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ON THE DETERMINANT OF (0,1) MATRICES

by

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I. Introduction

a) In the present paper we consider \( n \times n \) matrices with elements 0,1 and our purpose is to investigate the number of all non-singular ones. We shall prove that the singular matrices form a negligible percent asymptotically. More precisely, we shall prove the following

**THEOREM**

\[
\text{Let } A_n \text{ denote the number of } n \times n \text{ matrices with elements 0, 1 having determinant 0, then}
\]

\[
\lim_{n \to +\infty} \frac{A_n}{2^n^2} = 0.
\]

b) In other words let us choose at random a matrix from the set of \( n \times n \) (0, 1) matrices such that all matrices have the same probability \( (2^{-n})^2 \). If \( a_n \) means the probability of the event that the determinant of the chosen matrix equals 0, then \( \lim_{n \to +\infty} a_n = 0 \). It is easy to see that the following fact is equivalent to our theorem:

If \( e_{i,j} \) are independent random variables which take the values 0 and 1 with probabilities \( \frac{1}{2}, \frac{1}{2} \) and

\[
p_n = \mathbb{P} \left( \begin{vmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,n} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} \end{vmatrix} = 0 \right)
\]

then

\[
\lim_{n \to +\infty} p_n = 0.
\]

We shall use all versions at the same time. In the section VI. we deal with a generalization of this problem in the case of infinite matrices.

c) The proof goes as follows: We show that the probability of the event, that the rank of an \( n \times n \) (0, 1) matrix is \( k+2 \), where \( k \) denotes the rank of the \( (n-1) \times (n-1) \) matrix, consisting of its first \( n-1 \) rows and columns, or is equal to \( n \), tends to 1 if \( n \to \infty \).

Using this fact we prove that

\[
\lim \inf_{n \to +\infty} \frac{A_n}{2^n^2} = 0.
\]

Having proved this, we prove the convergence of the sequence \( A_n/2^n^2 \).

Before the proof of the theorem we give some definitions and lemmas.
II. Definitions and Lemmas

a) Let $A_{n,k}$ denote the number of $n \times n (0, 1)$ matrices whose rank is equal to $k$. Clearly

$$A_n = \sum_{k=1}^{n-1} A_{n,k} = 2^{n^2} - A_{n,n}.$$  

Then we have to prove that

$$\lim_{n \to +\infty} \frac{A_{n,n}}{2^{n^2}} = 1.$$  

First we give a known lemma.

**Lemma 1.** Let $a_1, a_2, \ldots, a_n$ be real numbers different from 0 and $c$ an arbitrary real number, then at most $\left(\begin{array}{c} n \\ \frac{n}{2} \end{array}\right)$ among the sums $\sum_{i=1}^{n} e_i a_i$ ($e_i$ is equal to 0 or 1) are equal to $c$.

**Proof.** Let us consider instead of the numbers $\sum_{i=1}^{n} e_i a_i$ the sums $2 \cdot \sum_{i=1}^{n} e_i a_i - \sum_{i=1}^{n} \varphi_i a_i$, where $\varphi_i = 2e_i - 1$, then $\varphi_i$ is equal to 1 or $-1$ if $e_i$ is equal to 1 or 0, respectively. The sum $\sum_{i=1}^{n} e_i a_i$ equals $c$ if the sum $\sum_{i=1}^{n} \varphi_i a_i$ equals $d = 2c - \sum_{i=1}^{n} a_i$. Then we can reformulate the lemma so that the numbers $e_i$ are equal to 1 or $-1$. In this case we can suppose without violating the generality, that the numbers $a_1, a_2, \ldots, a_n$ are all positive.

Then it is enough to prove the following: if $a_1, a_2, \ldots, a_n$ are positive numbers and $d$ is an arbitrary real number, then at most $\left(\begin{array}{c} n \\ \frac{n}{2} \end{array}\right)$ among the numbers $\sum_{i=1}^{n} e_i a_i$ ($e_i$ equals 1 or $-1$) are equal to $d$.

Let us correspond for every sum $\sum_{i=1}^{n} e_i a_i$ the set of those natural numbers $i$ for which $e_i = 1$ holds. If for two different sums $\sum_{i=1}^{n} e_i a_i = \sum_{i=1}^{n} \varepsilon'_i a_i$, then the corresponding sets of the two sums cannot contain each other.

The Sperner-theorem implies that the number of sums equal to any constant is at most $\left(\begin{array}{c} n \\ \frac{n}{2} \end{array}\right)$.

Clearly we can formulate the lemma as follows: if $a_1, a_2, \ldots, a_m$ are real numbers, among which $n$ are different from 0 and $c$ is an arbitrary real number
then among the numbers $\sum_{i=1}^{n} e_i a_i$ ($e_i$ equals 0 or 1) at most $\left(\begin{array}{c} n \\ \frac{n}{2} \end{array}\right) 2^{m-n} < \frac{2^m}{\sqrt{n}}$

are equal to $c$.

b) **Definitions.**

A system of $k$ linearly independent row (resp. column) vectors of a matrix of rank $k$ is called a row (resp. column) basis of the matrix.

We shall use that any row (resp. column) vector is a uniquely determined linear combination of the vectors of any fixed row (resp. column) basis.

1) The degree of a row (resp. column) vector with respect to a given row (resp. column) basis, is the number of those elements of the row (resp. column) basis, which have coefficients different from 0 in the above mentioned linear combination.

2) The degree of a row (resp. column) vector is the largest one among the degrees of this row (resp. column) vector with respect to all possible row (resp. column) bases.

3) The row (resp. column) degree of a matrix is the largest one among the degrees of its row (resp. column) vectors.

**Lemma 2.** If the row-degree of an $m \times n$ $(0, 1)$ matrix is $l$ and its rank is $k$, then we can add to the matrix a column vector (with components $0, 1$) so that the rank of the obtained $m \times (n+1)$ matrix is $k$ again, at most $\frac{2^m}{\sqrt{l}}$ different ways.

**Proof.** For the sake of simplicity let us suppose that the first $k$ row vectors form the basis, with respect to which the degree of the $i$-th row vector is equal to $l$.

Let us denote the $i$-th row vector by $a_i$, the $j$-th column vector by $b_j$ and the additional (the $(n+1)$-th) column vector by $b_{n+1}$, i.e.

$$a_i = (a_{i,1}; a_{i,2}; \ldots; a_{i,n}),$$

$$b_j = \begin{pmatrix} b_{1,j} \\ \vdots \\ b_{m,j} \end{pmatrix},$$

$$b_{n+1} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

The row vectors of the enlarged matrix are

$$a'_i = (a_{i,1}; a_{i,2}; \ldots; a_{i,n}, b_i).$$

So we have $a_i = c_1 a_1 + c_2 a_2 + \ldots + c_k a_k$ where among the constants $c_i$ $l$ are different from 0.

If the degree of the new $(m \times (n+1))$ matrix is also $k$ then (because the maximal numbers of linearly independent row and column vectors are equal to each other and clearly $a'_1, a'_2, \ldots, a'_k$ are also linearly independent)

$$a'_i = c_1 a'_1 + c_2 a'_2 + \ldots + c_k a'_k$$

hence

$$b_i = c_1 b_1 + c_2 b_2 + \ldots + c_k b_k.$$

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But \( b_i \) is equal to 0 or 1 and among the numbers \( c_i \) \( l \) are different from 0, so by Lemma 1 we can choose the vector \((b_1, b_2, \ldots, b_l)\) at most \( \frac{2k}{l} \) different ways such that \( b_i = 0 \) holds; similarly we can choose \((b_1, b_2, \ldots, b_l)\) at most \( \frac{2k}{l} \) different ways such that \( b_i = 1 \) holds. That is, we have at most \( \frac{2 \cdot 2^m}{\sqrt{l}} \) possibilities to choose the vector \( \mathbf{b}_{n+1} \). Q. e. d.

Similarly, if the column-degree of a matrix is \( l \), then we can construct to the matrix a row vector at most \( \frac{2 \cdot 2^m}{\sqrt{l}} \) different ways such that the maximal numbers of linearly independent vectors of both matrices are equal to each other.

c) **Lemma 3.** By \( k \) \( m \)-dimensional vectors (with elements 0, 1) we can construct at most \( 2^k \) different vectors (with components 0, 1) with linear combinations.

**Proof.** Let us consider a \( k \times m \) matrix with row vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \). It contains at most \( 2^k \) different column vectors (because it has only 0 or 1 components). If the \( i_1 \)-th, \( i_2 \)-th, \ldots, \( i_l \)-th column vectors are the different ones \( (i \equiv 2^k) \), so any of the others is equal to one of these, then in the linear combinations of the row vectors the \( i_1 \)-th, \( i_2 \)-th, \ldots, \( i_l \)-th components can arbitrarily vary. Then among the linear combinations, whose components are 0, 1, at most \( 2^l \equiv 2^k \) can be different. Q. e. d.

**Lemma 4.** There exists a natural number \( m_0 \) so that the number of those \( m \times n \) (0, 1) matrices whose row-degree is at most \( \log m \) but not equal to 1, is less than \( 2^{n(m-1)} \cdot 2^{m/5} \) if \( m > m_0 \).

**Proof.** Let us denote by \( D_l \) the number of those \( m \times n \) (0, 1) matrices, whose row-degree is \( l \) and by \( D_{l, i} \) the number of those (0, 1) matrices in which the \( i \)-th row vector has degree \( l \). Then

\[
D_l = \sum_{i=1}^{m} D_{l, i} = m \cdot D_{l, m}
\]

(because evidently \( D_{l, 1} = D_{l, 2} = \ldots = D_{l, m} \)).

We shall prove that

\[
D_{l, m} \leq m^l \cdot 2^{2^l} \cdot 2^{n(m-1)} \quad (l \geq 2),
\]

what proves our Lemma because the number of those matrices whose row-degree is at most \( \log m \) but is not equal to 1 is

\[
\sum_{l=2}^{\log m} \sum_{i=2}^{\log m} m \cdot m^l \cdot 2^{2^l} < 2^{n(m-1)} \cdot \log m \cdot m \cdot m^{\log m} \cdot 2^{2 \log m} < 2^{n(m-1)} \cdot 2^{m/5}
\]

if \( m > m_0 \) for some suitable natural number \( m_0 \).

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If we fill in the first \( m-1 \) rows of the matrix arbitrarily (it can be done by \( 2^{m(m-1)} \) different ways) we can construct the last row using a row-basis consisting of the first \( m-1 \) rows by a linear combination (because \( l \equiv 2 \)) but actually we use only \( l \) rows of the row-basis, because the coefficients of the other rows are equal to 0. We have \( \binom{m-1}{l} < m^l \) possibilities to choose the \( l \) vectors and by \( l \) vectors we can construct at most \( 2^{2^l} \) vectors as linear combinations according to Lemma 3, that is

\[
D_{l,m} < m^l \cdot 2^{2^l} \cdot 2^{n(m-1)} \quad (l \equiv 2).
\]

Q. e. d.

Similarly the number of \( m \times n \) (0, 1) matrices whose column-degree is at most \( \log \ n \) but is not equal to 1, is less than \( 2^{n(n-1)} \cdot 2^{n^{4/5}} \), if \( n > m_0 \).

If the row-(resp. column) degree of an \( m \times n \) matrix is equal to 1, then we have two possibilities: either there are two rows(resp. columns) which are equivalent (the number of such matrices is less than \( m^2 \cdot 2^{(m-1)n} \) (resp. \( n^2 \cdot 2^{m(n-1)} \)), or the rank of the matrix is \( m \) (resp. \( n \)) — these are the good cases for us.

d) Let us consider an \( n \times n \) (0, 1) matrix \( (n > m_0) \).

A) If its rank is \( n \), then any additional column vector is linearly dependent of the column vectors of the matrix.

B) 1. If its rank is \( k < n \) and its row-degree is \( l > \log n \) then by Lemma 2 we have at most \( \frac{2^2 \cdot 2^n}{\sqrt{l}} < \frac{2^2 \cdot 2^n}{\sqrt{\log n}} \) possibilities to add a column vector so that the rank of the obtained \( n \times (n+1) \) is also \( k \).

B) 2. The number of those \( n \times (n+1) \) (0, 1) matrices for which the row-degree of the \( n \times n \) matrix consisting of its first \( n \) columns is less than \( \log n \) but not equal to 1 — by Lemma 4 — is less than \( 2^{n(n+1)} \cdot \frac{2^{n^{4/5}}}{2^n} \).

B) 3. If an \( n \times (n+1) \) matrix has the property that the row-degree of the \( n \times n \) matrix consisting of its first \( n \) columns is equal to 1, then (because \( k < n \)) in the latter matrix there exist two rows which are equivalent. So the number of these matrices is less than \( 2^{n(n+1)} \cdot \frac{n^2}{2n} \).

Let \( \mathbf{B} \) denote the set of matrices of the types B)2. and B)3. The number of elements of \( \mathbf{B} \) is less than

\[
2^{n(n+1)} \left( \frac{2^{n^{4/5}}}{2^n} + \frac{n^2}{2^n} \right) < 2^{n(n+1)} \cdot \frac{1}{2^{n/2}}
\]

if \( n > n_0 \equiv m_0 \) for some suitable natural number \( n_0 \).

By a similar way we can prove that if we enlarge the obtained \( n \times (n+1) \) matrix by a row vector and if the matrix is not an element of the set \( \mathbf{B} \), then the probability of the event, that the rank of the new matrix is larger than the rank of the first matrix is at least

\[
1 - \frac{2}{\sqrt{\log n}}.
\]
a) So we have proved the following

**Lemma 5.** Let us consider an arbitrary \( n \times n \) (0, 1) matrix which is not element of the set \( B \). Let us enlarge the matrix by a column vector (with components 0, 1) and let us add to the new matrix a row vector in all possible ways. So we obtain \( 2^{2n+1} \) \((n + 1) \times (n + 1)\) matrix.

If the rank of the first matrix is \( k < n \), then the rank of the new matrices are \( k + 2 \) except for at most \( \frac{2}{\sqrt{\log n}} \cdot 2^{2n+1} \) matrices, and if the rank of the first matrix is \( k = n \), then the rank of the new matrices are \( n + 1 \) except for at most \( \frac{2}{\sqrt{\log n}} \cdot 2^{2n+1} \) matrices.

b) Using Lemma 5 we obtain

**Lemma 6.** There exists a sequence \( n_1, n_2, \ldots, n_k, \ldots \) of natural numbers such that

\[
A_{n_k, n_k} > 2^{n_k} \left( 1 - \frac{6}{\sqrt{\log n_k}} \right) \quad (k = 1, 2, \ldots),
\]

where \( A_{m,r} \) denotes the number of \( m \times m \) (0, 1) matrices whose ranks are equal to \( r \).

By other words

\[
\liminf_{n \to +\infty} p_n = \liminf_{n \to +\infty} \frac{A_n}{2n^2} = \limsup \left( 1 - \frac{A_{n,n}}{2n^2} \right) = 1 - 1 = 0.
\]

**Proof.** Let us put \( S_n = \sum_{k=0}^{n} A_{n,k} \cdot k \) and \( f(n) = \frac{S_n}{2n^2} \). The inequality \( S_n < \sum_{k=0}^{n} A_{n,k} \cdot n = n \cdot 2n^2 \) implies that \( f(n) < n \). Let \( \overline{A}_{n,k} \) denote the number of those \( n \times n \) matrices whose ranks are \( k \) and which are not elements of the set \( B \) and \( B_{n,k} = A_{n,k} - \overline{A}_{n,k} \). We can obtain all \((n + 1) \times (n + 1)\) matrices so that we enlarge the \( n \times n \) matrices by a column vector to the right and after it by a row vector upwards in all possible ways.

So we can obtain from the \( n \times n \) matrices of number \( \overline{A}_{n,k} \) and of rank \( k \) new \((n + 1) \times (n + 1)\) matrices the number of which is \( \overline{A}_{n,k} \cdot 2^{2n+1} \) and among them \( x_{n,k} \cdot \overline{A}_{n,k} \cdot 2^{2n+1} \) have rank smaller than \( \min(k + 2, n + 1) \). By Lemma 5,

\[
x_{n,k} < \frac{2}{\sqrt{\log n}} \quad (k = 0, 1, 2, \ldots, n).
\]
c) So we have

\[ S_{n+1} = \sum_{k=0}^{n+1} k \cdot A_{n+1,k} \equiv 2^{2n+1} \sum_{k=0}^{n-1} \bar{A}_{n,k} (1 - x_{n,k})(k+2) + \]

\[ + 2^{2n+1} \sum_{k=0}^{n-1} \bar{A}_{n,k} \cdot x_{n,k} \cdot k + 2^{2n+1} A_{n,n} (1 - x_{n,n})(n+1) + 2^{2n+1} \bar{A}_{n,n} \cdot x_{n,n} \cdot n = \]

\[ = 2^{2n+1} \left( \sum_{k=0}^{n-1} \bar{A}_{n,k} (k+2) + A_{n,n}(n+1) \right) - 2^{2n+1} \left( 2 \cdot \sum_{k=0}^{n-1} \bar{A}_{n,k} \cdot x_{n,k} + \bar{A}_{n,n} x_{n,n} \right) \]

\[ = 2^{2n+1} \left( \sum_{k=0}^{n} \bar{A}_{n,k} (k+2) - \bar{A}_{n,n} \right) - 2^{2n+1} \left( 2 \cdot \sum_{k=0}^{n} \bar{A}_{n,k} - \bar{A}_{n,n} \right) \frac{2}{\sqrt{\log n}} = \]

\[ = 2^{2n+1} \sum_{k=0}^{n} \bar{A}_{n,k} \cdot k + 2 \cdot 2^{(n+1)^2} \left( 1 - \frac{2}{\sqrt{\log n}} \right) - 2^{2n+1} \cdot \bar{A}_{n,n} \left( 1 - \frac{2}{\sqrt{\log n}} \right) - \]

\[ - 2 \cdot 2^{2n+1} \left( 1 - \frac{2}{\sqrt{\log n}} \right) \sum_{k=0}^{n} B_{n,k} = 2^{2n+1} \sum_{k=0}^{n} A_{n,k} \cdot k + 2 \cdot 2^{(n+1)^2} \left( 1 - \frac{2}{\sqrt{\log n}} \right) - \]

\[ - 2^{2n+1} \cdot A_{n,n} \left( 1 - \frac{2}{\sqrt{\log n}} \right) - 2^{2n+1} \left( 1 - \frac{2}{\sqrt{\log n}} \right) \sum_{k=0}^{n} B_{n,k} - \]

\[ - 2^{2n+1} \cdot A_{n,n} \left( 1 - \frac{2}{\sqrt{\log n}} \right) - 2^{2n+1} \cdot \bar{A}_{n,n} \left( 1 - \frac{2}{\sqrt{\log n}} \right) \equiv \]

\[ \equiv 2^{2n+1} \cdot S_n + 2 \cdot 2^{(n+1)^2} \left( 1 - \frac{2}{\sqrt{\log n}} \right) - 2^{2n+1} \cdot A_{n,n} \left( 1 - \frac{2}{\sqrt{\log n}} \right) - \]

\[ - 2^{2n+1} \cdot 2^{n+2} \cdot \frac{n+2}{2n^2} \equiv 2^{2n+1} \cdot S_n + 2 \cdot 2^{(n+1)^2} \left( 1 - \frac{3}{\sqrt{\log n}} \right) - 2^{2n+1} \cdot A_{n,n} \left( 1 - \frac{2}{\sqrt{\log n}} \right). \]

d) Dividing by \( 2^{(n+1)^2} \) we get

\[ f(n+1) \equiv f(n) + 2 \left( 1 - \frac{3}{\sqrt{\log n}} \right) - \frac{A_{n,n}}{2n^2} \left( 1 - \frac{2}{\sqrt{\log n}} \right). \]

If we suppose that there exists a number \( N_0 \) such that

\[ A_{n,n} < 2n^2 \left( 1 - \frac{6}{\sqrt{\log n}} \right) \]

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holds for all \( n \geq N_0 \) then we have

\[
f(n + 1) \equiv f(n) + 2 \left( 1 - \frac{3}{\sqrt{\log n}} \right) - \left( 1 - \frac{7}{\sqrt{\log n}} \right)
\]

that is

(1)

\[
f(n + 1) \equiv f(n) + \left( 1 + \frac{1}{\sqrt{\log n}} \right)
\]

for all \( n \geq N_0 \). But \( \sum_{k=N_0}^{\infty} \frac{1}{\sqrt{\log k}} = + \infty \) therefore using Relation (1) \((n - N_0) \) times we obtain

\[
f(n + 1) \equiv f(N_0) + (n - N_0) + \sum_{k=N_0}^{n} \frac{1}{\sqrt{\log k}}.
\]

for all \( n \geq N_0 \). If \( N \) is so large that \( \sum_{k=N_0}^{n} \frac{1}{\sqrt{\log k}} > N_0 + 1 \) holds, then we have \( f(N + 1) > N + 1 \) which is a contradiction. Q.e.d.

IV.

a) Lemma 7.

Let \( f(x, y) \) be a function defined for all pairs \( x, y \) of natural numbers with the following properties:

There exists a natural number \( n \) and a real number \( 0 < c < 1 \) such that

1° \( f(x, y) \equiv 0 \)

2° \( f(x, x) = 1 \)

3° \( f(x, y + 1) \equiv f(x, y) \)

4° \( f(n, n - 1) < c \)

5° \( f(m + 1, k) \equiv cf(m, k) + (1 - c)f(m, k - 2) + d_m \)

for all \( m \geq n \) and \( 0 \leq k \leq m \), where \( \{d_m\} \) is a sequence of positive numbers.

We show that these properties imply that

(2) \( f(m, m - 1) < 2c + \sum_{s=n}^{\infty} d_s \)

or all \( m \geq n \).

b) By a double application of 5° we get

(3) \( f(m + 2, k) \equiv c^2f(m, k) + 2c(1 - c)f(m, k - 2) + (1 - c)^2f(m, k - 4) + d_m + d_{m + 1} \)

\[
\begin{cases}
    m \geq n \\
    0 \leq k \leq m
\end{cases}
\]

and this inequality implies (as \( f(m, k - 4) \leq f(m, k - 2) \)):

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\[ f(m + 2, k) \equiv c^2 f(m, k) + (1 - c^2) f(m, k - 2) + d_m + d_{m+1}. \]

The relation
\[ f(N + 1, k) \equiv c f(N, k) + (1 - c) f(N, k - 2) + d_N \]
\[ \equiv f(N, k) \left[ c + (1 - c) \right] + d_N = f(N, k) + d_N \]
\[ m \geq n \]
\[ 0 \leq k \leq m \]

and Relation \( 1^o \) \((f(n, n - 1) < c)\) show that

\[ f(m, k) < c + \sum_{s=n}^{m-1} d_s \]
for all \( k \equiv n - 1 \) \((m \equiv n)\).

Now we prove by induction that the following inequality holds:

\[ f(n + t, n - 2 + t - i) \equiv c + \left[ \sum_{s=0}^{\lfloor \frac{t-i}{2} \rfloor - 1} \right. \left[ \begin{array}{c} i + s \\ \ \ \ \ \ \ \ s \end{array} \right] \left. \right] + \sum_{s=n}^{n+t-1} d_s \]
\[ (t \equiv 2, \ i \equiv 0). \]

If \( i > t - 2 \), then we have to prove that \( f(n + t, n - (i - t + 2)) \equiv c + \sum_{s=n}^{n+t-1} d_s \), but this is an immediate consequence of (5). Let us suppose that \( i \equiv t - 2 \).

c) In the case \( t = 2 \) (and so \( i = 0 \)) the inequality is
\[ f(n + 2, n) \equiv c + c^2 + d_n + d_{n+1}. \]

By (4) we have
\[ f(n + 2, n) \equiv c^2 f(n, n) + (1 - c^2) f(n, n - 2) + d_n + d_{n+1} \equiv c^2 + (1 - c^2) c + d_n + d_{n+1} < c + c^2 + d_n + d_{n+1}. \]

In the case \( t = 3 \) the inequality is (for \( i = 1 \) or \( i = 0 \))
\[ f(n + 3, n) \equiv c + c^3 + d_n + d_{n+1} + d_{n+2} \]
\[ f(n + 3, n + 1) \equiv c + c^2 + d_n + d_{n+1} + d_{n+2}. \]

Using Relation (4):
\[ f(n + 3, n) \equiv c^2 f(n + 1, n) + (1 - c^2) f(n + 1, n - 2) + d_{n+1} + d_{n+2} \equiv c^2 [c f(n, n) + (1 - c) f(n, n - 2) + d_n] + (1 - c^2) (c + d_n) + d_{n+1} + d_{n+2} \equiv c^3 + c^3 (1 - c) + c (1 - c^2) + d_n + d_{n+1} + d_{n+2} < c + c^3 + d_n + d_{n+1} + d_{n+2}. \]
or similarly:

\[ f(n+3, n+1) \leq c^2f(n+1, n+1) + (1-c^2)f(n+1, n-1) + d_{n+1} + d_{n+2} \leq c^2 + (1-c^2)(c + d_n) + d_{n+1} + d_{n+2} \leq c + c^2 + d_n + d_{n+1} + d_{n+2}. \]

That is the inequality is proved in the cases \( t=2 \) and \( t=3 \).

d) Let us suppose that the inequality is proved for \( t=T \) and let us prove it for \( t=T+2 \). Denote \( \left\lfloor \frac{T-i}{2} \right\rfloor = w \). Applying (3) we get, if \( i \geq 2 \left( \binom{n}{k} = 0 \right) \) and \( k > n \) or \( k < 0 \)

\[ f(n+T+2, n-2+(T+2)-i) = f(n+T+2, n+T-i) \leq c^2f(n+T, n+T-i) + 2c(1-c)f(n+T, n-2+T-i) + (1-c)^2f(n+T, n-4+T-i) + d_{n+T} + d_{n+T+1} \leq c^2 \left( 1 + \sum_{s=0}^{w-1} \binom{i+s-2}{s} c^{i+s} + \sum_{s=n}^{n+T-1} d_s \right) + 2c(1-c) \left( 1 + \sum_{s=0}^{w-1} \binom{i+s}{s} c^{i+s+2} + \sum_{s=n}^{n+T-1} d_s \right) + (1-c)^2 \left( 1 + \sum_{s=0}^{w-2} \binom{i+s+2}{s} c^{i+s+4} + \sum_{s=n}^{n+T-1} d_s \right) + d_{n+T} + d_{n+T+1} = c + \sum_{s=n}^{n+T+1} d_s + \sum_{s=0}^{w} \binom{i+s-2}{s} c^{i+s+2} + 2 \sum_{s=1}^{w} \binom{i+s-1}{s-1} c^{i+s+2} - 2 \sum_{s=2}^{w+1} \binom{i+s-2}{s-2} c^{i+s+2} + \sum_{s=4}^{w+1} \binom{i+s-3}{s-4} c^{i+s+2} = S. \]

Using the following identity

\[ \binom{i+s-2}{s} + 2 \binom{i+s-1}{s-1} - 2 \binom{i+s-2}{s-2} + \binom{i+s}{s-2} - 2 \binom{i+s-1}{s-3} + \binom{i+s-2}{s-4} = \binom{i+s}{s} \]

one can see, that

\[ S = c + \sum_{s=n}^{n+T+1} d_s + \sum_{s=0}^{w} \binom{i+s}{s} c^{i+s+2} - 2 \binom{i+w-1}{w-1} c^{i+w+3} - 2 \binom{i+w}{w-2} c^{i+w+3} + \binom{i+w}{w-2} c^{i+w+4} + \binom{i+w-1}{w-3} c^{i+w+3}, \]

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and as
\[
\begin{pmatrix} i+w \\ w-2 \end{pmatrix} + \begin{pmatrix} i+w-1 \\ w-3 \end{pmatrix} \equiv 2 \begin{pmatrix} i+w \\ w-2 \end{pmatrix},
\]
we get the relation
\[
f(n+(T+2), n-2+(T+2)-i) \equiv S \equiv c + \sum_{s=0}^{\left[ \frac{T-i}{2} \right] + \left[ \frac{T-s}{2} \right] + n+T+1} \binom{i+s}{s} c^{i+s+2} + \sum_{s=n}^{\infty} d_s
\]
what we had to prove.

If \( i=0 \) or \( i=1 \), then the estimate
\[
f(n+T, n+T-i) \equiv c + \sum_{s=0}^{w} \binom{i+s-2}{s} c^{i+s} + \sum_{s=n}^{n+T-1} d_s
\]
and also the identity was false. Instead of this estimate we write \( f(n+T, n+T-i) \equiv 1 \), and so we get for \( S \) the same formula as above.

e) Let us apply the proved inequality in the case \( i=0 \).

\[
f(n+t, n+t-2) \leq c + \sum_{s=0}^{\left[ \frac{t}{2} \right]-1} c^s + \sum_{s=n}^{n+t-1} d_s \quad (t \geq 2).
\]
Hence
\[
f(n+t+1, n+t) \equiv cf(n+t, n+t) + (1-c)f(n+t, n+t-2) + d_{n+t} \leq
\]
\[
\leq c + (1-c) \left( c + \sum_{s=0}^{\left[ \frac{t}{2} \right]-1} c^s + \sum_{s=n}^{n+t-1} d_s \right) + d_{n+t} \leq c + (1-c) \left( c + \sum_{s=0}^{\infty} c^s \right) + \sum_{s=n}^{\infty} d_s = 2c + \sum_{s=n}^{\infty} d_s
\]
for all \( t \geq 2 \).

But
\[
f(n+2, n+1) \equiv cf(n+1, n+1) + (1-c)f(n+1, n-1) + d_{n+1} \leq
\]
\[
\leq c + (1-c)(c+d_n) + d_{n+1} \leq 2c + \sum_{s=n}^{\infty} d_s
\]
and
\[
f(n+1, n) \equiv cf(n, n) + (1-c)f(n, n-2) + d_n \leq c + (1-c)c + d_n < 2c + \sum_{s=n}^{\infty} d_s,
\]
hence we proved that for all \( m \geq n \)
\[
f(m, m-1) < 2c + \sum_{s=n}^{\infty} d_s
\]
holds.

Q. e.d.
Now we can already prove the theorem:
Let \( \varepsilon \) be an arbitrary positive number. Let the integer \( N \) be so large that for the \( N \)-th element of the sequence \( n_k \) (defined in Lemma 6)

\[
\frac{13}{\sqrt{\log n_N}} < \varepsilon.
\]

Let us put

\[
f(m, k) = \sum_{i=0}^{k} A_{m,i} 2^{m^2},
\]

\[
c = \frac{6}{\sqrt{\log n_N}},
\]

\[
n = n_N,
\]

\[
d_m = \frac{1}{2^{m/2}}.
\]

It is easy to see that for the function \( f(m, k) \) 1°—2°—3° hold.

The fulfillment of 4° follows from the definition of the sequence \( \{n_k\} \) (in lemma 6). Let us prove that 5° holds.

From the \( \overline{A}_{m,k-1} \) matrices of rank \( k-1 \) except for at most \( c \cdot \overline{A}_{m,k-1} \cdot 2^{2m+1} \) ones, and from the \( \overline{A}_{m,k} \) matrices of rank \( k \) except for at most \( c \cdot \overline{A}_{m,k} \cdot 2^{2m+1} \) ones we get such matrices, which have at least \( k+1 \) as rank. So we have

\[
2^{(m+1)^2} f(m + 1, k) \leq \sum_{i=0}^{k-2} A_{m,i} \cdot 2^{2m+1} + c \cdot 2^{2m+1} (\overline{A}_{m,k-1} + \overline{A}_{m,k}) +
\]

\[
+ 2^{2m+1} \cdot d_m \cdot 2^{m^2} \leq \sum_{i=0}^{k-2} A_{m,i} \cdot 2^{2m+1} + c \cdot 2^{2m+1} (A_{m,k-1} + A_{m,k}) + d_m \cdot 2^{(m+1)^2} =
\]

\[
f(m, k-2) \cdot 2^{(m+1)^2} + c \cdot 2^{(m+1)^2} (f(m, k) - f(m, k-2)) + d_m \cdot 2^{(m+1)^2} =
\]

\[
= 2^{(m+1)^2} [cf(m, k) + (1 - c)f(m, k-2) + d_m].
\]

Dividing by \( 2^{(m+1)^2} \) we obtain

\[
f(m + 1, k) \equiv cf(m, k) + (1 - c)f(m, k-2) + d_m,
\]

that is 5° holds.

By lemma 7 we get:

\[
f(m, m-1) < 2c + \sum_{s=n}^{\infty} d_s
\]

for all \( m \geq n \). But

\[
\sum_{s=n}^{\infty} d_s = \sum_{s=n}^{\infty} \frac{1}{2^{s/2}} = \frac{4}{2^{n/2}} < \frac{1}{\sqrt{n}}
\]

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that is
\[ f(m, m - 1) < \frac{12}{\sqrt{\log n_N}} + \frac{1}{\sqrt{\log n_N}} < \varepsilon \]
for all \( m \geq n_N \) or in other terms
\[ A_{m,m} = 2^m (1 - \varepsilon) \]
what proves our theorem.

VI.

a) Professor EGYED asked whether the following generalization of this theorem is true:

Let us consider the matrices:
\[
\begin{bmatrix}
a_{1,1} & a_{1,2} & \ldots & a_{1,k} \\
a_{2,1} & a_{2,2} & \ldots & a_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{i,1} & a_{i,2} & \ldots & a_{i,k} \\
\end{bmatrix}
\]
where the elements \( a_{i,k} \) equal to 0 or 1. The set of those matrices in which the rows or the columns are not "linearly independent", has a measure 0.

First we have to agree in that what is the meaning of "linearly independent" in this case.

Let \( a_{i,k} \) \( (i = 1, 2, \ldots; k = 1, 2, \ldots) \) be mutually independent random variables which take on the values 0, 1 with probabilities \( \frac{1}{2}, \frac{1}{2} \). Let us form by these random variables the above matrix.

We make use of two definitions of the linear dependence of the rows of a matrix.

The rows of a matrix are *finitely linearly dependent*, if there exists a natural number \( i \), some natural numbers (finitely many) \( i_1 < i_2 < \ldots < i_s \) and real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_s \) with the properties:

\[ i_v \neq i \quad \text{for} \quad v = 1, 2, \ldots, s \]

and

\[ a_{i,k} = \sum_{v=1}^{s} \alpha_v a_{i_v,k} \quad \text{for} \quad k = 1, 2, \ldots. \]

The rows of a matrix are *infinitely linearly dependent*, if there exists a natural number \( i \) and real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i = 0, \alpha_{i+1}, \ldots \) such that

\[ a_{i,k} = \sum_{v=1}^{\infty} \alpha_v a_{r,v,k} \quad \text{for} \quad k = 1, 2, \ldots. \]

Let \( A \) denote the event that the rows of a random matrix are finitely linearly dependent and \( B \) the event that they are infinitely linearly dependent.

Making use of these definitions we can formulate the question as follows:

What are the probabilities \( P(A) \) and \( P(B) \) equal to?
b) The answer is:

(9) \( \mathbf{P}(A) = 0 \),

(10) \( \mathbf{P}(B) = 1 \).

The proofs of these relations are simple.

Proof of (9):
Let \( A_i \) denote the event that

\[
 a_{i_1,t} = a_{i_2,t} = \ldots = a_{i,t} = 0 \quad \text{and} \quad a_{i,t} = 1.
\]

Clearly \( A_1, A_2, \ldots, A_t, \ldots \) are mutually independent and \( 0 < \mathbf{P}(A_1) = \mathbf{P}(A_2) = \ldots = \mathbf{P}(A_t) = \ldots \). One can see from the relation (7) that \( A_t \cap A = \emptyset \), so

\[
 \bigcup_{t=1}^{\infty} A_t \cap A = \emptyset
\]

and thus \( \bigcup_{t=1}^{\infty} \bigcap_{t=1}^{\infty} A_t = A \). This fact implies that

\[
 \mathbf{P} \left( \bigcap_{t=1}^{\infty} A_t \right) \geq \mathbf{P}(A).
\]

But we have \( \mathbf{P} \left( \bigcap_{t=1}^{\infty} A_t \right) = \prod_{t=1}^{\infty} \mathbf{P}(A_t) \) because the events \( A_t \) are independent.
As \( \mathbf{P}(A_t) < 1 \) and

\[
 \mathbf{P}(A_t) = \mathbf{P}(A_{t+1}) = \ldots = \mathbf{P}(\bar{A}_t) = \ldots
\]

we get \( \prod_{t=1}^{\infty} \mathbf{P}(\bar{A}_t) = 0 \) whence \( \mathbf{P}(A) = 0 \).

Proof of (10):
Let \( B_t \) denote the event \( (t=1, 2, \ldots) \) that there exists a natural number \( i_t \) (different from \( i_1, i_2, \ldots, i_{t-1} \)) such that

\[
 a_{i_t,1} = a_{i_t,2} = \ldots = a_{i_t,t-1} = 0 \quad \text{and} \quad a_{i_t,t} = 1.
\]

Clearly \( \mathbf{P}(B_t) = 1 \), therefore \( \mathbf{P} \left( \bigcup_{t=1}^{\infty} B_t \right) = 1 \).

That is, a random matrix contains a triangular matrix, in which all diagonal elements are equal to 1, with probability 1. Clearly the matrix also contains an \( i \)-th row vector which is different from the rows of the triangular matrix, with probability 1. We show that such a row of the matrix is an infinite linear combination of the \( i_1 \)-th, \( i_2 \)-th, \( i_3 \)-th, \ldots, \( i_t \)-th, \ldots rows.

Put \( x_{i_1} = a_{i_1,1} \) and define the numbers \( x_{i_k} \) successively as

\[
 x_{i_k} = a_{i,k} - \sum_{y=1}^{k-1} a_{i,y} a_{i,y,k}.
\]

If \( t \) is not one of the numbers \( i_k \) then let \( x_i = 0 \).
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Since

\[ a_{i_0,k} = 1, \]

\[ a_{i_v,k} = 0 \quad \text{for} \quad v > k \]

and

\[ \alpha_v = 0 \quad \text{if} \quad v \neq i_t, \]

we have

\[ a_{i,k} = \alpha_{i_k} + \sum_{v=1}^{k-1} \alpha_v a_{i_v,k} = \sum_{v=1}^{k} \alpha_v a_{i_v,k} = \sum_{v=1}^{\infty} \alpha_v a_{i,v,k} = \sum_{\mu=1}^{\infty} \alpha_{\mu} a_{\mu,k}, \]

that is (10) holds.

The condition (8) says that

\[ \lim_{N \to \infty} \sum_{v=1}^{N} \alpha_v a_{v,k} = a_{i,k} \quad \text{for} \quad k = 1, 2, \ldots. \]

If we substitute this condition by the condition

\[ \lim_{N \to \infty} \sum_{v=1}^{N} \alpha_v a_{v,k} = a_{i,k} \quad \text{uniformly in} \quad k, \]

then the probability in question is equal to 0.

c)

In the proof of (10) we actually proved that the rows (and clearly the columns too) of a random matrix contain an infinite basis with probability 1.

(A subset of a set of vectors is called) “basis infinitely”, if any element of the set can uniquely be represented by an infinite linear combination of the elements of the subset.

A subset of a set of vectors is called to be “basis finitely” (it can contain infinitely many vectors) if any element of the set can uniquely be represented by a linear combination of finitely many vectors of the elements of the subset.

I do not know whether there exists a set of vectors (with countably many components) containing no “basises infinitely”.

(Finitely many vectors ever contain basis. It is easy to see that a set of countably many vectors also contain at least one “basis finitely” and we proved above that this basis is the whole set with probability 1.)

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(Received March 17, 1966.)

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