GEOMETRY OF COMPLEX MONGE-AMPÈRE EQUATION

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A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Jian Song

And approved by

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New Brunswick, New Jersey

May, 2020

ABSTRACT OF THE DISSERTATION

Geometry of Complex Monge-Ampère equation

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In this thesis, we study three problems related to Complex Monge-Ampère equations. After the introduction and preliminary, In chapter 3, we study Kähler Ricci flow on Fano bundle, with finite time singularity. we show that under the suitable assumption on the initial and ending Kähler class, the evolving Kähler metrics along Kähler Ricci flow have uniform diameter bound and moreover, if we assume the fiber of Fano bundle is $\mathbb{P}^n$ or $M_{m,k}$, the evolving metric will converge to a Kähler metric on the base of the Fano bundle in Gromov-Hausdorff sense, which generalizes the result of Song-Szekelyhidi-Weinkove [103] who study the Kähler Ricci flow on projective bundle.

In chapter 4, based on Kolodziej's fundamental result on C^0 estimate of complex Monge-Ampère equation, we study the geometric property of complex manifolds coupled with a family of Kähler metrics which come from solutions of a family of complex Monge-Ampère equations. As a application, on a minimal Kähler manifold with intermediate Kodaira dimension, we obtain uniform diameter bound of a family of collapsing Kähler metrics whose Kähler class is small perturbation of the canonical class. This is our first attempt to understand canonical metric on complex manifold with nef canonical class.

In chapter 5, we further study degeneration of Kähler-Einstein metrics with negative curvature on canonical polarized complex manifold. For this purpose, we construct complete Kähler-Einstein metric near isolated log canonical singularity through two different methods and for those log canonical singularity coupled with a model metric satisfying bounded geometry property roughly, we prove a rigidity result concerning complete Kähler-Einsteins near the singularity.

Acknowledgements

It's hard to imagine I could finish my Ph.D. work without the help of the following people.

First of all, I would like to thank my advisor Professor Jian Song for his guidance, constant encouragement, enlightening discussions and great support through the years. He lead me from the start to the various interesting topics of complex geometry. He taught me to hold different view points towards one problem and made a lot of suggestions which enable me to make progress in my research. I'm also grateful to his criticism throughout years which push me to improve my ability from many different aspects. He has been consistently sharing me his experience of research and life to me, which provide me enormous support. To me, he is not only a thesis advisor, but also a role model, from whom I will learn in my future life.

I would like to thank Professor Hans-Joachim Hein for teaching me various techniques on constructing complete Calabi-Yau metrics on complex varieties and also lot of discussion on his joint work with professor Song Sun, which sheds light on my proof of theorems in chapter 4.

I would like to thank Professor Xiaojun Huang for a lot of encouragement and help throughout the years, and for his great lecture on several complex variables, from which I begin to appreciate the beauty of complex analysis. I also would like to thank Professor Xiaochun Rong for his encouragement and many helpful discussion on Riemannian Geometry.

I am grateful to Professor Xiaowei Wang for his criticism through the years. His words "You should have a better understanding of the paper than the author", "What's point of the proof" push me to think harder and have a full understanding of the research paper. I'm also grateful to Professor Jacob Sturm for his encouragement and agreeing to write recommendation letter for me during my post-doc search. I would like to thank Professor Phong-Dung Phong for the organization of Informal complex geometry and PDE seminar at Columbia university every year, where is a fruitful environment for math research. I want to thank Professor Bin Guo for his wonderful seminar lectures, which help me to understand the most recent mathematical results.

I would like to thank my friend Dr. Hanlong Fang for our friendship. Hanlong is so generous who spends a lot of time helping me to solve various problems in my life and my research when I was a first year graduate student. And he keeps on encouraging and supporting me when I'm upset. Hanlong is a role model to me and I learn especially, how to be a true friend from him.

I would like to thank my friend Dr. Songyan Xie for our friendship and his hospitality when I was visiting AMSS. Songyan kept on encouraging me to be persistent on my math research when I planed to give up in academia. His broad knowledge and his optimism relieved my anxiety and pressure.

I would like to thank my high school classmates Guo Li for treating me the most delicious food each time when I visited Nanjing during summer vacation and for his encouragement when I was upset through the years.

I'm grateful to my friend Dr. Liming Sun who shared his life experience to me generously and helped me deal with many annoying and tedious problems patiently. I'm grateful to my friend Dr. Lun Zhang who taught me statistics and coding when I'm searching an industry job.

I'm grateful to my undergraduate advisor Professor Hourong Qin, Professor Jiaqiang Mei and Professor Yalong Shi for their beautiful introduction of modern mathematics to me. I would like to thank my undergraduate school Nanjing University, where is great place to study mathematics as an undergraduate student.

I'm grateful to Professor Shijin Zhang for collaboration, Professor Zhenlei Zhang for invitation to visit Capital Normal University, Professor Song Sun for his interest in my work and invitation to report my result at USTC. I am grateful to Professor Valentino Tosatti, Professor Ben Weinkove, Professor Julien Keller, Professor Gabor Szekelyhidi, Professor Julius Ross, Professor Xiangwen Zhang, Professor Chi li, Professor Ming Xiao, Professor Steven Lu, Professor Pengfei Guan, Professor Zhiqin Lu, Professor Lei Ni, Professor Gang Tian for their help in my post-doc search.

I would like to thank Professor Sagun Chanillo, Professor Yanyan Li, Professor Feng Luo, Professor Zhenchao Han for teaching me mathematics. I would like to thank Dr. Jiayin Pan, Dr. Xiaoping Zhu, Dr. Yukun Yao, Dr. Zhuohui Zhang, Dr, Han Lu, Dr. Zhuolun Yang, Dr. Fei Qi, Dr Weihong Xu, Dr Songhao Zhu for friendship at Rutgers. I would like to thank a lot of my other friends. I can not list all of their names here, but every moment of talking to, working or playing with them not only helps me to make progress in my career but also enriches my life outside mathematics.

I would like to thank the faculties at Rutgers who taught me much knowledge and the staff who helped me to get involved in the new environment quickly when I was a freshman in USA. I would like to thank the Graduate school of Rutgers for their generous financial support throughout the years.

Finally and most importantly, I would like to thank my parents for their love and constant support of my work.

Dedication

To my parents Meifen Chen and Xuzhong Fu

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Chapter 1

Introduction

1.1 Topic 1: Kähler Ricci flow with finite time singularity

1.1.1 Background

The Ricci flow, introduced by Hamilton, has become a powerful tool to study the topology and geometric structures of Riemannian manifolds.

In [104, 105, 106], Song-Tian introduced the analytic minimal model program which is parallel to Mori's birational minimal model program. On one hand, Kähler-Ricci flow with surgery can be viewed as the complex analogue of Thurston's three dimensional geometrization conjecture. On the other hand, the surgery is canonical and correspond to the birational surgery in Mori's program such as divisorial contraction or flip, see [7, 60].

1.1.2 Kähler-Ricci flow on Fano bundle

Consider the Kähler-Ricci flow $\omega = \omega(t)$ given by

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \qquad \omega(0) = \omega_0, \tag{1.1.1}$$

It's well-known that, from Tian-Zhang [130, 119], a maximal smooth solution to (1.1.1) exists on $[0, T)$ where $T > 0$ is given by

$$T = \sup\{t > 0 | [\omega_0] - 2\pi t c_1(X) > 0\}. \tag{1.1.2}$$

Song-Székelyhidi-Weinkove [103] studied the behavior of the Kähler-Ricci flow on the projective bundles. In our article [47], we generalize their result in the sense that we could have more types of Fano fibers other than projective spaces. For example the fiber could be $\mathbb{P}^m$ blown-up at one point or $M_{m,k}$ which is the weighted projective space $Y_{m,k}(1 \leq k < m)$ blown-up at the orbifold point.

Suppose we have a bundle $X \to Y$ with fiber F being a fano manifold. By (1.1.2), T is finite and if we further assume that the limiting Kähler class $[\omega_0] - 2\pi T c_1(X)$ satisfies

$$[\omega_0] - 2\pi T c_1(X) = [\pi^* \omega_Y] \tag{1.1.3}$$

for some Kähler metric ω_Y on Y. By lemma 3.2, we have a limiting form on ω_T,

$$\omega_T := \pi^* \omega_Y + \sqrt{-1}\partial\overline{\partial}\varphi_T \geq 0. \tag{1.1.4}$$

My first result of collaboration with Shijin Zhang is an estimate of ω_T.

Theorem 1.1. *([47]) Assume (X, Y, π, F) is a Fano bundle, ω_0 is the Kähler metric on X, ω_Y is a Kähler metric on Y satisfying (1.1.3) for some $T > 0$, ω_T is defined by (1.1.4). Then there exists a uniform constant $C > 0$ such that*

$$C^{-1}\pi^* \omega_Y \leq \omega_T \leq C\pi^* \omega_Y. \tag{1.1.5}$$

We further show that diameter of manifold X with metric $\omega(t)$ is finite and there exists a sequence of metrics along the Kähler-Ricci flow converge subsequentially to a metric on Y in the Gromov-Hausdorff sense as $t \to T$.

Theorem 1.2. *([47]) Let (X, Y, π, F) be a Fano bundle with F is $\mathbb{P}^m$ blown up at one point $(m \geq 2)$ or $F = M_{m,k}(1 \leq k < m)$, ω_Y be a Kähler metric on Y and ω_0 be a Kähler metric on X. Assume $\omega(t)$ is a solution of the Kähler-Ricci flow (1.1.1) for $t \in [0, T)$ with initial metric ω_0 and $[\omega_0] - 2\pi T c_1(X) = [\pi^* \omega_Y]$, then we have*

(1) $\mathrm{diam}(X, \omega(t)) \leq C$ *for some uniform constant $C > 0$;*

(2) There exists a sequence of times $t_i \to T$ and a distance function d_Y on Y (which is uniformly equivalent to the distance induced by ω_Y, such that $(X, \omega(t_i))$ converges to (Y, d_Y) in the Gromov-Hausdorff sense.

1.2 Topic 2: Geometric estimate of Monge-Ampère equation

1.2.1 background

Complex Monge-Ampère equations are a fundamental tools to study Kähler geometry. Consider the following complex Monge-Ampère equation

$$(\theta + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{-f}\theta^n, \tag{1.2.1}$$

on a Kähler manifold (X, θ). By deriving a priori estimate of equation (1.2.1) in his fundamental work ([136]), Yau solves the Calabi conjecture. After that, in Kolodziej's deep work [72], Yau's C^0-estimate for solutions of equation (1.2.1) is improved by applying the pluripotential theory. More precisely, suppose the right hand side of equation (1.2.1) satisfies the following L^p bound

$$\int_X e^{-pf}\theta^n \leq K, \text{ for some } p > 1,$$

then there exists $C = C(X, \theta, p, K) > 0$ such that any solution φ of equation (1.2.1) satisfies the the following L^∞-estimate

$$\|\varphi - \sup_X \varphi\|_{L^\infty(X)} \leq C.$$

Building on Kolodziej's work, a family of degenerating complex Monge-Ampère equations are intensively studied in [9, 72, 40, 37] and a notable application is the existence of Kähler-Einstein metric on canonical model of general type variety.

1.2.2 Geometric estimate of Monge-Ampère equation and application to generalized Kähler-Einstein metric on manifold with nef canonical bundle

We study the following Monge-Ampère equation on a Kähler manifold (X, θ) where θ is a fixed Kähler form:

$$(\theta + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\lambda\varphi}\Omega, \tag{1.2.2}$$

where $\lambda = 0$ or 1, and Ω is a smooth volume form satisfying $\int_X \Omega = \int_X \theta^n$. Moreover, we assume that

$$\int_X \left(\frac{\Omega}{\theta^n}\right)^p \theta^n \leq K, \; Ric(\Omega) = -\sqrt{-1}\partial\overline{\partial}\log\Omega \geq -A\theta, \tag{1.2.3}$$

for some $p > 1$, $K > 0$ and $A \geq 0$. By Kolodziej's work, we know that the solutions φ have uniform C^0 bound for different volume form Ω under condition (1.2.3). We further convert this analytic C^0 estimate to uniform diameter estimate hence bridging Monge-Ampère equation with geomtric compactness in Riemannian geometry. My first result of collaboration with Bin Guo and Jian Song is:

Theorem 1.3. *([46]) Let (X, θ) be an Kähler manifold, then under assumption (1.2.3), there exists $C = C(X, \theta, p, K, A)$ such that the solution φ of equations (1.2.2) and theKähler metric g associated to the Kähler form $\omega = \theta + \sqrt{-1}\partial\overline{\partial}\varphi$ satisfy the following estimates,*

1. $\|\varphi - \sup_X \varphi\|_{L^\infty(X)} + \|\nabla_g \varphi\|_{L^\infty(X,g)} \leq C$.

2. $Ric(g) \geq -Cg$.

3. $Diam(X, g) \leq C$.

Let $\mathcal{M}(X, \theta, p, K, A)$ be the space of all solutions of equation (1.2.2), where Ω satisfies assumption (1.2.3). One consequence of Theorem 1.3 is a uniform noncollapsing condition for $\mathcal{M}(X, \theta, p, K, A)$.

$$C^{-1} r^{2n} \leq Vol_g(B_g(x, r)) \leq C r^{2n}, \tag{1.2.4}$$

where $B_g(x, r)$ is the geodesic ball centered at q with radius r in (X, g).

We also study the Monge-Ampère equation with degenerating reference from χ and improve previous results in the sense that we derive estimates without assuming semi-positivity of limiting reference form χ. Consider a family of degenerate Monge-Ampère equation with limiting reference class χ being only nef and of numerical dimension κ.

$$(\chi + t\theta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = t^{n-\kappa} e^{\lambda\varphi_t + c_t}\Omega, \text{ for } t \in (0, 1], \tag{1.2.5}$$

where $\lambda = 0$, or 1, c_t is a normalizing constant. Our second result is:

Theorem 1.4. *([46]) Suppose the volume measures Ω in equation (1.2.5) satisfy L^p integrablility assumption (1.2.3). Then there exists a unique $\varphi_t \in \mathrm{PSH}(X, \chi + t\theta)$ solving* *equation (1.2.5) for all* $t \in (0,1]$*. Furthermore, there exists* $C = C(X, \chi, \theta, p, K) > 0$ *such that for all* $t \in (0,1]$*,*

$$\|(\varphi_t - \sup_X \varphi_t) - V_t\|_{L^\infty(X)} \le C,$$

where V_t *is the extremal function associated to* $\chi + t\theta$*.*

The refined C^0 estimate of theorem 1.4 can be applied to generalize Theorem 1.3, especially for minimal Kähler manifolds with nef canonical bundle in a geometric setting.

Theorem 1.5. *([46]) Suppose* X *is a smooth minimal model equipped with a smooth Kähler form* θ*. For any* $t > 0$*, there exists a unique smooth twisted Kähler-Einstein metric* g_t *on* X *satisfying*

$$Ric(g_t) = -g_t + t\theta. \tag{1.2.6}$$

There exists $C = C(X, \theta) > 0$ *such that for all* $t \in (0,1]$*,*

$$Diam(X, g_t) \le C.$$

Furthermore, for any $t_j \to 0$*, after passing to a subsequence, the twisted Kähler-Einstein manifolds* (X, g_{t_j}) *converge in Gromov-Hausdorff topology to a compact metric length space* $(\mathbf{Z}, d_{\mathbf{Z}})$*. The Kähler forms* ω_{t_j} *associated to* g_{t_j} *converge in distribution to a nonnegative closed current* $\widetilde{\omega} = \chi + \sqrt{-1}\partial\overline{\partial}\widetilde{\varphi}$ *for some* $\widetilde{\varphi} \in \mathrm{PSH}(X, \chi)$ *of minimal singularities, where* $\chi \in [K_X]$ *is a fixed smooth closed* $(1,1)$*-form.*

The diameter bound or non-collapsing condition we get is crucial in geometric compactness theory in particular in the Cheeger-Colding-Tian theory. Although so far we don't know too much about the limiting space especially in the case of theorem (1.5), we plan to study the tangent cone of the limiting space in the future. Our utimate goal in the future attempts to establish a geometric theory for canonical metric on minimal models of algebraic variety without assuming abundance conjecture. However if we assume the abundance conjecture, we can improve our understanding the limiting metric space $(\mathbf{Z}, d_{\mathbf{Z}})$ in our Theorem (1.5).

Theorem 1.6. *([46]) Suppose X is a projective manifold of complex dimension n equipped with a Kähler metric θ. If the canonical bundle K_X is semi-ample and $\nu(K_X) = \kappa \in \mathbb{N}$, then the following hold for the twisted Kähler-Einstein metrics g_t satisfying*

$$Ric(g_t) = -g_t + t\theta,\ t \in (0, 1].$$

(1) There exists $C > 0$ such that for all $t \in (0, 1]$,

$$Diam(X, g_t) \leq C.$$

(2) Let ω_t be the Kähler form associated to g_t. For any compact subset $K \subset\subset X \backslash \mathbf{S}$, we have

$$\|g_t - \Phi^* g_{can}\|_{C^0(K,\theta)} \to 0, \quad as\ t \to 0.$$

(3) The rescaled metrics $t^{-1}\omega_t|_{X_y}$ converge uniformly to a Ricci-flat Kähler metric $\omega_{CY,y}$ on the fibre $X_y = \Phi^{-1}(y)$ for any $y \in X_{can} \setminus \Phi(\mathbf{S})$, as $t \to 0$.

(4) For any sequence $t_j \to 0$, after passing to a subsequence, (X, g_{t_j}) converge in Gromov-Hausdorff topology to a compact metric space $(\mathbf{Z}, d_{\mathbf{Z}})$. Furthermore, $X_{can} \setminus S_{can}$ is embedded as an open subset in the regular part $\mathcal{R}_{2\kappa}$ of $(\mathbf{Z}, d_{\mathbf{Z}})$ and $(X_{can} \setminus S_{can}, \omega_{can})$ is locally isometric to its open image.

In particular, if $\kappa = 1$, $(\mathbf{Z}, d_{\mathbf{Z}})$ is homeomorphic to X_{can}, with the regular part being open and dense, and each tangent cone being a metric cone on $\mathbb{C}$ with cone angle less than or equal to 2π.

1.3 Topic 3: Kähler-Einstein geometry near log canonical singularity

1.3.1 Background

Kähler-Einstein metric has been the central topic in complex geometry for decades. For complex manifolds with X with $C_1(X) < 0$ and $C_1(X) = 0$, the existence of Kähler-Einstein metrics are confirmed by Aubin, Yau [4, 136] and Yau [136] separately. Also, rescent results of Chen-Donaldson-Sun [16, 17, 18] confirm the Yau-Tian-Donaldson conjecture for smooth Fano manifold. On the other hand, it will be interesting to understand the geometry of Kähler-Einstein metric on singular variety. In their pioneering work [59], Hein ans Sun study the asymptotic behaviour of diffenert Calabi-Yau metrics on singular varieties with special cone singularity by using the fundamental tool developed in Donaldson-Sun [38, 39]. On the other hand, in the recent work of Song [114], he proves that for a family of canonical polarized varieties, the negative Kähler-Einstein metrics of nearby fibers converge to a singular Kähler-Einstein on the central fiber which has complete end towards the locus of Non-Klt center. Hence it's a natural question to study the metric behaviour of Kähler-Einstein metric with negative curvature near log canonical singularity.

1.3.2 Complete Kähler-Einstein metrics near isolated log canonical singularity and their geometric rigidity

We study the Kähler-Einstein metric locally near an isolated log canonical singularity. We fix the geometric domains that will be discussed.

Setting: Let (X, p) be a germ of isolated normal log canonical $\mathbb{Q}$-Cartier singularity embedded in $(\mathbb{C}^N, 0)$. Our main interest will be neighbourhood of the singular point p. Using a bounded PSH function ρ on X, we cut a domain

$$\Omega := \{\rho < a\}$$

contained in X such that $\partial\Omega$ is strongly pseudoconvex. We also fix reference metric and volume form

$$\chi = \sqrt{-1}\partial\overline{\partial}\rho, \Omega_X = e^{\rho} V \wedge \bar{V}$$

on Ω, where V is local holomorphic volume form (up to taking root of multiple holomorphic volume form) on a neighbourhood of p in X. The complex Monge-Ampère equation of our interest in relation to the Kähler-Einstein equation on Ω (More precisely on $(\Omega \setminus p)$) is given by

$$\begin{cases} (\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\varphi}\Omega_X. \\ \varphi_{|\partial\Omega} = f \end{cases} \tag{1.3.1}$$

where f is an arbitrary smooth function. Our first result is:

Theorem 1.7. *([32]) Let (Ω, p) be a germ of isolated log canonical singularity as above. There exists solution $\varphi \in PSH(\chi)\bigcap C^\infty(\Omega \setminus p)$ of equation (1.3.1) satisfying the following conditions.*

(1) For any $\epsilon > 0$, there exists $C_{p,\epsilon} > 0$ such that

$$\varphi \geq \epsilon \log |\sigma_D|^2_{h_D} - C_{p,\epsilon}.$$

where σ_D is an effective divisor supporting on the exceptional locus when we blow up the singularity.

(2) $\varphi = -\infty$ on p.

We will also construct Kähler-Einstein metric on $(\Omega \setminus p)$ by using bounded geometry method. In order to use bounded geometry method, we need more assumptions for our singularity (X, p)

Property A: Let (X, p) a germ of isolated log canonical singularity embedded in $(\mathbb{C}^N, 0)$. If there is a complete metric $\chi = \sqrt{-1}\partial\overline{\partial}\rho$ defined on $(X \setminus p)$ satisfying (1) has a system of quasi coordinates. (2) $Ric(\chi) + \chi = \sqrt{-1}\partial\overline{\partial}M$ and $||\nabla^k_\chi M|| < C_k$ (Here the potential function M is not unique, we only require one of them satisfy the boundedness property, and in this article, the most interesting case is $M = 0$). Then we call (X, p) has property A and we define

$$\Omega := \{\rho < a\}, \partial\Omega := \{\rho = a\}$$

where a is a fixed constant (We can assume $\partial\Omega$ is smooth by adjusting constant a) .

Again we fix our geometric domain to be a triple (Ω, p, χ), and our second theorem is concerning the existence of Kähler-Einstein metric by using bounded geometry in a perturbation way.

Theorem 1.8. *([32]) Suppose (X, p) is a germ of log canonical admitting property A. Then for any smooth function ψ on the boundary $\partial\Omega$, the following Dirichlet problem*

$$\begin{cases} (\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\varphi + M}\chi^n \ \text{ on } \Omega \\ \\ \varphi|_{\partial\Omega} = \psi, \end{cases} \tag{1.3.2}$$

admits a solution in function space defined in Cheng-Yau [26] with $||\varphi||_{k,\alpha} < C(k, \psi, \rho, M)$.

There are a large class of log- canonical singularities admitting Kähler-Einstein uniformization which satisfies property (A). Especially, a complete picture of uniformization of isolated log canonical singularity in complex dimension 2 is obtained in [69, 70].

We proceed to compare two Kähler-Einstein metrics on $(\Omega \setminus p)$ which are complete towards p . First of all, we compare their volume forms. Suppose χ is a local complete Kähler Einstein metric on $(\Omega \setminus p)$ and χ' is another complete Kähler Einstein metric. Let $\varphi = \log \frac{\chi'^n}{\chi^n}$ be the ratio of volume forms. Also for any ϵ, define a punctured neighbourhood U_ϵ of p to be:

$$U_\epsilon := \{x | \mathrm{dist}_\chi(x, \partial\Omega) \geq \frac{2c(n)}{\epsilon} \quad and \quad \mathrm{dist}_{\chi'}(x, \partial\Omega) \geq \frac{2c(n)}{\epsilon}\}$$

Then our theorem in [32] concerning the comparison of volume ratio is:

Theorem 1.9. *([32]) For any* $\epsilon > 0$, *we have* $-\epsilon \leq \varphi \leq \epsilon$ *in* U_ϵ.

If both χ and χ' are complete towards p, then the above theorem shows that $f(x) \to 0$ when $x \to p$. If we further assume χ has bounded geometry property, we can even compare two different Kähler-Einstein metrics to higher order derivatives. If we write $\chi' = \chi + \sqrt{-1}\partial\overline{\partial}\varphi$ where $\varphi = \log \frac{\omega'^n}{\omega^n}$, then our last theorem in [32] is:

Theorem 1.10. *([32])Suppose* (Ω, p, χ) *is a metric with property* (A) *and* χ_1 *is another Kähler-Einstein metric on* Ω *and complete towards* p*. then for any positive number* ϵ *and any non negative integer* k*, we have* $\sum_{i=1}^{k} \|\nabla^i \varphi\|_\chi(q) \leq \epsilon$ *for* q *in* U_ϵ

Chapter 2

Preliminaries

In this chapter, we will collect some basic facts and definitions of Kähler geometry and Riemannian geometry. They are basically well-known from literature and will be stated without proofs. I include these material to make my presentation more self contained and part of them are taken from Bin Guo's Ph.D thesis [54].

2.1 Kähler geometry

Let (X, ω, J) be a compact complex manifold. The metric form ω is called Kähler if it is closed, i.e. $d\omega = 0$, or in local coordinates $(z_1, \dots, z_n)$,

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \quad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}_j}, \quad \forall i, j, k,$$

where $g_{i\bar{j}}$ is the components of ω in these coordinates, i.e.,

$$\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

The Kähler metric ω lies in a cohomology class $[\omega] \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$. By the $\sqrt{-1}\partial\overline{\partial}$-lemma ([48]) for any other $(1,1)$-form ω' in the same cohomology class as ω, there exists a smooth real function φ such that

$$\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\varphi.$$

Hence all the Kähler metrics in the Kähler class $[\omega]$ can be written as the form $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ for some $\varphi \in PSH(X, \omega)$ where

$$PSH(X, \omega) = \{\varphi \in C^\infty(X, \mathbb{R}) | \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\}.$$

The Riemannian curvature of ω is equal to (in locally coordinates $(z_1, \dots, z_n)$)

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l},$$

and the Ricci curvature is

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det g_{k\bar{l}}, \tag{2.1.1}$$

and the scalar curvature $R = g^{i\bar{j}} R_{i\bar{j}}$. Here and in the rest of the thesis, we denote $g^{i\bar{j}}$ the inverse of $g_{i\bar{j}}$, i.e., $g^{i\bar{j}} g_{k\bar{j}} = \delta^i_k$. The first and second Bianchi identities say that the indices with or without bar are all symmetric in the local components: $R_{i\bar{j}k\bar{l}}$, $R_{i\bar{j}k\bar{l},p}$ and $R_{i\bar{j}k\bar{l},\bar{p}}$.

The Ricci form

$$\mathrm{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz_i \wedge d\bar{z}_j = \frac{1}{2\pi} \sqrt{-1}\partial\bar{\partial} \log\det g_{i\bar{j}}$$

is a closed $(1,1)$-form, and its cohomology class (denoted by $C_1(X)$) is called the first Chern class of X.

A holomorphic line bundle L over the Kähler manifold X is a vector bundle over X with fiber $\mathbb{C}$, and the transition functions h_{ij} over $U_i \cap U_j$ are never zero holomorphic functions, where $L|_{U_i} \cong U_i \times \mathbb{C}$ is a local trivialization of L, and $X = \cup_i U_i$. The transition functions satisfy

$$h_{ij} h_{ji} = 1, \quad \text{on } U_i \cap U_j \neq \emptyset,$$

and

$$h_{ij} h_{jk} h_{ki} = 1, \quad \text{on } U_i \cap U_j \cap U_k \neq \emptyset.$$

These equations implies that $\{h_{ij}\}$ defines a 1 co-cycle hence a cohomology class the Cech group $H^1(X, \mathcal{O}^*)$.

A holomorphic section s of L is defined locally by $s = s_i e_i$ on each U_i, where e_i is a local frame of L over U_i and s_i is a holomorphic function on U_i. s is globally defined iff $s_i = h_{ij} s_j$ over $U_i \cap U_j$, since $e_i h_{ij} = e_j$. A Hermitian metric h on L is given by positive local functions $\{h_i\}$ over U_i such that $h_j = |h_{ij}|^2 h_i$ on $U_i \cap U_j$. Hence the $(1,1)$-form $-(2\pi)^{-1}\sqrt{-1}\partial\bar{\partial}\log h_i$ is globally defined, noting that $\sqrt{-1}\partial\bar{\partial}\log|h_{ij}|^2 = 0$. Actually, this $(1,1)$-form is called the curvature of the Hermitian metric h, and we will denote it by $\mathrm{Ric}(h)$. It represents a cohomology class in the Dolbeault cohomology group $H^{1,1}(X, \mathcal{O}) \cap H^2(X, \mathbb{Z})$. It is not hard to see for all Hermtian metrics on L, their curvatures lie in the same cohomology class, and we will denote this class by $C_1(L)$.

The set of holomorphic sections of L is denoted by $H^0(X,L)$, and the norm of $s \in H^0(X,L)$ with respect to a Hermtian metric h on L is defined by

$$|s|_h^2 := s_i \bar{s}_i h_i \quad \text{on } U_i,$$

and it is each to check this norm is globally defined by the transition laws of s_i and h_i.

The canonical line bundle K_X of X is defined to the determinant line bundle of $T^*_{(1,0)}M$, and $dz_1 \wedge \cdots \wedge dz_n$ is a local section of K_X on any local coordinates chart $(z_1, \ldots, z_n)$. The holomorphic sections of K_X are holomorphic n-forms. For any Kähler metric ω, $\frac{1}{\det g_{i\bar{j}}}$ defines a Hermtian metric on K_X. And its associated curvature is given

$$-(2\pi)^{-1}\sqrt{-1}\partial\bar{\partial}\log\frac{1}{\det g_{i\bar{j}}} = (2\pi)^{-1}\sqrt{-1}\partial\bar{\partial}\log\det g_{i\bar{j}} = -\mathrm{Ric}(\omega).$$

Hence we see that $-C_1(X) = C_1(K_X)$, or $C_1(X) = C_1(-K_X)$, where $-K_X$ is dual line bundle of K_X.

We recall a few notions about the line bundles.

Definition 2.1. *Given a holomorphic line bundle L over a compact Kähler manifold X, then*

(1) *L is called* ample, *if the linear system $|kL|$ for some $k \in \mathbb{N}$ gives an embedding of X to some projective space $\mathbb{CP}^N$, i.e., $kL = \mathcal{O}_{\mathbb{CP}^N}(1)$. In this case, X is necessarily projective by definition. The Kodaira embedding theorem implies that this is equivalent to the existence of a Hermtian metric h on L with curvature* $\mathrm{Ric}(h) > 0$.

(2) *L is called* numerical effective *(or* nef*) if for any irreducible curve $C \subset X$,* $\int_C C_1(L) \geq 0$.

(3) *L is* big, *if the Kodaira dimension $\kappa(L) = n$, or (in the* nef *case) equivalently*

$$\int_X C_1(L)^n > 0.$$

(4) *L is called* base point free, *if for any point $x \in X$, there exist a section $s \in H^0(X,L)$ such that $s(x) \neq 0$.*

Finally, we recall the notion the complex Monge-Ampëre equation and its relation with Kähler-Einstein metric:

(1) When $C_1(X) < 0$, if we fix a reference form ω in Kähler class $[K_X]$, any other Kähler metric g in class $[K_X]$ can be written as $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$. Then

$$Ric(g) = -g \Longleftrightarrow .(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\varphi}\Omega$$

(2) When $C_1(X) = 0$, if we fix a reference form ω on X, any Kähler metric g in class $[\omega]$ can be written as $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$. Then

$$Ric(g) = 0 \Longleftrightarrow (\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \Omega$$

where Ω is suitable smooth volume form on X.

2.2 Riemannian geometry and metric geometry

2.2.1 Riemannian geometry

Let (M, g) be a Riemannian manifold and $p \in M$ be a point. The cut-locus of p is defined to be the points $q \in M$ either q is a conjugate point of p or there exists at least two distinct minimal geodesics from p to q. It is known that the cut-locus has measure zero by an application of Sard's theorem. The exponential map $\exp_p : T_pM \to M$ is local diffeomorphism in the interior of cut-locus. Denote $\Omega = M\backslash\{$the cut-locus of $p\}$, then $\exp_p^{-1}(\Omega)$ is a star-shaped domain in $T_pM \cong \mathbb{R}^n$. It is also well-known that the distance function $d(x) = d(p, x)$ is smooth in $\Omega\backslash\{p\}$. The injectivity radius of p is defined to be

$$i_p = inj_g(p) := \sup\{r > 0 |\ B(p, r) \subset \Omega\}$$

where $B(p, r)$ is the geodesic ball centered at p. And it is clear that $\exp_p : B(0, i_p) \subset T_pM \to B(p, i_p) \subset M$ is a diffeomorphism.

The space forms are simply connected manifolds with constant sectional curvature, which by the uniformization theorem are S^n, $\mathbb{R}^n$ and $\mathbb{H}^n$, with curvatures normalized being $1, 0, -1$, respectively. The metric with constant sectional curvature K is given by (see [13])

$$dr^2 + \mathrm{sn}_K(r)^2 g_{S^{n-1}},$$

where $g_{S^{n-1}}$ is the standard metric on S^{n-1} with curvature 1, and

$$\mathrm{sn}_K(r) = \begin{cases} \frac{1}{\sqrt{K}}\sin(\sqrt{K}r), & \text{if } K > 0 \\ r, & \text{if } K = 0 \\ \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}r), & \text{if } K < 0. \end{cases}$$

Theorem 2.2 (Hessian comparison ([13]))**.** *Let (M, g) be a complete Riemannian manifold with dimension n and $p \in M$ be a fixed point. Suppose the sectional curvature of g satisfies*

$$\kappa \leq \mathrm{sect}_g \leq K$$

for some $\kappa, K \in \mathbb{R}$, then the Hessian of $r(x) = d(p, x)$ satisfies

$$\mathrm{Hess}_{r_K}(r(x)) \leq \mathrm{Hess}_r(x) \leq \mathrm{Hess}_{r_\kappa}(r(x)),$$

at x where $r(\cdot)$ is smooth, and r_κ and r_K are the distance functions on the space forms with constant curvature κ, K, respectively.

When we have only Ricci curvature lower bound, we have the Bonnet-Myers' theorem, Laplacian comparison theorem and Bishop-Gromov volume comparison theorem:

Theorem 2.3 (Bonnet-Myers' theorem)**.** *Suppose (M, g) is a complete Riemannian manifold with $\mathrm{Ric}(g) \geq (n-1)K > 0$, then the diameter of (M, g) is bounded above by $\frac{\pi}{\sqrt{K}}$.*

Theorem 2.4 (Laplacian comparison)**.** *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)K$ for some $K \in \mathbb{R}$, $r(x) = d(x, p)$ for some $p \in M$, then*

$$\Delta r(x) \leq \Delta_K r_K(r(x)),$$

smoothly when $r(\cdot)$ is smooth at x and globally in the sense of distributions, where r_K is the distance function in the space form S^n_K.

Theorem 2.5 (Volume comparison). *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)K$ *for some $K \in \mathbb{R}$, $r(x) = d(x, p)$ for some $p \in M$, then the function*

$$r \mapsto \frac{Vol_g(B(p, r))}{Vol_K(B_K(r))}$$

is non-increasing, where $Vol_K(B_K(r))$ is the volume of geodesic ball of radius r in the space form S^n_K.

2.2.2 Metric geometry

Definition 2.6. *Given any two compact metrics spaces (X, d_X) and (Y, d_Y), the Gromov-Hausdorff (GH) distance $d_{GH}(X, Y)$ of X, Y is defined to be the infimum of all $\epsilon > 0$ such that there is a map (continuous or not) $f : X \to Y$ which is called ϵ-Gromov-Hausdorff approximation (ϵ-GHA) such that*

(1) f is ϵ-onto, i.e., the image $f(X)$ is ϵ-dense in (Y, d_Y),

(2) f is ϵ-isometry, i.e., for any $x_1, x_2 \in X$,

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \leq \epsilon.$$

There are also other equivalent definitions of GH distance, for example, $d_{GH}(X, Y)$ can also be defined as the infimum of $\epsilon > 0$ over all compatible metrics on $X \sqcup Y$ such that both components are ϵ-dense. These two definitions may not be the same, but they are equivalent and hence do not affect our applications.

We say sequence of compact metric spaces (X_i, d_i) converges to (X_∞, d_∞) in GH topology, if $d_{GH}(X_i, X_\infty) \to 0$ as $i \to \infty$.

One of the fundamental results in metric geometry is the Gromov pre-compactness theorem:

Theorem 2.7 (Gromov pre-compactness). *The set $\mathcal{M}(n, \Lambda, D)$ of n dimensional compact Riemannian manifolds (M, g) such that*

$$\mathrm{Ric}(g) \geq \Lambda, \quad \mathrm{diam}(M, g) \leq D$$

is pre-compact in the GH topology.

In the case manifolds not having finite diameter, we consider the *pointed* -GH convergence. We say

$$(X_i, d_i, p_i) \xrightarrow{p-GH} (X_\infty, d_\infty, p_\infty),$$

if for any $R > 0$, the metric balls $B_i(p_i, R) \xrightarrow{GH} B_\infty(p_\infty, R)$. Hence by Gromov precompactness theorem, for any sequence of complete Riemannian manifolds (M_i^n, g_i, p_i) with $\mathrm{Ric}(g_i) \geq \Lambda$, there exists a subsequence which converges in pointed GH sense.

In general the GH limit space of a sequence of metric spaces does not have good regularities. Under some geometric assumptions, Cheeger-Colding prove that the limit space does have some regularities:

Theorem 2.8 ([10, 28])**.** *Let (M_i^n, g_i, p_i) be a sequence of smooth Riemannian manifolds with*

$$\mathrm{Ric}(g_i) \geq -(n-1), \quad Vol(B(p_i, 1)) \geq v_0 > 0,$$

then any GH limit of (M_i, g_i, p_i), $(M_\infty, d_\infty, p_\infty)$ satisfies

(1) Volume converges, $\lim_{i\to\infty} Vol_{g_i}(B(p_i, R)) = \mathcal{H}^n(B_\infty(p_\infty, R))$ for any $R > 0$, where $\mathcal{H}^n$ is a suitable n-dimensional Hausdorff measure on (M_∞, d_∞).

(2) M_∞ has a regular-singular decomposition, $M_\infty = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{R}$ is defined to be the points whose tangent cones are $\mathbb{R}^n$, and $\mathcal{S} = M_\infty \backslash \mathcal{R}$.

(3) The Hausdorff dimension of $\mathcal{S} \leq n-2$.

Recall a tangent cone at $q \in M_\infty$ is the GH limit of the spaces $(M_\infty, r_i^{-2} d_\infty, q)$ for a sequence $r_i \to 0$. The tangent cone at a point $q \in M_\infty$ may not be unique, and it depends on the choice of sequence $r_i \to 0$. We remark that by definition no tangent cone at $q \in \mathcal{S}$ can be $\mathbb{R}^n$. And if a tangent cone at some point splits off a Euclidean factor $\mathbb{R}^{n-1}$, then it must be $\mathbb{R}^n$, hence the point is regular.

If we assume Ricci curvature uniformly bounded, instead of lower bound, then Cheeger-Colding-Tian theory says more about the regularity of the limit space.

Theorem 2.9 ([12])**.** *Suppose a sequence of Riemannian manifolds (M_i, g_i, p_i) converges in GH sense to $(M_\infty, d_\infty, p_\infty)$. Suppose*

$$|\mathrm{Ric}(g_i)| \leq n-1, \quad Vol_{g_i}(B(p_i, 1)) \geq v_0 > 0,$$

then we have

(1) *In the regular-singular decomposition* $M_\infty = \mathcal{R} \cup \mathcal{S}$, $\mathcal{R}$ *is an open* $C^{2,\alpha}$ *manifold with a* $C^{1,\alpha}$ *Riemannian metric compatible with the distance* d_∞ *on* M_∞. $\mathcal{S}$ *is closed and of Hausdorff codimension* ≥ 2.

(2) *If* (M_i, g_i) *are Kähler, then* $\mathcal{S}$ *is of Hausdorff codimension* ≥ 4.

Chapter 3

Kähler Ricci flow on Fano bundle

This chapter is from the joint work [47] with Shijin Zhang.

3.1 Introduction

The Ricci flow, introduced by Hamilton ([58]), has become a powerful tool to study the topology and geometric structures of Riemannian manifolds. In general, the Ricci flow develops finite time singularities. Hamilton's program of Ricci flow with surgeries was carried out by Perelman [87, 88, 89] to prove Thurston's geometrization conjecture. The minimal model theory in birational geometry can be viewed as the complex analogue of Thurston's geometrization conjecture. Later in [21] Cao introduced the Kähler-Ricci flow and use it to prove the existence of Kähler-Einstein metrics on manifolds with negative or vanishing first Chern class ([136, 4]).

In this chapter we study the behavior of the finite time singularity of the Kähler-Ricci flow. Following Song-Tian's analytic minimal model programm, we study Käherl Ricci flow on Fano bundle. Before we prove our main result (1.1),(1.2) we introduce the necessary concepts needed.

Definition 3.1 (Fano bundle). *Let X, Y be compact Kähler manifolds with dimension n, m, respectively, F be a Fano manifold with dimension $n-m$ and a surjective holomorphic map $\pi : X \to Y$. We say X is a Fano bundle over Y with fiber F, if for any $y \in Y$, there exists a Zariski open set $y \in U \subset Y$ and a biholomorphism $\Phi : \pi^{-1}(U) \to U \times F$ such that the diagram*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \Phi\ } & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{Pr}_1} & \\ & U & \end{array}$$

commutes, where Pr_1 *is the projection map onto the first factor. We denote it as* (X, Y, π, F).

Since the fiber F is a Fano manifold, the solution $\omega(t)$ develops a singularity after a finite time. By (1.1.2), T is finite since $F \cdot c_1(X)^{n-m} > 0$ for every fiber F. Hence we assume that the limiting Kähler class $[\omega_0] - 2\pi T c_1(X)$ satisfies

$$[\omega_0] - 2\pi T c_1(X) = [\pi^* \omega_Y] \tag{3.1.1}$$

for some Kähler metric ω_Y on Y.

3.2 Proof of Theorem 1.1

In this section, we recall some estimates for the Kähler-Ricci flow, establish a estimate for $\omega(t)$ on the horizontal level set and prove the Theorem 1.1.

We define reference $(1,1)-$forms $\hat{\omega}_t$ on X for $t \in [0, T]$ by

$$\hat{\omega}_t = \frac{1}{T}((T-t)\omega_0 + t\pi^*\omega_Y). \tag{3.2.1}$$

Then $\hat{\omega}_t$ is a Kähler form in the cohomology class $[\omega(t)]$ for $t \in [0, T)$. Let Ω be the unique smooth volume form on X with $\sqrt{-1}\partial\overline{\partial}\log\Omega = \frac{\partial}{\partial t}\hat{\omega}_t =: \chi \in -2\pi c_1(X)$ and $\int_X \Omega = 1$. We also can write $\hat{\omega}_t$ as $\hat{\omega}_t = \omega_0 + t\chi$.

It's well-known that the Kähler-Ricci flow equation (1.1.1) is equivalent to the following complex Monge-Ampère equation

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} = \log \dfrac{(\frac{1}{T}((T-t)\omega_0 + t\pi^*\omega_Y) + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega} \\ \varphi(0) = 0, \end{cases} \tag{3.2.2}$$

where Ω is a smooth volume form, $\chi = \sqrt{-1}\partial\overline{\partial}\log\Omega \in -2\pi c_1(X)$, and $\omega(t) = \frac{1}{T}((T - t)\omega_0 + t\pi^*\omega_Y) + \sqrt{-1}\partial\overline{\partial}\varphi > 0$.

Then the following estimates are well known, see the Lemma 2.1 and Lemma 2.2 in [109]. In this paper we use C to denote a uniform constant, independent of time but possibly depending on ω_0, n, T, which may differ from line to line.

Lemma 3.2. *For any Kähler manifold* (X, ω_0) *and Kähler manifold* (Y, ω_Y)*. If there exists a surjective holomorphic map* $\pi : X \to Y$*, and the smooth solution* $\omega(t)$ *of the*

Kähler-Ricci flow (1.1.1) on X satisfying $\lim_{t\to T}[\omega(t)] = [\pi^*\omega_Y](T < +\infty)$*. Then we have*

1. *There exists a uniform constant $C > 0$ such that $||\varphi||_{L^\infty} \leq C$;*

2. *There exists a uniform constant $C > 0$ such that $\dot{\varphi} \leq C$;*

3. *As $t \to T$, $\varphi(t)$ converges pointwise on X to a bounded function φ_T satisfying*

$$\omega_T := \pi^*\omega_Y + \sqrt{-1}\partial\overline{\partial}\varphi_T \geq 0. \tag{3.2.3}$$

4. *There exists a uniform constant $c > 0$ such that*

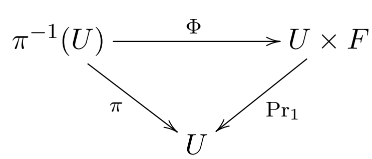
$$\omega(t) \geq c\pi^*\omega_Y. \tag{3.2.4}$$

By Lemma 3.2 above, we know there exists a bound function φ_T on X satisfying $\lim_{t\to T}\varphi(t) = \varphi_T$. We define

$$\omega_T := \pi^*\omega_Y + \sqrt{-1}\partial\overline{\partial}\varphi_T \geq 0. \tag{3.2.5}$$

Next motivated by the argument of Song, see subsection 3.1 in [102], which Song estimated the evolving metrics of the Kähler-Ricci flow in a well-chosen set of directions in the tangent space of each point on X instead of all directions, we estimate the metric $\omega(t)$ on the horizontal level set of the Fano bundle X.

Let (X, Y, π, F) be the Fano bundle (see Definition 3.1). Since for any $x \in X$, let $y = \pi(x)$, there exists a Zariski open set $(y \in)U \subset Y$, such that the diagram



commutes, where Pr_1 is the projection map onto the first factor. Let $f = \mathrm{Pr}_2 \circ \Phi(x)$, $H = \Phi^{-1}(U \times \{f\})$, where Pr_2 is the projection map onto the second factor. Let D be a some divisor such that $Y\backslash U \subset D$ and s be a holomorphic section on $[D]$ and let h be a Hermitian metric on $[D]$. Define $|s|^2 = hs\overline{s}$. Then on the horizontal level set H, we have the estimate for $\omega(t)$.

Lemma 3.3. *Assume $\omega(t)$ is the solution of the Kähler-Ricci flow (1.1.1) and* $\lim_{t\to T}[\omega(t)] = [\pi^*\omega_Y]$. *Fix any point $x \in X$, then there exists $U \subset Y$, let $f(x) = \mathrm{Pr}_2 \circ \Phi(x)$ and $H = \Phi^{-1}(U \times \{f(x)\})$. Then there exist uniform constants $C > 0$ and $\alpha > 0$, such that* $\omega(t)|_H \leq \frac{C}{\pi^*(|s|^{2\alpha})}(\pi^*\omega_Y)|_H$.

Proof. Since for any $x \in \pi^{-1}(U)$, $\pi(x) = y \in U$, and Φ is a biholomorphism from $\pi^{-1}(U)$ to $U \times F$, there exist constants $\alpha > 0$ and $C > 0$ such that $\pi^*\omega_Y|_H \geq \pi^*|s|^\alpha \omega_Y|_U$. On the other hand, for each time $t \in (0,T)$, $\omega(t)$ is equivalent to metric ω_0. Hence if we let

$$u(t,x) = \pi^*(|s|^{2\alpha})\mathrm{tr}_{\pi^*\omega_Y|_H}(\omega(t)|_H)(x),$$

we know $u \to 0$ along $X\backslash\pi^{-1}(U)$ and hence a positive maximum must occur in $\pi^{-1}(U)$ at each fixed time $t \in (0,T)$. We assume the maximum can be obtained at point $x_0 \in X$. Let $y_0 = \pi(x_0) \in Y$. We choose normal coordinate system $(z^i)_{i=1,\cdots,n}$ for $g(t)$ at x_0 and $(w^\alpha)_{\alpha=1,\cdots,m}$ for g_Y at y_0. For any holomorphic vector $\frac{\partial}{\partial w^\alpha}$, there exist holomorphic vector $\frac{\partial}{\partial x^\alpha} \in T_xX$ such that $d\pi_x(\frac{\partial}{\partial x^\alpha}) = \frac{\partial}{\partial w^\alpha}$ for any x in the local normal coordinate chart of x_0. The map π is given locally as $(\pi^1, \cdots, \pi^m)$ for holomorphic functions $\pi^\alpha = \pi^\alpha(z^1, \cdots, z^n)$. We write $\frac{\partial}{\partial x^\alpha}$ as $\frac{\partial}{\partial x^\alpha} = a^i_\alpha \frac{\partial}{\partial z^i}$ for holomorphic functions a^i_α. Then u can be expressed as $u(t,x) = |s|^{2\alpha}(\pi(x))g_Y^{\alpha\overline{\beta}} a^i_\alpha \overline{a^j_\beta} g_{i\overline{j}}$. For convenience, we denote $u_1 = g_Y^{\alpha\overline{\beta}} a^i_\alpha \overline{a^j_\beta} g_{i\overline{j}}$. Then at point x_0

$$\begin{aligned}
\Delta u_1 &= g^{k\bar{l}}\partial_k\partial_{\bar{l}}(g_Y^{\alpha\overline{\beta}} a^i_\alpha \overline{a^j_\beta} g_{i\overline{j}}) \\
&= \sum_{k,l=1}^{n} g^{k\bar{l}}\partial_k(\partial_{\bar{\delta}} g_Y^{\alpha\overline{\beta}} \overline{\pi^\delta_l} a^i_\alpha \overline{a^j_\beta} g_{i\overline{j}} + g_Y^{\alpha\overline{\beta}} a^i_\alpha \overline{\partial_l a^j_\beta} g_{i\overline{j}} + g_Y^{\alpha\overline{\beta}} a^i_\alpha \overline{a^j_\beta} \partial_{\bar{l}} g_{i\overline{j}}) \\
&= -\partial_\gamma\partial_{\bar{\delta}}(g_Y)_{\beta\overline{\alpha}}\pi^\gamma_k \overline{\pi^\delta_k} a^i_\alpha \overline{a^i_\beta} + |\partial_k a^i_\alpha|^2 - a^i_\alpha \overline{a^j_\alpha} R_{i\overline{j}} \\
&= (\mathrm{Rm}(g_Y))_{\gamma\bar{\delta}\beta\overline{\alpha}}\pi^\gamma_k \overline{\pi^\delta_k} a^i_\alpha \overline{a^i_\beta} + |\partial_k a^i_\alpha|^2 - a^i_\alpha \overline{a^j_\alpha} R_{i\overline{j}}
\end{aligned}$$

On the other hand,

$$\frac{\partial u_1}{\partial t} = g_Y^{\alpha\overline{\beta}} a^i_\alpha \overline{a^j_\beta} \frac{\partial}{\partial t} g_{i\overline{j}} = -a^i_\alpha \overline{a^j_\alpha} R_{i\overline{j}}.$$

Hence

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)\log u_1 &= \frac{1}{u_1}(-(\mathrm{Rm}(g_Y))_{\gamma\bar{\delta}\beta\overline{\alpha}}\pi^\gamma_k \overline{\pi^\delta_k} a^i_\alpha \overline{a^i_\beta} - |\partial_k a^i_\alpha|^2) + \frac{|\nabla u_1|^2}{u_1^2} \\
&\leq c_Y \mathrm{tr}_\omega \pi^*\omega_Y + \frac{1}{u_1}(\frac{|\nabla u_1|^2}{u_1} - |\partial_k a^i_\alpha|^2),
\end{aligned}$$

where $-c_Y$ is a lower bound for the bisectional curvature of ω_Y on Y. It is easy to get (see [104])

$$\frac{|\nabla u_1|^2}{u_1} - |\partial_k a^i_\alpha|^2 \le 0.$$

Hence we have

$$(\frac{\partial}{\partial t} - \Delta)\log u_1 \le c_Y \mathrm{tr}_\omega \pi^*\omega_Y. \tag{3.2.6}$$

Since $\sqrt{-1}\partial\overline{\partial}(\pi^*|s|^2)(x_0) = \sqrt{-1}\partial\overline{\partial}|s|^2(y_0)$, is bounded by some multiple of $\pi^*\omega_Y$. Combine Lemma 2.1, we have

$$(\frac{\partial}{\partial t} - \Delta)\log u \le C\mathrm{tr}_\omega \pi^*\omega_Y \le C'.$$

Hence by the maximum principle, we have $u \le C$. □

Now we prove the Theorem 1.1.

Proof of Theorem 1.1. Lower bound follows from (4) in Lemma 3.2. For any point $y \in Y$, each fiber $\pi^{-1}(y) = F$ is a closed Kähler manifold, and since $\pi^*\omega_Y|_{\pi^{-1}(y)} = 0$, we have

$$\sqrt{-1}\partial\overline{\partial}\varphi_T|_{\pi^{-1}(y)} = \omega_T|_{\pi^{-1}(y)} \ge 0,$$

since φ_T is bounded, φ_T must be constant on the fiber $\pi^{-1}(y)$. Hence there exists a bounded function ψ_T on Y satisfying

$$\varphi_T = \pi^*\psi_T.$$

Hence

$$\omega_T = \pi^*(\omega_Y + \sqrt{-1}\partial\overline{\partial}\psi_T).$$

Now for any $x \in X$, we may assume that $|s|^2(\pi(x)) = 1$, there exists an open set $\pi(x) \in U \subset Y$, such that Lemma 3.3 holds. Now we consider the open set $U_{1/2} := \{y \in U||s|^2(y) > 1/2\}$. Then by Lemma 3.3, there exists a constant $C > 0$ such that

$$\sqrt{-1}\partial\overline{\partial}\psi_T|_{U_{1/2}} \le C\omega_Y.$$

Since Y is a compact manifold, there exist a finite open set $\{U^i_{1/2}(1 \le i \le N)\}$ (N is a positive integer number) satisfying

$$\cup_{i=1}^N U^i_{1/2} = Y.$$

Hence we obtain that there exists a uniform constant $C > 0$ such that

$$\sqrt{-1}\partial\overline{\partial}\psi_T \leq C\omega_Y.$$

Hence we finish the proof of the theorem. □

We also study the Kähler-Ricci flow on the Fano bundles with the fiber F is $\mathbb{P}^m$ blown up at one point or $M_{m,k}$ which is the weighted projective space $Y_{m,k}$ (the definition see Section 4) blown up at the orbifold point.

And our Theorem 1.2 is a combination of Theorem 3.4 and Theorem 3.19 below.

Remark 1. *When the dimension of X is 2, our method basically can cover most del Pezzo surface. It will be more interesting when the complex structure of the fiber is changing and when the fiber is general Fano variety in higher dimension.*

3.3 F Is $\mathbb{P}^m$ Blown Up At One Point

In this section, we consider the case of F is $\mathbb{P}^m$ blown up at one point. One essential point of Song-Székelyhidi-Weinkove's proof [103] is that the projective space admits a metric which has positive holomorphic bisectional curvature. Although $\mathbb{P}^m$ blown up at one point doesn't admit a metric with nonnegative holomorphic bisectional curvature, but we have such metric with nonnegative bisectional curvature on outside of the divisor. Then we need to estimate the locally holomorphic vector field near the divisor under the evolving metrics, by using a idea of Song-Weinkove [109]. We also need Lemma 2.2, estimate of the evolving metrics along the Kähler-Ricci flow which were restricted to a horizontal set. We prove the following

Theorem 3.4. *Let (X, Y, π, F) be a Fano bundle with F is $\mathbb{P}^m$ blown up at one point $(m \geq 2)$, ω_Y be a Kähler metric on Y and ω_0 be a Kähler metric on X. Assume $\omega(t)$ is a solution of the Kähler-Ricci flow (1.1.1) for $t \in [0, T)$ with initial metric ω_0 and $[\omega_0] - 2\pi T c_1(X) = [\pi^*\omega_Y]$, then we have*

(1) $\operatorname{diam}(X, \omega(t)) \leq C$ *for some uniform constant $C > 0$;*

(2) There exists a sequence of times $t_i \to T$ and a distance function d_Y on Y (which is uniformly equivalent to the distance induced by ω_Y, such that $(X, \omega(t_i))$ converges to (Y, d_Y) in the Gromov-Hausdorff sense.

3.3.1 Key Estimates

Write $\pi_1 : F \to \mathbb{P}^m$ for the blow-down map, which is an isomorphism from $F \backslash E$ to $\mathbb{P}^m \backslash \{p\}$, where $p \in \mathbb{P}^m$ and $E = \pi_1^{-1}(p)$, which is biholomorphic to $\mathbb{P}^{m-1}$. For convenient, once and for all, a coordinate chart V centered at p, which we identify via coordinates $z^1, \cdots, z^m$ with the unit ball D_1 in $\mathbb{C}^m$,

$$D_1 = \{(z^1, \cdots, z^m) \in \mathbb{C}^m | \sum_{i=1}^{m} |z^i|^2 < 1\}. \tag{3.3.1}$$

Denote by g_e the Euclidean metric on D_1. Since g_e and g_{FS} are uniformly equivalent on D_1, it suffices to estimates for g_e on D_1. Write $D_r \subset D_1$ for the ball of radius $0 < r < 1$ with respect to g_e.

We recall the definition of the blow-up construction, following the exposition in [48]. We identify $\pi_1^{-1}(D_1)$ with the submanifold $\tilde{D_1}$ of $D_1 \times \mathbb{P}^{m-1}$ given by

$$\tilde{D_1} = \{(z, l) \in D_1 \times \mathbb{P}^{m-1} | z^i l^j = z^j l^i\}, \tag{3.3.2}$$

where $l = [l^1, \cdots, l^m]$ are homogeneous coordinates on $\mathbb{P}^{m-1}$. The map π_1 restricted to $\tilde{D_1}$ is the projection $\pi|_{\tilde{D_1}}(z, l) = z \in D_1$, with the exceptional divisor $E \cong \mathbb{P}^{m-1}$ given by $\pi_1^{-1}(0)$. The map π gives an isomorphism from $\tilde{D_1} \backslash E$ onto the punctured ball $D_1 \backslash \{0\}$.

On $\tilde{D_1}$ we have coordinate charts $\tilde{D_1}_i = \{l^i \neq 0\}$ with local coordinates $\tilde{z}(i)^1, \cdots, \tilde{z}(i)^m$ given by $\tilde{z}(i)^j = l^j/l^i = z^j/z^i$ for $j \neq i$ and $\tilde{z}(i)^i = z^i$. The divisor E is given in $\tilde{D_1}_i$ by $\{\tilde{z}(i)^i = 0\}$. The line bundle $[E]$ over $\tilde{D_1}$ has transition functions z^i/z^j on $\tilde{D_1}_i \cap \tilde{D_1}_j$. We can define a global section s of $[E]$ over F by setting $s(z) = z^i$ on $\tilde{D_1}_i$ and $s = 1$ on $F \backslash \pi_1^{-1}(D_{1/2})$. The section s_1 vanishes along the exceptional divisor E. We also define a Hermitian metric h_1 on $[E]$ as follows. First let h_2 be the Hermitian metric on $[E]$ over $\tilde{D_1}$ given in $\tilde{D_1}_i$ by

$$h_2 = \frac{\sum_{j=1}^{m} |l^j|^2}{|l^i|^2}, \tag{3.3.3}$$

and let h_3 be the Hermitian metric on $[E]$ over $F\backslash E$ determined by $|s_1|^2_{h_2} = 1$. Now define the Hermitian metric h_1 by $h_1 = \rho_1 h_2 + \rho_2 h_3$, where ρ_1, ρ_2 is a partition of unity for the cover $(\pi_1^{-1}(D_1), F\backslash\pi_1^{-1}(D_{1/2}))$ of F, so that $h_1 = h_2$ on $\pi_1^{-1}(D_{1/2})$. The function $|s_1|^2_{h_1}$ on F is given on $\pi_1^{-1}(D_{1/2})$ by

$$|s_1|^2_{h_1}(x) = \sum_{i=1}^{m} |z^i|^2 =: r^2, \tag{3.3.4}$$

for $\pi_1(x) = (z^1, \cdots, z^m)$. On $\pi_1^{-1}(D_{1/2}\backslash\{0\})$, the curvature $R(h_1)$ of h_1 is given by

$$R(h_1) = -\sqrt{-1}\partial\overline{\partial}\log(\sum_{i=1}^{m} |z^i|^2). \tag{3.3.5}$$

We have the following lemma (see [48], p.187).

Lemma 3.5. *For sufficiently small $\epsilon_0 > 0$,*

$$\omega_F = \pi_1^*\omega_{FS} - \epsilon_0 R(h_1) \tag{3.3.6}$$

is a Kähler form on F.

From now on we fix $\epsilon_0 > 0$ as in the Lemma 3.5, with ω_F defined in Lemma 3.5. In $\pi_1^{-1}(D_{1/2}\backslash\{0\})$, which we can identify with $D_{1/2}\backslash\{0\}$, the metric ω_F has the form:

$$\omega_F = \pi_1^*\omega_{FS} + \sqrt{-1}\frac{\epsilon_0}{r^2}\sum_{i,j=1}^{m}(\delta_{ij} - \frac{\overline{z^i}z^j}{r^2})dz^i d\overline{z^j}, \tag{3.3.7}$$

for r given by (3.3.4). It is easy to see that, in $D_{1/2}\backslash\{0\}$, $R(h_1) \leq 0$, and the following lemma holds (see [109]).

Lemma 3.6. *There exist positive constants C such*

$$\pi_1^*\omega_{FS} \leq \omega_F \leq C\frac{\pi_1^*\omega_{FS}}{|s_1|^2_{h_1}} \tag{3.3.8}$$

Since (X, Y, π, F) is a Fano bundle, for any $x \in X$, let $y = \pi(x)$, there exists a Zariski open set $(y \in)U \subset Y$, such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \Phi\ } & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{Pr}_1} & \\ & U & \end{array}$$

commutes, where Pr_1 is the projection map onto the first factor. Let D be a some divisor such that $Y\backslash U \subset D$ and s be a holomorphic section on $[D]$ and let h be a Hermitian metric on $[D]$. Define $|s|^2 = hs\bar{s}$, for simplicity, we also write $\pi^*|s|_h^2$ as $|s|_h^2$. On $\pi^{-1}(U)$, we denote $\tilde{\omega} = \Phi^*(\mathrm{Pr}_2^*\pi_1^*\omega_{FS} + \mathrm{Pr}_1^*\omega_Y)$, we also write $|s_1|_{h_1}^2$ to represent $\Phi^*\mathrm{Pr}_2^*(|s_1|_{h_1}^2)$, where Pr_2 is the projection map onto the second factor. Then we have the following

Lemma 3.7. *There exist uniform constants $C > 0$ and $\alpha > 0$ such that for $\omega = \omega(t)$ a solution of the Kähler-Ricci flow,*

$$\omega(t) \le \frac{C}{|s|_h^{2\alpha}|s_1|_{h_1}^2}\tilde{\omega}. \tag{3.3.9}$$

Proof. Fix $0 < \epsilon \le 1$. By Lemma 3.6, we know

$$\tilde{\omega} \ge C\Phi^*(|s_1|_{h_1}^2(\mathrm{Pr}_2^*\omega_F + \mathrm{Pr}_1^*\omega_Y)).$$

Since $\mathrm{Pr}_2^*\omega_F + \mathrm{Pr}_1^*\omega_Y$ is a fixed Kähler metric on $U \times F$ and Φ is a biholomorphism from $\pi^{-1}(U)$ to $U \times F$, for any fixed time t, there exists a constant $\alpha > 0$ such that

$$\mathrm{tr}_{\tilde{\omega}}\omega \le \frac{C}{|s|_h^{\alpha}|s_1|_{h_1}^2}.$$

Hence if we set

$$Q_\epsilon = \log(|s|_h^{2\alpha}|s_1|_{h_1}^{2+2\epsilon}\mathrm{tr}_{\tilde{\omega}}\omega). \tag{3.3.10}$$

For each fixed time $t \in (0, T)$. We know the maximum of Q_ϵ must be obtained at some point $x_0 \in \Phi^{-1}(U \times F\backslash E)$. Now we compute at point (x_0, t)

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)Q_\epsilon &= (\frac{\partial}{\partial t} - \Delta)\log \mathrm{tr}_{\tilde{\omega}}\omega + \alpha\mathrm{tr}_\omega R(h) + (1+\epsilon)\mathrm{tr}_\omega R(h_1)\\ &\le (\frac{\partial}{\partial t} - \Delta)\log \mathrm{tr}_{\tilde{\omega}}\omega + \alpha\mathrm{tr}_\omega R(h).\end{aligned} \tag{3.3.11}$$

From the argument in the proof of Lemma 3.3, there exists a uniform constant $C > 0$ such that

$$\alpha\mathrm{tr}_\omega R(h) \le C\mathrm{tr}_\omega\pi^*\omega_Y \le C'.$$

By a well-known computation (see [136, 4, 21]):

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\log \mathrm{tr}_{\tilde{\omega}}\omega &= \frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}(-g^{i\bar{j}}\tilde{g}^{k\bar{q}}\tilde{g}^{p\bar{l}}g_{k\bar{l}}\tilde{R}_{i\bar{j}p\bar{q}} - g^{i\bar{j}}\tilde{g}^{k\bar{l}}g^{p\bar{q}}\tilde{\nabla}_i g_{k\bar{q}}\tilde{\nabla}_{\bar{j}}g_{p\bar{l}} + \frac{|\nabla \mathrm{tr}_{\tilde{\omega}}\omega|^2}{\mathrm{tr}_{\tilde{\omega}}\omega})\\ &\le -\frac{1}{\mathrm{tr}_{\tilde{\omega}}\omega}g^{i\bar{j}}\tilde{g}^{k\bar{q}}\tilde{g}^{p\bar{l}}g_{k\bar{l}}\tilde{R}_{i\bar{j}p\bar{q}}\end{aligned} \tag{3.3.12}$$

Denote $\hat{g}$ as the product metric $\mathrm{Pr}_2^*\pi_1^*\omega_{FS} + \mathrm{Pr}_1^*\omega_Y$, then $\tilde{g} = \Phi^*\hat{g}$. We compute with metric $\hat{g}$, since the bisectional curvature of ω_{FS} is positive, we have

$$\hat{R}_{i\bar{j}p\bar{q}} = (\mathrm{Pr}_2^*\pi_1^*R(\omega_{FS}))_{i\bar{j}p\bar{q}} + (\mathrm{Pr}_1^*R(\omega_Y))_{i\bar{j}p\bar{q}} \geq (\mathrm{Pr}_1^*R(\omega_Y))_{i\bar{j}p\bar{q}}. \tag{3.3.13}$$

Since $\Phi^*\mathrm{Pr}_1^* = \pi^*$, pulling back via the map Φ, we have

$$g^{i\bar{j}}\tilde{g}^{k\bar{q}}\tilde{g}^{p\bar{l}}g_{k\bar{l}}\tilde{R}_{i\bar{j}p\bar{q}} \geq -C(\mathrm{tr}_{\tilde{\omega}}\omega)(\mathrm{tr}_\omega\pi^*\omega_Y)$$

for some uniform constant. Hence we obtain that

$$(\frac{\partial}{\partial t} - \Delta)\log \mathrm{tr}_{\tilde{\omega}}\omega \leq C\mathrm{tr}_\omega \mathrm{Pr}_1^*\omega_Y \leq C'.$$

Hence

$$(\frac{\partial}{\partial t} - \Delta)Q_\epsilon \leq C. \tag{3.3.14}$$

Then using the maximum principle and letting $\epsilon \to 0$, we obtain the lemma. □

We assume that $|s|_h(y) = 1$ and denote $U_{1/2} = \{\tilde{y} \in U | |s|_h^2(\tilde{y}) > 1/2\}$.

Consider the holomorphic vector field

$$\sum_i^m z^i \frac{\partial}{\partial z^i},$$

defined on the unit ball D_1. This defines via π_1 a holomorphic vector field V on $\pi_1^{-1}(D_1) \subset F$ which vanishes to order 1 along the exceptional divisor E. We can extend V to be a smooth $T^{1,0}$ vector field on the whole of F, and $\mathrm{Pr}_2^*(V)$ to be a smooth $T^{1,0}$ vector field on $U_{1/2} \times F$, then pull back by Φ and then extend it to a vector $\tilde{V}$ on whole of X which vanish on $X \backslash \pi^{-1}(U_{1/2})$. We then have the following lemma.

Lemma 3.8. *For $\omega = \omega(t)$ a solution of the Kähler-Ricci flow, we have the estimate*

$$|\tilde{V}|_\omega^2 \leq C|s_1|_{h_1}, \tag{3.3.15}$$

for a uniform constant C. Locally, in $D_{1/2}\backslash\{0\}$ we have

$$|W|_g^2 \leq \frac{C}{r}, \tag{3.3.16}$$

for

$$W = \sum_{i=1}^{m} (\frac{x^i}{r}\frac{\partial}{\partial x^i} + \frac{y^i}{r}\frac{\partial}{\partial y^i})$$

the unit length radial vector field with respect to g_e*, where* $z^i = x^i + \sqrt{-1}y^i$.

In the statement and proof of the lemma, we are identifying $\pi_1^{-1}(D_{1/2}\backslash\{0\})$ with $D_{1/2}\backslash\{0\}$ via the map π_1, writing g for the Kähler metric $(\pi_1^{-1})^*(\omega|_F)$.

Proof. In this proof, we denote ω_U for $\Phi^*(\mathrm{Pr}_1^*\omega_Y + \mathrm{Pr}_2^*\omega_F)$ on $\pi^{-1}(U)$ and a Hermitian metric $\tilde{\omega} = \rho_1\omega_0 + \rho_2\omega_U$, where ρ_1, ρ_2 is a partition of unity for the cover $(X\backslash\pi^{-1}(U_{1/2}), \pi^{-1}(U))$, so that $\tilde{\omega} = \omega_U$ on $\pi^{-1}(U_{1/2})$ and which is equivalent to metric ω_0.

From the Lemma 3.5 we have, in $D_{1/2}$,

$$|\tilde{V}|^2_{\tilde{\omega}} = |V|^2_{\omega_F} = |V|^2_{\pi^*\omega_{FS}}. \tag{3.3.17}$$

It follows that $|\tilde{V}|^2_{\omega_0}$ is uniformly equivalent to $|s_1|^2_{h_1} = r^2$ in $D_{1/2}$.

For any fixed point (x,t). We compute in a normal coordinate system for g at (x,t), we have

$$(\frac{\partial}{\partial t} - \Delta)\log|\tilde{V}|^2_\omega = \frac{1}{|\tilde{V}|^2_\omega}(-g^{i\bar{j}}g_{k\bar{l}}(\partial_i\tilde{V}^k)(\overline{\partial_j\tilde{V}^l}) + \frac{|\nabla|\tilde{V}|^2_\omega|^2_\omega}{|\tilde{V}|^2_\omega}) \leq 0. \tag{3.3.18}$$

Where we use the Cauchy-Schwarz inequality to get the above inequality (the detail, see the proof of Lemma 2.6 in [109]).

Then using the maximum principle, we obtain that there exists a positive constant C such that

$$|\tilde{V}|^2_\omega \leq C|s_1|^2_{h_1}. \tag{3.3.19}$$

Now define a Hermitian metric $\tilde{\omega}_F$ on $\mathbb{P}^m$ by

$$\tilde{\omega}_F = \omega_e \quad \text{on} \quad D_1,$$

and extending in an arbitrary way to be a smooth Hermitian metric on F. For small $\epsilon > 0$, we consider the quantity

$$Q_\epsilon = \log(|\tilde{V}|^{2+2\epsilon}_\omega |s|^{2\alpha+2\epsilon}_h \mathrm{tr}_{\Phi^*(\mathrm{Pr}_2^*\pi_1^*\tilde{\omega}_F + \mathrm{Pr}_1^*\omega_Y)}\omega) - At \tag{3.3.20}$$

where α is the constant in Lemma 3.7 and A is a constant to be determined. Since $\tilde{\omega}_F$ is uniformly equivalent to ω_{FS}, we see that for fixed t, using Lemma 3.7 and (3.3.19),

$$(|V|^{2+2\epsilon}_{\omega}|s|^{2\alpha+2\epsilon}_{h}\mathrm{tr}_{\Phi^*(\mathrm{Pr}_2^*\pi_1^*\tilde{\omega}_F+\mathrm{Pr}_1^*\omega_Y)}\omega)(x)$$

tends to zero as x tends to $X\backslash\pi^{-1}(U_{1/2})\cup\Phi^{-1}(U_{1/2}\times E)$ and thus $Q_\epsilon(x)$ tends to negative infinity. We now applying the maximum principle to Q_ϵ. Since Q_ϵ is uniformly bounded from above on $\Phi^{-1}(U_{1/2}\times F)\backslash\Phi^{-1}(U_{1/2}\times D_{1/2}\backslash\{0\})$, we only need to rule out the case when Q_ϵ attains its maximum at a point in $\Phi^{-1}(U_{1/2}\times D_{1/2}\backslash\{0\})$. Assume that at some point $(x_0,t_0)\in\Phi^{-1}(U_{1/2}\times D_{1/2}\backslash\{0\})\times(0,T)$, we have $\sup_{\Phi^{-1}(U\times F\backslash E)\times[0,t_0]}Q_\epsilon = Q_\epsilon(x_0,t_0)$.

As in the proof of Lemma 3.7, we have

$$(\frac{\partial}{\partial t}-\Delta)\log|s|^{2\alpha+2\epsilon}_h=(\alpha+\epsilon)\mathrm{tr}_\omega R(h)\le C\mathrm{tr}_\omega\pi^*\omega_Y. \tag{3.3.21}$$

By (3.3.12), pulling back by the biholomorphic map Φ, we have in $\Phi^{-1}(U_{1/2}\times D_{1/2}\backslash\{0\})$,

$$(\frac{\partial}{\partial t}-\Delta)\log(\mathrm{tr}_{\Phi^*(\mathrm{Pr}_2^*\pi_1^*\tilde{\omega}_F+\mathrm{Pr}_1^*\omega_Y)}\omega)\le C\mathrm{tr}_\omega\Phi^*\mathrm{Pr}_1^*\omega_Y. \tag{3.3.22}$$

Hence

$$(\frac{\partial}{\partial t}-\Delta)\log|s|^{2\alpha+2\epsilon}_h(\mathrm{tr}_{\Phi^*(\mathrm{Pr}_2^*\pi_1^*\tilde{\omega}_F+\mathrm{Pr}_1^*\omega_Y)}\omega)\le C\mathrm{tr}_\omega\pi^*\omega_Y\le C'. \tag{3.3.23}$$

Take $A=C'+1$. By (3.3.18), at (x_0,t_0), we obtain

$$0\le(\frac{\partial}{\partial t}-\Delta)Q_\epsilon\le -1, \tag{3.3.24}$$

a contradiction. Thus Q_ϵ is uniformly bounded from above. Letting ϵ tend to zero, since $|s|^2_h>1/2$ on $\pi^{-1}(U_{1/2})$, we obtain

$$(\mathrm{tr}_{\Phi^*(\mathrm{Pr}_2^*\pi_1^*\omega_{FS}+\mathrm{Pr}_1^*\omega_Y)}\omega)|\tilde{V}|^2_\omega\le C, \tag{3.3.25}$$

for some uniform constant C. By Lemma 3.6, we have

$$(\mathrm{tr}_{\omega_0}\omega)|\tilde{V}|^2_\omega\le C,$$

and since $|\tilde{V}|^2_\omega\le(\mathrm{tr}_{\omega_0}\omega)|\tilde{V}|^2_{\omega_0}$ this gives

$$|\tilde{V}|^4_\omega\le C|\tilde{V}|^2_{\omega_0}, \tag{3.3.26}$$

then the lemma follows from the fact that $|\tilde{V}|^2_{\omega_0}$ is uniformly equivalent to $|s_1|^2_{h_1}$ in $D_{1/2}$. □

Next we estimate on the lengths of spherical and radial paths in the punctured ball $D_{1/2}\backslash\{0\}$, which again we identify with its preimage in each fiber under π_1.

Lemma 3.9. *We have*

(1) For any $y \in Y$ and for $0 < r < 1/2$, the diameter of the $2m-1$ sphere S_r of radius r in D_1 centered at the origin with the metric induced from $\omega|_{\pi^{-1}(y)}$ is uniformly bounded from above, independent of r and y.

(2) For any $z \in D_{1/2}\backslash\{0\}$, the length of a radial path $\gamma(\lambda) = \lambda z$ for $\lambda \in (0,1]$ with respect to $\omega|_{\pi^{-1}(y)}$ is uniformly bounded from above by a uniform constant multiple of $|z|^{1/2}$.

Hence the diameter of $D_{1/2}\backslash\{0\}$ with respect to $\omega|_{\pi^{-1}(y)}$ is uniformly bounded from above and

$$\mathrm{diam}(\pi^{-1}(y), \omega|_{\pi^{-1}(y)}) \leq C.$$

Proof. For any $y \in Y$, we can choose $|s|^2_h(y) = 1$. Then using Lemma 3.7, consider the metric $\omega|_{\pi^{-1}(y)}$, we have

$$\omega|_{\pi^{-1}(y)} \leq \frac{C}{|s_1|^2_{h_1}} \pi_1^* \omega_{FS}. \tag{3.3.27}$$

Then using the same argument in the proof of Lemma 2.7 in [109], we obtain the lemma. □

Now we can prove the (1) of Theorem 3.4.

Proof of (1) in Theorem 3.4. For any $p, q \in X$. Denote $p_1 = \pi(p), q_1 = \pi(q)$. Then there exist two open subset $U_1, U_2 \subset Y$, such that $p_1 \in U_1, p_2 \in U_2$ and there exist holomorphic maps Φ_1, Φ_2 such that $\Phi_i : \pi^{-1}(U_i) \to U_i \times F (i = 1, 2)$ are the biholomorphic map. Denote $p_2 = \mathrm{Pr}_2\Phi_1(p), q_2 = \mathrm{Pr}_2\Phi_2(q)$. Since Y is compact, we may assume $U_1 \cap U_2 \neq \varnothing$. We assume $\tilde{p}_1 \in U_1 \cap U_2$, denote $\tilde{p} = \Phi_1^{-1}((\tilde{p}_1, p_2))$ and $\tilde{q} = \Phi_2^{-1}((\tilde{p}_1, q_2))$, by Lemma 3.3 we know $d_{\omega(t)}(p, \tilde{p}) \leq C$ and $d_{\omega(t)}(q, \tilde{q}) \leq C$.

Since $\tilde{p}, \tilde{q} \in \pi^{-1}(\tilde{p}_1)$, then by Lemma 3.9, we have $d_{\omega(t)}(\tilde{p}, \tilde{q}) \leq C$. Using the triangle inequality we finish the proof of (1) in Theorem 1.2. □

3.3.2 Diameter of fiber tends to zero

In this subsection, we prove that the diameter of fiber with $\omega(t)$ tends to zero as $t \to T$. Let $d_\omega = d_{\omega(t)}$ be the distance function on X associated to the evolving Kähler metric ω. Using the same argument in the proof of Lemma 3.2 and Lemma 3.3 in [109], we prove the following lemmas.

Lemma 3.10. *Fix a point $y_0 \in Y$. There exists a uniform constant C (independent of y_0) such that for any $p, q \in E$, and any $t \in [0, T)$,*

$$d_\omega(\Phi^{-1}(y_0, p), \Phi^{-1}(y_0, q)) \leq C(T - t)^{1/3}. \tag{3.3.28}$$

Proof. We can assume that $|s|_h^2(y_0) = 1$. We replace E by $\Phi^{-1}(\{y_0\} \times E)$ in the proof of Lemma 3.2 in [109], using Lemma 3.7 and using Lemma 3.9. See the argument of the proof of Lemma 3.2 in [109]. □

Combine Lemma 3.9 and Lemma 3.10, we have

Lemma 3.11. *Fix a point $y_0 \in Y$. There exists a uniform constant C (independent of y_0) such that for any $0 < \delta_0 < 1/2$ and for anyt $\in [0, T)$*

$$\mathrm{diam}_{\omega(t)}(\Phi^{-1}(\{y_0\} \times \pi_1^{-1}(D_{\delta_0}))) < C(|\delta_0|^{1/2} + (T - t)^{1/3}). \tag{3.3.29}$$

Proof. We also assume that $|s|_h^2(y_0) = 1$. For any $p, q \in \pi_1^{-1}(D_{\delta_0}))$. Since Lemma 3.10, we only consider $p \in \pi_1^{-1}(D_{\delta_0} \backslash \{0\}))$ and $q \in E$. By Lemma 3.9 (2), we know the length of a radial path $\gamma(\lambda) = \lambda p$ with respect to ω is uniformly bounded from above by $C|p|^{1/2} \leq C\delta_0^{1/2}$. Since $\gamma(\lambda)$ tends to a point $p_0 \in E$ as $\lambda \to 0^+$, we know $d_{\omega|_{\pi^{-1}(y_0)}}(p, p_0) \leq C\delta_0^{1/2}$. By Lemma 3.10, $d_\omega(\Phi^{-1}((y_0, p_0)), \Phi^{-1}((y_0, q))) \leq C(T-t)^{1/3}$. Hence we have

$$d_\omega(\Phi^{-1}((y_0, p)), \Phi^{-1}((y_0, q))) \leq C(\delta_0^{1/2} + (T - t)^{1/3}).$$

□

Lemma 3.12. *Fix a point $y_0 \in Y$. There exists a uniform constant C (independent of y_0) such that for any $p, q \in \pi^{-1}(y_0)$, and any $t \in [0, T)$,*

$$d_\omega(p, q) \leq C(T-t)^{1/15}. \tag{3.3.30}$$

Proof. We also assume that $|s|_h^2(y_0) = 1$. For each fixed t satisfying $T - t < 2^{-15}$, i.e., $2(T-t)^{2/15} < 1/2$. Take $\delta_0 = (T-t)^{2/15}$, by Lemma 3.11, we have

$$\mathrm{diam}_{\omega(t)}(\Phi^{-1}(\{y_0\} \times \pi_1^{-1}(D_{2\delta_0}))) < C((T-t)^{1/15} + (T-t)^{1/3}) \leq C'(T-t)^{1/15}.$$

We denote $p' = \mathrm{Pr}_2 \circ \Phi(p), q' = \mathrm{Pr}_2 \circ \Phi(p)$. Hence we only consider the case of $p', q' \in F \backslash \pi_1^{-1}(D_{\delta_0})$.

Since $\pi^{-1}(y_0)$ is biholomorphic to F, which is $\mathbb{P}^m$ blown up at one point, there exists a curve $\gamma \cong \mathbb{P}^1$, such that $p, q \in \gamma \cap (\pi^{-1}(y_0) \backslash \Phi^{-1}(\{y_0\} \times \pi_1^{-1}(D_{\delta_0})))$. We may assume that p, q lie in a fixed coordinate chart U whose image under the holomorphic coordinate $z = x + \sqrt{-1}y$ is a ball of radius 2 in $\mathbb{C} = \mathbb{R}^2$ with respect to the Euclidean metric ω_e. In this coordinate, by Lemma 3.7, we know

$$\omega(t)|_{\pi^{-1}(y_0)} \leq \frac{C}{(\delta_0)^2}(\pi_1)^* \omega_{FS} \leq \frac{C'}{(\delta_0)^2}\omega_e.$$

Since closed curve $\gamma \subset F$,

$$\int_\gamma \omega(t) = \int_\gamma [\frac{1}{T}((T-t)\omega_0 + t\pi^*\omega_Y)] = \frac{T-t}{T}\int_\gamma \omega_0 \leq C(T-t). \tag{3.3.31}$$

Write $\sigma = (T-t)^{1/3} > 0$, which we may assume is sufficiently small.

Moreover, we may assume that p is represented by the origin in $\mathbb{C} = \mathbb{R}^2$, that q is represented by the point $(x_0, 0)$ with $0 < x_0 < 1$, and that the rectangle

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq x_0, -\sigma \leq y \leq \sigma\} \subset \mathbb{R}^2 = \mathbb{C}$$

is contained in the image of U. Now in $\mathcal{R}$, the fixed metric $\hat{g}_0$ induced from the metric g_0 on X is uniformly equivalent to the Euclidean metric. Thus from (3.3.31),

$$\int_{-\sigma}^{\sigma}(\int_0^{x_0}(\mathrm{tr}_{\hat{g}_0} g)dx)dy = \int_{\mathcal{R}}(\mathrm{tr}_{\hat{g}_0} g)dxdy \leq C(T-t). \tag{3.3.32}$$

Hence there exists $y' \in (-\sigma, \sigma)$ such that

$$\int_0^{x_0}(\mathrm{tr}_{\hat{g}_0} g)(x, y')dx \leq \frac{C}{\sigma}(T-t) = C(T-t)^{2/3}. \tag{3.3.33}$$

Now let p'' and q'' be the points represented by coordinates $(0, y')$ and (x_0, y'). Then, considering the horizontal path $s \mapsto (s, y')$ between p'' and q'', we have

$$\begin{aligned} d_\omega(p'', q'') &\leq \int_0^{x_0} (\sqrt{g(\partial_x, \partial_x)})(x, y')dx \\ &= \int_0^{x_0} (\sqrt{\mathrm{tr}_{\hat{g}_0} g}\sqrt{\hat{g}_0(\partial_x, \partial_x)})(x, y')dx \\ &\leq (\int_0^{x_0} (\mathrm{tr}_{\hat{g}_0} g)(x, y')dx)^{1/2}(\int_0^{x_0} (\hat{g}_0(\partial_x, \partial_x))(x, y')dx)^{1/2} \\ &\leq C(T-t)^{1/3}. \end{aligned} \tag{3.3.34}$$

and

$$d_\omega(p, p'') \leq d_{\omega|_{\pi^{-1}(y_0)}}(p, p'') \leq \frac{C}{(\delta_0)^2}\sigma = C(T-t)^{1/15}. \tag{3.3.35}$$

Using the same argument we can prove

$$d_\omega(q, q'') \leq C(T-t)^{1/15}. \tag{3.3.36}$$

Hence by triangle inequality

$$d_{\omega(t)}(p, q) \leq C(T-t)^{1/15}. \tag{3.3.37}$$

□

3.3.3 Gromov-Hausdorff Convergence

In this subsection, we prove that there exists a sequence of metrics along the Kähler-Ricci flow converges sub-sequentially to a metric on Y in the Gromov-Hausdorff sense as $t \to T$.

Lemma 3.13. *Write $d_t : X \times X \to \mathbb{R}$ for the distance function induced by the metric $\omega(t)$. There exists a sequence of times $t_i \to T$, such that the functions d_{t_i} converge uniformly to a function $d_\infty : X \times X \to \mathbb{R}$.*

Moreover, if, for $p, q \in Y$, we let $d_{Y,\infty}(p, q) = d_\infty(\tilde{p}, \tilde{q})$, where $\tilde{p} \in \pi^{-1}(p)$ and $\tilde{q} \in \pi^{-1}(q)$, then $d_{Y,\infty}$ defines a distance function on Y, which is uniformly equivalent to that induced by ω_Y.

Proof. First note that the functions $d_t : X \times X \to \mathbb{R}$ are uniformly bounded. Indeed by (1) in Theorem 1.2, we have a constant $C > 0$ such that $d_t(x, y) < C$ for any $t < T$

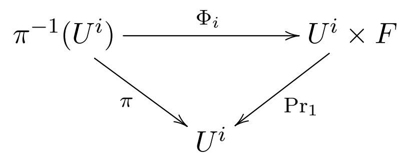
and $x, y \in X$. Next, we prove that the functions $d_t : X \times X \to \mathbb{R}$ are equicontinuous with respect to the metric on $X \times X$ induced by d_0.

For any $x, x', y, y' \in X$, we have

$$\begin{aligned}|d_t(x,y) - d_t(x',y')| &\le |d_t(x,y) - d_t(x,y')| + |d_t(x,y') - d_t(x',y')| \\ &\le d_t(y,y') + d_t(x,x').\end{aligned} \tag{3.3.38}$$

Define $|s|_h^2 = hs\bar{s}$, and we assume $|s|_h(y) = 1$.

Since Y is compact, there exist finite Zariski open sets $U^1, \cdots, U^N$ and biholomorphic map $\Phi_1, \cdots, \Phi_N$ such that the diagram



commutes, where $D_i = Y \backslash U^i$, and $\cup_{i=1}^N \pi^{-1}(U^i_{1/3}) = X$. Let s_i be the holomorphic sections on $[D_i]$ and let h_i be the Hermitian metrics on $[D_i]$. Here $U^i_r = \{\tilde{y} \in U^i | |s_i|_{h_i} > r\}$ for $0 < r \le 1$.

Claim 1. *There exists a uniform constant $\delta > 0$ such that if $x \in \pi^{-1}(U^{i_0}_{1/3})$ for some $i_0 \in \{1, \cdots, N\}$ and $d_0(x, x') < \delta$, then $x' \in \pi^{-1}(U^{i_0}_{2/3})$.*

Proof of Claim. Denote A^i_r be the boundary of $\pi^{-1}(U^i_r)$ and denote $d_0(A^i_{1/3}, A^i_{2/3}) = \delta_i > 0$. Let ρ^i_1, ρ^i_2 be the partition of unity for the cover $(X \backslash \pi^{-1}(U^i_{2/3}), \pi^{-1}(U^i))$. Then $\omega^i = \rho^i_1 \omega_0 + \rho^i_2 \Phi_i^*(\mathrm{Pr}_1^* \omega_Y + \mathrm{Pr}_2^* \omega_F)$ are the Hermitian metrics on X, which are equivalent to metric ω_0, and $\omega^i = \Phi_i^*(\mathrm{Pr}_1^* \omega_Y + \mathrm{Pr}_2^* \omega_F)$ on $\pi^{-1}(U^i_{2/3})$. Hence there exists a uniform constant $C > 0$ such that

$$C^{-1}\omega_0 \le \omega^i \le C\omega_0. \tag{3.3.39}$$

We take $\delta = C^{-2} \min\{\delta_1, \cdots, \delta_N\}$. Now we can prove $x' \in \pi^{-1}(U^{i_0}_{2/3})$. If not, we know

$$d_{\omega^{i_0}}(x,x') \ge d_{\omega^{i_0}}(A^{i_0}_{1/3}, A^{i_0}_{2/3}) \ge C^{-1} d_0(A^{i_0}_{1/3}, A^{i_0}_{2/3}) = C^{-1}\delta_{i_0}. \tag{3.3.40}$$

On the other hand,

$$d_{\omega^{i_0}}(x,x') \le C d_0(x,x') < C\delta. \tag{3.3.41}$$

It is a contradiction. Hence $x' \in \pi^{-1}(U^{i_0}_{2/3})$. We finish the proof of the claim. $\square$

Now if $x, x' \in X$ satisfying $d_0(x, x') < \delta$, by the above claim we have $x, x' \in U^{i_0}_{2/3}$ for some $i_0 \in \{1, \cdots, N\}$. Now we choose $q_0 \in F$ such that $|s|^2_h(q_0) = 1$. Then by the Lemma 3.7, we have

$$d_t(\Phi^{-1}_{i_0}(\pi(x), q_0), \Phi^{-1}_{i_0}(\pi(x'), q_0)) \le C d_{\omega_Y}(\pi(x), \pi(x')) \le C' d_0(x, x'). \tag{3.3.42}$$

By Lemma 3.12

$$\begin{aligned} d_t(x, x') &\le d_t(x, \Phi^{-1}_{i_0}(\pi(x), q_0)) + d_t(\Phi^{-1}_{i_0}(\pi(x'), q_0), x') + d_t(\Phi^{-1}_{i_0}(\pi(x), q_0), \Phi^{-1}_{i_0}(\pi(x'), q_0)) \\ &\le C[(T-t)^{1/15} + d_0(x, x')]. \end{aligned} \tag{3.3.43}$$

Now we prove the following lemma.

Lemma 3.14. *With the assumption of* (3.3.43)*, there exists a sequence of times $t_i \to T$, such that the functions d_{t_i} converges uniformly to a continuous function d_∞.*

Proof of Lemma 3.14. We denote $M = X \times X$, is a compact manifold. The first thing to recall is that any compact metric space has a countable dense subset. This follows directly from the definition of compactness. Namely, given any $k \in \mathbb{N}$, cover M by all the balls of radius $\frac{1}{k}$ (centred at all the points of M.) By compactness of M this has a finite subcover, let $Q_k \subset M$ be the set of centers of such a finite subcover. Then every point of M is in one of the balls, so it is distant at most $\frac{1}{k}$ from (at least) one of the points in Q_k. The union, $Q = \cup_k Q_k$, of these finite sets is (at most) countable and is clearly dense in M, i.e., any point in M is the limit of a sequence in Q.

Let $\{d_{t_n}\}$ be a sequence of d_t ($t_n \to T$ as $n \to \infty$). Take a point $q_1 \in Q$, then $\{d_{t_n}(q_1)\}$ is a bounded sequence in $\mathbb{R}$. So, by Heine-Borel Theorem, we may extract a subsequence of d_{t_n} so that $\{d_{t_{n_{1,j}}}(q_1)\}$ converges in $\mathbb{R}$. Since Q is countable we can construct successive subsequences, $d_{t_{n_{k,j}}}$ of the preceding subsequence $d_{t_{n_{k-1,j}}}$, so that the kth subsequence converges at the first kth point $\{q_1, \cdots, q_k\}$ of Q. Now, the diagonal sequence $d_{t_{n_i}} = d_{t_{n_{i,i}}}$ is a subsequence of d_{t_n}. So along this subsequence $d_{t_{n_i}}(q)$ converges for each point in Q. It is a subsequence of the original sequence $\{d_{t_n}\}$, we just denote it as $\{d_{t_n}\}$ and we want to show that it converges uniformly; it suffices to show that it is uniformly Cauchy.

For any given $\epsilon > 0$. By the assumption of (3.3.43), we can choose $T_\epsilon \in [0, T)$ such that for any $t \in [T_\epsilon, T)$ we have $C(T-t)^{1/15} \leq \epsilon/6$ and choose $\delta = \epsilon/6C > 0$ so that $|d_{t_n}(x) - d_{t_n}(y)| < \epsilon/3$ whenever $d_0(x,y) < \delta$ and $t_n \in [T_\epsilon, T)$. Next choose $k > 1/\delta$. Since there are only finitely many points in Q_k we may choose N so large that for any $n > N$ we have $t_n \in [T_\epsilon, T)$, then $|d_{t_n}(q) - d_{t_m}(q)| < \epsilon/3$ if $q \in Q_k$ and $n, m > N$. Then for a general point $x \in M$ there exists $q \in Q_k$ with $d_0(x,q) < 1/k < \delta$, so

$$|d_{t_n}(x) - d_{t_m}(x)| \leq |d_{t_n}(x) - d_{t_n}(q)| + |d_{t_n}(q) - d_{t_m}(q)| + |d_{t_m}(q) - d_{t_m}(x)| < \epsilon \quad (3.3.44)$$

whenever $n, m > N$. Thus the sequence is uniformly Cauchy, hence uniformly convergent. Hence the function d_∞ is continuous. We finish the proof of Lemma 3.14. □

It follows that d_∞ is nonnegative, symmetric and satisfies the triangle inequality.

Let $d_Y : Y \times Y \to \mathbb{R}$ be the distance function on Y induced by the metric ω_Y. From Lemma 3.2, we have a constant $c > 0$ such that $d_t(x,y) \geq \sqrt{c} d_Y(\pi(x), \pi(y))$. It follows that the limit d_∞ satisfies

$$d_\infty \geq \sqrt{c} d_Y(\pi(x), \pi(y)). \quad (3.3.45)$$

Now we want to prove the upper bound. When $\pi(x), \pi(y) \in U^{i_0}_{1/3}$ for some $i_0 \in \{1, \cdots, N\}$, by the inequality (3.3.43) and Lemma 3.9, we have

$$d_t(x,y) \leq C(T-t)^{1/15} + C d_Y(\pi(x), \pi(y)). \quad (3.3.46)$$

This implies that

$$d_\infty(x,y) \leq C d_Y(\pi(x), \pi(y)). \quad (3.3.47)$$

Now we consider the general case. Assume $\pi(x) \in U^1_{1/3}$ and $\pi(y) \in U^{i_0}_{1/3}$ for some $i_0 \neq 1$. Since $\cup_{i=1}^N U^i_{1/3} = Y$, we know for any minimal geodesic $\gamma(t)$ connecting $\pi(x)$ and $\pi(y)$ with metric ω_Y, there exist a finite points $y_0 = \pi(x), y_1 = \gamma(t_1), y_2 = \gamma(t_2), \cdots, y_L = \gamma(t_L), y_{L+1} = \pi(y) \in Y$ such that y_i and y_{i+1} are in the same $U^{j_0}_{1/3}$ for some $j_0 \in \{1, 2, \cdots, N\}$. We choose $x_0 = x, x_{L+1} = y$ and any $x_1, \cdots, x_L \in X$ satisfying $\pi(x_i) =$

y_i. Hence by

$$
\begin{aligned}
d_Y(\pi(x), \pi(y)) &= \sum_{i=0}^{L} d_Y(y_i, y_{i+1}) \\
&\geq C^{-1} \sum_{i=0}^{L} d_\infty(x_i, x_{i+1}) \\
&\geq C^{-1} d_\infty(x, y).
\end{aligned} \tag{3.3.48}
$$

For $p, q \in Y$, we now define $d_{Y,\infty}(p, q) = d_\infty(\tilde{p}, \tilde{q})$, where $\tilde{p} \in \pi^{-1}(p), \tilde{q} \in \pi^{-1}(q)$. This is independent of the choice of lifts $\tilde{p}$ and $\tilde{q}$ since if say $\tilde{p}'$ is a different lift of p, then by (3.3.48) and by the triangle inequality, we have

$$
d_\infty(\tilde{p}', \tilde{q}) \leq d_\infty(\tilde{p}, \tilde{q}) + d_\infty(\tilde{p}', \tilde{p}) = d_\infty(\tilde{p}, \tilde{q}) \tag{3.3.49}
$$

and by switching $\tilde{p}$ and $\tilde{p}'$ we get the reverse inequality. Moreover, it follows from (3.3.47) and (3.3.48) that $d_{Y,\infty}$ is uniformly equivalent to d_Y. Hence we finish the proof of Lemma 3.13. □

Theorem 3.15. *In the notation of Lemma 3.13 we have $(X, d_{t_i}) \to (Y, d_{Y,\infty})$ in the Gromov-Hausdorff sense, where we recall that d_{t_i} is the distance function induced by the metric $\omega(t_i)$.*

Proof. Using the same argument in the proof of Theorem 3.1 in [103]. Hence finish the proof of (2) in Theorem 3.4. □

3.4 F Is Some Weighted Projective Space Blown Up At The Orbifold Point

In this section, we will consider the case of the fiber F is the family of m-folds $M_{m,k}(1 \leq k < m)$, was introduced by Calabi [20], which as generalization of the Hirzebruch surfaces. $M_{n,k}$ is a compactification of the blow up of a $\mathbb{Z}_k$-orbifold point of the orbifold $Y_{m,k}$, is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{m-1}$. The detail of the construction of $M_{m,k}$ and $Y_{m,k}$, please see [110].

3.4.1 Definitions of $M_{m,k}$ and $Y_{m,k}$

We define $M_{m,k}$ to be the $\mathbb{P}^1$-bundle

$$M_{m,k} = \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O}) \tag{3.4.1}$$

over $\mathbb{P}^{n-1}$. We will assume in this paper that $m \geq 2$ and $1 \leq k < m$ (the latter implies that $M_{m,k}$ is a Fano manifold). Denote by E_0 and E_∞ the divisors in $M_{m,k}$ corresponding to sections of $\mathcal{O}(-k)\oplus\mathcal{O}$ with zero $\mathcal{O}(-k)$ and $\mathcal{O}$ component, respectively (the detail see Section 9 in [110]). E_0 is an exceptional divisor with normal bundle $\mathcal{O}(-k)$ of the type discussed above. The complex manifold $M_{m,k}$ can be described as

$$\begin{aligned} M_{m,k} =& \{([Z_1, \cdots, Z_m], (\sigma, \mu) \in \mathbb{P}^{m-1} \times ((\mathbb{C}^m \times \mathbb{C})\backslash\{(0,0)\})|\sigma \text{ is in} \\ & \text{the line } \lambda \mapsto (\lambda(Z_1)^k, \cdots, \lambda(Z_m)^k)\}/\sim, \end{aligned} \tag{3.4.2}$$

where

$$([Z_1, \cdots, Z_m], (\sigma, \mu)) \sim ([Z_1, \cdots, Z_m], (\sigma', \mu')) \tag{3.4.3}$$

if there exists $a \in \mathbb{C}^*$ such that $(\sigma, \mu) = (a\sigma', a\mu')$. Then E_0 and E_∞ are the divisors $\{\sigma = 0\}$ and $\{\mu = 0\}$, respectively.

The orbifold $Y_{m,k}$ is the weighted projective space

$$Y_{m,k} = \{(Z_0, \cdots, Z_m) \in \mathbb{C}^{n+1}\}/\sim . \tag{3.4.4}$$

where $(Z'_0, \cdots, Z'_m) \sim (Z_0, \cdots, Z_m)$ if there exists $\lambda \in \mathbb{C}^*$ such that

$$(Z'_0, \cdots, Z'_m) = (\lambda^k Z_0, \lambda Z_1, \cdots, \lambda Z_m). \tag{3.4.5}$$

We write elements of $Y_{m,k}$ as $[Z_0, \cdots, Z_m]$. Then $Y_{m,k}$ has a single $\mathbb{Z}_k$-orbifold point at $[1, 0, \cdots, 0]$.

We define the map $\pi_1 : M_{m,k} \to Y_{m,k}$ by

$$\pi_1(([Z_1, \cdots, Z_m], (\sigma, \mu))) = [\mu, b^{1/k} Z_1, \cdots, b^{1/k} Z_m], \tag{3.4.6}$$

where $b \in \mathbb{C}$ is defined by

$$\sigma = b((Z_1)^k, \cdots, (Z_m)^k). \tag{3.4.7}$$

π_1 is globally well-defined, surjective and injective on the complement of E_0.

If we identify the line bundle $\mathcal{O}(-k)$ with the open subset $\{\mu \neq 0\}$ of $M_{m,k}$ and $\mathbb{C}^m/\mathbb{Z}_k$ with the open subset $\{Z_0 \neq 0\}$ of $Y_{m,k}$ via $(z_1, \cdots, z_m) \mapsto [1, z_1, \cdots, z_m]$, then π_1 restricted to $M_{m,k} \backslash E_0$ is a biholomorphism onto $Y_{m,k} \backslash \{[1, 0, \cdots, 0]\}$ and $\pi_1(E_0) = [1, 0, \cdots, 0]$.

All of the manifolds $M_{m,k}$ admit Kähler metrics. Indeed, the cohomology classes of the line bundles $[E_0]$ and $[E_\infty]$ span $H^{1,1}(M_{m,k}; \mathbb{R})$ and every Kähler class α can be written uniquely as

$$\alpha = \frac{b}{k}[E_\infty] - \frac{a}{k}[E_0] \tag{3.4.8}$$

for constants a, b with $0 < a < b$. The first chern class

$$c_1(M_{m,k}) = \frac{m+k}{k}[E_\infty] - \frac{n-k}{k}[E_0]. \tag{3.4.9}$$

Hence if $1 \leq k \leq m-1$, then $M_{m,k}$ is a Fano manifold. He and Sun proved that any weighted projective space exists an orbifold Kähler (in fact is Kähler-Ricci soliton) metric ω_{orb} with positive bisectional curvature, see Theorem 1.2 in [62].

Let L be the $(-k)$ line bundle over $\mathbb{P}^{m-1}$, for $k \geq 1$. We give a description of the total space of L as follows. Writing $[Z_1, \cdots, Z_m]$ for the homogeneous coordinates on $\mathbb{P}^{m-1}$, we define

$$L = \{([Z_1, \cdots, Z_m], \sigma) \in \mathbb{P}^{m-1} \times \mathbb{C}^m | \sigma \text{ is in the line } \lambda \mapsto (\lambda(Z_1)^k, \cdots, \lambda(Z_m)^k)\}, \tag{3.4.10}$$

and let $P : L \to \mathbb{P}^{m-1}$ be the projection onto the first factor. Each fiber $P^{-1}([Z_1, \cdots, Z_m])$ is the line in $\mathbb{C}$. L can be given m complex coordinate charts

$$U_i = \{([Z_1, \cdots, Z_m], \sigma) \in L | Z_i \neq 0\} \text{ for } i = 1, \cdots, m.$$

On U_i, we have coordinates $\omega^j_{(i)}$ for $j = 1, \cdots, m$ with $j \neq i$ and $y_{(i)}$. The $\omega^j_{(i)}$ are defined by

$$\omega^j_{(i)} = Z_j/Z_i \text{ for } j \neq i,$$

and $y_{(i)}$ by

$$\sigma = \frac{y_{(i)}}{(Z_i)^k}((Z_1)^k, \cdots, (Z_m)^k).$$

On $U_i \cap U_l$ with $i \neq l$, we have

$$\omega^j_{(i)} = \frac{\omega^j_{(l)}}{\omega^i_{(l)}} \text{ for } j \neq i, l, \quad \omega^l_{(i)} = \frac{1}{\omega^i_{(l)}} \text{ and } y_{(i)} = y_{(l)}(\tfrac{Z_i}{Z_l})^k = y_{(l)}(\omega^i_{(l)})^k.$$

Now let E be the submanifold of L defined by the zero section of L over $\mathbb{P}^{m-1}$. Denote by $[E]$ the pull-back line bundle P^*L over L, which corresponds to the hypersurface E. Writing the transition functions of $[E]$ in $U_i \cap U_l$ as $t_{il} = (Z_i/Z_l)^k = y_{(i)}/y_{(l)}$, we have a section $\tilde{s}$ over $[E]$ given by

$$s_i : U_i \to \mathbb{C}, \quad s_i = y_i.$$

We can define a Hermitian metric $\tilde{h}$ on the fibers of $[E]$ by

$$h_i = \frac{(\Sigma^m_{j=1}|Z_j|^2)^k}{|Z_i|^{2k}} \text{ on } U_i.$$

Namely, $\tilde{h}$ is the pull-back of h^{-k}_{FS} where h_{FS} is the Fubini-Study metric on $\mathcal{O}(1)$. We have

$$|\tilde{s}|^2_{\tilde{h}} = |y_{(i)}|^2 \frac{(\Sigma^m_{j=1}|Z_j|^2)^k}{|Z_i|^{2k}} \quad \text{on } U_i. \tag{3.4.11}$$

If we denote $r^2 = \sum^m_{i=1} |z^i|^2$, then we have

$$\pi_1^* r^{2k} = |\tilde{s}|^2_{\tilde{h}} \tag{3.4.12}$$

on L.

Let ω_e be the standard orbifold metric on $\mathbb{C}^m/\mathbb{Z}_k$, which lifts to the Euclidean metric on $\mathbb{C}^m$, we write ω_e as

$$\omega_e = \frac{\sqrt{-1}}{2\pi}\Sigma_i dz^i \wedge d\overline{z^i}. \tag{3.4.13}$$

Denote ω_F be the metric ω_X in Lemma 2.3 in [110], it is a Kähler metric on $M_{m,k}$. We will work in a local uniformizing chart around the orbifold point $p \in Y_{n,k}$, which we identify with the unit ball D_1 in $\mathbb{C}^m$. Then we know that ω_{orb} is uniformly equivalent to the Euclidean metric ω_e on D_1. We write D_R for the ball in $\mathbb{C}^m$ of radius $R > 0$. Then from the section 2 in [110], on $\pi_1^{-1}(D_1\backslash\{0\})$ we have

$$k|\tilde{s}|^{2(k-1)/k}_{\tilde{h}} \pi_1^*\omega_e \leq \omega_F \leq \frac{C}{|\tilde{s}|^{2/k}_{\tilde{h}}}\pi_1^*\omega_e \tag{3.4.14}$$

for some uniform constant $C > 0$. Hence on $\pi_1^{-1}(D_1\backslash\{0\})$ there exists a uniform constant C such that

$$C^{-1}|\tilde{s}|^{2(k-1)/k}_{\tilde{h}} \pi_1^*\omega_{\text{orb}} \leq \omega_F \leq \frac{C}{|\tilde{s}|^{2/k}_{\tilde{h}}}\pi_1^*\omega_{\text{orb}}. \tag{3.4.15}$$

Now if we denote $\tilde{\omega} = \Phi^*(\mathrm{Pr}_2^*\pi_1^*\omega_{\mathrm{orb}} + \mathrm{Pr}_1^*\omega_Y)$, since the bisectional curvature of ω_{orb} is positive, using the same argument of Lemma 3.7, we obtain

Lemma 3.16. *There exist uniform constants $C > 0$ and $\alpha > 0$ such that for $\omega = \omega(t)$ a solution of the Kähler-Ricci flow,*

$$\omega(t) \leq \frac{C}{|s|_h^{2\alpha}|\tilde{s}|_{\tilde{h}}^{2/k}}\tilde{\omega}. \tag{3.4.16}$$

Using the same notations as in Lemma 3.8, then we have

Lemma 3.17. *For $\omega = \omega(t)$ a solution of the Kähler-Ricci flow. Then there exist uniform constants $C > 0$ and $R_0 = R_0(m, k) \in (0, 1)$ such that on $D_{R_0}\backslash\{0\}$:*

$$|\tilde{V}|_\omega^2 \leq Cr^{2k/k+1}. \tag{3.4.17}$$

Locally, in $D_{R_0}\backslash\{0\}$ we have

$$|W|_g^2 \leq \frac{C}{r^{2/(k+1)}}, \tag{3.4.18}$$

for

$$W = \sum_{i=1}^m (\frac{x^i}{r}\frac{\partial}{\partial x^i} + \frac{y^i}{r}\frac{\partial}{\partial y^i})$$

the unit length radial vector field with respect to g_e, where $z^i = x^i + \sqrt{-1}y^i$.

Proof. We using (3.4.14) and using the same argument in the proof of of Lemma 3.8, we can obtain the Lemma. □

Using the same argument in the proof of Lemma 3.9, we can estimate on the lengths of spherical and radial paths in the punctured ball $D_{R_0}\backslash\{0\}$.

Lemma 3.18. *We have*

(1) *For any $y \in Y$ and for $0 < r < R_0$, the diameter of the $2m-1$ sphere S_r of radius r in D_{R_0} centered at the origin with the metric induced from $\omega|_{\pi^{-1}(y)}$ is uniformly bounded from above, independent of r and y.*

(2) *For any $z \in D_{R_0}\backslash\{0\}$, the length of a radial path $\gamma(\lambda) = \lambda z$ for $\lambda \in (0, 1]$ with respect to $\omega|_{\pi^{-1}(y)}$ is uniformly bounded from above by a uniform constant multiple of $|z|^{k/(k+1)}$.*

Hence the diameter of $D_{R_0}\backslash\{0\}$ *with respect to* $\omega|_{\pi^{-1}(y)}$ *is uniformly bounded from above and*

$$\operatorname{diam}(\pi^{-1}(y), \omega|_{\pi^{-1}(y)}) \le C.$$

Then using the same argument in Section 3, we can prove the diameter of the fiber along the metrics $g(t)$ tend to zero, then we obtain

Theorem 3.19. *Let* (X, Y, π, F) *be a Fano bundle with* F *is* $M_{m,k}(1 \le k < m)$*,* ω_Y *be a Kähler metric on* Y *and* ω_0 *be a Kähler metric on* X*. Assume* $\omega(t)$ *is a solution of the Kähler-Ricci flow (1.1.1) for* $t \in [0, T)$ *with initial metric* ω_0 *and* $[\omega_0] - 2\pi T c_1(X) = [\pi^*\omega_Y]$*, then we have*

(1) $\operatorname{diam}(X, \omega(t)) \le C$ *for some uniform constant* $C > 0$*;*

(2) There exists a sequence of times $t_i \to T$ *and a distance function* d_Y *on* Y *(which is uniformly equivalent to the distance induced by* ω_Y*, such that* $(X, \omega(t_i))$ *converge to* (Y, d_Y) *in the Gromov-Hausdorff sense.*

Chapter 4

Geometric estimate of Monge-Ampère equation

The chapter is from joint work [46] with Bin Guo and Jian Song.

4.1 Introduction

As we mentioned in the introduction, complex Monge-Ampère equations are closely related to geometric equations of Einstein type, and in many geometric settings, one makes assumption on a uniform lower bound of the Ricci curvature. Therefore it is natural to consider the family of volume measures, whose curvature is uniformly bounded below. More precisely, we let $\Omega = e^{-f}\theta^n$ be a smooth volume form on X such that

$$Ric(\Omega) = -\sqrt{-1}\partial\overline{\partial}\log\Omega \geq -A\theta \tag{4.1.1}$$

for some fixed constant $A \geq 0$. This is equivalent to say,

$$\sqrt{-1}\partial\overline{\partial}f \geq -Ric(\theta) - A\theta,$$

or

$$f \in \mathrm{PSH}(X, Ric(\theta) + A\theta).$$

Let's explain one of the motivations for condition (4.1.1) by some examples. Let $\{E_i\}_{i=1}^I$ and $\{F_j\}_{j=1}^J$ be two families of effective divisors of X. Let σ_{E_i}, σ_{F_j} be the defining sections for E_i and F_j, respectively, and h_{E_i} and h_{F_j} smooth hermitian metrics for the line bundles associated to E_i and F_j respectively. In [136], Yau considers the following degenerate complex Monge-Ampère equations

$$(\theta + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \left(\frac{\sum_{i=1}^I |\sigma_{E_i}|_{h_{E_i}}^{2\beta_i}}{\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j}}\right)\theta^n, \tag{4.1.2}$$

where $\alpha_j, \beta_i > 0$, and various estimates are derived [136] assuming certain bounds on the degenerate right hand side of equation (4.1.2).

If we consider the following case

$$(\theta + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \frac{\theta^n}{\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j}} \ . \tag{4.1.3}$$

the volume measure will blow up along common zeros of $\{F_j\}_{j=1}^J$. If the volume measure on the right hand side of the equation (4.1.3) is L^p-integrable for some $p > 1$, i.e.,

$$\Omega = \left(\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j} \right)^{-1} \theta^n$$

satisfies

$$\frac{\Omega}{\theta^n} = \left(\sum_{j=1}^J |\sigma_{F_j}|_{h_{F_j}}^{2\alpha_j} \right)^{-1} \in L^p(X, \theta^n), \text{ for some } p > 1, \int_X \Omega = \int_X \theta^n,$$

then there exists a unique (up to a constant translation) continuous solution of (4.1.3). Furthermore, Ω can be approximated by smooth volume forms Ω_j (c.f. [31]) satisfying

$$Ric(\Omega_j) \geq -(A + A')\theta, \ \left\| \frac{\Omega_j}{\theta^n} \right\|_{L^p(X,\theta^n)} \leq \left\| \frac{\Omega}{\theta^n} \right\|_{L^p(X,\theta^n)}, \ \int_X \Omega_j = \int_X \theta^n$$

for some fixed $A' \geq 0$. Therefore condition (4.1.1) is a natural generalization of the above case. In the special case when $\{F_j\}_{j=1}^J$ is a union of smooth divisors with simple normal crossings and each $\alpha_j \in (0, 1)$, the solution of equation (4.1.3) has conical singularities of cone angle of $2\pi(1 - \alpha_j)$ along F_j, $j = 1, ..., J$.

Before proving our Theorem 1.3, we first make some remarks on this theorem.

Remark 2. *If we write $\Omega = e^{-f}\theta^n$, assumption (1.2.3) in Theorem 1.3 on Ω is equivalent to the following on f:*

$$e^{-f} \in L^p(X, \theta), \ \int_X e^{-f}\theta = [\theta]^n, \ f \in \mathrm{PSH}(X, Ric(\theta) + A\theta). \tag{4.1.4}$$

f is uniformly bounded above by the plurisubharmonicity and the Kähler metric g associated to $\omega = \theta + \sqrt{-1}\partial\overline{\partial}\varphi$ is bounded below by a fixed multiple of θ (see Lemma 4.3). However, one can not expect that g is bounded from above since f is not uniformly bounded above as in the example of equation (4.1.3). Fortunately, we can bound the diameter of (X, g) uniformly by Theorem 1.3.

Remark 3. *The gradient estimate in Theorem 1.3 is a generalization of the gradient estimate in [113]. The new insight in our approach is that one should estimate gradient and higher order estimates of the potential functions with respect to the new metric instead of a fixed reference metric for geometric complex Monge-Ampère equations such as those studied in Theorem 1.3. We refer interested readers to [6, 96] for gradient estimates for complex Monge-Ampère equations with respect to various background metrics.*

Remark 4. *Combining the lower bound of Ricci curvature and the non-collapsing condition (1.2.4), we can apply the theory of degeneration of Riemannian manifolds [10] so that any sequence of Kähler manifolds* $(X, g_j) \in \mathcal{M}(X, \theta, p, K, A)$*, after passing to a subsequence, converges to a compact metric space* (X_∞, d_∞) *with well-defined tangent cones of Hausdorff dimension* $2n$ *at each point in* X_∞*. In the case of equation (4.1.3), we believe the solution induces a unique Riemannian metric space homeomorphic to the original manifold* X *and all tangent cones are unique and biholomorphic to* $\mathbb{C}^n$*. If this is true, one might even be able to establish higher order expansions for the solution. The ultimate goal of our approach is to construct canonical domains and equations on the blow-up of solutions for geometric degenerate complex Monge-Ampère equations, by degeneration of Riemannian manifolds.*

We will also use similar techniques in the proof of Theorem 1.3 to obtain diameter estimates in more geometric settings. Before that, let us introduce a few necessary and well-known notions in complex geometry.

Definition 4.1.1. *Let* X *be a Kähler manifold of complex dimension* n *and* $\alpha \in H^2(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{R})$ *be nef. The numerical dimension of the class* α *is given by*

$$\nu(\alpha) = \max\{k = 0, 1, ..., n \mid \alpha^k \neq 0 \text{ in } H^{2k}(X, \mathbb{R})\}. \tag{4.1.5}$$

when $\nu(\alpha) = n$*, the class* α *is said to be big.*

The numerical dimension $\nu(\alpha)$ is always no greater than $\dim_{\mathbb{C}}(X)$.

Definition 4.1.2. *Let* X *be a Kähler manifold and* $\alpha \in H^2(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{R})$*. Then the class* α *is nef if* $\alpha + \mathcal{A}$ *is a Kähler class for any Kähler class* $\mathcal{A}$*.*

When the canonical bundle K_X is nef, X is said to be a minimal model. The abundance conjecture in birational geometry predicts that the canonical line bundle is always semi-ample (i.e. a sufficiently large power of the canonical line bundle is globally generated) if it is nef.

Definition 4.1.3. *Let ϑ be a smooth real valued closed $(1,1)$-form on a Kähler manifold X. The extremal function V associated to the form ϑ is defined by*

$$V(z) = \sup\{\phi(z) \mid \vartheta + \sqrt{-1}\partial\overline{\partial}\phi \geq 0, \ \sup_X \phi = 0\},$$

for all $z \in X$.

Any $\psi \in \mathrm{PSH}(X, \vartheta)$ is said to have minimal singularities defined by Demailly (c.f. [5]) if $\psi - V$ is bounded.

Let (X, θ) be a Kähler manifold of complex dimension n equipped with a Kähler metric θ. Suppose χ is a real valued smooth closed $(1,1)$-form and its class $[\chi]$ is nef and of numerical dimension κ. We consider the following family of complex Monge-Ampère equations

$$(\chi + t\theta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = t^{n-\kappa}e^{\lambda\varphi_t + c_t}\Omega, \text{ for } t \in (0,1], \tag{4.1.6}$$

where $\lambda = 0$, or 1, and c_t is a normalizing constant such that

$$\int_X t^{n-\kappa}e^{c_t}\Omega = \int_X (\chi + t\theta)^n. \tag{4.1.7}$$

Straightforward calculations show that c_t is uniformly bounded for $t \in (0,1]$. The following proposition generalizes the result in [9, 72, 40, 139] by studying a family of collapsing complex Monge-Ampère equations. It also generalizes the results in [37, 40, 75] for the case when the limiting reference form is semi-positive.

Proposition 4.1.1. *We consider equation (1.2.5) with the normalization condition (4.1.7). Suppose the volume measure Ω satisfies*

$$\int_X \left(\frac{\Omega}{\theta^n}\right)^p \theta^n \leq K$$

for some $p > 1$ and $K > 0$. Then there exists a unique $\varphi_t \in \mathrm{PSH}(X, \chi + t\theta)$ up to a constant translation solving equation (1.2.5) for all $t \in (0,1]$. Furthermore, there exists

$C = C(X, \chi, \theta, p, K) > 0$ *such that for all* $t \in (0, 1]$,

$$\|(\varphi_t - \sup_X \varphi_t) - V_t\|_{L^\infty(X)} \leq C,$$

where V_t *is the extremal function associated to* $\chi + t\theta$ *as in Definition 4.1.3.*

We will use proposition 4.1.1 can be applied to generalize Theorem 1.3, especially for minimal Kähler manifolds in a geometric setting. A Kähler manifold is called a minimal model if its canonical bundle is nef.

Both Theorem 1.3 and Theorem 1.5 are generalization and improvement for the techniques developed in [113] for diameter and distance estimates. With the additional bounds on the volume measure, we transform Kolodziej's analytic L^∞-estimate to a geometric diameter estimate. It is a natural question to ask how the metric space $(\mathbf{Z}, d_{\mathbf{Z}})$ is related to the current $\widetilde{\omega}$ on X. We conjecture $\widetilde{\omega}$ is smooth on an open dense set of X and its metric completion coincides with $(\mathbf{Z}, d_{\mathbf{Z}})$. However, at this moment, we do not even know the Hausdorff dimension or uniqueness of $(\mathbf{Z}, d_{\mathbf{Z}})$.

When X is a minimal model of general type, Theorem 1.5 is proved in [113] and the result in [128] shows that the singular set is closed and of Hausdorff dimension no greater than $2n - 4$.

We can also replace the smooth Kähler form θ in Theorem 1.5 by Dirac measures along effective divisors. For example, if $\{E_j\}_{j=1}^J$ is a union of smooth divisors with normal crossings and

$$\sum_{j=1}^J a_j E_j$$

is an ample $\mathbb{Q}$-divisor with some $a_j \in (0, 1)$ for $j = 1, ..., J$. Then Theorem 1.5 also holds if we let $\theta = \sum_{j=1}^J a_j [E_j]$. In this case, the metric g_t is a conical Kähler-Einstein metric with cone angles of $2\pi(1 - a_j)$ along each complex hypersurface E_j.

A special case of the abundance conjecture is proved by Kawamata [66] for minimal models of general type. When X is a smooth minimal model of general type, it is recently proved by the third named author [113] that the limiting metric space $(\mathbf{Z}, d_{\mathbf{Z}})$ in Theorem 1.5 is unique and is homeomorphic to the algebraic canonical model X_{can} of X. This gives an analytic proof of Kawamata's result using complex Monge-Ampère equations, Riemannian geometry and geometric L^2-estimates. Theorem 1.5 also provides a

Riemannian geometric model for the non-general type case. This analytic approach will shed light on the abundance conjecture if such a metric model is unique with reasonably good understanding of its tangle cones.

Theorem 1.5 can also be easily generalized to a Calabi-Yau manifold X equipped with a nef line bundle L over X of $\nu(L) = \kappa$.

Our final result assumes semi-ampleness for the canonical line bundle and aims to connect the algebraic canonical models to geometric canonical models. Let X be a Kähler manifold of complex dimension n. If the canonical bundle K_X is semi-ample, the pluricanonical system induces a holomorphic surjective map

$$\Phi : X \to X_{can}$$

from X to its unique canonical model X_{can}. In particular, $\dim_{\mathbb{C}} X_{can} = \nu(X)$. We let $\mathbf{S}$ be the set of all singular fibers of Φ and $\Phi^{-1}(S_{X_{can}})$, where $S_{X_{can}}$ is the singular set of X_{can}. The general fibre of Φ is a smooth Calabi-Yau manifold of complex dimension $n-\nu(X)$. It is proved in [113] that there exists a unique twisted Kähler-Einstein current ω_{can} on X_{can} satisfying

$$Ric(\omega_{can}) = -\omega_{can} + \omega_{WP}, \tag{4.1.8}$$

where $\Phi^*\omega_{can} \in -c_1(X)$ and ω_{WP} is the Weil-Petersson metric for the variation of the Calabi-Yau fibres. In particular, ω_{can} has bounded local potentials and is smooth on $X_{can} \setminus S_{can}$. We let g_{can} be the smooth Kähler metric associated to ω_{can} on $X_{can} \setminus S_{can}$.

We remark that a special case of Theorem 1.6 is proved in [142] with a different approach for $\dim_{\mathbb{C}} X = 2$. In general, the collapsing theory in Riemannian geometry has not been fully developed except in lower dimensions. In the Kähler case, one hopes the rigidity properties can help us understand the collapsing behavior for Kähler metrics of Einstien type as well as long time solutions of the Kähler-Ricci flow on algebraic minimal models. Key analytic and geometric estimates in the proof of (2) in Theorem 1.6 are established in [104, 105] for the collapsing long time solutions of the Kähler-Ricci flow and its elliptic analogues. The proof for (3) and (4) is a technical modification of various local results of [131, 50, 132, 51], where collapsing behavior for families of Ricci-flat Calabi-Yau manifolds is comprehensively studied. Theorem 1.6 should also hold for Kähler manifolds with some additional arguments.

Finally, we will also apply our method to a continuity scheme proposed in [78] to study singularities arising from contraction of projective manifolds. This is an alternative approach for the analytic minimal model program developed in [104, 105, 106] to understand birational transformations via analytic and geometric methods (see also [109, 110, 112, 102]). Compared to the Kähler-Ricci flow, such a scheme has the advantage of prescribed Ricci lower bounds and so one can apply the Cheeger-Colding theory for degeneration of Riemannian manifolds, on the other hand, it loses the canonical soliton structure for the analytic transition of singularities corresponding to birational surgeries such as flips.

Let X be a projective manifold of complex dimension n. We choose an ample line bundle L on X and we can assume that $L - K_X$ is ample, otherwise we can replace L by a sufficiently large power of L. We choose $\theta \in [L - K_X]$ to be a smooth Kähler form and consider the following curvature equation

$$Ric(g_t) = -g_t + t\theta, \quad t \in [0, 1]. \tag{4.1.9}$$

Let

$$t_{min} = \inf\{t \in [0, 1] \mid \text{ equation } (4.1.9) \text{ is solvable at } t\}. \tag{4.1.10}$$

It is straightforward to verify that $t_{min} < 1$ by the usual continuity method (c.f. [78]). The goal is to solve equation (4.1.9) for all $t \in (0, 1]$, however, one might have to stop at $t = t_{min}$ when K_X is not nef.

Theorem 4.1. *Let g_t the solution of equation (4.1.9) for $t \in (t_{min}, 1]$. There exists $C = C(X, \theta) > 0$ such that for any $t \in (t_{min}, 1]$,*

$$Diam(X, g_t) \le C. \tag{4.1.11}$$

Theorem 1.5 is a special case of Theorem 4.1 when $t_{min} = 0$ (c.f. [113]). When $t_{min} > 0$, Theorem 4.1 is also proved in [79] with the additional assumption that

$$t_{min}L + (1 - t_{min})K_X$$

is semi-ample and big. The diameter estimate immediately allows one to take a geometric limit as a compact metric length space. In particular, it is shown in [79] that the limiting space is homeomorphic to the projective variety from the contraction induced by the $\mathbb{Q}$-line bundle $t_{min}L+(1-t_{min})K_X$ when it is big and semi-ample. One can also use Theorem 4.1 to obtain a weaker version of Kawamata's base point free theorem in the minimal model theory (c.f. [56]).

4.2 Proof of Theorem 1.3

In this section, we prove our theorem 1.3 after some preparation. Throughout this section, we let $\varphi \in \mathrm{PSH}(X,\theta)$ be the solution of the equation (1.2.2) satisfying the condition (1.2.3) in Theorem 1.3. We let $\omega = \chi + \sqrt{-1}\partial\overline{\partial}\varphi$ and let g be the Kähler metric associated to ω.

Lemma 4.2. *There exists $C = C(X, \theta, p, K) > 0$ such that*

$$\|\varphi - \sup_X \varphi\|_{L^\infty(X)} \leq C.$$

Proof. The L^∞ estimate immediately follows from Kolodziej's theorem [72].

□

The following is a result similar to Schwarz lemma.

Lemma 4.3. *There exists $C = C(X, \theta, p, K, A) > 0$ such that*

$$\omega \geq C\theta.$$

Proof. There exists $C = C(X, \theta, A) > 0$ such that

$$\Delta_\omega \log \mathrm{tr}_\omega(\theta) \geq -C - C\mathrm{tr}_\omega(\theta),$$

where Δ_ω is the Laplace operator associated with ω. Then let

$$H = \log \mathrm{tr}_\omega(\theta) - B(\varphi - \sup_X \varphi)$$

for some $B > 2C$. Then

$$\Delta_\omega H \geq C\mathrm{tr}_\omega(\theta) - C.$$

It follows from maximum principle and the L^∞-estimate in Lemma 4.2 that $\sup_X \mathrm{tr}_\omega \theta \leq C$.

□

Lemma 4.3 immediately gives the uniform Ricci lower bound.

Lemma 4.4. *There exists* $C = (X, \theta, p, K, A) > 0$ *such that*

$$Ric(g) \geq -Cg.$$

Proof. We calculate

$$Ric(g) = -\lambda g + Ric(\Omega) + \lambda\theta \geq -\lambda g - (A - \lambda)\theta \geq -Cg$$

for some fixed constant $C > 0$ by Lemma 4.3.

□

We will now prove the uniform diameter bound.

Lemma 4.5. *There exists* $C = (X, \theta, p, K, A) > 0$ *such that*

$$Diam(X, g) \leq C.$$

Proof. We first fix a sufficiently small $\epsilon = \epsilon(p) > 0$ so that $p - \epsilon > 1$. Suppose $Diam(X, g) = D$ for some $D \geq 4$. Let $\gamma : [0, D] \to X$ be a normal minimal geodesic with respect to the metric g and choose the points $\{x_i = \gamma(6i)\}_{i=0}^{[D/6]}$. It is clear that the balls $\{B_g(x_i, 3)\}_{i=0}^{[D/6]}$ are disjoint, so

$$\sum_{i=0}^{[D/6]} \mathrm{Vol}_\theta\left(B_g(x_i, 3)\right) \leq \int_X \theta^n = V,$$

hence there exists a geodesic ball $B_g(x_i, 3)$ such that

$$\mathrm{Vol}_\theta\left(B_g(x_i, 3)\right) \leq 6VD^{-1}.$$

We fix such x_i and construct a cut-off function $\eta(x) = \rho(r(x)) \geq 0$ with

$$r(x) = d_g(x, x_i)$$

such that

$$\eta = 1 \text{ on } B_g(x_i, 1),\ \eta = 0 \text{ outside } B_g(x_i, 2),$$

and

$$\rho \in [0, 1], \quad \rho^{-1}(\rho')^2 \leq C(n), \quad |\rho''| \leq C(n).$$

Define a function $F > 0$ on X such that $F = 1$ outside $B_g(x_i, 3)$, $F = D^{\frac{\epsilon}{p(p-\epsilon)}}$ on $B_g(x_i, 2)$, and

$$\int_X F\Omega = [\theta]^n, \quad \int_X \Big(\frac{F\Omega}{\theta^n}\Big)^{p-\epsilon}\theta^n \leq \Big(\int_X F^{\frac{p(p-\epsilon)}{\epsilon}}\theta^n\Big)^{\frac{\epsilon}{p}}\Big(\int_X (\frac{\Omega}{\theta^n})^p\theta^n\Big)^{\frac{p-\epsilon}{p}} \leq C$$

for some $C = C(X, \theta, p, K) > 0$.

We now consider the equation

$$(\theta + \sqrt{-1}\partial\overline{\partial}\phi)^n = e^{\lambda\phi}F\Omega.$$

By similar argument as before, $\|\phi - \sup_X \phi\|_{L^\infty} \leq C = C(X, \theta, p, K)$. Let $\hat{g} = \theta + \sqrt{-1}\partial\overline{\partial}\phi$. Then on $B_g(x_i, 2)$,

$$Ric(\hat{g}) = -\lambda\hat{g} + Ric(\Omega) + \lambda\theta, \quad Ric(g) = -\lambda g + Ric(\Omega) + \lambda\theta.$$

In particular,

$$\Delta_g \log\frac{\hat{\omega}^n}{\omega^n} = -\lambda n + \lambda \mathrm{tr}_g(\hat{g}),$$

where $\Delta_g = \Delta_\omega$. Let

$$H = \eta\Big(\log\frac{\hat{\omega}^n}{\omega^n} - \big((\varphi - \sup_X\varphi) - (\phi - \sup_X\phi)\big)\Big).$$

On $B_g(x_i, 2)$, we have

$$\Delta_g H = -(\lambda+1)n + (\lambda+1)\mathrm{tr}_g(\hat{g}) \geq -2n + n\left(\frac{\hat{\omega}^n}{\omega^n}\right)^{1/n}.$$

In general, on the support of η, we have

$$\begin{aligned}\Delta_g H &\geq \eta\left(-2n + n\Big(\frac{\hat{\omega}^n}{\omega^n}\Big)^{1/n}\right) + 2\eta^{-1}Re\,(\nabla H\cdot\nabla\eta) - 2\frac{H|\nabla\eta|^2}{\eta^2} + \eta^{-1}H\Delta_g\eta \\ &\geq \eta^{-1}\Big(C\eta^2 e^{H/(n\eta)} + 2Re\big(\nabla H\cdot\nabla\eta\big) - 2\frac{H|\nabla\eta|^2}{\eta} + H\Delta_g\eta - 2n\eta^2\Big).\end{aligned}$$

We may assume $\sup_X H > 0$, otherwise we already have upper bound of H. The maximum of H must lie at $B_g(x_i, 2)$ and at this point

$$\Delta_g H \leq 0, \quad |\nabla H|^2 = 0.$$

By Laplacian comparison we have

$$\Delta_g\eta = \rho'\Delta r + \rho'' \geq -C, \quad \frac{|\nabla\eta|^2}{\eta} = \frac{(\rho')^2}{\rho} \leq C.$$

So at the maximum of H, it holds that

$$0 \geq C\eta^2 e^{H/(n\eta)} - CH - 2n \geq CH^2 - CH - 2n,$$

therefore $\sup_X H \leq C$. In particular on the ball $B_g(x_i, 1)$ where $\eta \equiv 1$, it follows that $\frac{\hat{\omega}^n}{\omega^n} \leq C$. From the definition of $\hat{\omega}$ and ω,

$$C \geq \frac{\hat{\omega}^n}{\omega^n} = D^{\frac{\epsilon}{p(p-\epsilon)}} e^{\lambda(\phi-\varphi)}.$$

Combined with the L^∞-estimate of ϕ and φ, we conclude that

$$D \leq C = C(n, p, \theta, A, K).$$

□

Lemma 4.6. *There exists $C = (X, \theta, p, K, A) > 0$ such that*

$$\sup_X |\nabla_g \varphi|_g \leq C.$$

Proof. Straightforward calculations show that

$$\Delta_g \varphi = n - \mathrm{tr}_g(\theta),$$

$$\begin{aligned} \Delta_g |\nabla \varphi|_g^2 &= |\nabla\nabla\varphi|^2 + |\nabla\bar{\nabla}\varphi|^2 + g^{i\bar{l}} g^{k\bar{j}} R_{i\bar{j}} \varphi_k \varphi_{\bar{l}} - 2\nabla\varphi \cdot \nabla \mathrm{tr}_g(\theta) \\ &\geq |\nabla\nabla\varphi|^2 + |\nabla\bar{\nabla}\varphi|^2 - C|\nabla\varphi|^2 - 2\nabla\varphi \cdot \nabla \mathrm{tr}_g(\theta), \end{aligned}$$

and

$$\Delta_g \mathrm{tr}_g\theta = \mathrm{tr}_g\theta \cdot \Delta_g \log \mathrm{tr}_g\theta + \frac{|\nabla \mathrm{tr}_g\theta|^2}{\mathrm{tr}_g\theta} \geq -C + c_0 |\nabla \mathrm{tr}_g\theta|^2$$

for some uniform constant $c_0, C > 0$. We choose constants α and B satisfying

$$\alpha > 4c_0^{-1} > 4, \ B > \sup_X \varphi + 1$$

and define

$$H = \frac{|\nabla\varphi|^2}{B - \varphi} + \alpha \mathrm{tr}_g\theta.$$

Then we have

$$\begin{aligned} \Delta H \geq & \frac{|\nabla\nabla\varphi|^2 + |\nabla\bar{\nabla}\varphi|^2}{B-\varphi} - C\frac{|\nabla\varphi|^2}{B-\varphi} - \frac{|\nabla\varphi|^2(\mathrm{tr}_g\theta - n)}{(B-\varphi)^2} - 2(1+\alpha)\frac{\langle\nabla\varphi, \nabla\mathrm{tr}_g\theta\rangle}{B-\varphi} \\ & - \alpha C + \alpha c_0 |\nabla \mathrm{tr}_g\theta|^2 + 2\langle \frac{\nabla\varphi}{B-\varphi}, \nabla H\rangle. \end{aligned} \tag{4.2.1}$$

We may assume at the maximum point z_{max} of H, $|\nabla\varphi| > \alpha$ and $H > 0$, otherwise we are done. At z_{max},

$$\nabla H = 0,\ \ \Delta H \leq 0$$

and so at z_{max}

$$\nabla|\nabla\varphi| = \frac{1}{2}\Big(- H\frac{\nabla\varphi}{|\nabla\varphi|} - \alpha(B-\varphi)\frac{\nabla \mathrm{tr}_g\theta}{|\nabla\varphi|} + \alpha\frac{\mathrm{tr}_g\theta\nabla\varphi}{|\nabla\varphi|}\Big).$$

By Kato's inequality $|\nabla|\nabla\varphi||^2 \leq \frac{|\nabla\nabla\varphi|^2+|\nabla\bar{\nabla}\varphi|^2}{2}$, it follows that

$$\begin{aligned}\frac{|\nabla\nabla\varphi|^2+|\nabla\bar{\nabla}\varphi|^2}{B-\varphi} \geq & \frac{1}{2(B-\varphi)}\Big(H^2 + \alpha^2(B-\varphi)^2(\mathrm{tr}_g\theta)^2 + \alpha^2\frac{|\nabla \mathrm{tr}_g\theta|^2}{|\nabla\varphi|^2} \\ & - 2\alpha H(B-\varphi)\mathrm{tr}_g\theta - 2\alpha H\frac{|\nabla \mathrm{tr}_g\theta|}{|\nabla\varphi|} - 2\alpha^2(B-\varphi)\mathrm{tr}_g\theta\frac{|\nabla \mathrm{tr}_g\theta|}{|\nabla\varphi|}\Big) \\ \geq & \frac{H^2}{4(B-\varphi)} - CH - \frac{|\nabla \mathrm{tr}_g\theta|^2}{B-\varphi} - C|\nabla \mathrm{tr}_g\theta|\end{aligned} \tag{4.2.2}$$

for some uniform constant $C > 0$. After substituting (4.2.2) to (4.2.1) and applying Cauchy-Schwarz inequality, we have at z_{max}

$$\begin{aligned}0 \geq & \frac{H^2}{4(B-\varphi)} - CH - C - \frac{2|\nabla \mathrm{tr}_g\theta|^2}{B-\varphi} - C|\nabla \mathrm{tr}_g\theta| + 4|\nabla \mathrm{tr}_g\theta|^2 \\ \geq & \frac{H^2}{4(B-\varphi)} - CH - C,\end{aligned}$$

for some uniform constant $C > 0$. Therefore $\max_X H \leq C$ for some $C = C(X, \theta, \Omega, A, p, K)$. The lemma then immediately follows from Lemma 4.2 and Lemma 4.3.

□

4.3 Uniform C^0 estimate in nef canonical class setting

In this section, we will prove Proposition 4.1.1 by applying the techniques in [9, 72, 40, 37]. We point out that our uniform C^0 estimate is modulo extremal function associated to a pseudoeffective class.

Let X be a Kähler manifold of dimension n. Suppose α is nef class on X of numerical dimension $\kappa \geq 0$. Let $\chi \in \alpha$ be a smooth closed $(1,1)$-form. We define the extremal function V_χ by

$$V_\chi = \sup\{\phi \mid \chi + \sqrt{-1}\partial\overline{\partial}\phi \geq 0,\ \phi \leq 0\}. \tag{4.3.1}$$

Let θ be a fixed smooth Kähler metric on X. Then we define the perturbed extremal function V_t for $t \in (0,1]$ by

$$V_t = \sup\{\phi \mid \chi + t\theta + \sqrt{-1}\partial\overline{\partial}\phi \geq 0,\ \phi \leq 0\}. \tag{4.3.2}$$

The above extremal functions were introduced in [9] when α is big.

We first rewrite the equation (1.2.5) for $\lambda = 0$ as follows

$$(\chi + t\theta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = t^{n-\kappa} e^{-f+c_t}\theta^n, \quad \sup_X \varphi_t = 0, \tag{4.3.3}$$

$t \in (0,1]$ by letting $\Omega = e^{-f}\theta^n$, where c_t is the normalizing constant satisfying

$$t^{n-\kappa}\int_X e^{-f+c_t}\theta^n = \int_X (\chi + t\theta)^n.$$

f satisfies the following uniform bound

$$\int_X e^{-pf}\theta^n \leq K,$$

for some $p > 1$ and $K > 0$.

The following definition is an extension of the capacity introduced in [72, 40, 37, 9].

Definition 4.3.1. *We define the capacity $Cap_{\chi_t}(\mathcal{K})$ for a subset $\mathcal{K} \subset X$ by*

$$Cap_{\chi_t}(\mathcal{K}) = \sup\Big\{ \int_{\mathcal{K}} (\chi_t + \sqrt{-1}\partial\overline{\partial}u)^n \mid u \in \mathrm{PSH}(X, \chi_t),\ 0 \leq u - V_t \leq 1\Big\}, \tag{4.3.4}$$

where $\chi_t = \chi + t\theta$ is the reference metric in (4.3.3). We also define the extremal function $V_{t,\mathcal{K}}$ by

$$V_{t,\mathcal{K}} = \sup\left\{u \in \mathrm{PSH}(X, \chi_t) \mid u \leq 0, \text{ on } \mathcal{K}\right\}. \tag{4.3.5}$$

If K is open, then we have

1. $V_{t,\mathcal{K}} \in \mathrm{PSH}(X, \chi_t) \cap L^\infty(X)$,

2. $(\chi_t + \sqrt{-1}\partial\overline{\partial}V_{t,\mathcal{K}})^n = 0$ on $X \setminus \overline{\mathcal{K}}$.

Lemma 4.7. *Let φ_t be the solution to (4.3.3). Then there exist $\delta = \delta(X, \chi, \theta) > 0$ and $C = C(X, \chi, \theta, p, K) > 0$ such that for any open set $\mathcal{K} \subset X$ and $t \in (0,1]$,*

$$\frac{1}{[\chi_t^n]}\int_{\mathcal{K}} (\chi_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n \leq Ce^{-\delta\left(\frac{[\chi_t^n]}{Cap_{\chi_t}(K)}\right)^{\frac{1}{n}}}. \tag{4.3.6}$$

Proof. Since $[\chi^m]=0$ for $\kappa+1\le m\le n$

$$[\chi_t^n]=\int_X \chi_t^n=\int_X\sum_{k=0}^n\binom{n}{k}\chi^k\wedge t^{n-k}\theta^{n-k}=\int_X\sum_{k=0}^{\kappa}\binom{n}{k}\chi^k\wedge t^{n-k}\theta^{n-k}=O(t^{n-\kappa}).$$

It follows that the normalizing constant c_t in (4.3.3) is uniform bounded. Let $M_{t,\mathcal{K}}=\sup_X V_{t,\mathcal{K}}$. Then we have

$$\begin{aligned}\frac{1}{[\chi_t^n]}\int_{\mathcal{K}}(\chi_t+\sqrt{-1}\partial\overline{\partial}\varphi_t)^n &= \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]}\int_K e^{-f}\theta^n\\ &\le \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]}\int_{\mathcal{K}} e^{-f}e^{-\delta V_{t,\mathcal{K}}/q}\theta^n, \quad \text{since } V_{t,\mathcal{K}}\le 0 \text{ on } K\\ &\le \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]}e^{-\delta M_{t,\mathcal{K}}/q}\int_X e^{-f}e^{-\delta(V_{t,\mathcal{K}}-M_{t,\mathcal{K}})/q}\theta^n\\ &\le \frac{t^{n-\kappa}e^{c_t}}{[\chi_t^n]}e^{-\delta M_{t,\mathcal{K}}/q}\Big(\int_X e^{-pf}\theta^n\Big)^{1/p}\Big(\int_X e^{-\delta(V_{t,\mathcal{K}}-M_{t,\mathcal{K}})}\theta^n\Big)^{1/q}\\ &\le Ce^{-\delta M_{t,\mathcal{K}}/q},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Obviously, there exists $\gamma=\gamma(X,\chi,\theta)>0$ such that for all $t\in(0,1]$,

$$V_{t,\mathcal{K}}\in \mathrm{PSH}(X,\gamma\theta).$$

We apply the global Hörmander's estimate ([125]) so that there exists $\delta=\delta(X,\chi,\theta)>0$ such that

$$\int_X e^{-\delta(V_{t,\mathcal{K}}-\sup_X V_{t,\mathcal{K}})}\theta^n\le C_\delta.$$

To complete the proof, it suffices to to show

$$M_{t,\mathcal{K}}+1\ge\Big(\frac{[\chi_t^n]}{Cap_{\chi_t}(\mathcal{K})}\Big)^{1/n}. \tag{4.3.7}$$

First we observe that by definition

$$\sup_X\Big((V_{t,\mathcal{K}}-\sup_X V_{t,\mathcal{K}})-V_t\Big)\le 0,$$

since $V_{t,\mathcal{K}}-\sup_X V_{t,\mathcal{K}}\in\mathrm{PSH}(X,\chi_t)$ is nonpositive. On the other hand, $V_{t,\mathcal{K}}\ge V_t$. This immediately implies that

$$0\le V_{t,\mathcal{K}}-V_t\le\sup_X V_{t,\mathcal{K}}=M_{t,\mathcal{K}}. \tag{4.3.8}$$

We break the rest of the proof into two cases.

• **The case when $M_{t,K} > 1$.** We let

$$\psi_{t,\mathcal{K}} = M_{t,\mathcal{K}}^{-1}(V_{t,\mathcal{K}} - V_t) + V_t.$$

Then

$$V_t \le \psi_{t,\mathcal{K}} \le V_t + 1$$

and by (4.3.8).

$$\begin{aligned}
\frac{1}{M_{t,\mathcal{K}}^n} &= \frac{1}{M_{t,\mathcal{K}}^n}\frac{\int_X(\chi_t+\sqrt{-1}\partial\overline{\partial}V_{t,\mathcal{K}})^n}{[\chi_t^n]} = \frac{1}{[\chi_t^n]}\int_{\overline{\mathcal{K}}}\Big(M_{t,\mathcal{K}}^{-1}\chi_t + \sqrt{-1}\partial\overline{\partial}(M_{t,\mathcal{K}}^{-1}V_{t,\mathcal{K}})\Big)^n \\
&\le \frac{1}{[\chi_t^n]}\int_{\overline{\mathcal{K}}}\Big(M_{t,\mathcal{K}}^{-1}\chi_t + \sqrt{-1}\partial\overline{\partial}(M_{t,\mathcal{K}}^{-1}V_{t,\mathcal{K}}) + (1-M_{t,\mathcal{K}}^{-1})(\chi_t+\sqrt{-1}\partial\overline{\partial}V_t)\Big)\Big)^n \\
&= \frac{1}{[\chi_t^n]}\int_{\overline{\mathcal{K}}}(\chi_t+\sqrt{-1}\partial\overline{\partial}\psi_{t,\mathcal{K}})^n \\
&\le \frac{Cap_{\chi_t}(\mathcal{K})}{[\chi_t^n]}.
\end{aligned} \tag{4.3.9}$$

• **The case when $M_{t,K} \le 1$.** By (4.3.8)

$$0 \le V_{t,\mathcal{K}} - V_t \le \sup_X V_{t,\mathcal{K}} = M_{t,\mathcal{K}} \le 1.$$

Now

$$[\chi_t^n] = \int_{\overline{\mathcal{K}}}(\chi_t+\sqrt{-1}\partial\overline{\partial}V_{t,\mathcal{K}})^n \le Cap_{\chi_t}(\overline{\mathcal{K}}). \tag{4.3.10}$$

So in this case $\frac{[\chi_t^n]}{Cap_{\chi_t}(\mathcal{K})} \le 1$.

Combining (4.3.9) and (4.3.10), (4.3.7) holds and we complete the proof of Lemma 4.7.

□

The following is an immediate corollary of Lemma 4.7.

Corollary 4.3.1. *There exists $C = C(X,\chi,\theta,p,K) > 0$ such that for all $t \in (0,1]$, we have*

$$\frac{1}{[\chi_t^n]}\int_{\mathcal{K}}(\chi_t+\sqrt{-1}\partial\overline{\partial}\varphi_t)^n \le C\Big(\frac{Cap_{\chi_t}(\mathcal{K})}{[\chi_t^n]}\Big)^2.$$

Proof. This follows from Lemma 4.7 and the elementary inequality that $x^2e^{-\delta x^{1/n}} \le C$ for some uniform $C > 0$ and all $x \in (0,\infty)$. □

Lemma 4.8. *Let $u \in \mathrm{PSH}(X,\chi_t) \cap L^\infty(X)$. For any $s>0$, $0 \le r \le 1$ and $t \in (0,1]$, we have*

$$r^n Cap_{\chi_t}(u - V_t < -s - r) \le \int_{\{u-V_t<-s\}} (\chi_t + \sqrt{-1}\partial\overline{\partial} u)^n. \tag{4.3.11}$$

Proof. For any $\phi \in \mathrm{PSH}(X,\chi_t)$ with $0 \le \phi - V_t \le 1$, we have

$$\begin{aligned} r^n \int_{\{u-V_t<-s-r\}} (\chi_t + \sqrt{-1}\partial\overline{\partial}\phi)^n &= \int_{\{u-V_t<-s-r\}} (r\chi_t + \sqrt{-1}\partial\overline{\partial}(r\phi))^n \\ &\le \int_{\{u-V_t<-s-r\}} (\chi_t + \sqrt{-1}\partial\overline{\partial}(r\phi) + \sqrt{-1}\partial\overline{\partial}(1-r)V_t)^n \\ &\le \int_{\{u-V_t<-s-r+r(\phi-V_t)\}} (\chi_t + \sqrt{-1}\partial\overline{\partial}(r\phi + (1-r)V_t - s - r))^n \\ &\le \int_{\{u<r\phi+(1-r)V_t-s-r\}} (\chi_t + \sqrt{-1}\partial\overline{\partial}u)^n \\ &\le \int_{\{u<V_t-s\}} (\chi_t + \sqrt{-1}\partial\overline{\partial}u)^n. \end{aligned}$$

The third inequality follows from the comparison principle and the last inequality follows from the fact that $r\phi + (1-r)V_t - s - r = r(\phi - V_t - 1) + V_t - s < V_t - s$.

Taking supremum of all $\phi \in \mathrm{PSH}(X,\chi_t)$ with $0 \le \phi - V_t \le 1$ we get (4.3.11).

□

Lemma 4.9. *Let φ_t be the solution to (4.3.3). Then there exists a constant $C = C(X,\chi,\theta,p,K) > 0$ such that for all $s > 1$*

$$\frac{1}{[\chi_t^n]} Cap_{\chi_t}\big(\{\varphi_t - V_t < -s\}\big) \le \frac{C}{(s-1)^{1/q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Applying Lemma 4.8 to $u = \varphi_t$ and $r = 1$, we have

$$\begin{aligned} &\frac{1}{[\chi_t^n]} Cap_{\chi_t}\big(\{\varphi_t - V_t < -s\}\big) \\ \le &\frac{1}{[\chi_t^n]} \int_{\{\varphi_t - V_t < -(s-1)\}} (\chi_t + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n \\ = &\frac{1}{[\chi_t^n]} \int_{\{\varphi_t - V_t < -(s-1)\}} t^{n-\kappa} e^{-f+c_t} \theta^n \\ \le &\frac{C}{(s-1)^{1/q}} \int_{\{\varphi_t - V_t < -(s-1)\}} (-\varphi_t + V_t)^{1/q} e^{-f} \theta^n \\ \le &\frac{C}{(s-1)^{1/q}} \Big(\int_{\{\varphi_t - V_t < -(s-1)\}} e^{-pf} \theta^n \Big)^{1/p} \Big(\int_{\{\varphi_t - V_t < -(s-1)\}} (-\varphi_t + V_t) \theta^n \Big)^{1/q} \end{aligned}$$

$$\leq \frac{C}{(s-1)^{1/q}}\Big(\int_X(-\varphi_t)\theta^n\Big)^{1/q},$$

where in the last inequality we use the assumption that $e^{-f}\in L^p(\theta^n)$, $V_t\leq 0$ and $\varphi_t\leq 0$. On the other hand, since $\varphi_t\in \mathrm{PSH}(X,\chi_t)\subset \mathrm{PSH}(X,C\theta)$ for some large $C>0$ and $\sup_X\varphi_t=0$, it follows from Green's formula that

$$\int_X(-\varphi_t)\theta^n\leq C$$

for some uniform constant C. The lemma follows by combining the inequalities above.

□

The following lemma is well-known and its proof can be found e.g. in [72, 40].

Lemma 4.10. *Let $F:[0,\infty)\to[0,\infty)$ be a non-increasing right-continuous function satisfying $\lim_{s\to\infty}F(s)=0$. If there exist $\alpha,A>0$ such that for all $s>0$ and $0\leq r\leq 1$,*

$$rF(s+r)\leq A\left(F(s)\right)^{1+\alpha},$$

then there exists $S=S(s_0,\alpha,A)$ such that

$$F(s)=0$$

for all $s\geq S$, where s_0 is the smallest s satisfying $(F(s))^\alpha\leq(2A)^{-1}$.

Proof of Proposition 4.1.1. Define for each fixed $t\in(0,1]$

$$F(s)=\Big(\frac{Cap_{\chi_t}\big(\{\varphi_t-V_t<-s\}\big)}{[\chi_t]^n}\Big)^{1/n}.$$

By Corollary 4.3.1 and Lemma 4.8 applied to the function φ_t, we have

$$rF(s+r)\leq AF(s)^2,\quad \text{for}\quad \text{all } r\in[0,1],\ s>0,$$

for some uniform constant $A>0$ independent of $t\in(0,1]$.

Lemma 4.9 implies that $\lim_{s\to\infty}F(s)=0$ and the s_0 in Lemma 4.10 can be taken as less than $(2AC)^q$, which is a uniform constant. It follows from Lemma 4.10 that $F(s)=0$ for all $s>S$, where $S\leq 2+s_0$. On the other hand, if $Cap_{\chi_t}\big(\{\varphi_t-V_t<-s\}\big)=0$, by Lemma 4.7 and the equation (4.3.3), we have

$$\int_{\{\varphi_t-V_t<-s\}}e^{-f}\theta^n=0,$$

hence the set $\{\varphi_t - V_t < -s\} = \emptyset$. Thus $\inf_X(\varphi_t - V_t) \geq -S$. Thus we finish the proof of Proposition 4.1.1.

□

Therefore we have proved Proposition 4.1.1 when $\lambda = 0$. We finish this section by proving the case when $\lambda = 1$. We consider the following complex Monge-Ampère equations for $t \in (0, 1]$,

$$(\chi + t\theta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = t^{n-\kappa} e^{\varphi_t - f + c_t}\theta^n. \tag{4.3.12}$$

where $f \in C^\infty(X)$ and c_t is the normalizing constant satisfying $t^{n-\kappa}\int_X e^{-f+c_t}\theta^n = \int_X(\chi + t\theta)^n$.

Corollary 4.3.2. *If*

$$\|e^{-f}\|_{L^p(X,\theta^n)} \leq K,$$

for $p > 1$ and $K > 0$, Then there exists $C = C(X, \chi, \theta, p, K) > 0$ such that

$$\|\varphi_t - V_t\|_{L^\infty} \leq C.$$

Proof. Since for each $t > 0$, it is proved in [5] that V_t is $C^{1,\alpha}(X, \theta)$, we can always find $W_t \in C^\infty(X)$ such that $\sup_X |V_t - W_t| \leq 1$. Furthermore, V_t is uniformly bounded above for all $t \in (0, 1]$. We let ψ_t be the solution of

$$(\chi_t + \sqrt{-1}\partial\overline{\partial}\psi_t)^n = t^{n-\kappa}e^{-f+c_t+W_t}\theta^n, \quad \sup_X \psi_t = 0.$$

and

$$u_t = \varphi_t - \psi_t.$$

Then

$$\frac{(\chi_t + \sqrt{-1}\partial\overline{\partial}\psi_t + \sqrt{-1}\partial\overline{\partial}u_t)^n}{(\chi_t + \sqrt{-1}\partial\overline{\partial}\psi_t)^n} = e^{u_t + \psi_t - W_t}.$$

Since $\sup_X |\psi_t - W_t| \leq \sup_X |\psi_t - V_t| + 1$, the maximum principle immediately implies that

$$\|u_t\|_{L^\infty(X)} \leq \|\psi_t - V_t\|_{L^\infty(X)} + 1$$

and so

$$\|\varphi_t - V_t\|_{L^\infty(X)} \le 2\|\psi_t - V_t\|_{L^\infty(X)} + 1.$$

□

4.4 Uniform diameter estimate of twisted Kähler-Einstein metrics in the nef canonical class setting

Let X be a Kähler manifold. X is said to be a minimal model if the canonical bundle K_X is nef. The numerical dimension of K_X is given by

$$\nu(K_X) = \max\{m = 0, ..., n \mid [K_X]^m \neq 0 \text{ in } H^{m,m}(X, \mathbb{C})\}.$$

Let θ be a smooth Kähler form on a minimal model X of complex dimension n. Let $\kappa = \nu(X)$, the numerical dimension of K_X. Let Ω be a smooth volume form on X. We let χ be defined by

$$\chi = \sqrt{-1}\partial\overline{\partial}\log\Omega \in K_X.$$

We consider the following Monge-Ampère equation for $t \in (0, \infty)$

$$(\chi + t\theta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = t^{n-\kappa}e^{\varphi_t}\Omega. \tag{4.4.1}$$

Since K_X is nef, $[\chi + t\theta]$ is a Kähler class for any $t > 0$. By Aubin and Yau's theorem, there exists a unique smooth solution φ_t solving (4.4.1) for all $t > 0$. Let $\omega_t = \chi + t\theta + \sqrt{-1}\partial\overline{\partial}\varphi$. Then ω_t satisfies

$$Ric(\omega_t) = -\omega_t + t\theta.$$

In particular, any Kähler metric satisfying the the above twisted Kähler-Einstein equation must coincide with ω_t.

Lemma 4.11. *There exists $C > 0$ such that for all $t \in (0, 1]$,*

$$C^{-1}t^{n-\kappa} \le [\chi + t\theta]^n \le Ct^{n-\kappa}.$$

Proof. First we note that $[\chi]^d \cdot [\theta]^{n-\kappa} > 0$ because $[\chi]^d \neq 0$ and $[\chi]$ is nef. Then

$$[\chi + t\theta]^n = t^{n-\kappa}\binom{n}{d}[\chi]^d \cdot [\theta]^{n-\kappa} + t^{n-\kappa+1}\Big(\sum_{j=d+1}^{n}\binom{n}{j}t^{j-d-1}[\chi]^j \cdot [\theta]^{n-j}\Big).$$

□

Lemma 4.12. *Let $V_t = \sup\{u \mid u \in \mathrm{PSH}(X, \chi + t\theta),\ u \le 0\}$. Then there exists $C > 0$ such that for all $t \in (0, 1]$,*

$$\|\varphi_t - V_t\|_{L^\infty(X)} \le C. \tag{4.4.2}$$

Proof. The lemma immediately follows by applying Proposition 4.1.1 to equation (4.4.1). □

We now prove the main result in this section.

Lemma 4.13. *There exists $C > 0$ such that for all $t \in (0, 1]$,*

$$Diam(X, g_t) \le C.$$

Proof. The proof applies similar argument in the proof of Theorem 1.3. Suppose $Diam(X, g_t) = D$ for some $D \ge 6$. Let $\gamma : [0, D] \to X$ be a smoothing minimizing geodesic with respect to the metric g_t and choose the points $\{x_i = \gamma(6i)\}_{i=0}^{[D/6]}$. It is clear that the balls $\{B_{g_t}(x_i, 3)\}$ are disjoint so

$$\sum_{i=0}^{[D/6]} \mathrm{Vol}_\Omega\left(B_{g_t}(x_i, 3)\right) \le \int_X \Omega = V,$$

where $\mathrm{Vol}_\Omega\left(B_{g_t}(x_i, 3)\right) = \int_{B_{g_t}}(x_i, 3)\Omega$. Hence there exists a geodesic ball $B_{g_t}(x_i, 3)$ such that

$$\mathrm{Vol}_\Omega\left(B_{g_t}(x_i, 3)\right) \le 6VD^{-1}.$$

We fix such x_i and construct a cut-off function $\eta(x) = \rho(r(x)) \ge 0$ with $r(x) = d_{g_t}(x, x_i)$ such that

$$\eta = 1 \text{ on } B_{g_t}(x_i, 1), \quad \eta = 0 \text{ outside } B_{g_t}(x_i, 2)$$

and

$$\rho \in [0, 1], \quad \rho^{-1}(\rho')^2 \le C, \quad |\rho''| \le C.$$

Define a function $F_t > 0$ on X such that

$$F_t = 1 \ \text{ outside } B_{g_t}(x_i, 3), \quad F_t = D^{1/2} \ \text{ on } \ B_{g_t}(x_i, 2)$$

and

$$C^{-1} \le \int_X F_t\Omega \le C, \quad \int_X F_t^2\Omega \le C.$$

We now consider the equation

$$(\chi + t\theta + \psi_t)^n = t^{n-\kappa} e^{\psi_t} F_t\, \Omega, \quad \text{for all } t \in (0,1].$$

Applying Corollary 4.3.2, there exists a uniform constant $C > 0$ such that for all $t \in (0,1]$,

$$\|\psi_t - V_t\|_{L^\infty(X)} \le C,$$

and so by Lemma 4.12

$$\|\varphi_t - \psi_t\|_{L^\infty(X)} \le C. \tag{4.4.3}$$

Let $\hat{g}_t = \chi + t\theta_t + \sqrt{-1}\partial\overline{\partial}\psi_t$. Then on $B_{g_t}(x_i, 2)$,

$$Ric(\hat{g}_t) = -\hat{g}_t + t\theta, \quad Ric(g_t) = -g_t + t\theta,$$

and so

$$\Delta_{g_t} \log \frac{\hat{\omega}_t^n}{\omega_t^n} = -n + \mathrm{tr}_{g_t}(\hat{g}_t) \ge -n + n\Big(\frac{\hat{\omega}_t^n}{\omega_t^n}\Big)^{1/n}.$$

Let $H = \eta \log \frac{\hat{\omega}_t^n}{\omega_t^n}$. We may suppose $\sup_X H = H(z_{max}) > 0$, otherwise we are done. z_{max} must lies in the support of η, and at z_{max} we have

$$\begin{aligned} 0 \ge \Delta_{g_t} H &\ge \frac{1}{\eta}\Big(H\Delta_{g_t}\eta + 2\langle \nabla\eta, \nabla H\rangle - 2\frac{H}{\eta}|\nabla\eta|^2 - n\eta^2 + n\eta^2 e^{\frac{H}{n\eta}}\Big) \\ &\ge \frac{1}{\eta}\Big(\frac{1}{2n}H^2 - CH\Big) \end{aligned}$$

for some uniform constant $C > 0$ for all $t \in (0,1]$. Maximum principle implies that $\sup_X H \le C(n)$, in particular on $B_{g_t}(x_i, 1)$ where $\eta \equiv 1$, there exists $C > 0$ such that for all $t \in (0,1]$,

$$\frac{\hat{\omega}_t^n}{\omega_t^n} = D^{1/2} e^{\psi_t - \varphi_t} \le C.$$

By the uniform L^∞-estimate(4.4.3), there exists $C = C(n, \chi, \Omega, \theta)$ such that $D \le C$.

□

Now we can complete the proof of Theorem 1.5. Gromov's pre-compactness theorem and the diameter bound in Lemma 4.13 immediately imply that after passing to a subsequence, (X, g_{t_j}) converges to a compact metric space. Since $\varphi_t - V_t$ is uniformly bounded and V_t is uniformly bounded below by V_0, φ_{t_j} always converges weakly to some

$\varphi_\infty \in \mathrm{PSH}(X, \chi)$, after passing to a subsequence. In particular, there exists $C > 0$ such that

$$||\varphi_\infty - V_0||_{L^\infty(X)} \le C,$$

where V_0 is the extremal function on X with respect to χ. $\square$

4.5 Convergence of twisted Kähler-Einsteins to canonical metric by assuming abundance conjecture

Our proof is based on the arguments of [104, 131, 132].

We fix some notations first. Recall X_{can} has dimension κ and χ is the restriction of Fubini-Study metric on X_{can} from the embedding $X_{can} \hookrightarrow \mathbb{CP}^{N_m}$, where $N_m + 1 = \dim H^0(X, mK_X)$. Hence $\Phi^*\chi$ is a smooth nonnegative $(1,1)$-form on X, and in the following we identify χ with $\Phi^*\chi$ for simplicity. Let θ be a fixed Kähler metric on X.

Define a function $H \in C^\infty(X)$ as

$$\chi^\kappa \wedge \theta^{n-\kappa} = H\theta^n$$

which is the modulus square of the Jacobian of the map $\Phi : (X, \theta) \to (X_{can}, \chi)$ and vanishes on S, the indeterminacy set of Φ, hence $H^{-\gamma} \in L^1(X, \theta^n)$ for some small $\gamma > 0$. We fix a smooth nonnegative function σ on X_{can} as defined in [131], which satisfies

$$0 \le \sigma \le 1, \quad 0 \le \sqrt{-1}\partial\sigma \wedge \bar{\partial}\sigma \le C\chi, \quad -C\chi \le \sqrt{-1}\partial\overline{\partial}\sigma \le C\chi, \tag{4.5.1}$$

for some dimensional constant $C = C(\kappa) > 0$. From the construction, σ vanishes exactly on $S' = \Phi(S)$. There exist $\lambda > 0$, $C > 1$ such that for any $y \in X^\circ_{can} = X_{can} \backslash S'$ (see [131])

$$\sigma(y)^\lambda \le C \inf_{X_y} H, \quad \text{here } X_y = \Phi^{-1}(y).$$

The twisted Kähler-Einstein metric g_t in (1.2.6) satisfies the following complex Monge-Ampère equation (with $\theta = \theta$)

$$(\chi + t\theta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = t^{n-\kappa} e^{\varphi_t}\Omega, \quad \text{for all } t \in (0, 1]. \tag{4.5.2}$$

In case K_X is semi-ample, $V_t = 0$ hence Corollary 4.3.2 implies: (see also [37, 72, 40])

Lemma 4.14. *There is a uniform constant $C>0$ such that $\|\varphi_t\|_{L^\infty(X)} \le C$.*

We have the following Schwarz lemma whose proof is similar to that of Lemma 4.3, so we omit it.

Lemma 4.15. *There exists a constant $C>0$ such that*

$$\mathrm{tr}_{\omega_t}\chi \le C, \quad \text{for all} \quad t\in(0,1].$$

We denote $\theta_y=\theta|_{X_y}$ for $y\in X^\circ_{can}$, the restriction of θ on the fiber X_y which is a smooth $(n-\kappa)$-dimensional Calabi-Yau submanifold of X. We will omit the subscript t in φ_t and simply write $\varphi=\varphi_t$, and define $\overline{\varphi}_y = ⨍_{X_y}\varphi\theta_y^{n-\kappa}$ to be the average of φ over the fiber X_y. Denote the reference metric $\hat{\omega}_t=\chi+t\theta$. We calculate

$$(\hat{\omega}_t+\sqrt{-1}\partial\overline{\partial}\varphi)|_{X_y} = \big(t\theta_y+\sqrt{-1}\partial\overline{\partial}(\varphi-\overline{\varphi}_y)\big)|_{X_y} = \omega_t|_{X_y},$$

hence

$$\big(\theta_y+t^{-1}\sqrt{-1}\partial\overline{\partial}(\varphi-\overline{\varphi}_y)|_{X_y}\big)^{n-\kappa} = t^{-n+\kappa}\omega_{t,y}^{n-\kappa}. \tag{4.5.3}$$

On the other hand,

$$\begin{aligned} t^{-n+\kappa}\frac{\omega_{t,y}^{n-\kappa}}{\theta_y^{n-\kappa}} &= t^{-n+\kappa}\frac{\omega_t^{n-\kappa}\wedge\chi^\kappa}{\theta^{n-\kappa}\wedge\chi^\kappa}\Big|_{X_y} \\ &\le C\big(\mathrm{tr}_{\omega_t}\chi\big)^\kappa\frac{\Omega}{\theta^{n-\kappa}\wedge\chi^\kappa}\Big|_{X_y} \\ &\le CH^{-1}\le C\sigma^{-\lambda}(y). \end{aligned}$$

Since the Sobolev constant of (X_y,θ_y) is uniformly bounded and Poincaré constant of (X_y,θ_y) is bounded by $Ce^{B\sigma^{-\lambda}(y)}$ for some uniform constants B, $C>0$ (see [131]), combined with the fact that

$$⨍_{X_y}(\varphi-\overline{\varphi}_y)\theta_y^{n-\kappa}=0,$$

Moser iteration implies ([136, 131])

Lemma 4.16. *There exist constants B_1, $C_1>0$ such that for any $y\in X^\circ_{can}$,*

$$\sup_{X_y} t^{-1}|\varphi-\overline{\varphi}_y| \le C_1e^{B_1\sigma^{-\lambda}(y)}, \quad \text{for all} \quad t\in(0,1].$$

Proposition 4.5.1. *On any compact subset $K \Subset X\backslash S$, there exists a constant $C = C(K) > 1$ such that for all $t \in (0,1]$*

$$C^{-1}\hat{\omega}_t \leq \omega_t \leq C\hat{\omega}_t, \quad \text{on } K.$$

Given the C^0-estimate in Lemma 4.16, Proposition 4.5.1 can be proved by the C^2-estimate ([136]) for Monge-Ampère equation together with a modification as in [104, 131, 132], so we omit the proof.

Let us recall the construction of the canonical metric ω_{can} on X°_{can} (see [104]). Define a function $F = \frac{\Phi_*\Omega}{\chi^\kappa}$ on X°_{can}, and F is in $L^{1+\varepsilon}$ for some small $\varepsilon > 0$ ([104]). The metric ω_{can} is obtained by solving the following complex Monge-Ampère equation on X_{can}

$$(\chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty)^\kappa = \binom{n}{\kappa} F e^{\varphi_\infty}\chi^\kappa,$$

for $\varphi_\infty \in \mathrm{PSH}(X_{can}, \chi) \cap C^0(X_{can}) \cap C^\infty(X^\circ_{can})$. Then $\omega_{can} = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty$, and in the following we will write $\chi_\infty = \omega_{can}$.

Any smooth fiber X_y with $y \in X^\circ_{can}$ is a Calabi-Yau manifold hence there exists a unique Ricci flat metric $\omega_{SF,y} \in [\theta_y]$ such that $\omega_{SF,y} = \theta_y + \sqrt{-1}\partial\overline{\partial}\rho_y$ for some $\rho_y \in C^\infty(X_y)$ with normalization $⨍_{X_y} \rho_y \omega^{n-\kappa}_{X,y} = 0$. We write $\rho_{SF}(x) = \rho_{\Phi(x)}$ if $\Phi(x) \in X^\circ_{can}$. ρ_{SF} is a smooth function on $X\backslash S$ and may blow up near the singular set S. Denote $\omega_{SF} = \theta + \sqrt{-1}\partial\overline{\partial}\rho_{SF}$ which is smooth on $X\backslash S$, and by [104] we know that $\frac{\Omega}{\omega_{SF}^{n-\kappa}\wedge\chi^\kappa}$ is constant on the smooth fibers X_y and is equal to $\Phi^* F$. For simplicity we will identify F with Φ^*F. Our arguments below are motivated by [104, 132].

Denote $\mathcal{F} = e^{-e^{A\sigma^{-\lambda}}}$ for suitably large constants $A, \lambda > 1$. From the proof of Proposition 4.5.1, we actually have that on $X\backslash S$ ([131])

$$C^{-1}\mathcal{F}\hat{\omega}_t \leq \omega_t \leq C\mathcal{F}^{-1}\hat{\omega}_t, \quad \text{for all } t \in (0,1].$$

Next we are going to show $\varphi_t \to \varphi_\infty = \Phi^*\varphi_\infty$ as $t \to 0$. Proposition 4.5.2 below can proved by following similar argument as in [132], but we present a slightly different argument in establishing **Claim 2** below.

Proposition 4.5.2. *There exists a positive function $h(t)$ with $h(t) \to 0$ as $t \to 0$ such that*

$$\sup_{X\backslash S} \mathcal{F}|\varphi_t - \varphi_\infty| \leq h(t). \tag{4.5.4}$$

Proof. Let $D \subset X_{can}$ be an ample divisor such that $X_{can} \backslash X_{can}^{\circ} \subset D$, $D \in \mu K_{X_{can}}$ for some $\mu \in \mathbb{N}$. Choose a continuous hermitian metric on $[D]$, $h_D = h_{FS}^{\mu/m} e^{-\mu\varphi_\infty}$ and a smooth defining section s_D of $[D]$, where h_{FS} is the Fubini-Study metric induced from $\mathcal{O}_{\mathbb{CP}^{N_m}}(1)$. Clearly $\sqrt{-1}\partial\overline{\partial}\log h_D = \mu(\chi + \sqrt{-1}\partial\overline{\partial}\varphi_\infty) = \mu\chi_\infty$. For small $r > 0$, let

$$B_r(D) = \{x \in X_{can} \mid d_\chi(x, D) \le r\}$$

be the tubular neighborhood of D under the metric d_χ, and denote $\mathcal{B}_r = \Phi^{-1}\big(B_r(D)\big) \subset X$.

Since both φ_t and φ_∞ are bounded in L^∞-norm, there exists r_ϵ with $\lim_{\epsilon\to 0} r_\epsilon = 0$ such that for all $t \in (0, 1]$

$$\sup_{\mathcal{B}_{r_\epsilon}\backslash S}(\varphi_t - \varphi_\infty + \epsilon\log|s_D|^2_{h_D}) < -1, \quad \inf_{\mathcal{B}_{r_\epsilon}\backslash S}(\varphi_t - \varphi_\infty - \epsilon\log|s_D|^2_{h_D}) > 1.$$

Let η_ϵ be a smooth cut-off function on X_{can} such that $\eta_\epsilon = 1$ on $X_{can}\backslash B_{r_\epsilon}(D)$ and $\eta_\epsilon = 0$ on $B_{r_\epsilon/2}(D)$. Write $\rho_\epsilon = (\Phi^*\eta_\epsilon)\rho_{SF}$, and $\omega_{SF,\epsilon} = \omega_{SF} + \sqrt{-1}\partial\overline{\partial}\rho_\epsilon$. Define the twisted differences of φ_t and φ_∞ by

$$\psi_\epsilon^\pm = \varphi_t - \varphi_\infty - t\rho_\epsilon \mp \epsilon\log|s_D|^2_{h_D}.$$

By similar argument in [104] we have

Claim 1: there exists an $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, there exists a τ_ϵ such that for all $t \le \tau_\epsilon$, we have

$$\sup_{X\backslash S}\psi_\epsilon^-(t,\cdot) \le 3\mu\epsilon, \quad \inf_{X\backslash S}\psi_\epsilon^+(t,\cdot) \ge -3\mu\epsilon.$$

Claim 2: We have

$$\int_X |\varphi_t - \varphi_\infty|\theta^n \to 0, \quad \text{as } t \to 0,$$

where φ_t is the Kähler potential of ω_t in (4.5.2).

Proof of Claim 2. For any $\eta > 0$, we may take $\mathcal{B}_{R_\eta} \subset X$ small enough so that $\int_{\mathcal{B}_{R_\eta}}\theta^n < \frac{\eta}{10}$. Take $\epsilon < \eta/10\mu$ small enough so that $r_\epsilon < R_\eta$. From **Claim 1** when $t < \tau_\epsilon$

$$\int_X |\varphi_t - \varphi_\infty|\theta^n = \int_{\mathcal{B}_{R_\eta}} |\varphi_t - \varphi_\infty|\theta^n + \int_{X\backslash\mathcal{B}_{R_\eta}} |\varphi_t - \varphi_\infty|\theta^n$$

$$\leq C\eta + \int_{X\backslash \mathcal{B}_{R_\eta}} \left(t|\rho_{SF}| + \epsilon |\log |s_D|^2_{h_D}|\right)\theta^n$$

$$\leq C\eta.$$

□

Given **Claim 2**, Proposition 4.5.2 follows similarly as in [132], so we skip it. □

We will apply an argument in [132] with a slight modification to show the lemma below:

Lemma 4.17.

$$\lim_{t\to 0} \mathcal{F} t\dot{\varphi}_t = 0.$$

Proof. Denote $s = \log t$ for $t \in (0, 1]$. We have $t\dot{\varphi} = \frac{\partial \varphi}{\partial s}$. Taking derivatives on both sides of the equation (4.5.2) and by maximum principle arguments we then get (see also [132])

$$\frac{\partial^2 \varphi}{\partial s^2} = t\dot{\varphi} + t^2\ddot{\varphi} \leq C, \quad \text{here } \ddot{\varphi} = \frac{\partial^2 \varphi}{\partial t^2}. \tag{4.5.5}$$

By the uniform convergence (4.5.4) of $\mathcal{F}\varphi(s) \to \mathcal{F}\varphi_\infty$ as $s \to -\infty$, for any $\epsilon > 0$, there is an S_ϵ such that for all s_1, $s_2 \leq -S_\epsilon$, we have $\sup_X |\mathcal{F}\varphi(s_1) - \mathcal{F}\varphi(s_2)| \leq \epsilon$. For any $s < -S_\epsilon - 1$ and $x \in X\backslash S$, by mean value theorem

$$\mathcal{F}\partial_s \varphi(s_x, x) = \frac{1}{\sqrt{\epsilon}} \int_s^{s+\sqrt{\epsilon}} \partial_s(\mathcal{F}\varphi) ds \geq -\sqrt{\epsilon}, \quad \text{for some } s_x \in [s, s+\sqrt{\epsilon}].$$

By the upper bound (4.5.5), it follows that $\mathcal{F}\partial_s\varphi(s, x) \geq -C\sqrt{\epsilon} - \sqrt{\epsilon}$. Similarly

$$\mathcal{F}\partial_s \varphi(\hat{s}_x, x) = \frac{1}{\sqrt{\epsilon}} \int_{s-\sqrt{\epsilon}}^{s} \partial_s(\mathcal{F}\varphi(\cdot, x)) ds \leq \sqrt{\epsilon}, \text{ for some } \hat{s}_x \in [s-\sqrt{\epsilon}, s],$$

from (4.5.5) we get $\mathcal{F}\partial_s\varphi(s, x) \leq C\sqrt{\epsilon} + \sqrt{\epsilon}$. Hence we show that for any $s \leq -S_\epsilon - 1$ or $t = e^s \leq e^{-S_\epsilon - 1}$, it holds that

$$\sup_{x\in X\backslash S} |\mathcal{F}\partial_s\varphi(s, x)| = \sup_{x\in X\backslash S} |\mathcal{F}t\partial_t\varphi(t, x)| \leq C\sqrt{\epsilon},$$

so the lemma follows.

□

Corollary 4.5.1. *There exists a positive decreasing function $h(t)$ with $h(t) \to 0$ as $t \to 0$ such that*

$$\sup_X \mathcal{F}\big(|\varphi_t - t\dot{\varphi}_t - \varphi_\infty| + t|\dot{\varphi}_t|\big) \le h(t).$$

From Corollary 4.5.1 a straightforward adaption of the arguments of [132], we have an improvement of local C^2-estimate:

Lemma 4.18. *On any compact subset $K \subset\subset X \backslash S$, we have*

$$\limsup_{t\to 0} \Big(\sup_K \big(\mathrm{tr}_{\omega_t}\chi_\infty - \kappa\big)\Big) \le 0.$$

With the local C^2 estimate (see Proposition 4.5.1), following standard local C^3-estimates ([136, 91, 100]), we have

Lemma 4.19. *For any compact $K \Subset X\backslash S$, there exists a $C = C(K) > 0$ such that*

$$\sup_K |\nabla_\theta \omega_t|^2 \le C t^{-1}.$$

We have built up all the necessary ingredients to prove Theorem 1.6, whose proof is almost identical to that of Theorem 1.3 in [132]. For completeness, we sketch the proof below.

Proof of Theorem 1.6. Fix a compact subset $K' \subset X^\circ_{can}$ and let $K = \Phi^{-1}(K')$. By the Calabi C^3 estimate in Lemma 4.19, it follows that

$$\|t^{-1}\omega_t|_{X_y}\|_{C^1(X_y,\theta_y)} \le C, \quad t^{-1}\omega_t|_{X_y} \ge c\,\theta_y,$$

for all $y \in K'$ and $\theta_y = \theta|_{X_y}$.

Step 1: Define a function f on X_y by

$$f = \frac{(t^{-1}\omega_t|_{X_y})^{n-\kappa}}{\omega_{SF,y}^{n-\kappa}} = \binom{n}{\kappa}\frac{(\omega_t|_{X_y})^{n-\kappa}\wedge \chi_\infty^\kappa}{\omega_t^n} e^{\varphi_t-\varphi_\infty} \le e^{h(t)}\Big(\frac{\mathrm{tr}_{\omega_t}\chi_\infty}{\kappa}\Big)^\kappa \le 1+\tilde{h}(t),$$

for some $\tilde{h}(t) \to 0$ as $t \to 0$ (here $\tilde{h}(t)$ depends on K), where in the first inequality we use the Newton-Maclaurin inequality. f also satisfies that

$$\int_{X_y}(f-1)\omega_{SF,y}^{n-\kappa} = 0, \qquad \lim_{t\to\infty}\int_{X_y}|f-1|\omega_{SF,y}^{n-\kappa} = 0. \tag{4.5.6}$$

The Calabi estimate implies that $\sup_{X_y} |\nabla f|_{\theta_y} \leq C$ for all $y \in K'$, and (X_y, θ_y) have uniformly bounded diameter and volume for $y \in K'$. So it follows that f converges to 1 uniformly on K as $t \to 0$. That is

$$\|\big(t^{-1}\omega_t|_{X_y}\big)^{n-\kappa} - \omega_{SF,y}^{n-\kappa}\|_{C^0(X_y,\theta_y)} \to 0, \text{ as } t \to 0,$$

uniformly on K'. Since $t^{-1}\omega_t|_{X_y}$ converges in $C^\alpha(X_y, \theta_y)$ topology to some limit metric $\omega_{\infty,y}$ which satisfies the Monge-Ampère equation (weakly) on X_y, $\omega_{\infty,y}^{n-\kappa} = \omega_{SF,y}^{n-\kappa}$, by the uniqueness of complex Monge-Ampère equations, it follows that $\omega_{\infty,y} = \omega_{SF,y}$ and $t^{-1}\omega_t|_{X_y}$ converge in C^α to $\omega_{SF,y}$, for any $y \in K'$. Next we show the convergence is uniform in K'.

Step 2: Define a new f on $X \backslash S$ which takes the form

$$f|_{X_y} = \frac{t^{-1}\omega_t|_{X_y} \wedge (\omega_{SF,y})^{n-\kappa-1}}{\omega_{SF,y}^{n-\kappa}} \geq \Big(\frac{(t^{-1}\omega_t|_{X_y})^{n-\kappa}}{\omega_{SF,y}^{n-\kappa}}\Big)^{1/(n-\kappa)},$$

and the RHS tends to 1 uniformly on K as $t \to \infty$. Then we have similar equations as in (4.5.6) for this new f. This implies

$$\Big\|\frac{1}{n-\kappa}\mathrm{tr}_{\omega_{SF,y}}(t^{-1}\omega_t)|_{X_y} - 1\Big\|_{L^\infty(K)} \to 0, \quad \text{as } t \to 0.$$

So $t^{-1}\omega_t|_{X_y} \to \omega_{SF,y}$ uniformly for any $y \in K'$.

Step 3: Define

$$\tilde{\omega} = t\omega_{SF} + \chi_\infty.$$

From a result of [132] (see the proof of Theorem 1.1 of [132]), we have $|\mathrm{tr}_{\omega_t}\big(\omega_{SF} - \omega_{SF,y}\big)| \leq Ct^{-1/2}$, then

$$\mathrm{tr}_{\omega_t}\tilde{\omega} \leq \mathrm{tr}_{\omega_t}\Big(t\omega_{SF,y} + \chi_\infty\Big) + C\sqrt{t} = n + \tilde{h}(t),$$

for some $\tilde{h}(t) \to 0$ when $t \to 0$. Moreover it can be checked that

$$\lim_{t\to 0}\frac{\tilde{\omega}^n}{\omega_t^n} = 1, \quad \text{on } K.$$

Hence we see that $\omega_t \xrightarrow{C^0(K)} \chi_\infty$ as $t \to 0$.

We finish the proof of (1), (2) and (3) of Theorem 1.6.

Remark 5. *From* **Steps 1, 2 and 3**, *we see that for any compact subset $K \subset X\backslash S$, there exists an $\varepsilon(t) = \varepsilon_K(t) \to 0$ as $t \to 0$ such that when t is small*

$$\Phi^*\chi_\infty - \varepsilon(t)\theta \leq \omega_t \leq \Phi^*\chi_\infty + \varepsilon(t)\theta, \quad \text{on } K, \tag{4.5.7}$$

and

$$\Phi^*\chi_\infty \leq (1+\varepsilon(t))\omega_t, \quad \text{on } K. \tag{4.5.8}$$

From the uniform convergence of $t^{-1}\omega_t|_{X_y}$ to $\omega_{SF,y}$ for any $y \in \Phi(K)$, we see that there is a uniform constant $C_0 = C_0(K) > 0$ such that

$$\omega_t|_{X_y} \leq C_0 t\omega_{SF,y}, \quad \text{for all } y \in \Phi(K). \tag{4.5.9}$$

Choose a sequence $t_k \to 0$. The metric spaces (X, ω_{t_k}) have $\mathrm{Ric}(\omega_{t_k}) \geq -1$ and $\mathrm{diam}(X, \omega_{t_k}) \leq D$ for some constant $D < \infty$. By Gromov's pre-compactness theorem up to a subsequence we have

$$(X, \omega_{t_k}) \xrightarrow{d_{GH}} (\mathbf{Z}, d_{\mathbf{Z}}),$$

for some compact metric length space $\mathbf{Z}$ with diameter bounded by D. The idea of the proof of (4) in Theorem 1.6 is motivated by [50], and we present below a slightly different argument from theirs.

Step 4: We will show **Claim 3:** There exists an open subset $\mathbf{Z}_0 \subset \mathbf{Z}$ and a homeomorphism $f : X^\circ_{can} \to \mathbf{Z}_0$ which is a local isometry.

Proof of **Claim 3**. By Lemma 4.15, the maps $\Phi = \Phi_k : (X, \omega_{t_k}) \to (X_{can}, \chi)$ are uniformly Lipschitz with respect to the given metrics, and the target space is compact, so up to a subsequence $\Phi_k \to \Phi_\infty : (\mathbf{Z}, d_{\mathbf{Z}}) \to (X_{can}, \chi)$ along the GH convergence $(X, \omega_{t_k}) \to (\mathbf{Z}, d_{\mathbf{Z}})$ which is also Lipschitz and the convergence is in the sense that for any $x_k \to (X, \omega_{t_k})$ which converges to $z \in \mathbf{Z}$, then $\Phi_\infty(z) = \lim_{k\to\infty} \Phi_k(x_k)$, and there is a constant $C > 0$ such that $d_\chi\big(\Phi_\infty(z_1), \Phi_\infty(z_2)\big) \leq C d_{\mathbf{Z}}(z_1, z_2)$ for all $z_i \in \mathbf{Z}$.

We denote $\mathbf{Z}_0 = \Phi_\infty^{-1}(X^\circ_{can})$ which is an open subset of $\mathbf{Z}$ since Φ_∞ is continuous. We will show that $\Phi_\infty|_{\mathbf{Z}_0} : \mathbf{Z}_0 \to X^\circ_{can}$ is a bijection and a local isometry. Hence $f = (\Phi_\infty|_{\mathbf{Z}_0})^{-1} : X^\circ_{can} \to \mathbf{Z}_0$ is the desired map.

• $\Phi_\infty|_{\mathbf{Z}_0}$ *is injective:* Suppose $\Phi_\infty(z_1) = \Phi_\infty(z_2)$ for $z_1, z_2 \in \mathbf{Z}_0 = \Phi_\infty^{-1}(X_{can}^\circ)$. Denote $y = \Phi_\infty(z_1) = \Phi_\infty(z_2) \in X_{can}^\circ$. Since $(X_{can}^\circ, \chi_\infty)$ is an (incomplete) smooth Riemannian manifold there exists a small $r = r_y > 0$ such that $\big(B_{\chi_\infty}(y, 2r), \chi_\infty\big)$ is geodesic convex. Choose sequences $z_{1,k}$ and $z_{2,k} \in (X, \omega_{t_k})$ converging z_1, z_2 respectively along the GH convergence. By definition of $\Phi_k = \Phi \to \Phi_\infty$ it follows that $d_\chi(\Phi(z_{1,k}), \Phi_\infty(z_1)) \to 0$ and $d_\chi(\Phi(z_{2,k}), \Phi_\infty(z_2)) \to 0$. Since d_χ and d_{χ_∞} are equivalent on $B_{\chi_\infty}(y, 2r)$, it follows that $d_{\chi_\infty}(\Phi(z_{1,k}), \Phi(z_{2,k})) \to 0$ and hence we can find minimal χ_∞-geodesics γ_k connecting $\Phi(z_{1,k})$ and $\Phi(z_{2,k})$ with $\gamma_k \subset B_{\chi_\infty}(y, r)$ and $L_{\chi_\infty}(\gamma_k) \to 0$. By the locally uniform convergence (4.5.7) on $\Phi^{-1}\big(\overline{B_{\chi_\infty}(y, 2r)}\big)$ there exists a lift of γ_k, $\tilde\gamma_k$ in $\Phi^{-1}\big(\overline{B_{\chi_\infty}(y, 2r)}\big)$, such that $L_{\omega_{t_k}}(\tilde\gamma_k) \le L_{\chi_\infty}(\gamma_k) + \epsilon(t_k) L_\omega(\tilde\gamma_k) \to 0$ as $t_k \to 0$. $\tilde\gamma_k$ connects $z_{1,k}$ and $z_{2,k}$ hence $d_{\omega_{t_k}}(z_{1,k}, z_{2,k}) \le L_{\omega_{t_k}}(\tilde\gamma_k) \to 0$, which implies by the convergence of $z_{i,k} \to z_i$ that $d_{\mathbf{Z}}(z_1, z_2) = 0$ and $z_1 = z_2$.

• $\Phi_\infty|_{\mathbf{Z}_0}$ *is a local isometry:* let $z \in \mathbf{Z}_0$ and $y = \Phi_\infty(z) \in X_{can}^\circ$. There is a small $r = r_y > 0$ such that $(B_{\chi_\infty}(y, 3r), \chi_\infty)$ is geodesic convex. Take $U = (\Phi_\infty|_{\mathbf{Z}_0})^{-1}\big(B_{\chi_\infty}(y, r)\big)$ to be an open neighborhood of $z \in \mathbf{Z}$. We will show that $\Phi_\infty|_{\mathbf{Z}_0} : (U, d_{\mathbf{Z}}) \to (B_{\chi_\infty}(y, r), \chi_\infty)$ is an isometry. Fix any two points $z_1, z_2 \in U$ and $y_i = \Phi_\infty(z_i) \in B_{\chi_\infty}(y, r)$ for $i = 1, 2$. As before we choose $z_{i,k} \in (X, \omega_{t_k})$ such that $z_{i,k} \to z_i$ along the GH convergence for $i = 1, 2$. It follows then from $\Phi_k = \Phi \to \Phi_\infty$ that $d_{\chi_\infty}\big(\Phi(z_{i,k}), y_i\big) \to 0$, and when k is large, $\Phi(z_{i,k})$ lie in $B_{\chi_\infty}(y, 1.1r)$. Choose ω_{t_k}-minimal geodesics γ_k connecting $z_{1,k}$ and $z_{2,k}$ such that $d_{\omega_{t_k}}(z_{1,k}, z_{2,k}) = L_{\omega_{t_k}}(\gamma_k) \to d_{\mathbf{Z}}(z_1, z_2)$. The curve $\bar\gamma_k = \Phi(\gamma_k)$ connects $\Phi(z_{1,k})$ with $\Phi(z_{2,k})$. If $\bar\gamma_k \subset B_{\chi_\infty}(y, 3r)$, from (4.5.8) it follows that

$$d_{\chi_\infty}(\Phi(z_{1,k}), \Phi(z_{2,k})) \le L_{\chi_\infty}(\bar\gamma_k) \le (1 + \epsilon(t_k)) L_{\omega_{t_k}}(\gamma_k) \to d_{\mathbf{Z}}(z_1, z_2).$$

In case $\bar\gamma_k \not\subset B_{\chi_\infty}(y, 3r)$, we have

$$d_{\chi_\infty}(\Phi(z_{1,k}), \Phi(z_{2,k})) \le 3.8r \le L_{\chi_\infty}(\bar\gamma_k \cap B_{\chi_\infty}(y, 3r)) \le (1+\epsilon(t_k)) L_{\omega_{t_k}}(\gamma_k) \to d_{\mathbf{Z}}(z_1, z_2).$$

Letting $k \to \infty$ we conclude that $d_{\chi_\infty}(y_1, y_2) \le d_{\mathbf{Z}}(z_1, z_2)$. To see the reverse inequality, we take χ_∞-minimal geodesics σ_k connecting $\Phi(z_{1,k})$ and $\Phi(z_{2,k})$. Clearly $\gamma_k \subset B_{\chi_\infty}(y, 3r)$. Take a lift of σ_k, $\tilde\sigma_k$ in $\Phi^{-1}\big(\overline{B_{\chi_\infty}(y, 3r)}\big)$ it follows from (4.5.7) that $d_{\omega_{t_k}}(z_{1,k}, z_{2,k}) \le L_{\omega_{t_k}}(\tilde\sigma_k) \le L_{\chi_\infty}(\sigma_k) + \epsilon(t_k) L_\omega(\tilde\sigma_k) \to d_{\chi_\infty}(y_1, y_2)$. Letting $k \to \infty$

we get $d_{\mathbf{Z}}(z_1,z_2) \leq d_{\chi_\infty}(y_1,y_2)$. Hence $d_{\mathbf{Z}}(z_1,z_2) = d_{\chi_\infty}(y_1,y_2)$ and $\Phi_\infty|_{\mathbf{Z}_0} : U \to B_{\chi_\infty}(y,r)$ is an isometry.

- $\Phi_\infty|_{\mathbf{Z}_0}$ *is surjective:* this is almost obvious from the definition. Take any $y \in X^\circ_{can}$ and any fixed point $x \in \Phi^{-1}(y) \subset (X,\omega_{t_k})$. Up to a subsequence $x \xrightarrow{d_{GH}} z \in (\mathbf{Z}, d_{\mathbf{Z}})$. It then follows from $\Phi_k \to \Phi_\infty$ that $d_\chi(y,\Phi_\infty(z)) = d_\chi\big(\Phi_k(x),\Phi_\infty(z)\big) \to 0$ as $k\to\infty$. Hence $\Phi_\infty(z) = y$ and $z \in \Phi_\infty^{-1}(X^\circ_{can}) = \mathbf{Z}_0$.

$\square$

Step 5: In this step we will show $\mathbf{Z}_0 \subset \mathbf{Z}$ is dense. Fix a base point $\bar{x} \in \mathbf{Z}_0$, upon rescaling if necessary we may assume the metric ball $B_{\chi_\infty}\big(f^{-1}(\bar{x}),2\big) \subset (X^\circ_{can},\chi_\infty)$ is geodesic convex. Choose a sequence of points $\bar{p}_k \in (X,\omega_{t_k})$ such that $\bar{p}_k \to \bar{x}$ along the GH convergence $(X,\omega_{t_k}) \to (\mathbf{Z}, d_{\mathbf{Z}})$. We define a function on $X\times[0,\infty)$ as the normalized volume ([10])

$$\underline{V}_k(x,r) = \frac{\mathrm{Vol}_{\omega_{t_k}}\big(B_{\omega_{t_k}}(x,r)\big)}{\mathrm{Vol}_{\omega_{t_k}}\big(B_{\omega_{t_k}}(\bar{p}_k,1)\big)},$$

by standard volume comparison it is shown in [10] that $\underline{V}_k(\cdot,\cdot)$ is equi-continuous and uniformly bounded hence they converges (up to a subsequence) to a function $\underline{V}_\infty : \mathbf{Z}\times[0,\infty)\to[0,\infty)$ in the sense that for any $x_k \to x$ along the GH convergence and $r \geq 0$,

$$\underline{V}_k(x_k,r) \to \underline{V}_\infty(x,r), \quad \text{as } k\to\infty.$$

And $\underline{V}_\infty$ satisfies similar estimates as in volume comparison, i.e. for $r_1 \leq r_2$, $\frac{\underline{V}_\infty(x,r_1)}{\underline{V}_\infty(x,r_2)} \geq \mu(r_1,r_2) > 0$ where $\mu(\cdot,\cdot)$ is the quotient of volumes of balls in a space form. The function $\underline{V}_\infty$ induces a Radon ν on $(\mathbf{Z}, d_{\mathbf{Z}})$. More precisely for any $K\subset\mathbf{Z}$, define

$$\hat{\nu}(K) = \lim_{\delta\to 0}\hat{\nu}_\delta(K) = \lim_{\delta\to 0}\inf\sum_i \underline{V}_\infty(x_i,r_i)$$

where the infimum is taken over all metric balls $B_{d_{\mathbf{Z}}}(x_i,r_i)$ with $r_i\leq\delta$ whose union covers K.

Claim 4: For any $x\in\mathbf{Z}_0$ and $r = r_x > 0$ such that $B_{\chi_\infty}\big(f^{-1}(x),2r\big)\subset X^\circ_{can}$ is geodesic convex, we have

$$\underline{V}_\infty(x,r) = v_0\int_{\Phi^{-1}\left(B_{\chi_\infty}(f^{-1}(x),r)\right)} e^{-\varphi_\infty}\theta^n$$

for a fixed constant $v_0 = \Big(\int_{\Phi^{-1}\big(B_{\chi_\infty}(f^{-1}(\bar{x}),1)\big)} e^{\varphi_\infty}\theta^n \Big)^{-1}$.

Proof of **Claim 4**. The proof is parallel to that in [50], so we only provide a sketch. For the given $x \in \mathbf{Z}_0$, we choose a sequence of points $p_k \in (X, \omega_{t_k})$ such that $p_k \to x$. As in [50], due to (4.5.7) and that the metrics ω_{t_k} and θ are equivalent in $\Phi^{-1}\big(B_{\chi_\infty}(f^{-1}(x), 2r)\big)$, it can be shown that

$$\Phi^{-1}\big(B_{\chi_\infty}(f^{-1}(x), r-\epsilon_k)\big) \subset B_{\omega_{t_k}}\big(p_k, r\big) \subset \Phi^{-1}\big(B_{\chi_\infty}(f^{-1}(x), r+\epsilon_k)\big) \tag{4.5.10}$$

when $k >> 1$ and here $\epsilon_k \to 0$ as $k \to \infty$. It follows then that

$$\lim_{k\to\infty} \int_{B_{\omega_{t_k}}(p_k,r)} e^{\varphi_{t_k}}\theta^n = \int_{\Phi^{-1}\big(B_{\chi_\infty}(f^{-1}(x),r)\big)} e^{\varphi_\infty}\theta^n.$$

From the equation $\omega_t^n = t^{n-\kappa}e^{\varphi_t}\theta^n$, we have

$$\begin{aligned}\underline{V}_k(p_k, r) =& \frac{\int_{B_{\omega_{t_k}}(p_k,r)} t^{n-\kappa}e^{\varphi_{t_k}}\theta^n}{\int_{B_{\omega_{t_k}}(\bar{p}_k,1)} t_k^{n-\kappa}e^{\varphi_{t_k}}\theta^n} \\ \to& \frac{\int_{\Phi^{-1}\big(B_{\chi_\infty}(f^{-1}(x),r)\big)} e^{\varphi_\infty}\theta^n}{\int_{\Phi^{-1}\big(B_{\chi_\infty}(f^{-1}(\bar{x}),1)\big)} e^{\varphi_\infty}\theta^n},\end{aligned}$$

where for the convergence of the denominators we use a similar relation as in (4.5.10) for $\bar{p}_k$, $\bar{x}$. From the definition that $\underline{V}_k(p_k, r) \to \underline{V}_\infty(x, r)$, we finish the proof of **Claim 4**. □

Since along the Gromov-Hausdorff convergence the diameters are uniformly bounded by $D < \infty$, $\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(p_k, D)) = \mathrm{Vol}(X, \omega_{t_k}^n)$. So

$$\underline{V}_\infty(x, D) = \lim_{k\to\infty} \frac{\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(p_k, D))}{\mathrm{Vol}_{\omega_{t_k}}(B_{\omega_{t_k}}(\bar{p}_k, 1))} = \lim_{k\to\infty} \frac{\int_X e^{\varphi_{t_k}}\theta^n}{\int_{B_{\omega_{t_k}}(\bar{p}_k,1)} e^{\varphi_{t_k}}\theta^n} = v_0 \int_X e^{\varphi_\infty}\theta^n.$$

Therefore from $\mathbf{Z} = B_{d_\mathbf{Z}}(x, D)$, we have

$$\hat{\nu}(\mathbf{Z}) \le v_0 \int_X e^{\varphi_\infty}\theta^n.$$

Assume $\mathbf{Z}_0 \subset \mathbf{Z}$ were not dense, then there exists a metric ball $B_{d_\mathbf{Z}}(z, \rho) \subset \mathbf{Z}\backslash\mathbf{Z}_0$, by volume comparison estimate for $\underline{V}_\infty$

$$\hat{\nu}\big(B_{d_\mathbf{Z}}(z, \rho)\big) \ge \underline{V}_\infty(z, D)\mu(\rho, D) =: \eta_0 > 0.$$

Then for any compact subset $K \subset \mathbf{Z}_0$, $\hat{\nu}(K) \leq \hat{\nu}(\mathbf{Z}) - \eta_0$. On the other hand, for any open covering $B_{d_{\mathbf{Z}}}(x_i, r_i)$ of K with $B_{\chi_\infty}(f^{-1}(x_i), 2r_i)$ geodesic convex in $(X^\circ_{can}, \chi_\infty)$ and $r_i < \delta$, we have

$$\sum_i \underline{V}_\infty(x_i, r_i) = \sum_i v_0 \int_{\Phi^{-1}\left(B_{\chi_\infty}(f^{-1}(x_i), r_i)\right)} e^{\varphi_\infty}\theta^n \geq v_0 \int_{\Phi^{-1}\left(f^{-1}(K)\right)} e^{\varphi_\infty}\theta^n$$

taking infimum over all such coverings and letting $\delta \to 0$, we get $\hat{\nu}(K) \geq v_0 \int_{\Phi^{-1}\left(f^{-1}(K)\right)} e^{\varphi_\infty}\theta^n$. If we take K large enough so that $f^{-1}(K) \subset X^\circ_{can}$ is large, we can achieve that

$$\hat{\nu}(K) \geq v_0 \int_{\Phi^{-1}(X^\circ_{can})} e^{\varphi_\infty}\theta^n - \frac{\eta_0}{10} = v_0 \int_X e^{\varphi_\infty}\theta^n - \frac{\eta_0}{10} \geq \hat{\nu}(\mathbf{Z}) - \frac{\eta_0}{10}.$$

Hence we get a contradiction, and $\mathbf{Z}_0 \subset \mathbf{Z}$ is dense since $\hat{\nu}(\mathbf{Z} \backslash \mathbf{Z}_0) = 0$.

□

4.5.1 Proof of Theorem 4.1

The proof of Theorem 4.1 is almost identical with that of Theorem 1.6. We give the sketch here. The solution g_t lies in the Kähler class $tL + (1-t)K_X$ for all $t \in (t_{min}, 1]$. By definition and straightforward calculations from estimates of Yau [136] and Aubin [4], for any $t \in (t_{min}, 1]$, the class $tL + (1-t)K_X$ is Kähler and so $t_{min}L + (1-t_{min})K_X$ is nef. We let Ω be a smooth volume form on X and $\chi \in [t_{min}L + (1-t_{min})K_X]$ be a smooth closed $(1,1)$-form defined by

$$\chi = \sqrt{-1}\partial\overline{\partial}\log\Omega + \theta.$$

Then the twisted Kähler-Einstein equation (4.1.9) is equivalent to the following complex Monge-Ampère equation for $t \in (t_{min}, 1]$

$$(\chi + (t - t_{min})\theta + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = (t - t_{min})^{n-\kappa} e^{\varphi_t}\Omega, \tag{4.5.11}$$

where $\kappa = \nu(t_{min}L + (1-t_{min})K_X)$, the numerical dimension of the line bundle $t_{min}L + (1-t_{min})K_X$. By Proposition 4.1.1, there exists $C = C(X, \chi, \theta) > 0$ such that for all $t \in (t_{min}, 1]$,

$$\|\varphi_t - V_t\|_{L^\infty(X)} \leq C,$$

where V_t is the extremal function associated to $\chi + (t - t_{min})\theta$. The rest of the proof for Theorem 4.1 is exactly the same as that of Theorem 1.5 and we leave it as an exercise for interested readers.

Chapter 5

Kähler-Einstein metric near isolated log canonical singularity

This chapter is from joint work [32] with Ved Datar and Jian Song.

5.1 Introduction

Existence of Kähler-Einstein metric on complex manifold has been the central topic in complex geometry for decades. Aubin and Yau [4, 136] establish the existence of Kähler-Einstein metric independently on canonically polarized compact manifold. And Yau [136] establishes the existence of Ricci flat metric on complex manifold with zero Chern class by solving the so called Calabi conjecture in [136]. Also, recent results of Chen-Donaldson-Sun [16, 17, 18] also Tian [127] confirm the Yau-Tian-Donaldson conjecture for smooth Fano manifolds. Also, there are intensive study of degenerate Monge-Ampère equations and construction of singular Kähler-Einstein metrics on singular varieties with Klt singularities, for example in [40, 139], based on Kolodziej's fundamental result in [72]. For canonical polarized variety with log canonical singularity, there are analytic difficulty to solve the Monge-Ampère equation. Berman and Guenancia [6] construct Kähler-Einstein metric on these varieties by a variational approach. However, little is known about the geometric property of these singular Kähler-Einstein metrics. Hence in our paper, we attempt to describe the geometry of the Kähler-Einstein metric with negative curvature on singular canonical polarized variety, especially its behaviour towards the log canonical locus.

On the other hand, understanding singular Kähler-Einstein metrics is crucial in terms of the compactness of complex manifolds coupled with Kähler-Einstein metrics. In their fundamental work [38, 39], Donaldson-Sun showed that the Gromov-Hausdorff limit of a sequence of none collapsed polarized Kähler-Einstein manifold is a $\mathbb{Q}$ -variety with Klt singularities and the tangent cone at any point of the limit metric space is unique. Motivated by uniqueness of tangent cone result, Li-Wang-Xu [83] further prove that on a Klt singularity with Kähler-Einstein metric, the Ricci flat tangent cone is independent of the metric structure based on a series of deep work in [81, 82]. And from a global scale, recent deep result of Hein-Sun [59] proved that for each Calabi-Yau variety with isolated cone singularity, for example, cone over a smooth Fano Einstein variety, the global Kähler-Einstein metric is asymptotically the same as the local Ricci flat metric constructed by Calabi Ansatz. Note that all the results mentioned above are concerned with non collapsed Kälher-Einstein metrics or more intrinsically Klt singularity, therefore it is natural to consider the Kähler-Einstein metric on log canonical singularity, which serves as the collapsed limit of complex manifold coupled with Kähler-Einstein metrics with negative curvature. In this paper, we prove a rigidity result concerning the Einstein metrics towards certain types of isolated log canonical singularities, which is an analogue of the result of Hein and Sun. They push the analysis of metrics to the tangent cone by blowing up the metrics at the singularity and we analyze the metrics by push them to infinite end.

Now we outline our results. In paper [114] of the second author, not only Kähler-Einstein metric is constructed on the canonical polarized variety with log canonical singularity, it is also shown that in a KSBA family of canonical polarized varieties, the Kähler-Einstein metric of nearby fiber will converge to the singular Kähler-Einstein metric on the central fiber in Gromov-Hausdorff sense and the singular metric on central fiber form complete end towards the log canonical locus. In this article we want to move one step further, aiming to have a more concrete description of the degeneration of Kähler-Einstein metrics towards the log canonical locus on the central fiber. We attack this problem by reducing it to a local question near the singularity. Roughly, we first construct infinite many local Kähler-Einstein metrics by solving related Monge-Ampère equation with Dirichlet boundary, which seems to be interesting itself. Then for certain type of log canonical singularity, we combine geometric argument and estimate from the Monge-Ampère equation to compare these different local Kähler-Einstein metrics.

It turns out that these different local Kähler-Einstein share the same metric behaviour towards the complete end. In particular, in complex dimension 2, we have a complete picture of the metric degeneration of canonical polarized surface based on the good local model metric constructed by Kobayashi and Nakamura in [69, 70].

We first introduce standard definition of log canonical singularity.

Definition 5.1. *Let X be a normal projective variety such that K_X is a $\mathbb{Q}$-Cartier divisor. Let $\pi : Y \to X$ be a log resolution and $\{E_i\}_{i=1}^p$ the irreducible components of the exceptional locus $Exc(\pi)$ of π. There there exists a unique collection $a_i \in \mathbb{Q}$ such that*

$$K_Y = \pi^* K_X + \sum_{i=1}^{p} a_i E_i.$$

Then X is said to have

- *terminal singularities if $a_i > 0$, for all i.*
- *canonical singularities if $a_i \geq 0$, for all i.*
- *log terminal singularities if $a_i > -1$, for all i.*
- *log canonical singularities if $a_i \geq -1$, for all i.*

We also want to fix the geometric domains that will be discussed throughout this paper. Recall our setting in the introduction.

Setting: Let (X, p) be a germ of isolated normal log canonical $\mathbb{Q}$-Cartier singularity embedded in $(\mathbb{C}^N, 0)$. Our main interest in this paper will be neighbourhood of the singular point p. Using a bounded PSH function ρ on X, we cut domains

$$\Omega := \{\rho < a\} \tag{5.1.1}$$

contained in X such that $\partial\Omega$ are strongly pseudoconvex. We also fix a Kähler metric χ and volume form Ω_X on X

$$\chi = \sqrt{-1}\partial\overline{\partial}\rho, \Omega_X = e^{\rho} V \wedge \bar{V}$$

where V is local holomorphic volume form (up to taking root of multiple holomorphic volume form) on a neighbourhood of p in X. The complex Monge-Ampère equation of our interest in relation to the Kähler-Einstein equation on Ω is given by

$$\begin{cases} (\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\varphi}\Omega_X. \\ \varphi_{|\partial\Omega} = f \end{cases} \tag{5.1.2}$$

where f is an arbitrary smooth function.

We have to prescribe singularities of the solution φ to obtain a canonical and unique Kähler-Einstein current on X. To do so, we lift all the data to a log resolution $\pi : Y \to X$. By definition of semi-log canonical singularities,

$$K_Y = \pi^* K_X + \sum_i a_i E_i - \sum_j b_j F_j,\ a_i \geq 0,\ 0 < b_j \leq 1.$$

We approximate equation (1.3.1) in the following way. We pull back all the data from X to Y. Let σ_E be the defining section for $E = \sum_{i=1}^{I} a_i E_i$ and σ_F be the defining section for $F = \sum_{j=1}^{J} b_j F_j$ (possibly multivalued). We equip the line bundles associated to E and F with smooth hermitian metric h_E, h_F on Y. Let Ω_Y be a smooth strictly positive volume form on $\pi^{-1}(\Omega)$, defined by

$$\Omega_Y = (|\sigma_E|^2_{h_E})^{-1} |\sigma_F|^2_{h_F} \Omega_X.$$

Let θ be a fixed smooth Kähler form on Y and we consider the following family of complex Monge-Ampère equations on Ω for $s \in (0, 1)$,

$$\begin{cases} (\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_s)^n = e^{\psi_s}(|\sigma_E|^2_{h_E} + s)(|\sigma_F|^2_{h_F} + s)^{-1}\Omega_Y. \\ \psi_{s_{|\partial\Omega}} = f \end{cases} \tag{5.1.3}$$

Abusing notation, we still denote the domain $\pi^{-1}(\Omega)$ by Ω. By the same argument as step 1 of theorem (1.8), we can assume $f = 0$ and there exists a unique smooth solution ψ_s solving equation (5.1.3) for $s > 0$. When $s = 0$, equation (5.1.3) coincides with equation (1.3.1). Next we want to use pluripotential theory to get uniform C^0 estimate with barrier of ψ_s. Similar C^0 estimate of degenerate Monge-Ampère equations are have been obtained in different settings such as on unit ball in [72] and singular variety with Klt singularity in [40, 37, 139]. The main differences of our geometric domain with

previous setting are twofold: Firstly, we consider the log canonical singularity, which means we only have L^1 integrability instead of $L^p(p>1)$ integrability of right hand side of equations (5.1.3), hence we don't have uniform C^0 control of solutions of a family of degenerate Monge-Ampère equations. Secondly, our geometric domain Ω is not globally strongly pseudoconvex after we blow up the isolated log canonical singularity.
Our theorems 1.7, 1.8 are concerning the construction of Kähler-Einstein metrics near isolated log canonical singularity.

We point out that a large class of log- canonical singularities admits uniformization with property (A) which is the key assumption in Theorem 1.8. Especially, a complete picture of uniformization of isolated log canonical singularity in complex dimension 2 is obtained in [69, 70]. Also, another interesting family of uniformization of high-dimension log canonical singularity (cone over abelian variety) is constructed in [41]. Concerning the existence of Kähler-Einstein metric, our first construction is more general than the second one. But our second construction will be more useful when we are comparing the model metric χ in property (A) with an arbitrary complete metric near the singular point p.

After we get the existence of many different local Kähler-Einstein metrics on $(\Omega \setminus p)$, we focus on investigating the geometry of these complete local metrics. It turns out that these different local Kähler-Einstein metrics are asymptotic close to each other at the infinity end. To achieve this, we first show that any complete Kähler-Einstein metric on $(\Omega \setminus p)$ comes from the solution of equation (1.3.2) by a geometric argument. Suppose we have two complete Kähler-Einstein metric χ and χ' on $(\Omega \setminus p)$, by Kähler-Einstein condition, we have

$$\chi = \sqrt{-1}\partial\overline{\partial}\log \chi^n, \chi' = \sqrt{-1}\partial\overline{\partial}\log \chi'^n, \chi' = \chi + \sqrt{-1}\partial\overline{\partial}\varphi$$

where $\varphi := \log \frac{\chi'^n}{\chi^n}$. The crucial thing is that we show that $\varphi(x) \to 0$ when $x \to p$. This seems to be none trivial even if we assume χ' comes from one solution of equation (1.3.2) corresponding to a choice of boundary function f, since we can have very huge perturbation of boundary condition f, from which we can only conclude boundedness of φ globally on $(\Omega \setminus p)$. Now we state our estimate of φ . For any ϵ, we define a neighbourhood U_ϵ of p to be $U_\epsilon := \{x|\mathrm{dist}_\chi(x,\partial\Omega) \geq \frac{2c(n)}{\epsilon}$ and $\mathrm{dist}_{\chi'}(x,\partial\Omega) \geq \frac{2c(n)}{\epsilon}\}$. Geometrically, this is a region consisting of points which are far away from the boundary $\partial\Omega$ measured in both metrics χ and χ', then

Theorem 5.2. *(Theorem 1.9) For any $\epsilon > 0$, $-\epsilon \leq \varphi \leq \epsilon$ in U_ϵ.*

Remark 6. *The above theorem is true as long as χ and χ' are complete. No other metric properties are required.*

With the above theorem proved, especially the boundedness of φ, if we further assume χ has bounded geometric property (A), we are able to show that any complete metric χ' on $(\Omega \setminus p)$ is one solution of equations (1.3.2) by showing the uniqueness in proposition (5.30) of smooth solution of equation (1.3.2) with the fixed boundary condition. Note that in theorem (1.8), for fixed f, we only find one solution in certain function space, but apriori we don't know whether the smooth solution is unique or not. Then use theorem (1.8) to control high-order derivatives of φ. Remember that we already prove the decay of φ towards the complete end in theorem (1.9), we are able to conclude that:

Theorem 5.3. *(Theorem 1.10) Suppose (Ω, χ) is a metric with property (A) and χ' is another complete Kähler-Einstein metric on Ω. Then for any positive number ϵ and any non negative integer k, we have $\sum_{i=1}^{k} \|\nabla^i \varphi\|_\chi(q) \leq \epsilon C(k,\chi,f)$ for $q \in U_\epsilon$ where C is a constant depends on the geometry of χ, k and f.*

As an important application of the above theorem (1.10), we have a detailed description of degeneration of Kähler-Einstein metrics on canonical polarized varieties with certain type log canonical singularity. In [114], the second author proves that

Theorem 5.4. *Let $\pi : \mathcal{X} \to B$ be a stable degeneration of smooth canonical models of complex dimension n over a disc $B \subset \mathbb{C}$. Suppose the central fibre $\pi^{-1}(0)$ is given by $\mathcal{X}_0 = \bigcup_{\alpha=1}^{\mathcal{A}} X_\alpha$, where $\{X_\alpha\}_\alpha$ are the irreducible components of $\mathcal{X}_0$. Let g_t be the* *unique Kähler-Einstein metric on $\mathcal{X}_t$ for $t \in B^*$ with*

$$Ric(g_t) = -g_t.$$

Then the following conclusions hold as $t \to 0$.

1. *There exist points $(p_t^1, p_t^2, ..., p_t^{\mathcal{A}}) \in \mathcal{X}_t \times \mathcal{X}_t \times ... \times \mathcal{X}_t$ such that $(\mathcal{X}_t, g_t, p_t^1, ..., p_t^{\mathcal{A}})$ converge in pointed Gromov-Hausdoff topology to a finite disjoint union of complete Kähler-Einstein metric spaces*

$$\mathbf{Y} = \coprod_{\alpha=1}^{\mathcal{A}} (Y_\alpha, d_\alpha, y_\alpha).$$

2. *Let $\mathcal{R}_{Y_\alpha}$ be the regular part of the metric space (Y_α, d_α) for each α. Then $(\mathcal{R}_{Y_\alpha}, d_\alpha)$ is a smooth Kähler-Einstein manifold of complex dimension n and the singular set $\mathcal{S}_\alpha = Y_\alpha \setminus \mathcal{R}_{Y_\alpha}$ is closed and has Hausdorff dimension no greater than $2n - 4$.*

3. *$\coprod_{\alpha=1}^{\mathcal{A}} Y_\alpha$ is homeomophic to $\mathcal{X}_0 \setminus \mathrm{LCS}(\mathcal{X}_0)$, where $\mathrm{LCS}(\mathcal{X}_0)$ is the non-log terminal locus of $\mathcal{X}_0$. $\coprod_{\alpha=1}^{\mathcal{A}} \mathcal{R}_{Y_\alpha}$ is biholomorphic to the nonsingular part of $\mathcal{X}_0$.*

4. *$\sum_{\alpha=1}^{\mathcal{A}} \mathrm{Vol}(Y_\alpha, d_\alpha) = \mathrm{Vol}(\mathcal{X}_t, g_t)$ for all $t \in B^*$, where $\mathrm{Vol}(Y_\alpha, d_\alpha)$ is the Hausdorff measure of (Y_α, d_α).*

Finally, theorem (1.10) and theorem (5.4) together give our last theorem:

Theorem 5.5. *In the same setting as theorem (5.4). Then towards the isolated log canonical singularities on the central fiber with property (A), the Kähler-Einstein metric on center fiber is asymptotic the same as the model metric χ defined in property (A).*

Proof. Item (3) in Theorem (5.4) gives the completeness of the unique Kähler-Einstein ω constructed in [114] towards the log canonical locus. Then theorem (5.5) gives the asymptotic closeness of local model metric χ and the global metric ω. ∎

5.2 Pluripotential theory and construction of Kähler-Einstein metrics: First Approach

The perturbed complex Monge-Ampère equation of our interest in relation to the Kähler-Einstein equation on Ω is given by (5.1.1)

$$\begin{cases} (\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_s)^n = e^{\psi_s}(|\sigma_E|^2_{h_E} + s)(|\sigma_F|^2_{h_F} + s)^{-1}\Omega_Y. \\ \psi_{s_{|\partial\Omega}} = f \end{cases} \tag{5.2.1}$$

We first generalize the deep results of [72] to our geometric domain.

Theorem 5.6. *Let ψ_s be the smooth solution of equation on Ω:*

$$\begin{cases} (\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_s)^n = e^{\psi_s} g\Omega_Y \\ \psi_{s_{|\partial\Omega}} = 0 \end{cases} \tag{5.2.2}$$

where Ω_Y is smooth positive volume on Ω and $g \in C^\infty(\Omega)$ satifying $\int_\Omega g^{1+\eta}\Omega_Y \le Q$, then we have $|\psi_s| \le C$, where $C = C(\Omega, \chi, \eta, Q)$.

We do some preparations for the proof.

Lemma 5.7. *Let Ω be as above and ω be a Kähler metric on Ω, then for $u, v \in PSH(\omega) \cap L^\infty(\Omega)$ satisfying $\lim_{\eta \to z}(u - v) \ge 0$ for any $z \in \partial\Omega$ we have*

$$\int_{u<v} (\omega + \sqrt{-1}\partial\overline{\partial}v)^n \le \int_{(u<v)} (\omega + \sqrt{-1}\partial\overline{\partial}u)^n.$$

We also introduce some standard concepts in pluripotential theory. For a compact set K in a domain Ω, here Ω is not necessary in $\mathbb{C}^N$. Define

$$Cap(K, \Omega) := sup\{\int_K (\sqrt{-1}\partial\overline{\partial}u)^n, u \in PSH(\Omega), -1 < u < 0\},$$

$$Cap_\omega(K, \Omega) := sup\{\int_K (\omega + \sqrt{-1}\partial\overline{\partial}u)^n, u \in \omega - PSH(\Omega), -1 < u < 0\}$$

$$U_{K,\Omega}(x) := sup\{u(x) | u \in PSH(\Omega), u|_K = -1, -1 \le u \le 0\},$$

$$U_{\omega,K,\Omega}(x) := sup\{u(x) | u \in \omega - PSH(\Omega), u|_K = -1, -1 \le u \le 0\}$$

The following lemma says that the capacity $Cap_\omega(K, \Omega)$ can be computed by extremal function $U_{\omega,K,\Omega}$. Similar results are proved on strongly pseudoconvex domain in $\mathbb{C}^n$ and compact complex manifold and a simple modification will give a version for our purpose.

Lemma 5.8. *For a compact set K in Ω, we have $cap_\omega(K,\Omega) = \int_K (\omega + \sqrt{-1}\partial\overline{\partial} u^*_{\omega,K})^n$, where $U^*_{\omega,K}$ is the upper semi regularization of function $U_{\omega,K}$.*

Proof. We sketch the proof here. First of all, by Proposition 4.1 of [53], $(\omega+\sqrt{-1}\partial\overline{\partial}U^*_{\omega,K})^n$ is supported on $K \cup \{U^*_{\omega,K=0}\}$ (In our setting, set $\{U^*_{\omega,K} = 0\}$ is larger than the set $\partial\Omega$). And since $U^*_{\omega,K}$ itself is ω- PSH with value between -1 and 0. Hence $\int_K (\omega + \sqrt{-1}\partial\overline{\partial}U^*_{\omega,K})^n \leq Cap_\omega(K,\Omega)$. On the other hand, fixing a ω- PSH function u with $-1 < u < 0$, we have

$$\int_K (\omega+\sqrt{-1}\partial\overline{\partial}u)^n \leq \int_{\{U^*_{\omega,K}<u\}} (\omega+\sqrt{-1}\partial\overline{\partial}u)^n \leq \int_{\{U^*_{\omega,K}<u\}} (\omega+\sqrt{-1}\partial\overline{\partial}U^*_{\omega,K})^n = \int_K (\omega+\sqrt{-1}\partial\overline{\partial}U^*_{\omega,K})^n$$

The first inequality is due to the facts that $(\omega + \sqrt{-1}\partial\overline{\partial}u)^n$ doesn't charge mass on $\{U_{\omega,K} \neq U^*_{\omega,K}\}$ and $K \subset \{U_{\omega,K} < u\}$. The second inequality is due to comparison principle and the third inequality is due to the facts that $\{U^*_{\omega,K} < u\} \cap \{U^*_{\omega,K} = 0\} = \emptyset$ and $(\omega + \sqrt{-1}\partial\overline{\partial}U^*_{\omega,K})^n$ is supported on $K \cup \{U^*_{\omega,K} = 0\}$. This will finish the proof. □

For fixed s, we are interested in the set where ψ_s is small. We define:

$$U(l) := \{\psi < -l\}, a(l) := Cap_\omega(U(l),\Omega), b(l) := \int_{U(l)} (\omega + \sqrt{-1}\partial\overline{\partial}\psi)^n$$

The following lemma roughly says that in our geometric settting, the $Cap_\omega(U(l),\Omega)$ can be controlled by $b(l+t)$ in some sense.

Lemma 5.9. *Fix $\omega := \omega_s = \chi + s\theta$ and let $\psi := \psi_s$ be the solution of equation of (5.2.2). Then for any $0 < t < 1, l > 2$, we have $t^n Cap_\omega(U(l+t),\Omega) \leq \int_{U(l)} (\omega + \sqrt{-1}\partial\overline{\partial}\psi)^n$.*

Proof. Consider any compact regular set $K \subset U(l+t)$, the $\omega - PSH$ function $W := \frac{1}{t}(\psi + l)$, and the set $V := \{W < U^*_{\omega,K}\}$. We can verify the inclusions $K \subset V \subset U(l)$. Once we have the inclusions, we can apply lemmas (5.7) and (5.8) to conclude:

$$Cap_\omega(K,\Omega) = \int_K (\omega + \sqrt{-1}\partial\overline{\partial}U^*_{\omega,K})^n \leq \int_V (\omega + \sqrt{-1}\partial\overline{\partial}U^*_{\omega,K})^n \leq \int_V (\omega + \sqrt{-1}\partial\overline{\partial}W)^n$$

$$\leq t^{-n} \int_V (\omega + \sqrt{-1}\partial\overline{\partial}\psi)^n \leq t^{-n} \int_{U(l)} (\omega + \sqrt{-1}\partial\overline{\partial}\psi)^n = t^{-n} b(l).$$

□

We also want to show the Monge-Ampère measure $\lambda(K) := \int_K (\omega + \sqrt{-1}\partial\overline{\partial}\psi)^n$ can by controlled by capacity.

Lemma 5.10. *For any compact set $K \subset \Omega$, we have $\lambda(K) \leq C_l Cap_{\omega_s}(K,\Omega)^l$ for constant l large.*

Proof. For simplicity, we assume that our domain Ω is part of compact complex manifold M without boundary and the metric $\omega := \chi + s\theta$ is the restriction of a Kähler metric $\tilde{\omega}$ on M. Using equation, we know $\lambda(K)$ is $L^p, p > 1$ integrable with respect to a fixed measure. And it is standard that $\lambda(K) < C_l Cap_{\tilde{\omega}}(K,M)^l$, where C_l is independent of s, see [37]. And it is easy to see from the definition that $Cap_{\tilde{\omega}}(K,M) \leq Cap_{\omega}(K,\Omega)$. This will finish the proof.

□

At last we need to show that the capacities have uniform decay.

Proposition 5.11. *Let ψ_s be the solution of equation (5.2.2), then $Cap_{\omega_s}(U(l+1),\Omega) < C\frac{1}{l^n}$ for some constant C independent of l and ω_s.*

Proof. The key observation of us is on Ω, χ can be represented by $\sqrt{-1}\partial\overline{\partial}\rho$, so $Cap_{\chi}(K,\Omega) \leq ACap(K,\Omega)$ whese A only depends on the norm of $|\rho|_{L^\infty(\Omega)}$ which is bounded. This enables us to compare $Cap_{\omega_s}(K,\Omega)$ with $Cap(K,\Omega)$ uniformly. Now we conclude that:

$$Cap_{\omega_s}(U(l+1),\Omega) \leq \int_{\bar{U}(l)} (\omega_s + \sqrt{-1}\partial\overline{\partial}\psi)^n = \lambda(\bar{U}(l)) \leq Cap_{\chi}(\bar{U}(l),\Omega)$$

$$\leq ACap(\bar{U}(l),\Omega) \leq ACap_{\frac{\omega_s}{l-\eta}}(\bar{U}(l),\Omega) = A\int_{\bar{U}(l)} (\frac{\omega_s}{l-\eta} + \sqrt{-1}\partial\overline{\partial}U^*_{\bar{U}(l)})$$

where $l >> \eta > 0$. Note $\psi = 0$ on $\partial\Omega$ and $\frac{\psi}{l-\eta} < -1$ on $\bar{U}(l)$, by comparison principle we get

$$\int_{\bar{U}(l)} (\frac{\omega_s}{l-\eta} + \sqrt{-1}\partial\overline{\partial}U^*_{\bar{U}(l)})^n \leq \int_{\bar{U}(l)} (\frac{\omega_s}{l-\eta} + \sqrt{-1}\partial\overline{\partial}\frac{\psi}{l-\eta})^n \leq \frac{1}{(l-\eta)^n}\int_{\Omega} (\omega_s + \sqrt{-1}\partial\overline{\partial}\psi)^n \leq C\frac{1}{(l-\eta)^n}$$

Now let $\eta \to 0$, then we finish the proof.

□

Remark 7. *Although in lemma (5.9) and Proposition (5.11), the estimtes are concerned with fix ψ_s, all the constants are independent of s.*

The following lemma is well-known and its proof can be found e.g. in [40].

Lemma 5.12. *Let $F : [0,\infty) \to [0,\infty)$ be a non-increasing right-continuous function satisfying $\lim_{l\to\infty} F(l) = 0$. If there exist $\alpha, A > 0$ such that for all $s > 0$ and $0 \le r \le 1$,*

$$rF(l+r) \le A\,(F(l))^{1+\alpha}\,,$$

then there exists $S = S(l_0, \alpha, A)$ such that

$$F(l) = 0$$

for all $l \ge S$, where l_0 is the smallest l satisfying $(F(l))^{\alpha} \le (2A)^{-1}$.

Proof of Theorem (5.6). Define for each fixed l large,

$$F(l) := Cap_{\chi}(U(l),\Omega)^{1/n}.$$

By lemma (5.9) and lemma (5.10) applied to the function ψ_s, we have

$$rF(l+r) \le AF(l)^2, \quad \text{for} \quad \text{all } r \in [0,1],\ l > 2,$$

for some uniform constant $A > 0$ independent of $r \in (0,1]$. Proposition (5.11) implies that $\lim_{l\to\infty} F(l) = 0$ and the l_0 in Lemma (5.12) can be taken as less than $(2AC)^q$, which is a uniform constant. It follows from Lemma (5.12) that $F(l) = 0$ for all $l > S$, where $S \le 2 + l_0$. On the other hand, if $Cap_{\chi}\{\psi_s < -l\} = 0$, by Lemma (5.9) , we have the integral $b(l) = 0$. Hence the set $\{\psi_s < -l\} = \emptyset$. Thus $\inf_{\Omega}(\psi_s) \ge -S$. Thus we finish the proof of Theorem (5.6).

□

We introduce two more parameter δ and ϵ in order to apply the maximum principle and consider the following family of complex Monge-Ampère equations

$$\begin{cases} \left((1+\delta)\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_{s,\delta,\epsilon}\right)^n = \dfrac{e^{\psi_{s,\delta,\epsilon}}(|\sigma_D|^{2\epsilon}_{h_D}+s)(|\sigma_E|^2_{h_E}+s)}{(|\sigma_F|^2_{h_F}+s)}\Omega_Y. \\ \psi_{s,\delta,\epsilon}{}_{|\partial\Omega} = 0 \end{cases} \tag{5.2.3}$$

Here both δ and ϵ are sufficiently small and we require $\epsilon > 0$.

By standard lemma, if (X, p) is $\mathbb{Q}$ factorial singularity

$$\chi - sD \tag{5.2.4}$$

is ample for $s > 0$ smaller than a fixed constant s_0, where the support of D coincides with the support of the exceptional divisors of a log resolution defined in (5.1). We can assume $s_0 = 1$ by adjusting the coefficients of D. Let σ_D be the defining section of D and choose a smooth hermitian metric h_D on the line bundle associated to D such that for any sufficiently small $s > 0$, and denote $\chi' = Ric(h_D)$

$$\chi - s\chi' > 0. \tag{5.2.5}$$

Lemma 5.13. *For any $\epsilon_0 > 0$, there exist $\delta_0 > 0$, $C > 0$ and $C' = C'(\epsilon_0) > 0$ such that for any $-\delta_0 \leq \delta \leq \delta_0$, $0 < s < 1$, and $0 < \epsilon < \epsilon_0/2$, the solution $\psi_{s,\delta,\epsilon}$ of equation satisfies the following estimate on Y,*

$$\epsilon_0 \log |\sigma_D|^2_{h_D} - C' \leq \psi_{s,\delta,\epsilon} \leq C. \tag{5.2.6}$$

Proof. We first obtain the upper bound of $\psi_{s,\delta,\epsilon}$. Since all $\psi_{s,\delta,\epsilon}$ are $A\theta$- PSH for some fixed large constant A and $\psi_{s,\delta,\epsilon} = 0$ on $\partial\Omega$, we can get the upper bound of $\psi_{s,\delta,\epsilon}$ by comparing $\psi_{s,\delta,\epsilon}$ with solution to $\Delta\varphi = -A, \varphi|_{\partial\Omega} = 0$ on Ω. Next we fix a sufficiently small $\delta_0 \geq 3\epsilon_0 > 0$ and consider the following family of equations on Y

$$\left((1+\delta)\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_{s,\delta,\epsilon_0}\right)^n = \frac{e^{\psi_{s,\delta,\epsilon_0}}(|\sigma_D|^{2\epsilon_0}_{h_D} + s)(|\sigma_E|^2_{h_E} + s)}{|\sigma_F|^2_{h_F} + s}\Omega_Y, \tag{5.2.7}$$

where $-\delta_0 \leq \delta \leq \delta_0$.

Since σ_D vanishes along F, there exist $\eta = \eta(\epsilon_0) > 0$ and $K = K(\epsilon_0, \delta_0) > 0$ such that for all $0 < s < 1$, we have

$$\left|\left|\frac{(|\sigma_{G_D}|^{2\epsilon_0}_{h_D} + s)(|\sigma_E|^2_{h_E} + s)}{|\sigma_F|^2_{h_F} + s}\right|\right|_{L^{1+\eta}(Y,\Omega_Y)} \leq K.$$

Theorem (5.6) implies that there exists $C_1 = C_1(\delta_0, \epsilon_0) > 0$ such that for all $3|\delta| \leq \delta_0$, $0 < s < 1$,

$$||\psi_{s,\delta,\epsilon_0}||_{L^\infty(Y)} \leq C_1.$$

Now we will compare $\psi_{s,\delta,\epsilon}$ to $\psi_{s,\delta',\epsilon_0}$ by applying the maximum principle. Let

$$\phi = \psi_{s,\delta,\epsilon} - \psi_{s,\delta',\epsilon_0} - \epsilon_0 \log |\sigma_D|^2_{h_D}.$$

Then ϕ satisfies the

$$\frac{\left((1+\delta')\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_{s,\delta',\epsilon_0} + (\delta - \delta' - \epsilon_0)\chi + \epsilon_0(\chi - \chi') + \sqrt{-1}\partial\overline{\partial}\phi\right)^n}{\left((1+\delta')\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_{s,\delta',\epsilon_0}\right)^n} = e^{\phi}\left(\frac{|\sigma_D|^{2\epsilon}_{h_D} + s}{1 + s|\sigma_D|^{-2\epsilon_0}_{h_D}}\right). \tag{5.2.8}$$

We choose $\delta' = -\delta_0$ and require $0 < \epsilon < \epsilon_0$. Since ϕ is smooth away from the zeros of D and ϕ tends to ∞ near zeros of D, we are able to apply the maximum principle to the minimum of ϕ and there exists $C_2 > 0$ such that

$$\inf_{\Omega} \phi \geq -C_2,$$

Since $\psi_{s,\delta',\epsilon_0}$ is bounded, there exists $C_3 > 0$ such that for all $3\delta \in (-\delta_0, \delta_0)$, $0 < s < 1$ and $\epsilon \in (0, \epsilon_0/2)$,

$$\psi_{s,\delta,\epsilon} \geq -C_3 + \epsilon_0 \log |\sigma_D|^2_{h_D}.$$

□

Next we prove the boundary C^1 estimate,

Lemma 5.14. *Let $\psi_{s,\delta,\epsilon}$ be the solution of equation (5.2.3), then $|\nabla_g \psi_{s,\delta,\epsilon}|_{\partial\Omega} \leq C$*

Proof. Noticing that $\psi_{s,\delta,\epsilon}$ and $\frac{e^{\psi_{s,\delta,\epsilon}}(|\sigma_D|^{2\epsilon}_{h_D}+s)(|\sigma_E|^2_{h_E}+s)}{(|\sigma_F|^2_{h_F}+s)} \frac{\Omega_Y}{((1+\delta)\chi+s\theta)^n}$ are uniformlly bounded in the neighboughood of the boundary $\partial\Omega$, we use the same argument as Step 4 in theorem (1.8) to get the estimate we want. □

We also prove the global C^1 estimate with suitable barrier function.

Proposition 5.15. *Let $\psi_{s,\delta,\epsilon}$ be the solution of equation (5.2.3), then $|\nabla_g \psi_{s,\delta,\epsilon}|^2 |\sigma_D|^N_{h_D} \leq C$ where N is a fixed constant, g is a fixed metric and C is independent of parameters δ, s, ϵ*

Proof. We first fix a constant η such that $\chi - \eta\sqrt{-1}\partial\overline{\partial}\log|\sigma_D|^2_{h_D} > 0$ and rewrite the equation (5.2.3) as

$$\big((1+\delta)\chi - \eta Ric(h_D) + s\theta + \sqrt{-1}\partial\overline{\partial}\phi_{s,\delta,\epsilon}\big)^n = \frac{e^{\phi_{s,\delta,\epsilon}}(|\sigma_D|^{2\eta}_{h_D})(|\sigma_D|^{2\epsilon}_{h_D}+s)(|\sigma_E|^2_{h_E}+s)}{(|\sigma_F|^2_{h_F}+s)}\Omega_Y.$$

where

$$\phi_{s,\delta,\epsilon} = \psi_{s,\delta,\epsilon} - \eta\log|\sigma|^2_{h_D}.$$

Using the fact that $\psi > -\frac{\eta}{2}\log|\sigma_D|^2_{h_D}$ in theorem (5.13), we know that $\phi > -C'$ where $-C'$ is uniform with respcet to all parameters δ, s, ϵ. Now our reference metrics $(1+\delta)\chi - \eta Ric(h_D) + s\theta$ in the above equation are uniformlly non degenerate as $\delta, s \to 0$, so can safely regard them as a fix metric g. Define

$$F := \frac{e^{\phi_{s,\delta,\epsilon}}(|\sigma_D|^{2\eta}_{h_D})(|\sigma_D|^{2\epsilon}_{h_D}+s)(|\sigma_E|^2_{h_E}+s)}{(|\sigma_F|^2_{h_F}+s)}\frac{\Omega_Y}{((1+\delta)\chi - \eta Ric(h_D) + s\theta)^n}$$

Now we define $H = \log|\nabla\phi|^2_g + \log|\sigma_D|^N_{h_D} - \gamma(\phi)$ where N is a constant, γ is a one variable monotone increase function to be determined. Here we also omit the parameters s, δ, ϵ for simplicity. We remark here that since the leading term of our function $\gamma(x)$ will be Ax and ϕ blows up in the rate of $-\eta\log|\sigma_D|^2_{h_D}$, we can conclude that $\log|\nabla\phi|^2_g + \log|\sigma_D|^N_{h_D} \le \gamma(\phi)$ when $A\eta > N+2$ and $z \to D$. So it's safe to assume H has a maximum in Ω. Direct computation shows that, see also [95] page 21,

$$\Delta'\log|\nabla\phi|^2_g \ge \frac{2\mathrm{Re}\nabla_m\log F\nabla^m\phi}{|\nabla\phi|^2_g} - \Lambda\, tr_{g'}g + \frac{|\nabla\nabla\phi|^2_{gg'} + |\bar\nabla\nabla\phi|^2_{gg'}}{|\nabla\phi|^2_g} - \frac{|\nabla|\nabla\phi|^2_g|^2_{g'}}{|\nabla\phi|^4_g} \quad (5.2.9)$$

where Δ' is taken with respect to metric $g' = (1+\delta)\chi - \eta Ric(h_D) + s\theta + \sqrt{-1}\partial\overline{\partial}\phi$ and Λ is the bound of bisectional curvature of metric g. We estimate

$$|\frac{2\mathrm{Re}\nabla_m\log F\nabla^m\phi}{|\nabla\phi|^2_g}| \le C + C|\sigma_E|^{-2}_{h_E} + |\sigma_F|^{-2}_{h_F} + |\sigma_D|^{-2}_{h_D} \le C|\sigma_D|^{-2}_{h_D} \quad (5.2.10)$$

According to a lemma in [95], we have

$$\frac{|\nabla\nabla\phi|^2_{gg'} + |\nabla\bar\nabla\phi|^2_{gg'}}{|\nabla\phi|^2_g} - \frac{|\nabla|\nabla\phi|^2_g|^2_{g'}}{|\nabla\phi|^4_g} \ge 2\Re\langle\frac{\nabla|\nabla\phi|^2_g}{|\nabla\phi|^2_g}, \frac{\nabla\phi}{|\nabla\phi|^2_g}\rangle_{g'} - 2\Re\langle\frac{\nabla|\nabla\phi|^2_g}{|\nabla\phi|^2_g}, \frac{\nabla\phi}{|\nabla\phi|^2_g}\rangle_g \quad (5.2.11)$$

At the maximum of H, we have

$$\nabla\log|\nabla\phi|^2 + \nabla\log|\sigma_D|^2_{h_D} - \gamma'\nabla\phi = 0$$

Hence we have

$$2\Re\langle\frac{\nabla|\nabla\phi|_g^2}{|\nabla\phi|_g^2},\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_{g'}-2\Re\langle\frac{\nabla|\nabla\phi|_g^2}{|\nabla\phi|_g^2},\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_g$$

$$=2\Re\langle-\nabla\log|\sigma_D|_{h_D}^2+\gamma'\nabla\phi,\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_{g'}-2\Re\langle-\nabla\log|\sigma_D|_{h_D}^2+\gamma'\nabla\phi,\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_g$$

$$\geq 2\Re\langle-\frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2},\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_{g'}+2\Re\langle\frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2},\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_g-2\gamma' \tag{5.2.12}$$

At the maximum of H, we can assume $|\sigma_D|_{h_D}^N|\nabla\phi|_g^2\geq 1$ otherwise we are done. Since $N\geq 4$, we have

$$|2\Re\langle-\frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2},\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_{g'}|\leq 2|\Re\langle\nabla|\sigma_D|_{h_D}^2,\frac{\nabla\phi}{|\nabla\phi|_g}\rangle_{g'}|\leq|\nabla|\sigma_D|_{h_D}^2|_{g'}^2+\frac{|\sigma_D|_{h_D}^2|\nabla\phi|_{g'}^2}{|\sigma_D|_{h_D}^2|\nabla\phi|_g^2}$$

$$\leq C|\nabla|\sigma_D|_{h_D}^2|_g^2\,tr_{g'}g+|\sigma_D|_{h_D}^2|\nabla\phi|_{g'}^2$$

$$|2\Re\langle-\frac{\nabla|\sigma_D|_{h_D}^2}{|\sigma_D|_{h_D}^2},\frac{\nabla\phi}{|\nabla\phi|_g^2}\rangle_g|\leq 2|\Re\langle\nabla|\sigma_D|_{h_D}^2,\frac{\nabla\phi}{|\nabla\phi|_g}\rangle_g|\leq C \tag{5.2.13}$$

On the other hand,

$$-\Delta'\gamma(\phi)=-\gamma'\Delta'\phi-\gamma''|\nabla\phi|_{g'}^2=\gamma' tr_{g'}g-n\gamma'-\gamma''|\nabla\phi|_{g'}^2,\Delta'\log|\sigma_D|_{h_D}^N\leq C\,tr_{g'}g.$$

Combine this equality with preceding estimates (5.2.10, 5.2.11, 5.2.12, 5.2.13), we have

$$\Delta' H\geq(\gamma'-\Lambda-C)tr_{g'}g-(n+2)\gamma'-(\gamma''+|\sigma_D|_{h_D}^2)|\nabla\phi|_{g'}^2-C|\sigma_D|_{h_D}^{-2} \tag{5.2.14}$$

Recall that $\phi>-C'$, now we construct our function γ as

$$\gamma(x)=(\Lambda+C+1)x-\frac{1}{x+C'+1}$$

Then by (5.2.14) we have

$$\Delta' H\geq tr_{g'}g-(n+2)(C+1+\Lambda)-C|\sigma_D|_{h_D}^{-2}+(\frac{1}{(\phi+C'+1)^3}-|\sigma_D|_{h_D}^2)|\nabla\phi|_{g'}^2$$

Noticing that $\phi\leq C-2\eta\log|\sigma_D|_{h_D}^2$, so we can safely assume that

$$(\frac{1}{(\phi+C'+1)^3}-|\sigma_D|_{h_D}^2)\geq|\sigma_D|_{h_D}^2$$

Finally we conclude that at the maximum of H, we have

$$tr_{g'}g\leq(|\sigma_D|_{h_D})^{-2}$$

Hence

$$|\nabla\phi|^2_{g'} \leq C(|\sigma_D|_{h_D})^{-4}, |\nabla\phi|^2_g \leq C(|\sigma_D|_{h_D})^{-6}$$

Choosing $N = 6$, we have $H_{max} \leq C$, and clearly we have $|\nabla\phi|^2_g|\sigma_D|^7 \leq C$ since γ is blowing up as $|\log|\sigma_D|_{h_D}|$. □

Lemma 5.16. *Let $\psi_{s,\delta,\epsilon}$ be the solution of equation (5.2.3), then $|\nabla^2_g\psi_{s,\delta,\epsilon}|_{\partial\Omega} \leq C$ where g is a fixed metric and C is independent of parameters δ, s, ϵ.*

Proof. Notice that our boundary is strictly pseudoconvex, and all data in the equation (5.2.3) is unifomrlly bounded near the boundary, we can use the local argument of CKNS [19] to conclude. □

Next, we will prove second order estimates with bounds from suitable barrier functions. There exists an effective Cartier divisor D on Y such that for any sufficiently small $s > 0$,

Lemma 5.17. *There exist A, δ_1, $\epsilon_1 > 0$ and $C = C(\delta_1, \epsilon_1) > 0$ such that for all $-\delta_1 < \delta < \delta_1$, $0 < \epsilon < \epsilon_1$ and $0 < s < 1$,*

$$\sup_{\Omega} \left(|\sigma|^A_{h_D}\right)(\Delta_\theta\psi_{s,\delta,\epsilon}) \leq C, \tag{5.2.15}$$

where Δ_θ is the Laplace operator with respect to the Kähler metric θ.

Proof. Let $\omega = (1+\delta)\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\psi_{s,\delta,\epsilon}$. Then we consider the quantity

$$H = \log tr_\theta(\omega) - A^3\psi_{s,\delta,\epsilon} + 2A^2\log|\sigma|^2_{h_D}$$

for some sufficiently large $A > 0$ to be determined. Straightforward calculations show that there exists $C > 0$ such that for all $\delta \in (-\delta_1, \delta_1)$, $\epsilon \in (0, \epsilon_1)$ and $0 < s < 1$,

$$\begin{aligned}
\Delta_\omega H &\geq \Delta_\omega \log tr_\theta(\omega) + 2A^2 tr_\omega(\chi - s_0 Ric(D)) - A^3 \\
&\geq A tr_\omega\theta - C\frac{tr_\theta Ric(\omega)}{tr_\theta(\omega)} - A^3 \\
&\geq A(tr_\theta(\omega))^{\frac{1}{n-1}}(\frac{\theta^n}{\omega^n})^{\frac{1}{n-1}} - C\frac{1}{tr_\theta\omega|\sigma_D|^{2\alpha}_{h_D}} - A^3 \\
&\geq A(tr_\theta(\omega))^{\frac{1}{n-1}}|\sigma_D|^{2\beta}_D - C\frac{1}{tr_\theta\omega|\sigma_D|^{2\alpha}_{h_D}} - A^3
\end{aligned}$$

where α, β are fixed constants only depending on the coefficient of exceptional divisor in the log resolution of singularity. For the third and fourth inequality, we use the equation. Noticing that

$$H < \log(tr_\theta(\omega)|\sigma_D|_{h_D}^{A^2})$$

We may assume H obtains maximum in the interior of Ω. Also, since our goal is to bound H, WLOG we assume

$$(tr_\theta(\omega)|\sigma_D|_{h_D}^{A^2}) > 1$$

otherwise we are done. Applying the maximum principle, at the maximal point x_{max} of H,

$$H(x_{max}) < tr_\theta(\omega)|\sigma_D|_{h_D}^{A^2} \leq nA^2.$$

On the other hand, since $\psi_{q,s,\delta,\epsilon} \leq C$, we have

$$tr_\theta \omega \leq \frac{1}{|\sigma_D|_{h_D}^{2A^2}}$$

This proves the lemma.

□

The following lemma on local high regularity of $\psi_{s,\delta,\epsilon}$ is established by the standard linear elliptic theory after applying Lemma (5.17) and linearizing the complex Monge-Ampère equation (5.2.7).

Lemma 5.18. *For any compact $K \subset\subset (\Omega \setminus p)$, there exist $\delta_2 > 0$, $\epsilon_2 > 0$ and $C = C(k, K, \delta_2, \epsilon_2) > 0$ such that for any $-\delta_2 \leq \delta \leq \delta_2$, $0 < \epsilon \leq \epsilon_2$ and $0 < s < 1$*

$$||\psi_{s,\delta,\epsilon}||_{C^k(K)} \leq C.$$

Before we take $\delta, \epsilon, s \to 0$, we derive a uniform estimate with respect to variations by the parameters δ, ϵ, and t.

Lemma 5.19. *For any compact $K \subset\subset (\Omega \setminus p)$, there exist $\delta_3 > 0$, $\epsilon_3 > 0$ and $C = C(K, \delta_3, \epsilon_3) > 0$ such that for any $-\delta_3 \leq \delta \leq \delta_3$, $0 < \epsilon \leq \epsilon_3$ and $0 < s < 1$, we have*

$$\left|\frac{\partial \psi_{s,\delta,\epsilon}}{\partial \delta}\right|_{L^\infty(K)} + \left|\frac{\partial \psi_{s,\delta,\epsilon}}{\partial \epsilon}\right|_{L^\infty(K)} + \left|\frac{\partial \psi_{s,\delta,\epsilon}}{\partial s}\right|_{L^\infty(K)} \leq C. \tag{5.2.16}$$

Proof. By the implicit function theorem, the solutions of (5.2.7) must be smooth with respect to the parameters δ, ϵ and s. Let $f = \frac{\partial \psi_{s,\delta,\epsilon}}{\partial \delta}$. Then $f \in C^\infty(Y)$ and

$$\Delta_{s,\delta,\epsilon} f = -tr_{\omega_{s,\delta,\epsilon}}(\chi) + f,$$

where $\Delta_{s,\delta,\epsilon}$ is the Laplace operator associated to the metric

$$\omega = (1+\delta)\chi + s\theta + \sqrt{-1}\partial\overline{\partial}\varphi_{s,\delta,\epsilon}.$$

The function $H = f - 10\psi_{s,\delta,\epsilon} + \log|\sigma_D|^2_{h_D}$ satisfies the following equation

$$\Delta_{s,\delta,\epsilon} H \geq f - 10n = H + 10\psi_{s,\delta,\epsilon} - \log|\sigma_D|^2_{h_D} - 10n.$$

Then for all sufficiently small δ and $\epsilon > 0$, H is uniformly bounded above and so f is uniformly bounded above on any compact subset in $\Omega \setminus p$. Estimates for $\left|\frac{\partial \psi_{s,\delta,\epsilon}}{\partial \epsilon}\right|$ and $\left|\frac{\partial \psi_{s,\delta,\epsilon}}{\partial s}\right|$ can be achieved similarly.

□

Now we are able to prove our Theorem (1.7).

Proof. we have uniform estimates for $\psi_{s,\delta,\epsilon}$ away from σ_D, for any sequence $s_j, \delta_j, \epsilon_j \to 0$, we can assume $\psi_{s_j,\delta_j,\epsilon_j}$ converges, after passing to a subsequence, to some

$$\varphi \in PSH(\chi) \cap C^\infty(\Omega \setminus p).$$

In particular, there exists $C > 0$ and for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\epsilon \log|\sigma_D|^2_{h_D} - C_\epsilon \leq \varphi \leq C.$$

(1), (2) and (3) can be proved from the above conclusion by passing the estimates of $\psi_{s,\delta,\epsilon}$ to the limit φ. Furthermore, φ solves equation (1.3.1) on $(\Omega \setminus p)$.

(4) can be reduced to the following statement: Suppose ϕ is a plurisubharmonic function on the unit ball $B \subset \mathbb{C}^n$ such that

$$\int_B |z_1|^{-2} e^{\phi} (\sqrt{-1})^n dz_1 \wedge d\overline{z_1} \wedge ... \wedge dz_n \wedge d\overline{z_n} < \infty,$$

then ϕ tends to $-\infty$ near $B \cap \{z_1 = 0\}$. Such a statement is proved by Berndtsson (c.f. Lemma 2.7 in [6]). From the left hand side of equation (5.2.3) and Stokes formula, we know that, $\varphi \to -\infty$ near the exceptional divisor with discrepancy -1. On the other hand, φ tends to $-\infty$ near $\pi^{-1}(p)$ in Y, otherwise there exists a curve C in exceptional divisor and C intersects at least one exceptional divisor with discrepancy -1, and so φ must be constant on C since it is pluriharmonic with singularities better than any log poles. This leads to contradiction and so φ must tend to $-\infty$ near $\pi^{-1}(p)$. Therefore the function φ can uniquely descend to $(\Omega \setminus p)$.

(6) can be proved as follows. Suppose $\varphi' \in PSH(Y,\chi) \cap C^\infty(Y \setminus \sigma_D)$ is a sequential limit of another sequence $\psi_{s_j,\delta_j,\epsilon_j}$. Then by the estimates in Lemma (5.19), on any compact set $K \subset\subset Y \setminus \sigma_D$, there exists $C > 0$ such that for sufficiently large $j > 0$,

$$\sup_K |\psi_{s_j,\delta_j,\epsilon_j} - \psi_{s_j',\delta_j',\epsilon_j'}| \leq C \left(|\delta_j - \delta_j'| + |\epsilon_j - \epsilon_j'| + |s_j - s_j'| \right).$$

This implies that

$$\varphi|_K = \varphi'|_K$$

and so $\varphi = \varphi'$ on Y after unique extensions over σ_D since both lie in $PSH(Y,\chi)$. The above argument implies that as $s, \delta, \epsilon \to 0$, the solution $\psi_{s,\delta,\epsilon}$ converges to the unique limit φ.

□

We will also prove a uniqueness result, which is different from the uniqueness theorem in [6].

Lemma 5.20. *There exists a unique solution $\varphi \in L^\infty_{loc}(\Omega \setminus p) \cap C^\infty(\Omega \setminus p)$ satisfying*

1. *$(\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^\varphi \Omega$ on $(\Omega \setminus p)$,*

2. *For any $\epsilon > 0$, there exist $C > 0$ and $C_{p,\epsilon} > 0$ with the following estimate*

$$\epsilon \log |\sigma_D|^2_{h_D} - C_\epsilon \leq \varphi \leq C,$$

where σ_D is an effective divisor supported on the locus of exceptional divisor.

In particular, $\varphi \in PSH(X,\chi)$ satisfies all the conditions in Lemma 1.7.

Proof. We first prove the uniqueness. Let φ be the Kähler-Einstein potential constructed in Lemma 1.7 as the limit of $\psi_{s,\delta,\epsilon}$ $(s, \delta, \epsilon \to 0)$. Suppose there exists another

φ' satisfying the conditions in the lemma and for any $\epsilon > 0$, there exist $C_1 > 0$ and $C_2 = C_2(\epsilon) > 0$ such that

$$\epsilon \log |\sigma_D|^2_{h_D} - C_2 \leq \varphi' \leq C_1$$

We consider the quantity

$$\phi = \psi_{s,-\delta,\epsilon} - \varphi' + \delta^3 \log |\sigma_D|^2_{h_D},$$

where σ_D and h_D are defined in (5.2.4) and (5.2.5). Then ϕ satisfies the following equation on the log resolution Y,

$$\frac{(\chi + \sqrt{-1}\partial\overline{\partial}\varphi' + s\theta - \delta\chi + \delta^3 Ric(h_D) + \sqrt{-1}\partial\overline{\partial}\phi)^n}{(\chi + \sqrt{-1}\partial\overline{\partial}\varphi')^n} = e^{\phi} \frac{(|\sigma_D|^{2\epsilon}_{h_D} + s)(|\sigma_E|^2_{h_E} + s)|\sigma_F|^2_{h_F}}{|\sigma_D|^{2\delta^3}_{h_D}|\sigma_E|^2_{h_E}(|\sigma_F|^2_{h_F} + s)}.$$

We pick $s << \delta << 1$, $\epsilon << \delta^2$ and apply the maximum principle to ϕ. There exists $C > 0$ such that for all $s << \delta << 1$, $\epsilon << \delta^2$,

$$\sup_X \phi \leq C.$$

Let $s, \delta, \epsilon \to 0$. We have

$$\varphi \leq \varphi'.$$

Similarly, we can prove $\varphi \geq \varphi'$ by applying the maximum principle to

$$\phi' = \psi_{s,\delta,\epsilon} - \varphi' - \delta \log |\sigma_D|^2_{h_D}.$$

□

5.3 Bounded geometry and construction of Kähler-Einstein metrics: Second Approach

In this section, we want to use bounded geometry methods of [69, 26, 129] to construct complete Kähler-Einstein metric on $\Omega \setminus \{p\}$. So our geometric domain of interest will be Ω with boundary $\partial\Omega$ and also one non compact end, which topologically is punctured neighbourhood of p. We also make the following convention:

Convention: In this section, when we talk about Kähler-Einstein metric on (Ω, ω), we assume its Ricci curvature is -1 and when we talk about complete metric, we assume the distance goes to ∞ when point in Ω goes to p in the Euclidean topology.

Before proving our theorem 1.8, we recall definitions of quasi-coordinate which are used by [25, 69, 129] to deal with complete Riemannian manifolds with bounded curvature but with shrinking injectivity radius.

Definition 5.21. *Let V be an open set in C^n with coordinates $(v^1, v^2, \cdots, v^n)$. Let X be an $n-$dimensional complex manifold and ϕ a holomorphic map of V into X. ϕ is called a quasi-coordinate map if it is of maximal rank everywhere. In this case, $(V, \phi, (v^1, v^2, \cdots, v^n))$ is called a quasi-coordinate of X.*

Definition 5.22. *Let $\widehat{\mathcal{U}}$ be a neighbourhood of p in Ω, being away from $\partial\Omega$ and ω is a complete Kähler metric towards p on $(\Omega \setminus p)$. A system of quasi-coordinates on $(\mathcal{U} := \widehat{\mathcal{U}} \setminus \{p\}, \chi)$ is a set of quasi-coordinates $\Gamma = \{(V_\alpha, \phi_\alpha, (v_\alpha^1, v_\alpha^2 \cdots, v_\alpha^n))\}$ of $\mathcal{U}$ with the following properties:*

(a) $\mathcal{U} \subset \bigcup_\alpha (\text{Image of } V_\alpha) \subset (\Omega \setminus p)$;

(b) The complement of certain open neighborhood $U \subset \mathcal{U}$ of the infinity point o is covered by a finite number of quasi coordinates which are coordinate charts in the usual sense;

(c) For each point $x \in U$, there is a quasi-coordinate V_β and $\tilde{x} \in V_\beta$, such that $\phi_\beta(\tilde{x}) = x$ and dist$(\tilde{x}, \partial V_\beta) \geq \epsilon_1$ in the euclidean sense, where ϵ_1 is constant independent of β;

(d) There are positive constant c and $A_k, k = 1, 2, \cdots$, independent of α, such that for each quasi coordinate $(V_\alpha, \phi_\alpha, (v_\alpha^1, v_\alpha^2 \cdots, v_\alpha^n))$, the following inequalities hold:

$$c^{-1}(\delta_{i\bar{j}}) \leq (g_{\alpha i\bar{j}}) \leq c(\delta_{i\bar{j}})$$

$$|\frac{\partial^{p+q}}{\partial v_\alpha^p \bar{v}_\alpha^q} g_{\alpha i\bar{j}}| < A_{p+q}, \forall p, q,$$

where $(g_{\alpha i\bar{j}})$ denote the metric tensor with respect to $(V_\alpha, \phi_\alpha, (v_\alpha^1, v_\alpha^2 \cdots, v_\alpha^n))$.

Roughly speaking, a set of quasi coordinates of metric domain $(\Omega \setminus p, \chi)$ is a set of coverings of $(\Omega \setminus p)$ coupled with the pull back metric satisfying uniform bounded metric properties. Before proceeding, we need introduce some standard definitions Cheng-Yau's function space, generalized maximum principle etc.

Definition 5.23. *We define the Hölder space of $C^{k,\alpha}$ function on $\mathcal{U} = \widehat{\mathcal{U}}\backslash p$ by exploiting the quasi-coordinate system. For any nonnegative integer $k, \alpha \in (0,1)$, we define*

$$\|u\|_{k,\alpha}(\mathcal{U}) = \sup_{V_\beta \in \Gamma}(\sup_{z \in V_\beta} \sum_{p+q \leq k} |\frac{\partial^{p+q}}{\partial v_\beta^p \partial \bar{v}_\beta^q} u(z)| + \sup_{z,z' \in V_\beta} \sum_{p+q=k} |z-z'|^{-\alpha} |\frac{\partial^{p+q}}{\partial v_\beta^p \partial \bar{v}_\beta^q} u(z) - \frac{\partial^{p+q}}{\partial v_\beta^p \partial \bar{v}_\beta^q} u(z')|)$$

Let's introduce one more compact set V with $\Omega \setminus \mathcal{U} \subset V \subset \Omega$ to cover whole Ω. Now define:

$$\|u\|_{k,\alpha}(\Omega) = \|u\|_{k,\alpha}(\mathcal{U}) + \|u\|_{k,\alpha}(V)$$

The function space $C^{k,\alpha}(\Omega)$ is the completion of $\{u \in C^k(\Omega) | \|u\|_{k,\alpha}(\Omega) < \infty\}$.

Remark 8. *The existence of quasi coordinate is crucially used in our proof. The classical interior Schauder estimate for a linear elliptic operator L, is as follows,*

$$\|u\|_{C^{k,\alpha}(V_1)} \leq C(\sup |u|_{V_2} + \|Lu\|_{C^{k-2,\alpha}(V_2)}), \text{ where } V_1 \subset\subset V_2 \subset R^m.$$

Notice that the constant C depends on the ellipticity of L, the $C^{k-2,\alpha}$ norms of the coefficients of L and the distance between V_1 and ∂V_2. If we have a quasi coordinate system defined above, the Schauder estimate on $\mathcal{U}$ is reduced to that on a fixed bounded domain in Euclidean space.

Before we proceed, we state and prove the following modified version of Yau's generalized maximal principle on noncompact manifold.

Lemma 5.24. *Suppose $(\mathcal{U}, \omega)$ is defined as above. f is a smooth function on $\mathcal{U}$, which is bounded from above, and $\sup f > \sup_{\partial \mathcal{U}} f$. Then there is a sequence $\{y_i\}$ in $\mathcal{U}$ such that $\lim_{i\to\infty} f(y_i) = \sup f, \lim_{i\to\infty} |\nabla_g f|(y_i) = 0$ and $\lim_{i\to\infty} \sup |\nabla_g^2 f|(y_i) = 0$, where the derivatives are taken with respect to metric ω.*

Proof. : Let $\sup f = L$. If $\sup f$ is obtained, the lemma is obvious. Otherwise we choose a sequence x_i with $\lim f(x_i) = \sup f$. It is easy to see $\{x_i\}$ must go to infinity. Now at each point we take a quasi coordinate chart V_i covering x_i. On each V_i, define a non-negative function $\rho^i : V_i \to \mathbb{R}$ such that

$$\rho^i(x_i) = 1, \rho^i = 0 \text{ on } \partial V_i, \rho^i \leq C, |\nabla \rho^i| \leq C, \text{ and } (\rho^i_{p\bar{q}}) \geq -C(\delta_{p\bar{q}}),$$

where C is positive number independent of i, and all norms are taken with respect to the Euclidean norm.

Now consider

$$\frac{L-f}{\rho^i}$$

as a function on V_i. Notice that $\frac{L-f}{\rho^i}$ blows up on the boundary of V_i, so it admits minimum at point y_i which is in the interior of V_i. Now

$$\frac{L-f}{\rho^i}(y_i) = \inf_{V_i} \frac{L-f}{\rho^i}.$$

Then

$$\begin{gathered} \tfrac{L-f}{\rho^i}(y_i) \leq \tfrac{L-f}{\rho^i}(x_i) = L - f(x_i), \\ \tfrac{d(L-f)}{L-f}(y_i) = \tfrac{d\rho^i}{\rho^i}(y_i), \\ \tfrac{(L-f)_{p\bar{q}}}{L-f}(y_i) \geq \tfrac{\rho^i_{p\bar{q}}}{\rho^i}(y_i). \end{gathered}$$

Using these inequalities and the property of ρ^i, we have

$$0 < L - f(y_i) \leq C(L - f(x_i)),$$

$$|df(y_i)| \leq C(L - f(x_i)),$$

$$(f_{p\bar{q}})(y_i) \leq C(L - f(x_i))(\delta_{p\bar{q}}).$$

By the bounded geometry of quasi coordinates, the above norms can also be take with respect to the metric ω. Hence sequence $\{y_i\}$ satisfies all the properties required in the lemma. □ □

5.3.1 Model metrics with property (A)

We provide some explicit examples of (X, p) with property (A).

Uniformizatoin of 2 dimensional isolated log canonical singularity by Kähler-Einstein metric.

Lemma 5.25. *[70, 86, 138]*

For any isolated normal surface singularity, they can be uniformaized by bounded symmetric domains with invariant Kähler-Einstein metric $\chi = \sqrt{-1}\partial\bar{\partial}\rho$ and classified as:

- $\mathbb{C}^2/\Gamma, \Gamma$ *a finite subgroup of* $\mathbb{U}(2)$ *possibly containing reflections. Invariant metric is defined by* $\sqrt{-1}\partial\overline{\partial}\rho, \rho = \log(\frac{1}{(1-\|Z\|^2)^2})$

- *One point compactification of* $\mathbb{H}\times\mathbb{H}/\Gamma, \Gamma$ *a parabolic discrete subgroup of* $Aut(\mathbb{H}\times\mathbb{H})$ *corresponding to a boundary point. Invariant metric is defined by* $\sqrt{-1}\partial\overline{\partial}\rho, \rho = \log(\frac{1}{y_1 y_2})$

- *One point partial compactification of* $\mathbb{B}^2/\Gamma, \Gamma$ *a parabolic discrete subgroup of* $Aut(\mathbb{B}^2)$. *Invariant metric is defined by* $\sqrt{-1}\partial\overline{\partial}\rho, \rho = \log\frac{1}{Imu-|v|^2}$

Remark 9. *In the Lemma above, the first item is actually Klt singularity. So in this section, we are mainly dealing with singularites of items 2 and 3. In other words, there are divisors with discrepancy* -1 *in the resolution. We include the nice construction in the appendix for reader's convenience.*

Another interesting example of isolated log canonical singularity unifomized by Bergman metric is proved in [41].

Lemma 5.26. *Let* A *be an abeliean variety with complex dimension* n *and* N *a negative line bundle on* A. *By contracting the zero section of* N, *one obtains a singular variety* $\widehat{X}$. *Let* $(\widehat{X}, o)$ *denote the germ of the isolated singularity of* $\widehat{X}$. *Then there is an open neighborhood (in Euclidean topology)* $\widehat{\mathcal{U}}$ *of* o *in* $\widehat{X}$ *such that* $\mathcal{U} := \widehat{\mathcal{U}}\backslash\{o\}$ *is a smooth quotient space of a unit complex ball* $\mathbb{B}^{n+1}$ *by a discrete subgroup of* $\mathrm{Aut}(\mathbb{B}^{n+1})$. *As a consequence,* $\mathcal{U}$ *has a negative Kähler- Einstein metric induced from the Bergman metric of the ball which is complete towards* o.

Remark 10. *The invariant Kähler-Einstein metrics in Lemma (5.25), [41] have a system of quasi coordinates in a punctured neighborhood of the isolated log canonical singularities. This is the main property we will use in the following theorem 1.8.*

5.3.2 Second approach of construction of local Kähler-Einstein metric

We first take the function space U to be an open set of $C^{k+\alpha}(\Omega)$, which is defined in (5.23), as follows:

$$U = \{\phi \in C^{k+\alpha}(\Omega) : \frac{1}{c}(g_{\alpha i\bar{j}}) \leq (g_{\alpha i\bar{j}} + \phi_{i\bar{j}}) \leq c(g_{\alpha i\bar{j}}), \text{in each quasi coordinate}\, V_\alpha\},$$

for some constant c, which however is not fixed.

We divide the proof into several steps.

5.3.3 Proof of Theorem 1.8

Step 1: Find an χ-PSH extension of function ψ to the domain Ω.

Choose an arbitrary smooth extension ψ_1 of ψ, which is supported on a neighborhood of $\partial\Omega$. Choose a convex increasing function $g : [-\infty, a] \to \mathbb{R}$ which is zero on $[-\infty, b]$ for some constant $b < a$. Now define

$$P = Ag(\rho) - Ag(a).$$

Notice that when A is large enough such that $\sqrt{-1}\partial\overline{\partial}P$ kills the negativity of $\chi + \sqrt{-1}\partial\overline{\partial}\psi_1$. Choose such $P + \psi_1$ as our new extension of ψ. By the construction we know that $\sqrt{-1}\partial\overline{\partial}(P + \psi_1)$ is supported on a neighborhood of $\partial\Omega$. This is crucial for our proof, since we need the new metric $\chi + \sqrt{-1}\partial\overline{\partial}(\psi_1 + P)$ to keep the behavior of χ in a neighborhood of the infinity.

In sum, we have

$$\begin{cases} (\chi + \sqrt{-1}\partial\overline{\partial}(\psi_1 + P))^n = e^{-F}\chi^n \text{ on } \Omega \\ \\ \psi_1 + P|_{\partial\Omega} = \psi. \end{cases} \tag{5.3.1}$$

where the function F is in Cheng-Yau's function space. Hence if we define $\tilde{\chi}$ by $\tilde{\chi} = \chi + \sqrt{-1}\partial\overline{\partial}(\psi_1 + P), \tilde{\varphi} = \varphi - (\psi_1 + P)$ and $\tilde{F} = F + \psi_1 + P$.

Simple calculation shows the Equation (1.3.2) is equivalent to

$$\begin{cases} (\tilde{\chi} + \sqrt{-1}\partial\overline{\partial}\tilde{\varphi})^n = e^{\tilde{\varphi}+\tilde{F}}\tilde{\chi}^n \text{ on } \Omega \\ \\ \tilde{\varphi}|_{\partial\Omega} = 0. \end{cases} \tag{5.3.2}$$

So from now on, we will focus on zero boundary value problem.

The rest of the proof is by continuity method, which is a combination of [19, 8, 25, 69]. We set up the continuity method as follows:

$$\begin{cases} (\chi+\sqrt{-1}\partial\overline{\partial}\varphi_t)^n = e^{\varphi_t+tM}\chi^n \\ \\ \varphi_t|_{\partial\Omega} = 0. \end{cases} \tag{5.3.3}$$

where M belongs to Cheng-Yau function space defined in (5.23).

Step 2: Prove the openess part in the continuity method. It will follow from the inverse mapping theorem. We need to show the linearized equation at χ_t

$$\begin{cases} \Delta_{\chi_t}h - h = v \ \text{ on } \ \Omega \\ \\ h|_{\partial\Omega} = 0, \end{cases} \tag{5.3.4}$$

has a unique solution in $C^{k,\alpha}(\Omega)$ with the estimate

$$\|h\|_{k,\alpha}(\Omega) \leq c\|v\|_{k-2,\alpha}(\Omega)$$

for some constant c independent of the function v.

We first remark here that $\chi_t := \chi+\sqrt{-1}\partial\overline{\partial}\varphi_t$ is a complete metric of bounded geometry up to $k-2$ covariant derivatives by the function choice of function space U at the beginning of the proof. Next take an exhaustion $\{\Omega_i\}$ of the domain Ω towards the infinity. (Here the boundary of our compact domain Ω_i has two components and one of them coincide with $\partial\Omega$). Following equation

$$\begin{cases} \Delta_{\chi_t}h_i - h_i = v \ \ on \ \ \Omega_i \\ \\ h|_{\partial\Omega_i} = 0, \end{cases} \tag{5.3.5}$$

has a unique solution h_i for each i. Maximum principle implies that $\sup_{\Omega_i}|h_i| \leq \sup|v|$. Interior Schauder estimate of our function space implies that

$$\|h_i\|_{k,\alpha}(V) \leq c\|v\|_{k-2,\alpha}(\Omega).$$

for any compact set V strict away from $\partial\Omega$. This inequality, combined with standard global Schauder estimate for a fixed compact set containing $\partial\Omega$ imply that $h_i \to h$ pointwise with

$$\Delta_{\chi_t}h - h = v, h|_{\partial\Omega} = 0$$

Moreover, we have

$$\|h\|_{k,\alpha}(\Omega) \leq c\|v\|_{k-2,\alpha}(\Omega).$$

Hence we establish the openness part.

Step 3: C^0estimate in closeness part.

We have the following equality:

$$\varphi+M = \log\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) - \log\det g_{i\bar{j}} = \int_0^1 \frac{\partial}{\partial t}\log\det(g_{i\bar{j}} + t\varphi_{i\bar{j}})dt = \int_0^1 (g+t\varphi)^{i\bar{j}}\varphi_{i\bar{j}}dt$$

Since at a point $x \in \Omega$ we may asume $g_{i\bar{j}} = \delta_{i\bar{j}}$ and $\varphi_{i\bar{j}} = \delta_{i\bar{j}}\varphi_{i\bar{i}}$, we have two inequalities as follows:

$$\varphi + M \leq \Delta_\chi\varphi,$$

$$\varphi + M \geq \Delta_{\chi_1}\varphi.$$

where $\chi_1 = \chi + \sqrt{-1}\partial\bar{\partial}\varphi_1$. By Lemma (5.24), we get the C^0 estimate.

Step 4: C^1 boundary estimate. On the one hand since $\varphi + M \leq \Delta_\chi\varphi$, we construct a barrier function h from above as follows. Take a neighborhood B of the boundary $\partial\Omega$, where $\partial B = \partial\Omega + C$ and C is the other side of the boundary ∂B. Then derive h by solving the following Dirichlet problem in B:

$$\begin{cases} \Delta_\chi h = c, \\ h|_{\partial\Omega} = 0 \text{ and } h|_C = d. \end{cases} \tag{5.3.6}$$

where d is a positive constant greater than $\sup|\varphi|$ and c is a constant smaller than $\inf(\varphi + M)$. Then maximal principle implies that $h \geq \varphi$ in B.

On the other hand, we construct a barrier function h_1 from below as follows. Take the global χ strictly PSH function P we constructed above and choose a constant b large enough such that

$$\begin{cases} (\chi + \sqrt{-1}\partial\bar{\partial}bP)^n \geq e^{sup\varphi+M}\chi^n \text{ on } \; B, \\ bP \leq \varphi \text{ on } C \; and \; bP = 0 \text{ on } \partial\Omega. \end{cases} \tag{5.3.7}$$

Then maximal principle of Monge-Ampère equation implies $h_1 := bP \leq \varphi$ on B. Noticing that h and h_1 coincide with φ on $\partial\Omega$, we get the boundary gradient estimate of φ.

Step 5: Global C^1 estimate. Since on noncompact manifold, we don't necessary have maximum point with gradient vanish etc. We will follow [8] to obtain the C^1 estimate.

Define $\phi = \log|\nabla\varphi|^2 - \gamma(\varphi)$ where γ is monotone increase function to be determined. Assume that $\sup_\Omega \phi$ is not obtained on $\partial\Omega$, then by the generalized maximum principle, we can find a point $q \in \Omega$ with

$$\phi(q) + \epsilon > \sup_\Omega \phi, |V|_\chi(q) < \epsilon, \Delta_\chi\phi(q) < \epsilon$$

where $V := \nabla_\chi\phi$. By (5.2.9), at q, we have

$$\Delta' \log\,|\nabla\varphi|_g^2 \geq \frac{2\mathrm{Re}\nabla_m \log F \nabla^m\varphi}{|\nabla\varphi|_g^2} - \Lambda\, tr_{g'}g + 2\Re\langle\frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\rangle_{g'} - 2\Re\langle\frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\rangle_g$$

Now using $V = \nabla\log|\nabla\varphi|^2 - \gamma'\nabla\varphi$, we get

$$\begin{aligned} & 2\Re\langle\frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\rangle_{g'} - 2\Re\langle\frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^2}, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\rangle_g \\ = {} & 2\Re\langle V + \gamma'\nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\rangle_{g'} - 2\Re\langle V + \gamma'\nabla\varphi, \frac{\nabla\varphi}{|\nabla\varphi|_g^2}\rangle_g > -\epsilon tr_{g'}g - \epsilon - \gamma' \end{aligned}$$

The inequality above with

$$\Delta'\gamma(\varphi) = \gamma'\Delta'\varphi + (\gamma'')|\nabla\varphi|^2$$

imply that

$$\epsilon tr_{g'}g > \Delta'\phi > (\gamma' - \epsilon - \Lambda)tr_{g'}g + (\gamma'')|\nabla\varphi|^2 - \gamma' - \epsilon - n - C$$

where C is the bound of gradient of function $\log F$. Now we construct our function γ as

$$\gamma(x) = (\Lambda + 2)x - \frac{1}{x + C' + 1}$$

where C' is the lower bound of φ. Then by standard argument we get global C^1 estimate.

Step 6: Boundary C^2 estimate. We notice the argument of [19] of boundary C^2 estimate is purely local around the boundary and our equation can be written as

$$\begin{cases} \det \varphi_{i\bar{j}} = e^{\varphi+f} \ \ on \ \ \Omega \\ \\ \varphi|_{\partial\Omega} = \psi, \end{cases} \tag{5.3.8}$$

locally, this is exactly one of the equation considered in [19], hence the estimate follows by the fact that $\sqrt{-1}\partial\overline{\partial}\rho$ is strictly positive in a neighbourhood of $\partial\Omega$.

Step 7: Standard C^2 estimate of Yau. We have the well-known inequality as follows:

$$\Delta' \log tr_g g' \geq -Btr_{g'}g - C$$

where B,C depends on the geometry of good background metric g and Ricci curvature of the volume form on the right hand side of the equation. Notice that

$$\Delta'\varphi = n - tr_{g'}g.$$

By setting A=B+C+1,we have the differential inequality

$$\Delta'(\log tr_g g' - A\varphi) = tr_{g'}g - An.$$

This inequality and the boundary C^2 estimate imply the global C^2 estimate. Then by Evans-Krylov Theorem, we derive higher order regularity. At last the metric upper bound we get now combined with C^0 bound of φ implies a lower bound of the metric, namely φ is in the function space U we introduced at the beginning of the proof.

5.4 Asymptotic analysis of different Kähler-Einstein metrics

In the following paragraph, we are concerned with the asymptotic behaviour of KE metrics we constructed above.

We first compare the volume forms of two different Kähler-Einstein metrics on Ω

5.4.1 Compare volume ratio of different Kähler-Einstein metrics

Fix a point q such that $\mathrm{dist}_\chi(q, \partial\Omega) \geq 2R$. We construct a cut-off function $\phi(x) = \rho(\frac{r(x)}{R}) \geq 0$ with

$$r(x) = d_\chi(x, q)$$

such that

$$\phi = 1 \text{ on } B_\chi(q, R),\ \phi = 0 \text{ outside } B_\chi(q, 2R),$$

and

$$\rho \in [0,1], \quad \rho^{-1}(\rho')^2 \leq C(n), \quad |\rho''| \leq C(n).$$

Let $H = \phi\varphi$. Since both χ and χ' are KE metrics, we have

$$tr_\chi \sqrt{-1}\partial\overline{\partial}\varphi = -n + tr_\chi \chi' \geq n(e^{\frac{f}{n}} - 1).$$

Assume H attain a positive maximum at point Q (otherwise $f(q) \leq 0$). Then at point Q, we have

$$\begin{aligned}
\Delta H &\geq \Delta_\chi \phi(\frac{H}{\phi}) + \phi\Delta_\chi\varphi + 2\Re\langle\nabla\phi, \nabla\frac{H}{\phi}\rangle \\
&\geq \frac{-H}{\phi}(R^{-2}(1+R)) + \phi n(e^{\frac{\varphi}{n}} - 1) + 2\Re\langle\nabla\phi, \frac{1}{\phi}\nabla H\rangle - 2H\Re\langle\nabla\phi, \frac{1}{\phi^2}\nabla\phi\rangle \\
&\geq \frac{-H}{\phi}(R^{-2}(1+R)) + n\phi\frac{\varphi^2}{n^2} + 2\Re\langle\nabla\phi, \frac{1}{\phi}\nabla H\rangle - 2\Re\frac{H}{\phi R^2} \\
&\geq \frac{-H}{\phi}(R^{-2}(1+R) - \frac{H}{n}) + 2\Re\langle\nabla\phi, \frac{1}{\phi}\nabla H\rangle - 2\Re\frac{H}{\phi R^2} \\
&\geq \frac{H}{\phi}(-R^{-1} - 2R^{-2} + \frac{H}{n}) + 2\Re\langle\nabla\phi, \frac{1}{\phi}\nabla H\rangle
\end{aligned}$$

By maximal principle on the ball of radius R, noticing that $\nabla H(Q) = 0$, we get

$$H \leq c(n)(\frac{1}{R} + \frac{1}{R^2}). \tag{5.4.1}$$

Hence $\varphi(q) = H(q) \leq 2c(n)\frac{1}{R}$ when R is large. Therefore we have $\varphi(q) \leq \epsilon$ when $\mathrm{dist}_\chi(q, \partial\Omega) \geq \frac{2c(n)}{\epsilon}$. Switch the role of χ and χ', by the same argument, we get $-\epsilon \leq \varphi \leq +\epsilon$ for $x \in U_\epsilon := \{x | \mathrm{dist}_\chi(x, \partial\Omega) \geq \frac{2c(n)}{\epsilon} \text{ and } \mathrm{dist}_{\chi'}(x, \partial\Omega) \geq \frac{2c(n)}{\epsilon}\}$. As a conclusion, we prove the theorem.

We prove some corollaries of Theorem (1.9)

Corollary 5.27. *Suppose we are in the setting of theorem (1.9) i.e, Ω admits a complete Kähler-Einstein metric χ with negative scalar curvature and $Vol_\chi(\Omega) < \infty$, then for any other complete Kähler-Einstein metric χ' with negative scalar curvature, $Vol_{\chi'(\Omega)} < \infty$.*

Proof. It's obvious, since $\varphi = \log \frac{\chi'^n}{\chi^n}$ is bounded by theorem (1.9). □

Another simple application is a quick proof of uniqueness of complete Kähler-Einstein metric on a complex manifold without boundary.

Corollary 5.28. *Let X be complex manifold without boundary. Suppose χ, χ' are two complete Kähler-Einstein metric with negative curvature on X, then $\chi = \chi'$*

Proof. From the Kähler-Einstein conditon, we know that

$$\chi = \sqrt{-1}\partial\bar{\partial}\log(\chi)^n, \chi' = \sqrt{-1}\partial\bar{\partial}\log(\chi')^n$$

Hence if we let $\varphi = \log \frac{\chi'^n}{\chi^n}$, then $\chi' = \chi + \sqrt{-1}\partial\bar{\partial}\log\frac{\chi'^n}{\chi^n}$ and φ satisfies the following equation:

$$(\chi + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{\varphi}\chi^n$$

From the proof theorem (1.9), at a fixed point $p \in X$, as long as we can get a large scale cut off function, which is always true on complete manifold, we have $\varphi(p) < 1$. Once we have bounded ness of φ, maximal principle will conclude the proof. □

By applying Yau's Schwarz Lemma, we have the following theorem concerning the comparison of two different Kähler-Einstein metrics.

Proposition 5.29. *Let χ, χ' be two complete Kähler Einstein metrics with negative scalar curvature. If moreover the bisectional curvature of χ is $\leq -K_2$. where K_2 is a positive constant. Then there is a constant c such that*

$$\frac{1}{c}g_1 \leq g \leq cg_1.$$

where g, g_1 are the Riemannian metrics corresponding to the Kähler forms χ, χ'.

Proof. Let $u = tr_{g_1}g$, by Chern-Lu's inequality we have

$$\Delta_{g_1}u \geq -K_1u + K_2u^2.$$

where K_1 is the Ricci curvature of ω_1. We still use the cut-off function ϕ as in theorem (1.9). Let $G = \phi u$, combine the Chern-Lu inequality and the same argument as as in theorem (1.9), we get the following inequality

$$G \leq \frac{K_1}{K_2} + c(K_1, K_2)R^{-\frac{1}{2}}$$

When R is larger, the estimate is better, hence we have $g \leq cg_1$ for some constant which depend on the metric g_1. Since we also prove the pointwise volume ratio estimate in theorem (1.9), hence we have

$$\frac{1}{c}g_1 \leq g \leq cg_1$$

□

Now we make use of property (A) of metric χ. We first prove bounded smooth solutions of Dirichlet problem in (1.3.2) is unique and any complete Kähler-Einstein metric on $(\Omega \setminus p)$ comes from the solution of equation (1.3.2).

Proposition 5.30. *Suppose Ω admits a complete metric $\chi = \sqrt{-1}\partial\overline{\partial}\rho$ with ρ bounded from above and goes to $-\infty$ towards p, then bounded smooth solution φ of*

$$\begin{cases} (\chi + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\varphi}\chi^n \quad on \quad \Omega \setminus p \\ \\ \varphi|_{\partial\Omega} = 0, \end{cases} \tag{5.4.2}$$

is unique i.e $\varphi = 0$. In particular, if χ is a complete metric towards p with property (A), then for any other complete Kähler-Einstein metric χ' with negative scalar curvature, χ' is one of the solutions from Theorem (1.3.2).

Proof. Let $\varphi_\epsilon = \varphi - \epsilon\rho$ and $\chi_\epsilon = (1+\epsilon)\chi$ then φ_ϵ safisfies the equation

$$\begin{cases} (\chi_\epsilon + \sqrt{-1}\partial\overline{\partial}\varphi_\epsilon)^n = \frac{e^{\varphi}}{(1+\epsilon)^n}\chi_\epsilon^n \quad on \quad \Omega \\ \\ \varphi_\epsilon|_{\partial\Omega} = -\epsilon\rho, \end{cases} \tag{5.4.3}$$

Since ρ goes to $-\infty$, φ_ϵ admits minimum in Ω. If minimum is on the boundary, $\varphi_\epsilon \geq -\epsilon\, inf_{\partial\Omega}\rho$. If the minimum is in the interior point Q, $\varphi_\epsilon(Q) = \varphi(Q) - \epsilon\rho(Q) \geq$

$\log(1+\epsilon)-\epsilon\, max\rho$. In both cases, let $\epsilon \to 0$, we get $\varphi \geq 0$. Similar argument showes that $\varphi \leq 0$. Hence 0 is the unique solution. For the second part of the theorem, on one hand, from the fact that χ' is complete and Theorem (1.9), we know that $\chi' = \chi + \sqrt{-1}\partial\overline{\partial}\varphi$ and φ is bounded, on the other hand, from Theorem (1.3.2) we can find $\tilde{\chi}' = \chi + \sqrt{-1}\partial\overline{\partial}\tilde{\varphi}$ with $\tilde{\varphi}|\partial\Omega = \varphi|\partial\Omega$. By the first part of this theorem, we conclude that $\varphi = \tilde{\varphi}, \chi' = \tilde{\chi}'$. □

At last, with the help of estimates of high order derivatives of φ, we prove the rigidity of local complete Kähler-Einstein metrics with negative scalar curvature near isoloated log canonical singularity. This is stronger than proposition (5.29).

5.4.2 Compare different Kähler Einstein Metrics

First of all, χ' is one of the solutions in theorem (1.7) by the completeness of χ', theorem (1.9) and theorem (5.4.2). For any point $q \in (\Omega \setminus p)$, we can choose a quasi coordinate $(\widehat{V}, \phi)$ covering q such that there is a point $\widehat{q} \in V \subset \widehat{V}$, $\phi(\widehat{q}) = q$ and $\mathrm{dist}(\widehat{q}, \partial\widehat{V}) \geq \mathrm{dist}(V, \partial\widehat{V}) \geq \epsilon_1$. Let ρ be the cut-off function we constructed in the proof of Lemma (5.24). Then we have the following inequalities:

$$\textstyle\sum_{i=1}^{k} |\rho^{(k)}|_{\mathrm{Euc}} \leq B_k,$$

where $B_k's$ are universal constants independent of p and V. This is true because under the construction of the system of quasi coordinates, we have $\mathrm{dist}(V, \partial\widehat{V}) \geq \epsilon_1 > 0$, hence controlling the derivatives of cut-off function unifomly. Actually, we can even assume the covering domains we choose are $B_{\frac{1}{4}\epsilon_1}, B_{\frac{1}{2}\epsilon_1}$ by subdividing the original coverings. By our previous proof of the A priori estimates of φ from equation (1.3.2), we also have the following inequalities, for any point $q \in (\Omega \setminus p)$ and any nonnegative integer k:

$$\textstyle\sum_{i=1}^{k} \|\nabla^{(k)}\varphi\|_{\chi}(q) \leq C_k.$$

When $k = 0$, the theorem is proved by Theorem (1.9). for $k \geq 1$, we do computations in the quasi coordinate as follows:

$$\int_{\widehat{V}} \rho\varphi\Delta\varphi = \int_{\widehat{V}} \rho|\nabla\varphi|^2 + \int_{\widehat{V}} \varphi\langle\nabla\varphi, \nabla\rho\rangle,$$

$$\int_V |\nabla\varphi|^2 \le \int_{\widehat{V}} \rho|\nabla\varphi|^2 \le C\cdot(C_2+B_1)\cdot \sup_{\widehat{V}}|\varphi|.$$

Similarly, we have

$$\int_V |\nabla^k\varphi|^2 \le C\cdot(B_k+C_{2k-1})\cdot \sup_{\widehat{V}}|\varphi|.$$

where C is the euclidean volume of V. Suppose $\text{dist}_\omega(p,\partial\Omega)\ge R$ and $\text{dist}_{\omega_1}(p,\partial\Omega)\ge R$, by the triangle inequality, we have

$$dist(\partial(\phi(\hat{V})),\partial\Omega)\ge dist(p,\partial\Omega)-dist(p,\partial(\phi(\hat{V})))\ge R-C'\epsilon_1\ge\frac{R}{2}.$$

where C' is a metric equivalence constant in quasi coordinates, where depends on the geometry of χ but independent of q. Hence by the C^0 estimate of φ in Theorem (1.9),

$$\int_V |\nabla^k\varphi|^2 \le C\cdot(C_{2k-1}+B_k)\cdot\frac{c(n)}{R}.$$

Now that we have L^2 norm control of all higher order derivatives, by Sobolev embedding on Euclidean space and property (d) of quasi-coordinate, we can conclude that $\sum_{i=1}^k \|\nabla^i\varphi\|_\chi(q)\le\epsilon C(k,\chi,f)$ for q $\in U_\epsilon$ where $C(k,f,\chi)$ is a constant depends on the geometry of χ, k and f.

To end section 3, we give an example of a family of canonical polarized varieties with central fiber equipped with log canonical singularity satisfying property (A).

5.4.3 Example: Degeneration of Godeaux surfaces

A surface X is called a Godeaux surface if $\pi_1(X)=\mathbb{Z}_5$ and universal cover is quintic hypersurface. A explicit construction could be as follows: Define $\mathbb{Z}_5$ on $\mathbb{P}^3$ in the following way:

$$\rho\bullet(X_0,X_1,X_2,X_3)=(X_0,\rho X_1,\rho^2X_2.\rho^3X_3)$$

Then there exists quintics (in $\mathbb{P}^3$) invariant and fixed point free under the $\mathbb{Z}_5$ action with 5 non degenerate triple points and no other singularities by a dimension count argument. (See [135] page 135). Then the $\mathbb{Z}_5$ quotient will give a family of Godeaux with central fiber a canonical polarized variety coupled with a single simple elliptic singularity (cone over elliptic curve).

Appendix A

Uniformization of lc singularity of complex dimension 2

We include the uniformization of 2 dimensional log canonical singularity for reader's convenience. This is basically taken from [69, 70]. Firstly let's us recall the classification of 2 dimensional isolated log canonical singularity in terms of the the configuration of exceptional curves. The first proof I know is due to Kawamata [65], page 141. See also [1].

Let (X, p) be a log-canonical surface singularity and let $\mu : Y \to X$ be the minimal resolution. Let G be the dual graph of the union of exceptional locus. Then of the following holds;

1. (x, p) is a quotient singularity (Klt singularity);
2. (x, p) is a simple elliptic singularity or a 2 dimensional cusp;
3. $G = (2, 2, 2, 2), (2, 4, 4), (2, 3, 6), (3, 3, 3)$;
4. $G =$

We will have a more detailed description of the terminology we used above in the discussion below. Here we just point out that simple elliptic singularity is a covering of case (3) and 2 dimensional cusp is covering of case (4).

Every cusp singularity is log canonical. The exceptional set in the minimal resolution is a cycle of $\mathbb{CP}^1$ or a double rational curve. It's is uniformized by by $\mathbb{H} \times \mathbb{H}$ with covering transformation group $G(M,V)$ which is a reflection-free discrete subgroup of $Aut(\mathbb{H}\times\mathbb{H})$ fixing the point (∞,∞) in the boundary.

More precisely, suppose we have a cycle of rational curves $\mathbb{CP}^1_i$ with self intersection $q_i, 0 \leq i \leq r-1$. For a integer k, let $\mathbb{C}^2_k$ be the k-th copy of $\mathbb{C}^2$ with the coordinate function (u_k, v_k). We put the identification defined by

$$u_k = u_{k-1}^{q_{k-1}} v_{k-1}, v_k = u_{k-1}^{-1}$$

on the disjoint union $\amalg_{k\in Z}\mathbb{C}^2_k$. Let the resulting manifold be Y'. In Y', the curve defined by $v_k = 0$ in $\mathbb{C}^2_k$ and $u_k = 0$ in $\mathbb{C}^2_{k+1}$ is a nonsingular curve with self-intersection $-q_k$. We denote these curves by S_k, hencee $S_k's$ is a chain of rational curves. The identification can also be written in the form as

$$(\log u_k, \log v_k) = (\log u_{k-1}, \log v_{k-1})\left(\begin{smallmatrix} q_{k-1} & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

$$= (\log u_0, \log v_0)\left(\begin{smallmatrix} q_0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} q_1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} q_{k-1} & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

Set

$$\left(\begin{smallmatrix} P_k & -P_{k-1} \\ Q_k & -Q_{k-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} q_0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} q_1 & -1 \\ 1 & 0 \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} q_{k-1} & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

Then $q_kP_k = P_{k-1} + P_{k+1}, P_0 = 1, P_1 = q_0$ and $q_kQ_k = Q_{k-1} + Q_{k+1}, Q_0 = 0, Q_1 = 1$, $\{P_k\}_{k\geq 1}$ and $\{Q_k\}_{k\geq 1}$ are determined by the continued fractions $[[q_0, q_1, q_2, \cdots, q_k]] = q_0 - (q_1 - (q_2 - \cdots - (q_{k-2} - q_{k-1}^{-1})^{-1}\cdots)^{-1})^{-1} = \frac{P_k}{Q_k}$ where P_k, Q_k are coprime positive integer. Also the infinite periodic continued fraction $[[q_0, q_1, \cdots, q_s, \cdots]] > 1$ represents a real quadtatic irrational number ω_0. For example if all $q_k = 4$, then $\omega_0 = 2 + \sqrt{3}$.

Let $\omega_s := [[q_s, q_{s+1}, \cdots]] > 1$ and $R_k = P_k - Q_k\omega_0$. Then $R_k's$ satisfy $q_kR_k = R_{k-1} + R_{k+1}$. From the definition, we get $R_0 = 1, R_1 = \omega_1^{-1}, \cdots, R_k = \omega_1^{-1}\omega_2^{-1}\cdots\omega_k^{-1}$, $M = \mathbb{Z} + \mathbb{Z}\omega_0$ is a free $\mathbb{Z}$- module of rank 2 and R_k, R_{k+1} for any k, is a basis of M. Since $\omega_k = \omega_{k+r}$ for any k,$R_kR_r = R_{k+r}$ holds. So $R_rM = M$. By the Hamilton-Cayley theorem, R_r and $R_{-r} = R_r^{-1}$ are both algebraic integers. In particular, $R_r^{-1} = R_r'$ where $'$ means to take the conjugate over $\mathbb{Q}$. Let $V = \{R_r^n\}_{n\in\mathbb{Z}} \cong \mathbb{Z}$ under the correspondence $R_r^n \leftrightarrow n$. Then $G(M,V) = \left\{\left(\begin{smallmatrix} \epsilon & \mu \\ 0 & 1 \end{smallmatrix}\right), \epsilon \in V, \mu \in M\right\}$ acts on $\mathbb{C}^2$ properly discontinuous and without fixed points as follows:

$$\left(\begin{smallmatrix} \epsilon & \mu \\ 0 & 1 \end{smallmatrix}\right) \bullet (z_1, z_2) := (\epsilon z_1 + \mu, \epsilon' z_2 + \mu') \tag{A.0.1}$$

The action of $G(M,V)$ can be restricted onto $\mathbb{H}^2$ where $\mathbb{H}$ is the upper half plane. Since $\epsilon\epsilon' = 1$, $Imz_1 Imz_2$ is invariant under the action of $G(M,V)$. We show there is a neighborhood Y^+ of $\bigcup_{k\in\mathbb{Z}} S_k$ in Y' such that $Y^+ \setminus \bigcup_{k\in\mathbb{Z}} S_k$ is biholomorphic to $\mathbb{H}^2/M$. Actually let

$$Y^+ = \{(u_k, v_k) \in \mathbb{C}^2_k, \infty \geq R'_{k-1}\log|u_k|^{-1} + R'_k \log|v_k|^{-1} > 0, \infty \geq R_{k-1}\log|u_k|^{-1} + R_k \log|v_k|^{-1} > 0\}$$

And the correspondence is given by

$$2\pi i(z_1, z_2) = (\log u_k, \log v_k)\begin{pmatrix} R_{k-1} & R'_{k-1} \\ R_k & R'_k \end{pmatrix} \tag{A.0.2}$$

It's easy to see that the above correspondence is well defined and that

$$\mathbb{H}^2/M \cong Y^+ \setminus \bigcup_{k\in\mathbb{Z}} S_k$$

Then we still need to put a periodic identification on Y^+. We consider the following $\mathbb{Z}$ action on Y^+. For $n \in \mathbb{Z}$ and (α, β) the coordinate of $\mathbb{C}^2_k$, $n \bullet (\alpha, \beta)$ is defined by (α, β) in terms of the $(k+nr)-$th coordinate. This $\mathbb{Z}$ restricts to the action on $Y^+ \setminus \bigcup_{k\in\mathbb{Z}} S_k$ and is compatible with the V action on $\mathbb{H}^2/M$ via (A.0.2). Indeed, the point of $Y^+ \setminus \bigcup_{k\in\mathbb{Z}} S_k$ expressed as (α, β) in the $(k+nr)-$th coordinate is written as $(\alpha^a\beta^b, \alpha^{-c}\beta^{-d})$ in the $k-$ coordinate, where

$$\begin{pmatrix} a & -c \\ b & d \end{pmatrix} = \left\{\begin{pmatrix} q_{k-r} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_{k-r+1} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} q_{k-1} & -1 \\ 1 & 0 \end{pmatrix}\right\}^n$$

So

$$\begin{aligned}(\log\alpha^a\beta^b, \log\alpha^{-c}\beta^{-d})\begin{pmatrix} R_{k-1} & R'_{k-1} \\ R_k & R'_k \end{pmatrix} &= (\log\alpha, \log\beta)\begin{pmatrix} R_{k-nr-1} & R'_{k-nr-1} \\ R_{k-nr} & R'_{k-nr} \end{pmatrix} \\ &= (\log\alpha, \log\beta)\begin{pmatrix} R_{k-1} & R'_{k-1} \\ R_k & R'_k \end{pmatrix}\begin{pmatrix} R^n_{-r} & 0 \\ 0 & R'^n_{-r} \end{pmatrix}\end{aligned}$$

This $\mathbb{Z}$ action on Y^+ is properly and discontinuous and without fixed points. Define $Y = Y^+/\mathbb{Z}$, then the image of S'_ks forms a cycle $B = \sum_{k=0}^{r-1} B_k$ of $\mathbb{CP}^1$ such that $B_k.B_k = -q_k$. Now we can conclude that

$$\mathbb{H}^2/G(M,V) \cong Y \setminus B$$

and the correspondence is given by (A.0.2) in the $k-$th coordinate of Y^+ and the Euclidean one of $\mathbb{H}^2$. The open set W_L of $\mathbb{H}^2$ defined by $\{(z_1, z_2) \in \mathbb{H}^2, Imz_1 Imz_2 > L\}$ is invariant under the action of $G(M,V)$ and its image in $Y\setminus B$ is a deleted neighborhood of B. Now we conclude that $\sqrt{-1}\partial\overline{\partial}\log Imz_1 Imz_2$ is invariant under the action of Γ and hence can descend to a Kähler metric on a punctured neighbourhood of singular point p.

(According to Kobayashi, under a further $\mathbb{Z}_2$ action, we can get case (4).

Remark 11. *A $\mathbb{Z}_2$ action on case (3) will give the case (4), but it seems unclear to us that how the metric $\chi = \sqrt{-1}\partial\overline{\partial}\log Imz_1 Imz_2$ can be invariant under the $\mathbb{Z}_2$ action. A naive action which keeps the metric invariant will be swapping z_1 and z_2, but this action has singularity along the line $z_1 = z_2$. We give an alternative argument to deal with case (4). By case (3), we already get a local Kähler-Einstein metric satisfying property A. The $\mathbb{Z}_2$ action will induce a automorphism of π of $(\Omega\setminus p)$, then by our theorem (5.5), $\|\nabla^j_\chi(\chi - \pi^*\chi)\| < C_j$ where C_j can be choosen as small as we want by shrinking the domain. Then $\chi + \pi^*\chi$ will be a model metric invariant under the $\mathbb{Z}_2$ action satisfying property A.*

Next we discuss simple elliptic singularity.

First let us recall the standard Bergman metric model on the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$, $(\mathbb{B}^2, -\sqrt{-1}\partial\overline{\partial}\log(1-|z|^2))$. Through the transformation $z_1 = \frac{u-i}{u+i}, z_2 = \frac{2v}{u+i}$, the Bergman metric model correspond to the Heisenberg model defined on the Siegel domain: $\Delta = \{(u,v)\in\mathbb{C}^2 | Imu - |v|^2 > 0, -\sqrt{-1}\partial\overline{\partial}\log(Imu - |v|^2)\}$. Let L be a lattice in the v plane, then the parabolic group P fixing the boundary point $p = (1,0)\in\partial\mathbb{B}^2$ is written in the Siegel domain expression as follows: (or you can at least verify $Imu - |v|^2$ is invariant under the action below through simple calculation)

$$P = \{(\mu,\gamma,r) | \mu\in\mathbb{U}(1), \gamma\in L, r\in\mathbb{R}\}$$

where (μ,γ,r) stands for the automorphism of Δ given by

$$\begin{pmatrix} 1 & 2i\mu\bar{\gamma} & r + i|\gamma|^2 \\ 0 & \mu & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

It's easy to see that the composition law is $(\mu, \gamma, r)(\mu', \gamma', r') = (\mu\mu', \mu\gamma' + \gamma, r + r' - 2Im(\mu\bar{\gamma}\gamma'))$.

Now we can start to talk about simple elliptic surface singularity. Let $A = \mathbb{C}/L, L = \{\mathbb{Z} + \mathbb{Z}\omega, Im\omega > 0\}$ and denote the projection $\mathbb{C} \to A$ by π. Let a be the area of the fundamental domain of L measured by the usual flat metric $|dz|^2$ of $\mathbb{C}$. There is a real closed form η on A such that $\pi^*\eta = i(2a)^{-1}dz \wedge d\bar{z}$.

Let ρ be the metric of Hermitian line bundle $N \to A$ whose curvature form is given by $-ib\pi(a)^{-1}dz \wedge d\bar{z}$. Since $H^1(\mathbb{C}, \mathcal{O}^*) = 1$, there is an isomorphism $\mathbb{C}^2 \cong \pi^*N$ between holomorphic line bundles over $\mathbb{C}$ where $\mathbb{C}^2$ is the trivial line bundle over $\mathbb{C}$. On the pull back bundle π^*N, we may regard $\pi^*\rho$ as a positive function on $\mathbb{C}$. There is an entire holomorphic function $\theta(z)$ such that $\pi^*\rho = \{exp(-|z|^2)|exp(\theta)|^2\}^{\frac{b\pi}{a}}$. The biholomomorphic map of $\mathbb{C}^2$ into itself defined by $(w, z) \to (exp(\frac{-b\pi\theta}{a})w, z)$ is an isomorphism of trivial line bundle over $\mathbb{C}$ and the Hermitian metric $exp(-|z|^2)^{\frac{b\pi}{a}}$ is pulled back to the Hermitian metric $\{exp(-|z|^2)|exp(\theta)|^2\}^{\frac{b\pi}{a}}$. So we may assume that $\pi^*N = \mathbb{C}^2, \pi^*\rho = (exp(-|z|^2))^{\frac{b\pi}{a}}$ through an automorphism of $\mathbb{C}^2$. Next we compute the transition function, let U be an open subset of $\mathbb{C}$ such that $\bar{U}$ is contained in a fundamental domain of L. Let γ be an arbitrary element in L. Since $\mathbb{C} \times U$ and $\mathbb{C} \times (U + \gamma)$ are local trivilization of $N|_{\pi(U)}$, there exists a non-vanishing holomorphic $g(z)$ defined in U such that $(w, z) \in \mathbb{C} \times U$ and $(w', z') \in \mathbb{C} \times (U_\gamma)$ represent the same point of $N|_{\pi(U)}$ if and only if $z' = z + \gamma$ and $w' = g(z)w$. Hence $g(z)$ must satisfy the following equality: $|w|^2(exp(-|z|^2))^{\frac{-b\pi}{a}} = |w'|^2(exp(-|z'|^2))^{\frac{-b\pi}{a}} = |g(z)|^2|w|^2(-exp(|z + \gamma|^2))^{\frac{-b\pi}{a}}$ for all $z \in U$ and $w \in \mathbb{C}$. Therefore $g(z)$ must be written in as

$$g(z) = exp\{-\frac{b\pi}{a}(z\bar{\gamma} + \frac{|\gamma|^2}{2}| + i\theta(\gamma))\}$$

where $\theta(\gamma)$ is a real number determined by $\gamma \in L$ modulo $(\frac{2a}{b})\mathbb{Z}$. If $z' = z + \gamma$ and $z'' = z' + \gamma'$, then $(w, z), (w', z')$ and (w'', z'') represent the same point if and only if the following three equalities hold:

$$w'' = exp\{-\frac{b\pi}{a}(z(\gamma \bar{+} \gamma') + \frac{|\gamma + \gamma'|^2}{2} + i\theta(\gamma + \gamma'))\}w$$

$$w'' = exp\{-\frac{b\pi}{a}(z'\bar{\gamma'} + \frac{|\gamma'|^2}{2} + i\theta(\gamma'))\}w'$$

$$w' = exp\{-\frac{b\pi}{a}(z\bar{\gamma} + \frac{|\gamma|^2}{2} + i\theta(\gamma))\}w$$

Hence $\theta(\gamma+\gamma') = \theta(\gamma)+\theta(\gamma') - Im(\bar{\gamma}\gamma')$ modulo $\frac{2a}{b}\mathbb{Z}$. Recall that $L = \{\mathbb{Z}+\mathbb{Z}\omega, Im\omega = a > 0\}$. It follows that $\theta(m+n\omega) = m\alpha + n\beta - mna$ modulo $\frac{2a}{b}\mathbb{Z}$ where α, β are fixed representativs of $\theta(1)$ and $\theta(\omega)$ respectively. Using this $\theta(\gamma)$, we define a group of 3×3 matrics as follows:

$$\Gamma = \{\begin{pmatrix} 1 & 2i\bar{\gamma} & i|\gamma|^2 - 2\theta(\gamma) \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}; \theta(m+n\omega) = m\alpha + n\beta - mna \quad modulo \quad \frac{2a}{b}\mathbb{Z}\}$$

Let B be the unit ball in $\mathbb{C}^2$ and Δ the domain in $\mathbb{C}^2$ defined by $\{(u,v) \in \mathbb{C}^2; Im(u) - |v|^2 > 0\}$. Then $z_1 = (u-i)(u+i)^{-1}$ and $z_2 = (2v)(u+i)^{-1}$ give a biholomorphic map of B to Δ. For a positive integer k, we consider the subdomain Δ_k of Δ defined by $\{(u,v) \in \mathbb{C}^2; Im(u) - |v|^2 > k\}$. Actually Δ_k corresponds to the horoball at $(1,0)$ of B with the Bergman metric. Γ is a discrete subgroup of the group of analytic automorphism of Δ, which also keep Δ_k invariant. This action is described as follows:

$$\begin{pmatrix} 1 & 2i\bar{\gamma} & i|\gamma|^2 - 2\theta(\gamma) \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} u + 2i\bar{\gamma}v + i|\gamma|^2 - 2\theta(\gamma) \\ v+\gamma \\ 1 \end{pmatrix}$$

The map $F : \Delta \to \mathbb{C}^2$ defined by $(u,v) \to (exp(\frac{b\pi i u}{2a}), v)$ maps Δ onto the set $V' = F(\Delta) = \{(w,z) \in \mathbb{C}^2; 0 < |w|^2(exp(-|z|^2))^{\frac{-b\pi}{a}}) < exp(-\frac{-b\pi k}{a}\}$. If we define $V = \{w \in N : 0 < \rho(w,w) < exp(\frac{-b\pi k a}{a})\}$, then $V' = \pi^{-1}(V)$. V is a deleted neighborhood of the zero-section of N and a punctured disk bundle over the elliptic curve A. It's easy to see that Δ/Γ is biholomorphic to V. And the Kähler form of the Bergman metric of unit ball B can be written in terms of the coordinate (u,v) of Δ as

$$-\sqrt{-1}\partial\bar{\partial}\log(Imu - |v|^2) = \frac{dv\wedge d\bar{v}}{Imu - |v|^2} + \frac{(-idu - 2\bar{v}dv)\wedge(id\bar{u} - 2vd\bar{v})}{4(Imu-|v|^2)^2} \tag{A.0.3}$$

This metric is invariant under the action of Γ, hence this projects down to a Kähler metric of V, whose Kähler form is given by

$$\frac{dz\wedge d\bar{z}}{(\frac{a}{b\pi})\log|w|^{-2} - |z|^2} + \frac{((\frac{a}{b\pi}(\frac{dw}{w}) + \bar{z}dz)\wedge((\frac{a}{b\pi})(\frac{d\bar{w}}{w}) + zd\bar{z})}{((\frac{a}{b\pi})\log|w|^{-2} - |v|^2)^2}$$

where $(w, z) = F(u, v) = (exp(\frac{b\pi iu}{2a}), v)$.

Case (c) is uniformized by simple elliptic singularity. We use the same notation as the case of simple elliptic singularity. The elliptic curve A has a nontrivial point group G, i.e., the corresponding lattice is invariant under the action of non-trivial finite subgroup of $\mathbb{U}(1)$. The central curve, namely A quotient, is an orbifold defined over $\mathbb{CP}^1$ described by $(b_1, b_2, \cdots)$ where $b_1, \cdots$ are branch indices. The only possible triads $(A, G, (b_1, \cdots))$ are

1. $L = \mathbb{Z} + \mathbb{Z}\omega$ (general lattice),$G = (-1), (2, 2, 2, 2)$
2. $L = \mathbb{Z} + \mathbb{Z}i, G = (i), (2, 4, 4)$
3. $L = \mathbb{Z} + \mathbb{Z}e^{\frac{2\pi i}{6}}, G = (e^{\frac{2\pi i}{6}}), (2, 3, 6)$
4. $L = \mathbb{Z} + \mathbb{Z}e^{\frac{2\pi i}{6}}, G = (e^{\frac{2\pi i}{3}}), (3, 3, 3)$

We can construct a discrete parabolic groups Γ corresponding to these triads, which fit into the exact sequence

$$1 \to \mathbb{Z} \to \Gamma \to E \to 1$$

where $\mathbb{Z}$ consists of automorphism $((\mu, \gamma, r))$ with $(\mu = 0, \gamma = 0, r \cong \frac{2a}{b}\mathbb{Z})$, Γ consists of automorphism $((\mu, \gamma, r))$ with $\mu \in G$ (a finite subgroup of $\mathbb{U}(1)$), $\gamma \in L$ (a lattice with a non-trivial point group G) and $r = r(\mu, \gamma) \in \mathbb{R}$ modulo $\frac{4a}{b}\mathbb{Z}$ obeying

$$r(\mu\mu', \mu\gamma' + \gamma) = r(\mu, \gamma) + r(\mu', \gamma') - 2Im(\mu\bar{\gamma}\gamma')mod\frac{4a}{b}\mathbb{Z}. \tag{A.0.4}$$

E is a discrete Euclidean motion group generatd by L and G. The map $\Gamma \to E$ is defined by forgetting R. For example, in the case L is general and $G = (-1)$ Define $r(-1, 0) = \frac{2a}{b}$ mod $\frac{4a}{b}\mathbb{Z}$ and $r(1, n + m\omega) = -2mna$ mod $\frac{4a}{b}\mathbb{Z}$. Actually, by (A.0.4), to construct such a group Γ, we only need to define $r(1, \gamma)$ and $r(e, \gamma)$ where e is a generator of the cyclic group G. And in this situation, the minimal resolution of the singularity is a central rational curve with 4 rational curves which don't intersection each other, sitting on it. If the branch index is 3, then in the resolution, there are two possibilities:

the first one is the Du Val singularity whose resolution is two rational curves with self-intersection -2, or it's a (3,1) singularity whose resolution is one rational curve with self-intersection -3. Similarly we can analyze the cases of branch index 4 and 6. An explicit resolution of the cyclic singularity can be found in [71].

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