quad (2.43)$$

where $\overline{I_p}$ is the closure of the open interval I_p . On each interval I_p we have a set of four matrices $T_\mu^{(p)} : I_p \rightarrow M_{n_p \times n_p}(\mathbb{C})$ that satisfy Nahm's equations on the interval.

To connect these solutions across the disjoint intervals we must impose boundary conditions at the s_i . Near the boundary $\overline{I_p} \cap \overline{I_{p+1}}$, we impose the conditions

$$T_a^{(p)} = \begin{pmatrix} -\frac{L_a^{(p)}}{s-s_{p+1}} + O(1) & O((s-s_{p+1})^{(n_p-n_{p+1}-1)/2}) \\ O((s-s_{p+1})^{(n_p-n_{p+1}-1)/2}) & T_a^{(p+1)} + O(s-s_{p+1}) \end{pmatrix} \quad , \quad (2.44)$$

when $n_p > n_{p+1}$ and

$$T_a^{(p+1)} = \begin{pmatrix} \frac{L_a^{(p+1)}}{s-s_{p+1}} + O(1) & O((s-s_{p+1})^{(n_{p+1}-n_p-1)/2}) \\ O((s-s_{p+1})^{(n_p-n_{p+1}-1)/2}) & T_a^{(p)} + O(s-s_{p+1}) \end{pmatrix} \quad , \quad (2.45)$$

when $n_p < n_{p+1}$. When $n_p - n_{p+1} = 0$ we need to make a choice of “jumping data” to specify the boundary conditions. The jumping data is given by a choice of a set of $2n_p$ vectors $\{a_{\alpha,r}^{(p)}\}$ where $\alpha = 1, 2$ and $r = 1, \dots, n_p$ [34].

Given a choice of jumping data, the boundary conditions at a $\overline{I_p} \cap \overline{I_{p+1}}$ when $n_p = n_{p+1}$ is

$$T_j^{(p-1)}(s_p) - T_j^{(p)}(s_p) = \frac{1}{2} a^\dagger \sigma_j a \quad (2.46)$$

Now we again construct the same Dirac operator as before (2.38), solve for the kernel, and construct the fields X, A_μ as in the case of $G = SU(2)$.

The only difference occurs when we have $n_p = n_{p+1}$. In this case, the boundary condition from the jumping data (2.46) descends to the behavior of the solutions to the

Dirac equation

$$\psi_b^{(p-1)} - \psi_b^{(p)} = S_b^{(p)} \cdot a^{(p)} \quad , \quad \delta_{ab} = \sum_{p=1}^{N-1} \int_{I_p} ds \, \psi_a^\dagger \psi_b + \sum_p S_a^\dagger S_b \quad , \quad (2.47)$$

where $S_b^{(p)}$ is a complex-valued matrix. In this case we must modify the construction of the fields X, A_μ :

$$\begin{aligned} X &= \sum_{p=1}^{N-1} \int_{I_p} s ds \, \psi^\dagger \psi + \sum_p s_p S^\dagger S \quad , \\ A_\mu &= -i \sum_{p=1}^{N-1} \int ds \, \psi^\dagger \partial_\mu \psi - i \sum_p S^\dagger \partial_\mu S \quad . \end{aligned} \quad (2.48)$$

Again, these fields are $SU(N)$ valued, satisfy the appropriate asymptotic boundary conditions, and solve the Bogomolny equation.

From Bogomolny to Nahm Data

We can also explicitly reconstruct the Nahm data from a monopole field configuration of charge $\gamma_m = \sum_I m_I H_I$ with Higgs vev $X_\infty = \text{diag}(\varphi_1, \dots, \varphi_N)$. Consider the Dirac operators coupled to a monopole field A_μ, X

$$\not{D} = i(\sigma^j D_j - X + s) \quad , \quad \not{D}^\dagger = i(\sigma^i D_i + X - s) \quad , \quad (2.49)$$

and their squares

$$\not{D}^\dagger \not{D} = -D_i^2 + (X - s)^2 \quad , \quad \not{D} \not{D}^\dagger = -D_i^2 + (X + s)^2 - \sigma_{\mu\nu} F^{\mu\nu} \quad . \quad (2.50)$$

The explicit form of the $\not{D}^\dagger \not{D}$ implies that $\text{Ker}[\not{D}] = 0$ due to a non-trivial potential $(X - s)^2$. However, we also see that $\not{D} \not{D}^\dagger$ have zero-modes, and in fact through the usual index computation see that

$$\dim [\text{Ker}[\not{D}^\dagger]] = \begin{cases} m_I & \varphi_I < s < \varphi_{I+1} \\ 0 & \text{else} \end{cases} \quad (2.51)$$

Since there are a finite number of zero-modes, which form a finite dimensional vector space of L^2 sections on \mathbb{R}^3 , we can write a local basis for the kernel $\{\psi_i\}_{i=1}^k$ where

$$\not{D}^\dagger \psi_i = 0 \quad , \quad \delta_{ij} = \int_{\mathbb{R}^3} d^3x \, \psi_i^\dagger(x, s) \psi_j(x, s) \quad . \quad (2.52)$$

Using this kernel we can construct the matrices

$$\begin{aligned} T_j(s) &= - \int_{\mathbb{R}^3} d^3x \, x_j \psi^\dagger \psi , \\ T_0(s) &= i \int_{\mathbb{R}^3} d^3x \, \psi^\dagger \frac{d\psi}{ds} , \end{aligned} \tag{2.53}$$

which satisfy Nahm's equations and have the pole structure specified above.

2.3.3 Singular Monopoles and Kronheimer's Correspondence

There also exists a construction for singular monopole configurations which is similar to the ADHMN construction of smooth monopole solutions. The construction of singular monopole field configurations is derived from the one-to-one connection between singular monopoles on \mathbb{R}^3 and certain instantons on the four-dimensional Taub-NUT space [108]. Then by using the construction of instantons on Taub-NUT space called the bow construction [37, 38, 41], one can give an explicit construct singular monopole configurations on \mathbb{R}^3 [19].

In this section we will review the construction of singular monopole field configurations. We will begin by reviewing Taub-NUT spaces and Kronheimer's correspondence between certain instantons on Taub-NUT and singular monopoles. Then we will describe the bow construction and show how it can be used to give explicit constructions of singular monopole configurations.

Taub-NUT Spaces

Taub-NUT is a 4D asymptotically locally flat (ALF) hyperkähler manifold. Topologically it is homeomorphic to \mathbb{R}^4 . Taub-NUT can be realized as an S^1 fibration over \mathbb{R}^3 where the restriction of the S^1 fibration to any 2-sphere S^2 in the base \mathbb{R}^3 surrounding the origin is the Hopf fibration of charge 1.

Taub-NUT has a metric which can be written in Gibbons-Hawking form as

$$ds^2 = V(\vec{x}) \, d\vec{x} \cdot d\vec{x} + V^{-1}(\vec{x}) \, \Theta^2 , \tag{2.54}$$

where

$$V(\vec{x}) = 1 + \frac{1}{2|\vec{x}|} \quad , \quad \Theta = d\xi + \omega \quad , \quad d\omega = *_3 dV \quad , \tag{2.55}$$

where ξ is the S^1 fiber coordinate and $*_3$ is the Hodge dual restricted to the base \mathbb{R}^3 .

Taub-NUT has a natural $U(1)$ isometry, denoted $U(1)_K$, given by translation of the fiber coordinate

$$f_k(\vec{x}, \xi) = (\vec{x}, \xi + \xi_0) \quad , \quad f^* ds^2 = ds^2 . \quad (2.56)$$

Taub-NUT has a single $U(1)_K$ fixed point where the S^1 fiber degenerates (in our convention at $\vec{x} = 0$) called the NUT center. Thus, $d\xi$ is not a globally well defined 1-form while $\Theta = d\xi + \omega$ is globally well defined.

Taub-NUT can be generalized to a manifold called multi-Taub-NUT (TN_k). This space is also a 4D ALF hyperkähler manifold which can be described by a S^1 fibration over \mathbb{R}^3 . However, TN_k has k points where the S^1 fiber degenerates. TN_k also has a natural $U(1)_K$ isometry which has k -fixed points where the S^1 fiber degenerates – hence, there are multiple NUT centers at $\vec{x}_i \in \mathbb{R}^3$. The metric on TN_k can also be written in Gibbons-Hawking form (2.54) where instead

$$V(\vec{x}) = 1 + \sum_{i=1}^k \frac{1}{2|\vec{x} - \vec{x}_i|} \quad , \quad d\omega = *_3 dV . \quad (2.57)$$

Again, $d\xi$ is globally ill-defined while $\Theta = d\xi + \omega$ is well defined.

Unlike Taub-NUT, TN_k is topologically non-trivial. It has a non-trivial (compact) cohomology group: $H_{cpt}^2(TN_k, \mathbb{Z}) = \Gamma[A_{k-1}]$ where $\Gamma[A_{k-1}]$ is the root lattice of the Lie group A_{k-1} . By Poincaré duality, the generators of $H_{cpt}^2(TN_k; \mathbb{Z})$ correspond to non-trivial 2-cycles in $H_2(TN_k, \mathbb{Z})$ which is generated by cycles that are homologous to the preimage of the lines running between any two NUT centers under the projection $\pi : TN_k \rightarrow \mathbb{R}^3$.

Review of Kronheimer's Correspondence

Consider a monopole configuration with k irreducible singular monopoles at positions $\vec{x}_n \in \mathbb{R}^3$ with charges P_n . Kronheimer's correspondence provides a one-to-one mapping between such a singular monopole configuration and a $U(1)_K$ -invariant instanton solution on TN_k [108]. Here, by a $U(1)_K$ -invariant instanton configuration we mean one in which the lift of the $U(1)_K$ action to the gauge bundle $P \rightarrow TN_k$ is equivalent to a

gauge transformation [61, 84]

$$f^* \hat{A} = g^{-1} \hat{A} g + i g^{-1} dg , \quad (2.58)$$

where f generates translations along ξ as in (2.56) and g is a gauge transformation that defines the lift of the $U(1)_K$ action to the gauge bundle. As shown in Appendix B, the lift of the $U(1)_K$ action is specified by the collection of 't Hooft charges $\{P_n\}$ which fixes the limiting behavior of the lift of the $U(1)_K$ action near the NUT centers $\lim_{\vec{x} \rightarrow \vec{x}_n} g = e^{-i P_n \xi}$.

Away from NUT centers, the connection can be put in a $U(1)_K$ invariant gauge

$$\hat{A} = A_{\mathbb{R}^3} + \psi(x)(d\xi + \omega) , \quad (2.59)$$

where $A_{\mathbb{R}^3}$ is a connection on the base $TN_k \rightarrow \mathbb{R}^3$ that has been lifted to the full TN_k . For \hat{A} to describe an instanton, it must satisfy the self-duality equation: $\hat{F} = *\hat{F}$ where \hat{F} is the curvature of \hat{A} . The curvature of \hat{A} can be written as

$$\hat{F} = (F_{\mathbb{R}^3} - \psi d\omega) - D\psi \wedge (d\xi + \omega) , \quad (2.60)$$

where $F_{\mathbb{R}^3}$ is the curvature of $A_{\mathbb{R}^3}$. Using the orientation form $\Theta \wedge dx^1 \wedge dx^2 \wedge dx^3$, we can then compute the dual field strength

$$*\hat{F} = - *_3 F_{\mathbb{R}^3} \wedge \left(\frac{d\xi + \omega}{V} \right) - V *_3 D\psi + \psi *_3 d\omega \wedge \left(\frac{d\xi + \omega}{V} \right) . \quad (2.61)$$

Now self-duality $\hat{F} = *\hat{F}$ reduces to the equation

$$*_3(F_{\mathbb{R}^3} - \psi d\omega) = V D\psi , \quad (2.62)$$

which can be re-expressed as the relation

$$*_3 F_{\mathbb{R}^3} = D(V\psi) . \quad (2.63)$$

Under the identification $X = V\psi$, we can recognize this as familiar Bogomolny equation (2.3).

As shown in Appendix B, the connection (2.59) can be extended globally iff $A_{\mathbb{R}^3}$ and $\psi(x)$ have the limiting forms

$$\lim_{\vec{x} \rightarrow \vec{x}_n} A_{\mathbb{R}^3} = P_n \omega \quad , \quad \lim_{\vec{x} \rightarrow \vec{x}_n} \psi(x) = -P_n . \quad (2.64)$$

In the setting of Kronheimer's correspondence, this limiting form of the $U(1)_K$ -invariant instanton configuration gives rise to the limiting behavior of the monopole configuration

$$\lim_{\vec{x} \rightarrow \vec{x}_n} F_{\mathbb{R}^3} = \frac{P_n}{2} d\Omega \quad , \quad \lim_{\vec{x} \rightarrow \vec{x}_n} X = \lim_{\vec{x} \rightarrow \vec{x}_n} V(x)\psi(x) = -\frac{P_n}{2|\vec{x} - \vec{x}_n|} . \quad (2.65)$$

Therefore, since $A_{\mathbb{R}^3}, X$ both satisfy the Bogomolny equation (2.63) and have the limiting behavior (2.65), a $U(1)_K$ invariant instanton on multi-Taub-NUT is in one-to-one correspondence with a singular monopole configuration on \mathbb{R}^3 where the 't Hooft defects $\{P_n, \vec{x}_n\}$ are encoded in the action of the $U(1)_K$ near the NUT centers (which by extension specifies it everywhere on TN_k). More technical details on Kronheimer's correspondence can be found in Appendix B.

2.3.4 Instantons on multi-Taub-NUT and Bows

Using Kronheimer's correspondence we we can construct a singular monopole field configuration from a $U(1)_K$ -invariant instanton on TN_k . In [36, 37, 38, 41], the authors provided a theoretical framework to find explicit instanton solutions on TN_k (and indeed for all gravitational instantons) that fundamentally relies on an object called a bow. The bow construction effectively reduces to solving Nahm's equations with a specific set of boundary conditions which naturally encodes the effect of singular monopoles (or NUT centers in the case of instantons on Taub-NUT).

Quick Review of Bows

A bow is a quiver where the nodes have been replaced by (wavy) intervals. These intervals support a vector bundle that can change rank at marked points and connect together along the quiver edges to form a connected space Σ . In our case $\Sigma = S^1$.

To a given bow, we can associate a set of differential equations that are analogous to the Nahm equations for the connection on E . It differs by including certain boundary/matching conditions at edges of intervals and at marked points which encode the data of the marked points and edges. A solution of the bow equations is called a representation of the bow.

The construction of an instanton configuration on multi-Taub-NUT requires two representations of a bow: a *small representation* and a *large representation*. The small representation encodes the geometry of multi-Taub-NUT and the large representation encodes geometry of the gauge bundle. These two representations can be used to construct a “Dirac-type” differential operator whose zero modes can be used to give an explicit instanton solution in analogy to the ADHM/ADHMN constructions [10, 138].

Now we will give precise definitions of bows and their representations and review how they can be used to give explicit instanton solutions on Taub-NUT space.

Bow Data: A *bow* is a directed linear (or ring) graph with nodes, where the nodes are replaced by a wavy line segments which hosts a collection of marked points. These marked points divide the wavy line segments into irreducible line segments. This is specified by:

1. A set of directed edges, denoted $\mathcal{E} = \{e_i\}$.
2. A set of continuous, irreducible wavy line segments, denoted $\mathcal{I} = \{\zeta_i\}$. We will additionally use \mathcal{I}_i to denote the set of $\zeta \in \mathcal{I}$ in between edges $e_i, e_{i+1} \in \mathcal{E}$.
3. A set of marked points denoted $\Lambda = \{x_i\}$. We will additionally use the notation Λ_i to be the set of marked points $x \in \Lambda$ which are at the end points of the $\zeta \in \mathcal{I}_i$.

See Figure 2.1 for an example of a bow.

Bow Representations: A *representation* of a bow consists of the following data

1. To each wavy interval $\zeta \in \mathcal{I}$, we associate a line segment σ_ζ with coordinate s such that $\sigma_\zeta = [o(\zeta), i(\zeta)]$ where $o(\zeta)$ and $i(\zeta)$ are the beginning and end of ζ respectively. The intervals σ_ζ connect along marked points and edges to form a single interval (or circle) $\Sigma = \bigcup_{\zeta \in \mathcal{I}} \sigma_\zeta$ according to the shape of the bow.
2. For each $x \in \Lambda$ we define a one-dimensional complex vector space \mathbb{C}_x with Hermitian inner product $\langle \cdot, \cdot \rangle$.

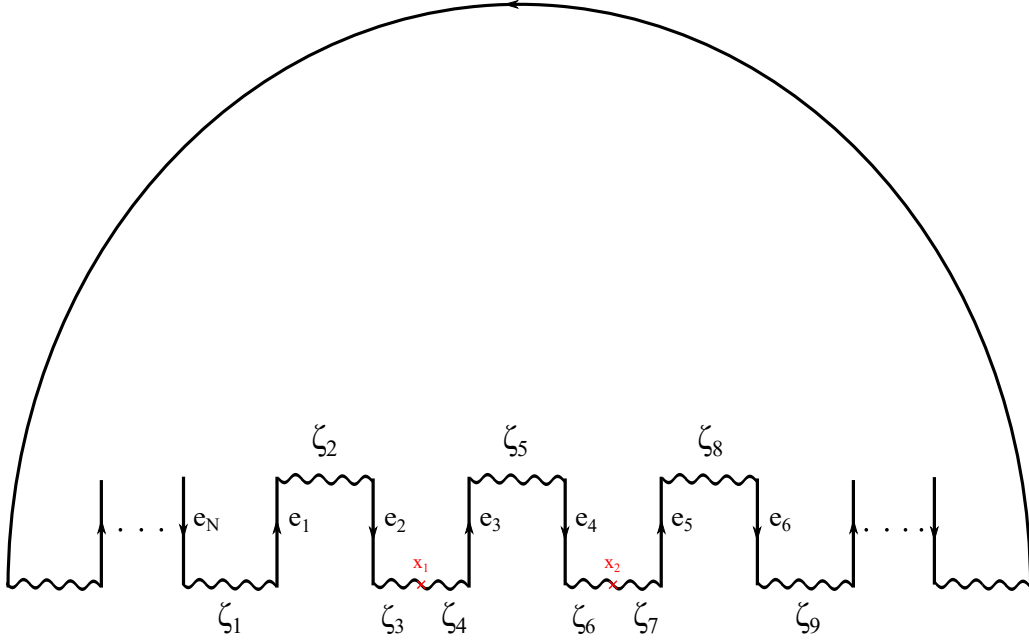


Figure 2.1: This figure is an example of a type A_n bow with edges $\{e_I\}$, segments $\{\zeta_i\}$, and marked points $\{x_n\}$.

3. For each $\zeta \in \mathcal{I}$, we assign a non-negative integer $R(\zeta) \in \mathbb{N}$ and for each point $x \in \Lambda$ we define $\Delta R(x) = |R(\zeta^-) - R(\zeta^+)|$ where ζ^\pm are the segments to the left and right of the point x .
4. For each $e \in \mathcal{E}$, we assign a vector $\vec{\nu}_e = (\nu_e^1, \nu_e^2, \nu_e^3) \in \mathbb{R}^3$.
5. For each $\zeta \in \mathcal{I}$, we define a vector bundle $E_\zeta \rightarrow \sigma_\zeta$ of rank $R(\zeta)$. And for each $x \in \Lambda$, we define an irreducible $\mathfrak{su}(2)$ representation of dimension $\Delta R(x)$ with generators $\{\rho_i\}$. This gives a representation of $(E_{\zeta^\pm}|_x)^\perp \subset (E_{\zeta^\mp})|_x$ for $R(\zeta^\mp) > R(\zeta^\pm)$, where ζ^\pm are the segments to the right/left of x .
6. For each $x \in \Lambda$ where $\Delta R(x) = 0$, we define a set of linear maps $I : \mathbb{C}_x \rightarrow E|_x$ and $J : E|_x \rightarrow \mathbb{C}_x$ and a set of linear maps $B_e^{LR} : E|_{t(e)} \rightarrow E|_{h(e)}$, $B_e^{RL} : E|_{h(e)} \rightarrow E|_{t(e)}$ for each $e \in \mathcal{E}$ where $h(e), t(e)$ is the head, tail of the arrow e respectively.
7. ∇_s - a Hermitian connection $\frac{d}{ds} + T_0$ and skew-Hermitian endomorphisms $\{T_i\}_{i=1}^3$

on E_ζ over the interval σ_ζ which have the pole structure

$$T_j(s) = \begin{pmatrix} \frac{1}{2} \frac{\rho_j}{s-x} + O((s-x)^0) & O((s-x)^{\frac{\Delta R-1}{2}}) \\ O((s-x)^{\frac{\Delta R-1}{2}}) & T_j^-(\lambda) + O(s-x) \end{pmatrix}, \quad (2.66)$$

near $x \in \Lambda$.

8. As in the ADHM and Nahm construction, there is a gauge symmetry of the instanton bundle E . These gauge transformations act on the various field as

$$g : \begin{pmatrix} T_0 \\ T_i \\ B_e^{LR} \\ B_e^{RL} \\ I_x \\ J_x \end{pmatrix} \mapsto \begin{pmatrix} g^{-1}(s)T_0g(s) - ig^{-1}\frac{d}{ds}g(s) \\ g^{-1}(s)T_i g(s) \\ g^{-1}(h(e))B_e^{LR}g(t(\zeta)) \\ g^{-1}(t(e))B_e^{RL}g(h(e)) \\ g^{-1}(x)I_x \\ J_x g(x) \end{pmatrix}. \quad (2.67)$$

9. If we reorganize these linear maps as

$$Q_x = \begin{pmatrix} J_x^\dagger \\ I_x \end{pmatrix}, \quad B_e^- = \begin{pmatrix} (B_e^{LR})^\dagger \\ -B_e^{RL} \end{pmatrix}, \quad B_e^+ = \begin{pmatrix} (B_e^{RL})^\dagger \\ B_e^{LR} \end{pmatrix},$$

$$\mathbf{T} = \mathbb{1} \otimes T_0 + i\sigma^j \otimes T_j, \quad \mathbf{T}^* = \mathbb{1} \otimes T_0 - i\sigma^j \otimes T_j, \quad \nu_{\mathbb{C}} = \nu_1 + i\nu_2, \quad (2.68)$$

then the linear maps are required to satisfy the ‘‘Nahm equation’’ [38]

$$\mu = \text{Im} \left(\frac{d}{ds} \mathbf{T} - i\mathbf{T}^* \cdot \mathbf{T} + \sum_{x \in \Lambda} \delta(s-x) Q_x \otimes Q_x^\dagger \right. \\ \left. + \sum_{e \in \mathcal{E}} \left(B_e^- \otimes (B_e^-)^\dagger \delta(s-t(e)) + B_e^+ \otimes (B_e^+)^\dagger \delta(s-h(e)) \right) \right), \quad (2.69)$$

where $\mu = \sum_j \nu_j(s) \sigma^j$ and $\vec{\nu}(s) = \vec{\nu}_e \delta(s-h(e)) + \vec{\nu}_e \delta(s-t(e))$.

This equation can be rewritten in a more familiar form as [36, 38, 37]

$$\begin{aligned}
0 = & \nabla_s T_3 + \frac{i}{2} [T_1 + iT_2, T_1 - iT_2] + \frac{1}{2} \sum_{x \in \Lambda} (J_x^\dagger J_x - I_x I_x^\dagger) \delta(s - x) \\
& + \frac{1}{2} \sum_{e \in \mathcal{E}} \left[\left((B_e^{LR})^\dagger B_e^{LR} - B_e^{RL} (B_e^{RL})^\dagger - \nu_3(s) \right) \delta(s - t(e)) \right. \\
& \quad \left. + \left((B_e^{RL})^\dagger B_e^{RL} - B_e^{LR} (B_e^{LR})^\dagger - \nu_3(s) \right) \delta(s - h(e)) \right], \\
0 = & \nabla_s (T_1 + iT_2) + i[T_3, T_1 + iT_2] - \sum_{x \in \Lambda} I_x J_x \delta(s - x) \\
& + \sum_{e \in \mathcal{E}} \left[(B_e^{RL} B_e^{LR} - \nu_{\mathbb{C}}(s)) \delta(s - t(e)) \right. \\
& \quad \left. + (B_e^{LR} B_e^{RL} - \nu_{\mathbb{C}}(s)) \delta(s - h(e)) \right]. \tag{2.70}
\end{aligned}$$

Note that this is simply the complexified Nahm equations with certain boundary terms.

Bow Construction of Instantons

Now, taking a small and large representation of a bow, we can construct instanton solutions on TN_k . The small representation is that of an A_{k-1} -type bow (a circular bow with k -edges and k -intervals) in which $\Lambda = \{\emptyset\}$ and $R_\zeta = 1, \forall \zeta \in \mathcal{I}$. The small representation specifies the geometry of TN_k . Here, we will denote the triple of skew-Hermitian endomorphisms of the small representation from condition (7.) as $\{t_i\}$ and the linear maps for each edge $e \in \mathcal{E}$, $b_e^{LR} : E_{t(e)} \rightarrow E_{h(e)}$ and $b_e^{RL} : E_{h(e)} \rightarrow E_{t(e)}$. In each interval away from the marked points and boundaries, the t_i satisfy $\frac{dt_i}{ds} = 0$ with boundary conditions defined by the b_e^{LR}, b_e^{RL} as in the Nahm equation (2.69).

The metric on the multi-Taub-NUT space can then be defined by reducing the “flat” metric

$$ds^2 = \sum_e \left[\frac{1}{2} d(b_e^{LR})^\dagger db_e^{LR} + \frac{1}{2} d(b_e^{RL})^\dagger db_e^{RL} + (dt_{e,0}^2 + d\vec{t}_e^2) \right], \tag{2.71}$$

by Nahm’s equations and gauge symmetry. Here, the angular coordinate on TN_k is determined by the gauge invariant data of t_0 : $\log(P \exp \oint ds t_0)$ [36, 37, 38, 41].

Now we can construct the instanton configuration from the large representation. The large representation is allowed to have non-empty Λ and generic data for the $R(\zeta)$. We will denote the maps of this representation as $\{T_i\}, B^{LR}, B^{RL}$.

Given a solution of the Nahm equations, we can define a Dirac operator

$$\mathcal{D}_t = \frac{d}{ds} + T_i \otimes \sigma^i - t_i(\mathbb{1} \otimes \sigma^i) . \quad (2.72)$$

Then, as in the ADHM and ADHMN constructions, we find the kernel of this operator

$$\mathcal{D}_t \psi_i = 0 , \quad (2.73)$$

and use it to construct a matrix

$$\Psi = \left(\begin{array}{c|c|c|c} \vdots & \vdots & & \vdots \\ \psi_1 & \psi_2 & \cdots & \psi_N \\ \vdots & \vdots & & \vdots \end{array} \right) , \quad (2.74)$$

of the linearly independent solutions. Using this, we can reconstruct the self-dual gauge field as in [36, 38, 37] by

$$A_\mu = i \int ds \Psi^\dagger D_\mu \Psi , \quad (2.75)$$

where

$$D_\mu = \frac{\partial}{\partial x^\mu} - i s a_\mu , \quad a_\mu = \frac{d\xi + \omega}{V(\vec{x})} , \quad (2.76)$$

and $V(\vec{x})$ is the harmonic function for multi-Taub-NUT and ω is the corresponding Dirac potential:

$$dV = *_3 d\omega . \quad (2.77)$$

As shown in [19], there is a special class of large bow representations, called *Cheshire bow representations*, that give rise to $U(1)_K$ -invariant instantons on multi-Taub-NUT. These bows have the special properties:

- A single sub-interval $\zeta \in \mathcal{I}$ such that $R(\zeta) = 0$,
- $R(\zeta_{L,e}) = R(\zeta_{R,e})$ where ζ_L, ζ_R are the intervals to the left and right of an edge $e \in \mathcal{E}$.

These bows give rise to $U(1)_K$ instantons because the action of $U(1)_K$ is determined by a non-trivial shift in $\oint ds t_0 \bmod 2\pi$. In the case of a Cheshire bow representation, we can use gauge symmetry to eliminate this shift since there is a $\zeta \in \mathcal{I}$ where $R(\zeta) = 0$ which means that Σ has effective endpoints on which the gauge transformations of E

are unrestricted. Thus, any shift of the fiber coordinate can be compensated by a gauge transformation resulting in $U(1)_K$ -invariant instantons [19, 37].

One such class bow and Cheshire representation that has a simple interpretation in terms of the corresponding singular monopole configuration are those that correspond to reducible monopoles. Given a bow which

- is circular,
- has $p+1$ wavy intervals separated by $p = \sum_I = p_I$ edges (to which we will identify the same FI-parameters $\vec{\nu}$),
- has N marked points $\{x_I\}$ distributed such that there are p_I edges in between x_I, x_{I+1} and no edges in between x_N, x_1 ,

we can construct a (reducible) singular monopole configuration from a small representation and a large Cheshire representation which has $R(\zeta) = m_I$ for wavy intervals between x_I, x_{I+1} and $R(\zeta) = 0$ for the interval between x_N, x_1 by

$$A_a = i \int ds \Psi^\dagger D_a \Psi \quad , \quad X = V(\vec{x}) \int ds \Psi^\dagger D_4 \Psi \quad , \quad (2.78)$$

where $V(\vec{x})$ is the harmonic function of the TN_k determined by the small bow representation. Such a singular monopole configuration will have

- Gauge group $G = SU(N)$
- Relative asymptotic magnetic charge ³

$$\tilde{\gamma}_m = \gamma_m - P^- = \sum_I m_I H_I \quad , \quad (2.79)$$

- $|\mathcal{E}|$ singular monopoles at $\vec{x}_n = \vec{\nu}$.

We will discuss the more general identification for irreducible monopoles later in Chapter 5.

³Here, P^- is the representative of P in the completely negative Weyl chamber.

2.4 Monopole Moduli Space

The set of solutions to the Bogomolny equation defines a smooth, finite dimensional space known as monopole moduli space: $\mathcal{M}(\gamma_m; X_\infty)$. This space notably has many properties:

1. \mathcal{M} is a hyperkähler manifold. This comes from the fact that we can combine A_i and X into a four-dimensional gauge field: $\hat{A}_a = (A_i, X)$ in which case the Bogomolny equation is equivalent to the self-duality equations for the four-dimensional (x^4 -invariant) gauge field \hat{A}_a :

$$\hat{F}_{ij} = \frac{1}{2} \epsilon_{ijkl} \hat{F}^{kl} . \quad (2.80)$$

The space of tangent vectors $T_{[\hat{A}]} \mathcal{M}$ at a point $[\hat{A}] \in \mathcal{M}$ is described by the functions which satisfy the linearized self duality equations:

$$\hat{D}_{[a} \delta \hat{A}_{b]} = \frac{1}{2} \epsilon_{ab}{}^{cd} \hat{D}_c \delta \hat{A}_d . \quad (2.81)$$

Since \hat{A}_a solves the self-duality equations, then a solution $\delta \hat{A}_a$ of the linearized self-duality equation will come with a triplet of such solutions $\bar{\eta}_{ab}^r \delta \hat{A}^b$ where $\bar{\eta}^r$ are the anti-self-dual 't Hooft symbols. This gives us a triplet of endomorphisms on $T\mathcal{M}$:

$$(\mathbb{J}^r \cdot \delta \hat{A})_a = (j^r)_{ab} \delta \hat{A}^b , \quad (j^r)_{ab} = (R_\kappa)_s^r (\bar{\eta}^s)_{ab} , \quad (2.82)$$

where R_κ is some choice of $\text{SO}(3)$ matrix. The endomorphisms satisfy the quaternionic algebra

$$\mathbb{J}^r \mathbb{J}^s = -\delta^{rs} \mathbb{1} + \epsilon^{rs}{}_t \mathbb{J}^t , \quad (2.83)$$

and hence \mathcal{M} is hyperkähler.⁴

2. $\mathcal{M} \neq \{\emptyset\}$ iff under a decomposition of γ_m in terms of simple coroots⁵

$$\gamma_m = \sum_I m^I H_I , \quad (2.84)$$

⁴This follows from the algebra of the $\{\bar{\eta}^r\}$: $\bar{\eta}^r \bar{\eta}^s = -\delta^{rs} - \epsilon^{rs}{}_t \bar{\eta}^t$.

⁵We define the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ relative to the asymptotic Higgs vev X_∞ which we assume to be generic. Here we then define $\mathfrak{t} \subset \mathfrak{g}$ to be the set of commuting elements of \mathfrak{g} . Then we can define the set of positive coroots by the set of coroots which have positive inner product with X_∞ with respect to the Killing form.

then the $m^I \geq 0$ for all I with at least one $m^I > 0$. In this case, the dimension of \mathcal{M} is:

$$\dim_{\mathbb{R}}[\mathcal{M}] = 4 \sum_X m^I . \quad (2.85)$$

3. The symmetry group of \mathcal{M} is given by $\mathbb{R}_{trans} \times SO(3)_{rot} \times T$. Here \mathbb{R}_{trans}^3 and $SO(3)_{rot}$ are translation and rotation symmetry of the spatial \mathbb{R}^3 respectively. The translation symmetry is generated by a triholomorphic killing fields while rotation symmetry is generated by killing vector fields which rotate the complex structures on \mathcal{M} . The factor T of the symmetry group is generated by global gauge transformations which is generated by triholomorphic killing vector fields.
4. We can separate out the orbits of the global symmetry, allowing us to realize the moduli space as a quotient

$$\mathcal{M} = \mathbb{R}_{cm}^3 \times \frac{\mathbb{R}_{X_\infty} \times \mathcal{M}_0}{\mathbb{D}} . \quad (2.86)$$

The factor \mathbb{R}^3 is the orbit of translation and \mathbb{R}_{X_∞} is the orbit of X_∞ in T . Here, \mathbb{D} is the group of deck transformations acting on the universal cover and hence $\mathbb{D} \cong \pi_1(\mathcal{M})$. \mathcal{M}_0 is called the strongly centered moduli space and is simply connected. See [25] for more details.

ADHMN Construction of Monopole Moduli Space

The ADHMN construction of monopole solutions also gives us an algebraic description of monopole moduli space. Given a solution of Nahm's equations (2.36), we can construct an explicit solution for the Higgs and gauge field describing the corresponding monopole configuration. However, these are both acted on by gauge transformations. In general, a gauge transformation is defined as an $(N - 1)$ -tuple of smooth maps

$$g = (g_1, \dots, g_{N-1}) \quad , \quad g_i : I_i \rightarrow U(k_i) , \quad (2.87)$$

that act as

$$T_0 \mapsto g^{-1} T_0 g - i g^{-1} dg \quad , \quad T_a \mapsto g^{-1} T_a g , \quad (2.88)$$

such that the boundary conditions (2.44) – (2.46) are preserved. This gives the description of monopole moduli space

$$\mathcal{M}(\gamma_m; X_\infty) = \left\{ \begin{array}{l} D_0 T_i^{(p)} + \frac{i}{2} \epsilon_{ijk} [T_j^{(p)}, T_k^{(p)}] = 0 \\ D_0 = \frac{d}{ds} + i[T_0, \cdot] \quad \forall i, j, k, p \end{array} \right\} \Bigg/ \mathcal{G}, \quad (2.89)$$

where \mathcal{G} is the group of gauge transformations as described above.

2.4.1 Singular Monopole Moduli Space

Similar to the case of smooth monopoles, the space of singular monopoles defines a moduli space of singular monopoles $\overline{\mathcal{M}}(P_n, \gamma_m; X_\infty)$. It also has many property similar to \mathcal{M} :

1. $\overline{\mathcal{M}}(P_n, \gamma_m; X_\infty)$ is hyperkähler with singularities. The space is conjecturally non-empty iff the relative magnetic charge $\tilde{\gamma}_m = \gamma_m - \sum_n P_n^-$ is dominant. Here P_n^- is the image of P_n under the Weyl group in the anti-fundamental chamber [131]. This means that under the decomposition

$$\tilde{\gamma}_m = \sum_X \tilde{m}^I H_I, \quad (2.90)$$

$\tilde{m}^I \geq 0 \quad \forall I$. If $\overline{\mathcal{M}} \neq \emptyset$, then the dimension of $\overline{\mathcal{M}}$ is given by:

$$\dim[\overline{\mathcal{M}}] = 4 \sum_X \tilde{m}^I. \quad (2.91)$$

2. $\overline{\mathcal{M}}$ does not factorize as in (2.86) since singular monopoles break translation symmetry.

It will be useful for our purposes to differentiate between the moduli space of irreducible and reducible singular monopoles. Consider a collection of minimal singular monopoles $\{P_i = h^{I(i)}, \vec{x}_i^{(n)}\}$ whose coincident limits $\{\vec{x}_i^{(n)}\} \mapsto \vec{y}_n$ produces the set of reducible singular monopoles $\{P_n, \vec{y}_n\}$. We will denote the corresponding singular monopole moduli space

$$\widehat{\overline{\mathcal{M}}}(\{P_n\}, \gamma_m; X_\infty) = \lim_{\vec{x}_i^{(n)} \rightarrow \vec{y}_n} \overline{\mathcal{M}}(\{h^{I(i)}\}, \gamma_m; X_\infty). \quad (2.92)$$

Explicitly, the i^{th} minimal singular monopole of charge $h^{I(i)}$ is inserted at $\vec{x}_i^{(n)}$ and contributes to the reducible singular monopole at \vec{y}_n which has total charge

$$P_n = \sum_{i: \vec{x}_i^{(n)} \rightarrow \vec{y}_n} h^{I(i)} . \quad (2.93)$$

Bow Construction of Singular Monopole Moduli Space

Kronheimer's correspondence tells us that singular monopole moduli space is equivalent to some moduli space of $U(1)_K$ -invariant instantons on multi-Taub-NUT. By using the explicit construction of the moduli space instantons on multi-Taub-NUT from the previous section, we see that singular monopole moduli space can be described as a bow variety corresponding to Cheshire bow representations [19].

As we discussed, singular monopole configurations correspond to a large representation of a bow with respect to a small representation that specifies the geometry of the Taub-NUT space. Thus, let us fix a type A_{k-1} bow with a small representation \mathfrak{r} . Further, fix the data of the instanton by choosing $\mathcal{I}, \mathcal{A}, \mathcal{E}, \{\vec{v}_e\}$, and $E \rightarrow \Sigma$ for the large representation \mathfrak{R} . The moduli space of singular monopoles is then given by set of all large representations modulo gauge equivalence. This is given by

$$\mathcal{M}_{bow}(\mathfrak{R}, \mathfrak{r}) = \left\{ \begin{array}{l} \mathbf{T} \in \mathbb{H} \otimes \text{End}(E) , \\ Q_x : \mathbb{C}_x \times \mathbb{C}_x \rightarrow E_x \times E_x , \\ B_e^+ : E_{h(e)} \times E_{h(e)} \rightarrow E_{t(e)} \times E_{t(e)} , \\ B_e^- : E_{t(e)} \times E_{t(e)} \rightarrow E_{h(e)} \times E_{h(e)} , \end{array} \right\} \left. \begin{array}{l} \text{Nahm's Equations} \\ (2.69) \end{array} \right\} / \mathcal{G} , \quad (2.94)$$

where Q_x, B_e^\pm, \mathbf{T} are defined as in (2.68) and $E_s = E|_s$ is the fiber of $E \rightarrow \Sigma$ at $s \in \Sigma$. This describes the moduli space of instantons on multi-Taub-NUT with fixed asymptotic data [36, 40].

Bow Variety Isomorphisms: Hanany-Witten Transitions

An interesting feature of bow varieties is that there are often many different, isomorphic formulations of the same bow variety. One such isomorphism that will be useful for us

is the *Hanany-Witten isomorphism* [142]. This allows us to exchange an adjacent edge and marked point in exchange for modifying the local values of $R(\zeta)$.

This isomorphism of representations is explicitly given by

where the $R(\zeta_i)$ obey the relation

$$R(\zeta_2) + R(\zeta'_2) = R(\zeta_1) + R(\zeta_3) + 1 . \quad (2.95)$$

As we will see, this will be intimately related to Hanany-Witten transitions of brane configurations.

Singularity Structure: Monopole Bubbling

The singular locus of $\overline{\mathcal{M}}$ has the special interpretation of describing monopole bubbling configurations. In the case of a single 't Hooft defect, singular monopole moduli space $\overline{\mathcal{M}}(P, \gamma_m; X_\infty)$ has the stratification [142]

$$\overline{\mathcal{M}}(P, \gamma_m; X_\infty) = \coprod_{|v| \leq P} \overline{\mathcal{M}}^{(s)}(v, \gamma_m; X_\infty) , \quad (2.96)$$

where $\overline{\mathcal{M}}^{(s)}(v, \gamma_m; X_\infty) \subset \overline{\mathcal{M}}(v, \gamma_m; X_\infty)$ is the smooth component of $\overline{\mathcal{M}}(v, \gamma_m; X_\infty)$ [142]. Here each component $\overline{\mathcal{M}}^{(s)}(v, \gamma_m; X_\infty)$ describes the degrees of freedom of the free (unbubbled), smooth monopoles in the bubbling sector with effective (screened) 't Hooft charge given by $v \in \Lambda_{cr} + P$. We will further denote the transversal slice of each component $\overline{\mathcal{M}}^{(s)}(v, \gamma_m; X_\infty) \subset \overline{\mathcal{M}}(P, \gamma_m; X_\infty)$ by $\mathcal{M}(P, v)$. As shown in [142, 23], in the case of reducible monopoles, $\mathcal{M}(P, v)$ is a quiver variety.⁶

Physically this should be thought of as follows. Singular monopole moduli space $\overline{\mathcal{M}}(P, \gamma_m; X_\infty)$ decomposes into a collection of nested singular monopole moduli spaces of decreasing charge and dimension: $\overline{\mathcal{M}}(v, \gamma_m; X_\infty)$ where $|v| \leq |P|$. Each lower-dimensional component describes the singular monopole moduli space that results when

⁶See Section 5.1.2 for the quivers $\Gamma(P, v)$ corresponding to $\mathcal{M}(P, v)$ for the cases of reducible singular monopoles.

a smooth monopole is absorbed into the defect. This reduces the charge of the 't Hooft defect and reduces the number of degrees of freedom in the bulk. The complicated structure of $\overline{\mathcal{M}}$ comes from how the nested components are glued together to form the total moduli space. This is determined by the transversal slice of each component which physically describes the moduli of smooth monopoles that were swallowed up by the 't Hooft defect. In the case of reducible defects, the transversal slice is particularly simple and is given by a quiver moduli space [142, 23, 24]. This indicates that quantum mechanically, bubbling of the smooth monopoles induces a corresponding quiver SQM on the world volume of the 't Hooft defect.

2.4.2 Triholomorphic Killing Vectors and Symmetries

A subject which will be crucial in later discussions is the realization of explicit triholomorphic killing vector fields that generate symmetries of monopole moduli space [75, 76, 77, 132, 133, 134]. Consider a local, real coordinate system $\{z^m\}$ on \mathcal{M} . These coordinates parametrize a smooth family of gauge-inequivalent solutions of the Bogomolny equations given by $\hat{A} = (A_i, X)$. We can describe the tangent space at $[\hat{A}] \in \mathcal{M}$ by

$$T_{[\hat{A}]} \mathcal{M} = \text{span}_{\mathbb{R}} \left\{ \delta_m \hat{A}_a = \partial_{z^m} \hat{A}_a - \hat{D}_a \epsilon_m \right\} , \quad (2.97)$$

where $\epsilon_m : \mathbb{R}^3 \rightarrow \mathfrak{g}$ projects onto a representative in gauge orbit of $[\hat{A}] \in \mathcal{M}$ which implies that

$$\hat{D}^a \delta_m \hat{A}_a = 0 \longrightarrow \hat{D}^2 \epsilon_m = 0 . \quad (2.98)$$

The $\delta_m \hat{A}_a$ form a local frame of $T\mathcal{M}$. This naturally extends to a covariant derivative on the moduli space

$$D_m = \partial_m + [\epsilon_m, \cdot] , \quad (2.99)$$

with curvature

$$\phi_{mn} = \partial_m \epsilon_n - \partial_n \epsilon_m + [\epsilon_m, \epsilon_n] = [D_m, D_n] . \quad (2.100)$$

This connection is called the *universal connection*. We will discuss this in more detail in the next section.

As we discussed above, the action of the torus T on the fields is generated by triholomorphic killing vector fields on \mathcal{M} . Such vector fields and hence their corresponding gauge transformations are generated by elements of the Cartan subalgebra \mathfrak{t} :

$$G : \mathfrak{t} \rightarrow \text{isom}_{\mathbb{H}}(\mathcal{M}) . \quad (2.101)$$

For any $H \in \mathfrak{t}$, there exists a unique solution to

$$\hat{D}^2 \epsilon_H = 0 \quad , \quad \lim_{r \rightarrow \infty} \epsilon_H(x) = H . \quad (2.102)$$

This generates a vector field $G(H)^m$ by decomposing the derivative in terms of the $\delta_m \hat{A}_a$

$$\hat{D}_a \epsilon_H = -G(H)^m \delta_m \hat{A}_a . \quad (2.103)$$

By nature of being triholomorphic killing vectors, translation along $G(H)^m$ preserves any complex structure

$$\mathcal{L}_{G(H)} \mathbb{J}^r = 0 . \quad (2.104)$$

Using the G -map, we can define a basis of the triholomorphic vector fields corresponding to the group of gauge transformations⁷

$$(K^I)^m = G(H_I)^m \quad I = 1, \dots, \text{rk}[\mathfrak{g}] . \quad (2.105)$$

We will use the notation $(K_0^I)^m = G_0(H_I)^m$ to denote the projection of the projection of the triholomorphic field $G(H_I)^m$ onto the component orthogonal to $G(X_\infty)^m$ on smooth monopole moduli space. This plays a special role in the analysis of smooth monopole moduli space which splits into a center of mass and $G(X_\infty)^m$ orbit.

⁷We are ignoring the subtlety associated with the possibility of the $m^I = 0$ for some I where $\gamma_m = \sum_I m^I H_I$. In this case, one must first reduce to the “effective Lie algebra” and then follow a similar story. For more details see [133].

2.4.3 Universal Bundle

The connection introduced in the last section, D_m , naturally arises as the connection on the universal bundle [11, 47] which parametrizes the principal G bundles over a space \mathcal{U} .

Suppose we want to consider a principal G bundle $P \rightarrow \mathcal{U}$ with a family of connections indexed by some continuous space T . Let Q be a principal G bundle over $\mathcal{U} \times T$ such that $Q|_{\mathcal{U} \times t} \cong P$, $\forall t \in T$. If these conditions hold, any choice of connection on $Q|_{\mathcal{U} \times t}$ which is continuous with respect to $t \in T$ is induced by pulling back the connection from a bundle \mathcal{Q} , with canonical connection D_m through a bundle map $\beta : Q \rightarrow \mathcal{Q}$. This bundle with connection (\mathcal{Q}, D_m) is the universal bundle whose connection is termed the universal connection.

We can define the universal bundle as follows. Let G be a compact, semisimple Lie group with a trivial center⁸ and let $P \rightarrow \mathcal{U}$ be a principal G bundle and let $\mathcal{G} = \left\{ \Phi : P \rightarrow P \mid \pi \circ \Phi = \pi, \pi : P \rightarrow \mathcal{U} \right\}$ be the group of gauge transformations. We want to construct a bundle with the spirit of $P \times \mathcal{A} \rightarrow \mathcal{U} \times \mathcal{A}/\mathcal{G}$. Naively one would expect this to be a principal $G \times \mathcal{G}$ bundle, however the action of $G \times \mathcal{G}$ is generically not free and does not act without fixed points.

The action of \mathcal{G} on $P \times \mathcal{A}$ is given by:

$$\Phi_g \cdot (p, A) = (\Phi_g(p), g^{-1}Ag + g^{-1}dg) \quad , \quad \Phi_g \in \mathcal{G} \quad (p, A) \in P \times \mathcal{A} \quad (2.106)$$

For generic $A \in \mathcal{A}$, the isotropy group $\Gamma_A = \{\Phi_g \in \mathcal{G} \mid \Phi_g(A) = A\}$ can be nontrivial. This means that the subspace of $(P \times \mathcal{A})/\mathcal{G}$ along a slice $[A] \in \mathcal{A}/\mathcal{G}$ is given by $P/\Gamma_A \times \{[A]\}$. Since \mathcal{G} acts freely on P , $(P \times \mathcal{A})/\mathcal{G}$ is a well defined, smooth space. However, this means that the action of G on this space will generically have fixed points. This can be solved by restricting to irreducible connections $\mathcal{A}^* = \{A \in \mathcal{A} \mid \Gamma_A = C(G) = \{1_G\}\}$ or to framed connections which we will define momentarily. In this paper, we will restrict to the space of framed connections following [11].

A framing is a choice of base point $x_0 \in \mathcal{U}$ and an isomorphism $\varphi : G \rightarrow P_{x_0}$ so

⁸If we were to include groups with a non-trivial center we would need to restrict P to be a principal G_0 bundle where $G_0 = G/Z(G)$.

that the set of gauge transformations are equivariant with respect to the G -action on the fiber P_{x_0} under this map φ . For our purposes we will pick x_0 to be the point at infinity. This means we can write the space of framed connections as: $\tilde{B} = \mathcal{A}/\mathcal{G}_0$ where $\mathcal{G}_0 = \{g \in \mathcal{G} \mid g_{x_0} = 1_G\}$ is the space of gauge transformations which act trivially on the space of framings. Therefore, instead of restricting to the space of framed connections, we can equivalently restrict to the gauge connections to be the group \mathcal{G}_0 . The action of G on $(P \times \mathcal{A})/\mathcal{G}_0$ is free because $\forall A \in \mathcal{A}, \mathcal{G}_0 \cap \Gamma_A = \{1_G\}$. Thus, the space $\mathcal{Q} = (P \times \mathcal{A})/\mathcal{G}_0 = P \times \tilde{B}$ forms a principal G bundle over $\mathcal{U} \times \mathcal{A}/\mathcal{G}_0$:

$$\begin{array}{ccc}
 & P \times \mathcal{A} & \\
 \mathcal{G}_0 \swarrow & & \searrow G \\
 \mathcal{Q} = P \times \mathcal{A}/\mathcal{G}_0 & & \mathcal{U} \times \mathcal{A} \\
 G \searrow & & \swarrow \mathcal{G}_0 \\
 & \mathcal{U} \times \mathcal{A}/\mathcal{G}_0 &
 \end{array} \tag{2.107}$$

The universal bundle \mathcal{Q} has a natural connection D_m (called the universal connection) which descends from tautological connection on $P \times \mathcal{A}$ and is compatible with the $(G \times \mathcal{G}_0)$ -invariant metric on $T(P \times \mathcal{A})$. At $(p, A) \in P \times \mathcal{A}$ the metric on $T_{p,A}(P \times \mathcal{A})$ is determined by the metric on \mathcal{U} , killing form on \mathfrak{g} , and L^2 norm.

2.5 Rational Map Construction of Monopole Moduli Space

There is also an additional algebraic formulation of monopole moduli spaces [54, 92, 93, 108, 30, 102]. The core idea is to study monopole configurations by the scattering of charged particles. Then, from studying the S-matrix, which relates a set of incoming states to a nontrivial set of outgoing states, we can reconstruct the field configuration. The physical data of the monopole configuration will be expressed in the structure of the bundle of final states over the plane at outgoing infinity, which is further identified as \mathbb{CP}^1 .

Consider a gauge theory with gauge group G on spatial \mathbb{R}^3 coupled an adjoint Higgs field X and a particle in a faithful representation⁹ of highest weight λ corresponding

⁹Note that any representation will work for this construction as long as the corresponding field

to a section of a vector bundle $H \rightarrow \mathbb{R}^3$. Pick a background field configuration with monopoles. This requires picking a connection ∇ and Higgs field configuration satisfying the Bogomolny equation

$$*F = \nabla X , \quad (2.108)$$

where F is the curvature 2-form of ∇ .

In order to examine the scattering of the charged particle along straight lines in \mathbb{R}^3 we will make use of the twistor method of [87] and work on the space of all oriented, straight lines¹⁰ $T\mathbb{P}^1 \rightarrow \mathbb{CP}_{cs}^1$.

Given a point $\eta \in \mathbb{CP}_{cs}^1$ in the base space, there is an identification $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$ given by the fiber $T_\eta \mathbb{CP}^1$, which we will parametrize by the coordinates (z_η, t_η) where z_η fixes a line in \mathbb{R}^3 whose direction is specified by η and t_η is the coordinate along this line. In these coordinates, the covariant derivative splits as $(\nabla_{t_\eta}, \nabla_{z_\eta}, \nabla_{\bar{z}_\eta})$ on each fiber of $T\mathbb{P}^1$.

Now we want to scatter a charged particle through the static monopole configuration. This is described by parallel transport along the lines $\ell \subset \mathbb{R}^3$, each of which is defined by some $(\eta, z_\eta) \in T\mathbb{P}^1$, by the connection

$$\nabla_\ell = (\nabla_{t_\eta} - iX)|_{z_\eta} . \quad (2.109)$$

Thus, let us consider scattering the charged particle of representation λ , which takes values in an associated vector bundle $H \rightarrow \mathbb{R}^3$, through the monopole configuration along a fixed line $\ell \subset \mathbb{R}^3$. The set of covariantly flat sections $\nabla_\ell s = 0$ of H restricted to ℓ defines a vector space:

$$E_\ell = \{s \in \Gamma(\ell, H|_\ell) \mid \nabla_\ell s = 0\} . \quad (2.110)$$

couples to all of the W-bosons (that is it is a faithful representation) and hence will capture the interactions with all monopoles.

¹⁰The space of oriented, straight lines in \mathbb{R}^3 can be identified with $T\mathbb{P}^1$ as follows. Fix the origin of \mathbb{R}^3 . A line in \mathbb{R}^3 is then specified by a choice of direction and displacement from the origin along a perpendicular plane intersecting the origin. Note that there is $S^2 \cong \mathbb{CP}_{cs}^1$ choices of directions in \mathbb{R}^3 and then there is an \mathbb{R}^2 space of displacement from the origin. Since this \mathbb{R}^2 is perpendicular to the choice of $v \in S^2 \cong \mathbb{CP}^1$, we can associate this with the tangent space and hence the space of oriented lines in \mathbb{R}^3 is given by $TS^2 \cong T\mathbb{P}^1$.

Since $(\vec{\nabla}, X)$ satisfy the Bogomolny equation, there is an operator relation

$$[\nabla_{\bar{z}_\eta}, (\nabla_{t_\eta} - iX)] = 0 , \quad (2.111)$$

for all $\eta \in \mathbb{CP}_{cs}^1$ [87]. Thus, E_ℓ extends to a holomorphic vector bundle over $T\mathbb{P}^1$

$$E \rightarrow T\mathbb{P}^1 \quad , \quad E\Big|_{\eta, z_\eta} = \left\{ s \in I(\ell_{z_\eta}, H|_{\ell_{z_\eta}}) \mid \nabla_{\ell(z, \eta)} s = 0 , \nabla_{\bar{z}_\eta} s = 0 \right\} . \quad (2.112)$$

This bundle has a pair of natural flag structures determined by the asymptotics of the set of solutions along the $t_\eta \rightarrow \pm\infty$. Let the vev X_∞ be a generic element of \mathfrak{t} . Without loss of generality, introduce an ordering of the weights of Δ_λ such that $x_i = \langle \mu_i, X_\infty \rangle \in \mathbb{R}$ are ordered: $x_i < x_{i+1}$, $\forall i$. Then the covariantly flat sections of are of the form

$$s(t) \sim |t_\eta|^{\pm k_p/2} e^{\pm x_p t} f(\eta, z_\eta) \text{ as } t \rightarrow \mp\infty , \quad (2.113)$$

where $k_p = \langle \mu_p, \gamma_m \rangle$. This allows us to define the two sets of subbundles

$$E_p^\pm = \left\{ s \in E \mid \lim_{t \rightarrow \pm\infty} |t|^{\pm k_p/2} e^{\pm x_p t} s(t) \text{ is finite} \right\} \subseteq E , \quad (2.114)$$

for which $\text{rk}[E_p^\pm] = p$. These subbundles have a flag structure:

$$0 = E_0^\pm \subset E_1^\pm \subset E_2^\pm \subset \dots \subset E_d^\pm = E , \quad (2.115)$$

given by the asymptotic behavior of the sections, which is determined by the ordering of the $\{x_p\}$. This flag structure gives a clear way to visualize the natural G -action on E . There is a maximal torus T that stabilizes the flag, and hence the space of inequivalent flags at fixed (η, z_η) is given by $G/T \cong G_{\mathbb{C}}/B$. Here T acts as a phase rotation of the sections at infinity. This encodes the gauge transformations of the conserved gauge group. Similarly, the action of B in the setting of $G_{\mathbb{C}}/B$ also corresponds to gauge transformations.

Now in order to obtain the standard construction of monopole moduli space, we can reduce this to a bundle over \mathbb{CP}^1 from a bundle over $T\mathbb{P}^1$ as follows [87]. First, make a choice of complex structure by picking a point $\eta \in \mathbb{CP}_{cs}^1$. Now define the bundles $E_I^\pm|_\eta \rightarrow T_\eta\mathbb{P}^1 \cong \mathbb{C}$ by restriction. These bundles can be canonically extended to $E_I^\pm|_\eta \rightarrow \mathbb{CP}^1$ by one-point compactification since the two flags of $E|_\eta \rightarrow \mathbb{C}$ are trivial

and isomorphic in the limit $|z| \rightarrow \infty$ for $z \in \mathbb{C}$. This is because physically, the scattering is trivial at infinity.

Now choose a local framing of $E_\eta \rightarrow \mathbb{CP}^1$ which trivializes the negative flag $\langle E_I^- \rangle$ as the standard flag of \mathbb{C}^d over \mathbb{CP}^1 . Physically, this trivialization corresponds to preparing the incoming scattering particle with no initial knowledge of the monopole configuration. This trivialization gives the flag $\{E_I^+\}$ the property

$$E_I^+/E_{I-1}^+ = \mathcal{O}(k_I) \otimes \mathbb{C}^{n_\lambda(\mu_I)} , \quad (2.116)$$

where $k_I = \langle \mu_I, \gamma_m \rangle$ [92]. This is due to the fact that the asymptotic solutions of $(\nabla_\ell - iX)s = 0$ are of the form $s(r) \sim r^{k_I/2} e^{-x_I r}$. Thus, trivializing $\{E_I^-\}$ means that the terms of E_I^+ will go as $s_I(r) \sim r^{k_I} e^{-x_I r}$ and hence the leading term in E_I^+ but not in E_{I-1}^+ is a degree k_I polynomial [93].

The key to this construction is that the data of the monopole is contained in the flag structure of $\{E_I^+\}$. This is because the flag structure encodes the amount to which the fiber of E rotates as the charged particle scatters through the monopoles. This non-trivial flag structure is expected because the non-trivial curvature of E can be attributed to Hecke modifications of the vector bundle as it scatters past monopoles as in [102].

Thus, we can associate the space of monopole configurations the space of flags $\{E_I^+\}$ subject to the condition (2.116). The space of these flags is described by the space of rational functions of degree $m = \{m^I\}_{I=1}^r$ into the flag variety

$$f : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B , \quad (2.117)$$

where $r = \text{rk } \mathfrak{g}$ and $\gamma_m = \sum_{I=1}^r m^I H_I$. This function the scattering matrix of physics.

Remark

1. Since scattering at infinity is trivial, the flag structures should match there and hence the rational functions are based at infinity: $\lim_{z \rightarrow \infty} f(z) = \mathbb{1}_{G_{\mathbb{C}}/B}$.
2. Since the bundle $E|_\eta \rightarrow \mathbb{CP}^1$ descends from the bundle $E \rightarrow T\mathbb{P}^1$, the rational map $f(z)$ descends from a rational map $\hat{f} : T\mathbb{P}^1 \rightarrow G_{\mathbb{C}}/B$ that is holomorphic with respect to $\eta \in \mathbb{CP}_{cs}^1$: $f(z, \eta)$. This encodes the hyperkähler structure of \mathcal{M} .

2.5.1 Explicit Construction of Flag Data and Rational Maps

We will now explicitly construct the rational map which encodes the data of monopoles and show how to explicitly give an algebro-geometric expression for monopole moduli space.

Define a set of functions $\{f_I(z)\}_{I=1}^r$ such that $f_I(z)$ is a rational function of degree m^I , $r = \text{rk } G$, and

$$\gamma_m = \sum_{I=1}^r m^I H_I . \quad (2.118)$$

Now define the map

$$f(z) = \exp \left\{ \sum_{\alpha \in \Phi^+} f_\alpha(z) E_\alpha \right\} , \quad f_\alpha = \sum_{I \in S_\alpha} f_I(z) , \quad (2.119)$$

where $S_\alpha = \{\alpha_I \in \Phi_{simple}^+ \mid \alpha = \sum_{I \in S_\alpha} \alpha_I\}$ is the decomposition of α in terms of simple roots, and $E_\alpha \in \mathfrak{g}^+$ are step operators: $[H_{\alpha_I}, E_{\alpha_J}] = C_{IJ} E_{\alpha_J}$. Clearly $f(z)$ takes values in $G_{\mathbb{C}}/B$. Such a map defines a flag of subbundles $\{E_I^+\} \subset E$ by acting as an upper triangular matrix on the E_I^+ , mapping each factor E_I^+ into the E_J^+ for $J \geq I$. This construction requires that $f(z)$ is a rational map of degree $\sum_{I \in S_\alpha} m^I$ for each E_α . This is equivalent to the condition that each pair $f_I(z) + f_J(z)$ is of degree $m^I + m^J$ for $C_{IJ} \neq 0$. This defines the space of flags, and hence monopole moduli space

$$\mathcal{M}(\gamma_m, \mathcal{X}_\infty) = \bigcap_I \left\{ |\Delta_{I,J}| \neq 0 : C_{IJ} \neq 0 \right\} , \quad (2.120)$$

where $\Delta_{I,J}$ is the resultant of the rational function $f_I(z) + f_J(z)$.

Remark

1. This space is naturally hyperkähler. In the rational functions $f_I(z)$, which are generically of the form

$$f_I(z) = \frac{\sum_{j=0}^{m^I-1} a_j^{(I)} z^j}{z^k + \sum_{k=0}^{m^I-1} b_k^{(I)} z^k} . \quad (2.121)$$

the coefficients $\{a_i^{(I)}, b_i^{(I)}\}$ are holomorphic functions of $\eta \in \mathbb{CP}_{c,s}^1$, which parametrizes the choice of complex structure on \mathcal{M} . The function f has this property because $f(z)$ naturally descend from functions $\hat{f} : T\mathbb{P}^1 \rightarrow G_{\mathbb{C}}/B$. Note that the action

of spatial rotations, $SO(3)_{rot}$, changes the direction of $\eta \in \mathbb{CP}_{cs}^1$ since it selects a direction in \mathbb{R}^3 and hence the action of $SO(3)_{rot}$ rotates the complex structure as expected.

2. Since \mathcal{M} is hyperkähler, it must have a real dimension which is a multiple of 4. Since the rational functions are generically of the form (2.121), a simple counting argument shows that specifying $f(z)$ requires m^I choices of $b_k^{(I)}$ and m^I choices of $a_j^{(I)}$ subject to $2r - 1$ constraint equations with $2r - 1$ free parameters from $|\Delta_{I,J}| \neq 0$. Thus, the total dimension is

$$\dim_{\mathbb{C}} \mathcal{M}(\gamma_m; \mathcal{X}_{\infty}) = 2 \sum_{I=1}^r m^I \implies \dim_{\mathbb{R}} \mathcal{M}(\gamma_m; \mathcal{X}_{\infty}) = 4 \sum_{I=1}^r m^I, \quad (2.122)$$

in agreement with (2.85).

3. Note that due to the choice of complex structure, this formulation only explicitly realizes $SO(2)_{rot} \times \mathbb{R} \times \mathbb{C} \times U(1)^r$ symmetry where $SO(2)_{rot} \cong U(1)_{rot}$ is spacetime rotation in the \mathbb{C} -plane, \mathbb{R} is translation in the \mathbb{R} -direction, \mathbb{C} is translation in the \mathbb{C} -plane, and $U(1)^r$ is a global gauge transformation along the unbroken maximal torus.

A generic element $(\lambda, \mu, \nu, \vec{\rho}) \in U(1)_{rot} \times \mathbb{R} \times \mathbb{C} \times U(1)^r$ where $\vec{\rho} = (\rho^1, \dots, \rho^r)$, acts on the rational maps as

$$f_I(z) \mapsto \lambda^{-2m^I} \mu^{-2} (\rho^I)^{-2} f_I(\lambda^{-1}(z - \nu)). \quad (2.123)$$

These symmetries can be used to fix $|\Delta_{I,J}| = 1$. The full $SO(3)_{rot}$ group acts by rotating the complex structure and hence is most natural in the setting of the rational map $\hat{f} : T\mathbb{P}^1 \rightarrow G_{\mathbb{C}}/B$ where $SO(3)_{rot}$ acts on \mathbb{CP}_{cs}^1 .

4. In the case of $G = SU(2)$, (2.120) simply reduces to the condition that $f_1(z)$ is a rational function or that $|\Delta_1| = 1$ reproducing the result from [12].

2.5.2 Physical Interpretation of Rational Map

It is important to understand how the physical data of monopole moduli space is contained in the rational map. Using the notation from before, we can decompose each

$f_I(z)$ as

$$f_I(z) = \sum_{i=1}^{m^I} \frac{a_i}{z - b_i} , \quad (2.124)$$

Each term in the sum corresponds to a monopole of charge $\gamma_m = H_I$ located at $\vec{x} = (-\frac{1}{2}\log|a|, b) \in \mathbb{R} \times \mathbb{C}$ with phase $\arg(a)$ [12]. Physically, this interpretation only makes sense physically as long as the monopoles are well separated as compared to the W-boson mass $m_W \sim \sqrt{\langle X^2 \rangle}$.

Example: 2 SU(2) Monopoles

The classic example of how the rational maps reproduce monopole moduli space is in the computation of the 2-monopole moduli space for $G = SU(2)$ as studied in [12]. In this case, monopole moduli space is determined by the condition that the rational map

$$f(z) = \frac{a_1 z + a_0}{z^2 + b_1 z + b_0} , \quad (2.125)$$

has degree 2. The corresponding moduli space is given by $|\Delta_f| = 1$ where we have used some of the symmetries to fix the resultant. This gives the equation

$$a_1^2 - b_2 a_0^2 - b_1 a_0 a_1 = 1 . \quad (2.126)$$

We can now exploit the full symmetries of \mathcal{M} to pick different coordinates so that the strongly centered moduli space \mathcal{M}_0 is defined by the variety

$$\{x^2 - zy^2 = 1\} \subset \mathbb{C}^3 . \quad (2.127)$$

This equation defines a 2-dimensional complex variety as a subset of \mathbb{C}^3 which is the famous Atiyah-Hitchin manifold [12]. Note that (2.127) defines a 4 complex dimensional space with 2 coordinates unrestricted. This means that the moduli space will be locally a direct product of this Atiyah-Hitchin space with a flat space describing the center of mass degrees of freedom. This is the famous result of [12].

Example: 2 SU(3) Monopoles

Now consider the case of 2 monopoles with total magnetic charge $\gamma_m = H_{\alpha_1} + H_{\alpha_2}$ in an SU(3) gauge theory. This means that we should consider two functions

$$f_1(z) = \frac{a_1}{z - b_1} , \quad f_2(z) = \frac{a_2}{z - b_2} . \quad (2.128)$$

Without loss of generality we can choose our coordinate center so that $b_1 = 0$. Then we have that the monopole moduli space is defined by the condition that $f_1(z) + f_2(z)$ is a degree 2 rational map. Again we will fix the phases so that monopole moduli space is described by $|\Delta_{1,2}| = 1$:

$$a_1 a_2 b_2^2 = -1 . \quad (2.129)$$

Relabeling into standard coordinates, we can rewrite this in the patch with $b_2 \neq 0$ as

$$x^2 + y^2 + z^2 = 0 , \quad (2.130)$$

which is simply the unresolved A_1 singularity. This tells us that the strongly centered charge (1,1)-monopole moduli space in $SU(3)$ is topologically equivalent to Taub-NUT. Restoring the center of mass degrees of freedom we removed by fixing $|\Delta_{1,2}| = 1$ and $b_1 = 0$ tells us that the moduli space should locally be of the form

$$\mathcal{M}(\gamma_m = H_1 + H_2; \mathcal{X}_\infty) \cong \mathbb{R}^3 \times S^1 \times TN . \quad (2.131)$$

This space is of the correct dimension and matches the result of [111].

Using this insight, we can see that the condition above is consistent with the general asymptotic metric of [111]. Consider the asymptotic limit of monopole moduli space where all monopoles are far separated relative to the W-boson mass $m_W \sim \sqrt{u}$. Now restrict to the subspace of \mathcal{M} where the location and phases of all monopoles except for two with non-trivial attractive magnetic force ($(\gamma_{m,1}, \gamma_{m,2}) < 0$) are fixed. This subspace is locally of the form $\mathbb{R}^3 \times S^1 \times TN$. This can be seen by separating the function

$$f_I(z) + f_J(z) = \frac{a_I}{z - b_I} + \frac{a_J}{z - b_J} + f_{fixed}(z) , \quad (2.132)$$

where $f_{fixed}(z)$ is a degree $m^I + m^J - 2$ rational function which has all fixed parameters $\{a_i, b_i\}$. The rational map degree condition now becomes approximately that for the case of the 2 $SU(3)$ monopoles above so that the subspace of strongly centered moduli space is approximately Taub-NUT, matching the behavior of the metric from [111].

2.5.3 Rational Map Formulation of Singular Monopole Moduli Space

This formulation can be extended to include the existence of singular monopoles. These correspond semiclassically to infinitely heavy magnetically charged particles and are used to describe 't Hooft defects. Therefore, it is also important to understand the moduli space of singular monopoles.

As pointed out in [108, 102], these are equivalent to having non-dynamical monopoles which are at fixed position and phase. In these references it is explicitly worked out that the flag $\{E_I^+\}$ should undergo a Hecke modification. This means

$$E_I^+ \rightarrow E_I^{+'} = E_I^+ \bigotimes_{n=1}^{N_{def}} \mathcal{O}_{z_n}(p_n^I) , \quad (2.133)$$

where $\mathcal{O}_{z_n}(1)$ is the sheaf of locally holomorphic functions with a simple pole at $z_n \in \mathbb{CP}^1$ [102] and

$$P_n = \sum_{I=1}^N p_n^I h^I , \quad p_n^I \in \mathbb{Z} , \quad (2.134)$$

where $\{h^I\}_{I=1}^r$ form a basis of Λ_{cochar} . This construction can further be motivated in analogy to [132], where singular monopoles were introduced by taking the limit of infinitely massive, fixed smooth monopoles.

Since we are simply adding magnetic sources without introducing additional moduli, the functions $f_I(z)$ are modified:

$$f_I(z) \rightarrow \tilde{f}_I(z) = f_I(z) + \sum_{n=1}^{N_{def}} f_{I,sing}^{(n)}(z) . \quad (2.135)$$

Here $\deg[f_I(z)] = \tilde{m}^I$ and $\deg[f_{I,sing}^{(n)}(z)] = p_n^I$ and has a p_n^I -order pole at z_n where

$$\tilde{\gamma}_m = \gamma_m - \sum_n P_n^- = \sum_I \tilde{m}^I H_I , \quad (2.136)$$

and P_n^- is the representative of P_n in the completely negative Weyl chamber. We will refer to $f_{I,sing}^{(n)}(z)$ as the part of $\tilde{f}_I(z)$ encoding the data of the singular monopole in analogy with the physical interpretation of Section 2.5.2. Singular monopole moduli space is then determined by the condition that

$$\overline{\mathcal{M}}(\{P_n\}, \gamma_m; \mathcal{X}_\infty) = \bigcap_{I,J} \left\{ |\tilde{\Delta}_{I,J}| \neq 0 : C_{IJ} \neq 0 \right\} , \quad (2.137)$$

where $\tilde{\Delta}_{I,J}$ is the resultant of the rational function $\tilde{f}_I(z) + \tilde{f}_J(z)$. This does not introduce any new degrees of freedom in the functions and hence the dimension of the moduli space is given by.

$$\dim_{\mathbb{R}} \overline{\mathcal{M}}(\{P_n\}, \gamma_m; \mathcal{X}_{\infty}) = 4 \sum_I \tilde{m}^I, \quad (2.138)$$

matching the results of [132].

Example: Singular Monopoles in $PSU(2)$

Consider the example of a single monopole interacting with a singular monopole in a $PSU(2)$ theory. This can be constructed from $SU(3)$ gauge theory with two monopoles of charge H_1, H_2 by taking the limit $X_{\infty}^2 \rightarrow \infty$ where $X_{\infty} = \sum_I X_{\infty}^I H_I$ [132]. It is known from [25] that taking this limit $\mathcal{M}(\gamma_m = H_1 + H_2) \rightarrow \overline{\mathcal{M}}(P = h^1; \gamma_m = H_1)$ will give rise to Taub-NUT space of charge 1. We will now show that this is exactly what we get from the construction given above.

As in [132], the projection procedure gives the charges

$$\tilde{\gamma}_m = H_1, \quad P = \frac{1}{2} H_1, \quad (2.139)$$

therefore as above, we simply have that the $\overline{\mathcal{M}}$ is defined by the condition that the rational map

$$\tilde{f}(z) = \frac{a_1}{z - b_1} + \frac{\tilde{a}_1^{(n)}}{z - z_n}, \quad (2.140)$$

is of degree 2. Let us pick a coordinate system such that the line defect is at the origin (that is $z_n = 0$). Then, following the example above, we find that the space is given by the variety

$$uvw^2 = -1, \quad (2.141)$$

which again is the relative moduli space of Taub-NUT. The difference between this and the case of smooth monopoles in $\gamma_m = 2H_1$ is that there is no center of mass term since the singular part of \tilde{f} does not provide additional degrees of freedom as expected.

Chapter 3

Semiclassical BPS States

Now we will turn to the role of monopoles in supersymmetric quantum field theories that have Lagrangian descriptions. Here, monopoles take a special role as “BPS states.” BPS states are $\frac{1}{2}$ -SUSY (multi-)particle states that saturate certain energy bounds. Since BPS states partially break SUSY, they form short (sub-)representations of the preserved SUSY (sub-)algebra. This property protects them from decay except along special loci in the moduli space because decay requires the BPS states to combine to form long representations of the full SUSY algebra. On these special loci, the BPS states decay in a very controlled way that is known explicitly [106, 107, 67, 50, 99, 150]. Thus, by studying the spectrum of BPS states, we can learn about a quantum theory’s strong coupling region.

Generically BPS states in the theories we are considering are dyons, which can be described by non-trivial bosonic field configurations. These are non-perturbative excitations that can be thought of as quantum lumps [43]. To describe the physics of such states one must take into account both the perturbative quantum excitations in the non-trivial bosonic background and the dynamics of the non-trivial bosonic background field along the moduli space of non-trivial field configurations. Because of this, the study of BPS states requires both a knowledge of perturbative and non-perturbative physics.

In this section we will discuss some of the basics of BPS states and derive an effective action that describes their dynamics in the semiclassical, adiabatic limit of a Lagrangian SUSY gauge theory. Specifically, we will show that they can be described by a SQM on monopole moduli space which, a.) when in the presence of ’t Hooft defects, is modified

This Chapter is based on material from my papers [22, 28].

to an SQM on singular monopole moduli space, b.) when in the presence of Wilson defects, couples to a vector bundle $\mathcal{E}_{\text{Wilson}}$, and c.) when coupled to 4D hypermultiplet matter, couples to a vector bundle $\mathcal{E}_{\text{matter}}$.

3.1 BPS States

A supersymmetric quantum field theory is one in which there is a conserved set of fermionic “supercharges” Q_A which anti-commute up to a conserved quantity. In 4D, the generators satisfy

$$\begin{aligned} \{Q_\alpha^A, \bar{Q}_{\dot{\alpha}, B}\} &= 2\delta_B^A \sigma_{\alpha\dot{\alpha}}^m P_m, \\ \{Q_\alpha^A, Q_\beta^B\} &= 2\epsilon_{\alpha\beta} \eta^{AB} \bar{Z}, \\ \{\bar{Q}_{\dot{\alpha}, A}, \bar{Q}_{\dot{\beta}, B}\} &= 2\epsilon_{\dot{\alpha}\dot{\beta}} \eta_{AB} Z, \end{aligned} \tag{3.1}$$

where P_m is the momentum operator, \bar{Q} is the complex conjugate of Q , $\alpha, \beta, \dot{\alpha}, \dot{\beta} = 1, 2$ are indices for the $\mathbf{2}_L$ and $\mathbf{2}_R$ representations of $SO(3, 1)$, $A, B = 1, \dots, \mathcal{N}$ where η^{AB} is an antisymmetric matrix, and Z is the central charge – an operator that commutes with the Hamiltonian (and supercharges).

Here, the positive integer $\mathcal{N} = 1, 2, 4$ is often referred to as the number of supersymmetries. This is because the supercharges are acted on by a $SU(\mathcal{N}) \times U(1)$ symmetry group which is called the R -symmetry group. We will focus on the case where $\mathcal{N} = 2$ so that η^{AB} is the standard epsilon symbol of the $\mathbf{2}$ representation of $SU(2)$. Note that we can only have non-trivial central charge when $\mathcal{N} \geq 2$.

Now let us consider how the supercharges act on massive particle states [129].¹ In this case, we can go to the particle’s rest frame so that the momentum operator acts as

$$P_\mu |\psi\rangle = M \delta_\mu^0 |\psi\rangle. \tag{3.2}$$

We can then reparametrize the SUSY algebra in terms of the linear combinations

$$\begin{aligned} \mathcal{R}_\alpha^A &= \zeta^{-1/2} Q_\alpha^A + \zeta^{1/2} \sigma_{\alpha\dot{\alpha}}^0 \bar{Q}^{\dot{\alpha}A}, \\ \mathcal{T}_\alpha^A &= \zeta^{-1/2} Q_\alpha^A - \zeta^{1/2} \sigma_{\alpha\dot{\alpha}}^0 \bar{Q}^{\dot{\alpha}A}, \end{aligned} \tag{3.3}$$

¹The massless particle states will follow analogously. See [129] for more details.

where $\zeta^{1/2} \in U(1)$ which satisfy

$$(\mathcal{R}_1^1)^\dagger = -\mathcal{R}_2^2 \quad , \quad (\mathcal{R}_1^2)^\dagger = \mathcal{R}_2^1 \quad , \quad (\mathcal{T}_1^1)^\dagger = \mathcal{T}_2^2 \quad , \quad (\mathcal{T}_1^2)^\dagger = -\mathcal{T}_2^1 . \quad (3.4)$$

In terms of the \mathcal{R}, \mathcal{T} , the SUSY anti-commutation relations acting on the single particle state $|\psi\rangle$ is given by

$$\begin{aligned} \{\mathcal{R}_\alpha^A, \mathcal{R}_\beta^B\} &= 4(M + \text{Re}(\zeta^{-1}Z))\epsilon_{\alpha\beta}\epsilon^{AB} , \\ \{\mathcal{T}_\alpha^A, \mathcal{T}_\beta^B\} &= 4(-M + \text{Re}(\zeta^{-1}Z))\epsilon_{\alpha\beta}\epsilon^{AB} , \\ \{\mathcal{R}_\alpha^A, \mathcal{T}_\beta^B\} &= 0 . \end{aligned} \quad (3.5)$$

The Hermiticity conditions (3.3) then imply that

$$\left(\mathcal{R}_1^1 + (\mathcal{R}_1^1)^\dagger\right)^2 = \left(\mathcal{R}_1^2 + (\mathcal{R}_1^2)^\dagger\right)^2 = 4(M + \text{Re}(\zeta^{-1}Z)) \geq 0 . \quad (3.6)$$

This is referred to as the Bogomolny bound. The states that saturate this bound are BPS states. Consequently a non-BPS state satisfies $M > -\text{Re}(\zeta^{-1}Z)$.

Consider a BPS state $|\psi_{BPS}\rangle$. The SUSY algebra then acts on this state following the commutation relations

$$\begin{aligned} \{\mathcal{R}_\alpha^A, \mathcal{R}_\beta^B\} &= 0 , \\ \{\mathcal{T}_\alpha^A, \mathcal{T}_\beta^B\} &= -8M\epsilon_{\alpha\beta}\epsilon^{AB} . \end{aligned} \quad (3.7)$$

By rescaling $\mathcal{T}_\alpha^A \mapsto \tilde{\mathcal{T}}_\alpha^A = \frac{1}{\sqrt{8M}}\mathcal{T}_\alpha^A$, we the commutation relations become

$$\begin{aligned} \{\mathcal{R}_\alpha^A, \mathcal{R}_\beta^B\} &= 0 , \\ \{\tilde{\mathcal{T}}_\alpha^A, \tilde{\mathcal{T}}_\beta^B\} &= -\epsilon_{\alpha\beta}\epsilon^{AB} . \end{aligned} \quad (3.8)$$

We can now see that the state must form a representation of the fermionic harmonic oscillator (generated by the $\tilde{\mathcal{T}}_\alpha^A$) and is annihilated by the $\tilde{\mathcal{R}}_\alpha^A$. Thus, a BPS state form a multiplet:

$$\begin{array}{ccc} & \tilde{\mathcal{T}}_1^1 \tilde{\mathcal{T}}_1^2 |\psi_{BPS}\rangle & \\ \swarrow & & \searrow \\ \tilde{\mathcal{T}}_1^1 |\psi_{BPS}\rangle & & \tilde{\mathcal{T}}_1^2 |\psi_{BPS}\rangle \\ \searrow & & \swarrow \\ & |\psi_{BPS}\rangle & \end{array} \quad (3.9)$$

where we used the Hermiticity condition (3.3) to eliminate the $\tilde{\mathcal{T}}_2^1, \tilde{\mathcal{T}}_2^2$.

This differs from the case of a non-BPS (fully supersymmetric) state whose SUSY algebra can be written

$$\begin{aligned} \{\tilde{\mathcal{R}}_\alpha^A, \tilde{\mathcal{R}}_\beta^B\} &= \epsilon_{\alpha\beta} \epsilon^{AB} , \\ \{\tilde{\mathcal{T}}_\alpha^A, \tilde{\mathcal{T}}_\beta^B\} &= -\epsilon_{\alpha\beta} \epsilon^{AB} , \end{aligned} \quad (3.10)$$

where $\tilde{\mathcal{R}}_\alpha^A = \frac{1}{\sqrt{8M}} \mathcal{R}_\alpha^A$ and $\tilde{\mathcal{T}}_\alpha^A = \frac{1}{\sqrt{8M}} \mathcal{T}_\alpha^A$. Non-BPS states are then not annihilated by either the $\tilde{\mathcal{R}}_\alpha^A$ or $\tilde{\mathcal{T}}_\alpha^A$ and thus form a long representation of the form

$$\begin{aligned} & \begin{array}{ccccccc} & & & & \mathcal{T}_1^1 \mathcal{T}_1^2 \mathcal{R}_1^1 \mathcal{R}_1^2 |\psi\rangle & & \\ & \nearrow & & \nwarrow & & \nearrow & \nwarrow \\ \mathcal{T}_1^2 \mathcal{R}_1^1 \mathcal{R}_1^2 |\psi\rangle & & \mathcal{T}_1^1 \mathcal{R}_1^1 \mathcal{R}_1^2 |\psi\rangle & & \mathcal{T}_1^1 \mathcal{T}_1^2 \mathcal{R}_1^2 |\psi\rangle & & \mathcal{T}_1^1 \mathcal{T}_1^2 \mathcal{R}_1^1 |\psi\rangle \\ \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow \\ \mathcal{R}_1^1 \mathcal{R}_1^2 |\psi\rangle & \mathcal{T}_1^2 \mathcal{R}_1^2 |\psi\rangle & \mathcal{T}_1^2 \mathcal{R}_1^1 |\psi\rangle & \mathcal{T}_1^1 \mathcal{R}_1^2 |\psi\rangle & \mathcal{T}_1^1 \mathcal{R}_1^1 |\psi\rangle & \mathcal{T}_1^1 \mathcal{T}_1^2 |\psi\rangle & \\ \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow & \nearrow \quad \nwarrow \\ \mathcal{R}_1^2 |\psi\rangle & \mathcal{R}_1^1 |\psi\rangle & \mathcal{T}_1^2 |\psi\rangle & \mathcal{T}_1^1 |\psi\rangle & & & \\ & \nwarrow & \nearrow & \nwarrow & \nearrow & & \\ & |\psi\rangle & & & & & \end{array} \end{aligned} \quad (3.11)$$

Since the representation corresponding to a BPS state (short) and a non-BPS state (long) are of different dimension, a BPS state cannot become non-BPS unless BPS states combine together to form a long representation of the SUSY algebra. This general property protects the mass of such states from quantum corrections that would potentially break the BPS bound

$$M \geq -\text{Re}(\zeta^{-1} Z) . \quad (3.12)$$

3.2 $\mathcal{N} = 2$ Supersymmetric Gauge Theory

For the rest of the paper we will consider $\mathcal{N} = 2$ supersymmetric gauge theories in 4D. Such a theory is specified by a gauge group G , some quaternionic representation of G , with couplings, masses, and Higgs vevs.

To the gauge group G we can associate a vectormultiplet superfield with fields (A_μ, ψ_A, Φ) . Here A_μ is a $\text{Lie}[G] = \mathfrak{g}$ -valued gauge field, ψ_A is an $SU(2)_R$ doublet of complex Weyl fermions that are in the adjoint representation of G , and Φ is a complex, adjoint-valued Higgs field. To this vector multiplet we can associate the complex coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} , \quad (3.13)$$

where g is the standard gauge field coupling and θ is the associated theta angle (instanton fugacity). Additionally, A_μ is uncharged under $U(1)_R$ while Φ has charge 2.

To the quaternionic representation G we can associate a hypermultiplet superfield field which has component fields (q_A, λ) where q_A is in the quaternionic representation of G which also furnishes a fundamental representation of $SU(2)_R$ and λ is a Dirac fermion which is uncharged under $SU(2)_R \times U(1)_R$. The outer automorphism group of the quaternionic representation which, commutes with the action of G , is a global symmetry (which may or may not be anomalous) that is referred to as the flavor symmetry group G_f . The hypermultiplet superfield comes with a specification of mass parameters which is valued in a Cartan subalgebra $m \in \mathfrak{t}_f \subset \mathfrak{g}_f = \text{Lie}[G_f]$. This breaks the flavor symmetry group down to the normalizer subgroup of m in G_f which we generically take to be a maximal torus T_f .

With this data one can specify the UV Lagrangian of the theory

$$\begin{aligned} \mathcal{L} = & \text{Re} \int d^2\theta \text{Tr} \left\{ \frac{-i\tau}{8\pi} W_\alpha W^\alpha \right\} + \frac{\text{Im}\tau}{4\pi} \int d^2\theta d^2\bar{\theta} \Phi^\dagger e^{2iV} \Phi \\ & + \frac{\text{Im}\tau}{4\pi} \left\{ \int d^2\theta d^2\bar{\theta} \left[Q^\dagger e^{2iV} Q + \tilde{Q}^T e^{-2iV} \tilde{Q}^* \right] + \text{Re} \int d^2\theta (\tilde{Q}^\dagger \Phi Q + m \tilde{Q}^\dagger Q) \right\} . \end{aligned} \quad (3.14)$$

In components this can be written as

$$\begin{aligned}
\mathcal{L} = & \frac{1}{g^2} \text{Tr} \left[-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + i(\psi_A \sigma^\mu D_\mu \bar{\psi}^A + \bar{\psi}^A \bar{\sigma}^\mu D_\mu \psi_A) + (M_n)^2 \right. \\
& \left. -i(\epsilon^{AB} \psi_A [\psi_B, \Phi^*] + \epsilon_{AB} \bar{\psi}^A [\bar{\psi}^B, \Phi]) + i\Phi^* [D, \Phi] \right] \\
& + \frac{1}{g^2} \left[-D_\mu q^{A\dagger} D^\mu q_A + i\bar{\lambda} \gamma^\mu D_\mu \lambda - |H_A|^2 + i q^{A\dagger} (\sigma^n)_A^B M_n q_B \right. \\
& -i(q^{A\dagger} \Psi_A \lambda + \bar{\lambda} \bar{\Psi}^A q_A) - \bar{\lambda} (\Phi_R + m_R) \lambda - i\bar{\lambda} (\Phi_I + m_I) \gamma^5 \bar{\lambda} \\
& \left. + i q^A \Phi H_A - i H^{A\dagger} \Phi^* q_A^* + m q^A H_A + m^* q_A^\dagger H^{A*} \right] \\
& + \frac{\theta}{32\pi^2} \text{Tr} [\epsilon_{\mu\nu\sigma\rho} F^{\mu\nu} F^{\sigma\rho}] ,
\end{aligned} \tag{3.15}$$

where M_n is a triplet of auxiliary fields $M_n = (g, f, D)$, $m = m_R + im_I$, $\Phi = \Phi_R + i\Phi_I$, and

$$\Psi_A = \begin{pmatrix} \psi_A, \epsilon_{AB} \bar{\psi}^B \end{pmatrix} , \tag{3.16}$$

is a Dirac fermion.

From this Lagrangian, one can show that the classical moduli space is described by solutions to the equations

$$\text{Tr} [T^a [\Phi, \Phi^\dagger]] + q^A \sigma_{AB}^3 T^a q^B = 0 \quad , \quad q^{A\dagger} \sigma_{AB}^\pm q^B = 0 \quad , \quad \Phi \cdot q^A + m \cdot q^A = 0 \quad , \tag{3.17}$$

where σ^3, σ^\pm are the standard combinations of Pauli matrices. We can generically distinguish between 3 different types of vacua: 1.) Coulomb-, 2.) Higgs-, and 3.) mixed-type vacua. These are defined by taking the vev's of Φ and q_A at infinity be 1.) $\langle q_A \rangle = 0$, 2.) $\langle \Phi \rangle = 0$, and 3.) $\langle q_A \rangle, \langle \Phi \rangle$ generic. We refer to the collection vacua in each class the Coulomb, Higgs, and mixed branches respectively.

While each vacua has its own interesting properties and interesting physical phenomena associated to it, we will only focus on the Coulomb branch here. For Coulomb type vacua, the vacuum equations reduce to

$$[\Phi, \Phi^\dagger] = 0 \quad , \tag{3.18}$$

and thus the Coulomb branch \mathcal{B} can be identified with $\mathcal{B} = (\mathfrak{t} \otimes \mathbb{C})/W$, where W is the Weyl group of $\mathfrak{t} \subset \mathfrak{g}$. Due to the quotient by the Weyl group, the Coulomb branch is parametrized by the casimirs of the vev of the scalar field.

At generic points on the Coulomb branch, the gauge group is classically broken to a maximal torus $T \subset G$ except along co-dimension ≥ 1 loci where $\Phi_\infty^I = 0$ where the Higgs vev decomposes into simple coroots as

$$\Phi_\infty = \sum_I \Phi_\infty^I H_I . \quad (3.19)$$

Quantum mechanically, the gauge symmetry breaks down to a maximal torus $T \subset G$ everywhere on \mathcal{B} .

3.2.1 Low Energy Theory

The low energy effective theory of 4D $\mathcal{N} = 2$ SYM theory can then be described as a 4D gauged non-linear sigma model on \mathcal{B} [155, 156]. $\mathcal{N} = 2$ supersymmetry implies that the metric on \mathcal{B} is hyperkähler . In the $\mathcal{N} = 1$ notation of [171], the low energy effective action can be written

$$\mathcal{L} = \text{Re} \left[\int d^2\theta \tau^{ij}(\Phi) W_i^\alpha W_{\alpha,j} \right] + \int d^4\theta K(\Phi, \bar{\Phi}) , \quad (3.20)$$

where Φ is a $\mathcal{N} = 1$ chiral superfield with components (a, ψ_2, F) and W_α is the curvature of the remaining massless vector superfield with field components (D, ψ_1, A_μ) . Here, τ^{ij} is a holomorphic function of Φ for each i, j , and $K(\Phi, \bar{\Phi})$ is the associated Kähler potential

$$\frac{\partial^2 K}{\partial a_i \partial \bar{a}_j} = \tau^{ij} , \quad (3.21)$$

where $i, j = 1, \dots, \text{rk } \mathfrak{g}$ run over the $U(1)$ subgroups of $T \cong U(1)^{\text{rk } \mathfrak{g}}$.

The physical structure of this theory can be seen more clearly by defining coordinates that are dual to a_i

$$a_D^i = \frac{\partial K}{\partial a_i} , \quad (3.22)$$

which allows us to define τ^{ij} as

$$\tau^{ij} = \frac{\partial a_D^i}{\partial a_j} , \quad (3.23)$$

Under this choice of coordinates, the metric of the effective action and Kähler form become

$$ds^2 = \frac{1}{2\pi} \text{Im} da_D^i da_i \quad , \quad \Omega = -\frac{1}{8\pi} (da_D^i \wedge d\bar{a}_i - da_i \wedge d\bar{a}_D^i) . \quad (3.24)$$

Note that there is an action of $SL(2; \mathbb{Z})$ that acts on the pairs (a_i, a_D^i) by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a_i \\ a_D^i \end{pmatrix} \longrightarrow \begin{pmatrix} a'_i \\ a_D^{i'} \end{pmatrix}, \quad (3.25)$$

such that the metric and Kähler form are preserved. Equation (3.22) then implies that τ^{ij} transforms under the same $SL(2; \mathbb{Z})$ as

$$\tau^{ij} \mapsto \frac{a\tau^{ij} + b}{c\tau^{ij} + d}. \quad (3.26)$$

We will identify the τ^{ij} that are related by $SL(2; \mathbb{Z})$ transformations as physically equivalent. Further, physical considerations require us to restrict us to the case where $g^2 > 0$ which implies that $\text{Im } \tau^{ij} > 0$. Thus, τ^{ij} takes values in the upper half plane quotiented by the action of $SL(2; \mathbb{Z})$: $\mathbb{H}^+/SL(2; \mathbb{Z})$.

A holomorphic function that takes values in $\mathbb{H}^+/SL(2; \mathbb{Z})$ where $SL(2; \mathbb{Z})$ acts as (3.26) (i.e. τ^{ij} is a modular form of weight 0) is uniquely specified. This function is the j -function of a torus. This allows us to encode the data of all of the τ^{ij} in an abelian variety X fibered over \mathcal{B} with a globally defined 1-form. Specifically, given a decomposition of the first homology group $H_1(X; \mathbb{Z}) = \text{span}\{\alpha_i, \beta^i\}$ into symplectically dual cycles, we can define the (a_i, a_D^i) by

$$a_i = \int_{\alpha_i} \lambda, \quad a_D^i = \int_{\beta^i} \lambda, \quad (3.27)$$

where λ is the globally defined 1-form which is often referred to as the Seiberg-Witten differential. The abelian varieties above a point $u \in \mathcal{B}$ can be encoded as the Prym variety of an N -fold cover of a Riemann surface ²

$$\Sigma_u \xrightarrow{N} C, \quad (3.28)$$

fibered over $u \in \mathcal{B}$.

More generally, for a 4D theory with hypermultiplets, Σ and C are allowed to have singularities at special points in \mathcal{B} . In this case we consider only the periods of the

²That is to say the Jacobian variety X is encoded as the kernel of the map of Jacobian varieties $J(\Sigma_u) \rightarrow J(C)$.

Prym variety associated to the fibration of closure of Σ_u ($\bar{\Sigma}_u$) over the closure of C (\bar{C})

$$X_u = \text{Prym}\left[\bar{\Sigma}_u \xrightarrow{N} \bar{C}\right] \quad , \quad u \in \mathcal{B} . \quad (3.29)$$

The explicit forms of the pairs (Σ_u, C) and the associated 1-form λ are known in a large number of cases. This allows one to explicitly compute the exact effective action in terms of the a_i, a_D^i . This was first done by Seiberg and Witten in their seminal papers [156, 155].

BPS states also have a natural interpretation in this description of the data of the 4D low energy effective theory. Specifically, they correspond to special closed 1-dimensional submanifolds of Σ . To each such submanifold $\mathcal{P} \subset \Sigma$ we can associate a mass and central charge

$$M_{\mathcal{P}} = \int_{\mathcal{P}} |\lambda| \quad , \quad Z_{\mathcal{P}} = \int_{\mathcal{P}} \lambda . \quad (3.30)$$

From this definition, it is clear that we have the identity

$$M_{\mathcal{P}} \geq |Z_{\mathcal{P}}| . \quad (3.31)$$

Thus, BPS states can be identified with 1-dimensional closed submanifolds $\mathcal{P} \subset \Sigma$ such that

$$M_{\mathcal{P}} = |Z_{\mathcal{P}}| . \quad (3.32)$$

This description of the low energy effective theory of a 4D $\mathcal{N} = 2$ gauge theory in terms of abelian varieties may seem quite esoteric, but it actually comes from the string theory construction of the 4D theory. This construction is called the class \mathcal{S} construction in which the 4D theory is given by wrapping M5-branes on Σ producing an effective 4D theory with $\mathcal{N} = 2$ SUSY. The BPS states then come from the boundary of M2-branes wrapping the one-dimensional submanifolds. We will discuss this construction further in Chapter 5.

3.3 Vanilla BPS States

BPS states in a 4D $\mathcal{N} = 2$ theory can generically be classified into two classes: smooth (or “vanilla”) BPS states and framed BPS states. Framed BPS states are BPS states

in a classical background with line defect insertions whereas vanilla BPS states are in a classical background without line defects.

In this section we will describe the dynamics of smooth BPS states in $\mathcal{N} = 2$ super-Yang-Mills theory.³ Here we will take G to be a semisimple, compact gauge group with trivial center.⁴ The classical configurations of smooth BPS states are generically described by smooth monopoles (which are allowed to additionally have electric charge). This is clear because the bosonic part of the action of SYM theory is identical to the Yang-Mills-Higgs theory studied in the previous chapter and hence the non-trivial background configurations (which we assume to be bosonic) are given by monopole configurations.

The bosonic part of the 4D $\mathcal{N} = 2$ SYM action is given by⁵

$$S_{bos} = -\frac{1}{g_0^2} \int d^4x \operatorname{Tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi D^\mu \Phi - \frac{1}{4} [\Phi, \Phi^\dagger]^2 \right\}, \quad (3.33)$$

where here we use g_0 (and later will use θ_0) to denote the bare coupling. This is of the same form as the YMH Lagrangian (2.1) with $V(\Phi) = [\Phi, \Phi^\dagger]^2$.

The bosonic part of the Hamiltonian of 4D $\mathcal{N} = 2$ SYM theory can be written

$$\begin{aligned} H_{bos} &= \frac{1}{g_0^2} \int d^3x \operatorname{Tr} \left\{ E_i^2 + B_i^2 + |D_0 \Phi|^2 + |D_i \Phi|^2 - \frac{1}{4} [\Phi, \Phi^*]^2 \right\} \\ &= \frac{1}{g_0^2} \int d^3x \operatorname{Tr} \left\{ \left| -E_i - iB_i + \zeta_{van}^{-1} D_i \Phi \right|^2 + \left| \zeta_{van}^{-1} D_0 \Phi + \frac{1}{2} [\Phi, \Phi^*] \right|^2 \right\} \\ &\quad - \operatorname{Re} \left(\zeta_{van}^{-1} Z^{cl} \right), \end{aligned} \quad (3.34)$$

where ζ_{van} is some phase and

$$Z^{cl} = \frac{2}{g_0^2} \int_{S_\infty^2} \operatorname{Tr} \{ \Phi (iF - *F) \}. \quad (3.35)$$

Thus, BPS states, which saturate the bound $M \geq -\operatorname{Re}[\zeta_{van}^{-1} Z^{cl}]$, must be solutions of the equations [152, 21]:

$$E_i = D_i Y, \quad B_i = D_i X, \quad D_0 X - [Y, X] = 0, \quad D_0 Y = 0 \quad (3.36)$$

³We will study the general case of $\mathcal{N} = 2$ SUSY gauge theories with matter in Section 3.6.

⁴We will for the remainder of the paper make these assumptions about the gauge group.

⁵We will adopt the conventions of [133].

where $\zeta_{van}^{-1}\Phi = Y + iX$. The solutions of these equations break $\frac{1}{2}$ supersymmetry. The choice of preserved supercharges is determined by the choice of $\zeta_{van} \in U(1)$.

In the gauge where $A_0 = Y$, the time-independent BPS equations simplify

$$E_i = D_i Y \quad , \quad B_i = D_i X \quad . \quad (3.37)$$

The first equation can be combined with Gauss's Law to

$$D^2 Y = 0 \quad , \quad D^2 A_0 = 0 \quad . \quad (3.38)$$

These equations have a unique solution once we specify asymptotic boundary conditions

$$\begin{aligned} Y &= Y_\infty - \frac{g_0^2}{4\pi} \frac{\gamma_e^{phys}}{2r} + O(r^{-(1+\delta)}) \\ E_i &= \frac{g_0^2}{4\pi} \frac{\gamma_e^{phys}}{2r^2} \hat{r}_i + O(r^{-(2+\delta)}) \end{aligned} \quad (3.39)$$

The second equation is the Bogomolny equation which, after fixing the asymptotic behavior

$$\begin{aligned} X &= X_\infty - \frac{\gamma_m}{2r} + O(r^{-(1+\delta)}) \quad , \\ B_i &= \frac{\gamma_m}{2r^2} \hat{r}_i + O(r^{-(2+\delta)}) \quad , \end{aligned} \quad (3.40)$$

has a moduli space of solutions given by monopole moduli space $\mathcal{M}(\gamma_m; X_\infty)$ as we discussed in Chapter 2. Therefore, given the asymptotic boundary conditions (3.39) and (3.40), the moduli space of BPS configurations is exactly the monopole moduli space $\mathcal{M}(\gamma_m; X_\infty)$.

It is clear from the pairing of E_i with $Y = \text{Re}[\zeta_{van}^{-1}\Phi]$ and B_i with $X = \text{Im}[\zeta_{van}^{-1}\Phi]$, that the choice of ζ_{van} that decides this splitting is very important and seemingly arbitrary. However there is a unique choice for ζ_{van} that maximizes the BPS bound for BPS states with fixed magnetic charge and finite electric charge:

$$\zeta_{van}^{-1} = - \lim_{g_0 \rightarrow 0} \frac{|Z_{cl}|}{Z_{cl}} \quad . \quad (3.41)$$

Given the expansion of Z_{cl} in terms as asymptotic data,

$$\zeta_{van}^{-1} Z_{cl} = - \left[\frac{4\pi}{g_0^2} (\gamma_m, X_\infty) + (\gamma_e^{phys}, Y_\infty) \right] + i \left[\frac{4\pi}{g_0^2} (\gamma_m, Y_\infty) - (\gamma_e^{phys}, X_\infty) \right] \quad (3.42)$$

where

$$\gamma_m = \frac{1}{2\pi} \int_{S_\infty^2} F \quad , \quad \gamma_e^{phys} = \frac{2}{g_0^2} \int_{S_\infty^2} *F = - \left(\gamma_e^* + \frac{\theta}{2\pi} \gamma_m \right) \quad , \quad (3.43)$$

the choice (3.41) of ζ_{van} implies that

$$\frac{4\pi}{g_0^2} (\gamma_m, Y_\infty) - (\gamma_e^{phys}, X_\infty) = 0 \quad . \quad (3.44)$$

Here γ_e^* is the dual element of the quantized electric charge. The shifted electric charge is derived by choosing a duality frame for the charge lattice above the moduli space and quantizing the electric and magnetic charges relative to this S-duality frame. See [133] for more details.

3.3.1 Collective Coordinate Expansion

As shown in [124], the dynamics of BPS states in the slow moving, semiclassical limit can be approximated by an SQM on the moduli space of BPS state configurations. The reason is that slow moving BPS states remain approximately BPS. Thus, by taking the fields to be functions of the coordinates on monopole moduli space, we can write the action as a functional on \mathcal{M} , thus describing an SQM on monopole moduli space.

In practice, we can reduce to the effective theory by perturbatively expanding the fields in the small coupling parameter g_0 and demand that the additional fields Y, A_0 , $\psi^A = \rho^A + i\eta^A$ solve their equations of motion to order $O(g_0^2)$ in the monopole background.⁶

Here we will collect spatial gauge field and Higgs field X into a single four-index vector field $\hat{A}_a = (A_i, X)$. The vector field \hat{A}_a can be associated with a self-dual connection which is invariant under the $a = 4$ direction. Since the monopole background is only determined by the fields in \hat{A}_a , we will assume that the first non-trivial term of all other fields is at higher order in g_0 . The equations of motion then imply that the lowest non-trivial order of the other fields are $\psi^A \sim O(g_0^{1/2})$ and $A_0, Y \sim O(g_0)$ [133].

⁶Here we have decomposed ρ^A, η^A in terms of symplectic-Majorana-Weyl spinors.

From the action

$$\begin{aligned}
S_{van} = & -\frac{1}{g_0^2} \int d^4x \operatorname{Tr} \left\{ \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu X D^\mu X + D_\mu Y D^\mu Y + [X, Y]^2 \right. \\
& - 2i\rho^A D_0 \rho_A - 2i\eta^A D_0 \eta_A - 2\eta^A \sigma^{0i} D_i \rho_A + 2\rho^A \sigma^{0i} D_i \eta_A \\
& \left. + 2i(\eta^A [Y, \eta_A] - \rho^A [Y, \rho_A]) - 2i(\rho^A [X, \eta_A] + \eta^A [X, \rho_A]) \right\} \\
& + \frac{\theta_0}{8\pi^2} \int \operatorname{Tr} F \wedge F ,
\end{aligned} \tag{3.45}$$

one can derive the equations of motion

$$\begin{aligned}
\hat{D}^a E_a + [Y, [Y, A_0]] + i([\rho^A, \rho_A] + [\eta^A, \eta_A]) &= 0 , \\
\hat{D}^2 Y - D_0^2 Y - i([\rho^A, \rho_A] - [\eta^A, \eta_A]) &= 0 , \\
i(D_0 \eta_A - [Y, \eta_A]) + \bar{\tau}^a \hat{D}_a \rho_A &= 0 , \\
i(D_0 \rho_A + [Y, \rho_A]) - \tau^a \hat{D}_a \eta_A &= 0 ,
\end{aligned} \tag{3.46}$$

where $\tau^a = (\sigma^0 \bar{\sigma}^i, -i\mathbb{1})$. Now let us consider the equations of motion. For ρ_A , the they are of the form

$$\bar{\tau}^a \hat{D}_a \rho_A = i[Y - A_0, \eta_A] , \tag{3.47}$$

whereas those for η_A are of the form

$$\tau^a \hat{D}_a \eta_A = i[Y + A_0, \rho_A] . \tag{3.48}$$

If we denote $L = i\bar{\tau}^a \hat{D}_a$, the fact that \hat{A}_a is anti-self dual implies that the kernel of L is non-trivial whereas the kernel of $L^\dagger = -i\tau^a \hat{D}_a$ is trivial. This means that given a solution of ρ_A , Y , and A_0 , there is a unique solution for η_A . The equation of motion then implies that $\eta_A \sim O(g^{3/2})$ and hence will lead to terms in the effective Lagrangian that are of higher order than we are considering.

However, there are non-trivial solutions for ρ_A . As shown in [133], there is a 2-1 mapping between vector bosons and Weyl fermions

$$\begin{aligned}
\rho^A &\longrightarrow \delta \hat{A}_a = 2\kappa_A \bar{\tau}_a \rho^A , \\
\delta \hat{A}_a &\longrightarrow \rho^A = -\frac{1}{4 \det \kappa} \delta \hat{A}_a \tau^a \kappa^A ,
\end{aligned} \tag{3.49}$$

where κ_A is a constant symplectic-Majorana-Weyl spinor characterizing the mapping between bosonic and fermionic zero modes.⁷ This mapping implies that [131]

$$\text{Ind}[L] = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{M}(\gamma_m; X_\infty) . \quad (3.50)$$

We will introduce a (local) basis for the fermionic zero modes of ρ^A by $\{\chi^m\}$. This allows us to expand the first non-trivial solution to ρ^A as

$$\rho^A = - \sum_n \frac{\delta_n \hat{A}_a \tau^a \kappa^A}{4 \det \kappa} \chi^n . \quad (3.51)$$

The fermionic zero modes χ^n form a local frame for the spin-bundle over the moduli space.

Using this form of ρ_A , the solutions for A_0, Y are given by [133]

$$\begin{aligned} A_0 &= -\dot{z}^m \epsilon_m + Y^{cl} + \frac{i}{4} \phi_{mn} \chi^m \chi^n + O(g_0^3) , \\ Y &= \epsilon_{Y_\infty} + Y^{cl} - \frac{i}{4} \phi_{mn} \chi^m \chi^n + O(g_0^3) , \end{aligned} \quad (3.52)$$

where ϕ_{mn} is the curvature of the pullback of the universal connection ϵ_m and

$$Y^{cl} = \tilde{\theta}_0(X - \epsilon_{X_\infty}) . \quad (3.53)$$

Here ϵ_H for $H \in \mathfrak{t}$ is the unique \mathfrak{g} -valued function that satisfies

$$\hat{D}^2 \epsilon_H = 0 \quad , \quad \lim_{r \rightarrow \infty} \epsilon_H = H \in \mathfrak{t} . \quad (3.54)$$

Note that Y^{cl} is the unique solution to $\hat{D}^2 Y^{cl} = 0$ with the appropriate pole structure and $\lim_{r \rightarrow \infty} Y^{cl} = 0$.

Substituting these into the action and integrating over the spatial \mathbb{R}^3 reduces the field theory to a particle moving on the spin bundle over \mathcal{M} . This is described by the Lagrangian

$$\begin{aligned} L_{c.c.} &= \frac{4\pi}{g_0^2} \left[\frac{1}{2} g_{mn} (\dot{z}^m \dot{z}^n + i \chi^m \mathcal{D}_t \chi^n - G(Y_\infty)^m G(Y_\infty)^n) - \frac{i}{2} \chi^m \chi^n \nabla_m G(Y_\infty)_n \right] \\ &\quad - \frac{4\pi}{g_0^2} (\gamma_m, X_\infty) + \frac{\theta_0}{2\pi} g_{mn} \dot{z}^m G(X_\infty)^n . \end{aligned} \quad (3.55)$$

⁷This choice of κ is irrelevant as long as $\det \kappa = -\frac{1}{2} \kappa_\beta^B \kappa_B^\beta \neq 0$.

where:

$$g_{mn} = \frac{1}{2\pi} \int_{\mathbb{R}^3} d^3x \operatorname{Tr} \left\{ \delta_m \hat{A}_a \delta_n \hat{A}^a \right\} \quad , \quad \Gamma_{mnp} = \frac{1}{2\pi} \int_{\mathbb{R}^3} d^3x \operatorname{Tr} \left\{ \delta_m \hat{A}^a D_p \delta_n \hat{A}_a \right\} \quad ,$$

$$\mathcal{D}_t \chi^n = \dot{\chi}^n + \Gamma_{mp}^n \dot{z}^m \chi^p \quad ,$$

and

$$G(H)_m = \delta_m \hat{A}^a \hat{D}_a \epsilon_H \quad , \quad H \in \mathfrak{t} \quad . \quad (3.56)$$

See [133, 78, 22] for more details.

3.3.2 Universal Hilbert Bundle

To reduce the 4D dynamics of BPS states in a supersymmetric QFTs to a SQM on monopole moduli space we are implicitly making use of a universal Hilbert bundle. This is similar to the universal bundle in the sense that it parametrizes families of Hilbert bundles of sections of a principal bundle over some Riemannian manifold.

Let $P \rightarrow \mathcal{U}$ be a principal G bundle over some Riemannian manifold \mathcal{U} and let $L^2(\mathcal{U}, P)$ be the Hilbert space of L^2 sections of P . Let \mathcal{A} be the space of all connections on P . There is an action of the gauge group \mathcal{G}_0 of framed gauge transformations (See Section 2.4.3) on this space which gives rise to the diagram:

$$\begin{array}{ccc} L^2(\mathcal{U}, P) & \longrightarrow & L^2(\mathcal{U}, P) \times \mathcal{A} \\ & & \downarrow \mathcal{G}_0 \\ & & \mathcal{H} = (L^2(\mathcal{U}, P) \times \mathcal{A}) / \mathcal{G}_0 \\ & & \downarrow \\ & & \mathcal{A} / \mathcal{G}_0 \end{array} \quad (3.57)$$

Analogous to the case of the universal bundle, the bundle \mathcal{H} can be thought of as a universal Hilbert bundle which has the universal connection induced by parallel transport along $\mathcal{A} / \mathcal{G}_0$. Now since \mathcal{M} injects into $\mathcal{A} / \mathcal{G}_0$ we can pull back the universal Hilbert bundle to a Hilbert bundle over the moduli space \mathcal{M} :

$$\begin{array}{ccccc} H & \xrightarrow{\iota^*} & \iota^*(H) & \longleftarrow & L^2(\mathcal{U}, P) \\ \downarrow & & \downarrow & & \\ \mathcal{A} / \mathcal{G}_0 & \xleftarrow{\iota} & \mathcal{M} & & \end{array} \quad (3.58)$$

Now define a hermitian operator $D_{[\hat{A}]}$ with trivial cokernel which acts fiber-wise on $\iota^*(H)$ determined by $[\hat{A}] \in \mathcal{M}$. We can then define the vector bundle $Ker[D_{[\hat{A}]}] \rightarrow \mathcal{M}$ which is the subspace of L^2 sections which are in the kernel of $D_{[\hat{A}]}$. The operator $D_{[\hat{A}]}$ also defines a projection map $P : \iota^*(H) \rightarrow Ker[D_{[\hat{A}]}]$:

$$\begin{array}{ccccc} H & \xrightarrow{\iota^*} & \iota^*(H) & \xrightarrow{P} & Ker[D] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}/\mathcal{G}_0 & \xleftarrow{\iota} & \mathcal{M} & & \mathcal{M} \end{array} \quad (3.59)$$

From this construction there is a connection on $Ker[D_{[\hat{A}]}]$ given by: $\nabla_{Ker[D]} = P(\iota^*\nabla_{\mathcal{A}/\mathcal{G}_0})$ which is the projected connection from $\iota^*(H)$.

Let us be more explicit. Consider a local basis of sections of $Ker[D_{[\hat{A}]}]$ given by $\{\delta_m \hat{A}_a\}_{m=1}^{\text{Ind}[D_{[\hat{A}]}]}$. Using the L^2 norm, Riemannian metric on \mathcal{U} , and Killing form on \mathfrak{g} we can write the metric on $Ker[D_{[\hat{A}]}]$ as:

$$g_{mn} = \frac{1}{2\pi} \int_{\mathcal{U}} d^n x \text{Tr} \left\{ \delta_m \hat{A}^a \delta_n \hat{A}_a \right\} . \quad (3.60)$$

This metric gives rise to a projected connection of the form

$$\Gamma_{pq}^m = \frac{1}{2\pi} g^{mn} \int_{\mathcal{U}} d^n x \text{Tr} \left\{ \delta_n \hat{A}^a \left(\frac{\partial}{\partial z^p} + [\epsilon_p, \cdot] \right) \delta_q \hat{A}_a \right\} , \quad (3.61)$$

where ϵ_p is the pullback of the universal connection form to $\iota^*(H)$. This can more generally be applied to associated bundles by changing the representation of the connection form ϵ_p as we will see later.

Collective Coordinate Symmetries

Since we are describing BPS particles in 4D, we expect that the symmetries of the four-dimensional theory are preserved in our collective coordinate theory. This theory has $\mathcal{N} = 4$ supersymmetry with the fields transforming as

$$\begin{aligned} \delta_\nu z^m &= -i\nu_a \chi^n (\tilde{\mathbb{J}}^a)_n^m + O(g_0^{5/2}) , \\ \delta_\nu \chi^m &= \nu_a [(z^p - G(Y_\infty)^p)(\mathbb{J}^a)_p^n \Gamma_{nq}^m \chi^q] + O(g_0^3) , \end{aligned} \quad (3.62)$$

where

$$\mathbb{J}^a = (\mathbb{J}^r, \mathbb{1}) \quad , \quad \tilde{\mathbb{J}}^a = (-\mathbb{J}^r, \mathbb{1}) , \quad (3.63)$$

and \mathbb{J}^r are a triplet of complex structures on \mathcal{M} as defined earlier. This leads to the supercharges

$$Q^a = \frac{4\pi}{g_0^2} \chi^m (\tilde{\mathbb{J}}^a)_m^n (\dot{z}_n - G(Y_\infty)_n) + O(g_0^{3/2}) . \quad (3.64)$$

There is also an $SU(2)_R$ symmetry of the four dimensional theory which is realized in the collective coordinate theory. Recall that under this $SU(2)_R$ group, the ρ^A, η^A transform as doublets whereas the bosons transform trivially. This leads to the variations

$$\delta_{I^r} z^m = 0 \quad , \quad \delta_{I^r} \chi^m = \frac{1}{2} \chi^n (\mathbb{J}^r)_n^m , \quad (3.65)$$

where we use δ_{I^r} to denote the transformation associated with the $\mathfrak{su}(2)$ generators: I^r . This leads to the conserved charges

$$I^r = \frac{i\pi}{g_0^2} (\omega^r)_{mn} \chi^m \chi^n , \quad (3.66)$$

where the ω^r are the triplet of Kähler forms associated with the complex structures \mathbb{J}^r .

Many of the symmetries of the four-dimensional theory can be expressed in the SQM as being generated by Killing vectors. Such a Killing vector K^E generates the transformations:

$$\delta_E z^m = (K^E)^m \quad , \quad \delta_E \chi^m = \chi^n \partial_n (K^E)^m . \quad (3.67)$$

This has a corresponding conserved Noether charge

$$N^E = -\frac{4\pi}{g_0^2} \left((K^E)^m g_{mn} (\dot{z}^n + \tilde{\theta}_0 G(X_\infty)^n) - \frac{i}{2} (\nabla_m (K^E)_n) \chi^m \chi^n \right) + O(g_0) . \quad (3.68)$$

Specifically, as we discussed in Chapter 2, the action of global gauge transformations on monopole moduli space is generated by triholomorphic Killing vectors. These are explicitly generated by the $G(H_I)_m$. When we quantize this theory, the operator associated with this conserved quantity will be the Lie derivative along K^E .

3.4 Line Defects

We would now like to generalize our discussion of vanilla BPS states to the framed case. This requires first reviewing general Wilson-'t Hooft operators in four dimensional $\mathcal{N} = 2$ field theories.

3.4.1 Wilson Lines

A Wilson line wrapped on a curve γ can be thought of semiclassically as an infinitely massive, charged particle which is coupled to the gauge (and Higgs) field whose world line is given by γ . Here we will consider such line operators that wrap the time direction at a fixed spatial coordinate which are called Wilson defects. Wilson defects source electric charge that is labeled by a weight $\lambda \in \Lambda_{wt}(G)$.

Wilson defects can be thought of as creating a Hilbert space of states that are localized at the insertion point. These defect Hilbert spaces are isomorphic to the highest weight representation V_λ corresponding to the λ .

In order to describe the contribution of Wilson defects to the collective coordinate theory it will be most convenient for our purposes to introduce spin impurity fields whose Hilbert space is exactly V_λ . Consider a four dimensional gauge theory with gauge group G . Let this theory be coupled to an N -component complex, fermionic field w_a which is in a representation $R_\lambda : G \rightarrow GL(V_\lambda)$ localized at \vec{x}_n with an action:

$$S_{def} = \int d^4x \delta^{(3)}(\vec{x} - \vec{x}_n) i w^\dagger D_t w \quad , \quad D_t = \partial_t + R_\lambda(A_0) . \quad (3.69)$$

This has the equations of motion

$$\frac{dw}{dt} = -R_\lambda(A_0)(t)w , \quad (3.70)$$

which has solutions

$$w(t) = P \exp \left[- \int_{t_0}^t dt' R_\lambda(A_0(t')) \right] w(t_0) . \quad (3.71)$$

To quantize this theory we need to impose

$$\{w_a^\dagger, w_b\} = \delta_{ab} , \quad (3.72)$$

where a, b are indices of the representation R_λ . This leads to a Hilbert space $\mathcal{H} = \Lambda^*(V_\lambda)$.

In order to describe a Wilson line in representation R_λ we need to project onto the first level of the tensor algebra. After this projection, the Lie group \mathfrak{g} acts on the Hilbert space $\mathcal{H}_{def} = \{w_a^\dagger | 0\rangle | a = 1, \dots, N\}$ by the matrices $\sum_{a,b=1}^N w_a^\dagger R_\lambda(T)_{ab} w_b$ for $T \in \mathfrak{g}$ where $\dim_{\mathbb{C}} V_\lambda = N$.

Restricting the tensor algebra can be achieved by projecting onto the $\frac{N+2}{2}$ -eigenstate of the operator

$$Q = \frac{1}{2} \sum_{a=1}^N (w_a^\dagger w_a - w_a w_a^\dagger) . \quad (3.73)$$

This can be accomplished by introducing an auxiliary scalar field $\alpha(t)$ to the action

$$S_{def} = \int d^4x \, \delta^{(3)}(x - x_n) i w_a^\dagger \left(D_t + \frac{N+2}{2} \alpha(t) \right) w_a . \quad (3.74)$$

Here, the auxiliary field α projects onto the Hilbert space $\Lambda^1(V_\lambda) \subset \Lambda^*(V_\lambda)$.

Upon insertion into the path integral

$$Z_{def}[A_0] = \int \mathcal{D}\alpha \mathcal{D}w \mathcal{D}w^\dagger w_a(+\infty) e^{iS_{def}} w_b^\dagger(-\infty) , \quad (3.75)$$

we can integrate out the w_a fields using the propagator $G(t - t') = \theta(t - t') \delta_{ab}$ with midpoint regularization $\theta(0) = \frac{1}{2}$. The integration over the w_a fields then yields

$$R_\lambda(\mathbb{1})_{aa} + \int dt_1 R_\lambda(A_0)_{aa}(t_1) - \int dt_1 dt_2 R_\lambda(A_0)_{ab}(t_1) \theta(t_1 - t_2) R_\lambda(A_0)_{ba}(t_2) + \dots , \quad (3.76)$$

which sums to

$$\text{Tr}_{R_\lambda} P \exp \left(- \int A_0(t) dt \right) = W_{R_\lambda}[A_0] . \quad (3.77)$$

See [165, 13] for more details.

Coadjoint Orbit Quantization

Another, equivalent method we may use to describe electrically charged line defects uses the geometric quantization of coadjoint orbits as in [5, 13]. This method relies on the geometric restrictions on the space of holomorphic sections of a line bundle on a certain flag manifold and the Borel-Weil-Bott theorem in order to construct the Hilbert space of states localized on the line defect as the highest weight representation of the gauge group.

The coadjoint orbit construction proceeds as follows. Consider a Wilson line inserted at the origin in \mathbb{R}^3 with representation R_λ where $\lambda \in \Lambda_{wt}$ is the associated highest weight. We can define the coadjoint orbit \mathcal{O}_λ which is the image of λ under the coadjoint

action of G . By the canonical pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$, one can identify $\mathcal{O}_\lambda \cong G/G_\lambda$ where $G_\lambda = \{g \in G | g \cdot \lambda g^{-1} = \lambda\}$ is the stabilizer of λ under the coadjoint action. We will restrict to generic λ in which case $\mathcal{O}_\lambda \cong G/T$.⁸

Now consider a line bundle over G/T . Such line bundles are classified by $H^2(G/T; \mathbb{Z})$. Due to the short exact sequence

$$1 \rightarrow T \rightarrow G \rightarrow \mathcal{O}_\lambda \rightarrow 1 , \quad (3.78)$$

there is an isomorphism

$$H^2(G; \mathbb{Z}) = H^1(T; \mathbb{Z}) = \text{Hom}(T, U(1)) \equiv \Lambda_{char}(G) \cong \Lambda_{wt} . \quad (3.79)$$

The Borel-Weil-Bott theorem then shows that the line bundle L_λ specified by $\lambda \in H^2(\mathcal{O}_\lambda; \mathbb{Z})$, has vanishing cohomology groups except for $H^0_{\bar{\partial}}(\mathcal{O}_\lambda; \mathbb{Z}) = V_\lambda$ which is the representation space associated to the associated weight $\lambda \in \Lambda_{wt}$.

Therefore, consider the line bundle $L_\lambda \rightarrow \mathcal{O}_\lambda$. In order to construct a Hilbert space $\mathcal{H}_{def} = H^0_{\bar{\partial}}(\mathcal{O}_\lambda, L_\lambda)$, we must quantize \mathcal{O}_λ . This requires a choice of symplectic form on \mathcal{O}_λ and a choice of polarization of \mathcal{O}_λ to define a set of “coordinates” and “momenta” and their commutation relations.

There is a natural choice of symplectic form coming from the a pre-symplectic form

$$\Theta_\alpha = -\langle \alpha, \theta \rangle = \text{Tr}(\lambda \theta) , \quad (3.80)$$

where:

$$\theta = g^{-1} dg \quad , \quad g \in G , \quad (3.81)$$

is the Maurer-Cartan 1-form on G . Using this we can define the 2-form

$$\nu_\alpha = d\Theta_\alpha = \frac{1}{2} \langle \alpha, [\theta, \theta] \rangle . \quad (3.82)$$

However, in order to define a symplectic form, ν_α must be non-degenerate. This requires that on $\mathcal{O}_\lambda \subset \mathfrak{g}^*$ we take $\alpha = \lambda$ so that ν_λ is the symplectic form defining the commutation relation. These structures are compatible with the metric defined by the Killing form on \mathcal{O}_λ and hence defines a Kähler manifold.

⁸For generic λ , $G_\lambda = T$ but more generally $T \subseteq G_\lambda$ and hence any line bundle over G/G_λ can be pulled back to G/T .

We can now define a polarization by a choice of complex structure on \mathcal{O}_λ . This can be achieved by making a choice of positive and negative roots for the lie algebra $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ with respect to $\lambda \in \mathfrak{t}$ which we assume to be a dominant weight.

With these structures in hand, we can now quantize \mathcal{O}_λ . From our choice of polarization, we allow our wavefunction to take values in the holomorphic line bundle $L_\lambda \rightarrow \mathcal{O}_\lambda$.⁹ The Hilbert space of states is thus given by the space of holomorphic sections of L_λ

$$\mathcal{H} = H^0(\mathcal{O}_\lambda, L_\lambda) \cong V_\lambda . \quad (3.83)$$

However, this line bundle L_λ has a non-trivial connection given by $i\Theta_\lambda$ which is subject to the restriction of the Bohr-Sommerfeld quantization condition

$$\frac{1}{h} \oint_{H(p,q)=E} p_i dq_i \in \mathbb{N} . \quad (3.84)$$

This is a consistency condition for defining a (projective) physical Hilbert space. It is equivalent to restricting to a physical Hilbert bundle that has only trivial monodromy around a closed paths of constant energy in phase space. Mathematically, this implies that

$$d\Theta_\lambda = \nu_\lambda \quad (3.85)$$

which is indeed the case for our Hilbert space.

Now that we have shown how to construct a defect Hilbert space we can use this formalism to incorporate a Wilson line into our path integral. Let us introduce the 1D Chern-Simons action component to the path integral:

$$Z_{def} = \int_{L\mathcal{O}_\lambda} \mathcal{D}U \exp \left[i \int_{\mathbb{R}} U^*(\Theta_\lambda) \right] = \int_{L\mathcal{O}_\lambda} \mathcal{D}U \exp \left[i \int_{\mathbb{R}} \text{Tr}(\lambda \cdot U^{-1} dU) \right] , \quad (3.86)$$

where $L\mathcal{O}_\lambda$ is the “line-space” of \mathcal{O}_λ so that

$$U : \mathbb{R} \rightarrow \mathcal{O}_\lambda , \quad (3.87)$$

⁹The condition that the wavefunction is holomorphic comes from the fact that our polarization of \mathcal{O}_λ is the complex structure. We could equivalently consider the antiholomorphic line bundle, but this choice is equivalent to picking coordinate versus momentum basis in the traditional construction of quantum mechanics.

is integrated over the space of maps from the world volume of the Wilson defect \mathbb{R} into \mathcal{O}_λ . The variation of the action is given by

$$\delta S_{def} = i \int_{\mathbb{R}} d\tau \nu_\alpha(U)_{mn} \delta U^m \frac{dU^n}{d\tau} , \quad (3.88)$$

where we have expanded in local coordinates on \mathcal{O}_λ where τ parametrizes the \mathbb{R} .

We can further couple this action to the gauge field as before by modifying the presymplectic form

$$\Theta_\lambda^A = -\langle \lambda, \theta_A \rangle \quad , \quad \theta_A = g^{-1} d_A g , \quad (3.89)$$

where $d_A = d + A \wedge$. This changes the action to:

$$S_{def} = i \int_{\mathbb{R}} U^* (\Theta_\lambda^A) = i \int_{\mathbb{R}} \text{Tr} (\lambda \cdot U^{-1} d_A U) . \quad (3.90)$$

Remark Note that this will lead to the same results as the quantization of the spin defect fields (with slightly different definitions of the same tensor fields) so we will continue with the spin defect fields since the formulas will be generally clearer.

SUSY Wilson Lines

In order to have a Wilson line that preserves maximal 1/2-supersymmetry, we must also couple the impurity fields to the Higgs field. This comes with the data of $\zeta \in U(1)$ which specifies the unbroken supersymmetry. Therefore in order to incorporate multiple line defects and preserve supersymmetry all defects must have the same ζ . Supersymmetry mandates the action:

$$S_{def} = \int d^4x (i w^\dagger (D_t - R_\lambda(Y)) w) \delta^{(3)}(x - x_n) , \quad (3.91)$$

which leads to supersymmetric Wilson lines of the form

$$W_{R_\lambda}[A_0 - Y] = \text{Tr}_{R_\lambda} P \exp \left(- \int R_\lambda(A_0 - Y) dt \right) . \quad (3.92)$$

3.4.2 't Hooft Defects

't Hooft defects can be thought of semiclassically as infinitely heavy magnetically charged particles, i.e. singular monopoles. In order to incorporate 't Hooft lines into

a four dimensional $\mathcal{N} = 2$ theory at a point \vec{x}_n , one must impose boundary conditions on the fields at \vec{x}_n just as in our discussion of singular monopoles. To make an 't Hooft defect supersymmetric one has to couple the field to $\text{Im}[\zeta^{-1}\Phi] = X$, complementary to the case for Wilson lines.¹⁰ The data for these defects is also given by a choice of ζ and \vec{x}_n , but instead of a representation, comes with a choice of magnetic charge $P_n \in \Lambda_{\text{cochar}}$ which by S-duality can be related to a weight of the Langlands dual group ${}^L G[100, 102]$.

An 't Hooft defect defined by the data (P_n, \vec{x}_n, ζ) has the corresponding boundary conditions

$$\begin{aligned}\zeta^{-1}\Phi &= \left(\frac{g_0^2 \theta_0}{8\pi^2} - i \right) \frac{P_n}{2r_n} + O(r_n^{-1/2}) , \\ F &= \frac{P_n}{2} \sin(\theta_n) d\theta_n \wedge d\phi_n - \tilde{\theta}_0 \frac{P_n}{2r_n^2} dt \wedge dr_n + O(r_n^{-3/2}) ,\end{aligned}\tag{3.93}$$

in local coordinates around \vec{x}_n as $r_n \rightarrow 0$. This can be expanded

$$\begin{aligned}B^i &= \frac{P_n}{2r_n^2} \hat{r}_n^i + O(r_n^{-3/2}) \quad , \quad E^i = -\tilde{\theta}_0 \frac{P_n}{2r_n^2} \hat{r}_n^i + O(r_n^{-3/2}) , \\ X &= -\frac{P_n}{2r_n} + O(r_n^{-1/2}) \quad , \quad Y = \tilde{\theta}_0 \frac{P_n}{2r_n} + O(r_n^{-1/2}) ,\end{aligned}\tag{3.94}$$

where $\tilde{\theta}_0 = \frac{g_0^2 \theta_0}{8\pi^2}$.

In order to have a well defined variational principle with these boundary conditions, we must include a boundary term in the Lagrangian

$$\begin{aligned}S_{\text{def}} &= \frac{2}{g_0^2} \int dt \sum_n \text{Re} \left\{ \zeta^{-1} \int_{S_n^2} \text{Tr} \{ (iF - *F) \Phi \} \right\} \\ &= -\frac{2}{g_0^2} \int dt \sum_n \int_{S_n^2} r_n^2 d\Omega_n \hat{r}_n^i \text{Tr} \{ X B_i + Y E_i \} ,\end{aligned}\tag{3.95}$$

where S_n^2 is the infinitesimal 2-sphere around the defect at \vec{x}_n . Additionally, when introducing 't Hooft lines, we must restrict to the gauge transformations that commute with the 't Hooft charge at the insertion point. This leads to a reduction of the structure group of the principal G bundle to $Z(P_n) = \{g \in G \mid g^{-1} e^{iP_n \phi} g = e^{iP_n \phi} , \forall \phi \in [0, 2\pi]\}$ at the defect at $\vec{x}_n \in \mathbb{R}^3$. We will denote the 't Hooft operator defined by (P, \vec{x}, ζ) as $L_{[P,0]}^\zeta(\vec{x})$. We will often suppress the dependence on ζ and \vec{x} .

¹⁰For $\theta_0 \neq 0$, we also have take into account the Witten effect.

't Hooft defects fall into three distinct classes: irreducible, minimal, and reducible 't Hooft defects as in the case of singular monopoles. An *irreducible* 't Hooft defect is defined by the data (P, \vec{x}, ζ) as discussed above. These are S-dual to the Wilson line with irreducible representation of highest weight $P \in \Lambda_{wt}(G^\vee)$ [102]. *Minimal* 't Hooft defects are irreducible 't Hooft defects with minimal charge – that is irreducible 't Hooft defects whose 't Hooft charge is a simple cocharacter $P = h^I$. These are S-dual to the Wilson line with the minimal irreducible representation of G^\vee .

A *reducible* 't Hooft defect is specified by a charge $P \in \Lambda_{cochar}$, position $\vec{x} \in \mathbb{R}^3$ and a phase $\zeta \in U(1)$. Such defects are the coincident limit of $N_{def} = \sum_I p_I$ minimal 't Hooft defects, each of charge $h^{I(i)}$ such that

$$P = \sum_{i=1}^{N_{def}} h^{I(i)} = \sum_{I=1}^{rk G} p_I h^I \quad , \quad p_I \geq 0 \quad , \quad \forall I \quad , \quad (3.96)$$

where $i = 1, \dots, N_{def}$ indexes the constituent minimal 't Hooft defects.¹¹ Thus, a reducible 't Hooft defect is the operator that results from taking the product of minimal 't Hooft operators. Consequently, they are S-dual to a Wilson line corresponding to a reducible representation given by the product of minimal representations of G^\vee . We will write reducible 't Hooft defects as

$$L_{\vec{p},0} = \prod_{I=1}^{rk G} \left(L_{[h^I,0]} \right)^{p_I} . \quad (3.97)$$

In generic theories with matter in representations $\{R_\mu\}$, we must further restrict that $\langle \mu, P \rangle \in \mathbb{Z}$ for all highest weights μ . This restricts P to take values in the magnetic weight lattice $P \in \Lambda_{mw} \subseteq \Lambda_{cochar}$ which is defined as the restriction $\Lambda_{mw} = \Lambda_{cochar}|_{\langle \mu, P \rangle \in \mathbb{Z}}$. We will more generally take \hat{h}^I to be the simple magnetic weights and consequently we we generally use the notation

$$L_{\vec{p},0} = \prod_{I=1}^{rk G} \left(L_{[\hat{h}^I,0]} \right)^{p_I} . \quad (3.98)$$

In many cases, (3.98) coincides with (3.97).

¹¹Here we use the notation where the i^{th} monopole is of charge $h^{I(i)}$. That is, $I(i) = 1, \dots, rk G$ according to the charge of the i^{th} monopole.

In $\mathcal{N} = 2$ supersymmetric theories, reducible 't Hooft operators are related to irreducible 't Hooft operators by the corresponding products of their associated representation of the Langlands dual group G^\vee :

$$L_{[P,0]} \cdot L_{[P',0]} = \bigoplus_{P''} R_{PP'}^{P''} L_{[P'',0]} \quad , \quad \mathcal{R}_P \otimes \mathcal{R}_{P'} = \bigoplus_{P''} R_{PP'}^{P''} \mathcal{R}_{P''} . \quad (3.99)$$

Here $\mathcal{R}_P, \mathcal{R}_{P'}, \mathcal{R}_{P''}$ are representations of G^\vee and $R_{PP'}^{P''}$ are its structure constants [102].

3.4.3 General Wilson-'t Hooft Defects

More generally, we can allow for the inclusion of Wilson-'t Hooft defects which source both electric and magnetic charge [100]. In order to include such defects, we must modify the boundary conditions of the 't Hooft defects to allow for an electric charge Q_n . From the equations of motion

$$\begin{aligned} \hat{D}_i E^i + i([\rho^A, \rho_A] + [\lambda^A, \lambda_A]) &= \frac{ig_0^2}{2} \sum_j w_j^\dagger R_j(T^a) w_j \delta^{(3)}(x - x_j) , \\ \hat{D}^2 Y - D_0^2 Y - i([\rho^A, \rho_A] - [\lambda^A, \lambda_A]) &= \frac{ig_0^2}{2} \sum_j w_j^\dagger R_j(T^a) w_j \delta^{(3)}(x - x_j) , \end{aligned} \quad (3.100)$$

we can see that the fields will have the local behavior

$$\begin{aligned} B^i &= \frac{P_n}{2r_n^2} \hat{r}_n^i + O(r_n^{-3/2}) \quad , \quad E^i = \frac{g_0^2}{4\pi} \frac{Q_n^*}{2r_n^2} \hat{r}_n^i - \frac{\tilde{\theta}_0 P_n}{2r_n^2} \hat{r}_n^i + O(r_n^{-3/2}) , \\ X &= -\frac{P_n}{2r_n} + O(r_n^{-1/2}) \quad , \quad Y = -\frac{g_0^2}{4\pi} \frac{Q_n^*}{2r_n} + \frac{\tilde{\theta}_0 P_n}{2r_n} + O(r_n^{-1/2}) , \end{aligned} \quad (3.101)$$

near the defect at \vec{x}_j . Such field configurations preserve $\frac{1}{2}$ -SUSY.

Here $Q_n \in \Lambda_{wt}(Z(P_n)) \subset \Lambda_{wt}(G)$ is the highest weight of a representation of $Z(P_n)$ ¹² because time independence of the background field implies

$$D_0 F_{ij} = 0 \quad (3.102)$$

and hence $[Q_n, P_n] = 0$. Again this restricts our space of gauge transformations which leave the boundary conditions invariant and hence take value in the stabilizer group

¹²Above we are using the notation $Q_n^* \in \mathfrak{t}$ to denote the dual under the canonical pairing $\langle \cdot, \cdot \rangle : \mathfrak{t}^* \times \mathfrak{t} \rightarrow \mathbb{R}$ with respect to an embedding of $\Lambda_{wt}(Z(P)) \hookrightarrow \Lambda_{wt}(G)$.

of (P_n, Q_n) at the defect at \vec{x}_n . Further, gauge invariance imposes that the Wilson and 't Hooft charges are only dependent on the Weyl orbit $[P, Q] \in \left(\Lambda_{cochar}(G) \times \Lambda_{wt}(Z(P)) \right) / W$.

In the case of multiple Wilson-'t Hooft defects, there is a non-trivial Dirac quantization condition. For any pair of line defects with charges (P, Q) and (P', Q') the Dirac quantization condition is

$$\langle P, Q' \rangle - \langle P', Q \rangle \in \mathbb{Z} \quad (3.103)$$

Due to the singularity in the gauge field coming from the 't Hooft defect, we must add a boundary term to the action to have a well defined variational principle

$$S_{def} = \frac{2}{g_0^2} \int dt \sum_n \int_{S_n^2} r_n^2 d^2 \Omega_n \hat{r}_n^i \text{Tr} \left\{ (E_i Y + B_i X) - \frac{g_0^2}{4\pi} \frac{Q_n^*}{2r_n^2} A_0 \hat{r}_{n,i} \right\}, \quad (3.104)$$

which leads to the boundary variation

$$\begin{aligned} \delta S_{bos} = \frac{2}{g_0^2} \int dt \sum_n \int_{S_n^2} d^2 \Omega_n r_n^2 \hat{r}_n^i \text{Tr} \left\{ \delta A_0 \left(E_i + \frac{g_0^2 \theta_0}{8\pi^2} B_i - \frac{g_0^2}{4\pi} \frac{Q_n^*}{2r_n^2} \hat{r}_{n,i} \right) \right. \\ \left. - \delta Y (D_i Y - E) + Y \left(\delta E_i + \frac{\theta_0 g_0^2}{8\pi^2} \delta B_i \right) \right. \\ \left. - \delta X (D_i X - B_i) - \delta A^j (F_{ij} - \epsilon_{ijk} D^k X) \right\}. \end{aligned} \quad (3.105)$$

This vanishes for the field configurations satisfying the BPS equations and Wilson-'t Hooft boundary conditions (3.101).

3.5 Framed BPS States

The presence of line defects significantly alters the Hilbert space of states and the spectrum of BPS states. These can be viewed in the “core-halo” picture where the BPS states divide into two types: those which binds tightly to the line defects to form a “core” and those which are only loosely bound to the line defects at large radius, forming a collection of “halo” clusters around the core in analogy with galaxies [69].

We now want to know how to describe the moduli space of framed BPS states in the semiclassical limit. We again have the same BPS equations as before, except except with new local boundary conditions at the insertion points of the Wilson-'t

Hooft defects. Thus we can conclude that the moduli space of framed BPS states is given by the singular monopole moduli space $\overline{\mathcal{M}}(\{P\}, \gamma_m; X_\infty)$ defined by taking only the magnetic part of all of the line defects.

3.5.1 Collective Coordinate Dynamics with Wilson-'t Hooft Defects

We now implement the same program as before, reducing the 4D description to a supersymmetric theory of collective coordinates. In addition to the Bogomolny equations, we want to satisfy the equations of motion for A_0 , Y , ψ^A and λ^A

$$\begin{aligned} \hat{D}^2(A_0 - \dot{z}^m \epsilon_m) + i([\rho^A, \rho_A] + [\lambda^A, \lambda_A]) - \frac{ig_0^2}{2} \sum_{j,a} T^a w_j^\dagger R_j(T^a) w_j \delta^{(3)}(x - x_j) &= 0 , \\ \hat{D}^2 Y - D_0^2 Y - i([\rho^A, \rho_A] - [\lambda^A, \lambda_A]) - \frac{ig_0^2}{2} \sum_{j,a} T^a w_j^\dagger R_j(T^a) w_j \delta^{(3)}(x - x_j) &= 0 , \\ i(D_0 \eta^A - [Y, \eta^A]) + \bar{\tau}^a \hat{D}_a \rho_A &= 0 , \\ i(D_0 \rho_A + [Y, \rho_A]) - \tau^a \hat{D}_a \eta_A &= 0 , \end{aligned} \tag{3.106}$$

to order $O(g_0^2)$.

Note that this leads to the same equations of motion for the ρ^A, λ^A . Thus, the moduli space will be $\overline{\mathcal{M}}$ and the dynamics will again couple to its spin bundle. The different solutions for A_0 and Y will again generate a superpotential. Generalizing the solution from earlier, the collective coordinate expansion for the fields above is given by

$$\begin{aligned} Y &= \epsilon_{Y_\infty} + Y^{cl} - \frac{i}{4} \phi_{mn} \chi^m \chi^n + O(g_0^2) , \\ A_0 &= -\dot{z}^m \epsilon_m + Y^{cl} + \frac{i}{4} \phi_{mn} \chi^m \chi^n + O(g_0^2) , \\ \rho^A &= \frac{1}{2\sqrt{\det \kappa}} \delta_m \hat{A}_a (-i\tau^a) \kappa^A \chi^m + O(g_0^2) , \\ \eta^A &= O(g_0^{3/2}) , \end{aligned} \tag{3.107}$$

where Y^{cl} is the classical solution for the fields

$$\hat{D}^2 Y^{cl} = 0 \quad , \quad \lim_{r \rightarrow \infty} Y^{cl} = 0 , \tag{3.108}$$

with the appropriate boundary conditions at the Wilson-'t Hooft defects (4.11).

Here we are using the definition of ϵ_H from the case of vanilla BPS states on \mathcal{M} . Specifically, to a generic point $[\hat{A}] \in \overline{\mathcal{M}}$ we can identify a connection $[\hat{A}_s] \in \mathcal{M}$ by subtracting the fields of the singular monopoles. Then we can define

$$\hat{D}_{\hat{A}_s}^2 \epsilon_H = 0 \quad , \quad \lim_{r \rightarrow \infty} \epsilon_H = H \quad , \quad (3.109)$$

as in [133].

After integrating over \mathbb{R}^3 the theory of collective coordinates can be described by the Lagrangian

$$\begin{aligned} L_{c.c.} = & \frac{4\pi}{g_0^2} \left[\frac{1}{2} g_{mn} (\dot{z}^m \dot{z}^n + i \chi^m \mathcal{D}_t \chi^n - G(Y_\infty)^m G(Y_\infty)^n) - \frac{i}{2} \chi^m \chi^n \nabla_m G(Y_\infty)_n \right] \\ & + \frac{\theta_0}{2\pi} (\gamma_m, Y_\infty) + \frac{\theta_0}{2\pi} (g_{mn} (\dot{z}^m - G(Y_\infty)^m) G(X_\infty)^n - i \chi^m \chi^n \nabla_m G(X_\infty)_n) \\ & - \frac{4\pi}{g_0^2} (\gamma_m, X_\infty) + i \sum_j w_j^\dagger (D_t - \epsilon_{Y_\infty}^{(j)} + \frac{i}{2} \phi_{mn} \chi^m \chi^n) w_j \quad , \end{aligned} \quad (3.110)$$

where

$$D_t w_j^b = \partial_t w_j^b - R(\epsilon_m^{(j)})^b{}_c w_j^c \quad , \quad (3.111)$$

and $\epsilon_m^{(j)}$ is the pullback of the universal connection evaluated at $\vec{x}_j \in \mathbb{R}^3$. The calculation of $L_{c.c.}$ is given in Appendix A.

There are a few notable differences from the case of vanilla BPS states that we wish to comment on:

1. Note that upon integrating out the w_j fields we get a Wilson line coupled to the universal connection on the moduli space. This is to be expected by naively plugging in the collective coordinate expansion of $A_0 - Y$ into the Wilson line in the four-dimensional theory. The SQM Wilson line arises from the term $A_0 = -\dot{z}^m \epsilon_m + \dots$

$$\begin{aligned} \text{Tr}_R P \exp e^{i \int (A_0 - Y) dt} & \rightarrow \text{Tr}_R P \exp e^{-i \int \epsilon_m \dot{z}^m dt + \dots} \\ & = \text{Tr}_R P \exp e^{-i \oint \epsilon_m dz^m + \dots} \quad , \end{aligned} \quad (3.112)$$

where the \dots is the supersymmetric completion. This means that the inclusion of a Wilson line at $\vec{x}_j \in \mathbb{R}^3$ couples the SQM collective coordinate theory to a vector

bundle which we will call the Wilson bundle $\mathcal{E}_{\text{Wilson}}(Q_j)$ which is an associated vector bundle of the universal bundle.

2. There are new terms proportional to θ_0 in the Lagrangian. The most important is

$$-\frac{i\theta_0}{2\pi}\chi^m\chi^n\nabla_m G(X_\infty)_n, \quad (3.113)$$

This term vanishes on \mathcal{M} , but is non-vanishing in the case of 't Hooft defects [133].

The SQM collective coordinate theory now couples to a vector bundle which we call the Wilson bundle

$$\mathcal{E}_{\text{Wilson}} = \bigotimes_j \mathcal{E}_{\text{Wilson}}(Q_j). \quad (3.114)$$

This theory is again supersymmetric with the same supersymmetry transformations as that of vanilla BPS states. This gives rise to the same supercharge

$$Q^a = \frac{4\pi}{g_0^2}\chi^m(\tilde{\mathbb{J}}^a)_m(\dot{z}_n - G(Y_\infty)_n). \quad (3.115)$$

The Wilson Bundle

The Wilson bundle, $\mathcal{E}_{\text{Wilson}}$ is different from bundle of vector multiplet zero modes in that it does not come from some pull back of the universal Hilbert bundle. Rather the Wilson bundle is roughly the pull back of the universal bundle, restricted to the defect point $\vec{x}_j \in \mathbb{R}^3$. One can construct the Wilson bundle by pulling back the principal G -bundle \mathcal{Q} through the diagram:

$$\begin{array}{ccccc} \mathcal{Q} = P \times \mathcal{A}/\mathcal{G}_0 & \xrightarrow{\iota^*} & \iota^*(\mathcal{Q}) & \xrightarrow{ev_{j*}} & ev_{j*}(\iota^*(\mathcal{Q})) \\ \downarrow & & \downarrow & & \downarrow \\ X \times \mathcal{A}/\mathcal{G}_0 & \xleftarrow{\iota} & X \times \mathcal{M} & \xleftarrow{ev_j} & \mathcal{M} \end{array} \quad (3.116)$$

where R_{Q_j} is the representation of highest weight Q_j and $ev_j : \mathcal{M} \rightarrow X \times \mathcal{M}$ where $ev_j : z^m \mapsto (\vec{x}_j, z^m)$. Then we can construct the associated bundle:

$$\mathcal{E}_{\text{Wilson}}(Q_j) = ev_{j*}(\iota^*(\mathcal{Q})) \times_{R_{Q_j}} \mathbb{C}^N \rightarrow \mathcal{M}. \quad (3.117)$$

This is why the connection on the Wilson Bundle $\mathcal{E}_{\text{Wilson}}(Q_j)$ is given by the universal connection evaluated at \vec{x}_j .

Hamiltonian Dynamics

We can now convert our Lagrangian formalism to the Hamiltonian formalism in preparation for quantization. The conjugate momenta of our fields are given by:

$$\begin{aligned} p_m &= \frac{4\pi}{g_0^2} g_{mn} \left[\dot{z}^m + \tilde{\theta} G(X_\infty)^m + \frac{i}{2} \chi^p \chi^q \Gamma_{pq}^m \right] + q_m , \\ (p_\chi)_m &= \frac{4\pi}{g_0^2} g_{mn} \chi^n , \quad (p_w)_a = i(w_j^\dagger)_a , \end{aligned} \quad (3.118)$$

where

$$q_m = i \sum_j w_j^\dagger R_j(\epsilon_m^{(j)}) w_j . \quad (3.119)$$

Introducing the notation

$$\pi_m = p_m - \frac{2\pi i}{g_0^2} \Gamma_{m,pq} \chi^p \chi^q - q_m , \quad (3.120)$$

we can write the Hamiltonian as

$$\begin{aligned} H_{c.c.} &= M^{cl} + \frac{g_0^2}{8\pi} \left\{ \pi_m g^{mn} \pi_n + g_{mn} G(\mathcal{Y}_\infty)^m G(\mathcal{Y}_\infty)^n + \frac{4\pi i}{g_0^2} \chi^m \chi^n \nabla_m G(\mathcal{Y}_\infty)_n \right\} \\ &+ i\tilde{\theta}_0 \left(iG(X_\infty)^m \pi_m + \frac{2\pi}{g_0^2} \chi^m \chi^n \nabla_m G(X_\infty)_n \right) + i \sum_j w_j^\dagger (\epsilon_{Y_\infty}^{(j)} - \frac{i}{2} \phi_{mn} \chi^m \chi^n) w_j , \end{aligned}$$

where

$$M^{cl} = \frac{4\pi}{g_0^2} (\gamma_m, X_\infty) + (\gamma_e^{phys}, Y_\infty) , \quad (3.121)$$

and the supercharge is given by:

$$Q^a = \chi^m (\tilde{\mathbb{J}}^a)_m^n (\pi_n - G(\mathcal{Y}_\infty)_n) . \quad (3.122)$$

3.5.2 Quantization

We now quantize the collective coordinate theory by elevating the coordinates and conjugate momenta to operators and imposing canonical commutation relations:

$$[z^m, p_n] = i\delta_n^m , \quad \{\chi^m, \chi^n\} = \frac{g_0^2}{4\pi} g^{mn} , \quad \{w_j^{a\dagger}, w_j^b\} = f^{ab} \quad (3.123)$$

We also want to impose $[z^m, \chi^n] = [z^m, w_j^a] = 0$. However, this implies that $\{\chi^m, p_n\} \neq 0$ and $\{w_j^a, p_n\} \neq 0$. In order to extract the z^m dependence from χ and w_j , we will

introduce (co-)frame fields $\mathcal{E}_{\underline{m}} = \mathcal{E}_{\underline{m}}^m \partial_m$ ($e^{\underline{m}} = e^{\underline{m}}_m dz^m$) on the (co-)tangent bundle and Wilson bundle with the standard properties

$$e^{\underline{m}}_{\underline{m}} = \delta^{\underline{mn}} \mathcal{E}_{\underline{n}}^n g_{nm} \quad , \quad e^{\underline{a}}_{\underline{a}} = \delta^{\underline{ab}} \mathcal{E}_{\underline{b}}^b f_{ab} . \quad (3.124)$$

Now let us to redefine the fields

$$\gamma^{\underline{m}} = \sqrt{\frac{g_0^2}{8\pi}} e^{\underline{m}}_m \chi^m \quad , \quad v^{\underline{a}}_j = e^{\underline{a}}_a w^a_j , \quad (3.125)$$

such that they obey the relations

$$[z^m, p_n] = i\delta_n^m \quad , \quad \{\gamma^{\underline{m}}, \gamma^{\underline{n}}\} = 2\delta^{\underline{mn}} \quad , \quad [z^m, \gamma^{\underline{n}}] = [p_m, \gamma^{\underline{n}}] = 0 . \quad (3.126)$$

Note that introducing the fields means that the connection term becomes the spin connection ¹³

$$\Gamma_{m,pq} \chi^p \chi^q \longrightarrow \omega_{m,\underline{pq}} \gamma^{\underline{p}} \gamma^{\underline{q}} . \quad (3.127)$$

Using this convention, the operator corresponding to π_m becomes

$$\hat{\pi}_m = -i\nabla_m - \frac{i}{2} \omega_{m,pq} \gamma^{\underline{p}} \gamma^{\underline{q}} - q_m , \quad (3.128)$$

which can also be rewritten as

$$\hat{\pi}_m = -ie^{-1/2} \mathcal{D}_m e^{1/2} , \quad (3.129)$$

where \mathcal{D}_m is the spin covariant derivative coupled to the Wilson bundle. Using this we can write the Hamiltonian operator as

$$\begin{aligned} \hat{H} = \frac{g_0^2}{8\pi} \left[-\frac{1}{\sqrt{g}} \mathcal{D}_m \sqrt{g} g^{mn} \mathcal{D}_n + g_{mn} G(\mathcal{Y}_{\infty}^{cl})^m G(\mathcal{Y}_{\infty}^{cl})^n + \frac{i}{2} \gamma^{mn} \nabla_m G(\mathcal{Y}_{\infty}^{cl})_n \right] \\ + M^{cl} + i\tilde{\theta}_0 \mathcal{L}_{G(X_{\infty})} + \frac{ig_0^2}{32\pi} \sum_j v_j^{\dagger} R(\phi_{mn}) \gamma^{mn} v_j + O(g_0^2) , \end{aligned} \quad (3.130)$$

where

$$\mathcal{L}_{K^E} = (K^E)^m \mathcal{D}_m + \frac{1}{4} \gamma^{mn} \nabla_m (K^E)_n , \quad (3.131)$$

¹³For the rest of the paper we will suppress the underline on the indices except when emphasizing the difference.

is the Lie derivative along K^E . Note that the defect degrees of freedom are missing from the Hamiltonian as they are incorporated into the mass term M^{cl} or are of higher order.

The Lie derivative operator can be related to the N^E which correspond to the transformation generated by the killing vector K^E

$$N^E = -\frac{4\pi}{g_0^2} \left((K^E)^m g_{mn} (\dot{z}^m + \tilde{\theta}_0 G(X_\infty)^n) - \frac{i}{2} \chi^m \chi^n \nabla_m (K^E)_n \right) . \quad (3.132)$$

Upon quantization the Noether charge becomes an operator associated to a conserved charge

$$\hat{N}^E = i \left((K^E)^m \mathcal{D}_m + \frac{1}{4} \gamma^{mn} \nabla_m (K^E)_n \right) = i \mathcal{L}_{K^E} . \quad (3.133)$$

The Noether charges associated to the triholomorphic vector fields $K^E = G(H_I)$ are related to the electric charge operator

$$\hat{N}^E |\Psi\rangle = \langle H_I, \hat{\gamma}_e \rangle |\Psi\rangle \quad , \quad K^E = G(H_I) . \quad (3.134)$$

Thus, the Dirac quantization of electric charges then implies that if we expand γ_e

$$\gamma_e = \sum_{I=1}^{rk[\mathfrak{g}]} n_e^I \alpha_I , \quad (3.135)$$

then

$$n_e^I = -g(G(\mathcal{Y}_\infty^{cl}), K^I) \in \mathbb{Z} , \quad (3.136)$$

where

$$\mathcal{Y} = \frac{4\pi}{g_0^2} Y + \frac{\theta_0}{2\pi} X . \quad (3.137)$$

Upon quantization, the supercharge operators becomes

$$\hat{Q}^a = -\frac{ig_0}{2\sqrt{2}\pi} \gamma^n (\tilde{\mathbb{J}}^a)_n^m \left(\mathcal{D}_m - iG(\mathcal{Y}_\infty^{cl})_m \right) , \quad (3.138)$$

which can be described as Dirac operators on the the bundle $S \otimes \mathcal{E}_{\text{Wilson}} \rightarrow \overline{\mathcal{M}}$ coupled to the triholomorphic killing field $G(\mathcal{Y}_\infty^{cl})$. Using the SUSY algebra for SQM

$$\{\hat{Q}^a, \hat{Q}^b\} = 2\delta^{ab} \left(\hat{H} + \text{Re}(\zeta^{-1} \hat{Z}) \right) , \quad (3.139)$$

and comparing with the above formula for the Hamiltonian, we can identify the central charge operator

$$Re(\zeta^{-1}\hat{Z}) = -M^{cl} = -\frac{4\pi}{g_0^2}(\gamma_m, X_\infty) - (\hat{\gamma}_e^{phys}, Y_\infty) . \quad (3.140)$$

From (3.138), we now see that the stable BPS states (those which saturate $M \geq -Re(\zeta^{-1}\hat{Z})$) must be in the kernel of the supercharge operators. The SUSY algebra implies that if a state is in the kernel of any one of the supercharge operators, it is in the kernel of all of the supercharge operators. Therefore, we can without loss of generality consider \hat{Q}^4 in which case the BPS states are given by the kernel of the twisted Dirac operator

$$i\gamma^m(\mathcal{D}_m - iG(\mathcal{Y}_\infty^{cl})_m)\Psi = 0 . \quad (3.141)$$

Comparison with Low Energy Limit

The standard formula for the central charge is given by:

$$\hat{Z} = (\hat{\gamma}_m, a_D) + \langle \hat{\gamma}_e, a \rangle . \quad (3.142)$$

Upon identifying a with Φ_∞ and a_D with $\tau_0\Phi_\infty$ (where $\tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}$) in the low energy limit (to first order in g_0) [133, 134] this becomes:

$$\begin{aligned} \zeta^{-1}\hat{Z}^{cl} &= \left(\gamma_e^* + \frac{\theta_0}{2\pi}\gamma_m, Y_\infty \right) - \frac{4\pi}{g_0^2}(\gamma_m, X_\infty) + i \left[\left(\gamma_e^* + \frac{\theta_0}{2\pi}\gamma_m, X_\infty \right) + \frac{4\pi}{g_0^2}(\gamma_m, Y_\infty) \right] \\ &= - \left[\frac{4\pi}{g_0^2}(\gamma_m, X_\infty) + (\gamma_e^{phys}, Y_\infty) \right] + i \left[\frac{4\pi}{g_0^2}(\gamma_m, Y_\infty) - (\gamma_e^{phys}, X_\infty) \right] , \end{aligned} \quad (3.143)$$

which is consistent with the semiclassical computation above.

3.5.3 1-loop Corrections

Thus far we have only considered terms coming from the perturbation series in g_0 coming from the variation of the collective coordinates. However, in quantizing the field theory, there are also loop corrections from the full quantum field theory. It has been shown in [133] that these terms give order $O(1)$ corrections to the effective SQM.

If we restrict to the vacuum of the soliton sector so that there are no incoming or outgoing perturbative states, then the 1-loop corrections are given by a sum over zero-point energies correcting the mass term. The correction is computed in [133] following [104] and given here for completeness

$$\begin{aligned} \Delta M_\gamma = & \frac{1}{2\pi} \sum_{\alpha \in \Delta^+} \langle \alpha, \gamma_m \rangle \langle \alpha, X_\infty \rangle \left\{ \ln \left(\frac{\langle \alpha, X_\infty \rangle^2}{2|A|^2} \right) + 1 \right\} \\ & + \frac{1}{\pi} \sum_{\alpha \in \Delta^+} \langle \alpha, \gamma_m \rangle \langle \alpha, Y_\infty \rangle \theta_\alpha , \end{aligned} \quad (3.144)$$

where

$$\langle \alpha, a \rangle = |\langle \alpha, a \rangle| e^{i\theta_\alpha} . \quad (3.145)$$

3.5.4 Extended Example

Now we will use the semiclassical analysis to compute the spectrum of framed BPS states in an example. Consider $\mathcal{N} = 2$ SYM theory with gauge group $SU(2)$. Let us try to use the formalism developed in this chapter to compute the spectrum of framed BPS states in the presence of a single Wilson line in the spin- j representation. We will restrict our attention to the framed BPS states with magnetic charge H_α . Here we will set $Y_\infty = 0$ and $\theta_0 = 0$.¹⁴

As discussed above, the framed BPS states with magnetic charge H_α are in the kernel of a Dirac operator on $\mathcal{M}(H_\alpha; X_\infty) \cong \mathbb{R}^3 \times S^1$. In this case, we can make a special choice of gauge for the universal connection $\epsilon_m = \hat{A}_m$. With this choice the supercharge Dirac operator becomes

$$\hat{Q}^4 = i\bar{\tau}^a \hat{D}_a + iqv , \quad (3.146)$$

where

$$X_\infty = i \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} , \quad (3.147)$$

and q is the eigenvalue of the electric charge operator

$$\hat{q} = \frac{1}{iv} \partial_\phi , \quad (3.148)$$

¹⁴This is the example considered in [165].

j	$ q $					
	0	1	2	3	4	5
$1/2$	1					
1		2				
$3/2$	4		3			
2		6		4		
$5/2$	9		8		5	
3		12		10		6

Table 3.1: This table displays the computed values of $\text{Ind}[\mathcal{D}^{\mathcal{Y}}]_{\mathcal{M}(\gamma_m=H_\alpha; X_\infty)}^{\gamma_e=\frac{q}{2}\alpha} = \underline{\mathcal{Q}}(W_j, \gamma = H_\alpha + \frac{q}{2}\alpha)$ from the index calculation as in [165, 133].

where ϕ is the coordinate on S^1 . Note that the Higgs vev leaves an unbroken $U(1)$ gauge symmetry which implies that framed BPS states are eigenstates of this operator.

Since \hat{A} satisfies the Bogomolny equations, \hat{Q}^4 has a trivial cokernel. Therefore the dimension of the kernel of \hat{Q}^4 is given by the index of \hat{Q}^4 . From a generalization of the usual index computation [31, 56, 165, 131, 169], the index of \hat{Q}^4 , and hence number of framed BPS states, is given by

$$N = \frac{1}{2} \sum_{\substack{|m| \leq 2j \\ m \in 2\mathbb{Z} + 2j}} m \text{sign}(m - q) . \quad (3.149)$$

Due to the Dirac quantization condition q must take the values [69]

$$q \in \begin{cases} 2\mathbb{Z} & j \in \mathbb{Z} + 1/2 \\ 2\mathbb{Z} + 1 & j \in \mathbb{Z} \end{cases} \quad (3.150)$$

This is to be expected since the dyon bound to a Wilson line always has even charge when the Wilson line has odd electric charge (i.e. when j is half integer) [165].

This computation for the index gives the multiplicity of BPS states (given by the framed BPS index $\underline{\mathcal{Q}}(W_j, \gamma = H_\alpha + \frac{q}{2}\alpha)$) is displayed in Table 1.

3.5.5 Sen Conjecture

We can also apply the identification of the supercharge with a Dirac operator on monopole moduli spaces to extend Sen's conjecture [158] to the singular space $\overline{\mathcal{M}}$ [133].

By the no exotics theorem, we know that (framed) BPS states form a trivial representation of the $SU(2)_R$ symmetry group. Thus, any BPS state Ψ should be annihilated by the generators of the $SU(2)_R$ group.

In order to examine the consequences of this statement, it is perhaps easiest to reformulate the wavefunction in terms of holomorphic differential forms. This makes use of the isomorphism between the Dirac spinor bundle and the space of $(0,*)$ -forms tensored with a square root of the canonical bundle [55]. In the case we are considering, where $\overline{\mathcal{M}}$ is hyperkähler, the canonical bundle is trivial and hence that the Dirac spinor bundle is simply isomorphic to the space of $(0,*)$ -forms on $\overline{\mathcal{M}}$.

Therefore, the wavefunction of a generic BPS state can be expressed as an L^2 section of the anti-symmetric tensor algebra of the holomorphic cotangent bundle

$$\Psi \in L^2 \left(\overline{\mathcal{M}}, \bigoplus_{q=1}^{2N} \Lambda^{(0,q)}(T^*\overline{\mathcal{M}}) \right) . \quad (3.151)$$

In terms of holomorphic differential forms, there exists a convenient choice of supercharge operator

$$\mathcal{Q} = \frac{i\sqrt{\pi}}{g_0} \left(\hat{Q}^3 + i\hat{Q}^4 \right) = \bar{\partial}_q - iG(\mathcal{Y}_\infty^{cl})^{(0,1)} \wedge , \quad (3.152)$$

where $\bar{\partial}_q = \bar{\partial} + q^{(0,1)} \wedge$ which naturally gives rise to a cochain complex

$$\mathcal{L}^2(\overline{\mathcal{M}}, \Lambda^{(0,0)}) \xrightarrow{\mathcal{Q}_0} \mathcal{L}^2(\overline{\mathcal{M}}, \Lambda^{(0,1)}) \xrightarrow{\mathcal{Q}_1} \dots \xrightarrow{\mathcal{Q}_{2N-1}} \mathcal{L}^2(\overline{\mathcal{M}}, \Lambda^{(0,2N)}) , \quad (3.153)$$

and an associated cohomology

$$H^q(\mathcal{Q}) = \ker \mathcal{Q}_q / \text{Im } \mathcal{Q}_{q-1} . \quad (3.154)$$

Now recall that the $SU(2)_R$ charges acting on the spinor bundle are given by

$$I_r = \frac{i\pi}{g_0^2} (\omega^r)_{mn} \chi^m \chi^n . \quad (3.155)$$

On a $(0,q)$ -form λ , the $SU(2)_R$ generators act by [133]

$$\hat{I}_+ \lambda = i\omega_+ \wedge \lambda , \quad \hat{I}_- \lambda = -i\iota_{\omega_-} \lambda , \quad \hat{I}_3 \lambda = \frac{1}{2}(q - N)\lambda , \quad (3.156)$$

where $\hat{I}_\pm = \hat{I}_1 \pm i\hat{I}_2$, $\omega_\pm \in \Lambda^2(T^*\overline{\mathcal{M}})$, and $\dim_{\mathbb{C}}[\mathcal{M}] = 2N$. Thus, we immediately see that the states that are annihilated by \hat{I}_r , and hence BPS states must be described by

primitive forms in the middle dimension: $q = N$ and $\hat{I}_\pm \lambda = 0$. In fact, since the $\{\hat{I}_r\}$ form a representation of $\mathfrak{sl}(2)$ and \hat{I}_\pm maps $\Lambda^{(0, N \mp 2)} \rightarrow \Lambda^{(0, N)}$, the \hat{I}_\pm must have only non-trivial action on non-primitive states. Therefore, the BPS states must be given by the subbundle of primitive states

$$\mathcal{H}_{BPS} = H^N(\mathcal{Q}) \cap \ker[I_\pm] . \quad (3.157)$$

This proves the Sen conjecture and extends it to the moduli space of singular monopoles: $\overline{\mathcal{M}}$. See [133] for more details.

3.6 BPS States and Hypermultiplets

Now we will turn to the general case of BPS states in $\mathcal{N} = 2$ SUSY gauge theory with arbitrary matter hypermultiplets and arbitrary line defects. We will take a general gauge group G and hypermultiplet with arbitrary quaternionic representation $\mathcal{R} = \bigoplus_i \mathcal{R}^{(i)}$ and flavor symmetry group G_F .

For the moment we will take the $\mathcal{R}^{(i)} = \pi_i \oplus \pi_i^*$ to decompose as a direct sum of two real representations so that we can construct the hypermultiplets from $\mathcal{N} = 1$ pairs of chiral superfields (Q^i, \tilde{Q}_i) . We will endow these hypermultiplets that have constituent fields (q_A^i, λ^i) with complex masses m_i . Here the index (i) denotes the flavor index which transforms under G_F . Note that q_A forms an $SU(2)_R$ doublet and λ is an $SU(2)_R$ singlet. The Lagrangian of the theory can be written as [110]

$$\begin{aligned} \mathcal{L} = & \text{Im} \left[\int d^2\theta \frac{\tau}{4\pi} W_\alpha W^\alpha \right] + \frac{\text{Im} [\tau]}{4\pi} \int d^4\theta \Phi^\dagger e^{2iV} \Phi \\ & + \frac{\text{Im} [\tau]}{4\pi} \left\{ \int d^4\theta \left(Q_i^\dagger e^{2iV} Q^i + \tilde{Q}^{i\dagger} e^{-2iV} \tilde{Q}_i \right) - \text{Re} \int d^2\theta \left(\tilde{Q}_i \Phi Q^i + m_i^j \tilde{Q}_j Q^i \right) \right\} . \end{aligned} \quad (3.158)$$

Note that the mass matrix m_i^j is generically a complex valued symmetric matrix. However, $SU(2)_R$ symmetry implies that $[m, m^\dagger] = 0$ [162].¹⁵ However, since Q^i and

¹⁵This can be checked by eliminating the auxiliary fields from the standard superspace Lagrangian. There is also a quick way to see this by an argument attributed to Seiberg. We can treat the mass matrix as the vev of a very weakly coupled scalar field which arises from gauging the flavor symmetry. Then by sending the coupling of the gauged flavor symmetry, we restrict to the vacuum $[m, m^\dagger] = 0$ and the dynamics freeze out, leaving us with a Lagrangian of the form above.

\tilde{Q}_i transform in conjugate representations of the flavor group G_F , we can diagonalize m_i^j . Hence we can generically take m to be diagonal. We will therefore suppress the hypermultiplet index (i) and representation maps $\mathcal{R}^{(i)} : G \rightarrow GL(N; \mathbb{C})$.

Following the same program as before, we will derive the BPS equations by considering the bosonic part of the action

$$\begin{aligned} H_{bos} = & \frac{1}{g^2} \int d^3x \operatorname{Tr} \left\{ E_i^2 + B_i^2 + |D_0\Phi|^2 + |D_i\Phi|^2 - \frac{1}{4}[\Phi, \Phi^*]^2 \right\} \\ & + \frac{1}{g^2} \int d^3x \left(|D_0q_A|^2 + |D_iq_A|^2 + |mh_A|^2 - mq^{\dagger A}\Phi^*q_A - m^*q^{\dagger A}\Phi q_A \right. \\ & \left. + \frac{1}{2}q^{\dagger A}\{\Phi^*, \Phi\}q_A - \frac{1}{4}(q^{\dagger A}\sigma_A^{n\ B}T^r q_B)^2 \right), \end{aligned} \quad (3.159)$$

where we are using notation similar to that of [133]. See Appendix A for conventions.

The supersymmetry transformations of the hypermultiplet fields are given by

$$\begin{aligned} \delta_\xi q_A &= 2(\xi_A \lambda_1 + \epsilon_{AB} \bar{\xi}^B \bar{\lambda}_2), \\ \delta_\xi \lambda_1 &= i\sigma^\mu \bar{\xi}^A D_\mu q_A - i(\Phi^* - m^*)\xi_A \epsilon^{AB} q_B, \\ \delta_\xi \bar{\lambda}_2 &= i\sigma^\mu \bar{\xi}^A D_\mu q_A - i(\Phi + m)\xi_A \epsilon^{AB} q_B. \end{aligned} \quad (3.160)$$

Using Gauss's Law

$$\hat{D}_a E^a = \frac{1}{2} ([\Phi, D_0\Phi^*] + [\Phi^*, D_0\Phi]) - \frac{T^r}{2} \left(q^{\dagger A} \mathcal{R}(T^r) D_0 q_A - q^A \mathcal{R}(T^r) D_0 q_A^* \right), \quad (3.161)$$

we can reduce the Hamiltonian to the form

$$\begin{aligned} H_{bos} = & \frac{1}{g_0^2} \int d^3x \operatorname{Tr} \left\{ |E_i + iB_i - \zeta^{-1} D_i \Phi|^2 + \left| \zeta^{-1} D_0 \Phi + \frac{1}{2} \left([\Phi, \Phi^*] - T^r q^{\dagger A} \sigma_A^{3\ B} \rho(T^r) q_B \right) \right|^2 \right\} \\ & + \frac{1}{g_0^2} \int d^3x \left(|D_i q_A|^2 + |D_0 q_1 + \bar{\zeta}^{-1} (\Phi^* - m^*) q_1|^2 + |D_0 q_2 + \zeta^{-1} (\Phi + m) q_2|^2 \right. \\ & \left. - \frac{1}{4} (q^{\dagger A} \sigma_A^{n\ B} \rho(T^r) q_B)^2 \right) + \operatorname{Re}\{\zeta^{-1} Z_{cl}\}, \end{aligned} \quad (3.162)$$

where $n = 1, 2$ and

$$Z_{cl} = -\frac{2m}{g^2} \int_{\mathcal{U}} d^3x (q^{\dagger 1} D_0 q_1 - D_0 q^{\dagger 2} q_2) + \frac{2}{g^2} \int_{\partial\mathcal{U}} \hat{n}^i d^2x \operatorname{Tr} [(E_i - iB_i)\phi]. \quad (3.163)$$

The new BPS equations for q_A are given by

$$D_i q_A = 0, \quad D_0 q_1 + \bar{\zeta}^{-1} (\Phi^* - m^*) q_1 = 0, \quad D_0 q_2 + \zeta^{-1} (\Phi + m) q_2 = 0, \quad (3.164)$$

which imply that $q_A = 0$. This is expected since we are considering the theory on the Coloumb branch. In fact, we will see that when we solve q_A perturbatively in g_0 , we find that $q_A = 0$ to the order we are considering.

Since bosonic zero modes only come from the vector multiplet, the moduli space is the same as before. Again there are fermionic zero modes and electric line defects leading to the spin and Wilson bundles over $\overline{\mathcal{M}}$ respectively. However, now there are additionally fermionic zero modes that will contribute an additional factor to the total bundle over $\overline{\mathcal{M}}$ that couples to the SQM which we will refer to as the matter bundle $\mathcal{E}_{\text{matter}}$. Because the hypermultiplets form a representation of the flavor group G_F , the matter bundle will have a structure group G_F .

In order to realize the matter bundle, we must solve the equations of motion

$$\begin{aligned}
\hat{D}^a E_a + [Y, [Y, A_0]] + i[\bar{\Psi}^A, \Psi_A] + 2iT^r \bar{\lambda} \rho(T^r) \lambda &= i \sum_j T^r w_j^\dagger R_j(T^r) w_j \delta^{(3)}(x - x_j) , \\
\hat{D}^2 Y - D_0^2 Y - i[\bar{\Psi}^A, \Psi_A] - 2iT^r \bar{\lambda} \rho(T^r) \lambda &= i \sum_j T^r w_j^\dagger R_j(T^r) w_j \delta^{(3)}(x - x_j) , \\
i[\not{D} \lambda + Y \lambda - iX \gamma_5 \lambda] + m_Y \lambda - im_X \gamma_5 \lambda &= 0 , \\
i(\not{D} \Psi_A - [Y, \Psi_A] + i\gamma_5 [X, \Psi_A]) &= 0 ,
\end{aligned} \tag{3.165}$$

where we have used the Dirac basis for γ^k and

$$\Psi^A = \begin{pmatrix} \psi^A \\ \bar{\psi}^A \end{pmatrix} , \quad \lambda = \begin{pmatrix} \lambda_1 \\ -\bar{\lambda}_2 \end{pmatrix} , \quad \zeta^{-1} m = m_Y + im_X . \tag{3.166}$$

Note that these equations lead to the same solution for the ρ_A . Here we take ζ to be either determined by the data of line defects in the case of framed BPS states or by the classical action in the vanilla case

$$\zeta_{van} = -\frac{Z^{cl}}{|Z^{cl}|} \equiv -\lim_{g_0 \rightarrow 0} \frac{Z}{|Z|} . \tag{3.167}$$

Typically, upon introducing hypermultiplets there are extra terms in the central charge Z , however they are not leading order in $g_0 \rightarrow 0$ since classically $q_A = 0$.

We can now solve for the hypermultiplet zero modes from the fermionic equations

of motion:

$$i[\not{D} + Y - iX\gamma_5]\lambda + m_R\lambda - im_I\gamma_5\lambda = 0 , \quad (3.168)$$

which in components is given by

$$\begin{aligned} (i\bar{\tau}^a \hat{D}_a - im_X)\lambda_1 &= i(A_0 - Y + im_Y)\sigma^0 \bar{\lambda}_2 , \\ (i\tau^a \hat{D}_a + im_X)\bar{\lambda}_2 &= i(A_0 + Y - im_Y)\lambda_1 . \end{aligned} \quad (3.169)$$

Note that the equations of motion for λ_2 is now given by:

$$(i\bar{\tau}^a \hat{D}_a - im_X)\lambda_2 = i(A_0 + Y - im_Y)\sigma^0 \bar{\lambda}_1 . \quad (3.170)$$

Since \hat{A}_a is self dual, the kernel of $i\bar{\tau}^a \hat{D}_a - im_X$ is non-trivial whereas the kernel of $i\tau^a \hat{D}_a + im_X$ is trivial. Hence, given λ_1 , Y , and A_0 , there is a unique solution for $\bar{\lambda}_2$. Again this implies that there are non-trivial zero modes associated to λ_1 which is $O(g_0^{1/2})$ while $\lambda_2 \sim O(g^{3/2})$ will contribute trivially to the effective action at order $O(g_0^2)$.

The non-trivial zero modes are thus described by solutions to the equation

$$\tilde{L}_{\mathcal{R}}\lambda_I = 0 \quad , \quad \tilde{L}_{\mathcal{R}} = i\bar{\tau}^a \hat{D}_a - im_X , \quad (3.171)$$

where \mathcal{R} is the full hypermultiplet representation. Self-duality of \hat{A}_a implies that $\text{coker}[\tilde{L}_{\mathcal{R}}] = \{0\}$ and hence $\text{Ker}[\tilde{L}_{\mathcal{R}}] = \text{Ind}[\tilde{L}_{\mathcal{R}}]$. Therefore the index of $\tilde{L}_{\mathcal{R}}$ determines the rank of the matter bundle $\mathcal{E}_{\text{matter}}(\mathcal{R}, m)$. A computation analogous to before shows that

$$\begin{aligned} \text{rk}_{\mathbb{C}}[\mathcal{E}_{\text{matter}}(\mathcal{R}, m)] \\ = \frac{1}{2} \sum_{\mu \in \Delta_{\mathcal{R}}} n_{\mathcal{R}}(\mu) \left\{ \langle \mu, \gamma_m \rangle \text{sign}(\langle \mu, X_{\infty} \rangle + \text{Re}[\zeta^{-1}m]) + \sum_{n=1}^{N_d} |\langle \mu, P_n \rangle| \right\} \end{aligned} \quad (3.172)$$

Here we employed an orthonormal basis of $V_{\mathcal{R}}$ associated with a weight space decomposition: $V_{\mathcal{R}} = \oplus_{\mu} V_{\mathcal{R}}[\mu]$ where $\mu \in \Delta_{\mathcal{R}} \subset \Lambda_{wt} \subset \mathfrak{t}^*$ are the weights of the representation and $n_{\mathcal{R}}(\mu) = \dim V_{\mathcal{R}}[\mu]$. In this decomposition, any vector $v \in V_{\mathcal{R}}[u]$, is acted on by $X_{\infty} \in \mathfrak{t}$ according to $i\mathcal{R}(X_{\infty})v = \langle \mu, X_{\infty} \rangle v$ where \langle , \rangle denotes the canonical pairing $\mathfrak{t}^* \otimes \mathfrak{t} \rightarrow \mathbb{R}$. See A for full details.

Multiple Hypermultiplets

In the case of multiple hypermultiplets of varying representations, we know that the mass can be chosen to be diagonal so that the hypermultiplet representation decomposes:

$$\mathcal{R} = \bigoplus_{i=1}^{N_f} \mathcal{R}^{(i)} \quad (3.173)$$

For generic values of $m = \text{diag}[m_1, \dots, m_{N_f}]$ flavor symmetry is broken to the maximal torus $U(1)^{N_f}$. Under this decomposition the matter bundle decomposes as a direct sum

$$\mathcal{E}_{\text{matter}}(\mathcal{R}, m) = \bigoplus_{i=1}^{N_f} \mathcal{E}(\mathcal{R}^{(i)}, m_i) \quad , \quad (3.174)$$

where each subbundle has rank $\text{rk}_{\mathbb{C}}[\mathcal{E}(\mathcal{R}^{(i)}, m_i)] = \text{Ind}[\tilde{L}_{\mathcal{R}^{(i)}, m_i}]$.

Example

As a consistency check, consider the case of an $SU(2)$ gauge theory with a single hypermultiplet in the fundamental representation ($n_{\rho}(\mu) = 1$) in the k -monopole background. Let us take $X_{\infty} = vH_{\alpha}$ where $v > 0$ and use the conventions, $\gamma_m = kH_{\alpha}$ and $\text{sign}(0)=0$. Then the rank of the matter bundle is given by

$$\text{Ind}[\tilde{L}_{\rho}] = \frac{1}{2} [\text{sign}(v + m_X) - \text{sign}(-v + m_X)] k = \begin{cases} 0 & |v| < |m_X| \\ \frac{k}{2} & m_X = \pm v \\ k & |m_X| < |v| \end{cases} \quad (3.175)$$

This computation exactly agrees with [132, 125, 76].

Note that the index changes rank based on the relative size of mass and Higgs vev. This change of the rank of the matter bundle is describing the phenomenon of wall crossing where the number of BPS states changes along certain loci in moduli space. We will discuss this further in the next chapter. In this scenario, it will have a clear, geometric interpretation in the string theory constructions that we will discuss in Chapter 5.

3.6.1 Matter Bundle and the Universal Connection

As it turns out the matter bundle can be described as a sub-bundle of the pullback of the universal Hilbert bundle to the moduli space $\overline{\mathcal{M}}$. As space-time fields, the hypermultiplet fermions are sections of the bundle $S \otimes E_{\mathcal{R}} \rightarrow \mathbb{R}^3$ where $E_{\mathcal{R}} = P \times_{\mathcal{R}} \mathbb{C}^N \rightarrow \mathbb{R}^3$ is an associated principal G bundle and we have restricted to only time independent fields as per the collective coordinate prescription. This means that we want to pull back the universal Hilbert bundle associated with sections $L^2(\mathbb{R}^3, S \otimes E_{\mathcal{R}})$.

After pulling back this bundle to the moduli space $\overline{\mathcal{M}}$, we can define the operator:

$$\tilde{L}_{\rho} = i\bar{\tau}^a \hat{D}_a - im_I, \quad (3.176)$$

and project onto its kernel.

Take a local trivialization of $Ker[\tilde{L}_{\rho}] \rightarrow \mathcal{M}$ over an open set $\mathcal{U} \subset \mathcal{M}$ with local coordinates $\{z^m\}$. We can pick a local frame in which the fibers are spanned by local sections $\{\lambda_a(x, z) \in L^2(\mathcal{U}, S^{\pm} \otimes E) | \tilde{L}_{\rho} \lambda_a(x, z) = 0, a = 1, \dots, \text{Ind}[\tilde{L}_{\rho}]\}$. In this local trivialization we can write the projected connection on $\mathcal{E}_{\text{matter}}$

$$A_{m,ab}(z) = \int_{\mathcal{U}} d^3x \left\langle \lambda_a(x, z), \left(\frac{\partial}{\partial z^m} + \mathcal{R}(\epsilon_m(x, z)) \right) \lambda_b(x, z) \right\rangle. \quad (3.177)$$

where $\langle \cdot, \cdot \rangle \rightarrow \mathbb{R}$ is the canonical hermitian connection on the fibers of $S^{\pm} \otimes E_{\rho} \rightarrow \mathcal{U}$ in this trivialization.

In the case of multiple hypermultiplets the bundle $Ker[\tilde{L}_{\rho}]$ decomposes as a direct sum

$$Ker[\tilde{L}_{\rho}, m_X] = \bigoplus_i Ker[\tilde{L}_{\mathcal{R}(i)}, Re(\zeta^{-1} m_i)]. \quad (3.178)$$

The bundle $Ker[\tilde{L}_{\rho}, m_X]$ exactly corresponds to the bundle of hypermultiplet zero modes: $\mathcal{E}_{\text{matter}}$.

3.6.2 Collective Coordinate Dynamics

Now we need to solve the equations of motion for the vector multiplet fields to reduce to the effective SQM theory. The equations of motion for the ρ^A, η^A are the same as

before because the coupling to the hypermultiplet zero modes are accompanied by a coupling to the hypermultiplet scalar which has trivial solution.

Thus, we only need to solve the equations of motion for A_0 and Y :

$$\begin{aligned}\hat{D}^2(A_0 - \dot{z}^m) + i[\rho^A, \rho_A] + 2iT^r \bar{\lambda} \rho(T^r) \lambda - i \sum_j T^r w_j^\dagger R_j(T^r) w_j \delta^{(3)}(x - x_j) &= 0, \\ \hat{D}^2 Y - D_0^2 Y - i[\rho^A, \rho_A] - 2iT^r \bar{\lambda} \rho(T^r) \lambda - i \sum_j T^r w_j^\dagger R_j(T^r) w_j \delta^{(3)}(x - x_j) &= 0.\end{aligned}\tag{3.179}$$

The equations of motion for all of the fields are solved by the collective coordinate expansion

$$\begin{aligned}A_0 &= -\dot{z}^m \epsilon_m + \frac{i}{4} \phi_{mn} \chi^m \chi^n + Y^{cl} - \frac{2i}{\hat{D}^2} T^r \bar{\lambda}_a \mathcal{R}(T^r) \lambda_b \psi^a \psi^b, \\ Y &= \epsilon_{Y_\infty} - \frac{i}{4} \phi_{mn} \chi^m \chi^n + Y^{cl} + \frac{2i}{\hat{D}^2} T^r \bar{\lambda}_a \mathcal{R}(T)^r \lambda_b \psi^a \psi^b, \\ \rho^A &= \frac{1}{2\sqrt{\det \kappa}} \delta_m \hat{A}_a (-i\tau^a) \kappa^A \chi^m, \\ \hat{A}_\mu &= \hat{A}_\mu(x, z(t)) \quad , \quad h_A = 0 \quad , \quad \eta^A = O(g_0^{3/2}), \\ \lambda &= \lambda_a(x; z) \psi^a \quad , \quad \lambda_a(x; z) = \begin{pmatrix} \lambda_a^\alpha(x; z(t)) \\ -\bar{\lambda}_{\dot{\alpha}a}(x; z(t)) \end{pmatrix},\end{aligned}\tag{3.180}$$

While the formulas for A_0, Y appear nonlocal due to the $\frac{1}{\hat{D}^2}$, here we simply use this notation to mean the Green's function as in [78].

By plugging in the solutions (3.180) into the action (3.15), we can reduce to the collective coordinate theory which has a Lagrangian

$$\begin{aligned}L_{c.c.} &= \frac{4\pi}{g_0^2} \left[\frac{1}{2} g_{mn} (\dot{z}^m \dot{z}^n + i \chi^m \mathcal{D}_t \chi^n - G(Y_\infty)^m G(Y_\infty)^n) + \frac{i}{2} \chi^m \chi^n \nabla_m G(Y_\infty)_n \right] \\ &+ \dot{z}^m q_m + \frac{4\pi}{g_0^2} \left(i h_{ab} \psi^a \mathcal{D}_t \psi^b - (m_Y h_{ab} + 2i T_{ab}) \psi^a \psi^b + \frac{1}{2} F_{mnab} \chi^m \chi^n \psi^a \psi^b \right) \\ &- \frac{4\pi}{g_0^2} (\gamma_m, X_\infty) + \frac{\theta_0}{2\pi} (\gamma_m, Y_\infty) + \frac{\theta_0}{2\pi} \left(g_{mn} (\dot{z}^m - G(X_\infty)^n) - i \chi^m \chi^n \nabla_m G(X_\infty)_n \right) \\ &- \Delta M_\gamma + i \sum_j w_j^\dagger (D_t - \epsilon_{Y_\infty}^{(j)} + \frac{i}{2} \phi_{mn} \chi^m \chi^n) w_j,\end{aligned}\tag{3.181}$$

where $\{\psi^a\}$ are the hypermultiplet zero modes with $\mathcal{D}_t\psi^a = \dot{\psi}^a + (A_m)^a_b \psi^b$ and

$$\begin{aligned} h_{ab} &= \frac{1}{2\pi} \int d^3x \bar{\lambda}_a \lambda_b \quad , \quad A_{mab} = \frac{1}{2\pi} \int d^3x \bar{\lambda}_a (\partial_m + \mathcal{R}(\epsilon_m)) \lambda_b \quad , \\ T_{ab} &= \frac{1}{2\pi} \int d^3x \bar{\lambda}_a \mathcal{R}(\epsilon_{Y_\infty}) \lambda_b \quad , \quad F_{mnab} = 2\partial_{[m} A_{n]ab} + A_{mac} A_n^c{}_b - A_{nac} A_m^c{}_b \quad . \end{aligned} \quad (3.182)$$

Here h_{ab} is the fiber metric and A_{mab} is associated the metric connection with curvature F_{mnab} on the matter bundle from the previous section and T_{ab} is the lift of the covariant spin derivative $\nabla_m G(Y_\infty)_n$ to the matter bundle. Additionally ΔM_γ is the 1-loop correction to the mass (similar to [104], see Appendix A) which is explicitly

$$\begin{aligned} \Delta M_\gamma &= \frac{1}{2\pi} \sum_{\alpha \in \Delta^+} \langle \alpha, \gamma_m \rangle \left\{ \ln \left(\frac{\langle \alpha, X_\infty \rangle^2}{2|A|^2} \right) + 1 \right\} + \frac{1}{\pi} \sum_{\alpha \in \Delta^+} \langle \alpha, \gamma_m \rangle \langle \alpha, Y_\infty \rangle \theta_\alpha \\ &\quad + \frac{1}{4\pi} \sum_{\mu \in \Delta_{\mathcal{R}}} n_{\mathcal{R}}(\mu) (\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle \ln \left(\frac{(\langle \mu, X_\infty \rangle + m_X)^2}{2|A|^2} \right) \quad . \end{aligned} \quad (3.183)$$

This theory again has $\mathcal{N} = 4$ supersymmetry with associated SUSY transformations

$$\begin{aligned} \delta_\nu z^m &= -i\nu_s (\tilde{\mathbb{J}}^s)^m_n \chi^n \quad , \\ \delta_\nu \chi^m &= (\mathbb{J}^s)^m_n (\dot{z}^n - G(Y_\infty)^n) \nu_s - i\nu_s \chi^k \chi^n (\mathbb{J}^s)_k^\ell \Gamma_{\ell n}^m \quad , \\ \delta_\nu \psi^a &= -A_{mb}^a \delta_\nu z^m \psi^b \quad , \end{aligned} \quad (3.184)$$

whose associated, conserved supercharge is given by

$$Q^s = \frac{4\pi}{g_0^2} \chi^m (\tilde{\mathbb{J}}^s)_{mn} (\dot{z}^n - G(Y_\infty)^n) \quad . \quad (3.185)$$

It is important to note that as a consequence of $\mathcal{N} = 4$ SUSY, this matter bundle is hyperholomorphic: F_{mnab} is type (1,1) in all complex structures on \mathcal{M} . [55].

Hamiltonian Formalism

Again we will need to convert to the Hamiltonian description in order to quantize the theory. A straightforward calculation shows that the conjugate momenta are given by

$$\begin{aligned} p_m &= \frac{4\pi}{g_0^2} g_{mn} \left[\dot{z}^m + \frac{i}{2} \chi^p \chi^q \Gamma_{pq}^m + i A_{ab}^m \psi^a \psi^b \right] + \frac{\theta_0}{2\pi} G(X_\infty)_m + q_m \quad , \\ (p_\chi)_m &= \frac{4\pi}{g_0^2} g_{mn} \chi^n \quad , \quad (p_\psi)_a = \frac{4\pi}{g_0^2} h_{ab} \psi^b \quad , \quad (p_{w_j})_a = i\delta(w_j^\dagger)_a \quad , \end{aligned}$$

where

$$q_m = i \sum_j w_j^\dagger R_j(\epsilon_m^{(j)}) w_j . \quad (3.186)$$

Again we introduce the notation

$$\pi_m = p_m - \frac{4\pi}{g_0^2} \left[\frac{i}{2} \chi^p \chi^q \Gamma_{pq}^m + i A_{ab}^m \psi^a \psi^b \right] - q_m . \quad (3.187)$$

The Hamiltonian can then be written

$$\begin{aligned} H_{c.c.} = & M^{1-lp} + \frac{g_0^2}{8\pi} \left\{ \pi_m g^{mn} \pi_n + g_{mn} G(\mathcal{Y}_\infty)^m G(\mathcal{Y}_\infty)^n + \frac{4\pi i}{g_0^2} \chi^m \chi^n \nabla_m G(\mathcal{Y}_\infty)_n \right\} \\ & + i\tilde{\theta}_0 \left(iG(X_\infty)^m \pi_m + \frac{2\pi}{g_0^2} \chi^m \chi^n \nabla_m G(X_\infty)_n \right) + \frac{4\pi}{g_0^2} \left(m_R h_{ab} - \frac{1}{2} F_{mnab} \chi^m \chi^n \right) \psi^a \psi^b \\ & + i \sum_j w_j^\dagger \left(\epsilon_{Y_\infty}^{(j)} - \frac{i}{2} \phi_{mn} \chi^m \chi^n \right) w_j , \end{aligned} \quad (3.188)$$

with associated supercharges

$$Q^a = \chi^m (\tilde{\mathbb{J}}^s)_m^n (\pi_n - iG(\mathcal{Y}_\infty)_n) . \quad (3.189)$$

3.6.3 Quantization

Again we can quantize the theory by elevating coordinates and momenta to operators and imposing canonical quantization conditions. As before we will need to introduce (co-)frame fields for our spin, hypermultiplet, and Wilson bundles. Then scaling the fields as

$$\chi^m = \frac{g_0}{2\sqrt{2\pi}} \gamma^m , \quad \psi^a = \frac{g_0}{2\sqrt{2\pi}} \theta^a , \quad (3.190)$$

using the same notation from the previous section.

Now we quantize the fields γ^m , θ^a , and $v_j^{\underline{a}}$ by imposing the $Cl(4N)$ algebra relations,

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn} , \quad \{\theta^a, \theta^b\} = 2\delta^{ab} , \quad \{v^{s\dagger}, v^t\} = 2\delta^{st} . \quad (3.191)$$

Note that we have a $Cl(4N)$ because $\dim_{\mathbb{R}}(\overline{\mathcal{M}}) = 4N$ for some $N \in \mathbb{N}$ and the associated moduli are fermionic. This implies that upon quantization we elevate from the vector bundle $\mathcal{E}_{\text{matter}} \rightarrow \overline{\mathcal{M}}$ to $Spin(\mathcal{E}_{\text{matter}}) \rightarrow \overline{\mathcal{M}}$. As before

$$\begin{aligned} \pi_m = & -i \left(\partial_m + \frac{1}{2} \Gamma_{nm}^n + \frac{1}{4} \omega_{m,\underline{pq}} \gamma^{\underline{pq}} + \frac{1}{2} \Omega_{m,\underline{ab}} \theta^{\underline{ab}} \right) - q_m , \\ = & -ie^{-1/2} \mathcal{D}_m e^{1/2} . \end{aligned} \quad (3.192)$$

From hereon out we will suppress the underline on the coordinates except to emphasize the flattened bundle. We can now write the supercharge operators as

$$\hat{Q}^a = -\frac{ig_0}{2\sqrt{2\pi}}\gamma^m(\tilde{\mathbb{J}}^a)_m^n \left(\mathcal{D}_m - iG(\mathcal{Y}_\infty^{cl})_m \right) . \quad (3.193)$$

Using the SUSY algebra

$$\{\hat{Q}^a, \hat{Q}^b\} = 2\delta^{ab}(\hat{H} + Re(\zeta^{-1}\hat{Z})) , \quad (3.194)$$

We see that the central charge operator is given by

$$-Re(\zeta^{-1}\hat{Z}_\gamma) = M_\gamma^{1-lp} + m_Y h_{ab}\theta^a\theta^b . \quad (3.195)$$

We should expect this additional term because our flavor charge is exactly given by $h_{ab}\theta^a\theta^b$ which descends from the QFT flavor charge $Q_f = \int d^3x \bar{\lambda}\lambda$. This reproduces the standard low energy formula

$$Z_{LE} = (\hat{\gamma}_m, a_D^{1-lp}) + \langle \hat{\gamma}_e, a \rangle + mQ_f , \quad (3.196)$$

where Q_f is the flavor charge

$$Q_f = \sum_i h_{ab}\theta_i^a\theta_i^b . \quad (3.197)$$

As before the framed BPS spectrum with hypermultiplets is

$$\mathcal{H}_{BPS} = \left\{ \Psi \in L^2(\overline{\mathcal{M}}, S \otimes \mathcal{E}_{\text{Wilson}} \otimes Spin(\mathcal{E}_{\text{matter}})) \mid i\gamma^m (\mathcal{D}_m - iG(\mathcal{Y}_\infty)_m) \Psi = 0 \right\} . \quad (3.198)$$

3.6.4 Generalized Sen Conjecture

It is now straightforward to analyze the cohomology of the total bundle of the effective SQM and relate it to the space of framed BPS states. From this we can derive a generalization of the Sen Conjecture which we will refer to as the *Generalized Sen Conjecture*. In fact, the proof follows nearly trivially because the new zero modes come from hypermultiplet fermions which are $SU(2)_R$ singlets.

Again we will identify the spin bundle with the holomorphic differential forms. In this language we can identify the differential operator coming from the suprcharges

$$\mathcal{Q} = \frac{i\sqrt{\pi}}{g_0^2} \left(\hat{Q}^3 + i\hat{Q}^4 \right) = \bar{\partial}_q - iG(\mathcal{Y}_\infty^{cl})^{(0,1)} \wedge , \quad (3.199)$$

and construct an associated differential cochain complex

$$\mathcal{L}^2(\overline{\mathcal{M}}, \Lambda^{(0,0)} \otimes S_m) \xrightarrow{\mathcal{Q}_0} \mathcal{L}^2(\overline{\mathcal{M}}, \Lambda^{(0,1)} \otimes S_m) \xrightarrow{\mathcal{Q}_1} \dots \xrightarrow{\mathcal{Q}_{2N-1}} \mathcal{L}^2(\overline{\mathcal{M}}, \Lambda^{(0,2N)} \otimes S_m) \quad (3.200)$$

where we have denoted $Spin(\mathcal{E}_{\text{matter}})$ as S_m . From this complex we can define the cohomology groups

$$H^q(\mathcal{Q}) = \ker \mathcal{Q}_q / \text{Im } \mathcal{Q}_{q-1} . \quad (3.201)$$

Since the hypermultiplet fermions are $SU(2)_R$ singlets, the $SU(2)_R$ generators act as before

$$\hat{I}_+ \lambda = i\omega_+ \wedge \lambda \quad , \quad \hat{I}_- \lambda = -i\iota_{\omega_-} \lambda \quad , \quad \hat{I}_3 \lambda = \frac{1}{2}(q - N)\lambda , \quad (3.202)$$

on some $Spin(\mathcal{E}_{\text{matter}})$ valued $(0, q)$ -form λ . The no exotics theorem implies that $q = N$ and that BPS states must be primitive elements

$$\lambda_{BPS} \in \Lambda_{\text{prim}}^{(0,N)}(\overline{\mathcal{M}}, Spin(\mathcal{E}_{\text{matter}})) , \quad (3.203)$$

where

$$\Lambda_{\text{prim}}^{(0,N)}(\overline{\mathcal{M}}, Spin(\mathcal{E}_{\text{matter}})) = \{\lambda \in \Lambda^{(0,*)}(T^*\overline{\mathcal{M}}) \otimes Spin(\mathcal{E}_{\text{matter}}) \mid \lambda \hat{I}_\pm \lambda = 0\} . \quad (3.204)$$

Thus, the set of BPS states is given by

$$\mathcal{H}_{BPS} = H_{\text{prim}, L^2}^N(\mathcal{Q}) , \quad (3.205)$$

where

$$H_{\text{prim}, L^2}^N(\mathcal{Q}) = \mathcal{H}^N(\mathcal{Q}) \cap \Lambda_{\text{prim}}^{(0,N)}(\overline{\mathcal{M}}, Spin(\mathcal{E}_{\text{matter}})) \Big|_{L^2} , \quad (3.206)$$

are the L^2 sections of the elements of the cohomology group $\mathcal{H}^N(\mathcal{Q})$ that are also primitive.

Conserved Charges

The conserved charge related to a (triholomorphic) killing vector K^A [133]

$$\hat{N}^A = i\mathcal{L}_{K^A} + O(g_0^2) . \quad (3.207)$$

From the explicit form of the supercharge operators, it is easy to show that the charge operators obey

$$[\hat{N}^A, \hat{Q}^4] = \frac{ig_0}{2\sqrt{2}\pi} \gamma^m [K^A, G(\mathcal{Y}_\infty^{cl})]_m, \quad (3.208)$$

where $[\cdot, \cdot]$ denotes the commutator of vector fields. When K^A is triholomorphic and killing, $[K^A, G(\mathcal{Y}_\infty^{cl})] = 0$. Thus, the electric charge generated by $G(\mathcal{Y}_\infty^{cl})$ is conserved.

It is also easy to show that the flavor charge in the effective SQM is conserved because it descends from a conserved charge in the full QFT. Therefore the L^2 kernel of \hat{Q}^4 and hence the BPS Hilbert space \mathcal{H}_{BPS} is graded by the electric and flavor charges

$$\mathcal{H}_{BPS} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{L,u,\gamma}^{BPS}, \quad (3.209)$$

where we can identify (in the semiclassical limit) $\gamma = \gamma_m \oplus \gamma_e \oplus \gamma_f$.

3.6.5 Summary of Collective Coordinate Analysis

Let us briefly summarize the results of our lengthy collective coordinate analysis. Adiabatically evolving BPS states in the semiclassical limit of a 4D $\mathcal{N} = 2$ theory with gauge group G and hypermultiplets with a quaternionic representation $\mathcal{R} = \bigoplus_i \mathcal{R}^{(i)}$ in the presence of general Wilson-'t Hooft defects is described by $\mathcal{N} = 4$ super quantum mechanics in the bundle

$$S_{Tot} = S \bigotimes_i \mathcal{E}_{\text{Wilson}}(Q_i) \bigotimes_{j=1}^{N_f} \text{Spin}[\mathcal{E}_{\text{matter}}(\mathcal{R}^{(j)}, m_j)] \rightarrow \overline{\mathcal{M}}(\{P_n\}, \gamma_m; X_\infty), \quad (3.210)$$

over singular monopole moduli space.¹⁶

¹⁶In the case of no magnetically charged line defects we take replace $\overline{\mathcal{M}} \rightarrow \mathcal{M}(\gamma_m; X_\infty)$.

The dynamics of this system can be described by the Lagrangian

$$\begin{aligned}
L_{c.c.} = & \frac{4\pi}{g_0^2} \left[\frac{1}{2} g_{mn} (\dot{z}^m \dot{z}^n + i \chi^m \mathcal{D}_t \chi^n - G(Y_\infty)^m G(Y_\infty)^n) + \frac{i}{2} \chi^m \chi^n \nabla_m G(Y_\infty)_n \right] \\
& + \sum_{i=1}^{N_f} \frac{4\pi}{g_0^2} \left(i h_{ab}^{(i)} \psi_{(i)}^a \mathcal{D}_t \psi_{(i)}^b - (m_Y^{(i)} h_{ab}^{(i)} + 2i T_{ab}^{(i)}) \psi_{(i)}^a \psi_{(i)}^b + \frac{1}{2} F_{mnab}^{(i)} \chi^m \chi^n \psi_{(i)}^a \psi_{(i)}^b \right) \\
& - \frac{4\pi}{g_0^2} (\gamma_m, X_\infty) + \frac{\theta_0}{2\pi} (\gamma_m, Y_\infty) + \frac{\theta_0}{2\pi} \left(g_{mn} (\dot{z}^m - G(X_\infty)^n) - i \chi^m \chi^n \nabla_m G(X_\infty)_n \right) \\
& - \Delta M_\gamma + i \sum_{j=1}^{n_W} w_j^\dagger (D_t - \epsilon_{Y_\infty}^{(j)} + \frac{i}{2} \phi_{mn} \chi^m \chi^n) w_j + \dot{z}^m q_m ,
\end{aligned} \tag{3.211}$$

where n_W is the number of electrically charged line defects and:

$$\begin{aligned}
g_{mn} &= \frac{1}{2\pi} \int_{\mathbb{R}^3} d^3x \text{Tr} \left\{ \delta_m \hat{A}_a \delta_n \hat{A}^a \right\} , \quad \Gamma_{mnp} = \frac{1}{2\pi} \int_{\mathbb{R}^3} d^3x \text{Tr} \left\{ \delta_m \hat{A}^a D_p \delta_n \hat{A}_a \right\} , \\
\mathcal{D}_t \chi^n &= \dot{\chi}^n + \Gamma_{mp}^n \dot{z}^m \chi^p , \quad \mathcal{D}_t \psi_{(i)}^a = \dot{\psi}_{(i)}^a + (A_m^{(i)})^a_b \dot{\psi}_{(i)}^b , \quad D_t w_j^b = \partial_t w_j^b - R(\epsilon_m^{(j)})^b_c w_j^c , \\
h_{ab}^{(i)} &= \frac{1}{2\pi} \int d^3x \bar{\lambda}_a^{(i)} \lambda_b^{(i)} , \quad A_{mab}^{(i)} = \frac{1}{2\pi} \int d^3x \bar{\lambda}_a^{(i)} (\partial_m + \mathcal{R}^{(i)}(\epsilon_m)) \lambda_b^{(i)} , \\
T_{ab}^{(i)} &= \frac{1}{2\pi} \int d^3x \bar{\lambda}_a^{(i)} \mathcal{R}^{(i)}(\epsilon_{Y_\infty}) \lambda_b^{(i)} , \quad F_{mnab}^{(i)} = 2\partial_{[m} A_{n]ab}^{(i)} + A_{mac}^{(i)} A_{nb}^{(i)c} - A_{nac}^{(i)} A_{mb}^{(i)c} .
\end{aligned} \tag{3.212}$$

Upon quantization the supercharge operators can be written

$$\begin{aligned}
\hat{Q}^a &= -\frac{i g_0}{2\sqrt{2}\pi} \gamma^m (\tilde{\mathbb{J}}^a)_m^\times \\
&\left(\partial_m + \frac{1}{4} \omega_{m,pq} \gamma^{pq} + \frac{1}{2} \sum_{i=1}^{N_f} \Omega_{m,ab}^{(i)} \theta_{(i)}^{ab} - \sum_{j=1}^{n_W} w_j^\dagger R_j(\epsilon_m^{(j)}) w_j - i G(\mathcal{Y}_\infty)_m \right) .
\end{aligned} \tag{3.213}$$

They satisfy the $\mathcal{N} = 4$ SQM algebra

$$\{\hat{Q}^a, \hat{Q}^b\} = 2\delta^{ab} (\hat{H} - \text{Re}(\zeta^{-1} \hat{Z})) , \tag{3.214}$$

where

$$\text{Re}(\zeta^{-1} \hat{Z}) = M_\gamma^{1-lp} + \sum_{i=1}^{N_f} m_R^{(i)} h_{ab}^{(i)} \theta_{(i)}^a \theta_{(i)}^b . \tag{3.215}$$

Therefore BPS states are L^2 -sections of the above bundle that are also in the kernel of the operator

$$\hat{Q}^4 = i \gamma^m \left(\partial_m + \frac{1}{4} \omega_{m,pq} \gamma^{pq} + \frac{1}{2} \sum_{i=1}^{N_f} \Omega_{m,ab}^{(i)} \theta_{(i)}^{ab} - \sum_{j=1}^{n_W} w_j^\dagger R_j(\epsilon_m^{(j)}) w_j - i G(\mathcal{Y}_\infty)_m \right) , \tag{3.216}$$

and further are invariant under $SU(2)_R$.

3.7 Rational Maps and Hyperholomorphic Vector Bundles

We will now describe how the hyperholomorphic bundles $\mathcal{E}_{\text{matter}}, \mathcal{E}_{\text{Wilson}}$ have a natural geometric interpretation in terms of rational maps.

3.7.1 Hypermultiplet Matter Bundle

Consider the matter bundle defined by the quaternionic representation \mathcal{R} and real mass $im_x \in \mathfrak{t}_f$.¹⁷ For generic values of im_x (as we will generally consider below), the flavor group is broken to a maximal torus $G_f \rightarrow T_f$. In this case, the quaternionic representation splits as a direct sum of quaternionic representations of the gauge group G

$$\mathcal{R} \rightarrow \bigoplus_i \mathcal{R}^{(i)} , \quad (3.217)$$

where each factor $\mathcal{R}^{(i)}$ corresponds to an eigenspace¹⁸ of im_x . Thus, the matter bundle splits as a direct sum over the eigenvalues of im_x . We will consider a generic factor in this sum.

Let us define the set of positive roots by $X_\infty \in \mathfrak{t}$. This specifies a splitting

$$\mathcal{R}^{(i)} = \pi_{\lambda^{(i)}} \oplus \pi_{\lambda^{(i)}}^* , \quad (3.218)$$

where $\pi_{\lambda^{(i)}}$ and $\pi_{\lambda^{(i)}}^*$ have corresponding highest weight $\lambda^{(i)}$ and $-\lambda^{(i)}$ where $\lambda^{(i)}$ is a dominant weight. From hereon we will suppress the index (i) when possible. As a warm up, we will construct a factor of matter bundle where the eigenvalue of im_x on the factor corresponding to $\mathcal{R}^{(i)}$ is zero.

Recall that the correspondence between rational maps and monopoles was derived in the previous chapter by studying the scattering of charged particles off of a monopole configuration. This can be rephrased as studying the fermionic zero modes in the presence of monopoles, similar to the semiclassical analysis of [22, 134, 133, 78]. The advantage of this approach is that it has a clear generalization to hypermultiplet matter zero modes.

¹⁷Note that the i is included here as naturally $\text{Lie}(U(1)) \cong i\mathbb{R}$.

¹⁸Note that there can be repeated factors of $\mathcal{R}^{(i)}$.

The hypermultiplet fermion zero modes are determined by the Dirac equation

$$(\sigma^a \nabla_a + iX)\psi = 0 \quad \Leftrightarrow \quad \begin{pmatrix} \bar{\nabla}_\ell & \bar{\nabla}_{\bar{z}} \\ \nabla_z & -\nabla_\ell \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 . \quad (3.219)$$

where the spin covariant derivatives ∇_a act in the $\mathcal{R}^{(i)}$ representation. This means that if we are to consider scattering a fermion of type ψ_2 , then the field satisfies the conditions

$$\nabla_\ell \psi_2 = 0 \quad , \quad \bar{\nabla}_{\bar{z}} \psi_2 = 0 . \quad (3.220)$$

And similarly for fermions of type ψ_1 , we have the complex conjugate equations. Note that ψ_1 couples to anti-monopoles while ψ_2 couples to monopoles. Thus, as usual for four-dimensions, only ψ_2 will have zero modes in the presence of monopoles. Therefore, we will only consider the flat sections of ψ_2 .

Recall that from the previous section that the flat sections of $\nabla_r + iX$ are in general of the form

$$s_p(r) \sim e^{-\varphi_\sigma r} r^{k_\sigma/2} f(\theta, \phi) \vec{v}_\sigma \quad , \quad \text{as } r \rightarrow \infty , \quad (3.221)$$

where $k_\sigma = \langle \sigma, \gamma_m \rangle$, $\sigma \in \Delta_{\lambda^{(i)}}$, and \vec{v}_σ is a vector in the weight space of corresponding weight σ in the \mathcal{R} -representation. This means that after picking the trivialization from before, the solutions of ψ_2 are of the form:

$$\psi_2(t, z) \sim e^{-2\varphi_\sigma t} r^{k_\sigma - n} z^n \vec{v}_\sigma \quad , \quad \text{as } t \rightarrow +\infty , \quad (3.222)$$

where $0 \leq n \leq k_\sigma$. This means that counting zero modes is the same as counting holomorphic sections of a line bundle corresponding to the flat sections of ∇_ℓ with exponential eigenvalue $e^{-2\varphi_\sigma t}$ in the limit as $t \rightarrow \infty$. Since scattering is trivial as $z \rightarrow \infty$, we can again trivially extend the line bundles to \mathbb{CP}^1 at $t \rightarrow \infty$.

In order to extract the physical information, we must eliminate the gauge redundancy given by the action of the Cartan subgroup T defined by X_∞ . Due to the natural G action with T redundancy, these line bundles have a natural construction as the pull back a line bundle from G/T via a meromorphic map. Recall from the Borel-Weil theorem that there exists a line bundle $L_\lambda \rightarrow G_{\mathbb{C}}/B$ such that $H_{\bar{\partial}}^0(L_\lambda; \mathbb{Z}) \cong V_\lambda$. Again, we trivialize the incoming hypermultiplet zero modes in the limit $t \rightarrow -\infty$ so that it is of

the form V_λ . Since $f : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$ is the scattering matrix which relates the trivial incoming states at $t \rightarrow -\infty$ to the non-trivial bundle of final states ($\mathcal{E}_{\text{matter}} \rightarrow \mathbb{CP}^1$) at $t \rightarrow +\infty$, a generic fiber of the matter bundle must be given by $H_{\mathbb{C}}^0(f^*L_\lambda; \mathbb{Z})$.

Matter Bundle Factor with Vanishing Mass

Formally, we can construct the factor of the matter bundle when $im_x^{(i)} = 0$ as follows. Consider the holomorphic line bundle $L_\lambda \rightarrow G_{\mathbb{C}}/B$ following from the Borel-Weil theorem, corresponding to the dominant weight λ . This line bundle L_λ can be constructed by the pullback of $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$ by the Plücker embedding

$$\begin{aligned} h_\lambda : G_{\mathbb{C}}/B &\rightarrow \mathbb{CP}_\lambda^1 , \\ gB &\mapsto g \cdot \mathbf{1}_\lambda . \end{aligned} \quad (3.223)$$

where $\mathbf{1}_\lambda$ is a 1-dimensional representation of B where

$$b \cdot \mathbf{1}_\lambda = e^{-\lambda(t)} \mathbf{1}_\lambda \quad , \quad b = e^t \in B . \quad (3.224)$$

Recall that locally, sections of L_λ are equivalent to B -equivariant maps: $\Gamma(G_{\mathbb{C}}/B, L_\lambda) \cong \{f : G_{\mathbb{C}} \rightarrow \mathbb{C} \mid f(gb) = e^{-\lambda(t)} f(g) , b = e^t \in B\}$. These can be expressed as sections of the pullback bundle $h_\lambda^* \mathcal{O}(1)$ which can be written explicitly as

$$\psi_{\vec{v}_\mu}(g) = \langle \pi_\lambda^{-1}(g) \vec{v}_\mu, \vec{v}_\lambda \rangle \quad , \quad \mu \in \Delta_\lambda , \quad (3.225)$$

where \vec{v}_λ is the highest weight vector in the π_λ highest weight representation of G , Δ_λ is the set of weights of the π_λ representation, and $\pi_\lambda^{-1}(g) = \pi_\lambda(g^{-1})$. Here the action of B acts as

$$\psi_{\vec{v}_\mu}(gb) = e^{-\lambda(t)} \psi_{\vec{v}_\mu}(g) , \quad (3.226)$$

as expected.

Now consider the pull back of this line bundle through the rational map

$$\begin{array}{ccccc} f^*(L_\lambda) = p_\lambda^* \mathcal{O}(1) & \xleftarrow{f^*} & L_\lambda = h_\lambda^* \mathcal{O}(1) & \xleftarrow{h_\lambda^*} & \mathcal{O}(1) , \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{CP}^1 & \xrightarrow{f} & G_{\mathbb{C}}/B & \xrightarrow[h_\lambda]{} & \mathbb{CP}_\lambda^1 \end{array} \quad (3.227)$$

where $p_\lambda = f \circ h_\lambda$. From this construction, it is clear that we are pulling back $\mathcal{O}(1) \rightarrow \mathbb{CP}^1$ via the rational map p_λ . The sections of $p_\lambda^* \mathcal{O}(1)$ are of the form

$$s(z) = \phi(z) \psi_{\vec{v}_\mu}(z) = \phi(z) \langle \pi^{-1}(f(z)) \vec{v}_\mu, \vec{v}_\lambda \rangle, \quad (3.228)$$

where $\phi : \mathbb{CP}^1 \rightarrow \mathbb{C}$ such that $s(z)$ is smooth and holomorphic. This is only non-zero for components of $f(z)$ such that $\langle \lambda - \mu, \pi^{-1}(f(z)) \rangle \neq 0$. Thus

$$\deg[p_\lambda] = \sum_{\mu \in \Delta_\lambda^+} \sum_I n_\lambda(\mu) \langle \lambda - \mu, \alpha_I \rangle \langle \alpha_I, \gamma_m \rangle = \sum_{\mu \in \Delta_\lambda^+} \langle \mu, \gamma_m \rangle, \quad (3.229)$$

and hence

$$f^*(\mathcal{L}_\lambda) = p_\lambda^*(\mathcal{O}(1)) \cong \bigoplus_{\mu \in \Delta_\lambda} \mathcal{O}(\langle \mu, \gamma_m \rangle)^{\otimes n_\lambda(\mu)}. \quad (3.230)$$

Now we can identify the matter bundle $\mathcal{E}_{\text{matter}} \rightarrow \mathcal{M}$ with the space of holomorphic sections of the bundle $f^*(L_\lambda)$. By expanding λ in terms of fundamental weights we can recover the dimension of the fiber of the matter bundle for $m_x = 0$ from [22] by counting the number of holomorphic sections

$$h^0(\mathbb{CP}^1, f^*(L_\lambda)) = \sum_{\mu \in \Delta_\lambda^+} n_\lambda(\mu) \langle \mu, \gamma_m \rangle = \frac{1}{2} \sum_{\mu \in \Delta_\lambda} n_\lambda(\mu) \text{sgn}(\langle \mu, X_\infty \rangle) \langle \mu, \gamma_m \rangle. \quad (3.231)$$

This gives the rank of a factor in the matter bundle with vanishing mass (3.172) [22].

Rational Map Formulation of Matter Bundle

With this inspiration we can recast the B -action of the line bundle $L_\lambda \rightarrow G_{\mathbb{C}}/B$ as

$$\psi_{\vec{v}_\mu}(gb) = e^{-\sum_{\sigma \in \Delta_\lambda} n_\lambda(\sigma) \text{sgn}(\langle \sigma, X_\infty \rangle) \langle \mu, t \rangle} \psi_{\vec{v}_\mu}(g). \quad (3.232)$$

This formulation is exactly identical to our previous definition. However, it is now clear how we should modify this story to take into account $im_x^{(i)} \neq 0$. Specifically, if we consider a component of $\mathcal{R} = \oplus_i \mathcal{R}^{(i)}$ where $im_x^{(i)}$ is nonzero, we should define the line bundle $L_\lambda(X_\infty, m_x^{(i)}) \rightarrow G_{\mathbb{C}}/B$ whose sections have the property

$$\tilde{\psi}_{\vec{v}_\mu}(gb) = e^{-\sum_{\mu \in \Delta_\lambda} n_\lambda(\mu) \text{sgn}(\langle \mu, X_\infty \rangle + m_x^{(i)}) \langle \mu, t \rangle} \tilde{\psi}_{\vec{v}_\mu}(g). \quad (3.233)$$

This can be defined again as the pullback bundle from \mathbb{CP}^1 where now we pullback by a modified Plücker embedding $\tilde{h}_\lambda : G_{\mathbb{C}}/B \rightarrow \mathbb{CP}^1$ where

$$b \cdot \mathbf{1}_\lambda = e^{\sum_{\mu \in \Delta_\lambda} n_\lambda(\mu) \text{sgn}(\langle \mu, X_\infty \rangle + m_x^{(i)}) \langle \mu, t \rangle} \mathbf{1}_\lambda. \quad (3.234)$$

We refer to the pullback bundle under the modified Plücker embedding $\tilde{h}_\lambda^* \mathcal{O}(1) = L_\lambda(X_\infty, m_x^{(i)})$ and we will define the function $\tilde{p}_\lambda = f \circ \tilde{h}_\lambda$. In this case the pullback bundle will be of the form

$$f^*(L_\lambda(X_\infty, m_x^{(i)})) = \bigotimes_{\substack{\mu \in \Delta_\lambda^+ \\ |\langle \mu, X_\infty \rangle| \geq |m_x^{(i)}|}} \mathcal{O}(\langle \mu, \gamma_m \rangle)^{\otimes n_\lambda(\mu)}. \quad (3.235)$$

The bundle over \mathcal{M} given by the space of holomorphic sections of $\tilde{p}_\lambda^* \mathcal{O}(1)$ to be

$$\begin{aligned} h_{\tilde{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda(X_\infty, m_x^{(i)}))) &= \sum_i \sum_{\substack{\mu \in \Delta_\lambda^+ \\ |\langle \mu, X_\infty \rangle| \geq |m_x^{(i)}|}} n_\lambda(\mu) \langle \mu, \gamma_m \rangle \\ &= \frac{1}{2} \sum_i \sum_{\mu \in \Delta_\lambda} n_\lambda(\mu) \operatorname{sgn}(\langle \mu, X_\infty \rangle + m_x^{(i)}) \langle \mu, \gamma_m \rangle. \end{aligned} \quad (3.236)$$

This is indeed the rank of the matter bundle as in equation (3.172) [22].

However, in order to match the matter bundle, we need to demonstrate that this bundle is given by the horizontal component of the universal connection acting in the π_λ representation. Since the gauge group is broken to $T \subset G$ by X_∞ , we have that gauge symmetry acts on sections of the matter bundle by phase rotation. This means that the universal connection on $\mathcal{E}_{\text{matter}}$ should project onto a single T -representative in the flag manifold G/T (or B -representative in $G_{\mathbb{C}}/B$).

Note that for the $m_x^{(i)} = 0$, the connection on $f^* L_\lambda \rightarrow \mathbb{CP}^1$ pulls back from the line bundle $L_\lambda(X_\infty, m_x^{(i)}) \rightarrow G_{\mathbb{C}}/B$:

$$\Theta_\lambda = -\langle \lambda, \theta \rangle \quad , \quad \theta = g^{-1} dg. \quad (3.237)$$

We can see this by noticing the equivalence relation on the fibers

$$\psi(bg) \sim e^{-\lambda(t)} \psi(g). \quad (3.238)$$

After choosing a representative, this redundancy is removed by the choice of connection

$$\nabla_{L_\lambda} = e^{\lambda(t)} d(e^{-\lambda(t)}) = -\lambda(g^{-1} dg)|_B = \Theta_\lambda. \quad (3.239)$$

This supports the fact that B acts on the line bundle L_λ as a gauge symmetry. This means that the connection ∇_{L_λ} projects onto a single B -representative, or rather a

single gauge representative of the zero mode bundle. Note that in the case of $m_x^{(i)} \neq 0$ the connection is slightly modified to

$$\tilde{\Theta}_\lambda = - \sum_i \sum_{\mu \in \Delta_\lambda} n_\lambda(\mu) \text{sgn}(\langle \mu, X_\infty \rangle + m_x^{(i)}) \langle \mu, dg^{-1}g \rangle . \quad (3.240)$$

This natural connection on f^*L_λ lifts to a connection the bundle of sections over \mathcal{M} which projects on a gauge equivalence class. Therefore, this lifted connection is exactly the parallel component of the universal connection and the bundle of sections of the pullback bundle is the matter bundle.

An important consistency check is that the connection on the matter bundle is hyperholomorphic. This can be seen as follows. Consider the connection on f^*L_λ for $f \in \mathcal{M}$. Using the choice of generic X_∞ , we have a canonical splitting of $\text{Lie}[G] = \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^+ \oplus \mathfrak{g}^-$ which comes from the splitting of the root lattice $\Phi = \Phi^+ \oplus \Phi^-$ into positive and negative weights with respect to X_∞ . This splitting defines a complex structure on $\text{Lie}[G/T]$ and hence on G/T . In this setting the Killing form defines a Hermitian metric which is only non-degenerate on $(\ , \) : \mathfrak{g}^- \times \mathfrak{g}^+ \rightarrow \mathbb{R}$. Further, since we have defined λ to be a dominant weight, $[\lambda, \cdot]$ acts diagonally on the splitting $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. Using this and the explicit form of the curvature of L_λ

$$F_{L_\lambda} = d(\Theta_\lambda) = d(\lambda, \theta) = -\frac{1}{2}(\lambda, \theta \wedge \theta) = -\frac{1}{2}(\theta, [\lambda, \theta]) , \quad (3.241)$$

we see that F_{L_λ} must be a type (1,1)-form on G/T . Since $G/T \cong G_{\mathbb{C}}/B$, this means that this also defines a holomorphic connection on $G_{\mathbb{C}}/B$. Since $f : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$ is a rational function, $f^*(F_{L_\lambda})$ must also be a (1,1)-form on \mathbb{CP}^1 and hence the connection on f^*L_λ is hyperholomorphic.

We can now construct the connection on $H_{\partial}^0(\mathbb{CP}^1, f^*(L_\lambda)) \rightarrow \mathcal{M}$. Consider the map: $ev : \mathbb{CP}^1 \times \mathcal{M} \rightarrow G_{\mathbb{C}}/B$ where $ev : (z, f) \mapsto f(z)$. Using this map, we can pull back the line bundle L_λ

$$\begin{array}{ccc} ev^*(L_\lambda) & \xleftarrow{ev^*} & L_\lambda \\ \downarrow & & \downarrow \\ \mathbb{CP}^1 \times \mathcal{M} & \xrightarrow{ev} & G_{\mathbb{C}}/B \end{array} . \quad (3.242)$$

The connection on $ev^*(L_\lambda)$ will also be a hyperholomorphic connection since pulling back a (1,1)-form by a rational function will still be a (1,1)-form. We can formally

construct the bundle of zero modes as follows

$$\begin{array}{ccccc}
 \text{hol}(\mathbb{CP}^1) \times \mathcal{A} & \longrightarrow & H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda)) & \xrightarrow{\iota^*} & H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda)) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{A} & & \mathcal{A} & \xleftarrow{\iota} & \mathcal{M}
 \end{array} \tag{3.243}$$

where \mathcal{A} is the space of all meromorphic maps from $\mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$ and $\text{hol}(\mathbb{CP}^1)$ is the space of all holomorphic functions on \mathbb{CP}^1 .

We see therefore that the connection on $H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda)) \rightarrow \mathcal{M}$ is given by the horizontal component of the connection on $ev^*(L_\lambda)$ along \mathcal{M} . Since $\mathbb{CP}^1 \times \mathcal{M}$ is a product space, we have that the parallel component of the connection will have a curvature of type (1,1) and hence is holomorphic.

Recall that $f : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$ descends from a rational map $\hat{f} : T\mathbb{P}^1 \rightarrow G_{\mathbb{C}}/B$. This means that the pullback connection $f^*(\Theta_\lambda)$ varies holomorphically with respect to $\eta \in \mathbb{CP}_{c.s}^1$ in the base and therefore that the holomorphic (1,1)-form curvature varies holomorphically with respect to the $\eta \in \mathbb{CP}_{c.s}^1$ and therefore is a type (1,1)-form with respect to all choices of $\eta \in \mathbb{CP}_{c.s}^1$. That is to say the connection on $H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda)) \rightarrow \mathcal{M}$ is a hyperholomorphic connection.

Proof that $H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda))$ has Hyperholomorphic Connection

Here we will summarize the details of the proof that $H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda))$ has a hyperholomorphic connection worked out in the previous section.

Proof. From explicit construction, we can see that the line bundle $L_\lambda \rightarrow G_{\mathbb{C}}/B$ has a holomorphic connection as in (3.241). By pulling back the bundle through the rational map: $ev : \mathbb{CP}^1 \times \mathcal{M} \rightarrow G_{\mathbb{C}}/B$, we know that the connection on $ev^*(L_\lambda)$ has a holomorphic connection. Therefore the component of the connection parallel to \mathcal{M} , which is the connection on $H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda)) \rightarrow \mathcal{M}$, will be holomorphic. And since the map $f : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$ descends from a map $\hat{f} : T\mathbb{P}^1 \rightarrow G_{\mathbb{C}}/B$, $ev : \mathbb{CP}^1 \times \mathcal{M} \rightarrow G_{\mathbb{C}}/B$ varies holomorphically with respect to the complex structure by construction, and hence the connection on $H_{\bar{\partial}}^0(\mathbb{CP}^1, f^*(L_\lambda)) \rightarrow \mathcal{M}$ must be hyperholomorphic. \square

Remark Note that the proof that $\tilde{\Theta}_\lambda$ is a hyperholomorphic connection follows with trivial modifications.

By putting together all of the different components of the quaternionic representation \mathcal{R} , we have that the total matter bundle is determined by that property that each fiber

$$\mathcal{E}_{\text{matter}} \Big|_{f \in \mathcal{M}} \cong \bigoplus_i H_{\bar{\partial}}^0 \left(f^* L_\lambda^{(i)}(X_\infty, m^{(i)}); \mathbb{Z} \right), \quad (3.244)$$

with connection

$$\nabla_{\text{matter}} = \sum_i \pi_{\lambda^{(i)}} \left(ev^* \tilde{\Theta}_{\lambda^{(i)}} \right) \Big|_{\mathcal{M}}. \quad (3.245)$$

Remark on $\mathcal{N} = 2^*$ Theory

It is interesting to think about the case of the $\mathcal{N} = 2^*$ theory. In this theory if we take the mass parameter $m \rightarrow 0$, then the theory should have $\mathcal{N} = 4$ SUSY. As shown in the seminal paper by Sen [158], this means that space of holomorphic sections of the line bundle $L_{\text{adj}}(X_\infty, 0) \rightarrow G_{\mathbb{C}}/B$ should be associated with the holomorphic tangent space so that the total bundle [133]

$$S\mathcal{M} \otimes H_{\bar{\partial}_A}^0(\mathcal{M}, L_{\text{adj}}(X_\infty, 0)) \cong T\mathcal{M}. \quad (3.246)$$

This implies that the spin bundle $S\mathcal{M} \rightarrow \mathcal{M}$ should be identified with the space of holomorphic sections of $L_{\text{adj}}(X_\infty, 0)$. This means that $S\mathcal{M}$ should be realized in a mathematically similar way.

In fact, because the coupling to the spin bundle comes from the zero modes of the vector-multiplet's (massless) adjoint fermions [22, 78], the above discussion shows that spin bundle must be isomorphic to $L_{\text{adj}}(X_\infty, 0)$. Roughly, this can be attributed to the correspondence between meromorphic functions on \mathbb{CP}^1 , which is used to define the geometry of \mathcal{M} and hence $S\mathcal{M}$, and sections of line bundles, which is used to define $L_{\text{adj}}(X_\infty, 0)$.

3.7.2 Wilson Bundle

The Wilson bundle is similar, yet fundamentally different from the matter bundle. In order to motivate the construction of this bundle, recall the essential properties of $\mathcal{E}_{\text{Wilson}}$ from [22]:

- The data of a Wilson line is given by (R_{Q_n}, \vec{x}_n) where R_{Q_n} representation of the Wilson defect with highest weight $Q_n \in \Lambda_{wt}$ and $\vec{x}_n \in \mathbb{R}^3$ is the position of the defect. The choice of ζ will have no effect here.
- The Wilson bundle will have the form

$$\mathcal{E}_{\text{Wilson}} = \bigotimes_n \mathcal{E}_{\text{Wilson}}^{(n)} = \bigotimes_n \mathcal{E}_{\text{Wilson}}(\vec{x}_n, Q_n) , \quad (3.247)$$

where each factor describes the degrees of freedom associated to the n^{th} Wilson line.

- The rank of the Wilson bundle is fixed

$$\text{rk}_{\mathbb{C}}[\mathcal{E}_{\text{Wilson}}] = \prod_n \dim_{\mathbb{C}}[R_{Q_n}] . \quad (3.248)$$

- The connection on $\mathcal{E}_{\text{Wilson}}^{(n)}$ is given by the universal connection in the representation R_{Q_n} evaluated at $\vec{x}_n \in \mathbb{R}^3$.

The fact that the dimension of the fibers does not change with γ_m demonstrates that the Wilson defect zero modes are fundamentally different than the hypermultiplet zero modes.

Coadjoint Orbit Quantization

In order to construct the Wilson bundle , we will give a construction of a generic factor $\mathcal{E}_{\text{Wilson}}^{(n)}$. This will require the use of the coadjoint orbit quantization of Wilson lines as we discussed earlier. In this construction we can identify the Hilbert space of defect states $\mathcal{H} = V_{Q_n}$ associated to the Wilson line of charge $Q_n \in \Lambda_{wt}$ by looking at the vector space of holomorphic sections of the associated Borel-Weil line bundle

$$\mathcal{H} = V_{Q_n} = H_{\bar{\partial}}^0(L_{Q_n}; \mathbb{Z}) \quad , \quad L_{Q_n} \rightarrow G_{\mathbb{C}}/B . \quad (3.249)$$

We then parametrize the defect degrees of freedom by the pull-backs of the holomorphic sections of L_{Q_n}

$$U : \mathbb{R} \rightarrow \mathcal{O}_{Q_n} = G_{\mathbb{C}}/B , \quad (3.250)$$

where \mathbb{R} parametrizes the time direction.

In this formalism, the space of the maps into $G_{\mathbb{C}}/B$ captures the different field configurations on the defect. Therefore, we can parametrize the space of fields on a Wilson line in the direction η at a fixed point $\vec{x} \in \mathbb{R}^3$ by the set of maps

$$\phi : \mathbb{R}_z \times \mathbb{CP}^1 \times \mathbb{R} \rightarrow G_{\mathbb{C}}/B , \quad (3.251)$$

where \mathbb{R}_z is line in \mathbb{R}^3 which is labeled by z_η and parametrized by t_η for the choice of complex structure $\eta \in \mathbb{CP}^1$.

We can then restrict to a particular (t_η, z_η) to fix the location of the Wilson line insertion. Thus, the Hilbert space of states of a Wilson line inserted at a point $\vec{x}_n = (t_\eta, z_\eta)$ is given by

$$\mathcal{H} = \varphi^* H_{\partial}^0(L_{Q_n}; \mathbb{Z}) \quad , \quad L_{Q_n} \rightarrow G_{\mathbb{C}}/B , \quad (3.252)$$

where $\varphi(z, t) = \lim_{t_\eta \rightarrow \infty} \phi(t_\eta, z_\eta = z, t = 0)$ where t parametrizes time (not to be confused with t_η which parametrizes lines in \mathbb{R}^3).

In our formalism, we are considering the time-independent configurations at $t_\eta \rightarrow +\infty$. Thus, we are considering maps ¹⁹

$$\phi : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B . \quad (3.253)$$

Since the Wilson defect can be realized as a collection of localized spin defect fields, we can again describe the Hilbert space of states as in equation (3.252).

Using the fact that pulling back a line bundle $L \rightarrow G_{\mathbb{C}}/B$ to \mathbb{CP}^1 couples the states in $\mathcal{H} = H_{\partial}^0(L; \mathbb{Z})$ to the gauge field, we can identify $\phi : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$ with $f : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$. This leads to a Wilson bundle which has fibers of the form

$$\mathcal{E}_{\text{Wilson}}^{(n)} \Big|_{f \in \mathcal{M}} = f^* H_{\partial}^0(L_\lambda; \mathbb{Z}) \Big|_{z=z_n} . \quad (3.254)$$

¹⁹Here we have implicitly extend the natural map $\tilde{\phi} : \mathbb{C}^1 \rightarrow G_{\mathbb{C}}/B$ to the map $\phi : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$.

We can explicitly check that this ansatz does indeed produce the Wilson bundle by showing that construction defines a bundle which matches the dimension and connection of the Wilson bundle. Specifically, the dimension is purely given by the dimension of the representation with highest weight Q_n and the connection will be the universal connection evaluated at the insertion point in the representation R_{Q_n} . This can be seen from the analysis of the rational map construction of the matter bundle from the previous section. In the case of the matter bundle, the universal connection was given by the pullback of the connection on L_λ . In this case, the connection on L_{Q_n} is given by

$$\Theta_{Q_n} = -\langle Q_n, \theta \rangle \quad , \quad \theta = g^{-1}dg \quad , \quad (3.255)$$

so that

$$\nabla_{\text{Wilson}} = R_{Q_n}(f^*\theta) \quad . \quad (3.256)$$

The formal construction of the connection and Wilson bundle is as follows. Let $L_{Q_n} \rightarrow G_{\mathbb{C}}/B$ the Borel-Weil line bundle of highest, dominant weight $Q_n \in \Lambda_{wt}$ (with connection $\Theta_{Q_n} = -\langle Q_n, \theta \rangle$) and $H_{\bar{\partial}}^0(L_{Q_n}; \mathbb{Z}) = V_{Q_n}$ be the vector space of holomorphic sections which is the representation of G with highest weight Q_n .

We can now construct a holomorphic vector bundle over \mathcal{M} . First choose a point $f \in \mathcal{M}$. Above this point, we have a vector space given by the pull back of the holomorphic sections of L_{Q_n} evaluated at the the corresponding insertion point $z_n \in \mathbb{C}$. This then extends to a vector bundle over \mathcal{M} where each factor of the fiber is of the form

$$\mathcal{E}_{\text{Wilson}}(\vec{x}_n, Q_n) \Big|_{f \in \mathcal{M}} = f^* \left(H_{\bar{\partial}}^0(L_{Q_n}; \mathbb{Z}) \right)_{z=z_n} \quad . \quad (3.257)$$

This vector bundle describes the Wilson bundle with connection given by $R_{Q_n} f^*(\theta) \Big|_{z=z_n}$. From the construction of the matter bundle, we know that this connection is the universal connection evaluated at \vec{x}_n in the R_{Q_n} representation and hence will again be hyperholomorphic.

The fact that $R_{Q_n} f^*(\theta) \Big|_{z=z_n}$ is hyperholomorphic can be seen by the fact that L_{Q_n} is a holomorphic line bundle with (1,1) curvature. This means that the pullback by a rational map, $f^* L_{Q_n}$ will also be a holomorphic line bundle. Since the pull back of the

sections will take value in $f^*L_{Q_n}$: $f^*\left(H_{\bar{\partial}}^0(L_{Q_n};\mathbb{Z})\right) \subset H_{\bar{\partial}}^0(f^*L_{Q_n};\mathbb{Z})$, the connection on $f^*\left(H_{\bar{\partial}}^0(L_{Q_n};\mathbb{Z})\right)$ will also be holomorphic. Further since $f(z)$ descends from a hyperholomorphic map $\hat{f} : T\mathbb{P}^1 \rightarrow G_{\mathbb{C}}/B$ where, this connection will naturally be hyperholomorphic.

We can directly construct the sections of $L_{Q_n} \rightarrow G_{\mathbb{C}}/B$,

$$\psi_{\vec{v}}(g) = \langle R_{Q_n}^{-1}(g)\vec{v}, \vec{v}_{Q_n} \rangle \quad , \quad \psi_{\vec{v}} \in H_{\bar{\partial}}^0(L_{Q_n};\mathbb{Z}) \quad , \quad (3.258)$$

where v_{Q_n} is the highest weight vector and $\vec{v} \in V_{Q_n}$. In this case, shifting the B -representative of g acts as

$$\psi_{\vec{v}}(g) \mapsto \psi_{\vec{v}}(b \cdot g) = e^{-\langle Q_n, t \rangle} \langle R_{Q_n}^{-1}(g)\vec{v}, \vec{v}_{Q_n} \rangle \quad , \quad b = e^t \in B \quad . \quad (3.259)$$

Thus, the connection on $f^*\left(H_{\bar{\partial}}^0(L_{Q_n};\mathbb{Z})\right)$ is given by

$$\nabla_{\text{Wilson}} = R_{Q_n}(f^*\theta) \quad , \quad (3.260)$$

and hence that the curvature of $f^*\left(H_{\bar{\partial}}^0(L_{Q_n};\mathbb{Z})\right)$ is given by

$$F_{\text{Wilson}} = R_{Q_n}(df^*\theta) = \frac{1}{2}R_{Q_n}([f^*\theta, f^*\theta]) \quad . \quad (3.261)$$

As before, we can realize this in terms of $G/T \cong G_{\mathbb{C}}/B$. Since $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g}$ is degenerate on $\mathfrak{g}^+ \times \mathfrak{g}^+$ and $\mathfrak{g}^- \times \mathfrak{g}^-$, we have that $[f^*\theta, f^*\theta] \in \mathfrak{t}$ and is a type (1,1) form. Therefore, F_{Wilson} is a type (1,1)-form and by the same logic as before, and therefore $\nabla_{\text{Wilson}} = R_{Q_n}(f^*\theta)$ is a hyperholomorphic connection.

By allowing for the insertion of multiple line defects, we get the complete Wilson bundle

$$\mathcal{E}_{\text{Wilson}} \Big|_{f \in \mathcal{M}} = \bigotimes_n \mathcal{E}_{\text{Wilson}, (\vec{x}_n, Q_n)} \Big|_{f \in \mathcal{M}} = \bigotimes_n f^* \left(H_{\bar{\partial}}^0(L_{Q_n};\mathbb{Z}) \right)_{z=z_n} \quad , \quad (3.262)$$

with the connection

$$\nabla_{\text{Wilson}} = \sum_n R_{Q_n} \left(f^* \theta_{Q_n} \right)_{z=z_n} \quad . \quad (3.263)$$

Remark

1. The different points $z \in \mathbb{CP}^1$ encode the data of how the system is affected by having a Wilson line at different points in space. In essence the bundle $f^*\left(H_{\bar{\partial}}^0(L_\lambda; \mathbb{Z})\right)$ contains all of the information of having Wilson lines in the presence of BPS monopoles.
2. An important question to ask is how this formalism encodes the information along the scattering direction. This can be answered by considering the action of translation on the rational maps. Translation in the plane perpendicular to the scattering direction is given by shifting $z \mapsto z + z_0$. By using the identification in Section 2.5.2, translation along the scattering direction is given by

$$t \mapsto t + t_0 \implies z \mapsto e^{2t_0} z . \quad (3.264)$$

This suggests that there is an equivalence of insertion positions for the Wilson defects. However, this is not surprising as the moduli space encodes all possible positions of the monopoles and hence only the difference in positions of fixed defects, such as Wilson-, 't Hooft-, and Wilson-'t Hooft defects, are physically relevant.

3.7.3 Rational Maps and General Bundles for Framed BPS States

The hyperholomorphic bundles from the previous section also exist as bundles over singular monopole moduli space. They are defined similarly to the case of smooth monopoles.

Matter Bundle for Framed BPS States

In this case we again define the matter line bundle by pulling back the Borel-Weil line bundle $L_\lambda \rightarrow G_{\mathbb{C}}/B$ via the rational map $\tilde{f} : \mathbb{CP}^1 \rightarrow G_{\mathbb{C}}/B$. However, the framed case is slightly different from the vanilla case. This is because the singular part of each

rational map \tilde{f}_I has no moduli. This means that

$$\begin{aligned} H_{\partial}^0(f^*(L_{\lambda}); \mathbb{Z}) &\cong V_{trivial} \oplus \mathcal{E}_{matter} \\ &\downarrow \\ &\mathcal{M} \end{aligned} \quad (3.265)$$

where $V_{trivial}$ is a trivial vector bundle. This trivial factor exists because of the holomorphic sections coming from the singular components of the rational map. This trivial factor is not an element of the bundle for several reasons

1. $\text{rk}[H_{\partial}^0(f^*L_{\lambda}; \mathbb{Z})] = \text{rk}[\mathcal{E}_{matter}] + \text{rk}[V_{trivial}]$ where $\text{rk}[V_{trivial}] = \sum_{n,I} p_n^I$,
2. There are generically no trivial factors of \mathcal{E}_{matter} ,
3. Physically, the factor $V_{trivial}$ corresponds to the collection of zero modes that are removed when we construct a singular monopole by taking the infinite mass limit of a smooth monopole. This comes from the fact that our construction of singular monopole moduli space is effectively taking the singular limit of a smooth monopole moduli space. See [131] for more details on this decoupling.

As before, this bundle will be a hyperholomorphic vector bundle over (singular) monopole moduli space of the appropriate rank

$$\text{rk}_{\mathbb{R}}[\mathcal{E}_{matter}] = \sum_{\mu \in \Delta_{\lambda}} n_{\lambda}(\mu) \left\{ \langle \mu, \gamma_m \rangle \text{sgn}(\langle \mu, X_{\infty} \rangle + m_x^{(i)}) + \sum_n |\langle \mu, P_n \rangle| \right\}. \quad (3.266)$$

Wilson Bundle for Framed BPS States

In the case of singular monopoles, we can have two types of electrically charged defects: Wilson defects and Wilson-'t Hooft defects. These both give rise to Wilson bundles as both require the insertion of a Wilson line into the path integral. In both cases, the Wilson bundle is again defined by the analogous construction where a generic fiber of a factor of $\mathcal{E}_{\text{Wilson}} \rightarrow \overline{\mathcal{M}}$ is of the form

$$\mathcal{E}_{\text{Wilson}}(\vec{x}_n, Q_n) \Big|_{f \in \overline{\mathcal{M}}} = \tilde{f}^* \left(H_{\partial}^0(L_{Q_n}; \mathbb{Z}) \right)_{z=z_n}. \quad (3.267)$$

This will be a hyperholomorphic vector bundle over singular monopole moduli space of the appropriate rank, $\text{rk } \mathcal{E}_{\text{Wilson}}^{(n)} = \dim[V_{Q_n}]$.

Chapter 4

Wall Crossing of Semiclassical Framed BPS States

We have now shown how BPS states in the semiclassical, adiabatic limit of a supersymmetric gauge theory can be described as solutions of the Dirac operator

$$\mathcal{D}^{\mathcal{Y}_0} = i (\mathcal{D} + i\mathcal{G}(\mathcal{Y}_0)) \quad , \quad (4.1)$$

on monopole moduli space. However, using this property to study the spectrum of BPS states is in general very difficult. The reason is that solving for the states in the kernel of $\mathcal{D}^{\mathcal{Y}_0}$ requires solving the Dirac equation on a $4N$ -dimensional manifold $\overline{\mathcal{M}}$ coupled to gauge fields and bundles of generic rank. This task is functionally impossible.

However, we can still learn something about BPS states by considering the Dirac operator in a certain asymptotic region. There the form of the metric simplifies to subleading order. While this does not allow us to determine the spectrum of BPS states in the asymptotic limit, it allows us to study *primitive wall crossing* which can be described simply as 2-body decay of cluster of BPS states. By studying $\mathcal{D}^{\mathcal{Y}_0}$ in this limit, we find that primitive wall crossing has a universal behavior that is controlled by a Dirac operator on single centered Taub-NUT.

4.1 Asymptotic Regions Of Moduli Space

Here we will consider a special subregion of the asymptotic region of monopole moduli space. This region is called the *two-galaxy region* in which the cluster of monopoles roughly separates into two, widely separated subclusters.

The two galaxy region can be defined as follows. The asymptotic region \mathcal{M}^{as} is region of monopole moduli space where all monopoles are widely separated relative to the mass of the lightest W -boson. Here we can identify the coordinates (\vec{x}_i, ψ_i) with a

collection of positions and phases that we can associate to individual monopoles. The two-galaxy region is then defined by the region in which we can partition the \vec{x}_i into two sets S_1, S_2 representing the two distinct galaxies of size N_1 and N_2

$$\{\vec{x}_i\}_{i=1}^N = S_1 \cup S_2, \quad S^1 \cap S^2 = \emptyset, \quad (4.2)$$

with

$$S_1 = \{\vec{x}_a\}_{a=1}^{N_1}, \quad S_2 = \{\vec{x}_s\}_{s=1}^{N_2}, \quad (4.3)$$

with $N_1 + N_2 = N$, such that $\min_{a,s}\{r_{as}\} \gg \max\{\max_{a,b}\{r_{ab}\}, \max_{s,t}\{r_{st}\}\}$. We will additionally use the notation $\gamma_{i,m}$ to denote the total magnetic charge of all of the monopoles in S_i . Here we will use the length scale $\Lambda \gg 1/m_W$ to denote the scale of the separation $\min_{a,s}\{r_{a,s}\} \sim O(\Lambda)$.

As in [25], we conjecture that the $\mathcal{D}^{\mathcal{Y}_0}$ on the asymptotic region, \mathcal{M}^{as} , of monopole moduli space is a Fredholm perturbation of the true Dirac operator on the full monopole moduli space. The reason is that wall crossing occurs where the Dirac operator fails to be Fredholm. This property is controlled by the subleading term in an expansion in $1/\Lambda$. Thus, on the asymptotic region of monopole moduli space, higher order perturbations are suppressed to $O(1/m_w\Lambda)$ and the Dirac operator is well approximated by its sub-leading order.

In the asymptotic region of monopole moduli space, there exists a local, canonical splitting of the electromagnetic charge lattice so that every particle can be assigned a charge

$$\gamma \in \Gamma = \Gamma_m \oplus \Gamma_e. \quad (4.4)$$

The metric on \mathcal{M}^{as} is given by the Lee-Weinberg-Yi/Gibbons-Manton metric [111, 79]

$$ds^2 = \mathbf{M}_{ij} d\vec{x}^i \cdot d\vec{x}^j + (\mathbf{M}^{-1})^{ij} \Theta_i \Theta_j, \quad (4.5)$$

where

$$\Theta_i = d\xi_i + \sum_{j \neq i} \vec{W}_{ij} \cdot d\vec{x}^j, \quad (4.6)$$

with

$$\mathbf{M}_{ij} = \begin{cases} m_i - \sum_{k \neq i} \frac{D_{ik}}{r_{ik}}, & i = j \\ \frac{D_{ij}}{r_{ij}}, & i \neq j, \end{cases} \quad \& \quad \vec{W}_{ij} = \begin{cases} -\sum_{k \neq i} D_{ik} \vec{w}_{ik}, & i = j, \\ D_{ij} \vec{w}_{ij}, & i \neq j \end{cases}. \quad (4.7)$$

Here m_i is the mass of the i^{th} monopole, $(r_{ij}, \theta_{ij}, \phi_{ij})$ are standard spherical coordinates on \mathbb{R}^3 centered on $\vec{r}_{ij} \equiv \vec{x}_i - \vec{x}_j$, \vec{w}_{ij} is the Dirac potential in terms of the relative coordinates \vec{r}_{ij} which is of the form

$$\vec{w}_{ij} \cdot d\vec{x}_{ij} = \frac{1}{2}(\pm 1 - \cos \theta_{ij})d\phi_{ij} , \quad (4.8)$$

and ξ_i is an angular coordinate of periodicity $p_i = 2/\alpha_{I(i)}^2$, which is the ratio of the length-squared of the long root to that of the root associated with monopole i . Note that the term $\sum_j \vec{W}_{ij} \cdot d\vec{x}^j$ can be rewritten as

$$\sum_j \vec{W}_{ij} \cdot d\vec{x}^j = \sum_{j \neq i} \frac{D_{ij}}{2} (\pm 1 - \cos(\theta_{ij})) d\phi_{ij} . \quad (4.9)$$

The mass and coupling parameters in the above formulas are

$$m_i = (H_{I(i)}, X_\infty) , \quad D_{ij} = (H_{I(i)}, H_{I(j)}) , \quad (4.10)$$

where to each constituent fundamental monopole we associate a simple co-root $i \mapsto H_{I(i)}$ describing its magnetic charge and the brackets $(\ , \)$ are a Killing form on $\mathfrak{g}, \mathfrak{g}^*$ such that the length-squared of long roots is two. These masses are related to the physical mass by a factor of $4\pi/g_0^2$, and the basis of simple co-roots is defined by X_∞ , such that X_∞ is in the fundamental Weyl chamber.

Now let us consider the asymptotic metric in two-galaxy region of monopole moduli space. Let us introduce the center-of-mass coordinates

$$\begin{aligned} \vec{X}_1 &= \frac{\sum_a m_a \vec{x}^a}{m_{\text{gal1}}} , & \vec{y}^a &= \vec{x}^a - \vec{x}^{a+1} , \quad a = 1, \dots, N_1 - 1 , \\ \vec{X}_2 &= \frac{\sum_p m_p \vec{x}^p}{m_{\text{gal2}}} , & \vec{y}^p &= \vec{x}^{p+1} - \vec{x}^{p+2} , \quad p = N_1, \dots, N - 2 , \end{aligned} \quad (4.11)$$

where $m_{\text{gal1}} = \sum_a m_a = (\gamma_{1,\text{m}}, \mathcal{X})$ is the mass associated with galaxy 1, *etc.* The indices a, b and p, q run over the relative coordinates within galaxies 1 and 2 respectively, and we've built in a shift in the numerical values that p, q run over so that these coordinates can be grouped together,

$$\vec{y}^i = (\vec{y}^a, \vec{y}^p) , \quad i, j = 1, \dots, N - 2 , \quad (4.12)$$

as will be convenient below. The inverse transformations to (4.11) are denoted

$$(\vec{x}^{ata}) = \mathbf{J}_1 \begin{pmatrix} \vec{y}^a \\ \vec{X}_1 \end{pmatrix} , \quad (\vec{x}^{ats}) = \mathbf{J}_2 \begin{pmatrix} \vec{y}^s \\ \vec{X}_2 \end{pmatrix} . \quad (4.13)$$

There is a corresponding change of phase variables given by

$$(\xi_a) = (\mathbf{J}_1^T)^{-1} \begin{pmatrix} \psi_a \\ \chi_1 \end{pmatrix}, \quad (\xi_p) = (\mathbf{J}_2^T)^{-1} \begin{pmatrix} \psi_p \\ \chi_2 \end{pmatrix}, \quad (4.14)$$

and we will denote by $\psi_i = \{\psi_a, \psi_p\}$ the collection of relative phases. See Appendix C for further details including the explicit form of the matrices $\mathbf{J}_{1,2}$.

Let us consider, for the moment, galaxy one in isolation. There is an associated moduli space $\mathcal{M}_1 = \mathcal{M}(\gamma_{m,1}, \mathcal{X})$. The coordinates $\{\vec{y}^a, \psi_a\}$ parameterize the asymptotic region of the strongly centered space $\mathcal{M}_{1,0} = \mathcal{M}_0(\gamma_{m,1}, \mathcal{X})$, while $\{\vec{X}_1, \chi_1\}$ parameterize the \mathbb{R}^4 of the universal cover $\widetilde{\mathcal{M}}_1 = \mathbb{R}_{(1)}^4 \times \mathcal{M}_{1,0}$. A similar story holds for galaxy two.

Now we return to the full picture where these two galaxies are interacting with each other. Using the center-of-mass coordinates for each galaxy we can construct the overall center-of-mass coordinates $\{\vec{X}, \chi\}$ and the relative-galaxy coordinates $\{\vec{R}, \psi\}$ as follows:

$$\begin{aligned} \vec{X} &= \frac{m_{\text{gal1}} \vec{X}_1 + m_{\text{gal2}} \vec{X}_2}{m_{\text{gal1}} + m_{\text{gal2}}}, & \vec{R} &= \vec{X}_1 - \vec{X}_2, \\ \chi &= \chi_1 + \chi_2, & \psi &= \frac{m_{\text{gal2}} \chi_1 - m_{\text{gal1}} \chi_2}{m_{\text{gal1}} + m_{\text{gal2}}}. \end{aligned} \quad (4.15)$$

Then it is the collection of position coordinates $\{\vec{y}^i, \vec{R}\}$ and phase coordinates $\{\psi_i, \psi\}$ that parameterize the strongly centered space $\mathcal{M}_0(\gamma_m, \mathcal{X})$ for the whole system.

By rewriting the metric (4.5) in terms of these new coordinates and expanding perturbatively in $1/R$ to order $O(1/R)$, where $R \sim |r_{as}|$ is the distance between the center of masses of the two galaxies and $Rm_W \gg 1$, we will find that the metric to order $O(1/R)$ can be written in the form

$$\begin{aligned} d\hat{s}^2 &:= M d\vec{X}^2 + \frac{1}{M} d\Xi^2 + (d\vec{Y}, d\vec{R}) \begin{pmatrix} \tilde{\mathbf{C}} + \frac{1}{R} \delta \mathbf{C} & \frac{1}{R} \mathbf{L} \\ \frac{1}{R} \mathbf{L}^T & \mu H(R) \end{pmatrix} \begin{pmatrix} d\vec{Y} \\ d\vec{R} \end{pmatrix} + \\ &+ (\Theta_0, \Theta_\psi) \begin{pmatrix} \tilde{\mathbf{C}} + \frac{1}{R} \delta \mathbf{C} & \frac{1}{R} \mathbf{L} \\ \frac{1}{R} \mathbf{L}^T & \mu H(R) \end{pmatrix}^{-1} \begin{pmatrix} \Theta_0 \\ \Theta_\psi \end{pmatrix}, \end{aligned} \quad (4.16)$$

where

$$M = M_1 + M_2, \quad \mu = \frac{M_1 M_2}{M_1 + M_2}, \quad (4.17)$$

are the total and reduced mass of the two galaxies.

Here we use the notation $\vec{Y} = (\vec{y}^a, \vec{y}^s)$ for the \mathbb{R}^3 coordinates for the constituents relative to the center of mass of each respective galaxy with respective phases

$$\Theta_0 = (\Theta_a, \Theta_s) = (d\psi_a + \sum_{b \neq a} \vec{w}_{ab} \cdot d\vec{x}_{ab}, d\psi_s + \sum_{t \neq s} \vec{w}_{st} \cdot d\vec{x}_{st}) , \quad (4.18)$$

and

$$\vec{R} , \Theta_\Psi = d\Psi + (\gamma_{1,m}, \gamma_{2,m}) \vec{w}(R) \cdot d\vec{R} , \quad (4.19)$$

are the coordinates for the relative moduli space, and (\vec{X}, Ξ) are the moduli coordinates for the center of mass. Further, the terms

$$H(R) = \left(1 - \frac{(\gamma_{1,m}, \gamma_{2,m})}{\mu R} \right) , \quad \tilde{\mathbf{C}} = \begin{pmatrix} (C_1)_{ab} & 0 \\ 0 & (C_2)_{st} \end{pmatrix} , \quad (4.20)$$

can be interpreted as describing the product LWY/GM metric of strongly centered moduli spaces of the different galaxies and the relative term: $\mathcal{M}_{0,1} \times \mathcal{M}_{0,2} \times \mathcal{M}_{0,rel}$. All of the other terms can be thought of as the $O(1/R)$ constant corrections which couple the different factors together. Note that the final term in the 4.16 has infinitely many terms in the expansion in $1/R$, however we only believe them to hold to order $O(1/R)$. See the Appendix C for the definition of the coordinates and undefined matrices.

Note that since the above metric is in Gibbons-Hawking form to $O(1/R)$ it is hyperkähler to order $O(1/R^2)$.

4.1.1 Triholomorphic Killing Vectors

In order to study the BPS spectrum we need the explicit form of $\mathcal{D}^{\mathcal{Y}_0}$. This requires the definition of the triholomorphic Killing field $G(\mathcal{Y}_0)_m$ in terms of the coordinates in the above metric on \mathcal{M}^{as} . This will have a decomposition in terms of the Killing fields of the LWY/GM metric: $\{\partial/\partial\psi^s\}$. In the two galaxy limit these become $\{\partial/\partial\psi^a, \partial/\partial\psi^s, \partial/\partial\Psi, \partial/\partial\Xi\}$.

The triplet of Kähler forms on \mathcal{M}^{as} are given by

$$\omega^a = \Theta_i \wedge dx_i^a - \frac{1}{2} \mathbf{M}_{ij} \epsilon^{abc} dx_i^b \wedge dx_j^c . \quad (4.21)$$

From this form it is clear that

$$\mathcal{L}_{\partial_{\xi_i}} \omega^a = 0 , \quad (4.22)$$

and hence the vector fields $\partial/\partial\xi^i$ and their linear combinations $\{\partial/\partial\psi^a, \partial/\partial\psi^s, \partial/\partial\Psi, \partial/\partial\Xi\}$ are triholomorphic Killing fields.

Remark In general, quantum corrections break the property that the $\{\partial/\partial\xi^i\}$ are triholomorphic Killing fields. However there are $\text{rank}[\mathfrak{g}]$ linear combinations which will remain triholomorphic and Killing. They are the vector fields described by

$$K_I = p_I \sum_{k=1}^{n_m^I} \frac{\partial}{\partial \xi_k^I} , \quad (4.23)$$

where there are n_m^I fundamental monopoles charged along H_I with corresponding angular coordinates ξ_k^I .

4.2 The Asymptotic Dirac Operator

Now we can compute the explicit form of the twisted Dirac operator on monopole moduli space. Because of the splitting (2.86), the spin bundle splits as a tensor product of the center of mass and the strongly centered part. This means that we can simply consider the kernel of the Dirac operator on the strongly centered moduli space which is of the form

$$\mathcal{D}^{\mathcal{Y}_0} = \mathcal{D} + i\mathcal{G}(\mathcal{Y}_0) . \quad (4.24)$$

Here $G(\mathcal{Y}_0)_m$ is a vector field which enacts a non-trivial gauge transformation along

$$\mathcal{Y}_0 = \mathcal{Y}_\infty - \frac{(\mathcal{Y}_\infty, X_\infty)}{(X_\infty, X_\infty)} X_\infty \quad , \quad \mathcal{Y}_\infty = \text{Im}[\zeta^{-1} a_D] , \quad (4.25)$$

which is the projection of \mathcal{Y}_∞ along the strongly centered moduli space. See [133, 134] for more details. This has the form:

$$G(\mathcal{Y}_0) = \sum_I \langle \alpha_I, \mathcal{Y}_0 \rangle K_I \quad , \quad K_I = p_I \sum_{k=1}^{n_m^I} \frac{\partial}{\partial \xi_k^I} . \quad (4.26)$$

After changing coordinates to those suitable for the two galaxy limit, we have that

$$G(\mathcal{Y}_0) = (\gamma_{1,m}, \mathcal{Y}_0) \frac{\partial}{\partial \Psi} + (\beta_a, \mathcal{Y}_0) \frac{\partial}{\partial \psi^a} + (\beta_s, \mathcal{Y}_0) \frac{\partial}{\partial \psi^s} , \quad (4.27)$$

where $(\beta_i, \mathcal{Y}_0) = |\mathbf{L}_i|$. See Appendix C for more details.

Now, using the spin connection (see Appendix C), we can construct the full Dirac operator. It is mostly simply written as

$$\mathcal{D}^{\mathcal{Y}_0} = \Lambda (\mathcal{D}_{12} + \mathcal{D}_{\text{Rel}}) \Lambda^{-1} , \quad (4.28)$$

where Λ is a frame rotation that absorbs most of the non-diagonal terms of the spin connection. See Appendix C. Here \mathcal{D}_{12} and \mathcal{D}_{Rel} is the splitting of $\mathcal{D}^{\mathcal{Y}_0}$ with respect to the splitting of the spin bundle.

Since the Hilbert space is graded by electric charge, we will consider the corresponding L^2 -harmonic spinors that are eigenvectors of the $U(1)^N$ symmetry:

$$\Psi = e^{i\nu^a \psi_a + i\nu^s \psi_s + i\Psi\nu} \Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}} . \quad (4.29)$$

We will now solve for the $\Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}}$ to $O(1/R^2)$. Due to the fact that the inter-cluster forces are suppressed by order $O(1/R)$ relative to the intra-cluster forces, we can make a Born-Oppenheimer type approximation for the BPS state dynamics. This leads us to make the ansatz

$$\Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}} = \left(\Psi_{12}^{(0)} + \frac{1}{R} \Psi_{12}^{(1)} + \mathcal{O}(R^{-2}) \right) \otimes \Psi_{\text{Rel}} , \quad (4.30)$$

where we further assume that Ψ_{Rel} has the form

$$\Psi_{\text{Rel}} = e^{-\alpha R} R^p \left(\hat{\Psi}_{\text{Rel}} + \mathcal{O}(R^{-1}) \right) , \quad (4.31)$$

where $\hat{\Psi}_{\text{Rel}}$ is independent of R .

With this ansatz, \mathcal{D}_{12} and \mathcal{D}_{Rel} acting on $\Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}}$ takes the form

$$\begin{aligned} \check{\mathcal{D}}_{12} &= \check{\mathcal{D}}_{12}^{(0)} + \frac{1}{R} \check{\mathcal{D}}_{12}^{(1)} + O(1/R^2) , \\ \check{\mathcal{D}}_{12} &= \Gamma^{ai} \left\{ (C^{-1/2})^i_{\underline{i}} \partial_{\underline{a}i} - (W_{\underline{a}} C^{-1/2} + w_{\underline{a}} \delta C \tilde{C}^{-1/2})^i_{\underline{i}} \partial_{4i} - w_{\underline{a}} (L^T \tilde{C}^{-1/2})_{\underline{i}} \partial_{4R} + \right. \\ &\quad \left. - \frac{1}{\mu R} (\tilde{C}^{-1/2} L)_{\underline{i}} \partial_{\underline{a}R} \right\} + \\ &\quad + \Gamma^{4i} \left\{ (C^{1/2})^i_{\underline{i}} \partial_{4i} - i(C^{-1/2})_{\underline{i}i} (\beta^i, \mathcal{Y}_0) + i \frac{(\gamma_{\text{m},1}, \mathcal{Y}_0)}{\mu R} (\tilde{C}^{-1/2} L)_{\underline{i}} \right\} + \\ &\quad + \frac{1}{4} \Gamma^{\rho k} \omega_{\underline{\mu} \underline{i} \underline{\nu} \underline{j}, \rho k} \Gamma^{\underline{\mu} \underline{i} \underline{\nu} \underline{j}} , \end{aligned} \quad (4.32)$$

and

$$\begin{aligned} \check{\mathcal{D}}_{\text{Rel}} &= \frac{1}{\sqrt{\mu}} \left(1 + \frac{(\gamma_{\text{m},1}, \gamma_{\text{m},2})}{2\mu R} \right) \left\{ \Gamma^{\underline{a}R} \partial_{\underline{a}R} - ip_{\mu} \Gamma^{\underline{a}R} w_{\underline{a}} - i\Gamma^{\underline{4}R} \left(x_{\mu} - \frac{p_{\mu}}{2R} \right) \right\} + \\ &+ O(1/R^2) , \end{aligned} \quad (4.33)$$

where

$$\begin{aligned} p_{\mu} &:= -[(\gamma_{\text{m},1}, \gamma_{\text{m},2})\nu - L_a \nu^a - L_s \nu^s] = \langle\langle \gamma_1, \gamma_2 \rangle\rangle , \\ x_{\mu} &:= (\gamma_{\text{m},1}, \mathcal{Y}_0) - \mu\nu = (\gamma_{\text{m},1}, \mathcal{Y}_0) + \left\langle \gamma_{\text{e},1} - \frac{\langle \gamma_{\text{e}}, X_{\infty} \rangle}{(\gamma_{\text{m}}, X_{\infty})} \gamma_{\text{m},1}^* , X_{\infty} \right\rangle . \end{aligned} \quad (4.34)$$

where $\langle\langle \gamma_1, \gamma_2 \rangle\rangle$ is the DSZ pairing of γ_1 with γ_2 and the $\check{\mathcal{D}}_{12}^{(0)}$ and $\check{\mathcal{D}}_{12}^{(1)}$ and $\check{\mathcal{D}}_{\text{Rel}}$ act only on the Ψ_{12} and Ψ_{Rel} parts respectively. Note that now $\check{\mathcal{D}}_{\text{Rel}}$ is now exactly the Dirac operator on Taub-NUT coupled to a vector field which can be translated to the Dirac operator from [131].

There are two types of terms that are affected by our ansatz 1.) the terms with angular derivatives and 2.) a term in $\check{\mathcal{D}}_{12}$ proportional to $\partial_{\underline{a}R}$. Since the angular coordinates are completely universal, we are allowed to replace all ∂_{4i} and ∂_{4R} terms by their eigenvalues. Similarly the term in $\check{\mathcal{D}}_{12}$ is of the form

$$\check{\mathcal{D}}_{12} = \dots - \frac{1}{\mu R} \Gamma^{\underline{a}i} (\tilde{C}^{-1/2} L)_{\underline{i}} \partial_{\underline{a}R} . \quad (4.35)$$

However, this only acts on the exponential $\Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}} \sim e^{-\alpha R R^p}$ to order $O(1/R)$ relative to the leading term. Therefore, this term can be replaced by $\frac{\alpha}{\mu R} \Gamma^{\underline{a}i} (\tilde{C}^{-1/2} L)_{\underline{i}} \frac{R_{\underline{a}}}{R}$.

With these simplifications, we see that

$$\left\{ \check{\mathcal{D}}_{12}, \check{\mathcal{D}}_{\text{Rel}} \right\} = O(1/R^2) . \quad (4.36)$$

This implies that $\Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}}$ must satisfy

$$\check{\mathcal{D}}_{12} \Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}} = 0 \quad , \quad \check{\mathcal{D}}_{\text{Rel}} \Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}} = 0 , \quad (4.37)$$

separately. Additionally, since the spin representations decouple and

$$\Gamma^{\underline{a}i} \partial_{\underline{a}R} \left(\Psi_{12}^{(0)} + \frac{1}{R} \Psi_{12}^{(1)} \right) = O(1/R^2) , \quad (4.38)$$

the separate factors of the wave function must also each satisfy a separate Dirac equation

$$\check{\mathcal{D}}_{12} \left(\Psi_{12}^{(0)} + \frac{1}{R} \Psi_{12}^{(1)} \right) = 0 \quad , \quad \check{\mathcal{D}}_{Rel} \Psi_{Rel} = 0 . \quad (4.39)$$

Since the leading term of the metric splits diagonally on $\mathcal{M}_{0,1} \times \mathcal{M}_{0,2}$, the leading term in the two galaxy Dirac operator, $\check{\mathcal{D}}_{12}$, also respects this splitting. Thus, the leading term $\Psi_{12}^{(0)}$ is simply the product of L^2 harmonic spinors for the separate galaxies. The $O(1/R)$ correction $\Psi_{12}^{(1)}$ is then determined by the leading order terms using degenerate perturbation theory. Therefore, we can say that the wavefunctions (and hence the degeneracies) are given by the product of L^2 -harmonic spinors for the separate galaxies with $O(1/R)$ corrections coming from interactions with the opposite galaxy.

Now using the calculations from [131, 133], we see that

$$\alpha = x_\mu = (\gamma_{m,1}, \mathcal{Y}_0) + \left\langle \gamma_{e,1} - \frac{\langle \gamma_e, X_\infty \rangle}{(\gamma_m, X_\infty)} \gamma_{m,1}^*, X_\infty \right\rangle , \quad (4.40)$$

$$p = \langle\langle \gamma_1, \gamma_2 \rangle\rangle .$$

This tells us that the relative wave function fails to be renormalizable exactly when α vanishes or when

$$(\gamma_{m,1}, \mathcal{Y}_0) + \left\langle \gamma_{e,1} - \frac{\langle \gamma_e, X_\infty \rangle}{(\gamma_m, X_\infty)} \gamma_{m,1}^*, X_\infty \right\rangle = 0 , \quad (4.41)$$

which we interpret as the location of the walls of marginal stability. This matches exactly with the field theory calculation from [133, 134].

Near the walls of marginal stability we can clearly see that the spectrum of $\check{\mathcal{D}}_{Rel}$ degenerates. Specifically, if we were to solve instead for the spectrum of L^2 -normalizable spinors solving

$$\check{\mathcal{D}}_{Rel} \Psi_{Rel,E} = iE \Psi_{Rel,E} , \quad (4.42)$$

we would see that the exponential behavior is now dictated by $\Psi_{Rel,E} \sim e^{-k_E r}$ where

$$k_E = \sqrt{x_\mu^2 - E^2} . \quad (4.43)$$

The L^2 normalizable spinors of this inhomogeneous Dirac equation are labeled by a

positive integer n which satisfies

$$p_\mu + 1 + \frac{(\gamma_{1,m}, \gamma_{2,m})}{2\mu\sqrt{x_\mu^2 - E^2}} (x_\mu^2 - (\gamma_1, \mathcal{Y}_0)^2 - E^2) = -n . \quad (4.44)$$

Conversely the states which have $E > x_\mu$ have no solution to (4.44) and reside in a continuum of scattering states. However, as we approach the wall of marginal stability by $x_\mu \rightarrow 0$, we see that the energy gap to the continuum of scattering states goes to zero: $E_{cont.} = x_\mu \rightarrow 0$. Thus we see at the wall of marginal stability where $x_\mu = 0$ the gap to the continuous spectrum vanishes and $\not{D}^{\mathcal{Y}_0}$ is no longer Fredholm. This implies that the spectrum of BPS states can change across such walls as we would expect from field theory.

The computation from [133] additionally shows that the space-time spin of the relative factor of the wavefunction is determined by the quantity p_μ :

$$j = \frac{1}{2} (|\langle\langle\gamma_1, \gamma_2\rangle\rangle| - 1) . \quad (4.45)$$

This reproduces the expected BPS degeneracies from field theory [133, 134].

Putting all of this together, we can conclude that a subspace of the L^2 kernel which disappears when we cross the wall of marginal stability has the form $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_j$. And further that the change in degeneracies are of the form

$$\Delta \overline{\mathcal{Q}}(\gamma_1 + \gamma_2; y) = \chi_{|\langle\langle\gamma_1, \gamma_2\rangle\rangle|}(y) \overline{\mathcal{Q}}(\gamma_1; y) \overline{\mathcal{Q}}(\gamma_2; y) , \quad (4.46)$$

where $\chi_{|\langle\langle\gamma_1, \gamma_2\rangle\rangle|}(y)$ is the character of the $SU(2)$ representation of dimension $|\langle\langle\gamma_1, \gamma_2\rangle\rangle|$ as a polynomial in y . This is exactly the primitive wall-crossing formula of [50]. It is also pleasing to note that the formulas for the walls of marginal stability and change in BPS index exactly match with the results of [161].

4.3 Framed BPS States and Haloes

We can now apply this analysis with some simple modifications to the study the wall crossing behavior of framed BPS states. This follows the construction of singular monopoles presented in [133, 134, 22]. In these papers, the authors showed that one can construct a singular monopole by sending the Higgs vev to infinity along a specific

direction in the Cartan subalgebra. This limiting procedure effectively reduces the rank of the gauge group and takes the mass of smooth monopoles to infinity, thus turning them into singular monopoles. Using this limiting procedure we can determine the metric in the asymptotic region of singular monopole moduli space from the metric in the asymptotic region of smooth monopole moduli space. This will allow us to generalize the above semiclassical analysis to the case of framed BPS states in the presence of pure magnetically charged line defects.

4.3.1 Pure 't Hooft Defects

In [132], the authors showed how one can obtain singular monopoles as a limit of smooth monopoles. This was motivated by using the string theory interpretation of $SU(N)$ smooth monopoles as (p, q) -strings stretched between N D3-branes and singular monopoles as semi-infinite D1-branes. We will discuss this procedure in the context of string theory later in Chapter 5.

In the context of quantum field theory, the procedure of [132] is as follows. Embed $SU(N)$ SYM theory with N_{def} singular monopoles into an $U(N+1)$ theory with smooth monopoles where the N_{def} defects are represented by smooth monopoles of charge along the H_N simple co-root whose charge we will denote by $\{P_n\}_{n=1}^{N_{def}}$. Without loss of generality, we will take these to be the negative Weyl-chamber representative (all other choices will be related by a hyperkähler isometry). Now restrict to the locus of far separated, fixed location and phase of the H_N -charged monopoles and then take the limit of $X_\infty^{(N)} \rightarrow \infty$ where

$$X_\infty = \sum_{I=1}^N X_\infty^{(I)} h^I \quad (h^I, H_J) = \delta_J^I . \quad (4.47)$$

This map sends the $U(N+1)$ theory to a $U(N)$ theory and corresponds to making the smooth monopoles charged along N^{th} simple coroot infinitely heavy, thereby producing singular monopoles in the $U(N)$ theory which we can then project to an $SU(N)$ theory.

However, there is a subtlety to this construction in projecting from $U(N) \rightarrow SU(N)$. Recall that in a $U(N)$ gauge theory, there is an extra $U(1)$ degree of freedom that usually trivially decouples from the dynamics of the system. However, in the presence

of singular monopoles, this decoupling is no longer trivial. In order to project out the center of mass properly, one must construct a map:

$$\Pi : \mathfrak{u}(1) \oplus \mathfrak{su}(N+1) \rightarrow \mathfrak{u}(1) \oplus \mathfrak{su}(N) \quad (4.48)$$

such that the diagram:

$$\begin{array}{ccc} \mathfrak{u}(N+1) & \xrightarrow{\mathcal{C}} & \mathfrak{u}(N) \\ \downarrow \rho & & \downarrow \rho \\ \mathfrak{u}(1) \oplus \mathfrak{su}(N+1) & \xrightarrow{\Pi} & \mathfrak{u}(1) \oplus \mathfrak{su}(N) \end{array} \quad (4.49)$$

commutes where \mathcal{C} is the natural projection and ρ is the natural splitting. This map is given explicitly as:

$$\Pi(H_I) = \begin{cases} H_I & I \neq N \\ H_N - \frac{N+1}{N} h^N & I = N \end{cases} \quad (4.50)$$

where h^N is the N^{th} cocharacter which satisfies $(h^I, H_J) = \delta_J^I$.

Now since $U(N)$ fits into the short exact sequence:

$$1 \rightarrow \mathbb{Z}_N \rightarrow U(1) \times SU(N) \rightarrow U(N) \rightarrow 1 \quad (4.51)$$

we have that projecting out the $U(1)$ degree of freedom in this way results in a theory whose gauge group is $PSU(N) = U(N)/U(1) = SU(N)/\mathbb{Z}_N$. See [132] for further details.

For our purposes, we will only need to know how to compute the coupling between the defect charges, $\{P_n = \Pi(H_N)\}$, and the smooth monopoles. These couple by terms proportional to

$$\begin{aligned} (P_n, H_{I(i)}) &= (\Pi(H_N), \Pi(H_{I(i)})) = (H_N, H_{I(i)}) - \frac{N+1}{N} (h^N, H_{I(i)}) \\ &= (H_N, H_{I(i)}) , \end{aligned} \quad (4.52)$$

for defects charged along a simple co-root. Therefore, the coupling is unchanged under the projection from $\mathfrak{su}(N+1) \rightarrow \mathfrak{su}(N)$.

Now we will apply this limiting procedure to determine the metric on the asymptotic region of singular monopole moduli space from the smooth LWY/GM metric. First, let us embed singular monopoles in an $SU(N)$ theory with fundamental charge into an

$SU(N+1)$ theory as smooth monopoles charged along the coroot H_N . Then, restrict to the subregion of fixed location and phase of the monopoles charged along H_N . Then take the limit as $X_\infty^{(N)} \rightarrow \infty$. This can be written as

$$ds_{\underline{\mathcal{M}}}^2 = \lim_{X_\infty^{(N)} \rightarrow \infty} \left(ds_{\mathcal{M}}^2 \Big|_{\vec{x}_{def}^{(n)} = \vec{x}_n} \right), \quad (4.53)$$

where the defects are located at $\{\vec{x}_n\}_{n=1}^{N_{def}}$.

A priori, it is not guaranteed that this procedure induces a well defined metric. Due to the restriction before taking the limit, any singular behavior that may arise from taking the limit would necessarily be contained in \mathbf{M}_{ij} and $(\mathbf{M}^{-1})^{ij}$. Let S_{sing} denote the set of indices that correspond to line operators in the projection. Since we will be restricting to the subspace $\vec{x}_i = 0$ for $i \in S_{sing}$, we only need to consider the behavior of $\mathbf{M}_{ij}, (\mathbf{M}^{-1})^{ij}$ for $i, j \notin S_{sing}$. By the form of \mathbf{M}_{ij} in equation 4.7, this clearly be well defined in the limit. We can now examine the form of $(\mathbf{M}^{-1})^{ij}$ by computing the cofactor matrix. If we use Λ_m to denote the mass scale which we take to approach infinity, $c_{ij} \sim O(\Lambda_m^{N^{(M)}})$ for $i, j \notin S_{sing}$ and $\det(M) \sim O(\Lambda_m^{N^{(M)}})$ where c_{ij} is the matrix of cofactors and $N^{(N)}$ is the number of monopoles charged under the N th simple coroot. Therefore $(M^{-1})^{ij} \sim O(1)$ for $i, j \notin S_{sing}$ and the limit of the metric is well defined.

The resulting metric is given by

$$ds_{\underline{\mathcal{M}}}^2 = \overline{\mathbf{M}}_{ij} d\vec{x}^i \cdot d\vec{x}^j + (\overline{\mathbf{M}}^{-1})^{ij} \overline{\Theta}_i \overline{\Theta}_j, \quad (4.54)$$

where

$$\overline{\Theta}_i = d\xi_i + \sum_{j \neq i} \frac{D_{ij}}{2} (\pm 1 - \cos(\theta_{ij})) d\phi_{ij} + \sum_{n=1}^{N_d} \frac{(P_n, H_j)}{2} (\pm 1 - \cos(\theta_{in})) d\phi_i, \quad (4.55)$$

and

$$\overline{\mathbf{M}}_{ij} = \begin{cases} m_i - \sum_{k \neq i} \frac{D_{ik}}{r_{ik}} - \sum_{n=1}^{N_d} \frac{(P_n, H_{I(i)})}{r_{in}} & , \quad i = j \\ \frac{D_{ij}}{r_{ij}} & , \quad i \neq j \end{cases} \quad (4.56)$$

Note that this construction implies

$$\lim_{X_\infty^{(N)} \rightarrow \infty} (\mathbf{M}^{-1})^{ij} \Big|_{\vec{x}_{def}^{(n)} = \vec{x}_n} = \left(\lim_{X_\infty^{(N)} \rightarrow \infty} M_{ij} \Big|_{\vec{x}_{def}^{(n)} = \vec{x}_n} \right)^{-1} \quad (4.57)$$

This means that the analysis used to write down the two galaxy limit of the smooth monopole moduli space can be implemented directly by substituting different values for \tilde{C} as in Appendix C.

We conjecture that this metric is the analog of the GM/LWY asymptotic metric for singular monopole moduli space. And similarly, inspired by Bielawski [17, 18] and Murray [137], and following the conjecture of Lee, Weinberg, and Yi in [111], we conjecture that this metric is exponentially close to the exact metric with corrections of order $e^{-m_i r_{ij}}$ for those \hat{i}, \hat{j} such that $I(\hat{i}) = I(\hat{j})$. In particular, if we have no more than one smooth monopole of each type, we conjecture that this asymptotic metric is the exact metric on the moduli space of singular monopoles.

We then may then go to two galaxy region of monopole moduli space. Collecting all of the line charges into a group we will call the *core* and the remaining into a group we will call the *halo*, two galaxy asymptotic metric may be written as

$$d\hat{s}^2 := \left(d\vec{Y}, d\vec{R} \right) \begin{pmatrix} \tilde{\vec{C}} + \frac{1}{R} \delta \vec{C} & \frac{1}{R} \vec{L} \\ \frac{1}{R} \vec{L}^T & M_h \vec{H}(R) \end{pmatrix} \begin{pmatrix} d\vec{Y} \\ d\vec{R} \end{pmatrix} + \\ + (\bar{\Theta}_0, \bar{\Theta}_\Psi) \begin{pmatrix} \tilde{\vec{C}} + \frac{1}{R} \delta \vec{C} & \frac{1}{R} \vec{L} \\ \frac{1}{R} \vec{L}^T & M_h \vec{H}(R) \end{pmatrix}^{-1} \begin{pmatrix} \bar{\Theta}_0 \\ \bar{\Theta}_\Psi \end{pmatrix}, \quad (4.58)$$

where

$$\begin{aligned} \bar{\Theta}_0 &= (\Theta_a, \bar{\Theta}_s) \quad , \quad \vec{H}(R) = 1 - \frac{(\gamma_{h,m}, \gamma_{c,m})}{M_h R} \quad , \\ \bar{\Theta}_\Psi &= d\Psi + \frac{(\gamma_{h,m}, \gamma_{c,m})}{2} (\pm 1 - \cos(\Theta) d\Phi) \quad , \\ \bar{\Theta}_s &= d\psi_s + \sum_{t \neq s} \frac{D_{st}}{2} (\pm 1 - \cos(\theta_{st})) d\phi_{st} + \sum_{n=1}^{N_d} \frac{(P_n, H_s)}{2} (\pm 1 - \cos(\theta_{sn})) d\phi_s \quad , \end{aligned} \quad (4.59)$$

and $M_h = (\gamma_{h,m}, X_\infty)$ is the mass of the halo galaxy. Here $\tilde{\vec{C}}$ is still the diagonal singular metric on $\overline{\mathcal{M}}_{0,h} \times \overline{\mathcal{M}}_{0,c}$ but has additional dependence on the $\{P_n\}$. And similarly \vec{L} and $\delta \vec{C}$ are both still constants with additional dependence on the $\{P_n\}$. This additional dependence is given in Appendix C.

The picture of this physical set up is very similar to before, but with a slightly different interpretation. Recall that in the case of framed BPS states, BPS states bound to a line defect, there is a generic core-halo structure. As discussed in the

previous chapter, this means that there is a “core” of “vanilla” BPS states which are tightly bound to the line defect and a second cluster or “halo” of vanilla BPS particles which are weakly bound to the core cluster, mirroring atomic and galactic structures. This halo is generally made up of many different constituents which bind together to form multiple different clusters which can be thought of similar to solar systems rotating around a galactic center. The physical picture we are investigating is the case with a single cluster of BPS states orbiting this core.

As we can see from of this metric, the Dirac operator will be of an identical form to the Dirac operator on the strongly centered moduli space for the case of vanilla BPS states which, by performing the same frame rotation, can be written

$$\mathcal{D}^{\mathcal{Y}_0} = \Lambda \left(\overline{\mathcal{D}}_{12} + \overline{\mathcal{D}}_{\text{Rel}} \right) \Lambda^{-1} . \quad (4.60)$$

Again we make the same ansatz for the wavefunction, $\underline{\overline{\Psi}}$, as before

$$\underline{\overline{\Psi}}_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}} = \left(\underline{\overline{\Psi}}_{12}^{(0)} + \frac{1}{R} \underline{\overline{\Psi}}_{12}^{(1)} + \mathcal{O}(R^{-2}) \right) \otimes \underline{\overline{\Psi}}_{\text{Rel}} , \quad (4.61)$$

with

$$\underline{\overline{\Psi}} = e^{i\nu^a \psi_a + i\nu^s \psi_s + i\Psi\nu} \underline{\overline{\Psi}}_{\nu^{(1)}, \nu^{(2)}, \nu}^{\text{LWY}} , \quad \underline{\overline{\Psi}}_{\text{Rel}} = e^{-\alpha R} R^p \left(\hat{\underline{\overline{\Psi}}}_{\text{Rel}} + \mathcal{O}(R^{-1}) \right) . \quad (4.62)$$

Again all of the same separability arguments hold. This again tells us that the wall crossing behavior is controlled by the relative part of the Dirac operator acting on the relative part of the moduli space which is given by:

$$\frac{1}{\sqrt{M_h}} \left(1 + \frac{(\gamma_{h,m}, \gamma_{c,m})}{2M_h R} \right) \left\{ \Gamma^{aR} \partial_{aR} - i p_\mu \Gamma^{aR} w_{\underline{a}} - i \Gamma^{4R} \left(x_\mu - \frac{p_\mu}{2R} \right) \right\} \underline{\overline{\Psi}}_{\text{Rel}} = 0 \quad (4.63)$$

where

$$\begin{aligned} p_\mu &:= -[(\gamma_{h,m}, \gamma_{c,m})\nu - L_a \nu^a - L_s \nu^s] = \langle\langle \gamma_h, \gamma_c \rangle\rangle , \\ x_\mu &:= (\gamma_{h,m}, \mathcal{Y}_0) - M_h \nu = (\gamma_{h,m}, \mathcal{Y}_0) + \langle \gamma_{h,e}, X_\infty \rangle . \end{aligned} \quad (4.64)$$

Again the location of the walls of marginal stability and change in BPS spectrum are controlled by x_μ and p_μ respectively. This gives the locus for the walls of marginal stability

$$(\gamma_{h,m}, \mathcal{Y}_0) + \langle \gamma_{h,e}, X_\infty \rangle = 0 , \quad (4.65)$$

which directly agrees with the computation from [133]. Note that this formula is simply the same as the vanilla case in the limit $X_\infty^{(N)} \rightarrow \infty$ with the restriction $\langle \lambda^N, \gamma_{i,e} \rangle = 0$ for $\langle \lambda^I, \alpha_J \rangle = \delta^I_J$.

In the the core-halo configuration of framed BPS states, the coordinate R has the interpretation of the distance from the center of mass of the halo-galaxy to the core-galaxy and line operator(s). Thus, the asymptotic form of the wave-function on the relative moduli space can be used to study what happens to the halo-galaxy near the walls of marginal stability. Using the form of the wavefunction

$$\underline{\Psi}_{Rel}(R) \sim (x_\mu)^{2p_\mu+3} R^{p_\mu} e^{-x_\mu R}, \quad (4.66)$$

we can see that near the wall of marginal stability (when $x_\mu \rightarrow 0$), the peak goes to infinity and broadens out. While the state is not itself decaying, this broadening comes from the halo-galaxy experiencing a weaker and weaker effective potential confining the center of mass to a single radius (where the binding energy balances with the rotational energy) as the center of mass goes out to infinity. This gives us the picture that as we cross a wall of marginal stability, BPS bound states go out to infinity and then come back to a stable boundstate radius as a (possibly) different state.

As before, in studying the solutions to the non-homogeneous Dirac equation

$$\overline{D}_{Rel} \underline{\Psi}_{Rel} = iE \underline{\Psi}_{Rel}, \quad (4.67)$$

we see that there is again an exponential dependence $\underline{\Psi}_{Rel} \sim e^{-kR}$ with $k = \sqrt{x_\mu^2 - E^2}$. Similarly there is an analogous bound state condition

$$p_\mu + 1 + \frac{(\gamma_{h,m}, \gamma_{c,m})}{2M_h \sqrt{x_\mu^2 - E^2}} (x_\mu^2 - (\gamma_{h,m}, \mathcal{Y}_0)^2 - E^2) = -n. \quad (4.68)$$

Again we see that as we approach a wall of marginal stability, $x_\mu \rightarrow 0$, the gap to the continuum of scattering states comes down to zero where the Dirac operator becomes non-Fredholm.

Analogous to the smooth case, we see that the same BPS degeneracy conditions hold for the framed case with degeneracy

$$j = \frac{1}{2} (|\langle \gamma_h, \gamma_c \rangle| - 1), \quad (4.69)$$

Again the wall crossing for framed BPS states is of the form

$$\Delta \overline{\Omega}(\gamma_h + \gamma_c; y) = \chi_{|\langle\langle \gamma_h, \gamma_c \rangle\rangle|}(y) \overline{\Omega}(\gamma_h; y) \overline{\Omega}(\gamma_c; y) , \quad (4.70)$$

thus reproducing the primitive wall crossing formula from [69].

4.3.2 Including Wilson Defects

Now we can consider coupling the theory to line defects that also carry electric charge. This couples the SQM on monopole moduli space to the Wilson bundle $\mathcal{E}_{\text{Wilson}} \rightarrow \mathcal{M}$ whose rank is given by the product of the dimensions of the highest weight representations associated to each Wilson defect's charge λ . Each factor of the Wilson bundle has a structure group $SU(r)$ with representation $\rho_\lambda : SU(r) \rightarrow GL(\mathcal{E}_{\text{Wilson}})$ and hence, the fiber of $\mathcal{E}_{\text{Wilson}}$ at a generic point $[\hat{A}] \in \mathcal{M}$ decomposes

$$\mathcal{E}_{\text{Wilson}} \Big|_{[\hat{A}]} = \bigotimes_n \bigoplus_{\mu \in \Delta_\lambda} V_\mu^{\otimes d_\lambda(\mu)} , \quad (4.71)$$

as a sum over weight spaces where $d_\lambda(\mu)$ is the degeneracy of the weight space associated to a weight $\mu \in \Delta_\lambda$.

Further, this vector bundle naturally has a connection given by the universal connection restricted to monopole moduli space. By supersymmetry this connection is hyperholomorphic.

As in [79], we can derive the asymptotic form of this connection by analyzing the Lagrangian for the associated SQM which is given in [22]. This Lagrangian encodes both the metric *and* connection, the latter of which describes the force of the Wilson-type defects on the BPS particles. Therefore, we can determine the hyperholomorphic connection from the quadratic contribution of the electromagnetic force between the Wilson line and distant dyons to the classical Lagrangian as in [79].

Consider the interaction between a pure Wilson line of charge $Q_n \in \Lambda_{\text{wt}} \subset \mathfrak{t}^*$ at $\vec{x}_n \in \mathbb{R}^3$ and a dyon (labeled by \hat{i}) of magnetic charge $H_{I(\hat{i})}$, electric charge $q_{\hat{i}} \in \Lambda_{\text{rt}}$, at position $\vec{x}^{\hat{i}}$, with velocity $\vec{v}_{\hat{i}}$:

$$L_{\hat{i}} = \dots + \langle q_{\hat{i}}, \vec{A} \rangle \cdot \vec{v}_{\hat{i}} - \langle q_{\hat{i}}, A_0 \rangle + (H_{I(\hat{i})}, \vec{\tilde{A}}) \cdot \vec{v}_{\hat{i}} - (H_{I(\hat{i})}, \tilde{A}_0) , \quad (4.72)$$

where the electric background field is produced by the vector potentials

$$A_0 = \frac{Q_n^*}{2|\vec{x}^{\hat{i}} - \vec{x}_n|} , \quad \vec{\tilde{A}} = Q_n^* \vec{w}(\vec{x}^{\hat{i}} - \vec{x}_n) , \quad \vec{A} = 0 , \quad \vec{\tilde{A}}_0 = 0 . \quad (4.73)$$

Here $\vec{\tilde{A}}, \vec{\tilde{A}}_0$ are the dual vector potential. We can without loss of generality couple this to all dyons by summing over the index \hat{i} , leading to the contribution to the full Lagrangian

$$L = \dots - \sum_{\hat{i}} \left(\frac{(q_i^*, Q_n^*)}{2|\vec{x}^{\hat{i}} - \vec{x}_n|} - (Q_n^*, H_{I(\hat{i})}) \vec{w}(\vec{x}^{\hat{i}} - \vec{x}_n) \cdot \vec{v}_{\hat{i}} \right) . \quad (4.74)$$

Now if we make the replacement as in [79]¹

$$q_i^* \rightarrow (M^{-1})^{\hat{i}\hat{j}} (\dot{\xi}_{\hat{j}} + \vec{w}(\vec{x}^{\hat{j}} - \vec{x}_n) \cdot \vec{v}_{\hat{j}}) , \quad (4.76)$$

then we get the contribution

$$L = \dots - \left(Q_n^*, \sum_{\hat{i}} H_{I(\hat{i})} \frac{1}{|\vec{x}^{\hat{i}} - \vec{x}_n|} (M^{-1})^{\hat{i}\hat{j}} (\dot{\xi}_{\hat{j}} + \vec{w}(\vec{x}^{\hat{j}} - \vec{x}_n) \cdot \vec{v}_{\hat{j}}) - H_{I(\hat{i})} \vec{w}(\vec{x}^{\hat{i}} - \vec{x}_n) \cdot \vec{v}_{\hat{i}} \right) . \quad (4.77)$$

This can be written in the form

$$L = \dots - \langle Q_n, q^{(n)}(\vec{x}^{\hat{i}}) \rangle , \quad (4.78)$$

where

$$q^{(n)} = \sum_{\hat{i}} H_{I(\hat{i})} \left(\frac{1}{|\vec{x}^{\hat{i}} - \vec{x}_n|} (M^{-1})^{\hat{i}\hat{j}} \Theta_{\hat{j}} - \omega_{\hat{i}n} \right) , \quad d\omega_{\hat{i}n} = *_{\mathbb{R}^3} d \left(\frac{1}{|\vec{x}^{\hat{i}} - \vec{x}_n|} \right) . \quad (4.79)$$

This leads to the connection on $\mathcal{E}_{\text{Wilson}}$

$$q(\vec{x}^{\hat{i}}) = \rho_{\otimes Q_n}(q^{(n)}(\vec{x}^{\hat{i}})) , \quad (4.80)$$

where each factor of $q(\vec{x}^{\hat{i}})$ describes the connection on the corresponding factor of (4.71).

This connection is indeed hyperholomorphic and is reminiscent of the hyperholomorphic connections on Taub-NUT written down in [19, 37, 38, 41, 125].

¹In going from the Lagrangian above to the metric we use an effective Lagrangian where there are N constants of motion

$$q_i^* = H_{I(\hat{i})} (M^{-1})^{\hat{i}\hat{j}} (\dot{\xi}_{\hat{j}} + \vec{w}(\vec{x}^{\hat{j}} - \vec{x}_n) \cdot \vec{v}_{\hat{j}}) , \quad (4.75)$$

which upon substitution into the Lagrangian, we get the full result of 4.74.

4.3.3 Wilson-'t Hooft Defects

We can now consider the case of general Wilson-'t Hooft defects. In this case we will again have a core-halo system as smooth monopoles will be required to screen the infinite self-energy of the magnetically charged defects. In order to study this system, we will couple the Dirac operator on singular monopole moduli space to the connection (4.79).

In the two galaxy limit the hyperholomorphic connection (4.79) projected onto the strongly centered moduli space takes the form

$$q^{(n)} = \left(\frac{H_{I(a)}}{|\vec{x}^a - \vec{x}_n|}, B_p(x), \frac{\gamma_{h,m}}{R} \right) \begin{pmatrix} \tilde{\mathbf{C}} + \frac{1}{R}\delta\mathbf{C} & \frac{1}{R}\bar{\mathbf{L}} \\ \frac{1}{R}\bar{\mathbf{L}}^T & m_{\text{halo}}\bar{H}(R) \end{pmatrix}^{-1} \begin{pmatrix} \bar{\Theta}_0 \\ \bar{\Theta}_\psi \end{pmatrix}, \quad (4.81)$$

where

$$B_p(x) = \left(\sum_{p=q+1}^{N-1} \frac{m_q}{m_{\text{halo}}} \right) \left(\sum_{q=N_{\text{core}}+1}^p \frac{H_{I(q)}}{|\vec{x}_q - \vec{x}_n|} \right) - \left(\sum_{q=N_{\text{core}}+1}^p \frac{m_q}{m_{\text{halo}}} \right) \left(\sum_{q=p+1}^{N-1} \frac{H_{I(q)}}{|\vec{x}_q - \vec{x}_n|} \right). \quad (4.82)$$

See Appendix C for more details.

This couples to the Dirac operator on $S^+ \otimes \mathcal{E}_{\text{Wilson}} \rightarrow \overline{\mathcal{M}}$ as

$$\not{D}_W^{\mathcal{Y}} = \dots + \Gamma^{\underline{\mu}I} E^{\nu J} \underline{\mu}_I q_{\nu J}. \quad (4.83)$$

Now the full twisted Dirac operator coupled to $\mathcal{E}_{\text{Wilson}}$ can be written in the form

$$\not{D}_W^{\mathcal{Y}_0} = \Lambda \left(\not{D}_{12,W}^{\mathcal{Y}_0} + \not{D}_{\text{Rel},W}^{\mathcal{Y}_0} \right) \Lambda^{-1}, \quad (4.84)$$

where

$$\not{D}_{\text{Rel},W}^{\mathcal{Y}_0} = \frac{1}{\sqrt{M_h}} \left(1 + \frac{(\gamma_{c,m}, \gamma_{h,m})}{2M_h R} \right) \left\{ \Gamma^{aR} \partial_{aR} - ip \Gamma^{aR} w_{\underline{a}} - i \Gamma^{4R} \left(x - \frac{p}{2R} \right) \right\}, \quad (4.85)$$

and

$$p = \langle \gamma_c^{\text{mono}}, \gamma_h \rangle + i \rho_\lambda(\gamma_{h,m}) \quad , \quad x = (\gamma_{h,m}, \mathcal{Y}_0) + \langle \gamma_{h,e}, X_\infty \rangle. \quad (4.86)$$

Again since the spin bundle and coordinates split according to $\mathcal{M} = \mathcal{M}_0 \times \mathcal{M}_{1,2} \times \mathcal{M}_{Rel}$, the separation of variables ansatz holds and hence the wave function can be written as

$$\Psi_{\nu^{(1)}, \nu^{(2)}, \nu}^{LWY} = \Psi_{12} \otimes \Psi_{Rel} = \left(\Psi_{12}^{(0)} + \frac{1}{R} \Psi_{12}^{(1)} + \mathcal{O}(R^{-2}) \right) \otimes \Psi_{Rel} \quad (4.87)$$

where

$$\overset{\vee}{D}_{12,W}^{\mathcal{Y}_0} \Psi_{12} = 0 \quad , \quad \overset{\vee}{D}_{Rel,W}^{\mathcal{Y}_0} \Psi_{Rel} = 0 \quad , \quad (4.88)$$

separately.

Now since all of the $\beta_i, \gamma_{i,m} \in \mathfrak{t}$ and we have that the Wilson vector bundle splits as a direct sum of weight spaces, on each factor, V_μ , the Dirac equation is of the form

$$\overset{\vee}{D}_{Rel,-}^{\mathcal{Y}_0} = \frac{1}{\sqrt{M_h}} \left(1 + \frac{(\gamma_{h,m}, \gamma_{c,m})}{2\mu R} \right) \left\{ \Gamma_{\underline{a}R} \partial_{\underline{a}R} - ip_\mu \Gamma_{\underline{a}R} w_{\underline{a}} - i\Gamma_{\underline{4}R} \left(x - \frac{p_\mu}{2R} \right) \right\} \quad , \quad (4.89)$$

for

$$p_\mu = \langle \gamma_c^{mono}, \gamma_h \rangle + \langle \mu, \gamma_h \rangle \quad . \quad (4.90)$$

This tells us that the walls of marginal stability are again at the same locations: $x_\mu = 0$. This is explicitly written as

$$(\gamma_{h,m}, \mathcal{Y}_0) + \langle \gamma_{h,e}, X_\infty \rangle = 0 \quad , \quad (4.91)$$

reproducing the results from [133, 134].

We now have to explain how wall crossing works for this situation – it is not so straight forward. The key is that the kernel of the Dirac operator is graded by the electric charge of the halo and the core

$$\text{Ker}_{L^2} \left[\overset{\vee}{D}_{Rel,W}^{\mathcal{Y}_0} \right] = \bigoplus_{\gamma_{h,e}, \gamma_{c,e}} \mathcal{H}_{\gamma_{c,e}, \gamma_{h,e}} \quad . \quad (4.92)$$

Here because of the Wilson bundle decomposition we have a further decomposition

$$\mathcal{H}_{\gamma_{c,e}, \gamma_{h,e}} = \bigoplus_{\substack{\mu \in \Delta_\lambda, \gamma_{c,e}^{mono} \in \Gamma_e \\ \mu + \gamma_{c,e}^{mono} = \gamma_{c,e}}} \mathcal{H}_{\gamma_{c,e}^{mono}, \gamma_{h,e}}[\mu] \quad . \quad (4.93)$$

where p_μ is a constant value on each factor of $\mathcal{H}_{\gamma_{c,e}, \gamma_{h,e}}$. Therefore, we have that across each wall of marginal stability, a spin- j multiplet on the relative moduli space \mathcal{M}_{Rel} , decays where

$$j = \frac{1}{2} (|\langle \gamma_c, \gamma_h \rangle| - 1) \quad . \quad (4.94)$$

Thus, across walls of marginal stability, we reproduce the primitive wall crossing formula of [69]:

$$\Delta \overline{\underline{\Omega}}(\gamma_h + \gamma_c; y) = \chi_{|\langle \gamma_h, \gamma_c \rangle|}(y) \overline{\underline{\Omega}}(\gamma_h; y) \overline{\underline{\Omega}}(\gamma_c; y) . \quad (4.95)$$

Pure Wilson Defects

In the case of pure Wilson lines, we have a fundamentally different physical picture as compared to the pure 't Hooft defects. Although the Wilson defects break translation invariance, they will not “freeze out” the center of center of mass component of smooth monopole moduli space. This is because they do not have an infinite self-energy. This additionally means that there is no core-halo system since the infinite self-energy does not conjure a core out of the vacuum to screen the defect. Rather we should consider the case of a single galactic halo far away from a (collection of) Wilson line(s).

An important feature of this system is that we cannot capture all of the bound states or wall crossing dynamics of this system. This is because by construction we assume that all states have magnetic charge and hence neglect all bound states of pure electric charge. However, it does mean that we can capture the bound magnetic states described in [69, 165].

The relevant Dirac operator for this system is the center of mass component of the full Dirac operator. This is given explicitly in the asymptotic limit by

$$q^{(s)} = \rho_\lambda(\gamma_m) \left(\frac{1}{2R} d\Xi + \vec{W}(\vec{X}) \cdot d\vec{X} \right) , \quad (4.96)$$

where $|\vec{X}| = R$, which is exactly the Dirac monopole connection on $\mathbb{R}^3 \times S^1$. This directly reproduces and generalizes the results of [165]. In this case the entire twisted Dirac operator can be written as

$$\not{D}_W^{\mathcal{Y}} = \left[\Gamma^i \partial_i - ip \Gamma^i W_i(\vec{X}) - i \Gamma^4 \left(x - \frac{p}{2R} \right) \right] , \quad (4.97)$$

where

$$p = \rho_\lambda(\gamma_m) \quad , \quad x = v + \frac{(\gamma_m, \mathcal{Y}_\infty)}{(\gamma_m, X_\infty)} = \frac{\langle \gamma^e, X_\infty \rangle}{(\gamma_m, X_\infty)} + \frac{(\gamma_m, \mathcal{Y}_\infty)}{(\gamma_m, X_\infty)} . \quad (4.98)$$

This is again the Dirac operator studied in [131].

As before, the walls of stability for these magnetic bound states are defined by the $x = 0$ locations and hence at the locus defined by

$$\langle \gamma^e, X_\infty \rangle + (\gamma_m, \mathcal{Y}_\infty) = 0 . \quad (4.99)$$

Again from the splitting of the Wilson bundle, the Dirac equation on each factor V_μ is of the form

$$\not{D}_W^{\mathcal{Y}} = \left[\Gamma^i \partial_i - i p_\mu \Gamma^i W_i(\vec{X}) - i \Gamma^4 \left(x - \frac{p_\mu}{2R} \right) \right] , \quad (4.100)$$

where

$$p_\mu = \langle \mu, \gamma_m \rangle . \quad (4.101)$$

Since p_μ is independent of γ_e , we have that the dimension of the kernel does not change across the walls as we can always find a unique γ_e such that $x > 0$ and the quantization condition

$$p_\mu + 1 + \frac{1}{2x} \left(x^2 - \frac{(\gamma_m, \mathcal{Y}_\infty)^2}{(\gamma_m, X_\infty)^2} \right) = -n \quad , \quad n \in \mathbb{N} . \quad (4.102)$$

This tells us that as we cross a wall of marginal stability, some states decay but then states come in from infinity with different electric charge. This reproduces exactly the results of [69].

Chapter 5

BPS States in String Theory

In this chapter we will study some string theory descriptions of BPS states in 4D $\mathcal{N} = 2$ $SU(N)$ gauge theories coming from certain brane configurations. Fundamentally, the brane constructions we will consider are primarily based on the embedding of 4D $SU(N)$ $\mathcal{N} = 2$ SYM theory into the world volume theory of D3-branes. This construction is as follows.

Take a stack of N parallel D3-branes that are localized at $x^{6,7,8,9} = 0$. The world volume theory of this stack is described by $U(N)$ $\mathcal{N} = 4$ gauge theory. Generally, the center of mass degree of freedom decouples so we can project to an $SU(N)$ $\mathcal{N} = 4$ gauge theory. By turning on a mass deformation via an Ω -deformation in the $x^{6,7,8,9}$ -directions we can break the $\mathcal{N} = 4$ SUSY down to $\mathcal{N} = 2$. The resulting theory is described by the $\mathcal{N} = 2$ $SU(N)$ gauge theory with a massive adjoint hypermultiplet. Then by sending the mass to infinity (making the Ω -deformation infinitely strong), the adjoint hypermultiplet is integrated out and the resulting theory is $\mathcal{N} = 2$ $SU(N)$ SYM theory.

This brane configuration allows for a simple interpretation of BPS states. In the semiclassical limit of the $\mathcal{N} = 2$ gauge theories that are engineered by this brane configuration there is a non-trivial Higgs vev X_∞ of one of the (real) scalar fields. This corresponds to separating the D3-branes in the x^4 -direction. Specifically, given a decomposition

$$X_\infty = \sum_I v^I H_I , \tag{5.1}$$

This chapter is based on material from my papers [22, 23, 24, 27].

the I^{th} D3-brane is localized at $x^6 = x_I^6$ such that $x_{I+1}^6 - x_I^6 = v^I$. In this configuration, W -bosons can be interpreted as fundamental strings stretching between the D3-branes. We can then see by an S-duality transformation that smooth monopoles/magnetically charged BPS states can be identified as D1-branes stretched between pairs of D3-branes [52] – or more generally that BPS states are described by (p, q) -strings stretched between D3-branes. Specifically, a (p, q) -string stretched between the I^{th} and $(I+1)^{th}$ D3-brane has a charge

$$\gamma = \gamma_m \oplus \gamma_e = pH_I \oplus q\alpha_I . \quad (5.2)$$

In [52] it is shown explicitly that the space of supersymmetric vacua is isomorphic to monopole moduli space. Further, in [132] it is shown that this brane configuration can be used to understand the wall crossing of these BPS states.

In this section we will study the more general brane configuration of [23, 35] which describes reducible 't Hooft defects and their associated framed BPS states. This brane configuration differs from the brane configuration of [52] for smooth monopoles by the inclusion of transverse NS5-branes. We will show that the introduction of these NS5-branes both gives rise to a singular monopole in the low energy effective theory on the D3-branes and that the supersymmetric vacua of the entire brane configuration is given by the appropriate singular monopole moduli space. We will show how this brane configuration can be used to study monopole bubbling, thereby allowing us to derive the singular geometry of $\widehat{\mathcal{M}}$. This will be crucial for computing the expectation value of 't Hooft defects in the next section.

5.1 Reducible 't Hooft Defects in String Theory

Now we will describe the brane configuration for reducible 't Hooft defects in a 4D $\mathcal{N} = 2$ $SU(N)$ SYM theory.

Consider flat spacetime $\mathbb{R}^{1,9} = \mathbb{R}^{1,3} \times \mathbb{R}^6$ with N D3-branes localized at $x^{5,6,7,8,9} = 0$ and $x^4 = v_I$ for $v_I \in \mathbb{R}$ and $I = 1, \dots, N$ such that

$$v_I < v_{I+1} \quad , \quad \sum_{I=1}^N v_I = 0 . \quad (5.3)$$

The low energy effective world volume theory of these branes is that of 4D $\mathcal{N} = 4$ $U(N)$ gauge theory. We then project to a 4D $\mathcal{N} = 2$ $SU(N)$ gauge theory with two real Higgs fields X, Y (corresponding to displacement in the x^4, x^5 -directions respectively) by projecting out the center of mass degree of freedom and adding a sufficiently large mass deformation as in [132, 156].

As we discussed, a smooth monopole with charge H_I is described by a D1-brane between the I^{th} and $(I + 1)^{th}$ D3-brane, localized at $x^{5,6,7,8,9} = 0$ and fixed location in $x^{1,2,3}$. For our purposes, we will consider the case of a general configuration with m^I smooth monopoles of charge H_I at distinct fixed points in the $x^{1,2,3}$ -directions. This is the standard construction of smooth monopoles in $SU(N)$ SYM theory with

$$\gamma_m = \sum_I m^I H_I \quad , \quad X_\infty = \sum_I (v_{I+1} - v_I) H_I \quad . \quad (5.4)$$

Now introduce k NS5-branes (indexed by $\sigma = 1, \dots, k$) localized at $\vec{x}_\sigma = (x_\sigma^1, x_\sigma^2, x_\sigma^3)$ at distinct points between the $I(\sigma)^{th}$ and $(I(\sigma) + 1)^{th}$ D3-branes.¹ As argued in [35], these NS5-branes introduce minimal/reducible singular monopoles and shifts the asymptotic magnetic charge so that the 't Hooft and relative magnetic charges are given by

$$P_n = \sum_{\sigma : \vec{x}_\sigma = \vec{x}_n} h^{I(\sigma)} = \sum_I p_I^{(n)} h^I \quad , \quad \tilde{\gamma}_m = \sum_I m^I H_I \quad . \quad (5.5)$$

See Figure 5.1.

To show that the NS5-branes introduce an 't Hooft defect into the D3-brane world volume theory we need to show that: 1.) it sources a magnetic field in the world volume theory of the D3-brane at a fixed location and 2.) it does not introduce any new degrees of freedom in the low energy theory. This brane configuration can be seen to reproduce these properties in the following manner.

As we know from [52], D1-branes ending on D3-branes source magnetic charge in the world volume theory of the D3-branes. While our brane configuration does not have any D1-branes connecting the NS5-branes to the D3-branes, it is Hanany-Witten dual to such a configuration. This can be seen as follows.

¹Here we index the NS5-branes by σ . To each NS5-brane we associate $\sigma \mapsto I(\sigma)$ to specify which pair of D3-branes it is sitting between in the x^4 -direction.

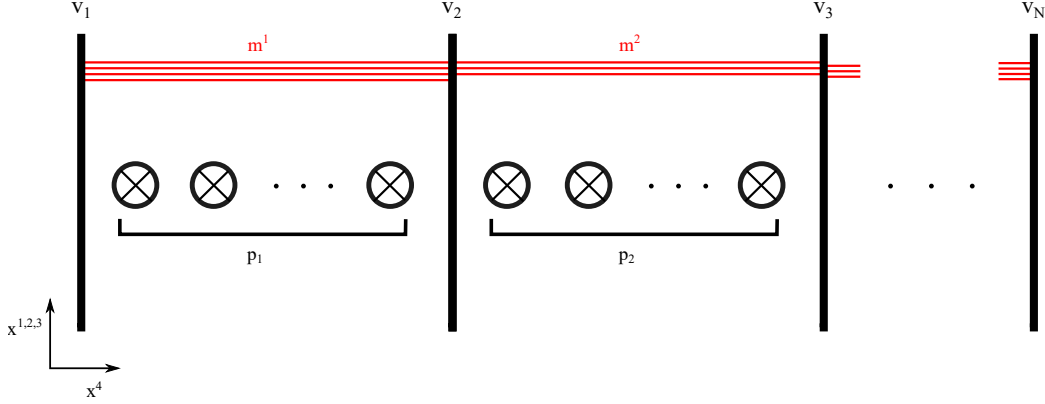


Figure 5.1: This figure shows the brane configuration of a single, reducible 't Hooft defect with 't Hooft charge $P = \sum_I p_I h^I$, relative magnetic charge $\tilde{\gamma}_m = \sum_I m^I H_I$, and Higgs vev $X_\infty = \sum_I (v_{I+1} - v_I) H_I$.

The D1/D3/NS5-brane configuration is the T-dual (along the $x^{6,7}$ -directions) to a brane configuration consisting of D3/D5/NS5-branes where D1-branes become D3-branes, D3-branes become D5-branes and NS5-branes are unaffected [33]. In the T-dual configuration, we can perform a Hanany-Witten transition which allows us to pull NS5-branes through an adjacent D5-brane and creating or destroying D3-branes connecting the D5-brane and NS5-brane so that the linking numbers

$$\begin{aligned} L_{NS5} &= -(\text{right})_{D5} + (\text{left})_{D3} - (\text{right})_{D3} , \\ L_{D5} &= (\text{left})_{NS5} + (\text{left})_{D3} - (\text{right})_{D3} , \end{aligned} \tag{5.6}$$

are preserved. Here we used the convention of [177] where $(\text{left})_{NS5/D5}$, $(\text{right})_{NS5/D5}$ are the number of NS5/D5-branes to the left, right of the given brane and $(\text{left})_{D3}$, $(\text{right})_{D3}$ are the number of D3-branes that end on the left, right side of the given brane respectively [83].

Similarly, Hanany-Witten transitions can be realized in the D1/D3/NS5-brane system by first T-dualizing to the D3/D5/NS5-brane configuration, performing the Hanany-Witten transformation, and then T-dualizing back to the D1/D3/NS5-brane configuration. The resulting transformation in the D1/D3/NS5-brane configuration allows us to pull an NS5-brane through a D3-brane while additionally changing the number of

connecting D1-branes to preserve the analogous linking numbers

$$\begin{aligned} L_{NS5} &= -(\text{right})_{D3} + (\text{left})_{D1} - (\text{right})_{D1} , \\ L_{D3} &= (\text{left})_{NS5} + (\text{left})_{D1} - (\text{right})_{D1} . \end{aligned} \tag{5.7}$$

Thus, by performing a sequence of Hanany-Witten transformations (for example sending the NS5-branes to positions $x_\sigma^4 < v_1$), one can transform to a dual frame where there are D1-branes connecting the NS5-branes to the D3-branes. There it is clear that the NS5-brane sources magnetic charge in the world volume theory of the D3-brane by nature of D1/D3-brane intersections.

Let us further consider the dual Hanany-Witten frame where D1-branes connect the D3- and NS5-branes. Here it may appear that the D1-branes can support local degrees of freedom, thereby introducing undesirable features to the low energy effective theory. However, the D3- and NS5-branes impose “opposite” boundary conditions on the D1-brane. This prevents the D1-brane from supporting any massless fields and hence does not introduce any new quantum degrees of freedom in the low energy theory [83]. Additionally, since the NS5-brane is heavy compared to all other branes in the system, its relative position in the $x^{1,2,3}$ directions will be fixed and hence will source magnetic charge will be sourced at a fixed location. Therefore, the NS5-brane configuration we have presented reproduces the “minimal” properties of a ’t Hooft operator in the world volume theory of the D3-branes.

Remark One may be curious how the phase $\zeta \in U(1)$ of a ’t Hooft defect operator is encoded in the geometry of this brane configuration. As shown in [22], this choice of phase is equivalent to a choice of direction in the $\mathbb{R}_{x^4} + i\mathbb{R}_{x^5}$ -plane in which to separate the D3-branes. Thus, the requirement that mutually supersymmetric ’t Hooft defects have the same choice of ζ is equivalent to the requirement that all NS5-branes be parallel to each other and are perpendicular to the separation of the D3-branes in the $x^{4,5}$ -directions. This is the geometric requirement for preserved supersymmetry.

Remark Note that this construction is fundamentally different from that of [83, 132] in which singular monopoles are obtained by taking an infinite mass limit of smooth monopoles – that is by sending a D3-brane with attached D1-branes off to infinity.

This procedure corresponds to embedding the $SU(N)$ SYM theory with singular monopoles as the worldvolume theory of the first N D3-branes in a stack of $N + 1$ D3-branes where the singular monopoles are identified with D1-branes stretched between the N and $(N + 1)^{th}$ D3-brane. The limit then corresponds to taking an outermost D3-brane in a stack of $N + 1$ D3-branes off to infinity. This creates a brane configuration with semi-infinite D1-branes that are connected to a stack of N D3-branes whose world volume theory is then described by $SU(N)$ SYM theory with singular monopole insertions at the semi-infinite D1-brane intersections. This process can be used to construct line defects with arbitrary charge, by taking the limit as multiple defects become coincident.

As we will discuss, the utility of this construction is that it is especially nice for studying monopole bubbling [23].

5.1.1 SUSY vacua

Now let us study the supersymmetric vacua of this brane configuration. We will take an approach similar to that of [57, 52] by analyzing the world volume theory of the D1-branes. See [40, 38] for similar analysis of a T-dual configuration.

Low Energy Effective Theory

Consider the brane configuration in the Hanany-Witten dual configuration in which D1-branes only end on NS5-branes.² This brane configuration has p NS5-branes which we will index by σ . These are localized at distinct points s_σ in the x^4 -direction and at points $\vec{x}_i^{(\sigma)}$ in the $x^{1,2,3}$ -directions. We then have m_σ D1-branes (indexed by $i = 1, \dots, m_\sigma$) stretching between the NS5-branes at s_σ and $s_{\sigma+1}$. Each interval in between pairs of NS5-branes contains some number of D3-branes (indexed by $I = 1, \dots, N$) which lie at

²It was proven in [23] that this frame exists if we satisfy

$$p_I \geq 2m^I \quad , \quad p_I = \sum_n p_I^{(n)} \quad , \quad p_I^{(n)} = \sum_{\substack{\sigma : \vec{x}_\sigma = \vec{x}_n \\ h^J(\sigma) = h^I}} (H_I, P_n) = \sum_{\substack{\sigma : \vec{x}_\sigma = \vec{x}_n \\ h^J(\sigma) = h^I}} 1 \quad , \quad (5.8)$$

While this is not a necessary condition, it will make the following analysis easier when considering monopole bubbling. For the rest of this paper we will specify to the case where this condition is satisfied.

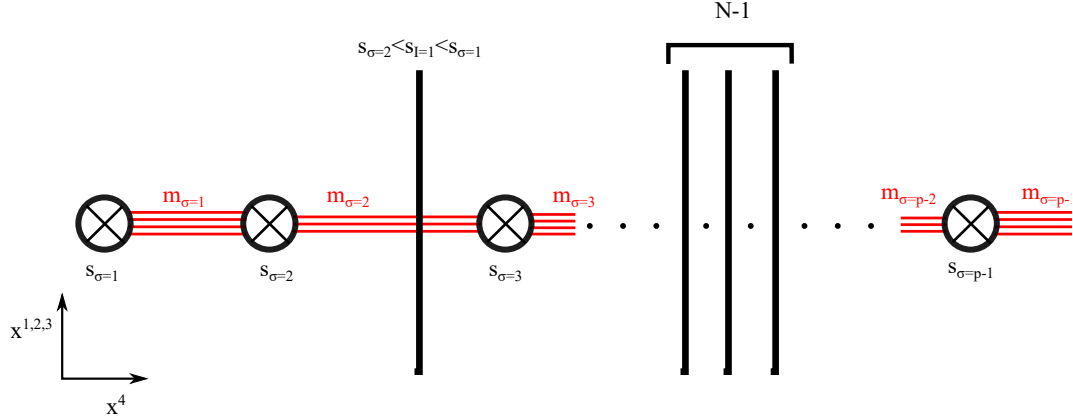


Figure 5.2: This figure illustrates the Hanany-Witten frame of the brane configuration in which we are studying the space of supersymmetric vacua of the world volume theory of D1-branes. Here there are m_σ D1-branes (red) that end on the NS5-branes (\otimes) at $x^4 = s_\sigma, s_{\sigma+1}$ and the D3-branes (black) give rise to fundamental domain walls at the intersection with D1-branes $x^4 = s_I$.

distinct points $x^4 = s_I$. See Figure 5.2. We will also use the notation q_σ to denote the number of D3-branes in between the σ^{th} and $(\sigma+1)^{th}$ NS5-branes. This is summarized in Table 5.1.

For purposes that will become clear later, we will wrap the x^4 -direction on a circle so that the D1-branes stretch along the circle direction but do not wrap all the way around. Thus, we will identify $\sigma \sim \sigma + p$ where $m_{\sigma=p} = 0$.

We want to study the low energy effective theory on the D1-branes. This theory is a two-dimensional $\mathcal{N} = (0, 4)$ quiver gauge theory with domain walls coming from the interactions with D3- and NS5-branes. The D3-brane intersections will give rise to *fundamental walls*, which introduce localized fundamental hypermultiplets from D1-D3 strings. Similarly, the NS5-brane intersections will give rise to *bifundamental walls*, which introduce localized bifundamental hypermultiplets from D1-D1 strings across the NS5-brane as in [83]. Here the data of the brane configuration maps to the 2D SUSY gauge theory as

	0	1	2	3	4	5	6	7	8	9
	\mathbb{R}_t	\mathbb{R}^3			S^1	\mathbb{R}^5				
Coordinates	x^0	\vec{x}			s	\vec{y}				
D3	—	—	—	—	s_I	0				
m_σ D1	—	$\vec{x}_i^{(\sigma)}$			$[s_\sigma, s_{\sigma+1}]$	0				
NS5	—	$\vec{\nu}_\sigma$			s_σ	—	—	—	—	—

Table 5.1: This table specifies the brane configuration whose moduli space of supersymmetric vacua is described by singular monopole moduli space.

- Gauge group: $G = \prod_\sigma U(m_\sigma)$ where each factor corresponds to an interval in the x^4 -direction bounded by NS5-branes,
- FI-parameters in each interval are given by the $\vec{\nu}_\sigma$,
- The Higgs vevs for the $U(m_\sigma)$ factor is given by

$$\vec{v}_\infty^{(\sigma)} = \begin{pmatrix} \vec{x}_1^{(\sigma)} & & \\ & \vec{x}_2^{(\sigma)} & \\ & & \ddots \end{pmatrix}, \quad (5.9)$$

up to a choice of ordering.

The action of the D1-brane world volume gauge theory is of the form

$$S = S_{bulk} + S_{FI} + S_f + S_{bf}, \quad (5.10)$$

where S_{bulk} is the bulk theory of the D1-branes, S_{FI} are FI-deformations, S_f is the contribution of fundamental walls (D3-branes), and S_{bf} is contribution of bifundamental walls (NS5-branes).

The $\mathcal{N} = (0, 4)$ SUSY of the theory comes from the fact that the bulk theory of the D1-branes (with the mass deformation) is described by a $\mathcal{N} = (4, 4)$ theory which is then broken to $\mathcal{N} = (0, 4)$ by the boundary conditions of the D3- and NS5-branes. The fact that the resulting SUSY is $\mathcal{N} = (0, 4)$ rather than $\mathcal{N} = (2, 2)$ can be deduced by noting that the truncation breaks the R -symmetry of the D1-brane theory from $Spin(8)_R \rightarrow Spin(4)_R \cong SU(2)_{R,1} \times SU(2)_{R,2}$ along the $x^{1,2,3,5}$ -directions. Then the introduction of D3- and NS5-branes breaks $Spin(4)_R \rightarrow Spin(3)_R \cong SU(2)_R$ along the

$x^{1,2,3}$ -directions. Then, since theories with $\mathcal{N} = (2, 2)$ SUSY have $U(1)_{R,1} \times U(1)_{R,2}$ symmetry whereas $\mathcal{N} = (0, 4)$ has $SU(2)_R$ symmetry, we can conclude that the total theory has $\mathcal{N} = (0, 4)$ SUSY. See [165] for a review of $\mathcal{N} = (0, 4)$ SUSY.

The bulk theory of the D1-branes is described by 2D $\mathcal{N} = (4, 4)$ SYM theory. This is composed of a $\mathcal{N} = (0, 2)$ vector-superfield V with superfield strength Σ , a Fermi multiplet Ψ , and two chiral multiplets in the adjoint representation $(\Phi, \tilde{\Phi})$. Here the vector and Fermi multiplets combine as a $\mathcal{N} = (0, 4)$ vector multiplet and the $(\Phi, \tilde{\Phi})$ combine into a $\mathcal{N} = (0, 4)$ twisted hypermultiplet. These supermultiplets can be written in terms of the component fields

$$\begin{aligned} V &= (v_0, v_s, \lambda_1, D) \quad , \quad \Psi = (\lambda_2, F, E_\Psi(\Phi)) \quad , \\ \Phi &= (\phi_1, \psi_1, G_1) \quad , \quad \tilde{\Phi} = (\bar{\phi}^2, \bar{\psi}_2, \bar{G}_2) \quad , \end{aligned} \quad (5.11)$$

where λ_A , and ϕ_A are $SU(2)_R$ doublets and $M^a = (\text{Re}[F], \text{Im}[F], D)$ is a real $SU(2)_R$ triplet. Here $E_\Psi(\Phi)$ is a superpotential which is a holomorphic function of all chiral superfields of the theory. It receives the contribution $E_\Psi = \dots + [\Phi, \tilde{\Phi}]$ from the twisted $\mathcal{N} = (0, 4)$ hypermultiplet $(\Phi, \tilde{\Phi})$.

The bulk contribution to the total action is

$$S_{bulk} = \frac{1}{8g^2} \int dt ds d^2\theta \text{Tr} \left\{ \bar{\Sigma} \Sigma + \bar{\Psi} \Psi + \bar{\Phi} (\mathcal{D}_-) \Phi + \tilde{\Phi} (\mathcal{D}_-) \tilde{\Phi} \right\} \quad , \quad (5.12)$$

where

$$\mathcal{D}_- = \partial_0 - \partial_1 - iV \quad , \quad (5.13)$$

and the superfields are written explicitly as [165]

$$\begin{aligned} V &= (v_0 - v_1) - i\theta^+ \bar{\lambda}^1 - i\bar{\theta}^+ \lambda_1 + \theta^+ \bar{\theta}^+ D \quad , \\ \Psi &= \lambda_2 + \theta^+ F - i\theta^+ \bar{\theta}^+ (D_0 + D_1) \lambda_2 - \bar{\theta}^+ E_\Psi(\phi) + \theta^+ \bar{\theta}^+ \frac{\partial E_\Psi}{\partial \phi^i} \psi_i \quad , \\ \Phi &= \phi_1 + \theta^+ \psi_1 - i\theta^+ \bar{\theta}^+ (D_0 + D_1) \phi \quad , \\ \tilde{\Phi} &= \bar{\phi}^2 + \theta^+ \bar{\psi}_2 - i\theta^+ \bar{\theta}^+ (D_0 + D_1) \bar{\phi}^2 \quad . \end{aligned} \quad (5.14)$$

Under $\mathcal{N} = (4, 4)$ SUSY, these fields reorganize themselves into a single $\mathcal{N} = (4, 4)$ vector-multiplet $\mathcal{V} = (v_0, v_s, X)_i, \chi_A, F, D)$ where the χ_A are a doublet of Dirac fermions

and X_i (for $i = 1, 2, 3, 5$) are real scalar bosons that encode the fluctuations of the D1-brane in the x^i direction. The SUSY transformations of these fields are given by [29]

$$\begin{aligned}
\delta A_\mu &= i\bar{\epsilon}_A \gamma_\mu \chi^A - i\bar{\chi}_A \gamma_\mu \epsilon^A , \\
\delta X_a &= i\bar{\epsilon}_A (\sigma^a)^A_B \chi^B - \bar{\chi}_A (\sigma^a)^A_B \epsilon^B , \quad a = 1, 2, 3 , \\
\delta X_5 &= \bar{\chi}_A \gamma_c \epsilon^A - \bar{\epsilon}_A \gamma_c \chi^A , \\
\delta \chi^A &= \gamma_{\mu\nu} F^{\mu\nu} \epsilon^A - i\gamma_c \gamma^\mu \epsilon^A D_\mu X_5 + \gamma^\mu (\sigma^a)^A_B \epsilon^B D_\mu X_a + M_a (\sigma^a)^A_B \epsilon^B .
\end{aligned} \tag{5.15}$$

Here γ_μ are the gamma matrices for Dirac fermions in 2D with $\gamma_c = i\gamma_0\gamma_1$ and $(\sigma^a)^A_B$ are the Pauli matrices for $SU(2)_{R,1} \subset Spin(4)_R$. See [29] for a review on 2D $\mathcal{N} = (4, 4)$ SYM.

In order to determine the vacuum equations of this theory, we will need to eliminate the auxiliary fields M_a , which are dependent on the interaction of the $\mathcal{N} = (0, 4)$ vector multiplet with all hypermultiplets in the theory. Here the $\mathcal{N} = (0, 4)$ (twisted) hypermultiplet in the $\mathcal{N} = (4, 4)$ vector multiplet $(\Phi, \tilde{\Phi})$, has a non-trivial coupling to the F - and D -fields given by

$$\frac{\delta S_{bulk}}{\delta M_a} = 2M_a + \epsilon_{abc}[X_b, X_c] . \tag{5.16}$$

Now we will consider the contribution to the action S_{FI} , which encodes the supersymmetric FI-deformations to the theory. This is given by

$$S_{FI} = \int dt ds \operatorname{Tr} \left\{ \nu_3(s) \int d^2\theta V + \bar{\nu}(s) \int d\theta^+ \Psi + c.c. \right\} , \tag{5.17}$$

where $\nu(s) = \nu_1(s) + i\nu_2(s) \in \mathbb{C}$ are constant on the interval $(s_\sigma, s_{\sigma+1})$. These couple to the F - and D -terms:

$$\frac{\delta S_{FI}}{\delta D} = \nu_3(s) \mathbb{1} \quad , \quad \frac{\delta S_{FI}}{\delta F} = \nu(s) \mathbb{1} . \tag{5.18}$$

Now let us consider the contributions to the action from the fundamental domain walls S_f . This contribution gives rise to $\mathcal{N} = (0, 4)$ hypermultiplets restricted to the world volume of the domain walls. By nature of preserving the $Spin(3)_R$ symmetry associated to the rotations of the $x^{1,2,3}$ -directions, this boundary theory preserves the $SU(2)_{R,1}$ R -symmetries. Take the $\mathcal{N} = (0, 4)$ hypermultiplet describing the I^{th}

fundamental domain wall theory to be described by a doublet of fundamental chiral superfields in conjugate gauge representations, $(\mathcal{Q}_{1I}, \mathcal{Q}_{2I})$, with constituent bosonic fields (Q_{1I}, J_{1I}) and (Q_{2I}, J_{2I}) respectively. The corresponding contribution to the total action is

$$S_f = \frac{1}{2} \sum_{I=1}^N \int dt \int d^2\theta \left(\bar{\mathcal{Q}}_{1I} \mathcal{D}_t \mathcal{Q}_{1I} + \bar{\mathcal{Q}}_{2I} \mathcal{D}_t \mathcal{Q}_{2I} \right), \quad (5.19)$$

where $\mathcal{D}_t = \partial_t \pm iV$ where V acts in the appropriate representation.

The localized hypermultiplet fields additionally contribute to the E -term for the Fermi superfield Ψ as

$$E_\Psi = \dots + \frac{1}{2} \sum_{I=1}^N \delta(s - s_I) \mathcal{Q}_{1I} \mathcal{Q}_{2I}. \quad (5.20)$$

The hypermultiplet fields also couple to the F- and D-terms as

$$\frac{\delta S_f}{\delta D} = \frac{1}{2} \sum_{I=1}^N (\bar{\mathcal{Q}}_{2I} \mathcal{Q}_{2I} - \mathcal{Q}_{1I} \bar{\mathcal{Q}}_{1I}) \delta(s - s_I), \quad \frac{\delta S_f}{\delta F_1} = \sum_{I=1}^N \mathcal{Q}_{1I} \mathcal{Q}_{2I} \delta(s - s_I), \quad (5.21)$$

which have the effect of adding boundary terms to the supersymmetry transformations and vacuum equations.

Now consider the contribution of bifundamental domain walls S_{bf} . In analogy with the fundamental domain walls, bifundamental domain walls give rise to $\mathcal{N} = (0, 4)$ bifundamental hypermultiplets on a domain wall preserving the same supersymmetry. This can be written in terms of two chiral superfields in conjugate representations $(\mathcal{B}_{1\sigma}, \mathcal{B}_{2\sigma})$ with constituent bosonic fields $(B_{1\sigma}, L_{1\sigma})$ and $(B_{2\sigma}, L_{2\sigma})$. These are described by the action

$$S_{bf} = \frac{1}{2} \sum_{\sigma=1}^p \int dt \operatorname{Tr} \int d^2\theta \left(\bar{\mathcal{B}}_{1\sigma} \tilde{\mathcal{D}}_t \mathcal{B}_{1\sigma} + \bar{\mathcal{B}}_{2\sigma} \tilde{\mathcal{D}}_t \mathcal{B}_{2\sigma} \right), \quad (5.22)$$

where $\tilde{\mathcal{D}}_t = \partial_t \pm i(V(s_\sigma^R) - V(s_\sigma^L))$ as appropriate to the representation. Here we use the notation $\Lambda(s_\sigma^{L,R}) = \lim_{s \rightarrow s_\sigma^\pm} \Lambda(s)$ for any superfield Λ .

The bifundamental hypermultiplets additionally contribute to the E -term for the Fermi superfield Ψ as

$$E_\Psi = \dots + \frac{1}{2} \sum_{\sigma=1}^p \mathcal{B}_{1\sigma} \mathcal{B}_{2\sigma} \delta(s - s_{\sigma-1}^R) + \mathcal{B}_{2\sigma} \mathcal{B}_{1\sigma} \delta(s - s_\sigma^L), \quad (5.23)$$

and couple to the F- and D-terms as

$$\begin{aligned}
\frac{\delta S_{bf}}{\delta D} &= \frac{1}{2} \sum_{\sigma=1}^p (\bar{B}_{1\sigma} B_{1\sigma} - B_{2\sigma} \bar{B}_{2\sigma} - \nu_{3\sigma}) \delta(s - s_{\sigma}^L) \\
&\quad + (B_{1\sigma} \bar{B}_{1\sigma} - \bar{B}_{2\sigma} B_{2\sigma} + \nu_{3\sigma}) \delta(s - s_{\sigma-1}^R) , \\
\frac{\delta S_{bf}}{\delta F} &= \sum_{\sigma=1}^p B_{1\sigma} B_{2\sigma} \delta(s - s_{\sigma-1}^R) + B_{2\sigma} B_{1\sigma} \delta(s - s_{\sigma}^L) .
\end{aligned} \tag{5.24}$$

Again, these contributions can be interpreted as adding boundary terms to the supersymmetry transformations and vacuum equations.

Vacuum Equations

Now we can determine the vacuum equations by examining the SUSY variations of the bulk fields as in (5.15). Since the domain walls break SUSY to $\mathcal{N} = (0, 4)$, we only impose half of supersymmetries of the bulk theory. The conserved supercharges are those generated by

$$\epsilon^A = \gamma^\mu \epsilon^A . \tag{5.25}$$

For these transformations, the contribution to the SUSY variation of the action from the FI-parameters away from the boundaries can be absorbed by making the shift

$$X^3 \mapsto X^3 - \int_{s_\sigma}^{s_{\sigma+1}} ds \nu_3(s) \quad , \quad X^1 + iX^2 \mapsto X^1 + iX^2 - \int_{s_\sigma}^{s_{\sigma+1}} ds \nu(s) . \tag{5.26}$$

This transformation, as in [40, 38], shifts the bulk dependence of the FI-parameters to boundary dependence at the bifundamental domain walls where the FI-parameter is discontinuous. By choosing the axial gauge $A_0 = 0$, the stationary vacuum equations become

$$(\sigma^a)^A_B \epsilon^B (D_1 X_a + M_a) = 0 . \tag{5.27}$$

By integrating out the auxiliary fields we see that this reduces to a triplet of equations which can be written as a real and complex equation:

$$\begin{aligned}
0 &= D_1 X_3 + \frac{i}{2}[X, \bar{X}] + \frac{1}{2} \sum_{I=1}^N (\bar{Q}_{2I} Q_{2I} - Q_{1I} \bar{Q}_{1I}) \delta(s - s_I) \\
&+ \frac{1}{2} \sum_{\sigma=1}^p (\bar{B}_{1\sigma} B_{1\sigma} - B_{2\sigma} \bar{B}_{2\sigma} - \nu_3^\sigma) \delta(s - s_\sigma^L) + (B_{1\sigma} \bar{B}_{1\sigma} - \bar{B}_{2\sigma} B_{2\sigma} + \nu_3^{\sigma-1}) \delta(s - s_{\sigma-1}^R) , \\
0 &= D_1 X + i[X_3, X] - \sum_{I=1}^N Q_{1I} Q_{2I} \delta(s - s_I) \\
&+ \sum_{\sigma=1}^p (B_{1\sigma} B_{2\sigma} - \nu^\sigma) \delta(s - s_\sigma^L) + (B_{2\sigma} B_{1\sigma} - \nu^\sigma) \delta(s - s_{\sigma-1}^R) ,
\end{aligned} \tag{5.28}$$

where $X = X_1 + iX_2$.

Under the identifications

$$\begin{aligned}
T_a &= X_a \quad , \quad I_x = Q_{1I} \quad , \quad J_x = Q_{2I} \quad , \\
\vec{\nu}_e &= \vec{\nu}_\sigma \quad , \quad B_e^{LR} = B_{1\sigma} \quad , \quad B_e^{RL} = B_{2\sigma} \quad ,
\end{aligned} \tag{5.29}$$

the SUSY vacuum equations (5.28) can be rewritten as the Nahm equations for the bow construction of instantons (2.70). Therefore, we can identify the moduli space of supersymmetric vacua \mathcal{M}_{vac} with a moduli space of instantons on multi-Taub-NUT \mathcal{M}_{bow} .

Now by studying the identification (5.29), we can determine the data of the corresponding bow variety. The ranks $R(\zeta)$ can be read off from the ranks of the $\{X_a\} = \{T_a\}$ which correspond to the ranks of the gauge group of the 2D theory in the different chambers. Further, we can identify the fundamental walls with $x \in \Lambda$ and similarly the bifundamental walls with $e \in \mathcal{E}$. By identifying the bow variety \mathcal{M}_{bow} with the moduli space of instantons on Taub-NUT, we see that the number of fundamental walls correspond to the rank of the 4D gauge group and the number of bifundamental walls correspond to the number of Taub-NUT centers. In this identification, the FI parameters are mapped to the position of the positions of the NUT centers.

In summary, we can match the data of the brane configuration to that of instantons on multi-Taub-NUT by specifying $(\mathcal{E}, \Lambda, \mathcal{I}, \{\vec{\nu}_e\}, \{R(\zeta)\})$ by identifying:

- The number of edges, $|\mathcal{E}| = p$, with the number NUT centers on multi-Taub-NUT: $p = \sum_{I,n} p_I^{(n)}$ where $p_I^{(n)} = |\{e_\sigma^{(I)} \in \mathcal{E} \mid \vec{\nu}^{(I,\sigma)} = \vec{x}_n\}|$,
- The total number of marked points, $|A| = N$, with (one plus) the rank of the gauge group G : $SU(N)$,
- The numbers $R(\zeta_\sigma^{(i)}) = m_\sigma$ with the Chern classes of the instanton bundle (note that one of the $R(\zeta_p^{(i)}) = 0$),
- The hyperkähler moment parameters $\vec{\nu}_\sigma = (\nu_1^{(\sigma)}, \nu_2^{(\sigma)}, \nu_3^{(\sigma)})$ with the positions of the different NUT centers: \vec{x}_σ ,
- The holonomy of the gauge field $\exp \left\{ \frac{1}{2\pi} \oint_{S_\infty^1} A \right\} = \exp \left\{ \frac{X_\infty}{2\pi R'} \right\}$, where R is the radius of the S^1 at infinity and $R' = 1/R$.

Note that this is simply the bow variety specified by identifying marked points x_I with D3-branes, edges e_σ with NS5-branes, and the wavy lines $\zeta_\sigma^{(i)}$ ($i = 1, \dots, 1 + q_\sigma$) with (stacks of) D1-branes in the dual Hanany-Witten frame. Further, the positions of the NS5-branes in the $x^{1,2,3}$ -directions are identified with the FI parameters $\vec{\nu}_\sigma = \vec{x}_\sigma$ and the numbers of D1-branes $\{m_\sigma\}$, are identified as the $R(\zeta_\sigma^{(i)}) = m_\sigma$. In this setting, Hanany-Witten bow isomorphisms correspond to Hanany-Witten transformations of the brane configuration. Thus, as we would expect, the bow variety that we have derived describes the moduli space of supersymmetric vacua of the original brane configuration as in Figure 5.1.

This bow variety is also the same bow variety derived in [23] that describes (reducible) singular monopole moduli space from using Kronheimer's correspondence. Therefore, we can indeed identify the moduli space of supersymmetric vacua of the brane configuration with reducible singular monopole moduli space with the data

$$P_n = \sum_I p_I^{(n)} h^I \quad , \quad \tilde{\gamma}_m = \sum_I m^I H_I \quad , \quad (5.30)$$

where the Higgs vev is defined by the holonomy

$$\exp \left\{ \frac{1}{2\pi} \oint_{S_\infty^1} A \right\} = \exp \left\{ \frac{X_\infty}{2\pi R'} \right\} \quad , \quad (5.31)$$

where $R' = 1/R$ is the dual radius of S^1_∞ . This leads us to the conclusion that the brane configuration we have presented does indeed describe 't Hooft defects in 4D $SU(N)$ $\mathcal{N} = 2$ gauge theories.

5.1.2 Monopole Bubbling

Thus far we have presented analysis showing that the moduli space of supersymmetric vacua of the brane configuration matches that of the moduli space of reducible singular monopoles. However, since there is very little known about the singularity structure of singular monopole moduli space, it is difficult to see that this analysis extends to include the bubbling configurations describing such singular configurations.

In this setup, monopole bubbling occurs when a D1-brane becomes spatially coincident with and intersects an NS5-brane. One may be worried that at the intersection with NS5-branes, the interpretation of the brane configuration breaks down. However, there are several reasons that suggest the opposite. First, the bubbling locus reproduces the correct effect on the bulk dynamics. Specifically, as argued in [23], one can adapt the computation from [39] to show that the 't Hooft charge is appropriately screened during monopole bubbling.

Additionally, although bubbling involves an intersection of a D1-brane with an NS5-brane, the bubbling configurations are actually non-singular. Specifically, we can go to the Hanany-Witten frame in which all of the NS5-branes are localized at distinct $x_\sigma^4 < v_1$. In this case, bubbling D1-branes will at most make them coincident with another D1-brane created by pulling NS5-branes through a D3-brane. See Figure 5.3. Further, notice that in studying the supersymmetric vacua, there is no obstruction to describing the singular locus of monopole moduli space. Therefore, it is not unreasonable to conjecture that this brane configuration gives a good description for monopole bubbling.

In fact, this brane configuration has actually been shown to reproduce some key data of singular monopole moduli spaces. In [23] it is shown that this brane configuration reproduces the structure of the bubbling locus (2.96) of reducible singular monopole moduli space $\widehat{\mathcal{M}}$ [142]. This can be seen as follows.

Consider the case of the $\mathcal{N} = 2$ $SU(2)$ SYM theory. This can be described by the

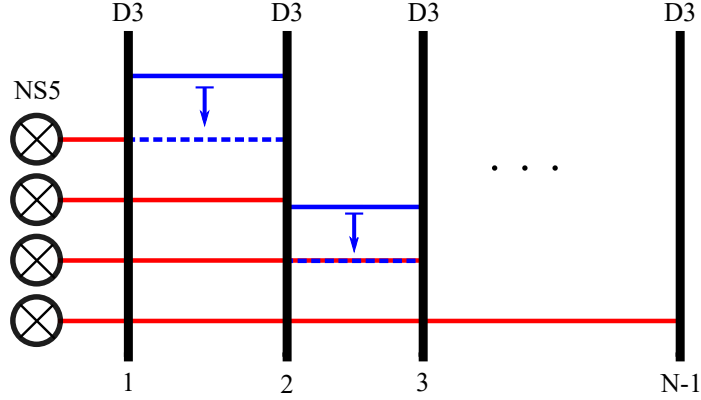


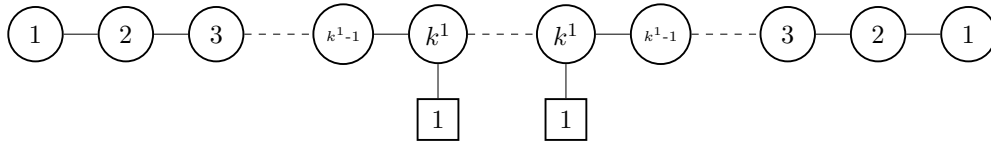
Figure 5.3: This figure describes a Hanany-Witten dual frame of the brane configuration in which monopole bubbling appears to be a non-singular process. In this figure we can see that bubbling of the finite D1-branes (blue) occur when they become spatially coincident with the NS5-brane (and associated D1-branes in red) in the $x^{1,2,3}$ -directions. Here, one can see that in this frame, bubbling is non-singular as it corresponds to at most coincident D1-branes.

above brane configuration as explained above by adding a large mass deformation. Now consider adding a single reducible 't Hooft defect localized at the origin with charge

$$P = p_1 h^1, \quad (5.32)$$

where there are $k^1 \leq m^1$ bubbled D1-branes such that $2k^1 \leq p_1$. Now to study monopole bubbling, consider *only* the bubbled D1-branes in addition to the D3- and NS5-branes. We can now perform a sequence of Hanany-Witten moves to go to the dual frame in which D1-branes only end on NS5-branes.³ See Figure 5.4.

Now the D1-brane world volume theory is given by a quiver SQM described by the quiver $\Gamma(P, \vec{v})$:



where the node of degree k^1 is repeated $p_1 - 2k^1 + 1$ -times. This SQM has a moduli

³Note that this exists because $2k^1 \leq p_1$. See [23] for a proof.

space of supersymmetric vacua given by $\mathcal{M}_{SQM}(\Gamma(P, \mathbf{v}))$.⁴

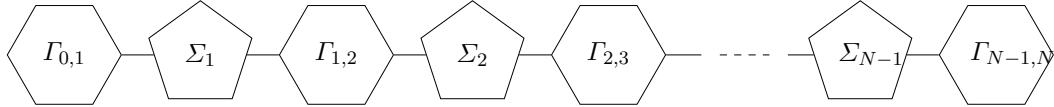
Similarly, one can go through the exercise to determine the quiver SQM for monopole bubbling in $\mathcal{N} = 2$ SYM theory. Let us consider monopole bubbling of the 't Hooft defect with charge

$$P = \sum_I p_I h^I, \quad (5.33)$$

in the case where the magnetic charge of the bubbled monopoles is given by

$$\gamma_m^{(bubbled)} = P - \mathbf{v} = \sum_I k_I H_I. \quad (5.34)$$

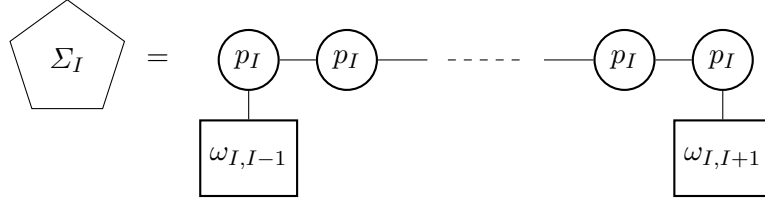
This brane configuration corresponds to a stack of N (separated) D3-branes with p_I spatially transverse NS5-branes in between the $D3_I$ - and $D3_{I+1}$ -branes. The bubbling corresponds to setting k_I D1-branes that run between the $D3_I$ - and $D3_{I+1}$ -branes to be spatially coincident (in the $x^{1,2,3}$ -directions) with the NS5-branes. In this case, the bubbling SQM can be determined by going to the dual Hanany-Witten frame in which the bubbled D1-branes only end on NS5-branes. In this frame the SQM is again a quiver gauge theory describing the effective world volume theory of the D1-branes given by the quiver $\Gamma(P, \mathbf{v})$:



where the sub-quivers Σ_I are given by

⁴There is also an additional special consideration when $p_1 = 2k^1$. In this case the quiver is given by

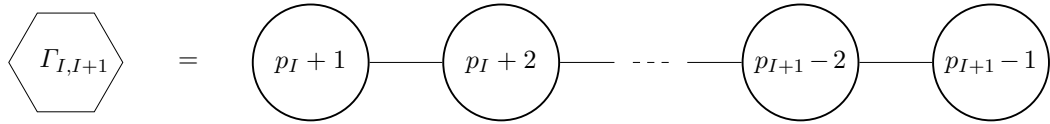




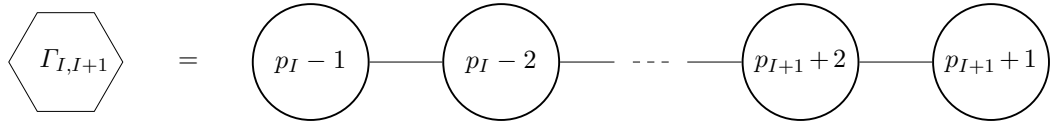
where Σ_I is of length

$$|\Sigma_I| = n_I + 1 - |p_{I+1} - p_I| \omega_{I,I+1} - |p_{I-1} - p_I| \omega_{I,I-1} \quad , \quad \omega_{i,j} = \begin{cases} 0 & p_I \leq p_j \\ 1 & p_I > p_j \end{cases} \quad (5.35)$$

while the sub-quiver $\Gamma_{I,I+1}$ is given by (with $p_0 = 0$ and $p_N = 0$)



when $p_i < p_{I+1}$ and



when $p_I > p_{I+1}$.⁵

Here the subquivers $\Gamma_{I,I+1}$ come from NS5-branes that change chambers in going to the magnetic Hanany-Witten frame and the subquivers Σ_I correspond to the NS5-branes which do not. Moving NS5-branes to the left or right across the $D3_{I+1}$ -brane (determined by the ordering of p_I, p_{I+1}) will give rise to an increasing or decreasing

⁵There are again some special cases for the above quiver:

- $p_I = p_{I+1}$: there is no $\Gamma_{I,I+1}$ quiver connecting Σ_I and Σ_{I+1} , but rather the last node of Σ_I is identified with the first node of Σ_{I+1} . Note that in this case $|\Sigma_I + \Sigma_{I+1}| = |\Sigma_I| + |\Sigma_{I+1}| - 1$.
- $p_I = p_{I+1} \pm 1$: $\Gamma_{I,I+1}$ is omitted and Σ_I is directly connected to Σ_{I+1} .
- $|\Sigma_I| = 1$: there is a single gauge node of magnitude p_I with two fundamental hypermultiplets.

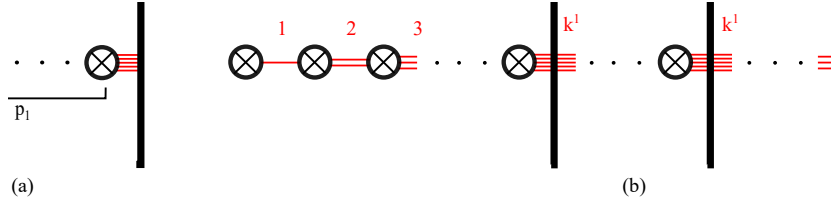


Figure 5.4: This figure shows the two Hanany-Witten frames of our brane configuration that we are considering: (a) the standard frame and (b) the Hanany-Witten “dual magnetic” frame (with the unbubbled monopoles removed).

$\Gamma_{I,I+1}$ respectively and additionally endows the Σ_{I+1} or Σ_I subquiver respectively with a fundamental hypermultiplet on the gauge node of the adjacent end. This combination of the ordering of p_I, p_{I+1} and p_I, p_{I-1} and their corresponding hypermultiplet nodes give rise to 4 different types of Σ_i subquivers.

Thus, this brane configuration shows that there is a SQM of bubbled monopoles living on the world line of the ’t Hooft defect which indicates how the singular strata in (2.96) are glued into the full moduli space. Specifically, the moduli space of supersymmetric vacua of this 1D quiver SQM $\mathcal{M}_{SQM}(\Gamma(P, \mathbf{v}))$ defines the transversal slice of each singular strata $\mathcal{M}(P, \mathbf{v})$ in (2.96).

Additionally, this construction has also been shown to reproduce exact quantum information about monopole bubbling by using localization. We will discuss the details of this computation in Chapter 6. However, the results therein provide a powerful verification that this brane configuration can be used to generally study monopole bubbling and further, it also suggests that monopole bubbling is itself a semiclassical effect.

5.1.3 Kronheimer’s Correspondence and T-Duality

Notice that the above identification of singular monopole moduli space with the moduli space of the supersymmetric vacua of the D1/D3/NS5-brane configuration relies crucially on Kronheimer’s correspondence. This suggests an interesting relationship between Kronheimer’s correspondence and T-duality which we will now explore.

Consider a general reducible singular monopole configuration with a reducible ’t

Hooft defect in $\mathcal{N} = 2$ $SU(N)$ SYM theory subject to the constraint (5.8). Now “resolve” the configuration by pulling apart all of the defects into minimal ’t Hooft defects localized at $\vec{x}_\sigma \in \mathbb{C}$ which are indexed by σ .

By Kronheimer’s correspondence this is dual to $U(1)_K$ -invariant instantons on Taub-NUT where the lift of the $U(1)_K$ action to the gauge bundle around any NUT center is given by

$$\lim_{\vec{x} \rightarrow \vec{x}_\sigma} g(\vec{x}; \alpha) = e^{ih^{I(\sigma)}\alpha} \quad , \quad \alpha \in U(1)_K \quad , \quad h^{I(\sigma)} \in A_{cochar} \quad , \quad (5.36)$$

where $P_\sigma = h^{I(\sigma)}$. Further, the first Chern class of the gauge bundle is given by

$$\tilde{\gamma}_m = \sum_I m^I H_I \quad , \quad (5.37)$$

and the Higgs vev is given in terms of the holonomy of the gauge field around the circle at infinity⁶

$$\exp \left\{ \frac{1}{2\pi} \oint_{S_\infty^1} \hat{A} \right\} = \exp \left\{ \frac{X_\infty}{2\pi R'} \right\} \quad , \quad (5.38)$$

where $R' = 1/R$ is the dual radius of S_∞^1 .

This gauge theory configuration can be embedded in the world volume theory of D4-branes wrapping TN_p . Having resolved the singularities coming from the coincident NUT centers, we can study the behavior of the T-dual brane configuration. T-dualizing along the circle fiber of TN_p then takes this configuration to a theory describing the world volume of D3-branes with some collection of NS5-branes (from NUT centers) and D1-branes from the D0-brane instantons. in the presence of D1- and NS5-branes.

We can then take the coincident limit of the NUT centers. Assuming that T-duality commutes with the resolving and taking the coincident limit of the NUT centers, we find that the T-dual brane configuration of D1/D3/NS5-branes coincides with the brane configuration that we have presented for reducible singular monopoles. However, before proceeding with the technical details of this calculation, we will first motivate this result.

⁶ Here this equation is only strictly true if we take X_∞ to be a periodic scalar field, which in decompactifying the T-dual $S^1 \rightarrow \mathbb{R}$ we allow to be a \mathfrak{t} -valued scalar.

Action of T-Duality on Fields

Let us consider the action of T-duality on gauge field configuration describing a $U(1)_K$ -invariant instanton on TN_p in the 4D $\mathcal{N} = 2$ SYM theory. Near each NUT center, the gauge field can be written in the $U(1)_K$ -invariant gauge

$$\hat{A} = A_{\mathbb{R}^3} + \psi(d\xi + \omega) , \quad (5.39)$$

where

$$\lim_{\vec{x} \rightarrow \vec{x}_\sigma} A_{\mathbb{R}^3} = P_\sigma \omega_{\vec{x}_\sigma} , \quad \lim_{\vec{x} \rightarrow \vec{x}_\sigma} \psi = -P_\sigma . \quad (5.40)$$

Again we will use the notation

$$ds^2 = V(\vec{x}) d\vec{x} \cdot d\vec{x} + V^{-1}(\vec{x}) (d\xi + \omega)^2 , \quad (5.41)$$

where

$$V(\vec{x}) = 1 + \sum_{\sigma} \frac{1}{2|\vec{x} - \vec{x}_\sigma|} , \quad d\omega = *_3 dV , \quad d\omega_{\vec{x}_\sigma} = *_3 d \left(\frac{1}{2|\vec{x} - \vec{x}_\sigma|} \right) . \quad (5.42)$$

In T-dualizing, Buscher duality tells us that the term $V^{-1}(\vec{x})(d\xi + \omega)^2$ in the metric generates a non-trivial B -field source at the positions of the NUT centers. This indicates the existence of NS5-branes in the transverse space at the position of the NUT centers in the $x^{1,2,3}$ -directions. Additionally, since the S^1 fiber has radius $1/\sqrt{V}$, under T-duality, the one form roughly transforms as

$$\psi(\vec{x}) d\xi = \psi(\vec{x}) \sqrt{V} \left(\frac{d\xi}{\sqrt{V}} \right) \mapsto \psi(\vec{x}) V d\xi' = X d\xi' . \quad (5.43)$$

This leads to the standard Higgs field X and connection $A_{\mathbb{R}^3}$ in (3+1)D that satisfy the Bogomolny equation. Additionally, from the limiting forms of $(A_{\mathbb{R}^3}, \psi)$ in (5.40) and the form of the harmonic function (5.42), one can see that these fields have the limiting form

$$\lim_{\vec{x} \rightarrow \vec{x}_\sigma} A_{\mathbb{R}^3} = P_\sigma \omega , \quad \lim_{\vec{x} \rightarrow \vec{x}_\sigma} V(\vec{x}) \psi(\vec{x}) = -\frac{P_\sigma}{2|\vec{x} - \vec{x}_\sigma|} , \quad (5.44)$$

which is exactly the 't Hooft boundary conditions at \vec{x}_σ . Therefore, from the field perspective, it clear that T-duality maps $U(1)_K$ -invariant instanton configurations on TN_p to singular monopole configurations on \mathbb{R}^3 .

Note that the other bosonic fields of the (3+1)D theory A_0, Y (where $\zeta^{-1}\Phi = Y + iX$ is the standard complex Higgs field) come from the five-dimensional gauge field along the x^0 -direction and the 5D Higgs field describing the D4-branes in the x^5 -direction and hence do not play a role in T-duality.⁷

String Theory Analysis

As described in [177], instantons in the world volume theory of a stack of D4-branes wrapping TN_p are T-dual to a brane configuration described by D1-, D3-, and NS5-branes. Here we will apply the analysis of [177] to the D1/D3/NS5-brane configuration to study how it behaves under T-duality. Here we will find that the brane configuration of D1/D3/NS5-branes proposed above is T-dual to the corresponding $U(1)_K$ -invariant instanton configuration on TN_p given by Kronheimer's correspondence [108].

Consider the D1/D3/NS5-brane configuration in the Hanany-Witten duality frame where D1-branes only end on NS5-branes as in Figure 5.2.⁸ In this case we have p NS5-branes (indexed by σ) with m_σ D1-branes running from the σ^{th} to the $(\sigma + 1)^{th}$ NS5-brane and q_σ D3-branes in between the σ^{th} and $(\sigma + 1)^{th}$ NS5-branes. Now wrap the x^4 -direction on a circle. T-duality along the x^4 -direction then maps: 1.) the collection of p NS5-branes into a transverse TN_p [147, 82], 2.) the stack of D3-branes to a stack of D4-branes wrapping the TN_p , and 3.) the D1-branes to some instanton configuration of the gauge bundle living on the D4-branes.

In order to specify the T-dual brane configuration we need to specify how the numbers and positions of the branes are reproduced by the instanton brane configuration. The number and positions of the NS5-branes are encoded in the B -field of the D1/D3/NS5-brane configuration. Since the NS5-branes are charged under the B -field, T-dualizing them give rise to a NUT center (due to Buscher duality) at the previous location of the NS5-brane in the $x^{1,2,3}$ -directions. Thus, the relative positions of the

⁷Note that we are truncating the standard 5D $\mathcal{N} = 2$ SYM theory to a $\mathcal{N} = 1$ theory by projecting out fluctuations in the $x^{6,7,8,9}$ -directions by including a large mass deformation as before.

⁸Recall that we are imposing the condition (5.8) so that there exists a magnetic Hanany-Witten duality frame.

NS5-branes on the T-duality circle is encoded by the cohomology class of the B -field.⁹ This can be measured by its period integrals

$$\theta_\sigma - \theta_{\sigma'} = \int_{C_{\sigma\sigma'}} \frac{B}{2\pi} , \quad (5.45)$$

where θ_σ is the position of the σ^{th} NS5-brane along the x^4 -direction [147, 82]. Here we have identified the homology cycles $C_{\sigma\sigma'}$ as follows. Given an ordering of the NS5-branes, there is a natural basis of $H_2(TN_p; \mathbb{Z})$ given by $\{C_{\sigma\sigma+1}\}$ where $C_{\sigma\sigma+1}$ is defined as the preimage under the projection map $\pi : TN_p \rightarrow \mathbb{R}^3$ of the line running between the NUT centers corresponding to the NS5 $_\sigma$ -brane and NS5 $_{\sigma+1}$ -brane in the base \mathbb{R}^3 . Here we identify $\sigma \sim \sigma + p$. We then define $C_{\sigma\sigma'}$ as the homology cycle

$$C_{\sigma\sigma'} = \sum_{\rho=\sigma}^{\sigma'-1} C_{\rho\rho+1} , \quad (5.46)$$

where we have assumed $\sigma > \sigma'$. This is topologically equivalent to the cycle defined by the preimage of the line running between the NUT centers corresponding to the NS5 $_\sigma$ -brane and the NS5 $_{\sigma'}$ -branes.

The rest of the data of the brane configuration is encoded in the gauge bundle through the instanton configuration [177]. In order to specify the class of the instanton bundle corresponding to the T-dual brane configuration, one must specify the first Chern class, second Chern class, and the holonomy of the connection.¹⁰ The first Chern class is valued in $H_{cpt}^2(TN_p; \mathbb{Z})$. These elements can be understood in the following fashion. $H_{cpt}^2(TN_p; \mathbb{Z})$ is naturally isomorphic to $H_2(TN_p; \mathbb{Z})$ by Poincaré duality. Using the basis of $H_2(TN_p; \mathbb{Z})$ above, we can identify the homology cycles $\{C_{\sigma\sigma+1}\}$ with basis elements $\{b_{\sigma\sigma+1}\}$ of $H_{cpt}^2(TN_p; \mathbb{Z})$. We can then identify a sequence of p numbers, $\{f_\sigma : \sigma = 1, \dots, p\}$ to an element of $H_{cpt}^2(TN_p; \mathbb{Z})$ as

$$B = \sum_{\sigma=1}^p (f_{\sigma+1} - f_\sigma) b_{\sigma\sigma+1} . \quad (5.47)$$

In this setup, [177] determined that the first Chern class of the instanton bundle is given by the corresponding element of $H_{cpt}^2(TN_p; \mathbb{Z})$ determined by the sequence of p

⁹Note that this is the relative positions as the absolute positions along the T-duality S^1 is a gauge dependent.

¹⁰That is to say, we specify the data of the relevant instanton moduli space. See [177] for more details.

numbers given by the linking numbers of the p NS5-branes: $\{\ell_\sigma : \sigma = 1, \dots, p\}$ where

$$\ell_\sigma = m_\sigma - m_{\sigma-1} + q_\sigma . \quad (5.48)$$

In [177], the author also computed the 2nd Chern character of the instanton bundle $V \rightarrow TN_p$

$$\int ch_2(V) = m_0 , \quad (5.49)$$

where m_0 is the number of D1-branes running between the NS5 _{p} -brane and the NS5₁-brane (recall that the NS5-branes are separated along a circle). In our case, we have $m_0 = 0$.

In order to completely specify the instanton bundle, we also need to specify the holonomy of the gauge connection. In the 5D gauge theory, the monodromy along the S^1 fiber at infinity encodes the positions of the D3-branes: ¹¹

$$U_\infty = \text{diag}(\exp(is_1/R), \exp(is_2/R), \dots, \exp(is_N/R)) . \quad (5.51)$$

Given this data of the instanton bundle and B -field configuration, we can completely determine the T-dual brane configuration of D1/D3/NS5-branes. Now by taking the coincident limit of the appropriate NUT centers, we arrive at the T-dual brane configuration for reducible 't Hooft defects.

In order to complete this discussion, we need to understand the action of $U(1)_K$ on the T-dual instanton configuration. Under T-duality, translation along the T-duality circle (the action of $U(1)_K$) maps to non-trivial abelian gauge transformations in the D1-brane world volume theory along the x^4 -direction in D1/D3/NS5-brane configuration. However, since the branes do not wrap all the way around the x^4 -direction, any such gauge transformation can be undone by a trivial gauge transformation. Therefore, this brane configuration will be dual to a $U(1)_K$ -invariant instanton configuration on TN_p .

¹¹Here when we take the decompactification radius we take $R \rightarrow \infty$ and the $s_I \rightarrow \infty$ such that $s_I/R \rightarrow v_I$ where the Higgs vev of the 4D theory is given by

$$X_\infty = \sum_{I=1}^{N-1} (v_{I+1} - v_I) H_I . \quad (5.50)$$

T-duality and Line Bundles

We will now show explicitly that T-duality exchanges singular monopole configurations with the $U(1)_K$ -invariant instanton solution given by Kronheimer's correspondence.

Consider $SU(N)$ $\mathcal{N} = 2$ SYM theory with a collection of reducible 't Hooft defects $\{P_n, \vec{x}_n\}$ such that

$$P_n = \sum_I p_I^{(n)} h^I, \quad (5.52)$$

where h^I are simple cocharacters. Additionally, let us allow for some collection of far separated smooth monopoles with total charge

$$\gamma_m = \sum_I m^I H_I, \quad (5.53)$$

that are indexed by $i = 1, \dots, \sum_I m^I$ with fixed positions $\vec{x}_i \in \mathbb{R}^3$ and charges $H_{I(i)}$ where $H_I, H_{I(i)}$ are simple coroots.

Now let us resolve the defects by pulling them apart into constituent minimal 't Hooft defects index by $\sigma = 1, \dots, p = \sum_{n,I} p_I^{(n)}$ with charges $h^{I(\sigma)}$ located at \vec{x}_σ . The corresponding brane configuration is T-dual to a gauge theory on multi-Taub-NUT with p NUT centers located at $\{\vec{x}_\sigma\}_{\sigma=1}^p$ and $U(1)_K$ invariant instantons that are far separated at positions $\{\vec{x}_i\}$. Due to the holonomy of the gauge bundle, the Chan-Paton bundle asymptotically¹² splits as a direct sum of line bundles

$$\mathcal{T} = \bigoplus_{I=1}^N \mathcal{R}_I. \quad (5.54)$$

These line bundles can be decomposed as a tensor product of line bundles that are each individually gauge equivalent to a canonical set of line bundles which can be defined as follows.

Choose the NUT center at position \vec{x}_σ . Now choose a line L_σ from \vec{x}_σ to ∞ which does not intersect any other NUT centers. Define $C_\sigma = \pi^{-1}(L_\sigma)$ to be the preimage of this line. To this infinite cigar we can identify a complex line bundle $\mathcal{L}_{\vec{x}_\sigma}$ with

¹²Here by asymptotically we mean at distances sufficiently far from any instanton. Specifically, we are interested in the behavior at infinity and arbitrarily close to the NUT centers. This can be seen from the perspective of singular monopole configurations because the gauge symmetry is broken at infinity by the Higgs vev and at the 't Hooft defects by their non-trivial boundary conditions [100].

connections

$$A_{\vec{x}_\sigma} = -\frac{d\xi + \omega}{2|\vec{x} - \vec{x}_\sigma|V(\vec{x})} + \frac{1}{2}\omega_{\vec{x}_\sigma} \quad , \quad d\omega_{\vec{x}_\sigma} = *_3 d\left(\frac{1}{|\vec{x} - \vec{x}_\sigma|}\right) . \quad (5.55)$$

This family of line bundles can be extended to include connections associated to arbitrary points $\vec{x}_i \neq \vec{x}_\sigma$

$$A_{\vec{x}_i} = -\frac{d\xi + \omega}{2|\vec{x} - \vec{x}_i|V(\vec{x})} + \frac{1}{2}\omega_{\vec{x}_i} \quad , \quad d\omega_{\vec{x}_i} = *_3 d\left(\frac{1}{|\vec{x} - \vec{x}_i|}\right) . \quad (5.56)$$

Under a B -field gauge transformation, the Chan-Paton bundle \mathcal{T} transforms as

$$\mathcal{T} \mapsto \mathcal{T} \otimes \mathcal{L}_A \quad , \quad B \mapsto B + d\Lambda , \quad (5.57)$$

where \mathcal{L}_A is the line bundle with connection given by Λ .

These connections have the property that

$$\int_{C_\sigma} \frac{d\Lambda_{\vec{x}_\rho}}{2\pi} = \mathbf{C}_{\sigma\rho} , \quad (5.58)$$

where $\mathbf{C}_{\sigma\rho}$ is the Cartan matrix of A_{p-1} .

We can additionally define the topologically trivial line bundle

$$\mathcal{L}_* = \bigotimes_{\sigma=1}^p \mathcal{L}_{\vec{x}_\sigma} , \quad (5.59)$$

where $\mathcal{L}_{\vec{x}_\sigma}$ is a line bundle with connection which is gauge equivalent to $A_{\vec{x}_\sigma}$ as above.

This line bundle is topologically trivial because its periods are trivial due to the properties of the Cartan matrix.

Since this is a topologically trivial line bundle,¹³ we can also define \mathcal{L}_*^t with connection

$$\Lambda_*^{(t)} = t \sum_{\sigma=1}^p A_{\vec{x}_\sigma} = t \frac{d\xi + \omega}{V(\vec{x})} \quad , \quad t \in \mathbb{R}/2\pi\mathbb{Z} . \quad (5.60)$$

These connections have the limiting forms

$$\begin{aligned} \lim_{\vec{r} \rightarrow \vec{x}_i} \Lambda_* &\rightarrow 0 \quad , \quad \lim_{\vec{r} \rightarrow \vec{x}_i} \Lambda_{\vec{x}_i} \rightarrow -\frac{1}{2}\omega_{\vec{x}_i} \quad , \quad \lim_{\vec{x} \rightarrow \vec{x}_\sigma} \Lambda_{\vec{x}_\sigma} \rightarrow -\frac{1}{2}\omega_{\vec{x}_\sigma} , \\ \lim_{\vec{r} \rightarrow \infty} \Lambda_* &\rightarrow (d\psi + \omega) \quad , \quad \lim_{\vec{r} \rightarrow \infty} \Lambda_{\vec{x}_i} \rightarrow \frac{1}{2}\omega \quad , \quad \lim_{\vec{r} \rightarrow \infty} \Lambda_{\vec{x}_\sigma} \rightarrow -\frac{1}{2}\omega , \end{aligned} \quad (5.61)$$

¹³This is trivial in the sense that the canonical pairing of the curvature with any closed 2-cycle is trivial.

where all other limits are finite. Here $\omega_{\vec{x}_i}$ is the Dirac potential centered at \vec{x}_i . This tells us that $\Lambda_*^{(t)}$ has non-trivial holonomy along the asymptotic circle fiber

$$\oint_{S^1_\infty} \Lambda_*^{(t)} = 2\pi t \quad , \quad t \sim t + 1 . \quad (5.62)$$

Therefore, this component of the Chan-Paton bundle describes the Higgs vev X_∞ of the T-dual brane configuration (5.68). Additionally, these asymptotic forms tell us that $\Lambda_{\vec{x}_i}$ is an asymptotically flat connection except near $\vec{x}_i \in \mathbb{R}^3$ where it can be smoothly continued in exchange for inducing a non-trivial first Chern class.

Using this, the factors of the Chan-Paton (gauge) bundle of the T-dual brane configuration are given by

$$\mathcal{R}_I = \mathcal{L}_*^{s_I/2\pi R} \bigotimes_{\sigma: I(\sigma)=I} \mathcal{L}_{\vec{x}_\sigma}^{-1} \bigotimes_{j: I(j)=I} \mathcal{L}_{\vec{x}_j} \bigotimes_{k: I(k)=I+1} \mathcal{L}_{\vec{x}_k}^{-1} , \quad (5.63)$$

where here the $j = 1, \dots, m^I$ and $k = 1, \dots, m^{I+1}$ index smooth monopoles with magnetic charge H_I and H_{I+1} respectively where $m^N = m^0 = 0$. Note that this reproduces the expression (5.51) where again s_I is the position of the i^{th} D3-brane along the x^4 circle before decompactifying.

The above decomposition of the Chan-Paton bundle is non-trivial and can be deduced by studying Hanany-Witten transformations. Consider the brane configuration where there is a single D3-brane localized at $s = 0$ along the x^4 circle with p NS5-branes at distinct, non-zero positions $\{s = y_\sigma \neq 0\}$ along the x^4 circle direction. We can choose a background B -field such that the Chan-Paton bundle of the D3-brane is trivial. Now move the D3-brane around the circle in the clockwise direction. Before the D3-brane intersects an NS5-brane, the Chan-Paton bundle is trivial and of the form

$$\mathcal{R} = \mathcal{L}_*^{s/2\pi R} . \quad (5.64)$$

As shown in [177], when the D3-brane intersects an NS5-brane at $s = y_\sigma$, the Chan-Paton bundle can jump by a factor of $\mathcal{L}_{\vec{x}_\sigma}^{-1}$. This reflects the fact that the Hanany-Witten transition creates a D1-brane which ends on the D3-brane (thus inducing the factor of $\mathcal{L}_{\vec{x}_\sigma}^{-1}$). Thus by moving the D3-brane around the circle to the point s , the Chan-Paton bundle is of the form

$$\mathcal{R} = \mathcal{L}_*^{s/2\pi R} \bigotimes_{\sigma: y_\sigma < s} \mathcal{L}_{\vec{x}_\sigma}^{-1} . \quad (5.65)$$

Note that when the D3-brane moves around the entire circle, the Chan-Paton bundle is again trivial because the $s \rightarrow s + 2\pi R$ is canceled by the overall factor of $\bigotimes_{\sigma} \mathcal{L}_{\vec{x}_{\sigma}}^{-1} = \mathcal{L}_{*}$. Therefore, each D1-brane that ends on a D3-brane contributes a factor of $(\mathcal{L}_{\vec{x}_{\sigma}})^{\pm 1}$ to its Chan-Paton bundle depending on orientation. This decomposition allows us to determine the cohomology classes of the line bundles in the asymptotic decomposition of the Chan-Paton/gauge bundle, thus giving the result (5.63).

This form of the Chan-Paton bundle corresponds to an instanton configuration with connection that is asymptotically of the form

$$\hat{A} = \text{diag}(\Lambda_1, \dots, \Lambda_N) , \quad (5.66)$$

where

$$\Lambda_I = \Lambda_*^{(s_I/2\pi R)} + \sum_{\sigma: I(\sigma)=I} \Lambda_{\vec{x}_{\sigma}} - \sum_{j: I(j)=I} \Lambda_{\vec{x}_j} + \sum_{j: I(j)=I+1} \Lambda_{\vec{x}_j} , \quad (5.67)$$

up to gauge equivalence. Because the connections are hyperholomorphic, this connection indeed describes an instanton configuration.

Now we can take the coincident limit of the appropriate NUT centers – this corresponds to reconstructing the reducible 't Hooft defects in the D1/D3/NS5-brane configuration. Since T-duality commutes with the movement of the NUT centers or NS5-branes appropriately, we can conclude that the coincident limit of NUT centers is the T-dual configuration corresponding to the D1/D3/NS5-brane configuration with reducible 't Hooft defects.

Using the asymptotic forms of the individual connections, we see that the connection \hat{A} has the limiting form exactly given by

$$\hat{A} = A + \psi(x)(d\xi + \omega) , \quad (5.68)$$

such that

$$dA = *_3 d(V\psi) \quad , \quad \lim_{\vec{x} \rightarrow \vec{x}_{\sigma}} V(x)\psi(x) = -\frac{P_{\sigma}}{2|\vec{x} - \vec{x}_{\sigma}|} \quad , \quad \lim_{r \rightarrow \infty} V(x)\psi(x) = X_{\infty} - \frac{\gamma_m}{2r} , \quad (5.69)$$

to leading order.¹⁴ This is an exact match with Kronheimer's correspondence [108].

¹⁴ Note that we had to take the decompactification limit as described in Footnote 6 which requires scaling the s_I with $R' = 1/R \rightarrow \infty$.

Therefore, Kronheimer's correspondence for our brane configurations acts as T-duality.

5.2 Irreducible Monopoles

Now by using the fact that Kronheimer's correspondence is equivalent to T-duality in the previous section, we can try to generalize this picture to include a description of non-minimal irreducible 't Hooft defects.¹⁵ The idea will be to first describe irreducible singular monopoles as $U(1)_K$ -invariant instantons on Taub-NUT through Kronheimer's correspondence, embed it into string theory as in the previous section, and then T-dualize to arrive at a brane configuration describing singular monopoles in \mathbb{R}^3 .

We expect this to work a priori because the field theoretic arguments we made before in Section 5.1.3 made no reference to whether the 't Hooft defect in question was reducible or irreducible. Thus we can expect that T-duality will more generally map $U(1)_K$ -invariant instantons with $U(1)_K$ -lift defined by $P \in \Lambda_{cochar}$ to singular monopole configurations with 't Hooft charge P . Further, the fact T-duality maps between families of configurations with isomorphic moduli spaces matches the fact that Kronheimer's correspondence states that the moduli space of $U(1)_K$ -invariant instantons whose action is defined by $P \in \Lambda_{cochar}$ is isomorphic to the singular monopole moduli space defined by the 't Hooft charge P .

However, we expect this to produce a different brane configuration as compared to reducible 't Hooft defects because $U(1)_K$ -invariant instantons on multi-Taub-NUT can differentiate between irreducible and reducible 't Hooft defects through the combined data of the $U(1)_K$ action and the NUT charge. The NUT charge is defined as the Hopf charge of the $TN_p|_{S^2_{\vec{\sigma}, \epsilon}} \rightarrow S^2_{\vec{\sigma}, \epsilon}$ over an infinitesimal 2-sphere of radius ϵ around a NUT center at \vec{x}_σ which can additionally be determined by the coefficient of the term $\frac{1}{2|\vec{x}-\vec{x}_\sigma|}$ in the harmonic function of the metric. Note that this changes as we take the limit as $\vec{x}_{\sigma'} \rightarrow \vec{x}_\sigma$ as in the case of reducible 't Hooft defects.

Now we can use the framework from the previous section to explicitly construct the Chan-Paton bundle in the case of an irreducible 't Hooft defect. This allows us to easily

¹⁵Here we mean 't Hooft defects associated to a 't Hooft charge $P \in \Lambda_{cochar}$ which are S-dual to a Wilson line of irreducible representation of highest weight $P \in \Lambda_{wt}(G^\vee)$.

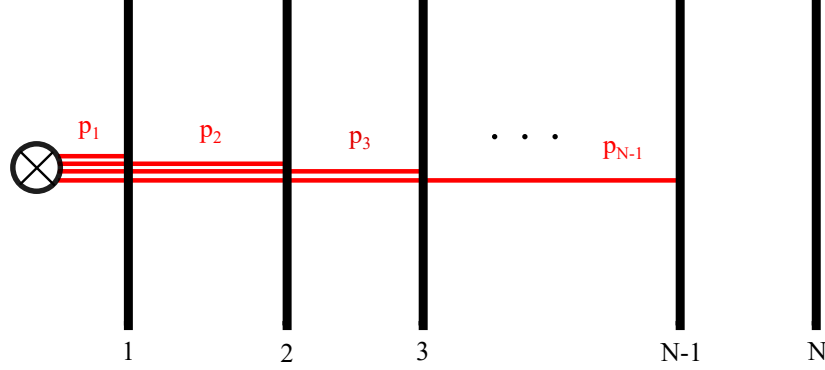


Figure 5.5: In this figure we show how to construct the string theory embedding of a $SU(N)$ irreducible 't Hooft operator of charge $P = \sum_I p_I h^I$.

control the lift of the $U(1)_K$ action and NUT charge separately and distinguish between the reducible and irreducible cases. This will allow us to give a complete description of the instanton configuration and its T-dual brane configuration for the case of generic NUT charge and $U(1)_K$ -action.

In summary, we will find that in a particular Hanany-Witten frame, an irreducible singular $SU(N)$ monopole at $\vec{x}_n \in \mathbb{R}^3$ with 't Hooft charge

$$P = \sum_I p_I h^I , \quad (5.70)$$

will be given by a single NS5-brane connected to the $(I + 1)^{th}$ D3-brane in a stack of N D3-branes by p_I D1-branes as in Figure 5.5.

5.2.1 $SU(2)$ Irreducible 't Hooft Defects

First let us consider the case of a single irreducible 't Hooft defect at the origin in $SU(2)$ $\mathcal{N} = 2$ SYM theory with 't Hooft charge, relative magnetic charge, and Higgs vev given by

$$P = p h^1 \quad , \quad \tilde{\gamma}_m = m H_1 \quad , \quad X_\infty = v H_1 . \quad (5.71)$$

By Kronheimer's correspondence this is dual to a $U(1)_K$ -invariant instanton on

Taub-NUT where the lift of the $U(1)_K$ action to the gauge bundle is given by

$$\lim_{\vec{x} \rightarrow \vec{0}} g(\vec{x}; \alpha) = e^{i p h^1 \alpha} \quad , \quad \alpha \in U(1)_K \quad , \quad (5.72)$$

the first Chern class of the instanton bundle is given by $(mH_1 - ph^1)$, and the Higgs vev is given in terms of the holonomy of the gauge field around the circle at infinity

$$\exp \left\{ \frac{1}{2\pi} \oint_{S_\infty^1} \hat{A} \right\} = \exp \left\{ \frac{X_\infty}{2\pi R'} \right\} \quad , \quad (5.73)$$

where $R' = 1/R$ is the dual radius of S_∞^1 . As in Section 2.3.3 we can locally write the connection \hat{A} as

$$\hat{A} = A + \psi(x)(d\xi + \omega) \quad , \quad (5.74)$$

such that

$$dA = *_3 d(V\psi) \quad , \quad \lim_{r \rightarrow 0} V(x)\psi(x) = -\frac{P}{2r} \quad , \quad \lim_{r \rightarrow \infty} V(x)\psi(x) = X_\infty - \frac{\gamma m}{2r} \quad . \quad (5.75)$$

Again, consider embedding this configuration of $U(1)_K$ -invariant instantons into string theory by wrapping a pair of D4-branes on Taub-NUT in the $x^{1,2,3,4}$ -directions (localized at $x^{5,6,7,8,9} = 0$) with fractional D0-branes.

As before consider the Chan-Paton bundle of the D4-branes. Due to the non-trivial holonomy, this splits asymptotically as a direct sum of line bundles

$$\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2 \quad . \quad (5.76)$$

Since the \mathcal{R}_i describe an instanton background in the D4-brane world volume theory along the Taub-NUT direction, the connection of these line bundles is hyperholomorphic (the curvature is a (1,1)-form in any complex structure).

As before, on Taub-NUT there are two families of $U(1)_K$ -invariant hyperholomorphic connections

$$A_* = \frac{d\xi + \omega}{V(x)} \quad , \quad A_{\vec{x}_i} = -\frac{d\xi + \omega}{2|\vec{x} - \vec{x}_i|V(x)} + \frac{1}{2}\omega_i \quad , \quad (5.77)$$

where ω_i is the Dirac potential centered at \vec{x}_i which solves

$$d\omega_i = *_3 d \left(\frac{1}{2|\vec{x} - \vec{x}_i|} \right) \quad . \quad (5.78)$$

Again we can define a line bundle with connection $\Lambda_*^{(t)} = t\Lambda_*$ which is asymptotically flat and has non-trivial holonomy at infinity

$$e^{i \oint_{S^1_\infty} \Lambda_*^{(t)}} = e^{2\pi i t} , \quad (5.79)$$

while $\Lambda_{\vec{x}_i}$ sources a non-trivial first Chern class centered around \vec{x}_i .

Now since there is a nontrivial Higgs vev X_∞ , the connection \hat{A} has nontrivial holonomy and hence asymptotically decomposes into two connections \hat{A}_i on the \mathcal{R}_i factors of the Chan-Paton bundle respectively. This can be written

$$\hat{A}_a = \begin{cases} A + \psi(x)(d\xi + \omega) & a = 1 \\ -A - \psi(x)(d\xi + \omega) & a = 2 \end{cases} \quad (5.80)$$

such that

$$dA = *_3 d(V\psi) \quad , \quad \lim_{r \rightarrow 0} V(x)\psi(x) = -\frac{p}{4r} \quad , \quad \lim_{r \rightarrow \infty} V(x)\psi(x) = v - \frac{m}{2r} . \quad (5.81)$$

Using this, we can write down the connections \hat{A}_a in terms of the $\Lambda_{\vec{x}_i}, \Lambda_*$ as

$$\hat{A}_1 = \frac{s_1}{2\pi R} \Lambda_* - p\Lambda_0 + \sum_i \Lambda_{\vec{x}_i} \quad , \quad \hat{A}_2 = \frac{s_2}{2\pi R} \Lambda_* - \sum_i \Lambda_{\vec{x}_i} , \quad (5.82)$$

in a certain choice of gauge where $s_1 - s_2 = v$. This gives rise to the decomposition of the Chan-Paton bundles as

$$\mathcal{R}_1 = \mathcal{L}_*^{s_1/2\pi R} \otimes \mathcal{L}_0^p \bigotimes_{i=1}^m \mathcal{L}_{\vec{x}_i}^{-1} \quad , \quad \mathcal{R}_2 = \mathcal{L}_*^{s_2/2\pi R} \bigotimes_{i=1}^m \mathcal{L}_{\vec{x}_i} , \quad (5.83)$$

where as before $\mathcal{L}_*^{(t)}$ is the line bundle with connection $t\Lambda_*$, $\mathcal{L}_{\vec{x}_i}$ is the line bundle with connection that is gauge equivalent to $\Lambda_{\vec{x}_i}$, and we have taken the positions of the monopoles to be at $\{\vec{x}_i\}$. Here we used the fact that flat gauge transformations of the B -field, $B \rightarrow B + d\Lambda$ act on the Chan-Paton bundle as [177]

$$\mathcal{R} \mapsto \mathcal{R} \otimes \mathcal{L}_\Lambda , \quad (5.84)$$

to make a choice of gauge such that $0 < s_1 < s_2 < 2\pi R$ and \mathcal{L}_0 only appears in \mathcal{R}_1 with integer power.

Now we wish to T-dualize this configuration along the S^1 fiber of Taub-NUT. Following the identification from the previous section, we can see that this configuration

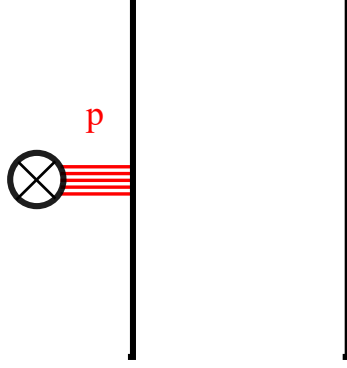


Figure 5.6: This figure shows the brane configuration for a reducible $SU(2)$ 't Hooft defect with charge $P = p h^1$.

will be T-dual to the brane configuration in Figure 5.6. This brane configuration is described by a pair of D3-branes localized at $x^{5,6,7,8,9} = 0$ and at definite values of $x_1^4, x_2^4 > 0$ so that $\Delta x^4 = v$ with an NS5-brane localized at $x^4 = 0$ and $x^{1,2,3} = 0$. There are then m D1-branes running between the D3-branes localized at positions $\vec{x}_n \in \mathbb{R}^3$ and p D1-branes connecting the NS5- and the D3₁-brane. These D1-branes emanating from the NS5-brane and ending on the D3-brane source a local magnetic charge which we identify with the 't Hooft defect. We will describe the 't Hooft charge P as specified by this configuration shortly.

5.2.2 $SU(N)$ Irreducible Monopoles

This story has a clear and straightforward generalization to the case of irreducible singular monopoles in an $SU(N)$ theory. Consider a single irreducible monopole configuration with 't Hooft charge, relative magnetic charge, and Higgs vev

$$P = \sum_I p_I h^I \quad , \quad \tilde{\gamma}_m = \sum_I m^I H_I \quad , \quad X_\infty = \sum_I v^I H_I \quad . \quad (5.85)$$

By Kronheimer's correspondence, this can be described by $U(1)_K$ -invariant instantons on Taub-NUT where the lift of the $U(1)_K$ action to the gauge bundle is given by

$$\lim_{\vec{x} \rightarrow \vec{0}} g(\vec{x}; \alpha) = e^{iP\alpha} \quad , \quad \alpha \in U(1)_K \quad , \quad (5.86)$$

the first Chern class is given by $\gamma_m = \tilde{\gamma}_m - P^-$. Again the holonomy of the gauge field around the S^1 fiber at infinity is dictated by the Higgs vev

$$\exp \left\{ \frac{1}{2\pi} \oint_{S_\infty^1} \hat{A} \right\} = \exp \left\{ \frac{X_\infty}{2\pi R'} \right\} , \quad (5.87)$$

where $R' = 1/R$ is the radius of S_∞^1 . Now embed this configuration into string theory by wrapping N D4-branes on Taub-NUT along the $x^{1,2,3,4}$ directions (that is they are localized at $x^{5,6,7,8,9} = 0$) with fractional D0-branes.

Now the Chan-Paton bundle of the D4-branes is a rank N bundle which asymptotically splits as the direct sum of line bundles:

$$\mathcal{R} = \bigoplus_{I=1}^N \mathcal{R}_I . \quad (5.88)$$

Again, the Chan-Paton bundles must decompose as a tensor product of line bundles with connections of the form $A_*^{(t)}$ and $A_{\vec{x}_i}$:

$$\mathcal{R}_I = \mathcal{L}_*^{s_I/2\pi R} \otimes \mathcal{L}_0^{p_I} \bigotimes_{n_I=1}^{m^I} \mathcal{L}_{\vec{x}_{n_I}} \bigotimes_{n_{I-1}=1}^{m^{I-1}} \mathcal{L}_{\vec{x}_{n_{I-1}}}^{-1} , \quad (5.89)$$

where $\{n_I\}$ indexes over the smooth monopoles with charge along H_I , $p_N = m^0 = 0$, and $0 < s_I < s_{I+1}$. Notice here that we have completely gauge fixed the B -field to a choice which is very convenient for matching to physical data.

T-dualizing this configuration will produce a configuration of D1/D3/NS5-branes as in Figure 5.5. In words, it will have a stack of N D3-branes separated at points $x_{i+1}^4 > x_i^4 > 0$ such that $x_{I+1}^4 - x_I^4 = v^I$,¹⁶ localized at $x^{5,6,7,8,9} = 0$ with a single NS5-brane localized at $x^4 = 0$ and at the origin in \mathbb{R}^3 . There will also be m^I D1-branes stretching from the D3 _{I} - to the D3 _{$I+1$} -brane and p_I D1-branes stretching from the NS5-brane to the D3 _{I} -brane. Again, the D1-branes emanating from the NS5-brane that end on the D3 _{I} -brane will source a local magnetic charge in the world volume theory of the D3-branes.

¹⁶See Footnote 14.

5.2.3 Physical 't Hooft Charges

This construction of singular monopoles is similar to that of [132] in the sense that they both introduce a Dirac monopole by having D1-branes in a way that couples to the center of mass of the stack of D3-branes which we have already projected out in going from a $U(N) \rightarrow SU(N)$ gauge theory. Thus, we also need to project out the part of the physical charges that couple to this center of mass degree of freedom. We take the natural projection map, given by:

$$\Pi(h) = h - (\text{Tr}_N h) \cdot \mathbb{1}_N , \quad (5.90)$$

for h an element of the Cartan subalgebra $h \in \mathfrak{t}$.

Now let us consider some example brane configurations to show that the 't Hooft charges match the field configurations we claim to describe.

Example 1 Consider again the case of $SU(2)$ singular monopoles as in the previous subsection. In this case, the brane configuration is described by the $U(2)$ 't Hooft charge

$$\tilde{P} = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} . \quad (5.91)$$

Under the projection map $\Pi : \mathfrak{u}(N) \rightarrow \mathfrak{su}(N)$, the 't Hooft charge becomes

$$P = \Pi(\tilde{P}) = \frac{1}{2} \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} = p h^1 . \quad (5.92)$$

This is exactly the charge of the field theory configuration (5.71).

Example 2 Now consider the case of singular monopoles in $SU(N)$ gauge theory. As in the previous subsection, take the brane configuration of Figure 5.6. This is described as follows.

Consider a stack of N D3-branes localized at $x^{5,6,7,8,9} = 0$ and at distinct values in the x^4 -direction which we will give an ordering from left to right. Now consider a *single* NS5-brane localized to the left of all of the D3-branes in the x^4 -direction and localized at $\vec{x}_n \in \mathbb{R}^3$. Now add p_I D1-branes which run from the NS5-brane to the $D3_I$ -brane

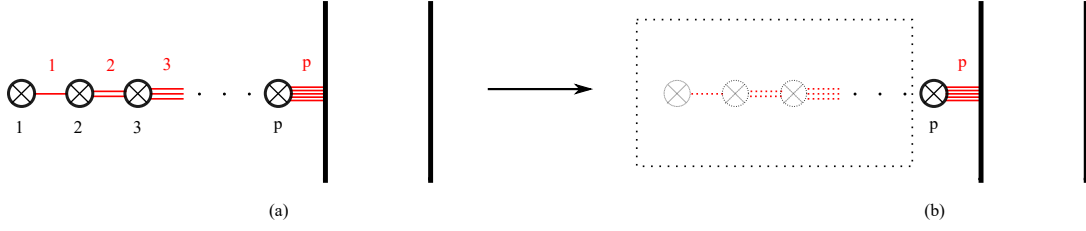


Figure 5.7: In this figure we show how to relate the $(L_{[h^1,0]})^p$ reducible 't Hooft operator (a) to the $L_{[P,0]}$ irreducible 't Hooft operator (b). This suggests that the tail of the reducible singular monopole configuration is analogous to the subleading terms in the OPE.

for $I \neq N$. This configuration will have a $U(N)$ 't Hooft charge

$$\tilde{P} = p_I \sum_{J=1}^I e_{J,J} , \quad (5.93)$$

where $e_{I,J}$ is the diagonal matrix with a single 1 in the (I, J) -component. Under the projection to $SU(N)$, this becomes

$$P = \Pi(\tilde{P}) = p_I \left(\sum_{J=1}^I e_{J,J} - \frac{1}{2} \mathbb{1}_N \right) = p_I h^I . \quad (5.94)$$

This matches the charge of the corresponding field configuration in (5.85).

Remark It is also interesting to note that heuristically one can think of irreducible 't Hooft defects as reducible defects where we have removed the “subleading terms” from the OPE. There is similarly a geometric interpretation to this procedure in terms of the brane construction. If we consider a reducible 't Hooft defect in $SU(2)$ SYM theory where we move all of the NS5-branes to distinct points to the left of the D3-branes, then we can think of removing the subleading terms of the OPE as removing all but the right most NS5-brane and the D1-branes connecting it to the D3-branes as in Figure 5.7.

Remark From this construction it is also clear how to insert multiple irreducible 't Hooft defects since the brane configuration only include local brane interactions. Therefore, this brane configuration can be used to describe general 't Hooft defect configurations in 4D $\mathcal{N} = 2$ supersymmetric gauge theories.

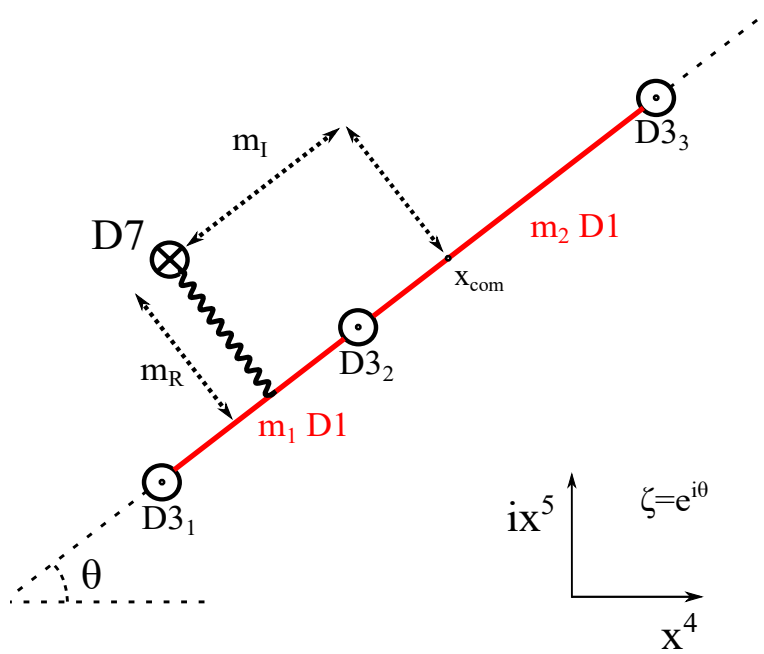
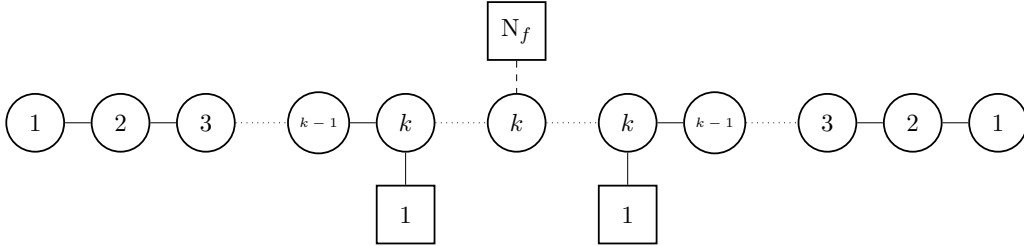


Figure 5.8: This figure illustrates the configuration of D-branes in Type IIB string theory corresponding to an $SU(3)$ gauge theory with total magnetic charge $\gamma_m = m_1 H_1 + m_2 H_2$ coupled to a single hypermultiplet in the fundamental representation with mass $\zeta^{-1}m = m_R + im_I$. Here we have identified the $\mathbb{R}_4 \oplus i\mathbb{R}_5 \cong \mathbb{C}$. Here there are m_1 D1-D7 strings which gives rise to vanilla BPS states that are charged under the flavor symmetry associated to the D7-brane.

5.3 Including Fundamental Hypermultiplets

We can also consider the case where we add fundamental hypermultiplets to the 4D $\mathcal{N} = 2$ $SU(N)$ SYM theory. Coupling to 4D fundamental matter changes the charge quantization of these theories satisfy $p_I \in 2\mathbb{Z}_+$ where $P = \sum_I p_I h^I$ for simple cocharacters p^I .¹⁷ This can be achieved in the brane configuration by adding D7-branes at fixed locations $x^4 + ix^5 = m^{(i)}$ where i indexes the fundamental hypermultiplets and $m^{(i)}$ is the complex mass of the corresponding fundamental hypermultiplet. This picture can be used to geometrically determine the spectrum of vanilla BPS states and its wall crossing as in [22]. See Figure 5.8.

As shown in [165], this couples the quiver SQM describing the low energy effective theory of the D1-branes to a short $\mathcal{N} = (0, 4)$ fundamental Fermi-multiplet. The quiver gauge theories are then of the form

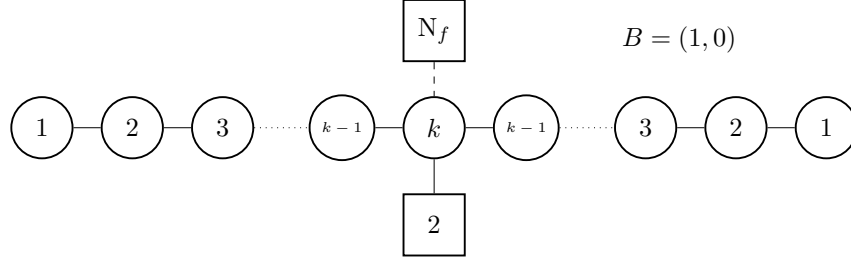


where the length of the quiver is $n - 1$ with k occuring $n - 2k + 1$ times where

$$P = n \hat{h}^1 \quad , \quad P - \vec{v} = k H_1 \quad , \quad n \in 2\mathbb{Z}_+ \quad , \quad \hat{h}^1 \in \Lambda_{cochar} \quad . \quad (5.95)$$

and the N_f fundamental Fermi-multiplets are coupled to the $(n/2)^{th}$ gauge node. Additionally, when $n = 2k$, $\Gamma(P, \vec{v})$ takes the special form

¹⁷This is a consequence of the fact that in a theory with fundamental matter $\Lambda_{mw}/\Lambda_{cochar} = \mathbb{Z}_2$. Hence, we will have $2p_I = n_I$ in these theories where the 't Hooft charge can be written as $P = \sum_I p_I h^I$ or $P = \sum_I n_I \hat{h}^I$ where $h^I \in \Lambda_{mw}$ and $\hat{h}^I \in \Lambda_{cochar}$.



See Figure 5.9 for examples.

5.3.1 Fundamental Hypermultiplets and Brane Webs

When we consider framed BPS states in the presence of fundamental hypermultiplets there are also interactions between NS5-branes and D7-branes in the full string theory [9]. Specifically, D7-branes are distinguished in type IIB string theory in that they are sources for the axio-dilaton. Thus, the holonomy around a D7-brane is exactly equivalent to an S-duality transformation $T \in SL(2; \mathbb{Z})$. This means that picking a fixed S-duality frame requires a specification of branch cuts in the transverse space. For a generic choice of branch cuts they will intersect the NS5-branes causing them to undergo an S-duality transformation.

To study the implications of this branch cut, we will have to first review some technology of (p, q) -Brane webs.

Brief Review on (p, q) -Brane Webs

A (p, q) -brane is a certain type of 5-brane in type IIB string theory that has p -units of NS-charge and q -units of RR-charge. This means that we can identify a $(1, 0)$ -brane with an NS5-brane and a $(0, 1)$ -brane with a D5-brane. As one would expect, the charge vector (p, q) of a (p, q) 5-brane transforms as a vector under the S-duality group $SL(2; \mathbb{Z})$. Further a tension of such a 5-brane is given by

$$T_{(p,q)} = |p + \tau q| T_{D5} , \quad (5.96)$$

where τ is the expectation value of the axio-dilaton and T_{D5} is the D5-brane tension.

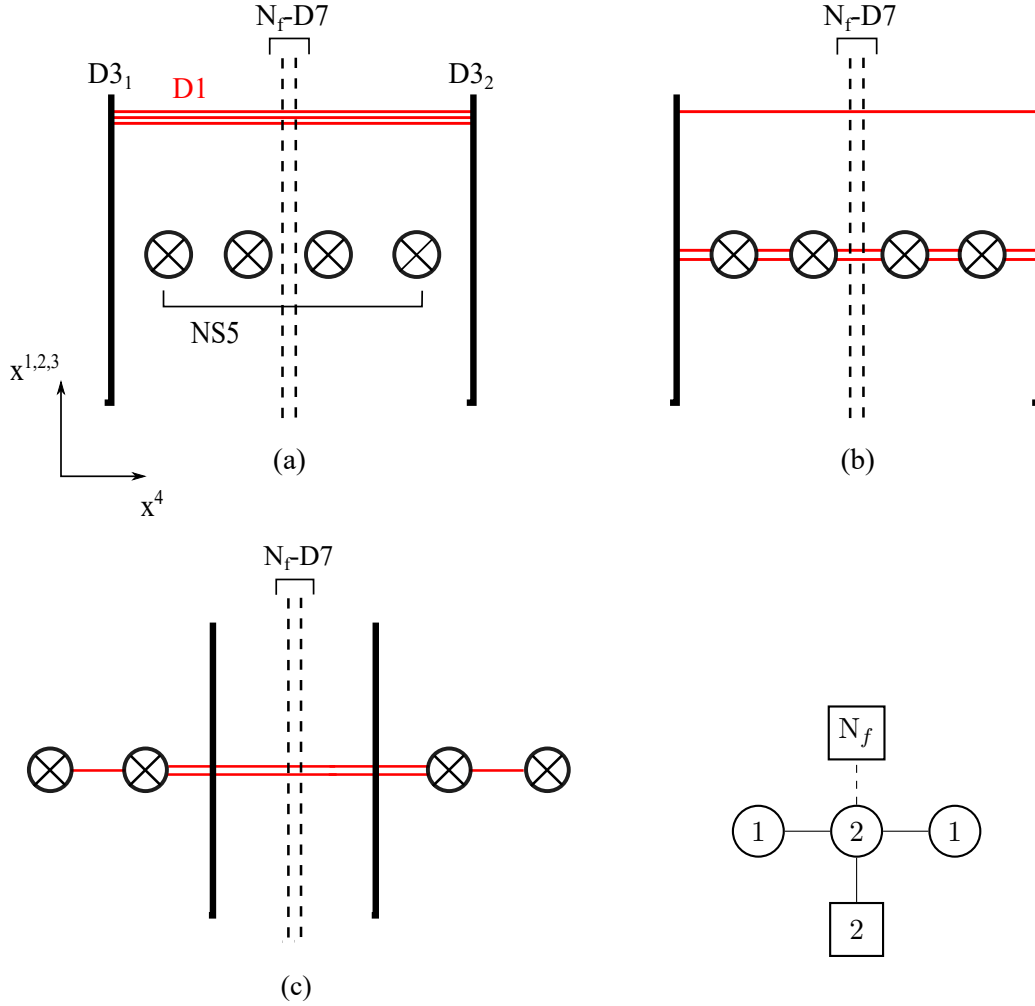


Figure 5.9: This figure shows many facets of the brane configuration describing singular monopoles and monopole bubbling in 4D $\mathcal{N} = 2$ gauge theory with N_f fundamental hypermultiplets for (a) the example of $SU(2)$ gauge theory with $\gamma_m = 3H_1$ and $P = 4h^1$. (b) displays an example of monopole bubbling where 2 monopoles have bubbled, screening the defect. By performing the Hanany-Witten transformations (c), we can see that the SQM living on the D1-branes is given by a quiver SQM (d).

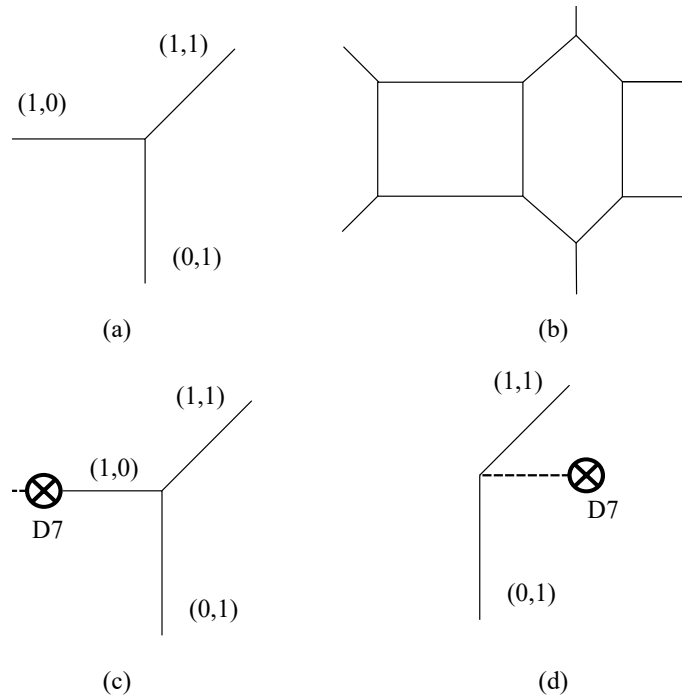


Figure 5.10: This figure shows two examples of (p, q) -brane webs in (a) and (b). (a) is the fundamental trivalent junction including an NS5- and D5-brane. (b) is an example of a generic (p, q) 5-brane web. Additionally, this figure shows in (c) and (d) how D7-branes can be combined with (p, q) 5-brane webs. (c) the brane web can end on them or (d) they can act on the brane web via the S-duality branch cut. These two are related by a Hanany-Witten-type transformation where the D7-brane is pulled through the 5-branes.

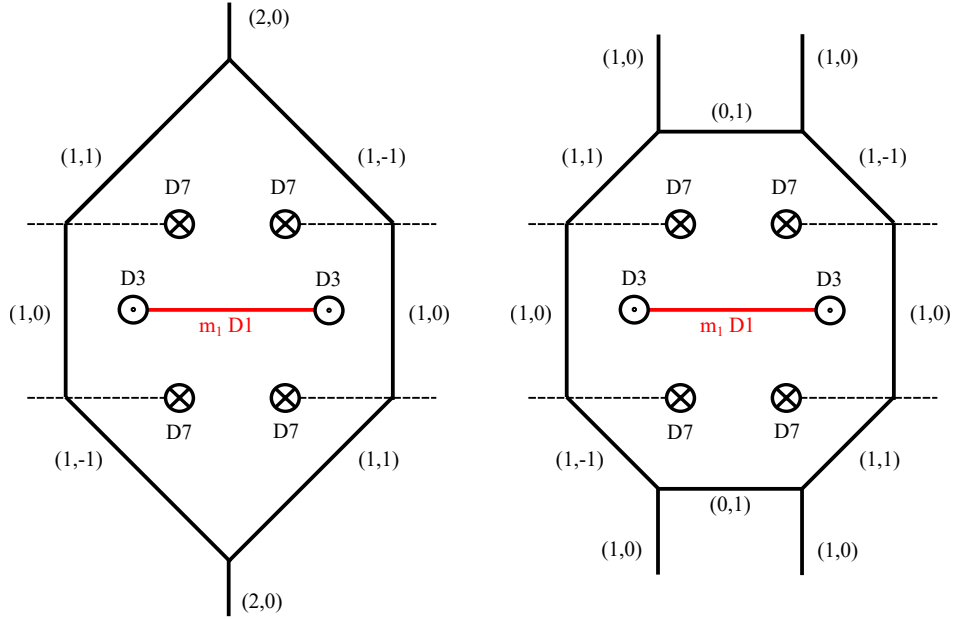


Figure 5.11: This figure shows the example of the improved brane configuration for the fundamental 't Hooft defect in $SU(2)$ $N_f = 4$ gauge theory. On the right is the resolved brane configuration suggested by [9].

Such (p, q) 5-branes can intersect in interesting ways to form a sort of web by taking their world volume to span the $x^{0,1,2,3,4}$ -directions and wrap straight lines in the $x^5 + ix^6$ \mathbb{C} -plane. See Figure 5.10.

(p, q) -brane webs can be realized as the T-dual of M-theory compactified on a Calabi-Yau 3-fold with toric singularities [114]. More straightforwardly, they can be fundamentally constructed from trivalent brane intersections. Charge conservation implies that for any trivalent vertex of (p, q) -branes with charges (p_i, q_i) , that the charges satisfy

$$\sum_{i=1}^3 p_i = \sum_{i=1}^3 q_i = 0 . \quad (5.97)$$

Preserving supersymmetry implies that up to an overall rotation, a (p, q) -brane must have a slope given by [1]

$$\Delta x^4 + i \Delta x^5 \parallel p + \tau q . \quad (5.98)$$

In the semiclassical limit we take $\tau \approx i$ so that D5- and NS5-branes are essentially perpendicular and a (p, q) brane has slope in the $x^4 + ix^5$ plane given by $m = q/p$.

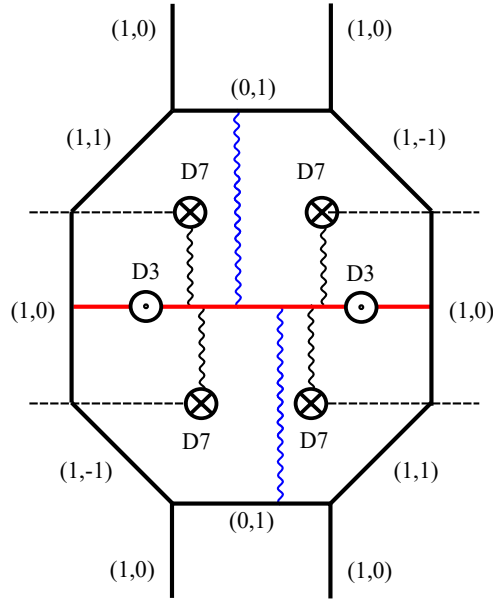


Figure 5.12: This figure shows the brane configuration for the bubbling locus of the $L_{1,0}$ 't Hooft defect in $SU(2)$ $N_f = 4$ gauge theory. Here the red line is a D1-brane, the black wavy lines are D1-D7 strings which give rise to fundamental Fermi multiplets, and the blue wavy lines are the D1-D5 strings that give rise to additional fundamental hypermultiplets.

Improved Brane Configuration

Now let us return to the D1/D3/NS5-brane configuration with additional D7-branes added in. It is pointed out in [9] that there is a non-trivial interaction between the NS5-branes that give rise to the 't Hooft defect and the D7-branes that give rise to the fundamental hypermultiplets. This implies that 5-brane webs are needed to fully realize monopole bubbling when coupling to fundamental hypermultiplets. The reason is that when the NS5-branes intersect the branch cuts from the D7-branes that they undergo an S-duality transformation to become a $(1, \pm 1)$ -brane. See Figure 5.10.

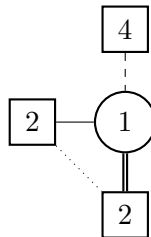
Preserving SUSY then implies that these transformed NS5-branes bend so that their world volume is along a sloped line in the $x^4 + ix^5$ plane. This means that the NS5-branes are no longer asymptotically parallel but rather intersect at some point along the x^5 -direction. This gives rise to the brane configuration on the left of Figure 5.11.

It is then shown in [9], that the resulting vertex in the brane-web configuration can

be “resolved” via a D5-brane as in the right side of Figure 5.11. However, by resolving the vertex of the 5-brane web via a D5-brane, one allows for D3-D5 strings that give rise to a Fermi-multiplet that is localized on the world volume of the ’t Hooft defect. This Fermi multiplet is coupled to the 4D gauge field and Higgs field of the $\mathcal{N} = 2$ vector multiplet in the fundamental representation. This Fermi-multiplet can be thought of as a spin defect field that gives rise to a Wilson line as shown in [165]. Thus, resolving the D5-brane interaction adds electric charge to the ’t Hooft defect. This will be important in the next section for computing the expectation value of ’t hooft defect operators.

Example Consider the example of the minimal ’t Hooft defect $L_{1,0}$ in $SU(2)$ $\mathcal{N} = 2$ gauge theory with $N_f = 4$ fundamental hypermultiplets. This can be described by a brane configuration realized by two parallel D3-branes, 4 D7-branes and 2 NS5-branes. Introducing the D7-branes requires $N_f = 4$ branch cuts which intersect the NS5-branes. The NS5-brane intersections can then be resolved as in [9] by introducing D5-branes such that there the D1-, D3-, and D7-branes are contained in an octagon.

The bubbling SQM can be read by going to the bubbling locus where we send a D1-brane to run between the NS5-branes as in Figure 5.12. The SQM is again a $\mathcal{N} = (0, 4)$ theory that is described by the quiver:



5.4 The Class \mathcal{S} Construction

Another brane configuration that can be used to study 4D $\mathcal{N} = 2$ gauge theories is called the class \mathcal{S} construction. This brane configuration goes back to the work of [155, 156, 176, 66, 68], explaining the geometric origin of Seiberg-Witten theory. There has been a great deal of technology for understanding 4D $\mathcal{N} = 2$ theories that have been developed via the class \mathcal{S} construction. In fact, this technology will be fundamental to

understanding our later chapters on localization and comparing the expectation values of line defects.

5.4.1 The 6D $\mathcal{N} = (2, 0)$ Theory

6D theories with $\mathcal{N} = (2, 0)$ supersymmetry have $\mathfrak{osp}(6, 2|4)$ superconformal symmetry which has a bosonic subgroup $SO(5, 1) \times Spin(5)_R$ of Lorentz and R -symmetry [138]. The field content is that of tensor multiplets which are comprised of a self-dual 2-tensor $B_{\mu\nu}$, 4 fermions Ψ^a , and 5 scalar fields φ^i where $a = 1, \dots, 4$ and $i = 1, \dots, 5$. As representations of $SO(5, 1) \times Spin(5)$, the fields transform as

$$B_{\mu\nu} : (\mathbf{3}, 1; 1) \quad , \quad \psi^a : (\mathbf{2}, 1; \mathbf{4}) \quad , \quad \varphi^i : (1, 1; \mathbf{5}) \quad . \quad (5.99)$$

The 6D $\mathcal{N} = (2, 0)$ theory can be constructed from string theory by compactifying type IIB on a K3-manifold [175].¹⁸ Consequently a corresponding 6D $\mathcal{N} = (2, 0)$ theory is labeled by an ADE Lie algebra \mathfrak{g} corresponding to the singularity structure of the K3-manifold.

In this construction, D3-branes that wrap non-trivial 2-cycles of $K3$ give rise to strings in the 6D theory that are charged under the B -field. Thus, a 6D $\mathcal{N} = (2, 0)$ theory of type \mathfrak{g} and has a spectrum of charged objects classified by elements of the integer lattice $\Lambda = \Lambda_{wt}(\mathfrak{g})$. The tension of the strings is proportional to the volume of the 2-sphere [175]. At singular point in K3 moduli space, where the non-trivial 2-cycles degenerate, the tension of these strings is identically zero. These are the “tensionless strings” in the $(2, 0)$ theory.

There is also another construction of the 6D $\mathcal{N} = (2, 0)$ theory of type $\mathfrak{g} = A_n$ as the low energy effective theory of a stack of M5-branes wrapped on $M_6 \subset X_{11}$. Here the $Spin(5)_R$ symmetry is manifest as symmetry group of the transverse 5-directions and allows us to identify the normal bundle of the world volume of M5-branes with the R -symmetry bundle in 6D $R \rightarrow M_6$.

In this construction, the B -field can be seen as the boundary field that trivializes the bulk 3-form gauge field C . When the M5-branes wrap a topologically trivial manifold

¹⁸This requires some subtlety with decoupling gravity. See [157] for more details.

M_6 , the B -field arises as the trivialization of the bulk C -field restricted to the world volume of the M5-branes M_6 .¹⁹

$$dB = C|_{M_6} . \quad (5.101)$$

The scalar fields of the 6D $\mathcal{N} = (2, 0)$ describe the transverse fluctuations of the M5-branes in 11D space time. Because the M5-branes are indistinguishable, there is an S_n exchange symmetry when they are coincident. This acts as the Weyl group on the lattice of charged objects. Thus, the 6D $\mathcal{N} = (2, 0)$ theory has a moduli space of supersymmetric vacua (sometimes called the tensor branch or Coulomb branch) that is parametrized by the independent Weyl-invariant operators constructed out of the vevs of scalar fields

$$\mathcal{M} = \mathbb{R}^{5r}/W . \quad (5.102)$$

Since the strings of the 6D $\mathcal{N} = (2, 0)$ theory are sources for the B -field, the fact that it descends from the bulk C -field implies that the strings arise from the ends of M2-branes that between M5-branes. These strings have a tension given by the separation between M5-branes which in the 6D theory is proportional to the vev of the scalar fields. Thus, at the singularity of the tensor branch where the M5-branes are all coincident the strings are “tensionless”.

In the M-theory construction of the 6D $\mathcal{N} = (2, 0)$ theory of type A_n , compactifying the stack of M5-branes on a circle then produces a stack of D4-branes in type IIA. These branes have a world volume theory is $U(N)$ maximal supersymmetric Yang-Mills theory (MSYM). Thus, we can conclude that the compactification the 6D $\mathcal{N} = (2, 0)$ theory is described by 5D non-abelian MSYM.

¹⁹Defining the precise relation between the bulk C -field and the B -field on the world volume of the M5-branes requires the use of differential cohomology.²⁰ Let us take the world volume of the M5-branes to be M_6 and the 11-dimensional spacetime to be X_{11} which locally forms a non-trivial \mathbb{R}^5 bundle over M_6 which we can identify with the normal bundle of M_6 . We can then identify the normal bundle with the R -symmetry bundle $R \rightarrow M_6$. From this we can construct the associated sphere bundle $S(R) \rightarrow M_6$. As shown in [135] the curvature of the B -field H can be explicitly written as

$$d\check{H} = \frac{1}{2}\pi_*(\check{C}_{S(R)} \cup \check{C}_{S(R)}) . \quad (5.100)$$

Here \check{H} is the lift of H to a differential cocycle and \check{C} is the restriction of the lift of C to a differential cocycle that is further restricted to the sphere bundle $S(R)$. The map π_* is then integration over the fibers of the map $\pi : S(R) \rightarrow M_6$. This shifts the cocycle $\check{C}_{S(R)}$ by a representative of the Wu-class $\lambda_{S(R)} = \frac{1}{4}p_1(TS(R))$. See [135, 136] for more details.

5.4.2 The Class \mathcal{S} Construction of $SU(N)$ Theories

Let us take spacetime to be $\mathbb{R}^7 \times Q$ where Q is a hyperkähler four-manifold. As is convention, we will take the flat directions to be the $x^{0,1,2,3,7,8,9}$ -directions and Q is spanned by the coordinates $x^{4,5,6,10}$. Let us endow Q with a complex structure where $v = x^4 + ix^5$ and $s = x^6 + ix^{10}$ are holomorphic and let $\Sigma \subset Q$ be a complex Riemann surface in Q . Presently we compactify the x^{10} -direction, so we will more generally define the complex coordinate $t = e^{-s} = e^{-(x^6 + ix^{10})/R}$.

Theories of class \mathcal{S} with 4D gauge group $G = \prod_{i=1}^r SU(N_i)$ can be described by compactifying the 6D $\mathcal{N} = (2,0)$ type A_N theory onto a Riemann surface C . Equivalently we can consider wrapping a single M5-brane on an N -branched cover $\Sigma \rightarrow C$ which we take to be defined by a polynomial:

$$F(t, v) = \prod_{i=1}^r (t - f_i(v)) , \quad (5.103)$$

where each $f_i(v)$ is a polynomial of order N_i .

Upon compactification along the x^{10} direction, we produce the type IIA brane configuration from [176] where there are r parallel NS5-branes localized $x^{7,8,9} = 0$ and certain values of x^6 (encoding the gauge coupling of the 4D theory) and N_i parallel D4-branes running between the NS5 $_i$ and NS5 $_{i+1}$ -branes that are localized at $x^{7,8,9} = 0$ and specific values of $x^{4,5}$ (encoding the Higgs vev). Specifically, if we hold v fixed and vary t , we will see that the roots are exactly the positions of the NS5-branes. Similarly if we hold t fixed and vary v , the roots will give us the position of the D4-branes. Here the $SU(N_i)$ gauge theory lives on the world volume theory of the stack of D4-branes stretched between the NS5 $_i$ and NS5 $_{i+1}$ -brane. See Figure 5.13.

So let us consider the 6D $\mathcal{N} = (2,0)$ theory of type A_{N-1} compactified on the Riemann surface C (with punctures). Since Riemann surfaces generally have a holonomy group containing $SO(2)$, we can construct a 4-dimensional $\mathcal{N} = 2$ theory by topologically twisting the compactified theory. In the 6-dimensional theory, the superconformal algebra is given by $\mathfrak{osp}(6, 2|4)$ which has bosonic part $\mathfrak{so}(5, 1) \oplus \mathfrak{so}(5)_R$. Under this

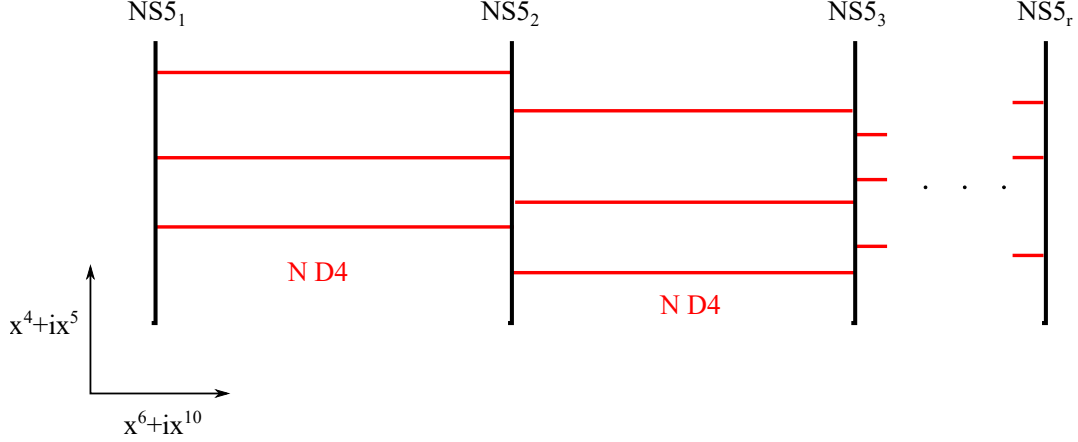


Figure 5.13: This is a figure of a system of D4- (red) and NS5- (black) branes as considered by Witten [176]. This brane construction arises as the compactification of an M5-brane wrapped on a Riemann surface in the class \mathcal{S} construction and describes a 4D $\mathcal{N} = 2$ quiver gauge theory with gauge group $SU(N)^r$.

compactification the algebra decomposes as

$$\mathfrak{so}(1, 5) \oplus \mathfrak{so}(5)_R \rightarrow \mathfrak{so}(1, 3) \oplus \mathfrak{so}(2)_C \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)_R , \quad (5.104)$$

which we can further twist by projecting to the diagonal component of $\mathfrak{so}(2)_C \oplus \mathfrak{so}(2)_R \rightarrow \mathfrak{so}(2)'_C$:

$$\mathfrak{so}(1, 3) \oplus \mathfrak{so}(2)_C \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)_R \rightarrow \mathfrak{so}(1, 3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2)'_{C'} . \quad (5.105)$$

Under this process, the 6-dimensional supercharges decompose:

$$(4, 4) \rightarrow (2, 1; 2)_{+1} + (1, 2; 2)_0 + (2, 1; 2)_0 + (1, 2; 2)_{-1} . \quad (5.106)$$

Under compactification, only the terms $(1, 2; 2)_0$ and $(2, 1; 2)_0$ survive, thus producing a 4D theory with $\mathcal{N} = 2$ SUSY.

The theory on such a Riemann surface can be understood more generally as follows. First consider a pants decomposition of such a theory given by a series of cuts $\{\gamma_i\}_{i=1}^{3g-3+n}$ that are not homotopic to the boundary components. For each cut γ_i there is an associated $SU(N)$ factor of the gauge group in 4D. Additionally, to each puncture, there is an associated hypermultiplet in 4D with representation specified by additional data at the puncture. See [162] for a full review of this identification.

The curve defined by (5.103) determines the entire brane configuration and in fact corresponds to the Seiberg-Witten curve that we discussed in Chapter 2 [176]. The corresponding Seiberg-Witten differential is given by the trivialization of the holomorphic symplectic form $\Omega = dx \wedge dz$ restricted to Σ :

$$\lambda = x \wedge dz . \quad (5.107)$$

In this setting, the closed one-dimensional submanifolds corresponding to BPS states are M2-branes wrapping non-trivial cycles in Σ as it stretches between different sheets of Σ .

5.4.3 't Hooft Defects in Theories of Class \mathcal{S}

The brane construction of theories of class \mathcal{S} also have a natural construction of 't Hooft defects in the resulting 4D $\mathcal{N} = 2$ theories. In such theories, a 4D line operator descends from a $\frac{1}{2}$ -SUSY surface defect/string operator in the 6D theory. Such a surface defect operator can be written in terms of the B -field and scalar field φ^i as

$$\mathcal{W}(\Sigma; n_i) = \exp \left\{ \int_{\mathcal{P} \times \mathbb{R}_t} (B + n_i \varphi^i \text{vol}(\mathcal{P} \times \mathbb{R}_t)) \right\} , \quad (5.108)$$

where Σ is the world volume of the surface defect/string operator and $n^i \in S^4$ is a vector determining the preserved SUSY. The surface defect/string operators of the 6D $\mathcal{N} = (2, 0)$ theory descend from the intersection of the M5-branes with M2-branes.

To produce a line operator in the 4D theory of class \mathcal{S} , we wrap the string operator on a 1-cycle $\mathcal{P} \subset C$ and the world volume of the line operator in 4D. Then by performing the topological twist along C and compactifying to 4D, this line operator becomes a line operator:

$$L = P_{\mathbb{R}_t}[\mathcal{W}(\mathcal{P} \times \mathbb{R}_t; n_i)] = \exp \left\{ \int_{\mathcal{P}} (B_t + n_i Y^i \text{vol}(\mathcal{P})) \right\} , \quad (5.109)$$

where $P_{\mathbb{R}_t}$ is the projection to the 4D theory. Therefore, the line defects can be labeled by $L_{\mathcal{P}}$ where \mathcal{P} is a smooth one-dimensional submanifold of C .

Isotopy classes of such submanifolds can be conveniently labeled, given a pants decomposition of C in terms of Dehn-Thurston parameters: ²¹

²¹The importance of being careful about connected components in the Dehn-Thurston theorem was

Theorem (Dehn-Thurston): [48, 164] Let C be an oriented Riemann surface with negative Euler characteristic that has genus g and n punctures. Let $\{\gamma_i\}_{i=1}^{3g-3+n}$ be a maximal set of non-intersecting curves defining a pants decomposition of C and let $\{\gamma_i\}_{i=3g-3+n+1}^{3g-3+2n}$ be a collection of simple closed curves near the punctures. There is a mapping

$$D : \mathcal{I}(C) \rightarrow \mathbb{Z}_{\geq 0}^{3g-3+2n} \times \mathbb{Z}^{3g-3+2n} , \quad (5.110)$$

$$\gamma \mapsto (\langle \gamma, \gamma_i \rangle, \vec{q})$$

where $\mathcal{I}(C)$ is the set of isotopy classes of closed one dimensional submanifolds, q_i is the twisting number with respect to γ_i , and \langle , \rangle is the intersection number. Elements in the image of D are denoted (\vec{p}, \vec{q}) and are called *Dehn-Thurston parameters*.

The choice of $\{\gamma_i\}_{i=1}^{3g-3+n}$ above correspond to a weak coupling decomposition of the UV curve C , and specifies a Lagrangian duality frame with gauge algebra $\mathfrak{su}(2)^{\oplus h}$ with $h = 3g - 3 + n$. Each curve corresponds to a weakly coupled $SU(2)$ gauge group in the 4D theory.

Now consider the line defect associated to a generic 1D submanifold $\gamma_{\vec{p}, \vec{q}}$ with Dehn-Thurston (DT) parameters $(\vec{p}, \vec{q}) = (p_1, \dots, p_h, q_1, \dots, q_h)$. This submanifold will have a set of connected components $\gamma_{\vec{p}, \vec{q}} = \bigoplus_{\alpha=1}^k \gamma_{\vec{p}, \vec{q}}^{(\alpha)}$ labeled by α , each of which has its own Dehn-Thurston parameters: $(\vec{p}^{(\alpha)}, \vec{q}^{(\alpha)}) = (p_1^{(\alpha)}, \dots, p_h^{(\alpha)}, q_1^{(\alpha)}, \dots, q_h^{(\alpha)})$. The line defect $L(\gamma_{\vec{p}, \vec{q}}, \zeta)$ then decomposes as a product of line defects

$$L_{\gamma_{\vec{p}, \vec{q}}} = \prod_{\alpha=1}^k L_{\gamma_{\vec{p}, \vec{q}}^{(\alpha)}} , \quad (5.111)$$

In [58] it is conjectured that the line defects $L(\mathcal{P})$ are the same as the 't Hooft-Wilson line defects of the Lagrangian theory with gauge algebra $\mathfrak{su}(2)^{\oplus h}$. Moreover, it is proposed that the Dehn-Thurston parameters should be identified with the 't Hooft-Wilson parameters characterizing the magnetic and electric charges. This cannot be

first made clear to us in joint work with Anindya Dey while checking predictions of S-duality in class S theories of type A_1 .

true in general, but it seems highly plausible for those Dehn-Thurston parameters that correspond to one-dimensional submanifolds $\gamma_{\vec{p},\vec{q}}$ with only one connected component. In this case the proposal of Drukker-Morrison-Okuda is that $L(\mathcal{P})$ corresponds to the 4D line operator $L_{[P^{(i)},Q^{(i)}]}$ which has 't Hooft-Wilson charges

$$P = \bigoplus_{j=1}^h p_j h^{I(j)} \quad , \quad Q = \bigoplus_{j=1}^h q_j \lambda^{I(j)} \quad , \quad (5.112)$$

where $h^{I(j)}$ is the simple magnetic weight, $\lambda^{I(j)}$ is the simple weight of the j^{th} factor of the gauge group, and $h = 3g - 3 + n$. It should be stressed that some more work is needed to make use of this conjecture: In mathematics *it is not known what conditions one should put on the Dehn-Thurston parameters (\vec{p}, \vec{q}) in order for $\gamma_{\vec{p},\vec{q}}$ to have a single connected component!* The only case where this is known is the once-punctured torus (corresponding to the $G = SU(2)$ $\mathcal{N} = 2^*$ theory) and the four-punctured sphere (corresponding to the $G = SU(2)$ $N_f = 4$ theory) [121]. In that case there are only a pair of DT parameters (p, q) and $\gamma_{(p,q)}$ has g connected components, where g is the greatest common denominator of p and q .

For example, in the case where the four-dimensional gauge group is $G = SU(2)$ we have only a pair of DT parameters (p, q) . Here the minimally charged 't Hooft defect corresponds to the line with DT parameters $(1, 0)$

$$L_{\gamma_{(1,0)}} = L_{[h^1,0]} \quad , \quad (5.113)$$

which can be identified with the highest weight representation R_{h^1} of $SU(2)^\vee$. Following the decomposition above, a line defect corresponding to DT parameters $(p, 0)$ is the p^{th} power of the simple 't Hooft defect

$$L_{\gamma_{(p,0)}} = (L_{[h^1,0]})^p \quad . \quad (5.114)$$

Thus, we see that the 't Hooft defect corresponding to $L_{\gamma_{(p,0)}}$ is reducible. This is the origin of our notation from (3.98)

$$L_{p,0} := L_{\gamma_{(p,0)}} \quad . \quad (5.115)$$

By contrast $L_{[ph^1,0]}$ corresponds to a trace in the representation R_{ph^1} .

Chapter 6

Expectation Value of 't Hooft Defects

In this section we will continue our discussion of 't Hooft defects. Again we will consider 4D $\mathcal{N} = 2$ gauge theories that have a Lagrangian description that are also theories of class \mathcal{S} . Theories that live in this intersection are amenable to many different techniques for studying 't Hooft defects, and in particular to compute their expectation values. In this chapter we will review the computation of the expectation value of 't Hooft defects in these theories by two such techniques: spectral networks and localization.

6.1 Line Defects in Theories of Class \mathcal{S}

In general, the expectation value of the supersymmetric line operators we are considering $\langle L \rangle$ is a holomorphic function on \mathcal{M} , the Hitchin moduli space, in a complex structure determined by the supersymmetry preserved by L . The preserved supersymmetry can be characterized by a phase ζ , which may be viewed as an element of the twistor sphere: ζ also determines a complex structure on \mathcal{M} . We will denote the space \mathcal{M} with complex structure determined by ζ as \mathcal{M}_ζ . $\langle L(\zeta) \rangle$ on \mathcal{M}_ζ can be computed, exactly, by using class \mathcal{S} techniques.

For theories of class \mathcal{S} , one exact method for computing the expectation value of $L(\zeta)$ expresses $\langle L(\zeta) \rangle$ in terms of “spectral network coordinates” on \mathcal{M}_ζ [69, 71, 70]. These coordinates are generalizations of well-known cluster, shear, and Fock-Goncharov coordinates. They are functions on the twistor space and, restricted to a fiber \mathcal{M}_ζ , are

This section is based on material from my papers [23, 26, 27].

holomorphic Darboux coordinates in complex structure ζ . We refer to the exact result for $\langle L(\zeta) \rangle$ in these coordinates as the “Darboux expansion.”

The collective work of [67, 68, 69, 70, 71] develops a technology to compute this expectation value. This relies on the fact that the expectation value of such a line operator can be computed by the trace of the holonomy of flat gauge connections along a path on the associated UV curve C

$$\langle L_{\mathcal{P}} \rangle = \text{Tr}_R \text{Hol}_{L_{\mathcal{P}}} P \exp \left(\oint_{\mathcal{P}} \mathcal{A} \right) . \quad (6.1)$$

This can be expressed as a Laurent polynomial in Darboux coordinates which is subordinate to a cell decomposition of C

$$\langle L_{\mathcal{P}} \rangle = \sum_{\gamma \in \Gamma_u} \overline{\Omega}(L_{\mathcal{P}}, \gamma; u) \mathcal{Y}_{\gamma} , \quad (6.2)$$

where Γ_u is the local charge lattice above $u \in \mathcal{B}$, $\overline{\Omega}(L_{\mathcal{P}}, \gamma; u)$ is the framed BPS index corresponding to the state with charge γ bound to the operator $L_{\mathcal{P}}$, and \mathcal{Y}_{γ} are “Darboux coordinates” on the moduli space of flat connections with the complexified gauge group $G_{\mathbb{C}}$: $\mathcal{M}_{flat}(C; G_{\mathbb{C}})$. These coordinates have the physical interpretation of the expectation value of a line defect of charge γ in the IR limit. Further, they satisfy the Poisson algebra

$$\mathcal{Y}_{\gamma} \mathcal{Y}_{\gamma'} = (-1)^{\langle\langle \gamma, \gamma' \rangle\rangle} \mathcal{Y}_{\gamma + \gamma'} , \quad (6.3)$$

where $\langle\langle \cdot, \cdot \rangle\rangle : \Gamma \times \Gamma \rightarrow \mathbb{Z}$ is the DSZ pairing on Γ .

The main tool for computing this holonomy of the flat connection on a vector bundle for generic Riemann surfaces is the method of *spectral networks* [70, 71]. This technique generalizes the method of trivializing the vector bundle over a triangulation with gluing conditions at the edges. And further, it makes use of a map between a vector bundle on C and a line bundle on the multi-sheeted cover Σ (the Seiberg-Witten curve). Each spectral network introduces a natural set of coordinates on the moduli space of flat connections, $\mathcal{M}_{flat}(C; G_{\mathbb{C}})$ corresponding to the Darboux coordinates of the above expansion \mathcal{Y}_{γ} .

6.1.1 Theories of Class \mathcal{S} and Spectral Networks

As we discussed in Chapter 5, theories of class \mathcal{S} are constructed by taking the six-dimensional $\mathcal{N} = (2, 0)$ theory and compactifying it along an oriented Riemann surface C with a topological twist [68, 66, 176]. For the type A_{N-1} theories of class \mathcal{S} , this can be described by string theory as the low energy effective action of a stack of N M5-branes wrapped on $C \times M_4$ with the same topological twist where M_4 is our 4D spacetime. In going to the low energy limit, the M5-branes wrap a Riemann surface Σ which is an N -branched cover $\Sigma \rightarrow C$. The vacuum equations describing the four-dimensional physics in M_4 are given by Hitchin's equations¹ on C

$$F_C + [\varphi, \varphi^\dagger] = 0 \quad , \quad \bar{\partial}_{A_C} \varphi = 0 \quad , \quad (6.4)$$

with gauge group $G = SU(N)$. Given a solution of these equations, we can identify the Seiberg-Witten curve and differential as

$$\Sigma = \{ \det(xdz - \varphi) = 0 \} \subset T^*C \quad , \quad \lambda_{SW} = xdz \quad , \quad (6.5)$$

where $(x, z) \mapsto xdz$ are coordinates on T^*C .

In these theories, a 4D line operator comes from an M2-brane whose boundary wraps a closed 1-cycle $\mathcal{P} \subset C$ times a path $\gamma \subset M_4$. These M2-branes couple to the 4D gauge field A associated flat $G_{\mathbb{C}}$ connection

$$\mathcal{A} = \zeta^{-1} \varphi + A_C + \zeta \varphi \quad , \quad (6.6)$$

so that their expectation value is given by the trace of the holonomy A along γ times the holonomy of \mathcal{A} along \mathcal{P} . Since this theory is topologically twisted, all supersymmetric quantities are independent of the sizes of M_4, Σ . Therefore, we can see that the expectation value of the 4D 't Hooft defect is given (semiclassically) by the holonomy of the flat complexified connection \mathcal{A} along \mathcal{P} by taking the limit where Σ is small so that there is no fluctuations in these spacetime dimensions [68, 69].

Spectral networks are a technique constructed in [67, 68, 69, 70, 71] which can be used to compute the trace of the holonomy of a complexified flat connection and hence

¹Here we use the notation A_C the G -connection on C .

the expectation value of 4D line operators in theories of class \mathcal{S} . Here we will only consider the case of 2-fold coverings which corresponds to the case of $G = SU(2)$ and $G_{\mathbb{C}} = SL(2; \mathbb{C})$. In this case we will label the sheets by an index $i = 1, 2$.

We define a *spectral network* \mathcal{W} subordinate to the covering $\Sigma \rightarrow C$ to be an oriented collection of open paths w on C called *walls* with the following properties:

- Generic walls, w , begin at branch points of $\pi : \Sigma \rightarrow C$ and end at punctures of C .
- Three walls begin at each branch point.
- Walls carry an ordered pair of the sheets of $\Sigma \rightarrow C$ – in our case: 12 or 21.
- Walls do not intersect, except at branch points.
- Each puncture of C has a decoration which encodes a trivialization and orientation of the covering $\Sigma \rightarrow C$ over the puncture.
- Walls can also end on other branch points in which case they pair with another wall to form a *double wall*.
- Each network comes with a resolution convention of double walls – American or British. These describe in which direction the walls are infinitesimally displaced in order to compute parallel transport across double walls.

A special class of spectral networks which arise naturally in theories of class \mathcal{S} are called *WKB spectral networks*. These are defined by a meromorphic, quadratic differential φ_2 on the closure \overline{C} of C . Locally, this is of the form

$$\varphi_2 = u(z)(dz)^2 . \quad (6.7)$$

We can now use φ_2 to define a spectral network as follows. Pick² a $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$. Now consider the foliation of C by curves γ which satisfy

$$e^{-2i\vartheta}\varphi_2(\gamma) \in \mathbb{R}_+ \quad \text{or} \quad e^{-2i\vartheta}u(\gamma(t)) \left(\frac{d\gamma}{dt} \right)^2 \in \mathbb{R}_+ , \quad (6.8)$$

²For our case we will want to pick $e^{i\vartheta} = \zeta$ where ζ is the phase of the line defect.

where we use the notation of 2ϑ following that of [89].

The spectral network $\mathcal{W}(\varphi_2, \vartheta)$ is then defined by the critical graph of the foliation – i.e. the limiting set of γ which divide the foliation into distinct sectors. In the class \mathcal{S} construction, this is given by the square of the scalar field in the Hitchin system

$$\varphi_2 = \lambda_{SW}^2 . \quad (6.9)$$

Spectral networks allow one to define a trivialization of the $SL(2; \mathbb{C})$ vector bundle over the complement $C \setminus \mathcal{W}$ and give gluing conditions across the walls. Consider the $rk = 2$, $SL(2; \mathbb{C})$ vector bundle $\pi : E \rightarrow C$ with connection ∇ . On each connected component $\sigma \in C \setminus \mathcal{W}$, we can trivialize the bundle $E \rightarrow C$ such that $E|_\sigma \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ where \mathcal{L}_i are line bundles on patches of C . Since the spectral network is subordinate to the covering $\Sigma \rightarrow C$, the trivialization of the vector bundle E over a connected component $\sigma \subset C \setminus \mathcal{W}$ lifts to a $GL(1; \mathbb{C})$ line bundle over Σ

$$E = \pi_* \mathcal{L} , \quad (6.10)$$

where \mathcal{L} is a line bundle on Σ such that the connection ∇ on E lifts to an abelian $GL(1; \mathbb{C})$ connection ∇^{ab} on \mathcal{L} $\nabla = \pi_* \nabla^{ab}$. This can also be written as

$$E|_z \cong \left(\mathcal{L}_1 \oplus \mathcal{L}_2 \right)|_z \cong \mathcal{L}|_{z^{(1)}} \oplus \mathcal{L}|_{z^{(2)}} , \quad (6.11)$$

where $z \in C$ and $\pi^{-1}(z) = z^{(1)} \oplus z^{(2)}$. This compatibility leads to the isomorphism [70, 71]

$$\mathcal{M}_{flat}(\Sigma, GL(1; \mathbb{C})) \cong \mathcal{M}_{flat}(C; SL(2; \mathbb{C})) . \quad (6.12)$$

This can be understood as a reflection of gauge group enhancement in a stack of D-branes.

This isomorphism allows us to compute holonomies of the non-abelian vector bundle $E \rightarrow C$ in terms of holonomies of the connection of a flat line bundle on Σ . This moduli space has a natural set of coordinates³:

$$\mathcal{X}_\gamma = \text{Hol}_\gamma \nabla^{ab} \in \mathbb{C}^\times , \quad \forall [\gamma] \in H_1(\Sigma; \mathbb{Z}) . \quad (6.13)$$

³Here we use the notation \mathcal{X}_γ for the Darboux coordinates where $\gamma \in H_1(\Sigma; \mathbb{Z})$ while we use \mathcal{Y}_γ for the Darboux coordinates where $\gamma \in \Gamma$ (the charge lattice).

These coordinates follow the multiplication rule

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'} , \quad (6.14)$$

and satisfy

$$\mathcal{X}_{\gamma_b} = -1 \quad , \quad \mathcal{X}_{\gamma+\omega^*\gamma} = 1 , \quad (6.15)$$

where γ_b is a small loop around a branch point b . Let us define $\Sigma' = \Sigma \setminus \{\text{branch points}\}$. For generic \mathcal{W} , we can fix a basis of $\{\gamma_i\} \in H_1(\Sigma'; \mathbb{Z}) / \langle \gamma + \omega^*\gamma \rangle$ to form our coordinate system on $\mathcal{M}(\Sigma, GL(1))$. These \mathcal{X}_{γ_i} are the Darboux coordinates related to the spectral network \mathcal{W} .

To each $[\gamma] \in H_1(\Sigma'; \mathbb{Z})$ we can associate a physical, conserved charge in the four dimensional theory $\gamma \in \Gamma$. In order to relate these to the physical charges of the 4D theory, these charges must be chosen so that the oriented intersection number of two curve classes is given by the charge DSZ pairing on the physical charges:

$$[\gamma_1] \# [\gamma_2] = \llbracket \gamma_1, \gamma_2 \rrbracket . \quad (6.16)$$

In general, there is not necessarily a unique choice of charge identification. These different choices correspond to different duality frames of the 4D theory.

After identifying the physical charge associated to $[\gamma] \in H_1(\Sigma; \mathbb{Z})$, the corresponding Darboux coordinate is of the form [67]

$$\log \mathcal{X}_\gamma = \frac{\pi R}{\zeta} Z_\gamma + \pi R \zeta \bar{Z}_\gamma + i\theta \cdot Q_\gamma + \left\{ \begin{array}{c} \text{non-perturbative} \\ \text{in } g_{4D} \end{array} \right\} , \quad (6.17)$$

where Z_γ is the central charge evaluated on the charge associated to γ and $\theta \cdot Q_\gamma$ is the Cartesian product of the vector of electric and magnetic theta angles with the vector of electromagnetic charges associated to Q_γ . It is important to note that these coordinates generically have non-perturbative corrections which, while complicated, are known and given in explicit formulas in [67].

We can compute the holonomy of the flat non-abelian gauge connection by decomposing the closed 1-cycle into open paths and computing the product of their associated holonomies. Since the spectral network provides a trivialization of the connection in

each of the cells, the holonomy along a path in a single cell can be written as

$$D_{\mathcal{P}} = \begin{pmatrix} \mathcal{X}_{\mathcal{P}} & 0 \\ 0 & \mathcal{X}_{\mathcal{P}}^{-1} \end{pmatrix} \quad \text{and} \quad \tilde{D}_{\mathcal{P}} = \begin{pmatrix} 0 & \mathcal{X}_{\mathcal{P}} \\ -\mathcal{X}_{\mathcal{P}}^{-1} & 0 \end{pmatrix}, \quad (6.18)$$

where $\tilde{D}_{\mathcal{P}}$ corresponds to when the path crosses a branch cut since the sheet order switches. Then in making a convenient choice of localization, we can write the holonomy across a generic wall [69, 70, 71]

$$\mathcal{S}_w = \begin{cases} \begin{pmatrix} 1 & S_w \\ 0 & 1 \end{pmatrix} & \text{for } w \text{ of type 21,} \\ \begin{pmatrix} 1 & 0 \\ S_w & 1 \end{pmatrix} & \text{for } w \text{ of type 12,} \end{cases}. \quad (6.19)$$

To compute the parallel transport across a double wall, one must infinitesimally displace the phase ζ (according to the resolution convention) so that the double wall is replaced by a pair of generic walls; then one can compute the holonomy using the rules above. These rules allow one to compute the holonomy of a complexified flat gauge connection along any path in terms of Darboux coordinates defined by the spectral network (spectral coordinates).

There are several consistency conditions that restrict the number of free spectral coordinates. These come from abelian gauge symmetry on open path segments and from imposing monodromy conditions around branch points and punctures. This gauge symmetry acts by rescaling the spectral coordinate $\mathcal{Y}_{\gamma_{ij}}$ by a function corresponding to the end points of the curve γ_{ij} with beginning and end points labeled by i and j respectively

$$\mathcal{X}_{\gamma_{ij}} \rightarrow g_i \mathcal{X}_{\gamma_{ij}} g_j^{-1}, \quad (6.20)$$

so that the trace of the holonomy around closed paths are gauge invariant.

The consistency conditions we impose for monodromy around a branch point b and puncture p is that

$$\text{Hol}_{\gamma_b} \nabla = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{Hol}_{\gamma_p} \nabla = \begin{pmatrix} \mathcal{X}_{m_p} & 0 \\ 0 & \mathcal{X}_{m_p}^{-1} \end{pmatrix}, \quad (6.21)$$

which come from the condition in 6.15 and the trivialization of the vector bundle at the punctures given in the data specifying the spectral network. Note that the data m_p associated to a puncture p is the mass parameter of the corresponding matter multiplet as in [66, 68, 69].

6.1.2 Wall Crossing in Spectral Networks

An important feature of spectral networks is that they give us an excellent tool for understanding wall crossing. In this setting, wall crossing is realized by changes of topology of the spectral network $\mathcal{W}(\varphi_2, \vartheta)$ as we scan the phase $\zeta = e^{i\vartheta}$ which can be lifted to $\hat{\zeta} \in \mathbb{C}^*$.

The locations of the critical phases $\zeta = \zeta_c$ where the spectral network undergoes topology changes lift to a co-dimension-1 “walls” in \mathbb{C}^* called and are called \mathcal{K} -walls[70]. Physically, each \mathcal{K} -wall corresponds to a wall of marginal stability $\widehat{W}(\gamma_k)$ where ζ is aligned with the phase of $\overline{Z_{\gamma_k}}$. Here, the change in topology of the spectral network causes the Darboux coordinates to undergo a cluster-like transformation [69]

$$K_{\gamma_k} : \mathcal{X}_{\gamma_i} \mapsto (1 + \sigma(\gamma) \mathcal{X}_{\gamma_k})^{-\ll \gamma_k, \gamma_i \gg \overline{\underline{\Omega}}(\gamma_k)} \mathcal{X}_{\gamma_i} \quad , \quad \gamma_k \in \Gamma \quad , \quad (6.22)$$

where

$$\sigma(\gamma) = (-1)^{\ll \gamma_e, \gamma_m \gg} \quad , \quad (6.23)$$

is a particular choice of quadratic refinement with respect to a choice of splitting of the charge lattice and $\gamma = \gamma_e \oplus \gamma_m$.⁴

However, since the expectation value of a line operator $L_{\mathcal{P}}$ is defined by a path $\mathcal{P} \subset C$ which is independent of the topology of the spectral network, the expectation value

$$\langle L_{\mathcal{P}} \rangle = \sum_{\gamma \in \Gamma} \overline{\underline{\Omega}}(\gamma, L_{\mathcal{P}}) \mathcal{Y}_{\gamma} \quad , \quad (6.24)$$

is wall crossing invariant. This means that the \mathcal{Y}_{γ} undergo coordinate transformations which exactly cancel the wall crossing of the framed BPS indices. Thus, by studying

⁴We will be working in the semiclassical limit so that there is always an almost canonical choice of charge lattice splitting.

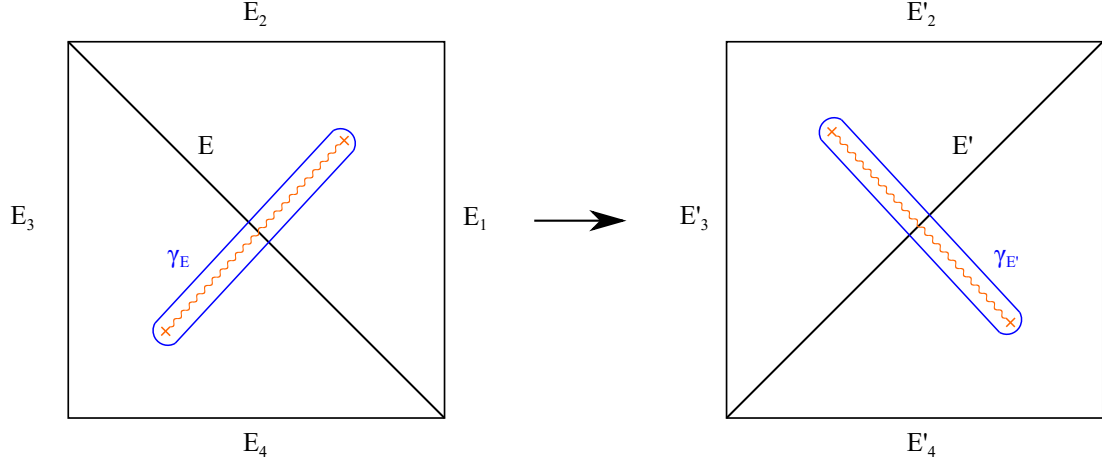


Figure 6.1: This figure shows the flip of an edge in a triangulation (left flips to right) giving rise to a Fock-Goncharov (shear) coordinate inside a quadrilateral with edges E_1, E_2, E_3, E_4 . This figure also demonstrates the projection of the paths in $\Sigma \rightarrow C$ corresponding to the Darboux coordinates \mathcal{X}_E and $\mathcal{X}_{E'}$.

the wall crossing properties of the \mathcal{Y}_γ , one can infer the wall crossing of framed BPS states.

A nice feature of generic WKB spectral networks is that the walls provide an ideal triangulation of C . In these networks, the associated Darboux coordinates have a natural identification with the edges of the triangulation. These coordinates are given by the holonomy along the lift under the projection $\pi : \Sigma \rightarrow C$ of a path running between the branch points of different triangles through a given edge of the triangulation. See Figure 6.1. We will use the notation where the Darboux coordinate associated to the edge E is denoted \mathcal{X}_E .

In such spectral networks, the fundamental topology shift that occurs in wall crossing is a flip of the triangulation. See Figure 6.1. Explicitly, in a generic WKB spectral network and consider a quadrilateral with edges E_1, E_2, E_3, E_4 with diagonal edge E , a flip on the edge $E \mapsto E'$ acts on the corresponding Darboux coordinates by:

$$\begin{aligned}
 \mathcal{X}_E &\mapsto \mathcal{X}_{E'} & , \quad \mathcal{X}_{E_1} &\mapsto \mathcal{X}'_{E_1} = \mathcal{X}_{E_1}(1 + \mathcal{X}_E) , \\
 \mathcal{X}_{E_2} &\mapsto \mathcal{X}'_{E_2} = \mathcal{X}_{E_2}(1 + \mathcal{X}_E^{-1})^{-1} & , \quad \mathcal{X}_{E_3} &\mapsto \mathcal{X}'_{E_3} = \mathcal{X}_{E_3}(1 + \mathcal{X}_E) , \\
 \mathcal{X}_{E_4} &\mapsto \mathcal{X}'_{E_4} = \mathcal{X}_{E_4}(1 + \mathcal{X}_E^{-1})^{-1} & ,
 \end{aligned} \tag{6.25}$$

where the signed intersection pairing of the edges is $\langle E_i, E_j \rangle = \delta_{i+1,j} - \delta_{i-1,j}$.

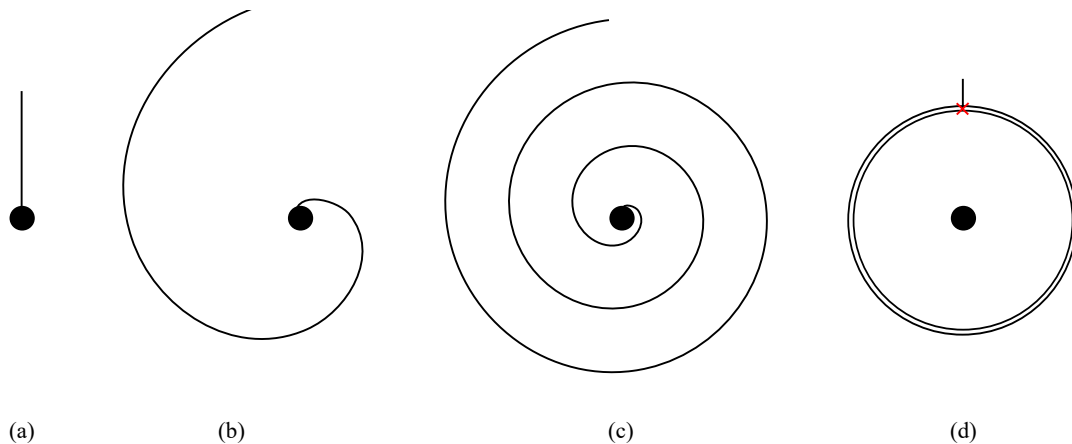


Figure 6.2: This figure shows how a wall running to a puncture (a) twists around the puncture in a sequence of flips (b),(c) and approaches the juggle in which the wall runs completely around the puncture (d).

In the case where there are punctures on C spectral networks can also undergo a topology change called a juggle.⁵ This can be understood as an infinite sequence of flips involving a puncture that has the effect of twisting a wall that runs to a puncture until it completely encircles it [68]. See Figure 6.2.

The juggle can be understood as follows [68]. Consider an annulus surrounding a puncture, P (which we replace by a disk with a marked point), with a single vertex V of the triangulation on the outer boundary. Now consider lifting the configuration to the simply connected cover which is a triangulated infinite strip as in Figure 6.3. In this covering there are an infinite number of images of the interior marked point ($P \rightarrow \{P_i\}$), exterior vertex ($V \rightarrow \{V_i\}$), and edges indexed by $i \in \mathbb{Z}$. We can define Darboux coordinates on the annulus as the Darboux coordinates on the triangulated strip corresponding to the different edges in the same preimage under the projection to the annulus.

If we choose an ordering of the lifted images of the vertices, we can define a winding number of an interior edge by the difference of the image number of the end points. Further, we can iteratively increase (decrease) the winding numbers of the interior edges by performing a sequence of simultaneous flips on all of the preimages of the the interior

⁵There is another transformation called a “pop” which has to do with changing the decoration of a given puncture, but this will not be important for our story. See [68] for more details.

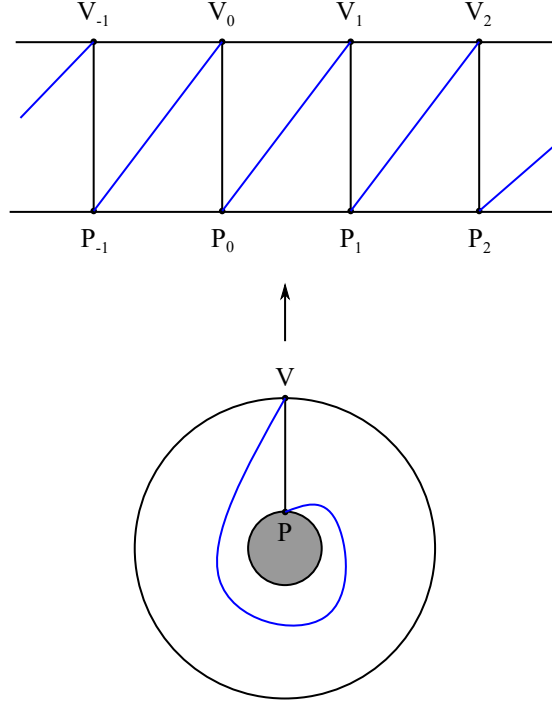


Figure 6.3: This figure shows how to lift a spectral network on an annulus surrounding a puncture on C to the simply connected cover. The puncture P , and exterior vertex V lift an infinite number of separated points, denoted $\{P_i\}$ and $\{V_i\}$ respectively, connected by interior edges giving a triangulation of the strip.

edge with the lowest (highest) winding number. See Figure 6.4. After n such flips, the interior edges run between the 0^{th} exterior vertex preimage to the n^{th} and $(n-1)^{th}$ interior preimage. We can now make sense of the corresponding Darboux coordinates in the limit as $n \rightarrow \infty$. First note that as $n \rightarrow \infty$ the interior edges approach a parallel line to the interior and exterior edges. This corresponds to a spectral network where there is a single, double wall circling the puncture of C under consideration. If we define $\mathcal{Y}_+^{(n)}$ and $\mathcal{Y}_-^{(n)}$ to be the edges with higher and lower winding number respectively after n flips, then in the $n \rightarrow \infty$ limit we can construct the well defined coordinates:

$$\mathcal{Y}_A^{(+)} = \lim_{n \rightarrow \infty} \mathcal{Y}_+^{(n)} \mathcal{Y}_-^{(n)} \quad , \quad \mathcal{Y}_B^{(+)} = \lim_{n \rightarrow \infty} (\mathcal{Y}_+^{(n)})^{-n} (\mathcal{Y}_-^{(n)})^{1-n} . \quad (6.26)$$

There exists an analogous coordinate system $\{\mathcal{Y}_A^{(-)}, \mathcal{Y}_B^{(-)}\}$ for the limit of sending the winding to $-\infty$ which is related

$$\mathcal{Y}_A^{(-)} = (\mathcal{Y}_A^{(+)})^{-1} \quad , \quad \mathcal{Y}_B^{(-)} = (\xi_+ - \xi_-)^{-4} (\mathcal{Y}_B^{(+)})^{-1} , \quad (6.27)$$

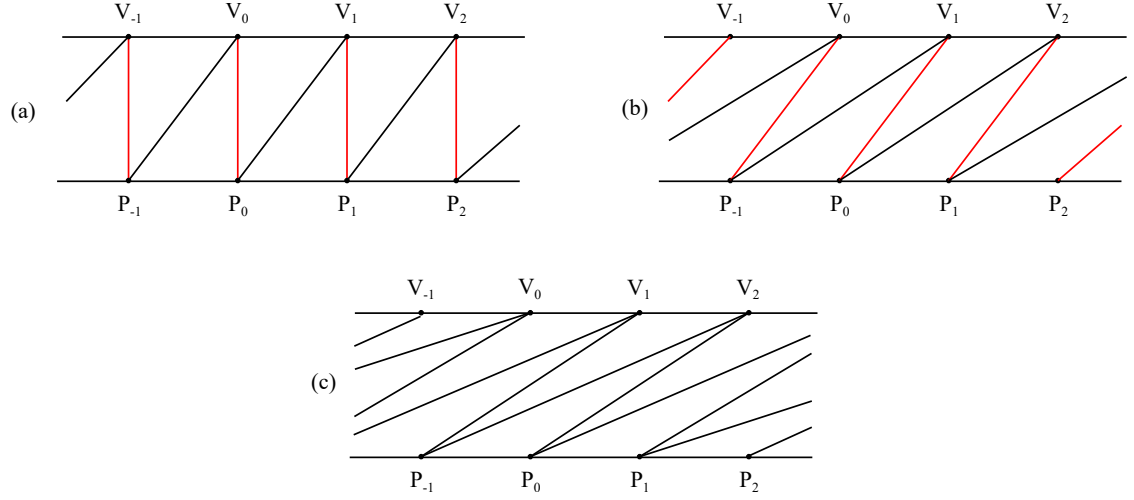


Figure 6.4: This figure demonstrates how flips in the spectral network on the annulus corresponds to increasing winding number by considering the flips of all of the preimages in the triangulated strip. Here the processes of going from (a) \rightarrow (b) and (b) \rightarrow (c) requires a sequence of 2 flips where the red edges undergo the flip.

where ξ_+ , ξ_- are the positive and negative eigenvalues of the monodromy matrix around the given puncture.

6.1.3 Line Defects and the AGT Correspondence

The expectation value of line defects in theories of class \mathcal{S} can also be exactly computed by using what is known as the AGT correspondence [3, 4].

Recall that theories of class \mathcal{S} are constructed by compactifying a corresponding 6D $\mathcal{N} = (2, 0)$ theory on a Riemann surface C with a topological twist that makes the theory independent of the scale of C . Because of this, the expectation value of SUSY operators in the 4D theory, which descend from SUSY operators in the 6D theory, are equal to the expectation value of a corresponding operator in the 2D theory on C [3, 4].

For theories of class \mathcal{S} with $SU(N)$ gauge group, the above construction is equivalent to wrapping a stack of N M5-branes on C with a topological twist. In this case, the corresponding 2D theory is A_{N-1} Toda theory on the closure of C , denoted \overline{C} . Here, punctures of C are associated with a flavor symmetry of 4D hypermultiplets and come with the data of a mass parameter specifying the 4D flavor symmetry. In the associated

2D Toda theory, each puncture corresponds to a vertex operator insertion in the path integral whose weight is determined by the associated mass parameter [69, 4, 59, 58].

We are interested in computing the expectation value of magnetically charged line defects in the 4D theory. Again, recall that in theories of class \mathcal{S} , line defects descend from strings in the 6D theory that wrap the 2-manifold $\gamma \times S_t^1 \subset C \times (\mathbb{R}^3 \times S_t^1)$ where γ is a closed 1-dimensional submanifold of C that does not go into the punctures. The electromagnetic charge of the associated 4D line defect in an S-duality frame is determined by the homology class of $\gamma \subset C$ with respect to the weak coupling cut decomposition of C corresponding to the S-duality frame [69, 58, 59, 4].

In the 2D Toda theory, a line defect associated to a closed curve γ corresponds to a loop operator \mathcal{L}_γ . This can be computed by [4, 168, 126]

$$\langle L_{\vec{p},0} \rangle_{T_4[SU(N),C]} = \left\langle \left(\prod_f V_{m_f} \right) \mathcal{L}_{\gamma_{\vec{p}}} \right\rangle_{\text{Toda}[A_{N-1}, \overline{C}]}, \quad (6.28)$$

where $T_4[SU(N), C]$ is the type $SU(N)$ 4D theory of class \mathcal{S} corresponding to the Riemann curve C , the $\{V_{m_f}\}$ are the vertex operators corresponding to the punctures of C with mass parameters $\{m_f\}$, and $\gamma_{\vec{p}}$ is the curve corresponding to the operator $L_{\vec{p},0}$ [4, 58, 59].

Complexified Fenchel-Nielsen Coordinates

The expectation value of 4D line defects computed using the AGT correspondence is naturally expressed in terms of Fenchel-Nielsen coordinates \mathbf{a}, \mathbf{b} . These can be defined as follows.

Choose a weak coupling region of the Coulomb branch. This defines a complex structure and comes with a maximal set of non-intersecting curves $\{\gamma_i\}_{i=1}^{3g-3+n}$ that are not isotopic to punctures on the UV curve C which correspond to weakly coupled gauge groups indexed by i .⁶ Each γ_i , corresponds to an $SU(N)$ factor of the gauge group of \mathfrak{g} , to which we can define the associated holomorphic coordinates $\{\mathbf{a}_i\} \in \mathfrak{t}_{\mathbb{C}}$ defined by

$$\langle L_{\gamma_i} \rangle = \text{Tr}_N e^{\mathbf{a}_i}. \quad (6.29)$$

⁶Here we are restricting to the case of Lagrangian theories of class \mathcal{S} with $SU(N)$ gauge group.

The $\{\mathfrak{a}_i\}$ are Poisson commuting with respect to the standard, symplectic $(2, 0)$ -form Ω_J on Seiberg-Witten moduli space [53, 102]

$$\Omega_J \left(\frac{\partial}{\partial \mathfrak{a}_i}, \frac{\partial}{\partial \mathfrak{a}_j} \right) = 0 , \quad (6.30)$$

and form a maximal set of Poisson commuting holomorphic functions.

Now we can define a set of symplectically dual coordinates $\{\mathfrak{b}_i\} \in \mathfrak{t}_{\mathbb{C}}$ with respect to Ω_J such that

$$\Omega_J = \frac{1}{\hbar} \sum_i \text{Tr}_N(d\mathfrak{a}_i \wedge d\mathfrak{b}_i) . \quad (6.31)$$

We can then fix the redundancy $\mathfrak{b}_i \rightarrow \mathfrak{b}_i + f_i(\mathfrak{a})$ where $\partial_{\mathfrak{a}_i} f_j = \partial_{\mathfrak{a}_j} f_i$ by specifying the semiclassical limit:

$$\begin{aligned} \mathfrak{a} &= i\theta_e^{(i)} - 2\pi\beta Y_{\infty}^{(i)} + \dots , \\ \mathfrak{b} &= i\theta_m^{(i)} + \frac{8\pi^2\beta}{g^2} X_{\infty}^{(i)} - \vartheta\beta Y_{\infty}^{(i)} + \dots , \end{aligned} \quad (6.32)$$

where $\theta_m^{(i)}$ and $\theta_e^{(i)}$ are the magnetic and electric theta angles of the i^{th} factor of the gauge group, $\zeta^{-1}\Phi_{\infty}^{(i)} = Y_{\infty}^{(i)} + iX_{\infty}^{(i)}$ are the real and imaginary parts of the phase rotated vev of the adjoint-valued Higgs field $\Phi^{(i)}$ of the $\mathcal{N} = 2$ vectormultiplet corresponding to the i^{th} factor of the gauge group, and ϑ, g define the real and complex part of the complex gauge coupling τ which we assume to be the same for all factors of the gauge group. Note that we will generally take Φ_{∞} to be fixed so that the $\mathfrak{a}, \mathfrak{b}$ have ζ -dependence via X_{∞}, Y_{∞} . Additionally, here β is the radius of the thermal circle and (...) correspond to non-perturbative corrections, which we will discuss later in Section 7.1.1.

For example, in the case of a single $SU(2)$ gauge group, the above discussion reduces to a single pair of Fenchel-Nielsen coordinates $\mathfrak{a}, \mathfrak{b}$.

6.2 Localization for $Z_{\text{mono}}(P, \mathfrak{v})$

Now we will discuss how to compute the expectation value of 't Hooft defects by using localization. To facilitate discussion, let us consider the the example of the 4D $\mathcal{N} = 2$ $SU(N)$ gauge theory on $\mathbb{R}^3 \times S^1$ with N_f fundamental hypermultiplets.⁷

⁷The analysis follows similarly for theories with higher rank gauge groups.

In such theories, the expectation values of line defects, and in particular 't Hooft defects, are holomorphic functions on Seiberg-Witten moduli space. In a weak coupling domain, the expectation value can be written in terms complexified Fenchel-Nielson coordinates \mathbf{a}, \mathbf{b} which are holomorphic, Darboux coordinates on Seiberg-Witten moduli space. Here, \mathbf{a} is canonically defined and its symplectic dual \mathbf{b} , while not canonically defined is uniquely fixed via the weak coupling expansion as in the previous section.

From general principles, the expectation value of the 't Hooft defects can be expressed in these coordinates as a Fourier expansion in \mathbf{b} . In general, this can be expressed as [97, 81]

$$\langle L_{P,0} \rangle = \sum_{\mathbf{v} \leq P} \cosh(\mathbf{v}, \mathbf{b}) (F(\mathbf{a}))^{|\mathbf{v}|} Z_{mono}(\mathbf{a}, m, \epsilon; P, \mathbf{v}) , \quad (6.33)$$

where the sum is over $\mathbf{v} = \sum_I v_I h^I$ such that $v_I \leq p_I$ for all I where $P = \sum_I p_I h^I$.

The expectation value above is expressed as a sum over monopole bubbling configurations where $\cosh(\mathbf{v}, \mathbf{b})F(\mathbf{a})$ encodes the contribution of bulk fields and $Z_{mono}(\mathbf{a}, m, \epsilon; P, \mathbf{v})$ describes the contribution from the SQM that arises on the 't Hooft defect from bubbling [23].

The field content of this theory consists of a $\mathcal{N} = 2$ vector multiplet (Φ, ψ_A, A_μ) with gauge group $SU(N)$ and N_f fundamental hypermultiplets $(q_A^{(f)}, \lambda^{(f)})$ with masses m_f where $f = 1, \dots, N_f$. We will express these hypermultiplets as a single hypermultiplet (q_A, λ) that transforms under the bifundamental representation of $G \times G_f = SU(N) \times SU(N_f)$ with a single mass parameter $m \in \mathfrak{t}_f \subset \mathfrak{g}_f$. Here q_A, ψ_A are a scalar- and Weyl fermion-doublets transforming under the $\text{spin-}\frac{1}{2}$ representation of $SU(2)_R$ and λ is a Dirac fermion.

This theory is described by the Lagrangian [133, 22]:

$$\begin{aligned}
L &= L_{vec} + L_{hyp} , \\
L_{vec} &= \frac{1}{g^2} \int d^3x \, \text{Tr} \left(\frac{1}{2} F^{\mu\nu} F_{\mu\nu} + |D_\mu \Phi|^2 - \frac{1}{4} [\Phi, \Phi^\dagger]^2 \right. \\
&\quad \left. - 2i\bar{\psi}^A \bar{\sigma}^\mu D_\mu \psi_A - i\psi^A [\Phi^\dagger, \psi_A] + i\bar{\psi}^A [\Phi, \bar{\psi}_A] \right) \\
&\quad + \frac{\vartheta}{8\pi^2} \int \text{Tr} (F \wedge F) , \\
L_{hyp} &= \frac{1}{g^2} \int d^3x \, \left(|D_\mu q_A|^2 + 2i\bar{\lambda} \not{D} \lambda + |mq_A|^2 - imq^{\dagger A} \Phi^\dagger q_A - im^* q^{\dagger A} \Phi q_A \right. \\
&\quad - 2m_R \bar{\lambda} \lambda + 2im_I \bar{\lambda} \gamma^5 \lambda - i\bar{\lambda} \Phi \lambda - i\lambda^T \Phi^\dagger \lambda^* \\
&\quad \left. + 2q^A \bar{\lambda} \Psi_A + 2\bar{\Psi}^A q_A^\dagger \lambda + \frac{1}{2} q^A \{\Phi, \Phi^\dagger\} q_A + \frac{1}{8} (q^{\dagger A} T^a (\tau_s)_A^B q_B)^2 \right) ,
\end{aligned} \tag{6.34}$$

where $s = 1, 2, 3$ is summed over, $(\tau_s)_A^B$ are the $SU(2)_R$ generators, $\Psi_A^T = (\psi_A, \bar{\psi}_A)$ is a Dirac fermion, and $m = m_R + im_I$.

The supersymmetry transformations of these fields are

$$\begin{aligned}
\delta_\xi \Psi_A &= -i\sigma^{\mu\nu} F_{\mu\nu} \xi_A + i\sigma^\mu D_\mu \bar{\xi}_A + \frac{i}{2} \xi_A [\Phi, \Phi^\dagger] , \\
\delta_\xi \Phi &= 2\xi^A \psi_A \quad , \quad \delta_\xi A_\mu = \xi^A \sigma_\mu \bar{\psi}_A + \bar{\psi}^A \bar{\sigma}_\mu \psi_A , \\
\delta_\xi q_A &= 2\bar{\Xi}_A \lambda \quad , \quad \delta_\xi \lambda = i\gamma^\mu \bar{\Xi}^A D_\mu q_A - (i\Phi^\dagger + m^*) q_A (\Xi^*)^A ,
\end{aligned} \tag{6.35}$$

where $\Xi_A^T = (\xi_A, \bar{\xi}_A)$ is a Dirac-fermion doublet of SUSY transformation parameters that transforms in the spin $\frac{1}{2}$ -representation of $SU(2)_R$.

As in the case of Seiberg-Witten theory, the space of SUSY vacua is given by the complexification of a Cartan subalgebra modulo the action of the Weyl group which is again parametrized by the vev of the complex scalar field. This generically breaks the conserved global symmetry group down to $\tilde{T} = T_{gauge} \times U(1)_\epsilon \times T_f$ where T_{gauge} is the maximal torus of the 4D gauge group, which describes the group of global gauge transformations, and T_f is the maximal torus of the flavor symmetry group.

Now let us include a (reducible) 't Hooft operator specified by the data $(P, \vec{x} = 0, \zeta)$. The gauge field singularity at $\vec{x} = 0$ requires adding a local boundary term to the action

specified by ζ :⁸

$$S_{bnd} = -\frac{1}{g^2} \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon^2(\vec{0})} \text{Tr} \left(\text{Im}[\zeta^{-1}\Phi]F + \text{Re}[\zeta^{-1}\Phi] * F \right), \quad (6.36)$$

where $S_\epsilon^2(\vec{0})$ is the 2-sphere of radius ϵ centered at $\vec{x} = \vec{0}$.

This insertion manifestly breaks $\frac{1}{2}$ -supersymmetry. The choice of $\zeta \in U(1)$ defines the conserved symmetries to be generated by a parameter ρ^A that is defined by

$$\xi^A = \zeta^{\frac{1}{2}}(\rho^A + i\pi^A), \quad (6.37)$$

where ρ^A, π^A are symplectic-Majorana-Weyl fermions.⁹ The conserved supercharges are given by the real combination

$$\mathcal{Q} = \rho^A Q_A + \bar{\rho}_A \bar{Q}^A, \quad (6.38)$$

where Q_A is the complex supercharge of the full $\mathcal{N} = 2$ SUSY algebra. Specifically, this means that $L_{p,0}$ is a \mathcal{Q} -invariant operator. This \mathcal{Q} satisfies the relation

$$\mathcal{Q}^2 = H + \mathfrak{a}Q_{\mathfrak{a}} + \epsilon_+ J_+ + m \cdot F, \quad (6.39)$$

where H is the Hamiltonian, $Q_{\mathfrak{a}}$ is the charge associated with global gauge transformations with fugacity \mathfrak{a} , J_+ is the charge associated to supersymmetric rotations¹⁰ in \mathbb{R}^3 that we associate with ϵ_+ in a $\frac{1}{2}$ - Ω background, and F is the set of conserved flavor charges.

6.2.1 Localization

Now we will compute the expectation value of the 't Hooft defect by using localization. The localization principle states that the expectation value of a \mathcal{Q} -invariant operator is invariant under a \mathcal{Q} -exact deformation of the Lagrangian

$$\mathcal{L} \rightarrow \mathcal{L} + t\mathcal{Q} \cdot V. \quad (6.40)$$

⁸Really we must take a sum of p boundary terms (where the charge of the reducible 't Hooft defect is $P = p h^1$), each centered at $\vec{x}^{(i)}$, and then take the limit as $\vec{x}^{(i)} \rightarrow 0$. Each of these corresponds to the boundary condition for a constituent minimal 't Hooft defect inserted at $\vec{x}^{(i)}$. To represent a single reducible 't Hooft defect, we require taking the limit $\vec{x}^{(i)}, \epsilon^{(i)} \rightarrow 0$ such that $|\vec{x}^{(i)}|/\epsilon^{(i)} \rightarrow 0$ where the physical boundary term for each minimal 't Hooft defect is inserted on a 2-sphere of radius $\epsilon^{(i)}$ surrounding $\vec{x}^{(i)}$. For simplicity, we will ignore this subtlety in the main discussion.

⁹Symplectic-Majorana-Weyl fermions satisfy: $\rho^A = \epsilon^{AB} \bar{\sigma}^0 \bar{\rho}_B$.

¹⁰These are spatial rotations with an R -charge rotation.

Then, by studying the limit as $t \rightarrow \infty$, we see that the path integral localizes to the zeros of V that are fixed under the action of \tilde{T} . As in [97, 81], if we make a choice

$$V = (\mathcal{Q} \cdot \bar{\lambda}, \lambda) + (\mathcal{Q} \cdot \bar{\psi}_A, \psi^A) , \quad (6.41)$$

then the path integral localizes to the zeros of $\mathcal{Q} \cdot \lambda$ and $\mathcal{Q} \cdot \psi^A$. This reduces the path integral to an integral over (the \tilde{T} -invariant subspace of) the moduli space of BPS equations. Note that since shifting t is a \mathcal{Q} -exact deformation of the action, the localization behavior of the path integral is independent of the value of t .

In our case, the associated BPS equations (before $\frac{1}{2}\Omega$ -deformation) are given by ¹¹

$$\begin{aligned} D_i X &= B_i \quad , \quad E_i = D_i Y \quad , \\ D_t Y &= 0 \quad , \quad D_t X - [Y, X] = 0 \quad , \\ D_i q &= 0 \quad , \quad D_0 q + (Y + m_I)q + i(X - m_R)q = 0 \quad , \end{aligned} \quad (6.42)$$

where B_i, E_i are the magnetic and electric field respectively and m is rotated by the phase ζ : $\zeta^{-1}m = m_R + im_I$. The solutions to these equations with respect to the 't Hooft defect (3.94) and asymptotic boundary conditions (3.39) – (3.40) are given exactly by singular monopole moduli space [97, 81, 23].

Thus, the expectation value of the 't Hooft defect localizes to an integral over the \tilde{T} -fixed locus of singular monopole moduli space with measure determined by the 1-loop determinant times the exponential of the classical action.¹² In this integral, the classical

¹¹To regularize the path integral, we will need to turn on a $\frac{1}{2}\Omega$ -deformation that modifies the BPS equations. However, the \tilde{T} -fixed locus of the moduli space of the deformed BPS equations will be identical to the \tilde{T} -fixed locus of the moduli space of the undeformed BPS equations. See [81, 23] for the explicit form and more details.

¹²To be precise, we are computing the expectation value of the 't Hooft defect with fixed electric and magnetic theta angle θ_e, θ_m . The electric theta angle is defined by fixing the holonomy of the gauge connection along the circle at infinity

$$\oint_{S^1_\infty} A_t dt = \theta_e . \quad (6.43)$$

The magnetic theta angle is defined as the Fourier dual of path integral with fixed magnetic charge $\langle L_{\vec{p},0} \rangle_{\gamma_m}$:

$$\langle L_{\vec{p},0} \rangle_{\theta_m} = \sum_m \langle L_{\vec{p},0} \rangle_{\gamma_m} e^{-i(\gamma_m, \theta_m)} . \quad (6.44)$$

Thus, by saying that the path integral “localizes to singular monopole moduli space”, we mean that each term in the Fourier sum (6.44) reduces to an integral over the reducible singular monopole moduli space $\overline{\mathcal{M}}(P, \gamma_m; X_\infty)$. Due to the universality of the geometry of the transversal slices/bubbling SQMs, we will find that this subtlety is irrelevant for the calculation of $Z_{mono}(P, v)$. See [23] for more details.

action is determined by the effective bulk charge sourced by the 't Hooft defect. Since singular monopole moduli space decomposes as the disjoint union of bubbling sectors with different effective charges, the expectation value of the line defect reduces to a sum of integrals over the \tilde{T} -fixed locus of different strata of the bubbling locus. Thus, the expectation value is of the form

$$\langle L_{\vec{p},0} \rangle = \sum_{\substack{|\mathbf{v}| \leq |P| \\ \mathbf{v} \in \Lambda_{cr} + P}} Z(\mathbf{a}, \mathbf{b}, m_f, \epsilon_+; P, \mathbf{v}) , \quad (6.45)$$

where $Z(\mathbf{a}, \mathbf{b}, m_f, \epsilon_+; P, \mathbf{v})$ is the reduction of the localized integral over $\widehat{\underline{\mathcal{M}}}(P, \gamma_m; X_\infty)$ to the strata $\widehat{\underline{\mathcal{M}}}_{\tilde{T}}^{(s)}(\mathbf{v}, \gamma_m; X_\infty)$ and the corresponding transverse slice $\mathcal{M}(P, \mathbf{v})$.

By integrating over the \tilde{T} -fixed subspace of each $\widehat{\underline{\mathcal{M}}}_{\tilde{T}}^{(s)}(\mathbf{v}, \gamma_m; X_\infty)$, the computation for $Z(\mathbf{a}, \mathbf{b}, m_f, \epsilon_+; P, \mathbf{v})$ can be further reduced to a \tilde{T} -equivariant integral over the transversal slice of each strata, $\mathcal{M}(P, \mathbf{v})$ [97, 81, 23]. As shown in [23], we can identify the universal coefficient of the integrand with $e^{(\mathbf{v}, \mathbf{b})} Z_{1-loop}(\mathbf{a}, m_f, \epsilon_+; \mathbf{v})$ and the remaining, integral dependent part as $Z_{mono}(\mathbf{a}, m_f, \epsilon_+; P, \mathbf{v})$. This will lead to the form of the expectation value of the 't Hooft defect

$$\langle L_{\vec{p},0} \rangle = \sum_{\substack{|\mathbf{v}| \leq |P| \\ \mathbf{v} \in \Lambda_{cr} + P}} e^{(\mathbf{v}, \mathbf{b})} Z_{1-loop}(\mathbf{a}, m_f, \epsilon_+; \mathbf{v}) Z_{mono}(\mathbf{a}, m_f, \epsilon_+; P, \mathbf{v}) , \quad (6.46)$$

where, $Z_{mono}(P, \mathbf{v})$ can be identified as:

$$Z_{mono}(\mathbf{a}, m_f, \epsilon_+; P, \mathbf{v}) = \int_{\mathcal{M}(P, \mathbf{v})} e^{\omega + \mu_{\tilde{T}}} \hat{A}_{\tilde{T}}(T\mathcal{M}) \cdot C_{\tilde{T}}(\mathcal{V}(\mathcal{R})) . \quad (6.47)$$

Here, $\hat{A}_{\tilde{T}}$ is the \tilde{T} -equivariant \hat{A} genus, $C_{\tilde{T}}$ is a \tilde{T} -equivariant characteristic class that depends on the matter content of the theory, $e^{\omega + \mu_{\tilde{T}}}$ is the equivariant volume form, and $\mathbf{a}, m_f, \epsilon_+$ enter the expression as the equivariant weights under the \tilde{T} -action. It will be crucial to us that $Z_{mono}(P, \mathbf{v})$ is *independent* of β .¹³ See [127, 97, 23] for more details.

6.2.2 Bubbling SQMs

As shown in [23], $Z_{mono}(P, \mathbf{v})$ can be physically interpreted as the contribution of an SQM localized on the 't Hooft defect of charge P that has an effective charge \mathbf{v} . This

¹³ \mathbf{a} is treated as independent of β .

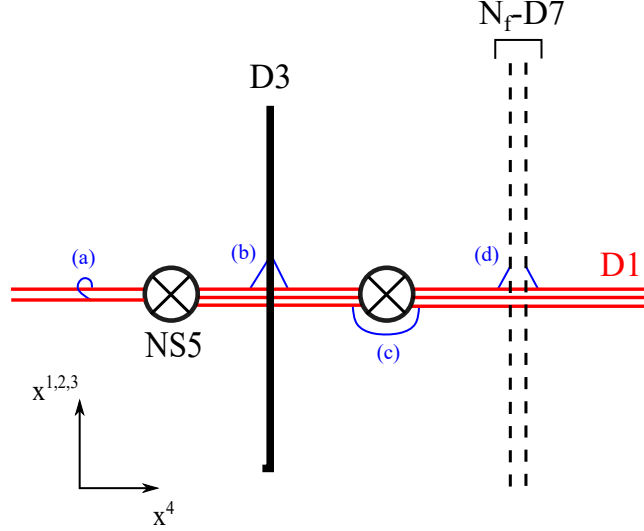


Figure 6.5: This figure illustrates which strings give rise to the various fields in the bubbling SQM. (a) describes the D1-D1 strings that give rise to a $\mathcal{N} = (0, 4)$ vector multiplet with fields (v_t, σ, λ^A) , (b) describes the D1-D3 strings that give rise to $\mathcal{N} = (0, 4)$ fundamental hypermultiplets with fields (ϕ^A, ψ_I) , (c) illustrates D1-D1' strings that give rise to $\mathcal{N} = (0, 4)$ bifundamental hypermultiplets with fields $(\underline{\phi}^A, \underline{\psi}_I)$, and (d) describes D1-D7 strings that give rise to the short $\mathcal{N} = (0, 4)$ Fermi multiplets with fields (η, G) .

leads to the interpretation of the integral in (6.47) as the localized path integral of the bubbling SQM. Then, since the (twisted) path integral of a SQM is formally equal to its Witten index, the monopole bubbling contribution, $Z_{mono}(P, \mathbf{v})$ can be expressed as the Witten index of the corresponding bubbling SQM specified by the quiver $\Gamma(P, \mathbf{v})$ as in Section 5.1.2:

$$Z_{mono}(P, \mathbf{v}) = I_W(\Gamma(P, \mathbf{v})) := \text{Tr}_{\mathcal{H}_{\Gamma(P, \mathbf{v})}} (-1)^F e^{-\frac{\beta}{2}\{\mathcal{Q}, \mathcal{Q}\} + \mathfrak{a}Q_{\mathfrak{a}} + \epsilon_+ Q_{\epsilon} + m \cdot F}, \quad (6.48)$$

where $Q_{\mathfrak{a}}$ is the charge for the flavor symmetry associated with 4D global gauge transformations, Q_{ϵ} is an R -charge associated to the $\frac{1}{2}\Omega$ -deformed background,¹⁴ and F is the set of conserved flavor charges.

The bubbling SQM specified by the quiver $\Gamma(P, \mathbf{v})$ is given by compactifying the 2D $\mathcal{N} = (0, 4)$ quiver gauge theory. Let us use the notation $G = \prod_{i=1}^{p-1} U(k^{(i)})$ for

¹⁴ Q_{ϵ} can also be understood as an R -symmetry charge in the bubbling SQM. The $\mathcal{N} = (0, 4)$ bubbling SQMs we are consider have an $SU(2)_R$ R -symmetry and an $SU(2)_r$ outer-automorphism “ R -symmetry”. Here the Q_{ϵ} is diagonal combination of the Cartans: $Q_{\epsilon} = Q_R - Q_r$. See [165] for more details.

the gauge group of the SQM such that the corresponding Lie algebra \mathfrak{g} decomposes as $\mathfrak{g} = \bigoplus_{i=1}^{n-1} \mathfrak{g}^{(i)} = \bigoplus_{i=1}^{n-1} \mathfrak{u}(k^{(i)})$ and its Cartan subalgebra $\mathfrak{t} = \bigoplus_{i=1}^{n-1} \mathfrak{t}^{(i)} = \bigoplus_{i=1}^{n-1} \mathfrak{u}(k^{(i)})$ where $P = n \hat{h}^1$, $\hat{h}^I \in \Lambda_{cochar}$.¹⁵

Each gauge node corresponds to a $\mathcal{N} = (0, 4)$ vector multiplet with constituent fields $(\sigma^{(i)}, \lambda^{A(i)}, v_t^{(i)})$, where $i = 1, \dots, n-1$ indexes the gauge nodes. In the string theory interpretation of Section 5.3, these vector multiplets arise from the $D1_i$ - $D1_i$ strings on the stack of $D1_i$ -branes stretched between the $NS5_i$ - and $NS5_{i+1}$ -branes. Additionally there are $\mathcal{N} = (0, 4)$ fundamental hypermultiplets with constituent fields $(\phi^{(i)}, \psi^{(i)}) \oplus (\tilde{\phi}^{(i)}, \tilde{\psi}^{(i)})$ that come from $D1_i$ - $D3$ strings and $\mathcal{N} = (0, 4)$ bifundamental hypermultiplets with constituent fields $(\underline{\phi}^{(i)}, \underline{\psi}^{(i)}) \oplus (\tilde{\underline{\phi}}^{(i)}, \tilde{\underline{\psi}}^{(i)})$ that come from the $D1_i$ - $D1_{i+1}$ strings at $NS5$ -branes. Also, in the case of theories with 4D fundamental hypermultiplets, there are additional $\mathcal{N} = (0, 4)$ short Fermi-multiplets with constituent fields $(\eta^{(i)}, G^{(i)})$ coming from $D3$ - $D7$ strings. See Figure 6.5. Additionally, see [165, 90] for more details on $\mathcal{N} = (0, 4)$ SQMs.

The bubbling SQM has a Lagrangian that decomposes as a sum of terms

$$L = L_{vec} + L_{Fermi} + L_f + L_{bf} , \quad (6.49)$$

which describe the contributions from vector multiplets, Fermi-multiplets, fundamental hypermultiplets, and bifundamental hypermultiplets respectively. These contributions can be found in Appendices D.1 and D.2. Here we will pick the convention where the gauge couplings for each factor in the gauge group are equal to e^2 by fixing a universal normalization of the Killing form for the SQM Lie algebra.

Now we wish to compute the Witten index of the bubbling SQM. This requires an understanding of the spectrum of the bubbling SQM. We can infer an approximate version of the spectrum from the classical moduli space and its surrounding potential.

¹⁵Recall that here we use the notation $L_{p,0}$ for $P = ph^1$, $h^1 \in \Lambda_{mw}$.

In this SQM, the potential energy is of the form

$$\begin{aligned}
U = & \frac{1}{e^2} \sum_i \left(|\sigma^{(i)} \cdot \phi^{(i)}|^2 + |\sigma^{(i)} \cdot \tilde{\phi}^{(i)}|^2 \right) \\
& + \frac{1}{e^2} \sum_i \left(|(\sigma^{(i+1)} - \sigma^{(i)}) \cdot \underline{\phi}^{(i)}|^2 + |(\sigma^{(i+1)} - \sigma^{(i)}) \cdot \underline{\tilde{\phi}}^{(i)}|^2 \right) \\
& + \frac{1}{2e^2} \sum_i \left(|\phi^{(i)}|^2 - |\tilde{\phi}^{(i)}|^2 - |\underline{\phi}^{(i)}|^2 + |\underline{\tilde{\phi}}^{(i)}|^2 + |\underline{\phi}^{(i-1)}|^2 - |\underline{\tilde{\phi}}^{(i-1)}|^2 \right)^2 \\
& + \frac{1}{e^2} \sum_i \left| \phi^{(i)} \tilde{\phi}^{(i)} - \underline{\phi}^{(i)} \underline{\tilde{\phi}}^{(i)} + \underline{\phi}^{(i-1)} \underline{\tilde{\phi}}^{(i-1)} \right|^2,
\end{aligned} \tag{6.50}$$

where here we are using scalar contraction. Thus, the moduli space is defined by the solutions to the equations:

$$\begin{aligned}
0 &= |\sigma^{(i)} \phi^{(i)}|^2, & 0 &= |\sigma^{(i)} \tilde{\phi}^{(i)}|^2, \\
0 &= |(\sigma^{(i+1)} - \sigma^{(i)}) \underline{\phi}^{(i)}|^2, & 0 &= |(\sigma^{(i+1)} - \sigma^{(i)}) \underline{\tilde{\phi}}^{(i)}|^2, \\
0 &= |\phi^{(i)}|^2 - |\tilde{\phi}^{(i)}|^2 - |\underline{\phi}^{(i)}|^2 + |\underline{\tilde{\phi}}^{(i)}|^2 + |\underline{\phi}^{(i-1)}|^2 - |\underline{\tilde{\phi}}^{(i-1)}|^2, \\
0 &= \phi^{(i)} \tilde{\phi}^{(i)} - \underline{\phi}^{(i)} \underline{\tilde{\phi}}^{(i)} + \underline{\phi}^{(i-1)} \underline{\tilde{\phi}}^{(i-1)},
\end{aligned} \tag{6.51}$$

for each i .

The solutions of these equations can be divided into Coulomb, Higgs, and mixed branches

$$\mathcal{M}_{vac} = \mathcal{M}_C \cup \mathcal{M}_H \cup \mathcal{M}_{mix}, \tag{6.52}$$

where

$$\begin{aligned}
\mathcal{M}_C &= \{ \sigma^{(i)} \in \mathfrak{t}^{(i)}, \sigma^{(i)} \neq \sigma^{(i+1)}, \phi^{(i)}, \tilde{\phi}^{(i)}, \underline{\phi}^{(i)}, \underline{\tilde{\phi}}^{(i)} = 0 \}, \\
\mathcal{M}_H &= \left\{ \begin{array}{l} |\phi^{(i)}|^2 - |\tilde{\phi}^{(i)}|^2 - |\underline{\phi}^{(i)}|^2 + |\underline{\tilde{\phi}}^{(i)}|^2 + |\underline{\phi}^{(i-1)}|^2 - |\underline{\tilde{\phi}}^{(i-1)}|^2 = 0 \\ \phi^{(i)} \tilde{\phi}^{(i)} - \underline{\phi}^{(i)} \underline{\tilde{\phi}}^{(i)} + \underline{\phi}^{(i-1)} \underline{\tilde{\phi}}^{(i-1)} = 0 \end{array} \right., \sigma^{(i)} = 0 \Bigg\} / \mathcal{G},
\end{aligned} \tag{6.53}$$

and \mathcal{G} is the group of gauge transformations. The mixed branch is significantly more complicated to write down in full generality, but it should be thought of as having asymptotic directions as in the Coulomb branch for some subset of directions of $\sigma \in \mathfrak{t}$ and some hypermultiplet scalars with non-zero expectation value. Because of this hybrid quality, the mixed branch, like the Coulomb branch, is non-compact and, like the Higgs branch, is a singular manifold.

We can additionally add an FI-deformation to the theory

$$L_{FI} = -(\xi, D) = -\sum_i \xi^{(i)} D^{(i)} . \quad (6.54)$$

This contribution changes the potential to

$$\begin{aligned} U = & \frac{1}{e^2} \sum_i \left(|\sigma^{(i)} \cdot \phi^{(i)}|^2 + |\sigma^{(i)} \cdot \tilde{\phi}^{(i)}|^2 \right) \\ & + \frac{1}{e^2} \sum_i \left(|(\sigma^{(i+1)} - \sigma^{(i)}) \cdot \underline{\phi}^{(i)}|^2 + |(\sigma^{(i+1)} - \sigma^{(i)}) \cdot \underline{\tilde{\phi}}^{(i)}|^2 \right) \\ & + \frac{1}{2e^2} \sum_i \left(|\phi^{(i)}|^2 - |\tilde{\phi}^{(i)}|^2 - |\underline{\phi}^{(i)}|^2 + |\underline{\tilde{\phi}}^{(i)}|^2 + |\underline{\phi}^{(i-1)}|^2 - |\underline{\tilde{\phi}}^{(i-1)}|^2 - e^2 \xi^{(i)} \right)^2 \\ & + \frac{1}{e^2} \sum_i \left| \phi^{(i)} \tilde{\phi}^{(i)} - \underline{\phi}^{(i)} \underline{\tilde{\phi}}^{(i)} + \underline{\phi}^{(i-1)} \underline{\tilde{\phi}}^{(i-1)} \right|^2 . \end{aligned} \quad (6.55)$$

This lifts the classical vacua associated to the Coulomb branch along with certain non-compact directions in the mixed branch by modifying the D-term vacuum equation to

$$e^2 \xi^{(i)} = |\phi^{(i)}|^2 - |\tilde{\phi}^{(i)}|^2 - |\underline{\phi}^{(i)}|^2 + |\underline{\tilde{\phi}}^{(i)}|^2 + |\underline{\phi}^{(i-1)}|^2 - |\underline{\tilde{\phi}}^{(i-1)}|^2 . \quad (6.56)$$

Consequently, when $\xi^{(i)} \neq 0$, the hypermultiplet scalar fields cannot all simultaneously satisfy $\phi^{(i)}, \tilde{\phi}^{(i)}, \underline{\phi}^{(i)}, \underline{\tilde{\phi}}^{(i)} = 0$.

Additionally, the FI-deformation resolves the singularities of the mixed and Higgs branches and lifts certain directions in the mixed branch. Now the Higgs branch can be written as a (resolved) hyperkähler quotient

$$\mathcal{M}_H = \vec{\mu}^{-1}(\vec{\xi}) / \mathcal{G} \quad , \quad \vec{\xi} = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) = (e^2 \xi, 0) , \quad (6.57)$$

where

$$\begin{aligned} \mu_{\mathbb{R}} = & |\phi^{(i)}|^2 - |\tilde{\phi}^{(i)}|^2 - |\underline{\phi}^{(i)}|^2 + |\underline{\tilde{\phi}}^{(i)}|^2 + |\underline{\phi}^{(i-1)}|^2 - |\underline{\tilde{\phi}}^{(i-1)}|^2 , \\ \mu_{\mathbb{C}} = & \phi^{(i)} \tilde{\phi}^{(i)} - \underline{\phi}^{(i)} \underline{\tilde{\phi}}^{(i)} + \underline{\phi}^{(i-1)} \underline{\tilde{\phi}}^{(i-1)} . \end{aligned} \quad (6.58)$$

Now in order to couple the Witten index to flavor fugacities, let us add masses for the hypermultiplet fields. These can be defined as flat connections coming from an associated flavor symmetry. We will choose to turn on mass parameters corresponding to the $\frac{1}{2}$ - Ω deformation with a mass parameter $\epsilon_+ = \text{Im}[\epsilon_+/\beta]$ and to a fugacity for 4D

global gauge symmetry with mass parameter $a = \text{Im}[\mathfrak{a}/\beta]$.¹⁶ These mass deformations modify the mass terms in the potential (6.55):

$$\begin{aligned}
U = & \sum_i \frac{1}{e^2} \left| (\sigma^{(i)} + aQ_{\mathfrak{a}} + \epsilon_+ Q_{\epsilon}) \cdot \phi^{(i)} \right|^2 + \frac{1}{e^2} \left| (\sigma^{(i)} + aQ_{\mathfrak{a}} + \epsilon_+ Q_{\epsilon}) \cdot \tilde{\phi}^{(i)} \right|^2 \\
& + \frac{1}{e^2} \sum_i \left| (\sigma^{(i+1)} - \sigma^{(i)} + aQ_{\mathfrak{a}} + \epsilon_+ Q_{\epsilon}) \cdot \underline{\phi}^{(i)} \right|^2 + \left| (\sigma^{(i+1)} - \sigma^{(i)} + aQ_{\mathfrak{a}} + \epsilon_+ Q_{\epsilon}) \cdot \underline{\tilde{\phi}}^{(i)} \right|^2 \\
& + \frac{1}{2e^2} \sum_i \left(|\phi^{(i)}|^2 - |\tilde{\phi}^{(i)}|^2 - |\underline{\phi}^{(i)}|^2 + |\underline{\tilde{\phi}}^{(i)}|^2 + |\underline{\phi}^{(i-1)}|^2 - |\underline{\tilde{\phi}}^{(i-1)}|^2 - e^2 \xi^{(i)} \right)^2 \\
& + \frac{1}{e^2} \sum_i \left| \phi^{(i)} \tilde{\phi}^{(i)} - \underline{\phi}^{(i)} \underline{\tilde{\phi}}^{(i)} + \underline{\phi}^{(i-1)} \underline{\tilde{\phi}}^{(i-1)} \right|^2.
\end{aligned} \tag{6.59}$$

where $Q_{\mathfrak{a}} \cdot \Phi$ and $Q_{\epsilon} \cdot \Phi$ encode the $Q_{\mathfrak{a}}, Q_{\epsilon}$ charges of the field Φ . See Appendices D.1 and D.2 for details about the charges of the fields.

The mass deformation lifts most of the Higgs and mixed branch vacua except at a collection of intersecting hyperplanes where hypermultiplet scalars become massless. This reduces the Higgs branch to a collection of points while reducing the mixed branch so that it only has non-compact directions coming from vector multiplet scalars. The mass deformations additionally give a mass of $4\epsilon_+$ to the fermionic component λ^2 which breaks SUSY $\mathcal{N} = (0, 4) \rightarrow \mathcal{N} = (0, 2)$ under which the $\mathcal{N} = (0, 4)$ vector multiplet decomposes as a $\mathcal{N} = (0, 2)$ vector multiplet $(v_t, \sigma, \lambda^1, D)$ and a $\mathcal{N} = (0, 2)$ Fermi-multiplet (λ^2, F) . With this choice, $Q = \rho^A Q_A$ is the preserved complex supercharge.

Due to the form of (6.59), the potential around each of the vacuum branches is quadratically confining. In the limit $e^2 \rightarrow 0$, this potential becomes infinitely steep and states become exactly localized on the moduli spaces. Since the Higgs branch is given by a collection of points, in the limit $e^2 \rightarrow 0$, this supports an infinite, discrete spectrum of harmonic oscillator-like states. However, the mixed branch, which has non-compact directions, supports both a discrete spectrum of bound states and a continuum of scattering states.

In addition, there are also states localized on the classically lifted Coulomb and mixed branches. Even though the potential energy on these branch is no longer zero, it

¹⁶This identification allows us to work with a unitary theory. We can then derive the full Witten index by analytic continuation. See [60] for more details.

is bounded. Again the potential in the normal direction is quadratically confining such that in the limit $e^2 \rightarrow 0$, the states become exactly localized on the lifted branches. Thus, the the Coulomb and lifted mixed branches constitute non-compact directions in field space with finite potential energy which can also support both a discrete spectrum of bound states and a continuum of scattering states. See Section 6.2.4 for further discussion of Higgs, mixed, and Coulomb Branch states.

Localization

Now we will attempt to compute the partition function of this theory by using localization. While parts of the following analysis have also been done using similar methods in [14, 90], we will find it instructive and physically insightful to present the full derivation of the localization computation.

The key to using localization in this setting is that the action of these theories is Q -exact. That is to say, we can rewrite the Lagrangian

$$L = \frac{1}{e^2} Q \cdot V_{vec} + \frac{1}{e^2} Q \cdot V_{matter} , \quad (6.60)$$

with

$$\begin{aligned} V_{vec} &= \sum_i \left(\bar{Q} \cdot \bar{\lambda}_A^{(i)}, \lambda^{(i)A} \right) , \\ V_{matter} &= \sum_i \left[\left(\bar{Q} \cdot \bar{\psi}^{(i)}, \psi^{(i)} \right) + \left(\bar{Q} \cdot \bar{\tilde{\psi}}^{(i)}, \tilde{\psi}^{(i)} \right) + \left(\bar{Q} \cdot \bar{\eta}^{(i)}, \eta^{(i)} \right) \right] . \end{aligned} \quad (6.61)$$

Thus, shifting the value of e is a supersymmetric deformation of the theory. This means that the result of localization should be independent of e and therefore we will take e to be generic and strictly positive.¹⁷

Now by the localization principle, the partition function reduces to an integral over the the moduli space of the time independent BPS equations:¹⁸

$$[\sigma, v_t] = 0 \quad , \quad M_r = 0 . \quad (6.62)$$

¹⁷Note that these are actually dimensionful quantities in the 1D SQM. These have dimension $[e^2] = \ell^{-3}$. Thus to take the “ $e \rightarrow 0$ ” limit, we must take $\ell^3 e^2 \rightarrow 0$ where ℓ is some fixed length scale. In our discussion we will use the FI-parameter ξ as our fixed length scale since in the upcoming discussion we want β to be variable.

¹⁸See (D.10)-(D.11) for the full SUSY transformations.

These BPS equations have a moduli space of solutions given by ¹⁹

$$\mathcal{M}_{BPS} = \left\{ \varphi = \beta(\sigma + iv_t)|_{t=0} \in (\mathfrak{t} \times T)/W \right\} \cong (\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr})/W = \widehat{\mathcal{M}}/W, \quad (6.63)$$

where \mathfrak{t} is the Lie algebra corresponding to the torus T of the SQM gauge group as defined by the quiver $\Gamma(P, \mathfrak{v})$. Note that this φ is not to be confused with the hypermultiplet fields $\phi_i, \tilde{\phi}_i, \underline{\phi}_i, \underline{\tilde{\phi}}_i$. Now as in [90, 14], the Wick rotated path integral is reduced to

$$Z^{(Loc)} = \int_{\mathcal{M}_{BPS}} \frac{d^{2r}\varphi}{(2\pi i)^r} Z_{det}(\varphi) = \frac{1}{|W|} \int_{\widehat{\mathcal{M}}} \frac{d^{2r}\varphi}{(2\pi i)^r} Z_{det}(\varphi), \quad (6.64)$$

where $r = \text{rk } \mathfrak{g}$ and \mathfrak{g} is the Lie algebra of the gauge group of the quiver SQM. The 1-loop determinant Z_{det} can now be computed in the background given by the zero-mode φ . For quiver SQMs this is of the form [90]

$$Z_{det}(\varphi) = \int_{\mathfrak{t}} d^r D Z_{int}(\varphi, D) \exp \left\{ -\frac{\pi\beta(D, D)}{e^2} + 2\pi i\beta(\xi, D) \right\}, \quad (6.65)$$

where

$$Z_{int}(\varphi, D) = Z_{vec}(\varphi) \cdot Z_{Fermi}(\varphi) \cdot Z_{hyp}(\varphi, D). \quad (6.66)$$

Here Z_{vec} , Z_{Fermi} , and Z_{hyp} are the 1-loop determinants from the vector-, Fermi-, and hyper- multiplet fields respectively.

First consider the vector multiplet contribution. This term originates solely from vector multiplet fermions. The reason is that there are no propagating modes of φ due to Gauss's law and there are no propagating modes of D due to the lack of a kinetic term [90]. Thus, the contributions to Z_{vec} come from integrating over the non-zero modes of λ^A . Note that λ^2 does not have any zero modes because it has a generic, non-zero mass due to the Ω -deformation.

Explicitly, the vector multiplet fermions give the contribution

$$Z_{vec} = \prod_{i=1}^{n-1} \prod_{\substack{\alpha \in \Delta_{adj}^{(i)} \\ \alpha \neq 0}} \sinh(\alpha(\varphi^{(i)} + q_i)) \prod_{\substack{\alpha \in \Delta_{adj}^{(i)} \\ \alpha \neq 0}} \sinh(\alpha(\varphi^{(i)})), \quad (6.67)$$

¹⁹Note that this rescaling enforces the periodicity condition $\varphi \sim \varphi + 2\pi i\lambda$ for $\lambda \in \Lambda_{cr}$.

where $\Delta_{adj}^{(i)}$ are the weights of the adjoint representation of the i^{th} simple summand of the gauge group and q_i represent the coupling to all global charges associated to the $\mathcal{N} = (0, 2)$ Fermi-multiplet of the $\mathcal{N} = (0, 4)$ vector multiplet (since only $\mathcal{N} = (0, 2)$ SUSY is preserved).

Similarly, the contribution from the Fermi-multiplet is given by only by the 1-loop determinant of the fermions which can be written as

$$Z_{Fermi} = \prod_{f=1}^{N_f} \prod_{\mu \in \Delta_{fund}^{(f)}} \sinh(\mu(\varphi^{(f)}) + q_f) , \quad (6.68)$$

where q_f encodes the coupling to all global charges and $\varphi^{(f)}$ is the complex vector multiplet scalar that couples to the f^{th} Fermi multiplet.

Now consider the contribution from hypermultiplets. This term can be divided into two parts

$$Z_{hyp} = Z_{hyp}^{(kin)} \cdot Z_{hyp}^{(Yuk)} , \quad (6.69)$$

where $Z_{hyp}^{(kin)}$ comes from kinetic terms of the hypermultiplet fields and $Z_{hyp}^{(Yuk)}$ comes from integrating out Yukawa interactions. Explicitly, these are of the form

$$\begin{aligned} Z_{hyp}^{(kin)} &= \prod_j \prod_{\mu \in \Delta_{hyp}^{(j)}} \prod_{m \in \mathbb{Z}} \frac{(\pi m - i\mu(\bar{\varphi}^{(j)}) - i\bar{q}_j)}{(|\pi m + i\mu(\varphi^{(j)}) + iq_j|^2 + i\mu(D))} \\ &= \prod_j \prod_{\mu \in \Delta_{hyp}^{(j)}} \frac{\sinh(\mu(\bar{\varphi}^{(j)}) + \bar{q}_j)}{\sinh(\alpha_{j,\mu}^+) \sinh(\alpha_{j,\mu}^-)} , \end{aligned} \quad (6.70)$$

$$Z_{hyp}^{(Yuk)} = \det h^{ab}(\varphi, D) ,$$

where j indexes the set of fundamental and bifundamental $\mathcal{N} = (0, 2)$ chiral multiplets making up the $\mathcal{N} = (0, 4)$ hypermultiplets and $\varphi^{(j)}$ is the complex vector multiplet scalar that couples to the j^{th} hypermultiplet where

$$\begin{aligned} \alpha_{j,\mu}^\pm &= i \operatorname{Im}[\mu(\varphi^{(j)}) + q_j] \pm \sqrt{\operatorname{Re}[\mu(\varphi^{(j)}) + q_j]^2 + i\mu(D)} , \\ h^{ab}(\varphi, D) &= \sum_j \sum_{\mu \in \Delta_{hyp}^{(j)}} \sum_{m \in \mathbb{Z}} \frac{\langle \mu, H_{I(a)} \rangle \langle \mu, H_{I(b)} \rangle}{(|\pi m + i\mu(\varphi^{(j)}) + iq_j|^2 + i\mu(D)) (\pi m - i\mu(\bar{\varphi}^{(j)}) - i\bar{q}_j)} , \end{aligned} \quad (6.71)$$

and $H_{I(a)}$ runs over the simple coroots of \mathfrak{g} .

Although the Yukawa coupling of is order $O(e)$, it is required to soak up the λ^1 zero modes. Thus, all other contributions from expanding the exponential of the Yukawa term will be suppressed by additional positive powers of e . Since these higher order terms do not contribute in the limit $e \rightarrow 0$, they must evaluate to zero by the localization principle.

Therefore, putting all of these elements together, the total 1-loop determinant is given by

$$\begin{aligned}
Z_{det}(\varphi) = & \int_{\mathfrak{t}} d^r D \prod_{f=1}^{N_f} \prod_{\mu \in \Delta_{fund}^{(f)}} \sinh(\mu(\varphi^{(f)}) + q_f) \exp \left\{ -\frac{\pi \beta(D, D)}{e^2} + 2\pi i \beta(\xi, D) \right\} \\
& \times \prod_{i=1}^{n-1} \prod_{\alpha \in \Delta_{adj}^{(i)}} \sinh(\alpha(\varphi^{(i)}) + q_i) \prod_{\substack{\alpha \in \Delta_{adj}^{(i)} \\ \alpha \neq 0}} \sinh(\alpha(\varphi^{(i)})) \\
& \times \prod_j \prod_{\mu \in \Delta_{hyp}^{(j)}} \frac{\sinh(\mu(\bar{\varphi}^{(j)}) + \bar{q}_j)}{\sinh(\alpha_{j,\mu}^+) \sinh(\alpha_{j,\mu}^-)} \cdot \det h^{ab}(\varphi, D) .
\end{aligned} \tag{6.72}$$

Regularization

As it turns out, this integral is singular and requires regularization. Physically, this arises because the bosonic part of the Euclidean action is of the form

$$S_{bos} = \frac{1}{e^2} \int dt \left(\bar{\phi} |\partial_t + \varphi + m|^2 \phi + iD(|\phi|^2 - e^2 \xi) + \frac{1}{2} D^2 \right) , \tag{6.73}$$

for a generic bosonic hypermultiplet field ϕ where m is its mass which is generically dependent on \mathfrak{a}, ϵ_+ . Thus, there is a bosonic zero mode when $\varphi = -m, D = 0$. This makes the path integral infinite due to the co-dimension $3r$ singularity.

Therefore, consider the local behavior near finite singularities. These singularities come from the hypermultiplet contribution to the 1-loop determinant where $m = 0$ and are given by the a collection of intersecting singular hyperplanes in $\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}$ located at

$$H_{\mu,j} = \left\{ \varphi \in \mathfrak{t}_{\mathbb{C}}/\Lambda_{cr} \mid D = 0 , \mu(\varphi^{(j)}) + q_j = 0 , \mu \in \Delta_{hyp}^{(j)} \right\} . \tag{6.74}$$

In order to see that this singularity leads to a divergent integral, it is sufficient to study the singularity from a single hyperplane in a transverse plane. In a local coordinate z

centered at the hyperplane, this singularity is of the form

$$\int_{B_R^2} d^2 z \int_{-L}^L dx \frac{1}{(|z|^2 + ix)^2} , \quad (6.75)$$

where $x = \mu(D^{(j)})$ and B_R^2 is an 2D ball of radius R around the origin and L is some finite cutoff. This integral is singular. Therefore, we need to regularize this integral.

One way we can regulate this expression is by shifting the contour of integration for D by $\mathfrak{t} \rightarrow \mathfrak{t} + i\eta$ for $\eta \in \mathfrak{t}$. In this case the singular integral becomes

$$\int_{B_R^2} d^2 z \int_{-L+i\eta}^{L+i\eta} dx \frac{1}{(|z|^2 + ix)^2} = \int_{B_R^2} d^2 z \int_{-L}^L dx \frac{1}{(|z|^2 - y + ix)^2} , \quad (6.76)$$

where $y = \mu(\eta^{(j)})$. This resolves the singularities where $\mu(\eta^{(j)}) < 0$. However, the integrand is still singular along a circle in the complex plane for the case for those μ such that $\mu(\eta^{(j)}) > 0$. This can further be regulated by cutting out the disks $B_\delta^{(sing)}$ of radius $\sqrt{|\mu(\eta^{(i)})|} + \delta$ around the ring singularity and then send $\delta \rightarrow 0$. There are subtleties associated with taking the limit $\delta \rightarrow 0$ which will also require taking $\eta \rightarrow 0$, however we will postpone a discussion until later. Now the regularized path integral is given by

$$I_W^{(Loc)} = \int_{(\mathfrak{t}_\mathbb{C}/\Lambda_{cr}) \setminus B_\delta^{(sing)}} \frac{d^{2r}\varphi}{(2\pi i)^r} \int_{\mathfrak{t}+i\eta} d^r D Z_{int} \exp \left\{ -\frac{\pi\beta(D, D)}{e^2} + 2\pi i\beta(\xi, D) \right\} , \quad (6.77)$$

where $B_\delta^{(sing)}$ is a union of δ -neighborhoods of the singularities of the integrand.

There can also be singularities arising from the infinite volume over \mathfrak{t} and $\mathfrak{t}_\mathbb{C}/\Lambda_{cr}$. Thus, let us examine the behavior of the integrand at $D \rightarrow \partial\mathfrak{t}$. Here, the Gaussian factor will exponentially suppress the integrand and hence there will be no singularity from the D -field.

Now let us examine the behavior of the integrand near $\partial(\mathfrak{t}_\mathbb{C}/\Lambda_{cr})$. Consider the integrand in the limit

$$\tau \rightarrow \infty \quad \text{where} \quad \varphi = \tau \mathfrak{u} \quad , \quad \mathfrak{u} \in \mathfrak{t} \quad (6.78)$$

where \mathfrak{t} is the Lie algebra of \mathfrak{g} which itself decomposes as $\mathfrak{t} = \bigoplus_{i=1}^{p-1} \mathfrak{t}^{(i)}$. As shown in Appendix D.3, the integrand Z_{int} has the limiting form

$$|Z_{int}| \underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathfrak{u}}}{\lesssim} \prod_{i=1}^{n-1} \exp \left\{ 2\tau \left(s(i) - 2 - 2\delta_{s(i),1} - 4\delta_{s(i),2} + \frac{N_f}{2} \delta_{i,i_m} \right) \sum_{a=1}^{k^{(i)}} |\mathfrak{u}_a^{(i)}| \right\} , \quad (6.79)$$

where

$$s(i) = 2k^{(i)} - k^{(i+1)} - k^{(i-1)} \quad , \quad i_m = \frac{n}{2} - 1 \quad . \quad (6.80)$$

Using the fact that $s(i_m) = 0$ or 2 and the fact that $N_f \leq 4$, we see that the exponential factors can at most completely cancel as $\tau \rightarrow \infty$. In this case, the behavior of the 1-loop determinant at infinity will be polynomially suppressed by the Yukawa terms for the hypermultiplet fields to order $O\left(\prod_i \tau^{-3k^{(i)}}\right)$. Therefore, since the measure goes as $\prod_i \tau^{2k^{(i)}-1}$, we have that the product of the integrand and measure will vanish as $O\left(\prod_i \tau^{-k^{(i)}-1}\right)$ and does not contribute infinitely to the localized path integral.

Therefore, excising the δ -neighborhoods $B_\delta^{(sing)}$ clearly resolves all singularities in the integrand and renders its integral finite. However, since we are making a choice of regularization, it is unclear how the resulting integral is related to the true path integral. Therefore, we will refer to this as the localized Witten index, $I_W^{(Loc)}$, to emphasize how it is distinct from the true Witten index.

Remark The D -contour deformation is physically well motivated because introducing a FI-parameter is equivalent to shifting the saddle point of the D integral to $ie^2\xi$. In our regularization prescription, the localization result will generically be dependent on η, ξ . This dependence even persists in the limit $\eta \rightarrow 0, \xi/\beta \rightarrow 0$ as the dependence on the chamber of $\eta, \xi \in \mathfrak{t}^*$ defined by the charges of the hypermultiplet scalars $\mu_i \in \mathfrak{t}^*$ as in the Jeffrey-Kirwan residue prescription.²⁰ This dependence encodes wall crossing in the SQM as studied in [90]. Thus, since the saddle point occurs at $\eta = e^2\xi$, we will restrict $\eta, \xi \in \mathfrak{t}^*$ to be in the same chamber. This is most easily achieved by assuming $\eta = c\xi$ for some positive constant $c \in \mathbb{R}^+$.

6.2.3 Reduction to Contour Integral

Now that we have a well defined volume integral over $\mathfrak{t}_\mathbb{C}/\Lambda_{cr} \times (\mathfrak{t} + i\eta)$, we can utilize the identity

$$\frac{\partial}{\partial \varphi_a} Z_{hyp}^{(kin)} = -iD_b h^{ab} Z_{hyp} \quad , \quad (6.81)$$

²⁰See 21 for more details.

where a, b are indices for a basis of simple coroots, to reduce the volume integral to a contour integral. This allows us to write the 1-loop determinant as a total derivative

$$Z_{int} = \left(\prod_a \frac{1}{iD_a} \bar{\partial}_{\bar{\varphi}_a} \right) Z_{int}^{(kin)} \quad , \quad Z_{int}^{(kin)} = Z_{vec} \cdot Z_{Fermi} \cdot Z_{hyp}^{(kin)} \quad , \quad (6.82)$$

such that the volume integral over $\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}$ can be reduced to a contour integral over the boundaries of the excised δ -neighborhoods and boundary $\partial\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}$

$$\begin{aligned} I_W^{(Loc)} &= \int_{(\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}) \setminus B_{\delta}^{(sing)}} \frac{d^{2r}\varphi}{(2\pi i)^r} \int_{\mathfrak{t}+i\eta} d^r D Z_{int} \exp \left\{ -\frac{\pi\beta(D, D)}{e^2} + 2\pi i\beta(\xi, D) \right\} \quad , \\ &= \int_{(\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}) \setminus B_{\delta}^{(sing)}} \frac{d^{2r}\varphi}{(2\pi i)^r} \int_{\mathfrak{t}+i\eta} \prod_a \frac{dD_a}{iD_a} \partial_{\bar{\varphi}}^{(r)} \left(Z_{int}^{(kin)} \right) \exp \left\{ -\frac{\pi\beta(D, D)}{e^2} + 2\pi i\beta(\xi, D) \right\} \quad , \\ &= \oint_{\substack{\partial B_{\delta}^{(sing)} \\ \cup (-\partial(\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}))}} \frac{d\varphi_1 \wedge \dots \wedge d\varphi_r}{(2\pi i)^r} \int_{\mathfrak{t}+i\eta} \prod_a \frac{dD_a}{iD_a} Z_{int}^{(kin)} \exp \left\{ -\frac{\pi\beta(D, D)}{e^2} + 2\pi i\beta(\xi, D) \right\} \quad , \end{aligned} \quad (6.83)$$

where $r = \text{rnk} \mathfrak{g}$ and a indexes the simple coroots of \mathfrak{t} . Here $B_{\delta}^{(sing)}$ is the neighborhood of radius $\sqrt{|\mu(\eta)|} + \delta$ surrounding each ring singularity in the integrand (where $\mu(\eta) > 0$) and $\partial\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}$ is the (asymptotic) boundary of $\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}$. The identity (6.81) is a consequence of supersymmetry [127, 128, 90, 14, 16, 15].

Consider the contributions from the contour integral around the excised $B_{\delta}^{(sing)}$. These terms are non-zero due to the poles in the 1-loop determinant from the bosonic fields of the hypermultiplets which are of the form:

$$Z_{int} \sim \prod_j \prod_{\mu \in \Delta_{hyp}(j)} \frac{1}{(|\mu(\varphi^{(j)}) + q_j|^2 + \mu(\eta) + i\mu(D'))^2} \quad , \quad D' = D + i\eta \quad , \quad (6.84)$$

for $\mu(\eta) > 0$. In this case, the contour integral over the excised disk of radius $\sqrt{|\mu(\eta)|} + \delta = r + \delta$ where $D' = 0$ is of the form

$$\oint_{\partial B_{\delta}} \frac{\bar{\varphi} d\varphi}{|\varphi|^2 - r^2} = 2\pi i \frac{(r + \delta)^2}{2r\delta + \delta^2} \quad . \quad (6.85)$$

Now we need to take $\delta \rightarrow 0$ as a regulator of the singularity at $|\varphi|^2 = |\mu(\eta)|$. Note that the integral above is infinite unless we take $\sqrt{|\mu(\eta)|} \rightarrow 0$ faster than δ . Therefore, we will define the regularization of the localized path integral with $\sqrt{|\mu(\eta)|} \rightarrow 0$, $\delta \rightarrow 0$, such that $\sqrt{|\mu(\eta)|}/\delta \rightarrow 0$. In this limit, we find that the boundary integrals are equivalent to computing the residue at the singularity with $\sqrt{|\mu(\eta)|}$, $D = 0$.

Now we can evaluate the terms in the integral (6.83) attributed to the poles $\partial B_\delta^{(sing)}$.

By using the fact

$$\lim_{|\eta_a| \rightarrow 0} \frac{1}{D_a + i\eta_a} = P\left(\frac{1}{D_a}\right) - i\pi \operatorname{sign}(\eta_a) \delta(D_a) , \quad (6.86)$$

we get

$$\begin{aligned} I_W^{(Loc),sing} &= \oint_{\partial B_\delta^{(sing)}} \frac{d\varphi_1 \wedge \dots \wedge d\varphi_r}{(2\pi i)^r} \int_{\mathfrak{t} + i\eta} \prod_a dD_a \delta(D_a) Z_{int}^{(kin)} \\ &\quad \times \exp \left\{ -\frac{\pi\beta(D, D)}{e^2} + 2\pi i \beta(\xi, D) \right\} \\ &\quad + \left\{ \text{Principal Terms in } 1/D \right\} , \end{aligned} \quad (6.87)$$

where principal terms are those that have a principal value of some D_a . Here, the principal value term vanishes because integrand does not have a singularity of sufficiently high codimension in φ and hence the contour integral over φ is identically zero.

Therefore, we find that the terms coming from the excised disks is exactly

$$I_W^{(Loc),sing} = \oint_{\partial B_\delta^{(sing)}} \frac{d\varphi_1 \wedge \dots \wedge d\varphi_r}{(2\pi i)^r} Z_{int}^{(kin)}(\varphi, D=0) , \quad (6.88)$$

which reduces to a sum over residues of $Z_{int}^{(kin)}(\varphi, D=0)$.

This sum over residues is equivalent to the Jeffrey-Kirwan residue prescription [98]. The reason is that the contour integral simply picks out tuples of poles for which $\mu(\eta) > 0$ – or equivalently it picks poles corresponding to given tuples of $\{\mu_p\}_{p=1}^{\text{rnkg}}$ such that $\mu_p(\eta) > 0, \forall p$. By mapping $\eta \in \mathfrak{t}$ to $\eta^\vee \in \mathfrak{t}^*$ by the Killing form, this is equivalent to the statement that the contour integral includes tuples of poles such that η^\vee is in the positive cone defined by the $\{\mu_p\}$. This is the definition of the JK residue prescription [98].²¹

²¹ The Jeffrey-Kirwan residue prescription selects a contour that such that the integral evaluates to a sum of residues corresponding a particular set of poles specified by a parameter $\xi \in \mathfrak{t}^*$. These are selected as follows. Consider a contour integral over an r -complex dimensional space. The poles of the integrand are solutions of the equations

$$Q_i(\varphi) + f_i(q) = 0 , \quad (6.89)$$

for some set of $Q_i \in \mathfrak{t}^*$ and $f_i(q)$ functions of some parameters q_j . Each of these poles defines a hyperplane in \mathfrak{t} along which the integrand is singular. To each hyperplane specified by the solution of (6.89), we associate the charge $Q_i \in \mathfrak{t}^*$. Any set of r linearly independent $\{Q_i\} \in \mathfrak{t}^*$ defines a positive cone in $\mathcal{C}_{\{Q_i\}} \subset \mathfrak{t}^*$. Each such cone corresponds to the intersection of r hyperplanes, which has a non-trivial residue.

The Jeffrey-Kirwan prescription specified by the $\xi \in \mathfrak{t}^*$ picks a contour such that the contour integral evaluates to the sum of residues associated to all cones $\mathcal{C}_{\{Q_i\}}$ such that $\xi \in \mathcal{C}_{\{Q_i\}}$ weighted by the sign of the determinant $\operatorname{sgn}(Q_{i_1} \wedge \dots \wedge Q_{i_r})$.

Boundary Terms at Infinity

Now consider the contributions to the contour integral from the boundary $\partial\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}$. For simplicity we will consider only the case of a $U(1)$ gauge theory as it is our main example. However, the following analysis in the next two sections generalizes to generic gauge groups. We will comment more on this later and continue to use notation that accommodates this generalization.

Here we are considering the integral

$$Z_{bnd} = \oint_{-\partial(\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr})} \frac{d\varphi}{2\pi i} \int_{\mathbb{R}+i\eta} \frac{dD}{iD} Z_{int}^{(kin)}(\varphi, D) e^{-\frac{\pi\beta D^2}{e^2} + 2\pi i\beta\xi D} \quad (6.90)$$

where ²²

$$Z_{int}^{(kin)}(\varphi, D) = 2 \sinh(2\epsilon_+) \prod_{f=1}^{N_f} \sinh(\mu_f(\varphi) - m_f) \times \prod_{j=1}^4 \frac{\sinh(\mu_j(\bar{\varphi}) + \bar{q}_j)}{\cosh(2i \operatorname{Im}[\mu_j(\varphi) + q_j]) - \cosh(2\sqrt{\operatorname{Re}[\mu_j(\varphi) + q_j]^2 + i\mu_j(D)})} . \quad (6.91)$$

Here j indexes over the representations of the 4 different $\mathcal{N} = (0, 4)$ fundamental chiral multiplets making up the two $\mathcal{N} = (0, 4)$ fundamental hypermultiplets.

In the limit $\operatorname{Re}[\varphi] \rightarrow \pm\infty$, the function $Z_{int}^{(kin)}(\varphi, D)$ is independent of D and is the 0th order coefficient of the Laurent expansion in e^φ . Thus, the boundary integral, which is evaluated in the limit $\operatorname{Re}[\varphi] \rightarrow \pm\infty$, is given by

$$\begin{aligned} Z_{bnd} &= \sum_{\pm} \lim_{\operatorname{Re}[\varphi] \rightarrow \pm\infty} \left(\pm Z_{int}^{(kin)} \right) \left(c(\eta) - \operatorname{erf} \left(\sqrt{\pi\beta} e\xi \right) \right) , \\ &= \sum_{\pm} \pm \lim_{\operatorname{Re}[\varphi] \rightarrow \pm\infty} \left(2 \sinh(2\epsilon_+) e^{(\sum_f |\mu_f| - \sum_j |\mu_j|) \operatorname{Re}[\varphi]} \right) e^{\pm \sum_f \operatorname{sign}(\mu_f) \bar{q}_f \mp \sum_j \operatorname{sign}(\mu_j) q_j} \\ &\quad \times \left(c(\eta) - \operatorname{erf} \left(\sqrt{\pi\beta} e\xi \right) \right) , \end{aligned} \quad (6.92)$$

where we have applied the formula in for the integral in Appendix D.4 and $\operatorname{erfc}^+(x)$ is the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x du e^{-u^2} . \quad (6.93)$$

²²Note that the contribution from the vector multiplet fermions is only given by $\sinh(2\epsilon_+)$ since the adjoint action is trivial for a $U(1)$ gauge group.

Here

$$c(\eta) = \begin{cases} 1 & \eta > 0 \\ -1 & \eta < 0 \end{cases} . \quad (6.94)$$

See Appendix D.4 for more details.

By using the fact that in our models

$$\sum_f |\mu_f| - \sum_i |\mu_i| = N_f - 4 , \quad (6.95)$$

we see that this boundary term is only non-zero when $N_f = 4$.²³

In summary, by carefully performing the localization computation of $I_W \rightarrow I_W^{(Loc)}$ by regularization, we find that

$$I_W^{(Loc)} = Z^{JK} + Z_{bnd} , \quad (6.97)$$

where Z^{JK} is the result from the Jeffrey-Kirwan residue prescription and Z_{bnd} is the boundary computation computed in (6.92).

Note that Z_{bnd} has explicit β, e, ξ dependence. Generically one would expect that the answer is independent of these parameters since the Lagrangian is Q -exact and hence variations of β, e, ξ are supersymmetric deformations of the action. However, this dependence can arise from a continuous spectrum of states which allows for a spectral asymmetry between bosonic and fermionic states [2]. As we previously discussed, our models have such a continuous spectrum of states arising from the non-compact directions in the mixed and Coulomb branches.

Remark Note that if we had instead identified $\eta = e^2 \xi$, regularity would have required us to take the limit $e^2 \rightarrow 0$. Then we would find that $I_W^{(Loc)} = Z^{JK} + Z_{bnd}(\beta = 0)$. This matches with the analysis of [113] in which the authors found that the localization computation of the Witten index, under a specific choice of regulator, can be identified with the computation of the Witten index in the limit $\beta \rightarrow 0$.

²³We can additionally consider the effect of including Chern-Simons terms as in Section 6.4. A Chern-Simons term with level k shifts the argument of (6.92) by a factor $e^{2k\text{Re}[\varphi]}$. Thus we have that this boundary term is only non-zero when

$$\sum_f |\mu_f| - \sum_i |\mu_i| + 2k = N_f - 4 + 2k = 0 . \quad (6.96)$$

Comparison with Literature

Let us take a moment to compare our results with that of the literature [14, 15]. In these papers, the authors give a physical derivation of the Jeffrey-Kirwan residue prescription for the elliptic genus of 2D $\mathcal{N} = (0, 2)$ gauge theories. Here the authors consider the localized path integral in the limit $e^2 \rightarrow 0$ over $(\mathfrak{t}_{\mathbb{C}}/\Lambda_r) \times \mathfrak{t}$ which they decompose as

$$\begin{aligned} Z &= \int_{\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}} \frac{d^{2r}\varphi}{(2\pi i)^r} \int_{\mathfrak{t}} d^r D Z_{int}(\varphi, D) e^{-\pi \frac{\beta D^2}{e^2} + 2\pi i \beta \xi D} \\ &= \int_{(\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}) \setminus B_{\delta}^{(sing)}} \frac{d^{2r}\varphi}{(2\pi i)^r} \int_{\mathfrak{t}} d^r D Z_{int}(\varphi, D) e^{-\frac{\pi \beta D^2}{e^2} + 2\pi i \beta \xi D} \\ &\quad + \int_{B_{\delta}^{(sing)}} \frac{d^{2r}\varphi}{(2\pi i)^r} \int_{\mathfrak{t}} d^r D Z_{int}(\varphi, D) e^{-\frac{\pi \beta D^2}{e^2} + 2\pi i \beta \xi D}, \end{aligned} \quad (6.98)$$

where $B_{\delta}^{(sing)}$ is the collection of δ -neighborhoods of the singularities as before.

After dropping the singular term, which the authors argue can be regularized to zero, the path integral to

$$Z = \int_{(\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}) \setminus B_{\delta}^{(sing)}} \frac{d^{2r}\varphi}{(2\pi i)^r} \int_{\mathfrak{t}} d^r D Z_{int}(\varphi, D) e^{-\frac{\pi \beta D^2}{e^2} + 2\pi i \beta \xi D}, \quad (6.99)$$

which is the same as (6.77). The authors then also deform the D -contour and reduce the path integral to a contour integral around $\partial B_{\delta}^{(sing)}$. They then show by contour integral methods that (6.99) reduces to a sum of residues according to the Jeffrey Kirwan residue prescription as above.

In our analysis we also take into account the possibility of an additional contribution coming from the asymptotic boundary of $\partial(\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr})$, while the models studied in [14, 15] have a compact target space so such terms do not arise. Similar boundary terms are also discussed for some models in [90, 16]. However, the analysis of these papers is not directly applicable to our model.

6.2.4 Coulomb and Higgs Branch States

In our discussion we often use the terminology such as ‘‘Higgs branch states’’ and ‘‘Coulomb branch states.’’ Here we will define this terminology precisely.

Pick a bubbling SQM and consider the family of quantum systems defined by the varying with respect to e, β, ξ . Due to the localization principle, the states that survive

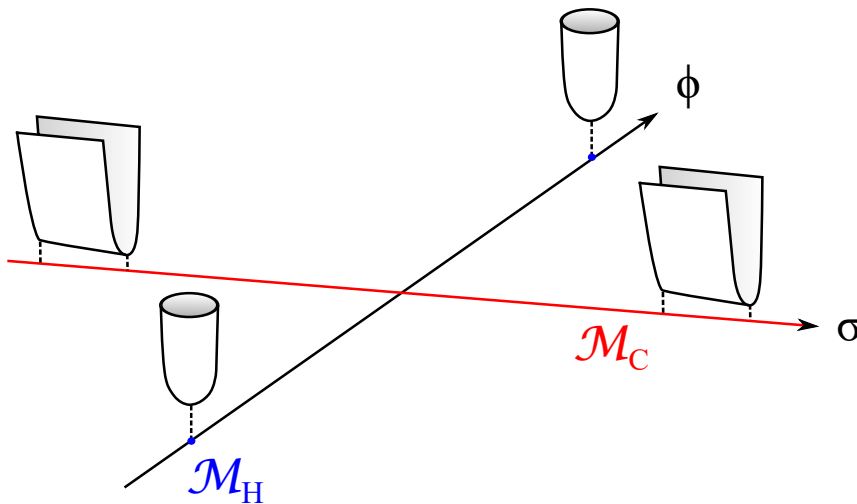


Figure 6.6: This figure illustrates the behavior of the potentials on the Higgs (blue) and Coulomb (red) branch.

in the limit $e^2|\xi|^3 \rightarrow 0$ should be the only ones that give non-canceling contributions to the Witten due to the localization principle.²⁴ As we monotonically approach $e^2|\xi|^3 \rightarrow 0$ with ξ fixed, the potential energy function of these families approaches an infinite value on all of field space except along the Higgs, Coulomb, and mixed branches as discussed in Section 6.2.2. See Figure 6.6.

As we decrease $e^2|\xi|^3 \rightarrow 0$, the potential around each component of the Higgs branch (which is topologically a collection of points) approaches an infinitely steep harmonic oscillator potential for all fields. In this limit, the potential additionally becomes flat along the Coulomb and mixed branches (which are non-compact) while simultaneously approaching an infinitely steep harmonic potential in the transverse directions.

Now consider the spectral decomposition of the Hilbert space defined by the family of Hamiltonians parametrized by e, ξ . Due to the behavior of the potential function near the different classical vacua, there will be orthogonal projection operators P^H, P^C, P^{mix} onto a space of eigenstates of the Hamiltonian such that the wave functions of states in this subspace have support localized near the classical Higgs, Coulomb, and mixed

²⁴Here we are taking our fixed length scale to be set by ξ in anticipation of the next section where we allow β to vary.

vacua respectively. We will refer to states in the image of P^H as “Higgs branch states,” those in the image of P^C as “Coulomb branch states,” and those in the image of P^{mix} as “mixed branch states.”

In fact, we believe there are projectors $P^{C,+} \geq P^C$ and $P^{mix,+} \geq P^{mix}$ to a subspace on which the Hamiltonian has a continuous spectrum but such that all the states in the image have wave functions with support localizing to a neighborhood of the Coulomb branch or mixed branch respectively.

Of course, in the spectral decomposition of the Hilbert space defined by the Hamiltonian, there will additionally be a projection operator P^M to a subspace on which the spectrum of the Hamiltonian is continuous and bounded below by a large constant M such that states in the image of P^M will have support throughout field space and are not in any sense localized near either branch. However, the mass gap M to the unlocalized, continuum of states goes to infinity as $e^2|\xi|^3 \rightarrow 0$ and hence gives vanishing contribution to the Witten index.

Physical Interpretation of Jeffrey-Kirwan Residues

The Jeffrey-Kirwan prescription for computing the path integral counts the BPS states that are localized on the Higgs branch. The reason is that the residues that are summed over in the JK residue prescription are in one-to-one correspondence with the unlifted Higgs branch vacua.

Consider the integrand Z_{int} in (6.66). This has poles along the hyperplanes

$$H_{\mu,j} = \{\varphi \in \mathfrak{t}_{\mathbb{C}}/\Lambda_{cr} \mid \mu_j(\varphi) + q_j = 0\} \ , \ \mu_j \in \Delta_{hyper}^{(j)} \ , \quad (6.100)$$

where q_j is the global charge for the j^{th} hypermultiplet (or equivalently its mass). The JK residue formula specified by the FI-parameter $\xi \in \mathfrak{t}^*$ then selects the residue given by r -tuples of poles corresponding to a codimension r intersection of r hyperplanes $H_{\mu,j}$ such that

$$(\xi, \mu_j) > 0 \ , \ \text{for each } \mu_j \ . \quad (6.101)$$

Physically, the hyperplanes $H_{\mu,j}$ define the locus in field space where the corresponding hypermultiplet field becomes massless:

$$0 = |(\mu_j(\sigma) + aQ_a + \epsilon_+ Q_\epsilon) \cdot \Phi_j|^2, \quad (6.102)$$

where j indexes over the fundamental and bifundamental scalar fields $\Phi_j \in \{\phi^{(i)}, \tilde{\phi}^{(i)}, \underline{\phi}^{(i)}, \underline{\tilde{\phi}}^{(i)}\}$ that have charge μ_j and $(\Phi_j, \bar{\Phi}_j)$ forms an $SU(2)_R$ doublet. Since $\text{rk} \mathfrak{g} = r$, and each Φ_i has a different mass, there can only be r simultaneously massless hypermultiplet fields. This corresponds to the statement that there are at most codimension r intersections of the hyperplanes $H_{\mu,j}$.

Now consider the D- and F-term equations for the Higgs branch. These can be written as

$$\begin{aligned} 0 &= - \sum_{j: (\xi^{(i)}, \mu_j) > 0} |\Phi_j|^2 + \sum_{j: (\xi^{(i)}, \mu_j) < 0} |\Phi_j|^2 + e^2 |\xi^{(i)}| \\ 0 &= \sum_{j: (\xi^{(i)}, \mu_j) > 0} \Phi_j \tilde{\Phi}_j - \sum_{j: (\xi^{(i)}, \mu_j) < 0} \Phi_j \tilde{\Phi}_j. \end{aligned} \quad (6.103)$$

As in the JK-prescription, the solutions of these equations where at most r Φ_j are massless are enumerated by an r -tuple Φ_j which obey

$$(\xi, \mu_j) > 0 \text{ for each } j. \quad (6.104)$$

This enumerates the entire resolved Higgs branch with respect to an FI-parameter $\xi \in \mathfrak{t}^*$. Therefore, the JK-residues are in one-to-one correspondence with the points on the Higgs branch.

Now note that the Jeffrey-Kirwan residue computation is independent of the value of e^2 . Thus, in the limit $e^2 \rightarrow 0$, we can identify states as being localized to a single vacuum branch in field space. Thus, we can identify each residue of the JK-prescription as counting the states that are localized on the corresponding point of the Higgs branch in the limit $e^2 \rightarrow 0$:

$$Z^{JK} = I_{Higgs}. \quad (6.105)$$

Remark Note that the interpretation of the Z^{JK} as an object counting the contribution of Higgs branch states matches with the previous analysis by taking a limit of

$e^2\beta^3 \rightarrow 0, \xi/\beta \rightarrow \infty$ with β fixed such that $e^2\xi = \xi'$ is constant.²⁵ In the effective SQM on the non-compact branches, the mass of the ground states is given by

$$m \sim \sqrt{e|\xi|} \rightarrow \infty \quad \text{as} \quad e^2\beta^3, \beta/\xi \rightarrow 0, \quad \text{with } e^2\xi = \xi', \beta \text{ fixed.} \quad (6.106)$$

Thus in taking this limit, all states on the non-compact branches are killed and $I_{asympt} \rightarrow 0$. Similarly, if we were to compute the standard Witten index, taking this limit kills the boundary terms. Thus

$$\lim_{\substack{e^2\beta^3 \rightarrow 0 \\ \xi/\beta \rightarrow \infty \\ e^2\xi = \xi'}} I_W = I_{Higgs} = Z^{JK}. \quad (6.107)$$

We stress that this is not the appropriate limit for computing $Z_{mono}(P, \mathbf{v})$. See [90] for more details regarding the computation of the Witten index in this limit.

6.2.5 Ground State Index

As shown in [160, 2, 178], the Witten index has to be handled with care in the case of a SQM with a continuous spectrum. As in our case, we have found that when there is a continuous spectrum, there can be a spectral asymmetry that gives rise to non-trivial β , e , and ξ dependence. Note that in order to compute the Witten index, we introduced the FI-parameter ξ . In the 4D picture this corresponds to separating the insertions of the minimal 't Hooft defects that make up the reducible 't Hooft defect. Thus, to compute $Z_{mono}(P, \mathbf{v})$, we want to take the limit “ $\xi \rightarrow 0$ ” which is formally given by the limit $\xi/\beta \rightarrow 0$. However, this is computationally indistinguishable from taking $\beta/\xi \rightarrow \infty$ with ξ fixed. Thus, $Z_{mono}(P, \mathbf{v})$ can be identified with the Witten index in the limit “ $\beta \rightarrow \infty$ ”

$$Z_{mono}(P, \mathbf{v}) = I_{\mathcal{H}_0} := \lim_{\beta \rightarrow \infty} \text{Tr}_{\mathcal{H}} (-1)^F e^{-\frac{\beta}{2}\{\mathcal{Q}, \mathcal{Q}\} + \mathfrak{a}Q_{\mathfrak{a}} + \epsilon_+ J_+ + m \cdot F}. \quad (6.108)$$

In the limit $\beta \rightarrow \infty$, contributions from all non-BPS states are completely suppressed. This effectively restricts the Witten index to a trace over the Hilbert space of BPS ground states. We will refer to the Witten index in this limit, $I_{\mathcal{H}_0}$, as the *ground*

²⁵Note that this is different from the rest of our analysis where we take ξ to be the fixed length scale.

state index. This matches with the fact that the result computed via AGT [3, 4] is independent of β , suggesting that we should only have contributions from BPS states.

As in the case of the Witten index, the ground state index can be computed as (the limit of) a path integral. Thanks to supersymmetry, one can attempt to compute that path integral by using localization. Again, using localization requires a choice of regularization²⁶ and hence we will refer to the result of the localization computation as $I_{\mathcal{H}_0}^{(Loc)}$ to distinguish it from the true ground state index.

This limit of the Witten index can be easily computed using our analysis from the previous section. Recall that $I_W^{(Loc)} = Z^{JK} + Z_{bnd}$. Since the Z^{JK} term is independent of β , taking the limit $\beta \rightarrow \infty$ only affects Z_{bnd} . The limit of the boundary term can be computed

$$\lim_{\beta \rightarrow \infty} Z_{bnd} = \lim_{\beta \rightarrow \infty} \sum_{\pm} \lim_{Re[\varphi] \rightarrow \pm\infty} \left(\pm Z_{int}^{(kin)} \right) \left(c(\eta) + \operatorname{erf} \left(\sqrt{\pi\beta} e\xi \right) \right) = 0 , \quad (6.109)$$

where

$$c(\eta) = \begin{cases} 1 & \eta > 0 \\ -1 & \eta < 0 \end{cases} . \quad (6.110)$$

Using the fact that

$$\lim_{\beta \rightarrow \infty} \left(c(\eta) - \operatorname{erf} \left(\sqrt{\pi\beta} e\xi \right) \right) = \begin{cases} 0 & \xi \times \eta > 0 \\ -2 & \xi > 0 > \eta \\ 2 & \eta > 0 > \xi \end{cases} , \quad (6.111)$$

we see that

$$\lim_{\beta \rightarrow \infty} Z_{bnd} = 2 \sinh(2\epsilon_+) \sinh \left(\sum_f m_f \right) \times \begin{cases} 0 & \xi \times \eta > 0 \\ -2 & \xi > 0 > \eta \\ 2 & \eta > 0 > \xi \end{cases} . \quad (6.112)$$

By identifying $\eta \sim e^2 \xi$, we find that

$$\lim_{\beta \rightarrow \infty} Z_{bnd} = 0 , \quad (6.113)$$

²⁶We will be taking the same choice of regularization as in the case of the Witten Index.

and hence that the localization computation of the ground state index is given by

$$I_{\mathcal{H}_0}^{(Loc)} = Z^{JK} . \quad (6.114)$$

Remark Although we have only shown that the boundary contribution vanishes for a SQM with a $U(1)$ gauge group, this result holds in general. One can see that the boundary contributions vanish more generally in the limit $\beta \rightarrow \infty$ as follows. Decompose $\partial\widehat{\mathcal{M}}$ into a disjoint union of open sets of increasing codimension $\partial\widehat{\mathcal{M}} = \partial\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr} = \coprod_i (\partial\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr})^{(i)}$. For each boundary component, the contour integral is of the form the contour integral is of the form

$$Z_{bnd}^{(i)} = \oint_{(\partial\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr})^{(i)}} \frac{d\varphi_1 \wedge \dots \wedge d\varphi_r}{(2\pi i)^r} \int_{\mathfrak{t}+i\eta} \prod_a \frac{dD_a}{D_a} Z_{int}^{(kin)}(\varphi, D) e^{-\frac{\pi\beta(D,D)}{e^2} + 2\pi i\beta(\xi, D)} , \quad (6.115)$$

where a indexes over the simple coroots of \mathfrak{t} . On each component, there exists a simple root $\alpha \in \Phi^+$ such that $|\langle \alpha, \varphi \rangle| \rightarrow \infty$ on $(\partial\mathfrak{t}_{\mathbb{C}}/\Lambda_{cr})^{(i)}$. Thus, in each such integral, $Z_{det}^{(kin)}$ will be independent of $\langle \alpha, D \rangle$ for some positive root $\alpha \in \Phi^+$. This means that each boundary integral will be proportional to a factor of

$$Z_{bnd}^{(i)} \sim \left(c(\langle \alpha, \eta \rangle) - \operatorname{erf} \left(\sqrt{\pi\beta} e \langle \alpha, \xi \rangle \right) \right) , \quad (6.116)$$

for some positive root $\alpha \in \Phi^+$ where $c(\langle \alpha, \eta \rangle)$ is defined in (6.110). By identifying $\eta \sim e^2 \xi$, this factor completely suppresses all boundary terms in the limit $\beta \rightarrow \infty$. Thus, the boundary terms vanish in the localization computation of the ground state index

$$\lim_{\beta \rightarrow \infty} Z_{bnd}^{(i)} = 0 \quad \forall i , \quad (6.117)$$

and therefore the ground state index is generally given by the Jeffrey-Kirwan residue formula

$$I_{\mathcal{H}_0}^{(Loc)} = Z^{JK} . \quad (6.118)$$

6.2.6 Summary

In this section we reviewed the localization computation for the Witten index of bubbling SQMs. In summary, we found:

1. The localized integral over the BPS moduli space is not well defined: it requires regularization. In general, the regularized Witten index, $I_W^{(Loc)}$, will differ from the true Witten index, I_W .
2. Under the choice of regularization we have presented, one arrives at the JK residue prescription plus a β -dependent boundary term that indicates the existence of a continuous spectrum of excited states: $I_W^{(Loc)} = Z^{JK} + Z_{bnd}$.
3. Since the AGT computation shows that $Z_{mono}(P, \mathbf{v})$ is independent of β , we conjecture that $Z_{mono}(P, \mathbf{v})$ should only count contributions from BPS states. Therefore, we identify $Z_{mono}(P, \mathbf{v})$ as the ground state index $I_{\mathcal{H}_0}$ which eliminates contributions from non-ground states by taking the limit as $\beta \rightarrow \infty$ of the Witten index. By direct computation, we find that in this limit, the localization computation of the ground state index is given by the Jeffrey-Kirwan residue prescription:

$$I_{\mathcal{H}_0}^{(Loc)} = \lim_{\beta \rightarrow \infty} (Z^{JK} + Z_{bnd}) = Z^{JK} , \quad (6.119)$$

which we identify as counting the states localized on the Higgs branch

$$I_{\mathcal{H}_0} = Z^{JK} = I_{Higgs} . \quad (6.120)$$

After regularization with $\eta \sim e^2 \xi$, we have

$$I_{\mathcal{H}_0}^{(Loc)} = \oint_{JK(\xi)} \frac{d\varphi_1 \wedge \dots d\varphi_r}{(2\pi i)^r} Z_{vec}(\varphi) Z_{Fermi}(\varphi) \tilde{Z}_{hyper}^{(kin)}(\varphi) , \quad (6.121)$$

where

$$\begin{aligned} Z_{vec} &= \prod_{i=1}^{n-1} \prod_{\alpha \in \Delta_{adj}^{(i)}} \sinh(\alpha(\varphi^{(i)}) + q_i) \prod_{\substack{\alpha \in \Delta_{adj}^{(i)} \\ \alpha \neq 0}} \sinh(\alpha(\varphi^{(i)})) , \\ Z_{Fermi} &= \prod_{f=1}^{N_f} \prod_{\mu \in \Delta_{fund}^{(f)}} \sinh(\mu(\varphi^{(f)}) + q_f) , \\ \tilde{Z}_{hyp}^{(kin)} &= \prod_j \prod_{\mu \in \Delta_{hyp}^{(j)}} \frac{1}{\sinh(\mu(\varphi^{(j)}) + q_j)} , \end{aligned} \quad (6.122)$$

and $\xi \in \mathfrak{t}^*$.

6.2.7 Examples: $SU(2)$ $N_f = 4$ Theory

Now we have eliminated the β dependence of the localization computation of $Z_{mono}(P, \mathbf{v})$ by identifying $Z_{mono} = I_{\mathcal{H}_0}$. Nevertheless, in general, the localization computation $I_{\mathcal{H}_0}^{(Loc)}$ still does not generically agree with $Z_{mono}(P, \mathbf{v})$ as computed from AGT. We will now illustrate this claim with several non-trivial examples in the $SU(2)$ $N_f = 4$ theory to show that the localization calculation for the ground state index $I_{\mathcal{H}_0}^{(Loc)} = Z^{JK}$ does not match with the results from the AGT computations [97]. These examples are an explicit realization of a generic feature of 't Hooft defects in $\mathcal{N} = 2$ $SU(N)$ gauge theories with $N_f = 2N$ fundamental hypermultiplets.

$$Z_{mono}(1, 0)$$

Consider the $L_{1,0}$ (minimal) 't Hooft defect in the $SU(2)$ $N_f = 4$ theory. This has 't Hooft charge

$$P = h^1 = 2\hat{h}^1 = \text{diag}(1, -1) \quad , \quad h^1 \in \Lambda_{mw} \quad , \quad \hat{h}^1 \in \Lambda_{cochar} \quad . \quad (6.123)$$

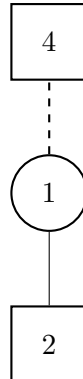
From general considerations, the expectation value of $L_{1,0}$ is of the form

$$\langle L_{1,0} \rangle = \left(e^{\mathbf{b}} + e^{-\mathbf{b}} \right) F(\mathbf{a}, m_f, \epsilon_+) + Z_{mono}(1, 0) \quad , \quad (6.124)$$

where $Z_{mono}(1, 0)$ corresponds to the bubbling with $\mathbf{v} = \text{diag}(0, 0)$. Here

$$F(\mathbf{a}, m_f, \epsilon_+) = \left(\frac{\prod_{\pm} \prod_{f=1}^4 \sinh(\mathbf{a} \pm m_f)}{\sinh^2(2\mathbf{a}) \prod_{\pm} \sinh(2\mathbf{a} \pm 2\epsilon_+)} \right)^{\frac{1}{2}} \quad . \quad (6.125)$$

In this example, the monopole bubbling contribution can be computed as the Witten index of the $\mathcal{N} = (0, 4)$ SQM described by the quiver:



Now the path integral from the previous section reduces to the contour integral

$$Z_{mono}^{(Loc)}(1, 0) = \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} Z_{vec}(\varphi, \mathbf{a}, \epsilon_+) Z_{hyper}(\varphi, \mathbf{a}) Z_{Fermi}(\varphi, \epsilon_+, m_i) . \quad (6.126)$$

where $JK(\xi)$ is the Jeffrey-Kirwan residue prescription specified by a choice of $\xi \in \mathfrak{t}^* \cong \mathbb{R}$ [98, 113, 90].

The general contributions of the different $\mathcal{N} = (0, 2)$ multiplets for a SQM labeled by gauge nodes $(k^{(1)}, \dots, k^{(n-1)})$ and fundamental hypermultiplet nodes (w_1, \dots, w_p) are given by [94, 23]

$$\begin{aligned} Z_{vec}(\varphi, \epsilon_+) &= \prod_{i=1}^n \left[\prod_{a \neq b=1}^{k^{(i)}} \right]' 2 \sinh(\varphi_{ab}^{(i)}) \times \prod_{i=1}^n \prod_{a \neq b=1}^{k^{(i)}} \sinh(\varphi_{ab}^{(i)} + 2\epsilon_+) , \\ Z_{Fermi}(\varphi, m_f, \epsilon_+) &= \prod_{f=1}^4 \prod_{a=1}^{k^{(f)}} 2 \sinh(\varphi_a^{(f)} - m_f) , \\ Z_{fund}(\varphi, \mathbf{a}, \epsilon_+) &= \prod_j \prod_{a=1}^{k^{(j)}} \prod_{\ell=1}^{w_j} \prod_{\pm} \frac{1}{2 \sinh(\pm(\varphi_a^{(j)} - \mathbf{a}_\ell) + \epsilon_+)} , \\ Z_{bifund}(\varphi, \mathbf{a}, \epsilon_+) &= \prod_{i=1}^{n-1} \prod_{a=1}^{k^{(i+1)}} \prod_{b=1}^{k^{(i)}} \prod_{\pm} \frac{1}{2 \sinh(\pm(\varphi_a^{(i+1)} - \varphi_b^{(i)}) + \epsilon_+)} , \end{aligned} \quad (6.127)$$

where the product $\left[\prod_{I,J} \right]'$ omits factors of 0, $\mathbf{a} = \text{diag}(\mathbf{a}_1, \mathbf{a}_2) \in \text{Lie}[SU(2)]$, the fundamental Fermi-multiplets couple to the f^{th} gauge group, and j indexes over the fundamental hypermultiplets (which couple to $\varphi^{(j)}$). For our SQM, this reduces to

$$Z_{mono}^{(Loc)}(1, 0) = \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} 2 \sinh(2\epsilon_+) \frac{\prod_{f=1}^4 \sinh(\varphi - m_f)}{\prod_{\pm} \sinh(\varphi \pm \mathbf{a} + \epsilon_+) \sinh(-\varphi \pm \mathbf{a} + \epsilon_+)} . \quad (6.128)$$

Here the JK residue prescription is determined by a choice of $\xi \in \mathbb{R}$ which corresponds to introducing an FI-parameter in the SQM. As shown in [90], the Witten index of an SQM can generically have wall crossing as ξ jumps between $\xi \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^-$.

Using this, the localization computation becomes

$$Z_{mono}^{(Loc)}(1, 0) = -4 \frac{\prod_f \sinh(\mathbf{a} - m_f \mp \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \mp 2\epsilon_+)} - 4 \frac{\prod_f \sinh(\mathbf{a} + m_f \pm \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \pm 2\epsilon_+)} : \quad \pm \xi > 0 . \quad (6.129)$$

This function is not symmetric under the action of the Weyl group of the flavor symmetry group ²⁷ and is not invariant under the choice of $\xi \in \mathbb{R}$. Therefore, this cannot

²⁷The Weyl group under the $SO(8)$ flavor symmetry is generated by $m_i \leftrightarrow m_{i+1}$ and $m_3 \leftrightarrow -m_4$ [97, 156].

be the correct form of $Z_{mono}(1, 0)$.

From the AGT result presented above, we know that the correct $Z_{mono}(1, 0)$ is given by

$$Z_{mono}(1, 0) = -4 \frac{\prod_f \sinh(\mathfrak{a} - m_f \mp \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \mp 2\epsilon_+)} - 4 \frac{\prod_f \sinh(\mathfrak{a} + m_f \pm \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \pm 2\epsilon_+)} + 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right) : \quad \pm \xi > 0 . \quad (6.130)$$

This answer for $Z_{mono}(1, 0)$ is surprisingly independent of the choice of ξ :

$$Z_{mono}(1, 0; \xi > 0) - Z_{mono}(1, 0; \xi < 0) = 0 . \quad (6.131)$$

Clearly the AGT result for $Z_{mono}(1, 0)$ in (6.130) does not match the localization result (6.129) due to the “extra term” $2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right)$. Therefore, as noted in [97], there is a discrepancy between the localization and AGT computation for $Z_{mono}(1, 0)$.

$Z_{mono}(2, 1)$

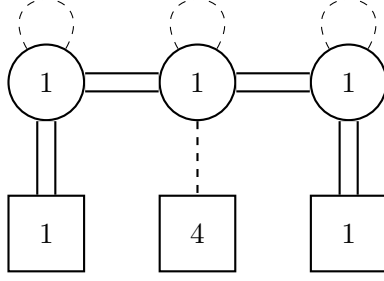
Now consider the $L_{2,0}$ line defect. This defect has ’t Hooft charge

$$P = 2h^1 = 4\hat{h}^1 = \text{diag}(2, -2) \quad , \quad h^1 \in A_{mw} \quad , \quad \hat{h}^1 \in A_{cochar} . \quad (6.132)$$

The expectation value of this line defect has two different monopole bubbling contributions:

$$\langle L_{2,0} \rangle = \left(e^{2\mathfrak{b}} + e^{-2\mathfrak{b}} \right) F(\mathfrak{a}, m_f)^2 + \left(e^{\mathfrak{b}} + e^{-\mathfrak{b}} \right) F(\mathfrak{a}, m_f) Z_{mono}(2, 1) + Z_{mono}(2, 0) , \quad (6.133)$$

where $Z_{mono}(2, v)$ is $Z_{mono}(\mathfrak{a}, m_f, \epsilon_+; P, \mathfrak{v})$ for $\mathfrak{v} = \text{diag}(v, -v)$. Here we will only be interested in the term $Z_{mono}(\mathfrak{a}, m_f, \epsilon_+; 2, 1)$. In this case, the relevant SQM is given by



The contour integral from localization of this SQM is of the form

$$Z^{JK}(2, 1) = \frac{1}{2} \oint_{JK(\vec{\xi})} \left(\prod_{i=1}^3 \frac{d\varphi_i}{2\pi i} \right) \sinh^3(2\epsilon_+) \frac{\prod_{f=1}^4 \sinh(\varphi_2 - m_f)}{\sinh(\pm\varphi_{21} + \epsilon_+) \sinh(\pm\varphi_{32} + \epsilon_+)} \times \frac{1}{\sinh(\pm(\varphi_1 - \mathfrak{a}_2) + \epsilon_+) \sinh(\pm(\varphi_3 - \mathfrak{a}_1) + \epsilon_+)} . \quad (6.134)$$

Evaluating the above contour integral requires a choice of parameter $\vec{\xi} \in \mathbb{R}^3$ that specifies the JK residue prescription. Due to the intricate dependence on the choice of $\vec{\xi}$, we will examine this in the simple sectors of $\xi_i > 0$ and $\xi_i < 0$.

For the choice of $\xi_i > 0$, the Jeffrey-Kirwan prescription sums over the residues associated to four poles specified by the triples:

$$\begin{aligned} \text{I.} \quad & \varphi_1 = -\mathfrak{a} - \epsilon_+ & \varphi_2 = -\mathfrak{a} - 2\epsilon_+ & \varphi_3 = \mathfrak{a} + \epsilon_+ \\ \text{II.} \quad & \varphi_1 = \mathfrak{a} - 3\epsilon_+ & \varphi_2 = \mathfrak{a} - 2\epsilon_+ & \varphi_3 = \mathfrak{a} - \epsilon_+ \\ \text{III.} \quad & \varphi_1 = -\mathfrak{a} - \epsilon_+ & \varphi_2 = \mathfrak{a} - 2\epsilon_+ & \varphi_3 = \mathfrak{a} - \epsilon_+ \\ \text{IV.} \quad & \varphi_1 = -\mathfrak{a} - \epsilon_+ & \varphi_2 = -\mathfrak{a} - 2\epsilon_+ & \varphi_3 = -\mathfrak{a} - 3\epsilon_+ \end{aligned} \quad (6.135)$$

Summing over the associated residues, the above contour integral evaluates to:

$$\begin{aligned} Z_{mono}^{(Loc)}(2, 1) = & -\frac{4 \prod_f \sinh(\mathfrak{a} + m_f + 2\epsilon_+)}{\sinh(2\mathfrak{a} + 2\epsilon_+) \sinh(2\mathfrak{a})} - \frac{4 \prod_f \sinh(\mathfrak{a} - m_f - 2\epsilon_+)}{\sinh(2\mathfrak{a} - 2\epsilon_+) \sinh(2\mathfrak{a} - 4\epsilon_+)} \\ & - \frac{4 \prod_f \sinh(\mathfrak{a} - m_f - 2\epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} - 2\epsilon_+)} - \frac{4 \prod_f \sinh(\mathfrak{a} + m_f + 2\epsilon_+)}{\sinh(2\mathfrak{a} + 2\epsilon_+) \sinh(2\mathfrak{a} + 4\epsilon_+)} , \end{aligned} \quad (6.136)$$

for $\xi_i > 0$. Similarly, for $\xi_i < 0$, the Jeffrey-Kirwan prescription sums over the residues

associated to the poles

$$\begin{aligned}
\text{I.} \quad & \varphi_1 = -\mathfrak{a} + \epsilon_+ & \varphi_2 = -\mathfrak{a} + 2\epsilon_+ & \varphi_3 = \mathfrak{a} - \epsilon_+ \\
\text{II.} \quad & \varphi_1 = \mathfrak{a} + 3\epsilon_+ & \varphi_2 = \mathfrak{a} + 2\epsilon_+ & \varphi_3 = \mathfrak{a} + \epsilon_+ \\
\text{III.} \quad & \varphi_1 = -\mathfrak{a} + \epsilon_+ & \varphi_2 = \mathfrak{a} + 2\epsilon_+ & \varphi_3 = \mathfrak{a} + \epsilon_+ \\
\text{IV.} \quad & \varphi_1 = -\mathfrak{a} + \epsilon_+ & \varphi_2 = -\mathfrak{a} + 2\epsilon_+ & \varphi_3 = -\mathfrak{a} + 3\epsilon_+
\end{aligned} \tag{6.137}$$

In the case $\xi_i < 0$, summing over the residues associated to these poles computes the contour integral to be

$$\begin{aligned}
Z_{mono}^{(Loc)}(2, 1) = & -\frac{4 \prod_f \sinh(\mathfrak{a} + m_f - 2\epsilon_+)}{\sinh(2\mathfrak{a} - 2\epsilon_+) \sinh(2\mathfrak{a})} - \frac{4 \prod_f \sinh(\mathfrak{a} - m_f + 2\epsilon_+)}{\sinh(2\mathfrak{a} + 2\epsilon_+) \sinh(2\mathfrak{a} + 4\epsilon_+)} \\
& - \frac{4 \prod_f \sinh(\mathfrak{a} - m_f + 2\epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} + 2\epsilon_+)} - \frac{4 \prod_f \sinh(\mathfrak{a} + m_f - 2\epsilon_+)}{\sinh(2\mathfrak{a} - 2\epsilon_+) \sinh(2\mathfrak{a} - 4\epsilon_+)} .
\end{aligned} \tag{6.138}$$

Now we can make use of the fact that the expectation value of line defects form a ring under the Moyal product [97]

$$\langle L_{2,0} \rangle = \langle L_{1,0} \rangle * \langle L_{1,0} \rangle , \tag{6.139}$$

with respect to the $(2,0)$ symplectic form Ω_J :

$$(f * g)(\mathfrak{a}, \mathfrak{b}) = e^{-\epsilon_+ (\partial_b \partial_{a'} - \partial_a \partial_b)} f(a, b) g(a', b') \Big|_{\substack{a, a' = \mathfrak{a} \\ b, b' = \mathfrak{b}}} , \tag{6.140}$$

to compute $Z_{mono}(2, 1)$ from the AGT computation for $\langle L_{1,0} \rangle$. This gives the result

$$\begin{aligned}
Z_{mono}(2, 1) = & -\frac{4 \prod_f \sinh(\mathfrak{a} \pm m_f + 2\epsilon_+)}{\sinh(2\mathfrak{a} + 2\epsilon_+) \sinh(2\mathfrak{a})} - \frac{4 \prod_f \sinh(\mathfrak{a} \pm m_f + 2\epsilon_+)}{\sinh(2\mathfrak{a} + 2\epsilon_+) \sinh(2\mathfrak{a} + 4\epsilon_+)} \\
& - \frac{4 \prod_f \sinh(\mathfrak{a} \mp m_f - 2\epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} - 2\epsilon_+)} - \frac{4 \prod_f \sinh(\mathfrak{a} \mp m_f - 2\epsilon_+)}{\sinh(2\mathfrak{a} - 2\epsilon_+) \sinh(2\mathfrak{a} - 4\epsilon_+)} \\
& + 2 \cosh \left(\sum_f m_f \pm 6\epsilon_+ \right) + 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right) , \quad \pm \xi_i > 0 , \quad \forall i .
\end{aligned} \tag{6.141}$$

As before, this answer is independent of the choice of sign of ξ_i .²⁸

Again, we see that this does not match the localization computations for $Z_{mono}(2, 1)$ for either choice of ξ due to the “extra term” $+2 \cosh \left(\sum_f m_f \pm 6\epsilon_+ \right) + 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right)$. Further, the localization result is not independent of the choice of $\vec{\xi}$ and for both choices

²⁸In fact, this is independent of the choice of $\vec{\xi} \in \mathbb{R}^3 \setminus \{0\}$.

of $\vec{\xi}$ (and indeed for all other choices of $\vec{\xi}$), $Z_{mono}(2, 1)$ is not invariant under Weyl symmetry of the $SO(8)$ flavor symmetry group. Therefore, as in the previous example, we find that the localization computation cannot be correct.

6.3 Proposed Resolution: Coulomb Branch States

As we have shown, there is a discrepancy between the localization and AGT result for the expectation value of 't Hooft defects in $SU(2)$ $N_f = 4$ supersymmetric gauge theory. Let us write the AGT result for the expectation value as

$$\langle L_{[P,0]} \rangle_{AGT} = \sum_{|\mathbf{v}| \leq |P|} e^{(\mathbf{v}, \mathbf{b})} (F(\mathbf{a}))^{|\mathbf{v}|} Z_{mono}(\mathbf{a}, m_i, \epsilon_+; P, \mathbf{v}) , \quad (6.142)$$

$$Z_{mono}(\mathbf{a}, m_i, \epsilon_+; P, \mathbf{v}) = Z_{mono}^{(Loc)}(\mathbf{a}, m_i, \epsilon_+; P, \mathbf{v}) + Z_{mono}^{(extra)}(\mathbf{a}, m_i, \epsilon_+; P, \mathbf{v}) ,$$

where $Z_{mono}^{(Loc)} = I_{\mathcal{H}_0}^{(Loc)}$ is the localization computation for $Z_{mono}(P, \mathbf{v})$ and $Z_{mono}^{(extra)}(P, \mathbf{v})$ is some extra term that is the difference between $Z_{mono}(P, \mathbf{v})$ and $Z_{mono}^{(Loc)} = Z^{JK}$.

We now would like to understand what is the origin of the extra term $Z_{mono}^{(extra)}(P, \mathbf{v})$ that we must add to the localization computation to give the full result for $Z_{mono}(P, \mathbf{v})$. As we will now show, these extra contributions come from states that are not counted by localization.

6.3.1 Witten Indices with Continuous Spectra

As shown in [160, 2, 178, 113], computing the Witten index is much more subtle for theories with a continuous spectrum of states. In that case, the supercharges are non-Fredholm operators and thus the Witten index, which is still well defined, cannot be understood as the index of a supercharge operator. In order to illustrate some features of the computation of the Witten index in these cases, we will take a brief aside to study a toy model that is closely related to the bubbling SQMs we are studying.

Toy Model on Semi-Infinite Line

Here we will examine a simplified model of the effective SQM on the Coulomb branch. Consider a supersymmetric particle on a semi-infinite line with a smooth potential $h(x)$.

This theory is described by the Hamiltonian [2, 90]

$$H = \frac{e^2 p^2}{2} + \frac{e^2}{2} h^2(x) + \frac{e^2}{2} [\psi^\dagger, \psi] h'(x) , \quad (6.143)$$

where x, ψ are superpartners that satisfy the commutation relations

$$[x, p] = i \quad , \quad \{\psi^\dagger, \psi\} = 1 . \quad (6.144)$$

These fields satisfy the supersymmetry transformations

$$\begin{aligned} \delta_\eta x &= \frac{1}{\sqrt{2}} \eta \psi + \frac{1}{\sqrt{2}} \eta^\dagger \psi^\dagger , \\ \delta_\eta \psi &= \frac{1}{\sqrt{2}} \eta^\dagger (ip + h(x)) , \end{aligned} \quad (6.145)$$

which are generated by the supercharge

$$Q = \frac{1}{\sqrt{2}} \psi^\dagger (ip + h) . \quad (6.146)$$

Let us consider a toy model of the effective SQM on an asymptotic Coulomb branch where $h(x) = h_0 + \frac{q}{x}$ where $q \neq 0$. In our applications, $2q$ will be an integer.

Generic states in this theory are described by wave functions of the form

$$\psi(x) = f_+(x)|0\rangle + f_-(x)\psi^\dagger|0\rangle \quad (6.147)$$

where we define the Clifford vacuum by $\psi|0\rangle = 0$.

We are interested in computing the ground state index of this theory. This can be derived from the spectrum of the Hamiltonian. The Hamiltonian can be written as a diagonal operator

$$H = -\frac{e^2}{2} \partial_x^2 + \frac{e^2(q^2 \pm q)}{2x^2} + \frac{e^2 q h_0}{x} + \frac{e^2}{2} h_0^2 , \quad (6.148)$$

on a basis of states $\{f_+(x)|0\rangle, f_-(x)\psi^\dagger|0\rangle\}$. Thus, eigenstates of the Hamiltonian solve the differential equation

$$\left(-\partial_x^2 + \frac{q^2 \pm q}{x^2} + \frac{2qh_0}{x} + h_0^2 - \frac{2E}{e^2} \right) f_\pm(x) = 0 . \quad (6.149)$$

The L^2 -normalizable solutions of this equation are given by ²⁹

$$f_s(x) = c e^{-\kappa x} x^{j_{s1}-1} F_1 \left(j_{s1} + \frac{s_2 |q| |h_0|}{\kappa}, 2s_1 j_{s1}, 2\kappa x \right) , \quad s, s_i = \pm 1 , \quad (6.152)$$

²⁹Note that we could also solve for the space of BPS states by writing

$$\psi(x) = f_+(x)|0\rangle + f_-(x)\psi^\dagger|0\rangle \longrightarrow \psi(x) = \begin{pmatrix} f_+(x) \\ f_-(x) \end{pmatrix} , \quad (6.150)$$

where c is a constant, $\kappa = \sqrt{h_0^2 - \frac{2E}{e^2}}$, $s_1 = s \times \text{sign}(q)$, $s_2 = \text{sign}(q) \times \text{sign}(h_0)$, and

$$j_{s_1} = |q| + \frac{1 + s_1}{2} \quad \text{or} \quad j_{s_1} = -|q| + \frac{1 + s_1}{2} . \quad (6.153)$$

Further, due to the large x -behavior of the confluent hypergeometric function of the first kind

$${}_1F_1(m, n, x) \underset{x \rightarrow \infty}{\sim} \Gamma(n) e^x \frac{x^{m-n}}{\Gamma(m)} , \quad m \notin \mathbb{Z}_+ , \quad (6.154)$$

L^2 -normalizability implies that

$$j_{s_1} > -\frac{1}{2} , \quad j_{s_1} + \frac{s_2 |q| |h_0|}{\kappa} = -n , \quad n \in \mathbb{Z}_+ . \quad (6.155)$$

The first condition comes from imposing regularity at $x = 0$ whereas the second condition comes from imposing regularity at $x \rightarrow \infty$. Together, these conditions imply that there is a global minimum of the potential energy at some $x > 0$ that supports a bound state.

From regularity at $x \rightarrow \infty$, we can solve for the discrete spectrum of the Hamiltonian for generic h_0, q

$$E_n = \left(1 - \frac{4q^2}{(1 + 2n + 2|q| + s_1)^2} \right) \frac{e^2 h_0^2}{2} . \quad (6.156)$$

In the case of BPS ground states ($E = 0$), the L^2 -normalizability constraints imply that the ground states of this theory solve

$$(1 + s_2)|q| + \frac{1}{2}(1 + s_1) = -n , \quad (6.157)$$

or

$$(s_2 - 1)|q| + \frac{1}{2}(1 + s_1) = -n \quad \text{and} \quad |q| < \frac{1}{2} . \quad (6.158)$$

This implies that the allowed solutions are those with $s_1, s_2 < 0$ or $s_1 < 0 < s_2$ with $|q| < \frac{1}{2}$.

As in the case of the Coulomb branch, this theory has a continuum of scattering states. These occur due to the non-compact direction in field space where the potential

so that the real supercharge $\mathcal{Q} = (Q + \bar{Q})$ acts as the differential operator

$$\mathcal{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \partial_x + h(x) \\ -\partial_x + h(x) & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (i\sigma^2 \partial_x + \sigma^1 h(x)) . \quad (6.151)$$

It is now straightforward to solve for the kernel of \mathcal{Q} and impose normalizability. When one further takes into account boundary conditions, equation (6.163) is reproduced.

energy approaches a finite value $\lim_{x \rightarrow \infty} U(x) = \frac{e^2 h_0^2}{2}$. In this case, the gap to the continuum is given by $E_{\text{gap}} = \frac{e^2 h_0^2}{2}$. Thus, the full spectrum of states is similar to that of the Hydrogen atom. There is a discrete spectrum of states for energies $E < E_{\text{gap}}$ that accumulate at E_{gap} and a continuous spectrum of states for energies $E \geq E_{\text{gap}}$.

However, there is an additional subtlety to this model. Due to the presence of the boundary at $x = 0$, we additionally have to worry about the real supercharge $\mathcal{Q} = Q + \bar{Q}$ being Hermitian:

$$\langle \Psi_1 | \mathcal{Q} \Psi_2 \rangle = \langle \mathcal{Q} \Psi_1 | \Psi_2 \rangle . \quad (6.159)$$

If we consider two generic states

$$|\Psi_i\rangle = f_i(x)|0\rangle + g_i(x)\psi^\dagger|0\rangle \mapsto |\Psi_i\rangle = \begin{pmatrix} f_i(x) \\ g_i(x) \end{pmatrix} , \quad (6.160)$$

then the real supercharge operator \mathcal{Q} acts as

$$\mathcal{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \partial_x + h(x) \\ -\partial_x + h(x) & 0 \end{pmatrix} \quad (6.161)$$

The constraint that \mathcal{Q} be Hermitian (6.159), then reduces to

$$[\bar{f}_1 g_2 - \bar{g}_1 f_2]_{x=0} = 0 . \quad (6.162)$$

There are 3 different types of restrictions we can impose on the Hilbert space so that \mathcal{Q} is Hermitian:

1. Impose $f(x) = 0, \forall x \in \mathbb{R}^+$. In this case the Hilbert space is reduced so that wave functions are only of the form $\mathcal{H} = \text{span}_{L^2(\mathbb{R}_+)} \{\psi^\dagger|0\rangle\}$.
2. Impose $g(x) = 0, \forall x \in \mathbb{R}^+$. In this case the Hilbert space is reduced so that wave functions are only of the form $\mathcal{H} = \text{span}_{L^2(\mathbb{R}_+)} \{|0\rangle\}$.
3. Impose $f(0) = 0, g(0) = 0$ or $f(0)$ and $g(0) = 0$. In this case we restrict the form of the wave functions allowed in the Hilbert space $\mathcal{H} = \text{span}_{\substack{L^2(\mathbb{R}_+) \\ \langle x|\psi\rangle|_{x=0}=0}} \{|0\rangle, \psi^\dagger|0\rangle\}$.

These three different choices give the different answers. They are each given by

$$\begin{aligned}
 1.) \quad I_{\mathcal{H}_0} &= \begin{cases} -1 & h_0 < 0, \quad q > -\frac{1}{2} \\ 0 & \text{else} \end{cases} & 2.) \quad I_{\mathcal{H}_0} &= \begin{cases} 1 & h_0 > 0, \quad q < \frac{1}{2} \\ 0 & \text{else} \end{cases} \\
 3.) \quad I_{\mathcal{H}_0} &= \begin{cases} 0 & h_0 \times q > 0 \\ -1 & q > 0 > h_0 \\ 1 & h_0 > 0 > q \end{cases} & & (6.163)
 \end{aligned}$$

Now let us try to use localization to compute the ground state index. This SQM can be described by the Lagrangian

$$L = \frac{1}{2e^2} (\dot{x}^2 + \psi^\dagger \dot{\psi} + D^2) + Dh(x) - \frac{e^2}{2} h'(x) [\psi^\dagger, \psi], \quad (6.164)$$

on the half space $x > 0$. As before, this Lagrangian is Q -exact. Thus, by studying the limit $e \rightarrow 0$, we see that the path integral localizes to the Q -fixed points $\dot{x} = 0$. This reduces the path integral to an integral over the line ³⁰

$$Z = \int_{\mathbb{R}} dD \int_{\mathbb{R}^+} dx \int [d\psi d\psi^\dagger] e^{-S}. \quad (6.165)$$

In this case, the 1-loop determinant comes from integrating over the fermion zero modes. The partition function then reduces to

$$Z = - \int_{\mathbb{R}} dD \int_{\mathbb{R}^+} dx \beta h'(x) e^{-\frac{\pi\beta D^2}{e^2} + 2\pi i\beta Dh}. \quad (6.166)$$

This can be evaluated by first integrating over D :

$$Z = - \int_{\mathbb{R}^+} dx \beta h'(x) \sqrt{\frac{e^2}{\beta}} e^{-\pi\beta e^2 h^2(x)}. \quad (6.167)$$

We can then evaluate the partition by making a change of variables. This produces the result

$$Z = \frac{1}{2} \begin{cases} -1 + \operatorname{erf}\left(\sqrt{\pi\beta e^2} h_0\right) & q > 0 \\ 1 + \operatorname{erf}\left(\sqrt{\pi\beta e^2} h_0\right) & q < 0 \end{cases} \quad (6.168)$$

³⁰Here we fix the normalization of the path integral so that the ground state index is an integer.

Now we can compute the ground state index by taking the limit as $\beta \rightarrow \infty$. In this case we find that the localization computation for the ground state index is exactly given by

$$I_{\mathcal{H}_0}^{(Loc)} = \begin{cases} 0 & h_0 \times q > 0 \\ -1 & q > 0 > h_0 \\ 1 & h_0 > 0 > q \end{cases} \quad (6.169)$$

Which matches with the explicit computation in the SQM for the third choice of boundary condition. Note that in the case $q = 0$ the Witten index is identically zero because the fermionic fields are non-interacting and massless. This is also reflected in the identically vanishing of the path integral since there are no fermion insertions. However, any correction can lift this exact degeneracy and give rise to a possibly non-trivial Witten index.

Effective Coulomb Branch SQM

Now let us apply this computation to the effective SQM on the Coulomb branch for the bubbling SQM of the minimal 't Hooft defect in the $SU(2)$ $\mathcal{N} = 2$ gauge theory with $N_f = 4$ fundamental hypermultiplets. See Section 6.3.3 and Appendix D.1. We will refer to this theory as $T_{\mathcal{M}_C}$. This theory is again a supersymmetric particle moving in a potential

$$h(\sigma) = \langle D \rangle(\sigma) = \xi + \frac{1}{2} \sum_{i=1}^2 \left(\frac{1}{\omega_i} - \frac{1}{\tilde{\omega}_i} \right) . \quad (6.170)$$

However, we are now taking the theory on two semi-infinite intervals $I_+ = \{\sigma > a + \epsilon\}$ and $I_- = \{\sigma < -a - \epsilon\}$ where $\omega_i, \tilde{\omega}_i$ are the effective masses of the integrated out hypermultiplet fields

$$\omega_i = |\sigma + (-1)^i a + \epsilon| \quad , \quad \tilde{\omega}_i = |\sigma + (-1)^i a - \epsilon| \quad , \quad (6.171)$$

for parameters $a, \epsilon \in \mathbb{R}_+$. On each of these intervals, the vacuum state has Fermion number -1 and has flavor charges $+1$ on I_+ and -1 on I_- . Note that this differs slightly from the effective SQM in Appendix D.1. They are related by a different choice

of normalization for the wave function describing the matter fields.³¹

The localization computation of the ground state index proceeds as before except that there are now two boundary contributions at $\sigma = \pm(a + \epsilon)$. Due to the limiting behavior of $h(\sigma)$, the localization result for the ground state index, analogous to (6.168), is given by

$$\begin{aligned} I_{\mathcal{H}_0}^{(Loc)}(T_{\mathcal{M}_C}) = & - \lim_{\beta \rightarrow \infty} \sinh(2\epsilon_+) e^{\sum_f m_f} \left(1 - \operatorname{erf} \left(\sqrt{\pi\beta} e\xi \right) \right) \\ & - \lim_{\beta \rightarrow \infty} \sinh(2\epsilon_+) e^{-\sum_f m_f} \left(1 + \operatorname{erf} \left(\sqrt{\pi\beta} e\xi \right) \right) . \end{aligned} \quad (6.172)$$

Here, the factor of $2\sinh(2\epsilon_+)$ comes from the decoupled Fermi-multiplet in the $\mathcal{N} = (0, 4)$ vector multiplet described by $\lambda^2, \bar{\lambda}_2$ and the $e^{\pm \sum_f m_f}$ comes flavor charge of the ground state on I_+ and I_- respectively. By using the explicit form of $h(\sigma)$ (6.170), we obtain the result

$$I_{\mathcal{H}_0}^{(Loc)}(T_{\mathcal{M}_C}) = -2 \sinh(2\epsilon_+) e^{\mp \sum_f m_f} , \quad \pm \xi > 0 . \quad (6.173)$$

Solving for the entire spectrum of this theory is much more difficult than in the previous example. However, only the BPS states contribute to the ground state index. These are computed in the Born-Oppenheimer approximation in Appendix D.1. In summary, we find that there are over 10 different types of restrictions on the Hilbert space in each interval that make \mathcal{Q} Hermitian: we will make a symmetric choice. Two distinguished choices lead to

$$\begin{aligned} 1.) \quad I_{\mathcal{H}_0}(T_{\mathcal{M}_C}) &= 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right) , \quad \pm \xi > 0 , \\ 2.) \quad I_{\mathcal{H}_0}(T_{\mathcal{M}_C}) &= -2 \sinh(2\epsilon_+) e^{\mp \sum_f m_f} , \quad \pm \xi > 0 . \end{aligned} \quad (6.174)$$

Now we see that the localization computation matches the explicit computation for the second choice of boundary condition.

However, now recall the localization expression for $Z_{mono}(1, 0)$ in the $SU(2)$ $N_f = 4$ theory in the expectation value of $L_{1,0}$ from Section 6.2.7. There we showed that

³¹See Appendix D.1 for more details.

$Z_{mono}^{(Loc)}(1, 0) \neq Z_{mono}(1, 0)$ as computed via AGT. In fact, they differed by a term

$$Z_{mono}^{(extra)}(1, 0) := Z_{mono}(1, 0) - Z_{mono}^{(Loc)}(1, 0) = 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right) \quad , \quad \pm \xi > 0 \quad . \quad (6.175)$$

However, this is exactly the computation of the ground state index of the BPS states localized on the Coulomb branch with the first choice (6.174). As it turns out, there is a unique choice of boundary conditions if we restrict to the case of pure Neumann or Dirichlet.

This result is in fact very natural. Recall that in Section 6.2.4 we explained that the Hilbert space of BPS states of a generic bubbling SQM theory can be decomposed into states localized on the Higgs, Coulomb, and mixed branches:

$$\mathcal{H}_{BPS} = \mathcal{H}_{BPS}^{(Higgs)} \oplus \mathcal{H}_{BPS}^{(Coulomb)} \oplus \mathcal{H}_{BPS}^{(mixed)} \quad . \quad (6.176)$$

Thus, the ground state index should similarly decompose as

$$I_{\mathcal{H}_0} = I_{Higgs} + I_{Coulomb} + I_{mixed} \quad . \quad (6.177)$$

In our case there is no mixed branch so that the summand $\mathcal{H}_{BPS}^{(mixed)}$ is trivial and $I_{mixed} = 0$. However, since we have identified the Jeffrey-Kirwan residue, $Z^{JK} = I_{Higgs}$, as counting the Higgs branch states, we have that $I_{\mathcal{H}_0}^{(Loc)} = I_{Higgs}$ has no contribution from Coulomb branch states. Therefore, it is clear that we need to add a term

$$I_{asympt} = I_{Coulomb} + I_{mixed} \quad , \quad (6.178)$$

which counts the BPS ground states on the non-compact Coulomb and mixed branches.

6.3.2 Proposal

Thus far we have been able to show that the localization computation of $I_{\mathcal{H}_0}$, $I_{\mathcal{H}_0}^{(Loc)}$, reproduces the JK -prescription for the path integral, but that this does not correctly reproduce $Z_{mono}(P, v)$ at least with the regularization procedure for localization that we have adopted. Further, by identifying Z^{JK} as counting Higgs branch states, we were able to conclude that $I_{\mathcal{H}_0}^{(Loc)}$ does not count any contributions from the ground states along the Coulomb and mixed branches.

Therefore, we propose that $Z_{mono}^{(extra)}(P, v)$ is the contribution of BPS states along the non-compact vacuum branches in the bubbling SQM. Mathematically, this can be phrased as

$$Z_{mono} = I_{\mathcal{H}_0} = I_{\mathcal{H}_0}^{(Loc)} + I_{asympt} = Z^{JK} + I_{asympt} , \quad (6.179)$$

where $I_{asympt} := I_{Coulomb} + I_{mixed}$ is the ground state index evaluated on the states localized along the Coulomb and mixed branches and $Z^{JK} = I_{Higgs}$ is the Jeffrey-Kirwan sum over residues [98].

Note that I_{asympt} is fundamentally distinct from the defect term $\delta I_{\mathcal{H}_0}$ which similarly can be appended to $\lim_{\beta \rightarrow \infty} I_W^{(Loc)}$ to correct the localization result [160, 178, 113]. As shown above, our computation of $I_{\mathcal{H}_0}^{(Loc)}$ has already taken the defect term into account. Rather, we propose that one must add an additional term I_{asympt} that corrects for the omitted ground states localized on the non-compact vacuum branches.

These states can be computed in the effective theory on the relevant vacuum branches in the Born-Oppenheimer approximation. As we will see, this definition is independent of the choice of ξ in all known examples. Since the Born-Oppenheimer approximation is only valid for $|\sigma/a|, |\sigma/\epsilon_+| \gg 0$, we must make a choice of effective boundary conditions at $|\sigma| = |a| + |\epsilon_+|$. Unitarity then restricts the types of allowed boundary conditions. In each of the following examples, there exists a (sometimes unique) boundary condition such that $I_{asympt} = Z_{mono}^{(extra)}$. We have chosen to use this boundary condition in all cases. The cases in which $I_{asympt} = 0$ do not require such a choice.

Relation to Defect Contribution

The correction of the ground state index by I_{asympt} at first glance appears to be similar to the work of [178, 160] in which the authors compute the ground state index by adding a “defect term” or “secondary term” to the Witten index. However, the two stories are quite different. The definition of the “defect term” relies on rewriting

$$I_{\mathcal{H}_0} = I_{\mathcal{H}_0}^{Bulk}(\beta_0) + \delta I_{\mathcal{H}_0}(\beta_0) , \quad (6.180)$$

where

$$\begin{aligned} I_{\mathcal{H}_0}^{Bulk} &= I_W(\beta_0) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta_0 H + \dots} , \\ \delta I_{\mathcal{H}_0} &= \int_{\beta_0}^{\infty} d\beta \partial_{\beta} \left(\text{Tr}_{\mathcal{H}}(-1)^F e^{-\frac{\beta}{2} \{\mathcal{Q}, \mathcal{Q}\} + \dots} \right) , \end{aligned} \quad (6.181)$$

which we will call the *bulk* and *defect* terms respectively. This is a trivial rewriting by making use of the fundamental theorem of calculus. When we write the path integral as an integral over field space, we can use supersymmetry to rewrite the ∂_{β} in the defect term as

$$\begin{aligned} \partial_{\beta} \left(\text{Tr}_{\mathcal{H}}(-1)^F e^{-\frac{\beta}{2} \{\mathcal{Q}, \mathcal{Q}\} + \dots} \right) &= -\text{Tr}_{\mathcal{H}}(-1)^F \mathcal{Q}^2 e^{-\frac{\beta}{2} \{\mathcal{Q}, \mathcal{Q}\} + \dots} \\ &= \text{Tr}_{\mathcal{H}} \mathcal{Q}(-1)^F \mathcal{Q} e^{-\frac{\beta}{2} \{\mathcal{Q}, \mathcal{Q}\} + \dots} . \end{aligned} \quad (6.182)$$

Then by integrating by parts inside the path integral [2, 160], this is equal to a derivative on field space

$$\begin{aligned} \text{Tr}_{\mathcal{H}} \mathcal{Q}(-1)^F \mathcal{Q} e^{-\frac{\beta}{2} \{\mathcal{Q}, \mathcal{Q}\} + \dots} &= \text{Tr}_{\mathcal{H}}(-1)^F \mathcal{Q}^2 e^{-\beta H + \dots} + \text{Tr}_{\mathcal{H}} \partial_{\phi^i} \left(\psi^i(-1)^F \mathcal{Q} e^{-\frac{\beta}{2} \{\mathcal{Q}, \mathcal{Q}\} + \dots} \right) \\ &= \frac{1}{2} \text{Tr}_{\mathcal{H}} \partial_{\phi^i} \left(\psi^i(-1)^F \mathcal{Q} e^{-\frac{\beta}{2} \{\mathcal{Q}, \mathcal{Q}\} + \dots} \right) , \end{aligned} \quad (6.183)$$

where $\psi^i \partial_{\phi^i}$ are the derivative terms in the supercharge which in turn can be written as a boundary integral in field space. One might therefore hope that the defect term is a feasible computation.

The utility of this rewriting is in the fact that the bulk term $I_{\mathcal{H}_0}^{Bulk}$ can be computed exactly in the limit as $\beta_0 \rightarrow 0$ by heat-kernel techniques. In this way, one might try to compute the ground state index $I_{\mathcal{H}_0}$.

In this paper we are using a different decomposition of $I_{\mathcal{H}_0}$. Here, we want to compute the ground state index directly by using localization. Since we can compute the Witten index via localization for generic β , we find that we can simply take the limit as $\beta \rightarrow \infty$ to obtain the ground state index

$$I_{\mathcal{H}_0}^{(Loc)} = \lim_{\beta \rightarrow \infty} I_W^{(Loc)}(\beta) . \quad (6.184)$$

Unfortunately, once we have used what appears to be the most natural way of regularizing the localized integral, we find that the localization expression for the ground state

index does not agree with AGT. Further, we find that this can be corrected by adding the contribution from BPS states that are localized along non-compact directions in field space with finite asymptotic potential.

In summary, the difference between our proposal and the defect term of [160, 178] is that the defect term $\delta I_{\mathcal{H}_0}$ is the difference between the Witten index at $\beta = 0$ and the ground state index whereas the asymptotic contribution I_{asympt} counts the BPS states that are omitted in the implementation of localization to compute the ground state index.

6.3.3 Examples

In this section, we will provide several non-trivial examples to show that $Z_{mono}^{(extra)}(P, \mathbf{v})$ is indeed reproduced by the ground state index of the Coulomb and mixed branch BPS states. In these examples we will study components of the expectation value of two line defects: $L_{1,0}$ and $L_{2,0}$. Specifically, we will be again interested in $Z_{mono}(1, 0)$ and $Z_{mono}(2, 1)$.

Although we are only performing the computation for examples of abelian gauge groups, there is no fundamental obstruction for performing the analogous computations for non-abelian gauge groups. The computation would be analogous to that of Appendices D.1 and D.2 with increased computational complexity. We believe that in the case of a non-abelian bubbling SQM, I_{asympt} may have a more interesting form and could potentially depend on the gauge fugacity \mathbf{a} .

$SU(2)$ $N_f = 4$ **Theory**

In this theory we will study components of the expectation value of two line defects: $L_{1,0}$ and $L_{2,0}$. Specifically, we will be interested in $Z_{mono}(1, 0)$ and $Z_{mono}(2, 1)$.

$\mathbf{Z}_{mono}(\mathbf{1}, \mathbf{0})$

In the full expression for the expectation value of the $L_{1,0}$ 't Hooft line defect, there

are two terms that contribute to $Z_{mono}(1, 0)$:

$$Z_{mono}(1, 0) = Z_{mono}^{(Loc)}(1, 0) + Z_{mono}^{(extra)}(1, 0) . \quad (6.185)$$

As shown in the previous section, the localization result for this $Z_{mono}(1, 0)$ is given by (6.129) whereas the full expression for $Z_{mono}(1, 0)$, as we know from AGT, is given by (6.130). This means that $Z_{mono}^{(extra)}(1, 0)$ is given by

$$Z_{mono}^{(extra)}(1, 0) = \begin{cases} 2 \cosh \left(\sum_f m_f + 2\epsilon_+ \right) & \xi > 0 \\ 2 \cosh \left(\sum_f m_f - 2\epsilon_+ \right) & \xi < 0 \end{cases} \quad (6.186)$$

We conjecture that this should be exactly reproduced by the Witten index of the ground states on the Coulomb branch.

As we have shown in Appendix D.1, this is indeed exactly reproduced by the Witten index of the asymptotic states on the Coulomb branch:

$$Z_{mono}^{(extra)}(1, 0) = I_{asympt}(1, 0) = \begin{cases} 2 \cosh \left(\sum_f m_f + 2\epsilon_+ \right) & \xi > 0 \\ 2 \cosh \left(\sum_f m_f - 2\epsilon_+ \right) & \xi < 0 \end{cases} \quad (6.187)$$

$Z_{mono}(2, 1)$

Again by comparing the localization expressions (6.136)-(6.138) with the full expression from AGT (6.141) for $Z_{mono}(2, 1)$, we find that

$$Z_{mono}^{(extra)}(2, 1) = 2 \cosh \left(\sum_f m_f \pm 6\epsilon_+ \right) + 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right) , \quad \pm \xi_i > 0 , \forall i . \quad (6.188)$$

As shown in Appendix D.2, $Z_{mono}^{(extra)}(2, 1)$ is exactly reproduced by the ground state index of the effective super quantum mechanics on the Coulomb branch (I_{asympt}):

$$I_{asympt} = 2 \cosh \left(\sum_f m_f \pm 6\epsilon_+ \right) + 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right) , \quad \pm \xi_i > 0 , \forall i . \quad (6.189)$$

By explicit computation, one can see that the Coulomb branch terms restore Weyl-invariance³² and invariance of $Z_{mono}(2, 1)$ under the choice of $\vec{\xi}$.

³²The Weyl group under the $SO(8)$ flavor symmetry is generated by $m_i \leftrightarrow m_{i+1}$ and $m_3 \leftrightarrow -m_4$.

Note that the bubbling SQM for this example has a non-trivial mixed branch. However, we conjecture that there are no states localized there. See D.2.3 for more details.

$SU(2)$ $N_f = 2$ **Theory**

Now consider the $L_{1,0}$ (minimal) 't Hooft defect in the $SU(2)$ $N_f = 2$ theory. As in the case of the $SU(2)$ $N_f = 4$ theory, this has 't Hooft charge

$$P = h^1 = 2\hat{h}^1 = \text{diag}(1, -1) \quad , \quad h^1 \in \Lambda_{mw} \quad , \quad \hat{h}^1 \in \Lambda_{cochar} \quad . \quad (6.190)$$

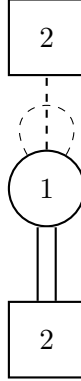
Similarly, the expression for its expectation value is of the form

$$\langle L_{1,0} \rangle = \left(e^b + e^{-b} \right) F(\mathbf{a}, m_f) + Z_{mono}(1, 0) \quad , \quad (6.191)$$

where

$$F(\mathbf{a}, m_f) = \left(\frac{\prod_{\pm} \prod_{f=1}^2 \sinh(\mathbf{a} \pm m_f)}{\sinh(2\mathbf{a}) \prod_{\pm} \sinh(2\mathbf{a} \pm 2\epsilon_+)} \right)^{\frac{1}{2}} \quad . \quad (6.192)$$

The monopole bubbling contribution can be computed as the Witten index of the $\mathcal{N} = (0, 4)$ SQM described by the $\mathcal{N} = (0, 2)$ quiver:



The Witten index of this quiver SQM reduces to the contour integral

$$Z_{mono}^{(Loc)}(1, 0) = \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} Z_{vec}(\varphi, \mathbf{a}, \epsilon_+) Z_{hyper}(\varphi, \mathbf{a}, \epsilon_+) Z_{Fermi}(\varphi, \mathbf{a}, \epsilon_+) \quad , \quad (6.193)$$

which is explicitly given by

$$Z_{mono}^{(Loc)}(1, 0) = \frac{1}{2} \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} \sinh(2\epsilon_+) \frac{\prod_{f=1}^2 \sinh(\varphi - m_f)}{\prod_{\pm} \sinh(\varphi \pm \mathbf{a} + \epsilon_+) \sinh(-\varphi \pm \mathbf{a} + \epsilon_+)} \quad . \quad (6.194)$$

This integral evaluates to ³³

$$Z_{mono}^{(Loc)}(1, 0) = \frac{\prod_f \sinh(\mathfrak{a} - m_f \mp \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \mp 2\epsilon_+)} + \frac{\prod_f \sinh(\mathfrak{a} + m_f \pm \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \pm 2\epsilon_+)} \quad : \quad \pm \xi > 0 . \quad (6.195)$$

From carefully taking the limit of $\langle L_{1,0} \rangle$ in the $N_f = 4$ theory to the $N_f = 2$ theory³⁴, we can see that the correct $Z_{mono}(P, \mathfrak{v})$ contribution is given exactly by

$$Z_{mono}(1, 0) = \frac{\prod_f \sinh(\mathfrak{a} - m_f \mp \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \mp 2\epsilon_+)} + \frac{\prod_f \sinh(\mathfrak{a} + m_f \pm \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \pm 2\epsilon_+)} \quad : \quad \pm \xi > 0 . \quad (6.196)$$

Thus, we see that

$$Z_{mono}^{(extra)}(1, 0) = 0 . \quad (6.197)$$

As seen in Appendix D.1, there are no Coulomb branch states and thus:

$$Z_{mono}^{(extra)}(1, 0) = I_{asympt}(1, 0) = 0 . \quad (6.198)$$

Note that the fact that $Z_{mono}^{(extra)}(1, 0) = 0$ is consistent with the fact that $Z_{mono}(1, 0)$ is actually invariant under the choice of ξ and under the Weyl symmetry of the flavor symmetry group.

$SU(2)$ SYM Theory

Here we will again be interested in the expectation value of the minimal 't Hooft defect that experiences monopole bubbling. This line defect has 't Hooft charge

$$P = 2h^1 = 2\hat{h}^1 = \frac{1}{2} \text{diag}(2, -2) \quad , \quad h^1 \in \Lambda_{mw} \quad , \quad \hat{h}^1 \in \Lambda_{cochar} \quad , \quad (6.199)$$

and hence is the next to minimal 't Hooft defect: $L_{2,0}$.

Its expectation value takes a similar form to $\langle L_{1,0} \rangle$ of the $SU(2)$ $N_f = 4$ theory:

$$\langle L_{2,0} \rangle = \left(e^{2b} + e^{-2b} \right) F(\mathfrak{a}, m_f) + Z_{mono}(2, 0) , \quad (6.200)$$

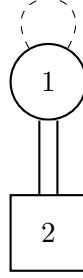
³³Note that this required fixing the overall sign of the Jeffrey-Kirwan residue computation. The reason is that the JK prescription does not give a derivation of the overall sign.

³⁴This requires taking $Re[m_4], Re[m_3] \rightarrow \infty$ such that $sgn(Re[m_4]/Re[m_3]) = -1$. This is a very subtle point that we will discuss in section 6.4. See [96] for more details about the analogous issues in 5D SYM.

where

$$F(\mathbf{a}, m_f) = \left(\frac{1}{\sinh(2\mathbf{a}) \prod_{\pm} \sinh(2\mathbf{a} \pm 2\epsilon_+)} \right)^{\frac{1}{2}} . \quad (6.201)$$

The monopole bubbling contribution (Z_{mono}) can be computed as the Witten index of the $\mathcal{N} = (0, 4)$ SQM described by the $\mathcal{N} = (0, 2)$ quiver:



The Witten index of this quiver SQM reduces to the contour integral

$$Z_{mono}^{(Loc)}(2, 0) = \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} Z_{vec}(\varphi, \mathbf{a}, \epsilon_+) Z_{hyper}(\varphi, \mathbf{a}, \epsilon_+) , \quad (6.202)$$

which is explicitly given by

$$Z_{mono}^{(Loc)}(2, 0) = \frac{1}{8} \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} \sinh(2\epsilon_+) \prod_{\pm} \frac{1}{\sinh(\varphi \pm \mathbf{a} + \epsilon_+) \sinh(-\varphi \pm \mathbf{a} + \epsilon_+)} . \quad (6.203)$$

This integral evaluates to

$$Z_{mono}^{(Loc)}(2, 0) = -\frac{1}{4 \sinh(2\mathbf{a}) \sinh(2\mathbf{a} \mp 2\epsilon_+)} - \frac{1}{4 \sinh(2\mathbf{a}) \sinh(2\mathbf{a} \pm 2\epsilon_+)} \quad : \quad \pm \xi > 0 . \quad (6.204)$$

From carefully taking the limit of $\langle L_{2,0} \rangle$ in the $N_f = 4$ theory to the $N_f = 0$ theory³⁵, we can see that the correct $Z_{mono}(2, 0)$ contribution is given exactly by

$$Z_{mono}(2, 0) = -\frac{1}{4 \sinh(2\mathbf{a}) \sinh(2\mathbf{a} \mp 2\epsilon_+)} - \frac{1}{4 \sinh(2\mathbf{a}) \sinh(2\mathbf{a} \pm 2\epsilon_+)} \quad : \quad \pm \xi > 0 . \quad (6.205)$$

Thus, we see that

$$Z_{mono}^{(extra)}(2, 0) = 0 . \quad (6.206)$$

³⁵As before, this requires a bit of care by taking $Re[m_i] \rightarrow \infty$ such that two masses go to $+\infty$ and two go to $-\infty$.

As seen in Appendix D.1, there are no Coulomb branch states and thus

$$Z_{mono}^{(extra)}(2, 0) = I_{asympt}(2, 0) = 0 . \quad (6.207)$$

This is again consistent with the fact that $Z_{mono}^{(Loc)}(2, 0)$ is independent of the choice of ξ and invariant under the action of the Weyl symmetry of the flavor symmetry group.

$SU(2)$ $\mathcal{N} = 2^*$ **Theory**

Here we will again be interested in the expectation value of the minimal 't Hooft defect that exhibits monopole bubbling. As in the case of $SU(2)$ SYM theory, this line defect has 't Hooft charge

$$P = 2h^1 = 2\hat{h}^1 = \frac{1}{2}\text{diag}(2, -2) \quad , \quad h^1 \in \Lambda_{mw} \quad , \quad \hat{h}^1 \in \Lambda_{cochar} \quad , \quad (6.208)$$

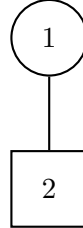
and hence is the next to minimal 't Hooft defect: $L_{2,0}$. As in the case of the $SU(2)$ SYM theory, this is of the form

$$\langle L_{2,0} \rangle = \left(e^{2b} + e^{-2b} \right) F(\mathfrak{a}, m_f) + Z_{mono}(2, 0) \quad , \quad (6.209)$$

where

$$F(\mathfrak{a}) = \left(\frac{\prod_{s_1, s_2 = \pm} \sinh(2\mathfrak{a} + s_1 m + s_2 \epsilon_+)}{\prod_{\pm} \sinh^2(2\mathfrak{a}) \sinh(2\mathfrak{a} \pm 2\epsilon_+)} \right)^{\frac{1}{2}} . \quad (6.210)$$

The monopole bubbling contribution (Z_{mono}) can be computed as the Witten index of the (mass deformed) $\mathcal{N} = (4, 4)$ SQM described by the quiver:



The Witten index of this quiver SQM reduces to the contour integral

$$Z_{mono}^{(Loc)}(2, 0) = \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} Z_{vec}(\varphi, \mathfrak{a}, \epsilon_+) Z_{hyper}(\varphi, \mathfrak{a}, \epsilon_+) \quad , \quad (6.211)$$

where the contributions of the different $\mathcal{N} = (0, 2)$ multiplets for this SQM are given by

$$Z_{mono}^{(Loc)}(2, 0) = \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} \frac{\sinh(2\epsilon_+)}{2 \prod_{\pm} \sinh(m \pm \epsilon_+)} \times \prod_{\pm} \frac{\sinh(\pm(\varphi + \mathfrak{a}) + m) \sinh(\pm(\varphi - \mathfrak{a}) + m)}{\sinh(\pm(\varphi + \mathfrak{a}) + \epsilon_+) \sinh(\pm(\varphi - \mathfrak{a}) + \epsilon_+)} . \quad (6.212)$$

Using this we can compute

$$Z_{mono}^{(Loc)}(2, 0) = \frac{\prod_{s=\pm} \sinh(2\mathfrak{a} + sm + \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} + 2\epsilon_+)} + \frac{\prod_{s=\pm} \sinh(2\mathfrak{a} + sm - \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} - 2\epsilon_+)} , \quad \pm\xi > 0 . \quad (6.213)$$

As shown in [97], the AGT computation produces

$$Z_{mono}(2, 0) = \frac{\prod_{s=\pm} \sinh(2\mathfrak{a} + sm + \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} + 2\epsilon_+)} + \frac{\prod_{s=\pm} \sinh(2\mathfrak{a} + sm - \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} - 2\epsilon_+)} , \quad \pm\xi > 0 , \quad (6.214)$$

and therefore that

$$Z_{mono}^{(extra)}(2, 0) = 0 . \quad (6.215)$$

As shown in Appendix D.1, there is a complete cancellation between Coulomb branch states such that $Z_{mono}^{(extra)}(2, 0)$ is reproduced by the Witten index of the asymptotic Coulomb branch states:

$$Z_{mono}^{(extra)}(2, 0) = I_{asympt}(2, 0) = 0 . \quad (6.216)$$

This is again consistent with the fact that $Z_{mono}^{(Loc)}(2, 0)$ is independent of the choice of ξ and invariant under the action of the Weyl symmetry of the flavor symmetry group.

Remark In general, the “extra” terms can be dependent on \mathfrak{a} as well as m_f and ϵ_+ . The reason is that because for non-abelian SQM gauge groups, there are generically non-trivial contributions from mixed branches which we expect can give rise to \mathfrak{a} dependence.

6.3.4 Comment on 4D $\mathcal{N} = 2$ $SU(2)$ Quiver Gauge Theories

It is interesting to ask how this analysis applies to 4D $\mathcal{N} = 2$ quiver gauge theories with gauge group $SU(2)$ at each node.³⁶ We believe there is no fundamental obstruction to

³⁶Here we consider only $SU(2)$ gauge groups due to additional subtleties with higher rank simple gauge groups with $N_f \geq 4$ fundamental hypermultiplets. See upcoming work for additional details.

applying this analysis to such theories beyond increasing computational complexity. There is also the additional complication that the brane configuration presented here does not simply generalize to the case of quiver gauge theories and hence can not be used to derive the quivers for the bubbling SQM.

However, the bubbling SQM can be deduced from the following arguments. Consider the bubbled defect in a quiver gauge theory with gauge group $G = \prod_i SU(2)_i$ specified by the data $(P, \mathbf{v}) \in \Lambda_{mw} \times \Lambda_{mw}$ which decomposes as a sum over gauge group factors

$$P = \bigoplus_i P_i \quad , \quad \mathbf{v} = \bigoplus_i \mathbf{v}_i \quad . \quad (6.217)$$

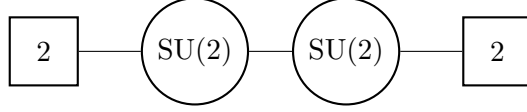
Further, let us define the quiver Γ_i which specifies the bubbling SQM associated to the pair (P_i, \mathbf{v}_i) with appropriate matter interactions. The full bubbling SQM is then derived by taking into account the 4D bifundamental hypermultiplets which lead to extra fermi and/or chiral multiplets connecting nodes between different Γ_i . The precise couplings can be obtained from demanding $U(1)_K$ -invariance of the full quiver.³⁷

Unfortunately, testing our hypothesis in this setting would be quite difficult as the necessary AGT computations also become increasingly difficult with increasing gauge group rank. The computation of the bubbling contribution to the expectation value of 't Hooft defects in quiver gauge theories is of interest for many reasons. One reason is the potential utility in exploring the deconstruction of the 6D $\mathcal{N} = (0, 2)$ theory [8].

Consider a $\mathcal{N} = 2$ superconformal ring quiver gauge theory with $G = \prod_{i=1}^N SU(2)_i$. The deconstruction hypothesis conjectures that in the limit $N \rightarrow \infty$ and $g_{YM} \rightarrow \infty$, the UV completion of this 4D theory is that of the 6D A_1 $\mathcal{N} = (0, 2)$ theory. In this limit the 't Hooft defects become surface defects that interact with tensionless strings [157]. Thus, the correct computation of expectation 't Hooft defects in quiver gauge theories can be used as a probe for understanding the 6D $\mathcal{N} = (0, 2)$ theory.

Therefore, let us demonstrate that our analysis applies to the computation of the monopole bubbling contribution of an 't Hooft defect in the simplest example of quiver gauge theory of higher rank. Consider the case of a superconformal $\mathcal{N} = 2$ quiver gauge theory with $G = SU(2)_1 \times SU(2)_2$ with fundamental matter:

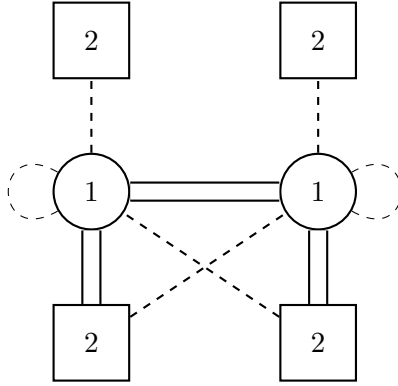
³⁷See upcoming work for additional details.



Now consider the bubbling sector where

$$P = \bigoplus_{i=1}^2 P_i \quad , \quad v = \bigoplus_{i=1}^2 v_i \quad , \quad (P_i, v_i) = (\text{diag}(1, -1), \text{diag}(0, 0)) \quad . \quad (6.218)$$

In this case, the $\mathcal{N} = (0, 2)$ bubbling SQM is of the form



The localization contribution to $Z_{mono}((1, 0) \oplus (1, 0))$ is then given by the contour integral

$$\begin{aligned} Z_{mono}^{(Loc)}((1, 0) \oplus (1, 0)) = & \\ \oint_{JK(\xi_1, \xi_2)} \frac{d\varphi_1 d\varphi_2}{(2\pi i)^2} & \frac{\sinh^2(2\epsilon_+) \prod_{f=1}^2 \sinh(\varphi_1 - m_f) \sinh(\varphi_2 - m_{f+2})}{\prod_{i=1}^2 \prod_{\pm} \sinh(\pm(\varphi_i - \mathbf{a}_i) + \epsilon_+) \sinh(\pm(\varphi_i + \mathbf{a}_i) + \epsilon_+)} \quad (6.219) \\ \times \frac{\prod_{\pm} \sinh(-\varphi_1 \pm \mathbf{a}_2 + m + \epsilon_+) \sinh(\varphi_2 \pm \mathbf{a}_1 + m + \epsilon_+)}{4 \sinh(\varphi_2 - \varphi_1) \sinh(\varphi_1 - \varphi_2 + 2\epsilon_+)} & . \end{aligned}$$

Let us choose $\xi_1, \xi_2 > 0$. In this case there are 8 poles contributing to this path integral:

$$\begin{aligned}
\text{I} : \varphi_1 &= \mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_2 - \epsilon_+ \quad , \\
\text{II} : \varphi_1 &= \mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_2 - \epsilon_+ \quad , \\
\text{III} : \varphi_1 &= -\mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_2 - \epsilon_+ \quad , \\
\text{IV} : \varphi_1 &= -\mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_2 - \epsilon_+ \quad , \\
\text{V} : \varphi_1 &= \mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_1 - \epsilon_+ \quad , \\
\text{VI} : \varphi_1 &= -\mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_1 - \epsilon_+ \quad , \\
\text{VII} : \varphi_1 &= \mathbf{a}_2 - 3\epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_2 - \epsilon \quad , \\
\text{VIII} : \varphi_1 &= -\mathbf{a}_2 - 3\epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_2 - \epsilon \quad .
\end{aligned} \tag{6.220}$$

See Appendix D.5 for the full expression of $Z_{mono}^{(Loc)}((1,0) \oplus (1,0))$ computed with $\xi_1, \xi_2 > 0$.

One can check that the localization result for $Z_{mono}((1,0) \oplus (1,0))$ from residues associated to these poles is not invariant under the Weyl symmetry of the $SU(2) \times SU(2)$ flavor symmetry group. Therefore, this cannot be the full, correct monopole bubbling contribution $Z_{mono}((1,0) \oplus (1,0))$ for this 't Hooft defect. Thus, we expect that the true $Z_{mono}((1,0) \oplus (1,0))$ has an extra contribution coming from Coulomb and mixed branch states that are missed in the standard localization computation.

6.4 Decoupling Flavors

Another way we can check our hypothesis is to see whether it is compatible with decoupling matter from $\mathcal{N} = 2$ theories. Consider decoupling fundamental hypermultiplets from the $SU(2)$ $N_f=4$ theory. The expressions for $\langle L_{p,0} \rangle_{N_f < 4}$ can be obtained by taking the limit as $m_i \rightarrow \infty$ while holding \mathbf{a}, \mathbf{b} fixed, provided we allow for a multiplicative renormalization of $\langle L_{p,0} \rangle_{N_f=4}$. This kind of limit was described in [69] Section 9. Note that this limit is not the decoupling limit in [156].³⁸

This allows us to compute the expectation value of the line defects in 4D $SU(2)$ gauge

³⁸The limit from [156] takes

$$A_{N_f=3} = 64q^{1/2}m_4 \quad , \tag{6.221}$$

theories with $N_f \leq 3$ fundamental hypermultiplets by taking the decoupling limit of the $N_f = 4$ theory. However, in order to use our prescription to compute I_{asympt} , we must take into account the effect of the decoupling limit on the bubbling SQM. When we decouple the fundamental hypermultiplets we are integrating out fundamental Fermi-multiplets coupled to the gauge field in bubbling SQM. This will generically introduce a Chern-Simons term (or in this case a Wilson line) determined by the way we decouple the masses [153, 154, 146, 6]

$$q = \frac{1}{2} \sum_{f=N_f+1}^4 \text{sgn}(\text{Re}[m_f]) . \quad (6.223)$$

This means that the $N_f = 1, 3$ theories must necessarily have a Chern-Simons term of level $\frac{2n+1}{2}$, $n \in \mathbb{Z}$ to be well defined.³⁹ The necessity of these half integer Chern-Simons terms is reflected in bubbling SQMs as a gauge anomaly for $q \in \mathbb{Z}$. In general, the allowed values of the Chern-Simons levels is consistent with the condition that the bubbling SQM be anomaly free $q \in \mathbb{Z} + \frac{N_f}{2}$ as noted in [90].

6.4.1 Examples

Now we can determine the value of $Z_{mono}(P, v)$ in the general N_f theory by taking certain limits of $Z_{mono}(P, v)$ in the $N_f = 4$ theory. This also gives an additional check of our hypothesis as we will now demonstrate in the case of the $N_f = 2, 3$ theories. Similar results also hold for the $N_f = 0, 1$ theories.

fixed with $u, m \rightarrow \infty$ with

$$u^{N_f=3} = u + \frac{1}{3}m_4^2 , \quad (6.222)$$

held constant where $\Lambda_{N_f=3}$ is the UV cutoff of the $N_f = 3$ theory. Further decoupling to the $N_f \leq 2$ proceeds analogously. The limit from [156] is different from the limit we take here because in their limit, the expressions for $\mathfrak{a}, \mathfrak{b}$ from (6.32) diverge (and non-perturbative corrections are small).

³⁹An analogous system was studied in 5D in [96].

$\langle L_{1,0} \rangle$ in the $N_f = 3$ Theory

Recall that for the expectation value of the $\langle L_{1,0} \rangle$ 't Hooft defect in the $N_f = 4$ theory, $Z_{mono}(1, 0)$ is of the form

$$Z_{mono}^{N_f=4}(1, 0) = -4 \frac{\prod_f \sinh(\mathbf{a} - \mathbf{m}_f \mp \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \mp 2\epsilon_+)} - 4 \frac{\prod_f \sinh(\mathbf{a} + \mathbf{m}_f \pm \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \pm 2\epsilon_+)} + 2 \cosh \left(\sum_f m_f \pm 2\epsilon_+ \right), \quad (6.224)$$

for $\pm \xi > 0$. Note that this is actually invariant under the choice of $\xi \in \mathbb{R}_+$.

By taking the limit $Re[m_4] \rightarrow \pm\infty$ we can decouple the 4^{th} fundamental hypermultiplet and find a result for the $N_f = 3$ theory. This produces the result

$$\begin{aligned} Z_{mono}^{N_f=3}(1, 0; q = \frac{s}{2}) &= \lim_{Re[m_4] \rightarrow s\infty} e^{sm_4} Z_{mono}(N_f = 4) \\ &= 2se^{-sa \pm s\epsilon_+} \frac{\prod_f \sinh(\mathbf{a} - \mathbf{m}_f \mp \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \mp 2\epsilon_+)} - 2se^{sa \pm s\epsilon_+} \frac{\prod_f \sinh(\mathbf{a} + \mathbf{m}_f \pm \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \pm 2\epsilon_+)}, \quad (6.225) \\ &+ e^{s \sum_f m_f \pm 2s\epsilon_+} : \quad \pm \xi > 0. \end{aligned}$$

Note that $Z_{mono}^{N_f=3}(1, 0; q = \frac{1}{2}) \neq Z_{mono}^{N_f=3}(1, 0; q = -\frac{1}{2})$.

In this case we see that $Re[m_4] \rightarrow \pm\infty$ corresponds to $q = \frac{s}{2}$. This modifies the localization integrand by including a factor of $e^{2q\varphi}$:

$$Z_{mono}^{(Loc)}(1, 0; q = \frac{s}{2}) = \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} \frac{2e^{s\varphi} \sinh(2\epsilon_+) \prod_{f=1}^3 \sinh(\varphi - m_f)}{\sinh(\varphi \pm \mathbf{a} + \epsilon_+) \sinh(-\varphi \pm \mathbf{a} + \epsilon_+)}. \quad (6.226)$$

As before, this can be evaluated as

$$\begin{aligned} Z_{mono}^{(JK)}(1, 0; q = \frac{s}{2}) &= 2se^{-sa \pm s\epsilon_+} \frac{\prod_f \sinh(\mathbf{a} - \mathbf{m}_f \mp \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \mp 2\epsilon_+)} \\ &- 2se^{sa \pm s\epsilon_+} \frac{\prod_f \sinh(\mathbf{a} + \mathbf{m}_f \pm \epsilon_+)}{\sinh(2\mathbf{a}) \sinh(2\mathbf{a} \pm 2\epsilon_+)}, \quad (6.227) \end{aligned}$$

where $s = \text{sign}(q)$.⁴⁰ This means that we have

$$Z_{mono}^{(extra)}(1, 0; q = \frac{s}{2}) = \begin{cases} e^{s(\sum_f m_f + \epsilon_+)} & \xi > 0 \\ e^{s(\sum_f m_f - \epsilon_+)} & \xi < 0 \end{cases} \quad (6.228)$$

This is exactly given by $I_{asympt}(q = \frac{s}{2})$ as shown in Appendix D.1.

⁴⁰Again this required fixing the overall normalization of the path integral.

$\langle L_{1,0} \rangle$ in the $N_f = 2$ Theory

Now we can go to the $N_f = 2$ theory by decoupling the 3^{rd} fundamental hypermultiplet. This can be done in two ways ($\text{Re}[m_3] \rightarrow \pm\infty$) which can induce a Wilson line of charge $q = -1, 0, 1$. The result for $q = 0$, which can be achieved by taking $\text{Re}[m_3] \rightarrow -\text{sgn}(q_{N_f=4}) \times \infty$, is given in the previous section. In the case of $q = \pm 1$, we have that $Z_{mono}(1, 0)$ is given by

$$\begin{aligned} Z_{mono}(N_f = 2) &= \lim_{\text{Re}[m_4], \text{Re}[m_3] \rightarrow q\infty} e^{-sm_4 - sm_3} Z_{mono}(N_f = 4) \\ &= -e^{-2q\mathfrak{a} \pm 2q\epsilon_+} \frac{\prod_f \sinh(\mathfrak{a} - m_f \mp \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \mp 2\epsilon_+)} - e^{2q\mathfrak{a} \pm 2q\epsilon_+} \frac{\prod_f \sinh(\mathfrak{a} + m_f \pm \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \pm 2\epsilon_+)} \\ &\quad + e^{s \sum_f m_f \pm 2s\epsilon_+}, \end{aligned} \quad (6.229)$$

where $s = \text{sign}[q]$. Again, by introducing the Wilson line in the SQM, this changes the localization computation to give

$$Z_{mono}^{(Loc)}(1, 0) = \frac{1}{2} \oint_{JK(\xi)} \frac{d\varphi}{2\pi i} \frac{e^{2q\varphi} \sinh(2\epsilon_+) \prod_{f=1}^2 \sinh(\varphi - m_f)}{\prod_{\pm} \sinh(\varphi \pm \mathfrak{a} + \epsilon_+) \sinh(-\varphi \pm \mathfrak{a} + \epsilon_+)}. \quad (6.230)$$

As before, this can be evaluated as ⁴¹

$$\begin{aligned} Z_{mono}^{(JK)}(1, 0) &= -e^{-2q\mathfrak{a} \pm 2q\epsilon_+} \frac{\prod_f \sinh(\mathfrak{a} - m_f \mp \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \mp 2\epsilon_+)} \\ &\quad - e^{2q\mathfrak{a} \pm 2q\epsilon_+} \frac{\prod_f \sinh(\mathfrak{a} + m_f \pm \epsilon_+)}{\sinh(2\mathfrak{a}) \sinh(2\mathfrak{a} \pm 2\epsilon_+)}. \end{aligned} \quad (6.231)$$

$$Z_{mono}^{(extra)} = \begin{cases} e^{s \sum_f m_f + 2s\epsilon_+} & \xi > 0 \\ e^{s \sum_f m_f - 2s\epsilon_+} & \xi < 0 \end{cases}, \quad s = \text{sign}(q). \quad (6.232)$$

Note that as shown in the previous section, $I_{asympt}(q = 0) = 0$. This is exactly given by $I_{asympt}(q)$ as appropriate as shown in Appendix D.1.

⁴¹Again this required fixing the overall normalization of the path integral.

Chapter 7

Index Theorems on \mathcal{M} and Characteristic Numbers on Kronheimer-Nakajima Spaces

Thus far we have discussed how the expectation value of an 't Hooft defect operator can be computed in theories of class \mathcal{S} by using spectral networks or in weakly coupled Lagrangian theories using localization. The class \mathcal{S} technique of spectral networks expresses the expectation value as a Laurent series in certain holomorphic “Darboux” coordinates on the Hitchin moduli space \mathcal{M} whose coefficients are framed BPS indices for IR framed BPS states. By the results of Chapter 2, we know that this can be related to the index of a Dirac operator

$$\overline{\mathcal{Q}}(\gamma, u; L(P, \zeta)) = \text{Ind} \left[\not{D}^{\mathcal{Y}} \right]_{\overline{\mathcal{M}}(P, \gamma_m; X_\infty)}^{(\gamma_e)}, \quad (7.1)$$

where $\gamma = \gamma_m \oplus \gamma_e$ can be identified in the semiclassical limit via its Lagrangian description.

On the other hand, localization of the weakly coupled Lagrangian description of 't Hooft defects computes the expectation value to be some rational function of the (exponentiated) complexified Fenchel-Nielsen coordinates $\mathfrak{a}, \mathfrak{b}$. These expressions encode the characteristic numbers of certain Kronheimer-Nakajima quiver varieties in the terms that we have thus far called Z_{mono} .

Thus, by comparing the line defect vev's computed by spectral networks and localization, we can derive unusual expressions for the L^2 -index of certain Dirac operators in terms of characteristic numbers of Kronheimer-Nakajima varieties. This can be used to yield an index theorem. In this chapter we will illustrate this principle in the example

This chapter is based on material from my publication [27].

of 4D $G = SU(2)$ $\mathcal{N} = 2$ SUSY gauge theories.

In the case of 4D $G = SU(2)$ $\mathcal{N} = 2$ supersymmetric gauge theories, there are two complexified-Fenchel-Nielsen coordinates \mathbf{a}, \mathbf{b} . From general principles, the expectation value of the 't Hooft defects can be expressed in Fenchel-Nielsen coordinates as a Fourier expansion in \mathbf{b} . More precisely, this can be written as [97, 81]

$$\langle L_{p,0}(\zeta) \rangle = \sum_{v \in \mathbb{Z}_+ : v \leq p} \cosh(\mathbf{v}, \mathbf{b}) (F(\mathbf{a}))^v Z_{mono}(\mathbf{a}, m, \epsilon; P, \mathbf{v}) , \quad (7.2)$$

where $\mathbf{v} = \text{diag}(v, -v)$ and $P = \text{diag}(p, -p)$. On the right hand side the ζ -dependence is captured by the use of complexified Fenchel-Nielsen coordinates on \mathcal{M}_ζ .¹

Here the expectation value above is expressed as a sum over monopole bubbling configurations where $\cosh(\mathbf{v}, \mathbf{b}) F(\mathbf{a})^v$ encodes the contribution of bulk fields and Z_{mono} describes the contribution from the SQM that arises on the 't Hooft defect from bubbling [23]. See [23, 24, 26] for more background and explanation of notation.

In the localization computation of $\langle L_{p,0} \rangle$, $Z_{mono}(P, \mathbf{v})$ is given by a characteristic number of a certain resolved Kronheimer-Nakajima space²

$$Z_{mono}(\mathbf{a}, m, \epsilon; P, \mathbf{v}) = \lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}_{KN}^\xi(P, \mathbf{v})} e^{\omega + \mu_T} \widehat{A}_T(T \widetilde{\mathcal{M}}_{KN}^\xi) \cdot C_{T \times T_F}(\mathcal{V}(\mathcal{R})) . \quad (7.3)$$

Here $\widetilde{\mathcal{M}}_{KN}^\xi(P, \mathbf{v})$ is a certain resolved Kronheimer-Nakajima space determined by the line defect charge (P) and core magnetic charge (\mathbf{v}), $e^{\omega + \mu_T}$ induces the T -equivariant volume form on $\widetilde{\mathcal{M}}_{KN}^\xi(P, \mathbf{v})$, $\widehat{A}_T(T \widetilde{\mathcal{M}}_{KN}^\xi)$ is the T -equivariant \widehat{A} -genus that describes the contribution from the $\mathcal{N} = 2$ vectormultiplet and $C_{T \times T_F}(\mathcal{V}(\mathcal{R}))$ is a characteristic class related to the matter hypermultiplets where T is the conserved global symmetry group of flavor, R -, and global gauge transformations. The equivariant integral can then be evaluated as a contour integral in an algebraic torus whose poles are enumerated by Young tableaux [143, 127, 128].

¹Usually complexified Fenchel-Nielsen coordinates are introduced as holomorphic coordinates, depending on a cutting system, of the character variety $\mathfrak{X} = \text{Hom}(\pi_1(C), G_{\mathbb{C}}) \backslash \text{conj.}$ for some complex gauge group $G_{\mathbb{C}}$. In our case, \mathcal{M}_ζ is isomorphic to \mathfrak{X} for all $\zeta \neq 0, \infty$ as a complex manifold, but not canonically. Our Fenchel-Nielsen coordinates will therefore also be functions on the twistor space of \mathcal{M} which, when restricted to a fiber \mathcal{M}_ζ , are holomorphic Darboux coordinates. It is in this way that they become comparable to spectral network coordinates.

²As discussed in Chapter 6, there is an additional subtlety with 4D $\mathcal{N} = 2$ $SU(N)$ theories with $N_f = 2N$.

On the other hand, using the class \mathcal{S} technology, the expectation value of a supersymmetric line defect can be computed by the trace of the holonomy of a flat $SL(N; \mathbb{C})$ connection along a corresponding curve γ in an associated Riemann surface C . Spectral networks express the expectation value of such 4D line defects as

$$\langle L_{p,0} \rangle_{u \in \mathcal{B}} = \sum_{\gamma \in \Gamma} \overline{\mathcal{Q}}(\gamma, L_{p,0}; u) \mathcal{Y}_\gamma, \quad (7.4)$$

where $\overline{\mathcal{Q}}(\gamma, L_{p,0}; u)$ are framed BPS indices, \mathcal{Y}_γ are Darboux functions on the moduli space of flat $SL(N; \mathbb{C})$ connections on C associated to the physical charge $\gamma = \gamma_m \oplus \gamma_e \oplus \gamma_f \in \Gamma$, and Γ is a torsor of the IR charge lattice [69]. Locally on moduli space we can decompose γ non-canonically into magnetic, electric, and flavor charge $\gamma = \gamma_m \oplus \gamma_e \oplus \gamma_f$.

In the semiclassical limit of the theories we are considering, the framed BPS indices of 't Hooft defects can be identified with the index of a twisted Dirac operator on singular monopole moduli space [124, 22, 133, 134, 165, 77, 78, 125]:

$$\overline{\mathcal{Q}}(\gamma, L_{p,0}; u) = \text{Ind}_{L^2} [\not{D}^{\mathcal{Y}}]_{\mathcal{E}_{\text{matter}} \otimes S\overline{\mathcal{M}}(P, \gamma_m, u)}^{\gamma_e \oplus \gamma_f}. \quad (7.5)$$

Here the superscript $\gamma_e \oplus \gamma_f$ denotes the associated eigenspace of the L^2 index of $\not{D}^{\mathcal{Y}}$, a Dirac operator modified by adding Clifford multiplication by a hyperholomorphic vector field defined by $\mathcal{Y} \in \mathfrak{t}_{\mathbb{C}}$. The Dirac operator acts on sections of $\mathcal{E}_{\text{matter}} \otimes S\overline{\mathcal{M}}(P, \gamma_m, u)$ where $S\overline{\mathcal{M}}(P, \gamma_m, u)$ is the spinor bundle on the singular monopole moduli space $\overline{\mathcal{M}}(P, \gamma_m, u)$ and $\mathcal{E}_{\text{matter}} \rightarrow \overline{\mathcal{M}}(P, \gamma_m, u)$ is a vector bundle over it related to hypermultiplet zero-modes. Additionally we take $\gamma = \gamma_m \oplus \gamma_e \oplus \gamma_f$ to be the asymptotic charge of the BPS state.

Thus, by comparing the expectation value of 't Hooft defects computed via localization and spectral network techniques in a weak coupling limit, we can derive a relation between characteristic numbers of Kronheimer-Nakajima spaces and indices of Dirac operators on singular monopole moduli space:

$$\begin{aligned} & \sum_{\gamma \in \Gamma} \overline{\mathcal{Q}}(\gamma, L_{p,0}; u) \mathcal{Y}_\gamma \\ &= \sum_{|\mathbf{v}| \leq |P|} e^{(\mathbf{v}, \mathbf{b})} (F(\mathbf{a}))^{|\mathbf{v}|} \lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}_{KN}^\xi(P, \mathbf{v})} e^{\omega + \mu T} \widehat{A}_T(T\widetilde{\mathcal{M}}_{KN}^\xi) \cdot C_{T \times T_F}(\mathcal{V}(\mathcal{R})). \end{aligned} \quad (7.6)$$

Since the formula is valid for an infinite number of line defects, we can use it both to express \mathcal{V}_γ in terms of $\mathfrak{a}, \mathfrak{b}$ (or vice versa) and to determine relations between Dirac indices and characteristic numbers on certain Kronheimer-Nakajima spaces.

7.1 Fenchel-Nielsen Networks

Generally, we will find that the coordinate transformations are quite complicated. However, as it turns out, there is a special class of spectral networks called *Fenchel-Nielsen networks* which is especially well suited to comparing with the localization results of $\langle L_{p,0} \rangle$ in terms of Fenchel-Nielsen coordinates [89]. Roughly, this is because the Darboux coordinates for Fenchel-Nielsen spectral networks exactly coincide with the Fenchel-Nielsen coordinates from localization.

These spectral networks have only double walls corresponding to a set of minimal cuts necessary to decompose the Riemann surface C into a disjoint product of punctured discs and annuli. This is a WKB spectral network where φ_2 is a Jenkins-Strebel differential — φ_2 gives a foliation of C by closed paths. Another way of saying this is that a Fenchel-Nielsen spectral network is given by a pants decomposition of C in which on each pair of pants, the spectral network is one of the two networks in Figure 7.1.

These spectral networks are referred to as Fenchel-Nielsen-type because the \mathfrak{a} -type Fenchel-Nielsen coordinate has a straightforward interpretation in terms of the associated spectral network coordinates. associated to these networks have a straightforward interpretation as complexified Fenchel-Nielsen coordinates.

Let us take a maximal set of non-intersecting curves $\{\gamma_i\}_{i=1}^{3g-3+n}$ that define a pants decomposition of C . On each pair of pants, there are classes of curves which are homotopic to a subset of the $\{\gamma_i\}$. The holonomy around a curve that is homotopic to

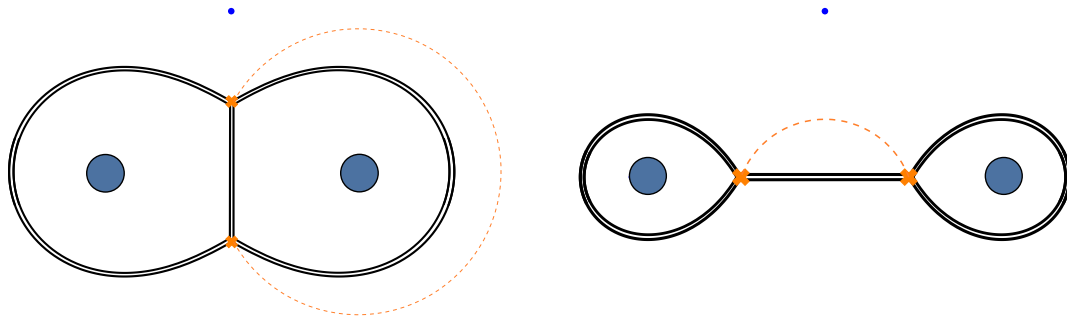


Figure 7.1: This shows the two possible types of Fenchel-Nielsen spectral networks on a single pair of pants where the dotted orange lines are a branch cut. These spectral networks are called “molecules” in [89].

such a γ_i is given in terms of the spectral network coordinates ³

$$\langle L_{\gamma_i} \rangle = \text{Tr}_2 \begin{pmatrix} \mathcal{Y}_{\gamma_i} & 0 \\ 0 & \mathcal{Y}_{\gamma_i}^{-1} \end{pmatrix} = \mathcal{Y}_{\gamma_i} + \mathcal{Y}_{\gamma_i}^{-1}. \quad (7.7)$$

However, we see from before, that this is simply the definition of the Fenchel-Nielsen coordinate \mathfrak{a} :

$$\mathcal{Y}_{\gamma_i} + \mathcal{Y}_{\gamma_i}^{-1} = \text{Tr}_2 e^{\mathfrak{a}}. \quad (7.8)$$

Wilson line vevs for a fundamental representation of a factor in the (four-dimensional) gauge algebra are usually expressed as three-term expressions in the functions \mathcal{Y}_{γ} . (See e.g. (10.33) from [69].) The relation to the above two-term expansion is clarified in equation (7.36) above.

In a large class of theories, such as the ones we study here, Fenchel-Nielsen spectral networks can be obtained from a generic WKB spectral network by performing a juggle. This requires changing ζ such that we cross an infinite number of \mathcal{K} -walls. In the theories we consider, there are infinite number of such walls which accumulate along co-dimension 1 “accumulation points” in the ζ -plane. See Figure 7.2. In our setting, sending ζ to an accumulation point is equivalent to undergoing the infinite number of

³Note that \mathcal{Y}_{γ} are defined for $\gamma \in H_1(\Sigma; \mathbb{Z})$ while L_{γ} is defined for $\gamma \subset C$. Here we use the loose notation where \mathcal{Y}_{γ} for $\gamma \subset C$ is defined as $\mathcal{Y}_{\pi^{-1}(\gamma)_i}$ the lift under the projection $\pi : \Sigma \rightarrow C$ onto one of the sheets. Due to (6.15), the two choices of lifting are related by inverses and thus are merely a choice of convention.

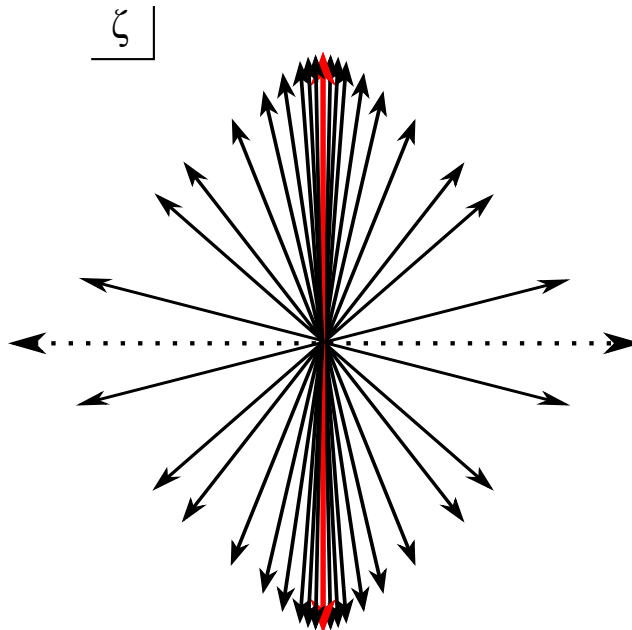


Figure 7.2: This figure shows the structure of the \mathcal{K} -walls in the ζ -plane. There are accumulation points (red) on the imaginary axis where the associated WKB spectral network becomes a Fenchel-Nielsen spectral network.

flips that occur in a juggle, leading to a Fenchel-Nielsen spectral network. See Section 7.1.1 for further discussion.

Remark Recall from the discussion of Section 6.2, that the spectral network coordinates \mathcal{Y}_γ is given in (6.17) and has a semiclassical expansion with an infinite number of non-perturbative corrections. Since, as we showed above, we can identify the complexified Fenchel-Nielsen coordinates with spectral network coordinates, the Fenchel-Nielsen coordinates $\mathfrak{a}, \mathfrak{b}$ must similarly have an infinite number of non-perturbative corrections to their semiclassical value. We will demonstrate this in the example of the $SU(2)$ $N_f = 0$ theory in Section 7.2.2 by computing the leading non-perturbative corrections.

7.1.1 Fenchel-Nielsen Spectral Networks and The Semiclassical Region

Now we can connect the formalism of semiclassical BPS states and spectral networks. Recall that the expectation value of line defects is determined by the framed BPS

index as in (6.2). Therefore, the index of the Dirac operator from the previous section can be used to determine the expectation value of an 't Hooft defect in the semiclassical limit.

Since we want to compare to the localization computation, which is naturally expressed in terms of Fenchel-Nielsen coordinates, one would hope to use Fenchel-Nielsen spectral networks and the associated Dirac operators. In order to implement this we need to know: 1.) if there exists Fenchel-Nielsen spectral networks in the semiclassical limit and 2.) where in parameter space these spectral networks exist so that we can compare to indices of Dirac operators. In this section we will show that such spectral networks exist in the semiclassical limit, but that they only exist in parameter space where the moduli space approximation breaks down.

The question of whether or not a Fenchel-Nielsen network exists is equivalent to the question of whether or not there exists a Jenkins-Strebel differential on C that encodes the data of the theory in some semiclassical limit. The data of the differential is $(u, \zeta, m) \in \mathcal{B} \times U(1) \times \mathfrak{t}_F$.

The existence of Jenkins-Strebel differentials on a Riemann surface C with punctures are studied by Liu [117, 116]. There, Liu shows that given a decomposition of C into a collection of punctured disks $\{D_m\}$ and annuli $\{R_k\}$, there exists a uniquely determined real Jenkins-Strebel differential with closed trajectory φ_2 with fixed monodromy $m_i \in \mathbb{R}$ around each puncture and height $h_k \in \mathbb{R}$ around each annuli where the height is defined as

$$h_k = \inf_{\gamma_k} \oint_{\gamma_k} |\operatorname{Im} \sqrt{\varphi_2}|, \quad (7.9)$$

where the infimum is taken over all paths that run between the boundaries of R_k . Note that the Fenchel-Nielsen spectral network is exactly given by the union of the boundaries of these component disks and annuli.

Now consider as an example the case of the 4D $SU(2)$ $\mathcal{N} = 2^*$ theory. This theory is constructed as a theory of class \mathcal{S} by taking C to be a torus with a single puncture. This theory comes with a complex 2-dimensional parameter space defined by $u \in \mathcal{B} \cong \mathbb{C}$ and the complex mass parameter of the hypermultiplet. C can be decomposed as an annulus R_a and a punctured disk D_m . See Figure 7.3 for the example of the 4D $SU(2)$

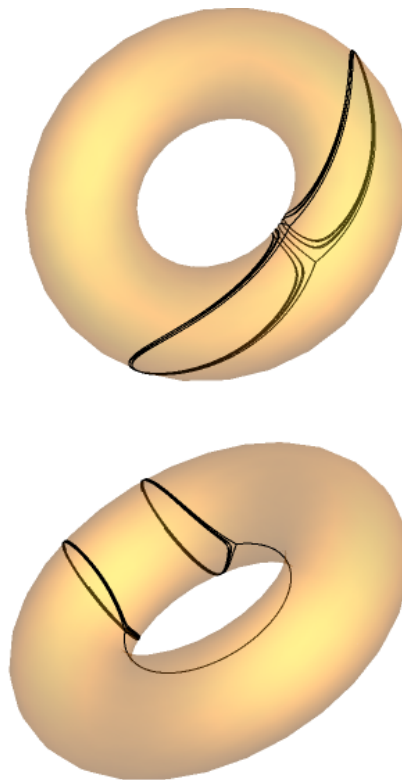


Figure 7.3: This figure shows the explicit decomposition of the UV curve $C = T^2 \setminus \{0\}$ into disks and annuli for the 4D $SU(2)$ $\mathcal{N} = 2^*$ theory in two different ways. Note that the boundary of these components give rise to the Fenchel-Nielsen spectral networks corresponding to both types of fundamental molecules. The type of Fenchel-Nielsen molecule describing the spectral network depends on the relative holonomies of the cuts. See [89] for details.

$\mathcal{N} = 2^*$ theory where $C = T^2 / \{0\}$.⁴ Thus, there is a 3 dimensional family (specifying m , γ , and ζ) of Jenkins-Strebel differentials which forms a real co-dimension 1 subspace of parameter space. This suggests that there could exist a Jenkins-Strebel differential in the semiclassical limit ($|u| \rightarrow \infty$) and therefore that there could exist a Fenchel-Nielsen spectral networks in the semiclassical limit. This has been confirmed by numerical computations.⁵

⁴We would especially like to thank Pietro Longhi for providing these figures.

⁵We would like to thank Pietro Longhi for sharing his numerical computation for the $SU(2)$ $\mathcal{N} = 2^*$ theory and for making the authors aware of Liu's work on Jenkins-Strebel differentials.

Now recall that for a WKB Fenchel-Nielsen spectral network, the *real* Jenkins-Strebel differential is related to the Seiberg-Witten differential as

$$\varphi_2 = \zeta^{-2} \lambda_{SW}^2 . \quad (7.10)$$

Asking that φ_2 as defined by this equation is a Jenkins-Strebel differential defines the Fenchel-Nielsen locus in $\mathcal{B}^* \times \mathbb{C}^*$.

As usual in Seiberg-Witten theory, the periods of λ_{SW} give the vev's of the Higgs field and mass parameters. In our case the UV curve is given by $C = T^2 \setminus \{0\}$. This means that if we pick a basis of $H_1(\bar{\Sigma}; \mathbb{Z}) = \text{span}_{\mathbb{Z}}\{A, B\}$,

$$\oint_A \lambda_{SW} = a \quad , \quad \oint_B \lambda_{SW} = a_D \quad , \quad \oint_{D_p} \lambda_{SW} = m_f , \quad (7.11)$$

where D_p is a loop circling the puncture and m_f is the mass of the adjoint hypermultiplet. In this notation, the condition that φ_2 is a Jenkins-Strebel differential (and hence gives rise to a Fenchel-Nielsen-type WKB spectral network) is that

$$\oint_A \zeta^{-1} \lambda_{SW} \in \mathbb{R} \quad , \quad \oint_{D_{m_i}} \zeta^{-1} \lambda_{SW} \in \mathbb{R} , \quad (7.12)$$

which can be rewritten as

$$\text{Im}[\zeta^{-1} a] = X_\infty = 0 \quad , \quad \text{Im}[\zeta^{-1} m_f] = m_x = 0 . \quad (7.13)$$

This locus in parameter space, which we will call the *Fenchel-Nielsen locus*, is an accumulation point of \mathcal{K} -walls in the ζ -plane and we will denote the associate phase in $U(1)$ as ζ_{FN} .

Unfortunately, the Fenchel-Nielsen locus is exactly where the moduli space approximation, which gives the identification between the framed BPS index and the index of a Dirac operator on singular monopole moduli space, breaks down. In the limit $X_\infty \rightarrow 0$, the space $\overline{\mathcal{M}}(P_n, \gamma_m; X_\infty)$ (and $\mathcal{M}(\gamma_m; X_\infty)$) are not defined. The reason is that the semiclassical expression for the central charge is given by

$$\zeta^{-1} Z_\gamma = - \left[\frac{4\pi}{g^2} (\gamma_m, X_\infty) - \langle \gamma_e, Y_\infty \rangle \right] + i \left[\frac{4\pi}{g^2} (\gamma_m, Y_\infty) + \langle \gamma_e, X_\infty \rangle \right] . \quad (7.14)$$

Thus the BPS mass $M_{BPS} = \text{Re}[\zeta^{-1} Z_\gamma]$ for a monopole goes to zero as we scan ζ such that $X_\infty \rightarrow 0$. However, we know that monopoles do not become massless in

the semiclassical limit. Thus, we can deduce that the non-perturbative quantum effects must become large and therefore the effective SQM description above must break down.

However, by taking $|X_\infty|, |Y_\infty| \rightarrow \infty$ as $|X_\infty|/|Y_\infty| \rightarrow 0$, we can still identify framed BPS indices with the index of a Dirac operator for phases which are arbitrarily close to the Fenchel-Nielsen locus. This will allow us to give an index theorem for the supercharge Dirac operators almost everywhere on the ζ -plane. See Figure 1 of [134] or Figure 4. of [133] for more details.

The above analysis makes it clear that there always exists $SU(2)$ Fenchel-Nielsen networks (and in fact all $SU(N)$ -type Fenchel-Nielsen spectral networks) in the semiclassical limit. These exist on the locus where all of the masses and $a_i = \oint_{A_i} \lambda_{SW}$ have the same phase. Such a spectral network can be constructed by gluing together pairs of pants with semiclassical Fenchel-Nielsen spectral networks on them by the procedure of [89]. The only condition here is that the Fenchel-Nielsen spectral networks all have the associated phase.

Remark Recall that a Fenchel-Nielsen spectral network corresponds to a WKB spectral network with a Jenkins-Strebel differential. This is defined by decomposing the Riemann surface C into a collection of annuli and punctured disks. On each component, the flow lines of φ give a foliation of curves that are homotopic to the boundary components. If we consider infinitesimally deforming the phase ζ associated to the quadratic differential, we find that the flow lines on each component are no longer homotopic to the boundary components, but rather spiral into them with a very large winding number. Thus, as we send $\zeta \rightarrow \zeta_{FN}$ the flow lines of φ_2 twist around the boundary components infinitely many times until they form closed paths, producing a Fenchel-Nielsen spectral network. This infinite spiraling indicates that Fenchel-Nielsen spectral networks can be achieved by performing a juggle on a WKB spectral network where all physical parameters have aligned phases. Using the procedure from Section 8.4 of [89], one can identify the limiting coordinates (6.26) with the Fenchel-Nielsen coordinates $\mathcal{Y}_A^{(+)} = e^a$, while $\mathcal{Y}_B^{(+)}$ defines a choice of e^b .⁶

⁶Note that we could also approach the Fenchel-Nielsen locus in the opposite direction. The procedure

Thus, the Darboux coordinates associated to Fenchel-Nielsen spectral networks in the cases we are studying can be obtained by acting on a generic set of spectral network coordinates by an infinite number of cluster coordinate transformations. The resulting spectral network coordinates are those that result from the flip (6.26). This gives a recursion formula for the Darboux coordinates that can be “integrated” to give a relation between the Darboux coordinates of a spectral network in any chamber and the Fenchel-Nielsen coordinates which are used in localization computations. This will be the primary computational tool that we will use to construct an index theorem and give a formula for the characteristic numbers in the next section.

7.2 Index Theorem and Characteristic Numbers

In this section we will compare the different methods of computing the expectation value of ’t Hooft defects in 4D $\mathcal{N} = 2$ $G = SU(2)$ asymptotically free theories with fundamental and hypermultiplet matter. We will outline how this comparison can be used to give an index theorem for Dirac operators on singular monopole moduli spaces and give the characteristic numbers of certain Kronheimer-Nakajima spaces⁷. We will explicitly show these for the $SU(2)$ $N_f = 0$ theory.

7.2.1 General Theory

Consider an $\mathcal{N} = 2$ $SU(2)$ Lagrangian theory of class \mathcal{S} with mass parameters of identical phase. Now pick a point in the semiclassical limit of the Coulomb branch away from the Fenchel-Nielsen locus. We are interested in computing the expectation values of a ’t Hooft defect which is specified by an integer p and a phase ζ .

Now consider comparing the localization and spectral network result for the expectation value of ’t Hooft defects. Localization requires introducing an IR regulating

from [89] in conjunction with the relation between the two limiting coordinates (6.27), correspond to two different choices of Fenchel-Nielsen coordinates. Equation (7.52) of [68] shows that \mathfrak{a} is well-defined and \mathcal{Y}_B^\pm define two choices of \mathfrak{b} that are related by (6.27).

⁷These are the transversal slice to the stratum of the bubbling locus of singular monopole moduli spaces. See [23, 142] for details.

$\frac{1}{2}\Omega$ -deformation and expresses the expectation value in terms of Fenchel-Nielsen coordinates. This coordinate expansion is well defined almost everywhere in a simply connected region on the Coulomb branch and is independent of the phase of ζ there due to trivial monodromy. The spectral networks computation however, is not independent of the phase ζ . Rather, it is different in each chamber $c_n \subset \mathbb{C}_\zeta$ of the ζ -plane.⁸ The reason is that the spectral network undergoes topology change at each \mathcal{K} -wall and hence has a different set of associated Darboux coordinates in each chamber $c_n \subset \mathbb{C}_\zeta$.

Away from the Fenchel-Nielsen locus, the spectral network coordinates are not Fenchel-Nielsen coordinates, but rather are Darboux coordinates which are related to the localization Fenchel-Nielsen coordinates by an infinite sequence of Kontsevich-Soibelman transformations.

Due to the “simple” transformation properties of the spectral network coordinates, these coordinate transformations can be integrated to determine the mapping between Fenchel-Nielsen coordinates and the Darboux coordinates in every chamber. This can be achieved as follows. First, solve for the expectation value of the minimal Wilson and ’t Hooft defects in a generic WKB spectral network of choice. We will assign the chamber in the ζ -plane in which we have computed these as the c_0 chamber. Now by tuning the phase of ζ , we will cross walls of marginal stability which takes us from the c_n chamber to the $c_{n\pm 1}$ chamber depending on the direction we tune ζ .

Now we can solve for the expectation values in all chambers by solving the recursive \mathcal{K} -wall crossing formulas [69]:

$$\langle L_{1,0} \rangle_{\zeta \in c_n}(\mathcal{Y}_{\gamma_i}) = \langle L_{1,0} \rangle_{\zeta \in c_{n-1}}(\mathcal{K}_{\gamma_n} \cdot \mathcal{Y}_{\gamma_i}) \quad , \quad \langle L_{0,1} \rangle_{\zeta \in c_n}(\mathcal{Y}_{\gamma_i}) = \langle L_{0,1} \rangle_{\zeta \in c_{n-1}}(\mathcal{K}_{\gamma_n} \cdot \mathcal{Y}_{\gamma_i}) \quad , \quad (7.15)$$

where $\langle L \rangle_{\zeta \in c_n}$ is the expectation value of L computed using the WKB spectral network associated to $\zeta \in c_n$ and the \mathcal{K} -wall between the c_n and c_{n-1} chamber is $\widehat{W}(\gamma_n)$.

After we set the $\frac{1}{2}\Omega$ deformation parameter $\epsilon_+ \rightarrow 0$, we can then compare the localization expression of the expectation value of the Wilson and ’t Hooft defects to their expression in terms of the spectral network coordinates in a generic chamber

⁸Here ζ changes the decomposition of u into X_∞, Y_∞ .

chamber c_i . Inverting these formulas allows us to solve for (\mathbf{a}, \mathbf{b}) in terms of the \mathcal{Y}_γ in some fixed chamber c_i .

Then, by combining this with the solution with the KS-wall crossing formulas, we then have an expression for the Fenchel-Nielsen coordinates (\mathbf{a}, \mathbf{b}) in terms of the \mathcal{Y}_γ in all chambers c_n . Inverting the formulas, one obtains an (admittedly complicated) expression for the \mathbf{a}, \mathbf{b} in terms of the \mathcal{Y}_γ .

We can then take these expressions and substitute the expression for \mathbf{a}, \mathbf{b} in terms of the \mathcal{Y}_γ into the localization expression above. By identifying the coefficients of the Laurent expansion with that of the spectral network computation we arrive at an expression for the framed BPS indices in every chamber. Then by using the relation of the index of the Dirac operator to the framed BPS indices in the semiclassical limit away from the Fenchel-Nielsen locus

$$\begin{aligned} \sum_{\gamma \in \Gamma} \text{Ind} \left[\mathcal{D}^{\mathcal{Y}} \right]_{\overline{\mathcal{M}}(P, \gamma_m; X_\infty)}^{\gamma_e \oplus \gamma_f} \mathcal{Y}_\gamma &= \sum_{\gamma \in \Gamma} \overline{\mathcal{Q}}(\gamma; L_{[P, 0]}, c_n) \mathcal{Y}_\gamma \\ &= \left\{ \sum_{|v| \leq |P|} e^{(v, \mathbf{b})} (F(\mathbf{a}))^{|v|} \left[\lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}_{KN}^\xi(P, v)} e^{\omega + \mu_T} \widehat{A}_T(T \widetilde{\mathcal{M}}_{KN}) \cdot C_{T \times T_F}(\mathcal{V}(\mathcal{R})) \right] \right\}_{\substack{\mathbf{a}(\mathcal{Y}_\gamma) \\ \mathbf{b}(\mathcal{Y}_\gamma)}}, \end{aligned} \quad (7.16)$$

we get an index formula for the associated Dirac operator in all chambers arbitrarily close to the Fenchel-Nielsen locus.

Similarly, we can substitute the expression for the Darboux coordinates \mathcal{Y}_γ in the c_n chamber in terms of the Fenchel-Nielsen coordinates \mathbf{a}, \mathbf{b} into the spectral network computation. Then, identifying the coefficients of the Laurent expansion in terms of the exponentiated Fenchel-Nielsen coordinates on both sides

$$\begin{aligned} \sum_{|v| \leq P} e^{\mathbf{b}}(F(\mathbf{a}))^{|v|} Z_{mono}(\mathbf{a}, m; P, v) &= \sum_{\gamma \in \Gamma} \overline{\mathcal{Q}}(\gamma; L_{[P, 0]}, c_n) \mathcal{Y}_\gamma(\mathbf{a}, \mathbf{b}), \\ Z_{mono}(\mathbf{a}, m; P, v) &= \lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}_{KN}^\xi(P, v)} e^{\omega + \mu_T} \widehat{A}_T(T \widetilde{\mathcal{M}}_{KN}) \cdot C_{T \times T_F}(\mathcal{V}(\mathcal{R})), \end{aligned} \quad (7.17)$$

allows us to express the characteristic numbers that determine $Z_{mono}(\mathbf{a}, m; P, v)$ as a rational function of exponentiated Fenchel-Nielsen \mathbf{a} -coordinates, masses, and framed BPS indices.

Remark Note that there is an additional subtlety in the case of the $\mathcal{N} = 2$ $SU(2)$ $N_f = 4$ theory. The reason is that $Z_{mono}(\mathfrak{a}, m; P, v)$ is not entirely given by a characteristic number but rather has an additional contribution from states on the Coulomb branch of an associated SQM [26].

7.2.2 Example: $SU(2)$ $N_f = 0$ Theory

Now we will apply the above, discussion to determine the framed BPS indices for the $SU(2)$ $N_f = 0$ theory. This will produce an index-like formula for a Dirac operator coupled to a hyperholomorphic vector field $G_n(\mathcal{Y}_\infty)$ on singular monopole moduli space.

The expectation value of the 't Hooft defect in the $N_f = 0$ theory is given in terms of Fenchel-Nielsen coordinates as

$$\langle L_{p,0} \rangle_{Loc} = \sum_{0 \leq v \leq p} \cosh(v, \mathfrak{b}) (F(\mathfrak{a}))^v Z_{mono}(\mathfrak{a}; P, v) , \quad (7.18)$$

where

$$P = \text{diag}(p, -p) \quad , \quad v = \text{diag}(v, -v) . \quad (7.19)$$

In the case where $\epsilon_+ = 0$, which is necessary for comparing with the Fenchel-Nielsen and Dirac operator expressions, these have the simple form

$$\langle L_{0,1} \rangle_{Loc} = e^{\mathfrak{a}} + e^{-\mathfrak{a}} \quad , \quad \langle L_{1,0} \rangle_{Loc} = \frac{e^{\mathfrak{b}} + e^{-\mathfrak{b}}}{2 \sinh(\mathfrak{a})} . \quad (7.20)$$

As shown in [69, 133], the expectation value of the 't Hooft defect of minimal charge in terms of Darboux coordinates in the chamber c_n is given by ⁹

$$\langle L_{1,0} \rangle_{\zeta \in c_n} = \frac{1}{\mathcal{X}_m \mathcal{X}_e^n} \left(U_n(f_n) - \frac{1}{\mathcal{X}_e} U_{n-1}(f_n) \right) \quad , \quad \langle L_{0,1} \rangle_{\zeta \in c_n} = 2f_n , \quad (7.21)$$

where

$$f_n = \frac{1}{2} \left(\mathcal{X}_e + \frac{1}{\mathcal{X}_e} \left(1 + \mathcal{X}_m^2 \mathcal{X}_e^{2n+2} \right) \right) , \quad (7.22)$$

and

$$\mathcal{X}_m = \mathcal{Y}_{\frac{1}{2}H_\alpha} \quad , \quad \mathcal{X}_e = \mathcal{Y}_{\frac{1}{2}\alpha} . \quad (7.23)$$

⁹Recall that $L_{1,0}$ is the minimal 't Hooft defect and $L_{0,1}$ is the minimal Wilson defect.

Here we take $n \in \mathbb{Z}_+$ to denote the chamber c_n in the ζ -plane and the notation U_n to denote the Tchebyshev polynomial of the second kind:

$$U_{n-1}(\cos(x)) = \frac{\sin(nx)}{\sin(x)} . \quad (7.24)$$

In this theory, the Fenchel-Nielsen locus is given by $\langle \alpha, X_\infty \rangle = 0$.¹⁰ Let us pick a $u \in \mathcal{B}$ such that $\Phi_\infty = \pm i X_\infty$ when $\zeta \in \mathbb{R}$ and $\Phi_\infty = \pm Y_\infty$ when $\zeta \in i\mathbb{R}$. We can now identify the imaginary axis as the Fenchel-Nielsen locus.

The spectrum of the vanilla BPS states in the semiclassical region are given by

$$\gamma = \pm \alpha \quad , \quad \gamma_n^\pm = \pm H_\alpha \oplus n \alpha \quad , \quad n \in \mathbb{Z} , \quad (7.25)$$

with BPS indices

$$\Omega(\gamma; u) = \begin{cases} -2 & \gamma = \pm \alpha \\ 1 & \gamma = \gamma_n^\pm \\ 0 & \text{else} \end{cases} \quad (7.26)$$

Thus, the phase of the central charge corresponding to a state with charge $\gamma = \gamma_m \oplus \gamma_e$ is given by

$$\begin{aligned} \text{phase}(Z_\gamma) &= -\arctan \left[\frac{(\gamma_m, Y_\infty)}{(\gamma_m, X_\infty)} + \frac{g^2}{4\pi} \left(\frac{\langle \gamma_e, X_\infty \rangle}{(\gamma_m, X_\infty)} + \frac{(\gamma_m, Y_\infty) \langle \gamma_e, Y_\infty \rangle}{(\gamma_m, X_\infty)^2} \right) \right] + O(g^4) , \\ &= -\arctan \left\{ \frac{(\gamma_m, Y_\infty)}{(\gamma_m, X_\infty)} \left[1 + \frac{g^2}{4\pi} \left(\frac{\langle \gamma_e, Y_\infty \rangle}{(\gamma_m, X_\infty)} + \frac{\langle \gamma_e, X_\infty \rangle}{(\gamma_m, Y_\infty)} \right) \right] \right\} + O(g^4) . \end{aligned} \quad (7.27)$$

Without loss of generality, we can restrict to the case ζ in the positive real half-plane (the other cases follow analogously). We are now only concerned with the phase of BPS states whose \mathcal{K} -walls are in the positive real half- ζ plane. These BPS states have charges γ_n^- with \mathcal{K} -walls along the phases

$$\text{phase}(Z_{\gamma_n^-}) = -\arctan \left\{ \frac{(H_\alpha, Y_\infty)}{(H_\alpha, X_\infty)} \left[1 - \frac{n g^2}{4\pi} \left(\frac{(H_\alpha, Y_\infty)}{(H_\alpha, X_\infty)} + \frac{(H_\alpha, X_\infty)}{(H_\alpha, Y_\infty)} \right) \right] \right\} , \quad (7.28)$$

to order $O(g^4)$. Note that the phases of the central charges are ordered

$$\text{phase}(Z_{\gamma_n^-}) > \text{phase}(Z_{\gamma_{n-1}^-}) , \quad (7.29)$$

¹⁰Note, that here we have defined $X_\infty \in \mathfrak{t}$ to lie in the positive chamber. We will thus assume that $\langle \alpha, X_\infty \rangle \geq 0$.

in the positive real half-plane. We can now define the chambers $c_n \subset \widehat{\mathbb{C}}^*$ as

$$c_n := \left\{ \zeta \in \widehat{\mathbb{C}}^* \mid \text{phase}(Z_{\gamma_n}) > \text{phase}(\zeta) > \text{phase}(Z_{\gamma_{n-1}}) \right\}. \quad (7.30)$$

At the \mathcal{K} -wall defined by $\widehat{W}(\gamma_n^-)$, the $X_{i,n-1}$ mutate as:

$$\begin{aligned} \mathcal{K}_{\gamma_n^-} X_{1,n-1} &= (1 + X_{1,n-1} X_{2,n-1}^n)^{-2n} X_{1,n-1}, \\ \mathcal{K}_{\gamma_n^-} X_{2,n-1} &= (1 + X_{1,n-1} X_{2,n-1}^n)^2 X_{2,n-1}. \end{aligned} \quad (7.31)$$

By comparing with the computation of $\langle L_{1,0} \rangle$ and $\langle L_{0,1} \rangle$ using localization, we can determine the coordinate transformation relating the Fenchel-Nielsen coordinates to the spectral network coordinates in the c_n chamber:

$$\mathcal{X}_m = -\frac{b(a^2 - 1)(1 + a^{2n+2}b^2)^n}{a(1 + a^{2n}b^2)^{n+1}}, \quad \mathcal{X}_e = \frac{a(1 + a^{2n}b^2)}{(1 + a^{2n+2}b^2)}, \quad (7.32)$$

where

$$a = e^a, \quad b = e^b. \quad (7.33)$$

We can now invert the expressions (7.32) to get

$$\begin{aligned} a &= f_n - \sqrt{f_n^2 - 1}, \quad b = \frac{\sqrt{a - \mathcal{X}_e}}{\sqrt{a^{2n+1}(a\mathcal{X}_e - 1)}}, \\ f_n &= \frac{1}{2} \left(\mathcal{X}_e + \frac{1}{\mathcal{X}_e} + \mathcal{X}_m^2 \mathcal{X}_e^{2n+1} \right). \end{aligned} \quad (7.34)$$

Note that both of these pairs of expressions requires matching the semiclassical expressions for $X_{1,0}, X_{2,0}, e^a, e^b$.

This can be used to show explicitly that the spectral network coordinates approach the Fenchel-Nielsen coordinates as we approach the Fenchel-Nielsen locus. Sending the phase $\zeta \rightarrow \zeta_{FN}$ can then be achieved by sending $n \rightarrow \infty$ or $n \rightarrow -\infty$. Going to the Fenchel-Nielsen locus rotates the phase of $\Phi_\infty = \zeta(Y_\infty + iX_\infty)$ so that, in the limit $n \rightarrow \pm\infty$, $\pm\langle\alpha, Y_\infty\rangle > 0$. Then from the expression for \mathcal{Y}_γ (6.17), we see that $\lim_{n \rightarrow \pm\infty} \mathcal{Y}_{\gamma_n}$ is exponentially suppressed

$$\lim_{n \rightarrow \pm\infty} \mathcal{Y}_{\gamma_n} \sim \lim_{n \rightarrow \infty} e^{-(|n|+1)\pi R|\langle\alpha, Y_\infty\rangle|} \times O(e^{-4\pi^2 R/g^2}) = 0. \quad (7.35)$$

Thus, sending $\zeta \rightarrow \zeta_{FN}$ reduces the standard three-term expansion of the value of the Wilson line to

$$\lim_{n \rightarrow \pm\infty} \langle L_{1,0} \rangle_n = \mathcal{Y}_{\frac{1}{2}\alpha} + \mathcal{Y}_{-\frac{1}{2}\alpha}. \quad (7.36)$$

This allows us to identify $\mathcal{Y}_{\frac{1}{2}\alpha} = e^{\pm \mathfrak{a}}$.¹¹

The coordinate identification above can also be seen from taking limits of the coordinate transformation expressions (7.32). As $\zeta \rightarrow \zeta_{FN}$, we pass through chambers with $n \rightarrow \pm\infty$. From the semiclassical expression for \mathfrak{a} in (6.32), we see that in this region

$$\left| e^{\pm \langle \alpha, \mathfrak{a} \rangle} \right| < 1 . \quad (7.37)$$

Therefore:

$$\lim_{n \rightarrow \pm\infty} \mathcal{X}_e = \lim_{n \rightarrow \pm\infty} \mathcal{Y}_{\frac{1}{2}\alpha} = e^{\pm \mathfrak{a}} \Big|_{\zeta=\zeta_{FN}} . \quad (7.38)$$

Similarly we can apply this method to $\mathcal{X}_m = \mathcal{Y}_{\frac{1}{2}H_\alpha}$ to find

$$\lim_{n \rightarrow \pm\infty} \mathcal{Y}_{\frac{1}{2}H_\alpha} = -\sinh(\mathfrak{a}) e^{\pm \mathfrak{b}} \Big|_{\zeta=\zeta_{FN}} . \quad (7.39)$$

Using the identification of the Fenchel-Nielsen coordinates with spectral network coordinates in c_n (7.34), it is possible to compute explicitly the non-perturbative corrections to $\mathfrak{a}, \mathfrak{b}$. From the results of [67], we know that the corrections to the semiclassical contribution of \mathcal{Y}_γ is given by solving the recursive formula

$$\begin{aligned} \log \mathcal{Y}_\gamma(u, \theta, \zeta) &= \log \mathcal{Y}_\gamma^{sf}(u, \theta, \zeta) \\ &+ \sum_{\gamma'} \Omega(\gamma'; u) \frac{\langle \gamma', \gamma \rangle}{4\pi i} \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 - \mathcal{Y}_{\gamma'}(u, \theta, \zeta')) , \end{aligned} \quad (7.40)$$

where \mathcal{Y}_γ^{sf} is the semi-flat term which is the semiclassical expression in (6.17) and $\ell_{\gamma'}$ is the ray in the ζ -plane along the \mathcal{K} -wall $\mathcal{K}_{\gamma'}$.

To compute the first order non-perturbative corrections to $\mathfrak{a}, \mathfrak{b}$, we need to use the fact that the semi-flat expressions for $\mathcal{X}_e, \mathcal{X}_m$ are of order $\mathcal{X}_e^{sf} \sim O(1)$ while $\mathcal{X}_m^{sf} \sim O\left(e^{-\frac{4\pi}{g^2}}\right)$. Using this, we can expand the expressions for $\mathfrak{a}, \mathfrak{b}$ as a Laurent series in \mathcal{X}_m :

$$\begin{aligned} e^{\mathfrak{a}} &= \mathcal{X}_e \left[1 + \frac{\mathcal{X}_e^{2n+1}}{\mathcal{X}_e - 1/\mathcal{X}_e} \mathcal{X}_m^2 + O(\mathcal{X}_m^4) \right] , \\ e^{\mathfrak{b}} &= -\frac{\mathcal{X}_m}{\mathcal{X}_e - \mathcal{X}_e^{-1}} \left[1 - \frac{\mathcal{X}_e^{2n+1} \mathcal{X}_m^2}{(\mathcal{X}_e - 1/\mathcal{X}_e)^3} f_n(\mathcal{X}_e) + O(\mathcal{X}_m^4) \right] , \end{aligned} \quad (7.41)$$

where

$$f_n(\mathcal{X}_e) = \mathcal{X}_e + n(\mathcal{X}_e - 1/\mathcal{X}_e) . \quad (7.42)$$

¹¹As we will see below, the signs are correlated with the different limits $n \rightarrow \pm\infty$.

Then by taking into account the non-perturbative corrections to $\mathcal{X}_e, \mathcal{X}_m$ and comparing orders in $e^{-\frac{4\pi}{g^2}}$, we can compute the first order non-perturbative corrections to the leading (semiclassical) expression for $\mathfrak{a}, \mathfrak{b}$ as

$$\begin{aligned}\mathfrak{a}_{n.p.}^{(1)} &= \log[\mathcal{X}_e^{n.p.(1)}] + \log \left[1 + \frac{(\mathcal{X}_e^{sf})^{2n+1} (\mathcal{X}_m^{sf})^2}{\mathcal{X}_e^{sf} - 1/\mathcal{X}_e^{sf}} \right], \\ \mathfrak{b}_{n.p.}^{(1)} &= \log[\mathcal{X}_m^{n.p.(1)}] + \left(\frac{\mathcal{X}_e^{sf} + 1/\mathcal{X}_e^{sf}}{\mathcal{X}_e^{sf} - 1/\mathcal{X}_e^{sf}} \right) \log[\mathcal{X}_e^{n.p.(1)}] \\ &\quad + \log \left[1 - \frac{(\mathcal{X}_e^{sf})^{2n+1} (\mathcal{X}_m^{sf})^2}{4(\mathcal{X}_e^{sf} - 1/\mathcal{X}_e^{sf})^2} f_n(\mathcal{X}_e^{sf}) \right],\end{aligned}\tag{7.43}$$

Here $\mathfrak{a}_{n.p.}^{(1)}, \mathfrak{b}_{n.p.}^{(1)}$ are the first of an infinite series of non-perturbative corrections to the semiclassical values of $\mathfrak{a}, \mathfrak{b}$

$$\mathfrak{a} = \mathfrak{a}_{s.c.} + \sum_{i=1}^{\infty} \mathfrak{a}_{n.p.}^{(i)}, \quad \mathfrak{b} = \mathfrak{b}_{s.c.} + \sum_{i=1}^{\infty} \mathfrak{b}_{n.p.}^{(i)},\tag{7.44}$$

and $\mathcal{X}_e^{n.p.(1)}, \mathcal{X}_m^{n.p.(1)}$ are the leading order non-perturbative corrections to the semi-flat expressions for $\mathcal{X}_m, \mathcal{X}_e$ which are given by [67]:

$$\begin{aligned}\mathcal{X}_e^{n.p.(1)}(u, \Theta, \zeta) &= \sum_{\gamma'=\gamma_n^\pm} \Omega(\gamma'; u) \frac{\langle\langle \gamma', \frac{1}{2}\alpha \rangle\rangle}{4\pi i} e^{i\Theta \cdot \gamma'} \int_{\mathbb{R}_+} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta e^{-i\alpha_{\gamma'}}}{\zeta' - \zeta e^{-i\alpha_{\gamma'}}} e^{-2\pi R|Z_{\gamma'}|(\zeta'+1/\zeta')}, \\ \mathcal{X}_m^{n.p.(1)}(u, \Theta, \zeta) &= \sum_{\gamma'=\gamma_n^\pm, \pm\alpha} \Omega(\gamma'; u) \frac{\langle\langle \gamma', \frac{1}{2}H_\alpha \rangle\rangle}{4\pi i} e^{i\Theta \cdot \gamma'} \\ &\quad \times \int_{\mathbb{R}_+} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta e^{-i\alpha_{\gamma'}}}{\zeta' - \zeta e^{-i\alpha_{\gamma'}}} e^{-2\pi R|Z_{\gamma'}|(\zeta'+1/\zeta')},\end{aligned}\tag{7.45}$$

Using the BPS indices (7.26), these integrals simplify to

$$\begin{aligned}\mathcal{X}_e^{n.p.(1)} &= \frac{i}{\pi} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}_+} \frac{d\zeta'}{\zeta'} \frac{\sin(\theta_m + n\theta_e)(\zeta'^2 + \zeta^2) + 2\zeta'\zeta \cos(\theta_m + n\theta_e) \sin(\alpha_{\gamma_n^+})}{\zeta'^2 + \zeta^2 - 2\zeta \cos(\alpha_{\gamma_n^+})} \\ &\quad \times e^{-2\pi R|Z_{\gamma'}|(\zeta'+1/\zeta')}, \\ \mathcal{X}_m^{n.p.(1)} &= \sum_{n \in \mathbb{Z}} \frac{n}{2\pi i} \int_{\mathbb{R}_+} \frac{d\zeta'}{\zeta'} \frac{\sin(\theta_m + n\theta_e)(\zeta'^2 + \zeta^2) + 2\zeta'\zeta \cos(\theta_m + n\theta_e) \sin(\alpha_{\gamma_n^+})}{\zeta'^2 + \zeta^2 - 2\zeta \cos(\alpha_{\gamma_n^+})} \\ &\quad \times e^{-2\pi R|Z_{\gamma'}|(\zeta'+1/\zeta')} \\ &\quad + \frac{2i}{\pi} \int_{\mathbb{R}_+} \frac{d\zeta'}{\zeta'} \frac{\sin(\theta_e)(\zeta'^2 + \zeta^2) + 2\zeta'\zeta \cos(\theta_e) \sin(\alpha_\alpha)}{\zeta'^2 + \zeta^2 - 2\zeta \cos(\alpha_\alpha)} e^{-2\pi R|Z_{\gamma'}|(\zeta'+1/\zeta')}.\end{aligned}\tag{7.46}$$

where above we have used the notation where the integral over ζ' has been mapped to the integral over the positive reals by the phase rotation $e^{i\alpha_{\gamma'}} = \text{phase}(Z_{\gamma'})$ and $\langle\langle\gamma, \gamma'\rangle\rangle$ is the DSZ pairing of charges.

Note that the \mathcal{Y}_γ are functions of $u, \theta_e, \theta_m, \zeta$ on the Hitchin moduli space. Because of the relation between the $\mathfrak{a}, \mathfrak{b}$ and the $\mathcal{X}_e, \mathcal{X}_m$ in (7.32) and (7.34), we clearly see that the $\mathfrak{a}, \mathfrak{b}$ must also be functions of $u, \theta_e, \theta_m, \zeta$. The explicit dependence of $\mathfrak{a}, \mathfrak{b}$ on ζ can be seen first fixing a point in Hitchin moduli space with fixed coordinates (u, θ_e, θ_m) , and then studying the Fenchel-Nielsen coordinates as functions of the complex structure ζ .

Index Theorem

We can now use these coordinate transformations to determine an index formula as follows:

1. Calculate the localization computation for the expectation value of the given line operator:

$$\langle L_{p,0} \rangle_{Loc} = \sum_{|v| < |P|} \frac{e^{(v, \mathfrak{b})}}{\sinh^{|v|}(\mathfrak{a})} \cdot \lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}^\xi(P, v)} e^{\omega + \mu_T} \widehat{A}(T\widetilde{\mathcal{M}}) , \quad (7.47)$$

where $\widetilde{\mathcal{M}}_{KN}^\xi(P, v) = \widetilde{\mathcal{M}}_{KN}^\xi(\vec{k}, \vec{w})$ is the corresponding Kronheimer-Nakajima quiver variety as described in Section 5.1.2. In this example it evaluates to

$$\langle L_{p,0} \rangle_{Loc} = \left(\frac{e^{\mathfrak{b}} + e^{-\mathfrak{b}}}{2 \sinh(\mathfrak{a})} \right)^p . \quad (7.48)$$

2. Perform the change of coordinates:

$$\mathfrak{a} \mapsto \log(f_n - \sqrt{f_n^2 - 1}) \quad , \quad \mathfrak{b} \mapsto \log \left(\frac{\sqrt{a - \mathcal{X}_e}}{\sqrt{a^{2n+1}(a\mathcal{X}_e - 1)}} \right) , \quad (7.49)$$

in the localization result, where f_n is given in equation (7.22) and $a = e^{\mathfrak{a}}$ as a function of $\mathcal{X}_m, \mathcal{X}_e$.

3. Expand the $\langle L_{p,0} \rangle_{Loc}$ as a Laurent series in $\mathcal{X}_m, \mathcal{X}_e$:

$$\langle L_{p,0} \rangle_{Loc} = \sum_{n_1, n_2} C_{n_m, n_e} \mathcal{X}_m^{n_m}, \mathcal{X}_e^{n_e} . \quad (7.50)$$

4. Identify the coefficient of the $\mathcal{X}_m^{n_m} \mathcal{X}_e^{n_e}$ term, C_{n_m, n_e} , with the index:

$$\text{Ind} \left[\mathcal{D}^{\mathcal{Y}} \right]_{\overline{\mathcal{M}}, c_n}^{\gamma_e} = C_{n_m, n_e} , \quad (7.51)$$

where

$$\zeta \in c_n \quad , \quad \overline{\mathcal{M}} = \overline{\mathcal{M}}(P, \gamma_m) \quad , \quad \gamma = \frac{n_e}{2} \alpha \oplus n_m H_\alpha \quad , \quad P = p H_\alpha . \quad (7.52)$$

After performing the Laurent expansion for $\langle L_{p,0} \rangle$ given in (7.48) in terms of the Darboux coordinates in the c_n chamber, we have an expression for the graded index of the twisted Dirac operator $\mathcal{D}^{\mathcal{Y}}$ on singular monopole moduli space:

$$\begin{aligned} \text{Ind} \left[\mathcal{D}^{\mathcal{Y}} \right]_{\overline{\mathcal{M}}(P, \gamma_m; X_\infty)}^{\gamma_e = \frac{n_e}{2} \alpha} &= \sum_{m=0}^{\infty} \sum_{j=0}^p \sum_{k=0}^{2p} \sum_{\ell=0}^{\infty} \sum_{i=0}^{\infty} \sum_{q=0}^k \sum_{d_1=0}^{j+m+q+i+(2n+1)(p-k)} \sum_{d_2=0}^{\lfloor d_1/2 \rfloor} \sum_{d_3=0}^{\infty} \binom{p+m-1}{m} \\ &\times \binom{2p}{k} \binom{p+\ell-1}{\ell} \binom{k+i-1}{i} \binom{p}{j} \binom{k}{q} \\ &\times \binom{j+m+q+i+(2n+1)(p-k)}{d_1} \binom{\lfloor d_1/2 \rfloor}{d_2} \binom{2d_3}{d_2} \frac{(-1)^{j+q+d_2} 2^{-2p}}{(1-2d_3)} \\ &\times \left\{ \begin{array}{ll} \sum_{i_1+i_2=2\ell-2d_2-2d_3} \binom{2\ell-2d_2-2d_3}{i_1, i_2} (-1)^{i_1} \Delta_{n_e, n_m} & \ell - d_2 - d_3 > 0 \\ \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1} \binom{i_1+2d_2+2d_3-2\ell-1}{i_1} \binom{i_1}{i_2} (-1)^{i_1} \Delta_{n_e, n_m} & \ell - d_2 - d_3 < 0 \end{array} \right. \end{aligned} \quad (7.53)$$

where Δ_{n_e, n_m} is a delta function that restricts the sum over the $\{m, j, k, \ell, i, q, d_i, i_i\}$ such that

$$\begin{aligned} n_m &= \begin{cases} i_2 & \ell - d_2 - d_3 > 0 \\ 2(\ell - d_2 - d_3) - i_1 & \ell - d_2 - d_3 < 0 \end{cases} \\ n_e &= \begin{cases} 2i_1 + (2n+2)i_2 + 2(d_2 + d_3 - \ell) + i + j + k - m - p & \ell - d_2 - d_3 > 0 \\ -(2n+2)i_1 + 2i_2 - 2(2n+1)(d_2 + d_3 - \ell) + i + j + m - p & \ell - d_2 - d_3 < 0 \end{cases} \end{aligned} \quad (7.54)$$

are fixed. Additionally,

$$P = \text{diag}(p, -p) \quad , \quad \zeta \in c_n \quad , \quad \gamma_m = n_m H_\alpha \quad . \quad (7.55)$$

This index formula for the case of $SU(2)$ SYM theory is also found in [133].

Characteristic Numbers

Now by expressing \mathcal{Y}_γ in terms of Fenchel-Nielsen coordinates, we can perform a Laurent expansion with respect to the exponential Fenchel-Nielsen coordinate e^b . This will allow us to isolate the characteristic number. By using the equations for the Darboux coordinates in terms of Fenchel-Nielsen coordinates (7.32), we get the expansion

$$\begin{aligned} \lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}_{KN}^\xi(P, \mathbf{v})} e^{\omega + \mu_T} \widehat{A}(T\widetilde{\mathcal{M}}_{KN}) = \sum_{0 \leq n_m, n_e \leq p} \left\{ \underline{\overline{\mathcal{Q}}}(n_m, n_e; c_n) Q_1^{(n_m, n_e)}(a; c_n) \right. \\ \left. + \underline{\overline{\mathcal{Q}}}(-n_m, n_e; c_n) Q_2^{(n_m, n_e)}(a; c_n) + \underline{\overline{\mathcal{Q}}}(-n_m, -n_e; c_n) Q_3^{(n_m, n_e)}(a; c_n) \right. \\ \left. + \underline{\overline{\mathcal{Q}}}(n_m, -n_e; c_n) Q_4^{(n_m, n_e)}(a; c_n) \right\} . \end{aligned} \quad (7.56)$$

where

$$\begin{aligned} P = \text{diag}(p, -p) \quad , \quad \mathbf{v} = \text{diag}(v, -v) \quad , \\ \underline{\overline{\mathcal{Q}}}(n_m, n_e; c_n) = \underline{\overline{\mathcal{Q}}}(\gamma; c_n) \quad , \quad \gamma = n_m H_I \oplus n_e \frac{1}{2} \alpha \quad , \end{aligned} \quad (7.57)$$

and the $Q_i^{(n_m, n_e)}(a, v; c_n)$ are different rational functions of a , defined as

$$\begin{aligned}
Q_1^{(n_m, n_e)}(a, v; c_n) &= \sum_{i_1=0}^{n_e} \sum_{j_1=0}^{\left\lfloor \frac{2n_m+v-i_1}{2} \right\rfloor} \sum'_{\substack{i_2+j_2= \\ \frac{v-n_m}{2}-i_1-j_1}} \begin{pmatrix} n_e \\ i_1 \end{pmatrix} \begin{pmatrix} i_2+n_e-1 \\ i_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2n_m n \\ j_1 \end{pmatrix} \begin{pmatrix} j_2+2n_m(n+1)-1 \\ j_2 \end{pmatrix} (-1)^{2n_m+i_2+j_2} \\
&\quad \times a^{2n(i_1+j_2)+(2n+2)(i_2+j_1)+n_e-2(n_m-v)} (1-a^2)^{2(n_m+v)}, \\
Q_2^{(n_m, n_e)}(a, v; c_n) &= \sum_{i_1=0}^{n_e} \sum_{j_1=0}^{\left\lfloor \frac{2n_m+v-i_1}{2} \right\rfloor} \sum'_{\substack{i_2+j_2= \\ \frac{n_m+v}{2}-i_1-j_1}} \begin{pmatrix} n_e \\ i_1 \end{pmatrix} \begin{pmatrix} i_2-n_e-1 \\ i_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2n_m(n+1) \\ j_1 \end{pmatrix} \begin{pmatrix} j_2+2n_m n-1 \\ j_2 \end{pmatrix} (-1)^{i_2+j_2+2n_m} \\
&\quad \times a^{2n(i_1+j_1)+(2n+2)(i_2+j_2)+n_e+2n_m-2v} (1-a^2)^{2(v-n_m)}, \\
Q_3^{(n_m, n_e)}(a, v; c_n) &= \sum_{i_1=0}^{n_e} \sum_{j_1=0}^{\left\lfloor \frac{2n_m+v-i_1}{2} \right\rfloor} \sum'_{\substack{i_2+j_2= \\ \frac{n_m+v}{2}-i_1-j_1}} \begin{pmatrix} n_e \\ i_1 \end{pmatrix} \begin{pmatrix} i_2-n_e-1 \\ i_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2n_m(n+1) \\ j_1 \end{pmatrix} \begin{pmatrix} j_2+2n_m n-1 \\ j_2 \end{pmatrix} (-1)^{i_2+2n_m+j_2} \\
&\quad \times a^{2n(i_1+j_1)+(2n+2)(i_1+j_2)+2n_m-n_e-2v} (1-a^2)^{2(v-n_m)}, \\
Q_4^{(n_m, n_e)}(a, v; c_n) &= \sum_{i_1=0}^{n_e} \sum_{j_1=0}^{\left\lfloor \frac{2n_m+v-i_1}{2} \right\rfloor} \sum'_{\substack{i_2+j_2= \\ \frac{v-n_m}{2}-i_1-j_1}} \begin{pmatrix} n_e \\ i_1 \end{pmatrix} \begin{pmatrix} i_2+n_e-1 \\ i_2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 2n_m n \\ j_1 \end{pmatrix} \begin{pmatrix} j_2+2n_m(n+1)-1 \\ j_2 \end{pmatrix} (-1)^{i_2+j_2+2n_m} \\
&\quad \times a^{2n(i_2+j_2)+(2n+2)(i_1+j_1)-2n_m-n_e-2v} (1-a^2)^{2(n_m+v)}.
\end{aligned} \tag{7.58}$$

Here we use the notation

$$\sum_{i=0}^{\begin{bmatrix} m \\ n \end{bmatrix}} = \sum_{i=0}^{\min[m,n]} , \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 1 , \quad \sum'_{i_2+j_2=\dots} = \begin{cases} \sum_{i_2+j_2=\dots} & n_e, n_m \neq 0 \\ \sum_{j_2=\dots} & n_e = 0 , n_m \neq 0 \\ \sum_{j_2=0}^{i_2=\dots} & n_m = 0 , n_e \neq 0 \\ \sum_{i_2, j_2=0} & n_e = n_m = 0 . \end{cases} \quad (7.59)$$

and the sums are restricted such that $\sum_{n=a}^b$ is identically zero for $b < a$.

Note that in both the formulas for the index of $\mathcal{D}^{\mathcal{Y}}$ and the characteristic numbers on \mathcal{M}_{KN} , there is a clear mixing of framed BPS states of magnetic charge γ_m among many characteristic numbers for different. This suggests that there is a very non-trivial relationship between framed BPS states and the geometry of singular monopole moduli space since the \mathcal{M}_{KN} are transversal slices to singular strata in $\overline{\mathcal{M}}(P, \gamma_m; X_\infty)$. It is an interesting challenge to differential geometers to try to prove such relations.

Explicit Example: $\langle L_{2,0} \rangle$ in $SU(2)$ SYM

We can illustrate the above formulas for the index of $\mathcal{D}^{\mathcal{Y}}$ and the characteristic numbers on $\mathcal{M}_{KN}(P, \mathbf{v})$ with the non-trivial example of the next-to-minimal 't Hooft defect: $L_{2,0}$.

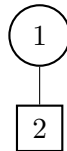
Let us first demonstrate the index theorem by calculating the index of $\mathcal{D}^{\mathcal{Y}}$. In our example, the expectation value from localization can be written

$$\langle L_{2,0} \rangle_{Loc} = \frac{2 \cosh(2\mathfrak{b})}{\sinh^2(\mathfrak{a})} + Z_{mono}(\mathfrak{a}; 2, 0) , \quad (7.60)$$

where

$$Z_{mono}(\mathfrak{a}; 2, 0) = \lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}^\xi(2,0)} e^{\omega + \mu_T} \widehat{A}(T\widetilde{\mathcal{M}}) , \quad (7.61)$$

is the characteristic number on the Kronheimer-Nakajima space defined by the quiver



as described in Section 5.1.2.

For this example, the characteristic number evaluates to [23]

$$Z_{mono}(\mathfrak{a}; 2, 0) = \frac{2}{4 \sinh^2(\mathfrak{a})} . \quad (7.62)$$

Let us compute the index theorem for the chamber c_1 . In this chamber, the coordinate transformation is of the form

$$\begin{aligned} a = e^{\mathfrak{a}} &= f_1 - \sqrt{f_1^2 - 1} \quad , \quad f_1 = \frac{1}{2}\mathcal{X}_e + \frac{1}{2\mathcal{X}_e} + \frac{\mathcal{X}_m^2 \mathcal{X}_e^{2n+1}}{2} , \\ b = e^{\mathfrak{b}} &= \sqrt{\frac{f_1 - \sqrt{f_1^2 - 1} - \mathcal{X}_e}{(f_1 - \sqrt{f_1^2 - 1})^4 \mathcal{X}_e - (f_1 - \sqrt{f_1^2 - 1})^3}} . \end{aligned} \quad (7.63)$$

Plugging this into the full expectation value

$$\langle L_{2,0} \rangle_{Loc} = \frac{2 \cosh(2\mathfrak{b})}{\sinh^2(\mathfrak{a})} + \frac{1}{2 \sinh^2(\mathfrak{a})} , \quad (7.64)$$

yields the Darboux expansion

$$\langle L_{2,0} \rangle_{Loc} \Big|_{\mathfrak{a}, \mathfrak{b} \mapsto \mathcal{X}_m, \mathcal{X}_e} = \frac{1}{\mathcal{X}_m^2} + \mathcal{X}_e^4 \mathcal{X}_m^2 + 2\mathcal{X}_e^2 , \quad (7.65)$$

in terms of the spectral network coordinates in the c_1 chamber. Note that this matches the direct computation from spectral networks (7.21) [133].

From this expansion we can read off the indices of the Dirac operator:

$$\text{Ind} \left[\mathcal{D}^{\mathcal{Y}} \right]_{\overline{\mathcal{M}}(P, \gamma_m; X_\infty)}^{\gamma_e = \frac{n_e}{2} \alpha} = \begin{cases} 1 & \gamma = H_\alpha \oplus 2\alpha \\ 1 & \gamma = -H_\alpha \\ 2 & \gamma = \alpha \\ 0 & \text{else} \end{cases} \quad (7.66)$$

where $P = \frac{1}{2} \text{diag}(2, -2)$ and $\zeta \in c_1$.

Now let us perform the inverse coordinate substitution to derive the characteristic number from the spectral network computation. Let us start with the expectation value of $L_{2,0}$ from the spectral network associated with $\zeta \in c_1$:

$$\langle L_{2,0} \rangle_{\zeta \in c_1} = \frac{1}{\mathcal{X}_m^2} + \mathcal{X}_e^4 \mathcal{X}_m^2 + 2\mathcal{X}_e^2 . \quad (7.67)$$

The coordinate transformation (7.32) now takes the form

$$\mathcal{X}_m = -\frac{b(a^2 - 1)(1 + a^4 b^2)}{a(1 + a^2 b^2)^2} \quad , \quad \mathcal{X}_e = \frac{a(1 + a^2 b^2)}{(1 + a^4 b^2)} . \quad (7.68)$$

Plugging this into (7.67) we find

$$\langle L_{2,0} \rangle_{\zeta \in c_1} \Big|_{\mathcal{X}_m, \mathcal{X}_e \mapsto \mathfrak{a}, \mathfrak{b}} = \frac{(b + 1/b)^2}{(a - 1/a)^2} , \quad (7.69)$$

which indeed matches with the localization computation. Expanding this in powers of $b = e^{\mathfrak{b}}$, we see that the 0^{th} order term ($Z_{mono}(\mathfrak{a}; 2, 0)$) is given by

$$Z_{mono}(\mathfrak{a}; 2, 0) = \frac{2}{(a - 1/a)^2} = \frac{2}{4 \sinh^2(\mathfrak{a})} . \quad (7.70)$$

We can also derive this result from the full formula for the characteristic number. Using the data

$$\overline{\Omega}(-1, 0; c_1) = \overline{\Omega}(1, 4; c_1) = 1 \quad , \quad \overline{\Omega}(0, 2; c_1) = 2 , \quad (7.71)$$

the characteristic number formula (7.56) reduces to

$$\begin{aligned} \lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}_{KN}^{\xi}(2,0)} e^{\omega + \mu_T} \widehat{A}(T\widetilde{\mathcal{M}}_{KN}) = \\ \overline{\Omega}(1, 4; c_1) Q_1^{(1,4)}(a; c_1) + \overline{\Omega}(0, 2; c_1) Q_1^{(0,2)}(a; c_1) + \overline{\Omega}(-1, 0; c_1) Q_2^{(1,0)}(a; c_1) . \end{aligned} \quad (7.72)$$

Evaluating the polynomials, we find

$$\begin{aligned} Q_1^{(1,4)}(a; c_1) &= 0 , \\ Q_1^{(0,2)}(a; c_1) &= \sum_{i_1=0}^2 \sum_{j_1=0}^{-i_1} \sum_{i_2=0}^2 \binom{2}{i_1} \binom{i_2+1}{i_2} (-1)^{i_2+j_2} a^{2(i_1+j_2)+(4(i_2+j_1)+2)} = a^2 , \\ Q_2^{(1,0)}(a; c_1) &= \sum_{j_1=0}^1 \sum_{j_2=1-j_1}^4 \binom{4}{j_1} \binom{j_2+1}{j_2} (-1)^{j_2} \frac{a^{2j_1+4j_2+2}}{(1-a^2)^2} = \frac{-2a^6 + 4a^4}{(1-a^2)^2} . \end{aligned} \quad (7.73)$$

Combining these results with the framed BPS indices (7.71), the full formula for the characteristic number evaluates to

$$\lim_{\xi \rightarrow 0} \int_{\widetilde{\mathcal{M}}_{KN}^{\xi}(2,0)} e^{\omega + \mu_T} \widehat{A}(T\widetilde{\mathcal{M}}_{KN}) = 2a^2 + \frac{-2a^6 + 4a^4}{(1-a^2)^2} = \frac{2}{(a - 1/a)^2} , \quad (7.74)$$

matching the result from direct computation.

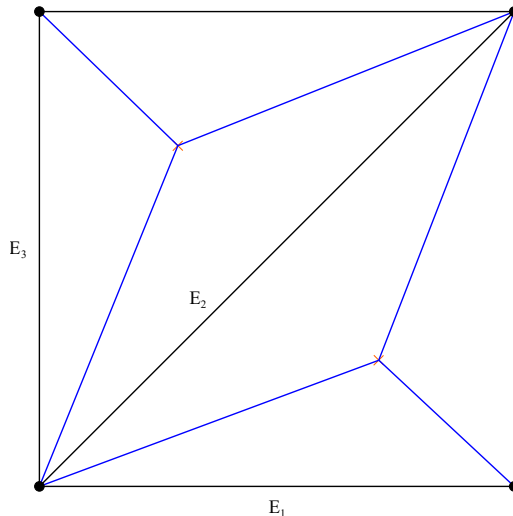


Figure 7.4: This figure shows a generic WKB spectral network (blue) on the punctured torus. This corresponds to the triangulation given by the (black) edges $E_{1,2,3}$ where the rectangle is periodically identified and the puncture is located at the identified corners.

7.2.3 Comments on the $\mathcal{N} = 2^*$ Theory

Here we would like to make some clarifying comments on the $SU(2)$ $\mathcal{N} = 2^*$ theory and the Fenchel-Nielsen locus in this theory. In the case of the $\mathcal{N} = 2^*$ theory the UV curve C is given by the once punctured torus. The algebra of line operators of this theory can be generated by the three simple line operators $L_{\gamma_{(1,0)}}$, $L_{\gamma_{(0,1)}}$, and $L_{\gamma_{(1,1)}}$. Note that there are three generating operators because the homology lattice is generated by a cycles that wrap the A -cycle, B -cycle, and the puncture.

A generic spectral network associated to the $SU(2)$ $\mathcal{N} = 2^*$ theory is given by an ideal triangulation of C as in Figure 7.4. In each chamber of the ζ -plane c , the charge lattice is spanned by three simple elements $\gamma_i[c]$ for $i = 1, 2, 3$ such that

$$\langle \gamma_i[c], \gamma_{i+1}[c] \rangle = 2 \quad , \quad \gamma_1[c] + \gamma_2[c] + \gamma_3[c] = \gamma_f \quad . \quad (7.75)$$

Given a particular choice of chamber c_0 we can identify

$$\gamma_1[c_0] = -\alpha \oplus \gamma_f \quad , \quad \gamma_2[c_0] = -H_\alpha \quad , \quad \gamma_3[c_0] = H_\alpha \oplus \alpha \quad . \quad (7.76)$$

In such a chamber, the expectation values of the line operators can be expanded in

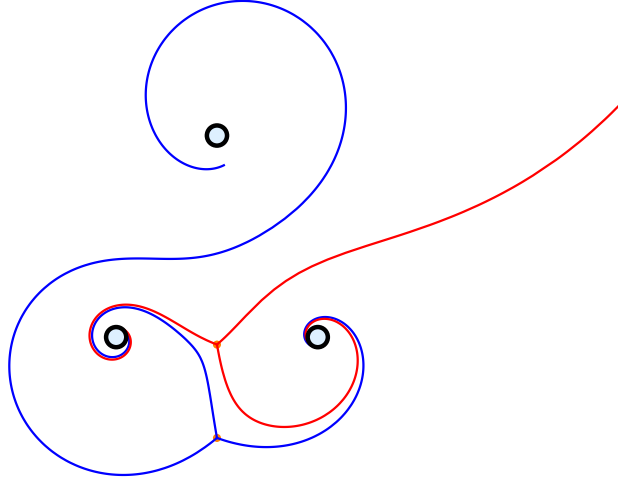


Figure 7.5: This figure shows a generic WKB spectral network on the punctured torus. The punctured torus is presented as a trinion with two boundary circles identified. These are the two lower circles in the above figure. We choose the A-cycle to be defined by the boundary of these circles. Here the lines are the walls of the corresponding WKB spectral network. While it is not drawn here due to computational limitations, the walls corresponding to the open paths run to the (upper) puncture.

terms as

$$\begin{aligned}
 \langle L_{\gamma_{(1,0)}} \rangle &= \sqrt{\mathcal{Y}_{\gamma_2} \mathcal{Y}_{\gamma_3}} + \frac{1}{\sqrt{\mathcal{Y}_{\gamma_2} \mathcal{Y}_{\gamma_3}}} + \sqrt{\frac{\mathcal{Y}_{\gamma_3}}{\mathcal{Y}_{\gamma_2}}} , \\
 \langle L_{\gamma_{(0,1)}} \rangle &= \sqrt{\mathcal{Y}_{\gamma_3} \mathcal{Y}_{\gamma_1}} + \frac{1}{\sqrt{\mathcal{Y}_{\gamma_3} \mathcal{Y}_{\gamma_1}}} + \sqrt{\frac{\mathcal{Y}_{\gamma_1}}{\mathcal{Y}_{\gamma_3}}} , \\
 \langle L_{\gamma_{(1,1)}} \rangle &= \sqrt{\mathcal{Y}_{\gamma_2} \mathcal{Y}_{\gamma_1}} + \frac{1}{\sqrt{\mathcal{Y}_{\gamma_2} \mathcal{Y}_{\gamma_1}}} + \sqrt{\frac{\mathcal{Y}_{\gamma_2}}{\mathcal{Y}_{\gamma_1}}} .
 \end{aligned} \tag{7.77}$$

Here \mathcal{Y}_{γ_i} is the spectral network coordinate corresponding to the edge E_i .

In the $SU(2)$ $\mathcal{N} = 2^*$ theory, the Fenchel-Nielsen locus is defined by

$$\bar{m} \int_A \lambda_{SW} \in \mathbb{R} \quad , \quad \int_A \zeta^{-1} \lambda_{SW} \in \mathbb{R} \quad , \tag{7.78}$$

where ζ is the phase defining the line operator (and corresponding WKB spectral network). As it turns out, this coincides with the exceptional locus

$$\mathcal{E} = \bigcup_i \mathcal{E}_i \quad , \quad \mathcal{E}_i = \{u \in \mathcal{B} \mid Z(\gamma_i; u)/m > 0 \text{ , } \text{Arg}[Z(\gamma_{i+1}; u)] < \text{Arg}[Z(\gamma_{i-1}; u)]\} \quad , \tag{7.79}$$

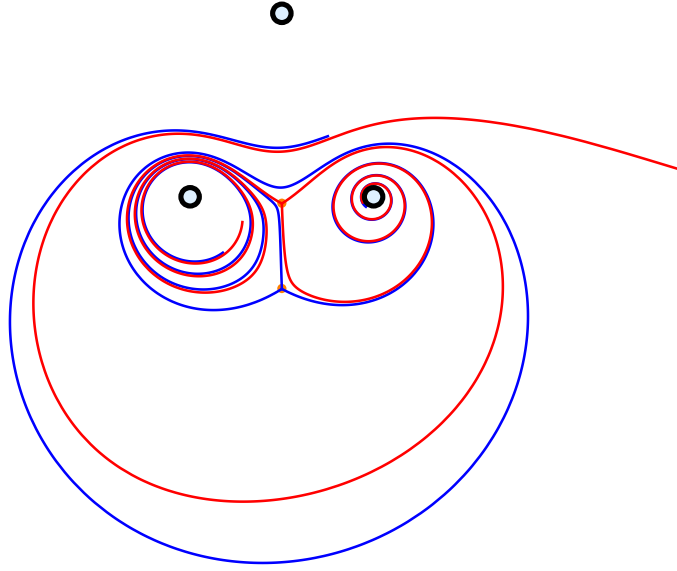


Figure 7.6: This figure shows the behavior of the generic WKB spectral network on the punctured torus from Figure 7.5 as it approaches the Fenchel-Nielsen spectral network (left in Figure 7.1). Again, while it is not drawn here due to computational limitations, the walls corresponding to the open paths run to the (upper) puncture.

from [120]. Here mathematical simplifications arise that allow for the exact computation of the spectrum generator which encodes the entire spectrum of BPS states.

As we approach to the Fenchel-Nielsen locus, we cross an infinite number of \mathcal{K} -walls in passing through the chambers c_n with increasing n . Mathematically, crossing the \mathcal{K} -wall going from chamber $c_n \rightarrow c_{n+1}$ corresponds to mutating along one of basis elements of the charge lattice in c_n , $\gamma_i[c_n]$. As discussed in [69], this transformation keeps the three-term expansion of the $\langle L_\gamma \rangle$ that have explicit \mathcal{Y}_{γ_i} dependence but increases the complexity of the $\langle L_\gamma \rangle$ that are independent of \mathcal{Y}_{γ_i} . This leads to a fairly simple change of variables between the complexified Fenchel-Nielsen coordinates and the \mathcal{Y}_{γ_i} given by [53]

$$\begin{aligned} \sqrt{\mathcal{Y}_{\gamma_1}} &= \frac{i}{\ell} \frac{\tilde{\beta} - \tilde{\beta}^{-1}}{\tilde{\beta}\lambda - (\tilde{\beta}\lambda)^{-1}} \quad , \quad \sqrt{\mathcal{Y}_{\gamma_2}} = i \frac{\tilde{\beta}\lambda - (\tilde{\beta}\lambda)^{-1}}{\lambda - \lambda^{-1}} \quad , \quad \sqrt{\mathcal{Y}_{\gamma_3}} = -i \frac{\lambda - \lambda^{-1}}{\tilde{\beta} - \tilde{\beta}^{-1}} \quad , \\ \lambda &= e^a \quad , \quad \ell = e^m \quad , \quad \tilde{\beta} = e^b \sqrt{\frac{e^{a+m} - e^{-a-m}}{e^{a-m} - e^{-a+m}}} \quad . \end{aligned} \quad (7.80)$$

Sending $\zeta \rightarrow \zeta_{FN}$ acts on the corresponding spectral network as in Figure 7.6. This makes it obvious that the Wilson line $\langle L_{\gamma_{(1,0)}} \rangle$, which is the holonomy around one of

the resolved punctures, keeps a three term expansion. And further, from the properties of a Fenchel-Nielsen spectral network, we see that the expression for $\langle L_{\gamma_{(1,0)}} \rangle$ becomes a two term expansion in the limit $\zeta \rightarrow \zeta_{FN}$. We believe mirrors the same behavior of the expectation value of the Wilson line in the $SU(2)$ $N_f = 0$ theory as discussed in the previous section.

Appendix A

Semiclassical Framed BPS States

A.1 Collective Coordinate Calculation for SYM

In this appendix we perform the calculation to reduce the four dimensional theory to the theory of collective coordinates. We will begin by expanding the fields:

$$\begin{aligned}
Y &= \epsilon_{Y\infty} + Y^{cl} - \frac{i}{4} \phi_{mn} \chi^m \chi^n , \\
A_0 &= -\dot{z}^m \epsilon_m + Y^{cl} + \frac{i}{4} \phi_{mn} \chi^m \chi^n , \\
E_a &= (\hat{D}_a A_0 - \partial_0 \hat{A}_a) = -\dot{z}^m \delta_m \hat{A}_a + \hat{D}_a Y^{cl} + \frac{i}{4} \hat{D}_a \phi_{mn} \chi^m \chi^n ,
\end{aligned} \tag{A.1}$$

and plugging them into the Lagrangian

$$\begin{aligned}
L &= \frac{1}{g_0^2} \int d^3x \operatorname{Tr} \left\{ E_i^2 - 2(\hat{D}_i X)^2 - (\hat{D}_i Y)^2 - 2i\rho^A (D_0 \rho_A + [Y, \rho_A]) + 2\tilde{\theta}_0 E_i B^i \right\} \\
&+ \frac{2}{g_0^2} \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \{ E_a Y + B_a X \} + \sum_j \int d^3x \delta^{(3)}(x - x_j) i w_j^\dagger (D_t - R(Y)) w_j ,
\end{aligned} \tag{A.2}$$

which can be rewritten as

$$\begin{aligned}
L &= \frac{1}{g_0^2} \int d^3x \operatorname{Tr} \left\{ (\hat{D}_a - \partial_0 \hat{A}_a)^2 - 2(\hat{D}X)^2 - (\hat{D}Y)^2 + 2i\rho^A (D_0 \rho_A + [Y, \rho_A]) \right\} \\
&+ \frac{2\tilde{\theta}_0}{g_0^2} \int d^3x \operatorname{Tr} \{ B^i E_i \} + 2 \sum_n \int_{S_n^2} d^2 \Omega_n r_n^2 \hat{r}_n^i \operatorname{Tr} \left\{ (E_i Y + B_i X) - \frac{g_0^2}{4\pi} \frac{Q_n}{2r_n^2} A_0 \hat{r}_n^i \right\} \\
&+ \frac{1}{g_0^2} \sum_j \int d^3x \delta^{(3)}(x - x_j) i w_j^\dagger \mathbb{D}_t w_j + O(g_0^4) .
\end{aligned} \tag{A.3}$$

This Appendix is material from my publication [22].

The result is given by

$$\begin{aligned}
g_0^2 L = & \int d^3x \operatorname{Tr} \left\{ \dot{z}^m \dot{z}^n \delta_m \hat{A}^a \delta_n \hat{A}_a - 2 \dot{z}^m \delta_m \hat{A}^a \hat{D}_a Y^{cl} - \frac{i}{2} \dot{z}^m \delta_m \hat{A}^a \hat{D}_a \phi_{mn} \chi^m \chi^n \right. \\
& + \frac{i}{2} \hat{D}_a Y^{cl} \hat{D}^a \phi_{mn} \chi^m \chi^n + (\hat{D}_a Y^{cl})^2 - \frac{1}{16} (\hat{D}_a \phi_{mn})^2 \chi^4 \\
& - (\hat{D}_a \epsilon_{Y_\infty})^2 - (\hat{D}_a Y^{cl})^2 + \frac{1}{16} (\hat{D}_a \phi_{mn})^2 \chi^4 - 2 \hat{D}_a \epsilon_{Y_\infty} \hat{D}^a Y^{cl} \\
& + \frac{i}{2} \hat{D}_a \phi_{mn} \hat{D}^a \epsilon_{Y_\infty} \chi^m \chi^n - \frac{i}{2} \hat{D}_a Y^{cl} \hat{D}^a \phi_{mn} \chi^m \chi^n - 2 (\hat{D}_a X)^2 \\
& + i \delta_m \hat{A}^a \delta_n \hat{A}_a \chi^m \dot{\chi}^n + i \chi^m \delta_m \hat{A}^a D_p \delta_n \hat{A}_a \dot{z}^p \chi^n \\
& - \frac{i}{2} (\epsilon_{Y_\infty} + 2 Y^{cl}) \hat{D}^2 \phi_{mn} \chi^m \chi^n \\
& \left. + 2 \tilde{\theta}_0 \hat{D}^a X \left(-\dot{z}^m \delta_m \hat{A}_a + \hat{D}_a Y^{cl} + \frac{i}{4} \hat{D}_a \phi_{mn} \chi^m \chi^n \right) \right\} \\
& + 2 \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ X \hat{D}_a X \right. \\
& - \frac{g_0^2}{4\pi r_n^2} Q_n^* \left(-\dot{z}^m \epsilon_m + Y^{cl} + \frac{i}{4} \phi_{mn} \chi^m \chi^n \right) \\
& + (\epsilon_{Y_\infty} + Y^{cl} - \frac{i}{4} \phi_{mn} \chi^m \chi^n) (-\dot{z}^m \delta_m \hat{A}_a \\
& \left. + \hat{D}_a Y^{cl} + \frac{i}{4} \hat{D}_a \phi_{mn} \chi^m \chi^n) \right\} \\
& + i g_0^2 \sum_j w_j^\dagger \dot{w}_j + i g_0^2 \sum_j w_j^\dagger (-\dot{z}^m \epsilon_m - \epsilon_{Y_\infty} + \frac{i}{2} \phi_{mn} \chi^m \chi^n) w_j .
\end{aligned} \tag{A.4}$$

For the rest of the calculation we will use the asymptotics and identities:

$$\begin{aligned}
\tilde{\theta}_0 &= \frac{\theta_0 g_0^2}{8\pi^2} , & \hat{D}^2 \phi_{mn} &= 2[\delta_m \hat{A}^a, \delta_n \hat{A}_a] , & \hat{D}_a \epsilon_H &= -G(H)^m \delta_m \hat{A}_a , \\
\phi_{mn} &\sim O_\infty(1/r), O_n(1) , & X &\sim O_\infty(1), O_n(1/r_n) , & Y^{cl} &\sim O_\infty(1/r^2), O_n(1/r_n) , \\
\delta_m \hat{A}_a &\sim O_\infty(1/r^2), O_n(r_n^{-1/2}) , & \epsilon_H &\sim O_{\infty,n}(1) ,
\end{aligned} \tag{A.5}$$

where O_n and O_∞ are the behavior as $r \rightarrow \infty$ and $r \rightarrow r_n$ respectively. Using these we

can rewrite the Lagrangian:

$$\begin{aligned}
g_0^2 L = & 2\pi g_{mn} (\dot{z}^m \dot{z}^n - G(Y_\infty)^m G(Y_\infty)^n + i\chi^m D_t \chi^n) \\
& + i \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ -\frac{1}{2} \dot{z}^m \delta_m \hat{A}_a \hat{D}^a \phi_{mn} + \frac{1}{2} \hat{D}^a \epsilon_{Y_\infty} \hat{D}_a \phi_{mn} + \frac{1}{2} \hat{D}^a Y^{cl} \hat{D}_a \phi_{mn} \right. \\
& \quad \left. + \frac{1}{2} \epsilon_{Y_\infty} \hat{D}^2 \phi_{mn} + Y^{cl} \hat{D}^2 \phi_{mn} + \frac{\tilde{\theta}_0}{2} \hat{D}^a X \hat{D}_a \phi_{mn} \chi^m \chi^n \right\} \\
& + i \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ \frac{1}{2} \dot{z}^m \delta_m \hat{A}_a \phi_{st} \chi^s \chi^t \right. \\
& \quad \left. - \frac{1}{2} (\hat{D}_a Y^{cl} \phi_{mn} - \epsilon_{Y_\infty} \hat{D}_a \phi_{mn} - Y^{cl} \hat{D}_a \phi_{mn}) \chi^m \chi^n \right\} \\
& + \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ -2\dot{z}^m \delta_m \hat{A}^a \hat{D}_a Y^{cl} - 2\hat{D}_a \epsilon_{Y_\infty} \hat{D}^a Y^{cl} \right. \\
& \quad \left. - 2\tilde{\theta}_0 \hat{D}^a X \delta_m \hat{A}_a \dot{z}^m + 2\tilde{\theta}_0 \hat{D}^a X \hat{D}_a Y^{cl} - 2(\hat{D}_a X)^2 \right\} \\
& + 2 \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ X \hat{D}_a X + \epsilon_{Y_\infty} \hat{D}_a Y^{cl} + Y^{cl} \hat{D}_a Y^{cl} \right. \\
& \quad \left. - \dot{z}^m \delta_m \hat{A}_a Y^{cl} - \dot{z}^m \delta_m \hat{A}_a \epsilon_{Y_\infty} \right\} \\
& - \frac{g_0^2}{4\pi r_n^2} \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ Q_n^* \left(-\dot{z}^m \epsilon_m + Y^{cl} + \frac{i}{4} \phi_{mn} \chi^m \chi^n \right) \right\} \\
& + i g_0^2 \sum_j w_j^\dagger D_t w_j + i g_0^2 \sum_j w_j^\dagger \left(-\epsilon_{Y_\infty}^{(j)} + \frac{i}{2} \phi_{mn}^{(j)} \chi^m \chi^n \right) w_j ,
\end{aligned} \tag{A.6}$$

where we have used:

$$\begin{aligned}
g_{mn} = & \frac{1}{2\pi} \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ \delta_m \hat{A}^a \delta_n \hat{A}_a \right\} \quad , \quad \Gamma_{n,pq} = \frac{1}{2\pi} \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ \delta_n \hat{A}^a D_p \delta_q \hat{A}_a \right\} \\
D_t \chi^n = & \dot{\chi}^n + \Gamma_{pq}^n \dot{z}^p \chi^q \quad , \quad D_t w_j = \dot{w}_j - R(\epsilon_m^{(j)}) \dot{z}^m w_j ,
\end{aligned} \tag{A.7}$$

This can be reduced to the following by canceling terms using asymptotics:

$$\begin{aligned}
g_0^2 L = & 2\pi g_{mn}(\dot{z}^m \dot{z}^n - G(Y_\infty)^m G(Y_\infty)^n + i\chi^m D_t \chi^n) + iw^\dagger D_t w + 4\pi\tilde{\theta}_0 G_m(X_\infty)\dot{z}^m \\
& + i \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ Y^{cl} \hat{D}^2 \phi_{mn} + \frac{1}{2} \tilde{\theta}_0 \hat{D}^a X \hat{D}_a \phi_{mn} + \frac{1}{2} \hat{D}^a Y^{cl} \hat{D}_a \phi_{mn} \right\} \chi^m \chi^n \\
& + i \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ \frac{1}{2} Y^{cl} \hat{D}_a \phi_{mn} - \frac{1}{2} \hat{D}_a Y^{cl} \phi_{mn} \right\} \chi^m \chi^n - (\gamma_m, X_\infty) \\
& + \frac{i}{2} \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ \epsilon_{Y_\infty} \hat{D}^2 \phi_{mn} \right\} \chi^m \chi^n + \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ -2 \hat{D}_a \epsilon_{Y_\infty} \hat{D}^a Y^{cl} \right\} \\
& + \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ 2 \epsilon_{Y_\infty} \hat{D}_a Y^{cl} \right\} - i \sum_j (i w_j^\dagger \epsilon_{Y_\infty} w_j) + \frac{i}{2} \sum_j (i w_j^\dagger \phi_{mn} w_j) \chi^m \chi^n \\
& + \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ 2 \tilde{\theta}_0 \hat{D}^a X \hat{D}_a Y^{cl} \right\} + \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ 2 Y^{cl} \hat{D}_a Y^{cl} \right\} \\
& - \frac{g_0^2}{4\pi} \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ Q_n \frac{\hat{r}_{n,a}}{r_n^2} \left(-\dot{z}^m \epsilon^m + Y^{cl} + \frac{i}{4} \phi_{mn} \chi^m \chi^n \right) \right\} .
\end{aligned} \tag{A.8}$$

By construction the purely classical components cancel which reduces to:

$$\begin{aligned}
= & 2\pi g_{mn}(\dot{z}^m \dot{z}^n - G(X_\infty)^m G(X_\infty)^n + i\chi^m D_t \chi^n) + i \sum_j w_j^\dagger D_t w_j + \dot{z}^m q_m \\
& + 4\pi\tilde{\theta}_0 G(X_\infty)_m \dot{z}^m - (\gamma_m, X_\infty) + \frac{ig_0^2}{2} \sum_j (i w_j^\dagger \phi_{mn}^{(j)} w_j) \chi^m \chi^n - g_0^2 \sum_j (i w_j^\dagger \epsilon_{Y_\infty}^{(j)} w_j) \\
& - 2 \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ \hat{D}_a \epsilon_{Y_\infty} \hat{D}^a Y^{cl} \right\} + 2 \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ \epsilon_{Y_\infty} \hat{D}_a Y^{cl} \right\} \\
& + \frac{i}{2} \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ \epsilon_{Y_\infty} \hat{D}^2 \phi_{mn} \right\} \chi^m \chi^n + i \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ Y^{cl} \hat{D}^2 \phi_{mn} \right\} \chi^m \chi^n .
\end{aligned} \tag{A.9}$$

We are now reduced to solving for the last two lines. The last line will turn out being related to identities equivalent to those in [133]. Consider the penultimate line:

$$\begin{aligned}
& -2 \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ \hat{D}_a \epsilon_{Y_\infty} \hat{D}^a Y^{cl} \right\} + 2 \sum_n \int_{S_n^2} d^2 \Omega_n^a \operatorname{Tr} \left\{ \epsilon_{Y_\infty} \hat{D}_a Y^{cl} \right\} \\
& = -2 \int_{S_\infty^2} d^2 \Omega_\infty^a \operatorname{Tr} \left\{ \epsilon_{Y_\infty} E_a \right\} = \frac{\theta_0 g_0^2}{2\pi} ((\gamma_m, X_\infty) - g_{mn} G(X_\infty)^m G(Y_\infty)^n) .
\end{aligned} \tag{A.10}$$

This makes use of the identities:

$$\begin{aligned}
\gamma_e^{phys} &= -\gamma_e^* - \frac{\theta_0}{2\pi} \gamma_m \quad , \quad \langle \gamma_e, h_A \rangle = -\frac{4\pi}{g_0^2} g_{mn} G(\mathcal{Y}_\infty)^m G(K_A)^n \\
E_a &= \hat{D}_a Y^{cl} + \hat{D}_a \epsilon_{Y_\infty} .
\end{aligned} \tag{A.11}$$

We can now use the same computation as in [133] to reduce the penultimate term:

$$\frac{i}{2} \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ \epsilon_{Y_\infty} \hat{D}^2 \phi_{mn} \right\} = -2\pi i \nabla_m G(Y_\infty)_n . \quad (\text{A.12})$$

The final term then evaluates to:

$$i \int_{\mathcal{U}} d^3x \operatorname{Tr} \left\{ Y^{cl} \hat{D}^2 \phi_{mn} \right\} \chi^m \chi^n . \quad (\text{A.13})$$

This results in the same collective coordinate Lagrangian as in [133] with additional terms for the defect degrees of freedom:

$$\begin{aligned} L_{c.c.} = & \frac{4\pi}{g_0^2} \left[\frac{1}{2} g_{mn} (\dot{z}^m \dot{z}^n + i \chi^m \mathcal{D}_t \chi^n - G(Y_\infty)^m G(Y_\infty)^n) - \frac{i}{2} \chi^m \chi^n \nabla_m G(Y_\infty)_n \right] \\ & + \frac{\theta_0}{2\pi} (g_{mn} (\dot{z}^m - G(Y_\infty)^m) G(X_\infty)^n - i \chi^m \chi^n \nabla_m G(X_\infty)_n) \\ & - \frac{4\pi}{g_0^2} (\gamma_m, X_\infty) + \frac{\theta}{2\pi} (\gamma_m, Y_\infty) + i \sum_j w_j^\dagger (D_t - \epsilon_{Y_\infty}^{(j)} + \frac{i}{2} \phi_{mn} \chi^m \chi^n) w_j . \end{aligned} \quad (\text{A.14})$$

Note that upon integrating out the w_j fields we again arrive at a Wilson line but this is on the moduli space. This is to be expected by naively plugging in the collective coordinate expansion of $A_0 - Y$ into the Wilson line in the 4-dimensional theory. This can be seen to be a Wilson line by looking at the term $-\dot{z}^m \epsilon_m$ in A_0 :

$$\operatorname{Tr}_{RP} \exp e^{i \int (A_0 - Y) dt} \rightarrow \operatorname{Tr}_{RP} \exp e^{-i \int \epsilon_m \dot{z}^m dt + \dots} = \operatorname{Tr}_{RP} \exp e^{-i \oint \epsilon_m dz^m + \dots} , \quad (\text{A.15})$$

where ... refers to the supersymmetric completion.

A.1.1 Wilson-'t Hooft Collective Coordinate Supersymmetry

It was shown in [133] that the supersymmetric variation of the first two lines of (A.20) are invariant under the supersymmetry transformations:

$$\begin{aligned} \delta_\nu z^m &= -i \nu_a \chi^n (\mathbb{J}^a)_n^m , \\ \delta_\nu \chi^m &= \nu_a (\dot{z}^n - G(Y_\infty)^n) (\mathbb{J}^a)_n^m - i \nu_a \chi^p (\mathbb{J}^a)_p^n \Gamma_{nq}^m \chi^q . \end{aligned} \quad (\text{A.16})$$

Therefore we only need to examine the supersymmetric variation of the terms containing the fields w_j . It is important to note that these w_j are sections of a associated principal G-bundle over \mathcal{M} with representation R_j . Note that we have been suppressing an index

for this representation which we will now include as w^a . This comes with a fiber metric f_{ab} for which $(\epsilon_m^{(j)})^a_b$ is the metric connection. This induces a supersymmetric variation:

$$\delta_\nu(w_j^a) = (\epsilon_m^{(j)})^a_b \delta_\nu z^m w_j^b . \quad (\text{A.17})$$

The computation of the supersymmetric variation of action is now identical to that of hypermultiplets:

$$\begin{aligned} \delta_\nu \left[i f_{ab} w^{a\dagger} D_t w^b \right] &= 2i f_{ab} \delta_\nu(w^{a\dagger}) D_t w^b - i w^{a\dagger} \delta_\nu z^m \partial_m (f_{ab} (\epsilon_n)^b_c) \dot{z}^n w^c \\ &\quad - i f_{ab} w^{a\dagger} (\epsilon_m)^b_c \delta_\nu \dot{z}^m w^c - i \delta_\nu z^m \partial_m (f_{ab}) w^{a\dagger} \dot{w}^b \\ &= 2i f_{ab} \delta_\nu(w_j^{a\dagger}) D_t w^b - i w^{a\dagger} \delta_\nu z^m \partial_m (f_{ab} (\epsilon_n)^b_c) \dot{z}^n w^c \\ &\quad + \dot{z}^m \partial_m (f_{ab} (\epsilon_m)^b_c) \delta_\nu^n w^{a\dagger} w^c - i \delta_\nu z^m \partial_m (f_{ab}) w^{a\dagger} \dot{w}^b \\ &= -i (\phi_{mn})_{ab} \delta_\nu z^m \dot{z}^n w^{a\dagger} w^b - \delta_\nu z^n (2(\epsilon_m)_{ab} + \partial_n f_{ab}) w^{a\dagger} \dot{w}^b \\ &= -i (\phi_{mn})_{ab} \delta_\nu z^m \dot{z}^n w^{a\dagger} w^b , \end{aligned} \quad (\text{A.18})$$

where we used the identities:

$$\begin{aligned} w^{\dagger a} D_t (\delta_\nu w^b) &= \delta_\nu w^{\dagger a} D_t w^b , \\ \partial_n f_{ab} &= -(\epsilon_n)_{ab} - (\epsilon_n)_{ba} , \end{aligned} \quad (\text{A.19})$$

from [133] and [78] respectively. We also have from [78] that the variation of the ϵ_{Y_∞} term will be given by:

$$\delta_\nu(w^{a\dagger} (\epsilon_{Y_\infty})_{ab} w^b) = \delta_\nu z^m w^{a\dagger} \nabla_m (\epsilon_{Y_\infty})_{ab} w^b = -(\phi_{mn})_{ab} \delta_\nu z^m G(Y_\infty)^n w^{a\dagger} w^b . \quad (\text{A.20})$$

This uses the identity

$$\begin{aligned} \nabla_m \epsilon_{Y_\infty} &= \frac{1}{\hat{D}^2} \hat{D}^2 \nabla_m \epsilon_{Y_\infty} = \frac{1}{\hat{D}^2} \left(\hat{D}^a \nabla_m \hat{D}_a \epsilon_{Y_\infty} + \hat{D}^a [\delta_m \hat{A}_a, \epsilon_{Y_\infty}] \right) \\ &= -2G(Y_\infty)^n \frac{1}{\hat{D}^2} [\delta_m \hat{A}^a, \delta_n \hat{A}_a] = -\phi_{mn} G(Y_\infty)^n . \end{aligned} \quad (\text{A.21})$$

Additionally, the variation of the ϕ_{mn} -term is given by

$$\begin{aligned} \delta_\epsilon \left[(\phi_{mn})_{ab} \chi^m \chi^n w^{a\dagger} w^b \right] &= \delta_\nu z^p \nabla_p (\phi_{mn})_{ab} \chi^m \chi^n w^{a\dagger} w^b + 2(\phi_{mn})_{ab} \chi^m \delta_\nu \chi^n w^{a\dagger} w^b \\ &= 2\nu_s (\phi_{mn})_{ab} (\mathbb{J}^{(s)})^m_p (\dot{z}^p - G(Y_\infty)^p) \chi^n w^{a\dagger} w^b , \end{aligned} \quad (\text{A.22})$$

which exactly cancels the other terms provided:

$$(\phi_{mn})_{ab} (\mathbb{J}^{(s)})^n_p = -F_{pnab} (\tilde{\mathbb{J}}^{(s)})^n_m , \quad (\text{A.23})$$

which is exactly the condition that the Wilson bundle $\mathcal{E}_{\text{Wilson}}(Q_j)$ is hyperholomorphic.

A.2 Hypermultiplet Index Calculation

In this appendix we will compute the index of the operator which determines the rank of the hypermultiplet matter bundle over the BPS moduli space. Consider the Dirac operator

$$\tilde{L}_\rho = i\bar{\tau}^a \hat{D}_a + im_X , \quad (\text{A.24})$$

acting on a fermion in the representation ρ where m_X is a real scalar.

A.2.1 Reduction to Boundary

Following [133], we will assume that $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V_\rho)$ to be a Lie algebra representation of \mathfrak{g} . We want to compute the index of \tilde{L}_ρ given by

$$\begin{aligned} \text{Ind}[\tilde{L}_\rho] &= \lim_{z \rightarrow 0^+} \text{tr}_{\mathbb{C}^2 \otimes V_\rho \otimes L^2(\mathcal{U})} \left\{ \frac{z}{\tilde{L}_\rho^\dagger \tilde{L}_\rho + z} - \frac{z}{\tilde{L}_\rho \tilde{L}_\rho^\dagger + z} \right\} = \lim_{z \rightarrow 0^+} B_{\rho,z} , \\ B_{\rho,z} &= \lim_{z \rightarrow 0} \text{tr}_{\mathbb{C}^2 \otimes V_\rho \otimes L^2(\mathcal{U})} \left\{ \frac{z}{\tilde{L}_\rho^\dagger \tilde{L}_\rho + z} - \frac{z}{\tilde{L}_\rho \tilde{L}_\rho^\dagger + z} \right\} . \end{aligned} \quad (\text{A.25})$$

Writing

$$i\tilde{D}_\rho = \begin{pmatrix} 0 & \tilde{L}_\rho \\ \tilde{L}_\rho^\dagger & 0 \end{pmatrix} , \quad (\text{A.26})$$

where again we use the convention

$$F^a = \begin{pmatrix} 0 & \tau^a \\ \bar{\tau}^a & 0 \end{pmatrix} , \quad \tau^a = (\sigma^0 \bar{\sigma}^i, -i\mathbb{1}_2) , \quad (\text{A.27})$$

the index can be expressed as

$$\begin{aligned} \text{Ind}[\tilde{L}_\rho] &= \lim_{z \rightarrow 0^+} \text{tr}_{\mathbb{C}^4 \otimes V_\rho \otimes L^2(\mathcal{U})} \left\{ \frac{z}{-i\tilde{D}_\rho + z} \bar{F} \right\} , \\ \bar{F} &= \prod F^i = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} . \end{aligned} \quad (\text{A.28})$$

If we now consider the Green's function

$$\begin{aligned}
G_\lambda &= \left(i\tilde{D}_\rho + \lambda \right)^{-1} , \\
G_{i\lambda} &= \left(i\tilde{D}_\rho - i\lambda \right) \left(-\tilde{D}^2 + \lambda^2 \right)^{-1} , \\
&= \begin{pmatrix} -i\lambda & \tilde{L}_\rho^\dagger \\ \tilde{L}_\rho & -i\lambda \end{pmatrix} \begin{pmatrix} \left(\tilde{L}_\rho^\dagger \tilde{L}_\rho + \lambda^2 \right)^{-1} & 0 \\ 0 & \left(L_\rho L_\rho^\dagger + \lambda^2 \right)^{-1} \end{pmatrix} .
\end{aligned} \tag{A.29}$$

Using this operator, one can express

$$B_{\rho,z} = -i\sqrt{z} \text{tr}_{\mathbb{C}^4 \otimes V_\rho \otimes L^2(\mathcal{U})} \left\{ \bar{\Gamma} G_{\rho, i\sqrt{z}} \right\} . \tag{A.30}$$

However, we can also compute $G_{\rho,z}$ by using the fact that it is the inverse of a differential operator. Specifically $G_{\rho,z}$ solves

$$\begin{aligned}
&i\Gamma^\mu \left(\frac{\partial}{\partial x^\mu} + \rho(A_\mu)(x) \right) G_{\rho,\lambda}(x, y) + i\Gamma^4 \rho(X)(x) G_{\rho,\lambda}(x, y) \\
&\quad + (\Gamma^4 m_X + \lambda) G_{\rho,\lambda}(x, y) = 0 , \\
&-i \frac{\partial}{\partial y} G_{\rho,\lambda}(x, y) \Gamma^\mu + i G_{\rho,\lambda}(x, y) \left(\Gamma^i \rho(A_i)(y) + \Gamma^4 \rho(X)(y) \right) \\
&\quad + G_{\rho,\lambda}(x, y) \Gamma^4 m_X + \lambda G_{\rho,\lambda}(x, y) = 0 .
\end{aligned} \tag{A.31}$$

where we used the fact that $G_{\rho,\lambda}^\dagger(x, y) = \overline{G_{\rho,\lambda}}^T(y, x)$ and that G^\dagger is the Green's function for $\tilde{L}_\rho^\dagger + \lambda^*$. We can now combine these equations as in [133]:

$$\begin{aligned}
2\lambda \text{tr}_{\mathbb{C}^4 \otimes V_\rho \otimes L^2(\mathcal{U})} \left\{ \bar{\Gamma} G_{\rho,\lambda} \right\} &= -i \left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu} \right) \text{tr}_{\mathbb{C}^4 \otimes V_\rho} \left\{ \bar{\Gamma} \Gamma^\mu G_{\rho,\lambda} \right\} \\
&\quad - i \text{tr}_{\mathbb{C}^4 \otimes V_\rho} \left\{ \bar{\Gamma} \Gamma^\mu (\rho(\hat{A}_\mu)(x) - \rho(\hat{A}_\mu)(y)) G_{\rho,\lambda}(x, y) \right\}
\end{aligned} \tag{A.32}$$

Following [31] we can write

$$\begin{aligned}
I_\rho(z) &= \text{tr}_{L^2(\mathcal{U})} B_{z,\rho} = \frac{i}{2} \int_{\mathcal{U}} d^3x \partial_i J_{\rho,z}^i(x, x) = \frac{i}{2} \int_{\mathcal{U}} d^3x \hat{r} \cdot \vec{J}_{\rho,z}(x, x) \\
J_{\rho,z}^i(x, y) &= \text{tr}_{\mathbb{C}^4 \otimes V_\rho \otimes L^2(\mathcal{U})} \left\{ \bar{\Gamma} \Gamma^i G_{i\sqrt{z}}(x, y) \right\}
\end{aligned} \tag{A.33}$$

This implies:

$$B_{z,\rho} = \frac{i}{2} \text{tr} \left\{ \begin{pmatrix} 0 & -\vec{\sigma} \cdot \hat{r} \\ \vec{\sigma} \cdot \hat{r} & 0 \end{pmatrix} \begin{pmatrix} -i\sqrt{z} & \tilde{L}_\rho^\dagger \\ \tilde{L}_\rho & -i\sqrt{z} \end{pmatrix} \begin{pmatrix} (R_{\rho,z} + \Delta R_\rho)^{-1} & 0 \\ 0 & R_{\rho,z}^{-1} \end{pmatrix} \right\} \tag{A.34}$$

where the trace is over $\mathbb{C}^4 \otimes V_\rho \otimes L^2(\mathcal{U})$ and we have defined

$$\begin{aligned}
R_{z,\rho} &= \tilde{L}_\rho \tilde{L}_\rho^\dagger = -D^2 - \rho(X)^2 + m_X^2 + 2im_X \rho(X) + z \\
&= -\partial^2 - \rho(X_\infty)^2 + m_X^2 + 2im_X \rho(X_\infty) + O(r^{-1}) , \\
\Delta R_\rho &= \tilde{L}_\rho^\dagger \tilde{L}_\rho - \tilde{L}_\rho \tilde{L}_\rho^\dagger = -2i\vec{\sigma} \cdot \rho(\vec{B}) , \\
\tilde{L}^\dagger - \tilde{L}_\rho &= 2\rho(X_\infty) - 2im_X + O(r^{-1}) ,
\end{aligned} \tag{A.35}$$

where \vec{B} is the magnetic field and $\lim_{r \rightarrow \infty} = X_\infty$. We will begin by computing the index at $r \rightarrow \infty$. In this case

$$\Delta R_\rho = \vec{\sigma} \cdot \hat{r} \frac{i\rho(\gamma_m)}{r^2} + O(r^{-(2+\delta)}) . \tag{A.36}$$

Note that at $r \rightarrow \infty$, ΔR_ρ can be used as an perturbative expansion coefficient so we can simplify

$$B_{\rho,z} = \frac{i}{2} \text{tr} \left\{ (\hat{r} \cdot \vec{\sigma})(\tilde{L}_\rho^\dagger - \tilde{L}_\rho) R_{\rho,z}^{-1} - (\hat{r} \cdot \sigma) L_\rho R_{\rho,z}^{-1} \Delta R_\rho R_{\rho,z}^{-1} \right\} + O(r^{-(2+\delta)}) . \tag{A.37}$$

It is clear that the first term will vanish in summing over the \mathbb{C}^2 representation since the term is proportional to $\vec{\sigma} \cdot \hat{r}$ which is of course traceless. We now have

$$\begin{aligned}
I_\rho(z) &= -\frac{i}{2} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) \int_{S_\infty^2} d^2x \int \frac{d^3k}{(2\pi)^3} \\
&\quad \text{tr}_{\mathbb{C}^2 \otimes V_\rho} \left\{ (\hat{r} \cdot \vec{\sigma}) \frac{(-X + im_X)(i\vec{\sigma} \cdot \hat{r} \rho(\gamma_m))}{r^2(k^2 - \rho(X_\infty)^2 + m_X^2 + 2im_X \rho(X_\infty) + z)^2} \right\} \\
&= \sum_{\mu \in \Delta_\rho} n_\rho(\mu) \int_{S_\infty^2} d^2\Omega \int \frac{d^3k}{(2\pi)^3} \frac{(\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle}{(k^2 + \langle \mu, X_\infty \rangle^2 + m_X^2 + 2m_X \langle \mu, X_\infty \rangle + z)^2} \\
&= \frac{1}{2} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) \frac{(\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle}{\sqrt{(\langle \mu, X_\infty \rangle + m_X)^2 + z}} ,
\end{aligned} \tag{A.38}$$

which agrees with [131] in the limit of $m_X \rightarrow 0$. Here we employed an orthonormal basis of V_ρ associated with a weight space decomposition: $V_\rho = \oplus_\mu V_\rho[\mu]$ where $\mu \in \Delta_\rho \subset \Lambda_G^V \subset \mathfrak{t}^*$ are the weights of the representation and $n_\rho(\mu) = \dim V_\rho[\mu]$. In this decomposition we have $\forall v \in V_\rho[u]$, that $i\rho(X_\infty)v = \langle \mu, X_\infty \rangle v$ where \langle , \rangle denotes the canonical pairing $\mathfrak{t}^* \otimes \mathfrak{t} \rightarrow \mathbb{R}$. Note that our representation ρ is a quaternionic representation since it acts on the h_A fields. This means it splits: $\rho = \pi \oplus \pi^\vee$ which has a quaternionic structure coming from $SU(2)_R$ symmetry.

Contribution from Defects

Now we can calculate the contribution from the defects and [56] in the case of no defects. As in [131] we can do this by using a spectral representation. This is nearly identical to the case as in [131] which includes a constant term for the scalar field X except that we will impose that this has a constant weight m_X in each weight space. This has the effect of modifying the solution

$$\tilde{\Psi}_{j,m,s}^{(i_\mu)}(E; x) \rightarrow \tilde{\Psi}_{j,m,s}^{(i_\mu)}(\sqrt{E^2 + m_X^2}; x) \quad , \quad \text{sgn}(E) \rightarrow \text{phase}(E - im) \quad . \quad (\text{A.39})$$

It is important to note that the normalization of the Dirac fermions in [131] will be unchanged. Because of this, the only difference is in the integrals:

$$\begin{aligned} \mathcal{I}_\nu^{(1)}(a) &= \int_{-\infty}^{\infty} dE \frac{r|E|}{E - ia/r} J_\nu(\sqrt{E^2 + m_X^2}r)^2 \\ &= 2ia \int_{mr}^{\infty} \frac{\xi d\xi}{(\xi^2 - m_X^2 r^2 + a^2)} J_\nu(\xi)^2 \quad , \\ \mathcal{I}_\nu^{(2)}(a) &= \int_{-\infty}^{\infty} \frac{r(E \pm im_X)|E|dE}{|E - im|(E + i\sqrt{z} + d(x))} J_\nu(\sqrt{E^2 + m_X^2}r) J_{\nu+1}(\sqrt{E^2 + m_X^2}r) \\ &= 2 \int_{mr}^{\infty} d\xi \frac{\xi^2 - mr^2}{\xi^2 - m_X^2 r^2 + a^2} J_\nu(\xi) J_{\nu+1}(\xi) \mp 2mra \int_{mr}^{\infty} \frac{J_\nu(\xi) J_{\nu+1}(\xi) d\xi}{(\xi^2 - m_X^2 r^2 + a^2)} \quad . \end{aligned} \quad (\text{A.40})$$

It is clear that for $a \neq 0$ that all of these integrands are bounded. We can evaluate these integrals by taking the limit inside the integral of $r \rightarrow 0$ and find that these all have bounded integrals. Then we have that $a \rightarrow 0$ in this limit so $\mathcal{I}_\nu^{(1)} \rightarrow 0$. And similarly the second term in $\mathcal{I}_\nu^{(2)}$ vanishes. However we find that the first term in $\mathcal{I}_\nu^{(2)}$ is non-vanishing and in fact approaches the value in [131] and hence we end up with the same result for the singular monopole contribution to the zero modes.

Therefore we have that the number of zero modes for the hypermultiplet fermions is given by

$$\text{Ind}[\tilde{L}_\rho] = \frac{1}{2} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) \left\{ \text{sgn}(\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle + \sum_{n=1}^{N_d} |\langle \mu, P_n \rangle| \right\} \quad . \quad (\text{A.41})$$

Thus, the rank of the matter bundle is given by

$$\text{rk}_{\mathbb{C}}[\mathcal{E}_{\text{matter}}] = \frac{1}{2} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) \left\{ \text{sgn}(\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle + \sum_{n=1}^{N_d} |\langle \mu, P_n \rangle| \right\} \quad . \quad (\text{A.42})$$

A.3 Hypermultiplet Collective Coordinates

Coupling to hypermultiplets in the quaternionic representation ρ give rise to the additional terms in the vector multiplet bosonic fields

$$\begin{aligned} A_0 &= -\dot{z}^m \epsilon_m - \frac{i}{4} \phi_{mn} \chi^m \chi^n + Y^{cl} - \frac{2i}{\hat{D}^2} T^r \bar{\lambda}_a \rho(T^r) \lambda_b \psi^a \psi^b , \\ Y &= \epsilon_{Y_\infty} + \frac{i}{4} \phi_{mn} \chi^m \chi^n + Y^{cl} + \frac{2i}{\hat{D}^2} T^r \bar{\lambda}_a \rho(T^r) \lambda_b \psi^a \psi^b , \end{aligned} \quad (\text{A.43})$$

where $\{\lambda_a\}$ are the Dirac spinors which span the kernel of \tilde{L}_ρ and T^r span the lie algebra \mathfrak{g} . Since these terms mirror the ϕ_{mn} terms, we only need to worry about a subset of the terms from (A.4). By asymptotic analysis, the only nonzero terms are of the form

$$\left\{ -\frac{2i}{g_0^2} \int_{\mathcal{U}} d^3x \bar{\lambda}_a \rho(\epsilon_{Y_\infty}) \lambda_b \right\} \psi^a \psi^b , \quad (\text{A.44})$$

which is the analogous term to the covariant derivative of the triholomorphic killing vector field: $\nabla_m G(Y_\infty)_n$. More concretely, we are looking at the terms

$$L = \frac{2}{g^2} \int d^3x \left(i \bar{\lambda} \not{D} \lambda - i \bar{\lambda} Y \lambda - i m_X \bar{\lambda} \gamma_5 \lambda - m_Y \bar{\lambda} \lambda \right) . \quad (\text{A.45})$$

This gives the collective coordinate Lagrangian

$$L_{ferm} = h_{mn} (i \psi^m \mathcal{D}_t \psi^n + m \psi^m \psi^n) + L_{int} . \quad (\text{A.46})$$

We now need to consider the interaction terms (that is terms with Y and A_0). Specifically we get term

$$T_{ab} = \frac{1}{2\pi} \int_{\mathcal{U}} d^3x \bar{\lambda}_a \rho(\epsilon_{Y_\infty}) \lambda_b , \quad (\text{A.47})$$

which upon comparing with the calculation in Appendix A is the term analogous to the covariant spin derivative of $G(Y_\infty)$.

A.4 1-Loop Mass Contribution from HM

Here we calculate the 1-loop mass contribution to the BPS states from the hypermultiplet fields. The key assumption we make is that we restrict to the “vacuum” of the solitonic sector (the vacuum of the quantum excitation Fock-space) so that there are no perturbative excitations in the incoming and outgoing states. Because of this, the 1-loop

contribution is simply the sum of the zero-point energies whose quadratically divergent parts we expect to vanish due to supersymmetry. We can calculate this following [104].

We will begin by considering the hypermultiplet Lagrangian

$$\begin{aligned}
g_0^2 \mathcal{L}_{HM} = & D_\mu h^{\dagger A} D^\mu h_A - i(\lambda_I \sigma^\mu D_\mu \bar{\lambda}_I + \bar{\lambda}_I \bar{\sigma}^\mu D_\mu \lambda_I) \\
& + 2i(h^{\dagger A} \psi_A \lambda_1 + \bar{\lambda}_1 \bar{\psi}^A h_A - h^{\dagger A} \bar{\psi}_A \bar{\lambda}_2 + \lambda_2 \psi^A h_A) \\
& - 2i(\lambda_2 \phi \lambda_1 - \bar{\lambda}_2 \phi^* \bar{\lambda}_1) - 2m \lambda_2 \lambda_1 - 2m^* \bar{\lambda}_1 \bar{\lambda}_2 \\
& + im h^{\dagger A} \phi^* h_A + im^* h^{\dagger A} \phi h_A + |m|^2 h^{\dagger A} h_A \\
& - \frac{1}{2} h^{\dagger A} \{\phi, \phi^*\} h_A + \frac{1}{4} (h^{\dagger A} \sigma_A^r B T^r h_B)^2 .
\end{aligned} \tag{A.48}$$

Since we are expanding around the classical solution: $h_A = \lambda_I = \psi_A = Y = A_0 = m_Y = 0$ we can reduce this using Dirac spinors to

$$\begin{aligned}
g_0^2 \mathcal{L}_{HM} = & D_\mu \delta h^{\dagger A} D^\mu \delta h_A - \delta h^{\dagger A} X^2 \delta h_A + 2im_X \delta h^{\dagger A} X \delta h_A + m_X^2 \delta h^{\dagger A} \delta h_A \\
& - 2(i\delta \bar{\lambda} \not{D} \delta \lambda + \delta \bar{\lambda} \gamma_5 X \delta \lambda - im_X \delta \bar{\lambda} \gamma_5 \delta \lambda) ,
\end{aligned} \tag{A.49}$$

where we used δh_A and $\delta \lambda$ to denote the quantum fluctuations about the classical solution. Using an eigen-function decomposition of the equations of motion

$$\begin{aligned}
(-D_i^2 - X^2 + 2im_X X + m_X^2) \delta h_A &= \omega_b^2 \delta h_A , \\
(i\gamma^i D_i + \gamma_5 X - im_X \gamma_5) \lambda &= \omega_f \gamma_0 \lambda .
\end{aligned} \tag{A.50}$$

The fermionic equation of motion squares to

$$\left(-D_i^2 - X^2 + m_X^2 + 2im_X X - i\gamma_5 \gamma^i B_i + i\sigma^k \otimes \mathbf{1}_2 B_k \right) \lambda = 0 . \tag{A.51}$$

This can further be decomposed as

$$\begin{aligned}
(-D_i^2 - X^2 + m_X^2 + 2im_X X) \eta &= \omega_\eta^2 \eta , \\
(-D_i^2 - X^2 + m_X^2 + 2im_X X + 2\sigma_k \bar{\sigma}^0 B_k) \chi &= \omega_\chi^2 \chi .
\end{aligned} \tag{A.52}$$

This means when we add the vacuum fluctuation energies

$$M_{1-loop} = \hbar \sum \omega_b - \frac{\hbar}{2} \sum \omega_\eta - \frac{\hbar}{2} \sum \omega_\chi , \tag{A.53}$$

this reduces to

$$M_{1-loop} = \frac{\hbar}{2} \sum (\omega_\chi - \omega_\eta) . \tag{A.54}$$

This simplification occurs because the δh_A are superpartners to the χ . Thus the spectrum of frequencies for ω_b, ω_χ match exactly. Note that the above quantity is non-zero because of the difference in the density of states between χ and η . This is exemplified by the fact that χ has zero modes while η does not. Following [104], we can use the index computation

$$\begin{aligned} I(z) &= \text{Tr} \left\{ \frac{z}{\mathcal{D}^\dagger \mathcal{D} + z} - \frac{z}{\mathcal{D} \mathcal{D}^\dagger + z} \right\} \\ &= \frac{1}{2} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) \left\{ \frac{(\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle}{\sqrt{(\langle \mu, X_\infty \rangle + m_X)^2 + z}} + \sum_{n=1}^{N_d} |\langle \mu, P_n \rangle| \right\}, \end{aligned} \quad (\text{A.55})$$

(where $\mathcal{D}^\dagger \mathcal{D} = -D_i^2 - X^2 + m_X^2 + 2im_X X + 4\sigma_k \bar{\sigma}^0 B_k$ and $\mathcal{D} \mathcal{D}^\dagger = -D_i^2 - X^2 + m_X^2 + 2im_X X$) to find this difference in the density of states

$$I(z) - I(0) = \int_0^\infty d\omega^2 \frac{z}{\omega^2 + z} \left(\frac{dn[\chi]}{d\omega^2} - \frac{dn[\eta]}{d\omega^2} \right). \quad (\text{A.56})$$

From these equations (drawing inspiration from [104]) we can see that the difference in the density of states is given by

$$\begin{aligned} \frac{dn[\chi]}{d\omega^2} - \frac{dn[\eta]}{d\omega^2} &= \\ &- \sum_{\mu \in \Delta_\rho} \frac{n_\rho(\mu) (\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle}{2\pi\omega^2 (\omega^2 - (\langle \mu, X_\infty \rangle + m_X)^2)^{1/2}} \theta(\omega^2 - (\langle \mu, X_\infty \rangle + m_X)^2). \end{aligned} \quad (\text{A.57})$$

Which we can use to evaluate the 1-loop contribution to the mass:

$$\begin{aligned} M_{1-loop} &= \frac{\hbar}{2} \int d\omega^2 \omega \left(\frac{dn[\chi]}{d\omega^2} - \frac{dn[\eta]}{d\omega^2} \right) \\ &= -\frac{\hbar}{2\pi} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) (\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle \int_0^\infty \frac{dk}{\sqrt{k^2 + (\langle \mu, X_\infty \rangle + m_X)^2}}, \end{aligned} \quad (\text{A.58})$$

which is logarithmically divergent.

We can regularize this by using a cutoff in the integral:

$$\begin{aligned} M_{1-loop} &= -\frac{\hbar}{2\pi} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) (\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle \int_0^{1/\epsilon} \frac{dk}{\sqrt{k^2 + (\langle \mu, X_\infty \rangle + m_X)^2}} \\ &= -\frac{\hbar}{2\pi} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) (\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle \log \left(\frac{1 + \sqrt{1 + (\langle \mu, X_\infty \rangle + m_X)^2 \epsilon^2}}{(\langle \mu, X_\infty \rangle + m_X) \epsilon} \right) \\ &= -\frac{\hbar}{2\pi} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) (\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle (-\log(\epsilon) - \log((\langle \mu, X_\infty \rangle + m_X)/2)) \\ &\quad + O(\epsilon). \end{aligned} \quad (\text{A.59})$$

In renormalizing, we expect to keep the non-divergent term

$$M_{1-loop} = \frac{\hbar}{4\pi} \sum_{\mu \in \Delta_\rho} n_\rho(\mu) (\langle \mu, X_\infty \rangle + m_X) \langle \mu, \gamma_m \rangle \log \left(\frac{(\langle \mu, X_\infty \rangle + m_X)^2}{2|\Lambda|^2} \right) , \quad (\text{A.60})$$

where Λ is the dynamically generated scale.

A.5 Hypermultiplet Collective Coordinate Supersymmetry

Here we compute the supersymmetry transformation of the collective coordinate Lagrangian following [78]. Since the collective coordinate theory for the supervector field has been shown to be supersymmetric [133, 78], we will only show that the hypermultiplet terms are supersymmetric:

$$\begin{aligned} \delta z^m &= -i\epsilon_s \tilde{\mathbb{J}}_n^{(s)m} \chi^n , \\ \delta \chi^m &= \mathbb{J}_n^{(s)m} (\dot{z} - G(Y_\infty))^n \epsilon_s - i\epsilon_s \chi^k \chi^n \mathbb{J}_k^{(s)\ell} \Gamma_{\ell n}^m , \\ \delta \psi^a &= -A_{mb}^a \delta z^m \psi^b = i\epsilon_s A_{nb}^a \chi^m \tilde{\mathbb{J}}_m^{(s)n} \psi^b . \end{aligned} \quad (\text{A.61})$$

Since the hypermultiplet bosonic field is vanishing, the supersymmetry transformations for the vectormultiplet fields will be the same as those from [133]. We simply need to derive the collective coordinate Lagrangian for the hypermultiplet fermionic fields. Since $h_A = 0$, the supersymmetric variation of λ is given by

$$\delta_\epsilon \lambda = 0 . \quad (\text{A.62})$$

Plugging in the collective coordinate expanded field

$$\lambda = \psi^a(t) \lambda_a(x, z(t)) , \quad (\text{A.63})$$

we have the supersymmetry transformation

$$\delta_\epsilon \lambda = \lambda_a \delta_\epsilon \psi^a + \partial_m \lambda_a \delta_\epsilon z^m \psi^a = 0 . \quad (\text{A.64})$$

After taking the $L^2(\mathcal{U})$ inner product with $\bar{\lambda}_b$ we get

$$\delta_\epsilon \psi^a = -A_{mb}^a \delta_\epsilon z^m \psi^b . \quad (\text{A.65})$$

We can now calculate the supersymmetric variation of the Lagrangian. We will start with the variation of the kinetic term:

$$\begin{aligned}
\delta_\epsilon(h_{ab}\psi^a\mathcal{D}_t\psi^b) &= 2h_{ab}\delta_\epsilon\psi^a\mathcal{D}_t\psi^b + \delta_\epsilon z^m\partial_m(h_{ab}A_{nc}^b)\dot{z}^n\psi^a\psi^c + h_{ab}\psi^a A_{nc}^b\delta_\epsilon\dot{z}^n\psi^c \\
&\quad + \delta_\epsilon z^m\partial_m h_{ab}\psi^a\dot{\psi}^b \\
&= 2h_{ab}\delta_\epsilon\psi^a\mathcal{D}_t\psi^b + \delta_\epsilon z^m\partial_m(h_{ab}A_{nc}^b)\dot{z}^n\psi^a\psi^c - \dot{z}^m\partial_m(h_{ab}A_{nc}^b)\delta_\epsilon\dot{z}^n\psi^a\psi^b \\
&\quad + \delta_\epsilon z^m\partial_m h_{ab}\psi^a\dot{\psi}^b \\
&= F_{nmab}\delta_\epsilon\dot{z}^n\dot{z}^m\psi^a\psi^b - \delta_\epsilon\dot{z}^n(2A_{nab}\psi^a\dot{\psi}^b - \partial_n h_{ab}\psi^a\dot{\psi}^b) \\
&= F_{nmab}\delta_\epsilon\dot{z}^n\dot{z}^m\psi^a\psi^b ,
\end{aligned} \tag{A.66}$$

where we used the identities

$$\psi^a\mathcal{D}_t(\delta_\epsilon\psi^b) = \delta_\epsilon\psi^a\mathcal{D}_t\psi^b \quad , \quad \partial_n h_{ab} = A_{nab} - A_{nba} , \tag{A.67}$$

from [133] and [78] respectively. We also have from [78] that the variation of the T_{ab} term will be given by

$$\delta_\epsilon(T_{ab}\psi^a\psi^b) = \delta_\epsilon z^m\nabla_m T_{ab}\psi^a\psi^b = F_{mnab}\delta_\epsilon z^m G(Y_\infty)^n\psi^a\psi^b , \tag{A.68}$$

where here ∇_m uses the connection A_{mab} . Finally we need to show that the variation of the mass terms is zero:

$$\begin{aligned}
\delta_\epsilon(h_{ab}\psi^a\psi^b) &= 2h_{ab}\psi^a\delta_\epsilon\psi^b + \delta_\epsilon z^m\partial_m h_{ab}\psi^a\psi^b = \delta_\epsilon z^m\psi^a\psi^b(-2A_{mab} + \partial_m h_{ab}) \\
&= 0 .
\end{aligned} \tag{A.69}$$

Therefore in order for our theory to be supersymmetric, we need to add a term which contains F_{mnab} . This term has to be $F_{mnab}\chi^m\chi^n\psi^a\psi^b$ following [78]. Computing the supersymmetric variation we find

$$\begin{aligned}
\delta_\epsilon(F_{mnab}\chi^m\chi^n\psi^a\psi^b) &= \delta_\epsilon z^p\nabla_p F_{mnab}\chi^m\chi^n\psi^a\psi^b + 2F_{mnab}\chi^m\delta_\epsilon\chi^n\psi^a\psi^b + O(g^{5/2}) \\
&= 2\epsilon_s F_{mnab}\mathbb{J}_p^{(s)m}(\dot{z} - G(Y_\infty))^p\chi^n\psi^a\psi^b .
\end{aligned} \tag{A.70}$$

Where we used the Bianchi identity in going from the first to the second line and again ∇_m is with respect to A_{mab} not Γ_{pq}^m . We now have the condition on F_{mnab} :

$$F_{mnab}\mathbb{J}_p^{(s)n} = -F_{pnab}\tilde{\mathbb{J}}_m^{(s)n} , \tag{A.71}$$

for $s = 1, 2, 3$. This equation implies that A_{mab} is a hyperholomorphic connection.

A.6 Conventions for Hypermultiplets

In this paper we use the following conventions from [133, 171] where the signature of the metric is given by $(-, +, +, +)$ and Weyl fermions obey the relations

$$\begin{aligned}\psi^\alpha &= \epsilon^{\alpha\beta} \psi_\beta \quad , \quad \psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta \quad , \quad \epsilon^{12} = \epsilon_{21} = 1 \quad , \\ \psi\chi &= \psi^\alpha \chi_\alpha = \chi^\alpha \psi_\alpha = \chi\psi \quad , \\ \bar{\psi}\bar{\chi} &= \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} \quad ,\end{aligned}\tag{A.72}$$

and

$$\begin{aligned}\sigma^{\mu\nu} &= \frac{1}{2}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \quad , \\ \bar{\sigma}^{\mu\nu} &= \frac{1}{2}(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \quad , \\ \epsilon_{\mu\nu\sigma\rho} \sigma^{\sigma\rho} &= \sigma_{\mu\nu} \quad , \\ \epsilon_{\mu\nu\sigma\rho} \bar{\sigma}^{\sigma\rho} &= -\bar{\sigma}_{\mu\nu} \quad ,\end{aligned}\tag{A.73}$$

as in [171]. We also have the identities

$$\begin{aligned}\sigma^{\mu\alpha\dot{\alpha}} &= \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma_{\dot{\beta}\beta}^\mu \quad , \\ \theta_\alpha \theta_\beta &= \frac{1}{2} \epsilon_{\alpha\beta} \theta^2 \quad , \\ \bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} &= -\frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^2 \quad ,\end{aligned}\tag{A.74}$$

which imply

$$\begin{aligned}\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} &= -\frac{1}{2}\theta^2\bar{\theta}^2\eta^{\mu\nu} \quad , \\ (\theta\psi)(\theta\lambda) &= -\frac{1}{2}\theta^2\psi\lambda \quad , \\ (\bar{\theta}\bar{\psi})(\bar{\theta}\bar{\lambda}) &= -\frac{1}{2}\bar{\theta}^2\bar{\psi}\bar{\lambda} \quad .\end{aligned}\tag{A.75}$$

We will take the notation that complex conjugation does not change the order of the fermions but rather acts as

$$(\psi_\alpha)^* = -\bar{\psi}_{\dot{\alpha}} \quad , \quad (\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}} \quad , \quad (\bar{\psi}_{\dot{\alpha}})^* = -\psi_\alpha \quad , \quad (\bar{\psi}^{\dot{\alpha}})^* = \psi^\alpha \quad .\tag{A.76}$$

This is chosen to preserve the inner product of two fermions and is used so that the complex conjugation does not affect the representation of the gauge group. Similarly when we have an $SU(2)$ doublet of fermions will transform as

$$(\psi^A)^* = \bar{\psi}_A \quad , \quad (\psi_A)^* = -\bar{\psi}^A \quad , \quad (\bar{\psi}_A)^* = \psi^A \quad , \quad (\bar{\psi}^A)^* = -\psi_A \quad .\tag{A.77}$$

These are chosen to be an involution, and we have associated the complex conjugate space of the spinor representation with the dual representation. Whenever spinor indices are suppressed we adopt canonical notation where contractions between spinor space and dual spinor space have the raised index first whereas contractions between complex conjugated spinor space and its dual space have the lowered index first:

$$\psi\chi = \psi^\alpha\chi_\alpha \quad , \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} . \quad (\text{A.78})$$

This notation also generalizes to the hypermultiplet scalar bosons as

$$(h^A)^\dagger = h_A^\dagger \quad , \quad (h_A)^\dagger = -h^{\dagger A} \quad , \quad (h_A^\dagger)^\dagger = h^A \quad , \quad (h^{\dagger A})^\dagger = -h_A . \quad (\text{A.79})$$

This means that we will follow the notation

$$|h_A|^2 = (h_A)^\dagger(h_A) = -h^{\dagger A}h_A . \quad (\text{A.80})$$

Note that since the hypermultiplet bosons form a complex doublet under $SU(2)_R$, the index of h^A can be raised and lowered with the ϵ_{AB} tensor.

A.7 $SU(2)_R$ Invariance of Hypermultiplets

Consider a single hypermultiplet in a representation $\rho = \pi \oplus \pi^*$ coupled to a gauge field with gauge group G . Since we are dealing with a single hypermultiplet with a single representation, we will suppress ρ . The Lagrangian for this theory is given by:

$$\begin{aligned} \mathcal{L} = & -\frac{i\tau}{4\pi} \int d^2\theta W_\alpha W^\alpha + c.c. + \frac{\text{Im } \tau}{4\pi} \int d^4\theta \Phi^\dagger e^{2iV} \Phi \\ & + \frac{\text{Im } \tau}{4\pi} \left\{ \int d^4\theta \left(Q^\dagger e^{2iV} Q + \tilde{Q} e^{-2iV} \tilde{Q}^\dagger \right) + \int d^2\theta (i\tilde{Q}\Phi Q + m\tilde{Q}Q) + c.c. \right\} . \end{aligned} \quad (\text{A.81})$$

Note that Q and \tilde{Q} are in conjugate representations of the gauge group (π vs. π^*) so that $\tilde{Q}\Phi Q$ and $\tilde{Q}Q$ are gauge invariant. If we denote the components of $V = (\phi, \psi_A, A_\mu)$, $\Phi = (\phi, \psi_2, F)$, $Q = (h_1, \lambda_1, H_1)$, and $\tilde{Q} = (-h^{\dagger 2}, \lambda_2, H^{\dagger 2})$. Here h_A and ψ_A are $SU(2)_R$

doublets. As in the notation of [171], the superfields can be written:

$$\begin{aligned}
V &= -\theta\sigma^\mu\bar{\theta}v_\mu - \theta^2\bar{\theta}\bar{\psi}^1 + \bar{\theta}^2\theta\psi_1 + \frac{1}{2}[2D - i\partial_\mu A^\mu]\theta^2\bar{\theta}^2, \\
W_\alpha &= \psi_{1\alpha} + [\delta_\alpha^\beta D - i(\sigma^\mu\bar{\sigma}^\nu)_\alpha^\beta\partial_{[\mu}A_{\nu]}]\theta_\beta - i\theta^2\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\bar{\psi}^{1\dot{\alpha}}, \\
\Phi &= \phi + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi + \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi + 2\theta\psi_2 - i\theta^2\partial_\mu\psi_2\sigma^\mu\bar{\theta} + \theta^2F, \\
Q &= h_1 + i\theta\sigma^\mu\bar{\theta}\partial_\mu h_1 + \frac{1}{4}\theta^2\bar{\theta}^2\Box h_1 + 2\theta\lambda_1 - i\theta^2\partial_\mu\lambda_1\sigma^\mu\bar{\theta} + \theta^2H_1, \\
\tilde{Q} &= -h^{\dagger 2} + i\theta\sigma^\mu\bar{\theta}\partial_\mu h^{\dagger 2} - \frac{1}{4}\theta^2\bar{\theta}^2\Box h^{\dagger 2} + 2\theta\lambda_2 - i\theta^2\partial_\mu\lambda_2\sigma^\mu\bar{\theta} - \theta^2H^{\dagger 2}.
\end{aligned} \tag{A.82}$$

The different terms of the Lagrangian can be written in terms of component fields as

$$\begin{aligned}
W_\alpha W^\alpha \Big|_{\theta^2} &= 2i\psi_1\sigma^\mu D_\mu\bar{\psi}^1 + D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon_{\mu\nu\sigma\rho}F^{\mu\nu}F^{\sigma\rho}, \\
\Phi^\dagger e^{2iV}\Phi \Big|_{\theta^2\bar{\theta}^2} &= -D_\mu\phi^\dagger D_\mu\phi + |F|^2 + i\phi^*[D, \phi] \\
&\quad + i(\psi_2\sigma^\mu D_\mu\bar{\psi}^2 + \bar{\psi}^2\bar{\sigma}^\mu D_\mu\psi_2) - i(\epsilon^{AB}\psi_A[\psi_B, \phi^*] + \epsilon_{AB}\bar{\psi}^A[\psi^B, \phi]), \\
Q^\dagger e^{2iV}Q \Big|_{\theta^2\bar{\theta}^2} &= D_\mu h^{\dagger 1}D^\mu h_1 - i(\lambda_1\sigma^\mu D_\mu\bar{\lambda}_1 + \bar{\lambda}_1\bar{\sigma}^\mu D_\mu\lambda_1) - ih^{1\dagger}Dh_1 \\
&\quad + 2i(h^{1\dagger}\psi_1\lambda_1 + \bar{\lambda}_1\bar{\psi}^1h_1) - H^{1\dagger}H_1, \\
\tilde{Q}e^{-2iV}\tilde{Q}^\dagger \Big|_{\theta^2\bar{\theta}^2} &= D_\mu h^{\dagger 2}D^\mu h_2 - i(\lambda_2\sigma^\mu D_\mu\bar{\lambda}_2 + \bar{\lambda}_2\bar{\sigma}^\mu D_\mu\lambda_2) + ih^{\dagger 2}Dh_2 \\
&\quad + 2i(\lambda_2\psi_1h_2 + h^{\dagger 2}\bar{\psi}^1\bar{\lambda}_2) - H^{\dagger 2}H_2, \\
i\tilde{Q}\Phi Q \Big|_{\theta^2} &= -ih^{\dagger 2}\phi H_1 - iH^{\dagger 2}\phi h_1 - 2i(\lambda_2\phi\lambda_1 + \lambda_2\psi_2h_1 - h^{\dagger 2}\psi_2\lambda_1) - ih^{\dagger 2}Fh_1 \\
\tilde{Q}Q \Big|_{\theta^2} &= -h^{\dagger 2}H_1 - H^{\dagger 2}h_1 - 2\lambda_2\lambda_1,
\end{aligned} \tag{A.83}$$

where the covariant derivatives are given by:

$$\begin{aligned}
D_\mu\phi^r &= \partial_\mu\phi^r + f^{rst}A_\mu^s\phi^t, \quad D_\mu\psi_A^r = \partial_\mu\psi_A^r + f^{rst}A_\mu^s\psi_A^t, \\
D_\mu h_A &= \partial_\mu h_A + T^a A_\mu^a h_A, \quad D_\mu\lambda_I = \partial_\mu\lambda_I + T^a A_\mu^a\lambda_I,
\end{aligned} \tag{A.84}$$

Here we used A, B as indices for the fundamental representation of $SU(2)_R$ and I, J as uncharged indices. We also assume that the representation of our gauge group is real (and hence elements of the lie algebra \mathfrak{g} are antisymmetric):

$$[T^r, T^s] = f^{rst}T^t, \quad \text{Tr}[T^r T^s] = \delta^{rs}, \quad (T^r)^T = -T^r. \tag{A.85}$$

Putting everything together we can write the Lagrangian:

$$\begin{aligned}
\mathcal{L} = & \frac{1}{g_0^2} \text{Tr} \left\{ \begin{aligned} & -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D_\mu \phi^* D^\mu \phi + i(\psi_A \sigma^\mu D_\mu \bar{\psi}^A + \bar{\psi}^A \bar{\sigma}^\mu D_\mu \psi_A) \\ & + D^2 + |F|^2 - i(\epsilon^{AB} \psi_A [\psi_B, \phi^*] + \epsilon_{AB} \bar{\psi}^A [\bar{\psi}^B, \phi]) + i\phi^* [D, \phi] \end{aligned} \right\} \\
& + \frac{\theta}{32\pi^2} \text{Tr} \{ \epsilon_{\mu\nu\sigma\rho} F^{\mu\nu} F^{\sigma\rho} \} \\
& + \frac{1}{g_0^2} \left[\begin{aligned} & D_\mu h^{\dagger A} D^\mu h_A - i(\lambda_I \sigma^\mu D_\mu \bar{\lambda}_I + \bar{\lambda}_I \bar{\sigma}^\mu D_\mu \lambda_I) - H^{\dagger A} H_A - i h^{\dagger A} \sigma_A^m{}^B M_m h^B \\ & + 2i(h^{\dagger a} \psi_A \lambda_1 + \bar{\lambda}_1 \bar{\psi}^A h_A - h^{\dagger A} \bar{\psi}_A \bar{\lambda}_2 + \lambda_2 \psi^A h_A) \\ & - i(H^{\dagger 2} \phi h_1 + h^{\dagger 2} \phi H_1 + h^{\dagger 1} \phi^* H_2 + H^{\dagger 1} \phi^* h_2) - 2i(\lambda_2 \phi \lambda_1 - \bar{\lambda}_2 \phi^* \bar{\lambda}_1) \\ & - im(h^{\dagger 2} H_1 + H^{\dagger 2} h_1 + 2\lambda_2 \lambda_1) + im^*(H^{\dagger 1} h_2 + h^{\dagger 1} H_2 + 2\bar{\lambda}_1 \bar{\lambda}_2) \end{aligned} \right]
\end{aligned} \tag{A.86}$$

where the F - and D -terms have been rewritten as $M_m = (f, g, D)$ where $F = f + ig$. This demonstrates the hyperkähler property of this theory. The auxiliary F - and D -terms can be eliminated from their equations of motion:

$$\begin{aligned}
2D + i[\phi, \phi^*] - iT^r (h^{\dagger A} \sigma_A^3{}^B T^r h_B) &= 0, \\
2g + iT^r (h^{\dagger A} \sigma_A^2{}^B T^r h_B) &= 0, \\
2f + iT^r (h^{\dagger A} \sigma_A^1{}^B T^r h_B) &= 0, \\
H_1 + i\phi^* h_2 - im^* h_2 &= 0, \\
H_2 + i\phi h_1 + im h_1 &= 0.
\end{aligned} \tag{A.87}$$

In order to emphasize the holomorphy of hypermultiplet fields we can write the equations for f, g as:

$$f + ig = -\frac{iT^r}{2} (h^{\dagger A} \sigma_A^+{}^B T^r h_B) \quad , \quad \sigma^\pm = \sigma^1 \pm i\sigma^2. \tag{A.88}$$

This is the complex moment map. Now, by eliminate D, f, g , and H_I , and combining h^1 and h^2 into an $SU(2)_R$ doublet h^A we can express the Lagrangian

$$\begin{aligned}
\mathcal{L} = & \frac{1}{g_0^2} \text{Tr} \left\{ \begin{aligned} & -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + i(\psi_A \sigma^\mu D_\mu \bar{\psi}^A + \bar{\psi}^A \bar{\sigma}^\mu D_\mu \psi_A) - D_\mu \phi D^\mu \phi^* \\ & -i(\epsilon^{AB} \psi_A [\psi_B, \phi^*] + \epsilon_{AB} \bar{\psi}^A [\bar{\psi}^B, \phi]) + \frac{1}{4} [\phi, \phi^*]^2 \end{aligned} \right\} \\
& + \frac{\theta}{32\pi^2} \text{Tr} \{ \epsilon_{\mu\nu\sigma\rho} F^{\mu\nu} F^{\sigma\rho} \} \\
& + \frac{1}{g_0^2} \left[\begin{aligned} & D_\mu h^{\dagger A} D^\mu h_A - i(\lambda_I \sigma^\mu D_\mu \bar{\lambda}_I + \bar{\lambda}_I \bar{\sigma}^\mu D_\mu \lambda_I) \\ & + 2i(h^{\dagger A} \psi_A \lambda_1 + \bar{\lambda}_1 \bar{\psi}^A h_A - h^{\dagger A} \bar{\psi}_A \bar{\lambda}_2 + \lambda_2 \psi^A h_A) \\ & - 2i(\lambda_2 \phi \lambda_1 - \bar{\lambda}_2 \phi^* \bar{\lambda}_1) - 2im\lambda_2 \lambda_1 + 2im^* \bar{\lambda}_1 \bar{\lambda}_2 \\ & - m h^{\dagger A} \phi^* h_A + m^* h^{\dagger A} \phi h_A + |m|^2 h^{\dagger A} h_A \\ & - \frac{1}{2} h^{\dagger A} \{ \phi, \phi^* \} h_A + \frac{1}{4} (h^{\dagger A} \sigma_A^n{}^B \text{Tr} h_B)^2 \end{aligned} \right] \tag{A.89}
\end{aligned}$$

where $n = 1, 2, 3$. Note that this Lagrangian is manifestly $SU(2)_R$ invariant.

Appendix B

Kronheimer's Correspondence

In this appendix we will give a more in depth review and proof of Kronheimer's correspondence. This correspondence gives a one-to-one mapping between singular monopole configurations on \mathbb{R}^3 with $U(1)_K$ invariant instantons on (multi-)Taub-NUT. Therefore, for completeness, we will first give a brief review of Taub-NUT spaces and their general properties.

B.1 Review of Taub-NUT Spaces

Taub-NUT is an asymptotically, locally flat (ALF) space. It has the natural structure of a circle fibration over \mathbb{R}^3 , $\pi : TN \xrightarrow{S^1} \mathbb{R}^3$ whose fiber degenerates at a single point (the NUT center), which we will take to be at the origin in \mathbb{R}^3 . For any finite, positive value of r , the restriction of the S^1 fibration of Taub-NUT to a 2-sphere of radius R , is the Hopf fibration of charge $\ell = 1$, $TN|_R \cong S^3 \xrightarrow{S^1} S_R^2$.

This space has a metric which can be expressed in Gibbons-Hawking form as

$$ds^2 = V(\vec{x}) d\vec{x} \cdot d\vec{x} + V^{-1}(\vec{x}) \Theta^2 , \quad (\text{B.1})$$

where

$$V(\vec{x}) = 1 + \frac{1}{2r} , \quad \Theta = d\xi + \omega , \quad (\text{B.2})$$

where ξ is the S^1 fiber coordinate with periodicity 2π and $|\vec{x}| = r$ is the radius in the base \mathbb{R}^3 space. Note that $V(\vec{x})$ is sometimes called the harmonic function. Further, $\omega \in \Omega^1(TN)$ is a 1-form on Taub-NUT and solves the equation

$$d\omega = *_3 dV , \quad (\text{B.3})$$

This Appendix is based on material from my publication [24].

where $*_3$ is the Hodge-star on the base \mathbb{R}^3 lifted to TN . Note that while ω and $d\xi$ are not globally well defined 1-forms, Θ is globally well defined. Additionally, Taub-NUT is homeomorphic to \mathbb{R}^4 under the coordinate transformation

$$x_1 + ix_2 = \sqrt{r} \cos\left(\frac{\theta}{2}\right) e^{i(\phi+\xi)} \quad , \quad x_3 + ix_4 = \sqrt{r} \sin\left(\frac{\theta}{2}\right) e^{i(\phi-\xi)} \quad . \quad (\text{B.4})$$

This space comes with a natural $U(1)$ action (which we will refer to as the $U(1)_K$ action) given by translation of the ξ coordinate. This means $\forall k \in U(1)_K$, there exists a $f_k \in \text{diff}(TN)$ such that in local coordinates $f_k : (\vec{x}_{\mathbb{R}^3}, \xi) \rightarrow (\vec{x}_{\mathbb{R}^3}, \xi + \hat{k})$ for $\hat{k} \in \mathbb{R}/2\pi\mathbb{Z}$. Note that the metric is invariant under this action

$$f_k^*(ds^2) = ds^2 \quad . \quad (\text{B.5})$$

Taub-NUT can also be extended to have multiple NUT centers, called multi-Taub-NUT (or TN_k for k -NUT centers). This space is also naturally a circle fibration over \mathbb{R}^3 : $TN_k \xrightarrow{S^1} \mathbb{R}^3$ where the S^1 fiber degenerates at k -points $\{\vec{x}_i\}_{i=1}^k$ in the base \mathbb{R}^3 . This space has a non-trivial topology given by $H_{cpt}^2(TN_k, \mathbb{Z}) = \Gamma[A_{k-1}]$ where $\Gamma[A_{k-1}]$ is the root lattice of the Lie group A_{k-1} . These non-trivial 2-cycles are homologous to the preimage of the lines running between any two NUT centers under the projection $\pi : TN_k \rightarrow \mathbb{R}^3$. This space has a metric given by

$$ds^2 = V(\vec{x}) d\vec{x} \cdot d\vec{x} + V^{-1}(\vec{x}) \Theta^2 \quad , \quad (\text{B.6})$$

where¹

$$V(\vec{x}) = 1 + \sum_{i=1}^k \frac{1}{2|\vec{x} - \vec{x}_i|} \quad , \quad \Theta = d\xi + \omega \quad , \quad (\text{B.7})$$

and again $d\omega = *_3 dV$. Again Θ is a globally defined 1-form and there is a natural $U(1)_K$ action given by translation along the S^1 fiber coordinate ξ .

B.2 Kronheimer's Correspondence for a Single Defect

Now we will derive Kronheimer's correspondence for the case of a single 't Hooft defect. Our setting is $U(1)_K$ -invariant instantons on single centered Taub-NUT space, TN .

¹More generally we can have $V(r) = 1 + \sum_{i=1}^n \frac{\ell_i}{2|\vec{x} - \vec{x}_i|}$ where $\ell_i \in \mathbb{Z}$. We can think of this as taking the case above and taking the limit $\vec{x}_i \rightarrow \vec{x}_j$ for some set of combination of i, j . Having $\ell_i \neq 1$ leads to orbifold-type singularities in the metric at the NUT centers.

Let us introduce a gauge field on Taub-NUT by introducing a principal G bundle, $\sigma : \mathcal{P} \rightarrow TN$ with a connection \hat{A} , for G a compact, simple Lie group. In order to study $U(1)_K$ -invariant instantons, we must first define a lift of the $U(1)_K$ action to \mathcal{P} . Due to the degeneration of the S^1 fiber, this must be defined on local patches and then extended globally by demanding $U(1)_K$ equivariant transition functions for \mathcal{P} .

In each simply connected patch $U_\alpha \subset \mathbb{R}^3 \setminus \{0\}$ not containing the origin which lifts to a patch $\mathcal{U}_\alpha = \pi^{-1}(U_\alpha)$ not containing the NUT center. In this patch, the lift of the $U(1)_K$ action is defined by a pair of choices $(f, \rho) : U(1)_K \rightarrow \text{Aut}(TN) \times \text{Aut}(\mathfrak{g})$ such that in local coordinates $\{x^\mu\}$,

$$k \cdot (x^\mu, g) \mapsto \left(f_k(x^\mu), (\rho_k(x^\mu))^{-1} g \rho_k(x^\mu) \right) \quad (\text{B.8})$$

and

$$\sigma(f_k(x^\mu), \rho_k^{-1} g \rho_k) = f_k(x^\mu) \in \mathcal{U}_\alpha, \quad (\text{B.9})$$

where $\pi : TN \rightarrow \mathbb{R}^3$. This means that a $U(1)_K$ -invariant connection \hat{A} in a patch \mathcal{U}_α must satisfy $f_k^* \hat{A} \cong \hat{A}$ up to a smooth gauge transformation

$$f_k^* \hat{A} = \rho_k^{-1} \hat{A} \rho_k + i \rho_k^* \theta, \quad (\text{B.10})$$

where θ is the Maurer-Cartan form [61, 84].

Now consider a self-dual, $U(1)_K$ -invariant connection above the patch $\mathcal{U}_\alpha = \pi^{-1}(U_\alpha)$, for $U_\alpha \subset \mathbb{R}^3 \setminus \{0\}$ simply connected. Without loss of generality, this can be written in the form [108]

$$\hat{A} = \pi^* A - \psi(x)(d\xi + \omega), \quad (\text{B.11})$$

where A is a \mathfrak{g} valued 1-form on the base \mathbb{R}^3 . We will refer to the form (B.11) of the connection as the $U(1)_K$ -invariant gauge.

Dropping the π^* notation, the curvature can be written as

$$\begin{aligned} \hat{F} &= \hat{D}\hat{A} = DA - \psi d\omega - D\psi \wedge (d\xi + \omega), \\ &= (F - \psi d\omega) - D\psi \wedge (d\xi + \omega), \end{aligned} \quad (\text{B.12})$$

where \hat{D} and D are the gauge-covariant derivatives with respect to the connection \hat{A} and A respectively.

Using the orientation form² $\Theta \wedge dx^1 \wedge dx^2 \wedge dx^3$, we can compute the dual curvature

$$*\hat{F} = - *_3 F \wedge \left(\frac{d\xi + \omega}{V} \right) - V *_3 D\psi + \psi *_3 d\omega \wedge \left(\frac{d\xi + \omega}{V} \right) . \quad (\text{B.13})$$

Self-duality $\hat{F} = *\hat{F}$ then reduces to the simple equation

$$*_3(F - \psi d\omega) = VD\psi , \quad (\text{B.14})$$

which can be written

$$*_3 F = D(V\psi) , \quad (\text{B.15})$$

which is equivalent to the Bogomolny equation under the identification $X = V\psi$.

Therefore a $U(1)_K$ -invariant connection on the patch \mathcal{U}_α in Taub-NUT is self-dual if and only if, the associated three dimensional connection and Higgs field $\hat{A} \mapsto (A, X = V\psi)$ satisfies the Bogomolny equation on $U_\alpha \subset \mathbb{R}^3$.

Now that we have shown that there is a local correspondence between $U(1)_K$ -invariant instantons on Taub-NUT and monopoles on \mathbb{R}^3 , we need to show that these solutions can be smoothly extended over all patches $\mathcal{U}_\alpha = \pi^{-1}(U_\alpha)$ for $U_\alpha \subset \mathbb{R}^3 \setminus \{0\}$.

Recall that in order to have a well defined principal G -bundle over a generic manifold M , on any two patches $\mathcal{U}_\alpha, \mathcal{U}_\beta$ with non-trivial intersection, the gauge fields must be related by some gauge transformation $g_{\alpha\beta}$. This is the data of the bundle and encodes its topology.

Let us define $\mathcal{U}_\alpha, \mathcal{U}_\beta = \pi^{-1}(U_\alpha), \pi^{-1}(U_\beta)$ for $U_\alpha, U_\beta \subset \mathbb{R}^3 \setminus \{0\}$. By comparing the definition of $U(1)_K$ -invariance in each patch with the gluing condition, the ρ_α satisfy

$$\rho_\alpha(\vec{x}, k) g_{\alpha\beta}(k \cdot \vec{x}) = g_{\alpha\beta}(\vec{x}) \rho_\beta(\vec{x}, k) , \quad (\text{B.16})$$

or rather

$$g_{\alpha\beta}(k \cdot \vec{x}) = \rho_\alpha^{-1}(\vec{x}, k) g_{\alpha\beta}(\vec{x}) \rho_\beta(\vec{x}, k) , \quad (\text{B.17})$$

and hence the transition functions are $U(1)_K$ -equivariant with respect to the lifted $U(1)_K$ action.

Now we want to extend the action over the NUT center where the S^1 fiber degenerates. Consider a generic open set $\mathcal{U}_\alpha = \pi^{-1}(U_\alpha)$ where $0 \notin U_\alpha \subset \mathbb{R}^3$. As before in this

²Note that this orientation form is the natural choice as $d\xi$ is not globally well defined.

patch we can write the connection \hat{A}_α in a $U(1)_K$ -invariant gauge (where $\rho_\alpha(\vec{x}; k) = \mathbb{1}_G$). Now take $\mathcal{U}_0 = \pi^{-1}(B_\epsilon^3)$ where B_ϵ^3 is the three dimensional ϵ -ball around the origin. Since the S^1 fiber degenerates at the origin, the $U(1)_K$ action has a fixed point in \mathcal{U}_0 and hence \hat{A}_0 cannot be written in the $U(1)_K$ -invariant gauge (B.11) in that patch .

However, we can determine the form of \hat{A}_0 in terms of a gauge transformation of a connection \hat{A}_α in the $U(1)_K$ -invariant gauge. Consider a \mathcal{U}_α as defined before such that $\mathcal{U}_\alpha \cap \mathcal{U}_0 \neq \emptyset$. The transition function $g_{0\alpha}$ between the $U(1)_K$ -invariant connection \hat{A}_α on \mathcal{U}_α and \hat{A}_0 on \mathcal{U}_0 has the limiting form $\lim_{\vec{x} \rightarrow 0} g(\vec{x}, \xi) \rightarrow e^{-iP\xi}$ for some choice of $P \in \Lambda_{cochar}$ [108], and hence $\lim_{\vec{x} \rightarrow 0} g^{-1}dg = -iPd\xi$.

The reason we have this limiting form of the gauge transformation is as follows. The component of any smooth gauge field along the fiber direction must go to zero at the NUT center. However, the $U(1)_K$ -invariant gauge is generically non-zero. Therefore, we must have that the transition between these two gauges must have the limit of a constant function

$$\lim_{\vec{x} \rightarrow 0} g^{-1}dg = -iPd\xi , \quad (\text{B.18})$$

which cancels the non-zero value of $\psi(0)$. Further since we must have a well defined gauge transformation, P is restricted to lie in $\Lambda_{cochar} = \{P \in \mathfrak{t} \mid \text{Exp}[2\pi P] = \mathbb{1}_G\}$ and hence the condition that \hat{A} be $U(1)_K$ -invariant and smooth requires that $\lim_{\vec{x} \rightarrow 0} \psi_\alpha(\vec{x}) \in \Lambda_{cochar}$. This results in the limiting form described above. Note that this gauge transformation $g_{0\alpha}(\vec{x}, \xi)$ is smooth for neighborhoods arbitrarily close to the NUT center, but is not globally smooth because of the degeneration of the ξ -fiber.

Using the limiting form of the gauge transformation above, the gauge field on \mathcal{U}_0 is of the form $\hat{A}_0 = g_{0\alpha}^{-1}\hat{A}_\alpha g_{0\alpha} + ig_{0\alpha}^{-1}dg_{0\alpha}$. It is clear from this form that we should identify $\rho_0(\vec{x}; k)$ with $g_{0\alpha}^{-1}(\vec{x}, k)$ since \hat{A}_α is $U(1)_K$ -invariant and all of the ξ -dependence of \hat{A}_0 is in the $g_{0\alpha}$ gauge transformation. Therefore, fixing the lift of the $U(1)_K$ action at the NUT center fixes the action globally in the case of a single singular monopole by gluing across the patches using (B.17) which is trivial due to the trivial topology of Taub-NUT ($H_2(TN; \mathbb{Z}) \cong 0$). Hence, gauge inequivalent $U(1)_K$ -invariant connections are defined by a choice of $P \in \Lambda_{cochar}$.

Using this we can see that the connection \hat{A}_0 has the limiting form

$$\lim_{\vec{x} \rightarrow 0} \hat{A}_0 \rightarrow [A + P\omega] - [\psi + P](d\xi + \omega) . \quad (\text{B.19})$$

By Uhlenbeck's Theorem, the gauge bundle can be smoothly continued over the NUT center if the action of the field \hat{A}_0 is finite [166]. This implies that

$$\lim_{\vec{x} \rightarrow 0} \psi(\vec{x}) = -P + O(r^\delta) \quad , \quad \lim_{x \rightarrow 0} A = -P\omega + O(r^{-1+\delta}) , \quad (\text{B.20})$$

and that all apparent singularities arising from higher order terms (as in $\delta \geq \frac{1}{2}$) can be gauged away. This means that the corresponding monopole solution will have the asymptotic behavior

$$F_{\mathbb{R}^3} = \frac{P_n}{2} d\Omega + O(r^{-2+\delta}) \quad , \quad X = -\frac{P_n}{2r} + O(r^{-1+\delta}) , \quad (\text{B.21})$$

in the limit $r \rightarrow 0$ for some $\delta > 0$.

Note that near $r \rightarrow \infty$, $V = 1$ and hence $X \rightarrow \psi(\vec{x})$. This means that the Higgs vev X_∞ is encoded in the holonomy of \hat{A} along the S^1 fiber at infinity – i.e. by the value of ψ as $r \rightarrow \infty$.

Therefore, by using the global lift of the $U(1)_K$ -action to the gauge bundle, local Kronheimer's correspondence can be extended globally. Hence, general singular monopoles configurations in \mathbb{R}^3 with one defect are in one-to-one correspondence with $U(1)_K$ -invariant instantons on Taub-NUT where the lift of the $U(1)_K$ action is defined by the 't Hooft charge of the singular monopole.

Remark It is worth commenting on the admissibility of subleading terms in the asymptotic behavior of the gauge/Higgs fields in the instanton/monopole solutions. From the analysis from [133], locally solving the gauge covariant Laplacian in the presence of a singular monopole imposes that

$$\begin{aligned} F_{\mathbb{R}^3} &= \frac{P_n}{2} d\Omega + O(r^{-2+\delta}) \\ \Phi &= -\frac{P_n}{2r} + O(r^{-1+\delta}) \end{aligned} \quad , \quad \delta = \begin{cases} \frac{1}{2} & \exists \mu \in \Lambda_{rt}^+ \text{ s.t. } \langle \mu, P \rangle = 1 \\ 1 & \text{else} \end{cases} , \quad (\text{B.22})$$

where Λ_{rt}^+ is the positive root lattice of the gauge group relative to $X_\infty \in \mathfrak{t}$. The $O(r^{-1/2})$ behavior can be explained by $\psi(r)$ having a subleading term going as $O(r^{1/2})$.

This is in fact the only subleading term allowed by the requirement of finite action by integrating

$$S_\epsilon = -\frac{1}{4g^2} \int_{\pi^{-1}(B_\epsilon^3)} \text{Tr} \left\{ \hat{F} \wedge * \hat{F} \right\} , \quad (\text{B.23})$$

in the singular gauge as in (B.19) where B_ϵ^3 is the solid 2-ball of radius $\epsilon > 0$ around the origin.

This fractional subleading behavior simply allows for the existence of non-smooth instantons which we should generally expect to contribute to any physical processes. The subleading behavior also seems to be a manifestation of the fact that singular monopoles with charge $P \notin \Lambda_{cr}$ cannot be fully screened by smooth monopoles which generally have charge in Λ_{cr} .

B.3 Generalization to Multiple Defects

Now we can ask how this generalizes to the case of multiple defects. This is accomplished by considering $U(1)_K$ -invariant instantons on multi-Taub-NUT, TN_k . Since the metric for this space can again be written in Gibbons-Hawking form as in (B.6), again locally self-dual $U(1)_K$ connections on TN_k are in one-to-one correspondence with local solutions of the Bogomolny equations on \mathbb{R}^3 . This follows from an identical calculation as in the previous section for a single defect by substituting the harmonic function and corresponding 1-form (V, ω) for those in the multi-Taub-NUT metric.

The proof of Kronheimer's correspondence for multiple defects is thus reduced to understanding how the $U(1)_K$ action extends across different patches. Due to this non-trivial topology, fixing the lift of the $U(1)_K$ action of a single NUT center does not specify the action completely as there are infinitely many gauge inequivalent ways to glue this action across different patches approaching different NUT centers.

Rather, the topology of the gauge bundle can be specified by the lift at a single NUT center and by a choice of Dirac monopole charge for each non-trivial homology 2-sphere – or equivalently one can specify the lift of the $U(1)_K$ action at each NUT center. However, we should ask whether or not specifying the topological class of the bundle in addition to the lift of the $U(1)_K$ action at a NUT center fixes the global lift of

the $U(1)_K$ action. We will momentarily argue that this is indeed the case. If this is true, then the set of all inequivalent choices of $U(1)_K$ action on the principal G -bundle on multi-Taub-NUT is in one-to-one correspondence with the set of all possible choices of 't Hooft charges in the corresponding singular monopole configuration. Together with local Kronheimer's correspondence, this global lift of the $U(1)_K$ action would imply that Kronheimer's correspondence also holds globally.

The question of whether or not we can construct this correspondence is now reduced to the question of whether or not there exists a gauge transformation on the intersection of different patches which is $U(1)_K$ equivariant such that the $U(1)_K$ action has the proper limiting form at the various NUT centers. As we reviewed in the previous section, in order to have a well defined principal G -bundle with $U(1)_K$ action, on any two intersecting patches $\mathcal{U}_\alpha, \mathcal{U}_\beta$, the gauge fields are related by some gauge transformation $g_{\alpha\beta}$ which satisfies the equivariance condition

$$g_{\alpha\beta}(k \cdot x) = \rho_\alpha^{-1}(x, k) g_{\alpha\beta}(x) \rho_\beta(x, k) . \quad (\text{B.24})$$

The new complication of defining the $U(1)_K$ action on multi-Taub-NUT is how it glues across patches containing different NUT centers. So, let us consider $\mathcal{U}_\alpha, \mathcal{U}_\beta = \pi^{-1}(U_\alpha), \pi^{-1}(U_\beta)$ for $U_\alpha, U_\beta \subset \mathbb{R}^3$ containing NUT centers at $\vec{x}_\alpha, \vec{x}_\beta$ respectively such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$. Using the limiting forms of the ρ_α , we can explicitly solve for the form of $g_{\alpha\beta}(x)$ in the patch $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$:

$$g_{\alpha\beta}(x) \cong \text{Exp} [i(P_\beta - P_\alpha)\psi] , \quad (\text{B.25})$$

up to a trivial gauge transformation, and hence specifies a class in $H^2(TN_n, \mathbb{Z})$.

The physical argument that this must be the correct class for the transition function is as follows. Consider a line ℓ_{ij} from \vec{x}_i to \vec{x}_j (two NUT centers). We know that the $U(1)_K$ action on a patch U_i near the \vec{x}_i goes as $\rho_i(k) \sim e^{iP_i k}$ and similarly on U_j near \vec{x}_j , $\rho_j(k) \sim e^{iP_j k}$. This means that on the transition $U_i \cap U_j$ along the line ℓ_{ij} , the transition function must be in the same cohomology class as $\text{Exp} [i(P_\alpha - P_\beta)\xi]$ because the winding number of $P \rightarrow \pi^{-1}(\ell_{ij}) \cong S^2$ is P_i on one hemisphere and P_j on the other hemisphere.

This is exactly analogous to the translation function along the equator on the sphere at infinity for a monopole in \mathbb{R}^3 . These transition functions, as in the picture of monopoles in \mathbb{R}^3 , have the interpretation of the gauge field having non-trivial flux on these spheres. This is necessary for the consistent lift of the $U(1)_K$ action across the entire space and in the corresponding singular monopole configuration on \mathbb{R}^3 , this flux is literally the physical magnetic flux between two singular monopoles (in the absence of smooth monopoles).

Therefore, the $U(1)_K$ action can be globally lifted to the gauge bundle over multi-Taub-NUT which can be used to globally extend local $U(1)_K$ -invariant instanton solutions. Hence, general singular monopole configurations in \mathbb{R}^3 are in one-to-one correspondence with $U(1)_K$ -invariant instantons on multi-Taub-NUT where the collection of 't Hooft charges specifies both the topology of the gauge bundle and the global lift of the $U(1)_K$ action.

Appendix C

Wall Crossing

C.1 The Two-galaxy Region for Smooth Monopoles

In describing the two-galaxy limit of the asymptotic region of monopole moduli space, we introduce center of mass and relative coordinates. Note that the difference of any two \vec{x}^a 's can be expressed as a linear combination of center of mass coordinates \vec{y}^a 's only. The coordinate transformations between the \vec{x}^a (absolute coordinates) and \vec{y}^a (relative coordinates) from (4.13) - (4.14) are given explicitly by

$$\vec{x}^a = \vec{X}_1 + (\mathbf{j}_1)^a_b \vec{y}^b, \quad \vec{x}^p = \vec{X}_2 + (\mathbf{j}_2)^p_q \vec{y}^q. \quad (\text{C.1})$$

where

$$\mathbf{j}_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_{N_1-1} \\ b_1 & a_2 & \cdots & a_{N_1-1} \\ b_1 & b_2 & \cdots & a_{N_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & b_2 & \cdots & b_{N_1-1} \end{pmatrix}, \quad \mathbf{j}_2 = \begin{pmatrix} a_{N_1} & a_{N_1+1} & \cdots & a_{N-2} \\ b_{N_1} & a_{N_1+1} & \cdots & a_{N-2} \\ b_{N_1} & b_{N_1+1} & \cdots & a_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N_1} & b_{N_1+1} & \cdots & b_{N-2} \end{pmatrix}, \quad (\text{C.2})$$

with

$$a_a = \frac{m_{a+1} + \cdots + m_{N_1}}{m_{\text{gal1}}}, \quad b_a = -\frac{(m_1 + \cdots + m_a)}{m_{\text{gal1}}}, \quad (a = 1, \dots, N_1 - 1),$$

$$a_p = \frac{m_{p+2} + \cdots + m_N}{m_{\text{gal2}}}, \quad b_p = -\frac{(m_{N_1+1} + \cdots + m_{p+1})}{m_{\text{gal2}}}, \quad (p = N_1, \dots, N - 2) \quad (\text{C.3})$$

This Appendix is based on material from my publication [25].

These fit into square Jacobian matrices that give the map $(\{\vec{y}\}, \vec{X})^T \mapsto \{\vec{x}\}$ as follows:

$$(\vec{x}^{\hat{a}}) = (\mathbf{j}_1 \mid \mathbf{1}_1) \begin{pmatrix} \vec{y}^a \\ \vec{X}_1 \end{pmatrix} \equiv \mathbf{J}_1 \begin{pmatrix} \vec{y}^a \\ \vec{X}_1 \end{pmatrix} , \quad (\text{C.4})$$

$$(\vec{x}^{\hat{p}}) = (\mathbf{j}_2 \mid \mathbf{1}_2) \begin{pmatrix} \vec{y}^s \\ \vec{X}_2 \end{pmatrix} \equiv \mathbf{J}_2 \begin{pmatrix} \vec{y}^s \\ \vec{X}_2 \end{pmatrix} . \quad (\text{C.5})$$

Here $\mathbf{1}_1, \mathbf{1}_2$ denote a length N_1, N_2 column vector with all entries equal to 1 respectively.

Similarly the angular coordinates transform as

$$\left(\frac{\partial}{\partial \xi^a} \right) = \mathbf{J}_1 \begin{pmatrix} \frac{\partial}{\partial \psi^a} \\ \frac{\partial}{\partial \chi_1} \end{pmatrix} , \quad \left(\frac{\partial}{\partial \xi^s} \right) = \mathbf{J}_2 \begin{pmatrix} \frac{\partial}{\partial \psi^s} \\ \frac{\partial}{\partial \chi_2} \end{pmatrix} . \quad (\text{C.6})$$

And finally we implement the global center of mass and relative coordinates (4.15).

In the limit that R is much greater than all of the y , the matrix M_{ij} has the structure

$$M_{ij} = \begin{pmatrix} M_{ab} & \frac{D_{aq}}{R} + O\left(\frac{y}{R^2}\right) \\ \frac{D_{pb}}{R} + O\left(\frac{y}{R^2}\right) & M_{pq} \end{pmatrix} , \quad (\text{C.7})$$

where

$$\begin{aligned} M_{ab} &= (M_1)_{ab} - \frac{1}{R} \delta_{ab} (H_{I(a)}, \gamma_{\text{m},2}) + O\left(\frac{y}{R^2}\right) , \\ M_{pq} &= (M_2)_{pq} - \frac{1}{R} \delta_{pq} (\gamma_{\text{m},1}, H_{I(p)}) + O\left(\frac{y}{R^2}\right) , \end{aligned} \quad (\text{C.8})$$

with

$$(M_1)_{ab} = \begin{cases} m_a - \sum_{c \neq a} \frac{D_{ac}}{r_{ac}} , & a = b , \\ \frac{D_{ab}}{r_{ab}} , & a \neq b , \end{cases} \quad (M_2)_{pq} = \begin{cases} m_p - \sum_{r \neq p} \frac{D_{pr}}{r_{pr}} , & p = q , \\ \frac{D_{pq}}{r_{pq}} , & p \neq q . \end{cases} \quad (\text{C.9})$$

The latter are the matrices that would appear in the GM/LWY metrics for galaxies one and two in isolation. Thus in the two-galaxy limit, the “mass matrix” takes the form

$$\begin{aligned} M_{ij} &= \begin{pmatrix} (M_1)_{ab} & 0 \\ 0 & (M_2)_{pq} \end{pmatrix} + \frac{1}{R} \begin{pmatrix} -\delta_{ab} (H_{I(a)}, \gamma_{\text{m},2}) & D_{aq} \\ D_{bp} & -\delta_{pq} (\gamma_{\text{m},1}, H_{I(p)}) \end{pmatrix} \\ &\quad + O\left(\frac{y}{R^2}\right) , \end{aligned} \quad (\text{C.10})$$

which we will write as

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_2 \end{pmatrix} - \frac{1}{R} \begin{pmatrix} \mathbf{D}_{11} & -\mathbf{D}_{12} \\ -\mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix} + O\left(\frac{y}{R^2}\right), \quad (\text{C.11})$$

where \mathbf{D}_{11} and \mathbf{D}_{22} are diagonal matrices and $\mathbf{D}_{21} = (\mathbf{D}_{12})^T$.

Then within each galaxy we make the change of variables (C.4). The relevant quantities to be computed are, at leading order,

$$\mathbf{J}_1^T \mathbf{M}_1 \mathbf{J}_1 = \begin{pmatrix} \mathbf{C}_1 & 0 \\ 0 & m_{\text{gal1}} \end{pmatrix}, \quad \mathbf{J}_2^T \mathbf{M}_2 \mathbf{J}_2 = \begin{pmatrix} \mathbf{C}_2 & 0 \\ 0 & m_{\text{gal2}} \end{pmatrix}, \quad (\text{C.12})$$

where $\mathbf{C}_1 = \mathbf{j}_1^T \mathbf{M}_1 \mathbf{j}_1$, *etc.* The $O(1/R)$ terms are

$$\begin{aligned} \mathbf{J}_1^T \mathbf{D}_{11} \mathbf{J}_1 &= \begin{pmatrix} \mathbf{j}_1^T \mathbf{D}_{11} \mathbf{j}_1 & \langle \beta_a, \gamma_{2,m} \rangle \\ \langle \beta_b, \gamma_{2,m} \rangle & (\gamma_{1,m}, \gamma_{2,m}) \end{pmatrix}, \\ \mathbf{J}_2^T \mathbf{D}_{22} \mathbf{J}_2 &= \begin{pmatrix} \mathbf{j}_2^T \mathbf{D}_{22} \mathbf{j}_2 & \langle \beta_p, \gamma_{1,m} \rangle \\ \langle \beta_q, \gamma_{1,m} \rangle & (\gamma_{1,m}, \gamma_{2,m}) \end{pmatrix}, \\ \mathbf{J}_1^T \mathbf{D}_{12} \mathbf{J}_2 &= \begin{pmatrix} \mathbf{j}_1^T \mathbf{D}_{12} \mathbf{j}_2 & \langle \beta_a, \gamma_{2,m} \rangle \\ \langle \beta_q, \gamma_{1,m} \rangle & (\gamma_{1,m}, \gamma_{2,m}) \end{pmatrix}, \end{aligned} \quad (\text{C.13})$$

where

$$\beta_a = (\mathbf{J}_1^T)_a{}^b H_{I(b)}^* = a_a \sum_{c=1}^a H_{I(c)}^* + b_a \sum_{c=a+1}^{N_1} H_{I(c)}^*, \quad a = 1, \dots, N_1 - 1, \quad (\text{C.14})$$

and similarly

$$\beta_p = (\mathbf{J}_2^T)_p{}^q H_{I(q)}^* = a_p \sum_{r=N_1+1}^{p+1} H_{I(r)}^* + b_p \sum_{r=p+2}^N H_{I(r)}^*, \quad p = N_1, \dots, N - 2, \quad (\text{C.15})$$

with the a and b coefficients given in (C.3). The key property to note of the $\beta_{a,p}$ is that they have zero pairing with \mathcal{X} ; for example we have $\langle \beta_a, \mathcal{X} \rangle = -a_a b_a + b_a a_a = 0$.

Thus far we have described the transformation of the quadratic form M_{ij} from the basis of differentials $d\vec{x}^i$ to the basis $(d\vec{y}^a, d\vec{X}_1, d\vec{y}^p, d\vec{X}_2)^T$. Next we implement the transformation $(\vec{X}_1, \vec{X}_2) \mapsto (\vec{X}, \vec{R})$ to center of mass coordinates. One finds that the quadratic form diagonalizes with respect to the overall center of mass coordinate. Then

collecting the remaining differentials into the block structure $(d\vec{y}, d\vec{R}) = (d\vec{y}^a, d\vec{y}^p, d\vec{R})$, one obtains the first line of (4.16) with

$$\begin{aligned}\tilde{\mathbf{C}} &= \begin{pmatrix} \mathbf{C}_1 & 0 \\ 0 & \mathbf{C}_2 \end{pmatrix}, \quad \delta\mathbf{C} = \begin{pmatrix} -\mathbf{j}_1^T \mathbf{D}_{11} \mathbf{j}_1 & \mathbf{j}_1^T \mathbf{D}_{12} \mathbf{j}_2 \\ \mathbf{j}_2^T \mathbf{D}_{21} \mathbf{j}_1 & -\mathbf{j}_2^T \mathbf{D}_{22} \mathbf{j}_2 \end{pmatrix}, \\ \mathbf{L} &= \begin{pmatrix} -\langle \beta_a, \gamma_{2,m} \rangle \\ \langle \beta_p, \gamma_{1,m} \rangle \end{pmatrix}, \quad H(R) = 1 - \frac{(\gamma_{1,m}, \gamma_{2,m})}{\mu R}. \end{aligned} \quad (\text{C.16})$$

Note that $\delta\mathbf{C}$ and \mathbf{L} are coordinate independent.

Now we turn to the connection one-forms on the N -torus, Θ_i . We change variables in the fiber coordinates according to (4.14). Denoting $\mathbf{J} = \text{diag}(\mathbf{J}_1, \mathbf{J}_2)$, the quantity we want to investigate therefore is $\mathbf{J}^T \vec{\mathbf{W}} \mathbf{J}$, where $\vec{\mathbf{W}}$ is the matrix with components \vec{W}_{ij} given in (4.7). The reason for the \mathbf{J}^T on the left is that we want Θ_i to transform like the legs along the fiber directions, (4.14). The overall factors of \mathbf{J}^T will then be of the right form to transform the inverse quadratic form, $(M^{-1})^{ij}$, to the y - X basis. The reason for the \mathbf{J} on the right of $\vec{\mathbf{W}}$ is that we will put a $\mathbf{J}\mathbf{J}^{-1}$ between \vec{W}_{ij} and $d\vec{x}^j$, using the \mathbf{J}^{-1} to map the $d\vec{x}^j$ to $(d\vec{y}^b, d\vec{X}_1, d\vec{y}^q, d\vec{X}_2)^T$.

However we only need \vec{W}_{ij} through $O(1/R)$, which takes the form

$$\begin{aligned}(\vec{W}_{ij}) &= \begin{pmatrix} (\vec{W}_1)_{ab} & 0 \\ 0 & (\vec{W}_2)_{pq} \end{pmatrix} + \begin{pmatrix} -\delta_{ab}(H_{I(a)}, \gamma_{m,2}) & D_{aq} \\ D_{bp} & -\delta_{pq}(\gamma_{m,1}, H_{I(p)}) \end{pmatrix} \vec{w}(\vec{R}) \\ &+ O\left(\frac{y}{R^2}\right), \end{aligned} \quad (\text{C.17})$$

where $\vec{W}_{1,2}$ are the corresponding \vec{W} 's for galaxies one and two in isolation. Wrapping the \mathbf{J}^T - \mathbf{J} around the first term, we observe that

$$\mathbf{J}_1^T \vec{\mathbf{W}}_1 \mathbf{J}_1 = \begin{pmatrix} \mathbf{j}_1^T \vec{\mathbf{W}}_1 \mathbf{j}_1 & \mathbf{0} \\ \mathbf{0}^T & 0 \end{pmatrix}, \quad (\text{C.18})$$

and similarly for $1 \mapsto 2$. In the text we denoted the upper-left $(N_{1,2} - 1) \times (N_{1,2} - 1)$ corners of these expressions by $\vec{\mathcal{W}}_{1,2}$ respectively:

$$\vec{\mathcal{W}}_1 = \mathbf{j}_1^T \vec{\mathbf{W}}_1 \mathbf{j}_1, \quad \vec{\mathcal{W}}_2 = \mathbf{j}_2^T \vec{\mathbf{W}}_2 \mathbf{j}_2. \quad (\text{C.19})$$

For the \mathbf{J} -transformation of the $O(1/R)$ terms we can make use of (C.13). We then make the final transformation to the center of mass coordinates on the relevant two-by-two block of $\mathbf{J}^T \vec{\mathbf{W}} \mathbf{J}$ — that is, the block whose rows correspond to χ_1, χ_2 and whose columns correspond to \vec{X}_1, \vec{X}_2 . Making use of the definitions (C.16), one eventually finds

$$\begin{aligned} \Theta_i (M^{-1})^{ij} \Theta_j &= \frac{d\chi^2}{(m_{\text{gal}1} + m_{\text{gal}2})} + (\Theta_0, \Theta_\Psi) \begin{pmatrix} \tilde{\mathbf{C}} + \frac{1}{R} \delta \mathbf{C} & \frac{1}{R} \mathbf{L} \\ \frac{1}{R} \mathbf{L}^T & \mu H(R) \end{pmatrix}^{-1} \begin{pmatrix} \Theta_0 \\ \Theta_\Psi \end{pmatrix} \\ &\quad + O\left(\frac{y}{R^2}\right), \end{aligned} \tag{C.20}$$

with (Θ_0, Θ_Ψ) defined as in (4.18), (4.19).

C.1.1 Hyperkählerity of the metric

Here we address the hyperkählerity of the asymptotic metric (4.16) in the two-galaxy region of the strongly centered moduli space. We collect the position and phase coordinates using indices $\tilde{i}, \tilde{j} = 1, \dots, N-1$ and writing

$$\vec{y}^{\tilde{i}} = (\vec{y}^a, \vec{y}^p, \vec{R}) = (\vec{\mathbf{y}}, \vec{R}) , \quad \psi_{\tilde{i}} = (\psi_a, \psi_p, \Psi) , \tag{C.21}$$

the metric has the form

$$ds_0^2 = G_{\tilde{i}\tilde{j}} d\vec{y}^{\tilde{i}} \cdot d\vec{y}^{\tilde{j}} + (G^{-1})^{\tilde{i}\tilde{j}} \left(d\psi_{\tilde{i}} + \vec{V}_{\tilde{i}\tilde{k}} \cdot d\vec{y}^{\tilde{k}} \right) \left(d\psi_{\tilde{j}} + \vec{V}_{\tilde{j}\tilde{l}} \cdot d\vec{y}^{\tilde{l}} \right) , \tag{C.22}$$

where the matrices $\mathbf{G}, \vec{\mathbf{V}}$ are given by

$$\begin{aligned} \mathbf{G} &= \begin{pmatrix} \tilde{\mathbf{C}} + \frac{1}{R} \delta \mathbf{C} & \frac{1}{R} \mathbf{L} \\ \frac{1}{R} \mathbf{L}^T & \mu H(R) \end{pmatrix} , \\ \vec{\mathbf{V}} &= \begin{pmatrix} \vec{\mathbf{W}} + \delta \mathbf{C} \otimes \vec{w}(\vec{R}) & \mathbf{L} \otimes \vec{w}(\vec{R}) \\ \mathbf{L}^T \otimes \vec{w}(\vec{R}) & -(\gamma_{1,\text{m}}, \gamma_{2,\text{m}}) \vec{w}(\vec{R}) \end{pmatrix} , \end{aligned} \tag{C.23}$$

where $\vec{\mathbf{W}} = \text{diag}(\vec{\mathbf{W}}_1, \vec{\mathbf{W}}_2)$. Note that $\mathbf{G}, \vec{\mathbf{V}}$ are of the form

$$\mathbf{G} = \tilde{\mathbf{G}} + \frac{1}{R} \mathbf{A} , \quad \vec{\mathbf{V}} = \vec{\tilde{\mathbf{V}}} + \mathbf{A} \otimes \vec{w}(\vec{R}) , \tag{C.24}$$

where $\tilde{\mathbf{G}} = \text{diag}(\tilde{\mathbf{C}}, \mu)$, $\vec{\tilde{\mathbf{V}}} = \text{diag}(\vec{\mathbf{W}}, 0)$, and \mathbf{A} is a constant matrix.

A metric of the form (C.22) is hyperkähler iff (letting $\alpha, \beta, \gamma = 1, 2, 3$)

$$\frac{\partial}{\partial y^{\alpha\tilde{i}}} V_{\beta\tilde{j}\tilde{k}} - \frac{\partial}{\partial y^{\beta\tilde{j}}} V_{\alpha\tilde{i}\tilde{k}} = \epsilon_{\alpha\beta}{}^{\gamma} \frac{\partial}{\partial y^{\gamma\tilde{i}}} G_{\tilde{j}\tilde{k}} \quad \& \quad \frac{\partial}{\partial y^{\alpha\tilde{i}}} G_{\tilde{j}\tilde{k}} = \frac{\partial}{\partial y^{\alpha\tilde{j}}} G_{\tilde{i}\tilde{k}} . \quad (\text{C.25})$$

These equations are satisfied on the leading order pieces, $(\mathbf{G}, \mathbf{V}) \rightarrow (\tilde{\mathbf{G}}, \tilde{\mathbf{V}})$ because this just gives a direct product metric on $\mathcal{M}_{1,0} \times \mathcal{M}_{2,0} \times \mathbb{R}_{\text{rel}}^4$ with the corresponding GM/LWY hyperkähler metrics on the first two factors, (the strongly centered moduli spaces for galaxies one and two in isolation), and a flat metric on the third. Since \mathbf{A} is constant, the only derivatives that do not annihilate the correction term are those involving derivatives with respect to the components of \vec{R} . It follows that the equations are indeed satisfied to order $O(1/R^2)$, and hence the metric (4.16) is hyperkähler to the relevant order.

C.2 The Dirac Operator in the Two-galaxy Region

Now we will construct the asymptotic Dirac operator based on the metric (4.16) to order $O(1/R)$. We will employ the coordinates $(y^{\alpha\tilde{i}}, \psi_{\tilde{i}})$ as in (C.21), and we will sometimes combine these together into $y^{\mu\tilde{i}}$, introducing $\mu, \nu = 1, \dots, 4$ with $y^{4\tilde{i}} \equiv \psi_{\tilde{i}}$. We refer to the components of the hatted metric on \mathcal{M}_0 with respect to these coordinates as $G_{\mu\tilde{i}, \nu\tilde{j}}$ so that

$$ds_0^2 = G_{\mu\tilde{i}, \nu\tilde{j}} dy^{\mu\tilde{i}} dy^{\nu\tilde{j}} . \quad (\text{C.26})$$

We will use underlined indices to refer to the corresponding tangent space directions.

C.2.1 Vielbein

The nonzero components of the vielbein, $e^{\mu\tilde{i}}_{\mu\tilde{i}}$, are taken to be

$$e^{\alpha\tilde{i}}_{\alpha\tilde{i}} = \delta^{\alpha}_{\alpha} e^{\tilde{i}}_{\tilde{i}} , \quad e^{4\tilde{i}}_{\alpha\tilde{i}} = (e_{\alpha})^{\tilde{i}}_{\tilde{i}} , \quad e^{4\tilde{i}}_{4\tilde{i}} = (e_4)^{\tilde{i}}_{\tilde{i}} , \quad (\text{C.27})$$

where the matrices are given by

$$\begin{aligned}
\mathbf{e} &= \begin{pmatrix} \tilde{\mathbf{C}}^{1/2} + \frac{1}{2R}\tilde{\mathbf{C}}^{-1/2}\delta\mathbf{C} & \frac{1}{R}\tilde{\mathbf{C}}^{-1/2}\mathbf{L} \\ 0 & \mu^{1/2}\left(1 - \frac{1}{2\mu R}(\gamma_{m,1}, \gamma_{m,2})\right) \end{pmatrix}, \\
\mathbf{e}_\alpha &= \begin{pmatrix} \left(\tilde{\mathbf{C}}^{-1/2} - \frac{1}{2R}\tilde{\mathbf{C}}^{-1/2}\delta\mathbf{C}\tilde{\mathbf{C}}^{-1}\right)\mathcal{W}_\alpha + w_\alpha\tilde{\mathbf{C}}^{-1/2}\delta\mathbf{C} & w_\alpha\tilde{\mathbf{C}}^{-1/2}\mathbf{L} \\ \mu^{-1/2}w_\alpha\mathbf{L}^T & -\mu^{-1/2}(\gamma_{m,1}, \gamma_{m,2})w_\alpha \end{pmatrix}, \\
\mathbf{e}_4 &= \begin{pmatrix} \tilde{\mathbf{C}}^{-1/2} - \frac{1}{2R}\tilde{\mathbf{C}}^{-1/2}\delta\mathbf{C}\tilde{\mathbf{C}}^{-1} & -\frac{1}{\mu R}\tilde{\mathbf{C}}^{-1/2}\mathbf{L} \\ 0 & \mu^{-1/2}\left(1 + \frac{1}{2\mu R}(\gamma_{m,1}, \gamma_{m,2})\right) \end{pmatrix}. \tag{C.28}
\end{aligned}$$

Similarly, the components of the inverse vielbein, $E^{\mu\tilde{i}}_{\underline{\mu\tilde{i}}}$, are given by

$$E^{\alpha\tilde{i}}_{\underline{\alpha\tilde{i}}} = \delta^\alpha_{\underline{\alpha}} E^{\tilde{i}}_{\underline{\tilde{i}}}, \quad E^{4\tilde{i}}_{\underline{4\tilde{i}}} = \delta_{\underline{4}}^\alpha (E_\alpha)^{\tilde{i}}_{\underline{\tilde{i}}}, \quad E^{4\tilde{i}}_{\underline{4\tilde{i}}} = (E_4)^{\tilde{i}}_{\underline{\tilde{i}}}, \tag{C.29}$$

with

$$\begin{aligned}
\mathbf{E} &= \begin{pmatrix} \tilde{\mathbf{C}}^{-1/2} - \frac{1}{2R}\tilde{\mathbf{C}}^{-1}\delta\mathbf{C}\tilde{\mathbf{C}}^{-1/2} & -\frac{1}{\mu^{1/2}R}\tilde{\mathbf{C}}^{-1}\mathbf{L} \\ 0 & \mu^{-1/2}\left(1 + \frac{1}{2\mu R}(\gamma_{m,1}, \gamma_{m,2})\right) \end{pmatrix}, \\
\mathbf{E}_\alpha &= \begin{pmatrix} -\mathcal{W}_\alpha\left(\tilde{\mathbf{C}}^{-1/2} - \frac{1}{2R}\tilde{\mathbf{C}}^{-1}\delta\mathbf{C}\tilde{\mathbf{C}}^{-1/2}\right) - w_\alpha\delta\mathbf{C}\tilde{\mathbf{C}}^{-1/2} & \frac{1\mathcal{W}_\alpha\tilde{\mathbf{C}}^{-1}\mathbf{L}}{\mu^{1/2}R} - \frac{w_\alpha\mathbf{L}}{\mu^{1/2}} \\ -w_\alpha\mathbf{L}^T\tilde{\mathbf{C}}^{-1/2} & \mu^{-1/2}(\gamma_{m,1}, \gamma_{m,2})w_\alpha \end{pmatrix}, \\
\mathbf{E}_4 &= \begin{pmatrix} \tilde{\mathbf{C}}^{1/2} + \frac{1}{2R}\delta\mathbf{C}\tilde{\mathbf{C}}^{-1/2} & \frac{1}{\mu^{1/2}R}\mathbf{L} \\ 0 & \mu^{1/2}\left(1 - \frac{1}{2\mu R}(\gamma_{m,1}, \gamma_{m,2})\right) \end{pmatrix}. \tag{C.30}
\end{aligned}$$

These satisfy the necessary relations to the order we are working:

$$E^{\mu\tilde{i}}_{\underline{\mu\tilde{i}}} e^{\mu\tilde{i}}_{\nu\tilde{j}} = \delta^\mu_{\nu} \delta^{\tilde{i}}_{\tilde{j}} + O(1/R^2), \quad \delta_{\underline{\mu\tilde{i}}}^{\mu\tilde{i}} \delta_{\underline{\nu\tilde{j}}}^{\nu\tilde{j}} e^{\mu\tilde{i}}_{\mu\tilde{i}} e^{\nu\tilde{j}}_{\nu\tilde{j}} = G_{\mu\tilde{i}, \nu\tilde{j}} + O(1/R^2). \tag{C.31}$$

C.2.2 Spin connection

Using the vielbein above, one can compute the spin connection. The discussion is organized according to how many of the $\tilde{i}, \tilde{j}, \tilde{k}$ indices take the last value, $N-1$, corresponding to $\vec{y}^{N-1} = \vec{R}$. By a slight abuse of notation we will refer to this index value as “ $\tilde{i} = R$ ” rather than $\tilde{i} = N-1$. The remaining indices running over the $N-2$ relative positions $\vec{y} = (\vec{y}^a, \vec{y}^p)^T$ are $i, j, k = 1, \dots, N-2$. We give the expressions below with all indices referred to the frame on the tangent space.

When there are no \underline{R} indices, there is a leading $O(1)$ piece and $O(1/R)$ corrections to it:

$$\omega_{\underline{\mu}\underline{i}\underline{\nu}\underline{j},\underline{\rho}\underline{k}} = \tilde{\omega}_{\underline{\mu}\underline{i}\underline{\nu}\underline{j},\underline{\rho}\underline{k}} + \delta\omega_{\underline{\mu}\underline{i}\underline{\nu}\underline{j},\underline{\rho}\underline{k}} + \cdots, \quad (\text{C.32})$$

where $\delta\omega = O(1/R)$. Explicitly, the nonzero leading-order spin connection is found to be

$$\begin{aligned} \tilde{\omega}_{\underline{\alpha}\underline{i}\underline{\beta}\underline{j},\underline{\gamma}\underline{k}} &= \frac{1}{2}(\tilde{C}^{-1/2})_{\underline{i}}^i(\tilde{C}^{-1/2})_{\underline{j}}^j(\tilde{C}^{-1/2})_{\underline{k}}^k \left(\delta_{\underline{\alpha}\underline{\gamma}}\delta_{\underline{\beta}}^{\underline{\beta}}\partial_{\underline{\beta}\underline{j}}\tilde{C}_{ik} - \delta_{\underline{\beta}\underline{\gamma}}\delta_{\underline{\alpha}}^{\underline{\alpha}}\partial_{\underline{\alpha}i}\tilde{C}_{jk} \right) - \\ &\quad - \frac{1}{2}\delta_{\underline{\alpha}\underline{\beta}}\delta_{\underline{\gamma}}^{\underline{\gamma}}(\tilde{C}^{-1/2})_{\underline{k}}^k \left[(\partial_{\underline{\gamma}k}\tilde{C}^{1/2})\tilde{C}^{-1/2} - \tilde{C}^{-1/2}(\partial_{\underline{\gamma}k}\tilde{C}^{1/2}) \right]_{ij}, \\ \tilde{\omega}_{\underline{4}\underline{i}\underline{\beta}\underline{j},\underline{\gamma}\underline{k}} &= \frac{1}{2}(\tilde{C}^{-1/2})_{\underline{i}}^i(\tilde{C}^{-1/2})_{\underline{j}}^j(\tilde{C}^{-1/2})_{\underline{k}}^k \delta_{\underline{\beta}}^{\underline{\beta}}\delta_{\underline{\gamma}}^{\underline{\gamma}} (\partial_{\underline{\beta}j}\mathcal{W}_{\underline{\gamma}ik} - \partial_{\underline{\gamma}k}\mathcal{W}_{\underline{\beta}ij}), \\ \tilde{\omega}_{\underline{4}\underline{i}\underline{4}\underline{j},\underline{\gamma}\underline{k}} &= -\frac{1}{2}(\tilde{C}^{-1/2})_{\underline{k}}^k \delta_{\underline{\gamma}}^{\underline{\gamma}} \left[(\partial_{\underline{\gamma}k}\tilde{C}^{1/2})\tilde{C}^{-1/2} - \tilde{C}^{-1/2}(\partial_{\underline{\gamma}k}\tilde{C}^{1/2}) \right]_{ij}, \\ \tilde{\omega}_{\underline{\alpha}\underline{i}\underline{\beta}\underline{j},\underline{4}\underline{k}} &= -\frac{1}{2}(\tilde{C}^{-1/2})_{\underline{i}}^i(\tilde{C}^{-1/2})_{\underline{j}}^j(\tilde{C}^{-1/2})_{\underline{k}}^k \delta_{\underline{\alpha}}^{\underline{\alpha}}\delta_{\underline{\beta}}^{\underline{\beta}} (\partial_{\underline{\alpha}i}\mathcal{W}_{\underline{\beta}kj} - \partial_{\underline{\beta}j}\mathcal{W}_{\underline{\alpha}ki}), \\ \tilde{\omega}_{\underline{\alpha}\underline{i}\underline{4}\underline{j},\underline{4}\underline{k}} &= \frac{1}{2}(\tilde{C}^{-1/2})_{\underline{i}}^i(\tilde{C}^{-1/2})_{\underline{j}}^j(\tilde{C}^{-1/2})_{\underline{k}}^k \partial_{\underline{\alpha}i}\tilde{C}_{jk}. \end{aligned} \quad (\text{C.33})$$

This is the spin connection on $\mathcal{M}_{1,0} \times \mathcal{M}_{2,0} \times \mathbb{R}_{\text{rel}}^4$, with the \mathcal{M}_0 factors equipped with their respective GM/LWY metrics. The $O(1/R)$ corrections to the above components are captured by the simple replacement rule

$$\tilde{\mathbf{C}} \rightarrow \mathbf{C} = \tilde{\mathbf{C}} + \frac{1}{R}\delta\mathbf{C}, \quad (\text{C.34})$$

leaving \mathbf{W}_{α} unchanged, (and expanding the result to first order in $1/R$). The reason this captures all $O(1/R)$ corrections to these components of the spin connection is that the contributions from the w_{α} terms in the vielbein cancel.

Note that the components involving the $\mathcal{W}_{\alpha ij}$ can be simplified using relations (C.25):

$$\tilde{\omega}_{\underline{\alpha}\underline{i}\underline{4}\underline{j},\underline{\beta}\underline{k}} = \tilde{\omega}_{\underline{\alpha}\underline{i}\underline{\beta}\underline{j},\underline{4}\underline{k}} = -\frac{1}{2}(\tilde{C}^{-1/2})_{\underline{i}}^i(\tilde{C}^{-1/2})_{\underline{j}}^j(\tilde{C}^{-1/2})_{\underline{k}}^k \epsilon_{\underline{\alpha}\underline{\beta}}^{\underline{\gamma}} \delta_{\underline{\gamma}}^{\underline{\gamma}} \partial_{\underline{\gamma}i}\tilde{C}_{jk}. \quad (\text{C.35})$$

Furthermore, this relation can be extended to the $O(1/R)$ corrections to these components using the replacement (C.34). The reason is that (C.34) is simply a shift of \tilde{C} by a constant as far as the $y^{\alpha i}$ are concerned, and \tilde{C} is always differentiated in the relations (C.25).

Next we consider the components of the spin connection with one \underline{R} index that have a non-vanishing $O(1/R)$ piece. With some effort they can all be related to the (C.33) in a rather simple way:

$$\begin{aligned}
\omega_{\underline{\alpha i \beta j}, \underline{\gamma R}} &= -\frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{k}} \tilde{\omega}_{\underline{\alpha i \beta j}, \underline{\gamma k}} + O(1/R^2) , \\
\omega_{\underline{\alpha i \beta R}, \underline{\gamma k}} &= -\frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{j}} \tilde{\omega}_{\underline{\alpha i \beta j}, \underline{\gamma k}} - \frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2})^{\underline{k}}_{\underline{k}} \delta_{\underline{\alpha \beta}} \delta_{\underline{\gamma}}^{\underline{\gamma}} \partial_{\gamma k} (\tilde{C}^{-1/2} L)_{\underline{i}} \\
&\quad + O(1/R^2) , \\
\omega_{\underline{4 i \beta j}, \underline{\gamma R}} &= -\frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{k}} \tilde{\omega}_{\underline{4 i \beta j}, \underline{\gamma k}} + O(1/R^2) , \\
\omega_{\underline{4 i \beta R}, \underline{\gamma k}} &= -\frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{j}} \tilde{\omega}_{\underline{4 i \beta j}, \underline{\gamma k}} + O(1/R^2) , \\
\omega_{\underline{\alpha i \beta R}, \underline{4 k}} &= -\frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{j}} \tilde{\omega}_{\underline{\alpha i \beta j}, \underline{4 k}} + O(1/R^2) , \\
\omega_{\underline{4 i \beta R}, \underline{4 k}} &= -\frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{j}} \tilde{\omega}_{\underline{4 i \beta j}, \underline{4 k}} + O(1/R^2) , \\
\omega_{\underline{4 i 4 j}, \underline{\gamma R}} &= -\frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{k}} \tilde{\omega}_{\underline{4 i 4 j}, \underline{\gamma k}} + O(1/R^2) .
\end{aligned} \tag{C.36}$$

Finally, one can show that components of the spin connection with two or three \underline{R} indices start at $O(1/R^2)$, and thus we can neglect them to the order we are working.

Now introduce gamma matrices $\Gamma^{\underline{\mu \tilde{i}}}$ satisfying the Clifford algebra

$$[\Gamma^{\underline{\mu \tilde{i}}}, \Gamma^{\underline{\nu \tilde{j}}}]_+ = 2\delta^{\underline{\mu \nu}} \delta^{\underline{\tilde{i} \tilde{j}}} , \tag{C.37}$$

and define $\Gamma^{\underline{\mu \tilde{i} \nu \tilde{j}}} = \frac{1}{2}[\Gamma^{\underline{\mu \tilde{i}}}, \Gamma^{\underline{\nu \tilde{j}}}]$ as usual. When we contract the spin connection components with the gamma matrices to construct the Dirac operator, we can absorb almost all effects of the ω with one \underline{R} index by introducing shifted gamma matrices:

$$\begin{aligned}
\Gamma^{\underline{\rho \tilde{k}}} \omega_{\underline{\mu \tilde{i} \nu \tilde{j}, \rho \tilde{k}}} \Gamma^{\underline{\mu \tilde{i} \nu \tilde{j}}} &= \Gamma^{\underline{\rho k}} \omega_{\underline{\mu \tilde{i} \nu \tilde{j}, \rho k}} \Gamma^{\underline{\mu \tilde{i} \nu \tilde{j}}} - 2\Gamma^{\underline{\gamma k}} \Gamma^{\underline{\alpha i \beta R}} \frac{1}{\sqrt{\mu R}} \delta_{\underline{\alpha \beta}} (\tilde{C}^{-1/2})^{\underline{k}}_{\underline{k}} \delta_{\underline{\gamma}}^{\underline{\gamma}} \partial_{\gamma k} (\tilde{C}^{-1/2} L)_{\underline{i}} \\
&\quad + O(1/R^2) ,
\end{aligned} \tag{C.38}$$

with

$$\Gamma^{\underline{\alpha i}} = \Gamma^{\underline{\alpha i}} - \frac{1}{\sqrt{\mu R}} (\tilde{C}^{-1/2} L)^{\underline{i}} \Gamma^{\underline{\alpha R}} , \quad \Gamma^{\underline{4 i}} = \Gamma^{\underline{4 i}} . \tag{C.39}$$

We also account for the $O(1/R)$ corrections to the $\omega_{\underline{\mu \tilde{i} \nu \tilde{j}, \rho \tilde{k}}}$ by working with the corrected \mathbf{C} as we discussed around (C.34). So above we account for both types of $O(1/R)$

corrections to the spin connection by working with $\omega_{\alpha i \beta j, \gamma k}$ which is expressed in terms of \mathbf{C} rather than $\tilde{\mathbf{C}}$, and working with the $\Gamma^{\alpha i}$.

These hatted gamma matrices can be realized by a frame rotation on the spin bundle. To do so we first complete the definition of the $\Gamma^{\underline{\mu i}}$ by setting

$$\Gamma^{\alpha R} = \Gamma^{\underline{\alpha R}} + \frac{1}{\sqrt{\mu}R}(\tilde{C}^{-1/2}L)_{\underline{i}}\Gamma^{\alpha i}, \quad \Gamma^{4R} = \Gamma^{\underline{4R}}. \quad (\text{C.40})$$

Then $\Gamma^{\underline{\mu i}} = \mathcal{R}_{\underline{\nu j}}^{\underline{\mu i}} \Gamma^{\underline{\nu j}}$, where the rotation is given by

$$\mathcal{R} = \prod_{\alpha, i} \mathcal{R}_{\theta^i}^{(\alpha)}, \quad (\text{C.41})$$

where $\mathcal{R}_{\theta^i}^{(\alpha)}$ is a local rotation in the $e^{\alpha i}$ - $e^{\alpha R}$ plane by angle

$$\theta^i = \frac{1}{\sqrt{\mu}R}(\tilde{C}^{-1/2}L)^i, \quad (\text{C.42})$$

and we work to linear order in the θ^i . This rotation on the frame index can in turn be implemented through an adjoint action on the spinor indices,

$$\Gamma^{\underline{\mu i}} = \mathcal{R}_{\underline{\nu j}}^{\underline{\mu i}} \Gamma^{\underline{\nu j}} = \Lambda \Gamma^{\underline{\mu i}} \Lambda^{-1}. \quad (\text{C.43})$$

with

$$\Lambda = \prod_{\alpha, i} \exp(al f \theta^i \Gamma^{\alpha i} \Gamma^{\alpha R}) = \mathbb{1} + al f \theta^i \Gamma_{\alpha i} \Gamma^{\alpha R} + O(\theta^2). \quad (\text{C.44})$$

C.2.3 Dirac operator

To construct the Dirac operator, $\not{D}^{\mathcal{Y}_0}$, on \mathcal{M}_0 we will also need $\not{C}(\mathcal{Y}_0)$. We first have

$$\begin{aligned} G(\mathcal{Y}_0) &= \sum_{I=1}^{\text{rk g}} \langle \alpha_I, \mathcal{Y}_0 \rangle G(h^I) = \sum_I \langle \alpha_I, \mathcal{Y}_0 \rangle p^I \sum_{k_I=1}^{n_m^I} \frac{\partial}{\partial \xi_{k_I}^I} + \{\text{exp. small}\} \\ &= \sum_I (H_I, \mathcal{Y}_0) \sum_{k_I=1}^{n_m^I} \frac{\partial}{\partial \xi_{k_I}^I} + \{\text{exp. small}\} \\ &= \sum_{a=1}^{N_1} (H_{I(a)}, \mathcal{Y}_0) \frac{\partial}{\partial \xi^a} + \sum_{p=N_1+1}^N (H_{I(p)}, \mathcal{Y}_0) \frac{\partial}{\partial \xi^p} + \{\text{exp. small}\}, \end{aligned} \quad (\text{C.45})$$

where we used that the exact triholomorphic Killing vectors $G(h^I)$ approach the linear combinations of those in the GM/LWY metric exponentially fast (in the same sense

that exact metric approaches the GM/LWY metric exponentially fast). Then using (C.6), (C.14), (C.15), followed by the change to center of mass coordinates (4.15), we find

$$\begin{aligned} G(\mathcal{Y}_0) &= \sum_a \langle \beta_a, \mathcal{Y}_0 \rangle \frac{\partial}{\partial \psi^a} + \sum_p \langle \beta_p, \mathcal{Y}_0 \rangle \frac{\partial}{\partial \psi^p} + (\gamma_{m,1}, \mathcal{Y}_0) \frac{\partial}{\partial \chi_1} + (\gamma_{m,2}, \mathcal{Y}_0) \frac{\partial}{\partial \chi_2} + \text{exp. small} \\ &= \sum_i \langle \beta_i, \mathcal{Y}_0 \rangle \frac{\partial}{\partial \psi^i} + (\gamma_{m,1}, \mathcal{Y}_0) \frac{\partial}{\partial \Psi} + \text{exp. small} , \end{aligned} \quad (\text{C.46})$$

where in the last step we also used that $(\gamma_m, \mathcal{Y}_0) = 0$, or equivalently $(\gamma_{m,1}, \mathcal{Y}_0) = -(\gamma_{m,2}, \mathcal{Y}_0)$.

Now we can compute

$$\mathcal{D}^{\mathcal{Y}_0} = \Gamma^{\rho\tilde{k}} \underline{E}_{\rho\tilde{k}}^{\rho\tilde{k}} \partial_{\rho\tilde{k}} + \frac{1}{4} \Gamma^{\rho\tilde{k}} \omega_{\mu\tilde{\nu}\tilde{j},\rho\tilde{k}} \Gamma^{\mu\tilde{\nu}\tilde{j}} - i \Gamma_{\rho\tilde{k}} e^{\rho\tilde{k}}_{\rho\tilde{k}} G(\mathcal{Y}_0)^{\rho\tilde{k}} , \quad (\text{C.47})$$

to $O(1/R^2)$ using (C.28), (C.30), (C.38), and (C.46). Given the simplifications in the spin connection afforded by (C.38), the goal will be to express everything in terms of the rotated Γ 's and then use (C.43). The final result will be an expression for $\mathcal{D}^{\mathcal{Y}_0}$, through to $O(1/R^2)$, in terms of the Λ -conjugation of another Dirac-type operator. The advantage of this approach is that the Λ -conjugation, which implements the frame rotation on the spin bundle, is sufficient to block-diagonalize the Dirac operator with respect to the $\Gamma^{\underline{\mu i}} - \Gamma^{\underline{\mu R}}$ decomposition of the Dirac spinor bundle. A key point is that the ‘extra’ term on the right-hand side of (C.38), which originates from the inhomogeneous term in the second line of (C.36), is exactly what is needed to account for the action of the derivative on Λ :

$$\begin{aligned} \Lambda \left[\Gamma^{\gamma k} \underline{E}_{\gamma k}^{\gamma k} \partial_{\gamma k} \right] \Lambda^{-1} &= \Gamma^{\gamma k} \underline{E}_{\gamma k}^{\gamma k} \partial_{\gamma k} \\ &\quad - \frac{1}{2} \Gamma^{\gamma k} \Gamma^{\alpha i \beta R} \frac{1}{\sqrt{\mu R}} \delta_{\alpha\eta} (\tilde{C}^{-1/2})_{\underline{k}}^k \delta_{\underline{\gamma}}^{\gamma} \partial_{\gamma k} (\tilde{C}^{-1/2} L)_{\underline{i}} + O(1/R^2) . \end{aligned} \quad (\text{C.48})$$

Thus, suppressing the details, we are able to bring the Dirac operator to the form quoted in the text:

$$\mathcal{D}^{\mathcal{Y}_0} = \Lambda \left(\mathcal{D}_{12}^{\mathcal{Y}_0} + \mathcal{D}_{\text{rel}}^{\mathcal{Y}_0} + O(1/R^2) \right) \Lambda^{-1} , \quad (\text{C.49})$$

where $\mathcal{D}_{12}^{\mathcal{Y}_0}$ consists of terms involving only $\underline{\mu}i$ -type gamma matrices, and \mathcal{D}_{rel} consists of terms involving only $\underline{\mu}R$ -type gamma matrices. For the first operator we have

$$\mathcal{D}_{12} = \mathcal{D}_{\mathcal{M}_{1,0} \times \mathcal{M}_{2,0}}^{\mathcal{Y}_0} + (\mathcal{D}_{12}^{\mathcal{Y}_0})^{(1)}, \quad (\text{C.50})$$

where the first term is precisely the $G(\mathcal{Y}_0)$ -twisted Dirac operator on $\mathcal{M}_{1,0} \times \mathcal{M}_{2,0}$, where each factor equipped with the GM/LWY metric, and the second term contains the $O(1/R)$ corrections. Explicitly,

$$\begin{aligned} \mathcal{D}_{\mathcal{M}_{1,0} \times \mathcal{M}_{2,0}}^{\mathcal{Y}_0} = & \Gamma^{\underline{\alpha}i} \delta_{\underline{\alpha}}^{\alpha} \left[(\tilde{C}^{-1/2})^i_{\underline{i}} \partial_{\alpha i} - (\mathcal{W}_{\alpha} \tilde{C}^{-1/2})^i_{\underline{i}} \partial_{4i} \right] + \Gamma^{\underline{4}i} (\tilde{C}^{1/2})^i_{\underline{i}} \partial_{4i} \\ & + \frac{1}{4} \Gamma^{\rho k} \tilde{\omega}_{\underline{\mu}i\nu j, \rho k} \Gamma^{\underline{\mu}i\nu j} - i \Gamma^{\underline{4}i} (\tilde{C}^{-1/2})^i_{\underline{i}} (\beta_i, \mathcal{Y}_0), \end{aligned} \quad (\text{C.51})$$

and

$$\begin{aligned} (\mathcal{D}_{12}^{\mathcal{Y}_0})^{(1)} = & \Gamma^{\underline{\alpha}i} \delta_{\underline{\alpha}}^{\alpha} \left\{ -\frac{1}{2R} (\tilde{C}^{-1} \delta C \tilde{C}^{-1/2})^i_{\underline{i}} \partial_{\alpha i} + ((\frac{1}{2R} \mathcal{W}_{\alpha} \tilde{C}^{-1} - w_{\alpha} \mathbb{1}) \delta C \tilde{C}^{-1/2})^i_{\underline{i}} \partial_{4i} \right. \\ & \left. - w_{\alpha} (L^T \tilde{C}^{-1/2})^i_{\underline{i}} \partial_{4R} - \frac{1}{\mu R} (\tilde{C}^{-1/2} L)^i_{\underline{i}} \partial_{\alpha R} \right\} \\ & + \Gamma^{\underline{4}i} \left\{ \frac{1}{2R} (\tilde{C}^{-1/2} \delta C)^i_{\underline{i}} \partial_{4i} - \frac{1}{2R} (\tilde{C}^{-1/2} \delta C \tilde{C}^{-1})^i_{\underline{i}} (\beta_i, \mathcal{Y}_0) \right. \\ & \left. + i \frac{(\gamma_{m,1}, \mathcal{Y}_0)}{\mu R} (\tilde{C}^{-1/2} L)^i_{\underline{i}} \right\} + \frac{1}{4} \Gamma^{\rho k} \delta \omega_{\underline{\mu}i\nu j, \rho k} \Gamma^{\underline{\mu}i\nu j}. \end{aligned} \quad (\text{C.52})$$

Meanwhile the second operator takes the form

$$\begin{aligned} \mathcal{D}_{\text{rel}}^{\mathcal{Y}_0} = & \frac{1}{\sqrt{\mu}} \left(1 + \frac{(\gamma_{m,1}, \gamma_{m,2})}{2\mu R} \right) \left\{ \Gamma^{\underline{\alpha}R} \delta_{\underline{\alpha}}^{\alpha} [\partial_{\alpha R} + (\gamma_{m,1}, \gamma_{m,2}) w_{\alpha} \partial_{4R} - w_{\alpha} L^i \partial_{4i}] \right. \\ & \left. + \Gamma^{\underline{4}R} \left[\left(\mu - \frac{(\gamma_{m,1}, \gamma_{m,2})}{R} \right) \partial_{4R} + \frac{L^i}{R} \partial_{4i} - i(\gamma_{m,1}, \mathcal{Y}_0) \right] \right\}. \end{aligned} \quad (\text{C.53})$$

(Strictly speaking, this expression contains some $O(1/R^2)$ terms when the R^{-1} multiplies w_{α} which should be dropped.)

C.3 Singular Monopole Moduli Space

In the case of singular monopoles, we have a core-halo system. Here we can choose our origin to be anywhere in the core, but to be explicit we choose it to be centered on one of the singular monopoles. In this case we need only go to center of mass and relative coordinates in the halo galaxy. We let indices $a, b = 1, \dots, N_{\text{core}}$ run over fundamental

(mobile) constituents in the core and indices $p, q = N_{\text{core}} + 1, \dots, N_{\text{core}} + N_{\text{halo}} = N$, run over fundamental constituents in the halo galaxy. We set

$$\vec{R} = \frac{\sum_p m_p \vec{x}^p}{m_{\text{halo}}} , \quad \vec{y}^p = \vec{x}^p - \vec{x}^{p+1} , \quad (\text{C.54})$$

where $m_{\text{halo}} = \sum_p m_p$ and we have introduced indices $p, q = N_{\text{core}} + 1, \dots, N - 1$ that run over the relative positions of halo constituents. The inverse is given by

$$\vec{x}^p = \vec{R} + (\mathbf{j}_h)^p_q \vec{y}^q , \quad (\text{C.55})$$

where \mathbf{j}_h has an identical form to \mathbf{j}_2 with the galaxy-two constituent masses replaced by halo constituent masses. Constructing \mathbf{J}_h by appending a column of 1's in the same way, we introduce the halo fiber coordinates

$$\begin{pmatrix} \psi_p \\ \Psi \end{pmatrix} = \mathbf{J}_h^T(\xi_p) . \quad (\text{C.56})$$

Note that (\vec{R}, Ψ) play the role here that was previously played by (\vec{X}_2, χ_2) . They parameterize the position of the center of mass of the halo galaxy relative to the *fixed* core.

The large R expansion of the quadratic form \overline{M}_{ij} , can be written in block form

$$(\overline{M})_{ij} = \begin{pmatrix} (\overline{M})_{ab} & \frac{D_{aq}}{R} + O\left(\frac{1}{R^2}\right) \\ \frac{D_{pb}}{R} + O\left(\frac{1}{R^2}\right) & (M)_{pq} , \end{pmatrix} \quad (\text{C.57})$$

with

$$(\overline{M})_{ab} = (\overline{M}_c)_{ab} - \delta_{ab} \frac{(H_{I(a)}, \gamma_{h,m})}{R} + O\left(\frac{1}{R^2}\right) , \quad (\text{C.58})$$

where

$$(\overline{M}_c)_{ab} = \begin{cases} m_a - \sum_{c \neq a} \frac{D_{ac}}{r_{ac}} - \sum_{n=1}^{N_{\text{def}}} \frac{(P_n, H_I(a))}{r_{na}} & a = b \\ \frac{D_{ab}}{r_{ab}} & a \neq b \end{cases} , \quad (\text{C.59})$$

and similarly

$$(M)_{pq} = (M_h)_{pq} - \delta_{pq} \frac{(H_{I(p)}, \gamma_{c,m})}{R} + O\left(\frac{1}{R^2}\right) , \quad (\text{C.60})$$

where

$$(M_h)_{pq} = \begin{cases} m_p - \sum_{u \neq p} \frac{D_{pu}}{r_{pu}} & p = u \\ \frac{D_{pq}}{r_{pq}} & p \neq q \end{cases} . \quad (\text{C.61})$$

In these expressions the core and halo magnetic charges are given by

$$\gamma_{c,m} = \sum_{a=1}^{N_{\text{core}}} H_{I(a)} + \sum_{n=1}^{N_{\text{def}}} P_n, \quad \gamma_{h,m} = \sum_{p=N_{\text{core}}+1}^N H_{I(p)}. \quad (\text{C.62})$$

Therefore in the limit of large R we can write $(\overline{M})_{ij}$ as

$$\begin{aligned} \overline{M}_{ij} = & \begin{pmatrix} (\overline{M}_c)_{ab} & 0 \\ 0 & (M_h)_{pq} \end{pmatrix} + \frac{1}{R} \begin{pmatrix} -\delta_{ab}(\gamma_{h,m}, H_{I(a)}) & D_{aq} \\ D_{bp} & -\delta_{pq}(\gamma_{c,m}, H_{I(p)}) \end{pmatrix} \\ & + O(1/R^2). \end{aligned} \quad (\text{C.63})$$

As before, we will write this as

$$\overline{\mathbf{M}} = \begin{pmatrix} \overline{\mathbf{M}}_c & 0 \\ 0 & \mathbf{M}_h \end{pmatrix} - \frac{1}{R} \begin{pmatrix} \mathbf{D}_{cc} & -\mathbf{D}_{ch} \\ -\mathbf{D}_{hc} & \mathbf{D}_{hh} \end{pmatrix} + O(1/R^2) \quad (\text{C.64})$$

Now we make the similarity transformation to the center of mass and relative coordinates in the halo, defining \mathbf{C}_h with components $(C_h)_{pq}$ such that

$$\mathbf{J}_h^T \mathbf{M}_h \mathbf{J}_h = \begin{pmatrix} \mathbf{j}_h^T \mathbf{M}_h \mathbf{j}_h & 0 \\ 0^T & m_{\text{halo}} \end{pmatrix} = \begin{pmatrix} \mathbf{C}_h & 0 \\ 0^T & m_{\text{halo}} \end{pmatrix}. \quad (\text{C.65})$$

We combine \mathbf{C}_h with the leading order core matrix, writing

$$\overline{\mathbf{C}} = \begin{pmatrix} \overline{\mathbf{M}}_c & 0 \\ 0 & \mathbf{C}_h \end{pmatrix}. \quad (\text{C.66})$$

The first order corrections to this are captured by

$$\delta \overline{\mathbf{C}} = \begin{pmatrix} -\mathbf{D}_{cc} & \mathbf{D}_{ch} \mathbf{j}_h \\ \mathbf{j}_h^T \mathbf{D}_{hc} & -\mathbf{j}_h^T \mathbf{D}_{hh} \mathbf{j}_h \end{pmatrix}. \quad (\text{C.67})$$

Finally the vector $\overline{\mathbf{L}}$ and the harmonic function $\overline{H}(R)$ appearing in (4.58) are

$$\overline{\mathbf{L}} = \begin{pmatrix} -(H_{I(a)}, \gamma_{h,m}) \\ \langle \beta_p, \gamma_{c,m} \rangle \end{pmatrix}, \quad \overline{H}(R) = \left(1 - \frac{(\gamma_{h,m}, \gamma_{c,m})}{m_{\text{halo}} R} \right). \quad (\text{C.68})$$

Here β_p is defined as in (C.15) but with \mathbf{J}_2 replaced by \mathbf{J}_h .

The terms in (4.58) involving the connection one-forms $(\overline{\Theta}_0, \overline{\Theta}_\psi)$ can be obtained by following analogous steps to those in the case of smooth monopoles.

Appendix D

SQM Localization

D.1 $U(1)$ $\mathcal{N} = (0, 4)$ SQM Analysis

In this appendix we will present a general analysis of the $\mathcal{N} = (0, 4)$ SQM with a $U(1)$ gauge group coming from monopole bubbling in 4D $SU(2)$ $\mathcal{N} = 2$, asymptotically free, supersymmetric gauge theories. See [165] for a review of $\mathcal{N} = (0, 4)$ SUSY.

The $\mathcal{N} = (0, 4)$ SQM we are considering consists of a $U(1)$ vector multiplet, two fundamental hypermultiplets and up to four $\mathcal{N} = (0, 2)$ Fermi-multiplets $\Psi^{(i)}$, that have been embedded into the $\mathcal{N} = (0, 4)$ theory by either embedding $\Psi_{\mathcal{N}=(0,2)}^{(i)} \hookrightarrow \tilde{\Psi}_{\mathcal{N}=(0,4)}^{(i)}$ as short $\mathcal{N} = (0, 4)$ Fermi-multiplets or by combining $(\Psi_{\mathcal{N}=(0,2)}^{(i)} \oplus \bar{\Psi}_{\mathcal{N}=(0,2)}^{(i+2)}) = \hat{\Psi}_{\mathcal{N}=(0,4)}$ to make long $\mathcal{N} = (0, 4)$ Fermi-multiplets. In the case of a 4D theory with N_f fundamental hypermultiplets we will have N_f short $\mathcal{N} = (0, 4)$ fundamental Fermi-multiplets. In the case of the 4D $SU(2)$ $\mathcal{N} = 2^*$ theory, the multiplets form $\mathcal{N} = (4, 4)$ SUSY multiplets which are then mass deformed to the $\mathcal{N} = (0, 4)$ theory. This means that in studying the $\mathcal{N} = 2^*$ theory, we must include a massive $\mathcal{N} = (0, 4)$ adjoint (twisted) hypermultiplet Γ and two long fundamental Fermi-multiplets in the SQM.

In general, these SQMs have a Lagrangian given by

$$L = L_{univ} + L_{theory} . \tag{D.1}$$

Here L_{theory} is the part of the Lagrangian that is dependent on the details of the 4D $\mathcal{N} = 2$ theory, and L_{univ} is the universal part of the Lagrangian that does not change

This Appendix is based on material from my publication [26].

with the matter content of the 4D Theory. Here we have

$$L_{univ} = L_{vec} + L_{hyp} + L_{FI} , \quad (D.2)$$

where

$$\begin{aligned} L_{vec} &= \frac{1}{e^2} \left[\frac{1}{2} (\partial_t \sigma)^2 + i \bar{\lambda}_A \partial_t \lambda^A + \frac{1}{2} D^2 + |F|^2 \right] , \\ L_{hyp} &= \frac{1}{e^2} \left(|D_t \phi^{A,f}|^2 - |\sigma \phi^{A,f}|^2 - \bar{\phi}_{A,f} (\sigma^r)^A_B M_r \phi^{B,f} \right) \\ &\quad + \frac{i}{2e^2} \left(\bar{\psi}_1^f (D_t + i\sigma) \psi_{1,f} + \psi_{1,f} (\tilde{D}_t - i\sigma) \bar{\psi}_1^f + \bar{\psi}_2^f (\tilde{D}_t - i\sigma) \psi_{2,f} + \psi_{2,f} (D_t + i\sigma) \bar{\psi}_2^f \right) \\ &\quad + \frac{i}{\sqrt{2}e^2} (\bar{\phi}_{A,f} \lambda^A \psi_1^f - \bar{\psi}_{1,f} \bar{\lambda}_A \phi^{A,f} + \bar{\phi}_{A,f} \bar{\lambda}^A \bar{\psi}_2^f - \psi_{2,f} \lambda_A \phi^{A,f}) , \\ L_{FI} &= -\xi D , \\ M_r &= (f , g , D) \quad , \quad r = 1, 2, 3 \quad , \quad f = \frac{1}{\sqrt{2}} (F + \bar{F}) \quad , \quad g = \frac{1}{\sqrt{2}i} (F - \bar{F}) . \end{aligned} \quad (D.3)$$

Here $A = 1, 2$ is an $SU(2)_R$ index, $f = 1, 2$ is an index for the global $SU(2)$ flavor symmetry, e is the gauge coupling, and $\sigma^{r=1,2,3}$ are the Pauli matrices acting on the $SU(2)_R$ indices. Here we use the convention

$$(\phi^A)^* = \bar{\phi}_A \quad , \quad \phi_A = \epsilon_{AB} \phi^B \quad , \quad \epsilon_{21} = \epsilon^{12} = 1 . \quad (D.4)$$

We additionally have that L_{theory} can have contributions from terms

$$L_{theory} = L_{Fermi} + L_{adj\,hyp} , \quad (D.5)$$

which are of the form

$$\begin{aligned} L_{Fermi} &= \frac{1}{e^2} \sum_j \left[\frac{i}{2} (\bar{\eta}_j (D_t + i\sigma) \eta_j + \eta_j (D_t - i\sigma) \bar{\eta}_j) + |G_j|^2 + m_j [\bar{\eta}_j, \eta_j] \right] , \\ L_{adj\,hyper} &= \frac{1}{e^2} \left[|\partial_t \rho^A|^2 + \frac{i}{2} \sum_{I=1}^2 (\bar{\chi}_I \partial_t \chi_I + \chi_I \partial_t \bar{\chi}_I) \right] , \end{aligned} \quad (D.6)$$

where in the case of a 4D theory with N_f fundamental hypermultiplets, we only include N_f fundamental Fermi multiplets and in the case of the $\mathcal{N} = 2^*$ theory we include both the adjoint hypermultiplet and 4 (short) fundamental Fermi multiplets.¹

¹Note that four short fundamental Fermi-multiplets is equivalent to two long fundamental Fermi-multiplets.

In the following analysis we will decompose the $\mathcal{N} = (0, 4)$ hypermultiplet that transforms in a quaternionic representation \mathcal{R} into two $\mathcal{N} = (0, 2)$ chiral multiplets transforming in conjugate representations $R \oplus \bar{R}$:

$$\text{Fundamental Hypermultiplet } \Phi = (\phi^A, \psi_I)_{\mathcal{R}} = \Phi_1 \oplus \bar{\Phi}_2 = (\phi, \psi)_R \oplus (\tilde{\phi}, \tilde{\psi})_{\bar{R}}, \quad (\text{D.7})$$

where $I = 1, 2$.

In this notation, the field content of this theory is given by

Lagrangian Term	Multiplet	Fields	Q_{Gauge}	Q_a	Q_ϵ	F_j
Universal	$\mathcal{N} = (0, 4)$ Vector-	σ	0	0	0	0
		λ^1	0	0	0	0
		λ^2	0	0	4	0
Universal	$\mathcal{N} = (0, 4)$ Fund. Hyper-	ϕ_1	1	-1	1	0
		$\tilde{\phi}_1$	-1	1	1	0
		ϕ_2	1	1	1	0
		$\tilde{\phi}_2$	-1	-1	1	0
		ψ_1	1	-1	1	0
		$\tilde{\psi}_1$	-1	1	1	0
		ψ_2	1	1	1	0
		$\tilde{\psi}_2$	-1	-1	1	0
4D N_f Fund. Hyper-	$\mathcal{N} = (0, 4)$ Fund. Fermi-	η_j	1	0	0	-2
$\mathcal{N} = 2^*$ Theory	$\mathcal{N} = (0, 4)$ Adjoint Twisted Chiral	ρ	0	0	1	1
		$\tilde{\rho}$	0	0	-1	1
		χ	0	0	0	1
		$\tilde{\chi}$	0	0	0	-1
$\mathcal{N} = 2^*$ Theory	$\mathcal{N} = (0, 4)$ Fund. Fermi	η_1	1	-1	1	0
		$\tilde{\eta}_1$	-1	1	1	0
		η_2	1	1	1	0
		$\tilde{\eta}_2$	-1	-1	1	0

This non- $SU(2)_R$ invariant notation is explicitly related to the $SU(2)_R$ -invariant notation by

$$\phi^{A,f} = \begin{pmatrix} \phi^f \\ \tilde{\phi}^f \end{pmatrix} \quad , \quad \psi_I^f = \begin{pmatrix} \psi^f \\ \tilde{\psi}^f \end{pmatrix} \quad , \quad \rho^A = \begin{pmatrix} \rho \\ \bar{\rho} \end{pmatrix} \quad , \quad \chi_I = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix} . \quad (\text{D.8})$$

Here we will use the convention $\epsilon^{12} = \epsilon_{21} = 1$ and will use a notation that is not flavor $SU(2)$ -invariant by mapping

$$\phi^f, \tilde{\phi}^f, \psi^f, \tilde{\psi}^f \mapsto \phi_f, \tilde{\phi}_f, \psi_f, \tilde{\psi}_f , \quad (\text{D.9})$$

so that the $\{\phi_f, \tilde{\phi}_f, \psi_f, \tilde{\psi}_f\}$ should be understood as normal complex valued scalar and fermion fields. It should be understood that all flavor and $SU(2)_R$ indices are contracted properly in the upcoming analysis. Additionally, note that in these theories, there are only J -type Fermi-multiplet interactions. See [90, 165] for a more complete discussion about this Lagrangian and the corresponding field content.

These fields transform under supersymmetry as

$$\begin{aligned} \delta v_t &= -\frac{1}{\sqrt{2}}\epsilon^A \bar{\lambda}_A - \frac{1}{\sqrt{2}}\bar{\epsilon}_A \lambda^A \quad , \quad \delta \sigma = \frac{1}{\sqrt{2}}\epsilon^A \bar{\lambda}_A + \frac{1}{\sqrt{2}}\bar{\epsilon}_A \lambda^A , \\ \delta \lambda^A &= \frac{i}{\sqrt{2}}\epsilon^A D_t \sigma + \frac{1}{\sqrt{2}e^2}\epsilon^B M_r (\sigma^r)^A_B , \end{aligned} \quad (\text{D.10})$$

and

$$\begin{aligned} \delta \phi^{A,f} &= -i \left(\epsilon^A \psi^f + \bar{\epsilon}^A \tilde{\psi}^f \right) , \\ \delta \psi^f &= \bar{\epsilon}_A \left(D_t - \frac{i}{e^2} \sigma \right) \phi^{A,f} \quad , \quad \delta \tilde{\psi}^f = -\epsilon_A \left(D_t - \frac{i}{e^2} \sigma \right) \phi^{A,f} , \\ \delta \eta_j &= i\epsilon_1 G_j + \bar{\epsilon}_1 \bar{F}_j \quad , \quad \delta G_j = -\frac{1}{2}\bar{\epsilon}_1 (D_t + i\sigma) \eta_j - \frac{1}{2}\epsilon_1 (\tilde{D}_t - i\sigma) \bar{\eta}_j , \\ \delta \rho^A &= -i \left(\epsilon^A \chi_1 + \bar{\epsilon}^A \chi_2 \right) \quad , \quad \delta \chi_1 = \bar{\epsilon}_A \partial_t \rho^A \quad , \quad \delta \chi_2 = -\epsilon_A \partial_t \rho^A , \end{aligned} \quad (\text{D.11})$$

The supercharges generating the supersymmetry transformations for the vector multiplet fields (σ, λ^A) and hypermultiplet fields $(\phi, \psi), (\tilde{\phi}, \tilde{\psi})$ are given by

$$Q_A = \psi^f \left(\pi_{A,f} + \frac{i\sigma}{e^2} \bar{\phi}_{A,f} \right) + \tilde{\psi}_f \left(\bar{\pi}_A^f - \frac{i\sigma}{e^2} \phi_A^f \right) - \frac{1}{\sqrt{2}} \bar{\lambda}_A (-ip_\sigma) + \frac{1}{\sqrt{2}e^2} (\sigma^r)_A^B \bar{\lambda}_B M_r . \quad (\text{D.12})$$

Additionally, when considering the 4D $\mathcal{N} = 2^*$ theory, we must add another hypermultiplet (ρ^A, χ_I) which contributes

$$Q_A = \dots + \chi_1 \left(\pi_{\rho,A} + \frac{im}{e^2} \bar{\rho}_A \right) + \bar{\chi}_2 \left(\bar{\pi}_{\rho,A} - \frac{im}{e^2} \rho_A \right) , \quad (\text{D.13})$$

where we have included possible mass terms.

Now consider adding the masses:²

$$\begin{aligned}
m_{\Phi_1} &= -\text{Im}[\mathbf{a}/\beta] + \text{Im}[\epsilon_+/\beta] = -2a + 2\epsilon , \\
m_{\tilde{\Phi}_1} &= -\text{Im}[\mathbf{a}/\beta] - \text{Im}[\epsilon_+/\beta] = -2a - 2\epsilon , \\
m_{\Phi_2} &= \text{Im}[\mathbf{a}/\beta] + \text{Im}[\epsilon_+/\beta] = 2a + 2\epsilon , \\
m_{\tilde{\Phi}_2} &= \text{Im}[\mathbf{a}/\beta] - \text{Im}[\epsilon_+/\beta] = 2a - 2\epsilon , \\
m_{\chi^2} &= 4\text{Im}[\epsilon_+/\beta] = 4\epsilon ,
\end{aligned} \tag{D.14}$$

Additionally, in the case of 4D theories with matter, we will add the masses

$$\begin{aligned}
m_{\rho^1} &= \text{Re}[m/\beta] + \text{Re}[\epsilon_+/\beta] = 2m + 2\epsilon , \\
m_{\rho^2} &= \text{Re}[m/\beta] - \text{Re}[\epsilon_+/\beta] = 2m - 2\epsilon , \\
m_{\hat{\psi}^{(i)}} &= \pm \text{Re}[\mathbf{a}/\beta] + \text{Re}[m/\beta] = \pm 2a + 2m , \\
m_{\tilde{\psi}^{(f)}} &= 2\text{Re}[m_f/\beta] = 2m_f ,
\end{aligned} \tag{D.15}$$

as appropriate. These masses break SUSY down to $\mathcal{N} = 2$ where Q_1, \bar{Q}^1 are the conserved supercharges. Since we know that the Witten index depends holomorphically on the masses [60], we will take the mass parameters to be real and positive with $a > \epsilon$ for simplicity and analytically continue in the final answer.³

²These masses come from turning on a flat gauge connection for a flavor symmetry $\mapsto a$, and for a $U(1)_R$ symmetry $\mapsto \epsilon$. The $U(1)_R$ symmetry comes from the diagonal combination of $SU(2)_R \times SU(2)_r$ where $SU(2)_R$ is an R -symmetry and $SU(2)_r$ is an outer automorphism symmetry. Turning on the ϵ mass corresponds to gauging the combination $Q_\epsilon = Q_R - Q_r$ where Q_R, Q_r are the generator of the Cartan for $SU(2)_R, SU(2)_r$ respectively.

³The analysis changes slightly for the case of mass parameters and when $\epsilon > a$, but the answer will be independent of these choices.

In terms of component fields, the universal Lagrangian is given by

$$\begin{aligned}
L_{univ} = & \frac{1}{e^2} \left[\frac{1}{2} (\partial_t \sigma)^2 + \frac{i}{2} (\bar{\lambda}_1 \partial_t \lambda^1 + \lambda^1 \partial_t \bar{\lambda}_1) + i \bar{\lambda}_2 (\partial_t - 4i\epsilon) \lambda^2 - \frac{1}{2} D^2 - |F|^2 \right] \\
& + \frac{1}{e^2} \left[|D_t \phi_1|^2 + |D_t \phi_2|^2 + |\tilde{D}_t \tilde{\phi}_1|^2 + |\tilde{D}_t \tilde{\phi}_2|^2 \right] \\
& - \frac{1}{e^2} \left[(\sigma - a + \epsilon)^2 |\phi_1|^2 + (\sigma - a - \epsilon)^2 |\tilde{\phi}_1|^2 + (\sigma + a + \epsilon)^2 |\phi_2|^2 + (\sigma + a - \epsilon)^2 |\tilde{\phi}_2|^2 \right] \\
& + \frac{i}{2e^2} \left(\bar{\psi}_1 (D_t + i(\sigma - a + \epsilon)) \psi_1 + \psi_1 (\tilde{D}_t - i(\sigma - a + \epsilon)) \bar{\psi}_1 \right. \\
& \quad + \bar{\psi}_2 (D_t + i(\sigma + a + \epsilon)) \psi_2 + \psi_2 (\tilde{D}_t - i(\sigma + a + \epsilon)) \bar{\psi}_2 \\
& \quad + \bar{\tilde{\psi}}_1 (\tilde{D}_t - i(\sigma - a - \epsilon)) \tilde{\psi}_1 + \tilde{\psi}_1 (D_t + i(\sigma - a - \epsilon)) \bar{\tilde{\psi}}_1 \\
& \quad \left. + \bar{\tilde{\psi}}_2 (\tilde{D}_t - i(\sigma + a - \epsilon)) \tilde{\psi}_2 + \tilde{\psi}_2 (D_t + i(\sigma + a - \epsilon)) \bar{\tilde{\psi}}_2 \right) \\
& + \frac{D}{e^2} (|\phi_1|^2 + |\phi_2|^2 - |\tilde{\phi}_1|^2 - |\tilde{\phi}_2|^2 - e^2 \xi) + \frac{\bar{F}}{e^2} (\phi_1 \tilde{\phi}_1 + \phi_2 \tilde{\phi}_2) + \frac{F}{e^2} (\phi_1 \tilde{\phi}_1 + \phi_2 \tilde{\phi}_2) \\
& - \frac{i}{\sqrt{2}e^2} \sum_i \left(\bar{\phi}_i \lambda^1 \psi_i + \tilde{\phi}_i \lambda^2 \psi_i + \bar{\phi}_i \bar{\lambda}_1 \bar{\tilde{\psi}}_i - \bar{\phi}_i \bar{\lambda}_2 \bar{\tilde{\psi}}_i \right. \\
& \quad \left. - \bar{\psi}_i \bar{\lambda}_1 \phi_i - \bar{\psi}_i \bar{\lambda}_2 \tilde{\phi}_i - \tilde{\psi}_i \lambda^1 \tilde{\phi}_i + \tilde{\psi}_i \lambda^2 \phi_i \right), \tag{D.16}
\end{aligned}$$

where $D_t = \partial_t + iv_t$ and $\tilde{D}_t = \partial_t - iv_t$.

Similarly, the L_{theory} will be of the form

$$L_{theory} = \frac{1}{e^2} \sum_{j=1}^{N_f} \left[\frac{i}{2} \left(\bar{\eta}_j (D_t + i\sigma) \eta_j + \eta_j (\tilde{D}_t - i\sigma) \bar{\eta}_j \right) - |G_j|^2 + m_j [\bar{\eta}_j, \eta_j] \right], \tag{D.17}$$

in the case of 4D $SU(2)$ with N_f -fundamental hypermultiplets and

$$\begin{aligned}
L_{theory} = & \frac{1}{e^2} \left[|\partial_t \rho|^2 + |\partial_t \tilde{\rho}|^2 - (m + \epsilon)^2 |\rho|^2 - (m - \epsilon)^2 |\tilde{\rho}|^2 \right. \\
& \quad \left. + \frac{i}{2} (\bar{\chi} \partial_t \chi + \chi \partial_t \bar{\chi} + \bar{\tilde{\chi}} \partial_t \tilde{\chi} + \tilde{\chi} \partial_t \bar{\tilde{\chi}}) - \frac{m}{2} ([\bar{\chi}, \chi] - [\bar{\tilde{\chi}}, \tilde{\chi}]) \right] \\
& + \frac{1}{e^2} \left[\frac{i}{2} \left(\bar{\eta}_1 (D_t + i(\sigma - a + \epsilon)) \eta_1 + \eta_1 (\tilde{D}_t - i(\sigma - a + \epsilon)) \bar{\eta}_1 \right. \right. \\
& \quad + \bar{\tilde{\eta}}_1 (\tilde{D}_t - i(\sigma - a - \epsilon)) \tilde{\eta}_1 + \tilde{\eta}_1 (D_t + i(\sigma - a - \epsilon)) \bar{\tilde{\eta}}_1 \\
& \quad + (\bar{\eta}_2 (D_t + i(\sigma + a + \epsilon)) \eta_2 + \eta_2 (\tilde{D}_t - i(\sigma + a + \epsilon)) \bar{\eta}_2 \\
& \quad \left. \left. + \bar{\tilde{\eta}}_2 (\tilde{D}_t - i(\sigma + a - \epsilon)) \tilde{\eta}_2 + \tilde{\eta}_2 (D_t + i(\sigma + a - \epsilon)) \bar{\tilde{\eta}}_2 \right) - |G_1|^2 - |G_2|^2 \right], \tag{D.18}
\end{aligned}$$

for the case of the 4D $SU(2)$ $\mathcal{N} = 2^*$ theory.

By defining the conjugate momenta to the elementary fields

$$\begin{aligned} p_\sigma &= \frac{1}{e^2} \partial_t \sigma, & p_{\lambda^A} &= \frac{i}{e^2} \bar{\lambda}_A, & p_{\psi_i} &= \frac{i}{e^2} \bar{\psi}_i, & \tilde{p}_{\psi_i} &= \frac{i}{e^2} \bar{\tilde{\psi}}_i, \\ \pi_i &= \frac{1}{e^2} \tilde{D}_t \bar{\phi}_i, & \tilde{\pi}_i &= \frac{1}{e^2} D_t \bar{\tilde{\phi}}_i, & p_{\eta_j} &= \frac{i}{e^2} \bar{\eta}_j, \\ p_\chi &= \frac{i}{e^2} \bar{\chi}, & \tilde{p}_\chi &= \frac{i}{e^2} \bar{\tilde{\chi}}, & \pi_\rho &= \frac{1}{e^2} \partial_t \bar{\rho}, & \tilde{\pi}_\rho &= \frac{1}{e^2} \partial_t \bar{\tilde{\rho}}, \end{aligned} \quad (\text{D.19})$$

we can compute the Hamiltonian and integrate out the auxiliary fields (D, F, G) :

$$H = \frac{e^2}{2} p_\sigma^2 - \frac{4\epsilon}{e^2} \bar{\lambda}_2 \lambda^2 + U + H_{matter} + H_I + v_t Q_{\text{Gauge}}, \quad (\text{D.20})$$

where

$$U = \frac{1}{2e^2} \left(|\phi_1|^2 + |\phi_2|^2 - |\tilde{\phi}_1|^2 - |\tilde{\phi}_2|^2 - e^2 \xi \right)^2 + \frac{1}{e^2} |\phi_1 \tilde{\phi}_1 + \phi_2 \tilde{\phi}_2|^2, \quad (\text{D.21})$$

and

$$\begin{aligned} H_{matter} &= e^2 \left[|\pi_1|^2 + |\pi_2|^2 + |\tilde{\pi}_1|^2 + |\tilde{\pi}_2|^2 \right] \\ &+ \frac{1}{e^2} \left[(\sigma - a + \epsilon)^2 |\phi_1|^2 + (\sigma - a - \epsilon)^2 |\tilde{\phi}_1|^2 + (\sigma + a + \epsilon)^2 |\phi_2|^2 + (\sigma + a - \epsilon)^2 |\tilde{\phi}_2|^2 \right] \\ &+ \frac{1}{2e^2} \left((\sigma - a + \epsilon) [\bar{\psi}_1, \psi_1] + (\sigma + a + \epsilon) [\bar{\psi}_2, \psi_2] \right. \\ &\quad \left. - (\sigma - a - \epsilon) [\bar{\tilde{\psi}}_1, \tilde{\psi}_1] - (\sigma + a - \epsilon) [\bar{\tilde{\psi}}_2, \tilde{\psi}_2] \right) \\ &+ H_{theory} \end{aligned} \quad (\text{D.22})$$

and

$$\begin{aligned} H_I &= -\frac{i}{\sqrt{2}e^2} \sum_i \left(\bar{\phi}_i \lambda^1 \psi_i + \tilde{\phi}_i \lambda^2 \psi_i + \tilde{\phi}_i \bar{\lambda}_1 \bar{\tilde{\psi}}_i - \bar{\phi}_i \bar{\lambda}_2 \bar{\tilde{\psi}}_i \right. \\ &\quad \left. - \bar{\psi}_i \bar{\lambda}_1 \phi_i - \bar{\psi}_i \bar{\lambda}_2 \tilde{\phi}_i - \tilde{\psi}_i \lambda^1 \bar{\tilde{\phi}}_i + \tilde{\psi}_i \lambda^2 \bar{\tilde{\phi}}_i \right), \end{aligned} \quad (\text{D.23})$$

where

$$Q_{\text{Gauge}} = Q_{theory} + \sum_i \left(i\phi_i \pi_i - i\bar{\phi}_i \tilde{\pi}_i - i\tilde{\phi}_i \pi_i + i\bar{\tilde{\phi}}_i \tilde{\pi}_i - \frac{1}{2e^2} [\bar{\psi}_i, \psi_i] + \frac{1}{2e^2} [\bar{\tilde{\psi}}_i, \tilde{\psi}_i] \right). \quad (\text{D.24})$$

Here, H_{theory} is of the form

$$H_{theory} = \frac{1}{2e^2} \sum_{j=1}^{N_f} (\sigma - 2m_j) [\bar{\eta}_j, \eta_j], \quad (\text{D.25})$$

for the 4D theory with N_f -fundamental hypermultiplets and

$$\begin{aligned} H_{theory} = & \frac{m}{2e^2} [\bar{\chi}_I, \chi_I] + e^2 |\pi_\rho|^2 + e^2 |\tilde{\pi}_\rho|^2 + \frac{(m+\epsilon)^2}{e^2} |\rho|^2 + \frac{(m-\epsilon)^2}{e^2} |\tilde{\rho}|^2 \\ & - \frac{(\sigma-a+m)}{2e^2} [\bar{\eta}_1, \eta_1] + \frac{(\sigma-a-m)}{2e^2} [\bar{\tilde{\eta}}_1, \tilde{\eta}_1] \\ & - \frac{(\sigma+a+m)}{2e^2} [\bar{\eta}_2, \eta_2] + \frac{(\sigma+a-m)}{2e^2} [\bar{\tilde{\eta}}_2, \tilde{\eta}_2] , \end{aligned} \quad (D.26)$$

for the $\mathcal{N} = 2^*$ theory. Additionally,

$$Q_{theory} = -\frac{1}{2} \sum_{j=1}^{N_f} [\bar{\eta}_j, \eta_j] , \quad (D.27)$$

or

$$Q_{theory} = -\frac{1}{2} ([\bar{\eta}_1, \eta_1] - [\bar{\tilde{\eta}}_1, \tilde{\eta}_1] + [\bar{\eta}_2, \eta_2] - [\bar{\tilde{\eta}}_2, \tilde{\eta}_2]) , \quad (D.28)$$

for the N_f -fundamental hypermultiplet and $\mathcal{N} = 2^*$ theory respectively. By Gauss's law we have that Q_{Gauge} must annihilate all physical states.

The classical vacuum equations for this theory are given by⁴

$$\begin{aligned} |\phi_1|^2 + |\phi_2|^2 - |\tilde{\phi}_1|^2 - |\tilde{\phi}_2|^2 - e^2 \xi &= 0 \quad , \quad \phi_1 \tilde{\phi}_1 + \phi_2 \tilde{\phi}_2 = 0 , \\ (\sigma - a + \epsilon)^2 |\phi_1|^2 + (\sigma - a - \epsilon)^2 |\tilde{\phi}_1|^2 + (\sigma + a + \epsilon)^2 |\phi_2|^2 + (\sigma + a - \epsilon)^2 |\tilde{\phi}_2|^2 &= 0 . \end{aligned} \quad (D.29)$$

The classical vacua of this theory are described by a Coulomb and Higgs branch which are defined by

$$\begin{aligned} \mathcal{M}_C &= \{\sigma \in \mathbb{R} , \phi_i, \tilde{\phi}_i = 0\} \cong \mathbb{R} , \\ \mathcal{M}_H &= \left\{ \begin{array}{l} \sigma = \pm a \pm' \epsilon , \\ |\phi_1|^2 + |\phi_2|^2 - |\tilde{\phi}_1|^2 - |\tilde{\phi}_2|^2 = -e^2 \xi , \phi_1 \tilde{\phi}_1 + \phi_2 \tilde{\phi}_2 = 0 \\ (\sigma - a + \epsilon)^2 |\phi_1|^2 + (\sigma - a - \epsilon)^2 |\tilde{\phi}_1|^2 \\ \quad + (\sigma + a + \epsilon)^2 |\phi_2|^2 + (\sigma + a - \epsilon)^2 |\tilde{\phi}_2|^2 = 0 \end{array} \right\} / U(1) . \end{aligned} \quad (D.30)$$

Note that in this case, the Higgs branch is given by a disjoint union of 4 points given by the 4-different choices of (\pm, \pm') in (D.30).

Now if we rescale the fields $\psi_i, \tilde{\psi}_i, \eta_i, \lambda_i$:

$$\psi_i, \tilde{\psi}_i, \eta_i \rightarrow \frac{1}{e} \psi_i, \frac{1}{e} \tilde{\psi}_i, \frac{1}{e} \eta_i \quad , \quad \lambda^A, \chi, \tilde{\chi} \rightarrow \frac{1}{e} \lambda^A, \frac{1}{e} \chi, \frac{1}{e} \tilde{\chi} , \quad (D.31)$$

⁴There is an additional vacuum equation for the $\mathcal{N} = 2^*$ theory, however it has only trivial solutions: $\rho^A = 0$.

such that the commutation relations become

$$\begin{aligned} \{\bar{\psi}_i, \psi_j\} = \delta_{ij} \quad , \quad \{\bar{\tilde{\psi}}_i, \tilde{\psi}_j\} = \delta_{ij} \quad , \quad \{\bar{\eta}_i, \eta_j\} = \delta_{ij} \quad , \quad \{\bar{\chi}, \chi\} = \{\bar{\tilde{\chi}}, \tilde{\chi}\} = 1 \quad , \\ [\phi_i, \pi_j] = i\delta_{ij} \quad , \quad [\tilde{\phi}_i, \tilde{\pi}_j] = i\delta_{ij} \quad , \quad [\rho, \pi_\rho] = i \quad , \quad [\tilde{\rho}, \tilde{\pi}_\rho] = i \quad . \end{aligned} \quad (\text{D.32})$$

with all other commutation relations (or anticommutation as appropriate) are trivial, we can define the oscillators

$$\begin{aligned} a_i &= \frac{1}{\sqrt{2}e} \left(\sqrt{\omega_i} \phi_i + \frac{ie^2 \pi_i}{\sqrt{\omega_i}} \right) \quad , \quad \bar{a}_i = \frac{1}{\sqrt{2}e} \left(\sqrt{\omega_i} \bar{\phi}_i + \frac{ie^2 \pi_i}{\sqrt{\omega_i}} \right) \quad , \\ \tilde{a}_i &= \frac{1}{\sqrt{2}e} \left(\sqrt{\tilde{\omega}_i} \tilde{\phi}_i + \frac{ie^2 \tilde{\pi}_i}{\sqrt{\tilde{\omega}_i}} \right) \quad , \quad \tilde{\bar{a}}_i = \frac{1}{\sqrt{2}e} \left(\sqrt{\tilde{\omega}_i} \tilde{\bar{\phi}}_i + \frac{ie^2 \tilde{\pi}_i}{\sqrt{\tilde{\omega}_i}} \right) \quad , \\ a_\rho &= \frac{1}{\sqrt{2}e} \left(\sqrt{\omega_\rho} \rho + \frac{ie^2 \pi_\rho}{\sqrt{\omega_\rho}} \right) \quad , \quad a_{\bar{\rho}} = \frac{1}{\sqrt{2}e} \left(\sqrt{\omega_\rho} \bar{\rho} + \frac{ie^2 \pi_\rho}{\sqrt{\omega_\rho}} \right) \quad , \\ \tilde{a}_\rho &= \frac{1}{\sqrt{2}e} \left(\sqrt{\tilde{\omega}_\rho} \tilde{\rho} + \frac{ie^2 \tilde{\pi}_\rho}{\sqrt{\tilde{\omega}_\rho}} \right) \quad , \quad \tilde{\bar{a}}_\rho = \frac{1}{\sqrt{2}e} \left(\sqrt{\tilde{\omega}_\rho} \tilde{\bar{\rho}} + \frac{ie^2 \tilde{\pi}_\rho}{\sqrt{\tilde{\omega}_\rho}} \right) \quad , \end{aligned} \quad (\text{D.33})$$

where

$$\begin{aligned} \omega_1 &= |\sigma - a + \epsilon| \quad , \quad \omega_2 = |\sigma + a + \epsilon| \quad , \quad \tilde{\omega}_1 = |\sigma - a - \epsilon| \quad , \quad \tilde{\omega}_2 = |\sigma + a - \epsilon| \quad , \\ \omega_\rho &= |m + \epsilon| \quad , \quad \tilde{\omega}_\rho = |m - \epsilon| \quad , \end{aligned} \quad (\text{D.34})$$

and all other (anti-) commutation relations have zero on the right hand side. Using this we can define a Fock space vacuum

$$\begin{aligned} a_i|0\rangle = \bar{a}_i|0\rangle = \tilde{a}_i|0\rangle = \tilde{\bar{a}}_i|0\rangle = a_\rho|0\rangle = a_{\bar{\rho}}|0\rangle = \tilde{a}_\rho|0\rangle = \tilde{\bar{a}}_\rho|0\rangle = 0 \quad , \\ \psi_i|0\rangle = \tilde{\psi}_i|0\rangle = \lambda^A|0\rangle = \eta_j|0\rangle = \chi|0\rangle = \tilde{\chi}|0\rangle = 0 \quad . \end{aligned} \quad (\text{D.35})$$

Using these creation and annihilation operators, H_{matter} and Q_{Gauge} can be written

$$\begin{aligned} H_{matter} &= \omega_1(a_1^\dagger a_1 + \bar{a}_1^\dagger \bar{a}_1 + 1) + (\sigma - a + \epsilon)(\bar{\psi}_1 \psi_1 - \frac{1}{2}) \\ &\quad + \omega_2(a_2^\dagger a_2 + \bar{a}_2^\dagger \bar{a}_2 + 1) + (\sigma + a + \epsilon)(\bar{\psi}_2 \psi_2 - \frac{1}{2}) \\ &\quad + \tilde{\omega}_1(\tilde{a}_1^\dagger \tilde{a}_1 + \tilde{\bar{a}}_1^\dagger \tilde{\bar{a}}_1 + 1) - (\sigma - a - \epsilon)(\tilde{\bar{\psi}}_1 \tilde{\psi}_1 - \frac{1}{2}) \\ &\quad + \tilde{\omega}_2(\tilde{a}_2^\dagger \tilde{a}_2 + \tilde{\bar{a}}_2^\dagger \tilde{\bar{a}}_2 + 1) - (\sigma + a + \epsilon)(\tilde{\bar{\psi}}_2 \tilde{\psi}_2 - \frac{1}{2}) + H_{theory} \quad , \end{aligned} \quad (\text{D.36})$$

and

$$\begin{aligned} Q_{\text{Gauge}} &= - \left[a_1^\dagger a_1 + a_2^\dagger a_2 - \bar{a}_1^\dagger \bar{a}_1 - \bar{a}_2^\dagger \bar{a}_2 - \tilde{a}_1^\dagger \tilde{a}_1 - \tilde{a}_2^\dagger \tilde{a}_2 + \tilde{\bar{a}}_1^\dagger \tilde{\bar{a}}_1 + \tilde{\bar{a}}_2^\dagger \tilde{\bar{a}}_2 \right] \\ &\quad - \frac{1}{2} \left[\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2 - \tilde{\bar{\psi}}_1 \tilde{\psi}_1 - \tilde{\bar{\psi}}_2 \tilde{\psi}_2 \right] + Q_{theory} \quad , \end{aligned} \quad (\text{D.37})$$

where

$$H_{theory} = \frac{1}{2} \sum_{j=1}^{N_f} (\sigma - 2m_j) [\bar{\eta}_j, \eta_j] , \quad (D.38)$$

or

$$\begin{aligned} H_{theory} = & \frac{m}{2} ([\bar{\chi}, \chi] - [\bar{\tilde{\chi}}, \tilde{\chi}]) + \omega_\rho [a_\rho^\dagger a_\rho + a_{\bar{\rho}}^\dagger a_{\bar{\rho}} + 1] + \tilde{\omega}_\rho [a_{\bar{\rho}}^\dagger a_{\bar{\rho}} + a_{\bar{\rho}}^\dagger a_{\bar{\rho}} + 1] \\ & + \frac{1}{2} (\sigma - a + m) [\bar{\eta}_1, \eta_1] - \frac{1}{2} (\sigma - a - m) [\bar{\tilde{\eta}}_1, \tilde{\eta}_1] \\ & + \frac{1}{2} (\sigma + a + m) [\bar{\eta}_2, \eta_2] - \frac{1}{2} (\sigma + a - m) [\bar{\tilde{\eta}}_2, \tilde{\eta}_2] , \end{aligned} \quad (D.39)$$

and

$$Q_{theory} = -\frac{1}{2} \sum_{j=1}^{N_f} [\bar{\eta}_j, \eta_j] \quad \text{or} \quad Q_{theory} = -\frac{1}{2} ([\bar{\eta}_1, \eta_1] - [\bar{\tilde{\eta}}_1, \tilde{\eta}_1] + [\bar{\eta}_2, \eta_2] - [\bar{\tilde{\eta}}_2, \tilde{\eta}_2]) , \quad (D.40)$$

for the N_f -fundamental hypermultiplet and $\mathcal{N} = 2^*$ theory respectively.

D.1.1 Matter Ground States

We can determine the asymptotic ground states by applying the Born-Oppenheimer approximation. In this approximation, we divide our fields into “slow” and “fast” fields. We then decompose the wave function as

$$|\Psi\rangle = |\psi_{\text{slow}}\rangle \otimes |\psi_{\text{fast}}\rangle , \quad (D.41)$$

and solve for the ground state of the fast degrees of freedom in the background determined by the slow degrees of freedom. This is described by $|\psi_{\text{fast}}\rangle$. Then we solve for the ground state of the slow degrees of freedom in the effective potential created by integrating out the fast degrees of freedom.

For our purposes, we want to study Coulomb branch states that stretch out into the asymptotic region of the Coulomb branch. Here, the fast degrees of freedom are described by the matter fields (the fundamental hypermultiplets and fundamental Fermi-multiplets) while the slow degrees of freedom are then described by the vector multiplet fields.⁵ Now determining the vacuum state of the fast fields requires minimizing H_{matter}

⁵In the case of the 4D $\mathcal{N} = 2^*$ theory, the slow degrees of freedom include the adjoint valued twisted hypermultiplet.

and U subject to the constraint that $Q_{\text{Gauge}}|\Psi\rangle = 0$. In order to construct a basis of states for the fast fields, let us define the operators

$$a_3 = a_\rho \quad , \quad \bar{a}_3 = a_{\bar{\rho}} \quad , \quad \tilde{a}_3 = \tilde{a}_\rho \quad , \quad \tilde{\bar{a}}_3 = \tilde{a}_{\bar{\rho}} \quad , \quad (\text{D.42})$$

and similarly

$$\omega_3 = \omega_\rho \quad , \quad \tilde{\omega}_3 = \tilde{\omega}_\rho \quad . \quad (\text{D.43})$$

Now, $H_{\text{matter}}, Q_{\text{Gauge}}$ can be written in terms of the complex creation and annihilation operators $a_i, \bar{a}_i, a_i^\dagger, \bar{a}_i^\dagger, \tilde{a}_i, \tilde{\bar{a}}_i, \tilde{a}_i^\dagger, \tilde{\bar{a}}_i^\dagger$ and the fermionic creation and annihilation operators $\psi_i, \tilde{\psi}_i, \eta_i, \bar{\psi}_i, \tilde{\bar{\psi}}_i, \bar{\eta}_i, \chi_i, \bar{\chi}_i$. Now let us pick a basis of states

$$\begin{aligned} & |(n_i, \bar{n}_i, \tilde{n}_i, \tilde{\bar{n}}_i, m_i, \tilde{m}_i, f_j)\rangle \\ &= \prod_{i=1}^2 \left(a_i^\dagger\right)^{n_i} \left(\bar{a}_i^\dagger\right)^{\bar{n}_i} \left(\tilde{a}_i^\dagger\right)^{\tilde{n}_i} \left(\tilde{\bar{a}}_i^\dagger\right)^{\tilde{\bar{n}}_i} \bar{\psi}_i^{m_i} \tilde{\bar{\psi}}_i^{\tilde{m}_i} \bar{\chi}_1^{m_3} \bar{\chi}^{\tilde{m}_3} \prod_{j=1}^4 \bar{\eta}_j^{f_j} |0\rangle \quad , \end{aligned} \quad (\text{D.44})$$

in the case of fundamental 4D matter and

$$\begin{aligned} & |(n_i, \bar{n}_i, \tilde{n}_i, \tilde{\bar{n}}_i, m_i, \tilde{m}_i, f_j)\rangle \\ &= \prod_{i=1}^3 \left(a_i^\dagger\right)^{n_i} \left(\bar{a}_i^\dagger\right)^{\bar{n}_i} \left(\tilde{a}_i^\dagger\right)^{\tilde{n}_i} \left(\tilde{\bar{a}}_i^\dagger\right)^{\tilde{\bar{n}}_i} \bar{\psi}_i^{m_i} \tilde{\bar{\psi}}_i^{\tilde{m}_i} \bar{\chi}_1^{m_3} \bar{\chi}^{\tilde{m}_3} \prod_{j=1}^2 \bar{\eta}_j^{f_j} \tilde{\bar{\eta}}_j^{\tilde{f}_j} |0\rangle \quad , \end{aligned} \quad (\text{D.45})$$

for the case of the $\mathcal{N} = 2^*$ theory. Note that this means that the quantum numbers are constrained

$$n_i, \bar{n}_i, \tilde{n}_i, \tilde{\bar{n}}_i \in \mathbb{Z}_+ \quad , \quad m_i, \tilde{m}_i, f_j, \tilde{f}_j = 0, 1 \quad . \quad (\text{D.46})$$

These quantum numbers have the interpretation of the eigenvalue of the number operator associated to the given fields. In this case, we have that the eigenvalues of

$H_{matter}, Q_{\text{Gauge}}$ of these states are given by

$$\begin{aligned}
E_{matter} = & \omega_1 (n_1 + \bar{n}_1 + 1) + (\sigma - a + \epsilon) \left(m_1 - \frac{1}{2} \right) \\
& + \omega_2 (n_2 + \bar{n}_2 + 1) + (\sigma + a + \epsilon) \left(m_2 - \frac{1}{2} \right) \\
& + \tilde{\omega}_1 (\tilde{n}_1 + \tilde{\bar{n}}_1 + 1) - (\sigma - a - \epsilon) \left(\tilde{m}_1 - \frac{1}{2} \right) \\
& + \tilde{\omega}_2 (\tilde{n}_2 + \tilde{\bar{n}}_2 + 1) - (\sigma + a - \epsilon) \left(\tilde{m}_2 - \frac{1}{2} \right) + E_{theory} , \\
q_{matter} = & -(n_1 - \bar{n}_1 + n_2 - \bar{n}_2 - \tilde{n}_1 + \tilde{\bar{n}}_1 - \tilde{n}_2 + \tilde{\bar{n}}_2 - m_1 - m_2 + \tilde{m}_1 + \tilde{m}_2) \\
& + q_{theory} .
\end{aligned} \tag{D.47}$$

Note that the m_i, \tilde{m}_i are quantum numbers and not masses. Here

$$E_{theory} = \sum_{j=1}^{N_f} (\sigma - 2m_j) \left(f_j - \frac{1}{2} \right) , \tag{D.48}$$

or

$$\begin{aligned}
E_{theory} = & m[m_3 - \tilde{m}_3] + \omega_3 (n_3 + \bar{n}_3 + 1) + \tilde{\omega}_3 (\tilde{n}_3 + \tilde{\bar{n}}_3 + 1) - (\sigma - a + m) \left(f_1 - \frac{1}{2} \right) \\
& + (\sigma - a - m) \left(\tilde{f}_1 - \frac{1}{2} \right) - (\sigma + a + m) \left(f_2 - \frac{1}{2} \right) + (\sigma + a - m) \left(\tilde{f}_2 - \frac{1}{2} \right) ,
\end{aligned} \tag{D.49}$$

and

$$Q_{theory} = - \sum_{j=1}^{N_f} \left(f_j - \frac{1}{2} \right) \quad \text{or} \quad Q_{theory} = -(f_1 + f_2 - \tilde{f}_1 - \tilde{f}_2) , \tag{D.50}$$

for the N_f -fundamental hypermultiplet and $\mathcal{N} = 2^*$ theory respectively.

Here we also need to define the flavor charges

$$\begin{aligned}
Q_a = & Q_a^{(theory)} + \frac{1}{2} [\bar{\psi}_1, \psi_1] - \frac{1}{2} [\bar{\tilde{\psi}}_1, \tilde{\psi}_1] - \frac{1}{2} [\bar{\psi}_2, \psi_2] + \frac{1}{2} [\bar{\tilde{\psi}}_2, \tilde{\psi}_2] \\
& - \frac{2}{e^2} \begin{cases} \omega_1 |\phi_1|^2 + \tilde{\omega}_1 |\tilde{\phi}_1|^2 - \omega_2 |\phi_2|^2 - \tilde{\omega}_2 |\tilde{\phi}_2|^2 & \sigma > a + \epsilon \\ -\omega_1 |\phi_1|^2 - \tilde{\omega}_1 |\tilde{\phi}_1|^2 + \omega_2 |\phi_2|^2 + \tilde{\omega}_2 |\tilde{\phi}_2|^2 & \sigma < -a - \epsilon \end{cases}
\end{aligned} \tag{D.51}$$

where $Q_a^{(theory)} = 0$ or

$$Q_a^{(theory)} = \frac{1}{2} ([\bar{\eta}_1, \eta_1] - [\bar{\tilde{\eta}}_1, \tilde{\eta}_1] - [\bar{\eta}_2, \eta_2] + [\bar{\tilde{\eta}}_2, \tilde{\eta}_2]) . \tag{D.52}$$

and

$$Q_\epsilon = Q_\epsilon^{(theory)} + 4\bar{\lambda}_2\lambda^2 + \frac{1}{2}[\bar{\psi}_1, \psi_1] + \frac{1}{2}[\bar{\tilde{\psi}}_1, \tilde{\psi}_1] + \frac{1}{2}[\bar{\psi}_2, \psi_2] + \frac{1}{2}[\bar{\tilde{\psi}}_2, \tilde{\psi}_2] \\ - \frac{2}{e^2} \sum_{i=1}^2 \begin{cases} \left(\omega_i |\phi_i|^2 - \tilde{\omega}_i |\tilde{\phi}_i|^2 \right) & \sigma > a + \epsilon \\ - \left(\omega_i |\phi_i|^2 - \tilde{\omega}_i |\tilde{\phi}_i|^2 \right) & \sigma < -a - \epsilon \end{cases} \quad (D.53)$$

and $Q_\epsilon^{(theory)} = 0$ or

$$Q_\epsilon^{(theory)} = \frac{2}{e^2} \left(-\omega_\rho |\rho|^2 + \tilde{\omega}_\rho |\tilde{\rho}|^2 \right) + \frac{1}{2} \sum_i ([\bar{\eta}_i, \eta_i] + [\bar{\tilde{\eta}}_i, \tilde{\eta}_i]) \quad , \quad \pm\sigma > 0 \quad . \quad (D.54)$$

In our basis of states, these can be written as

$$Q_a = Q_a^{(theory)} \\ + \begin{cases} n_1 + \bar{n}_1 + \tilde{n}_1 + \tilde{\bar{n}}_1 - n_2 - \bar{n}_2 - \tilde{n}_2 - \tilde{\bar{n}}_2 + m_1 - \tilde{m}_1 - m_2 + \tilde{m}_2 & \sigma > a + \epsilon \\ -n_1 - \bar{n}_1 - \tilde{n}_1 - \tilde{\bar{n}}_1 + n_2 + \bar{n}_2 + \tilde{n}_2 + \tilde{\bar{n}}_2 + m_1 - \tilde{m}_1 - m_2 + \tilde{m}_2 & \sigma < -a - \epsilon \end{cases} \quad (D.55)$$

with $Q_a^{(theory)} = 0$ or

$$Q_a^{(theory)} = (f_1 - \tilde{f}_1 - f_2 + \tilde{f}_2) \quad , \quad (D.56)$$

and

$$Q_\epsilon = Q_\epsilon^{(theory)} + 4\bar{\lambda}_2\lambda^2 \\ + \begin{cases} -n_1 - \bar{n}_1 + \tilde{n}_1 + \tilde{\bar{n}}_1 - n_2 - \bar{n}_2 + \tilde{n}_2 + \tilde{\bar{n}}_2 + m_1 + \tilde{m}_1 + m_2 + \tilde{m}_2 & \sigma > a + \epsilon \\ n_1 + \bar{n}_1 - \tilde{n}_1 - \tilde{\bar{n}}_1 + n_2 + \bar{n}_2 - \tilde{n}_2 - \tilde{\bar{n}}_2 + m_1 + \tilde{m}_1 + m_2 + \tilde{m}_2 & \sigma < -a - \epsilon \end{cases} \quad (D.57)$$

where $Q_\epsilon^{(theory)} = 0$ or

$$Q_\epsilon^{(theory)} = -(n_3 + \bar{n}_3 - \tilde{n}_3 - \tilde{\bar{n}}_3) + (f_1 + \tilde{f}_1 + f_2 + \tilde{f}_2) \quad . \quad (D.58)$$

The constraint for a supersymmetric ground state is now

$$(H_{matter} + \epsilon Q_\epsilon + a Q_a + \sum_f m_f F_f) |\Psi\rangle = 0 \quad , \quad Q_{\text{Gauge}} |\Psi\rangle = 0 \quad (D.59)$$

There are 5 distinct regions in σ space in which we can impose these conditions. The physically relevant ones are those for which $|\sigma| > |a| + |\epsilon|$. Therefore we will restrict to the regions in which $\sigma > a + \epsilon$ and $\sigma < -a - \epsilon$ where we are assuming $a > \epsilon > 0$.

It is actually more convenient to solve the equations

$$(H_{matter} + \epsilon Q_\epsilon + a Q_a + \sum_f m_f F_f + \sigma Q_{\text{Gauge}})|\Psi\rangle = 0, \quad (\text{D.60})$$

and then solve $Q_{\text{Gauge}}|\Psi\rangle = 0$.

As it turns out there are only solutions only for the case of $N_f = 4$ and the $\mathcal{N} = 2^*$ theory. For the general theory, the zero energy condition (D.59), can be written as

$$0 = \sum_{i=1}^2 \begin{cases} n_i + \bar{n}_i + m_i + (1 - \tilde{m}_i) & \sigma > a + \epsilon \\ -\bar{n}_i - \tilde{n}_i - \tilde{m}_i - (1 - m_i) & \sigma < -a - \epsilon \end{cases} \quad (\text{D.61})$$

For N_f fundamental hypermultiplet theories, the gauge invariance condition can be written

$$0 = \begin{cases} \bar{n}_1 + \bar{n}_2 + \tilde{n}_1 + \tilde{n}_2 + \left(2 - \frac{N_f}{2}\right) + \sum_{j=1}^{N_f} (1 - f_j) & \sigma > a + \epsilon \\ -n_1 - n_2 - \bar{n}_1 - \bar{n}_2 - \left(2 - \frac{N_f}{2}\right) - \sum_{j=1}^{N_f} f_j & \sigma < -a - \epsilon \end{cases} \quad (\text{D.62})$$

while for the $\mathcal{N} = 2^*$ theory, it can be written as

$$0 = \begin{cases} \bar{n}_1 + \bar{n}_2 + \tilde{n}_1 + \tilde{n}_2 + 2 - (f_1 + f_2 - \tilde{f}_1 - \tilde{f}_2) & \sigma > a + \epsilon \\ -n_1 - n_2 - \bar{n}_1 - \bar{n}_2 - 2 - (f_1 + f_2 - \tilde{f}_1 - \tilde{f}_2) & \sigma < -a - \epsilon \end{cases} \quad (\text{D.63})$$

These equations clearly have no solution for $N_f = 0, 1, 2, 3$.

The ground state solutions for $\mathcal{N} = 2^*$ and the $N_f = 4$ theory are given by:

$$\begin{aligned} \sigma > a + \epsilon & : \bar{n}_i, \tilde{n}_i = 0, \{f_j = 1 \text{ or } f_{1,2} = 1, \tilde{f}_{1,2} = 0\}, \\ \sigma < -a - \epsilon & : n_i, \bar{n}_i = 0, \{f_j = 0 \text{ or } \tilde{f}_{1,2} = 1, f_{1,2} = 0\}. \end{aligned} \quad (\text{D.64})$$

Now we can solve for the matter ground states in the regions $\sigma > a + \epsilon$ and $\sigma < -a - \epsilon$.

We will define the matter ground states in these regions as ⁶

$$\begin{aligned} |+\rangle &= \Theta(\sigma - a - \epsilon) | (n_i, \bar{n}_i, \tilde{n}_i, \bar{\tilde{n}}_i, m_i = 0, \tilde{m}_i = 1, \{f_j = 1 \text{ or } f_{1,2} = 1, \tilde{f}_{1,2} = 0\} \rangle , \\ |-\rangle &= \Theta(a + \epsilon - \sigma) | (n_i, \bar{n}_i, \tilde{n}_i, \bar{\tilde{n}}_i, \tilde{m}_i = 0, m_i = 1, \{f_j = 0 \text{ or } \tilde{f}_{1,2} = 1, f_{1,2} = 0\} \rangle . \end{aligned} \quad (\text{D.65})$$

D.1.2 Asymptotic States

Thus far we have computed $|\psi_{\text{fast}}\rangle = |\pm\rangle$ for $\pm\sigma > a + \epsilon$. Now we must find the state $|\psi_{\text{slow}}\rangle$ that is dependent on the adjoint valued fields only, such that the entire state $|\Psi\rangle = |\psi_{\text{slow}}\rangle \otimes |\pm\rangle$ is annihilated by the conserved supercharge operators Q_1, \bar{Q}^1 . To this effect, we can apply the Born rule to get an effective supercharge

$$Q_{\text{eff},A} = -\frac{\langle \psi_{\text{fast}} | Q_A | \psi_{\text{fast}} \rangle}{\langle \psi_{\text{fast}} | \psi_{\text{fast}} \rangle} = -\langle Q_A \rangle . \quad (\text{D.66})$$

Using the fact that $|\mp\rangle$ is in the harmonic oscillator ground state of all bosonic, hypermultiplet fields, ⁷ we find that $\langle F \rangle = 0$ and that the effective supercharges are of the form

$$\begin{aligned} Q_{\text{eff},A} &= \frac{e}{\sqrt{2}} \left(-i\langle p_\sigma \rangle \delta_A^B - \frac{1}{e^2} \langle D \rangle (\sigma^3)_A^B \right) \bar{\lambda}_B , \\ \bar{Q}_{\text{eff}}^A &= \frac{e}{\sqrt{2}} \left(i\langle p_\sigma \rangle \delta_A^B - \frac{1}{e^2} \langle D \rangle (\sigma^3)_A^B \right) \bar{\lambda}_B , \end{aligned} \quad (\text{D.70})$$

where

$$\begin{aligned} \langle p_\sigma \rangle &= -i\partial_\sigma + \frac{i}{2} \sum_{\pm} \left(\frac{1}{\sigma \pm a + \epsilon} + \frac{1}{\sigma \pm a - \epsilon} \right) , \\ \langle D \rangle &= -e^2 \xi + \frac{e^2}{2} \sum_i \left(\frac{1}{\omega_i} - \frac{1}{\tilde{\omega}_i} \right) . \end{aligned} \quad (\text{D.71})$$

⁶Note that we do not need to worry about the normalization of $|\pm\rangle$, so long as it is normalizable. The reason is that the only physically relevant thing is for the total wave function to have unit norm.

⁷Due to the form of the oscillators (D.33):

$$a \sim \frac{e}{\sqrt{2\omega}} \left(i\bar{\pi} + \frac{\omega}{e^2} \phi \right) = \frac{e}{\sqrt{2\omega}} \left(\partial_{\bar{\phi}} + \frac{\omega}{e^2} \phi \right) , \quad (\text{D.67})$$

the wave function $|\pm\rangle$ is of the form

$$|\pm\rangle \sim e^{-\frac{\omega}{e^2} |\phi|^2} . \quad (\text{D.68})$$

Note that this implies

$$\langle \pm | |\phi|^2 | \pm \rangle = \frac{e^2}{2\omega} . \quad (\text{D.69})$$

Explicitly, the complex supercharges are given by

$$\begin{aligned}
Q_{\text{eff},1} &= \frac{e}{\sqrt{2}} \bar{\lambda}_1 \begin{cases} \left(-\partial_\sigma + \frac{1}{\sigma+a-\epsilon} + \frac{1}{\sigma-a-\epsilon} + \xi \right) & \sigma > a + \epsilon \\ \left(-\partial_\sigma + \frac{1}{\sigma+a+\epsilon} + \frac{1}{\sigma-a+\epsilon} + \xi \right) & \sigma < -a - \epsilon \end{cases} \\
Q_{\text{eff},2} &= \frac{e}{\sqrt{2}} \bar{\lambda}_2 \begin{cases} \left(-\partial_\sigma + \frac{1}{\sigma+a+\epsilon} + \frac{1}{\sigma-a+\epsilon} - \xi \right) & \sigma > a + \epsilon \\ \left(-\partial_\sigma + \frac{1}{\sigma+a-\epsilon} + \frac{1}{\sigma-a-\epsilon} - \xi \right) & \sigma < -a - \epsilon \end{cases}
\end{aligned} \tag{D.72}$$

with similar expressions for the complex conjugate supercharges.⁸

Using these complex supercharges, we can construct the real supercharges

$$\begin{aligned}
Q_1 &= Q_{\text{eff},1} + \bar{Q}_{\text{eff}}^1, & Q_2 &= -i(Q_{\text{eff},1} - \bar{Q}_{\text{eff}}^1), \\
Q_3 &= Q_{\text{eff},2} + \bar{Q}_{\text{eff}}^2, & Q_4 &= -i(Q_{\text{eff},2} - \bar{Q}_{\text{eff}}^2),
\end{aligned} \tag{D.73}$$

Since we are deforming by a mass parameter ϵ , SUSY is broken from $\mathcal{N} = (0, 4) \rightarrow \mathcal{N} = (0, 2)$ such that Q_1, Q_2 are the conserved real supercharges. Therefore, supersymmetric ground states are in the kernel of Q_1, Q_2 or equivalently in the kernel of $Q_{\text{eff},1}$ and its complex conjugate operator.

Now let us consider the states that are killed by $Q_{\text{eff},1}$ and its complex conjugate on the semi-infinite interval $\sigma > a + \epsilon$.

Here the relevant supercharges are given by

$$\begin{aligned}
Q_{\text{eff},1} &= \frac{e}{\sqrt{2}} \bar{\lambda}_1 \begin{cases} \left(-\partial_\sigma + \frac{1}{\sigma+a-\epsilon} + \frac{1}{\sigma-a-\epsilon} + \xi \right) & \sigma > a + \epsilon \\ \left(-\partial_\sigma + \frac{1}{\sigma+a+\epsilon} + \frac{1}{\sigma-a+\epsilon} + \xi \right) & \sigma < -a - \epsilon \end{cases} \\
\bar{Q}_{\text{eff}}^1 &= \frac{e}{\sqrt{2}} \lambda^1 \begin{cases} \left(\partial_\sigma - \frac{1}{\sigma+a+\epsilon} - \frac{1}{\sigma-a+\epsilon} + \xi \right) & \sigma > a + \epsilon \\ \left(\partial_\sigma - \frac{1}{\sigma+a-\epsilon} - \frac{1}{\sigma-a-\epsilon} + \xi \right) & \sigma < -a - \epsilon \end{cases}
\end{aligned} \tag{D.74}$$

Since the ϵ -mass deformation breaks SUSY from $\mathcal{N} = (0, 4) \rightarrow \mathcal{N} = (0, 2)$ we have that only the $Q_{\text{eff},1}$ supercharge is preserved. These supercharges satisfy the supersymmetry algebra

$$\{Q_{\text{eff},1}, \bar{Q}_{\text{eff}}^1\} = H_{\text{eff}} - Z, \tag{D.75}$$

⁸Note that if we had normalized the matter wave functions $|\pm\rangle$ such that $\langle \pm | \pm \rangle = 1$, then we would have $\langle p_\sigma \rangle = -i\partial_\sigma$. However, we have made this choice of normalization such that when restricted to the Coulomb branch, all of the σ dependence is manifested in $|\psi_{\text{adjoint}}\rangle$. This will make the discussion of normalizability of the state along the Coulomb branch simpler.

where

$$Z = -\epsilon Q_\epsilon - \mathfrak{a} Q_\mathfrak{a} - \sum_j \mathfrak{m}_f F_j . \quad (\text{D.76})$$

This corresponds to the effective Hamiltonian

$$H_{\text{eff}} = \frac{e^2 \langle p_\sigma \rangle^2}{2} + \langle D \rangle^2 + \partial_\sigma \langle D \rangle [\lambda^1, \bar{\lambda}_1] . \quad (\text{D.77})$$

The wave functions dependent on the Clifford algebra spanned by the $\{\bar{\lambda}_1, \bar{\lambda}_2\}$ that are killed by both $Q_{\text{eff},1}$ and \bar{Q}_{eff}^1 span a 4-dimensional Fock space dependent on the vector multiplet zero fields

$$\begin{aligned} |\psi_{SUSY}\rangle &= \begin{cases} \tilde{\omega}_1 \tilde{\omega}_2 e^{\xi\sigma} (\alpha_1 |+\rangle + \beta_1 \bar{\lambda}_2 |+\rangle) + \omega_1 \omega_2 e^{-\xi\sigma} (\alpha_2 \bar{\lambda}_1 |+\rangle + \beta_2 \bar{\lambda}_1 \bar{\lambda}_2 |+\rangle) & \sigma > a + \epsilon \\ \omega_1 \omega_2 e^{\xi\sigma} (\alpha_1 |-\rangle + \beta_1 \bar{\lambda}_2 |-\rangle) + \tilde{\omega}_1 \tilde{\omega}_2 e^{-\xi\sigma} (\alpha_2 \bar{\lambda}_1 |-\rangle + \beta_2 \bar{\lambda}_1 \bar{\lambda}_2 |-\rangle) & \sigma < -a - \epsilon \end{cases} \end{aligned} \quad (\text{D.78})$$

where the α_i, β_i are undetermined coefficients and $\omega_i, \tilde{\omega}_i$ are given by (D.34).

There are some additional considerations for the case of the $\mathcal{N} = 2^*$ theory. The reason is that there is a decoupled $\mathcal{N} = (0, 4)$ adjoint valued hypermultiplet field that pairs with the $\mathcal{N} = (0, 4)$ vector multiplet to make a $\mathcal{N} = (4, 4)$ vector multiplet. Because of the representation theory of the supersymmetry algebra, the Witten index is identically zero.

This can be seen as follows. Since the $\mathcal{N} = (0, 4)$ adjoint valued hypermultiplet is completely decoupled, the vector multiplet state splits

$$|\psi_{\text{slow}}\rangle = |\psi_{\text{vector}}\rangle \otimes |\psi_{\text{hyper}}\rangle . \quad (\text{D.79})$$

Since we can write the Hamiltonian for the adjoint hypermultiplet fields in terms of simple harmonic oscillators, we can pick a basis of states for the hypermultiplet wave functions

$$|n_3, \bar{n}_3, \tilde{n}_3, \bar{\tilde{n}}_3, m_3, \tilde{m}_3\rangle = (a_3^\dagger)^{n_3} (\bar{a}_3^\dagger)^{\bar{n}_3} (\tilde{a}_3^\dagger)^{\tilde{n}_3} (\bar{\tilde{a}}_3^\dagger)^{\bar{\tilde{n}}_3} \bar{\chi}^{m_3} \tilde{\chi}^{\tilde{m}_3} |0\rangle_{\text{hyp}} , \quad (\text{D.80})$$

where

$$a_3 |0\rangle_{\text{hyp}} = \bar{a}_3 |0\rangle_{\text{hyp}} = \tilde{a}_3 |0\rangle_{\text{hyp}} = \bar{\tilde{a}}_3 |0\rangle_{\text{hyp}} = \chi |0\rangle_{\text{hyp}} = \tilde{\chi} |0\rangle_{\text{hyp}} = 0 . \quad (\text{D.81})$$

Because the hypermultiplet fields are completely decoupled (up to flavor symmetries), there are no constraints on the values of the n 's and m 's except

$$n_3, \bar{n}_3, \tilde{n}_3, \bar{\tilde{n}}_3 \in \mathbb{Z}_+ \quad , \quad m_3, \tilde{m}_3 = 0, 1 . \quad (\text{D.82})$$

These states have definite charge under Q_m and Q_ϵ . The eigenvalues of these charge operators is given by

$$q_\epsilon = (n_3 - \bar{n}_3 - \tilde{n}_3 + \bar{\tilde{n}}_3) \quad , \quad q_m = 2m(m_3 - \tilde{m}_3) + m(n_3 - \bar{n}_3 + \tilde{n}_3 - \bar{\tilde{n}}_3) . \quad (\text{D.83})$$

Now pick a state $|\psi_0\rangle = |n_3^*, \bar{n}_3^*, \tilde{n}_3^*, \bar{\tilde{n}}_3^*, m_3^*, \tilde{m}_3^*\rangle$. Now to this state we can identify another state with the same charges under Q_m and Q_ϵ with different fermion numbers. Specifically, we can make a shift depending on the value of $(m_3^* - \tilde{m}_3^*)$

$$(m_3^* - \tilde{m}_3^*) \mapsto (m_3' - \tilde{m}_3') = (m_3^* - \tilde{m}_3^*) + 1 \quad , \quad (\bar{n}_3^*, \bar{\tilde{n}}_3^*) \mapsto (\bar{n}_3', \bar{\tilde{n}}_3') = (\bar{n}_3^* + 1, \bar{\tilde{n}}_3^* + 1) , \quad (\text{D.84})$$

or

$$(m_3^* - \tilde{m}_3^*) \mapsto (m_3' - \tilde{m}_3') = (m_3^* - \tilde{m}_3^*) - 1 \quad , \quad (n_3^*, \tilde{n}_3^*) \mapsto (n_3', \tilde{n}_3') = (n_3^* + 1, \tilde{n}_3^* + 1) , \quad (\text{D.85})$$

depending on the value of $(m_3^* - \tilde{m}_3^*)$ where we only shift one of the m_3^*, \tilde{m}_3^* . The state $|\psi_0'\rangle = |n_3', \bar{n}_3', \tilde{n}_3', \bar{\tilde{n}}_3', m_3', \tilde{m}_3'\rangle$ will then have the same eigenvalues q_ϵ, q_m with different fermion number by ± 1 , hence canceling the contribution of $|\psi_0\rangle$ to the Witten index. Therefore, there is no contribution to the Witten index from asymptotic Coulomb branch states in bubbling SQM for the case of the 4D $\mathcal{N} = 2^*$ theory. Thus, from hereon out, we will only consider the bubbling SQM for the $N_f = 4$ theory.

It is a subtle point to define the fermion number of these states. As explained in [72], the bosonic Fermi vacuum should be defined relative to the lowest energy state of the fermions. The Fermi vacuum of the Fermi-multiplet and multiplet fermions is defined by their bare mass terms in the full Hamiltonian. However, since the fermions in these multiplets come in pairs, the fermion number $(-1)^F$ is only dependent on the vector multiplet fermions that do not come in a symmetric pair.

In our Born-Oppenheimer approximation, the vector multiplet fermion λ^2 is given

a bare mass while λ^1 is given a mass from 1-loop terms. The mass terms are given by

$$H_{mass} = -4\epsilon\bar{\lambda}_2\lambda^2 \mp e^2 \sum_i \left(\frac{1}{\omega_i^2} - \frac{1}{\tilde{\omega}_i^2} \right) [\lambda^1, \bar{\lambda}_1] \quad , \quad \pm\sigma > a + \epsilon . \quad (\text{D.86})$$

Since $\omega_i > \tilde{\omega}_i$ for $\sigma > a + \epsilon$ ($\omega_i < \tilde{\omega}_i$ for $\sigma < -a - \epsilon$) for $\epsilon > 0$ and similarly $\omega_i < \tilde{\omega}_i$ for $\sigma > a + \epsilon$ ($\omega_i > \tilde{\omega}_i$ for $\sigma < -a - \epsilon$) for $\epsilon < 0$, the physical, bosonic vacuum state is defined by

$$\begin{aligned} \bar{\lambda}_2|0\rangle_{phys} &= \lambda^1|0\rangle_{phys} = 0 \quad , \quad \epsilon > 0 , \\ \lambda^2|0\rangle_{phys} &= \bar{\lambda}_1|0\rangle_{phys} = 0 \quad , \quad \epsilon < 0 , \end{aligned} \quad (\text{D.87})$$

which differs from our ϵ -invariant choice of Fock vacuum is defined in (D.35):

$$\lambda^A|0\rangle_{ours} = 0 . \quad (\text{D.88})$$

These two choices are related by

$$|0\rangle_{phys} = \begin{cases} \bar{\lambda}_2|0\rangle_{ours} & \epsilon > 0 \\ \bar{\lambda}_1|0\rangle_{ours} & \epsilon < 0 \end{cases} . \quad (\text{D.89})$$

Thus, the fermion number of our vacuum states are given by

$$(-1)^F|0\rangle = -|0\rangle \quad \implies \quad (-1)^F|\pm\rangle = -|\pm\rangle . \quad (\text{D.90})$$

D.1.3 Hermitian Supercharge Operators and Boundary Conditions

Note that the real supercharge operators defined in (D.73) are not actually self-adjoint on the relevant semi-infinite interval because integration by parts picks up boundary terms. Therefore, we must restrict the Hilbert space of BPS states to those on which the above supercharges are self-adjoint. It will be sufficient to impose that $\mathcal{Q}_1 = Q_{\text{eff},1} + \bar{Q}_{\text{eff}}^1$ is Hermitian. In the seminifinite interval $\pm\sigma > a + \epsilon$, this has the form

$$\begin{aligned} \mathcal{Q} &= \frac{e}{\sqrt{2}}(\lambda^1 - \bar{\lambda}_1)D_\sigma \\ &+ \frac{e}{\sqrt{2}}(\lambda^1 + \bar{\lambda}_1) \left(\frac{1}{|\sigma + a + \epsilon|} + \frac{1}{|\sigma - a + \epsilon|} - \frac{1}{|\sigma + a - \epsilon|} - \frac{1}{|\sigma - a - \epsilon|} + \xi \right) , \end{aligned} \quad (\text{D.91})$$

where $D_\sigma = \langle ip_\sigma \rangle$.

Since this only has $\bar{\lambda}_1$ and λ^1 Clifford elements, it is natural to divide the Hilbert space as

$$\mathcal{H} = \text{span}\{\mathbb{1}, \bar{\lambda}_2\} \otimes \{|+\rangle, \bar{\lambda}_1|+\rangle\} . \quad (\text{D.92})$$

Now consider a generic state that is annihilated by λ^2 :

$$|\Psi\rangle = f(\sigma)|+\rangle + g(\sigma)\bar{\lambda}_1|+\rangle . \quad (\text{D.93})$$

In this subspace, the supercharge \mathcal{Q}_1 (which we choose to be our localizing supercharge) is the form of a Dirac operator:

$$\mathcal{Q} = \frac{e}{\sqrt{2}} \begin{pmatrix} 0 & D_\sigma + A(\sigma) \\ -D_\sigma + A(\sigma) & 0 \end{pmatrix} \quad \text{where} \quad |\Psi\rangle = \begin{pmatrix} f(\sigma) \\ g(\sigma) \end{pmatrix} , \quad (\text{D.94})$$

and

$$A(\sigma) = \left(\frac{1}{|\sigma + a + \epsilon|} + \frac{1}{|\sigma - a + \epsilon|} - \frac{1}{|\sigma + a - \epsilon|} - \frac{1}{|\sigma - a - \epsilon|} + \xi \right) . \quad (\text{D.95})$$

On these states, we have that

$$\langle \Psi_1 | \mathcal{Q}_1 \Psi_2 \rangle = \langle \mathcal{Q}_1 \Psi_1 | \Psi_2 \rangle - \left[\bar{f}_1 g_2 - \bar{g}_1 f_2 \right]_{\pm\sigma=a+\epsilon} . \quad (\text{D.96})$$

And therefore, for the \mathcal{Q}_i to be self-adjoint, we must impose

$$\left[\bar{f}_1 g_2 - \bar{g}_1 f_2 \right]_{\pm\sigma=a+\epsilon} = 0 . \quad (\text{D.97})$$

A similar argument holds for the pair of states

$$|\tilde{\Psi}\rangle = \tilde{f}(\sigma)\bar{\lambda}_2|+\rangle + \tilde{g}(\sigma)\bar{\lambda}_1\bar{\lambda}_2|+\rangle . \quad (\text{D.98})$$

Now we see that there are more than 10 different restrictions we can impose on the Hilbert space such that (D.97) is satisfied. We will impose the same condition on the Hilbert space for $\sigma = a + \epsilon$ and $\sigma = -a - \epsilon$. These choices are given by a combination of restricting wave functions and completely eliminating all wave functions in different factors of the Hilbert space under the decomposition

$$\mathcal{H} = \oplus_{n_1, n_2=0,1} \mathcal{H}_{n_1, n_2} = \oplus_{n_1, n_2=0,1} \text{span}_{L^2} \{ \bar{\lambda}_1^{n_1} \bar{\lambda}_2^{n_2} |0\rangle \} . \quad (\text{D.99})$$

These different conditions that we can impose are:

- Type I: restricting the wave functions in a factor of \mathcal{H}_{n_1, n_2} such that $\langle \sigma | \psi \rangle \big|_{\sigma=\pm(a+\epsilon)} = 0$ for $|\psi\rangle \in \mathcal{H}_{n_1, n_2}$
- Type II: eliminating a factor of \mathcal{H}_{n_1, n_2}

We will choose either purely Type I or Type II conditions.

Given our assumptions that the boundary conditions are symmetric and purely Type I or Type II, there is a unique such choice such that $I_{asympt} = Z_{mono}^{(extra)}$. If we choose any other boundary condition, then we have that $I_{asympt} \neq Z_{mono}^{(extra)}$. Therefore, we believe that the physics of relating $I_{\mathcal{H}_0}^{(Loc)}$ with a counting of Higgs branch states suggests that we should choose boundary conditions that restrict our wave functions to be of the form $|\Psi_f\rangle$:

$$\mathcal{H}_{BPS}^{\sigma > a+\epsilon} = \text{span} \left\{ N_1 \omega_1 \omega_2 e^{\xi \sigma} \bar{\lambda}_2 |+\rangle, N_2 \tilde{\omega}_1 \tilde{\omega}_2 e^{-\xi \sigma} \bar{\lambda}_1 |+\rangle \right\}. \quad (\text{D.100})$$

A similar computation shows that

$$\mathcal{H}_{BPS}^{\sigma < -a-\epsilon} = \text{span} \left\{ N_1 \tilde{\omega}_1 \tilde{\omega}_2 e^{\xi \sigma} \bar{\lambda}_2 |-\rangle, N_2 \omega_1 \omega_2 e^{-\xi \sigma} \bar{\lambda}_1 |-\rangle \right\}. \quad (\text{D.101})$$

Thus, what we have really shown is that in the Born-Oppenheimer approximation, there is a suitable boundary condition so that $I_{asympt} = Z_{mono}^{(extra)}$. Clearly this aspect of our proposal needs to be improved.

D.1.4 Extra Contribution to the Witten Index

Now we have found the BPS states for the semi-infinite intervals $\sigma > a + \epsilon$ and $\sigma < -a - \epsilon$. Interestingly, these states undergo wall crossing with the sign of ξ . Essentially, as is evident from equations (D.100)-(D.101), as one approaches the wall of marginal stability at $\xi = 0$, the states contributing to the Witten index go off to infinity as $1/\xi$ and become non-normalizable at $\xi = 0$. Then as we again increase $|\xi|$ from 0, another state comes in from infinity.

By using the results of (D.100) and (D.101), we have that there are only 2 normalizable BPS states for a given choice of $\xi > 0$ or $\xi < 0$. The corresponding (unnormalized,

but normalizable) states are given by:

$$\begin{aligned} |\Psi_1\rangle &= \begin{cases} \tilde{\omega}_1 \tilde{\omega}_2 e^{-\xi\sigma} \bar{\lambda}_1 |+\rangle & \xi > 0 \\ \omega_1 \omega_2 e^{\xi\sigma} \bar{\lambda}_2 |+\rangle & \xi < 0 \end{cases}, \\ |\Psi_2\rangle &= \begin{cases} \tilde{\omega}_1 \tilde{\omega}_2 e^{\xi\sigma} \bar{\lambda}_2 |-\rangle & \xi > 0 \\ \omega_1 \omega_2 e^{-\xi\sigma} \bar{\lambda}_1 |-\rangle & \xi < 0 \end{cases}. \end{aligned} \quad (\text{D.102})$$

Now we can ask how these contribute to the Witten index. Here the flavor charges associated to \mathfrak{a} , ϵ are given by equations (D.51) and (D.53). For our cases, these reduce to

$$Q_{\mathfrak{a}} = 0 \quad , \quad Q_{\epsilon} = 2 - 4\bar{\lambda}_2 \lambda^2. \quad (\text{D.103})$$

and similarly

$$Q_{\mathfrak{m}_f} = [\bar{\eta}_f, \eta_f]. \quad (\text{D.104})$$

This means that the flavor charges of the ground state are given by

	$Q_{\mathfrak{a}}$	Q_{ϵ}	F_j
$ +\rangle$	0	+2	+1
$ -\rangle$	0	+2	-1

Then, using the fact that $\bar{\lambda}_2$ has charge -4 under Q_{ϵ} with all other charges annihilating $\bar{\lambda}_1, \bar{\lambda}_2$, we see that the charges evaluated on the different states are given by

$$\begin{aligned} \begin{pmatrix} aQ_a \\ \mathfrak{m}_f F_f \\ \epsilon Q_{\epsilon} \\ (-1)^F \end{pmatrix} \bar{\lambda}_1 |+\rangle &= \begin{pmatrix} 0 \\ +\mathfrak{m}_f \\ +2\epsilon \\ +1 \end{pmatrix} \bar{\lambda}_1 |+\rangle \quad , \quad \begin{pmatrix} aQ_a \\ \mathfrak{m}_f F_f \\ \epsilon Q_{\epsilon} \\ (-1)^F \end{pmatrix} \bar{\lambda}_2 |-\rangle = \begin{pmatrix} 0 \\ -\mathfrak{m}_f \\ -2\epsilon \\ +1 \end{pmatrix} \bar{\lambda}_2 |-\rangle \quad , \\ \begin{pmatrix} aQ_a \\ \mathfrak{m}_f F_f \\ \epsilon Q_{\epsilon} \\ (-1)^F \end{pmatrix} \bar{\lambda}_2 |+\rangle &= \begin{pmatrix} 0 \\ +\mathfrak{m}_f \\ -2\epsilon \\ +1 \end{pmatrix} \bar{\lambda}_2 |+\rangle \quad , \quad \begin{pmatrix} aQ_a \\ \mathfrak{m}_f F_f \\ \epsilon Q_{\epsilon} \\ (-1)^F \end{pmatrix} \bar{\lambda}_1 |-\rangle = \begin{pmatrix} 0 \\ -\mathfrak{m}_f \\ +2\epsilon \\ +1 \end{pmatrix} \bar{\lambda}_1 |-\rangle \quad . \end{aligned} \quad (\text{D.105})$$

We are interested in the contribution of these states to the ground state index:

$$I_{\mathcal{H}_0} \Big|_{\mathcal{M}_C} = I_{asympt} = \lim_{\beta \rightarrow \infty} I_{\mathcal{H}_{\mathcal{M}_C}} (-1)^F e^{-\frac{\beta}{2} \{ \mathcal{Q}, \mathcal{Q} \} + \mathfrak{a} Q_a + \epsilon + Q_\epsilon + \sum_f m_f F_f} . \quad (\text{D.106})$$

By using the fact that BPS states are annihilated by \mathcal{Q} and the charges (D.105), the asymptotic Coulomb branch states as in (D.102) give a contribution to the Witten index:

$$I_{asympt} = \begin{cases} e^{-\sum_f m_f - 2\epsilon} + e^{\sum_f m_f + 2\epsilon} = 2 \cosh \left(\sum_f m_f + 2\epsilon \right) & \xi > 0 \\ e^{-\sum_f m_f + 2\epsilon} + e^{\sum_f m_f - 2\epsilon} = 2 \cosh \left(\sum_f m_f - 2\epsilon \right) & \xi < 0 \end{cases} = Z_{mono}^{(extra)} . \quad (\text{D.107})$$

This is exactly the contribution $Z_{mono}^{(extra)}(1, 0)$.

D.1.5 1D Wilson Lines

We can additionally add supersymmetric Wilson lines to the SQM. These are labeled by a parameter q that is quantized $q \in \mathbb{Z} + \frac{N_f}{2}$. This adds a term to the total Lagrangian:

$$L_{Wilson} = -q(v_t + \sigma) . \quad (\text{D.108})$$

Note that this is supersymmetric due to the fact that $\delta v_t = -\delta \sigma$. This only changes the above analysis by changing the gauge invariance condition (recall that we solved the condition $(H + Z + \sigma Q_{\text{Gauge}})|\Psi\rangle = 0$):

$$(Q_{\text{Gauge}} - q)|\Psi\rangle = 0 . \quad (\text{D.109})$$

This only changes the choice of matter ground states. Let us consider the N_f fundamental hypermultiplet theory. Here the gauge invariance condition is given by equation

$$0 = q - \begin{cases} \bar{n}_1 + \bar{n}_2 + \tilde{n}_1 + \tilde{n}_2 + \left(2 - \frac{N_f}{2}\right) + \sum_{j=1}^{N_f} (1 - f_j) & \sigma > a + \epsilon \\ -n_1 - n_2 - \bar{\tilde{n}}_1 - \bar{\tilde{n}}_2 - \left(2 - \frac{N_f}{2}\right) - \sum_{j=1}^{N_f} f_j & \sigma < -a - \epsilon \end{cases} \quad (\text{D.110})$$

Therefore, for $N_f < 4$, we only have solutions for $q = \pm \frac{4-N_f}{2} = \pm q_{crit}$ in the $\pm \sigma > a + \epsilon$ region.

Once we have the existence of the matter ground states, the analysis for the vector-multiplet part of the states carries over from the $N_f = 4$ theories. This leads to the

contribution to the ground state index (D.106) from the asymptotic states

$$I_{asympt} = \begin{cases} e^{\sum_f m_f + 2\epsilon_+} & q = q_{crit} \\ e^{-\sum_f m_f - 2\epsilon_+} & q = -q_{crit} \\ 0 & \text{else} \end{cases} \quad (\text{D.111})$$

for $\xi > 0$ and

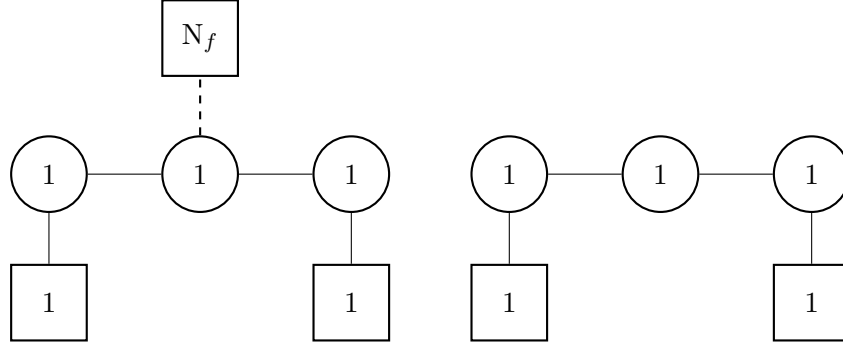
$$I_{asympt} = \begin{cases} e^{\sum_f m_f - 2\epsilon_+} & q = q_{crit} \\ e^{-\sum_f m_f + 2\epsilon_+} & q = -q_{crit} \\ 0 & \text{else} \end{cases} \quad (\text{D.112})$$

for $\xi < 0$.

Note that here the fermion number is always even due to the sign of the mass term of the Fermi-multiplets which is determined by the sign of σ . This relies on the fact that we are working in the limit where $\sigma \gg m_f, \forall f$.

D.2 $U(1)^3 \mathcal{N} = (0, 4)$ SQM Analysis

In this appendix we will analyze the ground states on the Coulomb branch of the $\mathcal{N} = (0, 4)$ SQM with gauge group $U(1)_1 \times U(1)_2 \times U(1)_3$ corresponding to the monopole bubbling term $Z_{mono}(\mathbf{a}, m_f, \epsilon; 2, 1)$. Here we have three $U(1)$ vector multiplets $(\sigma_i, \lambda_i^A, M_i^r)$ where $i = 1, 2, 3$, two fundamental hypermultiplets $(\phi_i^A, \psi_{i,I})$ where $i = 1, 2$, and two bifundamental hypermultiplets $(\underline{\phi}_i^A, \underline{\psi}_{i,I})$ where $i = 1, 2$. Additionally, dependent on the specific 4D theory, we have up to 4 fundamental (short) Fermi-multiplets (η_i, F_i) and 3 adjoint valued hypermultiplets $(\rho_i^A, \chi_{i,I})$. The quivers for the bubbling SQMs are given by:



in the case of the theory with N_f fundamental hypermultiplets (given by a $\mathcal{N} = (0, 4)$ quiver SQM) and the $\mathcal{N} = 2^*$ theory (given by a $\mathcal{N} = (4, 4)$ quiver SQM).

The total Lagrangian again decomposes as

$$L = L_{univ} + L_{theory} , \quad (\text{D.113})$$

where L_{univ} is the universal term describing 4D SYM field content and L_{theory} depends on the matter content of the 4D theory. The universal term decomposes as

$$L_{univ} = L_{vec} + L_{hyp} + L_{bf} + L_{FI} . \quad (\text{D.114})$$

After introducing notation analagous to that of Appendix D.1, the field content of this theory and their charges are given by

Lag. Term	$\mathcal{N} = (0, 4)$ Multiplet	Fields	$Q_G^{(1)}$	$Q_G^{(2)}$	$Q_G^{(3)}$	$Q_{\mathfrak{a}}$	Q_{ϵ}	F_j
Universal	Vector-	σ_1	0	0	0	0	0	0
		λ_1^1	0	0	0	0	0	0
		λ_1^2	0	0	0	0	4	0
		σ_2	0	0	0	0	0	0
		λ_2^1	0	0	0	0	0	0
		λ_2^2	0	0	0	0	4	0
		σ_3	0	0	0	0	0	0
		λ_3^1	0	0	0	0	0	0
		λ_3^2	0	0	0	0	4	0
Universal	Fund. Hyper-	ϕ_1	1	0	0	-1	1	0
		$\tilde{\phi}_1$	-1	0	0	1	1	0
		ϕ_2	0	0	1	1	1	0
		$\tilde{\phi}_2$	0	0	-1	1	1	0
		ψ_1	1	0	0	-1	1	0
		$\tilde{\psi}_1$	-1	0	0	1	1	0
		ψ_2	0	0	1	1	1	0
		$\tilde{\psi}_2$	0	0	-1	1	1	0
Universal	Bifund. Hyper-	$\underline{\phi}_1$	-1	1	0	0	1	0
		$\underline{\tilde{\phi}}_1$	1	-1	0	0	1	0
		$\underline{\phi}_2$	0	-1	1	0	1	0
		$\underline{\tilde{\phi}}_2$	0	1	-1	0	1	0
		$\underline{\psi}_1$	-1	1	0	0	1	0
		$\underline{\tilde{\psi}}_1$	1	-1	0	0	1	0
		$\underline{\psi}_2$	0	-1	1	0	1	0
		$\underline{\tilde{\psi}}_2$	0	1	-1	0	1	0

Lag. Term	$\mathcal{N} = (0, 4)$ Multiplet	Fields	$Q_G^{(1)}$	$Q_G^{(2)}$	$Q_G^{(3)}$	Q_a	Q_ϵ	F_j
N_f Fund. Hyper-	Fund. Fermi-	η_f	0	1	0	0	0	-2
$\mathcal{N} = 2^*$ Theory	Twisted Chiral	ρ_1	0	0	0	0	1	1
		$\tilde{\rho}_1$	0	0	0	0	-1	1
		χ_1	0	0	0	0	0	1
		$\tilde{\chi}_1$	0	0	0	0	0	-1
		ρ_2	0	0	0	0	1	1
		$\tilde{\rho}_2$	0	0	0	0	-1	1
		χ_2	0	0	0	0	0	1
		$\tilde{\chi}_2$	0	0	0	0	0	-1
		ρ_3	0	0	0	0	1	1
		$\tilde{\rho}_3$	0	0	0	0	-1	1
		χ_3	0	0	0	0	0	1
		$\tilde{\chi}_3$	0	0	0	0	0	-1
$\mathcal{N} = 2^*$ Theory	Fund. Fermi	η_1	1	0	0	1	-1	0
		$\tilde{\eta}_1$	-1	0	0	1	1	0
		η_2	0	0	1	1	1	0
		$\tilde{\eta}_2$	0	0	-1	-1	1	0

In terms of these component fields, the Lagrangian terms can be written as

$$\begin{aligned}
L_{vec} + L_{FI} &= \sum_{i=1}^3 \frac{1}{2e^2} \left[(\partial_t \sigma_i)^2 + i (\bar{\lambda}_{i,1} \partial_t \lambda_i^1 + \lambda_i^1 \partial_t \bar{\lambda}_{i,1}) \right. \\
&\quad \left. + 2i \bar{\lambda}_{i,2} (\partial_t - 4i\epsilon) \lambda_i^2 - (M_{i,r})^2 - 2e^2 \xi_i M_{i,3} \right] , \\
e^2 L_{hyp} &= |D_{t,1} \phi_1^A|^2 - |\sigma_1 \phi_1^A|^2 + \bar{\phi}_{1,A} (\sigma^r)^A_B M_{1,r} \phi_1^B + \frac{i}{2} \left(\bar{\psi}_1 (D_{t,1} + i\sigma_1) \psi_1 \right. \\
&\quad \left. + \psi_1 (\tilde{D}_{t,1} - i\sigma_1) \bar{\psi}_1 + \bar{\tilde{\psi}}_1 (\tilde{D}_{t,1} - i\sigma_1) \tilde{\psi}_1 + \tilde{\psi}_1 (D_{t,1} + i\sigma_1) \bar{\tilde{\psi}}_1 \right) \\
&\quad + \frac{i}{\sqrt{2}} \left(\bar{\phi}_{1,A} \lambda_1^A \psi_1 - \bar{\psi}_1 \bar{\lambda}_{1,A} \phi_1^A + \bar{\phi}_{1,A} \bar{\lambda}_1^A \tilde{\psi}_1 - \tilde{\psi}_1 \lambda_{1,A} \phi_1^A \right) \\
&\quad + |D_{t,3} \phi_2^A|^2 - |\sigma_3 \phi_1^A|^2 + \bar{\phi}_{2,A} (\sigma^r)^A_B M_{3,r} \phi_2^B + \frac{i}{2} \left(\bar{\psi}_2 (D_{t,3} + i\sigma_3) \psi_2 \right. \\
&\quad \left. + \psi_2 (\tilde{D}_{t,3} - i\sigma_3) \bar{\psi}_2 + \bar{\tilde{\psi}}_2 (\tilde{D}_{t,3} - i\sigma_3) \tilde{\psi}_2 + \tilde{\psi}_2 (D_{t,3} + i\sigma_3) \bar{\tilde{\psi}}_2 \right) \\
&\quad + \frac{i}{\sqrt{2}} \left(\bar{\phi}_{2,A} \lambda_3^A \psi_2 - \bar{\psi}_2 \bar{\lambda}_{3,A} \phi_2^A + \bar{\phi}_{2,A} \bar{\lambda}_3^A \tilde{\psi}_2 - \tilde{\psi}_2 \lambda_{3,A} \phi_2^A \right) , \\
e^2 L_{bf} &= |D_{t,21} \underline{\phi}_1^A|^2 - |\sigma_{21} \underline{\phi}_1^A|^2 + \bar{\underline{\phi}}_{1,A} (\sigma^r)^A_B M_{21,r} \underline{\phi}_1^B + \frac{i}{2} \left(\bar{\psi}_1 (D_{t,21} + i\sigma_{21}) \psi_1 \right. \\
&\quad \left. + \psi_1 (\tilde{D}_{t,21} - i\sigma_{21}) \bar{\psi}_1 + \bar{\tilde{\psi}}_1 (\tilde{D}_{t,21} - i\sigma_{21}) \tilde{\psi}_1 + \tilde{\psi}_1 (D_{t,21} + i\sigma_{21}) \bar{\tilde{\psi}}_1 \right) \\
&\quad + \frac{i}{\sqrt{2}} \left(\bar{\underline{\phi}}_{1,A} \lambda_{21}^A \psi_1 - \bar{\psi}_1 \bar{\lambda}_{21,A} \underline{\phi}_1^A + \bar{\underline{\phi}}_{1,A} \bar{\lambda}_{21}^A \tilde{\psi}_1 - \tilde{\psi}_1 \lambda_{21,A} \underline{\phi}_1^A \right) + |D_{t,32} \underline{\phi}_2^A|^2 \\
&\quad - |\sigma_{32} \underline{\phi}_2^A|^2 + \bar{\underline{\phi}}_{2,A} (\sigma^r)^A_B M_{32,r} \underline{\phi}_2^B + \frac{i}{2} \left(\bar{\psi}_2 (D_{t,32} + i\sigma_{32}) \psi_2 \right. \\
&\quad \left. + \psi_2 (\tilde{D}_{t,32} - i\sigma_{32}) \bar{\psi}_2 + \bar{\tilde{\psi}}_2 (\tilde{D}_{t,32} - i\sigma_{32}) \tilde{\psi}_2 + \tilde{\psi}_2 (D_{t,32} + i\sigma_{32}) \bar{\tilde{\psi}}_2 \right) \\
&\quad + \frac{i}{\sqrt{2}} \left(\bar{\underline{\phi}}_{2,A} \lambda_{32}^A \psi_2 - \bar{\psi}_2 \bar{\lambda}_{32,A} \underline{\phi}_2^A + \bar{\underline{\phi}}_{2,A} \bar{\lambda}_{32}^A \tilde{\psi}_2 - \tilde{\psi}_2 \lambda_{32,A} \underline{\phi}_2^A \right) .
\end{aligned} \tag{D.115}$$

where we used the notation

$$(D_{t,ij} \pm i\sigma_{ij}) = \partial_t + 2i(v_{i,t} - v_{j,t}) \pm i(\sigma_i - \sigma_j) \quad , \quad \lambda_{ij}^A = \lambda_i^A - \lambda_j^A . \tag{D.116}$$

L_{theory} can have contributions from terms of the form

$$L_{theory} = L_{Fermi} + L_{adj\,hyp} . \tag{D.117}$$

In components, this can be written

$$\begin{aligned}
e^2 L_{Fermi} &= \sum_j \left[\frac{i}{2} \left(\bar{\eta}_i (D_{t,2} + i\sigma_2) \eta_i + \eta_i (\tilde{D}_{t,2} - i\sigma_2) \bar{\eta}_i \right) + |G_i|^2 + m_j [\bar{\eta}_j, \eta_j] \right] , \\
L_{adj\ hyp} &= \sum_i \frac{1}{e^2} \left[|\partial_t \rho_i^A|^2 + (m + \epsilon) |\rho_i|^2 + (m - \epsilon)^2 |\tilde{\rho}_i|^2 \right. \\
&\quad \left. + \sum_I \left(\frac{i}{2} (\bar{\chi}_{i,I} \partial_t \chi_{i,I} + \chi_{i,I} \partial_t \bar{\chi}_{i,I}) - \frac{m}{2} [\bar{\chi}_{i,I}, \chi_{i,I}] \right) \right] ,
\end{aligned} \tag{D.118}$$

where $\chi_{i,I} = (\chi_i, \bar{\chi}_i)$.

Now we can go to the Hamiltonian formalism and integrate out the auxiliary fields. We will again scale our fermionic fields such that they obey anti-commutation relations of the form

$$\{\bar{\psi}_i, \psi_j\} = \delta_{ij} \quad , \quad \{\bar{\eta}_i, \eta_j\} = \delta_{ij} . \tag{D.119}$$

Now the total Hamiltonian will be form the form

$$H = H_{vm} + U + H_{matter} + H_I + \sum_i v_{t,i} Q_{Gauge}^{(i)} , \tag{D.120}$$

where

$$\begin{aligned}
H_{vm} &= \frac{e_1^2 p_{\sigma 1}^2}{2} + \frac{e_2^2 p_{\sigma 2}^2}{2} + \frac{e_3^2 p_{\sigma 3}^2}{2} - 4\epsilon \sum_i \bar{\lambda}_{i,2} \lambda_i^2 + H_{adj\ hyp} , \\
H_{matter} &= e^2 [|\pi_1|^2 + |\pi_2|^2 + |\tilde{\pi}_1|^2 + |\tilde{\pi}_2|^2 + |\underline{\pi}_1|^2 + |\underline{\pi}_2|^2 + |\tilde{\underline{\pi}}_1|^2 + |\tilde{\underline{\pi}}_2|^2] \\
&\quad + \frac{1}{e^2} \left(\omega_1^2 |\phi_1|^2 + \omega_2^2 |\phi_2|^2 + \tilde{\omega}_1^2 |\tilde{\phi}_1|^2 + \tilde{\omega}_2^2 |\tilde{\phi}_2|^2 + \underline{\omega}_1^2 |\underline{\phi}_1|^2 + \underline{\omega}_2^2 |\underline{\phi}_2|^2 + \tilde{\underline{\omega}}_1^2 |\tilde{\underline{\phi}}_1|^2 + \tilde{\underline{\omega}}_2^2 |\tilde{\underline{\phi}}_2|^2 \right) \\
&\quad + \frac{1}{2} \left((\sigma_1 - a + \epsilon) [\bar{\psi}_1, \psi_1] - (\sigma_1 - a - \epsilon) [\bar{\tilde{\psi}}_1, \tilde{\psi}_1] \right. \\
&\quad \left. + (\sigma_3 + a + \epsilon) [\bar{\psi}_2, \psi_2] - (\sigma_3 + a - \epsilon) [\bar{\tilde{\psi}}_2, \tilde{\psi}_2] \right) \\
&\quad - \frac{1}{2} \left((\sigma_{21} + \epsilon) [\bar{\underline{\psi}}_1, \underline{\psi}_1] - (\sigma_{21} - \epsilon) [\bar{\tilde{\underline{\psi}}}_1, \tilde{\underline{\psi}}_1] + (\sigma_{21} + \epsilon) [\bar{\underline{\psi}}_2, \underline{\psi}_2] - (\sigma_{32} - \epsilon) [\bar{\tilde{\underline{\psi}}}_2, \tilde{\underline{\psi}}_2] \right) \\
&\quad + H_{Fermi} ,
\end{aligned} \tag{D.121}$$

where

$$\begin{aligned}\omega_1 &= |\sigma_1 - a + \epsilon| \quad , \quad \tilde{\omega}_1 = |\sigma_1 - a - \epsilon| \quad , \quad \omega_2 = |\sigma_3 + a + \epsilon| \quad , \quad \tilde{\omega}_2 = |\sigma_3 + a - \epsilon| \quad , \\ \underline{\omega}_1 &= |\sigma_{21} + \epsilon| \quad , \quad \underline{\tilde{\omega}}_1 = |\sigma_{21} - \epsilon| \quad , \quad \underline{\omega}_2 = |\sigma_{32} + \epsilon| \quad , \quad \underline{\tilde{\omega}}_2 = |\sigma_{32} - \epsilon| \quad , \\ \omega_{i+2} &= |m + \epsilon| \quad , \quad \tilde{\omega}_{i+2} = |m - \epsilon| \quad , \quad \text{for } a, \epsilon > 0 \quad .\end{aligned}\tag{D.122}$$

and

$$H_{Fermi} = \frac{1}{2} \sum_{j=1}^{N_f} (\sigma_2 - 2m_f) [\bar{\eta}_j, \eta_j] \quad ,\tag{D.123}$$

or

$$\begin{aligned}H_{Fermi} &= (\sigma_1 - a + m) [\bar{\eta}_1, \eta_1] - (\sigma_1 - a - m) [\bar{\tilde{\eta}}_1, \tilde{\eta}_1] \\ &\quad + (\sigma_3 + a + m) [\bar{\eta}_2, \eta_2] - (\sigma_3 + a - m) [\bar{\tilde{\eta}}_2, \tilde{\eta}_2] \quad ,\end{aligned}\tag{D.124}$$

for the 4D theories with N_f -hypermultiplets or for the $\mathcal{N} = 2^*$ theory respectively and $H_{adj\ hyp}$ is only included for the $\mathcal{N} = 2^*$ theory and is given by

$$H_{adj\ hyp} = \sum_{i=1}^3 \left[e^2 |\pi_{i+2}|^2 + e^2 |\tilde{\pi}_{i+2}|^2 + \frac{\omega_{i+2}^2}{e^2} |\rho_i|^2 + \frac{\tilde{\omega}_{i+2}^2}{e^2} |\tilde{\rho}_i|^2 + \frac{m}{2} \sum_I [\bar{\chi}_{i,I}, \chi_{i,I}] \right] \quad .\tag{D.125}$$

Additionally,

$$\begin{aligned}U &= \frac{1}{2e^2} \left(|\phi_1|^2 - |\tilde{\phi}_1|^2 - |\underline{\phi}_1|^2 + |\underline{\tilde{\phi}}_1|^2 - e^2 \xi_1 \right)^2 \\ &\quad + \frac{1}{2e^2} \left(|\underline{\phi}_1|^2 - |\underline{\tilde{\phi}}_1|^2 - |\underline{\phi}_2|^2 + |\underline{\tilde{\phi}}_2|^2 - e^2 \xi_2 \right)^2 \\ &\quad + \frac{1}{2e^2} \left(|\underline{\phi}_2|^2 - |\underline{\tilde{\phi}}_2|^2 + |\phi_2|^2 - |\tilde{\phi}_2|^2 - e^2 \xi_3 \right)^2 \\ &\quad + \frac{1}{e^2} |\phi_1 \tilde{\phi}_1 - \underline{\phi}_1 \underline{\tilde{\phi}}_1|^2 + \frac{1}{e^2} |\underline{\phi}_1 \underline{\tilde{\phi}}_1 - \underline{\phi}_2 \underline{\tilde{\phi}}_2|^2 + \frac{1}{e^2} |\underline{\phi}_2 \underline{\tilde{\phi}}_2 + \phi_2 \tilde{\phi}_2|^2 \quad ,\end{aligned}\tag{D.126}$$

and

$$\begin{aligned}
H_I = & \frac{i}{\sqrt{2}} \left(\bar{\phi}_1 \lambda_1^1 \psi_1 + \tilde{\phi}_1 \lambda_1^2 \psi_1 + \tilde{\phi}_1 \bar{\lambda}_{1,1} \tilde{\psi}_1 - \bar{\phi}_1 \bar{\lambda}_{1,2} \tilde{\psi}_1 \right. \\
& \left. - \bar{\psi}_1 \bar{\lambda}_{1,1} \phi_1 - \bar{\psi}_1 \bar{\lambda}_{1,2} \tilde{\phi}_1 - \tilde{\psi}_1 \lambda_1^1 \tilde{\phi}_1 + \tilde{\psi}_1 \lambda_1^2 \phi_1 \right) \\
& + \frac{i}{\sqrt{2}} \left(\bar{\phi}_2 \lambda_3^1 \psi_2 + \tilde{\phi}_2 \lambda_3^2 \psi_2 + \tilde{\phi}_2 \bar{\lambda}_{3,1} \tilde{\psi}_2 - \bar{\phi}_2 \bar{\lambda}_{3,2} \tilde{\psi}_2 \right. \\
& \left. - \bar{\psi}_2 \bar{\lambda}_{3,1} \phi_2 - \bar{\psi}_2 \bar{\lambda}_{3,2} \tilde{\phi}_2 - \tilde{\psi}_2 \lambda_3^1 \tilde{\phi}_2 + \tilde{\psi}_2 \lambda_3^2 \phi_2 \right) \\
& + \frac{i}{\sqrt{2}} \left(\bar{\phi}_1 \lambda_{21}^1 \psi_1 + \tilde{\phi}_1 \lambda_{21}^2 \psi_1 + \tilde{\phi}_1 \bar{\lambda}_{21,1} \tilde{\psi}_1 - \bar{\phi}_1 \bar{\lambda}_{21,2} \tilde{\psi}_1 \right. \\
& \left. - \bar{\psi}_1 \bar{\lambda}_{21,1} \phi_1 - \bar{\psi}_1 \bar{\lambda}_{21,2} \tilde{\phi}_1 - \tilde{\psi}_1 \lambda_{21}^1 \tilde{\phi}_1 + \tilde{\psi}_1 \lambda_{21}^2 \phi_1 \right) \\
& + \frac{i}{\sqrt{2}} \left(\bar{\phi}_2 \lambda_{32}^1 \psi_2 + \tilde{\phi}_2 \lambda_{32}^2 \psi_2 + \tilde{\phi}_2 \bar{\lambda}_{32,1} \tilde{\psi}_2 - \bar{\phi}_2 \bar{\lambda}_{32,2} \tilde{\psi}_2 \right. \\
& \left. - \bar{\psi}_2 \bar{\lambda}_{32,1} \phi_2 - \bar{\psi}_2 \bar{\lambda}_{32,2} \tilde{\phi}_2 - \tilde{\psi}_2 \lambda_{32}^1 \tilde{\phi}_2 + \tilde{\psi}_2 \lambda_{32}^2 \phi_2 \right) .
\end{aligned} \tag{D.127}$$

Now by identifying $\phi_{i+2} = \rho_i$, we can define the operators

$$\begin{aligned}
a_i &= \frac{1}{\sqrt{2}e} \left(\omega_i \phi_i + \frac{ie^2 \bar{\pi}_i}{\omega_i} \right) , & \bar{a}_i &= \frac{1}{\sqrt{2}e} \left(\omega_i \bar{\phi}_i + \frac{ie^2 \pi_i}{\omega_i} \right) , \\
\tilde{a}_i &= \frac{1}{\sqrt{2}e} \left(\tilde{\omega}_i \tilde{\phi}_i + \frac{ie^2 \tilde{\pi}_i}{\tilde{\omega}_i} \right) , & \tilde{\bar{a}}_i &= \frac{1}{\sqrt{2}e} \left(\tilde{\omega}_i \tilde{\bar{\phi}}_i + \frac{ie^2 \tilde{\pi}_i}{\tilde{\omega}_i} \right) ,
\end{aligned} \tag{D.128}$$

for $i = 1, \dots, 5$ and

$$\begin{aligned}
\underline{a}_i &= \frac{1}{\sqrt{2}e} \left(\underline{\omega}_i \underline{\phi}_i + \frac{ie^2 \underline{\pi}_i}{\underline{\omega}_i} \right) , & \underline{\bar{a}}_i &= \frac{1}{\sqrt{2}e} \left(\underline{\omega}_i \underline{\bar{\phi}}_i + \frac{ie^2 \underline{\pi}_i}{\underline{\omega}_i} \right) , \\
\tilde{\underline{a}}_i &= \frac{1}{\sqrt{2}e} \left(\tilde{\underline{\omega}}_i \tilde{\underline{\phi}}_i + \frac{ie^2 \tilde{\underline{\pi}}_i}{\tilde{\underline{\omega}}_i} \right) , & \tilde{\underline{\bar{a}}}_i &= \frac{1}{\sqrt{2}e} \left(\tilde{\underline{\omega}}_i \tilde{\underline{\bar{\phi}}}_i + \frac{ie^2 \tilde{\underline{\pi}}_i}{\tilde{\underline{\omega}}_i} \right) ,
\end{aligned} \tag{D.129}$$

for $i = 1, 2$.

Now we can write the matter Hamiltonian as

$$\begin{aligned}
H_{matter} = & \omega_1(a_1^\dagger a_1 + \bar{a}_1^\dagger \bar{a}_1 + 1) + (\sigma_1 - a + \epsilon)(\bar{\psi}_1 \psi_1 - \frac{1}{2}) \\
& + \tilde{\omega}_1(\tilde{a}_1^\dagger \tilde{a}_1 + \bar{\tilde{a}}_1^\dagger \bar{\tilde{a}}_1 + 1) - (\sigma_1 - a - \epsilon)(\tilde{\psi}_1 \tilde{\psi}_1 - \frac{1}{2}) \\
& + \omega_2(a_2^\dagger a_2 + \bar{a}_2^\dagger \bar{a}_2 + 1) + (\sigma_3 + a + \epsilon)(\bar{\psi}_2 \psi_2 - \frac{1}{2}) \\
& + \tilde{\omega}_2(\tilde{a}_2^\dagger \tilde{a}_2 + \bar{\tilde{a}}_2^\dagger \bar{\tilde{a}}_2 + 1) - (\sigma_3 + a - \epsilon)(\tilde{\psi}_2 \tilde{\psi}_2 - \frac{1}{2}) \\
& + \underline{\omega}_1(\underline{a}_1^\dagger \underline{a}_1 + \bar{\underline{a}}_1^\dagger \bar{\underline{a}}_1 + 1) + (\sigma_2 - \sigma_1 + \epsilon)(\bar{\underline{\psi}}_1 \underline{\psi}_1 - \frac{1}{2}) \\
& + \underline{\tilde{\omega}}_1(\underline{\tilde{a}}_1^\dagger \underline{\tilde{a}}_1 + \bar{\underline{\tilde{a}}}_1^\dagger \bar{\underline{\tilde{a}}}_1 + 1) - (\sigma_2 - \sigma_1 - \epsilon)(\bar{\underline{\tilde{\psi}}}_1 \underline{\tilde{\psi}}_1 - \frac{1}{2}) \\
& + \underline{\omega}_2(\underline{a}_2^\dagger \underline{a}_2 + \bar{\underline{a}}_2^\dagger \bar{\underline{a}}_2 + 1) + (\sigma_3 - \sigma_2 + \epsilon)(\bar{\underline{\psi}}_2 \underline{\psi}_2 - \frac{1}{2}) \\
& + \underline{\tilde{\omega}}_2(\underline{\tilde{a}}_2^\dagger \underline{\tilde{a}}_2 + \bar{\underline{\tilde{a}}}_2^\dagger \bar{\underline{\tilde{a}}}_2 + 1) - (\sigma_3 - \sigma_2 - \epsilon)(\bar{\underline{\tilde{\psi}}}_2 \underline{\tilde{\psi}}_2 - \frac{1}{2}) \\
& + H_{Fermi} .
\end{aligned} \tag{D.130}$$

here again H_{Fermi} is given by (D.123) or (D.124) for the case of the corresponding 4D theory having N_f fundamental hypermultiplets or being the $\mathcal{N} = 2^*$ theory respectively.

These operators also allow us to write $Q_{\text{Gauge}}^{(i)}$ simply as

$$\begin{aligned}
Q_{\text{Gauge}}^{(1)} = & - \left(a_1^\dagger a_1 - \bar{a}_1^\dagger \bar{a}_1 - \tilde{a}_1^\dagger \tilde{a}_1 + \bar{\tilde{a}}_1^\dagger \bar{\tilde{a}}_1 - \underline{a}_1^\dagger \underline{a}_1 + \bar{\underline{a}}_1^\dagger \bar{\underline{a}}_1 + \tilde{a}_1^\dagger \tilde{a}_1 - \bar{\tilde{a}}_1^\dagger \bar{\tilde{a}}_1 \right) \\
& - \left(\bar{\psi}_1 \psi_1 - \tilde{\bar{\psi}}_1 \tilde{\psi}_1 - \bar{\underline{\psi}}_1 \underline{\psi}_1 + \bar{\underline{\tilde{\psi}}}_1 \underline{\tilde{\psi}}_1 \right) + Q_{theory}^{(1)} , \\
Q_{\text{Gauge}}^{(2)} = & - \left(\underline{a}_1^\dagger \underline{a}_1 - \bar{\underline{a}}_1^\dagger \bar{\underline{a}}_1 - \tilde{a}_1^\dagger \tilde{a}_1 + \bar{\tilde{a}}_1^\dagger \bar{\tilde{a}}_1 \right) + \left(\underline{a}_2^\dagger \underline{a}_2 - \bar{\underline{a}}_2^\dagger \bar{\underline{a}}_2 - \tilde{a}_2^\dagger \tilde{a}_2 + \bar{\tilde{a}}_2^\dagger \bar{\tilde{a}}_2 \right) \\
& - \left(\bar{\underline{\psi}}_1 \underline{\psi}_1 - \bar{\underline{\tilde{\psi}}}_1 \underline{\tilde{\psi}}_1 \right) + \left(\bar{\underline{\psi}}_2 \underline{\psi}_2 - \bar{\underline{\tilde{\psi}}}_2 \underline{\tilde{\psi}}_2 \right) + Q_{theory}^{(2)} , \\
Q_{\text{Gauge}}^{(3)} = & - \left(\underline{a}_2^\dagger \underline{a}_2 - \bar{\underline{a}}_2^\dagger \bar{\underline{a}}_2 - \tilde{a}_2^\dagger \tilde{a}_2 + \bar{\tilde{a}}_2^\dagger \bar{\tilde{a}}_2 \right) - \left(\underline{a}_2^\dagger \underline{a}_2 - \bar{\underline{a}}_2^\dagger \bar{\underline{a}}_2 - \tilde{a}_2^\dagger \tilde{a}_2 + \bar{\tilde{a}}_2^\dagger \bar{\tilde{a}}_2 \right) \\
& - \left(\bar{\underline{\psi}}_2 \underline{\psi}_2 - \bar{\underline{\tilde{\psi}}}_2 \underline{\tilde{\psi}}_2 \right) - \left(\bar{\underline{\psi}}_2 \underline{\psi}_2 - \bar{\underline{\tilde{\psi}}}_2 \underline{\tilde{\psi}}_2 \right) + Q_{theory}^{(3)} .
\end{aligned} \tag{D.131}$$

where

$$Q_{theory}^{(2)} = -\frac{1}{2} \sum_{j=1}^{N_f} [\bar{\eta}_j, \eta_j] \quad , \quad Q_{theory}^{(1)} = Q_{theory}^{(3)} = 0 \tag{D.132}$$

for the theory with N_f fundamental hypermultiplets and

$$\begin{aligned}
Q_{theory}^{(1)} = & -\frac{1}{2} [\bar{\eta}_1, \eta_1] + \frac{1}{2} [\bar{\tilde{\eta}}_1, \tilde{\eta}_1] , \\
Q_{theory}^{(3)} = & -\frac{1}{2} [\bar{\eta}_2, \eta_2] + \frac{1}{2} [\bar{\tilde{\eta}}_2, \tilde{\eta}_2] , \\
Q_{theory}^{(2)} = & 0 ,
\end{aligned} \tag{D.133}$$

for the case of the $\mathcal{N} = 2^*$ theory.

Let us again pick a basis of states for our Hilbert space

$$\begin{aligned} \left| (n_i, \bar{n}_i, \tilde{n}_i, \bar{\tilde{n}}_i, m_i, \tilde{m}_i) ; (\underline{n}_i, \underline{\bar{n}}_i, \underline{\tilde{n}}_i, \underline{\bar{\tilde{n}}}_i) ; (f_j) \right\rangle = \\ \left(\prod_{i=1}^5 (a_i^\dagger)^{n_i} (\bar{a}_i^\dagger)^{\bar{n}_i} (\tilde{a}_i^\dagger)^{\tilde{n}_i} (\bar{\tilde{a}}_i^\dagger)^{\bar{\tilde{n}}_i} \bar{\psi}_i^{m_i} \tilde{\psi}_i^{\tilde{m}_i} \right) \times \\ \left(\prod_{i=1}^2 (\underline{a}_i^\dagger)^{\underline{n}_i} (\underline{\bar{a}}_i^\dagger)^{\underline{\bar{n}}_i} (\underline{\tilde{a}}_i^\dagger)^{\underline{\tilde{n}}_i} (\underline{\bar{\tilde{a}}}_i^\dagger)^{\underline{\bar{\tilde{n}}}_i} \underline{\bar{\psi}}_i^{m_i} \underline{\tilde{\psi}}_i^{\tilde{m}_i} \right) \times \left(\prod_{j=1}^4 \bar{\eta}_j^{f_j} \right) |0\rangle , \end{aligned} \quad (\text{D.134})$$

for the case of 4D fundamental matter or

$$\begin{aligned} \left| (n_i, \bar{n}_i, \tilde{n}_i, \bar{\tilde{n}}_i, m_i, \tilde{m}_i) ; (\underline{n}_i, \underline{\bar{n}}_i, \underline{\tilde{n}}_i, \underline{\bar{\tilde{n}}}_i) ; (f_j) \right\rangle = \\ \left(\prod_{i=1}^5 (a_i^\dagger)^{n_i} (\bar{a}_i^\dagger)^{\bar{n}_i} (\tilde{a}_i^\dagger)^{\tilde{n}_i} (\bar{\tilde{a}}_i^\dagger)^{\bar{\tilde{n}}_i} \bar{\psi}_i^{m_i} \tilde{\psi}_i^{\tilde{m}_i} \right) \times \\ \left(\prod_{i=1}^2 (\underline{a}_i^\dagger)^{\underline{n}_i} (\underline{\bar{a}}_i^\dagger)^{\underline{\bar{n}}_i} (\underline{\tilde{a}}_i^\dagger)^{\underline{\tilde{n}}_i} (\underline{\bar{\tilde{a}}}_i^\dagger)^{\underline{\bar{\tilde{n}}}_i} \underline{\bar{\psi}}_i^{m_i} \underline{\tilde{\psi}}_i^{\tilde{m}_i} \right) \times \left(\prod_{j=1}^2 \bar{\eta}_j^{f_j} \bar{\tilde{\eta}}_j^{\tilde{f}_j} \right) |0\rangle , \end{aligned} \quad (\text{D.135})$$

for the case of $\mathcal{N} = 2^*$ theory where we have identified $\chi_{1,i} = \psi_{i+2}$ and $\chi_{2,i} = \tilde{\psi}_{i+2}$ and the vacuum state is defined as

$$\begin{aligned} a_i|0\rangle = \bar{a}_i|0\rangle = \tilde{a}_i|0\rangle = \bar{\tilde{a}}_i|0\rangle = \underline{a}_i|0\rangle = \underline{\bar{a}}_i|0\rangle = \underline{\tilde{a}}_i|0\rangle = \underline{\bar{\tilde{a}}}_i|0\rangle = 0 , \\ \psi_i|0\rangle = \tilde{\psi}_i|0\rangle = \underline{\psi}_i|0\rangle = \underline{\tilde{\psi}}_i|0\rangle = \eta_j|0\rangle = \tilde{\eta}_j|0\rangle = 0 . \end{aligned} \quad (\text{D.136})$$

Thus, the quantum numbers are constrained

$$n_i, \bar{n}_i, \tilde{n}_i, \bar{\tilde{n}}_i, \underline{n}_i, \underline{\bar{n}}_i, \underline{\tilde{n}}_i, \underline{\bar{\tilde{n}}}_i \in \mathbb{Z}_+ \quad , \quad m_i, \tilde{m}_i, f_j = 0, 1 . \quad (\text{D.137})$$

Now as before we want to solve for gauge invariant BPS states. These satisfy

$$(H - Z)|\Psi\rangle = 0 \quad , \quad Q_{\text{Gauge}}|\Psi\rangle = 0 , \quad (\text{D.138})$$

where

$$Z = -aQ_a - \epsilon Q_\epsilon - mF_m - \sum_{j=1}^{N_f} m_j F_j , \quad (\text{D.139})$$

where F_m and F_j are the flavor charges associated with the 4D adjoint hypermultiplet and 4D fundamental hypermultiplets respectively.

As in the $U(1)$ case, there are unique matter ground states for the regions⁹

$$S_+ = \{\sigma_1 > a + \epsilon, \sigma_3 > -a + \epsilon, \sigma_2 > \sigma_1 + \epsilon, \sigma_3 > \sigma_2 + \epsilon\} . \quad (\text{D.140})$$

and

$$S_- = \{\sigma_1 < a - \epsilon, \sigma_3 < -a - \epsilon, \sigma_2 < \sigma_1 - \epsilon, \sigma_3 < \sigma_2 - \epsilon\} , \quad (\text{D.141})$$

which have the quantum numbers

$$\begin{aligned} S_+ : N_i, \bar{N}_i, \tilde{N}_i, \bar{\tilde{N}}_i, \tilde{m}_i, \tilde{m}_1, \underline{m}_2 = 0, m_i, \underline{m}_1, \tilde{m}_2 = 1, f_j = 0, \\ S_- : N_i, \bar{N}_i, \tilde{N}_i, \bar{\tilde{N}}_i, m_i, \underline{m}_1, \tilde{m}_2 = 0, \tilde{m}_i, \tilde{m}_1, \underline{m}_2 = 1, f_j = 1, \end{aligned} \quad (\text{D.142})$$

where here we use the notation $\{N_i, \bar{N}_i, \tilde{N}_i, \bar{\tilde{N}}_i, M_i, \tilde{M}_i\}_{i=1}^4$ to collectively refer to the quantum numbers of all hypermultiplets where $i = 1, 2$ correspond to the fundamental hypermultiplets and $i = 3, 4$ correspond to the 1^{st} and 2^{nd} bi-fundamental hypermultiplets respectively.

We will denote the matter ground state wave functions in these regions as

$$\begin{aligned} |\Psi_+\rangle &= \delta_{S_+} |N_i, \bar{N}_i, \tilde{N}_i, \bar{\tilde{N}}_i, \tilde{m}_i, \tilde{m}_1, \underline{m}_2 = 0, m_i, \underline{m}_1, \tilde{m}_2 = 1, f_j = 0\rangle, \\ |\Psi_-\rangle &= \delta_{S_-} |N_i, \bar{N}_i, \tilde{N}_i, \bar{\tilde{N}}_i, m_i, \underline{m}_1, \tilde{m}_2 = 0, \tilde{m}_i, \tilde{m}_1, \underline{m}_2 = 1, f_j = 1\rangle, \end{aligned} \quad (\text{D.143})$$

where δ_S is the indicator function for the set S .

D.2.1 Effective Hamiltonian

In analogy with the procedure in Appendix D.1 we can compute the effective Hamiltonian by integrating out the fundamental hypermultiplet and Fermi-multiplet matter.

In this SQM, the supercharge is of the form:

$$\begin{aligned} Q_A &= Q_{\text{matter},A} - Q_{\text{vec},A}, \\ Q_{\text{vec},A} &= \sum_{i=1}^3 \frac{e}{\sqrt{2}} (-ip_{\sigma i} \bar{\lambda}_{i,A} + M_r(\sigma^r)_A{}^B \bar{\lambda}_B), \end{aligned} \quad (\text{D.144})$$

and $Q_{\text{matter},A}$ is analogous to the first terms of (D.12) which annihilate the harmonic oscillator wave functions of the matter fields. Now the effective supercharge is of the

⁹Recall that we are assuming $a, \epsilon > 0$.

form

$$Q_{\text{eff},A} = \langle Q_{\text{vec},A} \rangle = \sum_{i=1}^3 \frac{e}{\sqrt{2}} \left(-i \langle p_{\sigma i} \rangle \bar{\lambda}_{i,A} - \frac{1}{e^2} \langle D_i \rangle (\sigma^3)_A{}^B \bar{\lambda}_B - \frac{\sqrt{2}}{e^2} \langle F_i \rangle (\sigma^+)_A{}^B \bar{\lambda}_B - \frac{\sqrt{2}}{e^2} \langle \bar{F}_i \rangle (\sigma^-)_A{}^B \bar{\lambda}_B \right) , \quad (\text{D.145})$$

where

$$\begin{aligned} F_1 &= \left(\phi_1 \tilde{\phi}_1 - \underline{\phi}_1 \underline{\tilde{\phi}}_1 \right) , \quad F_2 = \left(\underline{\phi}_1 \tilde{\phi}_1 - \phi_2 \tilde{\phi}_2 \right) , \quad F_3 = \left(\underline{\phi}_2 \tilde{\phi}_2 + \phi_2 \tilde{\phi}_2 \right) , \\ D_1 &= \left(|\phi_1|^2 - |\tilde{\phi}_1|^2 - |\underline{\phi}_1|^2 + |\underline{\tilde{\phi}}_1|^2 - e^2 \xi_1 \right) , \\ D_2 &= \left(|\underline{\phi}_1|^2 - |\tilde{\phi}_1|^2 - |\underline{\phi}_2|^2 + |\underline{\tilde{\phi}}_2|^2 - e^2 \xi_2 \right) , \\ D_3 &= \left(|\underline{\phi}_2|^2 - |\underline{\tilde{\phi}}_2|^2 + |\phi_2|^2 - |\tilde{\phi}_2|^2 - e^2 \xi_3 \right) . \end{aligned} \quad (\text{D.146})$$

Now by using the form of (D.146), we see that $\langle F_i \rangle = \langle \bar{F}_i \rangle = 0$, $\forall i$. Again, due to having a non-zero ϵ, a , we have broken SUSY to $\mathcal{N} = (0, 2)$, preserving the supercharges $Q_{\text{eff}1}, \bar{Q}_{\text{eff}}^1$.

We can now compute the effective Hamiltonian by squaring the supercharges

$$H_{\text{eff}} = \{ \bar{Q}_{\text{eff}}^1, Q_{\text{eff},1} \} - Z . \quad (\text{D.147})$$

Using the fact that Gauss's law imposes $Q_{\text{Gauge}}^{(i)} = 0$, $\forall i$, we have that only flavor charges contribute to the central charge. This gives rise to the central charge:

$$Z = 4\epsilon \sum_i \bar{\lambda}_{i,2} \lambda_i^2 - 6\epsilon - \sum_{f=1}^4 \text{m}_f [\bar{\eta}_f, \eta_f] . \quad (\text{D.148})$$

This gives us the full effective Hamiltonian:

$$\begin{aligned} H_{\text{eff}} &= \sum_i \frac{e^2 \langle p_{\sigma i} \rangle^2}{2} + \frac{1}{2e^2} \langle D_i \rangle^2 - \frac{1}{2} [\bar{\lambda}_{i,1}, \lambda_i^1] \partial_{\sigma i} \langle D_i \rangle \\ &\quad - 4\epsilon \sum_i \bar{\lambda}_{i,2} \lambda_i^2 + 6\epsilon + 2 \sum_{f=1}^4 \text{m}_f [\bar{\eta}_f, \eta_f] , \end{aligned} \quad (\text{D.149})$$

where

$$\langle D_i \rangle = -e^2 \xi + \frac{e^2}{2} \begin{cases} \frac{1}{\omega_1} - \frac{1}{\bar{\omega}_1} - \frac{1}{\omega_1} + \frac{1}{\bar{\omega}_1} & i = 1 \\ \frac{1}{\omega_1} - \frac{1}{\bar{\omega}_1} - \frac{1}{\omega_2} + \frac{1}{\bar{\omega}_2} & i = 2 \\ \frac{1}{\omega_2} - \frac{1}{\bar{\omega}_2} + \frac{1}{\omega_2} - \frac{1}{\bar{\omega}_2} & i = 3 \end{cases} \quad (\text{D.150})$$

and

$$i\langle p_{\sigma i} \rangle = \partial_{\sigma i} - \frac{1}{2} \begin{cases} \frac{1}{\sigma_1 - a + \epsilon} - \frac{1}{\sigma_1 - a - \epsilon} - \frac{1}{\sigma_{21} + \epsilon} - \frac{1}{\sigma_{21} - \epsilon} & i = 1 \\ \frac{1}{\sigma_{21} + \epsilon} + \frac{1}{\sigma_{21} - \epsilon} - \frac{1}{\sigma_{32} + \epsilon} - \frac{1}{\sigma_{32} - \epsilon} & i = 2 \\ \frac{1}{\sigma_3 + a + \epsilon} + \frac{1}{\sigma_3 + a - \epsilon} + \frac{1}{\sigma_{32} + \epsilon} + \frac{1}{\sigma_{32} - \epsilon} & i = 3 \end{cases} \quad (\text{D.151})$$

where the $s_i = \text{sign}(\arg(\omega_i))$, $\tilde{s}_i = \text{sign}(\arg(\tilde{\omega}_i))$ where $\omega_i, \tilde{\omega}_i$ are treated as the absolute value function of its argument. This gives rise to the effective supercharges:

$$\begin{aligned} Q_{\text{eff},1} &= \frac{e}{\sqrt{2}} \lambda_1^1 \left(-\partial_{\sigma_1} - \frac{1}{\tilde{\omega}_1} + \frac{1}{\tilde{\omega}_1} + \xi_1 \right) + \frac{e}{\sqrt{2}} \lambda_2^1 \left(-\partial_{\sigma_2} - \frac{1}{\tilde{\omega}_1} + \frac{1}{\tilde{\omega}_2} + \xi_2 \right) \\ &+ \frac{e}{\sqrt{2}} \lambda_3^1 \left(\partial_{\sigma_3} - \frac{1}{\tilde{\omega}_2} - \frac{1}{\tilde{\omega}_2} + \xi_3 \right) \\ \bar{Q}_{\text{eff}}^1 &= \frac{e}{\sqrt{2}} \lambda_1^1 \left(-\partial_{\sigma_1} + \frac{1}{\omega_1} - \frac{1}{\omega_1} + \xi_1 \right) + \frac{e}{\sqrt{2}} \lambda_2^1 \left(\partial_{\sigma_2} + \frac{1}{\omega_1} - \frac{1}{\omega_2} + \xi_2 \right) \\ &+ \frac{e}{\sqrt{2}} \bar{\lambda}_3 \left(\partial_{\sigma_3} + \frac{1}{\omega_2} + \frac{1}{\omega_2} + \xi_3 \right), \end{aligned} \quad (\text{D.152})$$

for S_+ and:

$$\begin{aligned} Q_{\text{eff},1} &= \frac{e}{\sqrt{2}} \bar{\lambda}_{1,1} \left(-\partial_{\sigma_1} + \frac{1}{\omega_1} - \frac{1}{\omega_1} + \xi_1 \right) + \frac{e}{\sqrt{2}} \bar{\lambda}_{2,1} \left(-\partial_{\sigma_2} + \frac{1}{\omega_1} - \frac{1}{\omega_2} + \xi_2 \right) \\ &+ \frac{e}{\sqrt{2}} \bar{\lambda}_{3,1} \left(-\partial_{\sigma_3} + \frac{1}{\omega_2} + \frac{1}{\omega_2} \right), \\ \bar{Q}_{\text{eff}}^1 &= \frac{e}{\sqrt{2}} \lambda_1^1 \left(\partial_{\sigma_1} - \frac{1}{\tilde{\omega}_1} + \frac{1}{\tilde{\omega}_1} + \xi_1 \right) + \frac{e}{\sqrt{2}} \lambda_2^1 \left(\partial_{\sigma_2} - \frac{1}{\tilde{\omega}_1} + \frac{1}{\tilde{\omega}_2} + \xi_2 \right) \\ &+ \frac{e}{\sqrt{2}} \lambda_3^1 \left(\partial_{\sigma_3} - \frac{1}{\tilde{\omega}_2} - \frac{1}{\tilde{\omega}_2} + \xi_3 \right), \end{aligned} \quad (\text{D.153})$$

for S_- .

D.2.2 Ground States

Unfortunately, solving for the ground states of this system is significantly more complicated than the last section. We have to balance an unknown choice of boundary conditions, Born-Oppenheimer approximation, and solving a system of partial differential equations.

Recall that in the Born-Oppenheimer approximation, we can only truly make sense of the quantum physics away from the boundaries. Thus, we are working in the limit

$$\epsilon/\sigma_1, \epsilon/\sigma_3, \epsilon/\sigma_{21}, \epsilon/\sigma_{32} \ll 1. \quad (\text{D.154})$$

Therefore, we will solve for ground states that are to first order in these parameters.

In order to study this differential operator, we will introduce a basis for the Clifford algebra:

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{pmatrix}_{\pm} = \begin{aligned} & f_1|\pm\rangle + f_2\bar{\lambda}_{1,1}|\pm\rangle + f_3\bar{\lambda}_{2,1}|\pm\rangle + f_4\bar{\lambda}_{1,1}\bar{\lambda}_{2,1}|\pm\rangle + f_5\bar{\lambda}_{3,1}|\pm\rangle \\ & + f_6\bar{\lambda}_{1,1}\bar{\lambda}_{3,1}|\pm\rangle + f_7\bar{\lambda}_{2,1}\bar{\lambda}_{3,1}|\pm\rangle + f_8\bar{\lambda}_{1,1}\bar{\lambda}_{2,1}\bar{\lambda}_{3,1}|\pm\rangle \end{aligned} \quad (\text{D.155})$$

In this basis, the Dirac operator $\mathcal{Q}_1 = Q_{\text{eff},1} + \bar{Q}_{\text{eff}}^1$, can be written as

$$\mathcal{Q}_1 = \begin{pmatrix} 0 & \mathcal{D}_3^- & \mathcal{D}_2^- & 0 & \mathcal{D}_1^- & 0 & 0 & 0 \\ \mathcal{D}_3^+ & 0 & 0 & \mathcal{D}_2^- & 0 & \mathcal{D}_1^- & 0 & 0 \\ \mathcal{D}_2^+ & 0 & 0 & \mathcal{D}_3^- & 0 & 0 & \mathcal{D}_1^- & 0 \\ 0 & \mathcal{D}_2^+ & \mathcal{D}_3^+ & 0 & 0 & 0 & 0 & \mathcal{D}_1^- \\ \mathcal{D}_1^+ & 0 & 0 & 0 & 0 & \mathcal{D}_3^- & \mathcal{D}_2^- & 0 \\ 0 & \mathcal{D}_1^+ & 0 & 0 & \mathcal{D}_3^+ & 0 & 0 & \mathcal{D}_2^- \\ 0 & 0 & \mathcal{D}_1^+ & 0 & \mathcal{D}_2^+ & 0 & 0 & \mathcal{D}_3^- \\ 0 & 0 & 0 & \mathcal{D}_1^+ & 0 & \mathcal{D}_2^+ & \mathcal{D}_3^+ & 0 \end{pmatrix} \quad (\text{D.156})$$

where

$$\mathcal{D}_i = -i\langle p_{\sigma_i} \rangle \quad , \quad D_i = \langle D_i \rangle \quad , \quad \mathcal{D}_i^{\pm} = \pm \mathcal{D}_i + D_i \quad . \quad (\text{D.157})$$

Now by taking wave functions that are functionally of the form

$$|\psi\rangle = \prod_i \omega_i \tilde{\omega}_i \underline{\omega}_i \underline{\tilde{\omega}}_i |\chi\rangle \quad , \quad (\text{D.158})$$

Now we are reduced to finding zero-eigenvectors of this matrix.

Recall that

$$D_i = 2 \begin{cases} \frac{1}{\omega_1} - \frac{1}{\tilde{\omega}_1} - \frac{1}{\omega_1} + \frac{1}{\tilde{\omega}_1} + \frac{\xi_1}{2} & i = 1 \\ \frac{1}{\omega_1} - \frac{1}{\tilde{\omega}_1} - \frac{1}{\omega_2} + \frac{1}{\tilde{\omega}_2} + \frac{\xi_2}{2} & i = 2 \\ \frac{1}{\omega_2} - \frac{1}{\tilde{\omega}_2} + \frac{1}{\omega_2} - \frac{1}{\tilde{\omega}_2} + \frac{\xi_3}{2} & i = 3 \end{cases} . \quad (\text{D.163})$$

Using the properties of the ω_i 's, we have that

$$\frac{1}{\omega_i} - \frac{1}{\tilde{\omega}_i} \sim O(\epsilon/\sigma^2) \sim 0 . \quad (\text{D.164})$$

Therefore, in our approximation, we only need to cancel the ξ_i 's which are not parametrically small and hence we can effectively replace D_i by ξ_i .

We see that $(1, 0, 0, 0, 0, 0, 0, 0)$ is clearly a 0-eigenvector and hence is a normalizable SUSY ground state. Now by rescaling our basis of eigenvectors by factors of ξ_i , we can see that there are additional approximate 0-eigenvectors such that the full space of ground states is given by

$$\begin{aligned} & \text{span}_{\mathbb{C}} \{ |v_1^{(+)}\rangle, |v_2^{(+)}\rangle, |v_3^{(+)}\rangle \} \\ &= \text{span}_{\mathbb{C}} \{ (1, 0, 0, 0, 0, 0, 0, 0)^{\text{tr}} , (0, 1, -1, 0, 0, 0, 0, 0)^{\text{tr}} , (0, 1, 0, 0, -1, 0, 0, 0)^{\text{tr}} \} . \end{aligned} \quad (\text{D.165})$$

Ground States in S_-

We can similarly perform the same analysis in the negative wedge. Here the analysis changes by looking for states that are annihilated by $\partial_i - D_i$. These states are of the form

$$|\chi\rangle = \frac{(\sigma_1 - a - \epsilon)(\sigma_3 + a - \epsilon)}{(\sigma_1 - a + \epsilon)(\sigma_3 + a + \epsilon)} \frac{(\sigma_{21} + \epsilon)(\sigma_{32} - \epsilon)}{(\sigma_{21} - \epsilon)(\sigma_{32} + \epsilon)} e^{\xi_1 \sigma_1 + \xi_2 \sigma_2 + \xi_3 \sigma_3} |\hat{\chi}\rangle . \quad (\text{D.166})$$

Acting on these states, the supercharge operator \mathcal{Q}_1 is of the form

$$\hat{\mathcal{Q}}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2D_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2D_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2D_2 & 2D_3 & 0 & 0 & 0 & 0 & 0 \\ 2D_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2D_1 & 0 & 0 & 2D_3 & 0 & 0 & 0 \\ 0 & 0 & 2D_1 & 0 & 2D_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2D_1 & 0 & 2D_2 & 2D_3 & 0 \end{pmatrix} \quad (\text{D.167})$$

Again we find 3 approximate BPS states

$$\begin{aligned} & \{|v_1^{(-)}\rangle, |v_2^{(-)}\rangle, |v_3^{(-)}\rangle\} \\ &= \{(0, 0, 0, 0, 0, 0, 0, 1)^{\text{tr}}, (0, 0, 0, 0, 0, -1, 1, 0)^{\text{tr}}, (0, 0, 0, -1, 0, 0, 1, 0)^{\text{tr}}\}. \end{aligned} \quad (\text{D.168})$$

Hermiticity

Again we have to impose boundary conditions so that the supercharges are Hermitian. The minimal boundary conditions to impose hermiticity allows us to keep all BPS states in both sectors. We will make a choice that is symmetric between exchange of $\bar{\lambda}_{I,1}$ and $\bar{\lambda}_{I,2}$ in analogy with the $U(1)$ case, and that is symmetric under $\bar{\lambda}_{1,I}$ and $\bar{\lambda}_{3,I}$.

So let us define the (unnormalized) states

$$\begin{aligned} |\psi_1^{(+)}\rangle &= \left(\sum_i \bar{\lambda}_{i,2} \right) |v_1^{(+)}\rangle, \quad |\psi_2^{(+)}\rangle = |v_2^{(+)}\rangle + |v_3^{(+)}\rangle, \\ |\psi_1^{(-)}\rangle &= \prod_i \bar{\lambda}_{i,2} |v_1^{(-)}\rangle, \quad |\psi_2^{(-)}\rangle = \bar{\lambda}_{1,2} \bar{\lambda}_{2,2} |v_2^{(-)}\rangle + \bar{\lambda}_{2,2} \bar{\lambda}_{3,2} |v_3^{(-)}\rangle. \end{aligned} \quad (\text{D.169})$$

Then it is consistent to pick boundary conditions such that the supersymmetric ground states are given by

$$\{|\psi_1^{(+)}\rangle, |\psi_2^{(+)}\rangle, |\tilde{\psi}_1^{(+)}\rangle, |\tilde{\psi}_2^{(+)}\rangle\}_{S_+} \cup \{|\psi_1^{(-)}\rangle, |\psi_2^{(-)}\rangle, |\tilde{\psi}_1^{(-)}\rangle, |\tilde{\psi}_2^{(-)}\rangle\}_{S_-}, \quad (\text{D.170})$$

where

$$|\tilde{\psi}_i^{(\pm)}\rangle = \tilde{P} |\psi_i^{(\pm)}\rangle, \text{ replace } \xi_i \leftrightarrow -\xi_i, \quad \tilde{P} = \prod_{i,A} (\lambda_i^A + \bar{\lambda}_{A,i}). \quad (\text{D.171})$$

Under this choice of boundary conditions, the normalizable asymptotic Coulomb branch states for $\xi_i > 0$ and $\xi_i < 0$ are given by

$$\{|\Psi_{BPS}\rangle\} = \begin{cases} \{|\psi_1^{(\pm)}\rangle, |\psi_2^{(\pm)}\rangle\} & \xi_i > 0, \forall i \\ \{|\tilde{\psi}_1^{(\pm)}\rangle, |\tilde{\psi}_2^{(\pm)}\rangle\} & \xi_i < 0, \forall i \end{cases} \quad (\text{D.172})$$

D.2.3 Mixed Branch

As it turns out, there are no mixed branch states in this theory. The reason is the following. The localization principal states that only finite energy states that survive in the limit $e^2 \rightarrow 0$ contribute to the Witten index. Due to the form of the potential (6.59), we must simultaneously solve the mass equations

$$\begin{aligned} 0 &= (\sigma_1 - a + \epsilon)^2 |\phi_1|^2 = (\sigma_1 - a - \epsilon)^2 |\tilde{\phi}_1|^2 = (\sigma_{21} + \epsilon)^2 |\underline{\phi}_1|^2 = (\sigma_{21} - \epsilon)^2 |\tilde{\underline{\phi}}_1|^2, \\ 0 &= (\sigma_3 + a + \epsilon)^2 |\phi_2|^2 = (\sigma_3 + a - \epsilon)^2 |\tilde{\phi}_2|^2 = (\sigma_{32} + \epsilon)^2 |\underline{\phi}_2|^2 = (\sigma_{32} - \epsilon)^2 |\tilde{\underline{\phi}}_2|^2, \end{aligned} \quad (\text{D.173})$$

the F-term equations

$$\begin{aligned} 0 &= \phi_1 \tilde{\phi}_1 - \underline{\phi}_1 \tilde{\underline{\phi}}_1, \\ 0 &= \underline{\phi}_1 \tilde{\phi}_1 - \underline{\phi}_2 \tilde{\underline{\phi}}_2, \\ 0 &= \underline{\phi}_2 \tilde{\underline{\phi}}_2 + \phi_2 \tilde{\phi}_2, \end{aligned} \quad (\text{D.174})$$

and the D-term equations

$$\begin{aligned} 0 &= |\phi_1|^2 - |\tilde{\phi}_1|^2 - |\underline{\phi}_1|^2 + |\tilde{\underline{\phi}}_1|^2 - e^2 \xi_1, \\ 0 &= |\underline{\phi}_1|^2 - |\tilde{\underline{\phi}}_1|^2 - |\underline{\phi}_2|^2 + |\tilde{\underline{\phi}}_2|^2 - e^2 \xi_2, \\ 0 &= |\underline{\phi}_2|^2 - |\tilde{\underline{\phi}}_2|^2 + |\phi_2|^2 - |\tilde{\phi}_2|^2 - e^2 \xi_3, \end{aligned} \quad (\text{D.175})$$

to order $O(e)$.

Let us consider the case where $\xi_i > 0$ or $\xi_i < 0, \forall i$. Here there are only solutions to the D-term equations when there are light fundamental hypermultiplet fields with non-zero expectation value due to the repeated appearance of bifundamental hypermultiplet

fields. Therefore the mixed branches are

$$\begin{aligned}
\text{I}_+ &: \sigma_1 = a - \epsilon, |\sigma_2|, |\sigma_3| \gg 0, \\
\text{II}_+ &: \sigma_1 = \sigma_2 - \epsilon = a - \epsilon, |\sigma_3| \gg 0, \\
\text{III}_+ &: \sigma_3 = -a - \epsilon, |\sigma_2|, |\sigma_1| \gg 0, \\
\text{IV}_+ &: \sigma_3 = \sigma_2 - \epsilon = -a - \epsilon, |\sigma_1| \gg 0, \\
\text{V}_+ &: \sigma_1 = a - \epsilon, \sigma_3 = -a - \epsilon, |\sigma_2| \gg 0,
\end{aligned} \tag{D.176}$$

for $\xi_i > 0$ and the mixed branches

$$\begin{aligned}
\text{I}_- &: \sigma_1 = a + \epsilon, |\sigma_2|, |\sigma_3| \gg 0, \\
\text{II}_- &: \sigma_1 = \sigma_2 + \epsilon = a + \epsilon, |\sigma_3| \gg 0, \\
\text{III}_- &: \sigma_3 = -a + \epsilon, |\sigma_2|, |\sigma_1| \gg 0, \\
\text{IV}_- &: \sigma_3 = \sigma_2 + \epsilon = -a + \epsilon, |\sigma_1| \gg 0, \\
\text{V}_- &: \sigma_1 = a + \epsilon, \sigma_3 = -a + \epsilon, |\sigma_2| \gg 0,
\end{aligned} \tag{D.177}$$

for $\xi_i < 0$. We conjecture that there are no BPS states localized on these vacuum branches.¹⁰

D.2.4 Contribution to the Witten Index

It follows from our conjecture in Section D.2.3 that in the $U(1)^3$ bubbling SQM, only Coulomb branch states contribute to the non-compact index I_{asympt} . These states give rise to the results

$$\begin{aligned}
I_{asympt}(\xi_i > 0) &= e^{\sum_f m_f + 6\epsilon_+} + e^{-\sum_f m_f - 6\epsilon_+} + e^{\sum_f m_f + 2\epsilon_+} + e^{-\sum_f m_f - 2\epsilon_+} \\
&= 2 \cosh \left(\sum_f m_f + 6\epsilon_+ \right) + 2 \cosh \left(\sum_f m_f + 2\epsilon_+ \right),
\end{aligned} \tag{D.178}$$

or

$$\begin{aligned}
I_{asympt}(\xi_i < 0) &= e^{\sum_f m_f - 6\epsilon_+} + e^{-\sum_f m_f + 6\epsilon_+} + e^{\sum_f m_f - 2\epsilon_+} + e^{-\sum_f m_f + 2\epsilon_+} \\
&= 2 \cosh \left(\sum_f m_f - 6\epsilon_+ \right) + 2 \cosh \left(\sum_f m_f - 2\epsilon_+ \right).
\end{aligned} \tag{D.179}$$

¹⁰See upcoming dissertation of the first author for more details.

D.3 Behavior of Localized Path Integral at Infinity

In this appendix we will consider the behavior of the integrand of the localized path integral (6.66), $Z_{int}(\varphi)$, at $\varphi \rightarrow \partial \mathfrak{t}_{\mathbb{C}}/\Lambda_{cr}$. Let us take $G = \prod_{i=1}^{n-1} U(k^{(i)})$ to be the gauge group of the SQM such that the corresponding Lie algebra \mathfrak{g} decomposes as $\mathfrak{g} = \bigoplus_{i=1}^{n-1} \mathfrak{g}^{(i)} = \bigoplus_{i=1}^{n-1} \mathfrak{u}(k^{(i)})$. Consider taking the limit

$$\tau \rightarrow \infty \quad \text{where} \quad \varphi = \tau \mathfrak{u} \quad , \quad \mathfrak{u} \in \mathfrak{t} \quad (\text{D.180})$$

where \mathfrak{t} is the Lie algebra of \mathfrak{g} which itself decomposes as $\mathfrak{t} = \bigoplus_{i=1}^{n-1} \mathfrak{u}(k^{(i)}) = \bigoplus_{i=1}^{n-1} \mathfrak{t}^{(i)}$.

The element \mathfrak{u} can be written with respect to this decomposition as

$$\mathfrak{u} = \bigoplus_{i=1}^{n-1} \mathfrak{u}^{(i)} \quad , \quad \mathfrak{u}^{(i)} = \sum_{a=1}^{k^{(i)}} \mathfrak{u}_a^{(i)} e_a^{(i)} \quad , \quad \mathfrak{t}^{(i)} \subset \mathfrak{u}(k^{(i)}) \quad , \quad \mathfrak{t}^{(i)} = \text{span}_{\mathbb{R}} \{e_a^{(i)}\}_{a=1}^{k^{(i)}} \quad , \quad (\text{D.181})$$

and as a matrix $e_a^{(i)} = \delta_{a,a}$. The matter content of a generic bubbling SQM transforms under the representations

$$\begin{aligned} \text{bifundamental hyper} &: \bigoplus_{i=1}^{n-1} \left[\mathbf{k}^{(i)} \otimes \mathbf{k}^{(i+1)} \right] \oplus \left[\bar{\mathbf{k}}^{(i)} \otimes \bar{\mathbf{k}}^{(i+1)} \right] \quad , \\ \text{fundamental hyper} &: \bigoplus_{i=1}^{n-1} \left[\delta_{s(i),1} \mathbf{k}^{(i)} \oplus \bar{\mathbf{k}}^{(i)} \right] \oplus 2 \left[\delta_{s(i_m),2} \mathbf{k}^{(i_m)} \oplus \bar{\mathbf{k}}^{(i_m)} \right] \quad , \quad (\text{D.182}) \\ \text{fundamental Fermi} &: \mathbf{N}_f \mathbf{k}^{(i_m)} \quad , \end{aligned}$$

where

$$s(i) = 2k^{(i)} - k^{(i+1)} - k^{(i-1)} \quad , \quad i_m = \frac{1}{2}n - 1 \quad . \quad (\text{D.183})$$

Using this, we can compute the limiting form of the different terms in Z_{det} as $\tau \rightarrow \infty$.

Using (6.67), we can see that

$$\begin{aligned} |Z_{vec}| &\underset{\varphi=\tau \mathfrak{u}}{\overset{\tau \rightarrow \infty}{\sim}} \prod_{i=1}^{n-1} \exp \left\{ 2\tau \sum_{\alpha \in \Delta_{adj}^{(i)}} |\alpha(\mathfrak{u}^{(i)})| \right\} = \prod_{i=1}^{n-1} \exp \left\{ 4\tau \sum_{\alpha \in \Delta_{adj}^{(i)+}} \alpha(\mathfrak{u}^{(i)}) \right\} \\ &= \prod_{i=1}^{n-1} e^{4\tau \rho^{(i)} \cdot \mathfrak{u}} \quad , \end{aligned} \quad (\text{D.184})$$

where $\Delta_{adj}^{(i)+}$ are the set of positive weights of the adjoint representation with respect to the splitting of the weight lattice where \mathfrak{u} is in the fundamental chamber and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g})} \alpha = \sum_{i=1}^{n-1} \rho^{(i)} \quad , \quad \rho^{(i)} = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}^{(i)})} \alpha \quad , \quad (\text{D.185})$$

is the Weyl element of \mathfrak{g} and $\mathfrak{g}^{(i)}$ respectively. Then using the form of

$$\rho^{(i)} = \frac{1}{2} \sum_{a=1}^{k^{(i)}} (k^{(i)} - 2a + 1) e_a^{(i)} , \quad (\text{D.186})$$

we can rewrite the limiting form as

$$|Z_{vec}| \underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathbf{u}}}{\sim} \prod_{i=1}^{n-1} e^{4\tau \sum_{a=1}^{k^{(i)}} (k^{(i)} - 2a + 1) u_a^{(i)}} . \quad (\text{D.187})$$

The contribution from the bifundamental hypermultiplets (6.69) has the limiting form

$$|Z_{hyper;bf}| \underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathbf{u}}}{\sim} \prod_{i=1}^{n-1} e^{-2\tau (k^{(i-1)} + k^{(i+1)}) \sum_{a=1}^{k^{(i)}} |u_a^{(i)}|} . \quad (\text{D.188})$$

Similarly, the fundamental hypermultiplets (6.69) and (6.68) contributions have the limiting forms

$$\begin{aligned} |Z_{hyper;f}| &\underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathbf{u}}}{\sim} \prod_{i=1}^{n-1} e^{-2\tau (\delta_{s(i),1} + 2\delta_{s(i),2}) \sum_{a=1}^{k^{(i)}} |u_a^{(i)}|} , \\ |Z_{Fermi;f}| &\underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathbf{u}}}{\sim} e^{\tau N_f \sum_{a=1}^{k^{(i_m)}} |u_a^{(i_m)}|} . \end{aligned} \quad (\text{D.189})$$

Putting these factors all together, we find that

$$\begin{aligned} |Z_{int}| \underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathbf{u}}}{\sim} \prod_{i=1}^{n-1} \exp \left\{ 4\tau \sum_{a=1}^{k^{(i)}} (k^{(i)} - 2a + 1) u_a^{(i)} + \tau N_f \sum_{a=1}^{k^{(i_m)}} |u_a^{(i_m)}| \right. \\ \left. - 2\tau (k^{(i-1)} + k^{(i+1)} + \delta_{s(i),1} + 2\delta_{s(i),2}) \sum_{a=1}^{k^{(i)}} |u_a^{(i)}| \right\} . \end{aligned} \quad (\text{D.190})$$

This is bounded from above by

$$\begin{aligned} |Z_{int}| \underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathbf{u}}}{\lesssim} \prod_{i=1}^{n-1} \exp \left\{ 4\tau \sum_{a=1}^{k^{(i)}} (k^{(i)} - 1) |u_a^{(i)}| + \tau N_f \sum_{a=1}^{k^{(i_m)}} |u_a^{(i_m)}| \right. \\ \left. - 2\tau (k^{(i-1)} + k^{(i+1)} + \delta_{s(i),1} + 2\delta_{s(i),2}) \sum_{a=1}^{k^{(i)}} |u_a^{(i)}| \right\} , \end{aligned} \quad (\text{D.191})$$

which can further be simplified to

$$|Z_{int}| \underset{\substack{\tau \rightarrow \infty \\ \varphi = \tau \mathbf{u}}}{\lesssim} \prod_{i=1}^{n-1} \exp \left\{ 2\tau \left(s(i) - 2 - 2\delta_{s(i),1} - 4\delta_{s(i),2} + \frac{N_f}{2} \delta_{i,i_m} \right) \sum_{a=1}^{k^{(i)}} |u_a^{(i)}| \right\} . \quad (\text{D.192})$$

Using the fact that $s(i_m) = 0$ or 2 and the fact that $N_f \leq 4$, we see that the exponential factors can at most completely cancel as $\tau \rightarrow \infty$. In this case, the behavior of the 1-loop determinant at infinity will be polynomially suppressed by the Yukawa terms for

the hypermultiplet fields to order $O(\prod_i \tau^{-3k^{(i)}})$. Therefore, since the measure goes as $\prod_i \tau^{2k^{(i)}-1}$, we have that the product of the integrand and measure will vanish as $O(\prod_i \tau^{-k^{(i)}-1})$ and the integral is convergent.

D.4 A Useful Integral

Often in the text we make use of a non-standard integral identity which we will now precisely derive. Consider the integral

$$F(a, b, \eta) = \int_{\mathbb{R}+i\eta} \frac{dD}{D} e^{-aD^2+ibD} , \quad (\text{D.193})$$

where

$$a > 0 \quad , \quad b \in \mathbb{C} \quad , \quad \eta \in \mathbb{R}^* . \quad (\text{D.194})$$

We claim this integral is just

$$F(a, b, \eta) = +i\pi \operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right) - i\pi \operatorname{sign}(\eta) , \quad (\text{D.195})$$

where we choose the positive square root of a and

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dy e^{-y^2} . \quad (\text{D.196})$$

Proof: $F(a, b, \eta)$ is an entire function of b . Moreover, it satisfies the differential equation

$$\frac{\partial F}{\partial b} = i\sqrt{\frac{\pi}{a}} \exp\left\{-\frac{b^2}{4a}\right\} , \quad (\text{D.197})$$

so

$$\begin{aligned} F(a, b, \eta) &= \int_0^b i\sqrt{\frac{\pi}{a}} \exp\left\{-\frac{s^2}{4a}\right\} ds + F(a, 0, \eta) \\ &= F(a, 0, \eta) + i\pi \operatorname{erf}\left(\frac{b}{2\sqrt{a}}\right) . \end{aligned} \quad (\text{D.198})$$

It thus remains to determine

$$F(a, 0, \eta) = H(\eta/\sqrt{a}) , \quad (\text{D.199})$$

where

$$H(\eta/\sqrt{a}) = \int_{\mathbb{R}+i\eta} \frac{dD}{D} e^{-D^2} . \quad (\text{D.200})$$

Now by contour integration arguments $H(\eta)$ only depends on the sign of η . Let H_+ be the value for $\eta > 0$ and H_- the value for $\eta < 0$. We can take the limit as $\eta \rightarrow 0^+$ and use

$$\frac{1}{D + i\eta} \rightarrow P\left(\frac{1}{D}\right) - i\pi\delta(D) , \quad (\text{D.201})$$

for D real, where P is the principal part. But

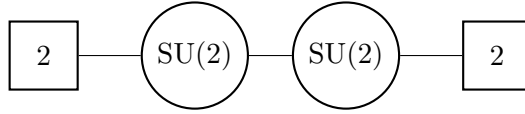
$$P \int \frac{dD}{D} e^{-D^2} = \lim_{\epsilon \rightarrow 0^+} \left[\int_{-\infty}^{-\epsilon} \frac{dD}{D} e^{-D^2} + \int_{\epsilon}^{\infty} \frac{dD}{D} e^{-D^2} \right] = 0 . \quad (\text{D.202})$$

Moreover, $H_-^* = H_+$, so

$$H(\eta) = \begin{cases} -i\pi & \eta > 0 \\ +i\pi & \eta < 0 \end{cases} \quad (\text{D.203})$$

D.5 Bubbling Contribution in the $SU(2) \times SU(2)$ SCFT

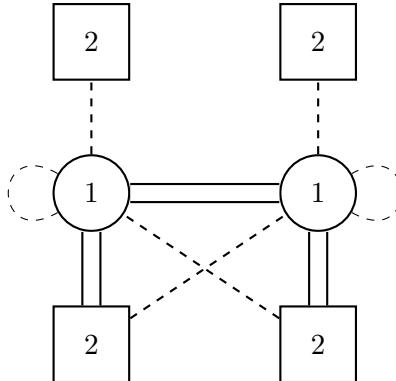
Consider the case of a superconformal $\mathcal{N} = 2$ quiver gauge theory with $G = SU(2)_1 \times SU(2)_2$ with fundamental matter:



Now consider the bubbling sector where

$$P = \bigoplus_{i=1}^2 P_i \quad , \quad \vec{v} = \bigoplus_{i=1}^2 \vec{v}_i \quad , \quad (P_i, \vec{v}_i) = (\text{diag}(1, -1), \text{diag}(0, 0)) . \quad (\text{D.204})$$

In this case, the $\mathcal{N} = (0, 4)$ bubbling SQM is of the form



The localization contribution to $Z_{mono}((1,0) \oplus (1,0))$ is then given by the contour integral

$$\begin{aligned}
Z_{mono}^{(Loc)} &= \sinh^2(2\epsilon_+) \\
&\times \oint_{JK(\xi_1, \xi_2)} \frac{d\varphi_1 d\varphi_2}{(2\pi i)^2} \frac{\prod_{f=1}^2 \sinh(\varphi_1 - m_f) \sinh(\varphi_2 - m_{f+2})}{\prod_{i=1}^2 \prod_{\pm} \sinh(\pm(\varphi_i - \mathbf{a}_i) + \epsilon_+) \sinh(\pm(\varphi_i + \mathbf{a}_i) + \epsilon_+)} \\
&\times \frac{\prod_{\pm} \sinh(-\varphi_1 \pm \mathbf{a}_2 + m + \epsilon_+) \sinh(\varphi_2 \pm \mathbf{a}_1 + m + \epsilon_+)}{4 \sinh(\varphi_2 - \varphi_1) \sinh(\varphi_1 - \varphi_2 + 2\epsilon_+)} .
\end{aligned} \tag{D.205}$$

Let us choose $\xi_1, \xi_2 > 0$. In this case there are 8 poles contributing this to this path integral:

$$\begin{aligned}
\text{I} : \varphi_1 &= \mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_2 - \epsilon_+ , \\
\text{II} : \varphi_1 &= \mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_2 - \epsilon_+ , \\
\text{III} : \varphi_1 &= -\mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_2 - \epsilon_+ , \\
\text{IV} : \varphi_1 &= -\mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_2 - \epsilon_+ , \\
\text{V} : \varphi_1 &= \mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_1 - \epsilon_+ , \\
\text{VI} : \varphi_1 &= -\mathbf{a}_1 - \epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_1 - \epsilon_+ , \\
\text{VII} : \varphi_1 &= \mathbf{a}_2 - 3\epsilon_+ \quad , \quad \varphi_2 = \mathbf{a}_2 - \epsilon_+ , \\
\text{VIII} : \varphi_1 &= -\mathbf{a}_2 - 3\epsilon_+ \quad , \quad \varphi_2 = -\mathbf{a}_2 - \epsilon_+ .
\end{aligned} \tag{D.206}$$

Using this set of poles as defined via the Jeffrey-Kirwan residue prescription, we find that the localization computation of $Z_{mono}^{(Loc)}((1,0) \oplus (1,0))$ is given by

$$\begin{aligned}
Z_{mono}^{(Loc)}((1,0) \oplus (1,0)) &= \frac{-\prod_{f=1}^2 \sinh(\mathbf{a}_1 - m_f - \epsilon_+) \sinh(\mathbf{a}_2 - m_{f+2} - \epsilon_+)}{\sinh(2\mathbf{a}_1) \sinh(2\mathbf{a}_1 - 2\epsilon_+) \sinh(2\mathbf{a}_2) \sinh(2\mathbf{a}_2 - 2\epsilon_+)} \\
&\times \frac{\sinh(\mathbf{a}_2 - \mathbf{a}_1 + m + 2\epsilon_+) \sinh(\mathbf{a}_1 + \mathbf{a}_2 - m - 2\epsilon_+)}{\sinh(\mathbf{a}_2 - \mathbf{a}_1) \sinh(\mathbf{a}_1 - \mathbf{a}_2 + 2\epsilon_+)} \\
&\times \sinh(\mathbf{a}_2 - \mathbf{a}_1 + m) \sinh(\mathbf{a}_2 + \mathbf{a}_1 + m) \\
&- \frac{\prod_{f=1}^2 \sinh(\mathbf{a}_1 - m_f - \epsilon_+) \sinh(\mathbf{a}_2 + m_{f+2} + \epsilon_+)}{\sinh(2\mathbf{a}_1) \sinh(2\mathbf{a}_1 - 2\epsilon_+) \sinh(2\mathbf{a}_2) \sinh(2\mathbf{a}_2 + 2\epsilon_+)} \\
&\times \frac{\sinh(\mathbf{a}_2 + \mathbf{a}_1 - m - 2\epsilon_+) \sinh(\mathbf{a}_1 - \mathbf{a}_2 - m - 2\epsilon_+)}{\sinh(\mathbf{a}_2 + \mathbf{a}_1) \sinh(\mathbf{a}_1 + \mathbf{a}_2 + 2\epsilon_+)} \\
&\times \sinh(\mathbf{a}_2 + \mathbf{a}_1 - m) \sinh(\mathbf{a}_2 - \mathbf{a}_1 - m) \\
&- \frac{\prod_{f=1}^2 \sinh(\mathbf{a}_1 + m_f + \epsilon_+) \sinh(\mathbf{a}_2 - m_{f+2} - \epsilon_+)}{\sinh(2\mathbf{a}_1) \sinh(2\mathbf{a}_1 + 2\epsilon_+) \sinh(2\mathbf{a}_2) \sinh(2\mathbf{a}_2 - 2\epsilon_+)} \\
&\times \frac{\sinh(\mathbf{a}_2 + \mathbf{a}_1 + m + 2\epsilon_+) \sinh(\mathbf{a}_1 - \mathbf{a}_2 + m + 2\epsilon_+)}{\sinh(\mathbf{a}_2 + \mathbf{a}_1) \sinh(\mathbf{a}_1 + \mathbf{a}_2 - 2\epsilon_+)} \\
&\times \sinh(\mathbf{a}_2 + \mathbf{a}_1 + m) \sinh(\mathbf{a}_2 - \mathbf{a}_1 + m) \\
&+ \frac{\prod_{f=1}^2 \sinh(\mathbf{a}_1 + m_f + \epsilon_+) \sinh(\mathbf{a}_2 + m_{f+2} + \epsilon_+)}{\sinh(2\mathbf{a}_1) \sinh(2\mathbf{a}_1 + 2\epsilon_+) \sinh(2\mathbf{a}_2) \sinh(2\mathbf{a}_2 + 2\epsilon_+)} \\
&\times \frac{\sinh(\mathbf{a}_2 - \mathbf{a}_1 - m - 2\epsilon_+) \sinh(\mathbf{a}_1 + \mathbf{a}_2 + m + 2\epsilon_+)}{\sinh(\mathbf{a}_2 - \mathbf{a}_1) \sinh(\mathbf{a}_1 - \mathbf{a}_2 - 2\epsilon_+)} \tag{D.207} \\
&\times \sinh(\mathbf{a}_2 - \mathbf{a}_1 - m) \sinh(\mathbf{a}_2 + \mathbf{a}_1 - m) \\
&- \sinh(m) \frac{\prod_{f=1}^2 \sinh(\mathbf{a}_1 - m_f - \epsilon_+) \sinh(\mathbf{a}_1 - m_{f+2} - \epsilon_+)}{\sinh(2\mathbf{a}_1) \sinh(2\mathbf{a}_1 - 2\epsilon_+)} \\
&\times \sinh(2\mathbf{a}_1 + m) \prod_{\pm} \frac{\sinh(\mathbf{a}_1 \pm \mathbf{a}_2 - m - 2\epsilon_+)}{\sinh(\mathbf{a}_1 \pm \mathbf{a}_2) \sinh(\mathbf{a}_1 \pm \mathbf{a}_2 - 2\epsilon_+)} \\
&+ \sinh(m) \frac{\prod_{f=1}^2 \sinh(\mathbf{a}_1 + m_f + \epsilon_+) \sinh(\mathbf{a}_1 + m_{f+2} + \epsilon_+)}{\sinh(2\mathbf{a}_1) \sinh(2\mathbf{a}_1 + 2\epsilon_+)} \\
&\times \sinh(2\mathbf{a}_1 - m) \prod_{\pm} \frac{\sinh(\mathbf{a}_1 \pm \mathbf{a}_2 + m + 2\epsilon_+)}{\sinh(\mathbf{a}_1 \pm \mathbf{a}_2 + 2\epsilon_+) \sinh(\mathbf{a}_1 \pm \mathbf{a}_2)} \\
&+ \sinh(m + 4\epsilon_+) \frac{\prod_{f=1}^2 \sinh(\mathbf{a}_2 - m_f - 3\epsilon_+) \sinh(\mathbf{a}_2 - m_{f+2} - \epsilon_+)}{\sinh(2\mathbf{a}_2) \sinh(2\mathbf{a}_2 - 2\epsilon_+)} \\
&\times \sinh(2\mathbf{a}_2 - m - 4\epsilon_+) \prod_{\pm} \frac{\sinh(\mathbf{a}_2 \mp \mathbf{a}_1 + m)}{\sinh(\mathbf{a}_2 \pm \mathbf{a}_1 - 2\epsilon_+) \sinh(\mathbf{a}_2 \pm \mathbf{a}_1 - 4\epsilon_+)} \\
&- \sinh(m + 4\epsilon_+) \frac{\prod_{f=1}^2 \sinh(\mathbf{a}_2 + m_f + 3\epsilon_+) \sinh(\mathbf{a}_2 + m_{f+2} + \epsilon_+)}{\sinh(2\mathbf{a}_2) \sinh(2\mathbf{a}_2 + 2\epsilon_+)} \\
&\times \sinh(2\mathbf{a}_2 + m + 4\epsilon_+) \prod_{\pm} \frac{\sinh(\mathbf{a}_2 \pm \mathbf{a}_1 - m)}{\sinh(\mathbf{a}_2 \pm \mathbf{a}_1 + 2\epsilon_+) \sinh(\mathbf{a}_2 \pm \mathbf{a}_1 + 4\epsilon_+)} .
\end{aligned}$$

One can check that the localization result for $Z_{mono}((1,0) \oplus (1,0))$ from residues associated to these poles is not invariant under the Weyl group of the flavor symmetry groups which is generated by the elements $W = \langle a_1, a_2, b_1, b_2 \rangle$ that act on the masses in the previous formula as

$$\begin{aligned}
a_1 : (m_1, m_2, m_3, m_4) &\mapsto (m_2, m_1, m_3, m_4) , \\
a_2 : (m_1, m_2, m_3, m_4) &\mapsto (m_1, m_2, m_4, m_3) , \\
b_1 : (m_1, m_2, m_3, m_4) &\mapsto (-m_2, -m_1, m_3, m_4) , \\
b_2 : (m_1, m_2, m_3, m_4) &\mapsto (m_1, m_2, -m_4, -m_3) .
\end{aligned} \tag{D.208}$$

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