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A Novel Approach to Generate Correctly Rounded Math Libraries for New Floating Point Representations

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Given the importance of floating-point (FP) performance in numerous domains, several new variants of FP and its alternatives have been proposed (e.g., Bfloat16, TensorFloat32, and Posits). These representations do not have correctly rounded math libraries. Further, the use of existing FP libraries for these new representations can produce incorrect results. This paper proposes a novel methodology for generating polynomial approximations that can be used to implement correctly rounded math libraries. Existing methods produce polynomials that approximate the real value of an elementary function \( f(x) \) and experience wrong results due to errors in the approximation and due to rounding errors in the implementation. In contrast, our approach generates polynomials that approximate the correctly rounded value of \( f(x) \) (i.e., the value of \( f(x) \) rounded to the target representation). This methodology provides more margin to identify efficient polynomials that produce correctly rounded results for all inputs. We frame the problem of generating efficient polynomials that produce correctly rounded results as a linear programming problem. Our approach guarantees that we produce the correct result even with range reduction techniques. Using our approach, we have developed correctly rounded, yet faster, implementations of elementary functions for multiple target representations. Our Bfloat16 library is 2.3× faster than the corresponding state-of-the-art while producing correct results for all inputs.

1 INTRODUCTION

Approximating real numbers. Every programming language has primitive data types to represent numbers. The floating point (FP) representation, which was standardized with the IEEE-754 standard [Cowlishaw 2008], is widely used in mainstream languages to approximate real numbers. For example, every number in JavaScript is a FP number! There is an ever-increasing need for improved FP performance in domains such as machine learning and high performance computing (HPC). Hence, several new variants and alternatives to FP have been proposed recently such as Bfloat16 [Tagliavini et al. 2018], Posit [Gustafson 2017; Gustafson and Yonemoto 2017], and TensorFloat32 [NVIDIA 2020].

Bfloat16 [Tagliavini et al. 2018] is a 16-bit FP representation with 8-bits of exponent and 7-bits for the fraction. It is already available in Intel FPGAs [Intel 2019] and Google TPUs [Wang and Kanwar 2019]. Bfloat16’s dynamic range is similar to a 32-bit float but has lower memory traffic and footprint, which makes it appealing for neural networks [Kalamkar et al. 2019]. Nvidia’s TensorFloat32 [NVIDIA 2020] is a 19-bit FP representation with 8-bits of exponent and 10-bits for the fraction, which is available with Nvidia’s Ampere architecture. TensorFloat32 provides the dynamic range of a 32-bit float and the precision of half data type (i.e., 16-bit float), which is intended for machine learning and HPC applications. In contrast to FP, posit [Gustafson 2017; Gustafson and Yonemoto 2017] provides tapered precision with a fixed number of bits. Depending on the value being represented, the number of precision bits varies. Inspired by posits, a tapered
precision log number system has been shown to be effective with neural networks [Bernstein et al. 2020; Johnson 2018].

Correctly rounded math libraries. Any number system that approximates real numbers needs a math library that provides implementations for elementary functions [Muller 2005] (i.e., $\log(x)$, $\exp(x)$, $\sqrt{x}$, $\sin(x)$). The recent IEEE-754 standard recommends (although it does not require) that the programming language standards define a list of math library functions and implement them to produce the correctly rounded result [Cowlishaw 2008]. Any application using an erroneous math library will produce erroneous results.

A correctly rounded result of an elementary function $f$ for an input $x$ is defined as the value produced by computing the value of $f(x)$ with real numbers and then rounding the result according to the rounding rule of the target representation. Developing a correct math library is a challenging task. Hence, there is a large body of work on accurately approximating elementary functions [Bris Barbé et al. 2006; Bruni et al. 2015; Bui and Tahar 1999; Chevillard et al. 2011, 2010; Chevillard and Lauter 2007; Gustafson 2020; Jeannerod et al. 2011; Kuprianova and Lauter 2014; LefÄvret et al. 1998; Lim et al. 2020], verifying the correctness of math libraries [Boldo et al. 2009; Daumas et al. 2005; de Dinechin et al. 2011; de Dinechin et al. 2006; Harrison 1997a,b; Lee et al. 2017; Sawada 2002], and repairing math libraries to increase the accuracy [Yi et al. 2019]. There are a few correctly rounded math libraries (for float and double types in the IEEE-754 standard) such as the IBM LibUltim (also known as MathLib) [IBM 2008; Ziv 1991], Sun Microsystem’s LibMCR [Microsystems 2008], CR-LIBM [Daramy et al. 2003], and MPFR math library [Fousse et al. 2007]. Widely used math library (i.e., libm in glibc) does not produce correctly rounded results for all inputs.

New representations lack math libraries. The new FP representations currently do not have math libraries specifically designed for them. One stop-gap alternative is to promote values from new representations to a float/double value and use existing FP libraries for them. For example, we can convert a Bfloat16 value to a 32-bit float and use the FP math library. However, this approach can produce wrong results for the Bfloat16 value even when we use the correctly rounded float library (see Section 2.6 for a detailed example). This approach also has suboptimal performance as the math library for float/double types probably uses a polynomial of a large degree with many terms than necessary to approximate these functions.

Prior approaches for creating math libraries. Most prior approaches use minimax approximation methods (i.e., Remez algorithm [Remes 1934] or Chebyshev approximations [Trefethen 2012]) to generate polynomials that have the smallest error compared to the value of an elementary function when evaluated with real numbers. The resultant polynomial can be evaluated with addition, subtraction, and multiplication operations. Typically, range reduction techniques are used to reduce the input domain such that the polynomial only needs to approximate the elementary function for a small input domain. Subsequently, the result of the polynomial evaluation on the small input domain is adjusted to produce the result for the entire input domain, which is known as output compensation. Polynomial evaluation, range reduction, and output compensation are implemented in some finite representation that has higher precision than the target representation. The approximated result is finally rounded to the target representation.

When the result of an elementary function $f(x)$ with reals is extremely close to the rounding-boundary (i.e., $f(x)$ rounds to a value $v_1$ but $f(x) + \epsilon$ rounds to a different value $v_2$ for very small value $\epsilon$), then the error of the polynomial must be smaller than $\epsilon$ to ensure that the result of the polynomial produces the correctly rounded value [LefÄvre and Muller 2001]. This probably necessitates a polynomial of a large degree with many terms. Another drawback of prior methods is that there can be round-off errors in the polynomial evaluation with a finite precision representation. Hence, the result produced may not be the correctly rounded result.
Our approach. This paper proposes a novel methodology to generate correctly rounded implementations of elementary functions by framing it as a linear programming problem. In contrast to prior approaches that generate polynomials by minimizing the error compared to the real value of an elementary function \( f(x) \), we propose to generate polynomials that directly approximate the correctly rounded value of \( f(x) \) inspired by the Minefield approach [Gustafson 2020]. Specifically, we identify an interval of values for each input that will result in a correctly rounded output and use that interval to generate the polynomial approximation. For each input \( x_i \), we use an oracle to generate an interval \([l_i, h_i]\) such that all real values in this interval rounds to the correctly rounded value of \( f(x_i) \). Using these intervals, we can subsequently generate a set of constraints, which is given to a linear programming solver, to generate a polynomial that computes the correctly rounded result for all inputs. The interval \([l_i, h_i]\) for correctly rounding the output of input \( x_i \) is larger than \([f(x_i) - \epsilon, f(x_i) + \epsilon]\) where \( \epsilon \) is the maximum error of the polynomial generated using prior methods. Hence, our approach has larger freedom to generated polynomials that produce correctly rounded results and also provide better performance.

Handling range reduction. Typically, generating polynomials for a small input domain is easier than a large input domain. Hence, the input is reduced to a smaller domain with range reduction. Subsequently, polynomial approximation is used for the reduced input. The resulting value is adjusted with output compensation to produce the final output. For example, the input domain for \( \log_2(x) \) is \((0, \infty)\). Approximating this function with a polynomial is much easier over the domain \([1, 2]\) when compared to the entire input domain \((0, \infty)\). Hence, we range reduce the input \( x \) into \( z \) using \( x = z \times 2^e \), where \( z \in [1, 2) \) and \( e \) is an integer. We compute \( y' = \log_2(z) \) using our polynomial for the domain \([1, 2]\). We compute the final output \( y \) using the range reduced output \( y' \) and the output compensation function, which is \( y = y' + e \). Polynomial evaluation, range reduction, and output compensation are performed with finite precision representation (e.g., double) and can experience numerical errors. Our approach for generating correctly rounded outputs has to consider the numerical error with output compensation. To account for rounding errors with range reduction and output compensation, we constrain the output intervals that we generated for each.
input $x$ in the entire input domain (see Section 4). When our approach generates a polynomial, it is guaranteed that the polynomial evaluation along with the range reduction and output compensation can be implemented with finite precision to produce a correctly rounded result for all inputs of an elementary function $f(x)$. Figure 1 pictorially provides an overview of our methodology.

**OURLIBM.** We have developed a collection of correctly rounded math library functions, which we call OURLIBM, for Bfloat16, Posits, and floating point using our proposed methodology. Concretely, OURLIBM contains ten elementary functions for Bfloat16, six elementary functions for 16-bit posits, and $\log_2(x)$ function in the domain $[1, 2)$ for a 32-bit float type. We have verified that our implementation produces the correctly rounded result for all inputs. In contrast, glibc’s $\log_2(x)$ function produces wrong results for more than a million inputs in the domain $[1, 2)$. Our library functions for Bfloat16 are on average $2.30 \times$ faster than the glibc’s double library and $1.79 \times$ faster than the glibc’s float library. We also observed that using glibc’s float library for Bfloat16 produces a wrong result.

**Contributions.** This paper makes the following contributions.

- Proposes a novel methodology that generates polynomials based on the correctly rounded value of an elementary function rather than minimizing the error between the real value and the approximation.
- Demonstrates that the task of generating polynomials with correctly rounded results can be framed as a linear programming problem while accounting for range reduction.
- Demonstrates OURLIBM, a library of elementary functions that produce correctly rounded results for all inputs for various new alternatives to floating point such as Bfloat16 and Posits. Our functions are on average $2.3 \times$ faster than state-of-the-art libraries.

## 2 BACKGROUND AND MOTIVATION

We provide background on the FP representation and its variants (i.e., Bfloat16), the posit representation, the state-of-the-art for developing math libraries, and a motivating example illustrating how the use of existing libraries for new representations can result in wrong results.

### 2.1 Floating Point and Its Variants

The FP representation $F_{n, |E|}$, which is specified in the IEEE-754 standard [Cowlishaw 2008], is parameterized by the total number of bits $n$ and the number of bits for the exponent $|E|$. There are three components in a FP bit-string: a sign bit $s$, $|E|$-bits to represent the exponent, and $|F|$-bits to represent the mantissa $F$ where $|F| = n - 1 - |E|$. Figure 2(a) shows the FP format. If $s = 0$, then the value is positive. If $s = 1$, then the value is negative. The value represented by the FP bit-string is a
When a real number $x$, when interpreted as an unsigned integer, satisfies $0 < E < 2^{|E|} - 1$. The normal value represented with this bit-string is $(1 + \frac{F}{2^{es}}) \times 2^{E-bias}$, where bias is $2^{|E|}-1 - 1$. If $E = 0$, then the FP value is a denormal value. The value of the denormal value is $(\frac{F}{2^{es}}) \times 2^{1-bias}$. When $E = 2^{|E|} - 1$, the FP bit-strings represent special values. If $F = 0$, then the bit-string represents $\pm \infty$ depending on the value of $s$ and in all other cases, it represents not-a-number (NaN).

IEEE-754 specifies a number of default FP types: 16-bit ($\mathbb{F}_{16,5}$ or half), 32-bit ($\mathbb{F}_{32,8}$ or float), and 64-bit ($\mathbb{F}_{64,11}$ or double). Beyond the types specified in the IEEE-754 standard, recent extensions have increased the dynamic range and/or precision. Bfloat16 [Tagliavini et al. 2018], $\mathbb{F}_{16,8}$, provides increased dynamic range compared to FP’s half type. Figure 2(b) illustrates the Bfloat16 format. Recently proposed TensorFloat32 [NVIDIA 2020], $\mathbb{F}_{19,8}$, increased both the dynamic range and precision compared to the half type.

2.2 The Posit Representation

Posit [Gustafson 2017; Gustafson and Yonemoto 2017] is a new representation that provides tapered precision with a fixed number of bits. A posit representation, $\mathbb{P}_{n,es}$, is defined by the total number of bits $n$ and the maximum number of bits for the exponents $es$. A posit bit-string consists of five components (see Figure 2(d)): a sign bit $s$, a number of regime bits $R$, a regime guard bit $\bar{R}$, up to $es$-bits of the exponent $E$, and fraction bits $F$. When the regime bits are not used, they can be re-purposed to represent the fraction, which provides tapered precision.

**Value of a posit bit-string.** The first bit is a sign bit. If $s = 0$, then the value is positive. If $s = 1$, then the value is negative and the bit-string is decoded after taking the two’s complement of the remaining bit-string after the sign bit. Three components $R$, $\bar{R}$, and $E$ together are used to represent the exponent of the final value. After the sign bit, the next $1 \leq |R| \leq n-1$ bits represent the regime $R$. Regime bits consist of consecutive 1’s (or 0’s) and are only terminated if $|R| = n - 1$ or by an opposite bit 0 (or 1), which is known as the regime guard bit ($\bar{R}$). The regime bits represent the super exponent. Regime bits contribute $useed^r$ to the value of the number where $useed = 2^{2es}$ and $r = |R| - 1$ if $R$ consists of 1’s and $r = -|R|$ if $R$ consists of 0’s.

If $2 + |R| < n$, then the next min($es, n-2 - |R|$) bits represent the exponent bits. If $|E| < es$, then $E$ is padded with 0’s to the right until $|E| = es$. These $|es|$-bits contribute $2^E$ to the value of the number. Together, the regime and the exponent bits of the posit bit-string contribute $useed^r \times 2^E$ to the value of the number. If there are any remaining bits after the es-exponent bits, they represent the fraction bits $F$. The fraction bits are interpreted like a normal FP value, except the length of $F$ can vary depending on the number of regime bits. They contribute $1 + \frac{F}{2^{es}}$. Finally, the value $v$ represented by a posit bit-string is,

$$v = (-1)^s \times (1 + \frac{F}{2^{es}}) \times useed^r \times 2^E = (-1)^s \times (1 + \frac{F}{2^{es}}) \times 2^{es+r+E}$$

There are two special cases. A bit-string of all 0’s represents 0. A bit-string of 1 followed by all 0’s’s represents Not-a-Real (NaR).

**Example.** Consider the bit-string 0000011011000000 in the $\mathbb{P}_{16,1}$ configuration. Here, $useed = 2^{2i} = 2^2$. Also $s = 0$, $R = 0000$, $\bar{R} = 1$, $E = 1$, and $F = 011000000$. Hence, $r = -|R| = -4$. The final exponent resulting from the regime and the exponent bits is $(2^2)^{-4} \times 2^1 = 2^{-7}$. The fraction value is 1.375. The value represented by this posit bit-string is $1.375 \times 2^{-7}$.

2.3 Rounding and Numerical Errors

When a real number $x$ cannot be represented in a target representation $\mathbb{T}$, it has to be rounded to a value $v \in \mathbb{T}$. The FP standard defines a number of rounding modes but the default rounding mode...
is the round-to-nearest-tie-goes-to-even (RNE) mode. The posit standard also specifies RNE rounding mode with a minor difference that any non-zero value does not underflow to 0 or overflow to NaR. We describe our approach with RNE mode but it is applicable to other rounding modes.

In the RNE mode, the rounding function \( v = RN_T(x) \), rounds \( x \in \mathbb{R} \) (Reals) to \( v \in \mathbb{T} \), such that \( x \) is rounded to the nearest representable value in \( \mathbb{T} \), i.e., \( \forall v' \in \mathbb{T} \), \( |x - v'| \leq |x - v''| \). In the case of a tie, where \( \exists v_1, v_2 \in \mathbb{T} \), \( v_1 \neq v_2 \) such that \( |x - v_1| = |x - v_2| \) and \( \forall v' \in \mathbb{T} \), \( |x - v_1| \leq |x - v''| \), then \( x \) is rounded to \( v_1 \) if the bit-string encoding the value \( v_1 \) is an even number when interpreted as an integer and to \( v_2 \) otherwise. Figure 3 illustrates the RNE mode with a 5-bit FP representation from Figure 2(c).

The result of primitive operations in FP or any other representation experiences rounding error when it cannot be exactly represented. Modern hardware and libraries produce correctly rounded results for primitive operations. However, this rounding error can get amplified with a series of primitive operations because the intermediate result of each primitive operation must be rounded. As math libraries are also implemented with finite precision, numerical errors in the implementation should also be carefully addressed.

### 2.4 Background on Approximating Elementary Functions

The state-of-the-art methods to approximate an elementary function \( f(x) \) for a target representation \( \mathbb{T} \) involves two steps. First, approximation theory (e.g., minimax methods) is used to develop a function \( A_\mathbb{R}(x) \) that closely approximates \( f(x) \) using real numbers. Second, \( A_\mathbb{R}(x) \) is implemented in a finite precision representation that has higher precision than \( \mathbb{T} \).

**Generating** \( A_\mathbb{R}(x) \). Mathematically deriving \( A_\mathbb{R}(x) \) can be further split into three steps. First, identify inputs that exhibit special behavior (e.g., \( \pm \infty \)). Second, reduce the input domain to a smaller interval, \([a', b']\), with range reduction techniques and perform any other function transformations. Third, generate a polynomial \( P(x) \) that approximates \( f(x) \) in the domain \([a', b']\).

There are two types of special cases. The first type includes inputs that produce undefined values or \( \pm \infty \) when mathematically evaluating \( f(x) \). For example, in the case of \( f(x) = 10^x \), \( f(x) = \infty \) if \( x = \infty \). The second type consists of interesting inputs for evaluating \( RN_T(f(x)) \). These cases include a range of inputs that produce interesting outputs such as \( RN_T(f(x)) \in \{\pm \infty, 0\} \). For example, while approximating \( f(x) = 10^x \) for \( Bfloat16(\mathbb{B}) \), all values \( x \in (-\infty,-40.5] \) produce \( RN_{\mathbb{B}}(10^x) = 0 \), inputs \( x \in [-8.46 \cdots \times 10^{-4}, 1.68 \cdots \times 10^{-3}] \) produce \( RN_{\mathbb{B}}(10^x) = 1 \), and \( x \in [38.75, \infty) \) produces \( RN_{\mathbb{B}}(10^x) = \infty \). These properties are specific to each \( f(x) \) and \( \mathbb{T} \).

**Range reduction.** It is mathematically simpler to approximate \( f(x) \) for a small domain of inputs. Hence, most math libraries use range reduction to reduce the entire input domain into a smaller domain before generating the polynomial. Given an input \( x \in [a, b] \) where \( [a, b] \subseteq \mathbb{T} \), the goal of range reduction is to reduce the input \( x \) to \( x' \in [a', b'] \), where \( [a', b'] \subset [a, b] \). We represent this
process of range reduction with $x' = RR(x)$. Then, the polynomial $P$ approximates the output $y'$ for the range reduced input (i.e., $y' = P(x')$). The output ($y'$) of the range reduced input ($x'$) has to be compensated to produce the output for the original input ($x$). The output compensation function, $OC(y', x)$, produces the final result by compensating the range reduced output $y'$ based on the range reduction performed for input $x$.

For example, consider the function $f(x) = \log_2(x)$ where the input domain is defined over $(0, \infty)$. One way to range reduce the original input is to use the mathematical property $\log_2(a \times 2^b) = \log_2(a) + b$. We decompose the input $x$ as $x = x_f \times 2^e$ where $x_f \in [1, 2)$ and $e$ is an integer. Approximating $\log_2(x)$ is equivalent to approximating $\log_2(x_f \times 2^e) = \log_2(x_f) + e$. Thus, we can range reduce the original input $x \in (0, \infty)$ into $x_f \in [1, 2)$. Then, we approximate $\log_2(x_f)$ using $P(x_f)$, which needs to only approximate $\log_2(x)$ for the input domain $[1, 2)$. To produce the output of $\log_2(x)$, we compensate the output of the reduced input by computing $P(x') + e$, where $e$ is dependent on the range reduction of $x$.

**Polynomial approximation $P(x)$.** A common method to approximate an elementary function $f(x)$ is with a polynomial function, $P(x)$, which can be implemented with addition, subtraction, and multiplication operations. Typically, $P(x)$ for math libraries is generated using the minimax approximation technique, which aims to minimize the maximum error, or $L_{\infty}$-norm,

$$||P(x) − f(x)||_{\infty} = \sup_{x \in [a,b]} |P(x) − f(x)|$$

where sup represents the supremum of a set. The minimax approach is attractive because the resulting $P(x)$ has a bound on the error (i.e., $|P(x) − f(x)|$). The most well-known minimax approximation method is the Remez algorithm [Remes 1934]. Both CR-LIBM [Darmay et al. 2003] and Metalibm [Kupriianova and Lauter 2014] use a modified Remez algorithm to produce polynomial approximations [Brisebarre and Chevillard 2007].

**Implementation of $A_\mathbb{R}(x)$ with finite precision.** Finally, mathematical approximation $A_\mathbb{R}(x)$ is implemented in finite precision to approximate $f(x)$. This implementation typically uses a higher precision than the intended target representation. We use $A_\mathbb{R}(x)$ to represent that $A_\mathbb{R}(x)$ is implemented in a representation with higher precision ($\mathbb{H}$) where $\mathbb{T} \subset \mathbb{H}$. Finally, the result of the implementation $A_\mathbb{H}(x)$ is rounded to the target representation $\mathbb{T}$.

### 2.5 Challenges in Building Correctly Rounded Math Libraries

An approximation of an elementary function $f(x)$ is defined to be a correctly rounded approximation if for all inputs $x_i \in \mathbb{T}$, it produces $RN_\mathbb{T}(f(x_i))$. There are two major challenges in creating a correctly rounded approximation. First, $A_\mathbb{T}(x)$ incurs error because $P(x)$ is an approximation of $f(x)$. Second, the evaluation of $A_\mathbb{H}(x)$ has numerical error because it is implemented in a representation with finite precision (i.e., $\mathbb{H}$). Hence, the rounding of $RN_\mathbb{T}(A_\mathbb{H}(x))$ can result in a value different from $RN_\mathbb{T}(f(x))$, even if $A_\mathbb{H}(x)$ is arbitrarily close to $f(x)$ for some $x \in \mathbb{T}$.

As $A_\mathbb{H}(x)$ uses a polynomial approximation of $f(x)$, there is an inherent error of $|f(x) − A_\mathbb{H}(x)| > 0$. Further, the evaluation of $A_\mathbb{H}(x)$ experiences an error of $|A_\mathbb{H}(x) − A_\mathbb{E}(x)| > 0$. It is not possible to reduce both errors to 0. The error in approximating the polynomial can be reduced by using a polynomial of a higher degree or a piece-wise polynomial. The numerical error in the evaluation of $A_\mathbb{H}(x)$ can be reduced by increasing the precision of $\mathbb{H}$. Typically, library developers make trade-offs between error and the performance of the implementation.

Unfortunately, there is no known general method to analyze and predict the bound on the error for $A_\mathbb{H}(x)$ that guarantees $RN_\mathbb{T}(A_\mathbb{H}(x)) = RN_\mathbb{T}(f(x))$ for all $x$ because the error may need to be arbitrarily small. This problem is widely known as *table-maker’s dilemma* [Kahan 2004]. It states
We will illustrate this methodology with an end-to-end example that creates correctly rounded libraries. We provide a high-level overview of our methodology to generate correctly rounded math libraries. An alternative to developing math libraries for new representations is to use existing libraries. We illustrate this behavior by generating an approximation to a high-degree polynomial, which causes it to be slower than the math library tailored for Bfloat16. This strategy is appealing if a correctly rounded math library for T exists and T’ has significantly more precision bits than T.

However, using a correctly rounded math library designed for T’ to approximate f(x) for T can produce incorrect results for values in T. We illustrate this behavior by generating an approximation for the function f(x) = 10^x in the Bfloat16 (B) representation (Figure 4). Let’s consider the input x = -0.0181884765625 ∈ B. The real value of f(x) ≈ 0.95898435797... (black circle in Figure 4). This oracle result cannot be exactly represented in Bfloat16 and must be rounded. There are two Bfloat16 values adjacent to f(x), b_1 = 0.95703125 and b_2 = 0.9609375. Since b_1 is closer to f(x), the correctly rounded result is RN_B(10^x) = b_1, which is represented by a black star in Figure 4.

If we use the correctly rounded float math library to approximate 10^x, we get the value, y’ = 0.958984375, represented by red diamond in Figure 4. From the perspective of a 32-bit float, y’ is a correctly rounded result, i.e. y’ = RN_{32.7}(10^x) = 0.958984375. Because y’ ∉ B, we round y’ to Bfloat16 based on the rounding rule, RN_B(y’)= b_2. Therefore, the float math library rounds the result to b_2 but the correctly rounded result is RN_B(10^x) = b_1.

Summary. Approximating an elementary function representation T using a math library designed for a higher precision representation T’ does not guarantee a correctly rounded result. Further, the math library for T’ probably requires higher accuracy than the one for T. Hence, it uses a higher degree polynomial, which causes it to be slower than the math library tailored for T.

3 HIGH-LEVEL OVERVIEW

We provide a high-level overview of our methodology to generate correctly rounded math libraries. We will illustrate this methodology with an end-to-end example that creates correctly rounded results for ln(x) with FP5 (i.e., a 5-bit FP type shown in Figure 2(c)).

3.1 Our Methodology for Generating Correctly Rounded Elementary Functions

Given an elementary function f(x) and a target representation T, our goal is to synthesize a polynomial that when used with range reduction (RR) and output compensation (OC) function produces the correctly rounded result for all inputs in T. The evaluation of the polynomial, range
reduction, and output compensation are implemented in representation $\mathbb{H}$, which has higher precision than $\mathbb{T}$.

Our methodology for generating correctly rounded elementary functions is shown in Figure 1. Our methodology consists of four steps. First, we use an oracle (i.e., MPFR [Fousse et al. 2007] with a large number of precision bits) to compute the correctly rounded result of the function $f(x)$ for each input $x \in \mathbb{T}$. In this step, a small sample of the entire input space can be used rather than using all inputs for a type with a large input domain.

Second, we identify an interval $[l, h]$ around the correctly rounded result such that any value in $[l, h]$ rounds to the correctly rounded result in $\mathbb{T}$. We call this interval the rounding interval. Since the eventual polynomial evaluation happens in $\mathbb{H}$, the rounding intervals are also in the $\mathbb{H}$ representation. The internal computations of the math library evaluated in $\mathbb{H}$ should produce a value in the rounding interval for each input $x$.

Third, we employ range reduction to transform input $x$ to $x'$. The generated polynomial will approximate the result for $x'$. Subsequently, we have to use an appropriate output compensation code to produce the final correctly rounded output for $x$. Both range reduction and output compensation happen in the $\mathbb{H}$ representation and can experience numerical errors. These numerical errors should not affect the generation of correctly rounded results. Hence, we infer intervals for the reduced domain so that the polynomial evaluation over the reduced input domain produces the correct results for the entire domain. Given $x$ and its rounding interval $[l, h]$, we can compute the reduced input $x'$ with range reduction. The next task before polynomial generation is identifying the reduced rounding interval for $P(x')$ such that when used with output compensation it produces the correctly rounded result. We use the inverse of the output compensation function to identify the reduced interval $[l', h']$. Any value in $[l', h']$ when used with the implementation of output compensation in $\mathbb{H}$ produces the correctly rounded results for the entire domain.

Fourth, we synthesize a polynomial of a degree $d$ using an arbitrary precision linear programming (LP) solver that satisfies the constraints (i.e., $l' \leq P(x') \leq h'$) when given a set of inputs $x'$. Since the LP solver produces coefficients for the polynomial in arbitrary precision, it is possible that some of the constraints will not be satisfied when evaluated in $\mathbb{H}$. In such cases, we refine the reduced intervals for those inputs whose constraints are violated and repeat the above step. If the LP solver is not able to produce a solution, then the developer of the library has to either increase the degree of the polynomial or reduce the input domain.

If the inputs were sampled in the first step, we check whether the generated polynomial produces the correctly rounded result for all inputs. If it does not, then the input is added to the sample and the entire process is repeated. At the end of this process, the polynomial along with range reduction and output compensation when evaluated in $\mathbb{H}$ produces the correctly rounded outputs for all inputs in $\mathbb{T}$.

3.2 Illustration of our Approach with $\ln(x)$ for FP5

We provide an end-to-end example of our approach by creating a correctly rounded result of $\ln(x)$ for the FP5 representation shown in Figure 2(c) with the RNE rounding mode. The $\ln(x)$ function is defined over the input domain $(0, \infty)$. There are 11 values ranging from 0.25 to 3.5 in FP5 within $(0, \infty)$. We show the generation of the polynomial with FP5 for pedagogical reasons. With FP5, it is beneficial to create a pre-computed table of correctly rounded results for the 11 values.

Our strategy is to approximate $\ln(x)$ by using $\log_2(x)$. Hence, we perform range reduction and output compensation using the properties of logarithm: $\ln(x) = \frac{\log_2(x)}{\log_2(e)}$ and $\log_2(x \times y^e) = \log_2(x) + e \log_2(y)$. We decompose the input $x$ as $x = x' \times 2^e$ where $x'$ is the fractional value represented by the mantissa, i.e. $x' \in [1, 2)$, and $e$ is the exponent of the value. We use $\log(x) = \frac{\log_2(x') + e}{\log_2(e)}$ for our

range reduction. We construct the range reduction function $RR(x)$ and the output compensation function $OC(y', x)$ as follows,

$$RR(x) = fr(x), \quad OC(y', x) = \frac{y' + \exp(x)}{\log_2(e)}$$

Fig. 5. Our approach for $\ln(x)$ with FP5. (a) For each input $x$ in FP5, we accurately compute the correctly rounded result (black circle) and identify intervals around the result so that all values round to it. (b) For each input and corresponding interval computed in (a), we perform range reduction to obtain the reduced input. The number below a value on the x-axis represents the reduced input. The reduced interval to account for rounding errors in output compensation is also shown. Multiple distinct inputs can map to the same reduced input after range reduction (intervals with the same color). In such scenarios, we combine the reduce intervals by computing the common region in the intervals (highlighted in bold for each color with dotted lines). (c) The set of constraints that must be satisfied by the polynomial for the reduced input. (d) LP formulation for the generation of a polynomial of degree one. (e) The coefficients generated by the LP solver for the polynomial. (f) Generated polynomial satisfies the combined intervals.
The next step is to identify the reduced input and the rounding interval for the reduced input such that it accounts for any numerical error in output compensation. Figure 5(b) shows the reduced intervals for a small subset of inputs. Let us suppose that we want to compute the rounding interval for $\log_2(x)$, which is the correctly rounded value of $\ln(x)$. Figure 5(a) shows the correctly rounded result for each input as a black dot.

**Step 2: Identifying the rounding interval $[l, h]$.** The range reduction, output compensation, and polynomial evaluation are performed with the double type. The double result of the evaluation is rounded to FP5 to produce the final result. The next step is to find a rounding interval $[l, h]$ in the double type for each output. Figure 5(a) shows the rounding interval for each FP5 output using the blue (upper bound) and orange (lower bound) bracket.

Let us suppose that we want to compute the rounding interval for $y = 1.0$, which is the correctly rounded result of $\log_2(2.5)$. To identify the lower bound $l$ of the rounding interval for $y = 1.0$, we first identify the preceding FP5 value, which is 0.75. Then we find a value $v$ between 0.75 and 1.0 such that values greater than or equal to $v$ rounds to 1.0. In our case, $v = 0.875$, which is the lower bound. Similarly, to identify the upper bound $h$, we identify the FP5 value succeeding 1.0, which is 1.25. We find a value $u$ such that any value less than or equal to $u$ rounds to 1.0. In our case, the upper bound is $h = 1.125$. Hence, the rounding interval for $y = 1.0$ is $[0.875, 1.125]$. Figure 6 shows the intervals for a small subset of FP5.

**Step 3-a: Computing the reduced input $x'$ and the reduced interval $[l', h']$.** We perform range reduction and generate a polynomial that computes $\log_2(x)$ for all reduced inputs in $[1, 2)$. The next step is to identify the reduced input and the rounding interval for the reduced input such that it accounts for any numerical error in output compensation. Figure 5(b) shows the reduced input (number below the value on the x-axis) and the reduced interval for each input.

To identify the reduced rounding interval, we use the inverse of the output compensation function, which exists if OC is continuous and bijective over real numbers. For example, for the input $x = 3.5 = 1.75 \times 2^1$, the output compensation function is,

$$OC(y', 3.5) = \frac{y' + 1}{\log_2(e)}$$

The inverse is

$$OC^{-1}(y, 3.5) = y\log_2(e) - 1$$

Thus, we use the inverse output compensation function to compute the candidate reduced interval $[l', h']$ by computing $l' = OC^{-1}(l, x)$ and $h' = OC^{-1}(h, x)$. Then, we verify that the output compensation result of $l'$ (i.e., $OC(l', x)$) and $h'$ (i.e., $OC(h', x)$), when evaluated in double lies in $[l, h]$. If it does not, then we iteratively refine the reduced interval by restricting $[l', h']$ to a smaller
We look for a polynomial that satisfies constraints for each reduced input (Figure 5(d)). We use an LP with our methodology while using a representation $H$. We combine all reduced intervals that correspond to the same reduced input by computing the value of $\ln$. This can map to the same reduced input after range reduction. In our example, both $x_1 = 1.25$ and $x_2 = 2.5$ reduces to $x' = 1.25$. However, the reduced intervals that we compute for $x_1$ and $x_2$ are $[l_1', h_1']$ and $[l_2', h_2']$, respectively. They are not exactly the same. In Figure 5(b), the reduced intervals corresponding to the original inputs that map to the same reduced input are colored with the same color. The reduced interval for $x_1 = 1.25$ and $x_2 = 2.5$ are colored in blue.

The reduced interval for $x_1$ indicates that $P(1.25)$ must produce a value in $[l_1', h_1']$ such that the final result, after evaluating the output compensation function in double, is the correctly rounded value of $\ln(1.25)$. The reduced interval for $x_2$ indicates that $P(1.25)$ must produce a value in $[l_2', h_2']$ such that the final result is the correct value of $\ln(2.5)$. To produce the correctly rounded result for both inputs $x_1$ and $x_2$, $P(1.25)$ must produce a value that is in both $[l_1', h_1']$ and $[l_2', h_2']$. Thus, we combine all reduced intervals that correspond to the same reduced input by computing the common interval. Figure 5(b) shows the common interval for a given reduced input using a darker shade. At the end of this step, we are left with one combined interval for each reduced input.

**Step 4: Generating the Polynomial for the reduced input.** The combined intervals specify the constraints on the output of the polynomial for each reduced input, which when used with output compensation in double results in a correctly rounded result for the entire domain. Figure 5(c) shows the constraints for $P(x')$ for each reduced input.

To synthesize a polynomial $P(x')$ of a particular degree (the degree is 1 in this example), we encode the problem as a linear programming (LP) problem that solves for the coefficients of $P(x')$. We look for a polynomial that satisfies constraints for each reduced input (Figure 5(d)). We use an LP solver to solve for the coefficients and find $P(x')$ with the coefficients in Figure 5(e). The generated polynomial $P(x')$ satisfies all the linear constraints as shown in Figure 5(f). Finally, we also verify that the generated polynomial when used with range reduction and output compensation produces the correctly rounded results for all inputs in the original domain.

## 4 OUR METHODOLOGY FOR GENERATING CORRECTLY ROUNDED LIBRARIES

Our goal is to create approximations for an elementary function $f(x)$ that produces correctly rounded results for all inputs in the target representation ($\mathbb{T}$).

**Definition 4.1.** A function that approximates an elementary function $f(x)$ is a correctly rounded function for the target representation $\mathbb{T}$ if it produces $y = \text{RN}_\mathbb{T}(f(x))$ for all $x \in \mathbb{T}$.

Intuitively, the result produced by the approximation should be the same as the result obtained when $f(x)$ is evaluated with infinite precision and then rounded to the target representation. It may be beneficial to develop precomputed tables with correctly rounded results of elementary functions for small data types (e.g., FP5). However, it is infeasible (due to memory overheads) to store such tables for every elementary function even with modestly sized data types.

We propose a methodology that produces polynomial approximation and stores a few coefficients for evaluating the polynomial. There are three main challenges in generating a correctly rounded result with polynomial approximations. First, we have to generate polynomial approximations that produce the correct result and are efficient to evaluate. Second, the polynomial approximation should consider rounding errors with range reduction and output compensation that are implemented in some finite precision representation. Third, the polynomial evaluation also is implemented with finite precision and can experience numerical errors.

We will use $A_{25}(x)$ to represent the approximation of the elementary function $f(x)$ produced with our methodology while using a representation $\mathbb{H}$ to perform polynomial evaluation, range...
Function CorrectlyRoundedPoly(f, T, H, X, RR, OC, d):
  L ← CalcRdIntervals(f, T, H, X)
  if L = ∅ then return (false, DNE)
  L′ ← CalcRedIntervals(L, H, RR, OC)
  if L′ = ∅ then return (false, DNE)
  Λ ← CombineRedIntervals(L′)
  if Λ = ∅ then return (false, DNE)
  S, P ← GeneratePoly(Λ, d)
  if S = true then return (true, P)
  else return (false, DNE)

Input Description:
  f: The oracle that computes the result of f(x) in arbitrary precision.
  T: Target representation of math library.
  H: Higher precision representation.
  X: Input domain of A(x).
  RR: The range reduction function.
  OC: The output compensation function.
  d: The degree of polynomial to generate.

Fig. 7. Our approach to generate a polynomial approximation P(x) that produces the correctly rounded result for all inputs. On successfully finding a polynomial, it returns (true, P). Otherwise, it returns (false, DNE) where DNE means that the polynomial Does-Not-Exist. Functions, CalcIntervals, CalcRedIntervals, CombineRedIntervals, and GeneratePoly are shown in Figure 8, Figure 9, and Figure 10, respectively.

Our approach has four main steps. First, we compute y ∈ T, the correctly rounded result of f(x), i.e. y = RN_T(f(x)) for each input x (or a sample of the inputs for a large data type) using our oracle. Then, we identify the rounding interval I = [l, h] ⊆ H where all values in the interval round to y. The pair (x, I) specifies that A(x) must produce a value in I such that A(x) rounds to y. The function CalcRdIntervals in Figure 7 returns a list L that contains a pair (x, I) for all inputs x.

Second, we compute the reduced input x′ using range reduction and a reduced interval I′ = [l′, h′] for each pair (x, I) ∈ L. The reduced interval I′ = [l′, h′] ensures that any value in I′ when used with output compensation code results in a value in I. This pair (x′, I′) specifies the constraints for the output of the polynomial approximation P(x) so A(x) rounds to the correctly rounded result. The function CalcRedIntervals returns a list L′ with such reduced constraints for all inputs x.

Third, multiple inputs from the original input domain will map to the same input in the reduced domain after range reduction. Hence, there will be multiple reduced constraints for each reduced input x′. The polynomial approximation, P(x′), must produce a value that satisfies all the reduced constraints to ensure that A(x) produces the correct value for all inputs when rounded. Thus, we combine all reduced intervals for each unique reduced input x′ and produce the pair (x′, Ψ)
where $\Psi$ represents the combined interval. Function CalcRndIntervals in Figure 7 returns a list $\Lambda$ containing the constraint pair $(x', \Psi)$ for each unique reduced input $x'$. Finally, we generate a polynomial of degree $d$ using linear programming so that all constraints $(x', \Psi) \in \Lambda$ are satisfied. Next, we describe these steps in detail.

### 4.1 Calculating the Rounding Interval

The first step in our approach is to identify the values that $A_{\mathbb{T}}(x)$ must produce so that the rounded value of $A_{\mathbb{T}}(x)$ is equal to the correctly rounded result of $y = f(x)$, i.e. $RN_{\mathbb{T}}(A_{\mathbb{T}}(x)) = RN_{\mathbb{T}}(y)$, for each input $x \in X$. Our key insight is that it is not necessary to produce the exact value of $y$ to produce a correctly rounded result. It is sufficient to produce any value in $\mathbb{H}$ that rounds to the correct result. For a given rounding mode and an input, we are looking for an interval $I = [l, h]$ around the oracle result that produces the correctly rounded result. We call this the rounding interval.

Given an elementary function $f(x)$ and an input $x \in X$, define an interval $I$ that is representable in $\mathbb{H}$ such that $RN_{\mathbb{T}}(v) = RN_{\mathbb{T}}(f(x))$ for all $v \in I$. If $A_{\mathbb{T}}(x) \in I$, then rounding the result of $A_{\mathbb{T}}(x)$ to $\mathbb{T}$ produces the correctly rounded result (i.e., $RN_{\mathbb{T}}(A(x)) = RN_{\mathbb{T}}(f(x))$). For each input $x$, if $A_{\mathbb{T}}(x)$ can produce a value that lies within its corresponding rounding interval, then it will produce a correctly rounded result. Thus, the pair $(x, I)$ for each input $x$ defines constraints on the output of $A_{\mathbb{T}}(x)$ such that $RN_{\mathbb{T}}(A_{\mathbb{T}}(x))$ is a correctly rounded result.

Figure 8 presents our algorithm to compute constraints $(x, I)$. For each input $x$ in our input domain $X$, we compute the correctly rounded result of $f(x)$ using an oracle and produce $y$. Next, we compute the rounding interval of $y$ where all values in the interval round to $y$. The rounding interval can be computed as follows. First, we identify $t_l$, the preceding value of $y$ in $\mathbb{T}$ (line 11 in Figure 8). Then we find the minimum value $l \in \mathbb{H}$ between $t_l$ and $y$ where $l$ rounds to $y$ (line 12 in Figure 8). Similarly for the upper bound, we identify $t_u$, the succeeding value of $y$ in $\mathbb{T}$ (line 13 in Figure 8), and find the maximum value $h \in \mathbb{H}$ between $y$ and $t_u$ where $h$ rounds to $y$ (line 14 in Figure 8). Then, $[l, h]$ is the rounding interval of $y$ and all values in $[l, h]$ round to $y$. Thus, the pair $(x, [l, h])$ specifies a constraint on the output of $A_{\mathbb{T}}(x)$ to produce the correctly rounded result for input $x$. We generate such constraints for each input in the entire domain (or for a sample of inputs) and produce a list of such constraints (line 7-9 in Figure 8).

### 4.2 Calculating the Reduced Input and Reduced Interval

After the previous step, we have a list of constraints, $(x, I)$, that need to be satisfied by our approximation $A_{\mathbb{T}}(x)$ to produce correctly rounded outputs. If we do not perform any range reduction,

then we can generate a polynomial that satisfies these constraints. However, it is necessary to perform range reduction (RR) in practice to reduce the complexity of the polynomial and to improve performance. Range reduction is accompanied by output compensation (OC) to produce the final output. Hence, $A(x) = OC(x_1) \cdot RR(x_2), x)$. Our goal is to synthesize a polynomial $P(x')$ that operates on the range reduced input $x'$ and $A(x) = OC(P(x)) \cdot RR(x_2), x)$ produces a value in $I$ for each input $x$, which rounds to the correct output.

To synthesize this polynomial, we have to identify the reduced input and the reduced interval for an input $x$ such that $A(x)$ produces a value in the rounding interval $I$ corresponding to $x$. The reduced input is available by applying range reduction $x' = RR(x)$. Next, we need to compute the reduced interval corresponding to $x'$. The output of the polynomial on the reduced input will be fed to the output compensation function to compute the output for the original input. For the reduced input $x'$ corresponding to the original input $x$, $y' = P(x'), A(x) = OC(y', x)$, and $A(x)$ must be within the interval $I$ for input $x$ to produce a correct output. Hence, our high-level strategy is to use the inverse of the output compensation function to compute the reduced interval, which is feasible when the output compensation function is continuous and bijective. In our experience, all commonly used output compensation functions are continuous and bijective.

However, the output compensation function is evaluated in $\mathbb{H}$, which necessitates us to take any numerical error in output compensation with $\mathbb{H}$ into account. Figure 9 describes our algorithm to compute reduced constraint $(x',I')$ for each $(x,I) \in L$ when the output compensation is performed in $\mathbb{H}$.

To compute the reduced interval $I'$ for each constraint pair $(x,[l,h]) \in L$, we evaluate the values $v_1 = OC^{-1}(l,x)$ and $v_2 = OC^{-1}(h,x)$ and create an interval $[\alpha,\beta] = [v_1,v_2]$ if $OC(y',x)$ is an increasing function (lines 5-6 in Figure 9) or $[v_2,v_1]$ if $OC(y',x)$ is a decreasing function (line 7 in

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**Figure 9.** CalcRedIntervals computes the reduced input $x'$ and the reduced interval $I'$ for each constraint pair $(x,I) \in L$. The reduced constraint pair $(x',I')$ specifies the bound on the output of $P(x')$ such that it produces the correct value for the input $x$. CombineRedIntervals combines any reduced constraints with the same reduced input, i.e. $(x_1',I_1')$ and $(x_2',I_2')$ where $x_1' = x_2'$ into a single combined constraint $(x,\Psi)$ by computing the common interval range in $I_1'$ and $I_2'$.
Figure 9). The interval \([\alpha, \beta]\) is a candidate for \(I'\). Then, we verify that the output compensated value of \(\alpha\) is in \([l, h]\) (i.e., \(I\)). If it is not, we replace \(\alpha\) with the succeeding value in \(\mathbb{H}\) and repeat the process until \(OC_{\mathbb{H}}(\alpha, x)\) is in \(I\) (lines 8-11 in Figure 9). Similarly, we verify that the output compensated value of \(\beta\) is in \([l, h]\) and repeatedly replace \(\beta\) with the preceding value in \(\mathbb{H}\) if it is not (lines 12-15 in Figure 9). If \(\alpha > \beta\) at any point during this process, then it indicates that there is no polynomial \(P(x')\) that can produce the correct result for all inputs. As there are only finitely many values between \([\alpha, \beta]\) in \(\mathbb{H}\), this process terminates. In the case when our algorithm is not able to find a polynomial, the user can provide either a different range reduction/output compensation function or increase the precision to be higher than \(\mathbb{H}\).

If the resulting interval \([\alpha, \beta] \neq \emptyset\), then \(I' = [\alpha, \beta]\) is our reduced interval. The reduced constraint pair, \((x', [\alpha, \beta])\) created for each \((x, I) \in L\) specifies the constraint on the output of \(P_{\mathbb{H}}(x')\) such that \(A_{\mathbb{H}}(x) \in I\). Finally, we create a list \(L'\) containing such reduced constraints.

4.3 Combining the Reduced Constraints
Each reduced constraint \((x'_i, I'_i) \in L'\) corresponds to a constraint \((x_i, I_i) \in L\). It specifies the bound on the output of \(P_{\mathbb{H}}(x'_i)\) (i.e., \(P_{\mathbb{H}}(x'_i) \in I'_i\) should be satisfied), which ensures \(A_{\mathbb{H}}(x_i)\) produces a value in \(I_i\). Range reduction reduces the original input \(x_i\) in the entire input domain of \(f(x)\) to a reduced input \(x'_i\) in the reduced domain. Hence, multiple inputs in the entire input domain can be range reduced to the same reduced input. More specifically, there can exist multiple constraints \((x_1, I_1), (x_2, I_2), \ldots \in L\) such that \(RR_{\mathbb{H}}(x_1) = RR_{\mathbb{H}}(x_2) = \hat{x}\). Consequently, \(L'\) contains reduced constraints \(\hat{x}, I'_1, \hat{x}, I'_2, \ldots \in L'\). The polynomial \(P_{\mathbb{H}}(\hat{x})\) must produce a value in \(I'_i\) to guarantee that \(A_{\mathbb{H}}(x_1) \in I_1\). It must also be within \(I'_i\) to guarantee \(A_{\mathbb{H}}(x_2) \in I_2\). Hence, for each unique reduced input \(\hat{x}\), \(P_{\mathbb{H}}(\hat{x})\) must satisfy all reduced constraints corresponding to \(\hat{x}\), i.e., \(P_{\mathbb{H}}(\hat{x}) \in I'_1 \cap I'_2\).

The function \(\text{CombineRedIntervals}\) in Figure 9 combines all reduced constraints with the same reduced input by identifying the common interval (\(\Psi\) in line 24 in Figure 9). If such a common interval does not exist, then it is infeasible to find a single polynomial \(P_{\mathbb{H}}(x')\) that produces correct outputs for all inputs before range reduction. Otherwise, we create a pair \((\hat{x}, \Psi)\) for each unique reduced interval \(\hat{x}\) and produce a list of constraints \(\Lambda\) (line 26 in Figure 9).

4.4 Generating the Polynomial using Linear Programming
Each reduced constraint \((x', [l', h']) \in \Lambda\) requires that \(P_{\mathbb{H}}(x')\) satisfy the following condition: \(l' \leq P_{\mathbb{H}}(x') \leq h'\). This constraint ensures that when \(P_{\mathbb{H}}(x')\) is combined with range reduction and output compensation, it produces the correctly rounded result for all inputs. When we are trying to generate a polynomial of degree \(d\), we can express each of the above constraints in the form:

\[
l' = c_0 + c_1 x' + c_2 (x')^2 + \ldots + c_d (x')^d \leq h'
\]

The goal is to find coefficients for the polynomial evaluated in \(\mathbb{H}\). Here, \(x', l'\) and \(h'\) are constants from perspective of finding the coefficients. We can express all constraints \((x'_i, [l'_i, h'_i]) \in \Lambda\) in a single system of linear inequalities as shown below, which can be solved using a linear programming (LP) solver.

\[
\begin{bmatrix}
  l'_1 \\
  l'_2 \\
  \vdots \\
  l'_{|\Lambda|}
\end{bmatrix} \leq
\begin{bmatrix}
  1 & x'_1 & \ldots & (x'_1)^d \\
  1 & x'_2 & \ldots & (x'_2)^d \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & x'_{|\Lambda|} & \ldots & (x'_{|\Lambda|})^d \\
\end{bmatrix}
\begin{bmatrix}
  c_0 \\
  c_1 \\
  \vdots \\
  c_{|\Lambda|}
\end{bmatrix}
\leq
\begin{bmatrix}
  h'_1 \\
  h'_2 \\
  \vdots \\
  h'_{|\Lambda|}
\end{bmatrix}
\]

Given a system of inequalities, the LP solver finds a solution for the coefficients with real numbers. The polynomial when evaluated in real (i.e., \(P_{\mathbb{H}}(x')\)) satisfies all constraints in \(\Lambda\). However, numerical
A Novel Approach to Generate Correctly Rounded Math Libraries for New Floating Point Representations

1 Function GeneratePoly($\Lambda$, $\mathbb{H}$, $d$):
2 $\Upsilon \leftarrow \Lambda$
3 while true do
4     $C \leftarrow \text{LPSolve}(\Upsilon, d)$
5     if $C = \emptyset$ then return (false, DNE)
6     $P_{\mathbb{H}} \leftarrow \text{CreateP}(C, d, \mathbb{H})$
7     $\Upsilon \leftarrow \text{Verify}(P_{\mathbb{H}}, \Lambda, \Upsilon, \mathbb{H})$
8     if $\Upsilon = \emptyset$ then return (true, $P_{\mathbb{H}}$)
9 end
10 Function Verify($P_{\mathbb{H}}$, $\Lambda$, $\Upsilon$, $\mathbb{H}$):
11     $Z \leftarrow \{(x', \Psi, \psi) | (x', \Psi) \in \Lambda, (x', \psi) \in\}$
12     foreach $(x', [l', h'), [\sigma, \mu]) \in Z$ do
13         if $P_{\mathbb{H}}(x') < l'$ then
14             $\Upsilon \leftarrow \Upsilon - \{(x', [\sigma, \mu])\}$
15             $\sigma' \leftarrow \text{GetSuccVal}(x', \mathbb{H})$
16             return $\Upsilon \cup \{(x', [\sigma', \mu])\}$
17         else if $P_{\mathbb{H}}(x') > h'$ then
18             $\Upsilon \leftarrow \Upsilon - \{(x', [\sigma, \mu])\}$
19             $\mu' \leftarrow \text{GetPrecVal}(x', \mathbb{H})$
20             return $\Upsilon \cup \{(x', [\sigma, \mu'])\}$
21     end
22 end
23 return $\emptyset$

Fig. 10. The function $\text{GeneratePoly}$ generates a polynomial $P_{\mathbb{H}}(x')$ of degree $d$ that satisfies all constraints in $\Lambda$ when evaluated in $\mathbb{H}$. If it cannot generate such a polynomial, then it returns $\text{false}$. The function $\text{LPSolve}$ solves for the real number coefficients of a polynomial $P_{\mathbb{H}}(x)$ using an LP solver where $P_{\mathbb{H}}(x)$ satisfies all constraints in $\Lambda$ when evaluated in real number. $\text{CreateP}$ creates $P_{\mathbb{H}}(x)$ that evaluates the polynomial $P_{\mathbb{H}}(x)$ in $\mathbb{H}$. The $\text{Verify}$ function checks whether the generated polynomial $P_{\mathbb{H}}(x)$ satisfies all constraints in $\Lambda$ when evaluated in $\mathbb{H}$ and refines the constraints to a smaller interval for each constraint that $P_{\mathbb{H}}(x)$ does not satisfy.

errors in polynomial evaluation in $\mathbb{H}$ can cause the result to not satisfy $\Lambda$. We propose a search-and-refine approach to address this problem. We use the LP solver to solve for the coefficients of $P_{\mathbb{H}}(x')$ that satisfies $\Lambda$ and then check if $P_{\mathbb{H}}(x')$ that evaluates $P_{\mathbb{H}}(x')$ in $\mathbb{H}$ satisfies the constraints in $\Lambda$. If $P_{\mathbb{H}}(x')$ does not satisfy a constraint $(x', [l', h']) \in \Lambda$, then we refine the reduced interval $[l', h']$ to a smaller interval. Subsequently, we use the LP solver to generate the coefficients of $P_{\mathbb{H}}(x')$ for the refined constraints. This process is repeated until either $P_{\mathbb{H}}(x')$ satisfies all reduced constraints in $\Lambda$ or the LP solver determines that there is no polynomial that satisfies all the constraints.

Figure 10 provides the algorithm used for generating the coefficients of the polynomial using the LP solver. $\Upsilon$ tracks the refined constraints for $P_{\mathbb{H}}(x')$ during our search-and-refine process. Initially, $\Upsilon$ is set to $\Lambda$ (line 2 in Figure 10). Here, $\Upsilon$ is used to generate the polynomial and $\Lambda$ is used to to verify that the generated polynomial satisfies all constraints. If the generated polynomial does not satisfy $\Lambda$, we restrict the intervals in $\Upsilon$.

We use an LP solver to solve for the coefficients of the $P_{\mathbb{H}}(x')$ that satisfies all constraints in $\Upsilon$ (line 4 in Figure 10). If the LP solver cannot find the coefficients, our algorithm concludes that it is not possible to generate a polynomial and terminates (line 5 in Figure 10). Otherwise, we create $P_{\mathbb{H}}(x')$ that evaluates $P_{\mathbb{H}}(x')$ in $\mathbb{H}$ by rounding all coefficients to $\mathbb{H}$ and perform all operations in $\mathbb{H}$ (line 6 in Figure 10). The resulting $P_{\mathbb{H}}(x')$ is a candidate for the correct polynomial for $A_{\mathbb{H}}(x)$.

Next, we verify that $P_{\mathbb{H}}(x')$ satisfies all constraints in $\Lambda$ (line 7 in Figure 10). If $P_{\mathbb{H}}(x')$ satisfies all constraints in $\Lambda$, then our algorithm returns the polynomial. If there is a constraint $(x', [l', h']) \in \Lambda$ that is not satisfied by $P_{\mathbb{H}}(x')$, then we further restrict the interval $(x', [\sigma, \mu])$ in $\Upsilon$ corresponding to the reduced input $x'$. If $P_{\mathbb{H}}(x')$ is smaller than the lower bound of the interval constraint in $\Lambda$ (i.e. $l'$), then we restrict the lower bound of the interval constraint $\sigma$ in $\Upsilon$ to the value succeeding $\sigma$ in $\mathbb{H}$ (lines 13-16 in Figure 10). This forces the next coefficients for $P_{\mathbb{H}}(x')$ that we generate using the LP solver to produce a value larger than $l'$. Likewise, if $P_{\mathbb{H}}(x')$ produces a value larger than the upper bound of the interval constraint in $\Lambda$ (i.e. $h'$), then we restrict the upper bound of the interval constraint $\mu$ in $\Upsilon$ to the value preceding $\mu$ in $\mathbb{H}$ (lines 17-20 in Figure 10).

We initially sample random 10\lo{lo} we have not exhaustively tested it. Our\Libm will also produce the correctly rounded outputs for many other functions. We do not report it because with our prototype.

This section describes our prototype for generating correctly rounded elementary functions and \Libm, a math library that we developed using our approach for B\float, \posit, and f\float data types. We present case studies for approximating elementary functions 10\times, ln(x), log2(x), and log10(x) with our approach for various types. We also evaluate the performance of our correctly rounded elementary functions with state-of-the-art approximations.

5 EXPERIMENTAL EVALUATION

This section describes our prototype for generating correctly rounded elementary functions and \Libm, a math library that we developed using our approach for B\float, \posit, and f\float data types. We present case studies for approximating elementary functions 10\times, ln(x), log2(x), and log10(x) with our approach for various types. We also evaluate the performance of our correctly rounded elementary functions with state-of-the-art approximations.

5.1 Prototype, \Libm, and Experimental Setup

Prototype. Our prototype for creating correctly rounded math libraries supports B\float, \posit (16-bit posit type in the Posit standard [Gustafson 2017]), and the 32-bit f\float type in the FP representation. The user can provide custom range reduction and output compensation functions. The prototype uses the MPFR library [Fousse et al. 2007] with 2,000 precision bits as the oracle to compute the real result of \f(x) and rounds it to the target representation. Although there is no bound on the precision to compute the oracle result (i.e., Table-maker’s dilemma) with the MPFR library, prior work has shown around 160 precision bits in the worst case is empirically sufficient for the double representation [LefÃĺvre and Muller 2001]. Hence, we determine that using 2,000 precision bits with the MPFR library is sufficient for the oracle result. The prototype uses SoPlex [Gleixner et al. 2015, 2012], an exact rational LP solver as the arbitrary precision LP solver for polynomial generation from constraints. Any arbitrary precision LP solver can be used with our prototype.

\Libm. We created \Libm, a math library that contains correctly rounded elementary functions for multiple data types. It contains 10 elementary functions for B\float and 6 elementary functions for \posit. The library produces correctly rounded outputs for all inputs. To show that our approach works for large data types, \Libm also includes a correctly rounded log2(x) for the 32-bit f\float type for inputs in [1, 2). Our approximation for log2(x) with the 32-bit f\float type is created by sampling a few oracle values rather than all the inputs. There are 223 f\float values in [1, 2).

We initially sample random 10,000 input points and generate the polynomial that produces the correctly rounded result for the sample inputs using our approach. Then, we verify that the generated polynomial produces the correctly rounded result for all inputs in [1, 2). We add any input that does not produce the correctly rounded result to the sample and re-generate the polynomial. We repeat this process until the generated polynomial produces the correct result for all inputs. Our approach will also produce the correctly rounded outputs for many other functions. We do not report it because we have not exhaustively tested it. \Libm performs range reduction and output compensation using the double type. We used state-of-the-art range reduction techniques for various elementary functions. Additionally, we split the reduced domain into multiple disjoint smaller domains using the properties of specific elementary functions to generate efficient polynomials. We evaluate all polynomials using the Horner’s method, i.e., \P(x) = c_0 + x(c_1 + x(c_2 + \ldots)) [Borwein and Erdelyi 1995], which reduces the number of operations in polynomial evaluation. Supplemental material provides the range reduction, output compensation, and the polynomial generated and the coefficients for each function for each data type.
with 2.10GHz processor and 32GB of RAM, running the Ubuntu 16.04 LTS operating system. We Table 1(a) shows that OurLibm produces correctly rounded results for all inputs. The third column and fourth column shows whether glibc’s float and double library used with Bfloat16 produces the correct result for all inputs. We use ✓ to indicate correctly rounded results and x, otherwise. (b) The list of Posit16 functions used. The second column shows whether OurLibm produces the correct results for all inputs. The third column shows whether the functions in SoftPosit-Math produces correctly rounded results for all inputs. N/A indicates that function is not available in SoftPosit-Math. (c) The float function used. First column indicates whether OurLibm produces the correctly rounded result for inputs \( x \in [1, 2] \). In the second and third column, we show whether glibc’s float and math library produce the correct result for all inputs.

<table>
<thead>
<tr>
<th>Bfloat16 Functions</th>
<th>Using OurLibm</th>
<th>Using Float mlib</th>
<th>Using Double mlib</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ln(x) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \log_2(x) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \log_{10}(x) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \exp(x) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \exp_2(x) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \exp_{10}(x) )</td>
<td>✓</td>
<td>x</td>
<td>✓</td>
</tr>
<tr>
<td>sinpi(x)</td>
<td>✓</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>cospi(x)</td>
<td>✓</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>sqrt(x)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>cbrt(x)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

(a) Correctly rounded results with Bfloat16

<table>
<thead>
<tr>
<th>Posit16 Functions</th>
<th>Using OurLibm</th>
<th>Using Float mlib</th>
<th>Using SoftPosit-Math</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ln(x) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \log_2(x) )</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \log_{10}(x) )</td>
<td>✓</td>
<td>✓</td>
<td>N/A</td>
</tr>
<tr>
<td>sinpi(x)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>cospi(x)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>sqrt(x)</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

(b) Correctly rounded results with Posit16

<table>
<thead>
<tr>
<th>float Functions</th>
<th>Using OurLibm</th>
<th>Using Float mlib</th>
<th>Using Double mlib</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_2(x) ) for ( x \in [1, 2] )</td>
<td>✓</td>
<td>x</td>
<td>✓</td>
</tr>
</tbody>
</table>

(c) Correct results in [1, 2) for a 32-bit float

The entire OurLibm and the prototype is written in C++. Both the prototype and OurLibm is available as open-source and publicly available. Although we have not optimized OurLibm for performance on a specific target, it already has better performance than the existing state-of-the-art approaches.

**Experimental setup.** We describe our experimental setup to check the correctness and performance of OurLibm. To the best of our knowledge, there is no publicly available math library specifically designed for Bfloat16. Hence, to compare the performance of our Bfloat16 elementary functions, we convert Bfloat16 input to float or double, use glibc’s float or double library function, and then convert the result back to Bfloat16. We use SoftPosit-Math library [Leong 2019] to compare against our Posit16 elementary functions. Finally, we compare our float functions to glibc’s float and double math library. For our performance experiments, we compiled our OurLibm with g++-9.2.1 and with -O3 optimizations. All experiments were conducted on a machine with 2.10GHz processor and 32GB of RAM, running the Ubuntu 16.04 LTS operating system. We count the number of cycles taken to compute the correctly rounded result for each input using hardware performance counters. We use both the average number of cycles per input and total cycles for all inputs to compare performance.

### 5.2 Correctly Rounded Elementary Functions in OurLibm

Table 1(a) shows that OurLibm produces correctly rounded results for all inputs with numerous elementary functions for the Bfloat16 representation. In contrast to OurLibm, we discovered that re-purposing existing glibc’s float library for Bfloat16 did not produce correctly rounded result for all inputs. The case with input \( x = -0.0181884765625 \) for \( \exp_{10}(x) \) was already discussed in Section 2.6. This case is interesting because the glibc’s float math library produces the correctly
We empirically compare the performance of the elementary functions in OurLibm. We found that glibc’s double library produces the correctly rounded results for all inputs in float16. Our experience during this evaluation illustrates that a correctly rounded function for T′ does not necessarily produce a correctly rounded library for T even if T′ has more precision than T.

Table 1(b) reports that OurLibm produces correctly rounded results for all but one function log10(x), which is not available in SoftPosit-Math library.

Finally, Table 1(c) reports that OurLibm produces the correctly rounded results for log2(x) for inputs in the domain [1, 2). The glibc’s double library produces the correct result for all inputs in the domain [1, 2). However, glibc’s float library does not produce the correctly rounded results for all inputs in [1, 2). We found 1.8 million input points in [1, 2) where glibc’s float library produces the wrong result. In summary, we are able to generate correctly rounded results for many elementary functions with various representations using our proposed approach.

5.3 Performance Evaluation of Elementary Functions in OurLibm

We empirically compare the performance of the elementary functions in OurLibm for float16, Posit16, and a 32-bit float type to the corresponding or similar glibc and SoftPosit-Math implementations.

5.3.1 Performance of float16 Functions in OurLibm. To measure performance, we measure the amount of time it takes for OurLibm to produce a float16 result given a float16 input for all inputs. Similarly, we measure the time taken by glibc’s float or double to produce float16 output given a float16 input, which involves a cast from float16 to a float or a double value. As sinpi(x) and cospi(x) are not available in glibc’s libm, we transform sinpi(x) = sin(\pi x) and cospi(x) = cos(\pi x) before using glibc’s sin and cos functions.

Figure 11(a) shows the speedup of OurLibm’s functions for float16 compared to glibc’s float math library (left bar in the cluster) and the double library (right bar in the cluster). On average, OurLibm’s functions are 1.79× faster when compared to the float library and 2.3× faster over the double math library. There are two reasons for the speedup. First, the polynomials used
for correctly rounded Bfloat16 elementary functions are simpler (i.e., they have a smaller degree or a few number of piece-wise polynomials) compared to the polynomials in the float or double’s math library. Second, using the float or double library requires casts from Bfloat16 to float (or double) and vice versa, for all inputs and the output, which incurs additional overhead.

To understand the speedup with polynomial evaluation, we conducted another experiment where we measure the time taken to perform polynomial evaluation with out any casts (i.e., with double inputs that produce double outputs) for both OURLIBM and glibc’ math libraries. Figure 11(b) shows the speedup of OURLIBM’s functions compared to the baseline of float’s (left bar in the cluster) and double’s library (right bar in the cluster) just for polynomial evaluation. On average, OURLIBM has 1.31× speedup over the float math library and 1.82× speedup over the double math library. For sqrt(x), OURLIBM’s version has a slowdown because both float and double math library utilizes the hardware instruction, FSQRT, to compute sqrt(x) whereas OURLIBM performs polynomial evaluation. Our cbrt(x) function is slightly slower than the float math library. The float library likely uses sophisticated range reduction and has a lower degree polynomial. In contrast, OURLIBM uses a 6th degree polynomial that approximates \( \sqrt{x} \) for the input domain [1, 8). Overall, OURLIBM’s functions for Bfloat16 not only produce correct results for all inputs but also are faster than the state-of-the-art glibc’s libraries re-purposed for Bfloat16.

### 5.3.2 Performance of Posit16 Elementary Functions in OURLIBM

Figure 12 shows the speedup of OURLIBM’s functions when compared to a baseline that uses SoftPosit-Math functions. The Posit16 input is cast to the double type before using OURLIBM. We did not measure the cost of this cast, which can incur additional overhead. SoftPosit-Math library does not have an implementation for log10(x). Hence, we do not report it. On average, OURLIBM has similar performance as SoftPosit-Math even when SoftPosit-Math is super-optimized for Posit16. SoftPosit-Math library performs all computations using integers. OURLIBM’s functions are 1.01× faster than SoftPosit-Math library’s functions excluding the sqrt function. SoftPosit-Math library computes sqrt(x) using the Newton-Raphson refinement method rather than using polynomial approximation functions, and thus produces a more efficient function. In summary, OURLIBM’s functions have similar performance to a highly optimized library with integer operations. We plan to explore integer operations for internal computation to further improve OURLIBM’s performance.

### 5.3.3 Performance Evaluation of Elementary Functions for Float

OURLIBM’s approximation of log2(x) for the 32-bit floating point type has a 1.02× speedup over glibc’s float math library, which produces wrong results for 1.8 million inputs in [1, 2]. Compared to glibc’s double math library, which produces the correctly rounded results (in float) for all float inputs in [1, 2), the float math library function in OURLIBM has 1.10× speedup.
5.4 Case Studies of Correctly Rounded Elementary Functions

We provide case studies to show that our approach (1) has more freedom in generating better polynomials, (2) generates different polynomials for the same underlying elementary function to account for numerical errors in range reduction and output compensation, and (3) generates correctly rounded results even when the polynomial evaluation is performed with the `double` type.

5.4.1 Case study with $10^x$ for Bfloat16. The $10^x$ function is defined over the input domain $(-\infty, \infty)$. There are four classes of special cases:

$$\text{Special cases of } 10^x = \begin{cases} 0.0 & \text{if } x \leq -40.5 \\ 1.0 & \text{if } -8.4686279296875 \times 10^{-4} \leq x \leq 1.68609619140625 \times 10^{-3} \\ \infty & \text{if } x \geq 38.75 \\ \text{NaN} & \text{if } x = \text{NaN} \end{cases}$$

A quick initial check returns their result and reduces the overall input that we need to approximate.

We approximate $10^x$ using $2^x$, which is easier to compute. We use the property, $10^x = 2^{\log_{2}(10) x}$ to approximate $10^x$ using $2^x$. Subsequently, we perform range reduction by decomposing $\log_{2}(10) x$ as $\log_{2}(10) = i + x'$, where $i$ is an integer and $x' \in [0, 1)$ is the fractional part.

Now, $10^x$ decomposes to

$$10^x = 2^{\log_{2}(10) x} = 2^{i + x'} = 2^i 2^{x'}$$

The above decomposition requires us to approximate $2^{x'}$ where $x' \in [0, 1)$. Multiplication by $2^i$ can be performed using integer operations. The range reduction, output compensation, and the function we are approximating $g(x')$ is as follows:

$$RR(x) = x' = x\log_{2}(10) - \lfloor x\log_{2}(10) \rfloor \quad OC(y', x) = y' 2^i = y' 2^{\lfloor x\log_{2}(10) \rfloor} \quad g(x') = 2^{x'}$$

Our approach generated a 4th degree polynomial that approximates $2^{x'}$ in the input domain $[0, 1)$. Our polynomial produces the correctly rounded result for all inputs in the entire domain for $10^x$ when used with range reduction and output compensation.

We are able to generate a lower degree polynomial because our approach provides more freedom to generate the correctly rounded results. We illustrate this point with an example. Figure 13 presents a reduced interval $[l', h']$ (in green region) for the reduced input $x' = 0.005626\ldots$ in our approach. The real value of $g(x')$ is shown in black circle. In our approach, the polynomial that approximates $g(x')$ has to produce a value in $[l', h']$ such that the output compensated value produces the correctly rounded result of $10^x$ for all input $x$ that reduces to $x'$. The value of $g(x')$ is extremely close to $l'$ with a margin of error $\epsilon = |g(x') - l'| \approx 1.31 \times 10^{-6}$. In contrast to our approach, if we approximated the real value of $g(x')$, then we must generate a polynomial with an error of at most $\epsilon$, i.e. the polynomial has to produce a value in $[g(x') - \epsilon, g(x') + \epsilon]$, which

Fig. 13. More freedom in generating a polynomial for $10^x$ with our approach. The reduced interval $[l', h']$ (in green box) corresponds to the reduced input $x' = 0.005626\ldots$ We show the real value of $g(x')$ (black circle) and the result produced by the polynomial generated with our approach (red diamond). If we approximated the real result $g(x')$ instead of the correctly rounded result, the margin of error for any such polynomial would be lower.
potentially necessitates a higher degree polynomial. The polynomial that we generate produces a value shown in Figure 13 with red diamond. This value has an error of $|P_{t}(x') - g(x')| \approx 7.05 \times 10^{-5}$, which is much larger than $e$. Still, the $4^{th}$ degree polynomial generated by our approach produces the correctly rounded value when used with the output compensation function for all inputs.

5.4.2 Case study with $\ln(x)$, $\log_{2}(x)$, and $\log_{10}(x)$ for Bfloat16. While creating the Bfloat16 approximations for functions $\ln(x)$, $\log_{2}(x)$, and $\log_{10}(x)$, we observed that our approach generates different polynomials for the same underlying elementary function to account for numerical errors in range reduction and output compensation. We highlight this observation in this case study.

To approximate these functions, we use a slightly modified version of the Cody and Waite range reduction technique [Cody and Waite 1980]. As a first step, we use mathematical properties of logarithms, $log_{b}(x) = \frac{log_{b}(x)}{log_{b}(b)}$ to approximate all three functions $\ln(x)$, $\log_{2}(x)$, and $\log_{10}(x)$ using the approximation for $\log_{2}(x)$. As a second step, we perform range reduction by decomposing the input $x$ as $x = t \times 2^{e}$ where $t \in [0, 1)$ is the fractional value represented by the mantissa and $e$ is an integer representing the exponent of the value. Then, we use the mathematical property of logarithms, $log_{b}(x \times y^{z}) = log_{b}(x) + z log_{b}(y)$, to perform range reduction and output compensation.

Now, any logarithm function $log_{b}(x)$ can be decomposed to $log_{b}(x) = \frac{log_{b}(t) + e}{log_{b}(b)}$.

As a third step, to ease the job of generating a polynomial for $log_{2}(t)$, we introduce a new variable $x' = \frac{t - 1}{t + 1}$ and transform the function $log_{2}(t)$ to a function with rapidly converging polynomial expansion:

$$ g(x') = log_{2}\left(\frac{1 + x'}{1 - x'}\right) $$

where the function $g(x')$ evaluates to $log_{2}(t)$.

The above input transformation, attributed to Cody and Waite [Cody and Waite 1980], enables the creation of a rapidly convergent odd polynomial, $P(x) = c_{1}x + c_{2}x^{2}...$, which reduces the number of operations. In contrast, the polynomial would be of the form $P(x) = c_{0} + c_{1}x + c_{2}x^{2}...$ in the absence of above input transformation, which has terms with both even and odd degrees.

When the input $x$ is decomposed into $x = t + e$ where $t \in [0, 1)$ and $e$ is an integer, the range reduction function $x' = RR(x)$, the output compensation function $y = OC(y', x)$, and the function that we need to approximate, $y' = g(x')$ are as follows,

$$ RR(x) = x' = \frac{t - 1}{t + 1}, \quad OC(y', x) = \frac{y' + e}{log_{2}(b)}, \quad g(x') = log_{2}\left(\frac{1 + x'}{1 - x'}\right) $$

Hence, we approximate the same elementary function for $\ln(x)$, $\log_{2}(x)$ and $\log_{10}(x)$ (i.e., $g(x')$). However, the output compensation functions are different for each of them.

We observed that our approach produced different polynomials that produced correct output for $\ln(x)$, $\log_{2}(x)$, and $\log_{10}(x)$ functions for Bfloat16, which is primarily to account for numerical errors in each output compensation function. We produced a $5^{th}$ degree odd polynomial for $\log_{2}(x)$, a $5^{th}$ degree odd polynomial with different coefficients for $\log_{10}(x)$, and a $7^{th}$ degree odd polynomial for $\ln(x)$. Our technique also determined that there was no correct $5^{th}$ degree odd polynomial for $\ln(x)$. Although these polynomials approximate the same function $g(x')$, they cannot be used interchangeably. For example, our experiment show that the $5^{th}$ degree polynomial produced for $\log_{2}(x)$ cannot be used to produce the correctly rounded result of $\ln(x)$ for all inputs.

5.4.3 Case study with $\log_{2}(x)$ for a 32-bit float. To show that our approach is scalable to data types with numerous inputs, we illustrate a correctly rounded $log_{2}(x)$ function for a 32-bit float type in the input domain $[1, 2)$. There are $2^{23}$ values in this input domain. When we generate a single polynomial for such a large number of points, it is likely that we generate a polynomial of a
higher degree. When a polynomial of a large degree is evaluated in a finite precision representation, it is likely to have numerical errors in polynomial evaluation. We wanted to test whether our approach can generate a polynomial that produces correctly rounded results for all inputs even when the polynomial is evaluated in some finite precision representation (i.e., double).

We used the Cody and Waite range reduction described above to reduce the input domain to $[0, \frac{1}{2})$. We used the oracle to generate rounding intervals, then identified reduced and combined intervals, and fed the constraints to the LP solver. We were able to generate a 15-degree odd polynomial that produces the correct result for all inputs! This 15-degree polynomial produces the correctly rounded results in the domain $[1, 2)$ for the float type when polynomial evaluation is performed in double. This experiment shows that our approach can generate polynomials of a large degree that produce correct results when evaluated in limited precision representation.

6 RELATED WORK

**Correctly Rounded Math Libraries for FP.** Since the introduction of the floating point standard [Cowlishaw 2008], a number of correctly rounded math library have been proposed including the IBM LibUltim (or also known as MathLib) [IBM 2008; Ziv 1991], Sun Microsystems’s LibMCR [Microsystems 2008], CR-LIBM [Daramy et al. 2003], and the MPFR math library [Fousse et al. 2007]. MPFR produces correctly rounded result for any arbitrary precision.

CR-LIBM [Daramy et al. 2003; LefÃĺvre et al. 1998] is a correctly rounded double math library developed using Sollya [Chevillard et al. 2010], which is a tool and a library for developing FP code. Given a degree $d$, a representation $\mathbb{H}$, and the elementary function $f(x)$, Sollya generates polynomials of degree $d$ with coefficients in $\mathbb{H}$ that has the minimum infinity norm [Brisebarre and Chevillard 2007]. Sollya uses a modified Remez algorithm with lattice basis reduction to produce polynomials. It also computes the error bound on the polynomial evaluation using interval arithmetic [Chevillard et al. 2011; Chevillard and Lauter 2007] and produces Gappa [Melquiond 2019] proofs for the error bound. Metalibm [Brunie et al. 2015; Kupriianova and Lauter 2014] is a math library function generator built using Sollya. MetaLibm is able to automatically identify range reduction and domain splitting techniques for some transcendental functions. It has been used to create correctly rounded elementary functions for the float and double types.

A number of other approaches have been proposed to generate correctly rounded results for different transcendental functions including square root [Jeannerod et al. 2011] and exponentiation [Bui and Tahar 1999]. A modified Remez algorithm has also been used to generate polynomials for approximating some elementary functions [Arzelier et al. 2019]. It generates a polynomial that minimizes the infinity norm compared to ideal elementary function and the numerical error in the polynomial evaluation. It can be used to produce correctly rounded results when range reduction is not necessary. Compared to prior techniques, our approach approximates the correctly rounded value $RN_{7}(f(x))$ and the margin of error is much higher, which generates efficient polynomials. Additionally, our approach also takes into account numerical errors in range reduction, output compensation, and polynomial evaluation.

**Posit math libraries.** SoftPosit-Math [Leong 2019] has a number of correctly rounded Posit16 elementary functions, which are created using the Minefield method [Gustafson 2020]. The Minefield method identifies the interval of values that the internal computation should produce and declares all other regions as a minefield. Then the goal is to generate a polynomial that avoids the mines. The polynomials in the minefield method were generated by trial and error. Our approach is inspired by the Minefield method. It generalizes it to numerous representations, range reduction, and output compensation. Our approach also automates the process of generating polynomials by encoding the mines as linear constraints and uses an LP solver. Recently, researchers have used the CORDIC method to generate a number of approximations to trigonometric functions for the
Posit32 type [Lim et al. 2020]. However, they do not provide correctly rounded results for all inputs.

**Verification of math libraries.** As performance and correctness are both important with math libraries, there is extensive research to prove the correctness of math libraries. Sollya verifies that the generated implementations of elementary functions produce correctly rounded results with the aid of Gappa [Daumas et al. 2005; de Dinechin et al. 2011; de Dinechin et al. 2006]. Typically, it uses interval arithmetic and has been used to prove the correctness of CR-LIBM. Recently, researchers have also verified that many functions in Intel’s math.h implementations have at most 1 ulp error [Lee et al. 2017]. Various elementary function implementations have also been proven correct using HOL Light [Harrison 1997a,b, 2009]. Similarly, CoQ proof assistant has been used to prove the correctness of argument reduction [Boldo et al. 2009]. Instruction sets of mainstream processors have also been proven correct using proof assistants (e.g., division and sqrt(x) instruction in IBM Power4 processor [Sawada 2002]). If the oracle includes all inputs, then OurLibm produces polynomial functions that produce correctly rounded results for all inputs. If used with sampling, then we can use prior approaches to prove the correctness of our implementations for all inputs.

**Rewriting tools.** Mathematical rewriting tools are other alternatives to create correctly rounded functions. If the rounding error in the implementation is the root cause of an incorrect result, we can use tools that detect numerical errors to diagnose them [Benz et al. 2012; Chowdhary et al. 2020; Fu and Su 2019; Goubault 2001; Sanchez-Stern et al. 2018; Yi et al. 2019; Zou et al. 2019]. Subsequently, we can rewrite them using tools such as Herbie [Panchekha et al. 2015] or Salsa [Damouche and Martel 2018]. Recently, a repair tool was proposed specifically for reducing the error of math libraries [Yi et al. 2019]. It identifies the domain of inputs that result in high error. Then, it uses piece-wise linear or quadratic equations to repair them for the specific domain. However, currently, these rewriting tools do not guarantee correctly rounded results for all inputs.

**Generating numerical code.** There is a body of work on generating numerical code that provides verified bounds on the numerical error [Darulova et al. 2018; Darulova and Kuncak 2017; Rocca et al. 2017]. Our approach and techniques produce accurate numerical code for elementary functions and can be extended to expressions that can be approximated using polynomials.

## 7 CONCLUSION

A library to approximate elementary functions is a key component of the FP representation and its alternatives. We propose a novel methodology to generate approximations for elementary functions that produces correctly rounded results for all inputs. The key insight is to identify the amount of freedom available to generate the correctly rounded result. Subsequently, we use this freedom to generate a polynomial using linear programming that produces the correct rounded results for all inputs. This paper shows that this approach generates polynomial approximations that are faster than existing libraries while producing correct results. Our approach can also allow designers of elementary functions to make pragmatic trade-offs with respect to performance and correct results. More importantly, it can enable standards to mandate correctly rounded results for elementary functions with new representations.
REFERENCES


A Novel Approach to Generate Correctly Rounded Math Libraries for New Floating Point Representations


A DETAILS ON OURLIBM

In the appendices, we describe the range reduction technique, special cases, and the polynomials we generated to create math library functions in OURLIBM. We use the same range reduction technique for each family of elementary functions across all types, i.e., $\ln(x)$, $\log_2(x)$, and $\log_{10}(x)$ for all $\textbf{float16}$, $\textbf{posit16}$, and $\textbf{float}$ use the same range reduction technique. Hence, we first describe the range reduction techniques that we use for each family of elementary functions in Appendix B. In each subsequent section, we describe the specific special cases, range reduction function, output compensation function, and the polynomial we use to create each function for $\textbf{float16}$ (Appendix C), $\textbf{posit16}$ (Appendix D), and $\textbf{float}$ (Appendix E).

B RANGE REDUCTION TECHNIQUES

In this section, we explain the general range reduction technique OURLIBM uses to reduce the input domain for each class of elementary functions.

B.1 Logarithm functions ($\log_b(x)$)

We use a slightly modified version of Cody and Waite’s range reduction technique [Cody and Waite 1980] for all logarithm functions. As a first step, we use the mathematical property of logarithms, $\log_b(x) = \frac{\log_b(x)}{\log_b(b)}$ to approximate logarithm functions using the approximation of $\log_2(x)$. As a second step, we decompose the input $x$ as $x = t \times 2^m$ where $t$ is the fractional value represented by the mantissa and $m$ is the exponent of the input. Then we use the mathematical property of logarithm functions, $\log_2(x \times y^2) = \log_2(x) + z\log_2(y)$ to decompose $\log_2(x)$. Thus, any logarithm function $\log_b(x)$ can be decomposed to,

$$\log_b(x) = \frac{\log_2(t) + m}{\log_2(b)}$$

As a third step, to ease the job of generating an accurate polynomial for $\log_2(t)$, we introduce a new variable $x’ = \frac{t-1}{t+1}$ and transform the function $\log_2(t)$ to a function with rapidly converging polynomial expansion:

$$g(x’) = \log_2 \left( \frac{1 + x’}{1 - x’} \right)$$

The function $g(x’)$ evaluates to $\log_2(t)$. The polynomial expansion of $g(x’)$ is an odd polynomial, i.e. $P(x) = c_1 x + c_3 x^3 + c_5 x^5 \ldots$. Combining all steps, we decompose $\log_b(x)$ to,

$$\log_b(x) = \frac{\log_2 \left( \frac{1 + x’}{1 - x’} \right) + m}{\log_2(b)}$$

When the input $x$ is decomposed into $x = t \times e$ where $t \in [1, 2)$ and $e$ is an integer, the range reduction function $x’ = RR(x)$, the output compensation function $y = OC(y’, x)$, and the function that we need to approximate, $y’ = g(x’)$ can be summarized as follows,

$$RR(x) = \frac{t - 1}{t + 1} \quad OC(y’, x) = \frac{y’ + m}{\log_2(b)} \quad g(x’) = \log_2 \left( \frac{1 + x’}{1 - x’} \right)$$

With this range reduction technique, we need to approximate $g(x’)$ for the reduced input domain $x’ \in [0, \frac{1}{2})$.

B.2 Exponential Functions ($a^x$)

We approximate all exponential functions with $2^x$. As a first step, we use the mathematical property $a^x = 2^{x\log_2(a)}$ to decompose any exponential function to a function of $2^x$. Second, we decompose the
With this range reduction technique, we need to approximate \( x^{log_2(a)} \) into the integral part \( i \) and the remaining fractional part \( x' \in [0, 1) \), i.e. \( x^{log_2(a)} = i + x' \). We can define \( i \) and \( x' \) more formally as:

\[
i = \lfloor x^{log_2(a)} \rfloor, \quad x' = x^{log_2(a)} - i
\]

where \( \lfloor x \rfloor \) is a floor function that rounds down \( x \) to an integer. Using the property \( 2^{x+y} = 2^x 2^y \), \( a^x \) decomposes to

\[
d^x = 2^{x^{log_2(a)}} = 2^{i+x'} = 2^i 2^{x'} = 2^{x^{log_2(a)} - \lfloor x^{log_2(a)} \rfloor} 2^{\lfloor x^{log_2(a)} \rfloor}
\]

The above decomposition allows us to approximate any exponential functions by approximating \( 2^x \) for \( x \in [0, 1) \). The range reduction function \( x' = RR(x) \), output compensation function \( y = OC(y', x) \), and the function we need to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = x^{log_2(b)} - \lfloor x^{log_2(b)} \rfloor \quad OC(y', x) = y'^{2^{x^{log_2(b)}}} \quad g(x') = 2^{x'}
\]

With this range reduction technique, we need to approximate \( 2^{x'} \) for the reduced input domain \( x' \in [0, 1) \).

### B.3 Square Root Function (\( \sqrt{x} \))

To perform range reduction on \( \sqrt{x} \), we first decompose the input \( x \) into \( x = x' \times 2^m \) where \( m \) is an even integer and \( x' = \frac{x}{2^m} \in [1, 4) \). Second, using the mathematical properties \( \sqrt{xy} = \sqrt{x} \sqrt{y} \) and \( \sqrt{2^{2x}} = 2^x \), we decompose \( \sqrt{x} \) to:

\[
\sqrt{x} = \sqrt{x' \times 2^m} = \sqrt{x'} \times 2^{\frac{m}{2}}
\]

The above decomposition allows us to approximate the square root function by approximating \( \sqrt{x} \) for \( x \in [1, 4) \). Since \( m \) is an even integer, \( \frac{m}{2} \) is an integer and multiplication of \( 2^{\frac{m}{2}} \) can be performed using integer arithmetic.

When the input \( x \) is decomposed into \( x = x' \times 2^m \) where \( x' \in [1, 4) \) and \( m \) is an even integer, the range reduction function \( x' = RR(x) \), output compensation function \( y = OC(y', x) \), and the function we need to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = x' \quad OC(y', x) = y'^{2^{\frac{m}{2}}} \quad g(x') = \sqrt{x}
\]

With this range reduction technique, we need to approximate \( \sqrt{x'} \) for the reduced input domain \( x' \in [1, 4) \).

### B.4 Cube Root Function (\( \sqrt[3]{x} \))

To perform range reduction on \( \sqrt[3]{x} \), we first decompose the input \( x \) into \( x = s \times x' \times 2^m \). The value \( s \in \{-1, 1\} \) represents the sign of \( x \), \( m \) is an integer multiple of 3, and \( x' = \frac{x}{2^m} \in [1, 8) \). Second, using the mathematical properties \( \sqrt[3]{xy} = \sqrt[3]{x} \sqrt[3]{y} \) and \( \sqrt[3]{2^{3x}} = 2^x \), we decompose \( \sqrt[3]{x} \) to:

\[
\sqrt[3]{x} = \sqrt[3]{s \times x' \times 2^m} = s \times \sqrt[3]{x'} \times 2^{\frac{m}{3}}
\]

The above decomposition allows us to approximate the cube root function by approximating \( \sqrt[3]{x} \) for \( x \in [1, 8) \). Since \( m \) is an integer multiple of 3, \( \frac{m}{3} \) is an integer and multiplication of \( 2^{\frac{m}{3}} \) can be performed using integer arithmetic.

When we decompose the input \( x \) into \( x = s \times x' \times 2^m \) where \( s \) is the sign of the input, \( x' \in [1, 8) \), and \( m \) is an integer multiple of 3, the range reduction function \( x' = RR(x) \), output compensation function \( y = OC(y', x) \), and the function we need to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = x' \quad OC(y', x) = s \times y'^{2^{\frac{m}{3}}} \quad g(x') = \sqrt[3]{x}
\]
With this range reduction technique, we need to approximate $\sqrt[3]{x'}$ for the reduced input domain $x' \in [1, 8)$.

**B.5 Sinpi Function ($sin(\pi x)$)**

To perform range reduction on $sin(\pi x)$, we use the property of $sin(\pi x)$ that it is a periodic and odd function. First, using the property $sin(-\pi x) = -sin(\pi x)$, we decompose the input $x$ into $x = s \times |x|$ where $s$ is the sign of the input. The function decomposes to $sin(\pi x) = s \times sin(\pi |x|)$.

Second, we use the properties $sin(\pi (x+2z)) = sin(\pi x)$ where $z$ is an integer and $sin(\pi (x+2z+1)) = -sin(\pi x)$. We decompose $|x|$ into $|x| = i + t$ where $i$ is an integer and $t \in [0, 1)$ is the fractional part, i.e. $t = |x| - i$. More formally, we can define $t$ and $i$ as,

$$i = [\lfloor x \rfloor], \quad t = |x| - i$$

If $i$ is an even integer, then $sin(\pi (t + i)) = sin(\pi t)$ (from the property $sin(\pi (x + 2z)) = sin(\pi x)$). If $i$ is an odd integer, then $sin(\pi (t + i)) = -sin(\pi t)$ (from the property $sin(\pi (x + 2z + 1)) = -sin(\pi x)$).

Thus, we can decompose the $sin(\pi x)$ function into,

$$sin(\pi x) = s \times sin(\pi |x|) = \begin{cases} s \times sin(\pi t) & \text{if } i \equiv 0 \pmod{2} \\ -s \times sin(\pi t) & \text{if } i \equiv 1 \pmod{2} \end{cases}$$

Third, we use the property $sin(\pi t) = sin(\pi (1 - t))$ for $0.5 < t < 1.0$ and introduce a new variable $x'$,

$$x' = \begin{cases} 1 - t & \text{if } 0.5 < t < 1.0 \\ t & \text{otherwise} \end{cases}$$

Since we perform the subtraction only when $0.5 < t < 1.0$, $x'$ can be computed exactly due to Sterbenz Lemma [Muller 2005]. The above decomposition reduces the input domain to $x' \in [0, 0.5]$ and requires us to approximate $sin(\pi x')$ for the reduced domain.

In summary, we decompose the input $x$ into $x = s \times (i + t)$ where $s$ is the sign of the input, $i$ is an integer, and $t \in [0, 1)$ is the fractional part of $|x|$, i.e. $|x| = i + t$. The range reduction function $x' = RR(x)$, the output compensation function $y = OC(y', x)$, and the function we need to approximate, $y' = g(x')$ are as follows:

$$RR(x) = \begin{cases} 1 - t & \text{if } 0.5 < t < 1.0 \\ t & \text{otherwise} \end{cases}, \quad OC(y', x) = \begin{cases} s \times y' & \text{if } i \equiv 0 \pmod{2} \\ -s \times y' & \text{if } i \equiv 1 \pmod{2} \end{cases} \quad g(x') = sin(\pi x')$$

With this range reduction technique, we need to approximate $sin(\pi x')$ for the reduced input domain $x' \in [0, 0.5]$.

**B.6 Cospi Function ($cos(\pi x)$)**

To perform range reduction on $cos(\pi x)$, we use the property of $cos(\pi x)$ that it is a periodic and even function. First, using the property $cos(-\pi x) = cos(\pi x)$, we decompose the input $x$ into $x = s \times |x|$ where $s$ is the sign of the input. The function decomposes to $cos(\pi x) = cos(\pi |x|)$.

Second, we use the properties $cos(\pi (x+2z)) = cos(\pi x)$ where $z$ is an integer and $cos(\pi (x+2z+1)) = -cos(\pi x)$. We decompose $|x|$ into $|x| = i + t$ where $i$ is an integer and $t \in [0, 1)$ is the fractional part, i.e. $t = |x| - i$. More formally, we can define $t$ and $i$ as,

$$i = [\lfloor x \rfloor], \quad t = |x| - i$$

If $i$ is an even integer, then $cos(\pi (t + i)) = cos(\pi t)$ (from the property $cos(\pi (x + 2z)) = cos(\pi x)$). If $i$ is an odd integer, then $cos(\pi (t + i)) = -cos(\pi t)$ (from the property $cos(\pi (x + 2z + 1)) = -cos(\pi x)$).
Thus, we can decompose \( \cos(\pi x) \) into,

\[
\cos(\pi x) = \cos(\pi|x|) = (-1)^{i \mod 2} \times \cos(\pi t)
\]

where \( i \mod 2 \) is the modulus operation in base 2.

Third, we use the property \( \cos(\pi t) = -\cos(\pi(1 - t)) \) for \( 0.5 < t < 1.0 \) and decompose \( t \) to,

\[
x' = \begin{cases} 
1 - t & \text{if } 0.5 < t < 1.0 \\
t & \text{otherwise}
\end{cases}
\]

Since we perform the subtraction only when \( 0.5 < t < 1.0 \), \( 1 - t \) can be computed exactly due to Sterbenz Lemma. Consequently, \( \cos(\pi x) \) function decomposes to,

\[
\cos(\pi x) = \begin{cases} 
-1 \times (-1)^{i \mod 2} \times \cos(\pi x') & \text{if } 0.5 < t < 1.0 \\
(-1)^{i \mod 2} \times \cos(\pi x') & \text{otherwise}
\end{cases}
\]

The above decomposition reduces the input domain to \( x' \in [0, 0.5] \).

In summary, we decompose the input \( x \) into \( s \times (i + t) \) where \( s \) is the sign of the input, \( i \) is an integer, and \( t \in [0, 1) \) is the fractional part of \( |x| \), i.e. \( |x| = i + t \). The range reduction function \( x' = RR(x) \), the output compensation function \( y = OC(y', x) \), and the function we need to approximate, \( y' = g(x') \) are as follows:

\[
RR(x) = \begin{cases} 
1 - t & \text{if } 0.5 < t < 1.0 \\
t & \text{otherwise}
\end{cases}
\]

\[
OC(y', x) = \begin{cases} 
-1 \times (-1)^{i \mod 2} \times y' & \text{if } 0.5 < t < 1.0 \\
(-1)^{i \mod 2} \times y' & \text{otherwise}
\end{cases}
\]

\[
g(x') = \cos(\pi x')
\]

With this range reduction technique, we need to approximate \( \cos(\pi x') \) for the reduced input domain \( x' \in [0, 0.5] \).

C DETAILS ON BFLOAT16 FUNCTIONS

In this section, we explain the bfloat16 functions in OURLIBM. More specifically, we describe the special cases, the range reduction and output compensation function, the function we must approximate, and the polynomials we generated for each bfloat16 math library function in OURLIBM.

C.1 \( \ln(x) \) for Bfloat16

The elementary function \( \ln(x) \) is defined over the input domain \((0, \infty)\). There are three classes of special case inputs:

\[
\text{Special case of } \ln(x) = \begin{cases} 
-\infty & \text{if } x = 0 \\
\infty & \text{if } x = \infty \\
NaN & \text{if } x < 0 \text{ or } x = NaN
\end{cases}
\]

We use the range reduction technique described in Appendix B.1. For \( \ln(x) \), the range reduction function \( x' = RR(x) \), the output compensation function \( y = OC(y', x) \), and the function to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = \frac{t - 1}{t + 1} \quad OC(y', x) = \frac{y' + m}{\log_2(e)} \quad g(x') = \log_2 \left( \frac{1 + x'}{1 - x'} \right)
\]
The value \( t \) is the fractional value represented by the mantissa of the input \( x \) and \( m \) is the exponent, i.e. \( x = t \times 2^m \). With this range reduction technique, we need to approximate \( g(x') \) for \( x' \in [0, \frac{1}{2}) \).

To approximate \( g(x') \), we use a 7th degree odd polynomial \( P(x) = c_1x + c_3x^3 + c_5x^5 + c_7x^7 \) with the coefficients,
\[
\begin{align*}
c_1 &= 2.885102725620722008414986703428439795970916748046875 \\
c_3 &= 0.974943826930012358289445728587452322446441650390625 \\
c_5 &= 0.39117252017394070751737444879836402833461761474609375 \\
c_7 &= 1.272215280788404902202682933420874178409576416015625
\end{align*}
\]

C.2 \( \log_2(x) \) for Bfloat16

The elementary function \( \log_2(x) \) is defined over the input domain \((0, \infty)\). There are three classes of special case inputs:
\[
\text{Special case of } \log_2(x) = \begin{cases} -\infty & \text{if } x = 0 \\ \infty & \text{if } x = \infty \\ \text{NaN} & \text{if } x < 0 \text{ or } x = \text{NaN} \end{cases}
\]

We use the range reduction technique described in Appendix B.1. For \( \log_2(x) \), the range reduction function \( (x' = RR(x)) \), the output compensation function \( (y = OC(y', x)) \), and the function to approximate \( (y' = g(x')) \) can be summarized as follows:
\[
RR(x) = \frac{t - 1}{t + 1} \quad OC(y', x) = y' + m \quad g(x') = \log_2 \left( \frac{1 + x'}{1 - x'} \right)
\]
The value \( t \) is the fractional value represented by the mantissa of the input \( x \) and \( m \) is the exponent, i.e. \( x = t \times 2^m \). With this range reduction technique, we need to approximate \( g(x') \) for \( x' \in [0, \frac{1}{2}) \).

C.3 \( \log_{10}(x) \) for Bfloat16

The elementary function \( \log_{10}(x) \) is defined over the input domain \((0, \infty)\). There are three classes of special case inputs:
\[
\text{Special case of } \log_{10}(x) = \begin{cases} -\infty & \text{if } x = 0 \\ \infty & \text{if } x = \infty \\ \text{NaN} & \text{if } x < 0 \text{ or } x = \text{NaN} \end{cases}
\]

We use the range reduction technique described in Appendix B.1. For \( \log_{10}(x) \), the range reduction function \( (x' = RR(x)) \), the output compensation function \( (y = OC(y', x)) \), and the function to approximate \( (y' = g(x')) \) can be summarized as follows:
\[
RR(x) = \frac{t - 1}{t + 1} \quad OC(y', x) = \frac{y' + m}{\log_2(10)} \quad g(x') = \log_2 \left( \frac{1 + x'}{1 - x'} \right)
\]
The value \( t \) is the fractional value represented by the mantissa of the input \( x \) and \( m \) is the exponent, i.e. \( x = t \times 2^m \). With this range reduction technique, we need to approximate \( g(x') \) for \( x' \in [0, \frac{1}{2}) \).
To approximate \( g(x') \), we use a 5\(^{th} \) degree odd polynomial \( P(x) = c_1 x + c_3 x^3 + c_5 x^5 \) with the coefficients,

\[
\begin{align*}
c_1 &= 2.88545942229525831797532974096946418285369873046875 \\
c_3 &= 0.956484867363945223672772044665180146694183349609375 \\
c_5 &= 0.6710954935542725596775426311069168150424957275390625
\end{align*}
\]

### C.4 \( e^x \) for Bfloat16

The elementary function \( e^x \) is defined over the input domain \((-\infty, \infty)\). There are four classes of special case inputs:

\[
\text{Special case of } e^x = \begin{cases} 
0.0 & \text{if } x \leq -93.0 \\
1.0 & \text{if } -1.953125 \times 10^{-3} \leq x \leq 3.890991 \times 10^{-3} \\
\infty & \text{if } x \geq 89.0 \\
\text{NaN} & \text{if } x = \text{NaN}
\end{cases}
\]

We use the range reduction technique described in Appendix B.2. For \( e^x \), the range reduction function \( x' = RR(x) \), output compensation function \( y = OC(y', x) \), and the function we have to approximate to approximate \( y' = g(x') \) is summarized below:

\[
RR(x) = x\log_2(e) - \lfloor x\log_2(e) \rfloor \quad OC(y', x) = y'2^{\lfloor x\log_2(e) \rfloor} \quad g(x') = 2^{x'}
\]

where \( \lfloor x \rfloor \) is a floor function that rounds down \( x \) to an integer. With this range reduction technique, we need to approximate \( 2^{x'} \) for \( x' \in [0, 1) \).

To approximate \( 2^{x'} \), we use a 4\(^{th} \) degree polynomial \( P(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 \) with the coefficients,

\[
\begin{align*}
c_0 &= 1.0000095976211798021182630691328085958957672119140625 \\
c_1 &= 0.69279247181322956006255253669223748147487640380859375 \\
c_2 &= 0.24256022458162823651761641485791187734117889404296875 \\
c_3 &= 5.014719237694532927296364732683287002146244049072265625 \times 10^{-2} \\
c_4 &= 1.45139853027161404297462610202273936010897159576416015625 \times 10^{-2}
\end{align*}
\]

### C.5 \( 2^x \) for Bfloat16

The elementary function \( 2^x \) is defined over the input domain \((-\infty, \infty)\). There are four classes of special case inputs:

\[
\text{Special case of } 2^x = \begin{cases} 
0.0 & \text{if } x \leq -134.0 \\
1.0 & \text{if } -2.8076171875 \times 10^{-3} \leq x \leq 2.8076171875 \times 10^{-3} \\
\infty & \text{if } x \geq 128.0 \\
\text{NaN} & \text{if } x = \text{NaN}
\end{cases}
\]

We use the range reduction technique described in Appendix B.2. For \( e^x \), the range reduction function \( x' = RR(x) \), output compensation function \( y = OC(y', x) \), and the function we have to approximate to approximate \( y' = g(x') \) is summarized below:

\[
RR(x) = x - \lfloor x \rfloor \quad OC(y', x) = y'2^{\lfloor x \rfloor} \quad g(x') = 2^{x'}
\]

where \( \lfloor x \rfloor \) is a floor function that rounds down \( x \) to an integer. With this range reduction technique, we need to approximate \( 2^{x'} \) for \( x' \in [0, 1) \).
To approximate $2^x$, we use a $4^{th}$ degree polynomial $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ with the coefficients,

\[c_0 = 1.0000091388165410766220020377659238874912261962890625\]
\[c_1 = 0.69265463004053107187729665383812971413135528564453125\]
\[c_2 = 0.2437159431234791218592576967494096609698337340234375\]
\[c_3 = 4.8046547014740259573528646797058172523975372314453125 \times 10^{-2}\]
\[c_4 = 1.557769641174900157518658484453125 \times 10^{-2}\]

C.6 $10^x$ for Bfloat16

The elementary function $10^x$ is defined over the input domain $(-\infty, \infty)$. There are four classes of special case inputs:

\[
\text{Special case of } 10^x = \begin{cases} 
0.0 & \text{if } x \leq -40.5 \\
1.0 & \text{if } -8.4686279296875 \times 10^{-4} \leq x \leq 1.68609619140625 \times 10^{-3} \\
\infty & \text{if } x \geq 38.75 \\
\text{NaN} & \text{if } x = \text{NaN}
\end{cases}
\]

**Range reduction.** We use the range reduction technique described in Appendix B.2. The range reduction function $x' = RR(x)$, output compensation function $y = OC(y', x)$, and the function we have to approximate to approximate $y' = g(x')$ is summarized below:

\[
RR(x) = x \log_2(10) - \lfloor x \log_2(10) \rfloor, \quad OC(y', x) = y'2^{x \log_2(10)}
\]
\[
g(x') = 2^x
\]

where $\lfloor x \rfloor$ is a floor function that rounds down $x$ to an integer. With this range reduction technique, we need to approximate $2^x$ for $x' \in [0, 1]$.

To approximate $2^x$, we use a $4^{th}$ degree polynomial $P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ with the coefficients,

\[c_0 = 1.0000778485054981903346060789772309362888336181640625\]
\[c_1 = 0.6917974008342254732539799884444828890435272216796875\]
\[c_2 = 0.2459833280009494360651700617381720803678035736083984375\]
\[c_3 = 4.575895298196537886865797872815164737403392791748046875 \times 10^{-2}\]
\[c_4 = 1.6390765806412448818418781115724414123595867156982421875 \times 10^{-2}\]

C.7 $\sqrt{x}$ for Bfloat16

The elementary function $\sqrt{x}$ is defined over the input domain $[0, \infty)$. There are three classes of special case inputs:

\[
\text{Special case of } \sqrt{x} = \begin{cases} 
0.0 & \text{if } x = 0.0 \\
\infty & \text{if } x = \infty \\
\text{NaN} & \text{if } x < 0 \text{ or } x = \text{NaN}
\end{cases}
\]

We use the range reduction technique described in Appendix B.3. The range reduction function $x' = RR(x)$, the output compensation function $y = OC(y', x)$ and the function we have to approximate $y' = g(x')$ can be summarized as follows:

\[
RR(x) = x' \quad OC(y', x) = y'2^{\frac{x'}{2}} \quad g(x') = \sqrt{x}
\]
where \( x' \) is a value in \([1, 4)\) and \( m \) is an even integer such that \( x = x' \times 2^m \) for the input \( x \). With this range reduction technique, we need to approximate \( \sqrt{x'} \) for \( x' \in [1, 4) \).

To approximate \( \sqrt{x'} \), we use a \( 4^{th} \) degree polynomial \( P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \) with the coefficients,

\[
\begin{align*}
c_0 &= 0.372021392608168022242409733735257757569028472900390625 \\
c_1 &= 0.7923315194006106398916244870633818209171295166015625 \\
c_2 &= -0.19923071993306279425794969029084313660860616455078125 \\
c_3 &= 3.80038460845395636988970812098705209791660308837890625 \times 10^{-2} \\
c_4 &= -3.08489157654257559404338528737753222308218479156494140625 \times 10^{-3}
\end{align*}
\]

\section*{C.8 \( \sqrt{x} \) for Bfloat16}

The elementary function \( \sqrt{x} \) is defined over the input domain \((-\infty, \infty)\). There are four classes of special case inputs:

\[
\text{Special case of } \sqrt{x} = \begin{cases} 
0.0 & \text{if } x = 0.0 \\
\infty & \text{if } x = \infty \\
-\infty & \text{if } x = -\infty \\
NaN & \text{if } x = NaN
\end{cases}
\]

We use the range reduction technique described in Appendix B.4. The range reduction function \( x' = RR(x) \), the output compensation function \( y = OC(y', x) \) and the function we have to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = x' \quad OC(y', x) = s \times y' 2^m \quad g(x') = \sqrt{x'}
\]

where \( s \) is the sign of the input \( x \), \( x' \) is a value in \([1, 8)\) and \( m \) is integer multiple of 3 such that \( x = s \times x' \times 2^m \). With this range reduction technique, we need to approximate \( \sqrt{x'} \) for \( x' \in [1, 8) \).

To approximate \( \sqrt{x'} \), we use a \( 6^{th} \) degree polynomial \( P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 \) with the coefficients,

\[
\begin{align*}
c_0 &= 0.56860957346246798760347473944420926272869110107421875 \\
c_1 &= 0.5752913905623990853399618572439067065715789794921875 \\
c_2 &= -0.180364291120356845521399691278929822146892547607421875 \\
c_3 &= 4.3868412288261666998057108912689727731049060821533203125 \times 10^{-2} \\
c_4 &= -6.5208421736825845915763721905022975988686084747314453125 \times 10^{-3} \\
c_5 &= 5.24108054614583814684314334763226448558270931243896484375 \times 10^{-4} \\
c_6 &= -1.7372029717709360593165601888898663673899136483669281005859375 \times 10^{-5}
\end{align*}
\]

\section*{C.9 \( \sin(\pi x) \) for Bfloat16}

The elementary function \( \sin(\pi x) \) is defined over the input domain \((-\infty, \infty)\). There are two classes of special case inputs:

\[
\text{Special case of } \sin(\pi x) = \begin{cases} 
NaN & \text{if } x = NaN or x = \pm \infty \\
0 & \text{if } x \geq 256 or x \leq -256
\end{cases}
\]

We use the range reduction technique described in Appendix B.5. We decompose the input \( x \) into \( x = s \times (i + t) \) where \( s \) is the sign of the input, \( i \) is an integer, and \( t \in [0, 1) \) is the fractional part.
of \(|x|\), i.e. \(|x| = i + t\). The range reduction function \(x' = RR(x)\), the output compensation function \(y = OC(y', x)\), and the function we need to approximate, \(y' = g(x')\) can be summarized as follows:

\[
RR(x) = \begin{cases} 
1 - t & \text{if } 0.5 < t < 1.0 \\
t & \text{otherwise}
\end{cases}, \quad OC(y', x) = \begin{cases} 
s \times y' & \text{if } i \equiv 0 \pmod{2} \\
s \times y' & \text{if } i \equiv 1 \pmod{2}
\end{cases}, \quad g(x') = \sin(\pi x)
\]

With this range reduction technique, we need to approximate \(\sin(\pi x')\) for \(x' \in [0, 0.5]\).

The \(\sin(\pi x)\) function exhibit a linear-like behavior around \(x = 0\). To approximate \(\sin(\pi x)\), we use a piecewise polynomial consisting of two polynomials:

\[
P(x) = \begin{cases} 
c_1 x & \text{if } x' \leq 6.011962890625 \times 10^{-3} \\
d_1 x + d_3 x^3 + d_5 x^5 + d_7 x^7 & \text{otherwise}
\end{cases}
\]

with the coefficients,

\[
c_1 = 3.14159292035398163278614447335712611675262451171875 \\
d_1 = 3.14151548702025307235885520640294998842010498046875 \\
d_3 = 5.16405991738943459523625278961844742298126220703125 \\
d_5 = 2.50692180297728217652775128954090178012847900390625 \\
d_7 = -0.44300851985643702191097759168769698590402069091796875
\]

### C.10 \(\cos(\pi x)\) for Bfloat16

The elementary function \(\cos(\pi x)\) is defined over the input domain \((-\infty, \infty)\). There are two classes of special case inputs:

\[
\text{Special case of } \cos(\pi x) = \begin{cases} 
\text{NaN} & \text{if } x = \text{NaN} \text{ or } x = \pm\infty \\
1 & \text{if } x \geq 256 \text{ or } x \leq -256
\end{cases}
\]

We use the range reduction technique described in Appendix B.6. We decompose the input \(x\) into \(x = s \times (i + t)\) where \(s\) is the sign of the input, \(i\) is an integer, and \(t \in [0, 1)\) is the fractional part of \(|x|\), i.e. \(|x| = i + t\). The range reduction function \(x' = RR(x)\), the output compensation function \(y = OC(y', x)\), and the function we need to approximate \(y' = g(x')\) can be summarized as follows:

\[
RR(x) = \begin{cases} 
1 - t & \text{if } 0.5 < t < 1.0 \\
t & \text{otherwise}
\end{cases}, \quad OC(y', x) = \begin{cases} 
-1 \times (-1)^{i \pmod{2}} \times y' & \text{if } 0.5 < t < 1.0 \\
(-1)^{i \pmod{2}} \times y' & \text{otherwise}
\end{cases}, \quad g(x') = \cos(\pi x')
\]

With this range reduction technique, we need to approximate \(\cos(\pi x')\) for \(x' \in [0, 0.5]\).

The \(\cos(\pi x)\) function exhibit a linear property around \(x = 0\). To approximate \(\cos(\pi x)\), we use the piecewise polynomial:

\[
P(x) = \begin{cases} 
c_0 & \text{if } x' \leq 1.98974609375 \times 10^{-2} \\
d_0 + d_2 x^2 + d_4 x^4 + d_6 x^6 & \text{if } 1.98974609375 \times 10^{-2} < x' < 0.5 \\
0.0 & \text{if } x' = 0.5
\end{cases}
\]
with the coefficients,

\[
\begin{align*}
    c_0 &= 1.00390625 \\
    d_0 &= 0.99997996859304827399483883709763176739215850830078125 \\
    d_2 &= -4.9324802047472200428046562592498958110809326171875 \\
    d_4 &= 4.02150995405109146219047033810056746006011962890625 \\
    d_6 &= -1.1640167711700171171429474270553328096866607666015625
\end{align*}
\]

D DETAILS ON POSIT16 FUNCTIONS

In this section, we explain the posit16 functions in OURLIBM. More specifically, we describe the special cases, the range reduction technique we used, how we split the reduced domain, and the polynomials we generated for each posit16 math library function in OURLIBM.

D.1 \( \ln(x) \) for Posit16

The elementary function \( \ln(x) \) is defined over the input domain \((0, \infty)\). There are two classes of special case inputs:

Special case of \( \ln(x) = \begin{cases} 
\text{NaR} & \text{if } x \leq 0 \\
\text{NaR} & \text{if } x = \text{NaR}
\end{cases} \)

We use the range reduction technique described in Appendix B.1. For \( \ln(x) \), the range reduction function \( x' = RR(x) \), the output compensation function \( y = OC(y', x) \), and the function to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = \frac{t - 1}{t + 1} \quad OC(y', x) = y' + m \log_2(e) \quad g(x') = \log_2 \left( \frac{1 + x'}{1 - x'} \right)
\]

The value \( t \) is the fractional value represented by the mantissa of the input \( x \) and \( m \) is the exponent, i.e. \( x = t \times 2^m \). With this range reduction technique, we need to approximate \( g(x') \) for \( x' \in [0, \frac{1}{3}] \).

To approximate \( g(x') \), we use a 9th degree odd polynomial \( P(x) = c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + c_9 x^9 \) with the coefficients,

\[
\begin{align*}
    c_1 &= 2.8853901812623563692659416992682963606954345703125 \\
    c_3 &= 0.96177728824005104257821585633791983127593994140625 \\
    c_5 &= 0.5780219285885953572901030383945908397436141967734375 \\
    c_7 &= 0.3944924321649024845370455134230665862560272216796875 \\
    c_9 &= 0.45254178489671204044242358577321283519268035888671875
\end{align*}
\]

D.2 \( \log_2(x) \) for Posit16

The elementary function \( \log_2(x) \) is defined over the input domain \((0, \infty)\). There are two classes of special case inputs:

Special case of \( \log_2(x) = \begin{cases} 
\text{NaR} & \text{if } x \leq 0 \\
\text{NaR} & \text{if } x = \text{NaR}
\end{cases} \)

We use the range reduction technique described in Appendix B.1. For \( \log_2(x) \), the range reduction function \( x' = RR(x) \), the output compensation function \( y = OC(y', x) \), and the function to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = \frac{t - 1}{t + 1} \quad OC(y', x) = y' + m \quad g(x') = \log_2 \left( \frac{1 + x'}{1 - x'} \right)
\]
The value $t$ is the fractional value represented by the mantissa of the input $x$ and $m$ is the exponent, \emph{i.e.} $x = t \times 2^m$. With this range reduction technique, we need to approximate $g(x')$ for $x' \in [0, \frac{1}{2})$.

To approximate $g(x')$, we use a $5^{th}$ degree odd polynomial $P(x) = c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + c_9 x^9$ with the coefficients,

$$
c_1 = 2.88539211096054075059046226670034229755401611328125
$$

$$
c_3 = 0.96158476800643521986700079651200212538242340087890625
$$

$$
c_5 = 0.583775666651582758603922296676598498284515380859375
$$

$$
c_7 = 0.330016589138880600540204568460467271506786346435546875
$$

$$
c_9 = 0.691650488349585102332639507949352264404296875
$$

\subsection*{D.3 $\log_{10}(x)$ for Posit16}

The elementary function $log_{10}(x)$ is defined over the input domain $(0, \infty)$. There are two classes of special case inputs:

$$
\text{Special case of } \log_2(x) = \begin{cases} 
NaR & \text{if } x \leq 0 \\
NaR & \text{if } x = NaR
\end{cases}
$$

We use the range reduction technique described in Appendix B.1. For $log_{10}(x)$, the range reduction function $(x' = RR(x))$, the output compensation function $y = OC(y', x)$, and the function to approximate $(y' = g(x'))$ can be summarized as follows:

$$
RR(x) = \frac{t - 1}{t + 1} \quad OC(y', x) = \frac{y' + m}{\log_2(10)} \quad g(x') = \log_2 \left( \frac{1 + x'}{1 - x'} \right)
$$

The value $t$ is the fractional value represented by the mantissa of the input $x$ and $m$ is the exponent, \emph{i.e.} $x = t \times 2^m$. With this range reduction technique, we need to approximate $g(x')$ for $x' \in [0, \frac{1}{2})$.

We approximate $g(x')$ with a $9^{th}$ degree odd polynomial $P(x) = c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + c_9 x^9$ with the coefficients,

$$
c_1 = 2.88539211096054075059046226670034229755401611328125
$$

$$
c_3 = 0.96158476800643521986700079651200212538242340087890625
$$

$$
c_5 = 0.583775666651582758603922296676598498284515380859375
$$

$$
c_7 = 0.330016589138880600540204568460467271506786346435546875
$$

$$
c_9 = 0.691650488349585102332639507949352264404296875
$$

\subsection*{D.4 $\sqrt{x}$ for Posit16}

The elementary function $\sqrt{x}$ is defined over the input domain $[0, \infty)$. There are two classes of special case inputs:

$$
\text{Special case of } \sqrt{x} = \begin{cases} 
0.0 & \text{if } x = 0.0 \\
NaR & \text{if } x < 0 \text{ or } x = NaR
\end{cases}
$$

We use the range reduction technique described in Appendix B.3. The range reduction function $x' = RR(x)$, the output compensation function $y = OC(y', x)$ and the function we have to approximate $y' = g(x')$ can be summarized as follows:

$$
RR(x) = x' \quad OC(y', x) = y' 2^{m} \quad g(x') = \sqrt{x}
$$

where $x'$ is a value in $[1, 4)$ and $m$ is an even integer such that $x = x' \times 2^m$ for the input $x$. With this range reduction technique, we need to approximate $\sqrt{x'}$ for $x' \in [1, 4)$. 

To approximate $\sqrt{x^7}$, we use a piecewise polynomial consisting of two $6^{th}$ degree polynomials

$$P(x) = \begin{cases} c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 & \text{if } x' \leq 2.14599609375 \\ d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + d_5 x^5 + d_6 x^6 & \text{otherwise} \end{cases}$$

with the coefficients,

$$c_0 = 0.2695935927094846307205955326935509219776566314697265625$$
$$c_1 = 1.129000960281488510247527301543235635757462890625$$
$$c_2 = -0.64843834364755160418866353211342357099056243896484375$$
$$c_3 = 0.3530868073027828568655195340397767722606658935546875$$
$$c_4 = -0.12717184127512942699169085269677452743053436279296875$$
$$c_5 = 2.62819293603375920567399106175798806361854076385498046875 \times 10^{-2}$$
$$c_6 = -2.3530402643644897538177462710268123191781342029517533203125 \times 10^{-3}$$
$$d_0 = 0.40915629885583498781542743927252926006496429443359375$$
$$d_1 = 0.743136217472554427843078883597627282142463916015625$$
$$d_2 = -0.1842527001546831189049413524116971350020236968994140625$$
$$d_3 = 4.3051395684769193647376846938641392625868320465087890625 \times 10^{-2}$$
$$d_4 = -6.6014424010839810319506426594671211205422878265380859375 \times 10^{-3}$$
$$d_5 = 5.74776888286255573622118841825567868918769395351409912109375 \times 10^{-4}$$
$$d_6 = -2.1374405303079146056961790112183052769978530704975128173828125 \times 10^{-5}$$

D.5 $\sin(\pi x)$ for Posit16

The elementary function $\sin(\pi x)$ is defined over the input domain $(-\infty, \infty)$. There is one special case input:

$$\sin(\pi x) = NaR \text{ if } x = NaR$$

We use the range reduction technique described in Appendix B.5. We decompose the input $x$ into $x = s \times (i + t)$ where $s$ is the sign of the input, $i$ is an integer, and $t \in [0, 1)$ is the fractional part of $|x|$, i.e. $|x| = i + t$. The range reduction function $x' = RR(x)$, the output compensation function $y = OC(y', x)$, and the function we need to approximate, $y' = g(x')$ can be summarized as follows:

$$RR(x) = \begin{cases} 1 - t & \text{if } 0.5 < t < 1.0 \\ t & \text{otherwise} \end{cases}, \quad OC(y', x) = \begin{cases} s \times y' & \text{if } i \equiv 0 \ (\text{mod } 2) \\ -s \times y' & \text{if } i \equiv 1 \ (\text{mod } 2) \end{cases}, \quad g(x') = \sin(\pi x)$$

With this range reduction technique, we need to approximate $\sin(\pi x')$ for $x' \in [0, 0.5]$.

The $\sin(\pi x)$ function exhibit a linear-like behavior around $x = 0$. To approximate $\sin(\pi x)$, we use a piecewise polynomial consisting of two polynomials:

$$P(x) = \begin{cases} c_1 x & \text{if } x' \leq 2.52532958984375 \times 10^{-3} \\ d_1 x + d_3 x^3 + d_5 x^5 + d_7 x^7 + d_9 x^9 & \text{otherwise} \end{cases}$$
with the coefficients,

\[
\begin{align*}
c_1 &= 3.141577060931899811890843920991756021976470947265625 \\
d_1 &= 3.141593069399674309494230328709818422794342041015625 \\
d_3 &= -5.1677486367595673044661452877335250377655029296875 \\
d_5 &= 2.5509842454171200983029929571785032749176025390625 \\
d_7 &= -0.6054711947334260324637966732552740722894685791015625 \\
d_9 &= 9.475996412214268693752217131986898125779628753662109375 \times 10^{-2}
\end{align*}
\]

\section*{D.6 \(\cos(\pi x)\) for Posit16}

The elementary function \(\cos(\pi x)\) is defined over the input domain \((-\infty, \infty)\). There are two classes of special case inputs:

\[
\cos(\pi x) = \text{NaR} \text{ if } x = \text{NaR}
\]

We use the range reduction technique described in Appendix B.6. We decompose the input \(x\) into \(x = s \times (i + t)\) where \(s\) is the sign of the input, \(i\) is an integer, and \(t \in [0, 1)\) is the fractional part of \(|x|\), i.e. \(|x| = i + t\). The range reduction function \(x' = RR(x)\), the output compensation function \(y = OC(y', x)\), and the function we need to approximate \(y' = g(x')\) can be summarized as follows:

\[
\begin{align*}
RR(x) &= \begin{cases} 
1 - t & \text{if } 0.5 < t < 1.0 \\
t & \text{otherwise}
\end{cases} \\
OC(y', x) &= \begin{cases} 
-1 \times (-1)^{i \mod 2} \times y' & \text{if } 0.5 < t < 1.0 \\
(-1)^{i \mod 2} \times y' & \text{otherwise}
\end{cases} \\
g(x') &= \cos(\pi x')
\end{align*}
\]

With this range reduction technique, we need to approximate \(\cos(\pi x')\) for \(x' \in [0, 0.5]\).

The \(\cos(\pi x)\) function exhibit a linear property around \(x = 0\). To approximate \(\cos(\pi x')\), we use the piecewise polynomial:

\[
P(x) = \begin{cases} 
c_0 & \text{if } x' \leq 3.509521484375 \times 10^{-3} \\
d_0 + d_2x^2 + d_4x^4 + d_6x^6 + d_8x^8 & \text{if } 3.509521484375 \times 10^{-3} < x' < 0.5 \\
0.0 & \text{if } x' = 0.5
\end{cases}
\]

with the coefficients,

\[
\begin{align*}
c_0 &= 1.0001220703125 \\
d_0 &= 1.000000009410458634562246515997685492038726806640625 \\
d_2 &= -4.93479863229652071510145106003619730472564697265625 \\
d_4 &= 4.05853647916781223869975292473100125783642333984375 \\
d_6 &= -1.3327362938689424343152722940430976450443267822265625 \\
d_8 &= 0.221533849576965868877209686615969857234954833984375
\end{align*}
\]

\section*{E \(\log_2(x)\) FOR FLOAT}

Our \(\log_2 x\) function for float in OurLibM is guaranteed to produce the correct results for the inputs in \([1, 2]\). For all other inputs, the result is undefined.

We use the range reduction technique described in Appendix B.1 to ease the job of creating the polynomial. For \(\log_2(x)\), the range reduction function \((x' = RR(x))\), the output compensation
function \( y = OC(y', x) \), and the function to approximate \( y' = g(x') \) can be summarized as follows:

\[
RR(x) = \frac{t - 1}{t + 1} \quad OC(y', x) = y' \quad g(x') = \log_2 \left( \frac{1 + x'}{1 - x'} \right)
\]

The value \( t \) is the fractional value represented by the mantissa of the input \( x \) when \( x \) is decomposed to \( x = t \times 2^m \) with an integer exponent \( m \). With this range reduction technique, we need to approximate \( g(x') \) for \( x' \in [0, \frac{1}{3}] \).

To approximate \( g(x') \), we use a 15\(^{th}\) degree odd polynomial,

\[
P(x) = c_1 x + c_3 x^3 + c_5 x^5 + c_7 x^7 + c_9 x^9 + c_{11} x^{11} + c_{13} x^{13} + c_{15} x^{15}
\]

with the coefficients,

\[
\begin{align*}
c_1 &= 2.88539008177725309067795935152991116046905517578125 \\
c_3 &= 0.9617966943187539197168689497630111873149871826171875 \\
c_5 &= 0.577077951509928688267363683818019926548004150390625 \\
c_7 &= 0.41220281933294511400589499316993164986522674560546875 \\
c_9 &= 0.32046296281382297133077940981914795935153961181640625 \\
c_{11} &= 0.264665103135787116439558985803159885108470916748046875 \\
c_{13} &= 0.1996122250113066820542684354222728870809078216552734375 \\
c_{15} &= 0.298387164422755202242143468538415618240833282470703125
\end{align*}
\]