EARLY DEVELOPMENT AND APPLICATION OF PROOF-LIKE REASONING: LONGITUDINAL CASE STUDIES

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ABSTRACT OF THE DISSERTATION

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This study traces three primary school students' longitudinal development of mathematical ideas and ways of reasoning while solving a strand of counting problems. The students worked on well defined, open-ended counting problems of variable difficulty in various settings: pairs, whole class settings, task-based interviews and small groups. Video-taped data, transcripts, and student work are analyzed for cognitive growth in reasoning, attentive to the social elements of collaboration in problem solving. Data include individual and group co-construction of justifications for solutions. Video narratives (VMCAnalytics) describe the students' learning progressions. Student dialogue and co-constructions that fostered their development are identified and displayed in the 13 published video narratives linked to the analyses. For each student, how do their recognition of patterns, use of strategies and representations, display of justifications and forms of reasoning about solutions to counting tasks develop over time and how might each journey be displayed with a learning progression using video data?

Analyses revealed local and global recognition for enumeration of outcomes (by recursive strategies), invention of composite operations, connection between tasks, rule generalization, and direct reasoning by cases, induction, controlling for variables. Particular forms of reasoning are identified for each student. The following cognitive and social factors revealed that learning occurred collaboratively, in a variety of settings. Students were attentive to the counter examples/arguments posed by others and worked to convince others about their arguments that were "proof like" in structure.

The longitudinal study showed how earlier ideas became the foundation for building later ideas, represented in more sophisticated ways. The results have implications for effective mathematical practices, such as collaborative learning, and attention to providing justifications for solutions. These pedagogical approaches can be incorporated in curriculum design, can supplement approaches to teacher professional development. The learning progressions can offer teachers an approach to formative assessment of student reasoning on solving counting tasks.

Keywords: proof-like justification, primary education, multiple case studies, socio-constructivism, learning progressions

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DEDICATION

To my parents and grandparents.

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Chapter 1 Introduction

1.1 Statement of the problem

There has been a widespread adoption of the *Common Core State Standards for Mathematics* (CCSSM) in the United States. NCTM (2000) takes the position that the adoption for many states is “an unprecedented opportunity for systemic improvement in mathematics education” in the US. CCSSM is backed by research of learning progressions about how understanding develops over time (NGA & CCSSO, 2010). The *Common Core State Standards Initiative* calls for education research on learning progressions, declaring the following:

> What students can learn at any particular grade level depends upon what they have learned before. Ideally then, each standard in this document [*CCSSM*] might have been phrased in the form, ‘Students who already know...should next come to learn....’ But at present this approach is unrealistic—not least because existing education research cannot specify all such learning pathways... One promise of common state standards is that over time they will allow research on learning progressions to inform and improve the design of standards to a much greater extent than is possible today (NGA & CCSSO, 2010, p. 4).

This research contributes to this call for research on progressions by examining how three students developed mathematical ideas and articulated or displayed support for their ideas, over time. This research differs from other learning progression research (e.g., Clements and Samara, 2004; Battista, 2007). The focus is on reasoning, argumentation, representations, strategies and heuristics, using the combinatorial domain as a vehicle for understanding how these constructs may develop over time, rather than studying how learning *of* the combinatorial domain develops. And therefore, this research reports on examples of progressions that align to the *CCSS for Mathematical Practices*, specifically the practices of articulating reasoning, justifying an argument, and using tools.

A hallmark of doing mathematics or understanding mathematics is the ability to reason about ideas, justify the correctness of an argument, and use various tools strategically (for supporting an argument or for problem solving) as early as primary school (NGA & CCSSO, 2010; NCTM, 2000). The mathematics education research community has emphasized providing opportunities for the aforementioned mathematical practices. Furthermore, the challenge for professional development and mathematics education programs is to prepare teachers to attend to cognitive development without imposing an “outside” mathematics/knowledge onto students, while creating conditions that foster various forms of reasoning that promote learning, such as justification and explanations. But how do students develop such mathematical behavior and is it the same for each student? Some mathematics education researchers have suggested instructional factors, such as attention to students’ explanations rather than imposition of ideas (e.g., (Maher, 1998), types of questioning (Maher & Martino, 1996a), attention to the underlying assumptions of student explanations and mathematical behavior (Davis, 1996), promotion of student negotiation (Yackel & Cobb, 1996), or task development and implementation (Stein, Grover, & Henningsen, 1996). Others suggested discourse factors, such as types of student-to-student interaction that serve different goals for problem solving (e.g., Yackel, Cobb, Wood, Wheatley, & Merkel, 1990; Yackel, Cobb, Wood, 1991), the type of meta-discursive rules present in discourse (e.g., Sfard, 2001), the structure of collaboration for deep learning (O’Donnell, 2006). Meanwhile, other researchers suggested studying cognitive factors, such as learners’ representational systems including affective systems about mathematical ideas (DeBellis & Goldin, 2006) or the process of accommodation and assimilation (Davis, 1984). Intuitively, theoretically, and backed by research each of the aforementioned factors play a role in the development of mathematical ideas and behavior. In research practice, the challenge lies in reporting the complexity of student development of ideas. With more research that attends to the process by which students develop their ideas, especially over time, perhaps educators, teachers, and researchers can better understand this complexity.

Accordingly, this research seeks to trace, over time, the origins and development of mathematical ideas of three learners as they worked on challenging open-ended counting problems of increasing difficulty in various settings: whole class, with a partner, one-on-one interviews, small-group formative assessment, and summative assessments. The study aims to understand what the learners' contributions and engagement in problem solving in these various settings reveal about their forms of reasoning and mathematical ideas as they make their ideas public, clarify and refine their contributions, take input from others, and build strong justifications for solutions to problems.

The construction and refinement of knowledge is a complex process and necessitates attention to both the cognitive and social actions of a learner. This includes the study of intellectual products (e.g., tools, idiosyncratic vocabulary, formal notation, the use of the mathematics register, drawings, and/or models), of argumentation brought forth in support of ideas, and of intellectual interactions with other participants and teacher/researchers. But these do not happen in isolation. Based on a social constructivist view, individual, social, and community learning are intertwined in the learner's experience. To understand the complexity of this experience, this qualitative research takes the form of *multiple case studies* (Yin, 2003; Baxter & Jack, 2008), viewing each learner's experience as interdependent with and not in isolation from the other learners, while maintaining attention to each learner's cognitive development. The investigation is longitudinal and microanalytic since it focuses in depth on the work of learners over a period of two to three years. Access to video files enabled the study. To this end, archived videos of students working in the counting/combinatorics strand from the Rutgers-Kenilworth Longitudinal Study (Maher, 2010) was accessed and analyzed. The nature of the Rutgers study lent itself to examine learning across a variety of perspectives because students who participated in the Longitudinal Study were followed doing mathematics in the elementary grades in various instructional settings.

The current research analyzes digitized video data stored in the Robert B. Davis Institute for Learning (RBDIL), making relevant aspects of the database to support the current research findings. New videos were ingested into the Video Mosaic Repository[1] so that new video narratives (VMCAnalytics) could be created tracing the development of study participants' ideas about related Tower Tasks, displaying problem solving, interactions with others, and attending to the representations, argumentation, and reasoning they use and/or modify over time.

This study contributes to the field by documenting multiple case studies longitudinally (over a two or three school-year period) of the development of primary students' ideas during their problem solving in task-based interview, small group, and classroom settings in the combinatorics strand. This research extends the literature on how children develop proof-like reasoning given appropriate conditions for creating convincing arguments for solutions to problems (Maher & Martino, 1996a; Maher & Martino, 1996b; Alston & Maher, 1993) from both a socio-constructivist and cognitive

[1] www.Videomosaic.org

theoretical perspective. The resulting video narratives are available, open source, worldwide, and enable researchers and teacher educators to bring users' attention to particular video data that support one's research agenda.

1.2 The Rutgers-Kenilworth Longitudinal Study (RKLS)

For more than twenty-five years, researchers at the Robert B. Davis Institute for Learning (RBDIL) of the Graduate School of Education at Rutgers University, led by researcher Carolyn A. Maher, studied the building of mathematical ideas and ways of reasoning about the ideas of students. A partnership with Harding Elementary School in Kenilworth, New Jersey formed in 1984 for professional development for teachers of mathematics (Maher, Powell, & Uptegrove, 2010). The Rutgers team assessed the intervention of the professional development by following a cohort of students in a class over a twelve-year period starting in their elementary grades as they participated in classroom mathematics.[2] The researchers designed and studied learning environments that invited student interaction and minimized researcher intervention while students engaged in cognitively challenging, well-defined open-ended problem tasks (Palius & Maher, 2011) that had familiar situations or objects (that did not require any complex, formal mathematical knowledge). In the RKLS, the *task design*, *contexts*, and *informality* were central to the design of the children's problem-solving activities. In addition, the researchers fostered a culture of independent thinking that gave children the opportunity to explore the tasks without giving direct instruction. The young children invented methods to convince each other of their ways of reasoning to solve the Towers Tasks,

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without being taught formal methods (Maher & Martino, 1996b). Students justified their reasoning in various instructional settings while working on problem solving tasks over a range of strands in mathematics including fractions, early algebra, combinatorics, geometry, probability, data analysis and pre-calculus. This research extends the study of the Rutgers Longitudinal Study project by considering the development of students' mathematical ideas in their early years, in detail, before those ideas were refined, longitudinally, while also using a historical and social lens to account for how these ideas developed.

This study focused on a cohort of three students that were followed by the Rutgers team in elementary school as they worked in individual, task-based interviews, in partner and small group problem solving, in individual assessments, and in whole class settings (Maher, Powell, Uptegrove, 2010). Maher (2010) traced the development of the counting and combinatorics mathematical strand in elementary and later grades with a cohort of students that included these three students. Stephanie, Michelle, and Milin were students of Harding Elementary School through first to fifth grade when the Rutgers team followed their mathematical activity in the rich, task-based learning environments. Researchers recorded all the activities using multiple cameras as the students worked on problem solving tasks of the combinatorics strand. The sessions are stored in the Robert B. Davis Institute for Learning (RBDIL) on more than 4,500 hours of videotape footage that form a collection of data that document how elementary through secondary and college-level students build important mathematical ideas.

1.3 Research Questions

For Michelle, Milin, and Stephanie,

1. How do their recognition of patterns, use of strategies and representations, display of justifications and forms of reasoning about solutions to counting tasks develop over time?
2. How might each journey be displayed with a learning progression using video data?

Learning progression is defined as a sequence of learning, which could include more complex ways of thinking about an idea, that followed one another in a student’s learning.

Chapter 2 Literature Review

The following review includes research that pertains to empirical studies regarding children’s problem solving and learning under several theoretical perspectives and the research that resulted from the Rutgers-Kenilworth Longitudinal Study (RKLS). The studies reviewed in this section emphasize the importance of advancing our understanding of the interplay between socio-cultural processes and individual cognitive development as learners are challenged to solve and justify solutions to problems.

2.1 Cognitive theories and empirical research

One prevalent cognitive theory is information-possessing theory (IP). The human mind and the ways in which an individual learns and gains knowledge can be described using the computer metaphor. IP theories rest on the premise that it is possible to model mathematical behavior of both experts and novices because there is a strong degree of consistency in those behaviors (Schoenfeld, 1982). That is, just as the processing parts of a computer (the CPU) that make it function can be designed (by a programmer), constructed (by a mechanic), executed (by a technician and stimulus), debugged (by a programmer), produced and stored (a completed version at the present moment awaiting updates and advancements in the technology of its parts), etc., so can the aspects of mathematical or other cognitive activities. The designing and constructing would be the learning and the constructing and building upon one’s schemas, the eventual storing in memory, etc.; the executions and debugging would be the activations and responses in behavior during mathematical activity that is stimulated by the environment (such as the task or the prompting of a teacher); finally, the “completed” version would be one’s understanding or one's expertise (the network of schemas or mental representations) at the present moment (that is, if you assume that learning and developing one's understanding is ongoing and never final). Cobb (1990), a constructivist, summarized a schema as "an elaborate production (Anderson, 1983) that consists of procedures for manipulating symbols and conditions that must be satisfied before the procedures can be carried out" (p. 71). A schema, like a program in a computer, is very important in IP theory because when researchers produce it (attempt to write it out like a computer program), from empirical evidence of students working and providing verbal data and from thorough task analysis, it implies the possibility that students' problem solving *could* be modeled, and to a significant degree of accuracy. It further implies, first, that one can predict students' incorrect behaviors, second, that one can model and simulate expert behavior, and, third, that one can train learners to behave like experts (see Schoenfeld's (1985) review of three exemplary studies).

Other researchers sought to describe how students reasoned about concepts without comparing their performance to an expert or to preconceived "truths." Davis (1976) and Ginsburg (1975) noted children as intuitive mathematicians, who cope with mathematical problems through interaction and develop "perceptual skills, patterns of thought, concepts, counting methods" (Ginsburg, 1975, p. 63).

Representations (external and internal) have been of interest in relation to mathematical problem solving (Goldin & Kaput, 1996; Goldin, 1998; Kaput, 1998; Vergnaud, 1998; Davis, 1984; Lesh, Post, & Behr, 1987; Arcavi, 2003). The research and education community call for students having opportunties and learning to make connections between different representations (Moschkovich, Schoenfeld & Arcavi,

1993; NCTM, 2000; NGA & CCSSO, 2010). *Internal* representations are concerned with a learner's mental activity of abstracting mathematical ideas. Davis (1984) explained that representations are mental models that allow humans to form associations between the properties of an idea and the idea. Davis posited that children assemble ideas like a jigsaw puzzle by building upon prior concrete experience. Each new idea can be assembled if it fits into the existing larger structure of earlier constructed ideas (known as an *assimilation paradigm*). Davis (1992) also explained that *cognitive obstacles* are roadblocks that occur when an assimilation paradigm is limited or incorrect. From a cognitive perspective, Davis' (1984) "paradigm teaching strategy" (p. 313) assumed learning occurred by children's building upon their powerful familiar ideas, modifying them when necessary. He also suggested that through simple tasks that have an "isomorphic image" to formal, abstract mathematics concepts learning occurred (p. 314).

External representations are external manifestations that represent mathematical ideas or concepts. External representations, which include concrete, visual, verbal, symbolic representations, can serve as "stimuli on the senses and help us understand these concepts" (Pape & Tchoshanov, 2001, p. 119). Representation research has indicated the use of representations to be related to sense-making, invention, deeper understanding, and communicating ideas or meaning, given appropriate tasks (e.g. Greeno & Hall, 1997; Nason & Woodruff, 2004). In contexts that promote building varied *external* representations and sharing of ideas, learners can modify and refine their existing mental representations, which in turn builds mathematical understanding (Maher, 2005).

Cifarelli (1998) studied representations “as conceptual organizations of actions” from a constructivist lens by interviewing eight problem solvers during their problem solving activity of related tasks. The researcher posited that the level of solution activity was an expression of solvers’ evolving conceptual structures. Levels of solution activity included levels of anticipations and the presence of reflection about prior, current, and/or potential solution activity. Conceptual structures was identified by three categories: recognition, re-presentation, and structural abstraction. Recognition was identified when solvers recognized the usefulness and relevance of prior acitivty when they attempted new tasks. Re-presentation was identified when solvers indicated having recognized prior activity and anticipated potential obstacles. Structural abstraction was indicated by organized cognitive actions where solvers were able to re-present their potential solution activity and operate on it, as well as aniticipate the results without needing to carry out the activity. The implications of Cifarelli’s constructivist perspective to cognitive development is as follows: “Cognitive theories have tended to adopt a single perspective in studying representation, with the result being an incomplete profile of what we know to be a very complex process” (Cifarelli, 1998, p. 261). The objective for a more precise explanation about how learners construct knowledge reaches beyond taking a constructivist perspective to cognitive development, discussed next.

2.2 Socio-constructivist theories and empirical research

Saxe (e.g., 2015, 1994) discovered that deeply interwoven in the cognitive development is the socio-cultural life. Specifically, the dimensions are cultural practices, social interactions, and sign-using activities. Saxe argued, that although there is no contradiction to the constructivist ideas of self-regulated internalization processes, they have not “led, in any rich sense, to advancing our understanding of the interplay between sociohistorical processes and cognitive developmental ones” (p. 8). Saxe found that while selling candy children were more advanced than their non-selling counterparts on mathematical activities, such as conservation, classification, and seriation, and that this could only be understood by studying socio-cultural processes. Similarly, Lave (1988) found that food shoppers engaged in mathematical activities for the goals of saving and monitoring money spending, while the same subjects could not solve similar mathematical problems presented in a traditional school sense. The pragmatic goals of selling candy or food shopping (i.e. socio-cultural activities) differed from school mathematics goals, where mathematical error takes on a harsher meaning in school.

The reports of such cases in which people are engaged in activity to reach a goal, successfully applying mathematical processes as a byproduct of their activity, has implications for research and practice. In terms of research, although the aforementioned socio-cultural dimensions are referring to different cultures geographically in the context of Saxe’s study, they teach us about the possibilities in the micro worlds of classrooms, in laboratory settings, among cohorts, within family and friend groups, and so on. Some key takeaways from his studies is that problem solving contexts, type of tasks, and the goal matter and can provide a different picture to children’s success in problem solving.

Studies have before attributed children’s problem-solving success to their related activities together and goal oriented interactions (Johnson & Johnson, 1994; Yackel, Wood, & Cobb, 1991; Powell A. , 2003; Francisco, 2013). For example, Yackel et al. (1991) demonstrated that collaborative dialogue among students, led to small-group problem solving opportunities of revisiting and developing one’s own solution based on another’s solution activity and modifying one’s own understanding of the solution to make sense of another’s solution activity for the purpose of reaching a consensus. Francisco (2013) suggested that the “socio-mathematical norm” (Yackel & Cobb, 1996) of justifying their ideas to each other, emphasized in the RKLS, helped foster students’ mathematical understanding. Moreover, Maher and Martino (1999) documented the types of teacher-researcher questioning that encourages justification and the “laboratory” classroom norms that promoted the development of particular ideas. Maher and Martino suggest that students will not naturally justify to each other how they arrived at a solution or why it is true, and so the teacher should interject questions that prompt the justification. McClain and Cobb (2001) analyzed learning in the social context across a school year as they adapted their laboratory classroom to negotiate norms that supported their students’ development of mathematical disposition and of intellectual autonomy. The aforementioned studies have in common that they designed (or adapted) the research conditions to foster a social problem-solving environment.

Some studies (e.g., Cobb et al., 1993) integrate social and individual contexts to account “for the messiness and complexity of mathematics learning and teaching as it occurs in classroom situations” (Cobb & Bauersfeld, 1995, p. ix). For example, Cobb, Yackel, and Wood (1992) analyzed a ten-minute segment of student interaction and learning. To make sense of the constraints on learning their analysis switched between individual and collective interpretations to show that individual and group development was reflexive, circular (rather than linear action and reaction), and mutually interdependent. They found evidence of *parallel* or *equivalent* interpretations. The former is the assumption that one’s understanding is shared by the others but as research observers there was evidence of different interpretations, whereas the latter is the lack of evidence of different interpretations. On occasion they found that when children became aware of a discrepancy, it became a topic of discussion, and in other occasions their interpretations were being taken-as-shared. They provided a rich account of the impact that resolution, irresolution, or non-detection of subjective discrepancies on the mathematical learning of each child. Jaworski (1994b) pointed out that through discourse individual students negotiating their ideas develop classroom meanings and their own personal meanings. Thus, the social interaction and the language used are directly related to individual student learning. However, the sole study of social interaction should not be taken for granted, as some studies showed that not all situations of discourse between students with or without a teacher contribute to learning (Steffe & Tzur, 1994; Sfard & Kieran, 2001).

2.3 Reasoning

Reasoning in mathematics is emphasized and embedded in the mathematical process and practice standards of both CCSS and NCTM. It is expected that students engage in reasoning in any content strand. Reasoning is displayed through communication and other forms of representation. The following empirical studies highlight what reasoning in various strands looks like, including the combinatorics strand.

Blanton & Kaput (2004, 2005) studied algebraic reasoning, which was classified as generalizing from particular instances, justifying generalizations through argumentation, and expressing generalizations in formal age-appropriate ways. Blanton & Kaput (2004) found that varying a single task parameter in order to generate, identify, and describe numerical patterns fostered the opportunity for students to engage in functional thinking, i.e. generalizing numerical patterns to describe functional relationships. Blanton & Kaput (2005) characterized other teacher classroom practices, in addition to task engineering, that promoted algebraic reasoning and presented a case study of a teacher's practices that affected student achievement. They showed that given opportunities to revisit themes within algebra students were able to reason about algebraic ideas in increasingly complex ways. These complex ways were identified as follows: using generalizations to solve tasks; justification, proof, and testing conjectures; and generalizing a mathematical process. They stated, "We conjecture that these categories reflect students' more evolved ability to reason algebraically and, because of their complexity, could indicate that algebraic reasoning was becoming a habit of mind" (p. 431).

The process of finding, describing, justifying, symbolizing mathematical ideas, and testing conjectures and generalization of processes are not limited to algebra. These reasoning activities are the conceptual underpinnings for mathematics in general. Reid (2002) characterized certain patterns of reasoning in a grade 5 mathematics class. The researcher argued that what is distinctly mathematical in mathematical reasoning, namely that conjectures must supported by deductive explanations or refuted by counterexample, has early patterns of reasoning. In the classroom he observed the following patterns of reasoning that was foundational for mathematical reasoning: looking for and studying regularities and patterns, making conjectures, and the necessity to support or refute statements. Carpenter et al. (2003) found the following classes of justification: appealing to authority; justifying by example; and providing generalizable arguments. The latter is the ability to provide a logical argument that applies to all cases. Ellis (2007) developed a taxonomy for generalizations that were divided into students' activity or "generalization actions" and students' statements or "reflective generalizations." By dividing into two parts, actions and statements, when a statement of generalization was made, Ellis was able to trace the reasoning to find the generalizing actions, identified as relating, searching and extending, that led to the final statement of reflection. The implications of tracing back to earlier student actions and reflections is that the range of sophistication expands to identify earlier actions that "originally appeared to be unproductive, but later were able to develop powerful results" (Ellis, 2007, p. 257). Drawing on the aforementioned research, with the intention of describing variation in reasoning for a task that was not typically used for eliciting generalization and justification, Vale et al. (2017) found "comparing and contrasting" actions for forming conjectures when generalizing, and "verifying that the common property holds for each member of the group" was an additional class for justification.

2.3.1 Combinatorial domain

In addition to eliciting conjectures, verification, and generalization, English (1990, 1991) noted that combinatorial tasks also facilitate the development of enumeration processes. English (1990) noted the following principles in children's knowledge construction of combinatorial ideas: the principle of difference (i.e., combinations are different if at least one element is different); the principle of systematic variation (i.e., if one element is varied systematically, then the combination will be different); the principle of constancy (i.e., if one element is held constant while the other is varied systematically, then the combination will be different). English (1991) found the following enumeration strategies (in increasing order of sophistication): random selection, trial-and-error procedure with random selection, emerging but incomplete patterns for selection, complete and cyclical patterns for selection, emerging but incomplete odometer strategy, and complete odometer pattern for selection. The trial-and-error procedure was indicated by scanning actions to retain or reject combinations. The odometer strategy was displayed by the control of one element while the other was varied in order to exhaust all combinations systematically.

Batanero, Godino, & Navarro-Pelayo (1997) summarized the following as operations and procedures of combinatorial ideas: (1) Basic combinatorial concepts and models: Combinatorial operations: combinations, arrangements, permutations, concept, notation, formulae; Combinatorial models: Sampling model: population, sample, ordered/non-ordered sampling, replacement; Distribution model: correspondence, application; Partition model: sets, subsets, union. (2) Combinatorial procedures: Logical procedures: classification, systematic enumeration, inclusion/exclusion principle, recurrence; Graphical procedures: tree diagrams, graphs; Numerical procedures: addition, multiplication and division principles, combinatorial and factorial numbers, Pascal's triangle, difference equations; Tabular procedures: constructing a table, arrays; Algebraic procedures: generating functions. (p. 240)

Janácková and Janácek (2006) studied permutation strategies that have not been identified in the work of the aforementioned researchers. Specifically, the strategies are as follows (in parentheses is an example or an explanation; see Appendix B for the Glossary of Terms): strategy of parallelism; group strategy; strategy of a constant beginning; strategy of symmetry; strategy of rotation; strategy of the complement of an exhausted subset.

The large-scale empirical studies of English, Batanero et al., and Janácková and Janácek provide the types of combinatorial ideas observed. One might ask do children develop the odometer strategy and, if so, how? What factors contribute to development? What happens when one uses strategies of symmetry and finding the complement? How do young children recognize, flag, or control for duplication?

Researchers have documented mathematical behavior during the RKLS, some longitudinally on mathematical understanding and representations (e.g., Teehan, 2017; Uptegrove, 2005) or on the development of reasoning (e.g., Sran, 2010; Tarlow, 2004), and others in one session, on the mathematical ideas and reasoning that emerged in the context of discourse (Powell, 2003). For example, Alston & Maher (1993) and Maher & Martino (1996a, 1996b) documented that participants developed convincing arguments in their early years to justify their solutions. Martino's (1992) dissertation reports the early years (grades one to three) of children building combinatorial ideas as they were challenged with various isomorphic and non-isomorphic counting tasks. It was the basis for a series of published articles with Dr. Carolyn Maher about students' use of "proof like" arguments to support their solutions (Maher & Martino, 1996b), how development could be traced to earlier combinatorial problems (Martino, 1992), how teacher questioning and minimal interventions promoted justification and generalization (Maher & Martino, 1992; Martino & Maher, 1999), how individual conceptual change occurs when the student is mentally invested for one's self (Maher & Martino, 2000), and how videotape study provide a microscope into the construction of mathematical knowledge (Davis, Maher, & Martino, 1992). The effects of interaction between student and researcher during the RKLS have also been previously studied (Martino & Maher, 1994;

Maher, 1988). Specifically, the researcher's role is described as a facilitator for promoting and encouraging students to actively build upon their own understanding of the underlying mathematical ideas found in the tasks. The current study contributes to the aforementioned studies by focusing on grades three through five, specifically focusing on the origins of the ideas that were reported in previous studies and the impact of research-student and/or student-student interactions on the development of those ideas.

Chapter 3 Theoretical Perspectives

The purpose of choosing several constructs for this research is articulated by Goldin (1998):

> There is a need to be able to use not just one construct in isolation (be it rule learning, algorithms, strategies, image schemata, visualization, heuristics, metacognition, metaphor, construction of meaning, affect, belief systems, or any other), but constructs in *combination with each other* (p. 142; emphasis in original).

Note that Goldin was describing cognitive constructs. The cognitive perspective accounts for learning through the observable features of these cognitive constructs, such as strategies, forms of reasoning, and representations used in support of problem solving of an individual. Davis (1984) and Davis & Maher (1990) posited that learners cycle through the following steps in mathematical situations: building a representation of the situation, retrieving or constructing a mental representation of relevant knowledge, constructing a mapping between the data representation and the knowledge representation, checking if the mapping is adequate, and applying, revisiting, and modifying the representations as new situations are encountered. An internal representation is a mental configuration of a learner that is not directly observable to others. The individual who may be doing the introspecting of his or her experience has direct access, however, internal representations cannot be observed by others. An external representation, on the other hand, can be observed. An external representation is a "physically embodied, observable configuration such as words, graphs, pictures, equations, and computer microworlds" (Goldin & Kaput, 1996, p. 400).

Davis and Maher (1990) presented evidence in which a student, Brian, built a representation for the mathematical situation by breaking the representation into parts.

They stated: “It [*the researcher’s evidence of the case study*] is particularly valuable for the way it shows the student breaking the representation-building task…This is typical of the behavior we find, both in students and in adult experts; people rarely try to take in an entire problem, but work instead to build representations for various separate pieces” (p. 73-4). What may be going on while the learner is building these pieces of representations? Goldin and Kaput (1996) contend that representations belong to “highly structured systems,” whether expressed internally or externally, “either personal and idiosyncratic or cultural and conventional” (p. 398). The representing relationship between the internal and external representation systems are not fixed, unidirectional, decontextualized, or isolated, but are “changeable”, “reversible”, complex, contextual, and interacting simultaneously (Goldin, 1998, p. 399). The theory that learners build connections between the internal and external by assigning meaning with these structured internal systems of representations has implications to attend to the representations that are observable (e.g. inscriptions, verbal, physical models, gestures, etc.). Furthermore, Davis & Maher (1990) posited that it is difficult, if not impossible, to build a solution directly from the problem statement and therefore, students should engage in “*explicit* construction of concrete representations of problem ‘input’ data” (p. 77; emphasis in original). Therefore, the study of observable strategies, heuristics, forms of reasoning, and other behavior (e.g., evidence of monitoring and control or evidence of imaging an object, idea, etc.) and the building, manipulation, modification of external representations serve as a part of the evidence for the formation of mathematical ideas.

Returning to Goldin’s (1998) reasoning about multiple construct analysis, other constructs can also strengthen the analysis of an individual’s cognitive growth. Social constructivist perspectives on learning posit that internal representations are built up over time and do not occur in isolation. Social constructivism considers both social interaction and individual meaning making as crucial to the learning of mathematical ideas and practices (Jaworski, 1994). Perspectives of socio-cultural (e.g. Cobb et al., 1997, Lampert & Cobb, 2003; Vygotsky, 1978), emergent (e.g., Cobb & Yackel, 1996), and situated cognition (e.g., Lave & Wenger, 1991) consider that doing and knowing mathematics is inherently a dynamic, social, and cultural activity (Cobb, Jaworski, Presmeg, 1996). Based on the design of the RKLS (and that it was a “laboratory” classroom study) the current research analyzed how participants worked together, what ideas were shared, what justifications were provided, what followed from researchers’ moves, or when convincing others about their solutions to a problem occurred. However it differed from other studies with a social perspective because the social constructs were used to explain for an individual’s learning progression, rather than to analyze consensus (Goldin, 1998) or the social constructs in and of themselves.

Chapter 4 Methodology

With several perspectives on learning and as a consequence of the design of the RKLS, the study traced, over time, the learning of an individual as he or she interacted with others, in a variety of settings. The current study was longitudinal because the individual's historical knowledge construction accounted later knowledge construction. The learning environments and longitudinal nature of the RKLS were designed with various opportunities to learn in variety of settings, such as in group and whole class sharing and discussion, in partner assessments, and in semi-structured interviews that were conducted beyond think-aloud protocols, in which the individual had the opportunity to share his or her thinking, revisit tasks, and at times explore a new idea. In-the-moment interaction between an individual and others was considered to explain for his or her individual learning and development. Participants that contributed ideas in the presence of the subject under study were also analyzed, supporting the notion of social constructivism. The origins and development of each study participant was traced in order to better understand the meaning of his or her contributions and then to understand how those ideas manifested to the other participants.

For the aforementioned reasons, this research study took the form of *qualitative, intrinsic, multiple, longitudinal case studies* (Stake, 1995; Shkedi, 2005) where multiple individuals were studied in detail, over time. *Intrinsic case study* refers to having an intrinsic interest in studying a particular case, with minimal goals of generalizing; in this case it was those who came together to share their ideas about related Tower Tasks in the RKLS. Even with intrinsic case studies, a researcher can make some conclusions about the themes that emerge during processes of development of certain mathematical ideas (e.g., certain types of reasoning or argumentation). The nature of the RKLS called for an intrinsic study to better understand how the individuals' mathematical ideas developed in a variety of settings and with particular laboratory classroom norms that have potential to extend to regular classrooms (see Martino and Maher, 1999).

4.1 Setting and participants

The data for this research study was situated in the longitudinal study in Harding Elementary School in Kenilworth, NJ that lasted for over two decades. Three elementary school students were the focus of this research. Video data related to three students working on various counting tasks ranging from 1990 to 1993 (see Table 4.1) were accessed and analyzed. Stephanie and Milin's activity timelines range from school years within 1990 to 1993, whereas Michelle's activity timeline ranges from school years within 1992 to 1993 (see Appendix A for each individual's timeline of activities). Participants worked individually with researchers, in dyads, in small group, and in whole class settings. Note that there were differences in the participation of particular activities across the study subjects. Researchers that facilitated or were present in the sessions under study were Alice S. Alston (R1), Carolyn A. Maher (R2), Amy Martino (R3), and classroom teacher-researcher, Mrs. O'Brian (R4). They are denoted as R1, R2, R3, and R4 throughout the dissertation report.

Table 4.1. Session timeline of student participation by session number, setting, grade level, and date of the session.

Session	*Setting*	*Grade*	*Date*	*Stephanie*	*Milin*	*Michelle*
I	Dyad	3	10/11/90	X		
II	Whole class	3	10/11/90	X	X	
III	Post-interview	3	10/11/90	X		
IV	Dyad	4	2/6/92	X	X	X
V	Whole class	4	2/6/92	X	X	X
VI	1st one-on-one interview	4	2/7/92	X	X	X
VII	2nd one-on-one interview	4	2/21/92	X	X	X
VIII	3rd one-on-one interview	4	3/6/92	X	X	
IX	Small group assessment interview	4	3/10/92	X	X	X
X	Dyad written assessment	4	6/15/92	X	X	X

XI	Individual written assessment	5	10/25/92	X	X	X
XII	Dyad, small group & whole class	5	2/26/93	X	X	X

4.2 Tasks

English (2005) asserted that combinatorial problems "facilitate the development of enumeration processes, as well as conjectures, generalizations, and systematic thinking…[and] supports children's development of beginning probability ideas" (p. 122). The task and classroom design of the RKLS enabled students to engage in such mathematical practices. Tasks were designed to be *well-defined, open-ended* (i.e., many formal justifications exist, and a solution exists), contextually *familiar* to children, and some of which were structurally or mathematically equivalent. Students worked on, or sometimes revisited, a task for at least 30 minutes; an individual or partner written assessment was given in some sessions and students recorded their solutions in most sessions. The particular design for the research called for minimal researcher intervention in finding solutions; however, researchers asked students for clarification, elaboration, to share with others, to listen to others, or to repeat ideas. The researcher design did not permit telling or showing students what to do (see Martino and Maher, 1999). This design encouraged students to explain their thinking and to convince others of the correctness or completeness of their solutions.

The following tasks (in chronological order) were given to the students:

Towers of Four Activity (Grade 3 – October 11, 1990)

Your group has two colors of Unifix Cubes. Work together and make as many different towers of four cubes as is possible using one or both color of your cubes. See if you and your partner can plan a good way to find all the four cube towers.

Extension of the Towers of Four Activity (Grade 3 – October 11, 1990)

Do you think there is more, less, or the same number of towers three cubes high than the four cubes high?

Towers of Five High Activity (Grade 4 – February 6, 1992)

Your group has two colors of Unifix Cubes for building towers. Work together and make as many different towers as you can that are five cubes high. See if you and your partner can plan a good way to find all the towers that are five cubes high and decide on a way to record what you find.

Tower Task Variations (Grade 4 interviews and assessments)

Find the total number of different towers that can be *n* cubes high selecting from *m* colors. Convince someone that they are all different and that you have found them all.

Guess My Tower Activity (Grade 5 – February 26, 1993)

You have been invited to participate in a Quiz Show and have the opportunity to win a vacation to Disney World. The game is played by choosing one of the four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be build when you select cubes of two colors, red and yellow.

You are given the following possibilities for a winning tower:

1. All cubes are exactly the same color;
2. There is only one red cube;
3. Exactly two cubes are red;
4. At least two cubes are yellow.

Q1. Which choice would you make and why would this choice be better than any of the others?

Q2. Assuming you won, you can play again for the Grand Prize which means you can take a friend to Disney World. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for a winning tower. Which choice would you make this time and why would this choice be better than any of the others?

4.3 Data sources and Validity

4.3.1 Video data

The video data include individual semi-structured interviews, small-group and dyad problem-solving sessions, and whole class sessions with Stephanie, Michelle, and Milin as participants. Metadata include transcripts of video data and the participants' written work. The interviews ranged between 30 and 120 minutes and other video data ranged between a few minutes to 120 minutes. The video recordings were captured by one to three cameras with microphones near the students. All video recordings for the data in this study were digitized from VHS, ingested[3], published to the Video Mosaic Collaborative repository (VMC), and stored in the Robert B. Davis Institute for Learning (RBDIL). New video narratives (VMCAnalytics[4]) were created using the newly ingested video data and the available video data in the repository. When video data was not applicable (e.g., individual written assessments) or if video data was missing, then other forms of data were analyzed, if available. In the latter case, if verified transcription with student written work was available, it was analyzed. In one case this occurred for the fifth-grade Session XII with Michelle and Milin; only metadata of the raw footage and video clips were found.

4.3.2 Students' Written Work

In addition to the video data (or in the case when video data was not applicable), written work and written assessments (one was administered at the end of the fourth grade and another in the beginning of the fifth grade) were analyzed. These data served an invaluable part to the analysis because it contained students' representations, models, and written justifications of the problem-solving activities under study.

4.3.2 **Validity**

Validity procedures included: validity in the *design* of a prolonged study in the field; *triangulation*; thick, rich *description*; and *peer debriefing* (Creswell & Miller, 2000). Each of these is described.

[3] The Guess My Towers raw footage of Milin and Michelle was not found in a VHS or a digitized format. Transcripts from Sran (2010) and student written work retrieved from www.videomosaic.org were used to analyze the session.

[4] see https://videomosaic.org/analytics

The design of the RKLS, including the data collection procedures, brought validity to the inferences that can or cannot be made about learning. A longitudinal study of the same participants cultivated trust and comfort (e.g. with researchers, with cameras surrounding them) and a particular culture for analysis (e.g., student-centered problem solving, minimal teacher intervention, sharing of ideas). Creswell & Miller (2000) stated that gaining credibility by “a tight and holistic case” can be built when the study is longer:

> This lens is focused on gaining a credible account by building a tight and holistic case. Being in the field over time solidifies evidence because researchers can check out the data and their hunches and compare interview data with observational data. It is not a process that is systematically established, but constructivists recognize that the longer they stay in the field, the more the pluralistic perspectives will be heard from participants and the better the understanding of the context of participant views. In practice, prolonged engagement in the field has no set duration, but ethnographers, for example, spend from 4 months to a year at a site (p. 128).

The data selected from the longitudinal study ranged from two to three years with the same participants; it also consists of multiple forms of participation to allow for a type of triangulation, which is explained next.

Multiple data sources allow for *triangulation* to ensure the validity of data analysis. Forming themes and testing hypotheses against multiple and different sources of information allows for a convergence of a more valid explanation. This occurred in three ways: *across data source*s, *theories*, and *methods* of data collection (Creswell & Miller, 2000). “Across data sources” refers to the study of three, rather than a single, participants’ development of mathematical ideas in solving the Tower Tasks, in a variety of settings, over time. The nature of the research questions, to trace three student’s development, hypothesizes that the development of each individual’s mathematical ideas could have been different and that peers, investigators, and the social culture that was present throughout the longitudinal study may have contributed to that development. Rather than studying one individual, the study of multiple participants, who came together in some sessions to share their ideas, allowed for explanation about the origins and construction processes of their ideas. "Theories" refers to taking multiple perspectives by assuming that learning does not occur in isolation. This stems from the assumption that there exists variation in learning at different points in different times within different environments. The "methods" of data collection refer to the following different forms of data (e.g., video, transcripts, written work) and the following session types: dyad and small group work sessions, whole class discussions, follow-up one-on-one interviews, a small group interview, and dyad and individual written assessments. A variety of forms, such as verbal (individual or group) and written (individual or group), allows one to corroborate claims made about the students' mathematical ideas. The variety of sessions also allowed RKLS researchers to revisit old ideas, providing further evidence to student reasoning, growth, and/or change.

This study considered "thick, rich description" as providing a detailed profile of each student's learning progression. Moreover, detailed VMCAnalytics were created to serve as a transparent trail or chronology of data collection and analyses (Creswell & Miller, 2000). These video narratives extended the use of description to include publicly available supporting data in the form of video narratives.

Lastly, "peer debriefing" refers to two independent researchers reviewing one's analysis through the peer review process required for publishing VMCAnalytics. Hence the three students' longitudinal VMCAnalytics served as analytic memos for others to view and participate in the analysis. The VMCAnalytics associated with this research study were peer reviewed, edited, and published online (see Appendix C).

4.4 Method of analysis

4.4.1 Seven phases for studying video data

This study followed the analytical model reported by Powell, Francisco, & Maher (2003) to analyze video data. The authors proposed the following nonlinear phases: attentively viewing the video data, describing the video data, identifying critical events before and after identifying pivotal mathematical strands, transcribing, constructing a storyline, and composing a narrative (p. 413).

Attentively viewing the video data entails, viewing and listening to it multiple times as a whole in order to become familiar with the contents. All video data had transcripts or had to be transcribed to enable a thorough analysis of the verbal data. All transcripts were verified by an independent researcher or graduate assistant. Verification entailed checking for accuracy by viewing and reading the transcripts as a whole at least twice and by someone other than the researcher. Furthermore, no lens is taken while doing this first phase.

An event is a connected sequence of mathematical behavior observable by utterances, actions, and inscriptions. Transcripts and other forms of data sources were then flagged for *critical events* based on the lens of the researcher and the research questions. A critical event is “a significant or contrasting change” from previous mathematical behavior (e.g., strategies, forms of reasoning) or “a conceptual leap from earlier understanding” (Powell et al., 2003, p. 416). Keeping the research questions in mind, a “critical event” was a moment in which a researcher intervened and what followed, another student proposed an idea or interacted with the subject and what followed, a strategy, argument, or supportive reasoning, a mathematical idea or behavior related to the Tower Tasks, or evidence of a change or addition to the subject's ideas about the solution to the Tower Tasks. An identification process to trace earlier strategies used and subsequent mathematical behavior was done to bring context to the critical event (Maher, 2002, p. 35). The collection of these events would serve as the evidence of the process by which a subject developed a particular idea. The trace of critical events in the past and in the future to obtain a set of related critical events that demonstrate the development of a thematic mathematical idea (e.g., Milin's development of reasoning by cases and induction) is called a pivotal mathematical strand (Kiczek, 2000; Steencken, 2001). Each event is reported in the Results chapters for each individual, displaying each student's learning progression of the thematic mathematical idea.

A methodology that combines social constructivist and cognitive perspectives is indirect in many seminal empirical studies or grounded by their data (e.g., see the collections of studies in Steffe et al., 1996). In *Theories of Mathematical Learning,* Cobb et al. (1996) called for the explicit coordination of the two theoretical perspectives. According to Cobb et al. (1996), a conclusion about an individual's cognition cannot be deduced solely from a social interactionist analysis of mathematical activity; however, it can richly inform cognitive analyses (Cobb et al., 1996, p. 5). The implication was to take several phases if one is to profile individual students while accounting for the ways in which individuals jointly participate in problem solving, argumentation, sharing of their ideas, convincing each other of the validity of ideas. The analysis consisted of three phases that included studying the data from each perspective and then a final coordination of the underlying themes or patterns.

The first phase of event identification was to separate the video and associated transcript data based on session type and then based on behaviors and other observable features (e.g., external representations) within that session. The behaviors included students' actions, strategies, external representations, explanations, and displays of justification that referred to aspects of a Tower Task or its solution. The following questions were used as the backdrop: how the student represented the problem and how the student expressed the relevant prior knowledge (Davis, 1984, p. 78). This process organized the large data set and provided preliminary insight into the themes of each student's mathematical ideas (i.e., pivotal mathematical strands). The pivotal strands served as the organization of the Results chapters. For example, the discovery of a doubling pattern in the solutions of related Tower Tasks and justification by inductive reasoning had its own long-term process, while the development of an argument by cases had a separate, although related, long-term process. This preliminary result led to multiple phases of tracing the past and future potential critical events that had factored into the development of each pivotal mathematical strand for each student.

The second phase identified social situations within a session surrounding the identified mathematical behaviors. Specifically, the situations included questioning, argumentation, explanation, persuasion, and/or reflection between study subjects and classmates and/or study subjects and researchers about Tower Tasks.

After building the descriptive narrative of the events for each individual, the third major phase was comparing longitudinally across events. Based on the research question, in a broad sense, the focus was evidence of change or no change in strategies, displays of justifications, forms of reasoning, and representations used and on surrounding contributing interactions. Situations in which no change in learning was evident were not excluded across sessions and within one session, and were grounded in the data. For example, if there was evidence of separate meanings by two or more participants in one session, interaction was traced to identify what kind of negotiation occurred and the results of that negotiation. At the level of interaction, taken-as-shared meaning may or may not emerge during the process of negotiation (Voigt, 1996, p. 34). These were indications where discord between actors and reactors may or may not occur and, consequently, may make apparent an individual's subjective knowledge on the topic.

The following example is an illustration of the identification process for a critical, the prior, and the antecedent events. In the fifth-grade Session XII Stephanie was referring to a doubling pattern that was observed for the number solutions of the Tower Tasks, but there was evidence of a missing or incomplete justification when the researcher probed for it. In Stephanie's first attempt in supporting the observed doubling pattern in the Tower Tasks, she used the already built towers of various heights to show that the number of taller towers was double the number of shorter towers. During this interaction between Stephanie and R2, the researcher intervened to ask Stephanie why the doubling pattern worked and to tell her that she had yet to justify "why." Stephanie attempted a second time, but the researcher stated that she did not include a justification for the pattern. Note that Stephanie had the opportunity to hear Milin's justification by inductive reasoning in a prior session and in the current session right before this instance. Then Matt intervened to explain his reasoning for the doubling pattern. Stephanie was silently listening, which was evidenced when she joined him in the explanation, using his reasoning and supporting the growth of towers with tower models. This instance was identified as a critical event and each of the aforementioned prior instances were identified as prior process events that contributed to Stephanie's experiences with the doubling pattern and its supportive reasoning. Moreover, a trace of prior sessions over the three school years occurred to identify what experiences she had with the doubling pattern, with a supporting inductive argument, or in general with comparisons of related Tower Tasks.. Returning to the fifth-grade instance, Stephanie had the opportunity to repeat the explanation to another group of students, which was further evidence that she had developed the idea of inductive reasoning in support of the doubling pattern. This was identified as an antecedent event. In all, a longitudinal narrative was created for Stephanie's longitudinal development of the inductive argument.

This example served to illustrate how critical events of the pivotal mathematical strand of Stephanie's justification by inductive reasoning for her observed number patterns of related Towers tasks were identified. Note that the critical event includes the *local* context within the session under study (i.e., in this example, Session XII), the past (events that explain the context of the critical event or the factors that may have played a role in the change), the present (the critical event with the conceptual leap), and the future (further evidence or confirmation that the change occurs). There was also a *global* context of development over time, or in other words, prior events of Stephanie's experience that may have contributed to the development of the particular pivotal mathematical strand. From the sessions where the participants revisited ideas and reconstructed solutions to various Tower Tasks, comparisons were made based on prior and future session events to account for learning.

4.4.2 A methodology for VMC ingestion/cataloging to support analysis through video narrative using the RUAnalytic tool

The Video Mosaic Collaborative (VMC) repository (www.videomosaic.org) houses 400+ hours of the video data from projects that includes the Rutgers-Kenilworth longitudinal study. There were over 4000+ hours of video data in the RBDIL digitized collection, some of which had not yet been ingested into the VMC for permanent storage. Part of the study was to analyze the relevant data that had not yet been studied and ingest it into the VMC (e.g. written assessments of some of the subjects in the cohort, interview data with the cohort before the “Gang of Four” video). The ingestion process and standards were determined by the Rutgers Libraries (https://rucore.libraries.rutgers.edu) team to ensure proper search of the contents in the VMC. A summary of the procedures follows.

In preparation for ingestion raw (full, unedited video), digitized data that was not in the VMC was identified, labeled, and verified that it existed in the RBDIL backup server. The video was then transcribed by a team of graduate students or the researcher of the current study. The transcripts were verified by the researcher of this study or another graduate candidate. The video data were then summarized. The description included the participants in the video, the problem, task or main questions posed, and a summary of the problem solving that occurred. A meta-data form was then completed, which included the description and all other information relevant for the search of the contents. The form was verified by the Rutgers Libraries team and the video was ingested into the VMC. Accompanying the ingested video were the student work and verified transcripts. The

Supplementary Materials provide hyperlink access to video and include transcript and student work data.

Once videos were ingested, it was possible to create video narratives (VMCAnaytics) using the RUAnalytic Tool[5]. The video narratives that were created to trace the development of ideas in study participants, displaying student written work, model building, problem solving processes, interactions with others, and attending to the representations they use and/or modify over time. The video narratives are published on the VMC after undergoing cycles of peer review and editing processes, and are publicly available, open access, and world-wide. The following VMCAnalytics and the corresponding hyperlink accompany the Results chapters:

- Chapters 5 – 7: Stephanie's Learning Progression in Reasoning by Cases to Solve Tower Tasks:
  - Part 1 of 3 (Grades 3 & 4): https://doi.org/doi:10.7282/t3-9zz5-za71
  - Part 2 of 3 (Grade 4): https://doi.org/doi:10.7282/t3-gj4x-wr97
  - Part 3 of 3 (Grade 4): https://doi.org/doi:10.7282/t3-4rx0-0p26
- Chapters 8 – 10: Stephanie's Learning Progression in Reasoning by an Inductive Argument to Solve Tower Tasks:
  - Part 1 of 3 (Grades 3 & 4): https://doi.org/doi:10.7282/t3-hfzp-8376
  - Part 2 of 3 (Grade 4): https://doi.org/doi:10.7282/t3-g20s-0d46
  - Part 3 of 3 (Grade 5): https://doi.org/doi:10.7282/t3-jv0q-p284
- Chapters 12 – 13: Milin's Learning Progression in Reasoning by Cases to Solve Tower Tasks:

[5] Retrieved from www.rucore.rutgers.edu

  - Part 1 of 2 (Grade 4): http://dx.doi.org/doi:10.7282/t3-vvfk-d817
  - Part 2 of 2 (Grade 4): http://dx.doi.org/doi:10.7282/t3-7kyt-1r45
- Chapters 14 – 15: Milin's Learning Progression in Reasoning by an Inductive Argument to Solve Tower Tasks:
  - Part 1 of 2 (Grade 4): http://dx.doi.org/doi:10.7282/t3-hrjs-jq34
  - Part 2 of 2 (Grades 4 & 5): http://dx.doi.org/doi:10.7282/t3-dwqy-mg03
- Chapter 16: Michelle's Longitudinal Problem Solving and Development of Reasoning About Tower Tasks:
  - Part 1 of 3 (Grade 4): https://doi.org/doi:10.7282/t3-c5ja-qn71
  - Part 2 of 3 (Grade 4): https://doi.org/doi:10.7282/t3-tp02-pw54
  - Part 3 of 3 (Grade 5): https://doi.org/doi:10.7282/t3-rjdv-ar64

These publicly available VMCAnalytics provide educators and researchers a video narrative to be followed along with the dissertation text. These data also served to provide transparency and insight into the analytical process because the particular video data selected, and the accompanying text written by the current study's researcher, are publicly available on the internet. These VMCAnalytics also serve to supplement teacher professional development for recognizing reasoning and the complex interplay of developing student-created or student-suggested ideas while maintaining the teacher's goals or agenda.

4.4.1 Overview of the Results

Chapters 5–16 present the results of Stephanie's, Milin's, and Michelle's work with the Tower Tasks with events from the video data and figures from the physical models built or inscriptional data written by the participants. Each event presented comes from the turns of speech that have been identified with respect to the pivotal mathematical strand. The analyses include conversational exchanges and the units of analyses involve turns of speech, relevant observable features in the video data, and student written work. Some conversational exchange extracts from a numbered transcript are presented. Note that each number indicates a turn of speech from the original transcript. A reference to a transcript line number associated with a particular turn of speech is labeled as "L#" throughout the chapters. Transcripts and student work can be found in the Supplementary Materials.

Video narratives (VMCAnalytics) accompany the events of each chapter; the titles and URLs are provided in each subsection, as well as the event numbers associated with that subsection.

Chapter 5 Results: Stephanie's early problem solving of Tower Tasks (Grades 3 & 4)

5.1 Grade 3

5.1.1 Dyad: Stephanie & Dana (Session I)

Date	October 11, 1990
Grade	3
Task	Towers (4-tall and then 3-tall)
Participants	Dyad pair: Stephanie & Dana; third-grade class
Researchers	R1, R3
VMCAnalytic	Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Events 1–2 https://doi.org/doi:10.7282/t3-9zz5-za71

On October 11, 1990, the third-grade class was organized so that students worked with a partner on the 4-tall Tower Task. Stephanie was paired with Dana. The researchers present were R2, R1, and R3. R1 presented the 4-tall Tower Task to the class verbally. R1 introduced the problem by illustrating with the red and blue Unifix cubes the definition of "tower." She demonstrated what a tower looks like for various "stories of tower" or heights and asked them to imagine that they would be building towers for "teeny tiny people" that would be 4-tall. She then verbally introduced the task, which was to find out how many different looking towers they can make that are 4-high and to convince someone that they had found them all. Then, the students were each given a paper with the statement of the problem and were asked to start working.

For every new tower Dana made, Stephanie compared it against existing towers (see Event 1). As they began the task, Stephanie and Dana built their towers separately. Stephanie ran out of blue cubes and suggested to Dana that they exchange their color cubes. When R3 asked if they were working together, Stephanie suggested to Dana that, "if we worked together then we would have more blocks and more combinations" (Clip 1 of 5; L29).

As they combined their towers, Stephanie stated to Dana: "Everything we make we have to check. Put that [*a newly built tower*] in line. Everything we make – let's make a deal – everything we make we have to check" (Clip 2 of 5; L4). Dana agreed that she would "make it" and Stephanie would "check it" (L5). Using a Guess and Check strategy, Dana built a new tower or gives Stephanie an existing tower from her own set and Stephanie compared it with each tower in their collection.

As they worked together, Stephanie was able to eliminate most of the duplicate towers from Dana's set. Note that neither Stephanie nor Dana checked Stephanie's original set for duplicates. Both Dana and Stephanie contributed ideas to generate new patterns and to check towers.

The process of developing certainty of a solution (see Event 2). This event illustrates how a *trial and error* procedure emerged for the purpose of being convinced of a solution set. While searching for a seventeenth tower, Stephanie and Dana attempted to build unique towers using a guess and check strategy. Four of the five attempts resulted in duplicate towers. When they find a seventeenth tower, Dana made a claim that they will not be able to find anymore. Stephanie suggested finding another unique tower, as illustrated by the following exchange:

17. D: I think that's the only one we're gonna get [*see Figure 5.1.1*].
18. S: Hang on Dana. We can always try more. We have to be almost positive [*makes RRRB*]. I got it, why don't we raise the blue just one…we have to raise the blue another one…now at the way top. Again, stumped! [*Dana makes a tower for her to check*]. Dana, I think we have this one. Yup, we do. Aw nuts we can't make anything. I'm almost positive.

Note that Stephanie began with RRRB and quickly glanced through the collection for duplicates. She identified the duplicate quickly. Then she suggested to "raise the blue just one; [*then*] another one; [*and*] now at the way top." She was building all towers with exactly one blue cube in a recursive manner and eliminated the candidates that were duplicates. Dana made a tower also that was a duplicate. As opposed to the previous Guess and Check procedure, Stephanie and Dana were checking to find additional unique towers. This was identified as Trial and Error because Stephanie indicated that she wanted to “try more [*patterns*]…to be almost positive” that they found all towers. After five attempts, when no unique towers were found (Stephanie’s four recursively built towers with exactly one blue cube and Dana’s tower), Stephanie concluded that she was “almost positive.”

B	R	R	B	R	R	R	B	B	B	B	R	B	R	R	R	B
B	R	R	B	B	R	B	R	R	B	R	B	B	B	R	R	R
R	B	R	B	R	B	B	R	B	R	B	R	B	B	R	B	R
R	B	R	B	B	R	B	R	B	B	R	R	R	R	B	R	B

Figure 5.1.1. Stephanie and Dana’s combined 17 towers, 4-tall.

Recognition of color opposite and inverse patterns. Twice during their *checking* phase, Dana looked for the color opposite tower when Stephanie built a new tower (Clip 3 of 5; L33-6; Clip 4 of 5; L18-9). For example, in one instance Stephanie referred to the tower BRRB and Dana asked her, “But do we have a red, blue, blue, red [RBBR]?” (Clip 3 of 5; L35). There is no verbal or physical evidence (because they were not all organized in color opposite pairs) whether the students checked all towers using the color opposite relationship. However, in several instances, there was evidence of awareness of pair relationships. In one instance, Stephanie related a pair of towers (RRRB and BRRR) as “cousins” (Clip 3 of 5; L38), which in this study are two towers that were *inverses* of each other. An inverse tower pair consists of two different towers that are duplicates when one is rotated vertically to match the other. R1 suggested Stephanie meant “cousins” to be the two color opposite towers (RBBB and BRRR). Stephanie indicated recognition of this relationship also: “Oh yeah…[*the two towers*] have sort of the same pattern: red one at the top and blue one on the bottom. And blue one at the top and red one at the bottom" (L43). When the class shared their ideas about opposite towers, Stephanie presented an example of two towers that were color opposites: "We have like a pattern, red, blue, red, blue, and then we have a pattern that's like blue, red, blue, red" (Clip 1 of 6; L17-8). Note that these towers are also inverses; however, Stephanie was responding to the R1's request to give an example of an opposite tower.

Certainty of 17 towers as a solution. R3 observed two trials in the previous event, where the girls obtained the seventeenth tower and then reported this solution. The researcher probed about their solution, asking, "Is every one of these different?" (L29) to which both Stephanie and Dana responded yes with confidence in their voices. When questioned how they were certain, Dana responded with, "Yeah, we build them and then checked them like this" [*she picked up a duplicate, put it across the build towers, and put it back in the eliminated set*] and Stephanie responded with, "Cause Dana built them, and I checked them" (L31-3). To the questions, "How can you be sure that you haven't made any of them twice, or that you have got them all? Is there a way you could be sure?" Stephanie offered the following strategy (L35):

> Well there is a way. We can take one. Like say we could take this one, this red with the blue on the bottom [*RRRB*]. And we could go and we could compare it to every one. And the ones that don't match, push back.

Stephanie responded to the question of duplication. The researcher asked them to double check, so Stephanie began to check. Dana exclaimed her certainty twice to Stephanie and then Stephanie stated, "Seventeen! I double-checked every single one" (L42-3). Although Stephanie said she would compare a tower to every other tower, in the video she only checked the last tower in the row.

Stephanie's attention to a counterexample. R1 questioned the girls about how many towers they had and if they saw any of them that were the same as each other (L1-3). Stephanie told the researcher that they had seventeen towers and demonstrated how she compared one tower by placing it against each tower in the set. The researcher picked up two towers, RRBR, from their set and silently showed them to Stephanie. Stephanie, smiling and slanting her head down, called for Dana's attention, stating the following: "I only checked one. [*Whispers inaudibly and Dana asserts "sixteen"*]. Sixteen. Let me check another one" (L4-8). Dana continued to write her original reasoning for the solution after changing the number solution from 17 to 16.

After the duplicate tower was presented, the R1 asked how they built the 16 towers. Stephanie named their process of building random towers as "making patterns" (L14). When she questioned what kind of patterns, Dana showed RBRR and Stephanie showed BRBR. Dana added her reasoning: "Because we used every single block...and we had a lot of them [*towers*]…And the ones that we had double, we would take one and if we had the double, we would take away and eliminate it" (L20). Dana provided this argument for the new solution of 16 and it was supported by the model representations and the process by which they were built. The models consisted, first, the two sets of towers that she and Stephanie built separately and, second, the towers that were created after the accumulation of new Unifix cubes from the duplicate towers, which she referred to as "a lot of them." Even when the researcher mentioned that they still had cubes available, Dana explained that "these [*cubes*] were the duplicates" (L25).

Stephanie concluded that she needed to check each tower again. She did not provide an answer (e.g., a claim, an argument, etc.) to the researcher's question. Rather, she suggested a method to check each tower to verify the new solution: “We take the first one [*tower in the model to her right*] and we check, and we put it back in its spot. Until we get down to the blue, red, red, blue [*the last tower in the model to her left*]. We could do that” (L33). Thus, after being presented with a counterexample to the argument that the 17 towers were verified by a method of “double-checking every one” [*tower*] (L4), Stephanie portrayed uncertainty about the new solution because she stated, “I don’t know if there are any more that match” (L19). As Stephanie was checking, Dana insisted, “We’re done Steph. Okay?” (L44) and Stephanie responded, “I’m just checking, Dana” (L45).

Trial and error as reasoning for the certainty of a complete solution. In this event Stephanie and Dana write their solution and supporting reasoning to the 4-tall Tower Task. After Stephanie completed her check for duplicates (a complete check of each tower), Stephanie wrote their solution of 16 towers and a supporting explanation. Their justification for their solution included that they used all (or most) of the given cubes to create “patterns” of towers (not specifying any method for the creation of each tower pattern) and that they eliminated the ones that matched by checking the towers against each other. When R3 asked, “Stephanie, what makes you so sure that you got everything?” Stephanie responded that she did not know, and Dana concluded, “We just checked it. Cause we used all of our blocks and then we had matches and the ones that matched we took one of them that matched, and we eliminated them” (L7-9). Dana responded once more that she was sure they did not miss any even after the researcher probed again (L13). Stephanie suggested that if she would “build one more” tower (L14), she and Dana could check if it matched any tower from their solution set. Thus, Stephanie tried to build one more (to verify that no new patterned tower could be made or to find a counterexample of a new pattern). Dana pointed out that they have the tower Stephanie built and its color opposite. Stephanie concluded, “I don’t think we can make another one. I really and truly don’t” (L20). The combination of multiple trials to find a new tower and the strategy of double-checking to identify duplicates consisted of their reasoning in their written and verbal responses.

The researcher asked them to organize the towers in “such a way so that when we share, it shows how you knew you had all of them” (L32). Stephanie indicated she did not understand what the researcher meant. The researcher rephrased her request as a question, “well, what convinced you that you had them all?” (L34). Stephanie responded, “we double-checked” (L35). The researcher accepted this and told the girls to record their result of sixteen towers in the same organization that they had originally built them (see Figure 5.2.1.1).

5.1.2 **Whole class (Session II)**

Date	October 12, 1990
Grade	3
Task	Tower (4-tall and then 3-tall)
Participants	Dyad pairs: Stephanie & Dana; Milin & Lauren; third-grade class
Researchers	R1, R3
VMCAnalytic	n/a

The solution as it’s related to the color opposite relationship of the towers. R1 led the group sharing session the following day on October 12, 1990. She asked the class for their results for different towers 4-tall. Most groups claimed that there were 16 towers. Jeff, on the other hand, thought the solution may be seventeen or nineteen, and that he and his partner, Brian, gave up after trying to find a different seventeenth tower (L7-8; L31-8). Jamie, from another group, claimed that if the total were 17, “then there must have been some of the patterns that you already have” [*she is standing next to her picture representation of the tower organized in color opposite pairs*; L10-4). This followed with a conversation about color opposite towers. The researcher asked several students (including Stephanie and Jeff) for explanations or different examples of opposites. For instance, Stephanie shared the example of BRBR and RBRB (which must be noted that this pair has both a color opposite and an inverse relationship; recall she called the latter "cousins"). In response to R1's query about even and odd numbers, Michael explained the difference in terms of sharing four versus three pieces of candy. When the researcher asked if it was important to have an even number of towers, Jamie explained that without the color opposites there would only be eight towers. She showed this by removing each of the eight color opposite towers from her pairs of towers on her desk (L42-4). This idea is revoiced by R1 in the form of a question, "So the opposites is what made the 'sixteen'? If you didn't have opposites, you would have only had eight?" (L45). Jamie agreed. The discussion that followed involved a comparison of the number of 4-tall towers with the number of 3-tall towers. The events of this discussion are presented in Chapter 8.1.1.

5.1.3 Interview (Session III)

Date	October 12, 1990
Grade	3
Task	Tower (4-tall and 3-tall)
Participants	Stephanie
Researcher	R3
VMCAnalytic	Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Event 3 https://doi.org/doi:10.7282/t3-9zz5-za71

Stephanie explains how double-checking provided certainty for the solution (see Event 3). Stephanie was interviewed by R3 following the class session on the same day. In this short clip Stephanie explained that she and Dana had to check their solution "a couple times" and that whenever they attempted to make a new tower, they found it was a duplicate of a tower in their solution set. Stephanie stated, "Well, we had to check a couple of times and we tried to make some different ones and we were checking and checking and they all came out the same" (L8). Recall, she was referring to the Trial and Error strategy they used to develop certainty of their solution.

5.2 Summary of Grade 3

Pattern recognition and use of strategies. In the third-grade dyad session (I), Stephanie's early class problem-solving began with non-systematic, exhaustive enumeration by a Guess and Check strategy to finding single towers. This process entailed building tower models using Unifix cubes. The availability of cubes was crucial to the existence and generation of more towers, as noted in two instances by Stephanie or Dana.

During partner work and whole class discussion, Stephanie named two inverse towers as "cousins" and recognized color opposites as towers that could also be cousins. These relationships were not part of Stephanie's problem-solving strategies or reasoning.

Representations. Physical models, drawings of the towers, and written explanations consisted of the representations of a solution. Towers were organized in a line in the order of which the towers were generated. Some towers were next to their color opposite towers, while others may have been random or part of an internal decision-making process that was not observable.

Displays of justification. Interactions between Stephanie and Dana or Stephanie and the researchers contributed to Stephanie adopting a Trial and Error strategy to verify the solution. Trial and Error arose as a strategy to gain certainty of *exhaustion* of all towers and to check that each tower was different from the rest of the towers. This served as part of the problem-solving process and the supportive reasoning in the written and post-interview justification of Stephanie's solution of 16 towers, 4-tall. For example, after

Dana claimed they found their solution, Stephanie disagreed and tested for more (missing) towers. During this verification, Stephanie recursively enumerated all towers with exactly one blue cube in every position (i.e., the recursive elevator strategy) to check if they existed within their solution set. After results from the test trials produced only errors (i.e., duplicate towers) Stephanie concluded their tower collection was verified. Trial and error strategy also arose to check that each tower was different. For example, during one of the trials, R1 showed Stephanie two duplicate towers. This served as a counter example to both Stephanie and Dana's claim that the solution was 17 towers. It also led Stephanie to exhaustively "double-check" each tower against the other towers, even upon Dana's insistence that they were finished. Beyond trial and error and elevator recursion during verification, no other systematic procedure during enumeration and verification was observable.

Reasoning and argumentation. For Stephanie organizing the towers for the purpose of conviction may have been the procedure and results of the trial and error. For example, when drawing their solution and writing their supportive reasoning Stephanie indicated not understanding what R3 meant when she asked the girls to organize the towers in the way that convinced them of their solution. Stephanie responded that they "double-checked." Interestingly, at the end of Session I when they were asked to organize in a way that gave them conviction of their solution, Stephanie indicated she did not understand. Organization may not have been a factor in conviction as was the physical action and results of Trial and Error.

5.3 Grade 4

5.3.1 Dyad: Stephanie & Dana (Session IV)

Date	February 6, 1992

Grade	4
Task	Towers (5-tall)
Participants	Dyad pair: Stephanie & Dana; fourth-grade class
Researchers	R2, R3, R1
VMCAnalytic	Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Events 4-5 https://doi.org/doi:10.7282/t3-9zz5-za71

On February 6, 1992 the fourth-grade class was divided into pairs to work on another Tower Task. Stephanie was paired with Dana, Michelle with Jeff, and Milin with Michael. The researchers present were R2, R1, and R3. A fourth-grade teacher and school principal were present as observers. At the beginning of the session, there were two sheets of paper posted on the board with the following statement:

> Building Towers: Your group has two colors of Unifix cubes for building towers. Work together and make as many different towers as you can that are five cubes high. See if you and your partner can plan a good way to find all the towers that are five cubes high and decide a way to record what you find.

R2 introduced the problem to the class; children were seated in pairs and had sets of Unifix cubes colored red and yellow at their desks. Before the students began their work on the problem, the researcher gave instructions and the class as a whole came to a consensus as to what was allowed when making different towers and what was not. The students in the class had the opportunity to work in pairs for about 40 minutes and then the entire class came together for a sharing session, which lasted for about 50 minutes, to include reports of findings from different groups.

Strategy of building and organizing towers by color opposite pair attributes. Stephanie and Dana started the task by building the case of the "easiest" towers, the towers with only one color. They generated towers of various patterns thereafter. For example, Stephanie suggested making a tower with "one [*red cube*] on the top" (RYYYY) and that Dana should make the tower with exactly "one yellow [*cube*] on the top" (YRRRR). Then, Stephanie suggested making a tower with one red cube in the middle (YYRYY) and asked Dana to make a tower with one yellow cube in the middle. Stephanie and Dana rapidly created towers and their color opposite towers based on ideas emerging from both girls. As one idea for a tower emerged, the other student made the color opposite tower. For example, when Dana makes the tower of alternating color cubes, Stephanie responded, “Tell me it so I can do the opposite. [*Dana shows the tower she makes*]. Oh ok, and I’ll do the opposite with red on top” (L63-7). Stephanie expressed pattern ideas by either specifying the positions of a color (e.g., “red in the middle…but no red on the bottom”) or specifying the number of positions to be taken up by a color without saying the color (e.g., “one on top,” “two in the middle,” “two bottom,” or “two on top and bottom”). The difference in this identification was that the former specifies the attribute of *one* tower, whereas the latter specifies the attribute of *a pair* of towers. In both cases Dana responded by building the color opposite tower.

They continued to build towers in this fashion. Stephanie asked Dana if they built the tower with “two in the middle”? (L95). She scanned the towers and noted that they did not make the pair yet and told Dana to make a “two in the middle” tower while she made the opposite. Dana built YRRYY and Stephanie built the color opposite (towers #15 and #16, respectively, in Figure 5.4.1). Stephanie offered a new idea: “one [*color*] on top, one [*of the same color*] in the middle, but not one [*of the same color*] on bottom.” She told Dana, “You do red and I’ll do the opposite” (L117). Dana built RYRYY and Stephanie built the opposite (towers #18 and 17, respectively). Note that they later built duplicates of this pair (towers #17, 18, 21, 22, 25, 26). In some instances, Stephanie announced the positions that the other color took on (e.g., “one skip”). Each child provided ideas for building towers very rapidly. Stephanie asked Dana not to “make another idea” until she caught up (L83).

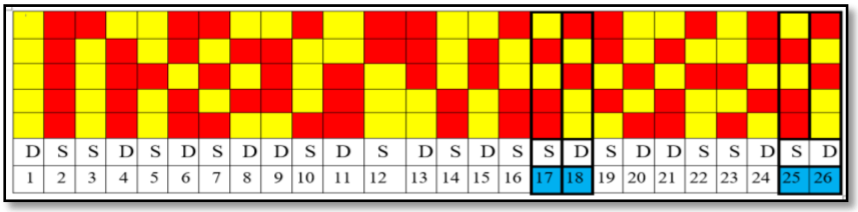


Figure 5.3.1. Stephanie (S) and Dana’s (D) 5-tall towers.

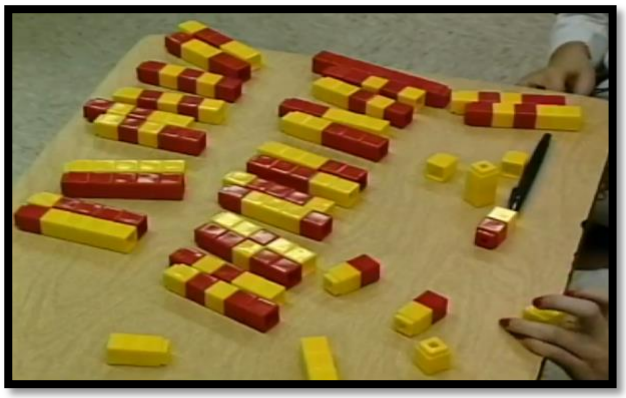


Figure 5.3.2. Stephanie and Dana’s towers paired by opposites.

As the number of towers built increased, the identification of patterns by individual towers or pair attributes began to generate duplicates that the girls did not immediately notice. The positions were described as “top,” “middle,” or “bottom,” where *top* referred to either the fifth level or both the fourth and fifth level (if she specified the number of cubes on top), *middle* refers to the third level or the middle position, and *bottom* refers to the first level or both the second and first levels. In one instance, the identified attribute focused on the positions of one or the other color. Recall Stephanie originally described the pairs RYRYY and YRYRR as, “one on top, one in the middle, but not on the bottom” (towers #17 and #18). This description focused on the position of two red (or yellow) cubes located on the top and in the middle and nowhere else. In another instance she describes the pair as “two bottom [*of the same color*], then skip one [*the middle*], and another one [*of that same color*].” This description referred to the positions of the three *other color* cubes that are located at the bottom and in the fourth position of RYRYY and its opposite YRYRR (towers #25 and #26). As the patterns of a tower became more complex (e.g., same colored cubes separated), the equivalence between towers with exactly X number of cubes of a color and towers with exactly 5–X number of cubes of the other color went unnoticed (e.g., towers with exactly two yellow cubes is equivalent to towers with exactly three red cubes).

Emergence of an inverse relationship. Dana recognized an *inverse* relationship between two towers. An *inverse* tower according to Dana was a tower that had the exact pattern of another tower when it is turned upside down (see Figure 5.4.3). In other words, the positions of the top two colors and the bottom two colors are reversed (the middle color stays in the middle). Dana called these towers “duplicates.” Dana recognized the relationship as an additional one to the color opposite relationship. She provided examples, showing: “These two match [*YYYRR and RRYYY*] and these two match [*YYRRR and RRRYY*]; this goes like that [*reverses YYYRR to show it matches RRYYY*].” (L132).

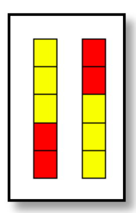


Figure 5.3.3. Dana’s example of an inverse relationship between towers.

Stephanie appeared to reject this idea for organizing the towers: “No, but Dana, the yellows are supposed to be two on the top [*as YYRRR*]. Look, these [*YYRRR*, *RRYYY*] go together, two on the top, three on the bottom, and these [*YYYRR and RRRYY*] go together, three on the top, two on the bottom.” Dana insisted that the inverted pairs “also go together" and responded, "I know," to acknowledge Stephanie's color opposite pairing (L137).

Using the idea of the inverse relationship to find missing tower pairs (see Event 4). Earlier, Dana defined an inverse relationship. Although Stephanie at first did not take up Dana's approach as a way to organize the towers, she later used it as a strategy to generate three new towers, namely the color opposite of the tower and two inverses to the original pair. Stephanie identified this generation as "reverse it" or the inverse tower as a "duplicate" (e.g., L159 or L173). Stephanie offers a new idea (YYRYR): "Did we ever do this? Dana, one on the bottom and one next to the bottom [*two on the bottom floors*] and just one here [*the same color on the fourth floor*]…and then we will just <u>reverse</u> it the other way so the two yellows on the top" [*referring to the inverse of RYRYY, namely YYRYR*] (L157-61). In Figure 5.3.4, from Stephanie's tower YYRYR, Dana built the opposite (on the right) and then flipped the pair upside down to find the inverse towers (first two on the left). Note that the towers YRYRR and RYRYY (towers #21 and 22) were duplicates (of towers #17 and 18) but the inverse pairs, RRYRY and YYRYR (towers #19 and 20), were new combinations. Stephanie and Dana continued to use this strategy to generate new towers, without realizing that there were some duplications.

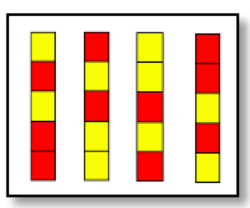


Figure 5.3.4. Stephanie and Dana's color opposite and inverse towers.

Explaining the relationship as a strategy to find missing towers. At one instance, Stephanie explained how to check towers using the relationship of inverse towers to R3: "That's how you check the duplicate pair, which is the same if you turn it upside down"

(L279). In another instance the girls explained their strategies for building new towers to R2. Stephanie called the inverse pairs "upside down duplicates" and compared it to another strategy: "Ok we take the design [*of a tower*] and, instead of just making new designs, we take one and we turn it upside down and we make the same design upside down" (L262). Dana explained it as follows: "We take one of them and we turn it upside down and as we see we get it – the same thing – but then we turn this back up right, it's different" (L260).

Using the two pair relationships as a strategy to organize the towers and eliminate duplicates. Earlier R2 asked them whether they believed they are done, given their 28 towers. Stephanie acknowledged that they needed to check their towers for duplicates. Stephanie suggested a method to group by color opposites and inverses. Although they overheard that Jeff and Michelle claimed to have 30 towers, Stephanie and Dana still pursued to check their current towers for duplicates (L306-320). Stephanie noticed that the towers "that won't have a group is the easy ones" [*all red- and all yellow-colored towers*] (L306; see Figure 5.3.5). After some struggle to find the two-pair relationship, they modified their search to organize only by color opposite pairs. They return to the strategy of organizing by the two pair relationships after being asked to prepare a presentation of their strategies. Stephanie suggested to Dana that they show the class their groups of opposite and inverse pairs. This process is described and illustrated next with several accompanying figures.

Stephanie organized the first set of towers by two relationships: opposite towers with exactly one of a color at the bottom floor and their inverse pairs (e.g., YYYYR and RYYYY; see Figure 5.3.5).

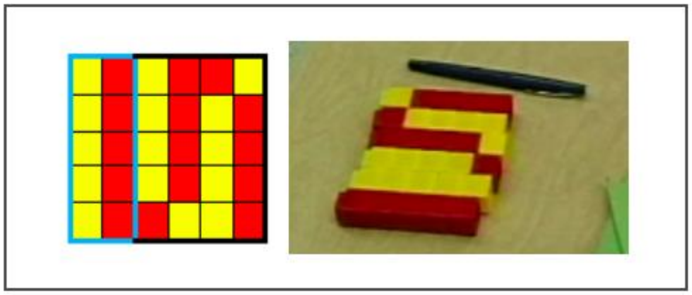


Figure 5.3.5. Dana and Stephanie's first group of opposite and inverse towers.

The next set that Stephanie organized consists of towers with exactly two of a color adjacent to each other at the top two positions and their inverses (see the circled set in Figure 5.3.6). Together they work to find the groups of opposites and inverses. As seen in the following excerpt Stephanie describes a way to check for inverses that requires turning the new tower over to compare that its pattern is the same as another tower. After Dana checks an incorrect tower she finds the correct one.

329. S: I will put it [*YYRRR and its opposite*] at the way end ok? There's a group [*of opposites*]. Where's its duplicate [*inverse pair*]?
320. D: Its duplicate? Let me see, here.
331. S: Does it have the duplicate? Is that the duplicate? That's not the duplicate. Turn it over it would be the same, but it's different.
332. D: Here you go [*gives her RRRYY and its opposite*]. Here's the duplicate of this.
333. S: Right. Here's the duplicate.

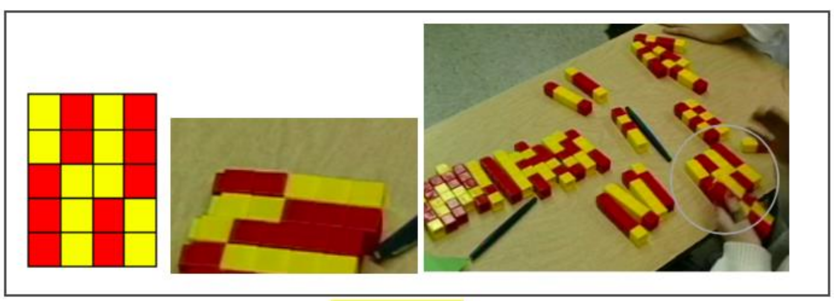


Figure 5.3.6. Dana and Stephanie's second group of color opposite and inverse towers.

The girls returned a second time to investigate an earlier attempt to generate a set of towers as indicated in Figure 5.4.7 (see YRYRR; RYRYY; RRYRY; YYRYR). Notice that they were successful not only in finding the set of towers, but also in eliminating a pair of duplicates.

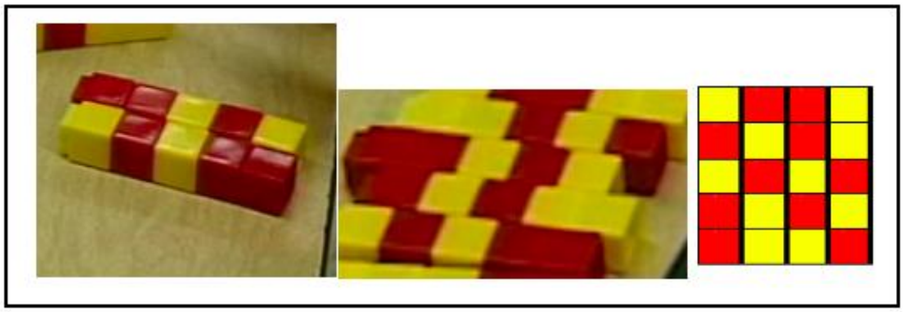


Figure 5.3.7. Dana and Stephanie's third group of opposite and inverse towers.

A fourth set had already existed before the girls chose not to pursue this method. In Figure 5.3.8 the left-most circled group was originally found in the first iteration of this organization method. It consisted of two of a color together on the second and third floors and their inverses. Without realizing that this set was already identified, Stephanie searched again for this set with the two duplicate towers of YRRYY and RYYRR. The rectangular border of Figure 5.3.8 consists of duplicate towers of the left-most circled group.

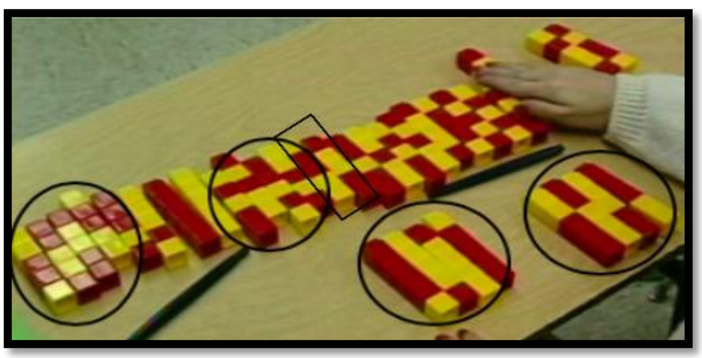


Figure 5.3.8. Towers that were grouped by opposites and inverses.

Then they continued to pursue finding towers by a color opposite relationship only. In fact, Stephanie restarted the process by checking each tower to ensure it was different from the collection and pairing it by its opposite. Some duplicates were found during their earlier organization by "groups" of opposite and inverse pairs. In Figure 5.3.9

duplicate towers boldened in blue. Others were found by comparing each tower against the whole collection and either eliminating the duplicate or locating and grouping with the opposite tower. Stephanie explained the process of ensuring uniqueness for each tower to R3: "I am looking for its pair [*YRYYR*]. First I check it [*holds the tower against the set of towers*] and then I am looking for its pair" [*opposite tower*].

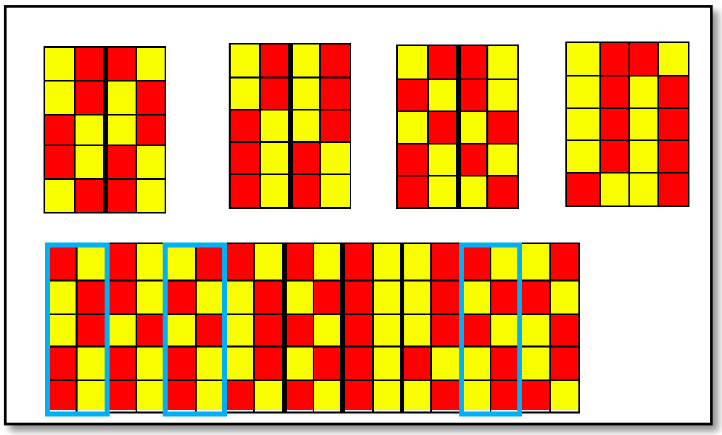


Figure 5.3.9. Final set of 34 towers after second iteration of organization strategy.

Reasoning about the solution of 28 versus 32. Stephanie and Dana found 28 towers. Stephanie described a strategy of Trial and Error in search of the four missing towers after Dana heard other groups had 32 towers. She explained to R3 that their quest resulted in duplication and their final solution was 28 towers. She supported her claim by referring to the duplication and that other groups might have duplicates. In other words, Stephanie held onto her solution because the Trial and Error tests led to duplication regardless of other groups' claims.

The solution of the number of towers has a limit but gaining certainty is not possible (see Event 5). In a conversation with R3 and then with R2 Stephanie showed evidence of uncertainty of finding all possible towers. First, she described being certain that a limit to the Tower Task existed and depended on how many colors were available and how tall were the towers. At this moment the total of 28 was her limit. However, she then offered the following possibility: “You have to think there are always more.” Stephanie explained to R2 and R3 in different instances that she could never be certain of the Tower solution (even though there was a limit) for two reasons: 1) Another person could create a new pattern, given a larger availability of red and yellow cubes and, 2) This problem was different from the Shirts and Pants Task (3rd grade task), which allowed for one to “imagine” the different outfits in their minds. For example, she stated (L631-3):

> There is no way, you could not go into your head and say I can figure this out in my head, you couldn’t. You always have to think this isn’t like the problem you have us like there were five shirts and four pairs of pants where you could go in your head and figure it out…because you could buy like, the biggest, you could have reds and yellows all over this room and people could still get ideas. You would not know that one person could have forty-four and other person could have, be having, and would be having fifty-eight and still going for more because they, you don’t know until you are finished, until you are absolutely positively sure.

Although these questions arose, Stephanie and Dana continued to search for the four missing towers after the class shared their numerical solutions. During the time the class explored one group’s odd number solution, Stephanie and Dana continued to work on their solution. They found four more towers, resulting in a total of 32 unique towers.

5.3.2 Whole class (Session V)

Date	February 6, 1992
Grade	4
Task	Towers (5-tall)
Participants	*Stephanie* & Dana, Jeff & *Michelle*, and *Milin* & Michael; fourth-grade class
Researchers	R2
VMCAnalytic	Stephanie’s Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Events 5–7 https://doi.org/doi:10.7282/t3-9zz5-za71

Opposites make the difference of an even solution. On February 6, 1992 the class shared their solutions to the 5-tall Tower Task. While students are working, R2 announced to the class to prepare a presentation of their findings. She also asked each group to share their numerical totals. Stephanie and Dana reported 28; Robert and Sebastian reported 35; others reported 32 including Milin's and Michelle's groups. R2 asked the class if it is possible to have an odd number of towers. Some students answered affirmatively whereas most argue it is not possible. Michael claimed each tower has a color opposite. Recall, as a third grader he used similar reasoning for the 3-tall Tower Task. Another student claims an odd number of towers is possible because a person has the choice to make or not make the color opposite. Students are invited to check Robert's set for duplicates. Some groups decide to continue to work on their own tower collections and to check for duplicates (e.g., Michelle and Jeff) or to find missing towers (e.g., Stephanie and Dana).

Duplication exists in different staircase patterns. The students are invited to study Ankur's patterns (see Figure 5.3.10; adapted from Sran (2010)). Ankur and his partner showed a yellow *staircase* pattern, which consists of six towers with no, one, two, three, four and five yellow cubes. The opposite color staircase pattern included four towers with exactly one, two, three, and four red cubes. Note that Ankur's yellow staircase pattern included the tower with five reds and five yellows while the opposite staircase did not include either tower to account for duplication in multiple staircase patterns. Ankur explained that they did not include a tower with all red cubes at the end of the red staircase pattern because it existed in the beginning of the yellow staircase pattern.

Figure 5.3.10. Ankur's staircase arrangements

R2 called attention to Ankur's *elevator* patterns that consists of towers with exactly one red cube in each position and towers with exactly one yellow in each position except for the top and bottom floors. She revoiced Ankur's earlier reasoning that he accounted for duplicates by excluding the aforementioned towers since they exist in the staircase patterns. She also noted that Jeff and other teams used similar elevator and staircase strategies while noticing duplicates within these patterns.

Exploration of exactly two red cubes "together." R2 asked students to find the towers with exactly two reds or two yellows, to study what they looked like, and to find their total. Stephanie collected some of these towers. R2 encouraged her to find all towers with exactly two reds cubes adjacent to each other. Stephanie acknowledged that some of her towers with exactly two red cubes have separation and began to search for the combinations with two adjacent red cubes. R2 challenged the class to find a convincing way to account for all possibilities of exactly two red cubes together without missing any towers (L331-3):

> You have exactly two reds together?...Convince me that you have to have all of them and there are no more…But you have to convince me by looking at a pattern that you have not missed any…Alex showed me this. Now these two reds are both on the bottom [*1st and 2nd*] floors, right? Is that right? So, I can keep track of this in

> my head easily these two reds are in the bottom floor. And he showed me that when we look at these two reds next to them they are on the second and third floor right? You see that? The first two reds are on the first and second floor these two are on the second and third floor. What is another possibility when I have these two reds together? Any ideas? Ankur?

Notice R2 prompted the class to convince her by “a pattern” and then she showed two examples of towers with two red cubes taking on positions recursively (i.e., “first and second floor…second and third floor”). She emphasized that she could “keep track of this” in her mind and challenged the class to find other possibilities. As a class they identified four towers with two red cubes together organized in an elevator pattern. Ankur stated the possibility with the red cubes on the third and fourth floors and Alex said fourth and fifth floors. R2 concluded, “Now we have here four towers with two reds together” (L339). She combined all towers in an elevator pattern to show the class (see Figure 5.4.11).

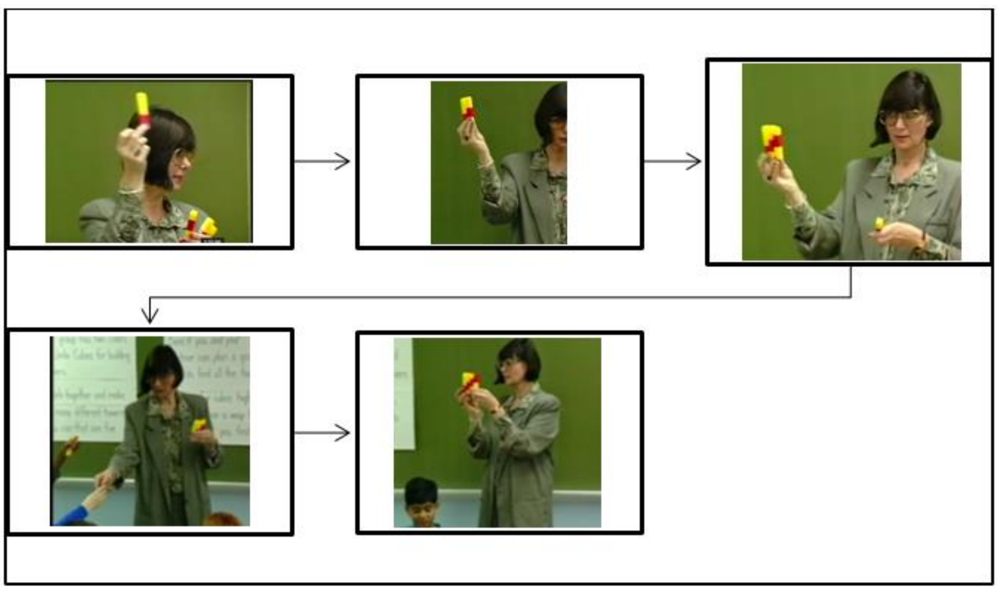


Figure 5.3.11. Four towers with two adjacent reds.

Direct reasoning for the justification of 10 towers with two red cubes “together” and “separated” by at least one yellow (see Event 6). The researcher pointed out that the towers with red cubes together were not the only towers with exactly two red cubes. R2 posed to the class a series of questions to which they replied in unison aloud. For example, R2 asked, “Do the two reds always have to be together?” The class responded, “No.” Then, she asked, “Can they be separated by a floor?... by two floors?...by three floors?...by four floors?” (L339-49). The students answered affirmatively except for the latter case, which produced different responses. R2 then challenged them to make her a tower “that is five high where there are two reds separated by four floors.” Michael and Ankur disconfirmed the possibility and stated that it is only possible when the tower was 6-tall.

Observing the footage of Stephanie and Dana, Stephanie was nodding or softly replying affirmatively when R2 prompted the class about how the red cubes could be separated in a 5-tall tower. When she asked about separation of four floors Stephanie replied, “No!” Together as a class they discussed why separation by four or more floors was not possible. As a class they completed the cases for exactly two red cubes together and two red cubes separated by at least one yellow cube. R2 claimed that they generated 10 total towers with exactly two reds and asked if they are sure there were no more. Stephanie summarized for the class how many towers existed with two red cubes (L395):

> I think so, because with the two [*reds*] together, you can make four [*gestures four fingers*]; with one [*yellow*] in between, you can make three [*gestures with three fingers*]*;* with two [*yellows*] in between, you can make two; with three [*yellows*] in between, you can make one. But you can’t make four [*yellows*] in between or five [*yellows*] in between or anything else because you can only use five blocks.

Notice Stephanie reasoned that it was not possible to have four or more yellow cubes separating the red cubes in a tower because there were only five cubes in total.

Imagining towers by using the color opposite and elevator strategy (see Event 7). Following Stephanie's summary, R2 asked if there was a set of towers that they could "imagine" in their minds and know its total (L398):

> Now that you have all possible ways for building your towers there are ten with exactly two reds, what do you automatically know the answer to? Look at the hands going up. You know some more towers without doing any building; you see them in your mind don't you? The minute you see them in your mind you didn't even have to make them. What do you see in your mind?

Some students offered that they could count the opposite towers with exactly two yellow cubes. In response to the total for the case of towers with exactly two of a color, the students concluded they had accounted for 20 towers thus far.

Next R2 prompted the class to *imagine* the towers with exactly one red and state the total. Stephanie responded, showing the elevator pattern (see Figure 5.3.12): "With exactly one red, five towers high? You can build [*pauses*], five."

Figure 5.3.12. Stephanie organizes all towers with exactly one red cube in an elevator pattern

Students calculated five towers and five more to account for the opposites. R2 asked what was remaining to make 32. Robert pointed out that it is the tower with "all red" and its opposite. The session ended with the class calculating a total of 32 towers 5-tall when selecting from two colors.

5.4 Summary of Grade 4 problem solving

Representations. Stephanie and Dana used physical towers to build and support their arguments about their number solution. Their final solution of 32 towers was organized by opposite pairs.

Pattern recognition and use of strategies. Stephanie and Dana applied a composite operation to generate three new towers from one tower. They used the operation as a method to organize tower pairs. Stephanie and Dana systematically generated and organized these sets of four towers after Dana recognized the inverse relationship as another relationship, while comparing opposite towers. This advancement led to the elimination of some duplicates and the emergence of others.

After enumeration slowed down, students used Trial and Error to search for missing towers (or verify that there were no missing towers) and to check whether each tower was different from the others. Stephanie explained that they checked each tower against the others and eliminated duplicates.

During whole class sharing the students' various local strategies and construction of the following cases: the towers with exactly two red cubes together, exactly two red cubes apart, the color opposite towers, exactly one red cube, the color opposite towers, and the single-colored towers. The elevator and opposite strategies were applied to generate and organize the towers with exactly one and exactly two red cubes. Controlling for duplicates was exemplified in this class discussion. For example, the researcher introduced Ankur's use of the elevator, staircase, and color opposite methods for particular cases. Jeff and Ankur noted that duplicates occurred when developing each of these patterns of towers. Ankur stated that he controlled for this by not including repeated towers when applying the color opposite pattern.

Forms of reasoning in support of the solution. Supporting reasoning for the *exhaustion* of the solution was the result of verification by Trial and Error that no new towers could be found (similar to Stephanie's third grade reasoning). Stephanie indicated that she did not find conviction in the contradictory totals of 28 (the number total she found after eliminating duplicates) and 32 (the number total Dana overheard from Jeff and Michelle). Stephanie further explained her reasoning that *there existed a limit*, but she thought certainty was not possible for two reasons: 1) others could always claim to find another pattern (conviction for the exhaustion of enumeration) and 2) the Tower Task was not similar to the Shirts and Pants Task where one could *imagine* the different outfits in their mind (complexity of the task). An interpretation of this reasoning is that the local strategies of color opposites and inverses did not provide Stephanie with a global organization to gain conviction. In addition, when Dana and Stephanie found the four missing towers (after a period of utilizing Guess and Check), this new data may have been further evidence for Stephanie that conviction of a solution may not have existed.

In whole class discussion Stephanie and others had an opportunity to share their thinking about the possibilities of a particular case and to be "convinced" (R2's language to the students). They also had the opportunity to use spatial reasoning (in application of the opposite and elevator strategies) to "imagine" without building other cases of towers.

Chapter 6 Results: Stephanie's development of the case argument to justify solutions to Tower Tasks in post-interviews (Grade 4)

6.1 First interview (Session VI)

Date	February 7, 1992
Grade	4
Task	Towers (5-tall)
Participants	Stephanie
Researchers	R2
VMCAnalytic	Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Events 8–10 https://doi.org/doi:10.7282/t3-9zz5-za71

This one-on-one interview between R2 and Stephanie was a 53-minute discussion that occurred in the fourth grade on the day after Stephanie and her partner, Dana, worked in their classroom on building towers of height five, selecting from two colors of Unifix cubes. The session opened with R2 asking Stephanie how she and Dana worked together on the problem. Stephanie recalled obtaining 32 towers and described how she and Dana built a tower and its "match." By *match*, Stephanie was referring to the opposite tower. Stephanie was then challenged to support her solution of 32 towers (L11):

> Suppose that there was something at stake here that someone came in, a fifth grader, and said I don't think there were thirty-two. Now you built thirty-two. You were convinced they were all different. So, if they said to you, "I think there were thirty." What would you say to that person?

Stephanie responded, "Well, I would say like what we did yesterday, when we were up at the board with the one block yellow, and then the two" [*blocks yellow*] (L12). The researcher asked her to show what she meant. With paper and available cubes, Stephanie set out to illustrate her reasoning. She chose to draw the representations of towers on paper. The first two events illustrate the two sets of towers that Stephanie produced in response to the prompt.

The staircase patterns of "1-to-5 blue" and its color opposite. Stephanie drew a representation of five towers in a recursive *staircase* pattern, such that the blue cubes increased by one in each adjacent tower following the first tower with exactly one blue cube in the bottom position until the final, fifth tower with exactly five blue cubes. Figure 6.1.1 illustrates the blue and yellow staircase patterns. R2 asked her to differentiate between the shaded and unshaded regions (L24), to write the total amount of towers (L28), and to give the set a name (L29). Stephanie responded by writing the letter “Y” on every unshaded region, articulating that the shaded region represents the blue, writing “5” for the total, and labeling it “1–5 [*one to five*] Blues.” Also, Stephanie accounted for the opposite set (L31). She wrote the label “1–5 yellows” and the total “10” for both staircases.

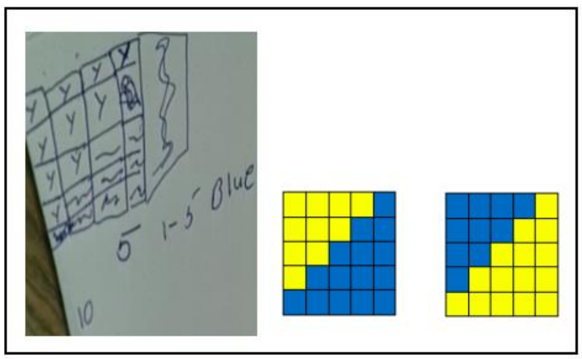


Figure 6.1.1. The “1-5” staircase.

The elevator patterns of “two blues together” and its opposite. Stephanie constructed a physical model of the next set of towers using Unifix cubes. For each tower she recursively arranged the two adjacent blue cubes in a position higher than its preceding tower (see Figure 6.1.2). The researcher suggested that Stephanie record her work and Stephanie then wrote the letter “B” in the empty adjacent spaces to represent what she called the “two blues at a time” (L49). The other spaces remained blank to represent the yellow cubes. The researcher showed her a tower with two blue cubes separated that she believed satisfied Stephanie’s case (YBYYB; L58). This prompted

Stephanie to change the name to “two blues together” (L59); thereby distinguishing between two blue cubes that could be together or apart in a tower. Stephanie named the opposite towers “two yellows together” (L63). She concluded there are a total of ***eight*** including color opposites.

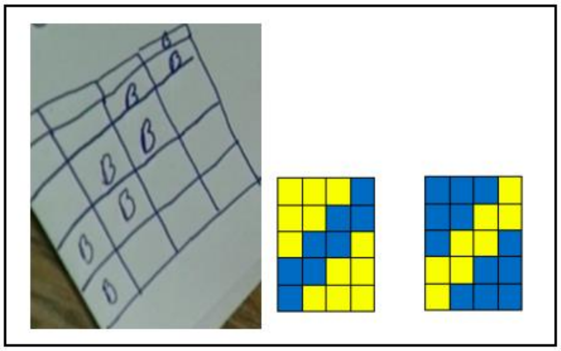


Figure 6.1.2. Towers with two blue together and opposites.

Argument in support of the difference between the two cases (see Event 8). In response to a prompt to justify the difference between the two sets she constructed – the staircase and the elevator – Stephanie noticed and removed a duplicate tower (YYYBB) from the latter case. She concluded that there were three towers with exactly two blues together or ***six*** towers when including their opposite towers. The researcher asked how she dealt with the issue of duplication in the previous day’s experience with Dana. This prompted a reflection about how she and Dana removed duplicate towers (L69-73):

> We ended up counting a lot over. We had thirty-four…so we subtracted I think three groups, because we were down to twenty-eight. Then we added two groups…We kept finding different patterns, but we didn’t check it with the other patterns.

Recall that Stephanie and Dana did not use the elevator and staircase patterns to build their sets; however, they created “different patterns” namely towers that had opposite and inverse relationships. Returning to the prompt, R2 asked Stephanie *if and why she was convinced that this case was different from the previous case* and *why there were no more* *within this case* (L74-87). Stephanie was asked if the case of towers with two adjacent blue cubes was *different* from the case showing a blue staircase pattern. She responded that they were different, and that she would monitor the next “groups” (L75): “Well, because we did these groups with the orange and the blues- the yellow and the blues. So, you know that this group is over, so you can’t make another group like this.”

The researcher challenged her to justify that there were *no more* towers in the case of the elevator pattern in both directions (building from top or from bottom), Stephanie responded by offering an upper bound for the elevator recursion (L84): “You can only build it five high. You’d have to have it [*a tower with two blues at the top BBYYYY*] so it would be seven high, no, six high in order to build another one.” Her reasoning for the upper bound was supported with an argument by contradiction that another tower using the same recursion would need to be taller. Implicit in her argument was the contradiction of the 5-tall tower assertion. As she explained, her thumb and index finger were moving the imaginary two blue cubes to a sixth level. R2 then asked, “What about low though? You can put it on the bottom.” (L85). Notice that Stephanie responded by recreating YBBYY (L86):

> Then you would be making it [*YBBYY*] over. If you put it here [*starts to redraw over tower BBYYY by writing B and B in the top two rows of her tower drawing*] you would be making what is here [*points to column three of the staircase set of towers*].

Figure 6.1.3 offers a flow chart of the conversation between Stephanie and R2.

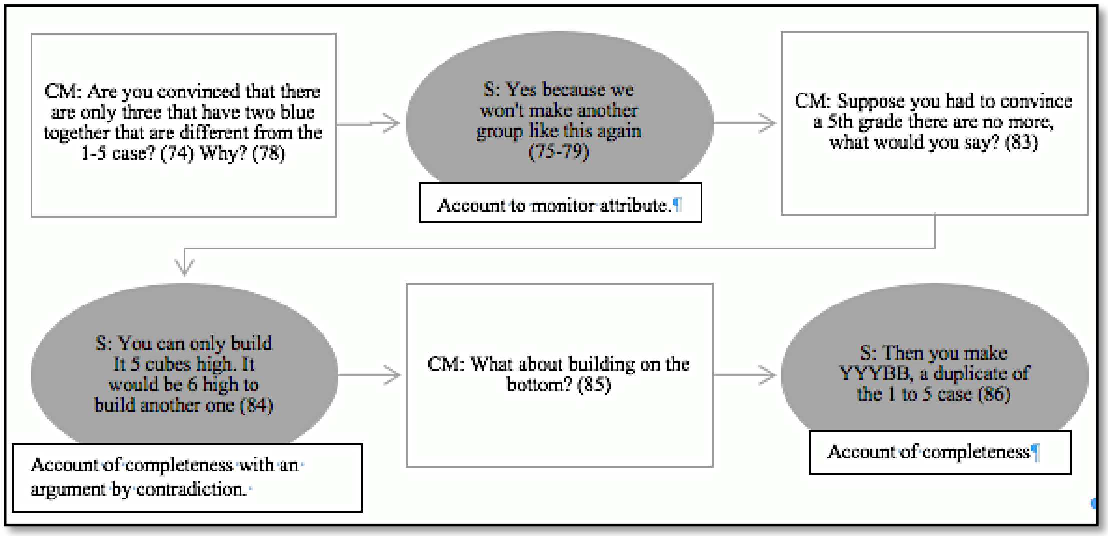


Figure 6.1.3. *Flow chart of a conversation about "two blues together" and staircases.*

Note Stephanie counted duplicate tower BBYYY (a duplicate of the third tower in the yellow staircase) and the opposite tower, YYBBB (a duplicate of the third tower in the blue staircase) in her final solution of thirty. However, there was a reflective discussion at the end of the interview about developing a method to account for differences between and completeness of each case. Figure 6.1.4 offers a reproduction of the final towers for the case of two adjacent blues with black bordering around two duplicate towers (YYBBB and BBYYY) that were left unnoticed in this interview (i.e., they are duplicates of towers in Stephanie's "one-to-five" staircase pattern).

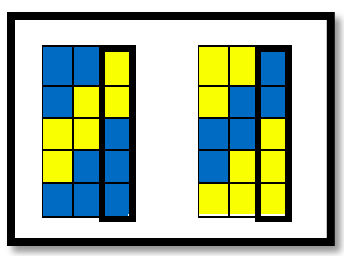


Figure 6.1.4. *A reproduction of the final towers in the case of exactly two blue together*.

Alternating pattern and supporting arguments. As Stephanie drew two towers (BYBYB and "the opposite one," YBYBY; L92) with an alternating color pattern, the researcher described it as "exactly three blues separated by an orange" [*yellow*] (L92).

This was the first instance she returned to the previous day's method of grouping by pairs with a color opposite relationship. She also drew the color opposite tower, rather than double counting as she did for previous opposites.

R2 asked her if there was another way to separate the blue cubes by a yellow cube. Stephanie provided three statements for why there are no more of those towers (L92-102; see Figure 6.1.5). She stated the conditions of the task and the attribute of the towers she generated: "We have three blues and two oranges, but we have five blocks" (L98). Without prompting, Stephanie also established a more specific condition that the three blue cubes must be separated. She provided two examples of towers (namely, BBYBY and YBYBB) that satisfy the three-blue condition; however, two of the blues were *not* separated. She stated, "But you can't put it so that these two [*blues*] are separate…Because there is only five blocks (L98). Note that these two examples of towers served as part of her argument and were not included in her final solution. She then argued for the uniqueness of the pair because the previous sets "had nothing to do with separating the blues and the yellows" (L116). Technically there was some separation in the other cases, but based on her previous examples, it was assumed that she was referring to every cube must alternate colors with its adjacent cube. In response to R2's prompt to name the pair, she labeled it "blue yellow" and wrote the numeral "2" for the total (L103-6, in contrast to her previous description of exactly three blue cubes separated by a yellow. Note that the alternating towers were not counted in the final numerical solution.

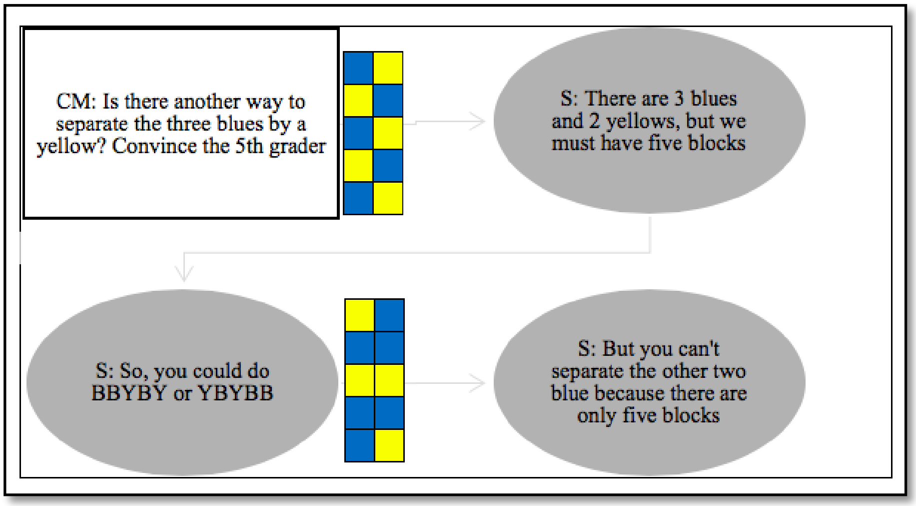


Figure 6.1.5. *Flow chart showing argument for alternating colors.*

The origins of the case of exactly one of a particular color in every "spot" (see Event 9). Stephanie continued to use the strategy of finding color opposites and inverses to find new towers. She drew towers with six more pairs of towers represented (L120-48). Excluding the pair with "three in the middle," the towers all had exactly one of a color. After the third pair, R2 stated, "Yeah, I thought you did something with ones" (L139). Stephanie continued creating new pairs with exactly one of a color. From one pair of towers with exactly one of a color, Stephanie used the *inverse* strategy ("reversing;" L61) from the previous day (see Figure 6.1.6).

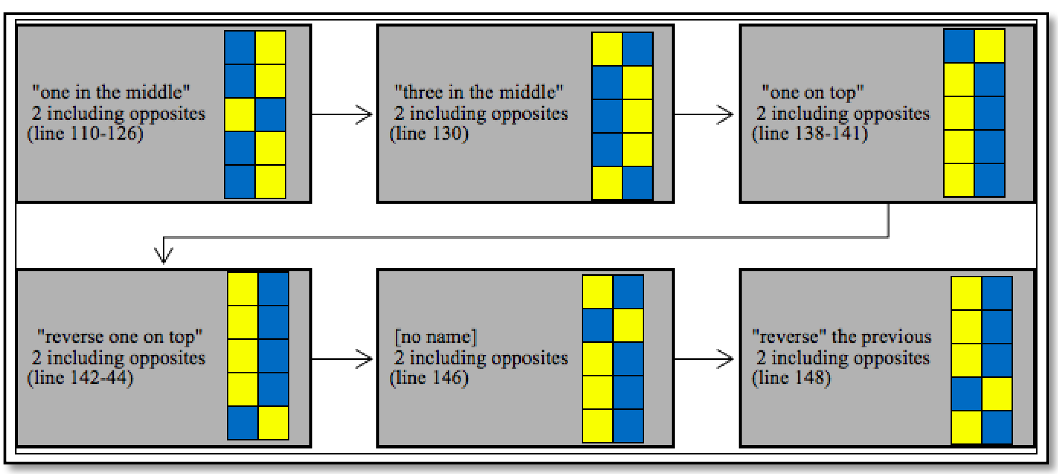


Figure 6.1.6. *A map of how Stephanie generated towers with exactly one of a color.*

R2 then asked Stephanie to discuss the tower pairs in relation to each other. Stephanie indicated that except for one pair, there was a common attribute between them. R2 asked her to record her observations. Stephanie categorized the towers by the position of the yellow cube as follows (L156-78):

> These two [*points to YBBBB and YYYBY*] are somewhat alike because they both have one but at a different place. We just moved it. [*R2 revoices*]. This has the yellow in the first spot [*from the top*] and this one has the yellow in the second spot. [*R2 suggests recording this*]. The other possibilities could be yellow here, yellow in the fourth [*points to the third position and writes "yellow in the third spot" and "yellow in the fourth"*], and we already did the yellow in the fifth here [*BBBBY*].

See Figure 6.1.7 for a summary of how she described and organized the towers with exactly one of a color.

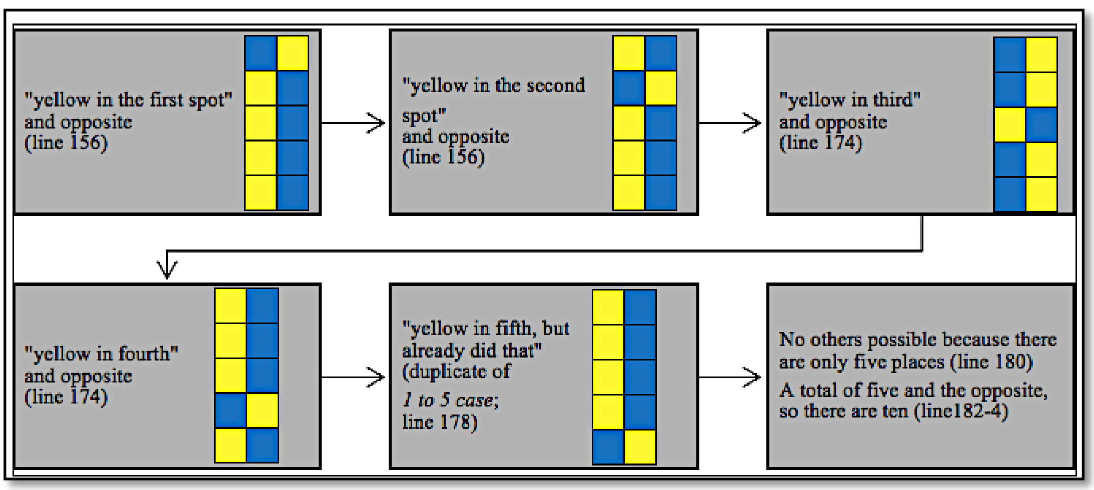


Figure 6.1.7. *The emergence of the elevator pattern.*

R2 asked Stephanie to justify why these types of towers were completed, to which Stephanie replied, "No, because there are only five places" (L180). In this event she developed a different way to find all towers with exactly one of a color, namely by recursively moving a cube to each of the positions.

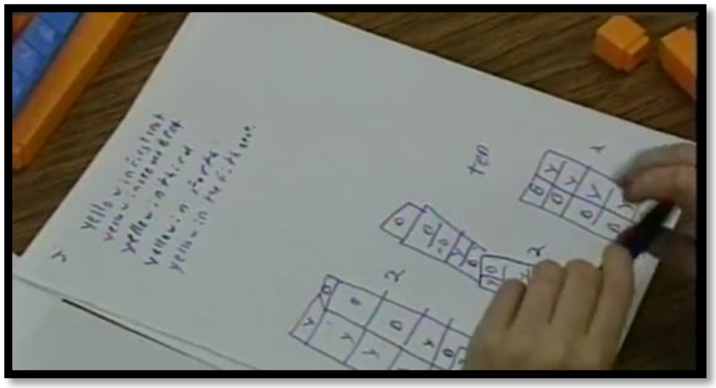


Figure 6.1.8. Still image of the video data of her written work.

Figure 6.1.9 show Stephanie's final 5-tall Tower solution of 30 towers. The boldened towers are duplicates that were not found during the interview. The total towers for each case are in parentheses.

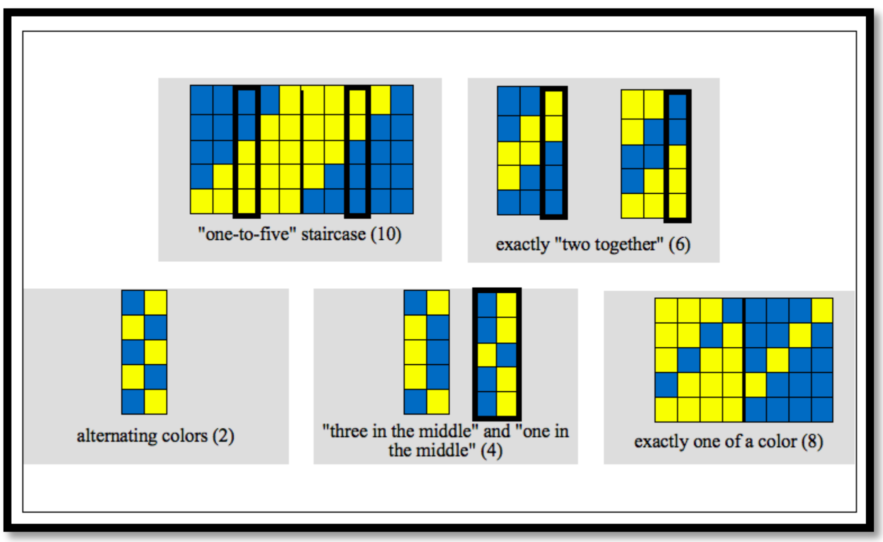
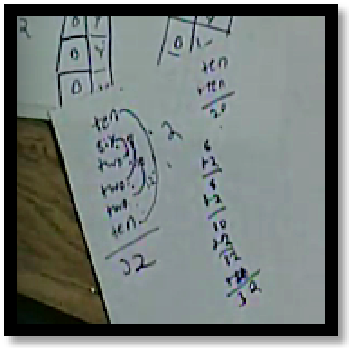


Figure 6.1.9. Stephanie's final 5-tall Tower solution.

Using the 5-tall Tower Task to solve the 4-tall Tower Task. Stephanie was next prompted to calculate the totals of each page and write it on a separate page (L187-99). Although the total is *28* unique towers, she calculates *32*, a total she indicated she expected (L129). Figure 6.1.10 illustrates her calculations and a reproduction of her written work. Note there were four duplicate towers. Some of these duplications were noticed by Stephanie as illustrated by this event.



6 (two together; 2 duplicates)
+2 (alternating)
8
+2 (three in middle)
10
+ 2 (one in middle; 2 duplicates)
12
+20 (one of a color & staircase)
32 (28 unique)

Figure 6.1.10. Still video image and reproduction of Stephanie's calculations.

R2 asked Stephanie to predict, first, and then to explain how many different towers there were 4-tall. Using each of her 5-tall tower drawings, she examined sets of possible 4-tall towers (L209-52). For the *staircase* she claimed there were ***eight***, including opposites (L218). For the case of the alternating pairs, she argued for the same amount as in 5-tall towers, namely ***two*** of them (L236). For the "two together" elevator case she claimed there were ***six*** including opposites. When R2 stated that the latter count seemed incorrect because it was the same amount as 5-tall, Stephanie recalled that there were two duplicates (namely YYBB and BBYY from the staircase pattern) and changed her claim to ***four***. There existed another pair of duplicates in the staircase and two together cases (The second towers of the blue staircase and yellow staircase patterns, YYBB and BBYY, were also towers with exactly two blue and yellow together). This was left unnoticed as in the 5-tall Tower solution.

For the case of "one in the middle" she argued it would not exist because there was no middle position in a 4-tall tower (L246-52). She evidenced that she was thinking about these towers by pairs rather than by a case. R2 asked her to consider the case of exactly one of a color.

The interactions of this event are presented as three chronological themes. The first was a discussion about the case of exactly one of a color 4-tall towers and how she imagined them “faster without building all of them” (L253). The second was a researcher-initiated consideration about possible duplications that led Stephanie to acknowledge that a pair of towers with exactly one of a color (in the bottom position) existed in her staircase patterns. The third was the actual construction of the 4-tall towers with exactly one of a color and the justification of its completeness.

R2 initiates consideration of the case of 4-tall towers with exactly one of a color: “Is there a way you can do that faster without building all of them – think about them in your head…I’d like you to tell me about all those [*pointing to the page of 5-tall towers with exactly one of a color*] together” (L253-7). Stephanie responded that it was the “same pattern in different places…it’s taking one and building on one pattern” (L258-60). She further explained this relationship: “You are taking that pattern, and then moving it down one, and then moving it down another, and another until you have all five patterns” (L262). She described recursively how a tower was generated from a previous tower starting at the bottom position. In response to the quantity of such 4-tall towers, Stephanie paused and then stated there would be four towers with exactly one blue cube.

Stephanie responded by acknowledging that the tower with “the one [*blue*] at the bottom [*YYYB*]…is the same as that” [*pointing to the first tower drawn on the paper with the blue staircase*] (L266-70). This was the second part of this event, which followed with R2 asking Stephanie to consider if there was a possibility of any duplication in her 5-tall Tower solution. She showed Stephanie the tower drawing of YYYYB. Stephanie agreed there were duplicates and then reduced the total to 30 towers (L271-4).

This discussion immediately followed the third part of this event, where she was asked to build all 4-tall towers with exactly one blue, using orange (representing the yellow cubes) and blue cubes, with the acknowledgement that the tower with the bottom blue was a duplicate (L282). When the researcher asked her if there were any more possibilities beyond the four towers built, she responded that “there’s only four blocks” (L292).

She counted a total of ***six*** towers with exactly one of a color and a total of ***20*** (***eight*** from the “one-to-five” staircase, ***four*** from the case of exactly two together, ***two*** from the towers with alternating colors, ***six*** from the case of exactly one of a color) 4-tall towers. Figure 6.1.11 offers a reproduction of her solution. The black boldened towers were duplicates not found during the interview. The totals of each case are in parentheses.

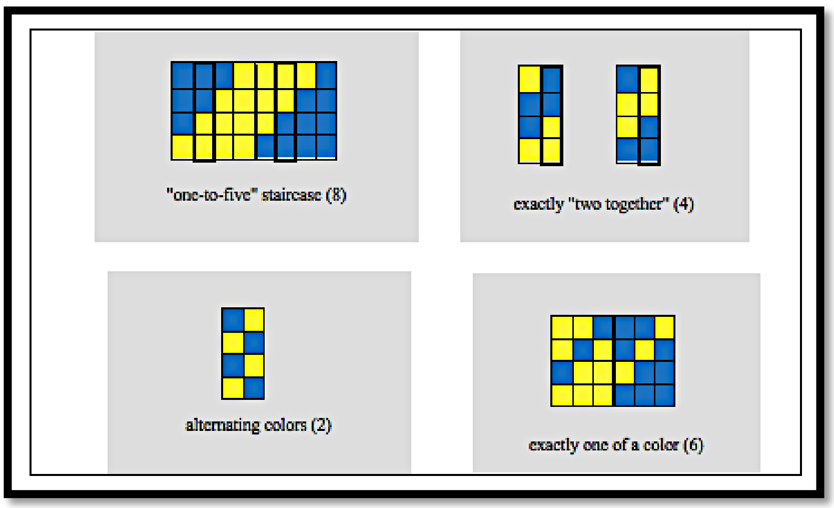


Figure 6.1.11. Final sets of cases for the 4-tall towers.

Note that duplicates BBYY and YYBB existed because they appeared in the elevator patterns of two blue adjacent and elevator patterns of two yellow adjacent, as well as in the staircase patterns. Thus, two copies of 5-tall tower and three copies of a 4-tall tower arise when using elevator, staircase, *and* color opposite strategies. Figure 6.1.12 illustrates an example of 4-tall towers where tower #1 from the case of two adjacent yellows is a duplicate of tower #4 from the case of two adjacent blues. Similarly towers

#2 and #3 are duplicates. Notice also that towers #1 and #2 appear in the yellow and blue staircase patterns, respectively.

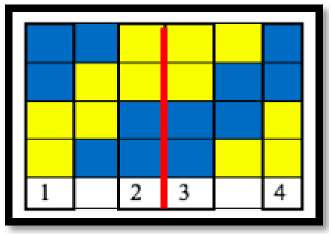


Figure 6.1.12. *Duplicates in the case of towers with two of a color adjacent.*

How to make the solution convincing (see Event 10). Towards the end of the interview, a discussion took place about how Stephanie might become certain that her 5-tall Towers solution did not have duplicate or missing towers. Stephanie claimed it took "a very lucky guess" (L312). R2 counterclaimed that, "But Math isn't a guess; in math you should be able to figure it out and be certain" (L313). She reminded Stephanie of her own convincing arguments about the bounds of the tower height and the recursive relationship among towers of a case, as well as the pattern organizations that afforded absolute certainty. For example, she asked Stephanie: "Why are you actually convinced there are only four [*of exactly one blue*] when you build towers of four? And if I build towers of five, how many are there" [*of exactly one blue*]? (L319). Stephanie replied, "five," and in response to towers of six, she replied immediately with, "six" (L320-2). The researcher pointed out, "If it's [*4-tall towers with one*] blue, [*there are*] four; yellow, [*also*] four. So, you don't have any doubt in your mind about that?" (L325). Stephanie responded in the affirmative when R2 made the point that Stephanie was certain for some cases and that she should be able to become certain for other cases (L326-8). However, Stephanie then stated, "You can't really be convinced because there is no absolute way that you can go and say, 'I'm right'" (L330). R2 again suggested that, "This is an absolute way…when you look at only one blue," referring to the 4-tall towers with exactly one blue in front of her in an elevator pattern, and Stephanie agreed, "Yeah, this is one of the absolute ways" (L331-4). The following exchange showed Stephanie's agreement.

335. R2: And I wonder if you could find other absolute ways of looking at maybe just two blues or just three blues or just four blues?
336. S: You can.
337. R2: This is an absolute way. How many ways can you do exactly four blues?
338. S: Once.
339. R2: You are convinced of that, right?
340. S: With the four [*tall*] towers.
341. R2: Yeah. You're convinced of that, right? No one can persuade you otherwise, right? Well, you are convinced of this, you are convinced to that, can you figure out ways of getting convinced to those middle cases, exactly two, or exactly three?
342. S: Yeah… It is possible to have a certain number and get it right by figuring out the number of two's you could have, threes, fours, ones [*of a color*] depending on the number of blocks [*height*] you have.

Stephanie claimed that it was possible to be certain of the exact total of towers in the case of exactly one, two, three, four of a color, depending on the height. In conclusion, Stephanie had an opportunity to reflect on the possibility of certainty for some cases. She was given the 6-tall Tower Task to solve for homework and to find a convincing method to prevent duplication and/or missing towers.

6.1.1 **Findings: Key developments of Session VI**

Stephanie had an opportunity to revisit the 5-tall Tower Task. She used *the elevator* and *staircase* strategies that were previously shared in the class discussion and also *folded back* to opposite and inverse tower pair generation and organization (e.g., for towers with exactly one of a color). When she was asked to discuss related towers, she developed the case of exactly one of a color and supported its exhaustion with an *argument by* *contradiction* using the elevator recursion and the 5-tall tower assertion. Finding duplication within her solution Stephanie indicated discouragement for gaining certainty in a Tower Task solution (i.e., the solution of the Tower Tasks required a "very lucky guess"). The researcher called attention to her conviction of the exhaustion of certain cases, such as exactly one or two of a color. Stephanie reflected on the possibility of certainty for some cases. Stephanie concluded that it was hard to convince someone of the overall solution but that it may be possible to find out how many "ones, twos, threes, fours" depending on the height given. R2 encouraged her to find all towers 6-tall at home utilizing strategies to generate and organize towers that made her have no doubt.

6.2 Second interview (Session VII)

Date	February 21, 1992
Grade	4
Task	Towers (6-tall)
Participants	Stephanie
Researchers	R2
VMCAnalytic	Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Part 2 of 3 (Grade 4); Events 1–5 https://doi.org/doi:10.7282/t3-gj4x-wr97

This one-on-one interview between R2 and Stephanie was a 55-minute conversation that occurred in the fourth grade, one week after Stephanie's first interview. For homework Stephanie was given the 6-tall Tower Task, selecting from two colors. The session opened with R2 asking about how Stephanie solved the problem. Stephanie provided the researcher with pages of written work, where each page consisted of a category of towers.

Exactly one and exactly two of a color together. The discussion began with Stephanie showing and explaining what she had written. The first page depicted a group of six towers that she called "one at a time" with exactly one blue color cube in each position and five orange colored cubes in the other positions (see Figure 6.2.1). She justified this case by reasoning about the conditions of the problem, that she "can only make six blocks high towers" (L11) and if she moved the blue any further down, she would "have to add another block" [*a seventh block*] (L13). Above the set of towers is the label "1 at a time = 6 Dubble opsite [*sic*] total = 12" referring to the six towers that has one blue and their six opposites, a total of ***12*** towers. The researcher accepted her explanation by stating, "I believe what you got there" (L14).

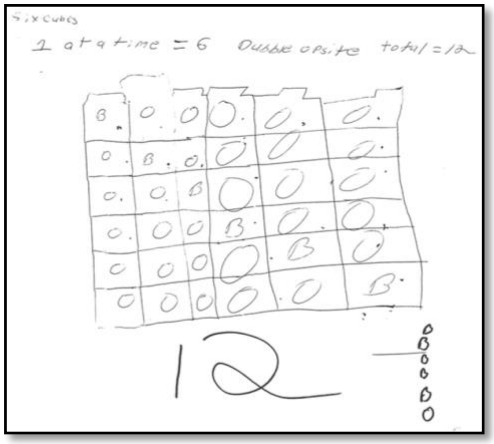


Figure 6.2.1. Stephanie's 6-tall tower drawing of one blue.

The second page (L21-7; Figure 6.2.2) had a group of five towers that she called "two at a time" and added the word "together" in a later conversation with R2 (L38-47) to indicate that the two blue cubes must be adjacent in each tower. She described a recursive (elevator) pattern with two blues starting at the top and then "cross[*ing*] over one" [*position down*] (e.g., the tower following BBYYYY is YBBYYY). Again, Stephanie indicated that she was imagining the opposite set, which resulted in a total of ***ten,*** and on her paper was the label "dobble opsite [*sic: double opposite*] total = 10."

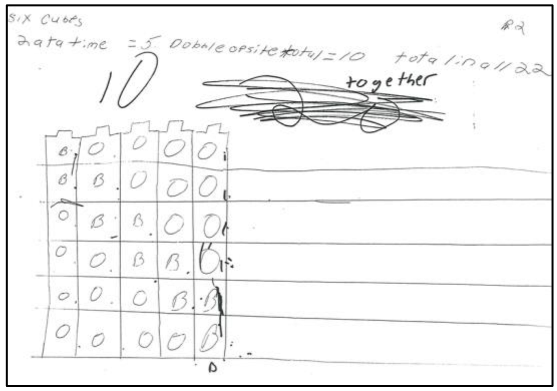


Figure 6.2.2. Stephanie's 6-tall tower drawing of two adjacent blues.

Argument and reasoning for the case of exactly two of a color together (see Event 1). Before getting to the third page, the researcher asked to review and record the name of the case and the corresponding total (L28-33). Their conversation returned to the "two-at-a-time" case to find out Stephanie's meaning behind "at a time" (L38-42). The researcher asked, "Are these the only two-at-a-time you can have?" (L38). Stephanie responded, "Ah! together" (L39). R2 asked her, "It's sort of like they can never be separated like they are glued or something?" (L40). Stephanie responded in affirmation, stating, "Yeah, because of the two at a time" (L42).

The researcher then asked her why she was "convinced that there are no more" for the case when they are together (L47-9). Stephanie noted, "Not like the two together and three together, but you can have two at a time…There are different ways you can have the two at a time" (L50-2). Notice that she used two descriptions to describe the case of exactly two of a color: "two together" and "two at a time." In this instance she was more specific about the case that the two same-color cubes must be "together," and that she acknowledged there were other ways to have the two same-color cubes arranged in a tower. The researcher revoiced her comment, "But when they're together [*towers with two adjacent blue and two adjacent orange*] these are the only ones" (L53). Stephanie affirmed. R2 then began a question, "How can you…" and Stephanie quickly completed it, "…Prove it?" (L55-7). R2 confirmed her question. To justify her claim, Stephanie argued that moving the two adjacent blue cubes any further, required the availability of a seventh position. Stephanie explained (L58–66):

> Once you get down to the last two blocks, you've used all the six [*positions*] and you're on your last two blocks. You can't go down here, blue, blue *[gesturing imaginary movement down to a seventh position]*, because you are missing a block. You need another block.

She described an imaginary tower that has two blue cubes in the sixth and seventh position, pointing to the missing seventh position of her 6-tall tower drawings. R2 asked her if she used "the same argument" (L67) for the case of exactly three of a color together, and Stephanie replied affirmatively as she pointed to her third page. Her third page had a group of ***four*** towers that she labeled as "3 at a time" and their ***four*** opposites, totaling ***eight*** towers.

Summary of the three cases thus far. The total number of towers in each case and some of the details about the color opposite towers were reviewed. As she calculated "four [*towers*] and four" towers (L68) that have three cubes of the same color together, the researcher asked a clarifying question: "When you said, 'four and four,' I'm supposed to imagine in my head that there could have been three orange here [*pointing to the top three BBB written on her paper*] and three blue [*pointing to the bottom OOO drawn on her paper*], right? When you do your partners, is that what you mean by that?" (L69). Stephanie agreed and provided another example of an opposite color tower, namely OBBBOO, the opposite of BOOOBB (L71–4). Then she counted and obtained ***30*** total towers, 6-tall. After R2 asked her, "And these thirty are all different?" Stephanie pointed to the three cases and described their common attribute: "Because this one is choosing three blocks [*adjacent of the same color*], this one is choosing two blocks [*adjacent of the same color*], and that, one–" (L84). The researcher completed her sentence, "–choosing one block, even when you switch them around" [*indicating color opposites*] (L85) and Stephanie responded affirmatively.

Exactly four of a color is equivalent to exactly two of a color (see Event 2). This event illustrates how Stephanie's case method (i.e., by *recursively* increasing the number of adjacent same-colored cubes one more than the previous case) posed a dilemma. She described the next case: "This one is four at a time…And then this is the same argument as up there and it's just adding an extra one" (L88-90). Stephanie provided an example of a tower and its opposite, namely BBBBOO and OOOOBB and stated there were ***six*** total towers of this category. The tower she picked was a duplicate. The researcher asked her if this opposite had ever been made before. Stephanie claimed that it had not, "Because we use the ones; we use the twos [*adjacent same-color cubes*]; we use three [*adjacent same-color cubes*]; and then we use four" [*adjacent same-color cubes*] (L94).

The researcher showed her the tower OOOOBB in the two adjacent blue case. Stephanie began to reduce her total as the researcher asked her why the duplicate occurred. R2 asked her about another tower, OBBBBO. Stephanie claimed, "No, I don't think anyone is gonna duplicate that so far because, look. So far, we've been doing them all so their together. So, these two oranges [*in <u>O</u>BBBB<u>O</u>*] are separated" (L100-2). Stephanie recalled the other set was "two at a time" (L106). R2 asked, "How can they be same if [*pointing at Stephanie's two papers*] this is 'two at a time' and this is 'four at a time'?" (L107). Stephanie utters, "Oh," pauses, and then began to count the orange cubes aloud in the two blue adjacent case (L110-2):

> One, two, three, four. That's [*case of two blue together*] also four at a time. That two at a time is also four at a time [*pointing to OOOOBB*]. Because look – one, two, three, four [*counting the number of oranges*] – there's four oranges together and it could be two [*blue*] at time too.

The researcher revoiced her explanation:

> So, when you think of the two blue at a time, you are thinking of four oranges at a time. Or when you are thinking of the two oranges at a time, you are thinking of the four blue at a time and that's how you get them?

Stephanie agreed. Stephanie noticed another duplicate, BBOOOO and OOBBBB. She identified an inverse relationship that it had with the other two towers OOOOBB and BBBBOO (they are "upside down" of each other; L124-6). R2 asked her, "Why do you suppose this one didn't come up [*pointing to the copy OBBBBO*] and the others did?" (L127). Stephanie explained, "This is the only one [*tower in the four together case*] where the two of them [*oranges*] aren't stuck together" (L128). She concluded that there were ***two*** towers from this case that were different from the other cases, namely OBBBBO and BOOOOB.

Recall in the previous interview that some of the duplicate towers obtained from a method by cases and a method by opposite cases were not found. (In combinatorics, the case of selecting exactly m of the *same available* things to be placed into n available positions is equivalent to the case of selecting n–m *opposite/other available* things to be placed into n available positions when there are two different things.) The simultaneous consideration of the case of towers with four adjacent blues and the case of the two adjacent oranges (color opposite case) caused the former case to have duplicate towers of the latter case. This is the first event in this interview where Stephanie considered duplication by associating the attributes of the other color in relation to the color of focus in the case.

Exactly five of a color together is a subset of exactly one of a color. This event provided further evidence of Stephanie considering the issue of equivalence between the five of a color case and one of the other color case and between the four of a color case and two of the other color case. Without prompting she presented the next case that she had originally drawn on another paper: “Then it’s just five at a time – I have just thought of something…this is the one at a time” (L134) while pointing to the tower BBBBBO. She used the drawing of the case of exactly one blue and wrote “BBBBBO” under the tower OOOOOB to illustrate the color opposite tower that she had already counted within her solution (see red circled region to the right of Figure 6.2.3). She pointed to another tower OBBBBB and stated, “And then the same with this one” (L142). Stephanie agreed with R2 when the researcher stated, “So you have no new ones on this [*case*]” as she pointed to Stephanie’s “five-at-a-time” paper. Stephanie wrote, “Have these” on the paper (see blue circled region to the left of Figure 6.2.3).

Figure 6.2.3. Stephanie’s 6-tall tower drawings of five adjacent blues (left) and one blue (right).

She continued with the case of “all the six-at-a-time” [*six of one color*] (L150), which added another ***two*** new towers. The next case Stephanie introduced were ***two*** towers with the color cubes alternating, which she called “patchworks” (L155–7). R2 described the case, using similar language as for the other cases: “So that’s *three* but *none* of them are glued *together*, huh?" (L158; emphasis added). Stephanie agreed. She had a total of ***36*** thus far (or 34 since the case of "four [*of a color*] together" will instead be included in the case described next).

The case of exactly two blue separated (see Events 3-5). The next events of Stephanie's fourth grade second interview detail the process by which she found the case of exactly two of a color separated by controlling for one variable while varying the other. The results are organized in three parts. The first event is a discussion between R2 and Stephanie analyzing the case that she previously created at home with exactly two blue separated where one blue cube was fixed at the top position. The second and third events consist of how she formed *an organized and completed version* of the general two blue separated case to account for all possibilities, guided by the researcher and using the Unifix cubes at first. The second event illustrates Stephanie's creation of the towers with one color *fixed* on the bottom-most position (the sixth floor) and *varying* the other cube of the same color in an elevator pattern, but also considering duplicates that occur when doing so. The third event shows Stephanie's creation of the remaining towers for the case with one color fixed in every other position other than the top and bottom positions (that is held constant) while varying the other cube of the same color again in a recursive manner and taking into consideration duplication.

Discussion of the case of exactly two blue separated with one blue cube fixed at the top position (see Event 3). Stephanie presented a sheet with five towers with one blue fixed at the top position (sixth floor) and with the other blue cube elevated from the bottom position (first floor) to the fifth position. The category was labeled as follows: "six cubes 2 seperated =5 dabble=10 total=54" (*sic*). Figure 6.2.4 presents Stephanie's original written work in lighter ink and Stephanie’s new written work during the interview discussion in darker ink. The following is noted about her representation of this category of towers: (1) Although she intended to draw 6-tall towers, her towers had seven positions (based on evidence that her label for the sheet indicated she was working on the 6-tall Tower Task, and also based on later events where she noted her own error); (2) She did not include the tower with the two blue cubes adjacent to each other (later in this event she would explain her reasoning why she purposefully omitted it).

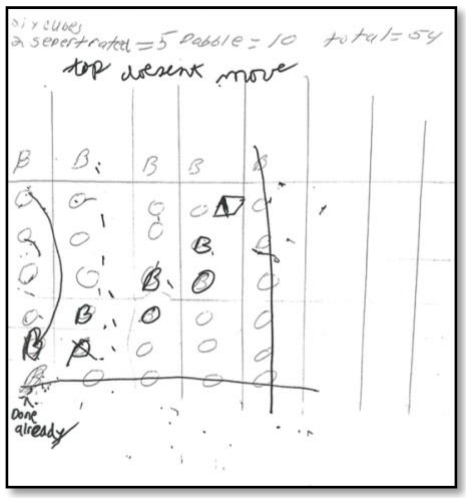


Figure 6.2.4. Stephanie’s representation for the case of two blue cubes separated.

Stephanie described the first tower as “two blue separated between the four.” Based on R2’s first question, “Now did you have that [<u>BOOOOOB</u>; *sic*] in any place?” she acknowledged it was a duplicate of her “four-at-a-time” case, and she wrote “Done already” under the tower (see Figure 6.2.4). R2 then asked her about her patterns: “How did– did you come up with some way to get all of these? I know you have two blue in all of these, right?” (L176). Stephanie described each tower by the number of orange cubes in between the two blue cubes using a recursive elevator strategy (L177): “I sort of like just went in order like we did that one-at-a-time… I started with the four in between... I mean five in between, then I went to the four in between, three in between, two in between and one in between." (Note that they were incorrectly drawn to be 7-tall towers, but intended to be 6-tall, and hence instead of four orange cubes in between the two blue cubes, she started with five orange cubes in between them.) The researcher observed that the "blue is on the top floor" for all towers. Stephanie added to R2's observation: "and then the blue from the bottom is moving up" (L184-9).

In the next part Stephanie introduced the idea that this method could be done another way: "Instead of having the blue from the bottom moving up to top, we could have had the blue from the top moving from up, down to the bottom…You can do it either way" (L207-9). The researcher probed to understand the method better and asked Stephanie to explain the original paper again. She replied (L220-7):

> Well, they've been separated. They're separated from the bottom…up. They'll go like this…you'd start out with the bottom and the top one, they're separated with five in between, but then the bottom one would move up one...[and] if it moved up five, it would be here [*the case of exactly two adjacent blue cubes*] and then we would have this [*shows BBOOOO*].

The researcher asked Stephanie to go over it again to make sure that she did not find any duplicates. She noticed that the tower with the blue cube fixed on the top and the other blue cube at the bottom (on the first level; BOOOOB) was already in the case of towers with exactly four orange together. Figure 6.2.5 presents Stephanie's representation of the case with the blue cube fixed at the bottom and the other blue cube varied recursively from top to bottom. She crossed off the first tower and wrote "Done" underneath. (The words "Done" was cut off from the copy of her original work and so, the researcher of this dissertation study transcribed over it.)

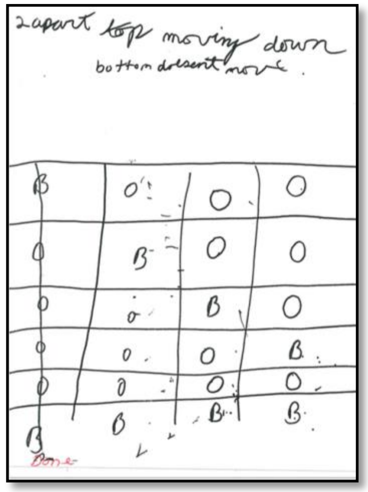


Figure 6.2.5. The 6-tall Tower case of two blue apart with a downward elevator pattern.

Then Stephanie was asked to calculate how many towers resulted from the case of the blue fixed at the top (She had five drawn on her paper) and the blue fixed at the bottom (She had four on her paper). She indicated that she was studying the discrepancy in the mismatch of the calculations and, after a pause, she explained that the original representation with the blue fixed at the top was seven cubes high rather than six cubes high (L245-59). Now she concluded there were ***three*** unique towers (not including BOOOOB and its opposite) with the blue fixed at the bottom, ***three*** unique towers with the blue fixed at the top, as well as ***six*** opposite color towers which were not drawn, and, thus a total of ***12*** unique towers.

Thus, Stephanie used her method of controlling one of the blue cubes once again (at a different position) to generate new towers. Her justification that each tower was different (after eliminating the duplicate of BOOOOB that was generated in both sets of towers) included: “We haven’t done two separated yet…because we did the two together, three together, four together, one” (L263-7). When asked to justify how the two cases with “two separated” (L268) were different from each other, she reviewed them silently and concluded that the towers were different, “because instead of moving the bottom, up, we are moving the top, down" (L269). She argued that the two cases differed because the second blue cube moved recursively upward in one case and moved recursively downward in the other set. She seemed to be focused on the varied blue cube, not the fixed blue cube. Stephanie counted ***48***, with 36 from the previous cases and 12 from the case of exactly two blue separated by at least one. Figure 6.2.6 is a reproduction of her 6-tall tower representations of exactly two blue (represented by the letter B) separated by at least one orange (represented by the empty white squares), where one blue was fixed at the top level in the first set and fixed at the bottom level in the second set. The duplicate tower that Stephanie removed (BOOOOB) is bolded. In the second set she already prevented herself from making that same tower (BOOOOB) and hence started with the blue in the second level from the top (OBOOOB).

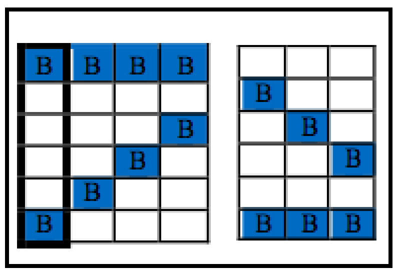


Figure 6.2.6. Stephanie's case of exactly two blue cubes.

In the third part of this event, Stephanie chose to pursue "three [*blues*] apart" (L281 & 299), However, R2 suggested that she provide an argument about the towers with two blue cubes apart: "How do you know you've done all the 'two aparts'? I am not sure I believe that; you haven't convinced me. Here I see that you have blues all in the bottom row and I see you have the blues all in the top row. Is that the only thing?" (L308). When Stephanie explained that she already did it "both ways" R2 refuted, "You have the blues on top and I believe that and you have the blues at the bottom. But there are other floors… you can have blue on the top floor blue on the bottom floor, why can't you have different floors of all blue?" (L328). Although several times R2 suggested to consider fixing the blue cube on another level of the towers, Stephanie recognized the duplications that would occur on some of the towers where the blue cubes would become adjacent (L323-5). It was evident that she was concerned about producing duplicates with the adjacent cube cases if she fixed the blue cube to other floors. The researcher, acknowledging her correct reasoning, suggested to consider one tower (OOBOOB) where the B of the first floor would now be fixed to the second floor (OOBOBO). This was the tower that Stephanie agreed might be new, responding, "Yeah, you can do that. Probably. You can do that" (L329). Here Stephanie agreed that this may be a possible new tower, but also showed some hesitation because she stated that someone could always come up with a new pattern (L343). At the end of this discussion, Stephanie agreed to pursue the pattern of changing the positions where the blue cube would be fixed (L344-7).

Controlling for one color while varying the other in the case of exactly two of a color separated by at least one of the other color (see Events 4-5). Previous to this event Stephanie hesitated to pursue the suggestion of the researcher claiming, "You are bound to come across someone who's gonna say 'Well, I don't believe you'" (L385). This event illustrates the result of the previous discussion about other possible towers with two blue cubes apart. With R2's encouragement, Stephanie built the first tower WRWRRR using red and white cubes that were provided. In response to what case the tower belonged, Stephanie pointed to her case of two blues apart with the top blue fixed at the top position (L413). The researcher helped her build while Stephanie designated how many positions the second white cube should move: "We go down two…three… You can go down four" (L429-49). The first set of towers that was built consisted of towers WRWRRR, WRRWRR, WRRRWR, and WRRRRW. Note they had one white cube fixed at the top while the other white cube was varied by at least one red cube apart from the fixed cube. She then built the next set of towers with the white cube fixed on the fifth floor while the other white cube was varied recursively with at least one red cube apart from the fixed cube. Then the next two sets consisted of the first white cube fixed on the fourth and third floors while the other white cube was varied at least one red cube apart from the fixed cube.

She drew a representation of the first set of physical towers that had one white cube fixed at the top while the other white cube was varied by at least one red cube apart from the fixed cube (use Figure 6.2.7 as a visual guide for the reader).

W	W	W	W						
				W	W	W			
W							W	W	
	W			W					W
		W			W		W		
			W			W		W	W

Figure 6.2.7. *A reproduction of Stephanie's case of two whites separated by at least one red.*

Note that the fourth tower (WRRRRW) was also the tower with exactly four red cubes together (the towers that were left over after duplicates were removed in the case of "four-at-a-time"). It is interesting to note that Stephanie was careful not to double count these towers towards the end of the interview (she did not include the case of four of a color cubes adjacent in her total). She then drew a representation of the next physical set with the white cube fixed on the fifth floor while the other white cube was varied recursively with at least one red cube apart from the fixed cube. Then the next two sets consisted of the first white cube fixed on the fourth and third floors while the other white cube was varied at least one red cube apart from the fixed cube. Figure 6.2.8 shows

Stephanie’s written work and physical models. Underneath the grid stated: “Two white separated by at least one red” and the total count of “20 with opposite.”

Figure 6.2.8. Stephanie’s two apart case in a grid and with physical models.

To describe the towers Stephanie stated, “It always has two whites separated…by one, two, three, four [*red cubes*]” (L453-7). The researcher asked her why she could not have white fixed on top and separated by five red cubes. Stephanie responded, “Because you need seven blocks, you need a seven-high tower” (L461). She extended this argument by restating the given assumptions that “there are only six places” (L481) in response to why there were no more than the towers she generated with the white cube fixed in the other positions. She summarized that her argument applied to each case (L485):

> The same argument is gonna come up every time the person asks, ‘can you make another one’ when you are already through. Because, if you have a six high tower [*explaining while using a RWRRRW tower*] and you wanna make it [*the two white cubes*] with apart and then one more [*red in between the whites*], you can't do it. The same argument will come up.

Stephanie used an argument by contradiction that no more towers could be generated using her recursive methods because if one were to separate the white cubes beyond the allotted positions with red cubes then one would violate the given assumption of a 6-tall Tower Task (L458-63; L478-85). Furthermore, when questioned about separating the white cubes by zero red cubes, she argued, “If you did white on top separated by none, it would be the two [*together*] patterns" (L463). Lastly, Stephanie observed, also, "If we build all the way back to the top from the bottom, like this [*builds W at the bottom and shows another W moving from the bottom positions to the top*], we will be doing the exact same thing" (L499). She explained that the same method of controlling for one variable and varying the other in the reverse direction would result in duplication of the set that she already created.

In the conclusion of the "two-apart" case exploration, Stephanie summarized the case and accounted for the color opposite towers. In response to the R2's question of how Stephanie would convince another researcher, R1, she summarized the case as follows (L517):

> Well, I would show her…the first pattern [*pointing to the recursive pattern of the second white cub*] with one at the top [*the fixed white cube on the sixth floor*], and I show that we can move it down one [*the fixed white cube on the fifth floor*], then we can move it down two [*the fixed white cube on the fourth floor*], and then we can move it down three [*the fixed white cube on the third floor*].

In response to R2's challenge, "Why can't you move down four" [*a tower with a fixed white cube on the second floor with the other white cube on the first floo*r)]? (L518) she stated, "Because we will be repeating the two together pattern" (L519).

She calculated a total of ***20*** different towers with two of a color separated (the white cubes represented the blue color in her original solution) and used the language of "at least" that was previously introduced by the researcher (C5, L524-543). She described it as, "two white separated by at least one red" (L555).

Summarizing the solution by cases and reflection. The conclusion of the interview consisted of reviewing the totals for each case (L563-683): ***12*** towers in the case of "one at a time;" ***10*** towers in the case of "two at a time;" ***20*** in the case of "two separated" by at least one of the other color; ***eight*** in the case of "three at a time;" ***two*** in the case of all towers with a single color; ***two*** in the case of "patchworks" or alternating color cubes; and finally the "three apart" case was given as a task to do on her own at home. Figure 6.2.9 is a reproduction of the cases discussed or found in this interview, selecting from two colors. In parentheses are a total count for each case, including the opposite towers, which are not represented in the figure. Note that she did not count the towers with exactly four of a color together because now they were included in the cases of exactly two of a color.

Moreover, she reflected on the case methodology as compared to problem solving methods used for the 5-tall Tower Task with Dana in the fourth-grade class session: "We just built towers [*with Dana*]… It's easier this way [*by cases*] because you have– you know it's organized. Me and Dana were just taking the things [*towers*] and going" [*using cubes to build a random tower for demonstration*] (L691-3). R2 asked her to try her "new plan" (method by cases) with 4-tall towers (L698). In the final conversation she compared the 5-tall solution of 32 to the 6-tall solution, which she now thought would be around 50 towers.

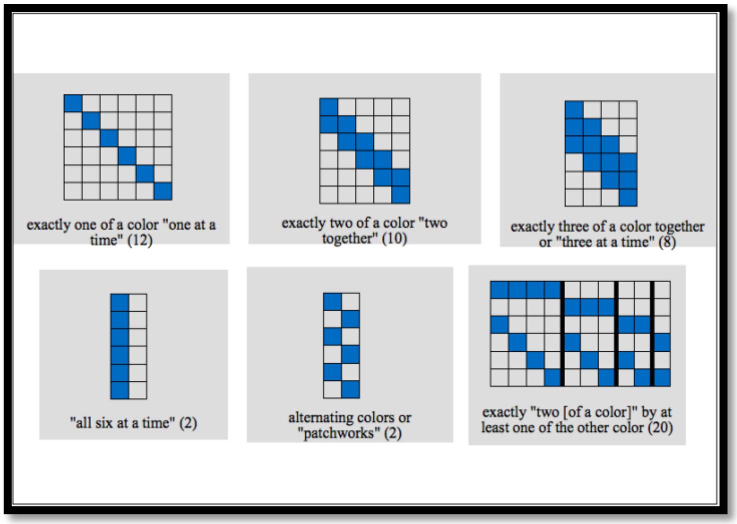


Figure 6.2.9. Stephanie's partial cases for 6-tall towers

6.2.1 Findings: Key developments of Session VII

Representations. In this interview Stephanie applied a method based on cases to partially solve the 6-tall Tower Task. Her solution was represented by drawings of 6-tall towers organized in a grid form with one case per page and a written indication of the common attribute (name of the case), the number (how many towers), and the number doubled to include the color opposite case (imagined rather than drawn). Her representations were in written form, even though she was offered cubes to take home. The cases she found consisted of towers with exactly one, two, three, four, five, six blue adjacent to each other, and two blue cubes separated from each other. Each case of towers was generated and organized using the *elevator* strategy in order to display each position that the adjacent or separated blue cubes took on recursively.

Recognition of patterns and use of strategies. During analysis of the cases with four and five blue cubes adjacent Stephanie eliminated duplicates that resulted from recursively enumerating cases of towers with adjacent blues in combination with the color opposite strategy (see *equivalent cases* in the Glossary of Terms). Recall the whole class discussion enumerated cases of adjacent reds of exactly two and applied the color opposite strategy, but why the cases of three, four, or five adjacent reds was not enumerated was implicit. Stephanie discovered the duplicates through the explicit enumeration of those equivalent cases.

It is interesting that in this session Stephanie generated and organized towers with exactly two blue cubes separated by at least one orange using the *odometer strategy*. She did not do this in isolation. Stephanie was challenged to consider other towers with the same attribute and to find all of them. As a counter argument to Stephanie's assertion that her case of two blue apart was complete and her concern of duplicating towers she had already found, R2 showed her the tower, OOBOBO, that had separation but was not limited to a fixed element at the top nor the bottom position. Extending her original idea of controlling for one blue at a particular position and varying the other blue using the elevator strategy, Stephanie exhibited control by ensuring that towers generated by the odometer strategy would not produce two *adjacent* blues and would not violate the conditions of the 6-tall Tower Task.

Forms of reasoning in support of the solution. She also argued by contradiction when justifying the completeness of some cases, asserting that the 6-tall condition would be violated if one were to continue the recursive pattern generation.

Stephanie attended to counterexamples multiple times in this session (and in sessions as early as third grade when R1 pointed out a duplicate as a counter example to a solution of 17 towers). Stephanie attended to the researcher's counterexample that pointed out a duplicate from the case of two adjacent blues. Stephanie eliminated the duplicate towers from the four adjacent blue case and the equivalent case of five adjacent blues (equivalent to the case of one blue).

Stephanie also portrayed uncertainty as she did in the first fourth-grade interview that someone may have another pattern for a tower. After completing the two-apart case she reflected on the organization, as compared to her strategies with Dana during Session IV and explained how she would convince another researcher.

6.3 Third interview (Session VIII)

Date	March 6, 1992
Grade	4
Task	Towers (1-through 4- tall towers)
Participants	Stephanie
Researchers	R2
VMCAnalytic(s)	Stephanie’s Development of Reasoning by Cases to Solve Tower Tasks: Part 2 of 3 (Grade 4); Events 6–11 https://doi.org/doi:10.7282/t3-gj4x-wr97

This one-on-one interview between R2 and Stephanie was an 85-minute discussion that occurred in the fourth grade about two weeks after Stephanie’s second interview. Prior to this interview Stephanie was given a home assignment to solve the 4-tall Tower Task, selecting from two colors of Unifix cubes, using her “new [*case*] method” that she had used to solve part of the 6-tall Tower Task.

Case of exactly one of a color in the 4-tall Tower Task (see Event 6). The session opened with R2 asking Stephanie what they had done in the previous interview. Stephanie said she was asked to work on the 4-tall Tower Task. She stated that her solution was ***20*** towers using her method. Stephanie’s argument was initiated by the researcher who reminded Stephanie to imagine she was asked to convince another student, “Stephen,” from her class who was not easy to convince (as per Stephanie’s claim in the previous interview), and that Stephen was not yet convinced that there were no more. In response, Stephanie presented towers using an elevator pattern in the reverse direction (moving the white cube from bottom to top), but cautioned, “Then you’ll just be repeating these things” [*the four towers*] (L15). She then discussed the possibility of moving the white block a higher, fifth position, but she remarked that the position did not exist (L15). She explained her reasoning in conversation with R2 for the completeness of the case of all different towers with exactly one white (refer to the event or transcript for the full conversation; L1-25):

> First, we have the towers with one white block, and the white block is on the top, and then it’s there and then [generating *towers recursively with exactly one white cube in each position from top to bottom*] it’s there, and then it’s there, and then it’s there. And there’s our first group of towers… Once you get down to the last one, at the bottom, you can’t move the white back up, because then you’ll just be repeating these things. But if you move the white down one, you’ll be missing a space [*a fifth position – lower bound*]. And if you can only use four blocks, you can’t have another one... Because if you move the white on the next position on top [*a fifth position – upper bound*], it’ll be like this [*shows a 5-tall tower*]. You’d need another block here, but you can’t do that... Because there’s only four blocks [*argument by contradiction*]... Because, well, the assignment said we have to use four [*reasoning about the assumptions of the task*]… [*counts and labels the case as “4 one moving down” and places the four physical towers on top of the paper under the label*]. Here’s the “one [*white*] moving down.”

When R2 asked her why, she supported this claim by alluding back to the assertions of the task that one could only “use four blocks” and so “you can’t have another one” (L15). This was an argument by contradiction. R2 imposed another case for her to consider: the case of moving the white cube to the next position from the top. Stephanie claimed that one would need another available position, but that was not possible. Stephanie provided a warrant for the researcher’s request for an explanation, arguing that there were only four available positions (L19). When R2 asked, “Why can’t there be five?” (L20), Stephanie backed this warrant, “Because well…the assignment said we have to use four” (L21). Therefore, Stephanie found ***four*** towers in this case.

In conclusion, the following consisted of the components of Stephanie's justification for the case of towers with exactly one white cube selecting from two colors: (1) She explained that the *heuristic* she used for generating the towers was by a recursive method of moving the white block from the top position one down from the till she reached the bottom-most position; (2) She *counted* the total distinct towers with exactly one white cube; (3) She provided an *argument* by contradiction that there were no others because, if another tower would be generated by the recursive method, then the towers would be five tall, thus violating the assumption of building 4-tall towers selecting from two colors; and (4) She defined a *name* for this case as "one moving down."

Cases of exactly two, three, and four of a color together (see Event 7). This event illustrates a discussion about three cases: "2 white glued together moving down with two black" (L33), "3 together whites with one black" (L45) and "4 whites together" (L53; labels in original). She found ***three*** towers of exactly two adjacent white cubes, ***two*** towers of exactly three adjacent white cubes, and ***one*** tower of exactly four adjacent white cubes.

In this event R2 asked her if the case of exactly two white cubes was complete when Stephanie began to present the case of exactly three white cubes together. Stephanie asserted, "No, these are not the only way you can do two white, but these are the only way you can do two white *together*" (emphasis in original; L39). She specified the attribute of the case, that there were two adjacent white cubes in each tower. This provided evidence that she was aware of other tower combinations with two white cubes, as well as a justification that the case under question was complete. She also explained that when she used her case method she found the cases of towers with white cubes "stuck together [*and*] then went back and went two white apart" (L41). R2 asked her why there would not be more towers in the cases of exactly two and three white together. Stephanie's argument again considered the 4-tall assertion (L45; L48). For example, for the case of exactly two white cubes together she stated there were no more tower combinations "because if you take the last one [*BBWW*] you can't move them [*the two together white cubes*] down another one [*a fifth position*] because you're only using four blocks" (L45).

Case of exactly two white separated by at least one black (see Event 8). For the case of exactly two white apart, she found ***three***, with WBWB and WBBW as one set and BWBW as its own set. She described reversing the tower WBWB upside down to find BWBW (L58). R2 and Stephanie engaged in a lengthy discussion about the two white apart case, specifically to convince hypothetical Stephen that the case of exactly two white apart had only three towers. This led to more specific questions about the relationship between the two towers WBWB and WBBW and the tower BWBW. Stephanie did not directly answer the question about how the two former towers were alike (L82-93); however, she indicated recognition of the differences between the towers: "This has the black at the top [*BWBW*] but these don't" [*WBWB and WBBW*] (L93). The researcher provided more detail to the differences and similarities between these towers and posed the original question to Stephanie (L94):

> If I asked how you these two are different than this, these two had whites at the top [*referring to WBWB and WBBW*] and then you had to skip [*referring to the one black between the whites in WBWB*], you have white here. And now the only way you skip is this white here [*pointing to the W in the first position of WBBW*]. And this starts here [*pointing to the top W in BWBW*] and you had to skip [*to place the second W in the first position*]. How do I know there's no more?

Stephanie argued that if one was to separate the white cubes in BWBW any further, there would need to be another black cube between the white cubes, and thus a 5-tall tower (L105):

> You have the black, white, black, white, and if you want to separate again, you'd have to have another piece [*another imaginary black in between the bottom B and W of BWBW*].

Then she gave the same argument for WBBW that no more black cubes could be between the white cubes (L107):

> We can't make another one of these [*WBBW*] because you can't fit any more than two [*black*] in between.

The researcher stated that she was describing individual attributes of each tower and that this was not convincing her why there were no more. So, in response, Stephanie demonstrates recursively, starting with tower WBWB, that accounted for how all three towers with exactly two white cubes separated can be generated (refer to Figure 6.3.1; L111):

> Then these two [*alternating color towers*], you have white, black, white, black [*WBWB; see #1*], and then you move the white down one [*both fingers on white move to the third floor and the other white to the first floor to produce BWBW; see #2*]. If I were to put the white here [*referring to the W on the third floor in BWBW and moving it to the second floor*]…I'd be getting this [*BBWW, a duplicate of the two white together group; see N*]. If I were to put the [*second floor*] white here [*to the first floor*] and…the [*other*] white here [*to the top floor*], I'd be getting that [*WBBW; see #3*]…[*She explores other positions of the white which returns only duplicates; see D*]…So I shifted to every single level and you can't make another one.

Figure 6.3.1 illustrates her display of justification. "D" refers to a "duplicate" tower, "N" refers to a tower that "does not belong to this case," "W" refers to the "white" cube, the black shading represents the "black" cube, and the numbers indicate how many unique towers were generated by this method. The arrows follow her movements with the white cube from one position to another to indicate another tower she was generating.

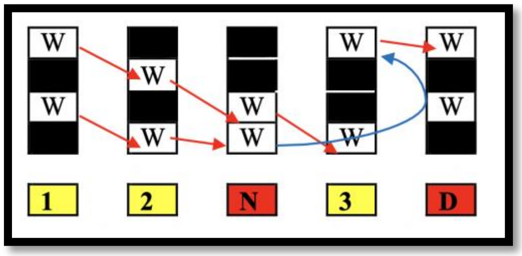


Figure 6.3.1. Stephanie's recursive method for the two-apart case.

Figure 6.3.2 illustrates her demonstration of moving the white cube to each position to argue that her recursive method accounted for all possible places where the two white cubes could be positioned.

Figure 6.3.2. Stephanie demonstration of the completion of the two-apart case.

Following this argument, R2 asked her to record the towers with exactly two white cubes separated alongside the recording of towers with exactly two white together.

Case of exactly three white separated by a black cube (see Event 9). The researcher summarized the cases thus far and posed a question for the next case of exactly three white, remarking, "So you've convinced me that you have **four** of exactly one whites; you've convinced me that there are **six** of exactly two whites. And are there **two** of exactly three whites?" Stephanie distinguished between the case of the whites together and the case of some white cubes separated by a black cube. Stephanie responded by trying to build new towers with exactly three whites separated by a black cube using a similar recursive method that she did for the towers with exactly two white separated (refer to Figure 6.3.3; L116-27):

> These are separate whites…let's say we'll start at the top. No, we can't start it at the top [*BWWW does not belong in this case*]. Okay there's your first one [*WBWW; see #1*]. There's your second one [*WWBW; see #2*] because you moved the black down one. And that's it. You can't make a third one…Because…if you move your black down to the last space you would have this [*builds WWWB and points to its duplicate*].

Figure 6.3.3 illustrates her display of justification. "N" represents to "does not belonging to this case," "W" represents the white cube, the black shading represents the black cube, and the numbers are counting how many different towers are generated by this method.

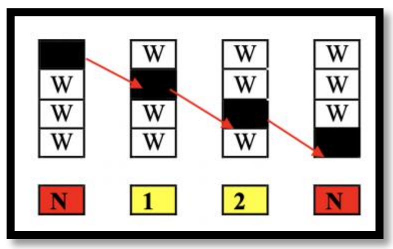


Figure 6.3.3. Stephanie's recursive method for the towers with exactly three white separated.

The researcher requested her to include the case of towers with three white cubes (with some separation) next to the case of three white cubes adjacent. Stephanie did so by placing the two cases adjacent to each other.

Cases of exactly four white cubes and exactly no white cubes (see Event 9). The next cases they discussed were the towers with exactly four white and exactly no white, which Stephanie immediately found to be a total of ***one*** and ***one***, respectively (L125-35). The researcher asked her to record the case of no white next to the case of exactly four white on her paper. Figures 6.3.4 and 6.3.5 show the representation in physical models. Thus far, Stephanie accounted for ***16*** towers.

Figure 6.3.4. 4-tall Tower solution organized by cases.

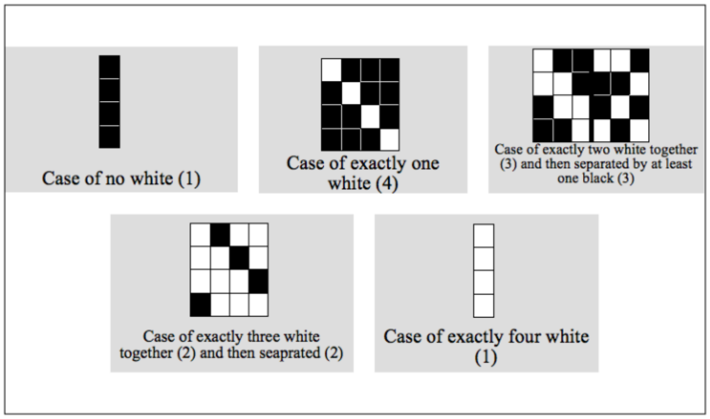


Figure 6.3.5. Stephanie's case-based organization for the 4-tall Tower solution.

The relationship between the case method and the color opposite towers (see Event 10). Earlier in the interview Stephanie described using the color opposite strategy with her cases of exactly one, two, three, and four black together, which led her to the solution of 20 (10 towers total for the cases of the white cubes, and 10 towers total for the cases of the black cubes; L61). She had not realized yet that the opposite cases would be duplicates of existing cases. This occurred because the case of towers selecting exactly *X* number of white cubes was equivalent to the ***color opposite*** case of tower combinations selecting exactly *4–X* number of black cubes (e.g., BBWW is a tower found when selecting two white cubes as well as a tower found in the color opposite case of two black cubes).

After Stephanie described the cases, focusing on white cubes, the researcher asked her to summarize what she found so far and calculate the total. R2 asked her the total of each case and the cumulative total. She found ***16***; ***one*** "no white," ***four*** "one white," ***six*** "exactly two whites," ***four*** "exactly three whites," and ***one*** "exactly four whites" (L137-49). R2 recalled she claimed a solution of ***20*** towers and questioned how that could be after she had "just convinced [*her*] there can't be any more" (L150). Again, Stephanie stated, "But we didn't do the opposite" (L151). R2 asked her to review what was the opposite of each case. Stephanie showed that the color opposite of the case of exactly no white was the case of exactly four white. Then the researcher asked her about the color opposite of the case of exactly one white. Stephanie found it to be the case of exactly three white. With the help of the researcher she also found the color opposite towers of the case of exactly two white to be within its own case. After a lengthy discussion Stephanie explained the following about how the color opposites were generated when using a method by cases (L204):

> Here you are using two blocks in the middle, two blocks separated [*the case of exactly two of a color*]. And like here, even if you didn't notice it until the end, you're using three blocks and one block and down here you're using three blocks and one block [*the case of exactly one of a color and the opposite case of exactly three of that same color*].

Notice that Stephanie did not specify the colors when stating the attributes of the cases that made apparent the equivalence in structure between a case and the color opposite case (e.g., "three blocks and one block…and down here…three blocks and one block"), a consequence that color opposites combinations would be redundant.

Cases focusing on the black cubes (see Event 11). R2 asked her if she could construct the towers with focus on the black cubes and Stephanie asserted, "That would just be doing the opposite way! You could do that!" (L209-10). The following is an excerpt of how Stephanie described what she called "the black system" (L226):

> 215. R2: So how would you start if I said exactly no blacks? What would you expect that Stephen would show you?
> 216. S: Exactly no blacks? He'd show me this [*all white tower*].

217. R2: I see, I see… and exactly one black; which group would he go to do you think?
218. S: He would show me one black, he would show me this [*exactly three white case*]
219. R2: Which was what group for you?
220. S: This was the three group [*of whites*].
221. R2: Exactly three whites. What about exactly two blacks? Which group would he be doing?
222. S: He would go to this group. This was my white group; my two white at the bottom [*BBWW*].
223. R2: So was it the same group?
224. S: Actually he was just doing the opposite as me. I would put it in the same group [*exactly two whites*].

In conclusion, R2 asked Stephanie to reflect on the opposite strategy that produced 20 towers and the cases strategy that produced 16. She stated that she would use the case methodology to generate the towers and the opposite strategy to check the solution. In response to R2's query about using solely the opposite strategy, Stephanie replied, "The opposite strategy can work, but I think it's better to go back and make the opposites" (L228).

3-tall Tower Task-solving strategy using the 4-tall solution strategy. Stephanie had just completed her case argument for the solution for the 4-tall Tower Task. Her cases consisted of exactly no white, one white, two white, three white, and four white. When questioned about her solution to the 3-tall Tower Task, Stephanie responded that there was a total of six, 3-tall towers, using cases from the 4-tall Tower solution as a base for comparison. She noticed she had *four* towers of exactly one white, *three* of exactly two white "glued together," *two* of exactly three white "glued together," and *one* of exactly four white. Using this (four, three, two, one) pattern, she conjectured that there would be *three* of exactly one white, *two* of exactly two white "together", and *one* of exactly three white, or a total of six towers. Note that she skipped the case of towers with no white cubes. R2 asked Stephanie to set up a “parallel [*case*] argument” (L269) for the 3-tall Tower Task to test her conjecture and was reminded to begin with “no whites” when she started with the case of exactly one white. She identified one tower “no white” and then three towers with exactly one white. After identifying these categories, she was asked if she still thought the total would be six. She now replied *seven* and explained that she, “didn’t include the no whites” (L284) in her conjecture. She was asked if it is possible to have seven, considering opposites. She replied it was possible, stating, “you don’t *always* have to have an opposite” (emphasis in her tone; L288). The researcher did not comment. For the case of exactly two white cubes “glued together,” Stephanie showed her earlier estimate of two towers was correct (L289-96). Then Stephanie identified one tower as exactly three white. Her tower model displayed her conjectured pattern of three, two, one and also included the one tower from the case of no whites. She concluded seven, but the researcher asked her if the whites have to always be glued together. Immediately Stephanie built one tower with exactly two white cubes separated (WBW). She summarized that she found eight 3-tall towers: ***one*** “no white,” ***three*** exactly “one white,” ***two*** exactly “two white glued together,” ***one*** exactly “three white,” and ***one*** exactly “two white separated” (see Figure 6.3.6 of her final work).

Figure 6.3.6. Stephanie’s 3-tall and 4-tall towers arranged by cases.

The researcher asked Stephanie what she thought about her new result, to which she replied:

304. S: I think that I found the opposite [*already*]. I think that it's going to turn out the same way that it turned out here [*points to the 4-tall cases of exactly one white and exactly two white*] with the opposites…Well you know how we found the opposites from here [*exactly one white*] to here [*exactly three white in 4-tall*]? [*Stephanie rearranges the 3-tall by opposites*]. Here its exactly two white cause here's your opposites [*puts BWB and WBW together*].
308. R2: Oh, so exactly one white, the opposite turns out to be in…
309. S: …The exactly two white.

Stephanie indicated that she was unsure why that may have occurred. R2 asked her how she changed her mind from six to seven and then from seven to eight:

315. R2: What made you change your mind from the six to the seven to eight?
316. S: I forgot to count the no whites
317. R2: What made you change your mind from the seven to the eight?
318. S: I saw that. I forgot about this one [*WBW*], the one with the two [*white*] separated

Stephanie acknowledged the two cases that she missed. Notice in Stephanie's continued problem solving her attention to a complete argument.

When R2 asked Stephanie to use the same "way of proof" (L329) for the two-tall problem, Stephanie immediately built black, black (BB) for the first case, named it "no white," and built the other towers silently. Her arrangement was similar to the 4- and 3-tall structure and it consisted of the cases of no white cubes, exactly one white cube, and exactly two white (see Figure 6.3.7). She found a total of *four* towers.

Figure 6.3.7. 2-tall, 3-tall, 4-tall towers arranged by cases.

Stephanie then conjectured the solution of towers one cube high would only consist of one tower (the white tower). The researcher suggested to her, “Okay, let’s do towers of one. Exactly no whites” (L335). Stephanie pulled of a black cube from a larger set of cubes and then corrected herself, “actually there’s two” (L336; see Figure 6.3.8).

Figure 6.3.8. 1- through 4-tall towers arranged by cases.

6.3.1 Findings: Key developments of Session VIII

Display of justifications and forms of reasoning in support of the solution. In this session Stephanie justified the solution to the 4-tall Tower Task with a complete argument by cases. She did so by cases, enumerating all towers, using recursive strategies, with no cubes, one, two, three, and four cubes white adjacent to each other, then two cubes and three cubes white separated by at least one black.

Each case was analyzed for its distinctiveness as compared to other cases and for its completeness given the generating strategy. For example, for the case of exactly one white cube, Stephanie argued by contradiction that the condition of the 4-tall Tower Task would be violated because, if the elevator pattern continued in either direction, it would generate towers beyond the given height (from the bottom-most and from the top-most position of the white cube). For other cases she provided an abbreviated argument by contradiction, which she phrased as “the same argument,” that R2 accepted.

Recognition of patterns and use of strategies. As compared to Session VII, in this session the odometer strategy did not emerge as a way to generate or to justify the completeness of the two-white apart case. However, she justified using recursion to move both white cubes in every position to explain how she eliminated duplicates and generated all unique possibilities. Perhaps towers 4-tall with separation were easier to find than towers 6-tall with separation and so the need to control for variables did not emerge. Also, she separated the cases with separation and adjacency of white cubes (so she did not originally organize the towers with exactly one black in an elevator pattern because these cases were separated).

Moreover, Stephanie had an opportunity to explain how the opposite strategy fit in with the case organization. Recall that she found 20 towers solving at home using a method by cases and opposites. The researcher asked her to find each opposite case within her global organization. She recognized that the case of exactly one white and exactly three white were opposites, since they both had one of a color in every position and three of the other color in the remaining positions. With help from the researcher she recognized the towers opposite of those with exactly two white existed within its own case and explained that it was due to the structure of “two blocks [*of one color*] and two blocks” [*of the other color*]. Due to the 4-tall condition she recognized that the structure of the color arrangement when choosing two whites in four positions is also choosing two blacks in four positions. She decided that the opposite strategy could be used after exhausting all towers by cases as a verification of the solution. She applied the case and opposite check methodology for the 3- and 2-tall Tower Tasks. She continued to separate the cases with white cubes separated and white cubes adjacent.

Furthermore, Stephanie had the opportunity to reflect on the case methodology when the researcher asked her how she would approach the task with the focus of black cubes instead of white. She claimed that the “black system” was the same, but in opposite order of the cases with white cubes as the focus color and showed this with the solution of towers in front of her.

6.4 Summary of Grade 4 interviews

Recognition of patterns and use of strategies. Throughout all interviews about 4-, 5-, and 6-tall Tower Tasks, R2 asked her how she would describe in written form the towers/sets of towers that she generated. The ideas discussed were the relationship of the *attributes* of a case (i.e., the number of cubes of a particular color and how those cubes were positioned in relation to each other and the other colored cubes), the *recursive rule for generation* (i.e., the recursive pattern used to permutate towers with particular attributes), the *differences* of each case (i.e., ensuring that towers were not repeated elsewhere), and the *exhaustion* of each case (i.e., the completion of a case based on the attribute and the pattern). In the second interview the discussion of a *global organization* emerged and by the end of the third interview Stephanie had developed a globally exhaustive systematic method of enumeration (by cases).

When Stephanie revisited the 5-tall Tower Task in Session VI, her solution was *systematic*; however, it was *not exhaustive*. She used ideas presented in whole class sharing: the staircase, the elevator, and doubling for the color opposite groups of towers within a particular case (by stating or drawing some examples that showed evidence that she was aware of these towers to the researcher). In Session VII when she solved the 6-tall task she had developed a method by partial cases. She experienced duplication and conflict between differing results in both sessions, but for different reasons. In Session VI it was her use of different recursive rules (i.e., staircase, elevator, composite operations of opposite and inverse) for different cases that required her to flag for duplicates, while in Session VII it was the enumeration by cases recursively (i.e., exactly one, exactly two, exactly three, exactly four, and so on) using the elevator pattern in combination with the color opposite operation that produced equivalent cases. She indicated uncertainty of the possibility of being convinced by a particular solution in Sessions VI and VII, but not in Session VIII. In Session VIII, she had solved the 1-, 2-, 3-, and 4-tall Tower task with a globally exhaustive method based on cases.

Stephanie also *folded back* to the color opposite and/or inverse pairing strategies that Dana and she used to generate towers with one of a color in every position (in Session VI) and two of a color apart (in Session VIII). Note that the case of towers with exactly one blue cube organized by the elevator pattern or generated by the elevator strategy was not her initial strategy, even after the presentation in the previous day's class discussion. After finding the towers and being asked their common attribute, Stephanie acknowledged the towers had one blue in every position. Stephanie then was asked to list the possibilities with exactly one blue, which she listed recursively. The idea of finding towers with a common attribute recursively, in an elevator pattern, emerged during this discussion.

Representations. The researcher encouraged Stephanie to record her solutions and ideas during the interviews. Towers were drawn in grid form. When drawing towers, she wrote a descriptive title for each case, using every day language. She also wrote the count of towers for each case and a double count for the opposites. In the third interview, she wrote the count for each case and considered the opposites as already within the cases. At some instances, she returned to using physical towers to explore new ideas, such as exploring the elevator strategy for towers with exactly one of a color in the first post-interview and exploring the odometer strategy for towers with exactly two of a color separated in the second post-interview.

When she was asked about the *attributes* of a set she described the commonality using everyday register. For example, the towers with “two blue at a time” referred to towers with exactly two blue cubes adjacent to each other. The researcher asked her to describe these towers more precisely by showing her an example of a tower with two blue cubes apart. Stephanie attended to counterexamples such as this and developed more precision about the attributes of each case. In Sessions VI and VII this occurred primarily through verbal precision rather than a change in the written label. In Session VIII she specified adjacent colors as “together” and explained that “two at a time” referred to towers beyond same colors that were together. Through explanations to the researcher, her use of everyday register terminology, such as “at-a-time,” “partners” or “doubles” (color opposite towers), “ones” (exactly one of a color), “twos” (exactly two of a color adjacent), “threes” (exactly three of a color adjacent), etc., became taken-as-shared between Stephanie and the researcher.

By the third interview Stephanie would use a specialized register (a combination of math, combinatorial, academic, and everyday register) to describe attributes with more precision. Academic and math register included words such as “exactly,” “at least,” “argument,” “prove it,” “category.” When using terms such as “argument” or “prove it” it would be associated with a situation in which, prompted or unprompted, Stephanie was providing a justification for the completeness of a case. Other words included “patterns” (This word evolved from a reference to a single tower’s attribute to a reference about a case’s attribute or about a numerical doubling pattern), “together” and “separated” (to distinguish between towers that had adjacency versus separation of a particular color), and “the two times thing” (the doubling pattern).

Forms of reasoning in support of the solution. Stephanie used direct or indirect reasoning to support her arguments for the exhaustion and the difference of a case (to other cases) using a particular rule. In each session she generated towers based on a rule and a color of focus (e.g., towers with two blues must be adjacent to each other); in other words, she generated towers by cases. She reasoned directly about the asserted rule to justify the exhaustion of a case. If the assertion was that all three blue cubes must be separated, then she reasoned by example that if one of the blue cubes was varied to any other position, it would cause two blues to be adjacent to each other. If the generating rule and the number of a particular color of focus for two cases was different, she directly reasoned that no duplicates were produced. In some of these instances, this reasoning was *complete* and in others it was *invalid*. For example, in Session VI, she used a blue staircase pattern for one case and an elevator pattern for the two adjacent blue case. She argued, invalidly, that these cases were different. However, she then realized that the blue staircase pattern and the elevator pattern of two blue adjacent consisted of duplicate towers. In Session VII, she no longer used the staircase strategy, perhaps due to this prior experience. Instead, she exhausted towers by partial cases for towers with one, two, three, four, five, and six blue cubes adjacent, and applied the opposite strategy to every case. In support of this, she reasoned that each of the cases had a different attribute of focus because of the differences in the number of blue (or yellow) cubes that were adjacent to each other. This reasoning was also invalid due to the focus of one attribute without relating it to the other (see *equivalent cases*). In the third interview, after generating 16 towers by a complete method based on cases, she discovered that opposites already existed within her solution (see *binomial symmetry*). In each of the aforementioned examples, she indicated acknowledgement of duplicates when it became a point of discussion.

In Sessions VII and VIII, she used a form of *indirect reasoning* to justify the exhaustion of some cases. For example, in Session VII, she argued that there could be no more towers with two adjacent blues, three adjacent blues, and exactly one blue using the elevator strategy once the color takes every position because, if one were to create another tower using the elevator rule, a taller tower would be generated that would violate the conditions of 6-tall towers. In another instance of Session VII, when justifying the case with two blue separated by at least one orange, she argued that if she continued the elevator strategy to the point that the two blue color cubes became adjacent (i.e., two blue cubes separated by zero orange cubes), she would then duplicate the towers in the two blue adjacent case. She also argued that if she continued the elevator strategy to the point that the two blue cubes were separated by five oranges, then she would then violate the conditions of the 6-tall towers. This was an example of a complete argument by contradiction in both directions of the elevator strategy. In Session VIII, she also argued by contradiction for each case that the continuation of the elevator strategy beyond the top and the bottom positions would violate the conditions of the height, which would violate the conditions of the 4-tall Tower Task. These were examples of complete, valid arguments in support of the exhaustion of cases. Some arguments in Sessions VII and VIII were found to be *abbreviated*, or in other words shortened or incomplete, but was valid and accepted by the researcher due to evidence of their form being taken-as-shared.

Chapter 7 Results: Stephanie's application of the case argument to justify solutions to Tower Tasks during assessments (Grades 4 & 5)

7.1 Small group formative assessment interview (Session IX)

Date	March 10, 1992
Grade	4
Task	Towers (3-tall)
Participants	Jeff, Michelle, Milin, Stephanie (i.e., "Gang of Four")
Researchers	R2
VMCAnalytic(s)	Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Part 3 of 3 (Grade 4); Events 1–4 https://doi.org/doi:10.7282/t3-4rx0-0p26

This small group assessment interview occurred on March 10, 1992 with four students, Milin, Michelle, Jeff, and Stephanie, and facilitated by R2 (R1, classroom teacher, and mathematics supervisor were observers). Milin, during his third interview, expressed interest to researchers about how other students were solving the problems. Each of the four students in this small group assessment had been interviewed at least once after they participated in the fourth-grade classroom session to solve the 5-tall Tower Task. Their approaches varied and so the researchers decided that sharing approaches with each other was timely. Stephanie had worked on towers up to 6-tall using case-based arguments to justify her solutions and organizations. Milin had worked on towers up to 6-tall, comparing his case-based and inductive arguments. Michelle had worked on the relationship between the Shirts and Pants problem and a Tower Task to inform her solutions to the Tower Tasks of various heights up to 5-tall. Jeff previously worked with his partner, Michelle, on the 5-tall Tower Task, organizing tower outcomes into various patterns. During this group assessment, Stephanie provided a justification by cases for the 3-tall Tower Task. She organized the eight tower outcomes into cases of no blue cubes, one blue cube, two blue cubes "stuck together," three blues cubes, and two blue cubes, separated.

The following events are divided into the main contributions of Stephanie that gave insight into her understanding and application of the case method. The order of the events is based on the chronological first occurrence of those contribution(s). The descriptions include other participants' contributions to the conversations. Their work on the Tower Tasks is presented in subsequent chapters.

Stephanie solving the 3-tall Tower Task by a case-based methodology (see Event 1). The session began with Milin's explanations of how taller towers can be generated from shorter ones. Discussion among members of the group about Milin's inductive method followed. R2 then invited Stephanie to share her justification. Her drawing consisted of a grid with horizontal and vertical lines delineating the color cubes and the towers, respectively. Figure 7.1.1 shows her first drawing of towers that were organized as follows: two towers with a single-color (RRR and BBB); three towers with exactly one red (or exactly two blues); and three towers with exactly two red (or exactly one blue). She began by naming each tower she found, explaining, "But that's what is different from mine. I just took the things and went– I just took one and went– Here is one red, red, red; blue, blue, blue and then I go like red, blue, blue; blue, red, blue–" (L238-40).

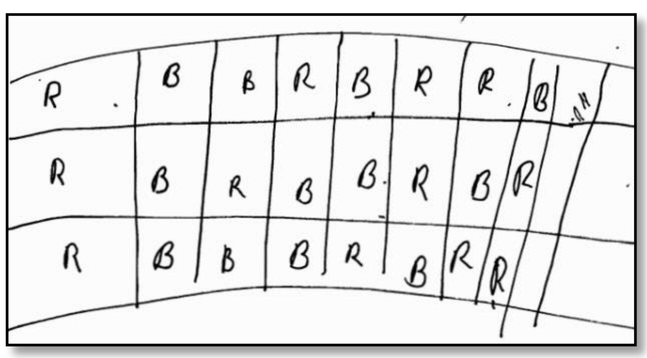


Figure 7.1.1. Stephanie's towers chart.

The researcher and Milin reacted:

241. R2: So, what I am hearing you say is that you're just–

242. Mil: –Guessing!
243. R2: You [*Stephanie*] believe there is eight. But you say guessing. Now, why does that sound like guessing?
244. Mil: Because what if you could make more?

Stephanie was challenged to provide a convincing argument by Milin, who asked, "Could you convince *her?*" [*referring to R2 with emphasis*] (L250).

Stephanie presented towers with no blue cubes and with exactly one blue cube (see Event 1). Stephanie reorganized her work. She drew a grid to show the following organization of towers: "All right, first you have without any blues, which is red, red, red" (L258). The researcher affirmed, "Okay, no blues" (L259). The following transcript describes Stephanie's presentation of the case of exactly one blue and the reactions that followed from the others:

262. S: Blue/red/red or red/blue/red or red/red/blue (BRR, RBR, RRB).
263. R2: Anything else?
264. Mic: And you would do the same pattern for–
265. S: No, not with the blue, not with 'one blue'–
266. Mic: You would do it, you would do it with one red and two blues.
267. J: You would alternate–
268. Mic: You would do it the other way around.
269. R2: That's not what she is doing. Let her finish. That's what you would do. You would alternate. Let's see what Stephanie does. Maybe she's not going to do that.
270. S: Well, there's no, there's no more of these because if you had to go down another one you'd have to have another block on the bottom…
271. R2: You buy that? That's all there is of those [*towers with one blue*]?
272. Mil, J: Yep, yeah.

Stephanie created the set BRR, RBR, and RRB *recursively,* showing the blue cube in every position on her grid drawing. R2asked her if there were any other towers. This sparked responses by Michelle and Jeff to create the color opposite towers. Stephanie disagreed that there were no more towers "with one blue" (L265), but she was interrupted by Michelle. The researcher sanctioned the alternative procedure as ideas/methods belonging to Jeff or Michelle, but asked the group to allow Stephanie to respond to her original question of “Anything else?” Stephanie argued that if the blue cube, “had to go down another one [*another position level lower*], you’d have to have another block on the bottom” (L270). In other words, if the blue cube moved one level lower, then the towers would be 4-tall rather than 3-tall. This was an abbreviated argument where the full argument by contradiction would include a warrant that the given tower height condition was violated when one continued the recursive pattern to one more level beyond the 3-tall assertion. By pointing to an imaginary lower position, she showed that her recursive method resulted in a tower taller than a 3-tall tower. Recall that this argument had been provided in full before in the previous interview (and had been present in all of the fourth-grade interviews) as a justification for many tower cases that she created for the 6-, 5-, and 4-tall Tower Tasks.

Towers with exactly two blues adjacent to each other (see Event 2): In this event Stephanie provided an argument for the case of “two blues” which were “stuck together.” First, she corrected herself when she suggested that the next case was towers with “three blues” (L270). R2 revoiced the cases that she had already completed: “You have no blues and now you have exactly one blue” (L274). Stephanie, rehearsing the researcher’s language, named the next case, “Now you have exactly two blues” (L275). She paused and recalled: “Wait, wait. Actually, yeah, that’s what I did last time I was here. I did exactly two blues” (L275). She drew BBR and RBB. The following excerpt portrays her explanation and Figure 7.1.2 shows her written work for the case in a red border:

278. S: Alright. You could put blue/blue/red *(BBR);* you could put red, blue, and blue *(RBB).*

279. Mil: You could put blue, red and blue *(BRB).* You could put...

280. S: Yeah, but that's not what I am doing. I'm doing it so that they're stuck together.

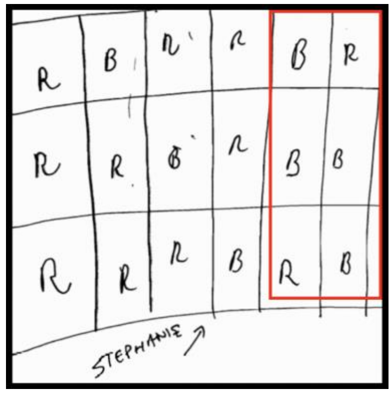


Figure 7.1.2. Stephanie's 3-tall towers by cases of "no blue," "one blue," "two blue."

Milin immediately pointed out that she missed the tower BRB and Stephanie counter argued that she was doing the case of exactly two blue cubes "stuck together" (L279-80). Stephanie stood by her organization even when Jeff twice suggested to capture other towers, "one red…two reds…three reds" (L282). Milin noted this, remarking, "Ah, but see, you did the same thing, but there's the blue" (L283). Stephanie claimed, "Well, that's not how I do it" (L285). The researcher asked the others to hear Stephanie explain her method. Stephanie started with "one blue, two blue," but the researcher added, "no blue," and Stephanie continued and repeated, "one blue, two blue" (L287-9). R2 then reminded her of Milin's challenge: "…one blue, and two blues, but Milin just said you don't have all two blues, and you said that – why is that?" (L290). It was at this point that Stephanie challenged Milin to draw her another tower with two blues, "With them stuck together, because that's what I'm doing" (L291). Milin acknowledged that there were no other towers in that category. Michelle posed a question, "What if you just had two blues and they weren't stuck together, you could–" (L294) and Stephanie repeated that this was not her method.

Exactly three blue and two blues apart (see Event 3). Then Stephanie continued to build one tower from the case of "three blues" and immediately followed with one tower, "blue, red, blue [*BRB*]," with "two blues stuck apart [*separated*]" (L297). Jeff noted that Milin and Michelle wanted to put the tower into the earlier two blues case. Milin alluded to an example in the case of exactly one blue where the tower RBR (with two reds separated) was in the same category with towers with two reds together and claimed that she was "following no pattern" (L303). Michelle added that within the case of exactly one blue, some red cubes were stuck together, while others were not. Stephanie provided backing to her organization (L305):

> Well, you are following your pattern, but my pattern goes no blue, one blue. This [*tower RBR*] was not meant to be like that [*referring to RBR did not belong in another category*]. It's in the category one blue [*the RBR*].. I could stick that in another category, but I want this to be in the category of one blue and not in the category of the opposite of this one [*of one blue*]. And then I have this one, red, red, blue [*RRB*]. So, to you, you might put that way at the end of the line, but I put it right here.

Stephanie explained that her towers with two red cubes and one blue cube were placed in the particular category of "one blue." She also acknowledged that the group might have put the towers in other categories.

Repeating the argument for Jeff (see Event 4). Jeff then asked the group, "Do you have to make a pattern?" (L306). Michelle pointed out that a pattern helped one to be sure there were no duplicate towers and that all towers were found. Stephanie called Jeff's attention to review her argument presented earlier. She pointed to the "one blue" case and challenged him "How could I build another one blue?" (L329). Jeff conceded it was not possible. Stephanie asked him if she "convinced [*him*] that there's no more one blue?" and Jeff responded affirmatively (L330-2). Stephanie added that if she were to put one of these towers in one place and another one of these towers on "another piece of paper" it would be harder to convince him. Jeff tried to challenge her that she was missing towers with one blue and Stephanie replied to his challenge:

> 347. S: But I have those three. Look blue/red/red, red/blue/red, red/red/blue. But then how am I supposed to make another one once that blue got down to the last block?
> 348. J: You can't.
> 349. S: Okay, so I've convinced you that there's no more 'one blue'?
> 350. J: Yeah.
> 351. S: All right, now we move on… 'two blue.'

Note that Stephanie described the towers recursively from top to bottom and pointed to the bottom cube when she asked Jeff if another one could be built. Jeff confirmed he was convinced. Stephanie then described the towers in the case of two blue cubes adjacent, which again sparked a debate:

> 353. S: Two blue. Here's one – right? Two blue – we have one, blue/blue/red, then we have red/blue/blue. How am I supposed to make another one?
> 354. J: Blue/red/blue.
> 355. S: No, this is together. Milin gave me that same argument.
> 357. J: But the thing is it doesn't matter if they're together
> 358. Mic: No, she means stuck together.
> 359. S: Stuck together, that means like –
> 360. J: I know.
> 361. S: Okay, so can I make any more of that kind?
> 362. J: No.

She concluded with the last two cases, with the help of Michelle:

> 363. Mic: Then you have to move to three, which you can only make one.
> 364. S: All right, yeah, you can only make one. And then you can make without blue, with the three red.
> 365. Mic: And then you can make two split apart.
> 366. S: Two split apart, which you can only make one of. And then…you can find the opposites right in this same group. All right, so I've convinced you that there's only eight?
> 367. J: Yeah.

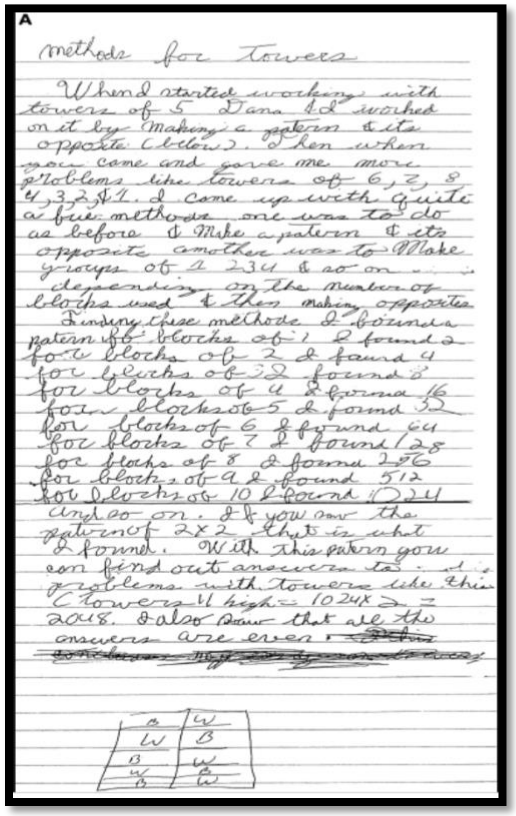
In a later discussion about the solution for 4- and 5-tall Tower Tasks, Stephanie noted it is possible to find these but, “the hard part is to make the pattern” (L407).

7.2 Individual written summative assessment

Date	May 15, 1992
Grade	4
Task	Towers
Participants	Stephanie
Researchers	R2
VMCAnalytic(s)	n/a

Recall on the last fourth grade individual interview R2 requested that Stephanie write an essay about what she learned in solving the Tower tasks. On May 15, 1992, the researchers followed up with Stephanie, who provided a written post-reflection of the fourth-grade interviews (Maher and Martino, 2000). Stephanie was asked to reflect upon and write about the methods she used for finding and justifying all possible towers. She wrote the following:

> When I started working with towers of 5 Dana & I worked on it by making a patern *[sic]* & its opposite (below). Then when you came and gave me more problems like towers of 6, 7, 8, 4, 3, 2 & 1 I came up with quite a few methods one was to do as before & make a patern & its opposite another was to make groups of 1 2 3 4 & so on depending on the number of blocks used & then making opposites. Finding these methods I found a patern
> For blocks of 1 I found 2
> For blocks of 2 I found 4
> […]And so on. If you saw the patern of 2x2 that is what I found. With this patern you can find out answers to problems with towers like this. Towers 11 high=1024x2=2048. I also saw that all the answers are even.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methods for towers

When I started working with towers of 5 Dana & I worked on it by making a patern & its opposite (below). Then when you came and gave me more problems like towers of 6, 7, 8 4, 3, 2, & 1. I came up with quite a few methods one was to do as before & make a patern & its opposite another was to make groups of 1 2 3 4 & so on ... depending on the number of blocks used & then making opposites
Finding these methods I found a patern for blocks of 1 I found 2
for blocks of 2 I found 4
for blocks of 3 I found 8
for blocks of 4 I found 16
for blocks of 5 I found 32
for blocks of 6 I found 64
for blocks of 7 I found 128
for blocks of 8 I found 256
for blocks of 9 I found 512
for blocks of 10 I found 1024
And so on. If you saw the patern of 2×2 that is what I found. With this patern you can find out answers to ... problems with towers like this (towers 11 high = 1024×2 = 2048. I also saw that all the answers are even.

B	W
W	B
B	W
W	B
B	W

Figure 7.2.1. Stephanie's handwritten reflection of Tower tasks.

She recalled her experience working with her partner, Dana, that they randomly built towers, discovering patterns that included finding tower opposites. She noted that she used another method, which "was to make groups of 1, 2, 3, 4 & so on depending on the number of blocks used & then making opposites[6]." She referred to a method based on cases that she had used when providing justifications during interviews. She stated the number of cases necessary to generate a solution depended on the "number of blocks used," which could be interpreted as the tower *height* because this is language she used before throughout the interviews. The "number of blocks used" could also be interpreted as the number of a particular color in a case (e.g., two blue). She also included in her

[6] Note that there was an equivalent relationship between the case with *X* number of a color and the *height minus X* number of the opposite color, and so, generating all consecutive cases up to the height would produce duplicate cases.

method the identification of opposites. This may be interpreted in two ways. Firstly, she could have been referring to doubling each case with their opposites, which she had done in all of the individual interviews with R2 and this caused duplications since some opposites existed in other cases. Secondly, she may have been referring to identifying opposites after creating cases as a way to check her solution. Recall, this is similar to her account in the third interview with R2 after she noticed the opposite method in combination with the case method created duplicates.

7.3 Dyad summative assessment (Session X)

Date	June 15, 1992
Grade	4
Task	Towers (3-tall)
Participants	Dyad pair: Stephanie & Milin
Researchers	Barnes (classroom teacher)
VMCAnalytic(s)	Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Part 3 of 3 (Grade 4); Event 5 https://doi.org/doi:10.7282/t3-4rx0-0p26

Written assessments were administered to the fourth-grade students on June 15, 1992 about the 3-tall Tower Task. Students were grouped and given the following written problem statement:

> Chris and Alex have been arguing about how many different towers can be built from Unifix cubes if there are two colors available and if each tower must be three cubes tall.
>
> Will you please settle the argument in a way that shows every possible tower and will convince Chris and Alex that you have not left any out and that there are no duplicates.
>
> Use whatever materials you like to work out the problem. Let us warn you, though, that we can send on to Chris and Alex only the pages on which you have recorded what you have done. We cannot send actual plastic cubes.
>
> Please be careful to write enough so that Chris and Alex will be convinced.

The classroom teacher introduced the problem, reading it aloud and explaining that they would work with their partners and write their solution so that it can be explained to the students who were not in the class. Unifix cubes were available if needed. She demonstrated a 3-tall tower asking students to give two examples. She then asked how many more different towers are possible.

Milin immediately responded he knew it was eight and Stephanie stated she would have to try it to be sure. Stephanie and Milin agreed to work on the problem individually. The teacher asked if anyone needed anything and Stephanie raised her hand to get Unifix cubes. Milin asked, “what colors do you want?” and Stephanie replied, “it doesn’t matter; black and blue?” They ended up picking green and black Unifix cubes. First, they built the towers using the cubes. Stephanie placed the physical towers on top of her grid. Then they both used the letter B to represent black cubes and G to represent green cubes in their written work on their individual tower drawings.

Stephanie drew a rectangular grid of four straight horizontal lines and added a vertical line after she completed each tower representation (see Figure 7.3.1 of Stephanie’s work).

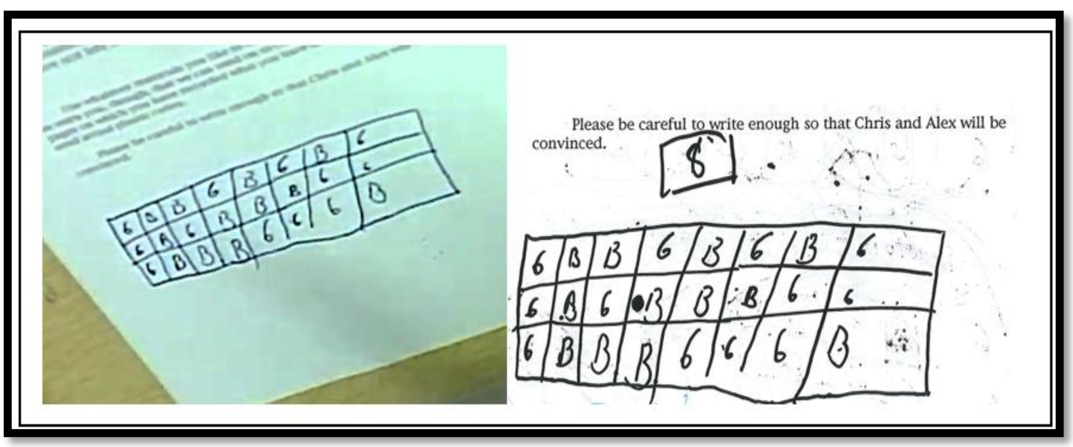

Figure 7.3.1. Stephanie’s first 3-tall solution.

Stephanie first considered the two towers of a single color (BBB and GGG) and recorded them as the first two towers in her grid. Stephanie took the lead in naming which color cubes followed each previous cube as they drew their tower representations. The next set of towers Stephanie drew was the case she called “two black,” (BGB, GBB, and BBG), which consisted of one tower with two black separated by one green cube and the other two towers with the two black cubes adjacent. Then she found the towers for the group “two green” (GBG, BGG, and GGB), similarly organized to the case of two black (see Figure 7.3.2). Notice that in this organization she included the towers with adjacent cubes and separate cubes of the same color in the same set. Although the cases of towers were not obvious on the written representation, Stephanie’s explanation provided evidence that she was grouping the towers in particular categories. Moreover, the color opposite sets were generated in juxtaposition, which was a suggestion that Jeff insisted on in the “Gang of Four” interview three months earlier.

Stephanie and Milin then decided to build the towers using green and black Unifix cubes and compare their model with their written solution. Milin was responsible for the two-black case and Stephanie for the two-green case. Stephanie took Milin’s two black towers, which were organized in an elevator pattern so that the single opposite color cube was positioned recursively from bottom to top, and she combined them with her two-green set, which she reorganized in the same pattern (see Figure 7.3.2).

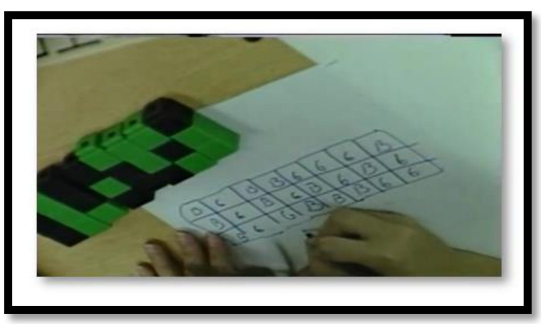

Figure 7.3.2. Stephanie’s second solution with physical towers and in written form.

Stephanie decided that they could convince Chris and Alex by drawing random towers below the solution representation that she created (see Figure 7.3.3) as a way to double check if her solution set included all of the random towers she would generate.

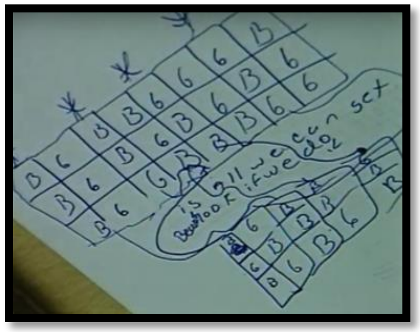

Figure 7.3.3. Stephanie's second solution by a variation of cases and a trial and error justification.

Milin suggested that she could see her solution above so Stephanie hid the solution above with a nearby textbook and began generating random towers. This was an example of a trial and error strategy to verify no towers were missing. She drew lines that showed how the random towers matched the towers within the solution set above it. She then wrote, "We have all of them. We just compared." Stephanie finalized her work by copying what she did on another piece of paper (see Figure 7.3.4). Then Milin and Stephanie discussed a doubling pattern that they would present to the hypothetical students. These results are presented in Chapters 9.4, 13.1.1, and 15.1.2.

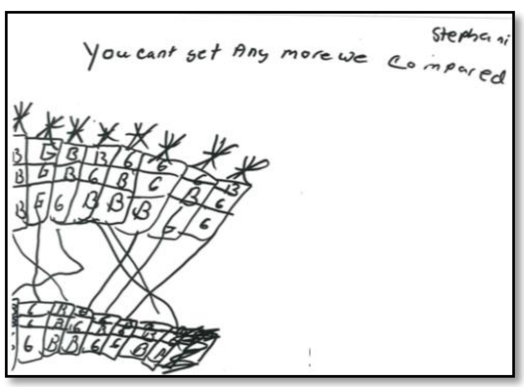

Figure 7.3.4. Stephanie's third recopied solution.

7.4 Individual written summative assessment (Session XI)

Date	October 25, 1992
Grade	5
Task	Towers (3-tall)
Participants	Stephanie
Researchers	R2
VMCAnalytic(s)	n/a

Written assessments were administered to the fifth-grade students on October 5, 1992 about the 3-tall Tower Task before they explored any more tasks using the towers that school year. Stephanie used an organization by cases for the 3-tall Tower Task (see Figure 7.4.1; adapted from Maher, 1998, p. 31):

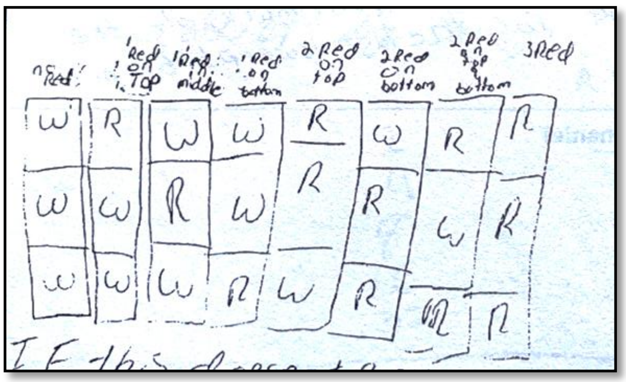

Figure 7.4.1. Stephanie's case-based organization for 3-tall towers.

In this organization she grouped her cases according to what she described as "color order." In other words, the cases were organized by the number of red cubes regardless of separation and adjacency for the colors in the case of exactly two of a color. She labeled the towers by specifying the number of red cubes and the location of the red cubes. They were labeled as follows: "no red, 1 red on top, 1 red in middle, 1 red on bottom, 2 red on top, 2 red on bottom, 2 red on top and bottom, 3 red." Notice that towers with exactly one red cube were organized in an elevator pattern from top to bottom, but towers with exactly two red cubes were organized by two red cubes adjacent, then by separation, and were all labeled as "2 red" with a specification of the position of the red cubes (see Figure 7.4.2; adapted from Maher & Martino, 1998, p. 31-2).

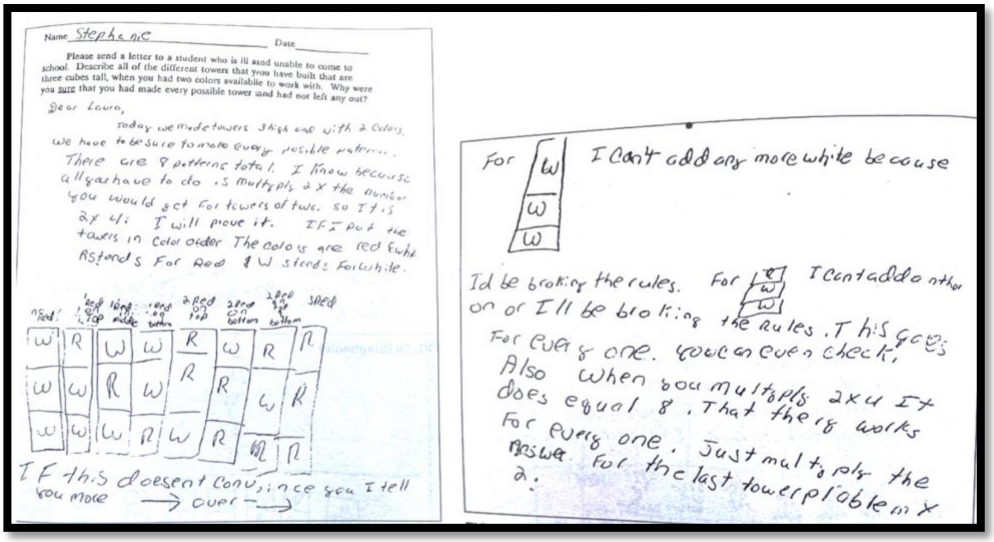

Figure 7.4.2. Stephanie's written explanation about her 3-tall tower solution

In her writing she argued why there were eight and only eight towers of height three when selecting from red and white cubes and she referred to a doubling "pattern" to support the results of her method:

> Dear Laura, today we made towers 3 high and with 2 colors, we have to be sure to make every possible pattern. There are 8 patterns total, I know because all you have to do is multyply [*sic*] 2 X the number you would get for towers of two. So, it is 2x4. I will prove it. If I put the towers in color order, the colors are red white. R stands for red and W stands for white [*draws eight towers organized by cases of the number of red cubes*]. If this doesn't convince you, I tell you more → over →
> For WWW [*drew the tower*], I can't add any more white because I'd be braking [*sic*] the rules. For RWW [*drew the tower*], I can't add another on or I'll be braking the rules. This goes for everyone. You can even check. Also, when you multyply 2x4 it does equal to 8. That thery [*sic*] works for everyone. Just multyply the answer for the last Tower Task x 2."

She argued that she could "prove" there were eight by organizing the towers in "color order." Her case-based organization supported that argument. She also argued that she could not add another white cube to a case, such as the single-colored tower WWW, because she would be "breaking the rules" of the three-high tower. Similarly, for the remaining cases she claimed, "I can't add another on or I'll be breaking the rules."

7.5 Summary of Grade 4 & 5 assessments

This chapter presented the results of how Stephanie applied what she learned about Towers tasks in formative and summative assessment sessions. In all assessment sessions, Stephanie attended to a global organization based on cases for the solution of the 3-tall Tower task. The descriptions of attributes were more precise and there was evidence of acknowledgement of the existence of opposite towers in her solutions based on cases. The recursive rule for generation of some cases was the elevator strategy. This is described in detail next.

Initially in Session IX, Milin challenged Stephanie's first solution by referring to it as "guessing." At the request of Milin and the researcher to convince them, this prompted Stephanie to present and justify how she had found all the towers 3-tall. Stephanie presented an argument by cases for finding all towers and an argument by contradiction to justify the towers of exactly one color. Her case organization was written in a grid format and consisted of the cases "without any blues" (or a tower with all red cubes), with exactly one blue cube, with two adjacent blues *("two blues stuck together")*, with three adjacent blues ("three blues"), and with two blues separated from each other ("*two blues separated*"). The cases of exactly one and exactly two adjacent blues followed the elevator pattern. She placed the "two blues separated" in a separate category which raised argumentation among group members that the tower combination of two blue cubes separated and stuck together all belonged to the same category of two blues. She declined their suggestions and considered "two blues stuck together" and "two blues separated" to be different categories. Recall this was how she built her cases in the individual interviews of Sessions VII and VIII.

When Jeff raised the question of the need for making a pattern, it stimulated a dialogue among the small group members who provided rationales for the value of looking for patterns. Stephanie repeated and refined her explanation, with Michelle's assistance, to convince Jeff of her case organization which included towers with exactly two of a color separated into two parts – two colors together and two colors separated.

In addition to completing her case argument, Stephanie claimed, "you can find the opposites right in the same group [*as she pointed to her solution*]." Tracing historically her interview experiences of finding duplicates when using the opposite strategy in combination with the case strategies, she may have been referring to the completion of her case organization when Jeff insisted on creating color opposite towers (in a different pattern) that already existed within her solution.

In the partner assessment of Session X, Stephanie and Milin solved the 3-tall task in multiple ways: 1) first written solution: the single-colored case; "two black" case with separation and then adjacent blacks; "two green" case with separation and then adjacent greens; 2) second solution with physical towers: the single-colored case; "two black" case with the green in an elevator pattern; "two green" case with the black in an elevator pattern; 3) "randomly" generated set of towers in written form. Maher and Martino (1993) indicated that the first solution by Stephanie was a refinement of her Session IX organization because she now incorporated the two of a color separated and adjacent towers in one group for both green and black. The second solution built with physical towers was also a refinement where Stephanie organized the "two green" case by varying the black in a recursive pattern, just as Milin did for the "two black" case. Although there was no indication of the cases with one black or one green (perhaps because they were equivalent cases to the "two green" and "two black" cases), the second arrangement provided evidence that the other color was acknowledged. A return to reasoning by *trial and error* was observed in the third solution for the purpose of ensuring all "randomly" generated (guessed by her) towers were not missing in the other solutions. This procedure for the purpose of conviction (her argument) was similar to her earlier argument for the 4-tall Tower Task in third grade, but her organization (recursive patterns and sets of opposites rather than single pairs of opposites) and language (e.g., "two green") were more sophisticated to ensure differences between towers.

In the individual assessment of Session XI, Stephanie solved the 3-tall task by cases recursively from the case with no red to the case with all three red. This was also a refinement of her Session IX organization because the two adjacent and separated red cases were organized together. Notice in Session X Stephanie modified her organization similar to Milin's arrangement by elevator pattern of the one color. However, in Session XI she returned to her organization of two adjacent reds (rather than of the one white) and then two separated reds (as she did in Sessions VIII and IX). Also, in the supporting argument, given her prior experiences, she may have been using an *abbreviated* indirect argument to justify that if one would continue the pattern in a case of towers, one would violate the assumptions of the 3-tall Tower Task. The difference in the current argument was that it was not complete because she did not warrant how the individual towers she picked as exemplars would violate "the rules" of the task.

This chapter concludes the results of Stephanie's learning progression on a method based on cases to solve Tower Tasks.

Chapter 8 Results: Stephanie's early connections between different Tower Tasks (Grade 3)

The following chapters (8-10) illustrate Stephanie's development and application of the doubling pattern for finding the number of taller towers and the supporting argument by induction. This chapter presents Stephanie's early experiences with and reasoning about various heights of towers.

8.1 Grade 3

8.1.1 Dyad: Stephanie & Dana (Session II)

Date	October 12, 1990
Grade	3
Task	Towers (3-tall vs. 4-tall)
Participants	Dyad pair: Stephanie & Dana; third-grade class
Researchers	R2, R1, and R3
VMCAnalytic(s)	Stephanie's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Events 1–2 https://doi.org/doi:10.7282/t3-hfzp-8376

On October 11, 1990, , the third-grade class, including students Stephanie and Dana, worked on and completed the 4-tall Tower Task. The next day the class shared their results. R1 then challenged the group to conjecture a solution for the 3-tall Tower Task.

Predictions for the 3-tall Tower task (see Event 1). R1 posed the question: "Suppose instead of towers that had four cubes, you could only have three cubes in each tower. Do you think there would be more towers, or do you think there would be fewer towers?" (Clip 2 of 6, L1). Stephanie conjectured that there would be the same number of towers. She justified her claim by explaining, "You are just taking one [*cube*] away from here [*pointing to the top cube of a tower*]. It's not like it's going to change the whole thing. It's gonna be one [*cube*] less" (Clip 3 of 6, L10–2). Other groups stated other predictions, which included that it would be fewer (e.g., 15) or that it would be more (e.g., Dana originally guessed 25). Matt suggested that they could settle this disagreement by removing the top cube from each tower and determining the solution. Each of the groups test their predictions using the 4-tall towers they built earlier.

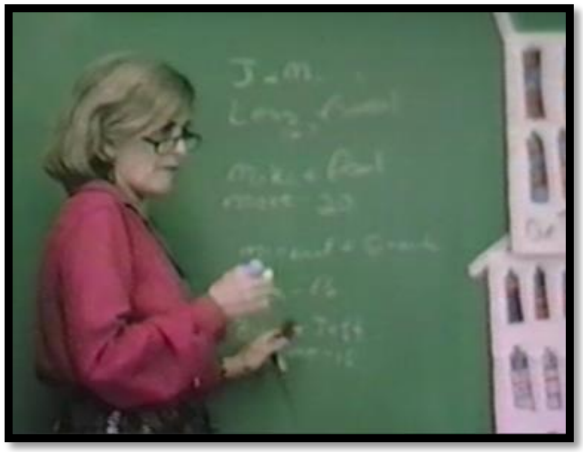

Figure 8.1.1. Predictions for the number of towers 3-tall.

Sharing their explanations for why towers 3-tall towers built from 4-tall towers were fewer (see Event 2). This episode portrays early exploration of comparing towers of different heights. Stephanie and Dana took the top cubes off of their 4-tall towers and attempt to build new towers using the available cubes, using a Guess and Check strategy. After several attempts to build new towers, Dana told R3 that they think there may be fewer towers (Clip 5 of 6, L13). In response for a justification Stephanie showed a tower and its duplicate: “Because we took one away. We had two of these [*RBB*]” (L15). R1 asked if there can be any more pairs that are duplicates of each other and Dana responded affirmatively. They continued to identify duplicates until they reach eight 3-tall towers.

Jeff, Milin, Stephanie and other students found eight towers 3-tall. R1 asked if any group wants to change their minds and for what reason. Stephanie responded, “It’s less. There is only eight...Because once you take these apart, you start to see that they match…one tooken [*sic*] off could mean a whole difference” (B2, L6–10). Dana stated there are eight that matched (L12). Brian stated that they attempted to build new towers, but found only duplicates (B2, L22). Jeff suggested (B2, L24–6):

> Well, because, first of all, you could choose to do it with a math problem. Sixteen minus eight is eight…or eight plus eight equals sixteen. And when you take one away from each, it would be one minus, one minus, and one minus because its sixteen minus eight or eight plus eight.

R1 revoiced his statement to the class.

8.1.2 Interview (Session III)

Date	October 12, 1990
Grade	3
Task	Towers (3- and 4-tall)
Participants	Stephanie
Researchers	R3
VMCAnalytic(s)	Stephanie's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Event 3 https://doi.org/doi:10.7282/t3-hfzp-8376

Stephanie explained why there were fewer 3-tall towers than 4-tall towers with imaginary towers (see Event 3). After the class sharing session, R3 interviewed Stephanie about what she learned. Stephanie claimed that there were fewer 3-tall towers and justified as follows (B3, L2):

> Wow, we learned the Unifix cubes, even though they [*3-tall height*] might be less, you might think there may be more [*3-tall towers*] because with the last block there is more combinations you can make – but there will be less [*towers*]. Because, once you take one block out– Say you have red, red, red, red [*RRRR*] and you have red, red, red, blue [*RRRB*]…Once you take one red away and one blue away [*gesturing removing the imagined red and blue cube*] it's [*the remaining 3-tall tower*] still the same.

Without any towers in front of her, Stephanie used generic reasoning to justify why there would be fewer 3-tall towers. She also explained that one might think there were more combinations of 3-tall towers because when cubes were removed from the 4-tall towers "there are more combinations you can make" with the available cubes. However, she pointed out that the remaining cubes, when used to make additional towers, generated duplicates, and then pointed out that there are duplicates in the remaining 3-tall towers.

She then described the experience as a "matching game" (B3, L16). Stephanie repeats her argument with another example: "Once you take these apart, you start to see that they match. Say you have blue, red, blue, blue [*BRBB*] and you have blue, red, blue, red [*BRBR*]. If you take off that red and that other blue…you have blue, red, blue [*BRB*]…blue, red, blue [*BRB*]" (B3, L18–26). By providing two different examples, which served as exemplars, she supported the claim in general that removing a cube from each 4-tall tower produced a pair of matching 3-tall towers.

8.2 Summary of Grade 3 problem solving and interview

Recognition of patterns and forms of reasoning. When solving the 3-tall Tower Task using the physical 4-tall towers, a relationship between the two tasks emerged. Stephanie used generic reasoning with physical and imaginary exemplars to justify why there were fewer 3-tall towers than 4-tall towers. Internal representations of tower patterns were evident at various instances (during each of the Sessions, I through III) when Stephanie fluently chose different examples of combinations of 4-tall towers that were duplicate combinations for towers 3-tall because the cubes in the three bottom positions "matched."

Chapter 9 Results: Stephanie's discovery and application of the doubling pattern to find number solutions to Tower Tasks (Grade 4)

9.1 Third Interview (Session VIII)

Date	March 6, 1992
Grade	4
Task	Towers (1-, 2-, 3-, 4-tall)
Participants	Stephanie
Researchers	R2
VMCAnalytic(s)	Stephanie's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 1 of 3 (Grades 3 & 4); Events 4–7 https://doi.org/doi:10.7282/t3-hfzp-8376

Stephanie had already worked on finding 3-tall towers from 4-tall towers (third grade) and 4-tall from 5-tall (fourth grade, first interview with R2 on 2/7/92) and compared totals between 6-tall and 5-tall (fourth grade, second interview on 2/21/92). In this third interview, Stephanie solved the 4-tall Tower Task first (refer to Chapter 6.3) and then continued to explore building shorter towers (3-tall, 2-tall, and 1-tall). This section reports on her exploration and discovery of relationships between solutions of consecutive heights of towers.

Stephanie's discovery of the doubling pattern after building towers 1-, 2-, 3-, 4-, and 5-tall (see Event 4). This event illustrates the first conversation between Stephanie and R2 about the emergence of a doubling pattern. At this point, Stephanie had generated all towers one, two, three, and four cubes high by cases. R2 asked her to record her findings on a piece of paper (see Figure 9.1.4). She wrote "1. 2" for the solution of 1-tall, "2. 4" for the solution of 2-tall, "3. 8" for the solution of 3-tall, "4. 16" for the solution of 4-tall.

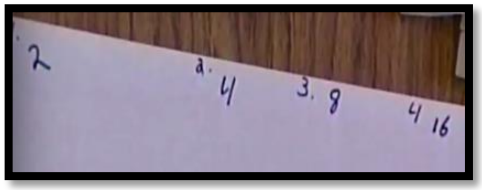

Figure 9.1.1. Stephanie's recording of number solutions for towers 1- through 4-tall.

Stephanie observed a doubling pattern and that the numerical solutions were even numbers (L350):

> That's weird… look, two times two is four [*pointing to the "2" and the "4"*], and four times two is eight [*pointing to the "4" and the "8"*], and eight times two is sixteen [*pointing to the "8" and "16"*]...It goes like in a pattern…it also turns out that every number is even.

The researcher asked her if the pattern holds, what would be her 5-tall tower prediction. Stephanie predicted 32 towers recalling her previous explorations with 5-tall towers in class with Dana, which also resulted in 32 towers.

R2 introduced the inductive argument for towers 1- and 2-tall (see Event 5). R2 asked her why she would expect four towers 2-tall from towers 1-tall. The researcher posed the following problem:

> Suppose I start with this [*shows the towers of one*] and now build towers of two. Why would I expect four rather than two? Now I'm building…Suppose I start with white on the bottom floor [*pointing to it*]. What kind of towers can I build with white on the bottom floor?

Stephanie showed that BW and WW were two towers that could be built from a white tower (W). Then she showed four towers (BBW, BWW, WBW, WWW) that are related to tower W and the other four towers (BBB, BWB, WBB, WWB) that were related to tower B. The researcher directed Stephanie's attention to the missing connections between the second and third tower levels (L380). Stephanie reiterated her observation (L381):

> You know what I'm saying? Look: with white on the bottom floor, these equal two [*towers 1-tall*] and with white on the bottom floor, you have two [*towers 2-tall]…*Now the total here [*for towers 2-tall*] is four [*2-tall towers*] and you get four [*3-tall towers*].

Stephanie observed a numerical pattern: from ***two*** 1-tall towers, ***two*** 2-tall towers with white on the bottom level and ***two*** *2-tall* towers with black on the bottom level can be found; and, from the total of ***four*** 2-tall towers, ***four*** 3-tall towers with white on the bottom level and ***four*** 3-tall towers with black on the bottom level can be found. She demonstrated by organizing the towers in the two groups as she spoke. R2 questioned Stephanie about the missing explanation regarding the second level, which could either have had a white or black cube. R2 reiterated to Stephanie that she did not account for the second level in the growth between 2- and 3-tall towers. Stephanie responded with "Yeah" and R2 posed another question to Stephanie after a few seconds of silence.

A relationship between the solutions of the Outfit and the Tower Tasks (see Event 6). Stephanie states that the Towers Tasks remind her of the Shirts and Pants Task. Stephanie explored the relationship between the outfits and the towers, using the towers to represent the possible outfit combinations where each level is another type of clothing. She began with building 1-tall towers (representing one type of clothing: pants) and continued building outfit representations with up to five options of clothing. For example, in response to the question about adding a black and a white feather to outfits with black and white pants, black and white shirts, and black and white hats, Stephanie stated, "Then we go to the fourth blocks [*4-tall*]…because we're adding another piece [*type*] of clothing" (L455-458). When exploring outfits with five types of clothing, she predicted 32 outfits by describing placing a black cube ("a black flower") on top of each 4-tall tower ("outfit") and placing a white cube on top of each 4-tall tower. She confirmed this by relating two of her previous activities where she found 32 towers, 5-tall. In response to, "What do you think you have here?" Stephanie replied that she has "a method...All you have to do is take the last number you had and multiply by two" (L500). She used the method to predict 64 towers, 6-tall. Notice she called the pattern "a method" and generalized its procedure for Tower Tasks (and for Outfit Tasks, selecting from two colors).

Using the doubling method to predict the number of towers 10-tall (see Event 10). This event displayed how Stephanie used her recognition of a pattern to find the number of towers of varying height. Stephanie was asked to think about the solution for 10-tall towers, using her new doubling rule. She used the strategy of building from the total of 64 towers, 6-tall, to find the total number of 10-tall towers:

> We know we have towers of six. Times it [*64*], six, seven, eight, nine, ten, that's four [*the difference between six and 10*]...by four– not by four– by eight because what happens is you multiply by two [*four times two is eight*]. It would give you [*writes '512'*]. Or you could just go like this: 64 times two equals [*writes '128'*]. That would be the seven *[tall]* towers. [*Writes '256'*]. That's the eight towers [*8-tall*]. Five twelve [*512*] is the nine [*tall towers; writes '1,024'*]. That's the ten towers [*10-tall towers*]. So, there is 1,024.

She reasoned that if the difference between 6- and 10-tall was four then she could multiply 64, the total 6-tall towers, by eight, since she was doubling four times. Stephanie attempted first to multiply by eight, which was the *double of the difference* between six and 10, and then checked her work by recursively multiplying by two.

Her reasoning, although invalid, was an attempt to find a relationship between towers of a height given towers of another height, when the heights were not necessarily consecutive numbers. Stephanie also checked her reasoning by using the recursive doubling rule for consecutive heights of towers until she obtained the total for 10-tall towers. The researcher encouraged her reasoning even when she found two conflicting results when using the doubling rule (1,024 towers) and using her shortcut method (512 towers). The researcher asked her to notice that she did not multiply by eight, but in fact multiplied by “two times two times two times two.”

Moreover, Stephanie noticed that her shortcut produced the result for 9-tall towers. R2 asked her to recalculate for 10-tall towers. She found that multiplying by two, four times, was the same as multiplying by 16. This event illustrates Stephanie applying the doubling rule and modifying it to create a shortcut.

9.2 Small group formative assessment interview (Session IX)

Date	March 10, 1992
Grade	4
Task	Towers (3-tall)
Participants	Stephanie, Milin, Michelle, Jeff (i.e., “Gang of Four”)
Researchers	R2 (R1, R3, and classroom teacher present)
VMCAnalytic	Stephanie’s Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 2 of 3 (Grade 4); Events 1–3 https://doi.org/doi:10.7282/t3-g20s-0d46

Participants claim various patterns for the solutions of Tower Tasks (see Event 1). During this group assessment interview that occurred on March 10, 1992 Milin presented to R2 and the other three participants, Stephanie, Jeff, and Michelle, an argument by induction for finding the number of towers 3-tall from towers 2-tall when selecting from red (R) and blue (B). Jeff challenged them: “If this was like a pattern, it would go two, four, six in between, and then eight” (L68). Stephanie responded that this was not the pattern we observed by stating, “The pattern that we saw was this: For one block at a time, we found two…four, and then eight alright? Two, four, and then eight” (L73-6). When R2 asked Stephanie to justify why eight, rather than six, towers three high, Milin volunteered to explain (L78-220). Note during his explanation, Stephanie was silent, writing her own solution. R2 asked Stephanie to contribute to the justification for towers 3-tall; however, the conversation continued by Jeff and Milin. When she was asked to contribute her thoughts again, she presented her solution. This solution was not analyzed in their group discussion. Rather, she produced another solution by cases when presenting to the group. R2 asked her if this method was different from Milin's. Her explanation gave insight into her understanding of his reasoning: "He built his towers up like this: he went red, blue, red, blue, red, blue, and so on" [*alternating writing the letters R and B to represent the bottom floors*] (L227-9). The researcher pointed out that she did not see Milin do that and Stephanie stated that she saw Michelle do it this way. Then R2 and Milin reviewed Milin's inductive argument as Stephanie watched. Finally, Stephanie concluded that her method is different and presents her reasoning for the eight towers 3-tall using an argument by cases (Refer to VMCAnalytic on Stephanie's Development of an Argument by Cases to Solve Tower Tasks: Part 3 of 3).

Stephanie discusses the theoretical method of obtaining the number of towers of any height (see Event 2). This event illustrates Stephanie offering a shortcut strategy that differed from the recursive doubling rule. After Stephanie presented her argument for the 3-tall tower, R2 asked about the solution to 4- and 5-tall Tower Tasks. The students offered the correct solutions. Then Stephanie explained her perspective for finding the solution (L407-9):

> The hard part is to make the patterns. Like, from now on, we know how to just – oh, you could give us a problem like how many in ten [*tall*] and we could just go– … I know the answer. I figured it out. It's one-thousand and twenty-four!...You could just give us a problem and we could go thirty-two times two–.

Stephanie referenced the doubling pattern to find the total combinations of towers (implicitly, when selecting from two colors) of any height by beginning with a known solution of a particular Tower Task, such as 5-tall. The reference for "patterns" is unclear and may be a reference to the case method.

Jeff challenged Stephanie by checking on his paper if the 10-tall solution was 1,024. Stephanie did not agree with his method and stated that she did the same earlier (L425):

> I just counted ahead. I counted ahead five or six [*difference in heights*] and I just multiply it [*the product of the difference in heights and two*] by that [*the solution to the smaller height*] and that would give me the same answer [*to the 10-tall towers*], but it didn't work…You have to figure out what's in between that [*the Tower number solutions in between a non-consecutive shorter and taller height*].

Then the observing researcher requested her to show the group how she found the solution to the 10-tall Towers task. She elaborated:

> You [*referring to R2*] wanted me to figure out ten, right? In order to figure out ten I was only up to five. So, what I had to do was I had to go and I had to say what was six [*tall Tower number solution*], what's seven [*tall Tower number solution*], what's eight [*tall Tower number solution*], what's nine [*tall Tower number solution*], and times that times the last one [*Tower number solution*] I had.

Stephanie described her idea about a shortcut method to find the solution of any height given the solution of another non-consecutive height. R2 showed the group Stephanie's written work from a previous interview. Stephanie elaborated on the shortcut strategy to solve the 10-tall Tower Task without recursively multiplying the previous height total by two (L433; see Figure 9.2.1):

> First, I thought…I don't wanna have to multiply seven, eight, nine before I can get ten. So, I figured six plus four equals ten. But since I am timesing [*sic*] times two, I multiply four times two, get eight, and then just multiply sixty-four times eight.

$$\boxed{\begin{array}{r} 6 \\ +\,4 \\ \hline 10 \end{array} \qquad\qquad \begin{array}{r} 64 \\ *\,(4*2) \\ \hline 512 \end{array}}$$

Figure 9.2.1. Stephanie's shortcut method to solve non-consecutive Tower Tasks.

Stephanie extends the recursive doubling rule to a shortcut for non-consecutive heights. Her strategy involved first finding the difference in the heights of the known solution and the unknown solution. Then she multiplied this difference by two and multiplied this result by the given solution in order to find the unknown solution. For this particular example, she knew that the solution to the 6-tall Tower Task was 64 and she wanted to find the solution to the 10-tall Tower Task. The difference between 10 and six is four, which she used as the "number of times" she should multiply by two (she mistakenly translated her correct reasoning into an addition of twos rather than factors of twos) to get to the 10-tall Tower number solution. In this case the number of times she wanted to multiply by two is four times, which she stated was eight. This result was then multiplied by 64 to obtain the solution for 10-tall towers. Since she was off by a factor of two, she found the result of 512. But Stephanie, Milin, and Jeff each agreed this was wrong. R2 asked the students, "Is that so very wrong?" (L460) and "When will this work? Why didn't the eight work?" (L467). Michelle did not think she was wrong, stating that "This [*method*] would work because if you multiply that [*512*] times two, you would have had a lot easier time than going times, times, times" [*by two recursively*] (L484). Figure 9.2.2 illustrates R2 presenting Stephanie's earlier calculations (from Session VIII) for the 10-tall towers.

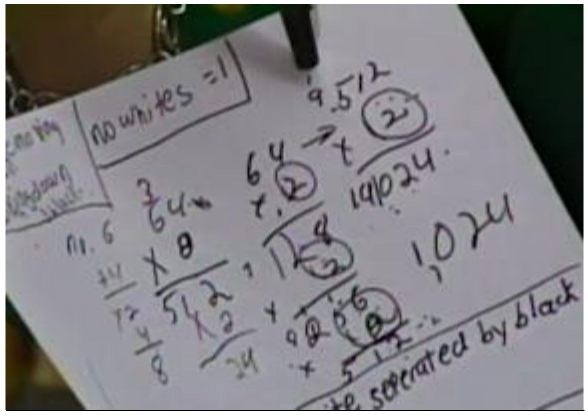

Figure 9.2.2. Stephanie's calculations for towers 6- through 10–tall.

As the students viewed her work, Stephanie acknowledged that her strategy gave the solution to 9-tall Tower Task. Michelle, Milin, and Jeff agreed that she needed to multiply by two once more. R2 ended the segment by leaving the students with a challenge to find the *single* number to multiply the solution for 6-tall towers to find the solution for 10-tall towers.

9.3 Individual written summative assessment

Date	May 15, 1992
Grade	4
Task	Towers
Participants	Stephanie
Researchers	R2
VMCAnalytic	n/a

Recall Stephanie was asked to reflect upon and write about the methods she used for finding and justifying all possible towers (see Chapter 7.2). She wrote the following about the doubling pattern:

> Finding these methods [*cases and opposites*] I found a patern [*sic*]:
> For blocks of 1 I found 2
> For blocks of 2 I found 4
> For blocks of 3 I found 8
> For blocks of 4 I found 16
> For blocks of 5 I found 32
> For blocks of 6 I found 64
> For blocks of 7 I found 128
> For blocks of 8 I found 256
> For blocks of 9 I found 512

> For blocks of 10 I found 512
> For blocks of 10 I found 1024
> And so on. If you saw the patern of 2x2 that is what I found. With this patern you can find out answers to problems with towers like this. Towers 11 high = 1024x2 = 2048. I also saw that all the answers are even.

Notice that she identified the solutions to the task up to 11-tall towers. Although she may have stated incorrectly “2x2,” it seems she was referring to a generalized rule of doubling.

9.4 Dyad summative assessment (Session X)

Date	June 15, 1992
Grade	4
Task	Towers (3-tall)
Participants	Dyad pair: Stephanie & Milin
Teacher	Barnes
VMCAnalytic	Stephanie’s Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 2 of 3 (Grade 4); Event 4 https://doi.org/doi:10.7282/t3-g20s-0d46

This section focuses on Stephanie and Milin’s discussions on June 15, 1992, during a dyad summative assessment on the 3-tall Tower Task. After solving the problem by enumerating combinations of towers Milin and Stephanie discuss a doubling rule for finding number solutions of towers of consecutive heights. Stephanie and Milin indicated familiarity to each other about the rule. Milin and Stephanie discussed (in a whisper) about their previous experiences with the researchers. Milin stated, “We did this with her” [*R1 or R2*] as he pointed to the Tower Task on his paper. Stephanie agreed that they had done these problems before. She also said to Milin that she wished she knew the doubling rule before, at which point Milin responded that he discovered it earlier. For example, Milin stated to the teacher, “We have another way. We knew this way before.” They both agreed to explain this method in their written assessments. Stephanie suggested, “If we showed them down to the ten, it would convince them” and wrote an explanation as indicated by Figure 9.4.1. The explanation was an additional argument for why there would only be eight 3-tall towers.

"There is an easyer way Muiltyply the no. For towers of 2 4 X the [cross out] no. two and you get 8 you can do this on any problem just find out the no. be for it."

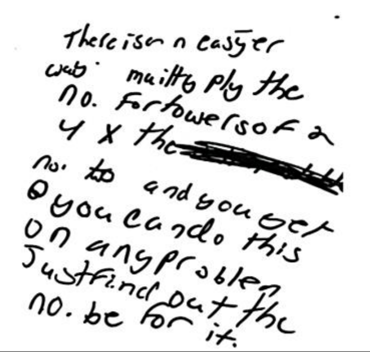

Figure 9.4.1 Stephanie's second page of written work.

The classroom teacher asked the students to elaborate and explain clearly how hypothetical students Chris and Alex would know how many towers there would be for any number high. In response Stephanie wrote, "All you have to [*do*] is find the no. [*'number'*] for the problem before and multiply by 2. Like this Just mulityply [*sic*] by **2**" (see Figure 9.4.2). Stephanie told Milin that they must also specify that the numbers one through 10 represent tower heights and that each previous number solution be multiplied by 2 to show the doubled solution of the next height. Listening to Stephanie, Milin adjusted his written work to include "x2 [*times 2*]" to each previous solution.

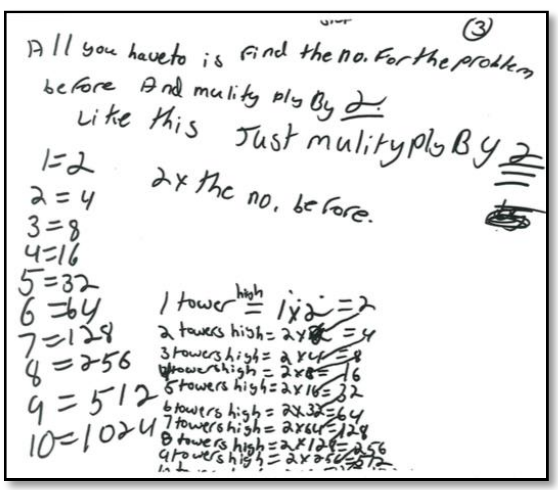

Figure 9.4.2. Stephanie's written explanation of the doubling pattern.

In this session Stephanie and Milin treated the doubling pattern that they observed in previous experiences as a generalized rule for finding the number solution to Tower Tasks of any height. They both showed how to apply the rule up to 10-tall towers.

9.5 Summary of Grade 4 interviews and assessments

Chapters 6 and 7 presented the results of the development and application of an argument based on cases in fourth-grade interview and assessment sessions. Chapter 9 presented results from the same sessions with the primary focus of the idea of the doubling pattern and how Stephanie applied it in her reasoning about the solutions to Tower Tasks.

Recognition of patterns and use of strategies. In Session VIII, after successfully solving the 4-tall Tower Task by cases, she used the same argument and organization to solve the 3-, 2-, and 1-tall Tower Tasks. By solving the related Tower Tasks using her case method, she had an opportunity to compare the numerical solutions. In this instance Stephanie first recognized a doubling relationship between consecutive tower height solutions and applied this doubling pattern to predict the solution of Tower Tasks up to 10-tall. This instance also began a new exploration for why the pattern held true. Stephanie explored why the doubling pattern was valid through an introduction to the inductive pattern, but she could not justify the connection between the doubling pattern and the growth of the towers. However, by using her tower models, she showed that outfits double when an additional article of clothing, selecting from two colors, is introduced to the Outfit Task. She showed the outfit possibilities doubled because one color of the article of clothing is placed on top of each outfit and then the other color of the article of clothing is placed on top of each outfit. The experience strengthened her observation of the doubling pattern, as she indicated explicitly that Outfit numerical solutions were the same as Tower numerical solutions. There was no evidence she made a connection of her inductive reasoning about the Outfit Task to the Tower Task.

In Session IX, she recognized the doubling pattern again and explained to the group how she generalized a rule to find the solutions for the number of towers of any height, selecting from two colors. In the following assessment sessions, Stephanie justified the number solution for a particular height by various organizations based on cases and supplemented her reasoning with an empirical argument of an observed doubling pattern to verify the number solution of Tower Tasks.

Chapter 10 Results: Stephanie's development of inductive reasoning to justify solutions to Tower Tasks (Grade 5)

10.1 Individual written summative assessment (Session XI)

Date	October 25, 1992
Grade	5
Task	Towers (3-tall)
Participants	Stephanie
Researchers	R2
VMCAnalytic	n/a

Recall Chapter 7.4 presented results of Stephanie's letter to an absent student on how to solve the 3-tall Tower Task using an argument by cases. The focus of this section is how she used the doubling rule as part of her argument in support her solution to the task:

> Dear Laura,
> Today we made towers 3 high and with 2 colors, we have to be sure to make every possible pattern. There are 8 patterns total, I know because all you have to do is multyply *[sic]* 2 X the number you would get for towers of two. So, it is 2x4. I will prove it *[draws towers in a case-based organization and justifies the combinations on the following page]...* Also, when you multyply *[sic]* 2x4 It does equal to 8. That thery *[sic]* works for every one. Just multyply *[sic]* the answer for the last tower problem x 2.

In this letter she began the argument by claiming there were eight because she doubled the numerical solution from the 2-tall Tower Task. She then drew representations of towers by cases of "no red," "1 red," 2 red," and "3 red" in order to "prove it *[her claim]*". She used a case argument to "prove" that the doubling rule matches the total combinations found by cases. She later concluded, "Also when you mulyply *[sic]* 2x4 It does equal 8." Stephanie reaffirmed that the total number of combinations made by cases matched the result of the doubling rule. Furthermore, she claimed: "That thery *[sic]* works for everyone. Just multyply [*sic*] the answer for the last tower problem x 2." She named the doubling pattern as a "theory" and claimed that one could obtain a solution to the total number of combinations of towers of any height by using the solution of the previous Tower Task.

10.2 Dyad: Stephanie & Matt; Small group, & Whole class (Session XII)

Date	February 26, 1993
Grade	5
Task	Guess My Tower (i.e., "GMT"; 3- and 4-tall)
Participants	Dyad pairs: Stephanie & Matt; Michelle & Milin; fifth-grade class
Researchers	R2, R1, and R3
VMCAnalytic	Stephanie's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 3 of 3 (Grades 5); Events 1–8 https://doi.org/doi:10.7282/t3-jv0q-p284

On February 26, 1993 fifth-graders, Stephanie, Michelle, Milin and their classmates, worked on the GMT Task (see Appendix D) in a class session, about a year after the "Gang of Four" group assessment. In this session Stephanie was partnered with Matt and Michelle was partnered with Milin. In this task students worked with towers 3-tall for the first part and towers 4-tall for the second part. Although the task was a variation of the Tower Task, it was an important part of this research study because the task required the outcomes of building towers of various heights in order to establish the sample space for all tower outcomes. Milin's inductive reasoning approach was disseminated to other students during this fifth-grade class session. The purpose of the next subsections was to explore Stephanie's learning of his idea. The events presented in this section were evidence of Stephanie's individual reasoning about the doubling pattern using the idea of inductive reasoning. The interactions between researchers and the children and/or the children with each other were considered.

Recalling and testing the conjectured doubling rule (see Events 1-2). In this event Stephanie and Matt were working together. Stephanie was attempting to explain to R3 and Matt a doubling rule that she recalled while Matt was participating in building towers by color opposites with her. She stated, “Whatever number you get from last one, you multiply by two, and then you get the number, how many they will be for the next one” (L19). R3 asked her why she thought the doubling rule worked for all tower heights, and Stephanie responded: “You know you cannot multiply by one, because say the last answer was three, and you multiply by one, you only get three again” (L27-9). The researcher left, and the students continued to solve the GMT questions. Then Stephanie explained to Matt (L117):

> I will explain to you what we did, we had last year when we were in fourth grade, they used to give us the Tower Task. Remember towers four high, how many can you make, towers five high, how many can you make towers six high. A couple of us figured a theory, because we used to see a pattern forming: if you multiply the last problem by two, you get the answer for the next problem, but you have to have *all* the answers [*combinations*]. This [*the GMT solution with 3- and 4-tall*] didn’t work out because we didn’t have all the answers up here [*referring to the 4-tall GMT outcomes*].

Notice Stephanie indicated a conflict in the 4-tall GMT outcomes and the solution to the 4-tall Tower Task, which she was recalling should be double the solution to the 3-tall Tower Task. R3 returned and overheard Stephanie claiming that the doubling pattern worked: “But I worked all the way to eleven [*tall*] with it [*towers*], and… I went all the way into thousands with it” (L134). R3 suggested that they build shorter towers and explore this idea. They agreed to work on the Tower Tasks starting with smaller heights using Unifix cubes to test the conjectured rule.

Matt suggested starting with 1-tall towers, but Stephanie already began building 2-tall towers and they continue with her suggestion. They built four 2-tall towers together. Stephanie then used the conjectured doubling rule to get a hypothetical total for towers 3-tall. They tested this rule by building towers 3- and 4-tall as seen in the following excerpt:

10. S: Alright. So, we have four [*2-tall towers*], okay? Now we multiply that [*four*] by two and we get eight. Okay, let's see if we get the next amount in eight [*3-tall towers total*].
11. M: Eight. okay. So what? Three?
12. S: Three. So really don't build again. Just add on [*to the shorter towers*]. Well, actually, yeah, build again. That way we can show that we multiplied it out… Cause I know there was a way. I just don't know which way it was.
[*They both contribute to creating different towers 3-tall using the opposite strategy.*]
21. S: 1, 2, 3, 4, 5, 6, 7, 8 [*counting the 3-tall towers*]. I knew it [*doubling*] worked!
22. M: Alright number four [*4-tall*].
23. S: All right, now the next is sixteen [*4-tall towers total*].

Stephanie applied a doubling rule where the number solution of towers for a previous height was multiplied by two to obtain the number solution of towers for the next height. She did so by using the number solution of 2-tall towers and "multiply that [*four*] by two and we get eight" (L10). Stephanie then suggested building 3-tall towers to test her rule. While building 3-tall Stephanie and Matt used the color opposite strategy. When Stephanie counted the number of towers, the result matches the conjectured total. Stephanie concluded that the doubling rule "worked." Notice Stephanie validated the doubling rule for 2- to 3-tall towers because of the match between the physical four towers, 2-tall, and eight towers, 3-tall. She then claimed there would be 16 4-tall towers and they began building them together.

Questioning how the doubling rule worked (see Event 3). Earlier R3 arranged the towers that Matt and Stephanie have built so that the camera view clearly showed 2- and 3-tall towers. They continued to build 4-tall towers while the researcher observes and asks them some questions. First, she asked, "How about ones [*1-tall*]? Did you do towers of one?" (L38). Stephanie acknowledged they did not. As they continued to build up to 4-tall towers, the conversation went as follows:

41. S: Yeah, we did towers of one. Oh, no we didn't!
42. R3: What are the possibilities for one?
43. S: One. two! Two, four, eight [*1-, 2-, and 3-tall tower totals]*.
[…]
52. R3: You do these so fast! How do you know when you have them all?
53. M: I don't know!
54. S: Well, we're trying to get up to sixteen. Because if you multiply these two *[1-tall towers]*, then it's four, and you get up to eight [*3-tall towers],* and then eight times two is sixteen, so you pretty much figure.
55. R3: That's very neat. I wonder why that works?
56. S: All right, um…I don't know! It just works sort of,.
57. R3: There must be a reason, don't you think?
58. S: Yeah. There must be. *[She is counting the 4-tall set]*. Ten. We need six more. *If* it works. Put the *[opposite]* pairs together.

Stephanie applied the doubling rule for determining how many 4-tall towers they need to construct. The researcher probed, "I wonder why that works" and "There must be a reason." When R3 questioned how they knew they had them all, Matt and Stephanie stated that they do not know. Stephanie made a qualification, "*If* it works" (emphasis in original) as she attempted to find six more 4-tall towers.

Michelle presents to Stephanie and Matt how and why the doubling pattern works (see Event 4). Milin stated that Stephanie might know his method. R2 invited Stephanie and Matt to listen to Michelle explain what Stephanie referred to as "the two times the number thing" (L594). Michelle pointed to an example of two 3-tall towers and described how "from this one you can add a red on top of it and yellow because there's two colors. And from this one you could add red and yellow" (L598). Matt stated, "It's like a sort of like a family tree" (L599) and Stephanie claimed, "See I knew I was right" (L601). Matt added, "You add a yellow or a red on top of that [*another 3-tall tower]*" and Michelle and Matt jointly demonstrate the process for the rest of the 3-tall towers.

Figure 10.2.1. Michelle's and Milin's tree organization of towers.

Matt recognized this organization in a pattern that showed the growth of towers of consecutive height. The researcher suggested that the students build the "tree" organization up to 4-tall, prepared to present to other students, and then have the new students extend the tree to 5-tall towers. Stephanie volunteered to do it. As they built it together on Milin's and Michelle's desk, Matt noticed a relationship between the towers: "These are the parents [*towers 1-tall*], their children [*2-tall*], their children" [*3-tall*] (L627-9). Matt connected the growth of the towers to the concept of a family tree.

Stephanie struggles to explain the connection between the rule and why it worked for Tower Tasks (see Event 5). Stephanie and Matt were asked to go to rebuild the argument to Michelle R. (different from the study subject, Michelle) and Bobby. Stephanie began with the two 1-tall towers, and asked Michelle R., "You're convinced this is really two [1-tall towers]?" After Michelle nods Stephanie continued: "Alright. Then we have to move on to the next one, okay? Now for towers of two [*tall*], there's only four. Are you convinced that there's only four?" R2 then asked Stephanie, "Why should she be convinced there are four?" [2-tall.] Notice Stephanie passed the question on to Michelle R. and challenged her to build the 2-tall towers. R2 returned to Stephanie: "You need to show me and Michelle. You're the teacher now, you've got to show us… I want to know how you are going to get to two high." Stephanie attempted several times, by stating the total number of 1- and 2-tall towers without giving the reason for why they grow from shorter towers. For example, she stated, "Okay. Once there's no more, there's absolutely, positively no more, you can't build any more with one. So, you build the next number. And that number is two [*tall*]...so you have four of two" [*tall*]. The following conversation took place between R2 and Stephanie:

6. S: Michelle [*R.*], you start out with two, okay? You're convinced this is really two?
[*Michelle nods.*]
8. S: All right. Then we have to move on to the next one, okay? Now for towers of two, there's only four. Are you convinced that there's only four?
9. R2: Why should she be convinced there are four?
10. S: Why are you convinced that there's only Oh, sorry. Why are you convinced that there is only four? [*Michelle smiles, but doesn't answer*.]…Show me! I mean, here! Take the blocks and build! Think of anything!
13. R2: But, I don't think that they're thinking of it. You need to show me and Michelle. You're the teacher now, you've got to show us, um... If you start with that yellow one...
[…]
24 S: Why are you convinced [*asking Bobby*]? Well, see, she's not going to let me go any further unless–
25. R2: I'm going to let you go further. I want to know how you are going to get to two high.
26 S: Okay. Once there's no more, there's absolutely, positively no more, you can't build any more with one. So you build the next number. And that number is two [*tall*]...so you have four of two [*tall*].
29. R2: That's a big jump for me, Stephanie. You're jumping too fast from four to two. I don't know how they change. I don't know how they grow.

Stephanie accounted for the number of 1- and 2-tall towers. Stephanie separately considered the various towers heights and transitioned from one height to another with the towers available. An argument about how the towers were built and organized did not emerge, perhaps because the towers were built before she made her presentation. In response to Stephanie's explanation, R2 intervened: "That's a big jump for me, Stephanie. You're jumping too fast from four to two. I don't know how they change. I don't know how they grow" (L29).

Matt helps Stephanie explain the doubling pattern "family tree" (see Event 6). As Stephanie struggled to answer R2's question of how the towers change from each height, Matt asked Stephanie to step aside and explained the following while dealing with the towers (L40-44):

> Now, from here [*1-tall red tower*], you did an opposite or the same color [*a red or a yellow 1-tall tower*]. So then you add the yellow or red on to the last one [*1-tall yellow tower*]...So you have a red on the bottom [*in YR*]...Same as this [*RR*]. So you have the same red on the bottom [*R on YR and RR*]. You add a red or a yellow on top [*on Y*]. You have the same yellow on the bottom, but you add a red or a yellow on the top [*RY and YY*].

Continuing, he explained how to build eight 3-tall towers and arranged the pairs in groups corresponding to their 1-tall and 2-tall "parent" towers that have the same color bases (the bottom cubes).

After Stephanie was interrupted by Matt, she stayed silent during his explanation. It appeared from her head facing downward that she might be uncomfortable, but it also may be that she was attentive to the nuance of the explanation of an idea that she was missing.

Stephanie and Matt jointly present the inductive argument to the group (see Event 7) After Matt finished building the corresponding 3-tall towers, Stephanie watched how he built the 4-tall tower, RRRY, and immediately followed his pattern to build the second tower, YRRY, which had the same bottom cubes. When R2 asked the other students if they saw how the towers grew, Stephanie responded positively. Stephanie and Matt jointly build the 4-tall towers using the argument of adding a yellow or a red cube on top of the previous towers. As Matt built one example she built the other. Then she stated, "Okay. You keep this one [*RRY*]. You can add a yellow on. [*Matt helps to build the other one with red on top*]. And then from this one [*YRY*] you can get yellow, red, yellow, yellow [*YYRY*]. Or yellow, red, yellow, red [*RYRY*]. You can get both of them" (L58-70).

Stephanie shares the tree pattern method with the class (see Event 8). Stephanie had the opportunity to explain the tree pattern to another group of children where Milin and Michelle (study subject Michelle) joined. The following event illustrates Stephanie's understanding of Milin's idea:

1. S: Alright, I have one red, okay? And I have a yellow, and from each of these [*2-tall towers*], you can make two [*2-tall towers*] because all you have to do is…you can add on a red to the red or a yellow to the red. And for the yellow, you can add on a red to the yellow and a yellow to the yellow, okay?
2. Mic: So, you don't have to look for duplicates.
3. S: Then each one of these has two. Like, okay if this is a family tree- the mother, the parents. [*Laughter of children.*]
4. S: Have kids and then, six kids, okay, well actually, no eight kids. Then they have eight kids, and each one of them has two kids. And this one, you can add one red, one yellow, one yellow, one red.

Notice in L1–3 she generalized the inductive procedure from 1- to 2-tall by stating, "from each of these [*2-tall towers*], you can make two [*2-tall towers*]" and compared it metaphorically to a "family tree" with "parents" and "kids." Recall Matt called the representation of growing tower organization as a family tree and Stephanie applied the metaphor to describe the generation of taller towers as a generation of parents and offspring: "Then each one of them [*parent towers*] has two kids." Also, Michelle provided backing that the procedure does not generate duplicates in L2. Moreover, the students joined in the explanation stating together, "And you keep on going on and on and on."

10.3 Summary of Grade 5

The results of Chapters 8-10 showed how recognition and justification of relationships within problems and across problems developed over the course of three years for Stephanie. The events in fourth grade illustrated her own reasoning for the doubling rule: an observed doubling pattern between two consecutive Tower Tasks supported by a complete case-based argument for towers up to 5-tall. Specifically, in the third interview session (VIII), the “Gang of Four” session (IX), and the individual and partner written assessments (X, XI) she justified that the results of the doubling rule matched the number of combinations for a *particular* height (1- to 6-tall) of towers using partial or complete arguments by cases. Although this reasoning was invalid for why there existed a relationship between *any* two consecutive Tower Tasks, it was a valid observation for her after her multiple experiences for gaining conviction of the total combinations from 1- to 5-tall Tower Tasks using her argument by cases.

Stephanie had the opportunity to listen to Milin’s inductive argument multiple times in the fourth grade in Session IX and twice in the fifth grade in Session XII when Michelle explained it to her and when Matt explained it to a different audience. She explored the notion during the Session VIII interview when she discovered the pattern, but did not fully take up R2’s argument. During those sessions she was asked to either explain how she understood Milin’s argument (Session IX) or why the doubling rule or “theory” may have worked (Session VIII & XII). It was not until the fifth-grade small group presentation that she was explicitly asked to justify why towers doubled by showing their growth using the physical towers in a “tree” pattern. In this instance Matt helped Stephanie when it was apparent Stephanie could not make the justification.

Evidencing hearing Matt's explanation, Stephanie joined Matt in displaying the inductive pattern by building two taller towers from each shorter tower. In a final event Stephanie presented a complete inductive argument for how and why the number of taller towers doubled from the previous shorter towers.

This chapter concludes the results of Stephanie's development and application of the doubling pattern for finding the number of taller towers and the supporting argument by induction.

Chapter 11 Results: Milin's early problem solving of Tower Tasks with a partner and in whole class discussion (Grades 3 & 4)

11.1 Grade 3

11.1.1 Whole class (Session II)

Date	October 12, 1990
Grade	3
Task	Tower (4-tall and 3-tall)
Participants	Dyad pair: Milin & Lauren; third-grade class
Researchers	R1, R3
VMCAnalytic(s)	n/a

Milin contributes to discussion about tower pairing relationships. Milin was a participant of the third-grade class sharing session, facilitated by R1. The discussion involved students sharing their problem-solving strategies, such as pairing by color opposites, for the 4-tall Tower Task, in which students offered examples. Milin, however, offered a different pairing strategy when he said: "They could be like switched like the other way around" (L14). R1 asked Milin what he meant by this, and Milin gave an example showing that BBBR, it could be RBBB, the inverse tower (as distinct from the color opposite tower of BBBR, which is RRRB). In another instance when students were sharing examples of 3-tall towers, Stephanie offered the tower, RBB, and Milin responded with the inverse tower, BBR (Clip 2 of 6; L38).

11.2 Grade 4

11.2.1 Dyad: Milin & Michael (Session IV)

Date	February 6, 1992
Grade	4
Task	Towers (5-tall)
Participants	Dyad pair: Milin & Michael; fourth-grade class
Researchers	R1, R2, R3
VMCAnalytic(s)	Milin's Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 2 (Grade 4); Events 1–2 http://dx.doi.org/doi:10.7282/t3-vvfk-d817

Guess and check, inverse, and opposite strategies (see Event 1). During the group work session, both Milin and Michael, partners, made a tower using a Guess and Check strategy by generating a random tower and checking it against the previously built towers. In addition to this strategy they made a second tower by color opposites, which Milin explained in an earlier instance to R1 (L171-3): “Because, see every time we make it like this [*holds a tower*], right? ...Then we change the color, like this [*opposites*]… So we get doubles of this and this and all this [*points to existing tower*]” (L171-3).

When they found 28 towers, R3 posed the following question: “Do you think there are any more [*towers*]?” (L200), to which both boys responded, “Maybe.” Then R3 followed up with, “Are you sure that they’re all different towers?…How do you know?” (L206-9). Both responded affirmatively and then Milin responded to the latter question by describing a guess and check strategy: “Because everything we get, we make it like this [*makes a tower*], right? …Right now I am going to check [*checking with existing set*]. See it’s not [*a duplicate*]” (L210-2). In a later instance, Milin shared with R1 a similar procedure about how they checked for duplicates: “because see we still keep on going like this [*checking new towers against the existing set*] and see this [t*he duplicate*]? It’s a duplicate of this so we can’t use this…if we find any duplicates [*within the existing set*] in our way…” (L296-301). He did not complete his sentence, but, like the previous instance, he demonstrated moving the new tower across the other towers and comparing them visually.

Returning to the dialogue with R3, she asked, “Also is there anything else that helps you to make sets or make towers?” (L220). Milin shared that they tried new patterns and did not duplicate old patterns: “Um, we just keep on checking to see if there's any [*pause*; *did not complete his sentence*]. And when we try to do it every way like [*pause; again, did not complete his sentence*] …We already know that we made five of these [*all yellow tower*] and five of the reds [*all red tower*] so we are not gonna try that again" (L221). Milin provided a first justification that they monitored by comparing with the existing towers. The second justification was that they would not make towers with all cubes of a single color since they have already been generated.

Furthermore, R3 asked Michael about their organization (L222). Notice Milin responded (L240-2): "See this [*cube*] turns to yellows from reds [*cube in the other tower*] and this [*next cube*] turns from... yellow to red… Like um when have this [*pointing to a cube in a tower*] we change the color to the other color" (L224-8). Michael noted, "That's how we got all of these" (L235). Notice that in addition to finding color opposite towers, the strategy for generating a tower or a pair of towers included inverting the pattern (e.g., refer to paired towers RYRRR and RRRYR or RRYYY and YYYRR). Michael called an inverse pair a "different match" (L81) when he stated that he did not agree with Milin to pair the towers this way (L75-81). Milin referred to an inverse tower as a "duplicate" (e.g., L326, L348, L380) because he would create a duplicate tower and flip it upside down. Moreover, Milin was using both strategies in combination to systematically find a set of four towers (similar to Stephanie and Dana's method; see Chapter 5.3.1). As shown in Figure 11.2.1 (adapted from Sran, 2010, p. 150), the top illustration was a way Milin used the heuristic was to find the color opposite and inverse towers and then find the color opposite of the inverse.

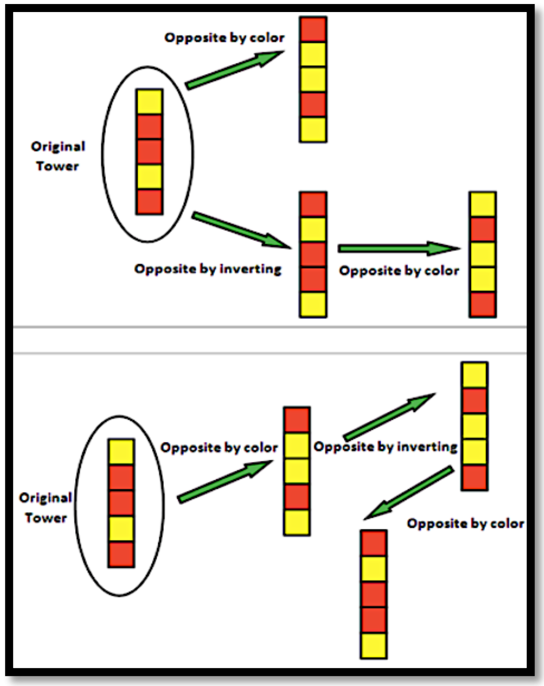

Figure 11.2.1. *An example of Milin's combination of color opposite and inverse opposite strategies.*

For example, in this session, he built tower RYRRR, found the color opposite tower YRYYY, inverted it to find RRRYR and then found the color opposite of the inverse (see Figure 11.2.3 for towers numbered as 23, 24, 29, 30). Another way he used the strategy was to find the color opposite of a tower and inverting the pairs (as showed in Figure 11.2.1 from the bottom part of the illustration). For example, in this session, he built tower RRYRY and its color opposite first and then inverted both towers to obtain their inverse pairs (see Figure 11.2.3 for towers numbered from 31–34).

Both Michael and Milin built towers based on the opposite strategy, but because Milin used the inverse strategy also it sometimes caused confusion similar to Stephanie and Dana's experience. The following exchange between Milin and Michael illustrates their use of both strategies:

247. Michael: [*builds YRYYY and places it down*]
248. Milin: [*points to RYRRR*]. We have its opposite! [*see Figure 11.2.2*]
249. Michael: We do?

250. Milin: No, I made this opposite [*shows YRYYY*]. Now make this' [*sic*] opposite [*pointing to tower RRRYR*]…
251. Michael: No…Go ahead. Turn it upside down. [*Milin turns RYRRR upside down to show it matches the other RRRYR*]
252. Milin: But, see? [*shows him the color opposite pairs by placing them together*]. Now we have to make an opposite for this [*RRRYR, which is originally paired with its inverse tower RYRRR*].
253. Michael: My eyes went weird. Whoa, whoa, whoa, what? Which was the one I just made?
254. Milin: This. [*YRYYY*]
255. Michael: What are you talking about?
256. Milin: See, I see look this is a perfect match [*color opposite pairs*]. Now we have to get a perfect match for this [*RRRYR; Milin builds YRYYY*].

Figure 11.2.2 shows Milin pointing to the tower RYRRR, which is the color opposite of the tower YRYYY that Michael had built and placed above the rest of the set from Milin's perspective (see circled tower).

Figure 11.2.2. Milin's opposite and inverse tower pairings.

In this excerpt there was some confusion about which tower was to be made, perhaps due to the inconsistent labels or vague language of the tower pair relationships; however, both Michael and Milin shared recognition of inverses and opposites. This was evident when Michael asked Milin to check if there was a tower that was the inverse of RRRYR regardless that he did not agree to pair it this way.

Supporting arguments for the solution (see Event 2). Towards the end of the partner work, when R1 asked them about their progress, Milin and Michael had 34 towers. During their conversation, R1 pointed out a duplicate tower and Milin removed two towers from their collection, claiming they now had 32 (L307–20). R1 questioned *why they removed two towers* automatically and Milin explained that each tower had a partner, implying that if one tower is a duplicate from another pair then its partner must be the duplicate of the opposite tower in that set. The researcher asked *if there is any way to tell whether they had found all the towers or not*. Milin responded that, “If we keep on doing *[building new towers]* and we keep on getting duplicates” (L328). Asked *how they knew their towers were different*, Milin pointed out that opposites are always different from their paired tower (L348). Milin’s reasoning was based on a Trial and Error strategy of Milin also indicated that time is another way of knowing whether they were done or not. He said, “So that’s more than ten minutes and we still didn’t find one” (L354). He indicated that there was a possibility of uncertainty if given “100 more hours” (L358) to work on the problem. Michael and Milin continued to use a Guess and Check strategy to build new towers. With each new tower, they moved it over the previously made towers to identify duplicates. Then they continued to utilize their reasoning about the relationship between color opposite towers because they did not check the second tower pair if the first one was different. Figure 11.2.3 (adapted from Sran, 2010, p. 49) illustrates their solution for the claim of 36 towers. The numbers indicate the tower order in which they were built.

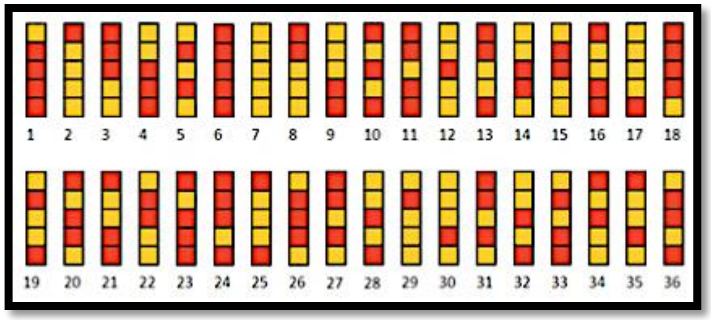

Figure 11.2.3. The 36 towers Milin and Michael made before identifying duplicates.

In the end of the session they removed another duplicate pair and found an extra missing pair. Figure 11.2.4 (adapted from Sran, 2010, p. 50) illustrates Milin and Michael's final collection of towers numbered in the order in which they were built. The missing numbers (e.g., 15, 16, 27, 28) are the removed duplicates.

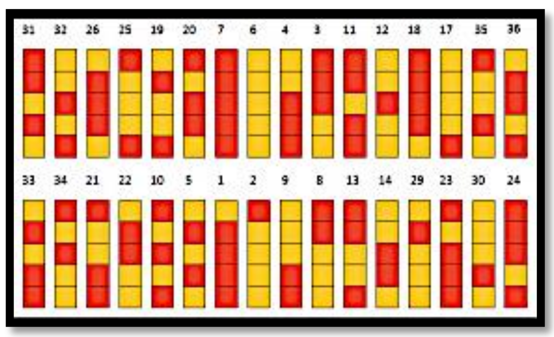

Figure 11.2.4. Milin and Michael's final collection of towers

11.2.2 **Whole class (Session V)**

Date	February 6, 1992
Grade	4
Task	Towers (5-tall)
Participants	Dyad pairs: Milin & Michael; Stephanie & Dana; Jeff & Michelle; fourth-grade class
Researchers	R2
VMCAnalytic(s)	Milin's Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 2 (Grade 4); Event 3 http://dx.doi.org/doi:10.7282/t3-vvfk-d817

On February 6, 1992 a fourth-grade class, facilitated by R2, discussed their findings for the 5-tall Tower Task (refer to Chapter 5.3.2 for the results of the discussion). This section presents the results of Milin's contribution to the discussion.

Justification for an even solution to the 5-tall Tower Task (see Event 3). The discussion began with every group announcing their solutions. One group had an odd number solution, while the rest had an even solution. R2 asked the class if the solution can be odd. Milin explained that you must have a color opposite tower for each tower (L28):

> We got thirty-six before, but then we found duplicates. But now we got thirty-two. And we keep on duplicating it by changing the color [*opposites*]. So, you can't get an odd number unless you don't duplicate it [*find opposites*] and get all of them.

Notice Milin did not directly make a connection between opposites and an even number solution; however, he discussed solutions that he and his partner obtained that were even numbers. In addition to his explanation, during group work with Michael, Milin explained to R1 that if they found a replica tower, they removed opposite tower pairs automatically. Therefore, in response to R2's query about whether the solution can be an odd number, Milin argued that a color opposite tower existed for each tower.

Then the students who reported they had 32 towers were invited to check for duplicates in the solution set of the group that got 35. Milin joined in this group activity. Milin and others from the class found the three duplicates in the set of 35. When they acknowledged 31 different towers, Milin agreed with another student that they needed to find the tower that does not have a "match" (L91). He applied this justification to help the group find duplicates and organize by "matches" (L127). During their search for duplicates, the researcher asked the students who used the terminology "match" to explain. Milin offered his understanding of what they meant: "I know what they mean. See this yellow [*on the original tower*] turns into red on this one [*on the second tower*] and all of these reds turn into yellow in this one" (L135). Then he offered another type of "match…if you put it the other way [*invert it*]" (L120) and student Matt disagreed with him. In this event, Milin provided reasoning that the solution must be even due to the existence of a color opposite tower for each tower. He applied this reasoning to help the group find duplicates and organize by "matches" (L127).

11.3 Summary of Grade 3 and 4 problem solving

Recognition of patterns and use of strategies. In third and fourth grades Milin recognized tower pair relationships, such as towers that could be "switched around" (inverses) and towers that were color "opposites." Third-grader Milin – who although was not a participant of the Longitudinal Study's cohort in Session II was captured on video participating in group discussions – indicated that he was reasoning about the solution through one of these tower relationships. Specifically, he claimed that towers could be inverted during a discussion about the relationship between an even solution and tower pair relationships. In Session IV, fourth-grader Milin used the two pair relationships through two invented composite operations to generate three towers from one tower. When students systematically combined the strategies of color opposite and inverses, Sran (2010) defined the composite move as an "opposite hybrid strategy." Sran found that Milin invented the strategy in a later interview with R1 (see Chapter 12.2 for Session VII). However, it was evident that Milin also used the hybrid strategy in Session IV.

Displays of justification. Milin used the pair relationship of color opposites as reasoning for his claim that the pair guaranteed *difference* between two towers. Milin also argued for an even solution of the 5-tall Tower Task, reasoning that an odd solution was not possible "unless you don't duplicate it [*find opposites*] and get all of them."

Milin and Michael used Trial and Error to search for missing towers (or verify that there were no missing towers) and to check whether each tower was different from the others. Supporting reasoning for the *exhaustion* of the solution was the result of trial and error verification that no new towers could be found. In addition, Milin used the length of time it took to search for a new combination as a factor in his conviction (Sran, 2010).

Representations. Michael and Milin used physical towers to build and argue about their solution. Milin terminology for color opposites included “duplicates,” “opposites”, or “match.” Milin also used “duplicate” and “match” for inverses and Michael specified that they were a “different match.”

Chapter 12 Results: Milin's Development of Reasoning by Cases to Solve Tower Tasks in Post-interviews (Grade 4)

12.1 First interview (Session VI)

Date	February 7, 1992
Grade	4
Task	Towers (5-tall)
Participants	Milin
Researchers	R1, R4
VMCAnalytic(s)	Milin's Development of Reasoning by Cases to Solve Tower Tasks: Part 1 of 2 (Grade 4); Events 4–8; http://dx.doi.org/doi:10.7282/t3-vvfk-d817 Part 2 of 2 (Grade 4); Event 1; http://dx.doi.org/doi:10.7282/t3-7kyt-1r45

The interview with R1, R4, and Milin occurred on February 7, 1992, the day after the classroom work with Michael on the 5-tall Tower Task. Earlier in the interview Milin was asked to recall the problem they had worked on and how he solved it with his partner, Michael.

Color opposite generating and pairing strategy. Milin's first response indicated use of a random strategy to build a tower pattern and the use of the color opposite strategy to generate the second in a pair: "Michael and I kept on building them and putting another one exactly like that but different colors…we looked at the colors and all the yellows turned to red, and all the reds turned to yellows" (L20-2). He claimed there was a total of 32 towers. In response to how they knew they were all different tower combinations, he explained that they monitored for duplicates by checking each tower against the collection of towers.

Generating exactly one of a color towers in an elevator pattern and supporting arguments (see Event 4). In this event, Milin referenced the class discussion of the previous day in response to how he thought he found all towers (L43-6). Recall that the class discussed the strategy offered by Ankur and Jeff to generate new towers by the use of patterns, where another tower is generated by varying the previous tower pattern in some recursive manner. Milin reports a strategy to organize towers with exactly one of a color and with exactly two of the same color adjacent, generated in an elevator pattern. He described the characteristics of these towers as follows (L47-60):

> That [*previous day's class discussion*] was about seeing if we had all of them…We found it like this [*uses a random tower to demonstrate*]: Starting from the reds, one red [*cube on the bottom position*], then another red on the next floor [*pointing to the first and second positions of a random tower indicating an imaginary red cube in each position*]…So they'll be five reds [*Milin builds YYYYR*]…Like this [*He builds YYYRY*] …And then three [*YYRYY which he did not build*]. See? This [*pointing to R in YYYRY*] goes in the staircase and keeps on going to the third, and fourth and fifth [*He points to the cubes in YYYRY indicating the imaginary red cube in the third, fourth, and fifth position*].

Note that Milin described the elevator pattern as a "staircase" pattern (*elevator* pattern in this research study is differentiated with a *staircase* pattern; see Appendix B for the Glossary of Terms). Milin was invited to build the towers he described, and he responded with incomplete representations using the red and yellow cubes. Aware that he only built part of the representation when he built YYYYR and YYYRY, he stated, "See this [*one red cube*] goes in the staircase and keeps on going to the third, and forth, and fifth" [*pointing to imaginary red cube in each position of tower YYYRY*] (L56). R1 asks him, "Why can't there be more?" (L61). Milin responds (L62-75):

> Because there's only five of these [*He points to YYYRY with five cubes in it*], so one [*red*] on each block [*position*]…Always four yellows if you're talking about one *[red]*, but if you're talking about two [*reds*] they'll be three yellows.

In response to the researcher's prompt why there could not be more than five, he reasoned about the 5-tall assertion and so, "one [*red cube*] on each block" (L60-2). R1 asked him how many there would be "if there were one yellow and four red cubes" (L76). He responds, "Five [*total*]…you could do the same thing but they'll be reds on this and yellows on this [*on opposite color cubes*]" (L83). Then he built the complete set of towers with exactly one red and declared, "See? It's like a staircase *[pointing to the red cube moving up along each position of the tower]*" (L95). He recorded on his sheet, "1 yellow and 4 reds are 5" and below it "1 red a [*sic*] 4 yellows are 5."

Exactly two of a color cubes adjacent and arguments (see Event 5). In this event Milin was asked, "And then after you did one yellow and four reds, and one red and four yellows, then what other possibilities were there?" (L110). He stated that, "there were ones [*towers*] that if you could use two of them [*two together*]" (L111) and there were "four for each [*color*]" (L113). He was asked to build the towers. Milin built towers with exactly two adjacent red in an elevator pattern. The following was their exchange about other possible towers with exactly two reds:

134. R1: Okay, and so this [*Milin builds the staircase pattern for exactly two red adjacent*] has two reds and three yellows for a staircase.
135. M: Two reds and three yellows on all of them [*two red adjacent towers*].
136. R1: On all of them? Is there any other way to have a tower that has two reds and three yellows except in a staircase [*except adjacent red*]?
137. M: No, [*he shakes his head*] there's not gonna be any because see, if you put [*pointing to the downward elevator pattern of the two red adjacent*] it'll only be one [*with two reds at the top RRYYY*] if you have three [*yellows*]. Because see, these two [*reds at the top*] could go [*shift down one level]* in there [*referring to two red cubes in the 4th and 5th floors of RRYYY*], these two [*YRRYY]*, these two [*YYRRY*] and these two [*YYYRR; signaling a shift in the towers as he points to the two adjacent reds in each tower in a downward red elevator pattern*]. That's it.

Without having built them yet, Milin counted four towers for the color opposites of towers with exactly two adjacent reds. He eventually built them and counted a total of 18 different towers with 10 towers of exactly one of a color and eight towers of exactly two adjacent color cubes. Figure 12.1.1 (adapted from Sran, 2010) illustrates Milin's use of the elevator pattern and color opposite strategy to generate towers with exactly one of a color and towers with exactly two of a color cubes adjacent. Notice Sran (2010) named the cases by “Groups”; therefore, this section adopts the labels.

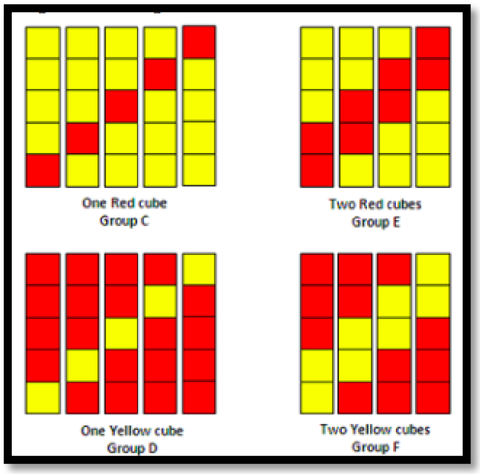

Figure 12.1.1. Color opposite towers in staircase pattern.

Recognizing equivalence between the cases of towers with three cubes of a color and with two cubes of the other color (see Event 6). Milin counted 18 towers because he doubled the cases of exactly one and exactly two adjacent reds to account for the color opposites. When he was asked about alternative patterns of two red cubed towers, he disagreed as he referred to the case of adjacent red cubes. This event centers around a discussion that took place with Milin’s initiation of the category of “threes,” and that, “on the threes [*exactly three adjacent of the same color*] there would be probably three on each” (L151). Earlier, two prompts (L152 & L160) asked him to show what towers with “three” would look like, but it was immediately redirected to look back at the towers he had (L169). Note that Milin had not yet considered that duplicates occur when accounting for cases of three adjacent yellows and two adjacent reds (e.g., YYYRR is in the category of exactly two red and exactly three yellow). This becomes evident after he builds some towers with three yellows:

185. M: And on the three's they'll probably be um—.
186. R4: *[Overlap speech*] —Show me what a three would look like.

187. M: On the yellows, I'll do it right now. [*He builds another YYYRR*]. Like this.
188. R1: But don't you already have that?
189. M: [*He looks at the duplicate*] Yeah. See, right here. [*He puts the new YYYRR on top of the old YYYRR*].
190. R4: So, are you gonna count that?
191. M: No. [*He pulls YYYRR apart*].

He built a tower that was already in his solution set (YYYRR; YYYRR is in both categories, exactly two red and exactly three yellow) and both R1 and R4 asked if he would count it into his set. Notice, upon prompting, he claimed that he would not and then he tried another tower, namely YYRRY. However, again it was a duplicate, so he concluded there were no unique towers with three of a color: "You can't make any others with three" (L206). R1 asked, "Does it have to be a staircase?" (L207) and Milin responded affirmatively. At the conclusion of the event, he permitted the possibility that "Maybe we had doubles of something?" (L210) in the previous day's solution.

Considering other patterns of towers with three of a color (see Event 7). Milin questioned his previous day's solution of 32 and suggested that "probably there would only be twenty" [*5-tall towers*; L218] after reviewing all the towers he had built (namely, ***10*** towers with exactly one of a color, ***two*** with a single color, ***eight*** with exactly two adjacent of a color). In addition to initiating the case of towers with "threes" [*towers with three of a color*], he also initiated the case of towers with "fours" [*four of a color*], but immediately realized, "But that [*towers with four*] would go with this" [*points to towers with one red*; L228-30]. This initiated a conversation to consider other patterns that may have remained:

233. R1: I wonder ... I wonder if you could take this tower right here? [*R1 stands up RYYRR*]. Is there a way you could rearrange those blocks in some way so that it looks different from all these other towers?
234. M: [*He builds YRRRY*] This? Yeah, this.

235. R1: [*Pause*] Yeah, so that's [*YRRRY*] one other one.
236. M: So, I guess there might be thirty-two still.
237. R1: Is there another way you could rearrange these [*R1 builds a second YRRRY and stands it next to the first YRRRY*] so that it would look different still?
238. M: [*He builds RYRYR*] This.
[…]
240. R4: You don't have that anywhere yet, do you?
242. M: Because all of these are together [*He points to the cases with the same colors adjacent; Groups C, E and F*].

The researcher encouraged Milin to consider a sample tower, RYYRR, and to rearrange it in a different way. He generated two new towers from it. He continued to find new towers, building the opposites of YRRRY and RYRYR, as well as other towers with same-color separation.

Noticing a relationship of cubes of one color in relation to the cubes of the other color (see Event 8). In the previous event, with probing by the researcher, Milin came to realize that more patterns existed. For about 5 minutes he searched for new towers using a Guess and Check strategy.

Figure 12.1.2 illustrates Milin's towers grouped by cases, starting with the case of one yellow cube (top left), followed by the color opposites (with one red; bottom left), the case with two adjacent reds in an elevator pattern (top second left), followed by their color opposites (with two yellow cubes together; bottom second left), then the case of single-colored towers (all red and all yellow; top third from the left), and lastly, the two cases of two of a color separated by at least one of the other color (far-right top and bottom). Note the black bordered tower RYYRR which was duplicated during his search.

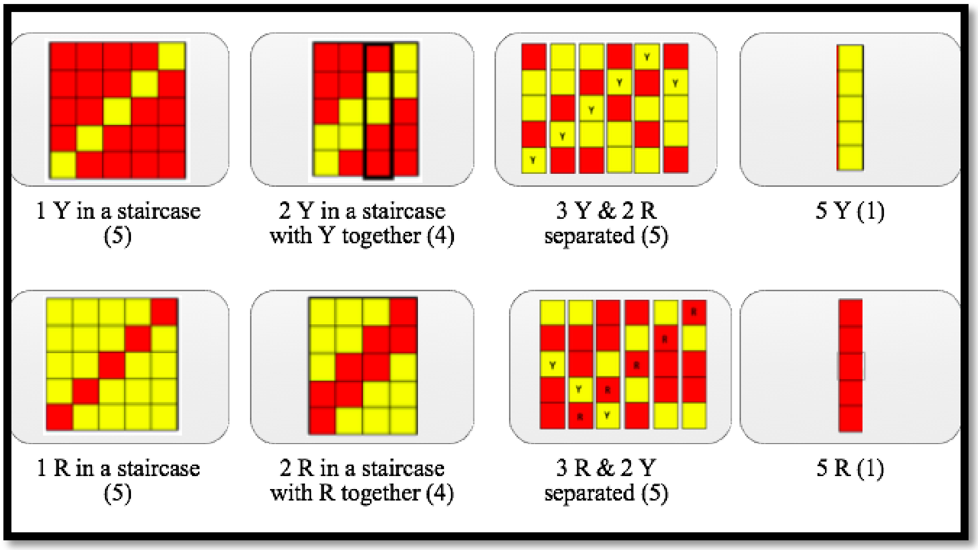

Figure 12.1.2. Milin's 5-tall towers grouped by cases.

Both facilitators asked Milin how he monitored different new towers, and what he considered to be "separated" versus "together." Milin justified that the towers he was generating were different because (L270-4):

> One of them [*the same colors in a tower*] is separated [*pointing to RYYRY*], so you can't make it like this *[pointing to towers with exactly two adjacent yellow*] … So, if one [*of the colors is*] separated [*he points to RYYRY]*. See on this [*RYYRY*] the red separated by–um [*points to the reds in RYYRY*]– I mean the yellows separated by the red [*points now to the yellows at the bottom separated by the red*]. So, this [*yellow on 1st floor*] and these two [*two yellows on 3rd and 4th floors*]. And this [*points to two adjacent yellow case*], they're just putting it like three [*points to the bottom reds in YYRRR*].

Notice that Milin pointed to tower YYRRR when explaining that the case of two adjacent yellow also had the attribute of three adjacent red. Although the reasoning is inaccurate (e.g., RYYRR), he continued to describe the difference between the new cases and previous cases: "They [*the two of a color cases with separation*] have to be separated, otherwise they'll be exactly like this, this, and this" [*pointing to towers in elevator patterns with exactly one and towers with exactly two adjacent*] (L296). Notice also that R1 suggested organizing the towers, as he built them, by the same number of yellows and reds (i.e., by cases): “So we'll put it here, and this partner right over here” [*She rearranges his patterns into two yellows separated and, their opposites, two reds separated*; L289]. Milin then built RYYRR and, in response to being asked to justify that it was different from the adjacent cases, he responded: (L328-34):

> Because on these [*he points to adjacency cases*] are all together, so right now I’m splitting these apart [*the red colors apart*]...On this [*pointing to tower YYRRR in two adjacent yellow case*], see they’re all three together [*three red cubes*] and these two are together [*two yellow cubes*].

Note that yet again he pointed to the tower YYRRR (that has three adjacent reds) as representative of the case with towers with two adjacent yellow. R4 then asked him: “What’s together over here?” [*indicating that the three red cubes are not always together in the towers of exactly two yellows*] (L333-5). Milin recognized that he had created a duplicate tower RYYRR (in Figure 12.1.3). Continuing forward, he looked at the separation features of both colors within each tower. For example, he built new towers: YRYRR and its color opposite, RYRYY (which contain the attributes of separation for each color) and compared them to all of his cases.

Figure 12.1.3. Milin identified duplicates.

There are no other cases, such as four of a color. Expecting 32 (L352), he searched for one more pair by building random patterns that consisted of three red cubes and two yellow cubes, which eventually would be the tower RRYRY and its color opposite. At first, he suggested that it “could have four yellows and one red” and immediately corrected himself “I mean no it can’t be” (L360). It was evident by his statement and his gaze toward Group C that he realized he already had the case of exactly four of a color because it was the equivalent case of exactly one of a color. He built RRYRY and immediately justified that it was a possible candidate by comparing it to Groups C, E, and F based on the same colored cubes were both separated (L364), as well as on the uniqueness of the “two reds on top” as compared to the towers within the local Group G (L366-82; see Figure 12.1.4). He then built the color opposite, YYRYR.

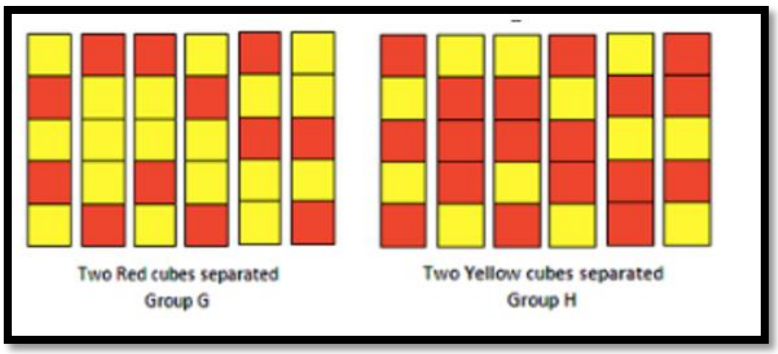

Figure 12.1.4. Complete set of different towers he built with some separation of the colors (adapted from Sran, 2010).

Arranging and looking for patterns within the case of three cubes of one color with some separation (see Event 9). Prior to this event Milin built RRYRY and immediately justified that it was a candidate because the yellow cubes were separated from each other, the red cubes were separated from each other, and it was the first tower with “two reds on top” (L364-382). He then built the color opposite, YYRYR.

R1 asked Milin if he thought he was finished, and if so, why. Milin responded that he could “make a staircase [*pattern*] out of this” [*two red separated and two yellow separated cases*] (L392) and began reorganizing the towers within Group G and H to justify its completeness. Figure 12.1.5 (adapted from Sran, 2010) illustrates his arrangement of Group G and H into Group I. The researcher pointed out that, “There’s lots of ways to fit those [*towers in Groups G and H*] together. But you really think you have them all?” (L409). In response, Milin justified his certainty by the length of time (“We took about 10 minutes and still didn’t find any” in L416) and by checking them locally and globally (“We got a couple of duplicates…and we checked them” in L418–20). Notice this was reasoning based on the results of Trial and Error, similar to his earlier reasoning in the class problem-solving session. He counted *20* towers from the case of two of one color and three of the other color altogether, *two* solid towers, *10* towers with four of a color and one of the other color, totaling *32* towers.

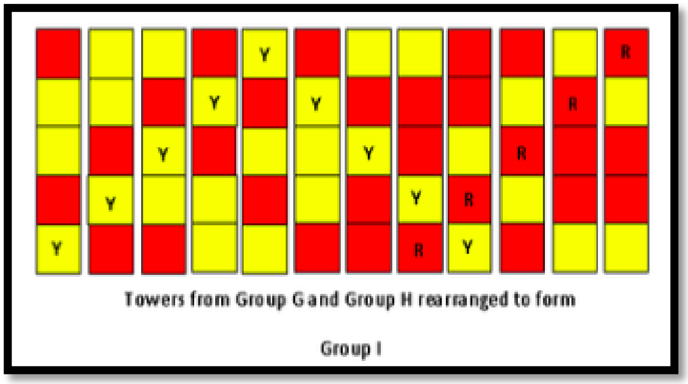

Figure 12.1.5. Milin’s imagined staircase pattern tower groups.

Applying the method based on cases of taller towers to find shorter towers (see Event 1; Part 2 of 2). The researcher asked Milin what strategy he would use to find 4-tall towers (L611). He responded that he would “try something out [*building a towe*r] and then I make the duplicate but a different way” [*color opposite tower*] (L612). He explained that he would generate towers of four from the sets of towers of five that he created earlier in this interview: “They’ll [*cases for 4-tall*] do the same thing [*as cases for 5-tall*]… But it’s gonna use up only less [*cubes or positions*]…So on this [*he points to the case of exactly two adjacent red*] if you take the top off it’ll do the same thing [*same elevator patterns]* and… Everything [*every case*], it’ll do the same thing” (L657, L659, L662). He referred to a strategy of taking the top cubes off of each 5-tall tower. He provided examples from each of the six cases he created earlier for the 5-tall towers. For example, from the case of exactly one red, he removed tower RYYYY from the set and, from the remaining four towers, he removes the top yellow cube. From the case of exactly two adjacent reds, he removes tower RRYYY from the set and, also, removes the top yellow cube from each of the remaining three towers. Similarly, from the single-colored towers, he removed the top cubes. For the case of two reds and two yellow adjacent he estimated, without explanation, there would be “about eight” instead of 12 towers using a similar procedure of removing the top cube from each tower. Although he initially guessed 16 4-tall towers, the interview concluded with his estimation of 24 (L678).

12.1.1 Findings: Key developments of Session VI

Recognition of patterns and use of strategies. Milin initially solved the 5-tall Tower Task by the color opposite strategy and making “staircase” patterns with physical towers. To justify the difference in the cases or towers he built, he focused on the attribute of adjacency or separation of one color. Like Stephanie, Milin had discovered duplication when focusing on one color or if that color was adjacent or separated to the same color. For example, he recognized that some 5-tall towers with exactly three adjacent yellow cubes can be identified as a tower with exactly two adjacent red (e.g., YYYRR or YYYRR but not RYYYR; see *Equivalent cases*). What followed was a dialogue about his consideration of other towers within the case of three of a color that were not accounted for in his earlier cases. At this initial stage he claimed that sets of towers must have an elevator pattern. The search for an elevator pattern limited the number of towers he was able to find (20 towers), and as a result he questioned the solution of 32 towers that he expected. With probing from the researchers, Milin used a Guess and Check strategy to find new towers. He also flagged an equivalent case (e.g., the case with four of a color already existed because it was the same as the case of one of the other color). Milin's 5-tall Tower solution was based on cases and opposites as follows: Towers with exactly one of a color and the opposite, with exactly two adjacent of a color and the opposite, with exactly three of a color with separation and the opposite, and with all of one color. With help from R1, towers were organized symmetrically by the cases of none, one, two, and three of one color and below the corresponding opposites cases. He also partially the 4-tall Tower Task based on the cases and opposites he obtained from removing a cube from the top of each 5-tall tower and eliminating duplicates. He predicted 24 towers, 4-tall.

Displays of justification and forms of reasoning in support of the solution. Milin formed elevator patterns with sets of towers and reasoned based on the length of time that resulted in flagging for duplicates and searching for missing towers (Trial and Error) as justification in support of the exhaustion of the cases. This reasoning was incomplete for the cases with separation of yellows and reds. On the other hand, for the cases with exactly one and two adjacent, he justified the exhaustion of the elevator pattern based on the number of positions available that a color could take on.

12.2 Second interview (Session VII)

Date	February 21, 1992
Grade	4
Task	Tower, selecting from two (1- to 4-tall) & three colors (1- to 2-tall)
Participants	Milin
Researchers	R1
VMCAnalytic(s)	Milin's Development of Reasoning by Cases to Solve Tower Tasks: Part 2 of 2 (Grade 4); Events 2–6 http://dx.doi.org/doi:10.7282/t3-7kyt-1r45

Grouping towers by opposites and elevator patterns (see Event 2). On February 21, 1992, R1 interviewed Milin for the second time in a three-part interview series, as was done with Stephanie and R2. At the onset R1 asked Milin if he had determined the answer to how many possible different 4-tall towers can be made selecting from two colors (L1). Milin replied that there were 16 total and reminded R1 how he changed his mind from the previous answer of 24 in the prior interview (L2; L34-7). He took out the towers, which were already built and mixed in the bag, and a paper with his written work in the previous interview. In response to the suggestion, “Let’s organize them in any way that is good for you” (L15), Milin rearranged the towers by color opposite pairs (see Figure 12.2.1).

Figure 12.2.1. Milin’s first arrangement by opposites.

When he was asked, “How am I going to know that we have everyone and that they’re all different?” (L13) Milin claimed the opposite pairs “are different, but which every way they are the same” (L18). When the researcher asked again how they were all different and if there were no more (L27), he argued there are a limited number of “ones” [*towers* with *exactly one of a color*] (L28). As in the previous interview, he elaborated about a “staircase” [*elevator*] pattern to find and organize the towers with exactly one of a color. Spontaneously he also built the case of towers with a single color or as he called it, “all of them [*the cubes*] the same” (L60) and placed it in the middle of the two cases of exactly one of a color (see Figure 12.2.2). He concluded there were ***10*** so far.

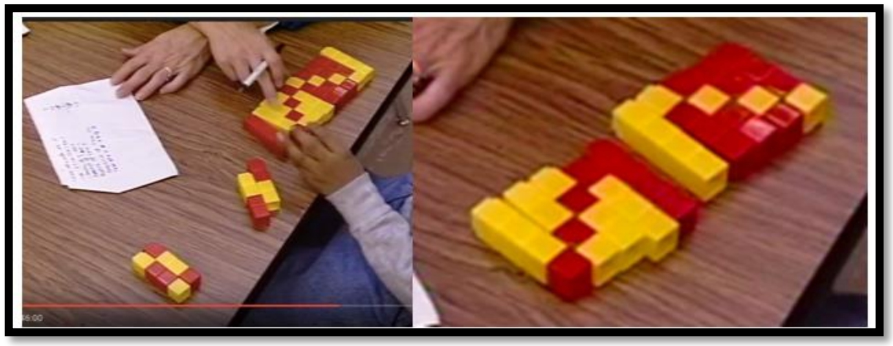

Figure 12.2.2. Milin's second arrangement by patterns.

Milin considers the case of "two of a color together" in an elevator pattern (see Events 3–4). In Milin's second arrangement a group of opposite tower pairs remained. R1 asked what they had in common, as in the following exchange:

82. M: These [*towers with two red and two yellow*] have in common is that they have some of– most of them have two yellows.
83. R1: And?
84. M: And two reds.
85. R1: Most of them?
86. M: Um, all of them, actually.
91. R1: Okay, so all of those [*towers with two red and two yellow*] have two yellows and two reds. What about all these [*towers with one red and three yellow*]? All of these have...?
92. M: They [*towers with exactly one red*] have three yellows and one red on this side.
93. R1: Uh huh.
94. M: And three reds and one yellow on this side [*towers with exactly one yellow*].
95. R1: Uh huh...and then in the middle [*single-colored towers*]?
96. M: In the middle are the two ones that—um—are all one color.

His response is noted here for his change in language from the beginning of the event towards the end. Note that he described the attributes of the cases using R1's language, attending to the number of cubes of one of the colors and the number of cubes of the other color (in L82, 84, 92, & 94). She probed his use of the word "most" (L85). This prompted Milin to modify his previous statement: "All of them, actually" (L86). Again, R1 asked Milin about the case of towers with exactly one of one color, to which he responded by providing the specific count in each color ("They have three yellows and one red on this side and three reds and one yellow on this side;" L92-94). R1 then asked about the single-color towers, to which Milin responded with language similar to her earlier description: "The two [*towers*] ones [*single-colored*] are all of one color."

R1 summarized, "So you have three of one color and one of the other and then you have four of one color and you have two and two. Is there anything else you can add?" (L97). In response to her query he replied (L98):

> If you want four of one color, it would be coming into these two [*points to two towers of same color, RRRR and YYYY*]. If you want three of this color it would come into these two columns [*points to one red and one yellow elevator pattern towers*]. But then on the twos you could only make six.

His response is noted here for the change in language from the beginning of the event towards the end. The phrase "coming into these" was his way of describing the kinds of patterns and relationships he noticed between and among sets of towers. Sometimes the expression referred to a tower belonging to a particular case/category, such as in this statement, and other times he seemed to search for relationships among towers in order to make them into a particular arrangement (e.g., elevator patterns) or to create a new category.

Returning to the R1's query: "So you have three of one color and one of the other…four of one color…two and two. Is there anything else you can add?" (L97), Milin rearranged the opposite pairs of exactly two of a color into a two red adjacent elevator pattern (see Figure 12.2.3). His reasoning for the case of exactly two of a color was extended to his use of the elevator arrangement and color opposite strategy. Milin explained his arrangement: "Only three of them can make a staircase [*either two red together or two yellow together*], but then if you want to start all over you need one more but that'll be this [*referring to a cyclical ascending elevator pattern of towers from red adjacent to yellow adjacent*]" (L114). Milin showed what he meant by referring to the two red adjacent elevator recursion. If, for example, one generated a two red adjacent elevator pattern of towers, there would only be three towers (YYRR, YRRY, RRYY). The last tower in this set (RRYY) was both a two red and a two yellow adjacent tower. So, one could continue the elevator for two yellow adjacent towers beginning with the last tower RRYY, creating a new tower RYYR, and then the last tower YYRR already existed in the two red adjacent elevator pattern. For the towers with exactly two adjacent cubes of the same color, one had the option of choosing either a red or a yellow elevator pattern. Then, this pattern became the focus of the cyclical explanation when considering the complete two adjacent red and two adjacent yellow elevator pattern. Supporting Milin's brief verbal explanation was his demonstration that illustrated the generation of duplicate towers when using the elevator pattern for each color. Consequently, with his brief verbal explanation and his demonstration with the towers, Milin justified to R1 that only one color should be chosen when generating towers elevator patterns with exactly two of a color adjacent. Therefore, his case for exactly two of a color adjacent consists of four unique towers.

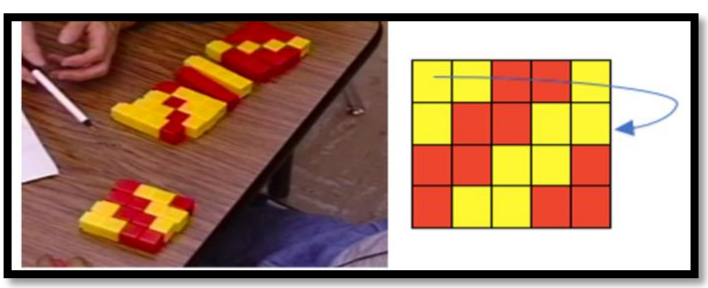

Figure 12.2.3. Milin's elevator pattern for the case of two of a color.

Milin described the last two towers with alternating color cubes as follows: "See, you want two of each [*opposite color alternating towers*] but you are separating them"

(L116). It is interesting that he left them separate from the other group to indicate that "you are separating" the colors. These towers were grouped by color opposites. He justified the difference among the towers within their own category due to color opposite differences and between other categories due to the same color cubes were not adjacent to each other. The case of "two reds and two yellows" totaled six new towers.

He created a third arrangement for his solution which he concluded: "They became sixteen" (L120), as illustrated in Figure 12.2.4.

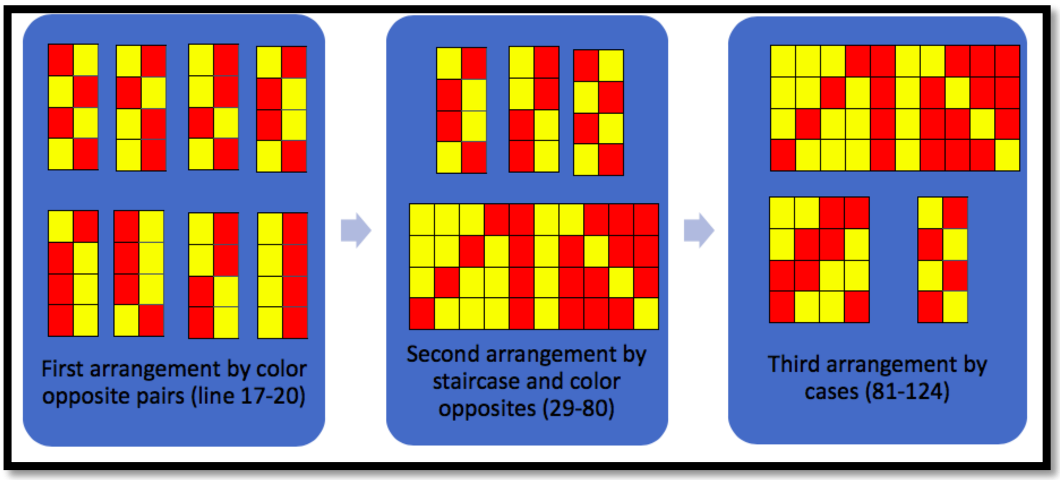

Figure 12.2.4. Milin's three versions of tower arrangements.

Figure 12.2.5 provides a summary of the conversation between Milin and R1 that developed his argument by cases to justify that there were 16 total 4-tall towers.

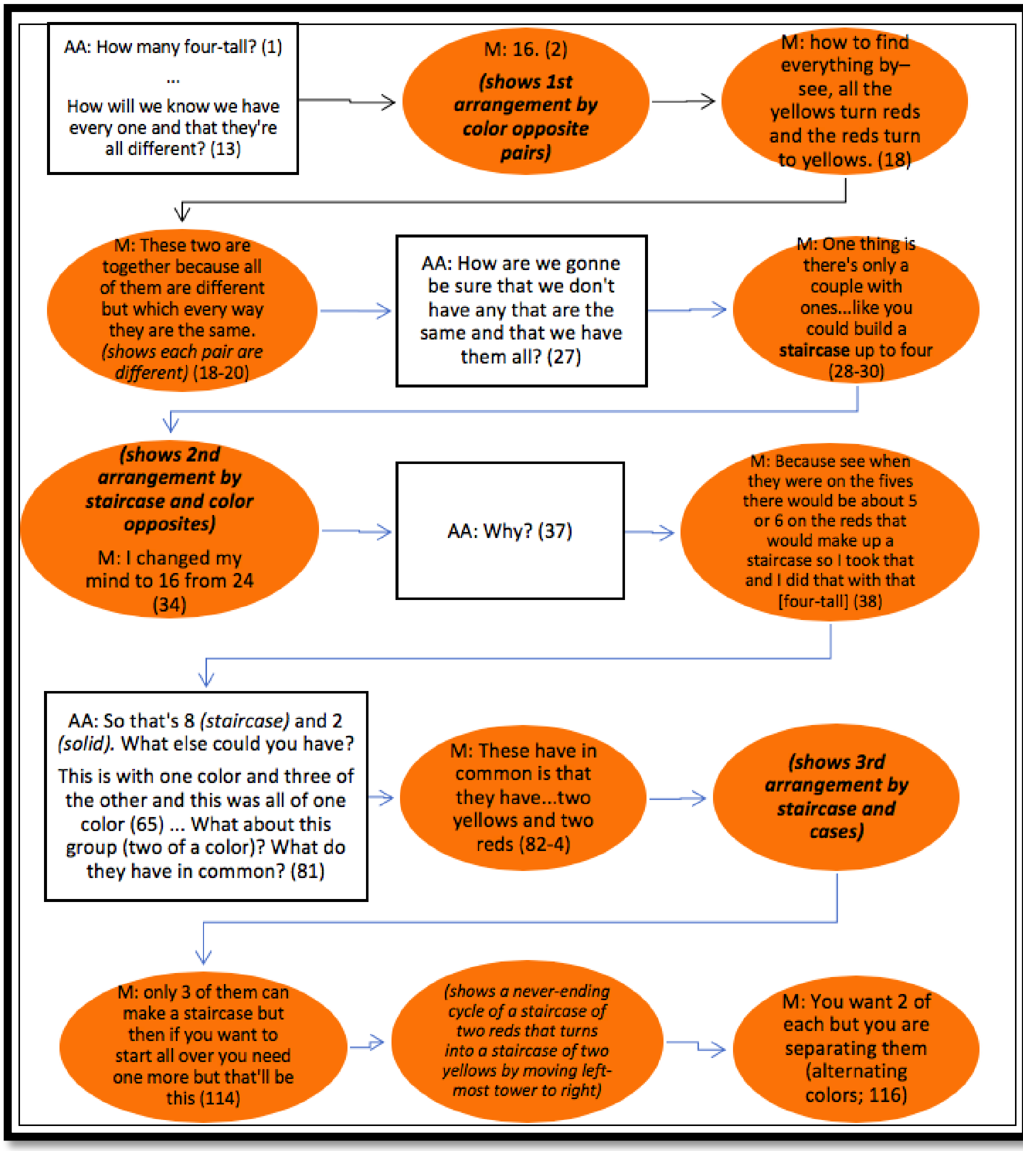

Figure 12.2.5. Overview of the development of Milin's argument by cases.

Discussion about the relationship between cases of towers 1-, 2-, and 4-tall (see Event 5). As Milin reviewed his written work with R1, a conversation evolved about the Tower solutions of various heights. He explained how he obtained the number of 1-, 2-, and 3-tall towers, and predicted the number of 6-tall towers. It began with 2-tall towers, which he claimed to be a total of four. Milin, when asked to provide an explanation, responded by constructing the color opposite towers (L129-133; see Figure 12.2.6). He argued that there were no more than four: "You can't make any more because see, these two” [*points to the color opposite pairs YR and RY*] (L140).

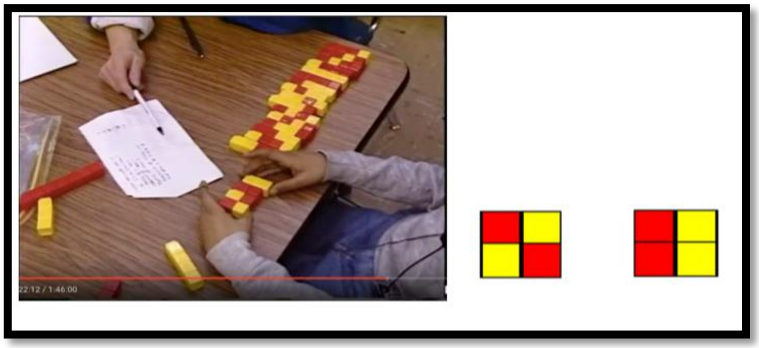

Figure 12.2.6. Milin’s 2-tall towers arranged as 4-tall towers.

The researcher pointed out that the towers with exactly one of a particular color are “like those two [*towers with*] ‘ones’ over there” [*points with her pen over the case of 4-tall towers with exactly one of a color*]. Then Milin showed the attribute of a single color occurring within the 2- and 4-tall towers. Milin mimicked her pointing gesture and stated, “And these two [*single-colored 2-tall towers*] are like exactly like these” [*gestures toward the single-colored 4-tall towers*]. He then organized the 2-tall towers in a similar arrangement to that of the 4-tall towers. Note that his arrangement included the cases with exactly one of a color on the left and right sides around the two single-colored towers (see Figure 12.2.7).

Figure 12.2.7. Milin’s 2-tall towers in a second arrangement similar to 4-tall towers.

For 1-tall towers, Milin quickly showed the two towers, Y and R (L144-147). He claimed that there were no more 1-tall towers because, “you can’t make any other design, like this” [*holding up a 2-tall tower of different colors*] (L148). He returned to his comparisons of one height to another height.

Milin used the color opposite strategy to build towers 3-tall (see Event 6). Milin claimed that, for 3-tall towers, there would be eight towers (L154). He started with the all red and all yellow towers, then the alternating colored-cube towers, and finally four more towers with exactly one of a color in the top position and then in the bottom position (see Figure 12.2.8).

Figure 12.2.8. 3-tall towers arranged by color opposites.

As Milin silently proceeded to build the last four towers, pairing them together as opposites, he commented about them: “See, these two [*pairs of exactly one of a color on top and exactly one of a color on bottom*] fall into the same hands because…see if you go like this [*turns one pair upside down to show they match the towers with exactly one of a color on the bottom*]…they’ll be the same… But, if you flip it [*rotates to original position*] over, they'll both be different” (L168-74). When he referred to two towers (or pairs of towers) as “falling into the same hands” he explained that they were duplicates of the other when one was a rotation of the other (L171-6). Tracing how he built the 5-tall towers in a classroom session (Session IV) a few weeks earlier, he also used inverse and opposite strategies to group tower pairs together (see Composite Operation). Perhaps this is why he exclaimed, “That’s what I *always* work on” (L178; Milin’s emphasis). He also claimed that these two pairs (YRR and RYY, YYR and RRY) were *not* related to the pairs with the single color and the alternating color cubes (L182):

> Uh huh, but then there's also these two [*shows YRY and RYR*]… But they don't fall into hands but these [*RRR and YYY*] and these two [*RRR and YYY*]. These fall into the same hands as these. [*Picks up RRR and YYY to compare with the all red and all yellow 4-tall towers*].

The second time Milin referred to something as "falling into the same hands" was in regard to a relationship between the single-colored and alternating-colored towers. Tracing how he referred to the parallel cases in 4-tall towers, he compared the same pairs as "coming into" each other (refer to Events 2–4). Both references related towers within one case or a case among other cases. When the researcher pointed out that the single-colored towers existed in every Tower Task, Milin added "except for zeros [*0-tall*]!" (L186), evidencing his own acknowledgement that he believed it to be true for all heights. In summary, his argument in support of the 3-tall Tower solution was dependent on the inverse and opposites relationships he recognized within his groups of towers.

Predicting the 6-tall Tower solution based on the 5-tall Tower solution. When prompted, Milin guessed "around forty something" (L191) for the 6-tall Tower Task. Two instances during this discussion the researcher redirected Milin to explore building from 1-tall towers to 2-tall towers. In the first instance he was reminded about the 5-tall Tower Task and prompted about the total 1-tall towers. Milin reviewed the number solutions for 2-, 3- and 4-tall Tower Tasks and R1 recalled the number solution for the 5-tall Tower Task (L195-200). He predicted that his elevator strategy would produce more towers for taller heights, stating (L200-204):

> And six, whew, that's even a bigger group [*more towers within a case*] because…see…every one [*the other height towers*] is smaller because…see if I had a seven one [*takes a large stack of all reds from the side and counts to make sure it has seven*]...a seven one like this…it would be more than sixes in all of these because it has more [*positions*] and you could change more stuff on it [*reaches for a set of yellows cubes to demonstrate by moving it up and down along the 7-tall red tower*].

Milin claimed that if the number of positions increased, then there would be a larger total of 6-tall towers, but not as large as the total 7-tall towers. The elevator pattern played a role in his reasoning and in his predictions about taller towers.

In the second instance he was asked directly how it would be possible "to go from towers of one and make them into towers of two?" [*R1 points to 1-tall towers*] (L207). Here Milin showed that the red cube could be put on top of the yellow cube (RY), before moving on to a new discussion. He was given for homework the task of finding 6-tall towers.

Milin explored 2-tall towers, selecting from three different colors. When asked if the two-color Tower Tasks reminded Milin of other problems, Milin mentioned the same task with three colors. His primary strategies were based on cases, grouping into "pairs of three," and elevator patterns. He argued that the number solutions for the three-color tasks would be larger than their two color counterparts because of the extra color that would generate an extra solid-colored tower (L232). This displayed another instance of his awareness of the case of the single-colored towers for any height (except 0-tall). He justified his claim by demonstration; he built 1- and 2-tall towers, selecting from three colors. He also argued that the 2-tall towers (selecting from three colors) he built would be in "pairs of three." He demonstrated this by grouping towers into sets and stating that towers that "fall into the same hands" would be in sets of three instead of two. He first grouped the towers into two sets of three (WY, RW, RY and WR, YW, YR) and described each set as "falling into the same kind of hands," or in other words, these were the towers with no repetition of the same color. When he rearranged them into pairs of color opposites (these 2-tall towers pairs would also be inverses; e.g., RY and YR) he described them as "falling into the same *exact* hands" (he emphasized the word "exact"). Tracing back to the two-color Tower Tasks, he grouped towers by their inverse or their color opposite relationships or both.

12.2.1 Findings: Key developments of Session VII

Display of justifications and forms of reasoning. Milin initially grouped his 4-tall towers (that he built for homework) by opposite pairs. When asked to organize them in a way to become certain of his solution, he used a globally exhaustive, systematic method of enumeration (by cases and the use of elevator patterns) to justify his solution. He also symmetrically organized them as follows: the case of exactly one of a color in a descending elevator pattern and the opposite in an ascending elevator pattern with the single-colored towers in between (for 2- and 4-tall towers) and the case of exactly two of a color on its own in an elevator pattern (for towers 4-tall). He used an elevator pattern to justify the exhaustion of the case of 4-tall towers with exactly one and exactly two of a color. When he was prompted by R1 to describe them, his language about his cases became more precise to include the attribute of each color. He also used his own language to describe a case or a tower that belonged to a case as "coming into these" or "falling into the same hands." After exploring towers shorter than 4-tall, Milin claimed that the number of shorter towers would be fewer than the number of taller towers because of the "bigger staircases."

He also created and solved a 2-tall Tower Task, selecting from three colors, by a method based on "pairs of three" and cases: towers with no color repeated and towers with a single color. He used similar phrasing of "falling into the same hands" to describe the relationship between towers within a case or to compare the single-colored case to the same case in other tower solutions. The three-color task events provided insight into Milin's reasoning of the Tower Tasks, selecting from two colors. Specifically, it reaffirmed the meaning of his language ("falling into the same hands") and that he was reasoning by cases by comparing cases of towers of different heights that were similar to each other or by inverse and opposite relationships within a case.

12.3 Third Interview (Session VIII)

Date	March 6, 1992
Grade	4
Task	Towers (1- through 6- tall towers)
Participants	Milin
Researchers	R1, R2
VMCAnalytic(s)	Milin's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 1 of 2 (Grade 4); Event 5 http://dx.doi.org/doi:10.7282/t3-hrjs-jq34

The interview between R1, Milin, and later joined by R2 (who, at the request of Milin, suggested a meeting with Stephanie and others to share their ideas) occurred on March 6, 1992. The interview elaborated on Milin's previous work of Tower Tasks of various heights, selecting from two or from three colors. He reported finding 50 towers, 6-tall.

Conflict between methods by "staircases" and by "families" (see Event 5). After exploring a new strategy that Milin called the "family strategy" (the amount of taller towers double the amount of shorter towers because two taller towers are generated from each shorter tower when selecting from two colors), Milin indicated conflict between the result of 50 towers built by partial cases and a predicted 64 towers by the family strategy (see Chapter 14.1.3 for his development of the "family strategy" in this interview). In this event, Milin began to doubt his old "staircase" strategy. When R1 suggested that Milin test the new strategy and compare it with his staircases, he responded that (L581-6):

> Building staircases are a wrong thing to do because maybe staircases don't have a couple of things… [*showed an alternating color tower in response to R2's request*

for an example of a tower not in an elevator pattern]. *S*ee, it won't be a staircase– at least not a nice one.

12.3.1 Findings: Key development of Session VIII

Recognition of patterns. The result in this section was reported to note the conflict in solutions when solving the 6-tall Tower Task by (partial) cases and predicting by a doubling pattern (see Chapters 14.1.3 – 15). Milin portrayed a lack of confidence in his method based on cases and displayed an example, albeit the example had three colors, when explaining that his method by cases may be missing towers. Recall, he would generate towers that could create an elevator pattern, which limited the towers he could find.

12.4 Summary of Grade 4 interviews

Chapter 12 presented Milin's development of reasoning about and organizing towers by cases.

Recognition of patterns and use of strategies. Milin relied on the elevator pattern to generate and exhaust a case of towers. This played a role in Milin's search for new combinations, especially when towers were taller and patterns were more complex (i.e., both colors had separation). Evidence showed that Milin was focused on the separation of one color, but not the other, which caused duplication in the cases of adjacent (since a tower can have cubes of the same color adjacent to each other and cubes of the other color separated from each other, like YRRRY). Like Stephanie, Milin indicated a conflict between the result he obtained by cases (30) versus the result he recalled from the class discussion (32) in the first interview and was encouraged to search for new towers. Milin *folded back* to Guess and Check and the color opposite and/or inverse pairing strategies (i.e., composite operations) to generate towers when he recognized towers with color separation existed. In Session VII, working on 4-tall towers, he indicated awareness of both colors and did not generate any duplicates (note that they were pre-built at home).

Forms of reasoning in support of the solution. Flagging for or resolution of duplicates based on his recognition of equivalent cases played an important role in how he solved the first task, 5-tall towers. Milin reasoned about 2-, 3-, 4-, and 5-tall towers through cases. His cases were not the same across all heights (e.g., 3-tall), but he indicated categorization for each Tower Task he solved and used those categories to justify completion of a solution or how towers were all different. Milin also directly reasoned about the number of available positions when justifying the completion of the cases of exactly one of a color and exactly two adjacent of a color for 5-tall towers.

He indicated a conflict between his "staircases" and "family" methods when predicting the solution to 6-tall Towers in Session VIII. The conflict is an important finding in Milin's transition from reasoning by cases to using a generalized doubling rule and then building. Chapters 14 – 15 provide results on the development of reasoning by induction.

Representations. Milin worked with physical towers and organized them in symmetrical patterns to represent his solution. Milin searched for visual patterns in his sets of towers in both Interview Sessions VI and VII to justify the cases he generated. When Milin put towers into elevator patterns, those towers became separate cases. As Milin continued to solve the 5-tall towers by cases, R1 organized the towers so that opposite color towers were below their partner. When he organized his solution to 4-tall Towers he created visual patterns. He used those patterns to solve 2-tall towers.

He recorded his numerical predictions or solutions and used them to explore number solution relationships. The use of his prior written work began new conversations about his predictions to other Tower Tasks. Milin described towers using idiosyncratic phrases to relate towers or cases across different heights of towers, such as “falling into the same hands,” or to name a case, such as “four and one” indicating towers with four of one color and one of the other color. When he noticed tower attributes of both colors, he began to describe local cases by both colors in order to justify the difference of each case in Session VI and Session VII. R4 prompted any vague descriptions and R1 demonstrated precision in describing both attributes. Milin used similar language to indicate which cases of towers were different and which were equivalent.

Chapter 13 Results: Milin's application of the method by cases to solve Tower Tasks during partner problem solving (Grades 4 & 5)

13.1 Grade 4

13.1.1 Dyad summative assessment (Session X)

Date	June 15, 1992
Grade	4
Task	Towers (3-tall)
Participants	Dyad pair: Milin & Stephanie
Teacher	Barnes
VMCAnalytic(s)	Milin's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 2 of 2 (Grades 4 & 5); Event 4 http://dx.doi.org/doi:10.7282/t3-dwqy-mg03

On June 15, 1992 Stephanie and Milin were partnered to work on the 3-tall Tower Task as an assessment of their reasoning. (Chapters 7.3 and 9.4 detail the session in Stephanie's perspective.)

Milin and Stephanie solve the 3-tall Tower summative assessment initially by cases (see Event 4). Once the problem was presented, Milin told Stephanie he knew the answer was eight and that they had seen this problem before (L20-27). Stephanie, unsure, stated that she will make sure because she forgot the solution. While drawing their first tower representations (see Figure 13.1.1), Milin claimed, "it's gonna be an even number" and Stephanie replied, "it always is" (L39-40). Milin drew squares for each cube in each of his tower drawings, while Stephanie drew a table and filled in columns as she created each tower. Milin drew the same towers that Stephanie called out in the same order.

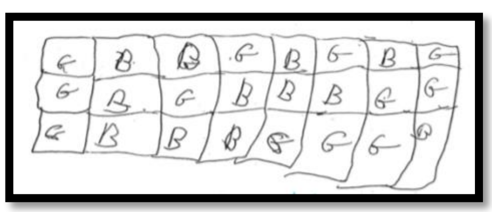

Figure 13.1.1. Milin's written solution of the towers

Once they finished drawing the tower combinations, they built the towers using green and black cubes. Stephanie told him to find all towers with two black cubes, naming them "two blacks." Milin found all three of them, as well as indicating to Stephanie that he found the single-colored towers. When Stephanie took Milin's share of the towers it was arranged in an elevator pattern, different from his written work, which was copied according to Stephanie's organization (see Figure 13.1.2).

Figure 13.1.2. Milin and Stephanie's 3-tall towers by cases.

Interestingly, Milin posed the question, "Can we make any more [*towers*]" when they put all eight towers together. In response Stephanie decided that they use a *guess and check* strategy to find the towers again and check if they were already built. Milin, on the other hand, reorganized the towers by opposites or by inverses. Figure 13.1.3 (adapted from Sran, 2010, p. 115) illustrates his reorganization by pairs, as a way of justification of his solution.

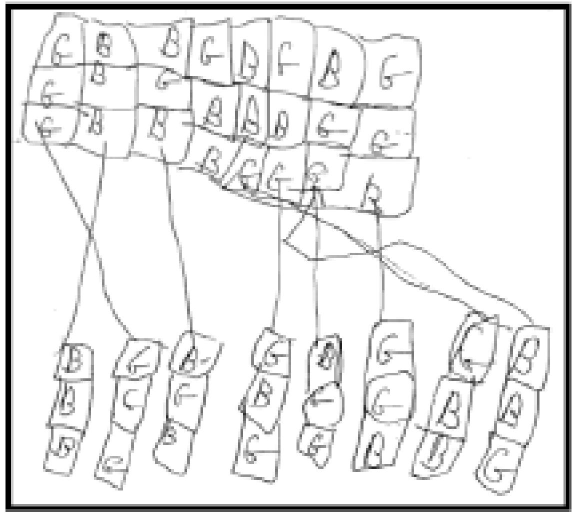

Figure 13.1.3. Milin's reorganization of the 3-tall towers by pairs.

Refer to Chapter 15.1.2 for the continuation of their problem-solving in this session, when they applied a doubling rule to justify their numerical solution.

13.2 Grade 5

13.2.1 Dyad: Milin & Michelle (Session XII)

Date	February 26, 1993
Grade	5
Task	Guess My Tower ("GMT"; 3- and 4-tall)
Participants	Dyad pairs: Michelle & Milin; Stephanie & Matt; fifth-grade class
Researchers	R2, R1, and R3
VMCAnalytic	n/a

Fifth graders Michelle and Milin were working together on the Guess My Tower task – an extension of the Tower Tasks – on February 26, 1993. R1 joined them to discuss their solutions, which prompted a conversation about how they were sure of their outcomes of each possibility given in the task without knowing their sample space or as they referred to the outcomes as, "the towers in the box." In this discussion they were negotiating the meaning of the tower representations that they created for the outcomes of the GMT and the sample space of all 3-tall towers selecting from two colors. During this negotiation, Michelle wanted to separate the tower representations of each outcome set and the set of towers that would be in the "box." After, R1 called the class' attention to the contents of the box: "Can we all agree that in this box there are no duplicate towers?" (see L204-33 for the complete class conversation).

Finding the 3-tall towers sample space. After the class is exposed to the rules of the "box" (the sample space of towers), Michelle and Milin found the contents of the first box, where towers had to be 3-tall. They used their tower outcomes from the given conditions of the GMT task to extract the duplicate towers and create their sample space. In other words, their strategies for finding the towers was to use the given conditions (which were the outcomes) of the task, build the possibilities and then combine those towers in order to find the contents of the box, using staircase patterns and color opposite relationships, as well as eliminating duplicate towers.

When Milin created the towers "with at least two yellow cubes," (the fourth outcome in GMT) the towers with exactly one red cube, he rearranged them in an elevator pattern in response to Michelle's earlier confusion of where Milin got his numerical solutions for the event outcomes very quickly. His organization for the contents of the box included the cases of towers with exactly one of a color *around* the two single-colored towers, just as the 4-tall towers were organized in Session VII in fourth grade (refer to the right in Figure 13.2.1). Milin's towers for each GMT outcome were also organized in elevator patterns (see #2 and #3 on the left in Figure 13.2.1), as well as color opposites (see #1) and inverse pairs (see #4). Figure 13.2.1 (adapted from Sran, 2010, p. 129-30) illustrates Milin's written work.

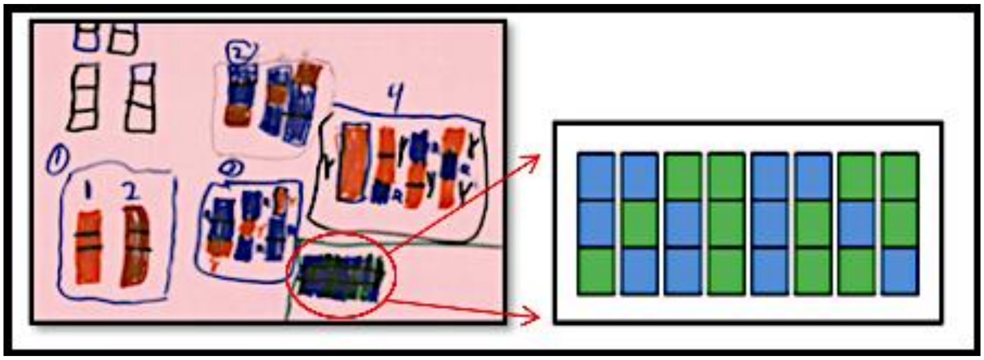

Figure 13.2.1. Milin's outcomes for GMT on the left and 3-tall towers on the right.

4-tall tower predictions. For part two of the GMT, Milin drew a grid of 16 empty slots of height four and filled them in with red or orange colors. He found the contents of the box, consisting of 4-tall towers, first. He claimed that there would be 16 4-tall towers because eight times two is 16. Milin described his method for finding the towers by "just using the staircase" (L466). He used elevator patterns and color opposites to generate the towers (see Figure 13.2.2). He organized them in a similar arrangement as the 3-tall towers.

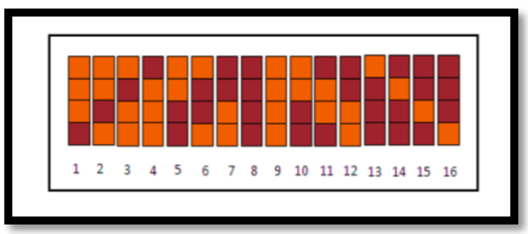

Figure 13.2.2. Milin's initial case representation.

In this session he generated towers with one of a color and two of a color adjacent in elevator patterns (for each color), and this resulted in two duplicate towers (towers #5 and #12 are duplicates of #10 and #7, respectively in Figure 13.2.3). He crossed out the duplicates after Michelle pointed him to them and abandoned his search for the complete solution (see Figure 13.2.3). Note, he was missing the alternating color towers.

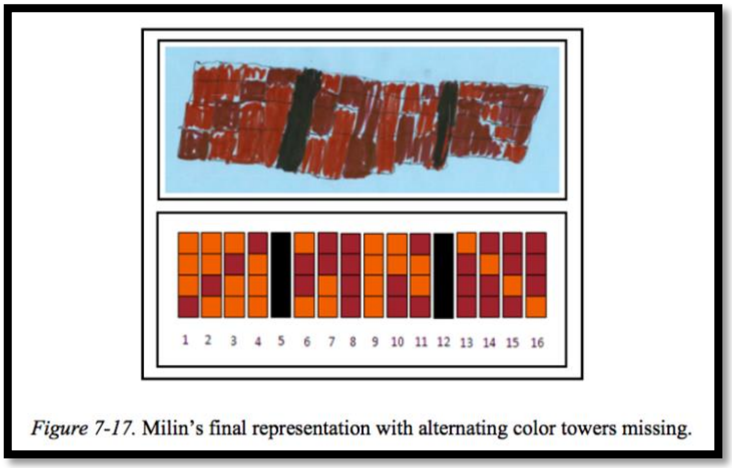

Figure 13.2.3. Milin's final 14 towers, 4-tall.

13.3 Summary of Grade 4 & 5 assessment and problem-solving sessions

Recognition of patterns and use of strategies. In Sessions X and XII, Milin found the 3-tall Tower Task number solution by applying a doubling rule and generating the towers using methods by cases and opposites. In Session X he also *folded back* to reorganizing by color opposites and inverses to display his justification to the solution. In Session XII he solved the 4-tall Tower Task in order to find information for the GMT Task. In both sessions he used the elevator strategy to generate and organize towers of exactly one of a color (for 3- and 4-tall) and exactly two of a color (for 4-tall).

Duplicates also emerged due to his reliance on the elevator patterns. In the 3-tall towers the case of one of a color is equivalent to the case of two of the opposite color, but not in the 4-tall towers. Due to this difference, Milin's way of constructing became evident in Session XII. Recall in Session VII, fourth grader Milin showed 4-tall towers with two of a color adjacent and their opposite towers could not be separated into two elevator patterns simultaneously due to duplication. In Session XII, fifth grader Milin created an elevator pattern for the cases of towers with exactly two red and exactly two orange on opposite sides of each other, which created 2 duplicates (e.g., RROO and OORR). Michelle noted these duplicates. Recall his elevator pattern limited his generation of more complex tower patterns where both colors were separated in some way from each other. For the third time (using a method by cases: for 5-tall he found 30; for 6-tall he found 50; for 4-tall he found 14 towers) he experienced the “staircases” failing to provide him with the doubling rule’s conjectured number solution.

Representations. Sran (2010) noted that Milin’s drawing of the 3-tall towers was organized by the same staircase pattern that he used in his earlier fourth grade work. Specifically, in Session XII he organized the 3- and 4-tall towers in a symmetric pattern similar to that of Session VII with the cases of opposite colors around the single-colored towers. In Sessions X and XII Milin freely alternated between physical towers and drawings. In Session XII he drew an empty grid with 16 places to draw his tower patterns, noting his recall and use of the number solution before generating the actual towers.

This chapter concludes the results of Milin’s development and application of a method based on cases to solve Tower Tasks. The next chapter presents the results of Milin’s development of reasoning by an inductive argument to solve Tower Tasks.

Chapter 14 Milin's development of reasoning by an inductive argument to solve Tower Tasks (Grade 4)

14.1 Post interviews

14.1.1 First interview (Session VI)

Date	February 7, 1992
Grade	4
Task	Towers (5-tall)
Participants	Milin
Researchers	R1, R4
VMCAnalytic(s)	n/a

After exploring 4-tall and 5-tall Tower Tasks in third and fourth grades, Milin began estimating and testing his conjectures of the number solutions to various heights in Session VI using various methods presented in this section.

Explorations of the relationship between 5-tall towers and other heights. The interview with R1, R4, and Milin occurred on February 7, 1992, the following day after the classroom work with Michael on the 5-tall Tower Task. After Milin reinvestigated the task with the researchers, R1 asked, "If I was going to say, 'Gosh, we really worked so hard to figure out towers of five,'…Could you have made towers of four instead of five?" (L449). Milin confirmed. R1 asked, "If you made towers of four, how many do you think there would have been?" (L451). In this instance he provided an estimate of 20 (L454–6); in other instances, he estimated 24 (L476–84), 18 (L502), and 16 (L616). Milin also claimed, "It [*4-tall*] has to be less than five's [*5-tall*] because five is a higher number" (L486). The researcher then asked about his prediction for the number solution for towers 3-tall (L497). He responded with similar reasoning: "It's [*3-tall*] probably going to be less than towers of four" (498). Similarly, he was asked about towers 2-tall and to imagine in his mind what they would look like (L505-7). He built them and stated, "about four" (L518). Not illustrated in this chapter (refer to Chapter 12.1 for Milin's method based on cases), in the first interview he tested his conjectures by building towers of different heights, using his method of "staircases" and opposites

1-, 2-, and 3-tall tower predictions. Spontaneously he stated, "on one's [*1-tall towers*] there would only be two" (L520). They reviewed the number solutions of towers of consecutive height from 1- to 3-tall. After finding towers 1- and 2-tall, he built six 3-tall towers and noticed that he could build more (L548-51). After making RYR Milin again made the claim that he could make "eighteen" or "six pairs" (L554; inequivalent values). R1 asked him to consider if the new tower RYR has an opposite pair, to which he responded by building YRY. Then R1 asked, "What else could you do?" Milin responded by looking around his set and the other cubes. The researcher intervened by having him recognize the pair with only one of a color and Milin responded with a new solution of eight towers. He elaborated: 1) He claimed that towers 4-tall would be greater, specifically they "would be about twelve," and 2) He grouped the towers by cases (see Chapter 12.1; L566-76). The researcher asked him to record the solutions for 1-, 2-, 3-, and 5-tall towers, which led to a conversation about the missing Tower Task of 4-tall. Milin estimated "about sixteen" (L598).

14.1.2 Second interview (Session VII)

Date	February 21, 1992
Grade	4
Task	Towers (1- to 4-tall), selecting from two colors; Towers (1- to 2-tall), selecting from three colors
Participants	Milin
Researchers	R1
VMCAnalytic(s)	n/a

On February 21, 1992, R1 interviewed Milin for the second time of a three-part series of interviews. Earlier in the interview Milin solved the 2-, 3-, and 4-tall Tower Tasks, selecting from two colors, by cases and color opposite strategies. He also proposed his own Tower Task, selecting from three colors.

Seeking numerical patterns for towers 1- and 2-tall, selecting from three colors and selecting from two colors. Milin built towers 1- and 2-tall selecting from three colors. When he was asked to justify the total of nine 2-tall towers selecting from three colors, he responded that he could continue building (i.e., *Trial and Error*). However, when R1 redirected him back to the solution of the 1-tall tower of three colors, Milin explained, "If they had this [*1-tall towers, three colors*] there'll be two, four, six, eight, nine for this, and I'm sure of that!" [*gesturing toward his group of 2-tall towers selecting from three colors*] (L258). He then elaborated on this claim:

> If you times it [*1-tall, selecting from two colors*] by two, you'd have four *[2-tall towers*], but if you had three [*colors*] you could times it by three, because, see, there's three of these *[holds up his towers of one with three colors*]...two, four, six, eight, nine.

He claimed there was a connection to the number of colors available and the total number of towers 2-tall. In addition to the claims that the 1-tall tower number solution was multiplied by two with two colors available, and the 1-tall number solution was multiplied by three with three colors available, he wrote, "2 x 2 = 4" and "3 x 3 = 9."

Early connections between the number of colors available and the solution to the Tower Tasks. Milin compared Tower solutions of different heights and searched for patterns among them. He estimated the number of 6-tall towers, selecting from two colors, was about 46 or 44. He provided an explanation for why the total combinations of towers selecting from two colors for any height had to be even (L360):

> Because two [*referring to 1-tall towers*] is an even number and, um, it's got to be even because you can make pairs of them [*of towers*]. But, if you had three [*colors*] you can't make pairs of them because of this: [*Holds up three cubes of different colors]*. If you make pairs of them [*towers selecting from two colors*], they'll be in twos maybe. But these three [*cubes of different colors*] would make a difference! [*Points to three 1-tall towers, selecting from three colors*].

Notice that Milin returned to his reasoning about grouping towers by pairs, when selecting from two colors (by opposites or inverses). He stated that the original 1-tall towers "makes a difference" to the number solution for other heights; that is, if two colors were available, the towers of the other heights would be "made up of twos [*pairs*]" (L308), whereas, if three colors were available, the towers of the other heights would be in "pairs of three."

He also conjectured that the extra color produced an odd solution that was one more or one less than the solution of the two-color task counterpart. For example, he estimated towers 4-tall, selecting from two and three colors: "I took this [*3-tall towers, two colors*] and I times it by two and added one more because of this [*picks up the third color, white*]…I could've subtracted one" (L312). He estimated that if there were 16 towers, 4-tall, selecting from two colors, then there would be 15 or 17, selecting from three colors. At the conclusion of the interview, he was asked to review the unanswered questions at home (the 6-tall Tower Task, selecting from two colors, and the 3-tall Tower Task, selecting from three colors).

14.1.3 Third interview (Session VIII)

Date	March 6, 1992
Grade	4
Task	Towers (1-through 4- tall towers)
Participants	Stephanie
Researchers	R2
VMCAnalytic(s)	Milin's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 1 of 2 (Grade 4); Events 1–6 http://dx.doi.org/doi:10.7282/t3-hrjs-jq34

On March 6, 1992, R1 conducted a third interview with Milin. R2 joined the session later (who, at the request of Milin, suggested a meeting with Stephanie and others to share their findings about the Tower Tasks). The interview elaborated on Milin's previous work with Tower Tasks of various heights, selecting from two and from three colors.

Milin generated towers 2-tall from towers 1-tall (see Event 1). In the beginning of the interview Milin recalled the number of towers he found for 1-, 2-, 3-, 4-, and 5-tall, selecting from two colors. R1 asked, "If I had towers of one, what would towers of two look like?" (L43). Milin showed a black (Bk) cube placed on top of the 1-tall black tower and a blue (Bl) cube placed on top of the 1-tall blue tower (BkBk and BlBl). As R1 asked if his 2-tall towers were complete, he built four 2-tall towers using new cubes, selecting from blue and black. The researcher questioned, "Why did that happen?" (L47) and suggested that he return to the 1-tall towers. He initially argued that it occurred "because you had to put something else on top of these [*points to towers 1-tall*]… because its gotta be two" [*2-tall*] (L48-50). Then R1 asked him what exactly he could place on top (L51). In this instance, Milin described how a 2-tall tower was generated from a 1-tall tower. He stated (L52-8):

> If you had a blue [*picks up Bl*] you could put another blue or a black on it [*moving BkBl near Bl*]…this and this [*show the two new towers*]. And black, you could put a blue or a black on [*demonstrates by putting the two 2-tall towers with the black bottom next to the black tower 1-tall*].

For the first time Milin showed how 2-tall towers were generated from 1-tall towers by placing an extra cube of each color available on top of the original 1-tall tower. He demonstrated this by building two towers with a black or a blue cube on top of a blue 1-tall tower, and, similarly, a black or a blue cube on top of a black 1-tall tower. Note the order that he built taller towers: by color opposites rather than by placing a blue or a black cube on top of each 1-tall tower. However, he showed the growth physically by placing each 1-tall tower alongside the two 2-tall towers that had the same color in the bottom position. R1 asked if Milin could have found any more towers and Milin responded, "Uh uh [*no*] because there's not enough Unifix cubes for this [*not enough* *colors*]. Like if there were three [*colors*]…for two [*2-tall*] they'll be around six or eight [*towers*] or something like that" (L62-73). The researcher asked him to continue selecting only from two colors.

Milin generated towers 3-tall from 2-tall (see Event 2). When he was asked how 3-tall towers were generated from 2-tall towers, he demonstrated a partial generalized reasoning through his actions on half of the generated towers: "You could always put another one on top of that [*2-tall towers*]*...*see, [*selects a blue cube*], just put another one on top of this, this, this" [*moves the blue cube over each 2-tall tower*] (L84-6). This event illustrates how the researcher aided in his articulation of what was occurring when towers were generated from shorter towers. He was asked to build 3-tall towers by beginning with a copy of each 2-tall tower (as per request for him to avoid using, and, in so doing, destroying the towers that he had built of smaller heights). The researcher commented: "This one grew to be this one" for two examples while spreading out the 2- and 3-tall towers and placing the appropriate taller towers next to their corresponding shorter towers. He began by building towers with a blue or a black cube on top of each duplicate 2-tall tower, but not in pairs. Note, as in the previous event, he made the single-colored towers first (placing a black on the black cubes and a blue on the blue cubes, and then varying the colors). However, R1 organized them and asked if anything else could be placed on top of shorter towers that only have one higher tower associated with it (e.g., L109-114):

> Okay, so this one grew to be this one [*blue/blue/blue with blue/blue*]. This one grew to be this one [*blue/blue/black with blue/black*]. This one grew to be this one [*black/black/black with black/black*]. [*R1 spreads the towers 2- and 3-tall*]. Wait, we need another one for that [*R1 points to the 2-tall tower that is not paired with a 3-tall tower*]*...* So this one turned into this one [*R1 points to the blue/black*

> *tower and then the blue/blue/black tower*]. Could it have turned into anything else?

Milin responded, “It [*blue/black*] could have turned into black [*black/blue/black*]” and built the tower. Note that he then rearranged the 3-tall towers by color opposites (“duplicates”) and took along the associated 2-tall shorter towers, but not the associated 1-tall towers. When he completed the 3-tall towers, he reasons that the new tower pairs belong to a particular “family” (L138):

> See, that [***blue****/black/black with blue cube on top*] would go into this family *[the black/black 2-tall tower]* because of this [*points to black/black 2-tall tower]*. And two blues *[builds black/blue/blue tower]* and if you wanted two of something on top, it will go here *[puts the black/blue/blue tower with the blue/blue family]*.

Two towers that belonged to a “family” of consecutively shorter towers had the same color cubes in every position, with the exception of the top, extra cube which contained one color on one 2-tall tower and the other color on a duplicate of the shorter tower. Note that the 2- to 3-tall family was disjoint from the 1-tall tower that started it all, due to Milin’s earlier rearrangement of the 2-tall towers as opposites.

Predicting the number of 4-tall towers from 3-tall towers and explains how 4- and 5-tall towers are generated from shorter towers (see Event 3). In this event, for 4- and 5-tall towers, he was asked to show one “family” that started from the black 1-tall tower. He separated the towers to make a clearer representation of the growth of the towers and showed how he generated two 4-tall towers from the blue/blue/black tower. Milin found 16 towers, 4-tall (L152-4; L164):

> You put either a black or a blue on it [*as R1 builds two black/blue/blue towers and hands them to Milin*]...and that would work for all of these too [*Milin points to the row of 3-tall towers*]...two for this, two for this, two for this, two for this, two for this, two for this [*as Milin points to each 3-tall towers*].

He then summarized that 5-tall would result in 32 towers from the 16 towers, 4-tall. Milin also claimed that the pattern “doesn’t work on towers of six” [*6-tall*] (L172). In response to how 6-tall were different, Milin explained, “Cause I got fifty. I made staircases and I made all of that” [*pointing to 6-tall towers in the bag*] (L180).

Milin conjectures that the doubling pattern breaks down at towers 6-tall (see Event 4). Throughout the interview there were a number of instances where he relied on his previous solutions to Tower Tasks of various heights to support his certainty that the doubling pattern broke down for the 6-tall Tower Task. For example, with his homework on 6-tall towers, he claimed the pattern broke down and stated, “Cause I got fifty. I made staircases and I made all of that” (L180). He explained his solution using the towers he built at home, but he could not find all the towers he made (L183-203). Another instance was when he referenced his written work and stated, “And once you get to sixteen [*Milin points to his paper where ‘5’ and ‘32’ are written*] you get all of them and you get thirty-two…but, it doesn’t work on six– towers of six” (L170-2). A third instance, after the researcher asked him to review the inductive pattern beginning from 1-tall to 5-tall for a “family” (see Figure 14.1.1).

Figure 14.1.1. Milin’s “family tree” doubling tower pattern.

When R1 asked if he was sure about 32 for the number of 5-tall towers, Milin responded, "We did it in class. That's one thing. Another thing is because if you follow the [doubling] pattern up to this" [*Milin points to his paper where '5' and '32' are written*] (L293-6). Milin claimed multiple times that the solution he found for the 6-tall Tower task was 50, even after he was asked to demonstrate if two 6-tall towers can be generated from a 5-tall tower. For example, he was asked to demonstrate for a 5-tall tower with four blue cubes on top and a black bottom and he responded as follows (L299-300):

> Him [*referring to a 5-tall tower with four blues on top and a black bottom*]? You just put another, either a black or a blue on [*Milin puts his hand over the 5-tall tower to indicate placing a sixth cube on top*].

R1 built each 6-tall tower as he described what it must have. He explained (L310-4):

> See it [*6-tall tower*] has to have this [*showing the 5-tall tower with 4 blues on top and a black bottom indicating the same pattern the 6-tall tower should have*] …On the bottom like this, see? [*Milin points to the bottom of the two 6-tall towers generated from the 5-tall tower*]…And then you put either a black or a blue on [*pointing to the top of a 6-tall tower*].

R1 revoiced his earlier claim, "But you're saying you couldn't do that for all of them going from five to six [*R1 points to the 5-tall towers*]" (L315). Milin responded with no and explained why (L316-24):

> Uh-uh [*indicates "No"*], because there's going to be less [*6-tall towers*]…Because some of the families can't actually afford them [*both laugh*]…You could only put a black or a blue on, but somewhere in there there's going to be this place where this one can't afford it [*Milin laughs*]…Unless I'm wrong.

Note how he continued to build a few examples of 6-tall towers from the 5-tall towers and to claim that the "family" pattern broke down because there were only 50 different 6-tall towers (e.g., L356). He did not explain why the pattern does not work after 5-tall even after building half of the family tree starting from a black 1-tall tower to some 6-tall towers. However, he suggested that he has doubt about his claim: "I might be wrong" (L440).With the researcher, he reviewed that from two towers, 1-tall, the family strategy produced four towers, 2-tall, eight towers, 3-tall, 16 towers, 4-tall, "to thirty-two [*5-tall*]…to fifty" [*6-tall towers*] (L395-9). Then he suggested that he had doubt about his own solution: "I might be wrong" (L440).

Pattern also breaks down "mysteriously" for the three-color Tower Tasks. Milin and R1 discussed the three-color Tower Tasks of 1-, 2-, and 3-tall. Milin recalled that 1-tall towers were three total and 2-tall towers were nine total. R1 and Milin applied his "family" strategy the three-color Tower Tasks for 1-, 2-, and partially for 3-tall. He became again concerned about the "family" strategy breaking down. Milin was able to verify that the inductive strategy for smaller heights of 1- to 2-tall worked, but he indicated that it was not possible to generate three towers from each 2-tall towers, selecting from three colors. In L535 (also in L543-69) he was certain that 3-tall was 25, but when he applied the family strategy he got 27 (from nine 2-tall towers multiplied by the three colors available). Even after physically placing a red, a yellow, and white cube on top of RY, he claimed, "But someplace it breaks up like thirty-two" (L551). Similar to the two-color problem, the physical manipulation did not convince him the pattern would work because of the discrepancy with his solutions (25 for the 3-tall three-color problem and 50 for the 6-tall two-color problem) and the results from the family strategy. The next event exposed what he meant when he stated earlier that the pattern "Doesn't work on towers of six" [*6-tall*] (L172).

Predicting the solution for the 6-tall Tower Task was 50, using his staircase and opposite strategies, or 64, using the inductive strategy (see Events 5-6). Before this event, while exploring the three-color Tower Task he described a doubling rule for the family strategy with two colors – that the pattern “was timesed [*sic*] by 2” (L537-9). Milin indicated, the “family strategy” was something he “found out like about today when I [*Milin*] was reading this paper” [*his written work from the previous interview*] (L573-8). When R1 suggested that Milin test the new strategy and compare it with his staircases, he responded that “building staircases are a wrong thing to do because maybe staircases don’t have a couple of things [*showed an alternating color tower in response to R2’s request for an example of a tower not in an elevator pattern*]. *S*ee, it won’t be a staircase– at least not a nice one (L581-6).

He indicated the possibility of being wrong about the family strategy breaking down for the two-color Tower Tasks numerous times (e.g., L440, L596-602). For example, he was asked to guess the solution of the 6-tall Tower Task with two colors (L596-615):

> But, I think I did something wrong on, um, from 32 to go to 6 [*6-tall, two colors*]. I think I did something wrong…Mmm, I dont think that pattern would break down like... My guesses are 50, if I was right the first time, or 64 [*6-tall*]…It’s probably not going to work with this [*points to towers with 3 colors*], but it’s going to work with this [*points to towers with 2 colors*]. I’m pretty sure of that.

He claimed he became certain that the inductive strategy will work for towers 6-tall. He was left with the same two tasks as the prior interview (the 6-tall Tower Task, selecting from two colors, and the 3-tall Tower Task, selecting from three colors), except this time to put his “family strategy” to the test against his claims.

14.2 Summary of Grade 4 interviews

Recognition of patterns and use of strategies. In this chapter Milin discovered a doubling pattern to the number solutions of the Tower Tasks and developed a justification by inductive reasoning. In the first and second interview sessions, he estimated various numerical solutions for towers up to 6-tall. For example, in Session VI, he applied his idea that the solution must be even when estimating the Tower Task numerical solution for heights less than 5-tall. He also conjectured that the 4-tall towers would be less than 5-tall, 3-tall would be less than 4-tall, and so on. He tested his conjectures by building towers 2- and 3-tall. For homework he solved the 4-tall and presented it in Session VII and he solved the 6-tall and discussed it in Session VIII. For both tasks he indicated using a method based on cases (4-tall) and partial cases (6-tall). In Session VII he sought numerical patterns between towers 1- and 2-tall, selecting from two and from three colors. He conjectured that the number of colors available made a difference. He conjectured a multiplicative relationship existed to obtain 2-tall towers: the product of the number of colors available and the number of 1-tall towers. For example, he noticed that three 1-tall towers, selecting from three colors, multiplied by three colors would result in nine 2-tall towers, selecting from three colors.

Forms of reasoning in support of the solution. His earlier experiences set the stage for Session VIII when he was asked to explore how to generate towers 2-tall from towers 1-tall and towers 3-tall from towers 2-tall. He demonstrated a partially generalized inductive reasoning, stating, “You could always put another one [*blue*] on top of that” [*each 2-tall tower*]. In discussion with the researcher about how the towers grew from shorter towers, he indicated that two new towers belonged to a “family” of shorter towers. He used a similar construction for 4-tall towers where the 4-tall towers belonged to a particular 3-tall tower and a particular “family” of towers. He continued to use an inductive method to generate a pair of 4-tall towers, stating, “You put either a black or a blue on it," when generating from a 3-tall tower. He explained the same inductive strategy would result in 32 towers, 5-tall.

Representations. The results showed also how in Session VIII Milin claimed that the "family" strategy broke down after 5-tall. His reasoning was based on his partial cases method for the 6-tall, which resulted in 50 towers. He also claimed the pattern broke down for the 3-tall, three-color task because he found 25 towers. Evidence showed that he may have relied heavily on earlier strategies (e.g., elevator and opposites) for guaranteeing the correct total number of towers. Also, it is important to note that he represented the growth by *physical* tower models. For example, in Session VIII, he only represented the growth of "one family" up to 6-tall and so his representation was incomplete. Perhaps when the combinations became taller or more complex, it was harder for Milin to imagine the different towers that would be generated by each family. Perhaps his prior tangible experiences with building towers gave him more certainty of an observed doubling pattern up to 5-tall and a cases and opposite based evidence of the 6-tall two-color solution and the 3-tall three-color solution. The use of a limited model of one family of towers may not have served as enough evidence to override the tangible experience of building the 6-tall towers. This was evidenced by his multiple references to his prior solution findings. When prompted about why the pattern did not work, his reasoning was vague, explaining that some families could not "afford" taller towers (even after R1's insistence that he could assume many more cubes were available).

Toward the end of the interview, he indicated doubt about his claim that the inductive strategy did not work after a certain height. He also suggested that his "staircases" may have missed some towers and justified his reasoning by showing an example of a tower with three colors alternating that would not have been generated by elevator patterns.

Chapter 15 Results: Milin's Application of the Inductive Argument to Justify Solutions to Tower Tasks (Grades 4 & 5)

15.1 Grade 4

15.1.1 Small group formative assessment interview (Session IX)

Date	March 10, 1992
Grade	4
Task	Towers (3-tall)
Participants	Milin, Stephanie, Michelle, Jeff (i.e., "Gang of Four")
Researchers	R2 (R1, R3, and classroom teacher present)
VMCAnalytic(s)	Milin's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 2 of 2 (Grade 4 & 5); Events 1–3 http://dx.doi.org/doi:10.7282/t3-dwqy-mg03

A small-group formative assessment interview facilitated by R2 occurred on March 10, 1992 with participants Jeff, Michelle, Milin, and Stephanie. In this session, the students were asked to convince the researcher and each other of their reasoning about the Tower Tasks.

Milin's version of an argument by induction in support of the doubling pattern (see Event 1). Milin first claimed that, from two 1-tall towers, four 2-tall towers could be generated. He was asked by R2 to explain, "Why from two you would get to four?" (L43-7):

> For each one of them you could add one– no two more– because there is a black, I mean a blue and a red [*colors available*]… For red [*on 1-tall tower R*]: You put a black on top and a red on top – I mean blue on top instead of black. And blue [*on 1-tall tower B*]: you put a blue on top and a red on top. You keep on doing that.

Notice that Milin self-corrected as he described the number of available color cubes to be added onto each shorter tower. Milin explained that, with two available colors, towers 2-tall could be made by placing a blue cube on top and a red cube on top of each shorter tower. This was the first of several instances in this session when he supported the doubling rule/pattern with an argument by induction.

Providing further support for why the pattern is a doubling pattern (see Event 2). The students seemed to agree, but it would become evident that they had not fully listened to or understood Milin's argument when the group discussed how eight 3-tall towers were generated from the four 2-tall towers. Again, R2 asked for an argument, but did not specify Milin to respond. Michelle claimed it was eight because of a rule she described as, "It would have to be times two times two equals four and four times two equals eight" (L60; L62). Jeff was not convinced and offered a different numerical pattern, namely counting by twos (i.e., two, four, six, eight, and so on), and so a different solution to the 3-tall Tower Task. Stephanie counterclaimed, "The pattern that we saw was this: for one block at a time we found two…four and then eight. Alright? [*speaking to Jeff*]. Two, four, and then eight?" (L73; L76).

R2 asked, "Why eight?" and asked Milin again to persuade Jeff (L80-8). Milin with the aid of the researcher, explained his reasoning:

80-4. Mil: For each one of those [*four towers 2-tall*]… you have to add one more color for each one… no, two more colors [*a red or blue*] for each one [*towers 2-tall*].
85. R2: So, this one with red on the bottom and blue on the top [*BR*]…
86. Mil: …You could put another blue or another red [*BBR or RBR*].
87. R2: You agree with that [*referring to other students*]? You can put a blue or red on top and that–
88. Mil: –So that will be two [*towers 2-tall*]. And then on this [*on RB*] you could put another red or blue on top that will be four [*accumulated towers 3-tall*].
[...]
90. Mil: See, now you see it? [*addresses Jeff*].
92. R2: And now here [*RR*] you could put…
93. Mil: …A red or blue [*finishes R2's sentence*]. And same thing with here [*on BB*].

Notice that the researcher played a role in clarifying what Milin states, either unclearly or inarticulately. They used Michelle's drawing of the four 2-tall towers as a reference point to discuss how the 3-tall towers were generated from the 2-tall towers. R2 asked clarifying questions when he used linguistic deictic, such as "those." Both R2 and Milin frequently pointed at the drawing to specify each tower. Milin explained that, for each 2-tall tower, a blue or a red can be placed on top of each 1-tall tower, while referring to the towers drawn on Michelle's paper.

However, the conversation sidetracked because Jeff was caught up with Michelle's process of drawing the 3-tall towers (which was not aligned with Milin's suggestions of putting each color on top of each tower) and because the representations used were Michelle's drawings it became the topic of discussion for a period of time.

Milin provided a complete version of an argument by induction (see Event 3). There was no observable evidence that Jeff, Michelle, and Stephanie understood in this instance when R2 asked for an explanation from the group. Perhaps in realizing this, R2 asked again how the eight 3-tall towers were generated. Again, it was Milin who offered a response using Michelle's representation in support of his claim. He also considered the conditions of the Tower Task as backing for his reasoning (L128-32):

> You have to keep on putting two for this [*on BR*] two for this [*on RB*] two for this [on *RR*] and two for this [on *BB*] and it will work out…Because there's only two colors you can't put any more on them.

In his argument, although he used vague linguistic deictic, he pointed to each 2-tall tower drawing. He also provided a warrant for his argument about the two-color condition of the Tower Task, which he implied limited no more than two towers from each shorter tower.

Jeff again agreed with this but was stuck on Michelle's unfinished drawing of the 3-tall towers, which again sidetracked the conversation. The researcher intervened, asking the students to reconsider Milin's idea after two times that the conversation was led away from his idea. She had asked the students to imagine the 2-tall towers selecting from two colors. Using Milin's and her own drawings of the towers, R2 guided the participants to attend to Milin's strategy with a demonstration of how to generate two 3-tall towers from a 2-tall tower RB:

> 189. R2: Now Milin is calling our attention to this first tower… and what is he asking us to do with it [*RB*]?
>
> 190. Mil: Put another blue and then make another thing exactly–
>
> 191. R2: Alright. Put another blue. Now, can you draw a picture of what that tower looks like? Now of three. This is a tower of three. He is putting another blue. …show us in the middle here what you just did with that one tower.
>
> […]
>
> 196. Mil: See so, I put the blue here the red on top of it [*draws a copy of RB*] and then I added one more that'd be red [*draws RRB*] but then I did like this blue then I put red back on top of it and then I put blue because there is only two colors [*see Figure 15.1.1*].

Milin responded to each of her questions and comments, even though in the beginning she was calling attention to the other students: "What is he asking us to do with it?" (L189). She accepted his response and, from then on, reiterated what he would say and asked him to draw for everyone to see. Figure 15.1.1 illustrates Milin's drawings while he explained how two 3-tall towers were generated from a 2-tall tower. On the left was his drawing accompanying his first explanation (L190-2) and on the right was accompanying his second explanation (L195-6).

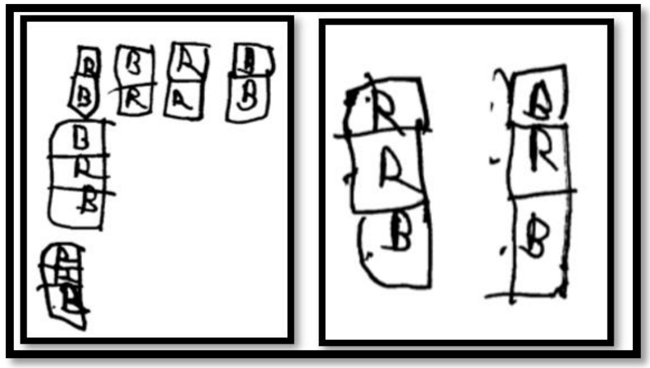

Figure 15.1.1. Milin's drawings of the two 3-tall towers generated from 2-tall tower RB.

Milin first drew a larger drawing of an exemplar tower RB with letter symbols for the colors red and blue, when the researcher asked him to in L191, and showed how it generated two 3-tall towers. She was providing commentary (e.g., "He is putting another blue;" "This is a tower of three,") as he was drawing the first tower BRB. Note Milin was completing the researcher's sentence to explain that he put a blue on top onto the first tower and commented as he drew the second tower with the red on top. R2 asked him to show this demonstration in the middle where he again repeated the process, adding support that "there is only two colors" (L196).

Milin's demonstration included many opportunities to share his idea to the group. First, he redrew the tower RB in a larger format on a cleaner part of the paper. Second, as he spoke about adding a blue cube on top of RB (see Figure 15.1.1 on the right), he drew the cube to show how and where it was placed. By providing side commentary, "Or you could put a red *there*" (emphasis added), showed that the choice of blue first was not special, but only part of the two-step process of putting both a blue or a red on each tower. He then drew a red cube on top while explaining, "And this one, you could put this way" and redrew a copy of the tower RB. He reiterated, "You could put a red instead of a blue."

He also engaged in the researcher's own representation of Milin's procedure by, for example, finishing her sentence:

197. R2: So, what you are telling me here if I could make my picture if I were doing what Milin asked me to do where we had a blue and a red what he is telling me to do is he is saying from this tower I am going to put blue on the top– [*see Figure 15.1.2*].
198. Mil: Or a red.
199. R2: Or from this tower I am going to put a red on the top.
200. Mil: Yeah.
201. R2: Is that what you are telling me to do? So, from this tower we get these two.

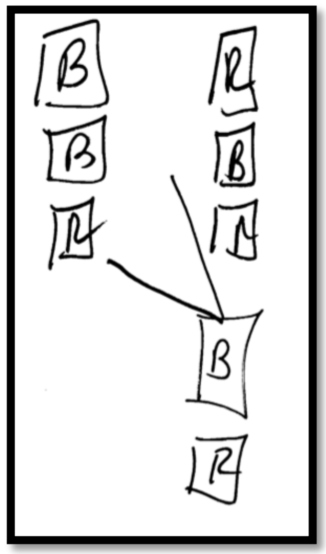

Figure 15.1.2. R2's drawing of Milin's explanation.

When she asked, "Is that what you are telling me to do?" Milin confirmed this was a correct procedure for generating the two 3-tall towers. Turning to the other students she asked if it made sense, when Jeff responded with "Yeah."

Immediately following the researcher's presentation, he generalized that the procedure would continue to each tower, that it would apply to 6-tall, and that it would result in 64 towers (L204). Milin stated: "And for each one [*shorter tower*] you keep on doing that [*inductive pattern*]. And for six [*6-tall*] you get sixty-four [*towers*]… It follows the pattern to five why can't it follow the pattern to six?" (L204-7).

Comparing Milin's inductive strategy to strategies used by others. This event includes the reactions of the students (L208-22). Note that R2 asked Jeff to "Write down these four [*2-tall towers*] and use his [*Milin's*] idea to see if you can build eight" (L153), which caused the other students to draw on their own papers also. When the researcher called everyone's attention again to Milin's procedure (L187) Stephanie and Jeff glanced a few times and continued to write on their own papers. For example, Jeff looked up above away from the table when Milin was demonstrating below on his paper and Stephanie leaned over Milin's demonstration, but her eyes were on her own paper. Both Jeff and Stephanie were attending to their own papers during this demonstration. Michelle began watching Milin, and very shortly after looked away from the table (during L196 when Milin demonstrated the example for a second time). When R2 demonstrated her own understanding of Milin's explanation (L197), Stephanie and Michelle watched while Jeff was writing on his own paper. Jeff was the only one to respond affirmatively when R2 asked if Milin's idea made sense, although still looking at his own paper.

Following Milin and R2's demonstrations, the researcher called Jeff's attention to his own paper where the tower BBR only had a blue cube on top. Jeff claimed he had that tower in his own collection, R2 pointed to his representation of tower BR as she explained to him that "from this one [*BR*], you could have put two things on top. You only put one" (L212). Everyone was looking over at this point and Jeff stated, "*Okay*, I understand…I am convinced" (Jeff's emphasis; L213-7). Michelle noted that she had already figured out the 3-tall Towers solution using a different strategy (by randomly drawing towers, their color opposites, and not duplicating any; see Chapter 16.1.4). When asked by the researcher, Stephanie began showing her solution using a case-based approach (see Chapter 7.1).

Milin presented (a second iteration of) his inductive argument to show formation of different tower combinations. Following previous arguments presented by the three other students in response to Jeff's question about the use of patterns, Jeff posed yet another question about how they were sure the patterns produced different tower combinations. Jeff asked: "How do you know there's different things in the pattern?" Milin responded by offering a justification for finding unique towers and draws a diagram illustrating his inductive reasoning. He showed a drawing of two towers 2-tall with a red and blue cube (BR and RB) and then drew a resulting tower, BRB, by adding B on top of the BR illustration. Michelle interrupted to elaborate on his reasoning: "Cause there's only two colors more, so you know you can't make more [*as she points to Milin's newly drawn tower BRB*]." Michelle provided a supporting warrant to justify that there are no more than two unique towers three cubes high generated from each of the two shorter towers. As Milin continued to illustrate generating two unique towers from the RB tower, he claimed, "You can't make any more from this one, so you go onto the next one" [*applying the inductive strategy to the next 2-tall tower*]. Jeff questioned him, "How do you know you can't make any more from that [*from RR*]?" Milin restated Michelle's earlier warrant, "Because there's not any more colors."

15.1.2 Dyad summative assessment (Session X)

Date	June 15, 1992
Grade	4
Task	Towers (3-tall)
Participants	Dyad Pair: Milin & Stephanie
Teacher	Barnes
VMCAnalytic(s)	Milin's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 2 of 2 (Grade 4 & 5); Events 4–5 http://dx.doi.org/doi:10.7282/t3-dwqy-mg03

On June 15, 1992 Stephanie and Milin were partnered to work on the 3-tall Tower Task as an assessment of their problem-solving strategies and supporting reasoning. Chapter 13.1.1. illustrated Milin's strategies to find the 3-tall towers. This chapter will focus on Milin's strategy for finding numerical solutions to various Tower Tasks and using it as a justification to the 3-tall Tower Task. Earlier, Milin told Stephanie he knew the answer was eight towers, 3-tall, and that they had seen and solved this problem before (L20-7). Stephanie, unsure, stated that she would make sure because she forgot the solution. While drawing their first few towers, Milin claimed, "It's gonna be an even number" and Stephanie replied, "It always is" (L39-40).

After they solved the problem by a method based on opposites and cases, Stephanie suggested, "Show the two times way, you know?" for further support and Milin agreed, "Oh yeah!" Milin and Stephanie wrote on their own sheets the numerical solutions for towers 1- through 10-tall (see Figure 15.1.3 for his written work), evidencing that they both were speaking of a doubling rule.

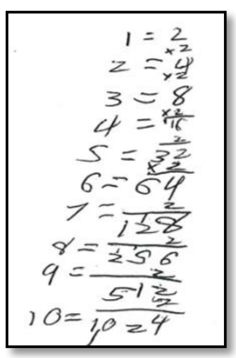

Figure 15.1.3. Milin's written work of a doubling pattern.

Milin wrote "1=2, 2=4, 3=8…"[*sic; 1-tall has 2 towers; 2-tall has 4 towers, etc.*] with "*x*2" between each line. Milin explained to Stephanie, "See, look there are only two colors. If there are three colors, you times by three…It depends how many colors. You have to times it by how many colors…If there is one color, you can only make two” (L118-26). Although his explanation was vague (he did not indicate what to multiply by the three or the two he referenced) and contained an error (one color available did not produce two towers from each shorter tower), he indicated a multiplicative relationship existed based on the number of colors available to find the total number for the higher height.

Stephanie suggested to Milin to modify his written work to be clearer for the hypothetical students who would read it. She stated, “If you didn’t have this information [*referring to her written work which included ‘1 tower = 1 x 2’ and a sentence before that explaining what it meant*]…and you didn’t have all those problems we have worked on before this, would you know what this means?” [*referring to Milin’s list of two columns of numbers with the equal sign between them*] (L144). Milin adjusted his written work similar to Stephanie’s that included the words “tower high” after a number and a multiplication sentence (see Figure 15.1.4).

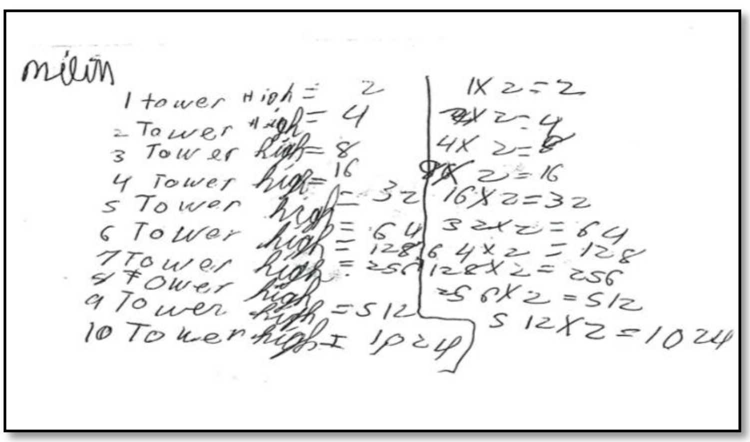

Figure 15.1.4. Milin’s adjusted written work of a doubling pattern.

15.2 Summary of Grade 4 interview and assessments

Recognition of patterns and use of strategies. In Session VIII Milin indicated a discrepancy between inductive and case strategies, whereas in Session IX an argument by induction became his main reasoning for the solutions to Tower Tasks. Recall in the previous session he indicated the doubling pattern would break down at 6-tall; however, in this interview he portrayed no doubt, as he did in the last interview, that this pattern would work for other heights. He evidenced certainty by the following claim: “It follows the pattern to five; why can’t it follow the pattern to six?”

During the assessment with Stephanie, he made an elevator pattern with a case of 3-tall towers and applied the doubling as a rule to identify the number solution for towers 1- through 10-tall. In this session they built their solution and then matched their physical tower solution to the numerical result of the doubling rule.

Forms of reasoning in support of the solution. At least six times during Session IX Milin presented an argument by induction for finding the number of towers 3-tall from towers 2-tall and towers 2-tall from towers 1-tall, when selecting from two colors. He demonstrated how from each tower of a given height two distinct towers of a taller, consecutive height were generate, when choosing from two colors. He generalized a recursive doubling rule of multiplying the previous height by two to get the solution for a taller height of towers. He supported his reasoning by using the condition of the task, “You have to add two more colors for each one” [*shorter towers*] and provided a warrant, “Because there’s only two colors, you can’t put any more [*than two cubes of different colors*] on them” [*shorter towers*].

Displays of justification. His demonstrations and explanations did not occur in isolation in Session IX. Milin tended to express his reasoning vaguely. R2 intervened to reiterate, show, or call attention to his idea over 20 times. In another instance, Michelle provided a backing to the claim that only two towers were generated from each shorter tower when Milin was convincing Jeff. In 26 instances, Milin was consistent with his argument for the number solutions to Tower Tasks and demonstrated his strategy through examples. His demonstrations were the supporting evidence for his claims of a particular number of tower combinations for a particular height. Through these demonstrations he generalized the procedure up to 6-tall. He generalized the argument by using the word "each" to indicate both colors and each tower in any set (e.g., L196; L204) and he repeated the action of adding one color and then the other (e.g., L192; L196; L198).

In Session XI, a justification for the doubling rule was not present. However, from his past experiences, the number sentences consisted of the product between the number of the previous total towers and the number two, that according to Milin was related to the number of available colors. Although Stephanie led Milin in writing, it was common for Milin to provide little insight into his reasoning unless probed. Perhaps because there was no probe for the validity of the rule, the children had not known that it would be necessary for the completeness of their argument. However, based on his historical experience with providing a complete argument through researchers' questions, one might infer that his written argument was an abbreviation of his reasoning. This inference was verified in the fifth-grade session when Milin provided a complete argument for why the doubling pattern worked to solve Tower Tasks.

15.3 Grade 5

15.3.1 Individual written summative assessment (Session XI)

Date	October 25, 1992
Grade	5
Task	Towers (3-tall)
Participants	Milin
Researchers	R1, R2
VMCAnalytic(s)	n/a

Fifth grader Milin took an individual assessment on October 25, 1992 before he saw any Tower Tasks that school year. The task was to send a letter to an absent student describing the different towers 3-tall and explaining why they were all the towers and that none were left out. Milin represented each tower individually with letters R and W indicating red and white, respectively. Figure 15.3.1 shows two pages of his written response.

He wrote the following text:

> 8
> To a person,
> I know how many their [*sic*] are because 1=2 2=4 3=8 4=16 5=32 [*sic*] and on the bottom is all of the ways. [*list of "2, 4, 8, 16, 32 and drawings of 3-tall towers RRR, WWW, RWR, RRW, WWR, WRW, WRR, RWW*].
> [*Next page begins with drawings of 1-tall towers R and W and vertical multiplication sentences of 2x2=4x2=8*]. You can only make two 1 tower towers [*drawings of 2-tall towers RW, WR, WW, RR*]. You can only make four 2 towers high. [*illegible: "You"*] could mix them up and 2 of the [*illegible: "same"*] soon you will see a patern [*sic*]. Keep on multiply 2.

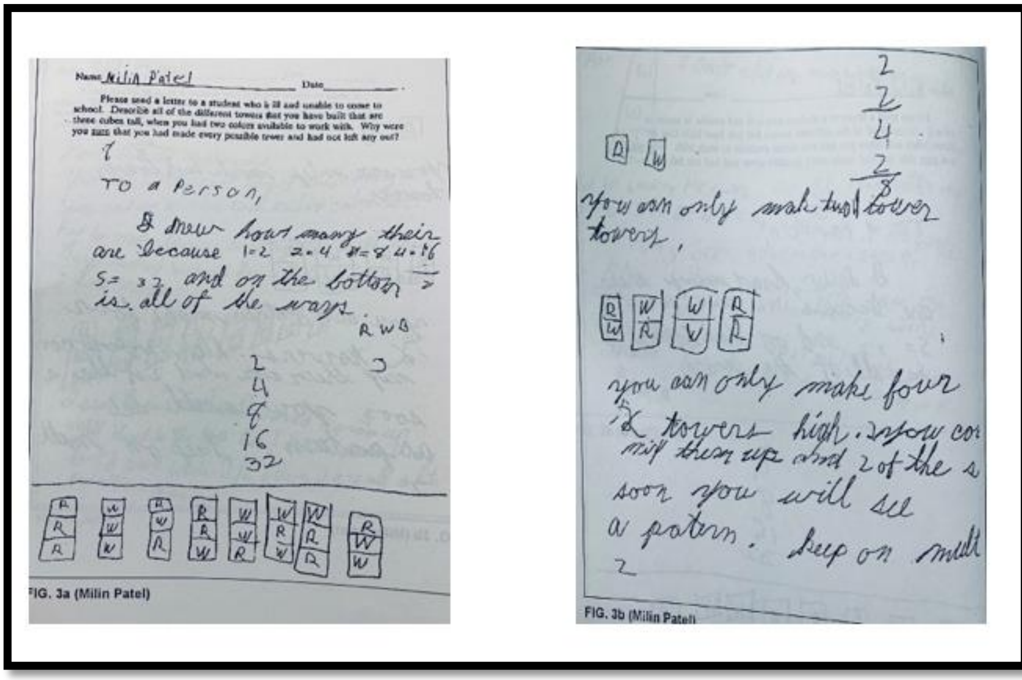

Figure 15.3.1. Milin's written solution to the 3-tall Tower summative assessment.

First, he wrote "8" under the problem statement indicating the solution to the 3-tall towers with a supporting picture of the different towers below. The organization of the towers can only be inferred. Note that six of the eight towers were organized by opposites or inverses (RWR and WRW were not). He then supported his reasoning by alluding to the following pattern: "1=2, 2=4, 3=8, 4=16, 5=32" [*sic: The number of 1-tall towers = 2 different towers, etc.*]. He listed the number solutions to tower heights in a column "2, 4, 8, 16, 32." Next to it he created a new column that included a list of three colors, "R W B," and the number "3," which he seemed to abandon. On the second page he wrote two multiplication number sentences "2x2=4x2=8." He drew and stated that only two 1-tall towers and only four 2-tall towers existed. Note that the 2-tall towers were organized by opposites. He generalized that there was a pattern that involved multiplying by two. Based on the resemblance of his previous arguments he was referring to a doubling rule for the total number of towers of consecutive heights.

15.3.2 Dyad: Milin & Michelle (Session XII)

Date	February 26, 1993
Grade	5
Task	Guess My Tower (GMT; 4-tall and 3-tall)
Participants	Dyad pairs: Michelle & Milin; Stephanie & Matt; fifth-grade class
Researchers	R2, R1, and R3
VMCAnalytic(s)	Milin's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Part 2 of 2 (Grade 4 & 5); Events 6–7 http://dx.doi.org/doi:10.7282/t3-dwqy-mg03

Milin justified the doubling rule using inductive reasoning to Michelle (see Event 6). Fifth graders Michelle and Milin were working together on the Guess My Tower Task – an extension of the Tower Tasks – on February 26, 1993. R2 asked them about their conjectures for the 4-tall Tower Task. Michelle said she did not know yet and Milin conjectured 16. He also described a rule he was using to think about it:

> Well it works like this, before, I don't know, I can't remember that. Before we found out that for blocks of two [*tall*] you have to multiply two from the first tower that's two times two. So the second one is four. Then two times four is eight, times one more would be times two would be sixteen.

Milin explained his reasoning behind the conjecture by explaining how he began with two 1-tall towers and multiplied by two to obtain a result of four 2-tall towers. He also explained that four multiplied by two gave eight 3-tall towers and, then, eight multiplied by two gave 16 4-tall towers. R2 focused on Michelle's comprehension, who was unsure after questioning, and asked Milin to convince her that this pattern worked for each height. He chose to build with Unifix cubes to explain his idea to Michelle.

His explanation began with R2 initiating the question of how many towers there were when they were 1-tall. Using the Unifix cubes, Milin showed two cubes, one yellow and one red (L489-92). Then he was asked about 2-tall (L493). He added a red cube on top of the red tower and a yellow cube on top of the yellow tower. He also placed a yellow cube on top of a red cube and a red cube on top of a yellow cube. When R2 asked him to explain his actions starting from the yellow and the red cubes (L497), Milin placed the towers with the yellow bottom cube in front of the yellow 1-tall tower and the towers with the red bottom cube in front of the red 1-tall tower. Milin again described the action of placing a yellow and a red cube on each shorter tower (L500-2), while R2 repeated his actions to give Michelle the opportunity to observe it. In this event there was no explanation yet for why this strategy was exhaustive and did not produce duplicate towers.

Milin generated towers 3-tall from towers 2-tall and supported it with inductive reasoning (see Event 7). Michelle indicated that she did not know why this strategy worked (L521). Milin provided a detailed explanation to Michelle:

539. Mil: Take these two off [*removes red and yellow cubes from the RYR and YYR, respectively*], okay? There is only two colors, in all. These two [*refers and points to YR and YR resulting from 3-tall towers with their top cubes off*] are the same thing as this [*points to original YR tower*], right? And if you put this to make it to a three, you put this on this [*adds a yellow cube on one of YR towers*]. And to make this one a three you put a red on this [*adds a red cube on the other YR tower*] and that wouldn't be a duplicate. [*Puts them together to show they are different 3-tall tower*].
540. R2: So, you are telling me when you have this one [*pointing to YR*] to make it one higher you can make it one higher [*pauses*]...
541. Mil: By putting a red and a yellow.
542. R2: By putting a red on it or by putting a yellow on it? [*Points to RYR and YYR*].
543.Mil: Yes.
544. R2: Does that make sense?
545. Mic: Yes.

To explain his reasoning to Michelle, Milin took the top cubes off from two examples of 3-tall towers, RYR and YYR, and compared them to the 2-tall tower, YR, that they were generated from. He demonstrated removing the top cubes to show the matching bottom cubes. He then stated the given, general condition of the Tower Task that there are two and only two colors to select from. Then he lined up the two duplicates of the original YR tower, stated that they were both the same as the original, and asked her, “Right?” Demonstrating placing a yellow cube and a red cube on top of each duplicate tower and explaining aloud his actions, Milin showed how the two different 3-tall towers are generated from the single 2-tall tower. He emphasized the differences between the two new towers to Michelle by placing them together and stating, “and that wouldn’t be a duplicate” (L539). R2 reviewed what he did (L540) and asked Michelle if it made sense. Michelle confirmed her understanding by repeating the process as she pointed to the towers.

15.4 Summary of Grade 5 assessment and problem-solving sessions

Forms of reasoning in support of the solution. In Session XI, Milin applied a doubling rule to support his reasoning for eight 3-tall towers. He justified the existence of a doubling pattern by drawing shorter 1- and 2-tall towers. His reasoning was incomplete because he did not explain how or why the number of taller towers doubled. Also, an inductive strategy of placing each color on top of each shorter tower would have a different organization than the one that was observed by Milin’s written work because the bottom colors did not match. However, perhaps he was building the towers from the shorter towers internally. Note that in the past he would build the 2-tall towers to make the single-colored towers first and other opposite towers by placing one color on top and then placing the remaining color to find the remaining taller towers. R1 in the past aided him in organizing them to show the inductive growth. Perhaps, he returned to the same constructions in the assessment of Session XI.

In Session XII, Milin, working with Michelle, recalled a doubling pattern and was asked to share with Michelle, who indicated uncertainty of the number solutions that Milin predicted very quickly. Facilitated by R2, Milin demonstrated how 2- and 3-tall towers were generated from shorter towers using an inductive procedure of placing each color on top of the shorter towers. He provided a warrant that because there were only two colors, only two towers could be created from each shorter tower. He further demonstrated by removing the top cubes from the 3-tall towers to show their bottom color cube patterns matched and the two different colors would ensure differences in the new taller tower patterns.

Milin's argument using inductive reasoning included: 1) Reasoning about the *conditions* of the task and the *relationship* between the bottom color cube patterns of the two taller towers and the shorter tower pattern from which they were generated; 2) A *demonstration* of each color cube placed on top of each shorter towers; 3) A *verification* that the new towers were not duplicates of each other due to the difference in the top position, and; 4) At three instances, checked with Michelle (e.g., "Okay?" [*indicating Michelle and pausing*]) while making the physical models visible for Michelle.

Chapter 16 Results: Michelle's problem solving and reasoning about the Tower Tasks in various settings (Grade 4)

16.1 Grade 4

16.1.1 Dyad: Michelle & Jeff (Session IV)

Date	February 6, 1992
Grade	4
Task	Towers (5-tall)
Participants	Dyad pair: Jeff & Michelle; fourth-grade class
Researchers	R2, R3, R1
VMCAnalytic	Michelle's Longitudinal Problem Solving and Development of Reasoning About Tower Tasks: Part 1 of 3 (Grade 4); Events 1–2 https://doi.org/doi:10.7282/t3-c5ja-qn71

Michelle working with Jeff on 5-tall towers (see Event 1). During the whole class problem-solving session on the 5-tall Tower Task in the fourth-grade, Michelle, with partner Jeff built towers (e.g., YYYYR and YRRRR) using a Guess and Check strategy. Michelle suggested that they check the existing set of towers and remove towers that did not have opposites yet in order to build and group by color opposite pairs (e.g., RRYYY & YYRRR). Jeff predicted 26 towers as they were rapidly generating towers. Jeff then built a yellow elevator pattern by organizing towers with exactly one yellow cube on the second, third, and fourth floors. Jeff and Michelle worked together using different strategies. For example, Jeff responded to Michelle's color opposite strategy by building an opposite tower, while still maintaining the strategy to complete the yellow *elevator* pattern. In another instance, Jeff asked Michelle to look at his pattern. She noticed that the tower with exactly one yellow cube on the bottom floor was missing from his set because it was removed and paired with its opposite tower. She suggested reorganizing the tower into his pattern. Jeff used the opposite strategy to build the opposite red elevator pattern. Michelle joined him by looking for existing towers that fit his pattern and by creating new towers when she could not find any within their set. Jeff organized the elevator pattern symmetrically starting with the yellow descending elevator to the red ascending elevator (see Figure 16.1.1). He called his organization a "design."

Figure 16.1.1. Jeff's symmetrical organization.

R3 asked how they generated their towers (see Event 2). When R3 asked, "How are you doing this?" (L175) Michelle said they were making opposites. Jeff explained that they made new towers and checked them against the existing set. In response to R3's question, "Will each tower have an opposite?" Jeff claimed, "It has to" and Michelle justified why: "Because if you did one way then you can do it a different way with another block. I think you can do it with a different color" (L184). After this dialogue, Michelle tried to dismantle Jeff's elevator pattern to match the opposites, but Jeff objected.

In a later instance, they obtained 30 towers and R2 challenged them, "I think there's more." Michelle used Guess and Check to find a new tower. When she found RYYRY she immediately built what she called a "duplicate" to complete the pair, another indicator of her use of the opposite strategy. They continued to search for towers. During this time Jeff developed a red (see on the left of Figure 16.1.2) and a yellow *staircase* pattern (see on the right of Figure 16.1.2), which created several duplicates.

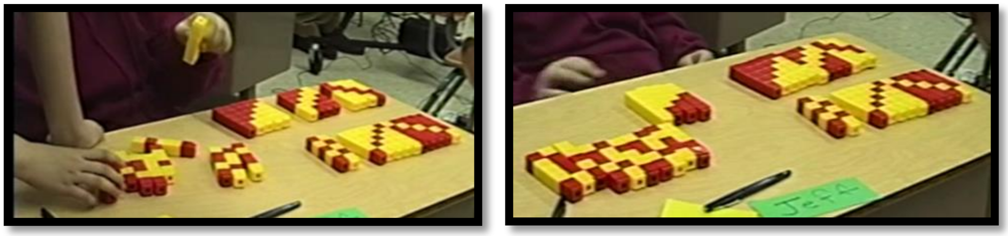

Figure 16.1.2. Jeff's staircase and elevator patterns.

During the sharing session and after prompting by the researcher to recheck their collections, Jeff and Michelle found 32 towers, eliminating the duplicates they had (see Figure 16.1.3). The towers were organized by color opposite pairs, as well as by opposite elevator patterns. Notice the staircase patterns and exactly two of a color elevator patterns were dismantled.

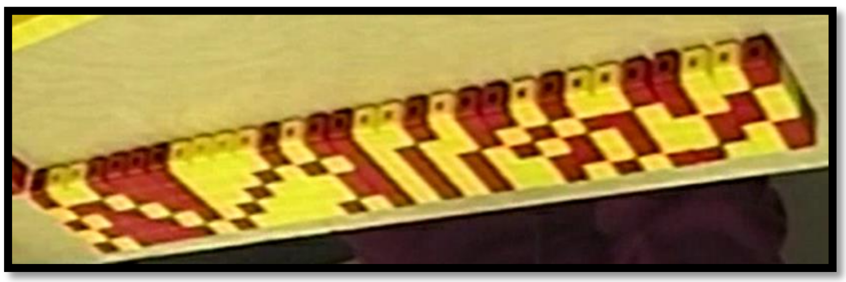

Figure 16.1.3. Jeff and Michelle's final 5-tall tower solution.

16.1.2 **First interview (Session VI)**

Date	February 7, 1992
Grade	4
Task	Towers (5-tall) & Outfits (2_n structure; n = 2)
Participants	Michelle
Researchers	R2, R1
VMCAnalytic	Michelle's Longitudinal Problem Solving and Development of Reasoning About Tower Tasks: Part 1 of 3 (Grade 4); Events 3–4 https://doi.org/doi:10.7282/t3-c5ja-qn71

Michelle's elevator and color opposite strategies for 5-tall Towers (see Event 3). The first post-interview with Michelle conducted by R2 (with R1 present) occurred on February 7, 1992, the day after Jeff and Michelle worked on the 5-tall Tower Task in class. Michelle reported that their solution was 32 towers. She was asked to convince the researchers that there were no more or fewer. Michelle began by building single-colored towers and two towers, RRRYR, RRRRY. She described a pattern: “looks like steps going up” (L20). R2 asked her to name the pattern in a way so that it would be descriptive enough for a hypothetical student who was on the phone. Michelle described the organization of an ascending yellow elevator pattern as follows: “You keep doing it until yellow gets to the top here” [*points to an imaginary yellow cube moving up each position in the built tower RRRRY*] (L24). Michelle claimed there were five towers in such a pattern (L38). When R2 asked her how she was sure, Michelle justified her reasoning by building the complete set (see Figure 16.1.4) and then stated, “Because it doesn’t go higher up. You would need more Unifix to make it six. [S*he made a tower 6-tall with one yellow*.]…Because it’s towers of five, not six” (L44).

Figure 16.1.4. Michelle’s yellow elevator pattern of 5-tall towers.

Michelle was then asked to record her findings. She wrote: “1 yellow goes diagonally” and a “5” in response to the name and total for this set. Without prompting, she described the opposite set: “And for the red ones it looks like steps” (L46). She wrote “5 1 red goes diagonally.” R2 acknowledged that she was convinced of those and asked if Michelle used “another pattern” (L75). Michelle built the tower of alternating colors and its color opposite pair. She was asked again to describe it, to which she responded, “Different colors” or “It is skipping colors” (L80-3) and wrote “2 [*towers with*] skips.”

Staircase patterns (see Event 3). In this event, Michelle built a set of towers that she described as “stairs” (see Figure 16.1.5) and was asked to describe it:

111. M: We also made this kind of pattern [*referring to a descending yellow staircase pattern of towers with five, four, three, two, one, and no yellow*]. I made the opposite [*referring to a red staircase pattern*].
112. R2: Describe this pattern.
113. M: Sort of like stairs and stairs upside down.
[…]
116. R2: What did you do if something like this happened? [*shows her the duplicate towers YRRRR of both the staircase and the elevator patterns*]
115. M: We would take it off.
[...]
131. M: I could say the first block is all red [*RRRRR*], and the second one has one yellow [*YRRRR*].
132. R2: How many yellow here [*referring to the all red tower*]*?*
133. M: None [*no yellow*].
134. R2: Oh, no yellow here [*pointing the RRRRR tower*] all red. I get that. [*skips one yellow*] Mhm. The third one I could say there is two yellows, then three yellows, then four yellows, then all yellows.

Michelle utilized staircase and elevator patterns that were discussed in the class session and that Jeff created in the previous day. She reported that she and Jeff removed duplicates from the patterns (e.g. towers with one red or one yellow in the staircase were duplicates of towers in the red or yellow elevator patterns). The discussion about the Tower Task ended with a challenge for Michelle to find a convincing way to show the total number of 5-tall towers.

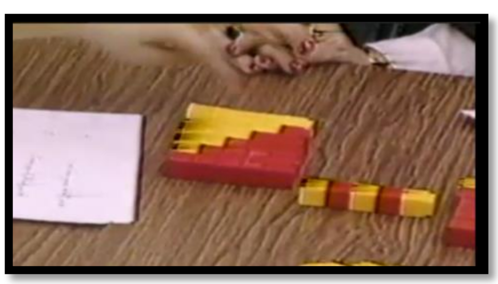

Figure 16.1.5. Michelle's yellow descending staircase pattern.

Making a connection to the Shirts and Pants Task (see Event 4). Following the discussion about towers, R1 asked Michelle if she recalled any previous tasks. Michelle responded that she recalled the Shirts and Pants Task of third grade. In response to what the task was about, Michelle spontaneously used the cubes to represent "outfits." She said, "Like, say if there was one yellow short [*selects a yellow cube*] and then one red top [*selects a red cube*] you will have to see how many pairs there was [*makes tower RY*] to make an outfit" (L154). R2 asked if the tower she made was an outfit and Michelle confirmed. R2 requested her to create a Shirts and Pants situation using the cubes. She did so by building four different towers, 2-tall, that consisted of red and yellow cubes, as seen in Figure 16.1.6.

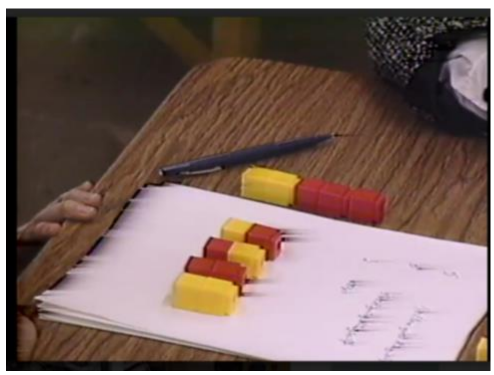

Figure 16.1.6. Michelle's "outfit" representation with 2-tall towers.

When asked if she could create a related Tower Task, she explained about a 5-tall Tower Task: "Like say there was like yellow for the first two floors [YY] and then two reds for the next two floors [RR] you have to see what kind of a pattern you could do" [*connects to make RRYY and then another R to make RRRYY*] (L166). R2 asked, "If you could make them any size you want, is possible to make a Tower Task that is like the Shirt and Pants problem you just made me?" She placed a yellow onto RY to make YRY. The researcher redirected her back to the 2-tall tower. Michelle explained how she thought about the outfits with respect to the towers as she rebuilt them:

174. M:	You could go like this: [*Builds RY*] Have one like this; [*Builds YR*] One like this; [*Builds RR*] You could just go two reds like this; and then two yellows [*builds YY*].
175. R2:	Okay. So, how will that be like Shirts and Pants that you just told me about?
176. M:	Well, you have to match up, let say it was two red bottoms [*pointing to the red at the bottom of YR and RR*] for the towers and two bottoms yellow [*pointing to the yellow at the bottom of RY and YY*] for the towers, and then there is two red tops for the towers and two yellow tops for the towers and you have to match them all.

Michelle called the bottom cubes in the towers as “bottoms” and the top cubes as “tops,” referring to pants and shirts, respectively. In the following excerpt Michelle responded affirmatively to R2’s questions about the relationship between the towers and the outfits:

177. R2:	Aha, that’s very interesting, very, very interesting. So, you’re telling me a tower then becomes like an outfit?
178. M:	Yeah.
179. R2:	And, you are telling the shirts and pants [*pointing at RY*] get to be like the floors of the tower?
180. M:	Yeah.

The interview ended with a request that Michelle record the new task.

16.1.3 **Second interview (Session VII)**

Date	February 21, 1992
Grade	4
Task	Towers (1- to 5-tall) and Outfits (2^n structure; n = 2, 3, 4, 5)
Participants	Michelle
Researchers	R1
VMCAnalytic	Michelle’s Longitudinal Problem Solving and Development of Reasoning About Tower Tasks: Part 1 of 3 (Grade 4); Events 5–8 https://doi.org/doi:10.7282/t3-c5ja-qn71

The second interview between R2 and Michelle occurred on February 21, 1992.

Reporting the relationship of 2-tall tower outcomes, selecting from two colors, to outfit outcomes with two articles of clothing (see Event 5). R2 asked Michelle about how the 2-tall towers were related to outfits. Michelle built the towers quickly using the color opposite strategy. She then drew *two* white shirts, *two* black shirts, *two* black pants, and *two* white pants. R2 posed a situation to her: If she went shopping, would her mother buy her two pairs of the same color pants and Michelle responded, “She would probably let me pick one” (L732). Michelle then crossed out the extra articles of clothing and drew four lines: one from a black shirt to black pants, a second from the same black shirt to white pants, a third from a white shirt to black pants, and a fourth from the same white shirt to white pants (see Figure 16.1.7).

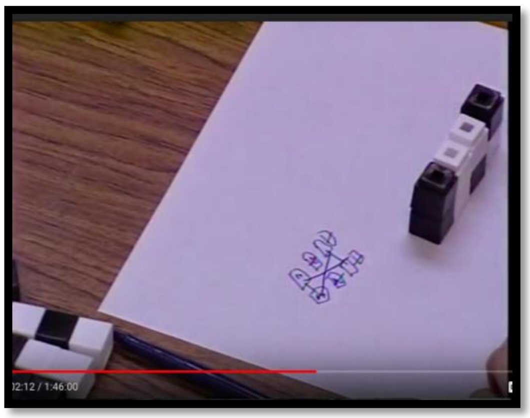

Figure 16.1.7. Michelle’s diagram of outfits.

She concluded that four outfits could be made. Note that she created a diagram that represented the matching of one type of article of clothing of each color with another type of each color, resulting in four outfits. Then when asked how the four 2-tall black and white towers related to this problem, she specified, “These would be the pants [*pointing to the bottom cube*] and these would be the shirts” [*pointing to the top cube*] (L738). To show further how her diagram related to the physical towers, she removed two bottom cubes, a black and a white, and separated them. She explained how two outfits were generated by pointing to imaginary black and white cubes being placed over each bottom cube to get two from each.

Relating 3-tall towers to outfits with three articles of clothing, selecting from two colors (*see Event 6*). R2 presented Michelle with an extension of the Outfit Task with two additional hats: "Suppose I said to you, when you bought your outfits [*black or white shirts and pants*], you also were able to buy two [*black and white*] hats…how many different combinations can you make?" (L35). Michelle responded by drawing a black and a white hat representation above her original diagram, along with two lines extending from each hat to the first two outfits, displaying four more combinations (L36):

> You could have one black hat and then one white hat [*drawings two hats above her outfit drawings*] and you would make more combinations. [*drawing 4 lines, 2 from each hat to each outfit representation*]. It would be eight [*outfits*].

When R2 asks her to explain further, she traces over the lines from each hat to each of the shirt and pants outfits (L38):

> Well, if you had four like this [*referring to the original four shirts and pants drawings*], you could do different ways like this [*traces over the lines*] and it would just go like that [*referring to new outfit combinations*] and you would have different hats for different outfits.

Note that she drew an extra line from the white hat to another outfit, illlustrating a total of five combinations; however, she counted "eight" outfits. R2 asked her to show her reasoning with towers. Michelle created four 3-tall towers with a white cube on the top of each original 2-tall tower (see Figure 16.1.8).

Figure 16.1.8. Michelle's 3-tall towers with a top white cube.

Then she built duplicate 2-tall towers with a black cube on top of each to make 3-tall towers. R2 then asked Michelle what outfit was represented by tower BBB. Michelle responded, “That’s the one with the black hat” (L46). Michelle continued by building BWW and noted it was the opposite of WBB. R2 asked, “Is this outfit [*referring to BBB*] the same as this outfit [*referring to WBB*] with a different hat?” (L55) and Michelle confirmed. R2 also placed WWW next to BWW and asked her if the towers were the same outfits with different hats. Michelle confirmed. Then Michelle built outfit, BBW, and in response to R2’s query about an associated outfit, Michelle showed WBW. R2 confrimed, “Same outfits but different hats– black hat outfit, right?” (L59). Michelle continued, building the last outfit representation of outfit BWB as a tower and placing it next to WWB tower (see Figure 16.1.9).

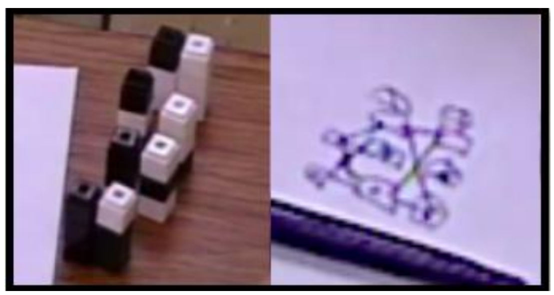

Figure 16.1.9. Michelle’s outfit diagram and outfit representation with towers.

Relating 4-tall towers to outfits with four types of articles of clothing selecting from two colors (see Event 7). R2 asked Michelle, “Now, suppose I let you have…two different feathers for your hats. Before you do it, what do you think? How many outfits would you have to have?” (L59). As Michelle pointed to the eight 3-tall towers, she answered, “I think there would be twelve [*outfits*]… Because there will be four more” [*four more outfits than represented in the eight 3-tall towers*] (L62-4). In response to R2’s request for a justification, she explained, “I would think twelve or sixteen because you could have different feathers for all these [*points to the row of towers with only white* *cubes on top*] and then different feathers for all these" [*points to the row of all black cubes on top*] (L66). The researcher posed an extension to the task: "What if I now had um, sunglasses, two different pairs of sunglasses for your outfits" (L69). Michelle conjectured 24 outfits, stating, "Because if you are adding– Well, you could have like sixteen more [*referring to the imaginary 16 towers 4-tall as she points to the 3-tall towers*] because you could have all different ones [*sunglasses*] for different outfits" (L72). R2 suggested that she write the calculuation. Michelle modified her answer after writing "16+16=32" in vertical addition format. R2 asked, "If you are making them like towers– feathers, right? What would your towers look like?" (L75). Michelle stated, "You could put the black feathers on top [*points to a 3-tall tower*] for another block [*referring to an extra cube on top of the 3-tall tower*]– and then the white ones [*feathers/cubes*] on top for another block" (L80).

Relating the Outfit Tasks to the Tower Tasks (see Event 8). R2, returning to earlier question in the interview, asked Michelle: "So, let me ask you again: If you are building towers of four, right? With exactly two colors, how many different towers of four do you think there would be?" (L81). After conjecturing 32 towers, in response to the request to justify the conjecture, Michelle justified, "Because if you added on like we did here [*points to the towers 3-tall*]– like we explained here [*referring to the 3-tall towers that represent the outfits with three articles of clothing*] then you would have like thirty-two– like that [*points to her calculations for the outfits with four articles of clothing*]" (L86). Pointing to the calculation, R2 asked for confirmation, "Is that the towers of four?" Michelle replied, "Because you have a feather on top [*refers to another cube on top of the 3-tall tower representations of the outfits with a hat, a shirt, and pants*], then you would have four on top" [*4-tall*] (L88). R2 asked Michelle to clear up the confusion by reviewing the sums she obtained from adding on two colors of hats, two colors of feathers, and then two colors of sunglasses. Michelle explained through gestures, such as pointing to the 3-tall towers, and descriptions of the article of clothing under question with its associated tower height. For example, Michelle explained: "If you had the sunglasses, it would be five up, so you have five of them" [*points to the 3-tall towers and refers to their growth, by pointing up toward the space above the 3-tall towers*] (L100). Note that Michelle used the 3-tall towers, which represent outfits with three articles of clothing, to make a connection to towers higher than 3-tall. When R2 asked her about 6-tall towers, Michelle replied with the solution of 64.

R2 asked, "Did you think these outfits and towers have anything to do with each other?" (L105). Michelle responded, "Well, sort of, because you have to make matches and for the towers you have to make different towers. So, like you have to make different outfits and different towers" (L106). R2 made a reference to her earlier claims: "You told me that if you are building towers of three, they'd be fewer than towers of four than towers of five. Do you think, now you could tell me how many towers of three they are gonna be with these two colors for sure?" (L109). Michelle responded, "Eight." R2 also asked about 4-, 5-, and 6-tall towers. Michelle accordingly responded with, "sixteen [*4-tall*],…thirty-two [*5-tall*],… sixty-four" [*6-tall*] (L111-116). R2 asked if she saw "a pattern in building towers." She responded, "Yeah…say, there's eight, you have to add eight more because you have the different feathers and then for the belts you have to have sixteen more" (L118).

Furthermore, in response to R1's question, whether it would be a different problem if instead of putting hats on she put shoes on, Michelle answered, "If you took off the hats and put the shoes on, I think it would be the same" (L122). The interview ended after R1 revoiced her last statement.

16.1.4 Small-group formative assessment interview (Session IX)

Date	March 10, 1992
Grade	4
Task	Towers (3-tall)
Participants	Jeff, Michelle, Milin, Stephanie (i.e., "Gang of Four")
Researchers	R2 (R1, R3, and classroom teacher present)
VMCAnalytic	Michelle's Longitudinal Problem Solving and Development of Reasoning About Tower Tasks: Part 2 of 3 (Grade 4); Events 1–4 https://doi.org/doi:10.7282/t3-tp02-pw54

Michelle's reasoning is presented in three parts: her individual reasoning about the Tower Task; how she took up Milin's inductive argument; and how she took up Stephanie's case argument.

Michelle's first prediction of the pattern for the solutions to Tower Tasks (see Event 1). In a small-group assessment interview that occurred on March 10, 1992, with Stephanie, Milin, Jeff, and facilitated by R2, Michelle made two claims prior to hearing the other children's arguments. First, Michelle recalled a pattern for the total number of towers 1-, 2-, 3-, 4-, and 5-tall. She explained:

> If you had only one block up [*1-tall tower*] and two colors, then you would have two towers, and we figured out that the other day that you keep on doing, like two times two would be four… There would be four towers for two high…And then for three high, you would have eight towers. And then for four high you would have twelve towers. And then you keep on doing it like that...and for five [*tall*] towers it would be twenty-five.

Michelle also noted a multiplicative relationship ("times two") building towers from 1- to 2-tall. She followed a doubling pattern for the number solution for towers 3-tall, but not for the number solution for 4- and 5-tall. In another instance, Michelle applied a doubling rule: "And then we went to four [*2-tall towers*] so it would have to be times: two times two equals four, and four times two would equals eight" (L62).

Relevance for the use of patterns with an example of the inductive strategy (see Event 2). After the initial responses from Milin and Stephanie, Jeff requested a justification for their use of patterns. Jeff posed the question, "Do you have to make a pattern?" to which Michelle responded, "No" (L306-367). Note that the children were using the word "pattern" to describe their number solutions to the Tower Tasks. Jeff asked for clarification, "Well then, why is everyone going by a pattern?" His question prompted discussion about the usefulness of patterns. The students offered several responses (e.g., L311, L317, L334):

307. Mil:	Because we like to.
308. S:	Yeah, it's easier.
309. Mic:	It's easier. [*referring to use of patterns*].
	[…]
311. Mic:	'Cause if you, 'cause if you, 'cause if you just keep on guessing like that, you're not sure if there's going to be more.
	[…]
316. S:	Because it's easier than just going "ooh-ooh" [*gestures as if selecting a tower from the air*]. There's a pattern.
317. Mic:	'Cause you might have a duplicate. And, and then you may not know.
318. S:	It's harder to check.

Notice Michelle agreed with Stephanie that a "pattern" was "easier." Milin offered, "Because we like to." However, Michelle further pointed out that "if you just keep on guessing like that, you're not sure if there's going to be more" (L311) and that "you might have a duplicate. And, and then you may not know" (L317). Stephanie and Milin were supported by Michelle, who argued that patterns also helped to control for duplicates. Recall, Michelle, working with Jeff as her partner on the 5-tall Tower Task, was assigned the task of finding duplicates in tower patterns created by Jeff (e.g., elevator and staircase).

Then Jeff posed another question, “How do you know there’s different things in the pattern?” (L320). The question raised discussion among the group. Milin and Stephanie reiterated earlier justifications and Michelle participated in the discussion. In this event, Milin showed how to account for each tower. Milin showed a drawing of two towers, 2-tall, with a red and blue cube (BR and RB) and then drew a resulting taller tower, BRB, by drawing “B” on top of the “BR” illustration (L323; L325; refer to Chapter 15.1.2). Michelle interrupted to elaborate on his action: “Cause there’s only two colors more, so you know you can’t make more [*pointing to Milin’s newly drawn tower BRB*]" (L324). Michelle provided a warrant to justify that there were no more than two unique 3-tall towers generated from each of the two shorter towers. This statement further supported Michelle’s earlier claim that without a “pattern” one may not be sure there were more missing towers.

Relevance for the use of patterns with an example of the organization by cases (see Event 3). In the previous event Milin, with Michelle’s support, provided an example of his earlier inductive strategy to convince Jeff of the relevance for the use of patterns. In this event Michelle engaged with Stephanie’s case argument to account for all 3-tall towers, as Stephanie finished her argument to convince Jeff about one of her categories, “one blue”. Michelle referred to the organization by an elevator pattern as a way to present a convincing argument: “But if you didn’t have that pattern, it would be harder to convince you [*Jeff*].” She acknowledged that a “pattern” had use for convincing others of the completeness of a case or solution (L331). As Stephanie continued her argument by cases, Michelle suggested, “Then you have to go to two blue” (L351). Stephanie agreed and presented the case of towers with two adjacent blues. Jeff challenged Stephanie that another unique tower could be made with two blue cubes. Stephanie offered a counter argument, emphasizing the conditions of her category that the two blue cubes must be adjacent to each other, and Michelle supported it:

335. S: No, this is together [*repeatedly taps on the drawing of her category of two blues adjacent to each other*]. Milin [*points to Milin*] gave me that same argument.
336. Mic: She means stuck together.

Jeff agreed those two towers were the only towers possible with the conditions of the case. Then Michelle and Stephanie jointly reviewed the remaining three cases as seen in the excerpt below:

363. Mic: Then you have to move to three, [*points to BBB*] which you can only make one.
364. S: Yeah, you can only make one…
365. Mic: Then you can make two split apart [*indicates index and fifth finger on the separated blues in BRB]*.
366. S: Two split apart [*repeats Michelle’s finger representation to show separation*], which you can only make one of.

Stephanie, with Michelle, reiterated how many towers could be found in the cases of “three blue” and “two [*blues*] split apart.” Notice that Michelle was the first to name which category came next after “one blue,” after “two blue stuck together” and after “three blue” (see L349-53 and L363-5). She also supported Stephanie’s argument for the case argument to include a category of “stuck together” (L355-59).

Michelle presented a solution for the 4-tall Tower Task using an argument by induction (see Event 4). R2 challenged the students to find the solution for 4-tall towers. In unison, Michelle and Milin immediately answer 16. The researcher asked Jeff if he agreed. Michelle responded by using Stephanie’s written solution, organized by cases, and extended Milin's inductive reasoning to show from each 3-tall tower, two unique 4-tall towers can be generated. Using Stephanie's drawing that was created during her demonstration of her case-based argument, Michelle wrote "2" above each 3-tall tower and said, "It's because, say you add a red or a blue, you can add a red or a blue here [on *RRR*]…you can put two colors here [on *BRR*], two colors there [*RBR*], two colors [*RRB*] – and keep on going" (L376; L379; see Figure 16.1.10). Jeff acknowledged understanding by joining her in the production of taller towers. In unison, Michelle and Jeff counted 16 towers.

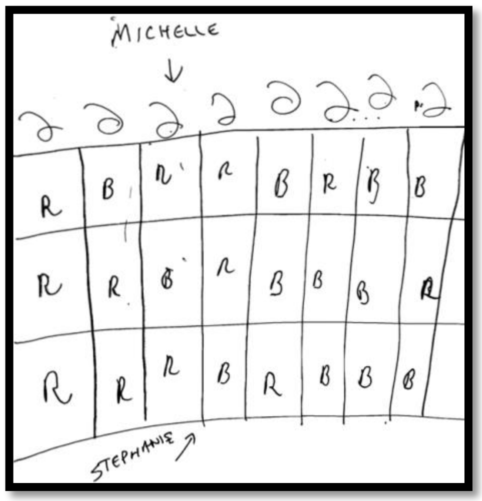

Figure 16.1.10. Michelle's written demonstration of Milin's inductive argument.

Milin returned to Michelle's earlier claim of 12 4-tall towers (refer to Event 1), insisting that she justify her initial claim. Jeff then provided a justification for the inductive argument ("You could do either a red or a blue"). Michelle agreed to 16 as the correct solution, as well as offered 32 as the solution for towers 5-tall, selecting from two colors.

16.1.5 Dyad summative assessment (Session X)

Date	June 15, 1992
Grade	4
Task	Towers (3-tall)
Participants	Dyad pair: Michelle & Jeff
Teacher	Barnes

VMCAnalytic	Michelle's Longitudinal Problem Solving and Development of Reasoning About Tower Tasks: Part 2 of 3 (Grade 4); Events 5–6 https://doi.org/doi:10.7282/t3-tp02-pw54

Fourth grader's Michelle & Jeff's solution of the 3-tall Tower assessment (see Event 5). Michelle and Jeff participated in a partner summative assessment of the 3-tall Tower Task on June 15, 1992. They built nine 3-tall towers using green and black Unifix cubes, and Michelle removed duplicates that she noticed. They examined the towers they found and then arranged them. Michelle grouped the towers with all green and all black, separating them from the rest of the collection. Jeff told her to group the towers with exactly one black and exactly one green in a pattern. Jeff created a descending elevator pattern of towers with the one green cube and Michelle created an ascending elevator pattern with the one black cube. She arranged the towers symmetrically so that the set of towers with the elevator patterns were placed between the tower with all green cubes and the tower with all black cubes. Michelle suggested that there may be more towers and Jeff questioned, "How?"

Figure 16.1.11 illustrates their original arrangement as they built the towers and their second arrangement where they created patterns. The first arrangement is organized by a green descending staircase, an ascending black staircase, and two towers with exactly one green. The second arrangement is organized by an elevator pattern by the black cube, the towers with all of one color, and an elevator pattern by the green cube.

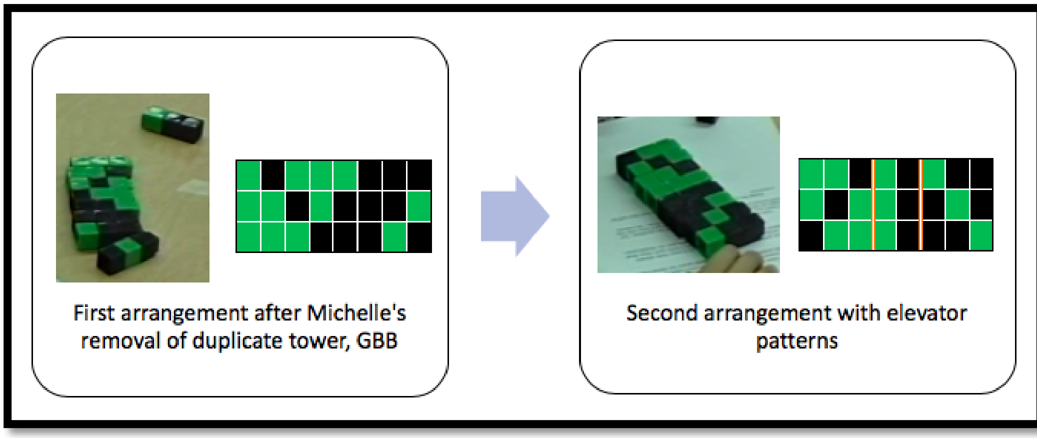

Figure 16.1.11. Jeff and Michelle's arrangements for the eight 3-tall towers.

Michelle asked Jeff whether he thought there were more towers, and Jeff responded, "I don't know" (L28). In response to Michelle's query of the way they could convince hypothetical Chris and Alex, Jeff responded, "Show them exactly what we did" (L30). Michelle observed Jeff drawing his solution.

Michelle and Jeff's joint written assessment (see Event 6). Next, Michelle joined Jeff in recording on her own paper. Jeff drew a key for the meaning of "G", "B" and "□" and Michelle did likewise. They discussed grouping the towers by "colors." Both Jeff and Michelle organized their towers in sets by elevator patterns and color opposite patterns. The towers within the case of exactly one of a color were grouped in elevator patterns. They grouped the two towers with all cubes of a single color as one set and Jeff labeled it as the "group of same color." Michelle wrote "only two here" for that set. They then grouped the towers with exactly one black cube in a descending elevator pattern. Jeff called it a "group of patterns where the black started at the top [*position*] and works its way down [*each position of the tower from the top floor to the bottom floor*]" and Michelle recorded the same label. Jeff then organized the towers with exactly one green cube in a descending elevator pattern, labeling it: "opposite of the one [*group*] with black working its way down" and drew an arrow to the previous set. Figures 16.1.12 and 16.1.13 illustrate a Jeff's individual written work.

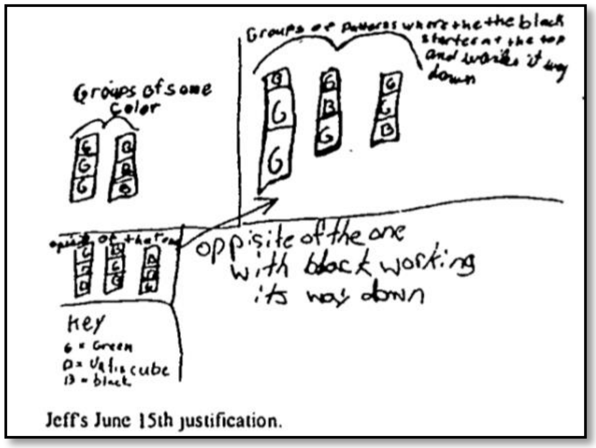

Figure 16.1.12. Jeff's towers grouped by cases.

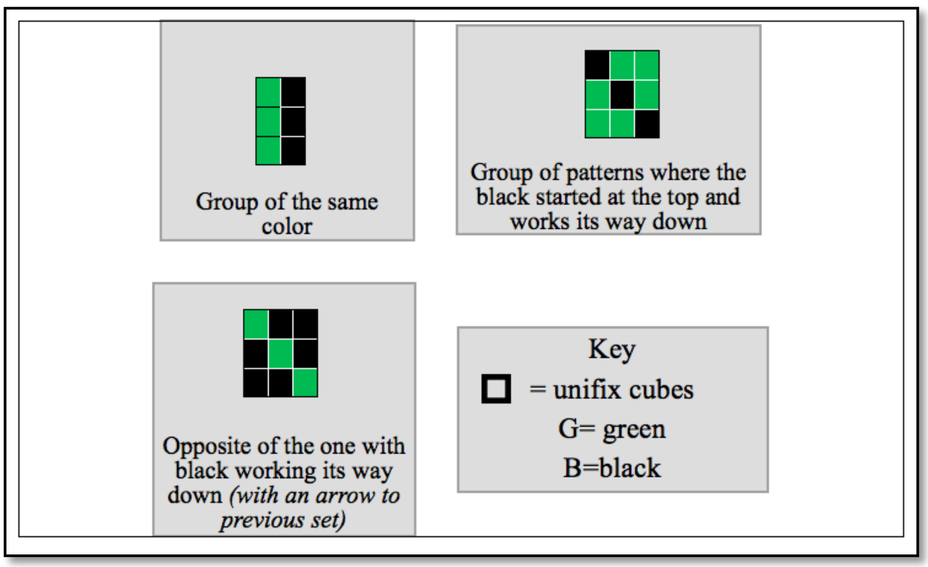

Figure 16.1.13. A reproduction of Jeff's written work.

Michelle did likewise (see Figure 16.1.14). Michelle included the following written explanation accompanying her solution as Jeff suggested how to describe the first two sets:

> The towers are 3 high. We started with simple all black and all green. Then we made patterns of black on top, black in the middle, and black on the bottom. Then we did green on top, green in the middle, and green on the bottom. We got an answer of 8. In pictures: [*referring to the drawings on the back of the paper that Jeff then draws of towers in groups*].

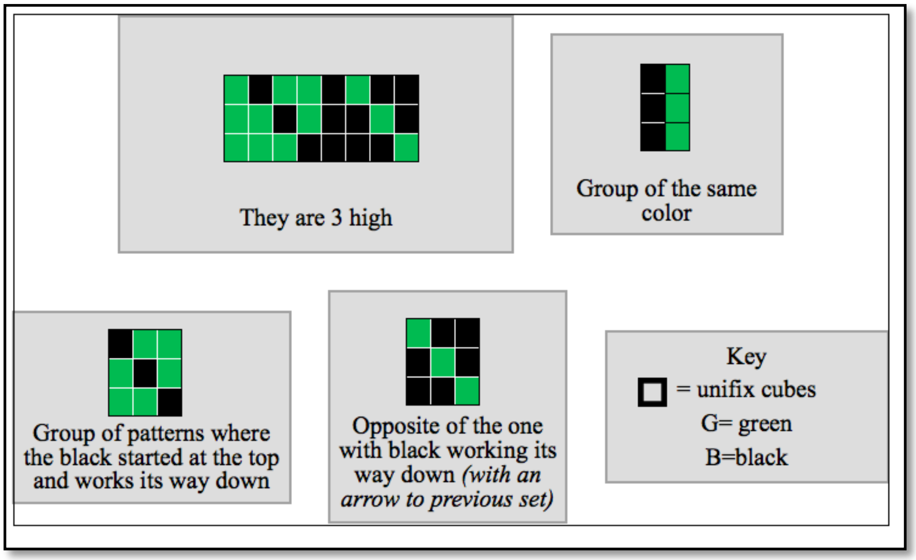

Figure 16.1.14. A reproduction of Michelle's written work.

Jeff redrew the towers below her written explanation, while Michelle finished her individual tower drawings by "groups" (refer to Figure 16.1.15).

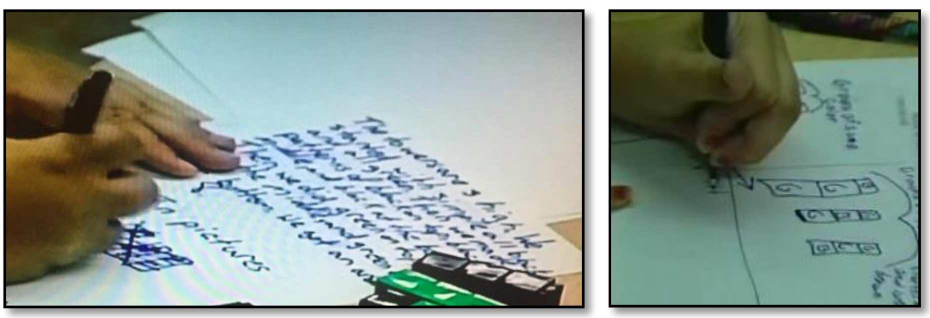

Figure 16.1.15. Jeff and Michelle's joint written assessment and drawings.

16.2 Summary of Grade 4

Recognition of patterns and use of strategies. In the dyad problem-solving session (IV) Michelle and Jeff were observed to use *elevator*, *staircase*, and *opposite* strategies for generation and organization of towers. The three strategies combined led to duplication. After enumeration slowed down, students used Guess and Check to search for missing towers (or verify that there were no missing towers) and to check whether each tower was different from the others. Jeff and Michelle showed the researchers that they noticed duplicates within elevator and staircase patterns. Jeff reasoned that he would continue to use patterns to generate towers and then check for duplicates after. Their final solution left the red and yellow elevator patterns and reorganized towers in the staircase patterns by opposites pairs and eliminated duplicates. Michelle noticed the duplicates that emerged from staircase and elevator patterns in the follow up interview.

In interview sessions, Michelle made early isomorphic connections between the Outfit Task and the Tower Task, selecting from two colors, up to 5-tall. She reasoned *analogically* that the position of a cube in the physical towers were representations of a type of article of clothing and the color was a representation of the color of the clothing item. For example, the cubes in the bottom position of the towers were representative to Michelle as the black or white pants in the outfit (e.g., a black cube in the bottom position was a black pair of pants), the second position as the black or white shirts in the outfit, the third position as the black or white hats, and so on. As the researcher presented a new type of clothing in black and in white to the Outfit Task, Michelle showed and explained how each of the two articles of clothing would be added to each outfit, creating half of the outfits with the black additional clothing item and half of the outfits with the white additional clothing item. This was evidence of *inductive* reasoning. She also noted an additive relationship between consecutive Outfit Tasks: the previous number of outfits was added twice (accounting for the addition of the clothing item in black to the outfits and the addition of the clothing item in white to the outfits) to obtain double the number of outfits with the addition of the new clothing item, selecting form two colors. In addition, she used analogical reasoning to reverse the direction of her thinking to predict the number of towers 4-, 5-, and 6-tall using the outfits. She noticed the same additive relationship for towers.

Displays of justifications and forms of reasoning in support of the solution. In Session IX, Michelle's role was seen as a supporter of the arguments of the other children because she provided further explanation, such as a backing, to their arguments. Recall she made the comment about one of Stephanie's cases, "But if you didn't have that pattern, it would be harder to convince you [*Jeff*]" (L334), Michelle provided a backing to the claim that only two taller towers were generated from each shorter tower. Specifically, she noted that in the Tower Task condition there were only two colors, so no more than two taller towers could be generated from each shorter tower (e.g., "There's only two colors more so you know you can't make more;" L324). When the researcher asked the group to predict 4-tall towers, Michelle showed how it was sixteen by using and extending Milin's argument by induction. When Stephanie first presented her argument by cases, Michelle, like the others, argued that like the "one blue" category, that contained both the two adjacent and the two separated reds, Stephanie's two blue category should have been similarly organized. Nevertheless, when it was Stephanie's turn to present her idea a second time, Michelle supported Stephanie's use of two different categories for two adjacent and two separated blue cubes.

It was evident that Michelle used multiple opportunities to explain her understanding of Milin's and Stephanie's arguments for the Tower Tasks. The events of Session IX also show her understanding of the value of the two different arguments toward solving the problem. Although she did not provide a convincing argument for no more or no less and no duplicate towers to the researcher earlier in her first interview session, after listening to others' ideas on the insistence of the researcher, she may have begun to notice that providing an organized procedure may be used to convince others.

On the contrary, one may counter argue that her isomorphic connection between the Outfit and Tower Tasks provided her a method to find the solution to Tower Tasks, and so in the interview sessions this became the focus and alternative methods were not covered. In the small-group interview there was little time left when the researcher asked if any tasks (e.g., Outfits) were related to the Tower Tasks, so Michelle (and Stephanie, who explored this in a previous interview sessions) did not have the opportunity to explain her analogical findings.

16.3 Grade 5

16.3.1 Dyad: Michelle & Milin (Session XII)

Date	February 26, 1993
Grade	5
Task	Guess My Tower ("GMT;" 3- and 4-tall)
Participants	Dyad pairs: Michelle & Milin; Stephanie & Matt; fifth-grade class
Researchers	R2, R1, and R3
VMCAnalytic	Michelle's Longitudinal Problem Solving and Development of Reasoning About Tower Tasks: Part 3 of 3 (Grade 5); Events 1–4 https://doi.org/doi:10.7282/t3-rjdv-ar64

Michelle explored outcomes from building 3-tall towers. The game, Guess My Tower, required selecting certain sets of towers from a box that contained all tower outcomes, first 3-tall, and then 4-tall, selecting from cubes available in two colors. Note that calculating the total number of towers that could be built was insufficient. The task required a knowledge of all outcomes, and thus having knowledge of sets of outcomes for building 3- and 4-tall towers.

Michelle summarized the tower possibilities for each given condition in the GMT task (obtaining a total of 12 towers). She supported her work by representing the complete set of towers as evidenced by some of her written work (illustrating some towers in an elevator pattern) and by her verbal descriptions and physical manipulation of the towers. For example, Michelle stated, "one could be on the bottom, one could be on the top, and one could be in the center," (L123) and, in two other instances, both Milin and Michelle rearranged the towers with exactly one red cube in an elevator pattern (L236-7; L330-5).

Both Milin and Michelle considered the tower that satisfied both conditions: "all cubes are exactly the same color and at least two cubes are yellow." Michelle realized that although the all yellow cube tower applied to both conditions, only one of this type would be in the box. R1 asked them to record their explanations, which resulted in creating tower representations for the four possibilities and the contents of the box. This excerpt shows the process by which they built the 3-tall towers:

241. Mic:	Okay. This is for number one [*YYY and RRR*] this is for number two [*three towers with one red*]. This is for number three [*three towers with two reds*] and this is for number four [*she puts YYY with the three towers with exactly one red cube*].
242. Mil:	This is for number four. Since there can't be any duplicates that means that these would–
243. Mic:	That means if I draw [*Michelle groups eight different towers*], that would be in the box.
244. Mil:	Yeah, but–
245. Mic:	It would be, because you can't have duplicates and there is no other duplicate of this [*YYY*] or anything else so, that's what would be in the box.
246. Mil:	Okay.

Michelle created the sample space of the 3-tall towers using the outcomes they obtained from solving the GMT. This was evidenced several times as she compared the results of each outcome and extracted the duplicates to find eight unique 3-tall towers. Michelle found eight physical towers, which then was reproduced as drawings.

Michelle explored outcomes building 4-tall towers. Michelle indicated that she wanted to find the outcomes for solving the GMT and then remove the duplicates to find the 4-tall tower outcomes, as she had done earlier. Milin, on the other hand, was finding the contents of the box that contained tower outcomes by cases and color opposite using elevator patterns. Michelle challenged him, stating, “Because there may be duplicates or something, we don’t know what is exactly in the box yet; we have to see what we have first” (L456). She wanted to find the different possibilities for a winning tower first, and then to determine the tower contents of the box. Recall Milin used a doubling strategy to know the *total* in the box; he used elevator patterns and opposite arrangements to identify the towers, yielding sixteen outcomes that consisted of two pairs of duplicates and two missing towers (Note that the two alternating color towers were missing and two pairs of towers with exactly two yellow adjacent were duplicated.). Michelle observed Milin’s approach for making tower representations and found two duplicates within his collection; she suggested to reconstruct the 4-tall towers using the GMT conditions. This followed with Milin abandoning his strategies to use hers.

Milin explained to Michelle and R2 the doubling strategy using inductive reasoning from 1- to 2-tall towers to find the tower outcomes for the GMT Task (see Event 1). R2 asked them how many towers can be made when they were 4-tall. Michelle responded, “I don’t know yet” (L481), because, as explained above, she was reconstructing the solution from the GMT conditions. Milin, on the other hand, stated, “I think sixteen” (L482). In response to R2’s request for justification, Milin explained:

> Well it works like this, before, I don’t know, I can’t remember. Before we found out that for blocks of two you have to multiply two from the first tower that’s two times two so the second one is four. Then two times four is eight times one more would be times two would be sixteen.

R2 asked Milin to find a way to convince Michelle, who indicated uncertainty about his explanation. Milin showed an inductive strategy after beginning with a yellow and a red cube, illustrating two, 1-tall towers. R2 revoiced Milin’s explanations and repeated

Milin's actions by placing each taller tower next to each shorter tower. In the following excerpt Michelle joined R2 by offering that Milin placed "a yellow and a red" on top of each 1-tall tower:

497. R2:	So, tell me again. You started with a yellow [*Y*] and you started with a red [*R*], right?
	[…]
500. Mil:	I put another yellow on top of it [*on the 1-tall yellow tower Y to make YY*].
501. R2:	You put another yellow on top of this [*the 1-tall yellow tower Y*] and got this [*YY*]. I could see that. What else?
502. Mil:	I put a red on top of that one [*RR*] and a red on top of this one [*RY – he builds one taller tower from each shorter tower and then goes back to build the second taller tower*], and a yellow on top of this one [*YRY*].
503. R2:	You put a red on top of this one [*returns to RY*]. Oh, so on this [*bottom*] yellow one you put
504. Mic:	A yellow and a red [*on Y*]
505. R2:	Oh. And on this [*on R*]?
506. Mic:	A yellow and a red too.

Note that as R2 revoiced his explanations by repeating Milin's actions of placing each taller tower next to each shorter tower, Michelle finished the researcher's sentences to place "a yellow and a red" on top of each 1-tall tower (L504-6).

Before Milin showed the production of the 3-tall towers, he claimed that they had already found the eight towers (L508). The researcher redirected the task by asking him to to justify to Michelle how and why the approach produces eight 3-tall towers. Michelle silently observed Milin, who was quickly building new towers using his inductive strategy. As he generated new towers from shorter towers, R2 pointed out how he generated them.

Note that Milin had a unique way of generating new towers. Sometimes, he generated one (rather than two) taller tower from a shorter tower, then moved on to a different shorter tower to build another taller tower, before returning to find the second taller towers. His approach to build one tower generated from one set and another tower generated from another set was not easy for the students (or facilitators) who were observing to follow. Note that he used the same method in an earlier interview with R1, in this session when generating the 2-tall towers, and in the next part of this event when generating 3-tall towers.

Milin demonstrated how new, taller towers, 3-tall, emerged from a shorter tower. For example, he placed a red cube on top of the RR tower and a yellow cube on top of the YY tower. R2 asked him to focus on one 2-tall tower and to explain what he was doing. Instead Michelle selected a yellow cube and attempted to place it on the RR tower. R2 asked her what she was trying to do:

516. R2:	What are you doing Michelle? On the red one [*RR*] he put a red [*RRR*], and what are you doing with that red one [*RR*]?
517. Mic:	We could add a yellow one to that [*YRR*].
518. R2:	So this red one [*RR*], if you are making it three [*tall*], you put a red on it [*RRR*]?
519. Mic:	Yeah and put a yellow on top of it [*YRR*]
520. R2:	Oh, why are you doing that? Why does that work?
521. Mic:	I don’t know.

Michelle again showed that she understood the procedure of generating two towers from a shorter tower. However, it was unclear whether she understood why the method accounted for the complete set of taller towers. She continued to organize the tower sets to help Milin show the generation of towers that evolved from the smaller towers.

Michelle joins in building 3-tall towers from 2-tall towers using inductive reasoning (see Event 2). Michelle’s recognition of Milin’s inductive generation of new towers became evident in this event. Milin provided an explanation to Michelle, who indicated earlier she did not understand how he generated the 3-tall towers (e.g., L539). Then Michelle continued with an explanation (L549-51):

> You could add yellow on from there [*on YY*] and you could also add red on [*on YY*]. See, I understand but like its everything times two… Like from here [*1-tall towers Y & R*] from the two [*1-tall towers*] if you times by two you got four [*moving the 2-tall towers next to 1-tall towers*]"

It was evident in Michelle's explanation that she recognized a doubling relationship between the number of towers of different heights. The researcher probed twice for a complete explanation of why this procedure worked: "But can you tell me why that works?" (L552). In the second explanation Michelle described a process: "You multiply by two because there is [*sic*] two colors of them and you could add two colors on top of each one" [*tower*] (L559). Michelle claimed that the previous number of towers were multiplied by two. She explained that this was due to the availability of two colors of cubes that get placed on top of each previous tower (e.g., she used the word "each"). The researcher revoiced Michelle's explanation in her own words (L560-6):

> So in other words, if I start with this yellow by making it higher the only way I can make it higher was either to put a red, that's one choice, [*Mic: "Yeah"*] or a yellow, that's a second choice [*Mic: "Yeah"*], so that's how I get two from this one [*Mic: "Yeah"*] and then I get two from this one.

Michelle extended the demonstration to the rest of the 3-tall towers by constructing missing 3-tall towers, rearranging the built towers to correspond to the previous stages of shorter towers, and arguing why it works (L573-5):

> This [*YR*] should be like this yellow/red/and then yellow [*YRY*]. Cause then you will have from the yellow and the red [*YR*], and you already added the red [*already existing RRY*]. And you have two yellows [*YY*], you added the yellow [*YYY*] and you added the red [*RYY*], cause those are the two colors. From this [*RR*] you added a red [*RRR*] and a yellow [*YRR*] and from this [*YR*] you added a red [*RYR*] and a yellow [*YYR*].

Notice that Michelle supported her reasoning with an inductive argument that became more general because she considered the conditions of the two-color availability and repeatedly demonstrated placing a red and a yellow on each previous tower.

Michelle took up Milin's doubling strategy using inductive reasoning from 2- to 3-tall towers (see Event 3). Michelle was explaining and placing each 3-tall tower next to the 2-tall towers from which they were generated. R2 commented, "Oh I see. Then I should be able to know how many I would have if they are four high and how to make them" (L576). This prompted a response from Michelle about Milin's method for finding taller towers from shorter towers (L578-80):

> This is a lot simpler than the last time you [*Milin*] explained it… Because last time we like- we didn't do it like this. It is easier to explain it when you have it like this... Like I think the answer is sixteen because eight times two is sixteen. From every one of these [*3-tall towers*] you add on two [*colors*].

Not only did Michelle apply the multiplication rule to get the total count of 4-tall towers, she also supported her reasoning using an inductive argument to explain how 4-tall towers were generated from each 3-tall tower.

Michelle explains the doubling idea using the inductive reasoning to Stephanie and Matt (see Event 4). Next, Milin claimed Stephanie knew the doubling idea. R2 asked Stephanie and Matt to listen to how doubling works. R2 encouraged Michelle to explain to the students. She explained as follows(L598 & L603):

> For this one [*RR 2-tall*] you can add a red on top of it and yellow [*to make 3-tall, points to each RRR & YRR*] because there's two colors and this one [*YY 2-tall]* you could add red and yellow [*to make 3-tall, points to each RYY & YYY*] so its– I don't know how to explain it ... And then it's here too [*holds up RY*] and here [*holds up YR*] and that how you find out how– [*Matt interrupts and Michelle stops speaking*].

Notice Michelle considered the conditions of the two-color availability and then repeatedly pointed to the possible red cube and yellow cube to be placed onto each 2-tall tower. Matt made an analogy between the inductive strategy and a family tree, and then joined Michelle in indicating from which 2-tall tower was each 3-tall pair generated.

16.4 Summary of Grade 5

Recognition of strategies and forms of reasoning in support of those strategies. In Session XII, Michelle found the outcomes of the GMT to find all towers 3- and 4-tall. It was evidence that it was easier to use the outcomes to find the eight 3-tall towers but the case was different for finding all 4-tall towers. When R2 arrived Michelle directly stated that she was not sure the total number of towers 4-tall, while Milin claimed the number was 16 towers, irrespective of the duplicates Michelle flagged within his case-based solution. R2 prompted Milin to explain his reasoning. During Milin's demonstration and explanations, Michelle was observed to join, first, in the action of generating inductively two taller towers from a shorter tower. R2 focused on prompting Michelle, rather than Milin, for justification of the inductive growth. Michelle's explanation became more general when she considered the condition of the two-color availability and demonstrated repeatedly the placement of a red and a yellow to each shorter tower to generate the whole collection of the 3-tall towers. Michelle also extended her reasoning to the 4-tall towers by applying the multiplication rule to get the total count of 4-tall towers, as well as explained that two 4-tall towers were generated from each 3-tall tower. She also referenced an earlier instance when Milin explained the inductive growth (perhaps this was Session IX, since that was the last time they were together) and that the current instance it was easier for her to understand. Recall in Session IX Milin drew generic examples to explain his reasoning for the doubling pattern, while in this session he used physical models and actions to generate two towers from each shorter tower (from 1- to 4-tall).

Chapter 17 Findings

17.1 Overview

The findings presented in this chapter derive from analyses of the video and inscriptive data (see Glossary of Terms in Appendix A3). Michelle, Milin, and Stephanie were the focus students of this study who participated in a formative assessment interview on March 10, 1992. Along with Jeff, the group became known as the “Gang of four” see Session IX). A goal of this study was to trace the journeys of Michelle, Milin, and Stephanie mathematical learning using socio-constructivist and cognitive lenses. The following question guided the study: How did a student’s recognition of patterns, use of strategies, display of justifications, and forms of reasoning in their argumentation with supporting representations about solutions to the counting tasks develop and change over time? Each category is discussed in their respective subsections. Student learning progressions are summarized at the end of this chapter.

17.2 Pattern recognition and development

Students’ pattern recognition and the recognition of patterns to solve a task developed over time by identifying local and global enumerative patterns. This section is organized by categories of local pattern recognition, local pattern applications, global pattern recognition, and global pattern application during Tower Task investigations over time. Subsections are chronological by the task, grade level, and participants to display the longitudinal and social nature of student recognition and growth.

17.2.1 Local Pattern Recognition

17.2.1.1 Towers 4-tall, selecting from 2 colors, Grade 3, Dyad Pair Problem Solving and Whole Class Discussion (Sessions I – II): Stephanie and Dana; Milin and Lauren

In the third-grade dyad session (I), Stephanie and her partner, Dana, built with red and blue Unifix cubes 16 different 4-tall towers. Their strategy was to make a 4-tall tower and then to check to see if the tower was already built. Satisfied that they could find no more, the students presented their solution. It's important to note that the girls did not begin solving the task by looking for patterns. Rather, their strategy was Guess and Check. However, when they explained to the researchers their solution, they offered some evidence of pattern recognition. For example, Stephanie and Milin recognized patterns of opposite colored tower pairs and inverse tower pairs and gave names to them (Stephanie: "cousins;" Milin: "switched around") – in later grades using both of these patterns as strategies.

17.2.2 Local Pattern Application

17.2.2.1 Towers 5-tall, selecting from 2 colors, Grade 4, Dyad Pair Problem Solving and Whole Class Discussion (Sessions IV – V): Stephanie and Dana; Michelle and Jeff; Milin and Michael

A common finding in all three dyads during the generation of physical towers and the organization of these towers (Session IV) was a search for new ways to group new and old towers and generate towers fitting their patterns. For example, all students constructed towers using a random tower pattern and then replicated the opposite pattern to generate pairs of towers. Stephanie, Milin, and their partners used the opposite and inverse patterns they recognized in third grade as strategies to generate and organize towers. Jeff noticed an elevator pattern among related towers with exactly one of a color, which prompted Jeff and his partner, Michelle, to build towers with other attributes according to such a pattern. In all dyad groups, the search for patterns was local to a set of towers.

During the class discussion of Session V, R2 displayed patterns of the elevator and the staircase, as well as portrayed an exhaustive strategy for enumerating towers with exactly two red cubes. Students were asked to imagine or build specific sets/cases of 5-tall towers and justify that their solution was complete. The students helped R2 group the towers in an elevator pattern. Stephanie explained to the class which towers of specific attributes could and could not be enumerated, such as the cases with exactly one of a color, two of a color together, and two of a color separated by at least one cube and no more than three cubes of the other color.

17.2.2.2 Towers 5-tall, selecting from 2 colors, Grade 4, 1-on-1 Interview (Session VI): Michelle, Milin, and Stephanie

During the first one-on-one interview session, the day after the class problem solving session, each student revisited the 5-tall Tower Task and were asked to justify the solution was 32 towers. Each student presented, reinvestigated, and/or rebuilt the idea of recursive enumeration by forming a local pattern among a set of towers, such as an elevator or a staircase pattern. Each student applied these patterns as a way of generating a case of towers (e.g., exactly one of a color) and demonstrating its completeness. An example of rebuilding the idea of a recursion to locally exhaust a set of towers occurred when Stephanie was asked to consider the common attributes among her towers with exactly one of a color, which she built by a composite operation using opposite and inverse strategies. This resulted in her noticing a recursive relationship ("It's the same pattern but in different places"). Michelle, Stephanie, and Milin encountered duplication when creating patterns for their cases of towers due to a focus on the spatial relations of a particular color (and not of the second color). For example, when generating cases of towers and the opposite cases, Stephanie and Milin discovered some cases were equivalent (e.g., exactly one of a color is equivalent to exactly three of the other color in a 4-tall tower). Michelle acknowledged that the patterns Jeff sought to create with towers in the previous session generated duplicates that had to be removed, thus affecting his use of the patterns.

17.2.3 Global enumerative pattern recognition and application

17.2.3.1 Towers 4-, 5-, and 6-tall, selecting from 2 colors, Grade 4, 1-on-1 Interviews (Sessions VI – VIII): Milin; Stephanie

Recognizing a global pattern among the cases of towers and how students built the idea differed for each student. Stephanie noticed a pattern among the *cases of towers* by the number of "ones, twos, threes [*of a color together*], and so on" in Session VI and applied this to the 6-tall Tower Task, sharing work from home in Session VII. Stephanie also displayed the heuristic of "controlling for variables" in which she kept the top/bottom row constant and identified elevator patterns. This heuristic was extended in the interview to enumerate the complete case of exactly two of a color. Milin similarly noticed this relationship in Session VI. Both students found 6-tall towers that fit in an elevator pattern, and hence, partially solved the 6-tall Tower Task by cases. For both students, the application of a global enumerative pattern resulted in acknowledging that there were other cases that did not necessarily have the same color cubes adjacent and in finding ways to resolve this issue.

As a result of this experience, when solving the 4-tall Tower Task in the next session (Session VIII), Stephanie refined her method by generating and organizing the cases of towers with adjacency of the same color cubes and then the cases of towers with separation of the same color cubes, all of which still utilized the elevator pattern. Milin used Guess and Check to search for 5-tall towers with same color cube separation in

Session VI, while searching for an elevator pattern among them to group them. In Session VII, Milin applied the same approach of searching for an elevator pattern among all cases in a simpler, 4-tall, Tower Task. Specifically, he organized his pre-built 4-tall towers in a symmetric pattern that consisted of some cases on one side and the opposite cases on the other, organized as if they were mirror images of each other, while separating the cases of towers with some separation of colors (e.g., alternating colors). He noticed such an organizational pattern existed for other heights and applied it to find the solutions to shorter Tower Tasks.

17.2.4 Noticing structure in related Tower Tasks

17.2.4.1 Towers 3-tall, selecting from 2 colors, Grade 3, Whole Class Discussion and Interview (Sessions II – III): Stephanie; Milin

Initially, both Stephanie and Milin predicted that the total number of 3-tall towers would be the same as the number of 4-tall towers. After exploration, the class discussed noticing duplicates when constructing towers, 3-tall, by removing one cube from each 4-tall tower. Specifically, Stephanie explained, using generic examples, that the one cube removed from each 4-tall tower resulted in pairs of duplicate towers.

17.2.4.2 Towers 1- and 2-tall, selecting from 2 and from 3 colors, Grade 4, 1-on-1 Interviews (Sessions VI – VIII): Milin

In Session VII Milin noticed that the two-color problem produced opposites while the three-color problem produced “pairs of three” and used these differences to find patterns among the 2-tall towers. He also recognized that the number of colors available produced different number patterns, such as three 1-tall towers multiplied by three colors available equaled nine 2-tall towers and two 1-tall towers multiplied by two colors available equaled four 2-tall towers.

In Session VIII, when Milin explored the growth of taller towers from shorter towers, he acknowledged that each shorter tower had the same pattern as two taller towers with the difference of the top color cube. This, later, would become a warrant for his justification as to why all towers were different using his inductive strategy in Session IX.

17.2.4.3 Towers 2-, 3-, 4-, 5-, and 6-tall and Outfits, selecting from 2 colors, Grade 4, 1-on-1 Interviews (Sessions VII – VIII): Michelle

Michelle also noticed that the taller towers generated from shorter towers had the same pattern with the exception of the top cube, which varied by the two colors available, when she explored Outfit Tasks, using tower representations. Specifically, Michelle began with the Outfit Task that had two types of clothing, selecting from two colors of each type (i.e., a two by two outfit structure) and followed with Outfit Tasks that were increased by an additional type of clothing, selecting from two colors of each type (i.e., a 2_n outfit structure where n=3, 4, and 5). Noticing a relationship between the number solutions of the Outfit Tasks, she generalized a procedure of doubling the number solution of the previous Outfit Task to identify the solution of the next Outfit Task with an additional article of clothing. Likewise, she extended the procedure to identify the number solutions of isomorphic Tower Tasks up to 6-tall.

17.2.5 Seeking numerical patterns

All three students estimated and explored numerical solutions to Tower Tasks of various heights in their final interviews (Sessions VII or VIII). Michelle and Milin recognized that the solutions for the 2-color problem should be an even number due to the opposite pairing relationship. Michelle, Milin, and Stephanie each recognized a doubling pattern among Tower Tasks, but from differing original methods: Stephanie recognized it after solving several tasks by cases and earlier methods; Milin recognized it after solving tasks by cases and opposites and then comparing numerical solutions, selecting cubes from two and three colors; and Michelle recognized it for the Outfit Task after generating outfits using tower models, inductively, and predicted it would be the same for the Tower Task. Michelle, Milin, and Stephanie presented their observations of a doubling pattern among Tower Tasks and explained their reasoning in a small-group interview with Jeff and R2 (Session IX).

During Session IX Jeff prompted a discussion about the necessity for “patterns” (the students’ reference to their methods for generating towers *systematically*). The students recognized the importance of using both local and global patterns in problem solving and in justification to others. For example, Stephanie repeated her method based on cases of towers and Milin repeated his method based on inductive growth of towers to demonstrate the necessity of “patterns” to Jeff. Michelle added to their arguments, providing warrants for why Stephanie’s or Milin’s local “patterns” (e.g., elevator pattern and the inductive growth of a shorter tower) were complete and valid (see the section on “displays of justification”).

The following sections present a summary of findings on the development of students’ use of strategies, forms of reasoning, and displays of justification.

17.3 Use of Strategies

The next subsections categorize the use of Guess and Check, opposite and inverse pairing, composite operation, and recursive strategies.

17.3.1 Guess and Check: Stephanie, Milin, Michelle

In problem-solving Session I, third graders Dana and Stephanie were observed building a tower pattern in a random order (or with no particular observable method) and then double-checking for duplicate towers (Maher & Martino, 1996). This strategy was used to generate tower outcomes to obtain a solution.

Guess and check strategy provided the children with quick generation of towers while leaving some duplicates unnoticed. After enumeration slowed down and researchers prompted students for justification of their solution, students referenced the result of their Guess and Check strategy that no more new towers could be found. For example, in Session I, Dana and Stephanie tested their solution by a trial and error procedure to generate new towers and to re-check the solution for duplicates. Stephanie used a Trial and Error procedure for two purposes: 1) to verify that there were no missing towers (or to search for a missing tower), by guess and check strategy, and 2) to verify a solution set contained no duplicates (or to flag and eliminate duplicates), by comparison. Interestingly, Stephanie recursively generated all towers with exactly one blue in each position to check if those towers were missing from their proposed solution. As the procedure resulted in trials with “no error” (no new different towers could be found using Guess and Check and all existing towers were different from each other) Dana and Stephanie claimed to have gained certainty of their solution of 16 explaining that they double-checked each new tower against the solution set.

The strategy of guess and check was also observed in the fourth-grade problem solving session (Session IV) across all three dyads when they generated a random first tower, checked it against the set, and then generated the color opposite or inverse second tower (see next section). Although the students moved away from the guess and check strategy toward generating towers by patterns, they *folded back* to guessing and checking when researchers challenged them to justify how they knew there were no missing towers. For example, they returned to a guess and check strategy to determine if their solutions were certain (i.e., trial and error).

17.3.2 **Opposite and inverse pairing strategies**

In the fourth-grade dyad session (IV), initial problem-solving strategies for the 5-tall Tower Task was *locally exhaustive* enumeration across all three dyads (Stephanie with Dana, Milin with Michael, and Michelle with Jeff). Specifically, once one random tower was found, the color opposite strategy was applied to generate a second tower and organize each tower pair. The strategy of inverse was also part of Milin and Stephanie's problem solving. The strategies evolved to other forms as discussed in the next two sections.

17.3.3 **Composite operations**

Composite operations were invented and applied when generating and/or organizing towers. Stephanie and Milin applied the opposite strategy, then inverted the pair, or vise-versa, to generate three new towers from one tower or to check if any towers were missing. Milin and his partner, Michael, generated some groups of towers by applying the composite operation. Stephanie and her partner, Dana, also generated some groups of towers in this way, as well as systematically organizing as many towers that were not generated in this way as they could, creating sets with four opposite and inverse pairs. This organization allowed the students to find duplicates and caused other unnoticed duplicates to emerge. They returned to comparing towers, opposite pairing, and guess and check to become convinced of their solutions.

17.3.4 Recursion

As early as third-grade Stephanie was observed to generate the towers with exactly one blue recursively when she moved the blue cube to every position to create different towers. Recursive patternmaking was used by Jeff and Michelle. It was presented by several students as a local strategy and as an organization during whole class discussion (Session V). Specifically, the use of elevator and color opposite strategies for generating and organizing by a common attribute (a case) and controlling for duplicates was exemplified in this class discussion. Stephanie used the idea of recursion when she explained how members of a set of towers were exhausted after the specific color took on every position.

In an interview with R2 Stephanie used a recursive method of controlling for one variable and varying the other (e.g., in the case of exactly two blue cubes apart from each other with one blue cube fixed at the top and the other blue cube varied recursively in every position). Stephanie, with prompting from R2 to find a way to exhaust the two blue case, used the *odometer* strategy (by controlling for variables and recursively varying the others) to generate all towers with exactly two of a color separated by at least one cube of the other color.

Recursion was also seen as a global strategy to enumerate cases of towers (see reasoning by cases). Stephanie built 3-tall towers with “no blue,” “one blue,” “two blue together”, “three blue”, and “two blue separated” and in another session put the two blue cases into one set.

17.4 Displays of justification and forms of reasoning in their argumentation

The following subsections are partitioned into the forms of reasoning that emerged in the learning progressions when considering all forms of reasoning displayed by the three students. Note that every student did not display each form of reasoning, and for those that did display it, the order was not necessarily the same. Nevertheless, the forms of reasoning are ordered by sophistication and/or comprehensiveness.

17.4.1 Justification by Trial and Error results

When children were asked how they could be sure they found all possible towers of a given height, they reasoned about their *trial and error* procedures and results in third- and fourth-grade problem solving sessions (I; IV). Across all three dyads the process of verification by *trial and error* served as supportive reasoning for the exhaustion of the tower outcomes because no new towers could be found and because no existing towers were duplicates of one another. For example, Milin used the length of time and the results of duplicates during trial and error as a measure for certainty of the completion of a solution to the task. Interestingly, Stephanie made the conclusion that a solution to the task was not possible because someone could also generate another new tower that no one has thought of. This occurred after she and Dana removed duplicates and compared their solution of 28 towers to Michelle and Jeff's of 32 towers, 5-tall.

17.4.2 Justifications by visual aids

As discussed earlier the children formed patterns with collections of towers. For many instances they did so to demonstrate their reasoning for a completion of a case of towers. The earliest of reasoning based on local patterns was observed when students claimed the opposite pattern for a tower was a guaranteed different tower. For example, Milin used this reasoning to justify that after checking one tower for duplication against a set, then the opposite tower was also different from the set. Milin's justification for an even number solution for Tower Tasks, selecting from two colors, also used knowledge of opposite patterns.

Each student also organized towers in an elevator pattern to justify the number of towers that were possible with exactly one of a color, exactly two of a color, exactly three of a color, and so on. For example, Jeff, with Michelle's help in Session IV, created patterns to convince R2 of towers with exactly one and exactly two of a color adjacent. In Session IX Michelle noted the importance of Stephanie's diagram where she recursively generated all towers with exactly two adjacent blue and exactly one blue in an elevator pattern: "If you didn't have that pattern, it would be harder to convince you."

Milin searched for symmetry among his cases to visually demonstrate the completion of his 4-tall solution in the fourth grade and the fifth grade. Symmetry, such as color opposite case symmetry, did not guarantee prevention of duplication, which Michelle noticed in Milin's fifth grade solution of 4-tall towers and Jeff's fourth-grade solution of 5-tall towers and which Milin and Stephanie noticed in their fourth-grade one-on-one interviews with researchers for 5- and 6-tall towers, respectively. The acknowledgment that patterns did not guarantee a valid or complete solution was associated with the development of other strategies or adjustment of one's reasoning about the solution. For example, it was associated with Stephanie's refinement of the method based on cases *without* opposites (in Session VIII) and Milin's move from a "staircase" (cases that could be organized in elevator patterns) and opposite method to an inductive method for solving Tower Tasks.

17.4.3 **Rule generalization**

Students noticed that the number solutions for consecutive Tower Tasks was double the previous outcome of total towers for one cube shorter. Milin and Stephanie became convinced of a doubling relationship after they had gained experience in solving several Tower Tasks using the method based on cases. For Milin and Stephanie, the doubling pattern was recognized as taller towers were built for the solutions that *matched* those for Tower Tasks that had previously solved. During fourth grade partner and individual assessments Milin and Stephanie generalized their observation of a doubling pattern as a rule for determining or verifying a number solution for Tower Tasks from 1- to 10-tall. For Stephanie, she applied the rule to taller Tower Tasks that she had not solved using other methods. Milin, on the other hand, at first did not consider the doubling relationship to work for all Tower Tasks, but later convinced himself otherwise. Like Stephanie, it became a rule that Milin applied to verify or find the number solutions for Tower Tasks up to 10-tall. In fifth grade, after hearing Milin present an inductive strategy and argument for towers up to 3-tall, Michelle provided an argument by induction for why the doubling relationship could be generalized to taller towers.

17.4.4 **Argument by contradiction**

Stephanie argued by contradiction to justify the completeness of a case of towers using a particular recursion (e.g., elevator or odometer patterns) in the fourth-grade individual and group interviews (Sessions VII, VIII, IX). Michelle similarly justified by contradiction for the case of exactly one red, for towers 5-tall. She built a tower 6-tall by continuing the elevator recursion for the red cube to show it produced a contradictory tower to the 5-tall Tower Task condition in the first fourth-grade interview (Session VI).

This reasoning by both Stephanie and Michelle was associated with a challenge by R2 to convince her of the completion of a case of towers.

17.4.5 Reasoning by analogy

Stephanie and Michelle noticed a similarity in the number solutions of the Tower Tasks to the solutions of the Outfit tasks when constructing tower models to represent outfit combinations. As previously indicated, Michelle used the number solutions to the Outfit Tasks of different outfit structures to predict the number solutions to the isomorphic Tower Tasks (in Session VII). In Session VIII, Stephanie reaffirmed her earlier discovery of the doubling pattern by analogously finding the same pattern for Outfit Tasks of varying outfit structures, selecting from two colors. Stephanie's knowledge was associated with her double rule generalization for all Tower Tasks of varying heights, selecting from two colors.

17.4.6 Reasoning by cases

Reasoning by cases was first presented (indirectly) during the fourth-grade class discussion in Session V when R2 asked students to consider Ankur's elevator pattern and apply it to other cases. Students considered ways to generate and organize towers with particular attributes (e.g., exactly two of a color and exactly one of a color) systematically. Stephanie showed evidence of noticing early elements of organizing by cases. During post-interviews, Michelle, Milin, and Stephanie used partial reasoning by cases of towers that could be organized by elevator and staircase patterns. Each student *folded back* to generating a tower and the color opposite tower when the attributes of tower patterns became more complex (i.e., same color cubes were separated from each other in some way). For Milin and Stephanie reasoning by cases was associated with researchers' prompting to convince them of the solution for Tower Tasks of various heights. For example, Milin would first organize his towers by color opposites and then use a method based on cases to convince R1 that he had found all towers. When Milin built 5-tall towers by cases (in Session VI), he used the number of both colors present in a set of towers to identify attributes of a case: "one and four" and "two and three," representing the number of possible yellow and red in a 5-tall tower. Milin focused on both color attributes when finding cases and opposite cases to control for equivalent cases and justify the differences. He applied similar reasoning by cases to build towers shorter than 5-tall in the follow-up interview (Session VII). His solution and supporting argument depended on the elevator patterns (what he called "staircases") that he was able to create with the sets of towers (e.g., For the 6-tall Tower Task, he argued that his method by cases produced 50 towers.).

Stephanie reflected on the method based on cases as a way of convincing someone of one's solution when she revisited the 5-tall Tower Task in a first interview. It was evident that she applied reasoning by cases in the next task of 6-tall towers because she found towers with common attributes of one blue, two adjacent blue, three adjacent blue, four adjacent blue, and others (e.g., the case of exactly two blue cubes apart from each other with one blue cube fixed at the top and the other blue cube varied recursively in every position). Her method by cases began with all towers with attributes of the same color adjacent to each other and then with the same color separated from each other and all the color opposite cases. She dealt with and recognized the equivalent cases that emerged when she found opposite cases. In Session IX she enumerated the cases of 3-tall towers from no cubes of a color of focus to all cubes in the color of focus and the case of two of a color separated. Stephanie no longer found opposite cases and she acknowledged they already existed in within these cases (see Reasoning about combinatorial ideas). In Session XI, Stephanie directly reasoned by cases, with a refinement in her method to include the cases of two of a color separated and together as one set.

17.4.7 Reasoning about combinatorial ideas

Solving Tower Tasks gave the students concrete experiences for constructing knowledge in the combinatorial domain. Within this domain, the application of a method based on cases has its complexities. For example, one must know when to begin and when to stop enumerating the cases, as well as the combinations within each case. This requires a *recognition of symmetry of the attributes* among sets of combinations (i.e., cases of towers, in this context) for efficiency and prevention of redundancy in enumeration. As early as the first Tower Task in third (for Stephanie and Milin) and fourth grades (for Michelle), students used the color opposite or inverse relationship to enumerate 4-tall towers by pairs. During interviews, Stephanie and Milin recognized symmetry in the opposite sets of towers. For example, in Session VIII, Stephanie analyzed an example with physical 4-, 3-, 2-, and 1-tall towers that dealt with what is known as *binomial symmetry* in combinatorics. Binomial symmetry is the equivalent structure and equal cardinality of the set of combinations when selecting r from n choices, and the complement set when selecting $n–r$ from n choices (Note that for an even number n there exists one set and its complement that are exactly the same set.). This is because each set has r of one type of element and $n–r$ of the other type of element (e.g., selecting one white (or three black) to be arranged in four positions, and the complement, selecting three white (or one black) to be arranged in four positions has the same structure and equal number of combinations. The symmetry of these numbers can be represented by Pascal's Triangle). Findings of Stephanie's earlier reasoning serves as a foundation to her later learning as described in Teehan's (2017) research study of Stephanie's eighth-grade exploration of combinatorial ideas about binomial identities using tower models.

17.4.8 Reasoning by induction

Fourth-grader Milin developed the idea of placing the same color cubes onto 1-tall towers and then different color cubes onto copies of the 1-tall towers to generate four different 2-tall towers. While being interviewed, R1 asked Milin to demonstrate how taller towers may have evolved from shorter ones. Milin responded by showing that each shorter tower generated two different taller towers, varied by a different color cube on top. Although Milin was able to utilize this method for taller Tower Tasks, he claimed this method could not be generalized for towers taller than 5-tall, selecting from two colors, and 2-tall, selecting from three colors. His reasoning was based on the tower models that he generated by building physical towers. Michelle, Milin, and Stephanie recognized that a doubling pattern existed for the number of towers of consecutive heights. It should be noted that it was Milin who provided a justification and a process for demonstrating how the doubling pattern evolved building taller towers from shorter ones. When Milin presented his method to Jeff, Michelle, R2, and Stephanie, he generalized the procedure and provided backing for why the generalization was true, with the help of Michelle and R2. Milin's reasoning by induction, in support of the existence of a doubling number pattern, was informal (as opposed to formal proof by mathematical induction), that included complete examples of how 3-tall and 2-tall towers grew from shorter towers, and by his gestures of placing each available color on top of each shorter tower, and his everyday language (words and phrases, such as “for each” tower and “only two colors” available).

In the fifth grade, Michelle, Milin, and Stephanie justified why the doubling pattern worked, using an argument by induction. They did so by showing how taller towers grew from shorter towers in a tree pattern. First Michelle encountered Milin applying the doubling pattern to find the number of 3- and 4-tall towers and was asked by R2 to justify why it was true. Michelle, using physical models of towers, watched Milin demonstrate the inductive procedure in support of the doubling pattern. Then Michelle demonstrated it for taller towers and explained why it was true in general (i.e., from each tower, two colors are available to place on top of the shorter towers, and hence the solution doubled). Then Michelle demonstrated the procedure to Stephanie and Matt, who were asked to show other students. Stephanie struggled initially to differentiate between a claim about a doubling pattern and the reasoning in support for why it was true. She watched Matt demonstrate how the pattern evolved, and then was successful in generalizing the procedure and explaining why the number of towers doubled as the towers grew for larger heights.

17.5 Student learning progressions

Figure 17.5.1 summarizes how the three student’s journeys were related with respect to their investigations of Tower Tasks before and after the “Gang of Four” session (“GI” in the grey triangle) and with respect to each other (refer to Appendix A for more detail about their timelines). Stephanie (in yellow with green arrows), Milin (in blue with black arrows), and Michelle’s (in orange with pink arrows) journeys are displayed illustrating key sessions with the partners with whom they worked (names of partners in the circles) and the type of session (indicated with triangles or circles). The arrows display a journey from one session to the next, showing where the students participated in common sessions and where they engaged in task-based interviews individually with researchers on Tower Tasks over two or three school years.

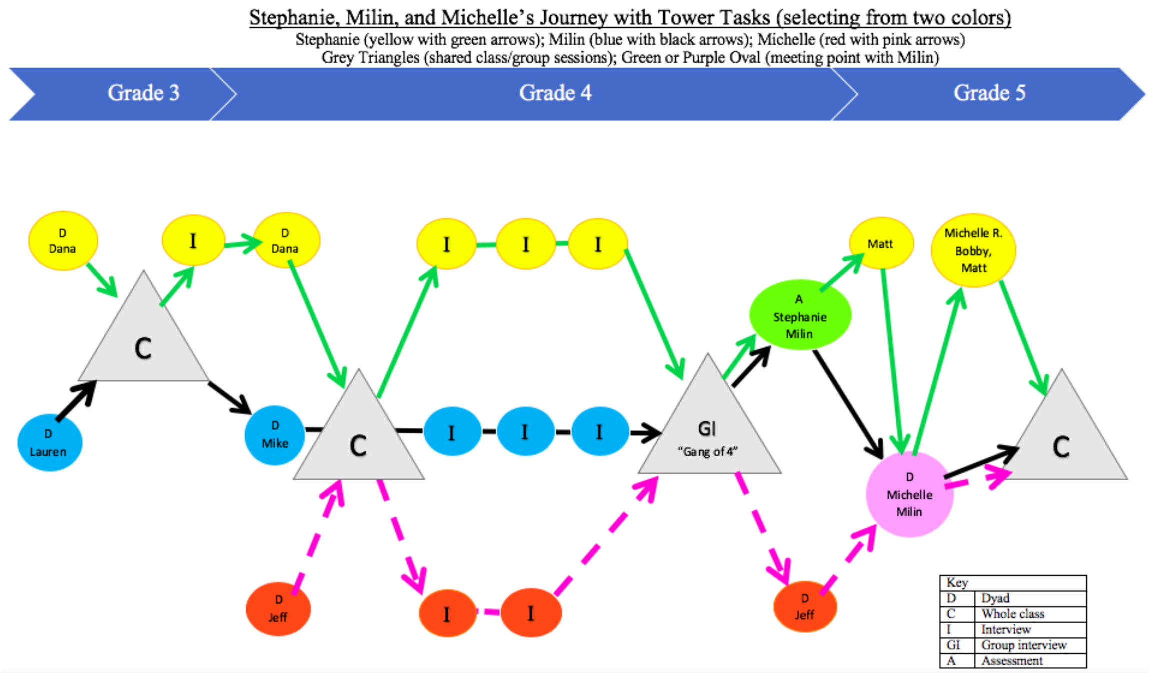

Figure 17.5.1. Student journeys with Tower Tasks.

Milin and Stephanie participated in two sessions and Michelle in one session (Grade 3 and 4), where they were challenged to investigate the same Tower Task (4- and 5-tall, selecting from two colors, respectively) with other partners and then discussed their solutions in a whole class setting. Each student then participated in a task-based interview, individually with researchers, in two or three sessions, before coming together in the "Gang of Four" group formative assessment where they shared their strategies and representations for the Tower Task solution. The three focus students participated in that session (along with a fourth student, Jeff). Data analysis indicated the forms of reasoning offered by the students along with the representations of their solutions, showing how the ideas of others contributed to individual reasoning. An outcome of this session was convincing each other of the solution using arguments by cases and induction. Lastly, in a fifth-grade whole class setting, the students worked on an application of the Tower Task, selecting from two colors. In this session, ideas were again shared among the three students, who displayed their understanding. An important outcome of this session was rebuilding and sharing the idea of an inductive argument to justify how and why the doubling relationship between shorter and taller towers worked. Development of student learning is described next.

During interviews (Sessions VI–VIII) the fourth-grade students worked to offer convincing arguments of their solutions of Tower Tasks of various heights. After the whole class discussion (Session IV), which took a case-based approach, it was evident that Michelle, Milin, and Stephanie took a partial case-based approach to explain to researchers of their reasoning. Each case created by Stephanie, Milin, and Michelle was reflected upon for its completion and for its comparison with other cases. The development of solution strategies that served as displays of justification was associated with researchers requesting for elaboration or justification of the methodology used to generate a set of towers, its completion, and its difference between other sets. As a result of prompting for justification, Michelle, Stephanie, and Milin noticed, removed, and continued to flag for duplicates. For example, when researchers prompted for justification about the choices of certain cases, Milin and Stephanie each found themselves creating duplicates (due to generating equivalent cases) and readjusted their methods to control for duplication (e.g., making sure tower patterns had both colors separated in some way so not to duplicate the cases with towers with adjacent same-colored cubes). Milin and

Stephanie searched for new ways to organize complex attributes of towers to convince the researchers that they found all towers and that they were all different. Specifically, they developed and used new strategies, such as elevator patterns, and modified or refined old strategies for local sets of towers, such as controlling for variables, to build their argument based on cases (or partial cases) in an attempt to convince the researchers of their solutions of Tower Tasks. Stephanie had developed a method to enumerate all cases of towers with the number of the same color cubes adjacent then separated from each other, using elevator patterns, to convince others of her solution to simpler Tower Tasks.

When asked to explain his opposite tower pairs, Milin developed a cases-based approach to finding the solution for a Tower Task which he displayed in a symmetrical pattern of cases and their opposites and applied his approach to shorter Tower Tasks (e.g., “falling into the same hands” was his reference to similar cases that existed in different Tower Tasks). While exploring the solutions of shorter Tower Tasks based on a method by cases (in Session VIII) Milin argued that the number of colors available made a difference in the number solutions. Milin’s inductive approach to show how taller towers grew from shorter towers was associated with the researcher asking how from the available colors (i.e., the 1-tall towers) could 2-tall towers be formed. Milin’s approach to convince others changed from a method by cases to a method by induction.

Attention to the presentation of a counterargument and counterexample offered by either a researcher or student was also associated with changes in student displays of justification. In Stephanie’s case, there were several instances in which she modified her claim due to a counterexample or counterargument. In Session I, R1 presented her a duplicate in the solution for the 4-tall Tower Task. This prompted Stephanie to reconsider the solution of 17 and check the new solution of 16 very carefully through several trials until each tower was checked for duplication. In Session VI, R2 presented her duplicate pairs in her solution of 32, reducing her solution to 30. This prompted Stephanie to reconsider her method by cases, when she acknowledged uncertainty in the existence of a complete, convincing argument. In Session VII, Stephanie showed a method by partial cases. As the cases became more complex and duplicates emerged, Stephanie again displayed uncertainty of the existence of a convincing argument. R2 prompted her to reconsider the case of towers with exactly two of a color separated by at least one of the other color, where she developed a way to control for one variable and vary the other until she exhausted all possibilities and justified using an argument by contradiction to show the completion of the case.

After Michelle was asked if the Tower Task reminded her of previous tasks she did with the researchers, an exploration of how the Outfit Task was related followed and early isomorphic connections developed, which aided her in solving the Tower Task another way. She drew shirts and shorts. When asked what they had to do with the towers, she used towers to represent the same outfits she drew. The next interview followed and built upon her idea about the relationship between outfits and towers. She used multiple representations and reasoning by analogy to solve Tower Tasks inductively up to 4-tall and predicted the outcomes for 5- and 6-tall towers.

The students' strategies and the backing to the claims that they provided formed their arguments. Their work can be explained, at least in part, by their personal constructions and the social interaction between and among students and researchers, challenged to convince one another of their solution or to elaborate on their reasoning. Specifically, Michelle, Milin, and Stephanie refined their supporting arguments, providing backing to their claims when questioned further. For example, in the second interview (Session VII), Stephanie provided justifications for finding all towers with exactly one of a color with an argument by contradiction, explaining that if she continued the elevator recursion to build more towers with exactly one of a color, she would violate the given height condition. She developed and refined the argument by contradiction for the cases of towers with an elevator recursion, providing backing to her argument when R2 asked for elaboration. In another example (Session IX), when challenged by Jeff, Milin and Stephanie presented a second iteration of their methods based on induction and cases, respectively, as a way to justify the relevance for the use of patterns and Michelle elaborated and demonstrated how Milin and Stephanie's arguments serve to convince someone of the solution to the 3-tall Tower Task. Specifically, Michelle heard the ideas of others, explained them in her words, and provided backing for Stephanie's and Milin's claims. After the "Gang of four" interview, during individual (Session XI and XII) and partner assessments (Session X) of the 3-tall Tower Task, the results show Jeff, Michelle, Milin, and Stephanie modifying their displays of justification. Stephanie (in Session X, XI, XII) and Michelle (in Session X and XI) solved 3-tall tower by a different version of a cases argument. Stephanie modified the cases to include towers with two of a color adjacent and separated together and Michelle, with Jeff, for the first time used a complete method by cases that incorporated ideas of opposite and elevator patterns. Milin (in Session XII) used an abbreviated method by induction, and Milin and Stephanie displayed the number solutions of Tower Tasks up to 10-tall to show a doubling pattern and to predict outcomes for taller Tower Tasks.

The following tables summarize each students' Tower activities over time with the findings of their associated representations, strategies, and forms of reasoning. Findings in a blue color represent the origins of recognition of doubling and development of an inductive argument for each student. The key for each table is provided below.

Key (see Appendix B for glossary of terminology):

M	Physical models
EL	Everyday Language ("family" = doubling strategy vs "staircases" = partial cases)
AL	Academic language (mathematics register)
D	Drawings
SG	Small group
(#)	Number solution to a task

Table 17.1. Summary of Stephanie's Tower activities from 1990 to 1992

Date	*10/11/90*	*10/12/90*	*2/6/92*	*2/6/92*	*2/7/92*
Session Type	Dyad	Whole Class	Dyad	Whole Class	1-on-1 Interview I
Task	4-tall (16)	3-tall (8) fewer than 4-tall (16)	5-tall (32)	5-tall (32)	5-tall (30)
Representations	M	M	M	EL	D; EL
Pattern recognition & Strategies	Guess & check Exhaust one blue	Opposite & inverse patterns Flag for duplicates	Composite operation Guess & check Flag for duplicates	Staircase, elevator, & opposite patterns Exhaustion of one and two red cases	Staircase, elevator, & opposite patterns Composite operations Flag for duplicates
Forms of Reasoning	Trial & Error results	4- to 3-tall creates duplicate pairs	Trial & Error results Solution uncertainty	Argument by contradiction for exactly 1 and 2R cases Spatial reasoning for 1R case	By partial cases By contradiction Estimate 4-tall (20) & 6-tall (50)

						Certainty for solution may be possible by cases

In blue – The origins of recognition of a doubling pattern and development of the inductive argument

Table 17.2. Summary of Stephanie's Tower activities from 1992 to 1993 (continued)

Date	*2/21/92*	*3/6/92*	*3/10/92*	*6/15/92*	*10/25/92*	*2/26/93*
Session Type	1-on-1 Interview II	1-on-1 Interview III	SG Assessment Interview	Dyad Summative Assessment	Individual Written Summative Assessment	Dyad and Small Group in Class
Task	6-tall (50)	1 to 10-tall (2, 4, 8, 16, 32, …1,024); Outfits	1 to 10-tall (2, 4, 8, 16, 32, …1,024)	1 to 10-tall (2, 4, 8, 16, 32, …1,024)	3-tall (8)	GMT
Representa tions	D; M; EL; AL	M; EL	D	D; M; EL; AL	D; AL	D; M; EL; AL
Strategies	Elevator & opposite Flag for equivalent cases Controlling for 2 variables	Elevator Flag for equivalent cases Opposite as check	Elevator	Elevator & opposite		Guess & check Generating towers inductively
Forms of Reasoning	By partial[7] cases By contradiction	By cases By contradiction Double pattern recognition Rule generalize Reasoning by analogy (Outfit	By cases By contradiction Solution certainty Rule generalize and application	By cases Solution certainty Rule application	Abbrev. argument by contradictio n By cases Solution certainty	
	Rule application				Rule application	

[7] Stephanie included all cases of towers with no, 1, 2, 3 blue adjacent, 3 blue cubes separated, 2 blue separated, and their opposite cases, and was given for home problem-solving to try the case of 3 blue separated by the odometer strategy

Inductive reasoning to build taller towers

Argument by induction for Doubling Rule

In blue – The origins of recognition of a doubling pattern and development of the inductive argument

Table 17.3. Summary of Milin's Tower activities from 1990 to 1992

Date	*10/12/90*	*2/6/92*	*2/6/92*	*2/7/92*	*2/21/92*
Session Type	Whole Class	Dyad in Class	Whole Class	1-on-1 Interview I	1-on-1 Interview II
Task	3-tall (8) fewer than 4-tall (16)	5-tall (32)	5-tall (32)	5-tall (32)	4-tall (16)
Representations	M	M	EL	M; EL	M; EL; AL
Pattern recognition & Strategies	Inverse & opposites patterns	Composite operation Guess & check	Staircase, elevator, & opposite patterns Exhaustion of 1R and 2R cases	Elevator & opposite Flags for duplicates	Elevator Composite operations
Forms of Reasoning	Prediction 3-tall (16)	Trial & Error Time factor	Opposites → Solution an even number	By partial Cases Estimate 4-tall (24)	By cases Problem-posing Comparing counting tasks

In blue – The origins of recognition of a doubling pattern and development of the inductive argument

Table 17.4. Summary of Milin's Tower activities from 1992 to 1993 (continued)

Date	*3/6/92*	*3/10/92*	*6/15/92*	*10/25/92*	*2/26/93*
Session Type	1-on-1 Interview III	SG Formative Assessment Interview	Dyad Summative Assessment	Individual Written Summative Assessment	Dyad and Small Group in Class
Task	1 to 6-tall (2, 4, 8, 16, 32, 50)	1 to 10-tall (2, 4, 8, 16, 32, …1,024)	1 to 10-tall (2, 4, 8, 16, 32, …1,024)	3-tall Sol: 8	GMT
Representations	M; EL	D	D; M; EL; AL	D; AL	D; M; EL; AL
Strategies	Generating towers inductively	Generating towers inductively	Elevator & opposite patterns	Generating towers inductively	Elevator & symmetry patterns

					Case organization
Forms of Reasoning	Inductive reasoning to build taller towers Double pattern recognition Estimate 6-tall by partial cases (50) vs. by induction (64)	Rule generalize Argument by induction for Doubling Rule and for 3-tall towers	By cases Rule generalize and application	Rule generalize and application Inductive argument for Doubling Rule	By partial[8] cases after Rule application Inductive reasoning to build taller towers Argument by induction for Doubling Rule

In blue – The origins of recognition of a doubling pattern and development of the inductive argument

Table 17.5. Summary of Michelle's Tower activities in 1992

Date	2/6/92	2/6/92	2/7/92	2/21/92
Session Type	Dyad	Whole Class	1-on-1 Interview I	1-on-1 Interview II
Task	5-tall (32)	5-tall (32)	5-tall (32) 1- to 3-tall & Outfits (2, 4, 8)	1- to 6-tall & Outfits (2, 4, 8, 16, 32, 64)
Representations	M	EL	M	D; M
Strategies	Staircase, elevator, & opposites Flag for duplicates	Staircase, elevator, & opposites Tower patterns generating duplicates Exhaustion of 1R and 2R cases	Staircase, elevator, & opposites Flag for duplicates	Generating outfits inductively using towers
Forms of Reasoning	Justification by Trial & Error results Every tower has an opposite		Estimate 4-tall (12, 16) Certainty for solution may be possible by cases	Double pattern recognition Reasoning by analogy (Outfits to Towers) Estimate 6-tall (64)

8 Milin included all cases of towers with no, 1, 2 red cubes, and their opposite cases, but he missed one tower with 3 red cubes separated and its opposite

In blue – The origins of recognition of a doubling pattern and development of the inductive argument

Table 17.6. Summary of Michelle's Tower activities from 1992 to 1993 (continued)

Date	*3/10/92*	*6/15/92*	*2/26/93*
Session Type	SG Formative Assessment Interview	Dyad Summative Assessment	Dyad and Small Group in Class
Task	1 to 10-tall (2, 4, 8, 16, 32, …1,024)	3-tall (8)	GMT
Representations	D; AL	D; M	D; M; EL; AL
Strategies	Building randomly and some by opposites to match Doubling Rule By cases	Elevator & opposites	GMT outcomes to build towers Generating towers inductively Rule application
Forms of Reasoning	Rule application and generalization By induction for Doubling Rule	By cases & opposites Rule application	Inductive reasoning to build taller towers Argument by induction for Doubling Rule

In blue – The origins of recognition of a doubling pattern and development of the inductive argument

Chapter 18 Discussion

18.1 Learning progressions through multiple lenses

The documentation of Michelle, Milin, and Stephanie's mathematical behavior, such as their utterances, gestures, manipulation of physical models, and inscriptions that supported their reasoning, formed the trace of their cognitive growth. These included ways in which these students represented their ideas over time that enabled a trace of their cognitive growth as they offered justifications for Tower Problem solutions with various proof-like arguments by cases, contradiction, and induction. Tracing students' cognitive growth revealed their building arguments, folding back to earlier ideas (Pirie & Kieran, 1992), and rebuilding new ideas based upon earlier ones. Students' cognitive journey in supporting solutions to problems led to proof-like justifications that emanated from local and global pattern recognition, use of recursive reasoning, and their reconciling data with equivalent cases and unreliable patterns.

In creating individual student learning progressions, video narratives were published on www.videomosaic.org/analytics, to be made available, open access, worldwide. Analyses of these data suggest that student learning pathways for individual learners are unique. Attention to the opportunities for students to work in a variety of settings led to socio-constructivist analyses of how students contributed individually to the learning and built on the ideas of others. Hence, student development of ideas depended not only on their particular growth, but also of the setting in which they worked in groups, with an opportunity to be challenged by others. The research setting called for opportunities for the students to revisit earlier solutions and provide new ideas for refining, modifying, or justifying earlier ideas. The setting also called for individual and group interviews where the students discussed solutions with an adult researcher and were further challenged to build on their existing solutions.. For each child, development towards creating "proof-like" justifications occurred unevenly as revealed in the learning progressions. Their movement to creating convincing arguments are shown in different task sequences. These show students with who began their problem solving with different pattern recognitions and different initial intuitive ideas later learning from each other with different takeaways after co-constructing solutions. The learning progressions trace how student original ideas are later refined and/or modified, influenced by input from others. Michelle, for example, partnered with Jeff on the 5-Tall Tower problem in which he created staircase and elevator patterns for generating towers. Follow-up interviews with Michelle captured Michelle's sharing of Jeff's patterns and the obstacle created with some used by Jeff to produced duplicate towers. Michelle made use of her knowledge of her solution to the Outfit Tasks in earlier grades and exploring her own conjecture that towers reminded her of outfits. Her recognition of the equivalent structure of a version of the Outfit Task and Tower Task served her well in not only recognizing an isomorphism but also for conjecturing a rule to find the numerical solutions to the Tower Tasks up to 6-tall, noticing a doubling relationship between Tower Tasks, and in building supporting physical tower models up to 4-tall.

In grade 4, in a classroom activity, the students were challenged to solve the 5-tall Tower Task after working on 4-tall in grade 3 and the class discussion where R2 made public different student's patterns of towers, introducing a case-based approach to justifying their solution to the problem. Nevertheless, student "take aways" from the "whole class" discussion differed as indicated in individual learning progressions. For example, during a post-interview Stephanie rebuilt the idea of recursively generating towers after folding back to earlier methods of opposites and inverses. The investigation of the completeness of a set of towers and the differences among towers within a case or between cases was associated with the development of solving Tower Tasks and reasoning by cases. Also, Milin, who immediately recalled the recursive patterns in interviews, investigated which cases to select, which cases were equivalent to each other, and how to deal with complex tower patterns. Eventually Milin came to realize that the taller the towers the more challenging it was to arrange the towers into patterns or to find missing towers. As a result of the need for a more efficient strategy, Milin came to recognize an inductive approach to generate taller towers. Stephanie also investigated the same issues of equivalent cases, associated with her taking a complete case-based approach without the use of opposites, and complex tower patterns, associated with the development of a controlling method to exhaust more complex cases of towers. We can infer vulnerability in inferences made about cognitive growth from a single event, session, or problem task.

The study of Stephanie's journey and how she developed her ideas required further study of the students who worked with her on several occasions to examine how their ideas interacted with her ideas (e.g., how and when she modified or refined her arguments by cases or by induction). Learning, did not occur in isolation. The multiple case studies provide insight into how the ideas of others influenced the learning of others and shed light on how students co-constructed together, questioning each other and in response to these questions, provided support for their reasoning. A social constructivist lense helped trace how students' ideas influenced each other. For example, Stephanie learned in an environment in which researchers asked her to “think-aloud” and explain her actions or reasoning behind those actions; she was influenced by those with whom she worked, by engaging in large group, whole class, and researcher discussions, as well as by working to achieve the goals of the tasks or to convince others of her ideas and to listen to their feedback, questions, and ideas, thus defining the learning environment within the longitudinal nature of the Rutgers Kenilworth study.

Another example of how the input of others contributed to Stephanie’s cognitive growth occurred as Stephanie recognized a need for a justification for the doubling pattern she noticed in Tower Task solutions. Stephanie expressed confidence in recognizing a doubling pattern; she expressed confidence that the doubling pattern was valid, backed by her earlier recognition from case arguments that there was a doubling pattern. The challenge to provide justification for from “why” there seemed to be a doubling pattern encouraged further exploration. In the literature of older mathematics students, Balacheff (1988) identified the conflict as naïve empiricism Stephanie, when was asked explicitly to explain why the pattern worked (with R2’s intervention), responded with an empirical argument (a existence claim; evidenced by the reasoning she offered), and, when another student, Matt, explained it to another group with Stephanie observing (social interaction), she had yet another opportunity to recognize the reasoning for the pattern, and shared the justification successfully to yet another group of students (evidenced by multiple instances of supportive reasoning). These experiences were learning opportunities for Stephanie to distinguish between an empirical argument and a justification for that argument. Multiple perspectives provided an account for how Stephanie and the other students developed their reasoning in a session and over time.

Simon (2012; p. 45) presented the following matrix to distinguish between different theoretical lenses for the study of an individual or a group.

Analyses by Nature and Subject of the Analysis

Cognitive analysis	Social analysis
Individual	
Cognitive analysis of individual	Social analysis of individual
Group	
Cognitive analysis of group	Social analysis of group

Figure 18.1.1. Simon's (2012) theoretical units of analysis of learning.

Each lense had its purpose for this research study. Simon (2012) asserted that when using *social* theoretical constructs in analysis of individuals (upper right quadrant) and *cognitive* theoretical constructs in analysis of groups (lower left quadrant) "maximized the constructs available for [*individual/group*] data collection and analysis" (p. 48). This study attests to the benefit of using multiple lenses in various settings. For example, in addition to a cognitive lens, a social lens was used in 1-on-1 interview data to account for learning through the interactions between researcher and student. In addition to a social analysis, a cognitive analysis was used in studying the small group and whole class data by using observable mathematical behaviors (e.g., language, representations, or heuristics used by participants). The multiple analyses provided depth into tracing the mathematical ideas and arguments that students brought to the conversation about a particular idea (e.g., Stephanie's argument by cases for the 3-tall Tower Task or Milin's argument by induction for the doubling pattern) and provided finer distinctions of each student's development in strategies and arguments that occurred in a particular session and in those later sessions. How the ideas of others contributed to a student's reasoning were not intitially visible. For example, evidence of the influence of the ideas of others came later. It sometimes appeared that children were not accepting or making use of an idea offered by another student at a particular moment in time (e.g., Dana's or Milin's inverse relationship; Jeff's insistence on putting the two blue adjacent and separated cases as one; Stephanie's 5-case organization for 3-tall towers). The longitudinal and detailed nature of this study showed that students may not have taken up the idea right away or in that session; however, the idea emerged later. This might be explained, at least in part, that when asked to express how they justified a solution, students interpreted this as "how did YOU solve it?:" When Stephanie was prompted to modify her five cases to include only one category of exactly two blue 3-tall towers, selecting from two colors, she may have appeared to reject the suggestions of Jeff, Milin, and Michelle. She justified her solution by explaining that she was reporting how she did it. A few weeks later, in a written summative assessment, Stephanie's case organization represented the suggestions of Jeff, Milin and Michelle, with a more elegant version of a case organization for the 3-tall towers. Moreover, although each student offered their own ideas in Session IX, the students were completing each other's sentences as they provided explanations. At times, the explanations showed that they were modifying their own ideas to include the ideas of others in the summative assessments (e.g., Jeff, Michelle, and Milin used a case-based approach). The examples suggest that children do listen to each other when it may appear they are not because they are giving their voice to how they are originally constructing their own knowledge and presenting their own creative ideas.

The Rutgers Kenilworth Longitudinal Study design, in which ideas were made public and there were opportunities to revisit tasks over years, offers some understanding as to how children may later take up the ideas of others. Michelle, who solved the 3-tall towers in a different way before, during, and after the "Gang of Four," was engaged in the review, evaluation, and articulation of others' ideas in several instances in two different school years. She pointed out warrants for claims and affirmed their validity in the small-group fourth- and fifth-grade sessions. Although the results show Michelle's absence of talk during the partner work with Jeff (in the fourth-grade dyad, whole class, and partner assessment sessions) one might consider that it was not a lack of engagement on the part of Michelle (she was busy finding duplicate towers created in Jeff's multiple patterns); nor can we infer that she was not taking up the ideas of others. However, analysis of data from later episodes with Michelle, suggests that Michelle and others, when asked to explain their own solutions, first display their own original thinking. See Chapter 15.1.1 for an example of Jeff, Michelle, and Stephanie's reactions to Milin's inductive argument. Later, revisiting tasks, there was evidence of listening when they took up the ideas of others.

18.2 The fragile nature of building foundational ideas

The three students in this study built and rebuilt foundational ideas on which more abstract, sophisticated, general ideas can later be built, This longitudinal multi-case study attests to the fragile nature of each student's emerging knowledge. Analyses show students folding back (Pirie & Kieran, 1992) to build upon already constructed ideas in order to move forward. In so doing, their knowledge increases. The design and sequence of the related counting tasks contributed to providing students the opportunity to build schema around the solving of counting tasks. Also, these tasks were shown to be appropriately challenging and later, appropriately modified; the environment for working on the task strand was supportive, creating a space for making ideas public and for building upon the ideas offered by students. Students revisited more challenging versions of the tasks by justifying tower solutions of taller heights. Note that the larger data set of the RKLS research evidence the rebuilding and deepening of mathematical knowledge for each student (e.g., in later sessions (e.g., Stephanie's interviews in eighth grade and later sessions with Stephanie and Michelle; Maher & Speiser, 1997; Maher et al., 2010).

18.3 Implications and limitations

This research offers a context in which a school mathematics enrichment program can offer their students (general education students, honors students, etc.) opportunities to solve and justify sequences of mathematics investigations and build foundational ideas for later more abstract ideas. For example, the condition of revisiting related tasks individually or in a group or to reflect upon one's work individually with a teacher allows each student to build ideas uniquely. The foundation is important for later building. For example, Teehan (2017) showed how Stephanie used towers as representations to build abstract combinatorial ideas and identify relationships to isomorphic problems. This research shows how Stephanie's early work of building a deep understanding of the solutions to early Tower Tasks provided a strong foundation on which to build later, more abstract, combinatorial ideas.

This study shows it is also possible for children to work together and to pay attention to the ideas of others. It serves as an example of how a formative assessment can inform student mathematical learning, in this case, of justifying solutions to tasks. There are implications for teacher education and teacher professional development in terms of some successful practices. For example, researchers encouraged students to record their ideas and to revisit the written ideas that were collected in prior sessions. For example, in the "Gang of Four" session R2 displayed Stephanie's written work to discuss the generalization of the doubling pattern and her shortcut idea. In another example, Milin revisited his recorded estimates of Tower Tasks to compare the solutions using a case-based approach to the "family" inductive approach, and then to explore why the numerical solutions doubled for consecutive height Tower Tasks by inductive reasoning.

The study of the development of mathematical ideas in the combinatorics strand occurred in a research setting, with "best practices" employed in studying how mathematical ideas and ways of reasoning develop in students.. The development of mathematical ideas in the combinatorics strand can vary according to individual student approaches, choice and sequence of tasks, and context for learning. For example, Torkildsen (2006) reported fourth graders successful solution of a counting task dealing with Taxicab Geometry. Teachers and researchers are encouraged to include the task with explorations with younger students. The mathematics strand accessible to the research students, under different conditions, may show that ideas and justifications develop differently, especially as teachers struggle with constraints such as curriculum requirements, time, and class size. The development of ideas in combinatorial reasoning may vary with task conditions. Although it was beyond the scope of the study, further research might study how analogical reasoning develops for isomorphic and non-isomorphic tasks. It may be of interest to explore whether the same or similar strategies, such as a method by cases or inductive reasoning emerge; do students fold back to non-systematic, non-enumerative methods when building up the representations of a new context or new task; or are approaches revealed with even greater complexity? In another mathematics strand and under different conditions, development of ideas may occur differently, especially as teachers try to implement with particular constraints.

The data used video-taped sessions of particular pairs, groups, and class discussions, and some learning may have been missed. For example, in third grade Milin participated in a whole class discussion, but his partner work was not captured because he was not in the cohort who had cameras facing their table. Detail about how he solved the 4-tall Tower Task the first time he was exposed to it may have offered insight into his earlier thinking.

The learning progressions reported in this study are specific to the development of ideas of one type of task across three children across two or more years of a research study. The sample size, a research classroom context, or the singular strand and task limitations also have their contributions.

For teachers who teach under the Common Core State Standards (CCSSO, 2010) or any other curriculum that promotes standards of mathematical practices, this report details how facilitators prompted children to make their ideas public (but without a mandatory requirement to do so), aided in elaboration, clarification, slowing down, and repetition of explanations, revoiced student thinking, introduced more precise language, requested ideas to be recorded, invited students to move to different groups or settings to hear someone's idea, and probed for justification. There is evidence that these interventions worked. Without being shown how to reason directly, students in a natural way developed their ways of articulating and supporting their arguments in response to these research norm interventions.

18.4 Future Work

Jeff's journey could be traced to understand more fully the origins of his questions in the "Gang of four" session. He was absent during some of the time between actual problem solving and task-based interviews. However, an analysis of his earlier problem solving with a partner may shed more light into his learning. This was the first case study about Michelle's early development of Tower Task ideas. Further study could trace Michelle's reasoning about related counting tasks in later grades, when she and Stephanie deal with isomorphic tasks in the eleventh grade (see Maher et al, 2010). Also, Michelle proved to be an important player in articulation of others' ideas. An interactionist analysis of engagement or learning is for further study.

Moreover, facilitators did not prescribe what tools students were to use. Unifix cubes were available, as well as paper and pencil. The findings show how students moved back and forth between multiple modes of representation to support their reasoning and how they folded back to re-construct some ideas. This research offers examples of how students "used tools strategically" (from the CCSS for Mathematical Practices), in various settings and conditions. For example, students were offered cubes in the Gang of Four session and chose not to use them to express their argument; instead, they used symbols, pictures and charts to show their justification. Multiple representations were encouraged in every lesson and further research could explore what it looked like when a student chose a unique way of representing an idea or justification of a problem solution. This study reports which representations were used in which instance and how that represented the mathematical ideas of the student. Further study could analyze in detail the complexity of the use of manipulatives in creating various representations that were generated by the students at different instances. Different settings also may have played a role in the way students represented their ideas. For example, what role might the provision of written justifications for solutions play in student responses?

Some areas of further study are teacher-research role in soliciting and playing attention to the ideas and reasoning of students and how they responded to and modified their moves accordingly. Multiple instances in this study illustrated associations of student growth in reasoning with a researcher move, such as asking a student to reflect on one's argument for the certainty of a solution (e.g., Stephanie in Sessions VI through IX). A research agenda might be based on multiple learning progressions of the development of mathematical practices, such as argumentation, using the models of the Rutgers Kenilworth Longitudinal Study, such as teacher-moves, to trace how these practices manifested, over time, across various mathematical strands.

The learning progressions and the supporting VMCAnalytics display the young students using various heuristics, forming arguments, expressing their reasoning, having to put their reasoning in writing, choosing their own representations, presenting their justification for a solution, and other critical practices that transcend mathematics education and are vital for adulthood. The students were randomly selected (from general education classes, not from a gifted class) for the longitudinal study from a school in a working-class community (see Maher et al., 2010). The takeaway is that children in schools can develop sophisticated mathematical practices without being directly taught how to, given some of the circumstances as described in this study.

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Appendix A: Timelines of Michelle, Milin, and Stephanie's Sessions and Tower Activities

Table 18.1 Session timeline of student participation by session number, setting, and date

Session	*Setting*	*Grade*	*Date*	*Stephanie*	*Milin*	*Michelle*
I	Dyad	3	10/11/90	X		
II	Whole class	3	10/12/90	X	X	
III	3rd-grade INT	3	10/12/90	X		
IV	Dyad	4	2/6/92	X	X	X
V	Whole class	4	2/6/92	X	X	X
VI	1st 4th-grade INT	4	2/7/92	X	X	X
VII	2nd 4th-grade INT	4	2/21/92	X	X	X
VIII	3rd 4th-grade INT	4	3/6/92	X	X	
IX	SG Assessment INT	4	3/10/92	X	X	X
X	Dyad WA	4	6/15/92	X	X	X
XI	I-WA	5	10/25/92	X	X	X
XII	Dyad & SG	5	2/26/93	X	X	X

Key	
INT	One-on-one interview
SG	Small group
WA	Written assessment
I	Individual

Table 18.2 Stephanie's Tower activities timeline from 1990–1993

Session	*Setting*	*Date*	*Task and Description*	*Grade*	*Participants*	*Length (min)*	*Transcript Supplemental Material Appendix*
I	Dyad	10/11/90	4-tall Tower & Explain what convinced you of your solution	3	Partner: Dana	30	B1
II	Class	10/12/90	Discuss 4-tall Tower solution & Predict if there is >, <, or the same number of 3-tall towers	3	R1; whole class	41	B2
III	INT	10/12/90	Explain reasoning for solutions to 3- & 4-tall Tower	3	R3	10	B3
IV	Dyad	2/6/92	5-tall Tower & justify solution	4	Partner: Dana	49	C1
V	Class	2/6/92	Discuss 5-tall Tower solution	4	R2; whole class	53	C4
VI	1st INT	2/7/92	Reconstruction of 5-tall Tower solution	4	R2	50	C5
VII	2nd INT	2/21/92	Reconstruction of 6-tall Tower solution	4	R2	50	C5
VIII	3rd INT	3/6/92	Reconstruction of 4-tall Tower solution	4	R2	85	C5
IX	SG Assessment INT	3/10/92	3-tall Tower & Doubling pattern Justification	4	R2; Milin, Michelle, Jeff	45	C8

X	Dyad WA	6/15/92	3-tall Tower	4	Partner: Milin	n/a	
XI	WA	10/25/92	3-tall Tower	5	Individual	n/a	
XII	Dyad; SG; Class	2/26/93	Guess My Tower; “Families” of towers from 1- to 5-tall	5	Partner: Matt	114	D1

Table 18.3 Michelle’s Towers activities timeline from 1992–1993

	Setting	*Date*	*Task and Description*	*Grade*	*Participants*	*Length (min)*	*Transcript Supplemental Material Appendix*
IV	Dyad	2/6/92	5-tall Tower & justify solution	4	Partner: Jeff	49	C2
V	Class	2/6/92	Discuss 5-tall Tower solution	4	R2; whole class	53	C4
VI	1st INT	2/7/92	Reconstruction of 5-tall Tower solution; Shirts & Pants	4	R2	20	C6
VII	2nd INT	2/21/92	1- through 6-tall; Outfits	4	R2	25	C6
IX	SG Assessment INT	3/10/92	3-tall Tower & Doubling pattern Justification	4	R2; Milin, Stephanie, Jeff	38	C8
X	Dyad WA	6/15/92	3-tall Tower	4	Partner: Jeff	30	C10
XI	WA	10/25/92	3-tall Tower	5	N/A	n/a	
XII	Dyad; SG; Class	2/26/93	Guess My Tower; “Families” of towers from 1- to 5-tall	5	Partner: Milin	120	D2

Table 18.4 Milin’s “Tower” activities timeline table from 1990–1993

	Setting	*Date*	*Task & description*	*Grade*	*Participants*	*Length (min)*	*Transcript Supplemental Material Appendix*
II	Class	10/12/90	Discuss 4-tall Tower solution & Predict if there is >, <, or the same number of 3-tall towers	3	R1; whole class	41	B2
IV	Class	2/6/92	5-tall Towers & Justify solution	4	Partner: Michael	49	C3
V	Class	2/6/92	Discuss 5-tall Tower solution	4	R2; whole class	53	C4
VI	1st INT	2/7/92	Reconstruction of 5-tall Tower solution	4	R1; R4	85	C7
VII	2nd INT	2/21/92	Extension problem with three colors	4	R1	33	C7
VIII	3rd INT	3/6/92	“Families” of towers from 1- to 5-tall	4	R1	38	C7
IX	SG Assessment INT	3/10/92	3-tall Tower & Doubling pattern Justification	4	R2; Michelle, Stephanie, Jeff	38	C8
X	Dyad WA	6/15/92	3-tall Tower	4	Partner: Stephanie	n/a	
XI	WA	10/25/92	3-tall Tower	5	Individual	n/a	

XII	Dyad; SG; Class	2/26/93	Guess My Tower; "Families" of towers from 1- to 5-tall	5	Partner: Michelle	120	D2

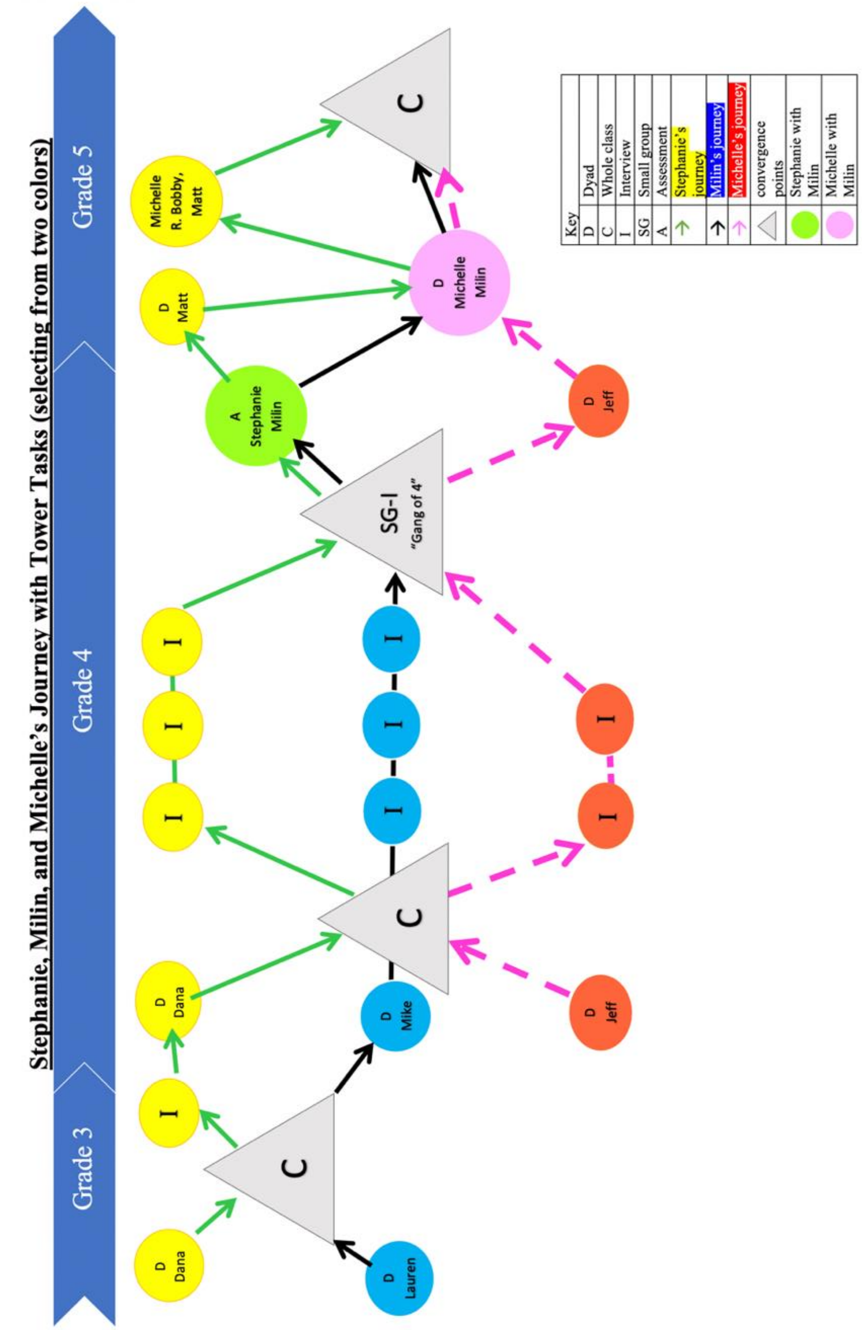


Figure 18.4.1. Michelle, Milin, and Stephanie's journey with Tower Tasks

Appendix B: Glossary of Terms

Duplicate

This occurs when there is an error of repetition in the possible combinations. For the case of towers, this would be two towers that have exactly the same order of colors from the first position to the last position.

Nonsystematic enumeration

This consists of trying to solve the problem by enumeration using a *guess and check* strategy, without any recursive procedure leading to the formation of all possibilities (Batanero et al., 1997).

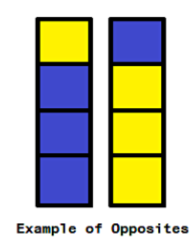
*Guess and Check**

The strategy of guess and check involves first guessing a possibility then checking that the possibility is applicable to the solution. Students can be observed using the guess and check method when building a tower pattern in a random order (or with no particular observable method) and then double-checking for duplicate towers (Maher & Martino, 1996). This occurred during the construction and generation of possibilities to obtain a solution.

Trial and Error

This strategy involves testing of a solution. It can involve checking if a combination is missing from a solution set or it can involve checking the correctness of the existing possibilities within a solution set. The *trial and error* strategy within the example of the Tower problems is: the trial can have two results: “error” or “no error.” In the situation of searching for missing combinations, “no error” occurs when the tower generated is a duplicate of a tower in the solution set. In the same situation, “error” occurs when a counterexample to the tested solution is found - the tower generated is a new tower pattern that did not formerly exist in the solution set. In the situation when checking for uniqueness or duplication of the combinations within a solution, “no error” occurs when each tower combination that is checked against the solution set is unique. In the same situation, “error” occurs when two pairs of combinations are duplicates and one is eliminated. In the latter case, this is a counterexample to the proposed solution.

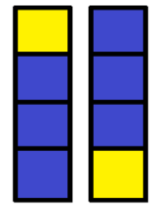
Strategies of locally exhaustive, systematic enumeration:



Color “Opposites” (children’s language)* - *Strategy of symmetry* (Janackova & Janacek, 2006)

Each element in a combination is replaced with the opposite element. The opposite of a tower in two colors is a tower of the same height where each position holds the opposite color of the cube in the corresponding position of the first tower. For example, a four-tall tower with yellow, blue, blue, blue and one with blue, yellow, yellow, yellow are opposites (Maher, Sran, & Yankelewitz, 2011).

Inverse towers (pairs) or "Cousins" (Stephanie's language) - Strategy of rotation (*Janackova & Janacek, 2006)

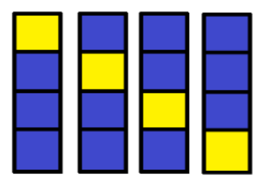


Example of Cousins

Two towers are said to be inverses of each other, inverse pairs, or "cousins" if one tower can be rotated vertically (180 degrees) to form the second tower. For example, a four-tall tower with yellow, blue, blue, blue and a tower with blue, blue, blue, yellow are cousins (Maher & Martino, 1996).

*Recursion**

Recursion is defined as an operation on one or more preceding elements according to a rule or formula involving a finite number of steps (Merriam-Webster, 2015). In this study a recursive technique includes generating a tower by building upon a preceding tower (or set of towers) by varying the cubes to one position lower or higher than the preceding tower, meanwhile still keeping some element of the case of towers fixed. An example of recursion (or generating combinations *recursively*) in terms of the Towers Task can be seen when one tower is generated from another to fulfil a rule (the pattern of towers as a whole). Generating each tower by an elevator or staircase pattern are examples of recursion that follow a particular rule (see strategies of parallelism, constant beginning, odometer for various examples of recursion).



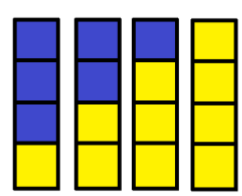
Example of the elevator pattern

The elevator pattern is used when finding all possible towers containing one cube of one color and the remaining cubes of the other color. The single colored cube is placed in the first position of the first tower. To create a second tower, the cube is then moved to the second position. The cube is continuously lowered one position to create new towers until it is placed in the final position (Maher, Sran & Yankelewitz, 2011).

*Strategy of a constant beginning (*Janackova & Janacek, 2006)

The symbols in the beginning positions remain identical up to the highest possible position, such that if the symbol in the highest position is not changed, the combination would be repeated. Staircase is an example of the strategy of a constant beginning.

"Staircase" (Ankur's Language)*



Example of the staircase pattern

The staircase pattern is named as such due to its resemblance to a staircase. In towers of two colors, the first tower begins with the first three positions as the same color followed by the 2nd color in the last position. In each new tower, the number of cubes of the 2nd color increases from the bottom by one cube until the final tower is a solid tower of that color (Maher, Sran & Yankelewitz, 2011).

*Group strategy (*Janackova & Janacek, 2006)

A preceding subset of combinations (with two elements at least) is used as a model for creating new combinations using some of the presented strategies. This subset becomes a model group. For example, when exhausting cases of towers, the case of two of a color adjacent becomes a model group for generating new combinations for three, four, five, etc. of a color adjacent using similar strategies (e.g., by parallelism or other forms of recursion) and having similar attributes (e.g., adjacency).

*Strategy of the odometer (English, 1991; 1993) – Controlling for variables**

W	W	W	W						
				W	W	W			
W							W	W	
	W			W					W
		W			W		W		
			W			W		W	W

Controlling for variables is a method in which one variable is held constant in position x while the other variable progressively occupies all remaining positions from the highest to the lowest or the lowest to the highest (but not both) without repeating previously discovered combinations (English, 1993; Janackova & Janacek, 2006). After exhausting all possibilities, next position x+1 is chosen for the constant variable and the process repeats (Janackova & Janacek, 2006). This strategy ends when all possibilities for the choice of the constant variable are exhausted. An example of this when building towers is when one color of the tower is held constant in one position while the color arrangements in all other positions are varied recursively (Maher & Martino, 1996).

Strategy of doubling combinations based on symmetry

The total count of tower combinations in a set/case is doubled to account for the number of color opposite towers (see Symmetry strategy). This strategy can be used either to enumerate the imaginary combinations or to guide generation of those combinations physically (see Group strategy).

Equivalent cases

"two Blues stuck together" "three reds stuck together"

The case when selecting m of a ***particular*** color to place into n positions (nCm) is equivalent to the case when selecting n–m of the ***opposite*** color into n positions (nCn–m). In the example for Towers combinations, *selecting two blues to place into five* available positions is equivalent to the case of *selecting three reds to place into five* available positions. The combinations with the attribute of exactly two of a color in towers five high and the combinations with the attribute of exactly three of the opposite color result in duplicate towers (e.g., RRBBB have both two reds and three blues). In the figure the first and last tower of each set are duplicates and occur when the strategy of (color opposite) symmetry is applied for each case of adjacent blues and then exhausting all adjacent blue cases, thereby repeating combinations. To avoid repetition, the strategies *in combination* must be taken with caution.

Strategies of and arguments for globally exhaustive systematic enumeration (Batanero et al., 1997):

*Case organization and/or argument**

In an organization and/or argument by cases, a statement is demonstrated by showing all of the smaller subsets of statements that make up the whole. For example, the solution to the 3-tall Tower Task when selecting from two colors (i.e. blue and yellow) can be justified by separating the towers into cases using a characteristic of the tower. One such characteristic is the number of cubes of a specific color that the towers contain. In this situation, the towers can be broken down into four cases: 1) towers containing no red (towers with a single color); 2) three towers containing one red (towers with exactly one of color); 3) three towers containing 2 red (within cases (2) and (3) can be cubes of same color adjacent to or separated from each other); 4) one tower containing 3 red or all red. An argument by cases would include an exhaustive enumeration of the total number of towers in each case.

*Argument by Contradiction**

When a situation arises that is inconsistent or contrary to known or inherent facts/assumptions, a contradiction has been reached. In the 4-tall Tower Task when selecting from two colors (i.e. yellow and blue), an argument by contradiction can be used to prove the total number of towers that can be built in the case of exactly one yellow cube. The yellow cube can be placed in either first, second, third, or fourth position. If other towers can be built with one yellow cube, the yellow cube would have to be in a different position, say, the fifth position or below the first (zeroth) position. Placing a cube in the fifth or zeroth position would require the tower to be a height of at least five. This is a contradiction of the requirement that the tower has a height four (Maher & Martino, 1996).

*Argument by Induction**

The general solution to the Towers Task, 2^n where *2* represents the number of colors available (or the total number of objects to choose from) and *n* represents the height of the tower (or the total positions available for a tower) can be justified by an inductive argument. An argument by induction involves three main steps.

The first step is to establish that the result is true for a basic case (usually $n = 0$ or $n = 1$). In the case of towers, we demonstrate the basis case n=1 or towers of one cube in height. Since there are only two cubes from which to select, (e.g., yellow or blue), there are only two towers that can be built. Thus, the requirement is demonstrated for the case of $n = 1$.

In the second step, we make an induction hypothesis in which we assume the result is true for $n=k$. Therefore, we assume the total number of different towers of height k would be 2^k. In the third step, we use this assumption to prove the next case $n = k+1$. The total number of towers that are $k + 1$ tall can be found by adding a cube to the top of all of the 2^k number of towers that are *k* tall. That additional cube can take on one of the two colors, i.e. yellow or blue. Therefore, for each of the existing 2^k towers, two new towers of height k+1 can be created; one with a yellow cube added to the top and one with a blue cube added to the top. Therefore, the total number of towers that can be created of height *k+1* is

$2^k \times 2 = 2^k \times 2^1 = 2^{k+1}$.

Thus, the argument is justified for the case of $n = k+1$.

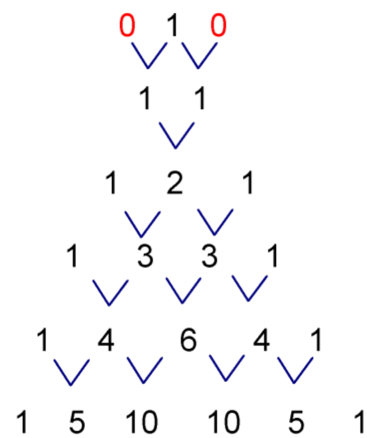
Identifying inductive reasoning for this study occurred when a student demonstrated the establishment of the following: (1) Building from the two, 1-tall towers (n=1), selecting from two colors, and then placing one yellow cube and one blue cube on the top of each, to generate two, 2-tall towers, and (2) demonstrating a continuation of the recursive pattern for generating taller towers from towers that are one cube shorter. Some students built from 2-or 3-tall towers, initially.

*Doubling rule**

The total number of different tower combinations of height *k* would be double the total number of tower combinations of height *k–1*.

*Pascal's Triangle**

Pascal's Triangle is a triangle of numbers in which the first row begins with the number 1. Each entry of the row that follows is defined by the sum of the two numbers above it. The diagram to the left will be used to explain Pascal's triangle in more detail. The first row again is defined as 1. The first entry of the 2nd row can be found by adding the number above and to the left of the new entry to the number above and to the right of the new entry. Since there is no number above and to the left, we consider it 0. Therefore, the first entry of the second row becomes 1. Similarly, the second entry becomes 1 + 0, which is 1. The first entry of the third row can be found the same way 0 (above and to the left) + 1 (above and to the right). The second entry of the third row is found by adding 1 (above and to the left) + 1 (above and to the right) and so on. The first six rows of Pascal's triangle are shown to the right.



Complete Argument

A provided idea that is logical or mathematically-sound using the Toulmin model (1958) to determine the components of the argument. A complete argument would include data, a conclusion, a warrant and, in some cases, a backing (see Van Ness, 2017). The researcher of the study may make this claim about work provided by students.

Abbreviated argument

A provided idea that is logical or mathematically-sound; however, a component of the Toulmin model, such as the warrant, is missing. This evaluation is given when any argument has been used by the student before and the student uses an abbreviated version to refer to the argument, as well as one of the participants, such as a researcher, takes it as shared.

Invalid Argument
A provided idea was not logical or mathematically-sound. When one of the components of Toulmin model of an argument do not logically or mathematically follow from the other, it will be determined that the argument is invalid. The researcher of this study may make this claim about work provided by students.

Undetailed Description
Not enough information is provided to determine whether or not the argument is convincing. The researcher of this study may make this claim about work provided by students or other participants when the components of the Toulmin model are missing to make a conclusion.

Taken-as-shared Heuristic/Strategy/Argument
Based on observable evidence, this is a claim by the researcher of this study that students recognize a particular Heuristic/Strategy/Argument provided by another student or researcher.

Not taken-as-shared Heuristic/Strategy/Argument
Based on observable evidence, this is a claim that students did not recognize a particular heuristic, strategy, or argument provided by another student or researcher.

*terminology adapted from McGowan (2016) and Cipriani (2017).

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Appendix C: VMCAnalytics information

1. Analytics on Stephanie's Development of Reasoning by Cases to Solve Tower Tasks: Grades 3, 4, & 5

Event #	*Date; Session Type; Task*	*Participants present with Stephanie*	*Event title*	*Title of original raw video/clip*	*Start-End Time & Transcript L#*
			PART 1 OF 3		
1.	10/11/1990 Dyad 4-tall	Dana	Every new tower Dana made, Stephanie compared against existing	Towers with Stephanie and Dana, Clip 2 of 5: Finding seventeen towers and checking for duplicates	00:00:08 - 00:01:08 L1-6
2.	10/11/1990 Dyad 4-tall	Dana; R1	The process of developing certainty of a solution	same as event 2	03:16 – 05:41 L17–8
3.	10/12/1992 Stephanie interview with R3	R3	Stephanie explains how double-checking provided certainty for the solution	Stephanie Grade 3 Towers interview excerpts	00:51 – 01:05 L5 – 6
4.	2/6/1992 Dyad 5-tall	Dana	Using the idea of the inverse relationship to find missing tower pairs	B59,Stephanie and Dana-Class work of the 5 tall-Tower Task (Work view)	00:19:28 - 00:19:44 L:155-178
5.	2/6/1992 Whole class 5-tall	Class; R2	The solution of the number of towers has a limit but gaining certainty is not possible	same as event	01:08:56 - 01:11:05 L: 628-42
6.	2/6/1992 Whole class 5-tall	Class; R2	Direct reasoning for the justification of ten towers with two red cubes "together" and "separated" by at least one yellow	same as event	1:32:20 – 1:33:10 L: 339-395
7.	2/6/1992 Whole class 5-tall	Class; R2	Imagining towers by using the color opposite and elevator strategy	same as event	01:36:15 - 01:36:58 L: 409-429
8.	2/7/1992 1st interview 5-tall	R2	Argument in support of the uniqueness of the two together case	B61, Stephanie revisits the 5-tall Tower Task (work view), Grade 4, February 6, 1992, raw footage	00:08:10 - 00:10:24 L49-87
9.		R2	A critical event of the origins of the case of exactly one of a particular color in every "spot"	same as event	20:01- 23:59 L120-80

10		R2	How to make the solution convincing	same as event	00:40:12 - 00:42:27 L307 – 45
			PART 2 OF 3		
1.	2/21/1992 2nd 1-on-1 interview	R2	Argument and reasoning for the case of exactly two of a color together	B62, Stephanie's and Milin's second of three interview sessions and Michelle's second of two interview sessions revisiting 5-tall Towers and other heights (work view), Grade 4, Feb 21, 1992, raw footage	00:03:43 - 00:05:40 L43-75
2.	2/21/1992 2nd 1-on-1 interview	R2	Exactly four of a color is equivalent to exactly two of a color and exactly five of a color together is equivalent of exactly one of a color	same as event	05:40 – 09:44 L47-145
3.	2/21/1992 2nd 1-on-1 interview	R2	Discussion of the case of exactly two blue separated with one blue cube fixed at the top position	same as event	13:19 – 14:44 L195–208
4.	2/21/1992 2nd 1-on-1 interview	R2	Controlling for one color while varying the other in the case of exactly two of a color separated by at least one of the other color and arguments by contradiction	same as event	00:35:18 - 00:37:56 L406–88
5.	2/21/1992 2nd 1-on-1 interview	R2	Reflecting on the organization as an argument for the two blue separated case	same as event	40:00 – 41:43 L506-23
6.	3/6/1992 3rd 1-on-1 interview	R2	Case of exactly one of a color in the 4-tall Tower task	B64, Stephanie third of three interview sessions when she used a case-based method for all heights below and including 4-tall Tower Tasks (work view), Grade 4, March 6, 1992, raw footage	03:20 – 04:29 L: 12 – 21

7.	3/6/1992	R2	Cases of exactly two, three, and four of a color together	same as event	08:21 – 08:44 L: 39 – 45
8.	3/6/1992	R2	Case of exactly two white cubes separated by at least one black cube	same as event	21:45 – 24:15 L1:02-111
9.	3/6/1992 3rd 1-on-1 interview	R2	Case of exactly three white separated by a black cube & Cases of exactly four white cubes and exactly no white cubes	same as event	25:22 – 28:25 L:116-36
10	3/6/1992 3rd 1-on-1 interview	R2	The relationship between the case method and the color opposite towers	same as event	28:50 – 32:40 L:150-182 OR 204
11	3/6/1992 3rd 1-on-1 interview	R2	Solving by cases focusing on the black cubes	same as event	37:22 – 40:38 L: 199 or 208 – 225
			PART 3 OF 3		
1.	3/10/1992 Small Group Assessment Interview	Milin, Michelle, Jeff, R2	Stephanie solving the 3-tall Tower task by a case-based methodology	B41, The Gang of Four (Jeff and Stephanie view), Grade 4, March 10, 1992, raw footage	17:30 – 19:49 L: 241 – 256
2.	3/10/1992	Milin, Michelle, Jeff, R2	Exactly 2 blue 'stuck together'		19:54-21:27
3.	3/10/1992	Milin, Michelle, Jeff, R2	Exactly 3 blue and exactly 2 blue separated		21:27-22:41
4.	3/10/1992	Milin, Michelle, Jeff, R2	Repeats her cases argument to Jeff		22:40-26:10 L: 256-
5.	6/15/1992 Dyad Assessment.	Milin	Two black and two green	B75, Combinatorics, Towers Assessment, WV, Steph Dana, Grade 3, 1992-06-15, Raw	01:25:52-01:26:27 L: 38-64

2. Analytics on Stephanie's Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Grades 3, 4, & 5

Event #	*Date; Session Type; Task*	*Participants present with Stephanie*	*Event title*	*Title of original raw video/clip*	*Start-End Time & Transcript L#*
			PART 1 OF 3: Grade 3–4		
1.	10/12/1990 Dyad 3-tall	Dana	Predictions for the 3-tall Tower Task	Towers Group Sharing, Clip 3 of 6: Guessing how many towers can be built	02:08 – 02:35 L: 10–12

				three cubes high, continued	
2.	10/12/1990 Whole class 3-tall	Class	Sharing their explanations for why towers 3-tall towers built from 4-tall towers are fewer	Towers Group Sharing, Clip 6 of 6:	00:42 – 01:18 L: 6–12
3.	10/12/1990 Interview	R3	Stephanie explains why there are fewer 3-tall towers than 4-tall towers with imaginary towers	Stephanie 3rd Grade Towers interview excerpts	01:57 –02:16 L: 15–16
4.	3/6/1992 3rd interview 4, 3, 2, 1-tall	R2	Stephanie's discovery of the doubling pattern after building towers 1, 2, 3, 4, and 5-tall	B64, Stephanie (4th grade) 3rd of 3 interview sessions when she used a case-based method for all heights below and including 4-tall Tower Tasks	55:26 – 57:00 L: 341-380
5.	3/6/1992 1 and 2-tall	R2	R2 introduces the inductive argument for towers 1 and 2-tall	Same as event 4	57:01 – 01:00:05 L: 362-383
6.	3/6/1992 Shirts & Pants	R2	A relationship between the solutions of the Shirts and Pants tasks and the Towers tasks	Same as event 4	1:16:06 – 1:18:19 L:
7.	3/6/1992 10-tall	R2	Using the doubling method to predict the number of towers 10-tall	Same as event 4	1:20:20 – 1:23:52 L: 496–538
			PART 2 OF 3: Grade 4		
8.	3/10/1992 Group Assessment Interview	Milin, Michelle, Jeff, R2	Stephanie claims a doubling pattern exists with Towers	B41, The Gang of Four (Jeff and Stephanie view), 4th Grade	04:30 –05:25 L: 68–79
9.	3/10/1992 Generalizing for Towers Tasks	Milin, Michelle, Jeff, R2	Stephanie discusses the theoretical method of obtaining the number of towers of any height	Same as event 8	27:17 – 28:08 L: 398–571
10	3/10/1992 10-tall	Milin, Michelle, Jeff, R2	Stephanie discusses a shortcut method for obtaining the number of towers of any height	Same as event 8	28:24 – 30:49 L: 448–481
11	6/15/1992 Dyad Assessment	Milin	Stephanie and Milin using the doubling rule to	B75, Combinatorics, Towers Assessment, WV, Steph-Milin,	1:40:50 – 1:43:29

	3-tall		solve the 3-tall Towers task	Grade 3, 1992-06-15, Raw	Or 1:38:18-1:40:50 L:
			PART 3 OF 3: Grade 5		
12	2/26/1993 Dyad GMT 2 to 3-tall	Matt; R3	Recalling and Testing the conjectured doubling rule.	Building Towers, Selecting from two colors for Guess My Tower (5th grade), Clip 2 of 5: Does the Number Double?	00:44 – 01:55 L: 6– 28
13	2/26/1993 4-tall	Matt; R3	Questioning how the doubling rule works	Same as event 12	02:29–03:43 L: 28–58
14	2/26/1993 1 to 4-tall	Michelle, Milin, Matt, R2	Michelle presents to Stephanie and Matt how and why the doubling pattern work	Building Towers, Selecting from two colors for Guess My Tower, Clip 3 of 5: Milin introduces an inductive argument	09:17 –10:17 L: 594–601
15	2/26/1993 1 to 2-tall	Matt; Michelle R.; Bobby; R2	Stephanie struggles to explain the connection between the rule and why it worked for Tower tasks	Building Towers, Selecting from two colors for Guess My Tower, Clip 4 of 5: Stephanie and Matt Rebuild the Argument	01:04 – 03:25 L:2–29
16	2/26/1993 1 to 3-tall	Matt; Michelle R.; Bobby; R2	Matt helps Stephanie explain the doubling pattern "family tree"	Same as event 15	00:03:25 - 00:04:09 L:
17	2/26/1993 3 to 4-tall	Matt; Michelle R.; Bobby; R2	Stephanie and Matt jointly present the inductive argument to the group	Same as event 15	00:06:19 - 00:07:31 L:
18	2/26/1993 1 to 4-tall	Class	Stephanie shares the tree pattern with the class	Building Towers, Selecting from two colors for Guess My Tower, Clip 5 of 5: Sharing with the Group	00:01 – 00:58 L: 1–9

3. Analytics on Milin's Development of Reasoning by Cases to Solve Tower Tasks: Grades 4 & 5

Event #	*Date; Session Type; Task*	*Participants present with Milin*	*Event title*	*Title of original raw video/clip*	*Start-End Time & Transcript L#*
			PART 1 OF 2: Grade 4		
1.	2/6/1992 Dyad 5-tall	Michael; R3	Explaining meaning of opposites and justifying how they are all different	B60, Milin and Michael classwork of the 5-tall Tower Task (work view), Grade 4, Feb 6, 1992, raw footage	21:20–22:37 L: 220–236

2.	2/6/1992	Michael; R1	Time and duplicate argument	B60	33:10–35:11 L: 320–350
3.	2/6/1992 Whole class 5-tall	Class; R2	Justification for an even number solution to the 5-tall Tower Task	B74, Combinatorics,T5T, Stephanie-Dana,(People view),Grade 4,Feb 6,1992,Raw Footage	47:35 – 48:05 L: 25 - 29
4.	2/7/1992 1st 1-on-1 interview 5-tall	R1	Generating exactly one of a color towers in a staircase pattern and arguments	B76, Milin's first of three interviews with researcher Alston on the 5-tall Tower Task (Work view), Grade 4, February 7, 1992, Raw footage	39:26 – 42:09 L: 43-96
5.	2/7/1992 5-tall	R1	Exactly two of a color cubes adjacent and arguments	B76	45:20 – 47:42 L: 110-137
6.	2/7/1992 5-tall	R1	Recognizing equivalence between the cases of towers with three cubes of a color and with two cubes of the other color	B76	50:27 – 52:23 L: 185-211
7.	2/7/1992 5-tall	R1	Considering that other towers with three of a color exist	B76	54:02 – 55:51 L: 228-251
8.	2/7/1992 5-tall	R1	Noticing relationship of cubes of one color in relation to the cubes of the other color	B76	57:51 – 01:02:20 L: 295 – 337
9.	2/7/1992 5-tall	R1	Arranging and looking for patterns within the case of three cubes with some separation	B76	1:06:29 – 1:09:47 L: 387 – 422
			PART 2 of 2: Grade 4		
1.	2/7/1992 4-tall	R1	Applying the strategy by cases of taller towers to find 4-tall Tower	B76	01:25:53 – 01:28:04 L: 651-678
2.	2/21/1992 2nd 1-on-1 interview 4-tall	R1	Milin groups towers by color opposites and elevator patterns	B62, Stephanie's and Milin's second of three interview sessions and Michelle's second of two interview sessions revisiting 5-tall Towers and other heights (4th grade	01:14:01 – 1:15:35 L: 11-38
3.	2/21/1992 4-tall	R1	Milin review the cases using more precise language	B62	01:18:00 – 1:19:45 L: 73-98

4.	2/21/1992 4-tall	R1	Milin considers the case of “two of a color together” in an elevator pattern	B62	01:19:47 – 01:21:09 L: 100-120
5.	2/21/1992 2 & 1-tall	R1	Milin discusses and relates towers 2-tall to towers 4- tall	B62	01:21:20 – 01:22:55 L: 129-148
6.	2/21/1992 3-tall	R1	Milin uses the color opposite strategy to build towers 3-tall	B62	1:23:33 – 1:25:42 L: 153-187

4. Analytics on Milin’s Development of Reasoning by an Inductive Argument to Solve Tower Tasks: Grades 4 & 5

Event #	*Date; Session Type; Task*	*Participants present with Milin*	*Event title*	*Title of original raw video/clip*	*Start-End Time & Transcript L#*
			PART 1 OF 2: Grade 4		
1.	3/6/1992 3rd 1-on-1 interview 1 to 2-tall	R1	Milin generates towers 2-tall from towers 1-tall	B66,Combinatorics, Steph & Milin interviews, 4th grade, 5-tall, March 6,1992,Workview, Raw	00:47 – 04:00 L: 29-77
2.	3/6/1992 2 to 3-tall	R1	Milin generates towers 3-tall from 2-tall	B66	04:28 – 07:47 L: 100-164
3.	3/6/1992 3 to 4-tall	R1	Predicting the number of towers 4-tall from towers 3-tall.	B66	08:00 – 09:49 L: 170-203
4.	3/6/1992 6-tall	R1	Milin conjectures that the doubling pattern breaks at towers 6-tall	B66	17:13 – 18:22 L: 187-201
5.	3/6/1992 4-tall	R1; R2	Milin doubts his staircase strategy	B66	32:36 – 33:36 L: 570-586
6.	3/6/1992 6-tall Towers	R1	Predicting that the solution to the 6-tall Tower Task is fifty, using cases, or sixty-four, using the “family” strategy	B66	34:39 – 35:45 L: 610-636
			PART 2 OF 2: Grades 4 - 5		
1.	3/10/1992 Small-group Assessment Interview 2- to 3-tall	Stephanie, Michelle, Jeff; R2	Milin presents a version of an argument by induction in support of the doubling pattern.	The Gang of Four (Michelle & Milin view), Grade 4, March, 10, 1992, raw footage	03:14 – 04:01 L: 299-332

2.	3/10/1992 From 2-tall to 3-tall	Stephanie, Michelle, Jeff; R2	Providing a further support for why the pattern is doubling pattern.	Gang of Four	04:21 – 05:48 L: 43-47
3.	3/10/1992 From 2-tall to 3-tall	Stephanie, Michelle, Jeff; R2	Milin provides a complete version of an argument by induction	Gang of Four	08:00 – 08:25 L: 80-88
4.	6/15/1992 Dyad Assessment 3-tall.	Stephanie	Milin and Stephanie solve the 3-tall Towers Assessment by cases initially	B75, Combinatorics, Towers Assessment, WV, Stephanie & Dana, Grade 3,1992-06-15,Raw	1:26:15-01:27:44 L: 39-52
5.	6/15/1992 Dyad Assessment 3-tall	Stephanie	Applying the doubling rule on a 3-tall assessment with Stephanie	Same as event 4	1:41:50 – 1:44:11 L:
6.	2/26/1993 GMT 1 to 2-tall	Michelle; R2	Milin explaining his inductive reasoning to Michelle	Building Towers, Selecting from two colors for Guess My Tower, Clip 3 of 5: Milin introduces an inductive argument.	0:00-2:24 L:
7.	2/26/1993 2 to 3-tall	Michelle; R2	Milin summarizes the doubling strategy from 2-tall to 3-tall towers and supports it with inductive reasoning	GMT Clip 3 of 5	4:11-6:27 L:

5. Analytics on Michelle's Longitudinal Problem Solving and Development of Reasoning About Tower Tasks: Grades 4 & 5

Event #	*Date; Session Type; Task*	*Participants present with Michelle*	*Event title*	*Title of original raw video/clip*	*Start-End Time & Transcript L#*
			Part 1 of 3: Grade 4		
1.	2/6/1992 Partner work 5-tall	Jeff	Michelle, working with Jeff in class on 5-tall Towers on February 6, 1992	B65, Jeff & Michelle class work on 5-tall Towers (WV), Grade 4, Feb 6, 1992, raw	00:07:27 - 00:08:47 L: 110–124
2.	2/6/1992 5-tall	Jeff R3	Researcher Martino asks how they generate towers	same as event 1	00:12:35–00:13:23 L: 175–185
3.	2/7/1992 1st 1-on-1 Interview 5-tall	R2	Michelle's elevator and color opposite strategies for 5-tall Towers on February 7, 1992 in her first one-on-one interview	B74, Combinatorics: Towers, work view, Grade 4, February 7, 1992, Raw footage. Retrieved from:	1:59-4:44 L: 19-46
4.	2/7/1992 Shirts & Pants	R2 R1	Making a connection to the Shirts and Pants Task	same as event 3	15:46–19:06

					L: 149-180
5.	2/21/1992 2nd 1-on-1 Interview Shirts & Pants	R2	Relating 2-tall towers to outfits with **two** articles of clothing selecting from two colors on February 21, 1992 in her second one-on-one interview	B62, Michelle's 2 of 2 interviews revisiting Towers (WV), Grade 4, Feb 21, 1992, raw	59:25–1:02:04 L: 17-34
6.	2/21/1992 Shirts, Pants, Hats	R2	Relating 3-tall towers to outfits with **three** articles of clothing selecting from two colors	same as event 5	1:02:10 – 1:05:12 L: 35-59
7.	2/21/1992 Shirts, Pants, Hats, Feathers	R2	Relating 4-tall towers to outfits with **four** articles of clothing selecting from two colors	same as event 5	1:05:12 – 1:06:30 L: 59-80
8.	2/21/1992 Outfit Task	R2	Relating the outfit combinations to the Towers Task	same as event 5	1:07:10 – 1:10:18 L: 81-124
			Part 2 of 3: Grade 4		
9.	3/10/1992 Small-Group Assessment 1 to 5-tall	R2 Stephanie Milin Jeff	Fourth grader Michelle's initial prediction of the pattern for the solutions to Towers tasks on March 10, 1992 in a group interview	B42, The Gang of Four (Michelle View), Grade 4, Mar 10, 1992, raw	01:31–03:27 L: 10-25
10.	3/10/1992 2 to 3-tall	R2 Stephanie Milin; Jeff	Relevance for the use of patterns	same as event 9	22:40–24:14 L: 306-324
11.	3/10/1992 3-tall	R2, Stephanie, Milin; Jeff	Reasoning by cases	same as event 9	24:30–26:10 L:331-367
12.	3/10/1992 4-tall	R2 Stephanie Milin; Jeff	Michelle presented a solution for the 4-tall Towers task applying Milin's inductive reasoning	same as event 9	26:21–27:20 L:368-399
13.	6/15/1992 Partner Assessment 3-tall	Jeff	Fourth grader's Michelle & Jeff's solution of the 3-tall Towers Assessment on June 15, 1992 and arrangement by elevator patterns	B75, Towers Assessment, WV, Grade 4, Jun 15, 1992, raw	10:33–12:13 L: 20–30
14.	6/15/1992 3-tall	Jeff	Michelle and Jeff's joint written assessment	Same as event 13	29:25–30:47 N/A
			Part 3 of 3: Grade 5		
15.	2/26/1993 Partner work 1 to 2-tall GMT	Milin R2	Milin explains to Michelle and Researcher Maher the doubling strategy using inductive reasoning from 1-tall to 2-tall towers to find the tower outcomes for the task, Guess My Tower (GMT), on February 26, 1993	Building Towers, Selecting from two colors for Guess My Tower, Clip 3 of 5: Milin introduces an inductive argument	01:43 – 03:30 L: 497–522

16.	2/26/1993 2 to 3-tall	Milin R2	Michelle joins in building towers from 2-tall to 3-tall towers using inductive reasoning	Same as event 15	00:04:00 - 00:06:45 L: 530 – 567
17.	2/26/1993 3 to 4-tall	Milin R2	Evidence of Michelle taking up Milin's doubling idea using inductive reasoning from 3 to 4-tall towers	Same as event 15	00:07:20 – 00:08:37 L: 573 – 582
18.	2/26/1993 1 to 3-tall	Milin R2 Stephanie Matt	Michelle explains the doubling idea using the inductive reasoning to Stephanie and Matt	Same as event 15	00:09:14 – 00:10:05 L: 594– 604