ON EXCEPTIONAL COLLECTIONS OF LINE BUNDLES ON TORIC DELIGNE-MUMFORD STACKS

By

CHENGXI WANG

A dissertation submitted to the

School of Graduate Studies

Rutgers, The State University of New Jersey

In partial fulfillment of the requirements

For the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Lev Borisov

And approved by

New Brunswick, New Jersey

May, 2020

ABSTRACT OF THE DISSERTATION

On Exceptional Collections of Line Bundles on Toric Deligne-Mumford Stacks

By CHENGXI WANG

Dissertation Director: Lev Borisov

We study strong exceptional collections of line bundles on Fano toric Deligne-Mumford stacks $\mathbb{P}_{\mathbf{\Sigma}}$. We prove that when the rank of Picard group is no more than two, any strong exceptional collection of line bundles generates the derived category of $\mathbb{P}_{\mathbf{\Sigma}}$, as long as the number of elements in the collection equals the rank of the (Grothendieck) K-theory group of $\mathbb{P}_{\mathbf{\Sigma}}$.

Moreover, we consider generalized Hirzebruch surfaces $\mathbb{F}_{\alpha,n}$ which are not Fano and have Picard rank two. We give a classification of all (strong) exceptional collections of line bundles of maximum length and show they generate the derived category, which is a generalization for the results of Hirzebruch surfaces. We show that any exceptional collections of line bundles on $\mathbb{F}_{\alpha,n}$ can be extend to maximum length $2(\alpha+1)$ which is the rank of K-theory. We give examples of strong exceptional collections of line bundles on $\mathbb{F}_{\alpha,n}$ which cannot be extended to strong exceptional collections of line bundles of length $2(\alpha+1)$, but can be extend to exceptional collections of line bundles of maximum length $2(\alpha+1)$.

Acknowledgements

I am deeply indebted to my advisor, Professor Lev A. Borisov for teaching me a lot of beautiful mathematics as well as his guidance, enlightening discussions, constant encouragement and support during my graduate study. I am very grateful to him for generously sharing with me his ideas and visions in mathematics. His suggestions on finding and proposing problems were tremendously valuable.

I would like to thank Professor Anders S. Buch for guiding me to an interesting research project which broadens my research horizon. I also owe a great deal to him for generously sharing ideas and helpful discussions during our joint work. I am very grateful to Professor Angela Gibney for numerous instructive suggestions, help and encouragement.

To Professor Aise Johan de Jong, Professor Lev A.Borisov, Professor Anders S. Buch and Professor Angela Gibney, I would like to thank them for being the Committee Member of my Thesis Defense. To Professor Chuck Weibel and Professor Daniel Krashen, I thank for their helps and the excellent courses taught at Rutgers. I would like to thank the faculty and staff of Mathematics Department at Rutgers, especially Professor Vladimir Retakh, Professor Michael Saks and Ms. Katie Guarino, for their patience, help and support.

I would like to thank Rutgers University for providing excellent academic environment and financial support; to the organizers of the conferences, workshops and seminars I spoke on and attended.

My thanks also go to my friends and fellow graduate students for their help and useful discussions in math. I am thankful to my parents for their support and encouragement.

Table of Contents

List of Figures

Chapter 1

Introduction

The branch of mathematics known as algebraic geometry mainly studies the sets of common zeros of systems of polynomials, which are referred to as algebraic varieties. It occupies a central place in modern mathematics and has multiple interactions with diverse fields such as complex analysis, topology, physics and number theory.

The development of algebraic geometry proceeded in waves with their own methods and viewpoints. The nineteenth century saw the development of non-Euclidean geometry and Abelian integrals in order to bring the old algebraic ideas back into the geometric fold. This later lead to the classification of algebraic surfaces up to birational isomorphism by members of the 20th century Italian school of algebraic geometry and the development of Riemann surfaces by Bernhard Riemann. Simultaneously, the algebraization of the algebraic geometry through commutative algebra lead to the crucial results such as Hilbert's basis theorem, Hilbert's Nullstellensatz. Then came the twentieth- century "American" school of Chow, Weil, and Zariski, which solidified the foundations of algebraic geometry of the Italian school. In the 1950s and 1960s, Serre and Grothedieck initiated the French school, which recast the foundations making use of sheaf theory and homological techniques. As a key achievement of this abstract algebraic geometry, Grothendieck's scheme theory allows one to use sheaf theory to study algebraic varieties in a way which is very similar to its use in the research of differential and analytic manifolds.

For every algebraic variety, we have the abelian category of coherent sheaves. However, the functors, induced by morphisms between varieties, are not exact; they do not take exact sequences to exact sequences. To give necessary corrections to non-exact functors, Cartan and Eilenberg [9] introduced the notion of derived functors using techniques developed by Grothendieck in [17]. This subsequently led to the development of derived categories and derived functors between them. Derived categories, in contrast to Abelian categories, do not have short exact sequences. It is formalized by Verdier as the notion of triangulated category [30]. The bounded derived category carries a very rich structure and encodes information which might not directly be visible from geometry [8, 20]. The first example of two non-isomorphic algebraic varieties having equivalent derived categories of coherent sheaves was uncovered by Mukai [26]. In the 1990s, Bondal and Orlov found an unexpected parallelism between the derived categories and birational geometry.

Further motivation for the study of derived categories is provided by Kontsevich's homological mirror symmetry conjecture [24] and the use of derived categories for D-branes in superstring theory [12]. Mirror Symmetry was discovered in string theory as a duality between families of 3-dimensional Calabi-Yau manifolds. It also reveals the relations between symplectic structures and complex structures. Homological mirror symmetry of Kontsevich is an open-string version of mirror symmetry. It asserts the equivalence between the derived category of coherent sheaves on one compact Calabi-Yau variety X and the derived Fukaya category a mirror compact Calabi-Yau variety Y.

It is important to explore the derived categories by constructing suitable generating sets. The framework of exceptional collections provides the simplest possible set up for this. Moreover, toric varieties provide rich examples of algebraic varieties, admitting an action of torus and fully determined by the combinatorics of its associated fan. Also, the toric DM stacks gives more freedom than toric varieties since the lattice points on each one dimensional ray no longer need to be primitive. Derived categories of coherent sheaves on toric varieties and DM stacks provide examples of combinatorially defined triangulated categories. A lot of work has been done over the years aimed at finding exceptional objects and collections in these categories.

Kawamata constructed exceptional collections in the bounded derived categories of of coherent sheaves on smooth Deligne-Mumford stacks in [22]. Alastair King conjectured in [23] that every smooth toric variety has a full strong exceptional collection of line bundles. Although the conjecture was proved to be false in [18], rich and varied results related to the conjecture were proved in [6, 19, 25, 13, 21, 28]. In particular, it was proved in [6] that there exist full strong exceptional collections of line bundles on smooth toric Fano DM stacks of Picard number no more than two, and of any Picard number in dimension two.

The full strong exceptional collections of line bundles constructed in [6] have length equal the rank of the (Grothendieck) K-theory group, which is known to be a necessary condition, see for example [16]. It is natural to ask whether any strong exceptional collection of line bundles of this length is a full strong exceptional collection. That is to say that the subcategory generated by all elements in the strong collection equals $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$, and there is no orthogonal complement phantom category. We propose the following conjecture.

Conjecture 1.1. *Any strong exceptional collection of line bundles of maximum length on a Fano toric DM stack is a full strong exceptional collection.*

In this thesis, we prove Conjecture 1.1 for $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$ (Theorem 3.7) and $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$ (Theorem 3.20). Our main idea is to "shrink" the strong exceptional collection by moving some specific elements successively and eventually obtain a standard full strong exceptional collection given in [6].

These results are also meaningful from the perspective of constructing phantom and quasi-phantom subcategories of the derived category of coherent sheaves on smooth projective varieties which has attracted considerable interest over the years. A quasi-phantom subcategory is an admissible subcategory with trivial Hochschild homology and a finite Grothendieck group. A phantom subcategory is an admissible subcategory with trivial Hochschild homology and a trivial Grothendieck group.

Some quasi-phantom subcategories are constructed in [3, 1, 15] as semiorthogonal complements to exceptional collections of maximum possible length on certain surfaces of general type for which $q = p_g = 0$. Moreover, the Grothendieck group of a quasiphantom is isomorphic to the torsion part of the Picard group of a corresponding surface. It is natural to ask whether there exists a phantom as a semiorthogonal complement to an exceptional collection of maximum length on a simply connected surface of general type with $q = p_g = 0$ like a Barlow surface. Böhning, H-Ch. Graf von Bothmer, L. Katzarkov, and P. Sosna achieved it in [2] by showing that in a small neighbourhood of the surface constructed by Barlow in the moduli space of determinantal Barlow surfaces, the generic surface has a semiorthogonal decomposition of its derived category into a length 11 exceptional sequence of line bundles and a category with trivial Grothendieck group and Hochschild homology.

Moreover, geometric phantom categories are constructed by S. Gorchinskiy and D. Orlov in [16] by considering admissible subcategories generated by the tensor product of two quasi-phantoms for which orders of their (Grothendieck) K-theory groups are coprime. They also show that these phantom categories have trivial K-motives and, hence, all their higher K-groups are trivial too. This result has implications for the structure of the Chow motive of a variety admitting a phantom category under certain assumptions on the semi-orthogonal decomposition [29].

However, it was showed in [27] that there are no quasi-phantoms, phantoms or universal phantoms in the derived category of smooth projective curves over a field k. Furthermore, it is impossible to build a phantom as a semiorthogonal complement to an exceptional collection of line bundles of maximum length in the derived category of a Fano toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$ if Conjecture 1.1 is confirmed. This is shown by the main result in Chapter 3 in the case of Picard rank less or equal to two.

We consider generalized Hirzebruch surfaces $\mathbb{F}_{\alpha,n}$ which are not Fano. We classify the (strong) exceptional collections of line bundles of maximum length to finitely many classes and give criterion for when the exceptional collection is strong, which generalize the results given by the toric systems of Hirzebruch surfaces in [19]. We show that any exceptional collection of line bundles can be extended to exceptional collection of line bundles of maximum length $2(\alpha + 1)$ (Theorem 4.15). Any SEC of line bundles with two vertical lines can be extended to a SEC of line bundles of maximum length with two vertical lines. When there exist strong exceptional collections (SEC) of line bundles on $\mathbb{F}_{\alpha,n}$ of maximum length with three vertical lines, then any SEC of line bundles with three vertical lines can be extended to a SEC of line bundles of maximum length (Theorem 4.33 and Theorem 4.35). However, when there exist no SEC of line bundles on $F_{\alpha,n}$ of maximum length with three vertical lines, then any SEC of line bundles with three vertical lines can be extended to a SEC of line bundles of length $3\alpha+3-n$ (Theorem 4.33). We also obtain that any exceptional collection of line bundles of maximum length generates the bounded derived category of coherent sheaves on $\mathbb{F}_{\alpha,n}$.

Since the requirement $\mathrm{Ext}^i(\mathcal{L}_1, \mathcal{L}_2) = 0$ in the definition of exceptional collections of line bundles translates into $\mathrm{H}^i(\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}) = 0$, it is natural to study line bundles with trivial cohomology spaces which are defined to be H−trivial line bundles in [32] and are discussed in [32, 7]. We use the forbidden sets defined in [6] to get all H-trivial line bundles (Definition 2.9) and acyclic line bundles (Definition 2.10). The key methods in the proofs is using H-trivial line bundles to analyze when a collection of line bundles is an exceptional collection, and using acyclic line bundles to determine when an exceptional collection of line bundles is a strong exceptional collection.

We now discuss the content of each chapter briefly.

The thesis is organized as follows. Chapter 2 introduces the concepts and definitions which will be used through the thesis. Section 2.1 recalls basic knowledge of toric DM stacks and (strong) exceptional collections of line bundles on $\mathbb{P}_{\boldsymbol{\Sigma}}$. In Section 2.2, we remind the reader how to calculate the cohomology of a line bundle $\mathcal{L}$ on $\mathbb{P}_{\boldsymbol{\Sigma}}$. In Section 3.1, we prove Conjecture 1.1 for the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\boldsymbol{\Sigma}})) = 1$. In Section 3.2, Conjecture 1.1 for the case of the rank of $\mathrm{Pic}(\mathbb{P}_{\boldsymbol{\Sigma}})$ equals two is settled. Section 3.3 contains brief discussion of further directions. Chapter 4 studies exceptional collections and strong exceptional collections of line bundles on certain generalized Hirzebruch surfaces (GHS). Section 4.1 contains the definition of generalized Hirzebruch surfaces and in it we calculate the rank of various K-theory groups. Section 4.2 gives a classification of all exceptional collection of line bundles on Generalized Hirzebruch Surfaces. In Section 4.3, we show that any exceptional collection of line bundles on generalized Hirzebruch surfaces can be extended to an exceptional collection of line bundles of maximum length. Section 4.4 determines when the exceptional collections of line bundles are strong exceptional collections. We conclude that our results match the known results for Hirzebruch surfaces in Section 4.5. In Section 4.6, we consider extending the strong exceptional collections of line bundles on generalized Hirzebruch surfaces. In Section 4.7, we show that any exceptional collection of line bundles on a generalized Hirzebruch surface of maximum length generates the bounded derived category of coherent sheaves on the surface.

Chapter 2

Background

In this chapter, we introduce the concepts and definitions which will be used through the thesis.

2.1 (Strong) exceptional collections of line bundles on toric DM stacks

In this section, we give an overview of toric Deligne-Mumford stacks $\mathbb{P}_{\mathbf{\Sigma}}$, the corresponding Grothendieck group and (strong) exceptional collections of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$.

Let Σ be a complete fan with m one-dimensional cones in a lattice N which is a free abelian group of finite rank. The assumption that N has no torsion allows us to refrain from the technicalities of the derived Gale duality of [4]. We pick a lattice point v in each of the one-dimensional cones of Σ and get a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^m)$, see [4]. The toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$ associated to the stacky fan $\mathbf{\Sigma}$ is constructed in [4] as a stack version of the homogeneous coordinate ring construction of a toric variety [10]. Line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ are described in [5, 6] similar to the scheme case of [11, 14].

Proposition 2.1. *The Picard group of $\mathbb{P}_{\mathbf{\Sigma}}$ is generated by $\{E_i\}_{i=1}^m$ with relations $\sum_{i=1}^m (w_i \cdot v_i) E_i$ for all w in the character lattice $M = N^*$.*

Proof. See [6]. □

Definition 2.2. *An object F in $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$ is exceptional if* $\mathrm{Hom}(F, F) = \mathbb{C}$ *and* $\mathrm{Ext}^t(F, F) = \mathrm{Hom}(F, F[t]) = 0$ *for $t \neq 0$. A sequence of exceptional objects*

$$(F_1, F_2, \ldots, F_n)$$

in $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$ *is called an exceptional collection if*

$$\mathrm{Ext}^t(F_i, F_j) = \mathrm{Hom}(F_i, F_j[t]) = 0$$

for all $i > j$ *and all* $t \in \mathbb{Z}$*. An exceptional collection is further called a strong exceptional collection if*

$$\mathrm{Ext}^t(F_i, F_j) = 0$$

for all $i < j$ *and all* $t \in \mathbb{Z} \setminus \{0\}$*.*

Remark 1. *A subset* $\mathcal{T}$ *of* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ *can be indexed to form a strong exceptional collection if and only if* $\mathrm{Ext}^t(\mathcal{L}_1, \mathcal{L}_2) = 0$ *for any* $\{\mathcal{L}_1, \mathcal{L}_2\} \in \mathcal{T}$ *and any* $t > 0$*. The reason is that the existence of nonzero* $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)$ *induces a partial order on the set* $\mathcal{T}$ *which can be extended to a linear order.*

Definition 2.3. *[6] Let* $\mathcal{T}$ *be a finite set of line bundles on* $\mathbb{P}_{\mathbf{\Sigma}}$ *(which are always exceptional objects on* $\mathbb{P}_{\mathbf{\Sigma}}$*). We call* $\mathcal{T}$ *a full strong exceptional collection if*

$$\mathrm{Ext}^t(\mathcal{L}_1, \mathcal{L}_2)$$

for any $\{\mathcal{L}_1, \mathcal{L}_2\} \in \mathcal{T}$ *and any* $t > 0$ *and the derived category of* $\mathbb{P}_{\mathbf{\Sigma}}$ *is generated by the line bundles in* $\mathcal{T}$*.*

Definition 2.4. *A toric DM stack* $\mathbb{P}_{\mathbf{\Sigma}}$ *is called Fano if the chosen points* v_i *are precisely the vertices of a simplicial convex polytope in* $N_{\mathbb{R}}$*.*

Definition 2.5. *[5] Let* $\mathbb{P}_{\mathbf{\Sigma}}$ *be a smooth DM stack. The (Grothendieck) K-theory group* $K_0(\mathbb{P}_{\mathbf{\Sigma}})$ *is defined to be the quotient of the free abelian group generated by coherent sheaves* $\mathcal{F}$ *on* $\mathbb{P}_{\mathbf{\Sigma}}$ *by the relations* $[\mathcal{F}_1] - [\mathcal{F}_2] + [\mathcal{F}_3]$ *for all exact sequences* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$*.*

Lemma 2.6. *[16] Let* $\mathbb{P}_{\mathbf{\Sigma}}$ *be a Fano toric DM stack and* $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n)$ *be an exceptional collection of objects in* $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$*. If* $n = \mathrm{rank} K_0(\mathbb{P}_{\mathbf{\Sigma}})$*, then* $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ *is a basis of* $K_0(\mathbb{P}_{\mathbf{\Sigma}})$*.*

Corollary 2.7. *Let* $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n)$ *be an exceptional collection of objects in* $\mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$*. Then* $n \leq \mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$*.*

2.2 Cohomology of line bundles on toric DM stacks

Now we remind the reader how to calculate the cohomology of a line bundle $\mathcal{L}$ on $\mathbb{P}_{\mathbf{\Sigma}}$. For each $\mathbf{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$, we define $Supp(\mathbf{r})$ to be the simplicial complex on n vertices $\{1, \ldots, n\}$ as follows

$$\begin{aligned} Supp(\mathbf{r}) = &\{J \subseteq \{1, \ldots, n\} | r_i \geq 0 \text{ for all } i \in J \\ &\text{and there exists a cone of } \Sigma \text{ containing all } v_i, i \in J\}. \end{aligned}$$

The following proposition gives a description of the cohomology of a linear bundle $\mathcal{L}$ on $\mathbb{P}_{\mathbf{\Sigma}}$.

Proposition 2.8. *[6] Let $\mathcal{L} \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. Then*

$$\mathrm{H}^j(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L}) = \bigoplus \mathrm{H}^{red}_{rkN-j-1}(Supp(\mathbf{r})),$$

where the sum is over all $\mathbf{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$ such that $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$.

Proof. See [6]. □

Remark 2. *We have the cohomology $\mathrm{H}^0(\mathcal{L}) \neq 0$ if and only if there exists $\mathbf{r} \in \mathbb{Z}^n_{\geq 0}$ such that $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$. Another extreme case is that $\mathrm{H}^{rk(N)}(\mathcal{L})$ only appears when the simplicial complex $Supp(\mathbf{r}) = \{\emptyset\}$, i.e. when $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$ with all $r_i \leq -1$.*

Remark 3. *Let $\mathcal{L} \cong \mathcal{O}(\sum_{i=1}^n a_i E_i)$ be a line bundle in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. Assume there is another expression $\mathcal{L} \cong \mathcal{O}(\sum_{i=1}^n r_i E_i)$. Then by Proposition 2.1, there exists an element $f \in N^*$ such that $r_i = a_i + f(v_i)$ for $i = 1, \ldots, n$, where $f(v_i) = (f.v_i)$. Thus the cohomology of $\mathcal{L}$ can also be written as following:*

$$\mathrm{H}^j(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L}) = \bigoplus_{f \in N^*} \mathrm{H}^{red}_{rkN-j-1}(Supp(\mathbf{r}_f)),$$

where $\mathbf{r}_f = (a_i + f(v_i))_{i=1}^n$.

We give the definition of H$-$trivial line bundles and acyclic line bundles as follows.

Definition 2.9. *Let $\mathcal{L}$ be a line bundle in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. We say that $\mathcal{L}$ is* H$-$*trivial iff $\mathrm{H}^j(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L}) = 0$ for all $j \geq 0$.*

Definition 2.10. *Let $\mathcal{L}$ be a line bundle in* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. *We say that $\mathcal{L}$ is acyclic iff* $\mathrm{H}^j(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L}) = 0$ *for all $j > 0$.*

A combinatorial criterion for H$-$triviality is given in terms of forbidden sets introduced below, see [6].

Definition 2.11. *For every subset $I \subseteq \{1, \dots, n\}$, we denote C_I to be the simiplicial complex $Supp(\mathbf{r})$ where $r_i = -1$ for $i \notin I$ and $r_i = 0$ for $i \in I$. Let $\Delta = \{I \subseteq \{1, \dots, n\} | C_I$ has nontrivial reduced homology $\}$. By Remark 2, Δ contains $\{1, \dots, n\}$ and $\emptyset$. For each $I \in \Delta$, the forbidden set associated to I is defined by*

$$FS_I := \{\mathcal{O}(\sum_{i \notin I}(-1 - r_i)E_i + \sum_{i \in I} r_i E_i) | r_i \in \mathbb{Z}_{\geq 0} \text{ for all } i\}.$$

Proposition 2.12. *Let $\mathcal{L}$ be a line bundle on $\mathbb{P}_{\mathbf{\Sigma}}$. Then $\mathcal{L}$ is* H$-$*trivial if and only if $\mathcal{L}$ does not lie in FS_I for any $I \in \Delta$.*

Proof. This follows immediately from Proposition 2.8. □

Proposition 2.13. *Let $\mathcal{L}$ be a line bundle on $\mathbb{P}_{\mathbf{\Sigma}}$. Then $\mathcal{L}$ is acyclic if and only if $\mathcal{L}$ does not lie in FS_I for any $I \in \Delta$ and $I \neq \{1, \dots, n\}$.*

Proof. This follows immediately from Proposition 2.8. □

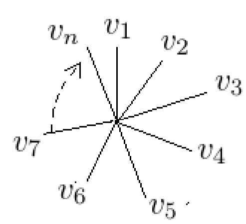

Figure 2.1:

Remark 4. *When the toric DM stacks is of dimension two, i.e. $N = \mathbb{Z}^2$, then we have a complete simplicial fan Σ in $\mathbb{Z}^2$ with n one-dimensional cones and n lattice points $\{v_i\}_{i=1}^n$ chosen in each of the one-dimensional cones of Σ, see Figure 2.1. The maximum cones of Σ are $\mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2, \mathbb{R}_{\geq 0}v_2 + \mathbb{R}_{\geq 0}v_3, \dots, \mathbb{R}_{\geq 0}v_n + \mathbb{R}_{\geq 0}v_1$. In dimension 2 case, we describe $\Delta = \{\emptyset, \{1, \dots, n\}\} \cup \{I \subset \{1, \dots, n\} | C_I$ is disconnected$\}$. For example, we* *have* $\{1,3\} \in \Delta$ *if* $n > 3$, $\{n,2,3\} \in \Delta$ *if* $n > 4$, *but* $\{1,2\} \notin \Delta$, $\{n,1,2\} \notin \Delta$ *for all* $n > 2$.

Chapter 3

Strong exceptional collections of line bundles of maximum length on Fano toric Deligne-Mumford stacks

3.1 The case when the rank of Picard group equals one

In the section, we prove Conjecture 1.1 when the rank of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ is one.

Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a Fano toric DM stack such that $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ has no torsion and rank one. In this case $\mathbb{P}_{\mathbf{\Sigma}}$ is a weighted projective space which we denote by $W\mathbb{P}(w_1, \ldots, w_m)$, where $gcd(w_1, \ldots, w_m) = 1$. [1] The rank of $K_0(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}))$ is $\sum_{i=1}^{m} w_i$. The Picard group $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ is $\{\mathcal{O}(d) | d \in \mathbb{Z}\}$, where $\mathcal{O}(E_i) = \mathcal{O}(w_i)$. By [6], we know that $\mathbb{P}_{\mathbf{\Sigma}}$ possesses a full strong exceptional collection of line bundles.

Proposition 3.1. *[6] Let $\mathcal{T} = \{\mathcal{O}(w) | -\mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}})) + 1 \leq w \leq 0\}$. Then $\mathcal{T}$ forms a full strong exceptional collection in the derived category of $W\mathbb{P}(w_1, \ldots, w_m)$.*

Proof. See [6]. □

From [6], for any $d_1, d_2 \in \mathbb{Z}$, we know that

$$\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(d_1), \mathcal{O}(d_2)) \neq 0 \Leftrightarrow d_2 - d_1 = \sum_{i=1}^{m} a_i w_i, \text{ for some } a_i \in \mathbb{Z}_{<0};$$
$$\mathrm{Hom}(\mathcal{O}(d_1), \mathcal{O}(d_2)) \neq 0 \Leftrightarrow d_2 - d_1 = \sum_{i=1}^{m} a_i w_i, \text{ for some } a_i \in \mathbb{Z}_{\geq 0}.$$

Remark 5. *In the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$, any exceptional collection on $X = \mathbb{P}_{\mathbf{\Sigma}}$ is a strong exceptional collection. Indeed, let*

$$\mathcal{T} = (\mathcal{O}(s_1), \ldots, \mathcal{O}(s_n))$$

[1] This condition comes from our assumption that N has no torsion.

be an exceptional collection on $\mathbb{P}_{\boldsymbol{\Sigma}}$*. We have* $\mathrm{Hom}(\mathcal{O}(s_j),\mathcal{O}(s_i)) = 0$ *for* $j > i$*. Then* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_i),\mathcal{O}(s_j)) = 0$ *for* $j > i$*. Otherwise, we get* $s_j - s_i = \sum_{i=1}^m a_i w_i$*, where* $a_i \in \mathbb{Z}_{<0}$*. This implies* $s_j - s_i = \sum_{i=1}^m b_i w_i$*, where* $b_i = -a_i \in \mathbb{Z}_{\geq 0}$*, which contradicts* $\mathrm{Hom}(\mathcal{O}(s_j),\mathcal{O}(s_i)) = 0$*.*

Main idea. Starting from an exceptional collection $\mathcal{T}$ of line bundles of maximum length, i.e., with $\sum_{i=1}^m w_i$ elements, we construct other exceptional collections of maximum length in $\mathcal{D}(\mathcal{T})$, the subcategory generated by elements in $\mathcal{T}$. Eventually, we will get to the exceptional collection in Proposition 3.1 given in [6]. This allows us to conclude that $\mathcal{D}(\mathcal{T}) = \mathbf{D}^b(coh(\mathbb{P}_{\boldsymbol{\Sigma}}))$.

The main step is to "move" the smallest element of the exceptional collection $\mathcal{T}$ by $\sum_{i=1}^m w_i$, see Figure 3.1.

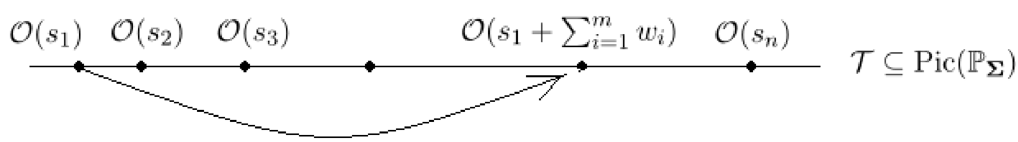

Figure 3.1:

Specifically: If line bundles $\mathcal{O}(s_1),\ldots,\mathcal{O}(s_n)$, where $s_1 < s_2 < \cdots < s_n$, form a strong exceptional collection $\mathcal{T}$ of maximum length, then

1. $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$ is not in the strong exceptional collection $\mathcal{T}$ (Lemma 3.3);

2. By replacing $\mathcal{O}(s_1)$ with $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$ and reordering, we get another strong exceptional collection (Lemma 3.4);

3. $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \in \mathcal{D}(\mathcal{T})$, so the new collection generates a subcategory of $\mathcal{D}(\mathcal{T})$ (Corollary 3.6).

Once we know these that these moves are possible, we can "shrink" the exceptional collection to make it one from Propostion 3.1 (Theorem 3.7).

Example 3.2. *We consider an exceptional collection on* $W\mathbb{P}(5,6)$

$$(\mathcal{O}(-15),\mathcal{O}(-13),\mathcal{O}(-10),\mathcal{O}(-9),\mathcal{O}(-8),\mathcal{O}(-7),\mathcal{O}(-6),\mathcal{O}(-5),\mathcal{O}(-3),\mathcal{O}(-1),\mathcal{O})$$

of maximum length 11*. We replace* $\mathcal{O}(-15)$ *by* $\mathcal{O}(-15+11)=\mathcal{O}(-4)$ *to get another strong exceptional collection*

$$(\mathcal{O}(-13),\mathcal{O}(-10),\mathcal{O}(-9),\mathcal{O}(-8),\mathcal{O}(-7),\mathcal{O}(-6),\mathcal{O}(-5),\mathcal{O}(-4),\mathcal{O}(-3),\mathcal{O}(-1),\mathcal{O}).$$

Then we replace $\mathcal{O}(-13)$ *by* $\mathcal{O}(-13+11)=\mathcal{O}(-2)$ *to get*

$$(\mathcal{O}(-10),\mathcal{O}(-9),\mathcal{O}(-8),\mathcal{O}(-7),\mathcal{O}(-6),\mathcal{O}(-5),\mathcal{O}(-4),\mathcal{O}(-3),\mathcal{O}(-2),\mathcal{O}(-1),\mathcal{O})$$

which is a full strong exceptional collection in Proposition 3.1 given in [6].

Lemma 3.3. *Let* $\mathcal{T}=\{\mathcal{O}(s_1),\dots,\mathcal{O}(s_n)\}$ *be a strong exceptional collection. Then* $\mathcal{O}(s_1+\sum_{i=1}^m w_i)\notin\mathcal{T}$*.*

Proof. If $\mathcal{O}(s_1+\sum_{i=1}^m w_i)\in\mathcal{T}$, then $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_1+\sum_{i=1}^m w_i),\mathcal{O}(s_1))\neq 0$ since $s_1-(s_1+\sum_{i=1}^m w_i)=-\sum_{i=1}^n w_i$. This contradicts the assumption that $\mathcal{T}$ is a strong exceptional collection. □

Lemma 3.4. *Let* $\mathcal{T}=\{\mathcal{O}(s_1),\dots,\mathcal{O}(s_n)\}$ *be a strong exceptional collection of maximum length on* $\mathbb{P}_{\mathbf{\Sigma}}$*, where* $s_1<s_2<\cdots<s_n$*. By replacing* $\mathcal{O}(s_1)$ *with* $\mathcal{O}(s_1+\sum_{i=1}^m w_i)$ *and reordering, we get another strong exceptional collection.*

Proof. Let $\mathcal{T}^1$ be a collection obtained by replacing $\mathcal{O}(s_1)$ with $\mathcal{O}(s_1+\sum_{i=1}^m w_i)$. For any $i\in\{2,\dots,n\}$, we have $s_i-s_1-\sum_{i=1}^m w_i>-\sum_{i=1}^m w_i$. Thus $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_1+\sum_{i=1}^m w_i),\mathcal{O}(s_i))=0$. Also for any $i\in\{2,\dots,n\}$, we have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_i),\mathcal{O}(s_1+\sum_{i=1}^m w_i))=0$. Otherwise, we get $s_1+\sum_{i=1}^m w_i-s_i=\sum_{i=1}^m a_iw_i$, where $a_i\leq -1$. Thus $s_1-s_i=\sum_{i=1}^m b_iw_i$, where $b_i<-1$. This implies $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_i),\mathcal{O}(s_1))\neq 0$, which contradicts the assumption that $\mathcal{T}$ is an exceptional collection. □

Lemma 3.5. *Let* $\mathcal{T}=\{\mathcal{O}(s_1),\dots,\mathcal{O}(s_n)\}$ *be a strong exceptional collection of maximum length on* $\mathbb{P}_{\mathbf{\Sigma}}$*, where* $s_1<s_2<\cdots<s_n$*. Then* $\mathcal{O}(s_1+\sum_{j\in J}w_j)$ *is in* $\mathcal{T}$ *for any proper subset* $J\subsetneqq\{1,2,\dots,m\}$*.*

Proof. Let $s=s_1+\sum_{j\in J}w_j$. We have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s),\mathcal{O}(s_k))=0$ for all $k\in\{1,2,\dots,n\}$. Otherwise, we have $s_k-s\in\sum_{i=1}^m\mathbb{Z}_{<0}w_m$ for some k. However, we have $s_k-s_1\geq 0$. So $s_k-s=s_k-s_1-\sum_{j\in J}w_j>-\sum_{j=1}^m w_j$, which leads to contradiction.

We have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_k), \mathcal{O}(s)) = 0$ for all $k \in \{1, 2, \dots, n\}$. Otherwise, we get $s_1 + \sum_{j\in J} w_j - s_k = s - s_k = \sum_{i=1}^m a_i w_i$ for some k, where $a_i \leq -1$. Thus $s_1 - s_k = \sum_{i=1}^m b_i w_i$, where $b_i \leq -1$. Therefore $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(s_k), \mathcal{O}(s_1)) \neq 0$, which contradicts that $\mathcal{T}$ is an exceptional collection.

If $\mathcal{O}(s)$ is not in $\mathcal{T}$, we can get another exceptional collection with $\sum_{i=1}^m w_i + 1$ elements by inserting $\mathcal{O}(s)$ into $\mathcal{T}$. This is impossible by Corollary 2.7. □

Corollary 3.6. *Let $\mathcal{T} = \{\mathcal{O}(s_1), \dots, \mathcal{O}(s_n)\}$ be a strong exceptional collection of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$, where $s_1 < s_2 < \cdots < s_n$. Then we have $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \in \mathcal{D}(\mathcal{T})$.*

Proof. We consider the Koszul complex [6]

$$0 \to \mathcal{O}(-\sum_{i=1}^m w_i) \to \cdots \to \bigoplus_{i=1}^m \mathcal{O}(-w_i) \to \mathcal{O} \to 0.$$

Then we tensor this complex by $\mathcal{O}(s_1 + \sum_{i=1}^m w_i)$ and get

$$0 \to \mathcal{O}(s_1) \to \cdots \to \bigoplus_{i=1}^m \mathcal{O}(-\sum_{j\neq i} w_j + s_1) \to \mathcal{O}(s_1 + \sum_{i=1}^m w_i) \to 0.$$

By Lemma 3.5, we have that $\mathcal{O}(s_1 + \sum_{j\in J} w_j)$ is in $\mathcal{T}$ for any proper subset $J \subsetneq \{1, 2, \dots, m\}$. Thus $\mathcal{O}(s_1 + \sum_{i=1}^m w_i) \in \mathcal{D}(\mathcal{T})$. □

Theorem 3.7. *Let $X = \mathbb{P}_{\mathbf{\Sigma}}$ be a Fano toric DM stack with $\mathrm{rank}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$. Assume $\mathcal{T} = \{\mathcal{O}(s_1), \dots, \mathcal{O}(s_n)\}$ is a strong exceptional collection of maximum length. Then $\mathcal{T}$ is a full strong exceptional collection.*

Proof. Without loss of generality, we assume $s_1 < s_2 < \cdots < s_n$. If $s_1 + \sum_{i=1}^m w_i \geq s_n$, then $\sum_{i=1}^m w_i \geq s_n - s_1$. Then $(s_1, \dots, s_n) = (s_1, s_1 + 1, \dots, s_1 + \sum_{i=1}^m w_i)$. So $\mathcal{T}$ is a twist of the collection of [6] and is therefore full. If $s_1 + \sum_{i=1}^m w_i < s_n$, we get a new strong exceptional collection

$$\mathcal{T}^1 = \{\mathcal{O}(s_2), \dots, \mathcal{O}(s_1 + \sum_{i=1}^m w_i), \dots, \mathcal{O}(s_n)\}$$

in $\mathcal{D}(\mathcal{T})$ by Lemma 3.4 and Corollary 3.6.

This process decreases $s_n - s_1$ and therefore terminates. So eventually we will be in the situation $s_1 + \sum_{i=1}^m w_i \geq s_n$. □

Remark 6. *When* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ *has torsion, the arguments go without significant changes. The details are left to the reader.*

3.2 The case when the rank of Picard group equals two

In this section, we consider Fano toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$ associated to a stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^m)$ in the lattice N with $\mathrm{rk}(N) = m - 2$. In this case, the rank of Picard group $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}))$ equals 2. Our aim is to prove Conjecture 1.1 in this case. We first assume that $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ has no torsion for ease of exposition.

We recall the results of [6].

Proposition 3.8. *[6] There exists a unique up to scaling collection of rational numbers* α_i *such that* $\sum_{i=1}^m \alpha_i = 0$ *and* $\sum_{i=1}^m \alpha_i v_i = 0$*. Moreover, all* α_i *in this relation are nonzero.*

Proof. See [6]. □

We pick one such relation $\sum_{i=1}^m \alpha_i v_i = 0$. Let $I_+ = \{i|\alpha_i > 0\}$ and $I_- = \{i|\alpha_i > 0\}$. Then we have $\{1, \ldots, m\} = I_+ \sqcup I_-$. Let $E_+ = \sum_{i \in I_+}(E_i)$ and $E_- = \sum_{i \in I_-}(E_i)$. We consider a linear function α on $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ with $\alpha(E_i) = \alpha_i$ from Proposition 3.8. Then $\alpha(E_+) + \alpha(E_-) = 0$.

Moreover, from [6], we can pick and fix a collection of positive numbers r_i, $i = 1, \ldots, m$ such that $\sum_i r_i = 1$ and $\sum_i r_i v_i = 0$. This collection of positive numbers gives a linear function f on $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ with $f(E_i) = r_i > 0$.

Let P be a parallelogram in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ given by

$$|f(x)| \leq \frac{1}{2}, \quad |\alpha(x)| \leq \frac{1}{2} \sum_{i \in I_+} \alpha_i.$$

Pick a generic point $p \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ so that the lines along the sides of the parallelogram $p + P$ do not contain any points from $\mathrm{Pic}_{\mathbb{Q}}(\mathbb{P}_{\mathbf{\Sigma}})$. Then we have the following.

Proposition 3.9. *[6] The set* S *of line bundles in* $p + P$ *forms a full strong exceptional collection on* $\mathbb{P}_{\mathbf{\Sigma}}$*.*

Proof. See [6]. □

Notation: The following notation will be used in our arguments and proofs. Let $\mathcal{T} = \{\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)\}$ be a collection of line bundles, we will abuse the notation slightly and denote by $\max(\alpha(\mathcal{T}))$ the maximum value of $\alpha(D_i)$ for $O(D_i)$ in $\mathcal{T}$ (and similarly, for min and f). We denote $\mathcal{T}_{\min(f)} = \{D_i \in \mathcal{T} | f(D_i) = \min(f(\mathcal{T}))\}$.

Main idea. The idea of the proof is similar to the case $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$. Starting from an exceptional collection $\mathcal{T}$ of line bundles of maximum length, we construct other exceptional collections of maximum length in $\mathcal{D}(\mathcal{T})$, the subcategory generated by elements in $\mathcal{T}$. Eventually, we get to the exceptional collection in Proposition 3.9.

Step 1. The first step is to "move" the largest elements in terms of the linear function α in the strong exceptional collection by $-E_+$ or E_- to construct a new strong exceptional collection in $\mathcal{D}(\mathcal{T})$, see Figure 3.2.

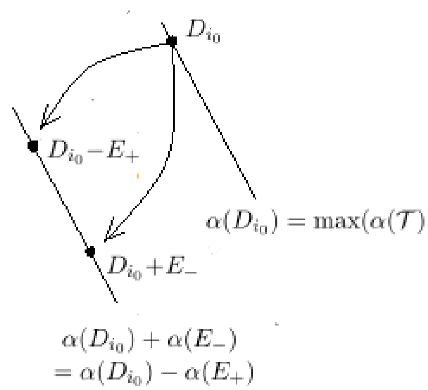

Figure 3.2:

Specifically: let $\mathcal{T} = (\mathcal{O}(D_1), \dots, \mathcal{O}(D_n))$ be a strong exceptional collection of line bundles of maximum length. We pick $i_0 \in \{1, \dots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Then

1. Both $\mathcal{O}(D_{i_0} - E_+)$ and $\mathcal{O}(D_{i_0} + E_-)$ are not in the strong exceptional collection $\mathcal{T}$ (Lemma 3.10);

2. Either replacing $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or with $\mathcal{O}(D_{i_0} + E_-)$, we get another strong exceptional collection after reordering (Lemma 3.13, Lemma 3.16 and

Lemma 3.17);

3. The new exceptional collection in (2) is in $\mathcal{D}(\mathcal{T})$ (Lemma 3.14 and Lemma 3.15).

By repeating the above step (Theorem 3.18), we can reduce the problem to the strong exceptional collection $\mathcal{S}$ in $\mathcal{D}(\mathcal{T})$ such that all the line bundles in $\mathcal{S}$ are within a strip of width less than $\alpha(E_+)$, i.e., $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+) = \alpha(-E_-)$.

Step 2. From now on, we consider a strong exceptional collection

$$\mathcal{T} = (\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n))$$

of maximum length within a strip of width less than $\alpha(E_+)$. If $\max(f(\mathcal{T}))-\min(f(\mathcal{T})) < f(E_+ + E_-) = 1$, then $\mathcal{T}$ is a full strong exceptional collection in Proposition 3.9. This allows us to conclude that $\mathcal{D}(\mathcal{T}) = \mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$.

Now, we assume $\max(f(\mathcal{T}))-\min(f(\mathcal{T})) \geq f(E_+ + E_-) = 1$. We pick $j_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{j_0}) = \max(\alpha(\mathcal{T}))$. Then we can replace $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$ to get another strong exceptional collection $\mathcal{T}'$ such that (Proposition 3.19):

1. $\max(f(\mathcal{T}')) \leq \max(f(\mathcal{T}))$;

2. $\min(f(\mathcal{T}')) \geq \min(f(\mathcal{T}))$;

3. $\sharp(\mathcal{T}'_{\min(f)}) \leq \sharp(\mathcal{T}_{\min(f)})$ if $\min(f(\mathcal{T}')) = \min(f(\mathcal{T}))$;

4. $\sharp(\{D_i \in \mathcal{T}' | f(D_i) = \min(f(\mathcal{T}))\}) < \sharp(\mathcal{T}_{\min(f)})$ if $f(D_{i_0}) = \min(f(\mathcal{T}))$.

By repeating the above step (Theorem 3.20), we get a new strong exceptional collection $\mathcal{S}$ such that $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+) = \alpha(-E_-)$ and $\max(f(\mathcal{S})) - \min(f(\mathcal{S})) < f(E_+ + E_-) = 1$ which is one in Proposition 3.9. This allows us to conclude that $\mathcal{D}(\mathcal{T}) = \mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$.

Details of proof. For a divisor class D in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$, we write $D = \sum_{i\in I}(\geq 0)E_i$ if D can be written as $D = \sum_{i\in I} a_i E_i$ with $a_i \in \mathbb{Z}_{\geq 0}$ for all i in a subset $I \subseteq \{1, \ldots, m\}$. We use similar notation for other inequalities.

The nonzero Ext groups between line bundles have been calculated in [6]. We denote by Ext^+, Ext^- the groups associated to sets I_+, I_-. Specifically, for any $D_1, D_2 \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$, we have

$$\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 = \sum_{i\in\{1,\dots,m\}} (<0)E_i;$$

$$\mathrm{Ext}^{+}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 = \sum_{i\in I_-} (<0)E_i + \sum_{i\in I_+} (\geq 0)E_i;$$

$$\mathrm{Ext}^{-}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 = \sum_{i\in I_+} (<0)E_i + \sum_{i\in I_-} (\geq 0)E_i;$$

$$\mathrm{Hom}(\mathcal{O}(D_1), \mathcal{O}(D_2)) \neq 0 \Leftrightarrow D_2 - D_1 = \sum_{i\in\{1,\dots,m\}} (\geq 0)E_i.$$

Lemma 3.10. *Let $\mathcal{T} = (\mathcal{O}(D_1), \dots, \mathcal{O}(D_n))$ be a strong exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. If $i_0 \in \{1, \dots, n\}$, then both $\mathcal{O}(D_{i_0} - E_+)$ and $\mathcal{O}(D_{i_0} + E_-)$ are not in $\mathcal{T}$.*

Proof. If $\mathcal{O}(D_{i_0} - E_+) \in \mathcal{T}$, we have $\mathrm{Ext}^-(\mathcal{O}(D_{i_0}), \mathcal{O}(D_{i_0} - E_+)) \neq 0$ since $D_{i_0} - E_+ - D_{i_0} = -E_+$. If $\mathcal{O}(D_{i_0} + E_-) \notin \mathcal{T}$, we have $\mathrm{Ext}^+(\mathcal{O}(D_{i_0} + E_-), \mathcal{O}(D_{i_0}) \neq 0$ since $D_{i_0} - D_{i_0} - E_- = -E_-$. These contradict that $\mathcal{T}$ is a strong exceptional collection. $\square$

For any subset $I \subseteq \{1, \dots, m\}$, we denote $E_I = \sum_{i\in I} E_i$.

Lemma 3.11. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. We pick $i_0 \in \{1, \dots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Then for any proper subset J of I_+ and any $k \in \{1, \dots, n\}$, we have*

$$\mathrm{Ext}^*(\mathcal{O}(D_{i_0} - E_J), \mathcal{O}(D_k)) = 0, \ \textit{where } * = \mathrm{rk}(N), +, -;$$

$$\mathrm{Ext}^*(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0, \ \textit{where } * = +, -.$$

Proof. (1) We have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0} - E_J), \mathcal{O}(D_k)) = 0$. Otherwise, we get $D_k - D_{i_0} + E_J = \sum_{i\in\{1,\dots,m\}}(<0)E_i$. Thus $D_k - D_{i_0} = \sum_{i\in\{1,\dots,m\}}(<0)E_i - E_J = \sum_{i\in\{1,\dots,m\}}(<0)E_i$. This implies $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0}), \mathcal{O}(D_k)) \neq 0$ which contradicts the assumption that $\mathcal{T}$ is a strong exceptional collection.

(2) We have $\mathrm{Ext}^+(\mathcal{O}(D_{i_0} - E_J), \mathcal{O}(D_k)) = 0$. Otherwise, we get $D_k - D_{i_0} + E_J = \sum_{i\in I_-}(<0)E_i + \sum_{i\in I_+}(\geq 0)E_i$. So we get $D_k - D_{i_0} = -E_- - E_J + \sum_{i\in I_-}(\leq 0)E_i +$

$\sum_{i\in I_+}(\geq 0)E_i$. We have $\alpha(-E_-) = \alpha(E_+) > \alpha(E_J)$ since $J \subsetneq I_+$. Also, $\alpha(\sum_{i\in I_-}(\leq 0)E_i) \geq 0$ and $\alpha(\sum_{i\in I_+}(\geq 0)E_i) \geq 0$. Thus $\alpha(D_k - D_{i_0}) > 0$ which contradicts the assumption that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$.

(3) We have $\mathrm{Ext}^-(\mathcal{O}(D_{i_0} - E_J), \mathcal{O}(D_k)) = 0$. Otherwise, we have $D_k - D_{i_0} + E_J = \sum_{i\in I_+}(<0)E_i + \sum_{i\in I_-}(\geq 0)E_i$. Thus $D_k - D_{i_0} = \sum_{i\in I_+}(<0)E_i - E_J + \sum_{i\in I_-}(\geq 0)E_i = \sum_{i\in I_+}(<0)E_i + \sum_{i\in I_-}(\geq 0)E_i$. This implies $\mathrm{Ext}^-(\mathcal{O}(D_{i_0}), \mathcal{O}(D_k)) \neq 0$, contradiction.

(4) We have $\mathrm{Ext}^+(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0$. Otherwise, we have $D_{i_0} - E_J - D_k = \sum_{i\in I_+}(\geq 0)E_i + \sum_{i\in I_-}(<0)E_i$. Thus $D_{i_0} - D_k = \sum_{i\in I_+}(\geq 0)E_i - \sum_{i\in I_-}(<0)E_i + E_J = \sum_{i\in I_+}(\geq 0)E_i - \sum_{i\in I_-}(<0)E_i$. This implies $\mathrm{Ext}^+(\mathcal{O}(D_k), \mathcal{O}(D_{i_0})) \neq 0$, contradiction.

(5) We have $\mathrm{Ext}^-(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0$. Otherwise, we have $D_{i_0} - E_J - D_k = \sum_{i\in I_+}(<0)E_i + \sum_{i\in I_-}(\geq 0)E_i$. Thus $D_{i_0} - D_k = \sum_{i\in I_+}(<0)E_i + E_J + \sum_{i\in I_-}(\geq 0)E_i$. We get $\alpha(\sum_{i\in I_+}(<0)E_i) = \sum_{i\in I_+}(<0)\alpha_i \leq \sum_{i\in I_+}(-1)\alpha_i < \sum_{i\in J}(-1)\alpha_i = \alpha(-E_J)$ since $J \subsetneq I_+$. So $\alpha(\sum_{i\in I_+}(<0)E_i + E_J) < 0$. Also, $\alpha(\sum_{i\in I_-}(\geq 0)E_i) \leq 0$. This implies $\alpha(D_{i_0} - D_k) < 0$ which contradicts the assumption that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. □

Lemma 3.12. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Then for any proper subset L of I_- and any $j \in \{1, \ldots, n\}$, we have*

$$\mathrm{Ext}^*(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) = 0, \text{ where } * = +, -;$$

$$\mathrm{Ext}^*(\mathcal{O}(D_j), \mathcal{O}(D_{i_0} + E_L)) = 0, \text{ where } * = \mathrm{rk}(N), +, -;$$

Proof. The proof is analogous to the proof of Lemma 3.11 and is left to the reader. □

Note that Lemmas 3.11, 3.12 only cover vanishing of five out of possible six $\mathrm{Ext}^{>0}$ spaces. The next Lemma addresses the remaining space.

Lemma 3.13. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$. We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Then either $\mathrm{Ext}^{\mathrm{rk}(N)}(D_k, D_{i_0} - E_J) = 0$ for all $k \in \{1, \ldots, n\}$ and all $J \subseteq I_+$ or $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0} + E_L, D_j) = 0$ for all $j \in \{1, \ldots, n\}$ and all $L \subseteq I_-$, or both.*

Proof. If $\text{Ext}^{\text{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) \neq 0$ for some k and some $J \subseteq I_+$, then

$$D_{i_0} - D_k - E_J = \sum (< 0) E_i = -E_- + \sum_{I^-} (\leq 0) E_i + \sum_{I^+} (< 0) E_i.$$

If $\text{Ext}^{\text{rk}(N)}(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) \neq 0$ for some j and some $L \subseteq I_-$, then

$$D_j - D_{i_0} - E_L = \sum (< 0) E_i = -E_+ + \sum_{I^+} (\leq 0) E_i + \sum_{I^-} (< 0) E_i.$$

We add the two equations to get

$$D_j - D_k - E_J - E_L = -E_+ - E_- + \sum_{I^+} (< 0) E_i + \sum_{I^-} (< 0) E_i.$$

Thus

$$\begin{aligned} D_j - D_k &= (-E_+ + E_J) + (-E_- + E_L) + \sum_{I^+} (< 0) E_i + \sum_{I^-} (< 0) E_i \\ &= \sum_{I^+} (< 0) E_i + \sum_{I^-} (< 0) E_i \end{aligned}$$

since $J \subseteq I_+$ and $L \subseteq I_-$. This implies $\text{Ext}^{\text{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_j)) \neq 0$ which contradicts the assumption that $\mathcal{T}$ is a strong exceptional collection. $\square$

Lemma 3.14. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$. We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Assume $\text{Ext}^{\text{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0$ for all $k \in \{1, \ldots, n\}$ and all proper subsets $J \subsetneq I_+$. Then $\mathcal{O}(D_{i_0} - E_+) \in \mathcal{D}(\mathcal{T})$.*

Proof. We have $\mathcal{O}(D_{i_0} - E_J) \in \mathcal{T}$ for all $J \subsetneq I_+$. Otherwise, there is $J \subsetneq I_+$ such that $\mathcal{O}(D_{i_0} - E_J) \notin \mathcal{T}$. By Lemma 3.11, we can add $\mathcal{O}(D_{i_0} - E_J)$ to $\mathcal{T}$ to get a strong exceptional collection with more than $\text{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$ elements. This is impossible by Corollary 2.7.

Now we consider the Koszul complex

$$0 \to \mathcal{O}(-E_+) \to \cdots \to \bigoplus_{i \in I_+} \mathcal{O}(-E_i) \to \mathcal{O} \to 0.$$

We tensor the complex by $\mathcal{O}(D_{i_0})$ to get

$$0 \to \mathcal{O}(D_{i_0} - E_+) \to \cdots \to \bigoplus_{i \in I_+} \mathcal{O}(-E_i + D_{i_0}) \to \mathcal{O}(D_{i_0}) \to 0.$$

Since $\mathcal{O}(D_{i_0} - E_J) \in \mathcal{T}$ for all $J \subsetneq I_+$, we get $\mathcal{O}(D_{i_0} - E_+) \in \mathcal{D}(\mathcal{T})$. $\square$

Lemma 3.15. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$. We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Assume* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) = 0$ *for any $j \in \{1, \ldots, n\}$ for any subset $L \subsetneq I_-$. Then $\mathcal{O}(D_{i_0} + E_-) \in \mathcal{D}(\mathcal{T})$.*

Proof. Analogous to Lemma 3.14. □

Lemma 3.16. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$. We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Assume* $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_J)) = 0$ *for any $k \in \{1, \ldots, n\}$ and any subset $J \subseteq I_+$. Then we can get a new strong exceptional collection by replacing $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ and reordering.*

Proof. We will carefully check vanishing of all six $\mathrm{Ext}^{>0}$ spaces with the new element of the collection.

(1) We have $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0} - E_+), \mathcal{O}(D_k)) = 0$ by the same argument as in (1) of Lemma 3.11.

(2) We have $\mathrm{Ext}^{+}(\mathcal{O}(D_{i_0} - E_+), \mathcal{O}(D_k)) = 0$. Otherwise, we get $D_k - D_{i_0} + E_+ = \sum_{i\in I_-}(<0)E_i + \sum_{i\in I_+}(\geq 0)E_i$. So $D_k - D_{i_0} = -E_- - E_+ + \sum_{i\in I_-}(\leq 0)E_i + \sum_{i\in I_+}(\geq 0)E_i$. We have $\alpha(-E_- - E_+) = 0$. Also, the coefficients in $\sum_{i\in I_-}(\leq 0)E_i + \sum_{i\in I_+}(\geq 0)E_i$ cannot be all zero. Otherwise, we have $D_k - D_{i_0} = -E_- - E_+$. This implies $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0}, D_k) \neq 0$ which contradicts that $\mathcal{T}$ is a strong exceptional collection. Now we get $\alpha(\sum_{i\in I_-}(\leq 0)E_i + \sum_{i\in I_+}(\geq 0)E_i) > 0$. Thus $\alpha(D_k - D_{i_0}) > 0$ which contradicts the assumption that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$.

(3) We have $\mathrm{Ext}^{-}(\mathcal{O}(D_{i_0} - E_+), \mathcal{O}(D_k)) = 0$ by the same argument as in (3) of Lemma 3.11.

(4) By assumption, $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_+)) = 0$ for all $k \in \{1, \ldots, n\}$.

(5) We have $\mathrm{Ext}^{+}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_+)) = 0$ by the same argument as in (4) of Lemma 3.11.

(6) We have $\mathrm{Ext}^{-}(\mathcal{O}(D_k), \mathcal{O}(D_{i_0} - E_+)) = 0$. Otherwise, we have $D_{i_0} - E_+ - D_k = \sum_{i\in I_+}(<0)E_i + \sum_{i\in I_-}(\geq 0)E_i$. Thus $D_{i_0} - D_k = \sum_{i\in I_+}(<0)E_i + E_+ + \sum_{i\in I_-}(\geq 0)E_i$.

If one of the coefficients in $\sum_{i\in I_+}(<0)E_i$ is less than -1, then $\alpha(\sum_{i\in I_+}(<0)E_i) = \sum_{i\in I_+}(<0)\alpha_i < \sum_{i\in I_+}(-1)\alpha_i = \alpha(-E_+)$. So $\alpha(\sum_{i\in I_+}(<0)E_i + E_+) < 0$. If all the coefficients in $\sum_{i\in I_+}(<0)E_i$ equal -1, then $D_{i_0} - D_k = \sum_{i\in I_-}(\geq 0)E_i$. Since $D_{i_0} \neq D_k$, the coefficients in $\sum_{i\in I_-}(\geq 0)E_i$ cannot be all zero. Thus $\alpha(\sum_{i\in I_-}(\geq 0)E_i) < 0$. Now, we obtain that either $\alpha(\sum_{i\in I_+}(<0)E_i + E_+) < 0$ or $\alpha(\sum_{i\in I_-}(\geq 0)E_i) < 0$. Therefore $\alpha(D_{i_0} - D_k) = \alpha(\sum_{i\in I_+}(<0)E_i + E_+ + \sum_{i\in I_-}(\geq 0)E_i) < 0$ which contradicts the assumption that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$.

We have verified that there are no $\mathrm{Ext}^{>0}$ spaces between the new element and other elements of the collection. □

Lemma 3.17. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$. We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Assume $\mathrm{Ext}^{\mathrm{rk}(N)}(\mathcal{O}(D_{i_0} + E_L), \mathcal{O}(D_j)) = 0$ for any $j \in \{1, \ldots, n\}$ for any subset $L \subseteq I_-$. Then we can get a new strong exceptional collection in $\mathcal{D}(\mathcal{T})$ by replacing $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} + E_-)$ and reordering.*

Proof. Analogous to Lemma 3.16. □

Proposition 3.18. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$. We can construct a new strong exceptional collection $\mathcal{S}$ in $\mathcal{D}(\mathcal{T})$ such that $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+) = \alpha(-E_-)$.*

Proof. The argument is similar to that of Theorem 3.7. Let $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. By Lemma 3.13, we have either $\mathrm{Ext}^{\mathrm{rk}(N)}(D_k, D_{i_0} - E_J) = 0$ for any $k \in \{1, \ldots, m\}$ and any $J \subseteq I_+$ or $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0} + E_L, D_j) = 0$ for any $j \in \{1, \ldots, m\}$ and any $L \subseteq I_-$. By Lemma 3.14, Lemma 3.15, Lemma 3.16 and Lemma 3.17, we get a new strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}(\mathcal{T})$ by replacing $\mathcal{O}(D_{i_0})$ by $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$, and reordering. See Figure 3.2.

We have $\max(\alpha(\mathcal{T}')) \leq \max(\alpha(\mathcal{T}))$ since $\alpha(D_{i_0} - E_+) = \alpha(D_{i_0} + E_-) < \alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. After a finite number of steps, we replace successively all $\mathcal{O}(D_i)$ such that $\alpha(D_i) = \max(\alpha(\mathcal{T}))$ by $\mathcal{O}(D_i - E_+)$ or $\mathcal{O}(D_i + E_-)$ to get a new strong exceptional collection $\mathcal{T}^1$ in $\mathcal{D}(\mathcal{T})$ such that $\max(\alpha(\mathcal{T}^1)) < \max(\alpha(\mathcal{T}))$.

If $\min(\alpha(\mathcal{T}^1)) < \min(\alpha(\mathcal{T}))$, there exists some D such that $\alpha(D_i) = \max(\alpha(\mathcal{T}))$ and $\alpha(D_i \mp E_\pm) = \min(\alpha(\mathcal{T}))$. Now we have

$$\max(\alpha(\mathcal{T}^1)) - \min(\alpha(\mathcal{T}^1)) < \max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T}^1)) = \alpha(D_i) - \alpha(D_i \mp E_\pm) = \alpha(E_+).$$

If $\min(\alpha(\mathcal{T}^1)) \geq \min(\alpha(\mathcal{T}))$, then $\max(\alpha(\mathcal{T}^1)) - \min(\alpha(\mathcal{T}^1)) < \max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T}))$.

This process decreases $\max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T}))$. Eventually we will be in the situation that $\max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T})) < \alpha(E_+) = \alpha(-E_-)$. □

Proposition 3.19. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles with length $n = \mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$. Assume*

$$\max(f(\mathcal{T})) - \min(f(\mathcal{T})) \geq f(E_+ + E_-) = 1.$$

We pick $i_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{i_0}) = \max(\alpha(\mathcal{T}))$. Then we can replace $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$ to get another strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}(\mathcal{T})$ such that:

1. $\max(f(\mathcal{T}')) \leq \max(f(\mathcal{T}))$*;*
2. $\min(f(\mathcal{T}')) \geq \min(f(\mathcal{T}))$*;*
3. $\sharp(\mathcal{T}'_{\min(f)}) \leq \sharp(\mathcal{T}_{\min(f)})$ *if* $\min(f(\mathcal{T}')) = \min(f(\mathcal{T}))$*;*
4. $\sharp(\{D_i \in \mathcal{T}' | f(D_i) = \min(f(\mathcal{T}))\}) < \sharp(\mathcal{T}_{\min(f)})$ *if* $f(D_{i_0}) = \min(f(\mathcal{T}))$*.*

Proof. If

$$\min(f(\mathcal{T})) < f(D_{i_0} - E_+) < f(D_{i_0} + E_-) \leq \max(f(\mathcal{T})), \tag{3.1}$$

by Lemma 3.13, Lemma 3.16 and Lemma 3.17, we can replace $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$ to reach the result.

If Equation 3.1 fails, there are several cases to consider.

Case $f(D_{i_0} - E_+) \leq \min(f(\mathcal{T}))$. We have $f(D_{i_0} + E_-) \leq \max(f(\mathcal{T}))$ by the assumption that $\max(f(\mathcal{T})) - \min(f(\mathcal{T})) \geq f(E_+ + E_-) = 1$. We show that replacing $\mathcal{O}(D_{i_0})$ with $\mathcal{O}(D_{i_0} + E_-)$ is possible and will achieve our goal, see (2) of Figure 3.3.

We have $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0}+E_L, D_j)=0$ for all $j\in\{1,\dots,m\}$ and $L \subsetneq I_-$. Otherwise, we get $D_j - D_{i_0} - E_L = \sum_{i\in\{1,\dots,m\}}(<0)E_i = -E_+ - E_- + \sum_{i\in\{1,\dots,m\}}(\leq 0)E_i$ for some j and some $L \subsetneq I_-$. Thus $D_j - D_{i_0} + E_+ = (E_L - E_-) + \sum_{i\in\{1,\dots,m\}}(\leq 0)E_i$. Then $f(D_j - D_{i_0} + E_+) = f((E_L - E_-) + \sum_{i\in\{1,\dots,m\}}(\leq 0)E_i) < 0$ which contradicts $f(D_{i_0} - E_+) \leq \min(f(\mathcal{T}))$. Then by Lemma 3.14, the line bundle $\mathcal{O}(D_{i_0}+E_-)\in\mathcal{D}(\mathcal{T})$.

Also, we have $\mathrm{Ext}^{\mathrm{rk}(N)}(D_{i_0}+E_-, D_j)=0$ for all $j\in\{1,\dots,m\}$. Otherwise, we get $D_j - D_{i_0} - E_- = \sum_{i\in\{1,\dots,m\}}(<0)E_i = -E_+ - E_- + \sum_{i\in\{1,\dots,m\}}(\leq 0)E_i$ for some j. Thus $D_j - D_{i_0} = -E_+ + \sum_{i\in\{1,\dots,m\}}(\leq 0)E_i$. If the coefficients in $\sum_{i\in\{1,\dots,m\}}(\leq 0)E_i$ are not all zero, then $f(D_j - D_{i_0} + E_+) = f(\sum_{i\in\{1,\dots,m\}}(\leq 0)E_i) < 0$, which contradicts that $f(D_{i_0} - E_+) \leq \widetilde{f}(\mathcal{T})$. If the coefficients in $\sum_{i\in\{1,\dots,m\}}(\leq 0)E_i$ are all zero, then $D_j - D_{i_0} = -E_+$. This implies $\mathrm{Ext}^-(\mathcal{O}(D_{i_0}), \mathcal{O}(D_j)) \neq 0$ which contradicts the assumption that $\mathcal{T}$ is a strong exceptional collection.

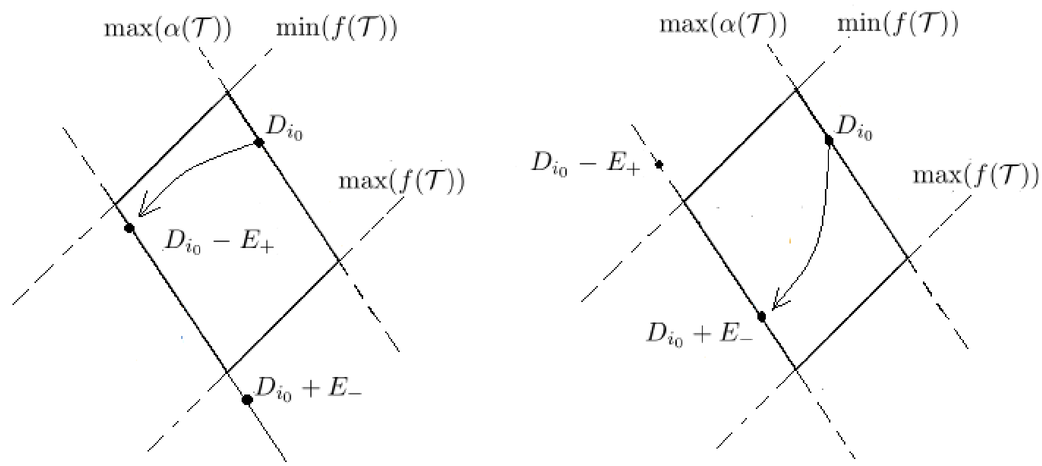

Figure 3.3:

Then by Lemma 3.16, we get a strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}(\mathcal{T})$ by replacing D_{i_0} with $\mathcal{O}(D_{i_0}+E_-)$ which satisfies (2), (3) and (4) of this Proposition. Since $f(D_{i_0}+E_-) \leq \max(f(\mathcal{T}))$, then $\max(f(\mathcal{T}')) \leq \max(f(\mathcal{T}))$.

Case $f(D_{i_0}+E_-) > \max(f(\mathcal{T}))$. We have $f(D_{i_0}-E_+) > \min(f(\mathcal{T}))$. By the same arguments, we can get a strong exceptional collection $\mathcal{T}'$ in $\mathcal{D}(\mathcal{T})$ by replacing D_{i_0} with $\mathcal{O}(D_{i_0}-E_+)$ which satisfies (1), (3) and (4) of this Proposition, see (1) of Figure 3.3.

Since $f(D_{i_0} - E_+) > \min(f(\mathcal{T}))$, then $\min(f(\mathcal{T}')) \geq \min(f(\mathcal{T}))$. □

Remark 7. *Let $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles with length $n = \mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$. Assume all line bundles in $\mathcal{T}$ are within a strip of α with width less than $\alpha(E_+)$ and $\max(f(\mathcal{T})) - \min(f(\mathcal{T})) \geq f(E_+ + E_-) = 1$. After doing the move in Proposition 3.19, we can guarantee that all line bundles in the new strong exceptional collection is within a strip of α with width less or equal to $\alpha(E_+)$. After replacing all D_j in $\mathcal{T}$ such that $\alpha(D_j) = \max(\alpha(\mathcal{T}))$, we get the width of the strip of α to be less than $\alpha(E_+)$.*

Theorem 3.20. *Let $\mathbb{P}_{\mathbf{\Sigma}}$ be a Fano toric DM stack with $\mathrm{rank}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$. Assume $\mathcal{T} = \{\mathcal{O}(D_1), \ldots, \mathcal{O}(D_n)\}$ be a strong exceptional collection of line bundles with length $n = \mathrm{rk}(K_0(\mathbb{P}_{\mathbf{\Sigma}}))$. Then $\mathcal{T}$ is a full strong exceptional collection.*

Proof. Without of loss of generality, we can assume that $\max(\alpha(\mathcal{T})) - \min(\alpha(\mathcal{T})) < \alpha(E_+) = \alpha(-E_-)$ by Proposition 3.18.

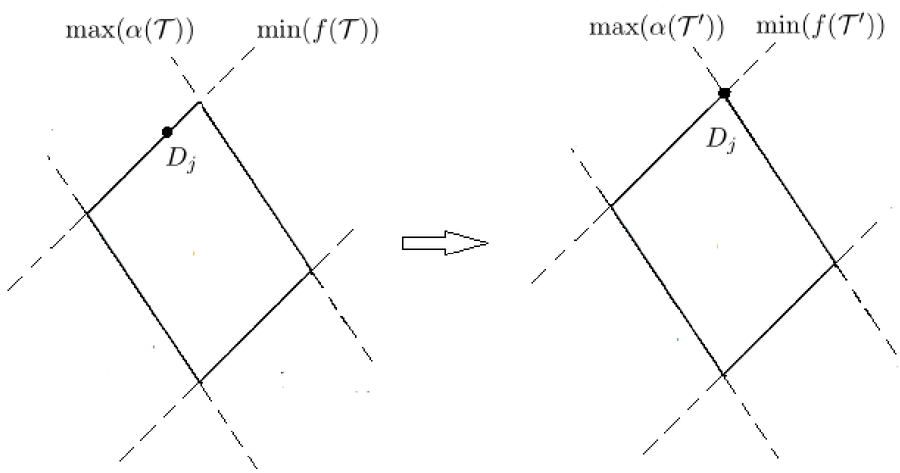

Figure 3.4:

Let D_j be an element in $\mathcal{T}$ such that $f(D_j) = \min(f(\mathcal{T}))$. If $\alpha(D_j) = \max(\alpha(\mathcal{T}))$, then by Proposition 3.19, after replacing $\mathcal{O}(D_j)$ with $\mathcal{O}(D_j - E_+)$ or $\mathcal{O}(D_j + E_-)$, we get another strong exceptional collection $\mathcal{T}'$ such that $\sharp(\{D_i \in \mathcal{T}' | f(D_i) = \min(f(\mathcal{T}))\}) < \sharp(\mathcal{T}_{\min(f)})$. If $\alpha(D_j) < \max(\alpha(\mathcal{T}))$, then by repeating the process in Proposition 3.19 several times, we will get to the situation that α takes maximum value at D_j, see Figure

3.4.

After replacing all elements in $\mathcal{T}_{\min(f)}$, we get $\min(f(\mathcal{T}))$ increase. Then we continue to apply Proposition 3.19. During the process, we assure that $\max(f(\mathcal{T}))$ does not increase and $\min(f(\mathcal{T}))$ increases. Thus $\max(f(\mathcal{T})) - \min(f(\mathcal{T}))$ decreases. Therefore, we will eventually be in the situation $\max(f(\mathcal{T})) - \min(f(\mathcal{T})) < 1$.

Also, by Remark 7, we get a new strong exceptional collection $\mathcal{S}$ of line bundles in $\mathcal{D}(\mathcal{T})$ such that $\max(\alpha(\mathcal{S})) - \min(\alpha(\mathcal{S})) < \alpha(E_+)$ and $\max(f(\mathcal{S})) - \min(f(\mathcal{S})) < 1$. So $\mathcal{S}$ is a full strong exceptional collection by Proposition 3.9. Thus $\mathcal{D}(\mathcal{T}) \supseteq \mathcal{D}(\mathcal{S}) = \mathbf{D}^b(coh(\mathbb{P}_{\mathbf{\Sigma}}))$. □

Remark 8. *When* $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ *has torsion, the arguments of this section go through without significant change. The details are left to the reader.*

3.3 Future directions

We expect our main result to be valid without the assumption on the rank of Picard group, as stated in Conjecture 1.1. Also, in the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$, we know that any exceptional collection of line bundles is a strong exceptional collection by Remark 5. However, in the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$, Theorem 3.20 does not tell us that every exceptional collection of maximum length is a full exceptional collection. Thus we hope we can drop the strong assumption to ask whether every exceptional collection of maximum length is a full exceptional collection. The possible future directions include dimension two $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 3$ Fano case, and dimension two non-Fano case. We hope that techniques of this thesis can be modified to settle them.

Moreover, in our proofs when we replace $j_0 \in \{1, \ldots, n\}$ such that $\alpha(D_{j_0}) = \max(\alpha(\mathcal{T}))$ with $\mathcal{O}(D_{i_0} - E_+)$ or $\mathcal{O}(D_{i_0} + E_-)$, the strong exceptional collection "shrinks" in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. We would like to find a more geometric meaning of this phenomenon.

Remark 9. *In [31], Špela Špenko, Michel Van den Bergh and Jason P. Bell show that in the case of* $rk(Pic) = 1$, *every strong exceptional collection of line bundles on a toric variety or stack can be extended to a strong exceptional collection of line bundles with length equal to rank of* K*-theory. However, in [13], Efimov constructs infinitely* *many examples of toric Fano varieties with Picard number three, which do not admit full exceptional collections of line bundles. It is meaningful to consider, whether we have the same result in the case of* $rk(Pic) = 2$ *as in the case of* $rk(Pic) = 1$.

Chapter 4

Exceptional collections and strong exceptional collections on generalized Hirzebruch surfaces

In this chapter we study in complete detail exceptional collections and strong exceptional collections of line bundles on certain generalized Hirzebruch surfaces (GHS).

4.1 Hirzebruch surfaces and generalized Hirzebruch surfaces

In this section, we give a brief review of Hirzebruch surfaces and define certain generalized Hirzebruch surfaces.

The Hirzebruch surface $\mathbb{F}_n$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ associated to the sheaf $\mathcal{O} \oplus \mathcal{O}(n)$. That is to say that $\mathbb{F}_n$ is the projectivization of the direct sum of line bundles $\mathcal{O} \oplus \mathcal{O}(n)$ on $\mathbb{P}^1$ which we denote by $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n))$. In terms of toric geometry, the corresponding fan Σ has rays $v_1 = (1,0), v_2 = (0,-1), v_3 = (-1,n), v_4 = (0,1)$ (see Figure 4.1). The

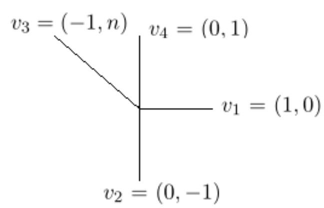

Figure 4.1:

Picard group is generated by E_1, E_2, E_3, E_4 with relations

$$E_1 = E_3, E_4 = E_2 - nE_3.$$

Thus Picard group is isomorphic to $\mathbb{Z}^2$ with basis E_2, E_3.

Definition 4.1. *The Generalized Hirzebruch surface $\mathbb{F}_{\alpha,n}$ is given to be the toric DM stack $\mathbb{P}_{\mathbf{\Sigma}}$ associated to the stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^4)$, where $v_1 = (1,0), v_2 =$*

$(0,-1), v_3 = (-\alpha, n), v_4 = (0,1)$, $\alpha \in \mathbb{Z}_{>0}$ *and* $n \geq \alpha$ *(Figure 4.2).*

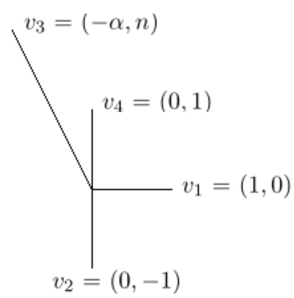

Figure 4.2:

Remark 10. *The reason to add the assumption* $n \geq \alpha$ *is that we have similar* H-*trivial line bundles when* $n \geq \alpha$*, so there is a common way to classify the exceptional collections of line bundles of maximum length. However, when* $n < \alpha$*, the structure of the set of* H-*trivial line bundles varies, so the exceptional collections of line bundles of maximum length do not have a common pattern. Note that the stack* $\mathbb{P}_{\mathbf{\Sigma}}$ *is Fano when* $n = \alpha$*, the stack* $\mathbb{P}_{\mathbf{\Sigma}}$ *is not Fano when* $n > \alpha$*, and the stack* $\mathbb{P}_{\mathbf{\Sigma}}$ *is nef-Fano iff* $n = \alpha + 1$.

Lemma 4.2. *The rank of the* K*-theory of* $\mathbb{P}_{\mathbf{\Sigma}} = \mathbb{F}_{\alpha,n}$ *equals* $2(\alpha + 1)$.

Proof. In general, the rank of K-theory of the proper toric smooth DM stack is the sum of normalized volumes of maximum cones. The absolute values of the determinants of the following four matrices are $1, 1, \alpha, \alpha$ respectively.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -\alpha & n \end{pmatrix}, \begin{pmatrix} -\alpha & n \\ 0 & 1 \end{pmatrix}$$

Then the rank of the K-theory equals $1 + 1 + \alpha + \alpha$. □

4.2 Classification of exceptional collections of line bundles on generalized Hirzebruch surfaces

In this section, we give a classification of all exceptional collection of line bundles on generalized Hirzebruch surfaces $\mathbb{F}_{\alpha,n}$.

The Picard group of $\mathbb{F}_{\alpha,n}$ is generated by E_1, E_2, E_3, E_4 with relations

$$E_1 = \alpha E_3, E_4 = E_2 - nE_3.$$

Let E_2, E_3 be basis of the Picard group. We have $E_1 = (0, \alpha), E_2 = (1, 0), E_3 = (0, 1), E_4 = (1, -n)$. By Remark 4 and Definition 2.11, we have

$$\Delta = \{\{1, 3\}, \{2, 4\}, \emptyset, \{1, 2, 3, 4\}\}.$$

The corresponding forbidden sets are computed as follows.

$$\begin{aligned}
F_{13} &= -E_2 - E_4 + \mathbb{Z}_{\geq 0}E_1 + \mathbb{Z}_{\geq 0}E_3 + \mathbb{Z}_{\geq 0}(-E_2) + \mathbb{Z}_{\geq 0}(-E_4) \\
&= -E_2 - E_2 + nE_3 + \mathbb{Z}_{\geq 0}E_3 + \mathbb{Z}_{\geq 0}(-E_2) + \mathbb{Z}_{\geq 0}(-E_4) \\
&= (-2, n) + \mathbb{Z}_{\geq 0}(0, 1) + \mathbb{Z}_{\geq 0}(-1, 0) + \mathbb{Z}_{\geq 0}(-1, n) \\
F_{24} &= -E_1 - E_3 + \mathbb{Z}_{\geq 0}E_2 + \mathbb{Z}_{\geq 0}E_4 + \mathbb{Z}_{\geq 0}(-E_1) + \mathbb{Z}_{\geq 0}(-E_3) \\
&= -\alpha E_3 - E_3 + \mathbb{Z}_{\geq 0}E_2 + \mathbb{Z}_{\geq 0}E_4 + \mathbb{Z}_{\geq 0}(-E_3) \\
&= (0, -(\alpha + 1)) + \mathbb{Z}_{\geq 0}(1, 0) + \mathbb{Z}_{\geq 0}(1, -n) + \mathbb{Z}_{\geq 0}(0, -1) \\
F_{\emptyset} &= -E_1 - E_2 - E_3 - E_4 + \mathbb{Z}_{\geq 0}(-E_1) + \mathbb{Z}_{\geq 0}(-E_2) + \mathbb{Z}_{\geq 0}(-E_3) + \mathbb{Z}_{\geq 0}(-E_4) \\
&= -(\alpha + 1)E_3 - 2E_2 + nE_3 + \mathbb{Z}_{\geq 0}(-E_3) + \mathbb{Z}_{\geq 0}(-E_2) + \mathbb{Z}_{\geq 0}(-E_4) \\
&= (-2, -(\alpha + 1) + n) + \mathbb{Z}_{\geq 0}(0, -1) + \mathbb{Z}_{\geq 0}(-1, 0) + \mathbb{Z}_{\geq 0}(-1, n) \\
F_{\{1,2,3,4\}} &= \mathbb{Z}_{\geq 0}E_1 + \mathbb{Z}_{\geq 0}E_2 + \mathbb{Z}_{\geq 0}E_3 + \mathbb{Z}_{\geq 0}E_4 \\
&= \mathbb{Z}_{\geq 0}(0, 1) + \mathbb{Z}_{\geq 0}(0, 1) + \mathbb{Z}_{\geq 0}(1, -n)
\end{aligned}$$

A convenient feature of $\mathbb{F}_{\alpha,n}$ is that the forbidden sets contain all lattice points in their respective convex hulls.

Lemma 4.3. *All lattice points in the following cones are in the forbidden sets.*

$$\begin{aligned}
FC_{13} &= (-2, n) + \mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(-1, 0) + \mathbb{R}_{\geq 0}(-1, n), \\
FC_{24} &= \big((0, -(\alpha + 1)) + \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(1, -n) + \mathbb{R}_{\geq 0}(0, -1), \\
FC_{\emptyset} &= (-2, -(\alpha + 1) + n) + \mathbb{R}_{\geq 0}(0, -1) + \mathbb{R}_{\geq 0}(-1, 0) + \mathbb{R}_{\geq 0}(-1, n), \\
FC_{\{1,2,3,4\}} &= \mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(0, 1) + \mathbb{R}_{\geq 0}(1, -n).
\end{aligned}$$

Proof. We have

$$F_{13} \subseteq FC_{13} \cap \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}), F_{24} \subseteq FC_{24} \cap \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}),$$

$$F_{\emptyset} \subseteq FC_{\emptyset} \cap \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}), F_{\{1,2,3,4\}} \subseteq FC_{\{1,2,3,4\}} \cap \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}).$$

Since the absolute value of the determinant of the matrix $\begin{pmatrix} -1 & n \\ 0 & 1 \end{pmatrix}$ is 1, any lattice point in cone FC_{13} can be generated by $(0,1)$ and $(-1,n)$, thus in the forbidden sets F_{13}. The arguments are similar for other cones. □

Corollary 4.4. *All the* H*-trivial line bundles on* $\mathbb{P}_{\mathbf{\Sigma}} = \mathbb{F}_{\alpha,n}$ *are*

$$(-2, n-1), \ldots, (-2, n-\alpha), \{(-1, y) | y \in \mathbb{Z}\}, (0, -1), \ldots, (0, -\alpha).$$

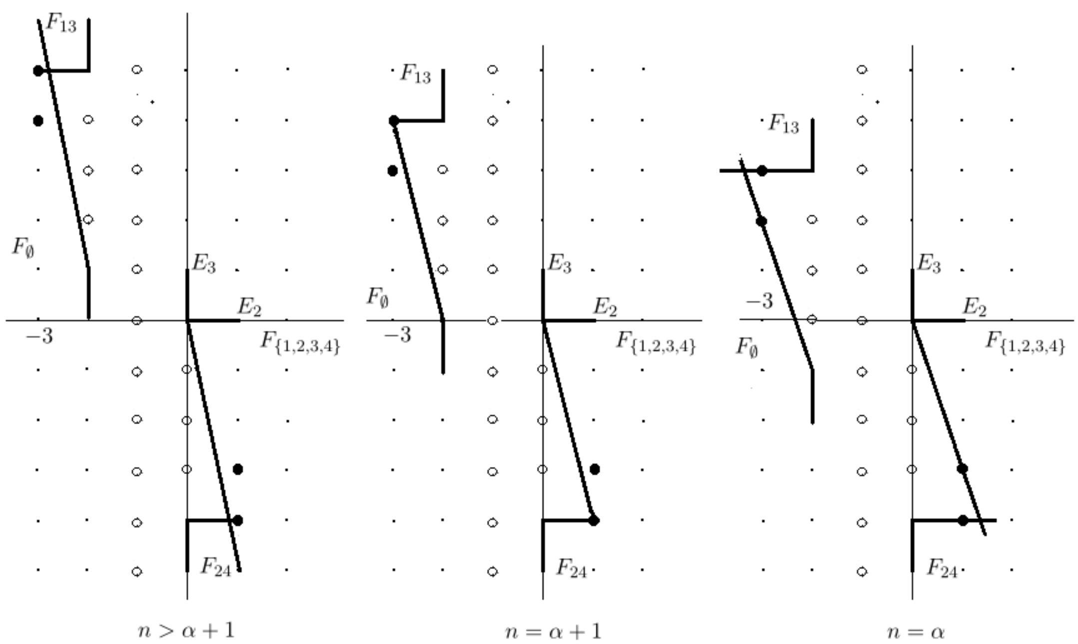

Figure 4.3:

Proof. The top boundary of $FC_{\emptyset}$ is $(-2, , -(\alpha+1)+n) + \mathbb{R}_{\geq 0}(-1, n)$. The bottom boundary of FC_{13} is $(-2, n) + \mathbb{R}_{\geq 0}(-1, 0)$. They intersect at $(-2 - \frac{\alpha+1}{n}, n)$. If $n > \alpha+1$, then this intersection point is to right of $(-3, \mathbb{Z})$ (see Figure 4.3). If $n = \alpha+1$, then this intersection point is on $(-3, \mathbb{Z})$ (see Figure 4.3). So when $n \geq \alpha+1$, there are no H-trivial line bundles with first coordinates less than -2. If $n = \alpha$, then it is easy to see that all lattice points with first coordinates -3 are still in the two cones (see

Figure 4.3). The same argument applies for positive first coordinate. Then the result is implied by Proposition 2.12 and Lemma 4.3. □

Remark 11. *The fan associated to* $\mathbb{P}_{\mathbf{\Sigma}} = \mathbb{F}_{\alpha,n}$ *has a collinear pair of rays* v_2, v_4 *(see Figure 4.2). Thus, by the main results in [32], there are infinitely many* H*-trivial line bundles on* $\mathbb{P}_{\mathbf{\Sigma}} = \mathbb{F}_{\alpha,n}$ *and the set of* H*-trivial line bundles has a tubes + ball description. That is to say that the set of* H*-trivial line bundles can be depicted in the form "finite set + finite set of lines ", which is consistent with the Corollary 4.4.*

In the next Lemma, we state the basic properties of any exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$.

Lemma 4.5. *Let*

$$\mathcal{T} = \{(a_1, b_1), (a_2, b_2), \dots, (a_r, b_r)\}$$

be an exceptional collection. Then it satisfies

1. $a_1 \leq a_2 \leq \dots \leq a_r$ *and* $x \in \{a_r - 2, a_r - 1, a_r\}$ *for any line bundle* $(x, y) \in \mathcal{T}$*.*
2. *There are no more than* $\alpha + 1$ *points in* $\mathcal{T}$ *on the same vertical line and the second coordinates increase for the points on the same vertical line.*

Proof. We have

$$\mathrm{H}^t(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{O}((a_i, b_i) - (a_j, b_j))) = \mathrm{Ext}^t(\mathcal{O}((a_j, b_j)), \mathcal{O}((a_i, b_i))) = 0$$

for any $i < j$ and $t \geq 0$. So $(a_i, b_i) - (a_j, b_j)$ is H-trivial. Thus $a_i - a_j \in \{0, -1, -2\}$ which implies *1.*

When $a_i = a_j$ for $i < j$, since $(a_i, b_i) - (a_j, b_j) = (0, b_i - b_j)$ is H-trivial, we get $b_i - b_j \in \{-1, \dots, -\alpha\}$. This implies $b_i < b_j$ and there are no more than $\alpha + 1$ points in $\mathcal{T}$ having the first coordinate, i.e., on the same vertical line. Thus we get *2.* □

Corollary 4.6. *Ordering of line bundles in an exceptional collection on* $\mathbb{F}_{\alpha,n}$ *is uniquely determined.*

Now we will give a classification of all exceptional collection of maximum length $\mathrm{rk}(K(\mathbb{P}_{\mathbf{\Sigma}})) = 2(\alpha+1)$.

Let $\mathcal{T} = \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$ be an exceptional collection. We first "fill in the holes" in the collection.

Lemma 4.7. *If there exists (a_i, b_i) and (a_{i+1}, b_{i+1}) such that $a_{i+1} = a_i$ and $b_i+1 < b_{i+1}$, then the collection obtained by inserting (a_i, b_i+1) between (a_i, b_i) and (a_i, b_{i+1}) is still an exceptional collection.*

Proof. For any other element $\mathcal{O}((a,b))$ in the collection, we have that $(a,b)-(a_i,b_i+1)$ is in the same vertical line as $(a,b)-(a_i,b_i)$ and $(a,b)-(a_i,b_{i+1})$ and is in between them. Since the set of H-trivial bundles in the same vertical line is a segment in $\mathbb{Z}$, the H-triviality of $(a,b)-(a_i,b_i+1)$ follows. The same holds for $(a_i,b_i+1)-(a,b)$. The only thing left to check is that the two Extension spaces from (a_i,b_i+1) to (a_i,b_i), and from (a_i,b_{i+1}) to (a_i,b_i+1) are zero. This is automatically implied by $b_i < b_i+1 < b_{i+1}$. □

Corollary 4.8. *After inserting line bundles between line bundles on the same vertical lines in $\mathcal{T}$ to make the second coordinates to be consecutive numbers, we get another exceptional collection with longer length.*

Corollary 4.9. *For any maximal exceptional collection, the second coordinates are consecutive numbers for points on the same vertical line.*

Case 1. We first consider the case that $a_r - a_1 = 2$. We assume that $a_1 = \cdots = a_{r_1} = a_r - 2, a_{r_1+1} = \cdots = a_{r_1+r_2} = a_r - 1, a_{r_1+r_2+1} = \cdots = a_r$, where $1 \le r_1, r_3 \le \alpha+1$, $0 \le r_2 \le \alpha+1$ and $r = r_1 + r_2 + r_3$. By Corollary 4.8, we can assume the second coordinates are consecutive numbers for points on the same vertical line.

Lemma 4.10. *We have $r_1 + r_3 \le \alpha + 1$ and $n - \alpha \le b_1 - b_r \le b_{r_1} - b_{r_1+r_2+1} \le n-1$.*

Proof. By assumption and *2.* in Lemma 4.5, we know that the second coordinates of the points on the left vertical line and right vertical line are $b_1 \le \ldots \le b_{r_1}$ and $b_{r_1+r_2+1} \le \ldots \le b_r$ respectively (see Figure 4.4). Since the second coordinates are consecutive numbers for points on the same vertical line, we have $b_{r_1} - b_1 = r_1 - 1$ and

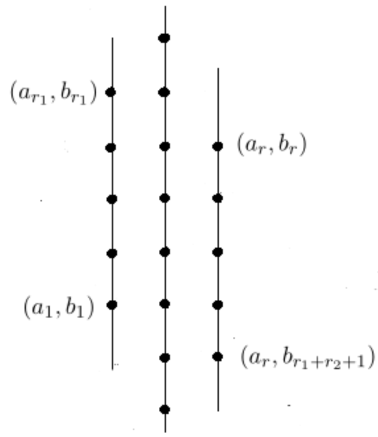

Figure 4.4:

$b_r - b_{r_1+r_2+1} = r_3 - 1$. Since $(a_1, b_1) - (a_r, b_r) = (-2, b_1 - b_r)$ is H-trivial, $b_1 - b_r \in \{n-\alpha, \dots, n-1\}$. Similarly, we get $b_{r_1} - b_{r_1+r_2+1} \in \{n-\alpha, \dots, n-1\}$. Then we have $(b_{r_1} - b_{r_1+r_2+1}) - (b_1 - b_r) = r_1 - 1 + r_3 - 1 \leq \alpha - 1$, which implies $r_1 + r_3 \leq \alpha + 1$. □

Now we assume $\mathcal{T}$ is of maximum length, i.e, $r = 2(\alpha + 1)$.

Corollary 4.11. *For an exceptional collection $\mathcal{T}$ of maximum length, we have $r_2 = \alpha + 1 = r_1 + r_3$, $b_1 - b_r = n - \alpha$ and $b_{r_1} - b_{r_1+r_2+1} = n - 1$.*

Proof. By Lemma 4.10 and $2(\alpha + 1) = r = r_1 + r_2 + r_3$, we have $r_2 \geq \alpha + 1$. By *2.* in Lemma 4.5, we have $r_2 \leq \alpha + 1$. Thus $r_2 = \alpha + 1$ and $r_1 + r_3 = r - r_2 = \alpha + 1$. The property $b_1 - b_r = n - \alpha$ and $b_{r_1} - b_{r_1+r_2+1} = n - 1$ follows from the proof of Lemma 4.10. Indeed, by the proof Lemma 4.10 and $r_1 + r_3 = \alpha + 1$, we get $(b_{r_1} - b_{r_1+r_2+1}) - (b_1 - b_r) = r_1 - 1 + r_3 - 1 = \alpha - 1$. Since $b_{r_1} - b_{r_1+r_2+1} \geq b_1 - b_r$ and both are in $\{n-\alpha, \dots, n-1\}$. Thus $b_1 - b_r = n - \alpha$ and $b_{r_1} - b_{r_1+r_2+1} = n - 1$. □

Proposition 4.12. *The exceptional collection of line bundles of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$ with elements in three vertical lines is, up to tensoring with a line bundle, are given by*

$$\begin{aligned}\mathcal{H}_{r_1,r_3} = \big(&(-2, n-\alpha), \dots, (-2, n-\alpha+r_1-2), (-2, n-\alpha+r_1-1),\\ &(-1, b_{r_1+r_2} - \alpha), \dots, (-1, b_{r_1+r_2} - 1), (-1, b_{r_1+r_2}),\\ &(0, -(r_3-1)), \dots, (0, -1), (0, 0)\big),\end{aligned}$$

for a pair $r_1, r_3 \in \{1, \ldots, \alpha\}$ *such that* $r_1 + r_3 = \alpha + 1$ *and* $b_{r_1+r_2} \in \mathbb{Z}$. *Since we have*

$$\#\{(r_1, r_3)|r_1, r_3 \in \{1, \ldots, \alpha\}, r_1 + r_3 = \alpha + 1\} = \alpha.$$

Now we have α *classes of exceptional collections of line bundles of maximum length on* $\mathbb{P}_{\mathbf{\Sigma}}$.

Proof. By Corollary 4.11, for any pair $r_1, r_3 \in \{1, \ldots, \alpha\}$ such that $r_1 + r_3 = \alpha + 1$, we get a class of exceptional collection of line bundles of maximum length

$$\begin{aligned}\mathcal{T}_{r_1,r_3} = \big(&(a_r - 2, b_r + (n-\alpha)), \ldots, (a_r - 2, b_{r_1} - 1), (a_r - 2, b_r + (n-\alpha) + (r_1 - 1)),\\ &(a_r - 1, b_{r_1+r_2} - \alpha), \ldots, (a_r - 1, b_{r_1+r_2} - 1), (a_r - 1, b_{r_1+r_2}),\\ &(a_r, b_r - (r_3 - 1)), \ldots, (a_r, b_r - 1), (a_r, b_r)\big),\end{aligned}$$

where $b_{r_1}, b_{r_1+r_2} \in \mathbb{Z}$. After tensoring with $(-a_r, -b_r)$ and replacing $b_{r_1+r_2} - b_r$ by $b_{r_1+r_2}$ to simplify notation, we get

$$\begin{aligned}\mathcal{H}_{r_1,r_3} = \big(&(-2, n-\alpha), \ldots, (-2, b_{r_1} - 1 - b_r), (-2, (n-\alpha) + (r_1 - 1)),\\ &(-1, b_{r_1+r_2} - \alpha), \ldots, (-1, b_{r_1+r_2} - 1), (-1, b_{r_1+r_2}),\\ &(0, -(r_3 - 1)), \ldots, (0, -1), (0, 0)\big),\end{aligned}$$

where $b_{r_1}, b_{r_1+r_2} \in \mathbb{Z}$.

To show that this collection is exceptional, we check that the appropriate line bundles are H-trivial. For elements in the same vertical line, the result follows from $r_1, r_3 \leq \alpha + 1$. For elements on the first and the third vertical lines, the result follows from $b_1 - b_r = n - \alpha - 0 = n - \alpha$ and $b_{r_1} - b_{r_1+r_2+1} = (n-\alpha) + (r_1 - 1) - (-(r_3 - 1)) = n - 1$. □

Case 2. We consider the case $a_r - a_1 < 2$. If $a_r = a_1$, then $\mathcal{T}$ lies on one vertical line. It can never be of maximum length. If $a_r - a_1 = 1$, then $\mathcal{T}$ lies on two consecutive lines. We assume that $a_1 = a_2 = \ldots = a_{r_1} = a_r - 1$ and $a_{r_1+1} = \ldots = a_{r_1+r_2}$, where $1 \leq r_1, r_2 \leq \alpha + 1$. So if $\mathcal{T}$ has maximum length, then $r_1 = r_2 = \alpha + 1$.

Now we can assume an exceptional collection of line bundles of maximum length lies on two consecutive vertical lines to be

$$\mathcal{T}_0 = \big((a_r - 1, b_{r_1} - \alpha), \ldots, (a_r - 1, b_{r_1} - 1), (a_r - 1, b_{r_1}), (a_r, b_r - \alpha), \ldots, (a_r, b_r - 1), (a_r, b_r)\big),$$

where $b_{r_1} \in \mathbb{Z}$. After tensoring with $(-a_r, -b_r)$ and replace $b_{r_1} - b_r$ by b_{r_1} to simplify the notations, we get

$$\mathcal{H}_0 = \big((-1, b_{r_1} - \alpha), \ldots, (-1, b_{r_1} - 1), (-1, b_{r_1}), (0, -\alpha), \ldots, (0, -1), (0, 0)\big), \quad (4.1)$$

where $b_{r_1} \in \mathbb{Z}$. Therefore, every exceptional collection of line bundles of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$ with elements in two vertical lines is, up to tensoring with a line bundle, of this form in Equation 4.1.

Proposition 4.13. *We can classify all exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ of maximum length into $\alpha + 1$ types.*

Proof. By Proposition 4.12, we have α types. And $\mathcal{H}_0$ in Equation 4.1 gives us one more type. □

We illustrate this classification in the case $\alpha = 3, n = 4$.

Example 4.14. *We have $v_1 = (1, 0), v_2 = (0, -1), v_3 = (-3, 4), v_4 = (0, 1)$. The Picard group is generated by E_1, E_2, E_3, E_4 with relations*

$$E_1 = 3E_3, E_4 = E_2 - 4E_3.$$

Let E_2, E_3 be basis. $E_1 = (0, 3), E_2 = (1, 0), E_3 = (0, 1), E_4 = (1, -4)$ We have forbidden sets (See Figure 4.5):

$$F_{13} = (-2, 4) + \mathbb{Z}_{\geq 0}(0, 1) + \mathbb{Z}_{\geq 0}(-1, 0) + \mathbb{Z}_{\geq 0}(-1, 4)$$

$$F_{24} = (0, -4) + \mathbb{Z}_{\geq 0}(1, 0) + \mathbb{Z}_{\geq 0}(1, -4) + \mathbb{Z}_{\geq 0}(0, -1)$$

$$F_{\emptyset} = (-2, 0) + \mathbb{Z}_{\geq 0}(0, -1) + \mathbb{Z}_{\geq 0}(-1, 0) + \mathbb{Z}_{\geq 0}(-1, 4)$$

$$F_{\{1,2,3,4\}} = \mathbb{Z}_{\geq 0}(0, 1) + \mathbb{Z}_{\geq 0}(0, 1) + \mathbb{Z}_{\geq 0}(1, -4)$$

All the H*-trivial line bundles are*

$$(-2, 3), (-2, 2), (-2, 1), \{(1, y) | y \in \mathbb{Z}\}, (0, -1), (0, -2), (0, -3)$$

which are represented by hollow dots in Figure 4.5.

There are $\alpha + 1 = 4$ classes of exceptional collection of line bundles with maximum length:

$$\mathcal{H}_{1,3} = \big((-2, 1), (-1, b_5 - 3), (-1, b_5 - 2), (-1, b_5 - 1), (-1, b_5), (0, -2), (0, -1), (0, 0)\big),$$

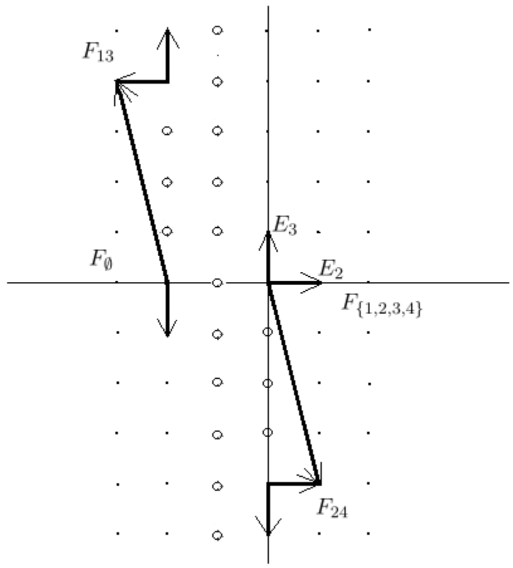

Figure 4.5:

$$\mathcal{H}_{2,2} = \big((-2,1),(-2,2),(-1,b_6-3),(-1,b_6-2),(-1,b_6-1),(-1,b_6),(0,-1),(0,0)\big),$$

$$\mathcal{H}_{3,1} = \big((-2,1),(-2,2),(-2,3),(-1,b_7-3),(-1,b_7-2),(-1,b_7-1),(-1,b_7),(0,0)\big),$$

$$\mathcal{H}_{0} = \big((-1,b_4-3),(-1,b_4-2),(-1,b_4-1),(-1,b_4),(0,-3),(0,-2),(0,-1),(0,0)\big),$$

where $b_4, b_5, b_6, b_7 \in \mathbb{Z}$.

4.3 Extensions to collections of maximum length

As mentioned in Remark 9, we want to see whether every strong exceptional collection of line bundles on a toric variety or stack with Picard number two can be extended to a strong exceptional collection of line bundles of maximum length. In this section, we don't assume that the collections are strong exceptional. For generalized Hirzebruch surfaces $\mathbb{P}_{\mathbf{\Sigma}} = \mathbb{F}_{\alpha,n}$, we will show that any exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ can be extended to an exceptional collection of line bundles of maximum length on $\mathbb{P}_{\mathbf{\Sigma}}$ (Theorem 4.15).

Theorem 4.15. *Any exceptional collection of line bundles on generalized Hirzebruch surfaces* $\mathbb{P}_{\mathbf{\Sigma}} = F_{\alpha,n}$ *can be extended to an exceptional collection with maximum length* $2(\alpha+1)$.

We will first show that any exceptional collection of line bundles

$$\mathcal{T} = \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$$

with $a_r - a_1 = 2$ can be extended to an exceptional collection of line bundles of maximum length on $\mathbb{P}_{\boldsymbol{\Sigma}}$.

By *1.* and *2.* in Lemma 4.5 and Lemma 4.9, we assume an exceptional collection of line bundles with $a_r - a_1 = 2$ to be

$$\begin{aligned}\mathcal{T} =\big(&(a_r - 2, b_{r_1} - (r_1 - 1)), \ldots, (a_r - 2, b_{r_1} - 1), (a_r - 2, b_{r_1}),\\ &(a_r - 1, b_{r_1+r_2} - (r_2 - 1)), \ldots, (a_r - 1, b_{r_1+r_2} - 1), (a_r - 1, b_{r_1+r_2}),\\ &(a_r, b_r - (r_3 - 1)), (a_r, b_r), \ldots, (a_r, b_r)\big),\end{aligned}$$

where $1 \le r_1, r_3 \le \alpha + 1$, $0 \le r_2 \le \alpha + 1$, $r = r_1 + r_2 + r_3$ and the second coordinates of the points on the same vertical line increase.

After tensoring with $(-a_r, b_r)$ and replacing $b_{r_1} - b_r$ by d, $b_{r_1+r_2} - b_r$ by c, we rewrite the collection

$$\begin{aligned}\mathcal{T} =\big(&(-2, d - (r_1 - 1)), \ldots, (-2, d - 1), (-2, d),\\ &(-1, c - (r_2 - 1)), \ldots, (-1, c - 1), (-1, c),\\ &(0, -(r_3 - 1)), \ldots, (0, -1), (0, 0)\big),\end{aligned}$$

where $c, d \in \mathbb{Z}$.

In the following lemma, we show that the exceptional collection is still an exceptional collection after extending the middle vertical line.

Lemma 4.16. *Let*

$$\begin{aligned}\mathcal{T}_1 =\big(&(-2, d - (r_1 - 1)), \ldots, (-2, d - 1), (-2, d),\\ &(-1, c - \alpha), \ldots, (-1, c - r_2), (-1, c - (r_2 - 1)), \ldots, (-1, c - 1), (-1, c),\\ &(0, -(r_3 - 1)), \ldots, (0, -1), (0, 0)\big),\end{aligned}$$

be the collection obtained by extending the second line in $\mathcal{T}$ to maximum length $\alpha + 1$. Then $\mathcal{T}_1$ is also exceptional collection.

Proof. New differences of the elements of the collection are in the line $(-1, \mathbb{Z})$ or the segment $(0, [-1, -\alpha])$. □

Remark 12. *Let* $l = 0, \ldots, (r_3 - 1)$ *and* $t = 0, \ldots, (r_1 - 1)$. *Then* $l - t = 1 - r_1, \ldots, r_3 - 1$. *Thus* $d + l - t = d - r_1 + 1, \ldots, d + r_3 - 1$. *Since* $\mathcal{T}$ *is exceptional collection, then* $(-2, d - t) - (0, -l) = (-2, d + l - t)$ *is* H*-trivial. This implies* $n - \alpha \leq d - r_1 + 1$ *and* $d + r_3 - 1 \leq n - 1$ *(see Figure 4.6).*

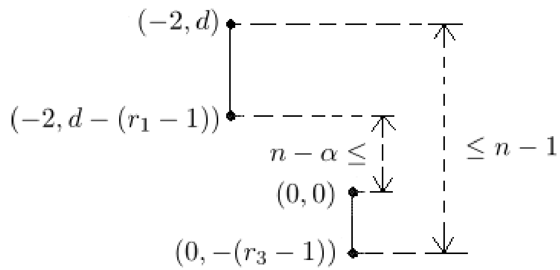

Figure 4.6:

Let

$$s_3 = n - 1 - (d + r_3 - 1), s_1 = d - r_1 + 1 - (n - \alpha)$$

be two nonnegative integers. Then we consider extend the first vertical line below by s_1 and extend the first vertical line below by s_3 (see Figure 4.7).

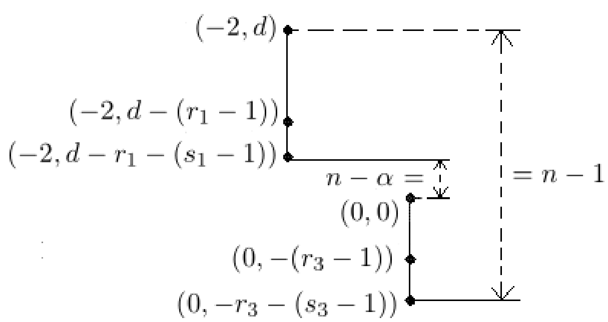

Figure 4.7:

In the following lemma, we claim that the exceptional collection is still an exceptional collection after extending as above.

Lemma 4.17. *Let*

$$\begin{aligned}\mathcal{T}_2 =\big(&(-2, d-r_1-(s_1-1)), \dots, (-2, d-r_1), (-2, d-(r_1-1)), \dots, (-2, d),\\ &(-1, c-\alpha), \dots, (-1, c-r_2-1), (-1, c-r_2), (-1, c-(r_2-1)), \dots, (-1, c),\\ &(0, -r_3-(s_3-1)), \dots, (0, -r_3-1), (0, -r_3), (0, -(r_3-1)), \dots, (0, -1), (0, 0)\big),\end{aligned}$$

be the collection obtained by extending the first and third vertical lines in $\mathcal{T}_1$ to length r_3+s_3 and r_1+s_1 respectively (see Figure 4.7). Then $\mathcal{T}_2$ is also an exceptional collection of length $s_1+r_1+s_3+r_3+(\alpha+1) = 2(\alpha+1)$.

Proof. This is an exceptional collection of $\mathcal{H}_{r_1+s_1, r_3+s_3}$ of Proposition 4.12. □

Proof of Theorem 4.15. For an exceptional collection of line bundles with $a_1 - a_r < 2$, we can extend each vertical line to length $\alpha+1$ to get an exceptional collection of maximum length. For an exceptional collection of line bundles with $a_1 - a_r = 2$, the result follows from Lemma 4.7, 4.16, 4.17. □

4.4 Strong exceptional collections of line bundles

In this section, we will determine when the exceptional collections of line bundles are strong exceptional collections. We first have the following criterion.

Lemma 4.18. *Let* $\mathrm{T} = \big((a_1, b_1), (a_2, b_2), \dots, (a_r, b_r)\big)$ *be an exceptional collection of line bundles. Then the collection* T *is further a strong exceptional collection if and only if $(a_i, b_i) - (a_j, b_j)$ is not in any of the Forbidden cones $F_{13}, F_\emptyset$ and F_{24} for any $i < j$.*

Proof. By Definition 2.2, the collection T is further a strong exceptional collection if and only if

$$\mathrm{H}^t(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{O}((a_j, b_j) - (a_i, b_i))) = \mathrm{Ext}^t(\mathcal{O}((a_i, b_i)), \mathcal{O}((a_j, b_j))) = 0$$

for any $i < j$ and $t > 0$. By Proposition 2.13, it is equivalent to that $(a_j, b_j) - (a_i, b_i)$ is not in any of the Forbidden cones $F_{13}, F_\emptyset$ and F_{24} for any $i < j$. □

Now we consider the exceptional collections of line bundles of maximum length.

$$\begin{aligned}\mathcal{H}_{r_1,r_3} =&\big((-2,n-\alpha),\dots,(-2,n-\alpha+r_1-2),(-2,n-\alpha+r_1-1),\\&(-1,b-\alpha),\dots,(-1,b-1),(-1,b),\\&(0,-(r_3-1)),\dots,(0,-1),(0,0)\big);\\ \mathcal{H}_0 =&\big((-1,d-\alpha),\dots,(-1,d-1),(-1,d),(0,-\alpha),\dots,(0,-1),(0,0)\big),\end{aligned}$$

where $b,d\in\mathbb{Z}$ and $r_1+r_3=\alpha+1$. The following two propositions give the criterions for the exceptional collection $\mathcal{H}_{r_1,r_3}$ and $\mathcal{H}_0$ to be strong.

Proposition 4.19. *The exceptional collection $\mathcal{H}_{r_1,r_3}$ is a strong exceptional collection only in the following two cases:*

1. *$n=\alpha$ and $b=r_1,r_1-1$;*

2. *$n=\alpha+1$ and $b=r_1$.*

Proof. For any two points $(x,b_i),(x,b_j)$ on the same vertical line in $\mathcal{H}_{r_1,r_3}$ such that $i<j$, we have $b_j-b_i>0$ since the second coordinates increase for points on the same vertical line. Thus $(x,b_j)-(x,b_i)=(0,b_j-b_i)$ is not in any of $F_{13},F_\emptyset$ and F_{24}. Therefore $\mathcal{H}_{r_1,r_3}$ is a strong exceptional collection if and only if it satisfies that

$$(0,t)-(-1,l)=(1,t-l),(-1,l)-(-2,k)=(1,l-k),(0,t)-(-2,k)=(2,t-k)$$

are not in any of $F_{13},F_\emptyset$ and F_{24} for $t=0,-1,\dots,-(r_3-1)$, $l=b,b-1,\dots,b-\alpha$ and $k=n-\alpha+(r_1-1),n-\alpha+(r_1-2),\dots,n-\alpha$.

Note that $(1,y),(2,y),y\in\mathbb{Z}$ are not in any of $F_{13},F_\emptyset$ and F_{24} if and only if $y\geq-\alpha$. Since the second coordinates of points in the same vertical line increase, we only have to check $t=-(r_3-1),l=b$ and $k=n-\alpha+(r_1-1)$. Therefore $\mathcal{H}_{r_1,r_3}$ is strong exceptional collection if and only if it satisfies (see Figure 4.8):

1. $-(r_3-1)-b\geq-\alpha$,

2. $b-\alpha-(n-\alpha+(r_1-1))=b-n-r_1+1\geq-\alpha$,

3. $-(r_3-1)-(n-\alpha+(r_1-1))\geq-\alpha$.

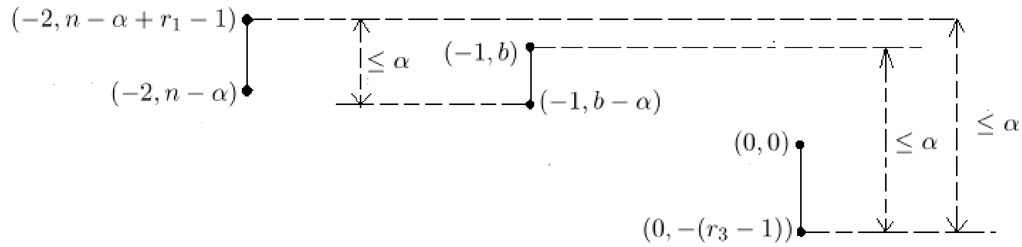

Figure 4.8:

The condition *1.* is equivalent to $\alpha - r_3 + 1 \geq b$. Since $r_1 + r_3 = \alpha + 1$, it is equivalent to $r_1 \geq b$. The condition *2.* is equivalent to $b \geq n - \alpha + r_1 - 1$. Since $r_1 + r_3 = \alpha$, we get $-(r_3-1)-(n-\alpha+(r_1-1)) = -(r_1+r_3)-n+\alpha+2 = -(\alpha+1)-n+\alpha+2 = -n+1$. Thus condition *3.* is equivalent to $\alpha + 1 \geq n$. Since $n \geq \alpha$, it is equivalent to $n \in \{\alpha, \alpha+1\}$.

When $n = \alpha$, we have (2) equivalent to $b \geq r_1 - 1$. Thus $\mathcal{H}_{r_1,r_3}$ is strong exceptional collection if and only if $b = r_1, r_1 - 1$. When $n = \alpha + 1$, condition *2.* is equivalent to $b \geq r_1$. Thus $\mathcal{H}_{r_1,r_3}$ is a strong exceptional collection if and only if $b = r_1$. □

Proposition 4.20. *The exceptional collection $\mathcal{H}_0$ is strong exceptional if and only if $d \leq 0$.*

Proof. With the similiar argument in Propostion 4.19, we get that $\mathcal{H}_0$ is strong if and only if $-\alpha - d \geq -\alpha$ which equailent to $0 \geq d$. □

We illustrate the results in the case $\alpha = 3, n = 4$.

Example 4.21. *Let us consider the toric DM stack in Example 4.14. We have $n = 4 = 3 + 1 = \alpha + 1$. We get*

$$\mathcal{H}_{1,3} = \big((-2,1),(-1,b_5-3),(-1,b_5-2),(-1,b_5-1),(-1,b_5),(0,-2),(0,-1),(0,0)\big)$$

is a strong exceptional collection if and only if $b_5 = 1$;

$$\mathcal{H}_{2,2} = \big((-2,1),(-2,2),(-1,b_6-3),(-1,b_6-2),(-1,b_6-1),(-1,b_6),(0,-1),(0,0)\big)$$

is a strong exceptional collection if and only if $b_6 = 2$;

$$\mathcal{H}_{3,1} = \big((-2,1),(-2,2),(-2,3),(-1,b_7-3),(-1,b_7-2),(-1,b_7-1),(-1,b_7),(0,0)\big)$$

is a strong exceptional collection if and only if $b_7 = 3$*;*

$$\mathcal{H}_0 = \big((-1, b_4 - 3), (-1, b_4 - 2), (-1, b_4 - 1), (-1, b_4), (0, -3), (0, -2), (0, -1), (0, 0)\big)$$

is a strong exceptional collection if and only if $b_4 \leq 0$*.*

4.5 Relation to toric systems on Hirzebruch surfaces

In this section, we will see that when $\alpha = 1$, the results in above sections match well the computation of so-called toric systems in [19] for Hirzebruch surfaces.

We first introduce the definition of toric systems.

Definition 4.22. *[19] A set of divisors* $\{A_i\}_{i=1}^n$ *on a projective rational surface* S *is called a toric system if it satisfies the following the conditions.*

1. $A_i \cdot A_{i+1} = 1$ *for* $1 \leq i < n$ *and* $A_1 \cdot A_n = 1$*;*
2. $A_i \cdot A_j = 1$ *for* $i \neq j$*,* $\{i, j\} \neq \{1, n\}$*, and* $\{i, j\} \neq \{k, k+1\}$ *for any* $1 \leq k < n$*;*
3. $\sum_{i=1}^n A_i = -K$*, where* K *is a canonical divisor of* S*.*

Next proposition gives the correspondence of toric systems and exceptional collection of line bundles.

Lemma 4.23. *[19] Let* $D_1, \dots, D_n$ *be divisors on a projective rational surface* S *such that* $\mathcal{O}(D_1), \dots, \mathcal{O}(D_n)$ *form an exceptional collection. Then* $A_i = D_{i+1} - D_i$ *for* $1 \leq i \leq n$ *and* $A_n = -K - \sum_{i=1}^{n-1} A_i$ *form a toric system.*

Proof. See [19]. □

When $\alpha = 1$, we have $v_1 = (1, 0), v_2 = (0, -1), v_3 = (-1, n), v_4 = (0, 1)$, where $n \geq 1$. The toric DM stacks $\mathbb{P}_{\mathbf{\Sigma}}$ associated to the stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\}_{i=1}^4)$ are the Hirzebruch surfaces $\mathbb{F}_n$. The Picard group is generated by E_1, E_2, E_3, E_4 with relations

$$E_1 = E_3, E_4 = E_2 - nE_3.$$

Let E_2, E_3 be basis, we have $E_1 = (0,1), E_2 = (1,0), E_3 = (0,1), E_4 = (1,-n)$ and the canonical divisor $K = -\sum_{i=1}^{4} E_4 = (-2, n-2)$. All forbidden sets are as follows.

$$F_{13} = (-2,n) + \mathbb{Z}_{\geq 0}(0,1) + \mathbb{Z}_{\geq 0}(-1,0) + \mathbb{Z}_{\geq 0}(-1,n)$$

$$F_{24} = (0,-2) + \mathbb{Z}_{\geq 0}(1,0) + \mathbb{Z}_{\geq 0}(1,-n) + \mathbb{Z}_{\geq 0}(0,-1)$$

$$F_{\emptyset} = (-2,n-2) + \mathbb{Z}_{\geq 0}(0,-1) + \mathbb{Z}_{\geq 0}(-1,0) + \mathbb{Z}_{\geq 0}(-1,n)$$

$$F_{\{1,2,3,4\}} = \mathbb{Z}_{\geq 0}(0,1) + \mathbb{Z}_{\geq 0}(0,1) + \mathbb{Z}_{\geq 0}(1,-n)$$

When $n = 2$, the forbidden sets are shown in Figure 4.9 For general n, all the H-trivial

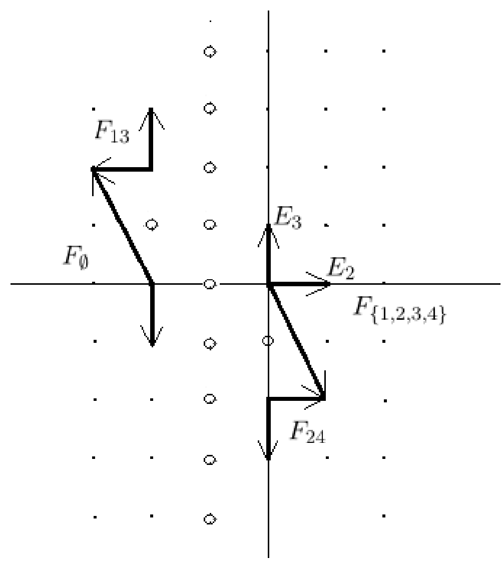

Figure 4.9:

line bundles are

$$(-2, n-1), \{(1,y) | y \in \mathbb{Z}\}, (0,-1).$$

By Proposition 4.12 and Equation 4.1, there are only two classes of exceptional collections of line bundles with maximum length:

$$\mathcal{H}_{1,1} = \big((-2,1), (-1,c-1), (-1,c), (0,0)\big),$$

$$\mathcal{H}_0 = \big((-1,b-1)(-1,b), (0,-1), (0,0)\big),$$

where $b, c \in \mathbb{Z}$.

This matches the results about toric systems on Hirzebruch surfaces $\mathbb{F}_n$ in [19]. The Proposition 5.2 in [19] is stated as follows with the notations in this thesis.

Proposition 4.24. *[19] There are the following toric systems on a Hirzebruch surface $\mathbb{F}_n$:*

1. $E_2, sE_2 + E_1, E_2, -(n+s)E_2 + E_1$ *for* $s \in \mathbb{Z}$ *and any* $n \geq 1$*;*

2. $-(n/2)E_2 + E_1, E_2 + s(-(n/2)E_2 + E_1), -(n/2)E_2 + E_1, E_2 - s(-(n/2)E_2 + E_1)$ *for* $s \in \mathbb{Z}$ *and* n *even.*

Toric systems of type (1) *are always exceptional. They are strongly exceptional for* $s \geq -1$*. They are cyclic strong exceptional if and only if* $s \geq -1$ *and* $n + s \leq 1$*.*

Toric system of type (2) *are almost never exceptional. The exceptions are for* $n = 0$*, where type* (2) *is symmetric to type* (1) *by exchanging* E_2 *and* E_1*, and for* $n = 2$ *and* $s = 0$*, which then coincides with a toric system of type* (1) *and is cyclic strongly exceptional.*

Let D_1, D_2, D_3, D_4 be the collection of divisors corresponding to the toric systems of type (1). By the Lemma 4.23, the collection falls into the following two cases:

1. $D_2 - D_1 = E_2, D_3 - D_2 = sE_2 + E_1, D_4 - D_3 = E_2$ for $s \in \mathbb{Z}$

2. $D_2 - D_1 = sE_2 + E_1, D_3 - D_2 = E_2, D_4 - D_3 = -(n+s)E_2 + E_1$ for $s \in \mathbb{Z}$ and any $n \geq 1$.

Without loss of generality, we let $D_4 = (0,0)$. Then we have the following two class of collection:

1. $\mathcal{W}_0 = \big(D_1 = (-1, -2-s), D_2 = (-1, -1-s), D_3 = (0, -1), D_4 = (0,0)\big)$ for $s \in \mathbb{Z}$;

2. $\mathcal{W}_1 = \big((D_1 = (-2, n-1), D_2 = (-1, n+s-1), D_3 = (-1, n+s), D_4 = (0,0)\big)$ for $s \in \mathbb{Z}$ and any $n \geq 1$.

By Proposition 4.24, we know there are exactly two classes of exceptional collections on $\mathbb{F}_n$ which are $\mathcal{W}_0$ and $\mathcal{W}_1$. In face $\mathcal{W}_0$ is the same class as $\mathcal{H}_0$ if let $b = -s-1$ and $\mathcal{W}_1$ is the same class as $\mathcal{H}_{11}$ if let $c = s+n$. By Proposition 4.20, the exceptional collection $\mathcal{W}_0$ is a strong exceptional collection if and only if $b = -s-1 \leq 0$ which is equivalent to $s \geq -1$. By Proposition 4.19, the exceptional collection $\mathcal{W}_1$ is a strong exceptional collection only at two cases that $n = 1, s = -1, 0$ and $n = 2, s = -1$. These match with the results in Proposition 4.24.

In [19], the authors introduced cyclic (strongly) exceptional collections.

Definition 4.25. *An infinite collection of sheaves* $\ldots, \mathcal{F}_i, \mathcal{F}_{i+1}, \ldots$ *is called a cyclic (strongly) exceptional collection if there exists an* m *such that* $\mathcal{F}_{i+m} \cong \mathcal{F}_i \otimes \mathcal{O}(-K)$ *for every* $i \in \mathbb{Z}$ *and if every winding (i.e. every subinterval* $\mathcal{F}_i, \ldots, \mathcal{F}_{i+m}$*) forms a (strongly) exceptional collection.*

From the Proposition 5.1 in [19], we know both $\mathcal{W}_0$ and $\mathcal{W}_1$ are cyclic exceptional collections. We consider the infinite collection

$$\ldots, (-3, -3-s+n), (-2, -3+n), (-2, -2+n), (-1, -2-s), (-1, -1-s), (0, -1), (0, 0), \ldots$$

constructed as Definition 4.25, where $(-2, -2+n) - K = (0, 0)$. The exceptional collection $\big((-2, -2+n), (-1, -2-s), (-1, -1-s), (0, -1)\big)$ is strong if and only if $-1-(-1-s) \geq -1, -1-(-2+n) \geq -1, -2-s-(-2+n) \geq -1$. We have that $-1-(-1-s) \geq -1$ is equivalent to $s \geq -1$. Also $-2-s-(-2+n) \geq -1$ is equivalent to $n+s \leq 1$. And $-1-(-2+n) \geq -1$ is equivalent to $n = 1, 2$. Thus the exceptional collection $\big((-2, -2+n), (-1, -2-s), (-1, -1-s), (0, -1)\big)$ is strong only at two cases that $n = 1, s = -1, 0$ and $n = 2, s = -1$. Actually, $\big((-2, -2+n), (-1, -2-s), (-1, -1-s), (0, -1)\big)$ is in the same class as $\mathcal{W}_1$ and $\mathcal{H}_{11}$.

Thus, the infinite collection

$$\ldots, (-3, -3-s+n), (-2, -3+n), (-2, -2+n), (-1, -2-s), (-1, -1-s), (0, -1), (0, 0), \ldots$$

is cyclic (strongly) exceptional collection only at two cases that $n = 1, s = -1, 0$ and $n = 2, s = -1$. This matches very well with the sufficient and necessary conditions in Proposition 4.24 for the infinite collection to be cyclic strongly exceptional. Indeed, the conditions $s \geq -1$ and $n + s \leq 1$ are equivalent to $-s \leq 1$ and $n \leq 1 - s$, which implies $n \leq 1 - s \leq 2$. Since $n \geq 1$, then $n = 1, 2$. When $n = 1$, the conditions $s + n \leq 1$ and $s \geq -1$ are equivalent to $-1 \leq s \leq 0$, i.e., $s = 0, -1$. When $n = 2$, the conditions $s + n \leq 1$ and $s \geq -1$ are equivalent to $-1 \leq s \leq -1$, i.e., $s = -1$.

4.6 Extending strong exceptional collections to maximum length

In this section, we analyze when a strong exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}} = \mathbb{F}_{\alpha,n}$ can be extended to a strong exceptional collection of line bundles of length $\mathrm{rk}(K(\mathbb{P}_{\mathbf{\Sigma}})) = 2(\alpha+1)$. If a strong exceptional collection of line bundles cannot be extend to length $2(\alpha+1)$, we consider what is the maximum length it can extend to.

We first consider to extend the strong exceptional collection of line bundles

$$\mathcal{T} = \{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$$

with $a_r - a_1 = 2$. By *1.*, *2.* in Lemma 4.5, we can assume

$$\begin{aligned}\mathcal{T} = \big(&(a_r-2, b_1), \ldots, (a_r-2, b_{r_1-1})(a_r-2, b_{r_1}),\\ &(a_r-1, b_{r_1+1}), \ldots, (a_r-1, b_{r_1+r_2-1}), (a_r-1, b_{r_1+r_2}),\\ &(a_r, b_{r_1+r_2+1}), \ldots, (a_1-2, b_{r-1}), (a_r, b_r)\big),\end{aligned}$$

where $1 \leq r_1, r_3 \leq \alpha+1$, $0 \leq r_2 \leq \alpha+1$, $r = r_1 + r_2 + r_3$ and the second coordinates of the points on the same vertical line increase.

Remark 13. *With the same argument in Proposition 4.19, exceptional collection $\mathcal{T}$ is a strong exceptional collection if and only if it satisfies:*

1. $b_{r_1+1} - b_{r_1} \geq -\alpha$;
2. $b_{r_1+r_2+1} - b_{r_1} \geq -\alpha$;
3. $b_{r_1+r_2+1} - b_{r_1+r_2} \geq -\alpha$.

Similarly, by *1.*, *2.* in Lemma 4.5, we can assume an exceptional collection of line bundles with $a_r - a_1 < 2$ to be

$$\begin{aligned}\mathcal{R} = \big(&(a_r-1, b_1), (a_r-1, b_2), \ldots, (a_r-1, b_{r_1}),\\ &(a_r, b_{r_1+1}), (a_r, b_{r_1+2}), \ldots, (a_r, b_r)\big),\end{aligned}$$

where $0 \leq r_2 \leq \alpha+1$, $1 \leq r_1 \leq \alpha+1$, $r = r_1 + r_2$ and the second coordinates of the points on the same vertical line increase. With the same argument in Proposition 4.20, exceptional collection $\mathcal{R}$ is a strong exceptional collection if and only if it satisfies $b_{r_1+1} - b_{r_1} \geq -\alpha$.

Corollary 4.26. *After inserting line bundles between line bundles on the same vertical lines in the strong exceptional collection to make the second coordinates to be consecutive numbers, we get an another strong exceptional collection with longer length.*

Proof. By Corollary 4.8, we know the collection is still an exceptional collection after inserting. The three conditions *1.*, *2.*, *3.* in Remark 13 still hold after inserting, which implies the new collection is a strong exceptional collection. □

Now we consider the case when there exist exceptional collections of line bundles of length $2(\alpha+1)$ on $\mathbb{F}_{\alpha,n}$, i.e., $n = \alpha, \alpha+1$. We will show at this case, any strong exceptional collection of line bundles with $a_r - a_1 = 2$ can be extended to a strong exceptional collection of line bundles of length $2(\alpha+1)$.

Remark 14. *Without loss of generality, we can assume a strong exceptional collection of line bundles with $a_r - a_1 = 2$ to be*

$$\begin{aligned}\mathcal{T} =\big(&(-2, c-(r_1-1)), \dots, (-2, c-1), (-2, c),\\ &(-1, b-(r_2-1)), \dots, (-1, b-1), (-1, b),\\ &(0, -(r_3-1)), \dots, (0, -1), (0, 0)\big),\end{aligned}$$

where $c, d \in \mathbb{Z}$, $0 \le r_2 \le \alpha+1$ and $1 \le r_1, r_3 \le \alpha+1$, which satisfies three conditions: $-(r_3-1) - b \ge -\alpha$, $-(r_3-1) - c \ge -\alpha$ and $b-(r_2-1)-c \ge -\alpha$.

Since $\mathcal{T}$ is an exceptional collection, we have that $(-2, c-(r_1-1))$ and $(-2, c)$ are H-trivial. Thus we have $n - \alpha \le c - (r_1 - 1)$ and $c \le n-1$. This implies $c - (n-\alpha) + 1 \le n - 1 - n + \alpha + 1 = \alpha$. So it is reasonable to extend the first vertical line to length $c - (n-\alpha) + 1$ in the following lemma.

Lemma 4.27. *Let*

$$\begin{aligned}\mathcal{T} =\big(&(-2, n-\alpha), \dots, (-2, c-r_1-1), (-2, c-r_1), (-2, c-(r_1-1)), \dots, (-2, c),\\ &(-1, b-(r_2-1)), \dots, (-1, b-1), (-1, b),\\ &(0, -(r_3-1)), \dots, (0, -1), (0, 0)\big),\end{aligned}$$

be the collection obtained by extend the first vertical line in $\mathcal{T}$. Then $\mathcal{T}_1$ is also a strong exceptional collection.

Proof. We first show $\mathcal{T}_1$ is an exceptional collection. It is sufficient to show $(-2,l)-(0,t)=(-2,l-t)$ is H-trivial for $l=c-r_1,\dots,n-\alpha$ and $t=0,-1,\dots,-(r_3-1)$. We have $l-t=n-\alpha,\dots,c-r_1+r_3-1$. We claim that $c-r_1+r_3-1\le n-1$. Since $\mathcal{T}$ is strong and by the three conditions in Remark 14, we have $c\le -r_3+1+\alpha$. This implies $c-r_1+r_3-1\le -r_1+r_3-1-r_3+1+\alpha=\alpha-r_1$. If $c-r_1+r_3-1>n-1$, we get $\alpha-r_1>n-1$. When $n=\alpha$, we get $r_1<1$, which contradicts to the assumption. When $n=\alpha+1$, we get $r_1<0$, which contradicts to the assumption. So we have $n-\alpha\le l-t\le n-1$ which implies $(-2,l-t)$ is H-trivial.

Since extending the third line will not change the three conditions in Remark 14, the exceptional collection $\mathcal{T}_1$ is a strong exceptional collection. □

Case 1 We first consider the case that $n=\alpha+1$. In this case, the strong exceptional collection

$$\begin{aligned}\mathcal{T}_1=\big(&(-2,1),\dots,(-2,c-1),(-2,c),\\&(-1,b-(r_2-1)),\dots,(-1,b-1),(-1,b),\\&(0,-(r_3-1)),\dots,(0,-1),(0,0)\big),\end{aligned}$$

where $c,b\in\mathbb{Z}$ and $1\le r_3,r_2,c\le\alpha+1$, which satisfies three conditions: $-(r_3-1)-b\ge-\alpha$, $-(r_3-1)-c\ge-\alpha$ and $b-(r_2-1)-c\ge-\alpha$.

We consider two subcases that $b\le c$ and $b>c$ separately.

If $b\le c$, we consider extending the middle vertical line as follows:

$$\big((-1,b-(r_2-1))\dots,(-1,b-1),(-1,b),(-1,b+1),\dots,(-1,c-1),(-1,c)\big).$$

This vertical line has length $c-b+r_2$. Since $b-(r_2-1)-c\ge-\alpha$, we get $c-b+r_2\le\alpha+1$. Thus it is reasonable to consider further extending the vertical line to length $\alpha+1$ as follows:

$$\big((-1,b-r_2-(s_2-1)),\dots,(-1,b-r_2)(-1,b-(r_2-1))\dots,(-1,b),(-1,b+1),\dots,(-1,c)\big),$$

where $s_2=\alpha+1-(c-b+r_2)$ (see Figure 4.10). By Lemma 4.10, we have that the total length of the first and third vertical lines is $c+r_3\le\alpha+1$. It is reasonable to consider extending the third vertical line to length $\alpha+1-c$ as follows

$$\big((0,-r_3-(s_3-1)),\dots,(0,-r_3),(0,-(r_3-1)),\dots,(0,-1),(0,0)\big),$$

where $s_3 = \alpha + 1 - r_3 - c$ (see Figure 4.10).

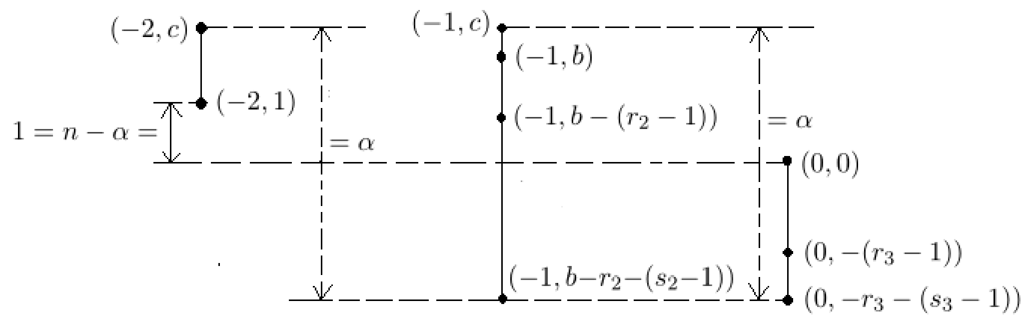

Figure 4.10:

In following proposition, we show that the strong exceptional collection is still a SEC after extending as above.

Proposition 4.28. *If $b \leq c$, let $s_2 = \alpha + 1 - (c - b + r_2)$ and $s_3 = \alpha + 1 - r_3 - c$. Then the collection*

$$\begin{aligned}\mathcal{T}_2 = \big(&(-2,1), \dots, (-2, c-1), (-2, c),\\ &(-1, b - r_2 - (s_2 - 1)), \dots, (-1, b - r_2)(-1, b - (r_2 - 1)) \dots, (-1, b), \dots, (-1, c),\\ &(0, -r_3 - (s_3 - 1)), \dots, (0, -r_3), (0, -(r_3 - 1)), \dots, (0, -1), (0, 0)\big)\end{aligned}$$

is a strong exceptional collection with maximum length.

Proof. We first show $\mathcal{T}_2$ is an exceptional collection. It is sufficient to show $(-2, l) - (0, -r_3+t) = (-2, l+r_3-t)$ is H-trivial for $l = 1, \dots, c$ and $t = -(s_3-1), \dots, 0$. We have $1+r_3 \leq l+r_3 \leq c+r_3$ and $0 \leq -t \leq s_3-1$. Thus $1 \leq 1+r_3 \leq l+r_3-t \leq c+r_3+s_3-1 = \alpha$. When $n = \alpha + 1$, we have $n - 1 = \alpha$ and $n - \alpha = 1$. So $(-2, l + r_3 - t)$ is H-trivial.

Then we claim that $\mathcal{T}_2$ is a strong exceptional collection. By the definition of s_3, we have $-r_3 - (s_3 - 1) - c = -\alpha$. By the definition of s_2, we have $b - r_2 - (s_2 - 1) - c = -\alpha$ (see Figure 4.10). Thus $\mathcal{T}_2$ satisfies the three conditions *1.*, *2.*, *3.* in 13, which implies our claim.

The length of the collection equals $r_3 + s_3 + (\alpha + 1) + c = 2(\alpha + 1)$. □

If $b > c$, we consider extending the third vertical line of $\mathcal{T}_1$ to length b as follow:

$$\big((-2, 1), (-2, 2), \dots, (-2, c-1), (-2, c), (-2, c+1), \dots, (-2, b-1), (-2, b)\big).$$

Since $\mathcal{T}_1$ is a strong exceptional collection, we have $-(r_3-1)-b \geq -\alpha$. Then we get $b \leq \alpha - r_3 + 1 \leq \alpha$ since $r_3 \geq 1$. It is reasonable that the length of the new vertical line is not more than α. Then we consider extending the middle vertical line to length $\alpha+1$ as follows:

$$\big((-1, b-r_2-(s_2-1)), \dots, (-1, b-r_2), (-1, b-(r_2-1)), \dots, (-1, b-1), (-1, b)\big),$$

where $s_2 = \alpha + 1 - r_2$. Also since $-(r_3-1)-b \geq -\alpha$, so the sum of the lengths of the new third and the first vertical line in $\mathcal{T}_1$ is $b + r_3 \leq \alpha + 1$. It is reasonable to consider extending the third vertical line in $\mathcal{T}_1$ as follows:

$$\big((0, -r_3-(s_3-1)), \dots, (0, -r_3), (0, -(r_3-1)) \dots, (0, -1), (0, 0)\big),$$

where $s_3 = \alpha + 1 - b - r_3$.

In following proposition, we show that the strong exceptional collection is still a SEC after extending as above.

Proposition 4.29. *If $b > c$, let $s_3 = \alpha + 1 - b - r_3$ and $s_2 = \alpha + 1 - r_2$.*

$$\begin{aligned}\mathcal{T}_3 = &\big((-2,1), (-2,2), \dots, (-2, c-1), (-2, c), (-2, c+1), \dots, (-2, b-1), (-2, b),\\ &(-1, b-r_2-(s_2-1)), \dots, (-1, b-r_2), (-1, b-(r_2-1)), \dots, (-1, b-1), (-1, b),\\ &(0, -r_3-(s_3-1)), \dots, (0, -r_3), (0, -(r_3-1)) \dots, (0, -1), (0, 0)\big)\end{aligned}$$

is a strong exceptional collection with maximum length.

Proof. We first show $\mathcal{T}_3$ is an exceptional collection. It is sufficient to show $(-2, t) - (0, -r_3+l) = (-2, t+r_3-l)$ is H-trivial for $l = -(s_3-1), \dots, 0$ and $t = 1, \dots, b$. We have $1+r_3 \leq t+r_3 \leq b+r_3$ and $0 \leq -l \leq s_3-1$. Thus $1 \leq 1+r_3 \leq t+r_3-l \leq b+r_3+s_3-1 = \alpha$ by the definition of s_3 and $r_3 \geq 1$. When $n = \alpha+1$, we have $n-1 = \alpha$ and $n - \alpha = 1$. Thus $(-2, t+r_3-l)$ is H-trivial.

Then we claim that $\mathcal{T}_3$ is a strong exceptional collection. By the definition of s_2, we have $b - r_2 - (s_2-1) - b = -\alpha$. By the definition of s_3, we have $-r_3-(s_3-1)-b = -\alpha$. Thus $\mathcal{T}_3$ satisfies the three conditions *1.*, *2.*, *3.* in 13, which implies our claim. ∎

Case 2 We consider the case that $n = \alpha$. In this case, the strong exceptional collection

$$\begin{aligned}\mathcal{T}_1 = \big(&(-2,0),\ldots,(-2,c-1),(-2,c),\\ &(-1,b-(r_2-1)),\ldots,(-1,b-1),(-1,b),\\ &(0,-(r_3-1)),\ldots,(0,-1),(0,0)\big),\end{aligned}$$

where $c, b \in \mathbb{Z}$ and $1 \leq r_3, r_2, c+1 \leq \alpha+1$, which satisfies three conditions: $-(r_3-1)-b \geq -\alpha$, $-(r_3-1)-c \geq -\alpha$ and $b-(r_2-1)-c \geq -\alpha$.

We consider two subcases that $b \leq c$ and $b > c$ separately. The arguments are similar as that of **Case 1**. When $b > c$, we consider extending the first vertical of $\mathcal{T}_1$ to length b as follow:

$$\big((-2,0),\ldots,(-2,c-1),(-2,c),(-2,c+1),\ldots,(-2,b-1)\big),$$

which is a little different from **Case 1**. The details of proof are left to the reader.

Theorem 4.30. *When $n = \alpha, \alpha+1$, any strong exceptional collection of line bundles with $a_r - a_1 = 2$ can be extended to a strong exceptional collection of line bundles of maximum length $2(\alpha+1)$.*

Proof. The result is implied by Remark 14, Proposition 4.28,4.29 and **Case 2**. □

Remark 15. *By Proposition 4.19, we know when $n > \alpha+1$, there are no strong exceptional collections of line bundles of length $2(\alpha+1)$ with three vertical lines on generalized Hirzebruch surfaces $\mathbb{F}_{\alpha,n}$. It is natural to ask what is the maximum length of strong exceptional collection of line bundles with three vertical lines on generalized generalized Hirzebruch surfaces.*

We will show that when $n > \alpha+1$, any strong exceptional collections of line bundles with $a_r - a_1 = 2$ on generalized Hirzebruch surfaces $\mathbb{F}_{\alpha,n}$ can be extend to a strong exceptional collections of line bundles of length $3\alpha+3-n$.

We need some preparations to prove it. By Remark 14, we can assume a strong exceptional collection of line bundles with $a_r - a_1 = 2$ to be

$$\begin{aligned}\mathcal{T} =&\big((-2, c-(r_1-1)), \dots, (-2, c-1), (-2, c),\\ &(-1, b-(r_2-1)), \dots, (-1, b-1), (-1, b),\\ &(0, -(r_3-1)), \dots, (0, -1), (0, 0)\big),\end{aligned}$$

where $c, d \in \mathbb{Z}$, $0 \le r_2 \le \alpha + 1$ and $1 \le r_1, r_3 \le \alpha + 1$, which satisfies three conditions: $-(r_3 - 1) - b \ge -\alpha$, $-(r_3 - 1) - c \ge -\alpha$ and $b - (r_2 - 1) - c \ge -\alpha$.

It is easy to see that the three conditions are equivalent to the vertical distances between the following three pairs of points are no more than α (see Figure 4.11).

$$\{(0, -(r_3-1)), (-1, b)\}, \{(0, -(r_3-1)), (-2, c)\}, \{(-1, b-(r_2-1)), (-2, c)\}$$

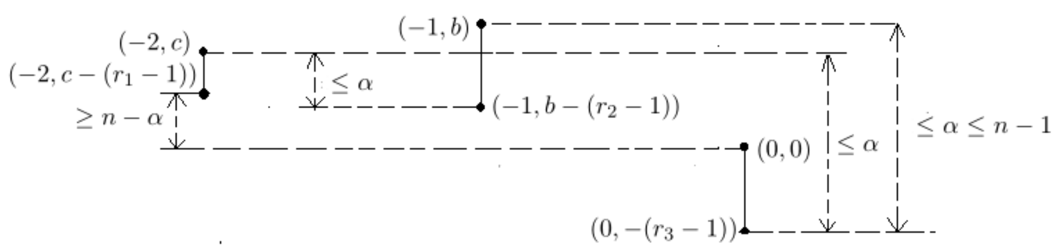

Figure 4.11:

We consider two subcases that $b > c$ and $b \le c$ separately. We first consider the subcase of $b > c$ (see Figure 4.11).

Proposition 4.31. *If $b > c$, let $s_1 = (c - (r_1 - 1)) - (n - \alpha)$, $s_2 = (\alpha + 1) - r_2$ and $s_3 = \alpha - (r_3 - 1 + b)$. Then the collection (see Figure 4.12)*

$$\begin{aligned}\mathcal{T}_1 =&\big((-2, c-r_1-(s_1-1)), \dots, (-2, c-r_1)), (-2, c-(r_1-1)), \dots, (-2, c), \dots, (-2, b),\\ &(-1, b-r_2-(s_2-1)), \dots, (-1, b-r_2)), (-1, b-(r_2-1)), \dots, (-1, b-1), (-1, b),\\ &(0, -r_3-(s_3-1)), \dots, (0, -r_3)), (0, -(r_3-1)), \dots, (0, -1), (0, 0)\big),\end{aligned}$$

obtained by extending $\mathcal{T}$ is a strong exceptional collection of line bundles with length $3\alpha + 3 - n$. And $\mathcal{T}_1$ cannot extend any more.

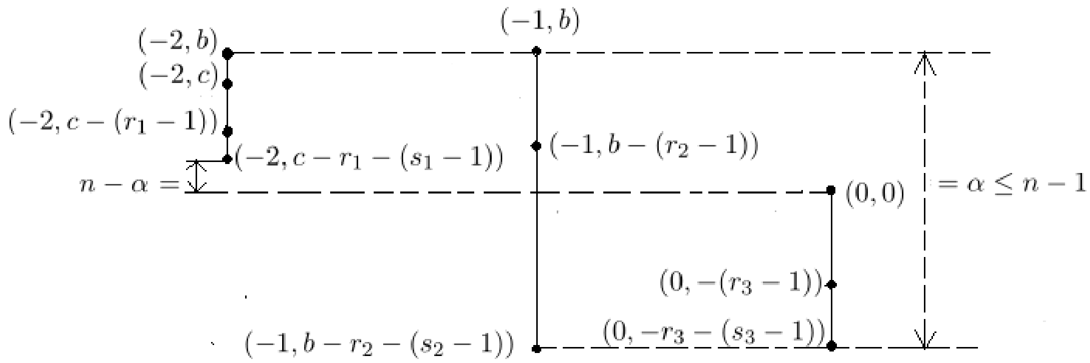

Figure 4.12:

Proof. We first show $\mathcal{T}_1$ is an exceptional collection. It is sufficient to show $(-2,t)-(0,l)=(-2,t+r_3-l)$ is H-trivial for $l=-r_3-(s_3-1),\dots,0$ and $t=c-r_1-(s_1-1),\dots,b$. Then we have $t-l=c-r_1-(s_1-1),\dots,b+r_3+(s_3-1)$. The vertical distance of the two points $(-2,c-r_1-(s_1-1))$ and $(0,0)$ is $c-r_1-(s_1-1)=n-\alpha$ by the definition of s_1. And the vertical distance of the two points $(-2,b)$ and $(0,-r_3-(s_3-1))$ is $b+r_3+(s_3-1)=-\alpha$ by the definition of s_1. Thus the first and third vertical line cannot extand anymore. Also $n-\alpha\le t-l\le-\alpha\le n-1$ since $n\ge\alpha+1$. This implies that $(-2,t)-(0,l)=(-2,t+r_3-l)$ is H-trivial.

Then we claim that $\mathcal{T}_1$ is a strong exceptional collection. By the definition of s_2, we have $b-r_2-(s_2-1)-b=-\alpha$. By the definition of s_3, we have $-r_3-(s_3-1)-b=-\alpha$. Thus $\mathcal{T}_1$ satisfies the three conditions *1., 2., 3.* in 13 and the second vertical cannot extend anymore, which implies our claim.

The total length of $\mathcal{T}_1$ is $s_1+r_1+(b-c)+r_2+s_2+r_3+s_3=3\alpha+3-n$ □

Now we consider the subcase of $b\le c$.

Proposition 4.32. *When $b\le c$ the collection $\mathcal{T}$ can be extend to a strong exceptional collection of line bundles on $\mathbb{F}_{\alpha,n}$ with length $3\alpha+3-n$ which cannot be extend further*

Proof. The result is obtained by similar arguments in Proposition 4.31. □

Theorem 4.33. *For $n=\alpha,\alpha+1$, any strong exceptional collection of line bundles on generalized Hirzebruch surfaces $\mathbb{F}_{\alpha,n}$ with $a_r-a_1=2$ can be extended to a strong* *exceptional collection of line bundles on* $\mathbb{F}_{\alpha,n}$ *of maximum length* $2\alpha+2$*. For* $n > \alpha+1$*, any strong exceptional collection of line bundles on generalized Hirzebruch surfaces* $\mathbb{F}_{\alpha,n}$ *with* $a_r - a_1 = 2$ *can be extended to a strong exceptional collection of line bundles on* $\mathbb{F}_{\alpha,n}$ *of length* $3\alpha + 3 - n$ *which cannot be extend anymore.*

Proof. The result is implied by Theorem 4.33, Proposition 4.31 and Proposition 4.32 □

Now let us consider a strong exceptional collection of line bundles with $a_r - a_1 < 2$ for $n \geq \alpha$.

By Corollary 4.26, we can assume a strong exceptional collection of line bundles with $a_r - a_1 < 2$ to be

$$\begin{aligned}\mathcal{R} =\big(&(a-1, c-(r_2-1)), \ldots, (a-1, c-1), (a-1, c),\\ &(0, b-(r_1-1)), \ldots, (a, b-1), (a, b)\big),\end{aligned}$$

where $a, b, c \in \mathbb{Z}$, $0 \leq r_2 \leq \alpha+1$ and $1 \leq r_1, r_2 \leq \alpha+1$, which satisfies $b-(r_1-1)-c \geq -\alpha$.

Proposition 4.34. *Let* $s_1 = \alpha + 1 - r_1$ *and* $s_2 = \alpha + 1 - r_2$*. Then the collection*

$$\begin{aligned}\mathcal{R}_1 =\big(&(a-1, c-r_2-(s_2-1)) \ldots, (a-1, c-r_2), (a-1, c-(r_2-1)), \ldots, (a-1, c),\\ &(0, b-(r_1-1)), \ldots, (a, b-1), (a, b), (a, b+1) \ldots, (a, b+s_1)\big)\end{aligned}$$

is a strong exceptional collection for maximum length $2\alpha + 2$*.*

Proof. Since $(-1, y)$ is H-trivial for any $y \in \mathbb{Z}$, so $(a-1, l) - (a, t) = (-1, l-t)$ is H-trivial for any $l, t \in \mathbb{Z}$. Thus $\mathcal{R}_1$ is an exceptional collection. Also, the condition $b-(r_1-1)-c \geq -\alpha$ still hold after extending, which implies our result. □

Theorem 4.35. *For* $n \geq \alpha$*, any strong exceptional collection of line bundles on on* $\mathbb{F}_{\alpha,n}$ *with* $a_r - a_1 < 2$ *can be extended to a strong exceptional collection of line bundles on* $\mathbb{F}_{\alpha,n}$ *of maximum length* $2\alpha + 2$*.*

Proof. The result is implied by Proposition 4.34. □

4.7 Exceptional collections of maximal length generate the derived category

In this section, we show that any exceptional collection of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ of maximum length generates $\mathcal{D}$, the bounded derived category of coherent sheaves on $\mathbb{P}_{\mathbf{\Sigma}}$.

It is shown in [5] that the bounded derived category of coherent sheaves on $\mathbb{P}_{\mathbf{\Sigma}}$ is generated by the invertible sheaves. Also since any such collection is a shift of one of $\mathcal{H}_{r_1,r_3}, \mathcal{H}_0$, it is sufficient to show that each of $\mathcal{H}_{r_1,r_3}, \mathcal{H}_0$ generates all line bundles.

Lemma 4.36. *If the points $(a,b),(a,b-1),\ldots,(a,b-\alpha)$ on the same vertical line are in the derived category, then all the other points $\{(a,y)|y\in\mathbb{Z}\}$ are also in the derived category.*

Proof. We consider the Koszul complex

$$0\to\mathcal{O}(-E_1-E_3)\to\mathcal{O}(-E_1)\oplus\mathcal{O}(-E_3)\to\mathcal{O}\to 0. \tag{4.2}$$

We tensor the complex by $\mathcal{O}(a,b)$ to get

$$0\to\mathcal{O}(a,b-\alpha-1)\to\mathcal{O}(a,b-\alpha)\oplus\mathcal{O}(a,b-1)\to\mathcal{O}(a,b)\to 0. \tag{4.3}$$

This implies $\mathcal{O}(a,b-\alpha-1)$ is in the derived category. To get that $\mathcal{O}(a,b-\alpha-2)$ is in the derived category, we further tensor the complex 4.3 by $\mathcal{O}(0,-1)$. By repeating the process, we obtain that $\mathcal{O}(a,b-\alpha-y)$ is in the derived category for any $y\in\mathbb{Z}_{\geq 0}$.

Then we tensor the complex 4.2 by $\mathcal{O}(a,b+1)$ to get

$$0\to\mathcal{O}(a,b-\alpha)\to\mathcal{O}(a,b-\alpha+1)\oplus\mathcal{O}(a,b)\to\mathcal{O}(a,b+1)\to 0.$$

This implies $\mathcal{O}(a,b-\alpha+1)$ is in the derived category. By similar argument, we can show it is true for $\mathcal{O}(a,b-\alpha+y)$ for any $y\in\mathbb{Z}_{\geq 0}$. This leads to our result. □

The following example illustrate the lemma more concretely.

Example 4.37. *Let us consider the toric DM stack in Example 4.14. We have $n = 4 = 3+1 = \alpha+1$ and $E_1=(0,3), E_2=(1,0), E_3=(0,1), E_4=(1,-4)$. The Koszul* *complex 4.2 in Lemma 4.36 is*

$$0 \to \mathcal{O}(0,-4) \to \mathcal{O}(0,-3) \oplus \mathcal{O}(0,-1) \to \mathcal{O} \to 0.$$

We consider an exceptional collection

$$\mathcal{H}_{3,1} = \big((-2,1),(-2,2),(-2,3),(-1,b-3),(-1,b-2),(-1,b-1),(-1,b),(0,0)\big),$$

where $b \in \mathbb{Z}$*. The middle vertical line in* $\mathcal{H}_{3,1}$ *contains* $(-1,b),(-1,b-1),\dots,(-1,b-3)$*. After tensor the Koszul complex with* $\mathcal{O}(-1,b)$*, we get the complex in Lemma 4.36 as follows:*

$$0 \to \mathcal{O}(-1,b-4) \to \mathcal{O}(-1,b-3) \oplus \mathcal{O}(-1,b-1) \to \mathcal{O}(-1,b) \to 0.$$

Thus $\mathcal{O}(-1,b-4)$ *in the derived category generated by* $\mathcal{H}_{3,1}$*. We further tensor the complex with* $(0,1)$ *to get*

$$0 \to \mathcal{O}(-1,b-3) \to \mathcal{O}(-1,b-2) \oplus \mathcal{O}(-1,b) \to \mathcal{O}(-1,b+1) \to 0.$$

This implies $\mathcal{O}(-1,b+1)$ *in the derived category generated by* $\mathcal{H}_{3,1}$*. With the argument in the proof of Lemma 4.36, we get* $\mathcal{O}(-1,y)$ *in the derived category for all* $y \in \mathbb{Z}$*.*

Proposition 4.38. *All line bundles are in the derived category generated by* $\mathcal{H}_0$*.*

Proof. Let $D(\mathcal{H}_0)$ be the derived category generated by $\mathcal{H}_0$. By Lemma 4.36, we get $\mathcal{O}(0,y), \mathcal{O}(-1,y) \in D(\mathcal{H}_0)$ for any $y \in \mathbb{Z}$. Then we consider the Koszul complex

$$0 \to \mathcal{O}(-E_2-E_4) \to \mathcal{O}(-E_2) \oplus \mathcal{O}(-E_4) \to \mathcal{O} \to 0. \tag{4.4}$$

This implies $\mathcal{O}(-2,n) = \mathcal{O}(-E_2-E_4) \in D(\mathcal{H}_0)$. We tensor the Koszul complex 4.4 by $\mathcal{O}(0,-1)$ and get $\mathcal{O}(-2,n-1) \in D(\mathcal{H}_0)$. Similarly, we tensor the Koszul complex by $\mathcal{O}(0,-t)$ and get $\mathcal{O}(-2,n-t) \in D(\mathcal{H}_0)$ for $t = 2,\dots,\alpha$. Then by Lemma 4.36, we get $\mathcal{O}(-2,y) \in D(\mathcal{H}_0)$ for any $y \in \mathbb{Z}$. With the similar argument for any given $x \in \mathbb{Z}$, we have $\{(x,y) | y \in \mathbb{Z}\} \subseteq D(\mathcal{H}_0)$. This leads to our conclusion. □

The following example helps us understand the lemma more concretely.

Example 4.39. *Let us consider the exceptional collection*

$$\mathcal{H}_0 = \big((-1, c-3), (-1, c-2), (-1, c-1), (-1, c), (0, -3), (0, -2), (0, -1), (0, 0)\big),$$

where $c \in \mathbb{Z}$*, on the toric DM stack in Example 4.14. We have* $n = 4 = 3 + 1 = \alpha + 1$*,* $E_1 = (0, 3), E_2 = (1, 0), E_3 = (0, 1), E_4 = (1, -4)$*. By Lemma 4.36, we get* $\mathcal{O}(0, y), \mathcal{O}(-1, y) \in D(\mathcal{H}_0)$ *for any* $y \in \mathbb{Z}\}$*. The Koszul complex 4.4 in Proposition 4.38 is*

$$0 \to \mathcal{O}(-2, 4) \to \mathcal{O}(-1, 0) \oplus \mathcal{O}(-1, 4) \to \mathcal{O} \to 0.$$

See the parallelogram in Figure 4.13. This implies $\mathcal{O}(-2, 4) \in D(\mathcal{H}_0)$*. We tensor the complex with* $(0, -1), (0, -2), (0, -3)$ *to get* $(-2, 1), (-2, 2), (-2, 3) \in D(\mathcal{H}_0)$*. Then by Lemma 4.36 again, we have* $\mathcal{O}(-2, y) \in D(\mathcal{H}_0)$ *for any* $y \in \mathbb{Z}$*.*

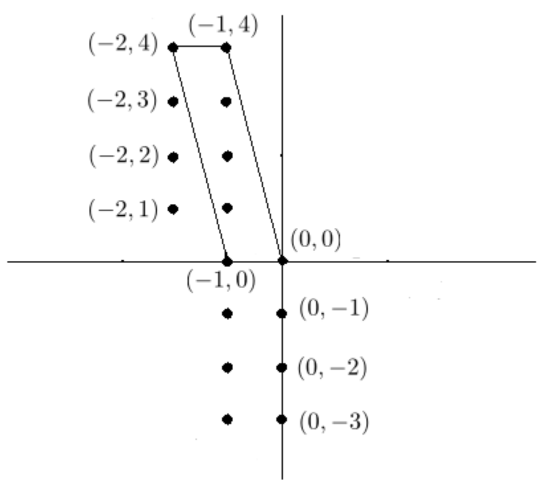

Figure 4.13:

Proposition 4.40. *For any given pair* $r_1, r_3 \in \{1, \dots, \alpha\}$ *such that* $r_1 + r_3 = \alpha + 1$*, all line bundles are in the derived category generated by* $\mathcal{H}_{r_1, r_3}$*.*

Proof. We have

$$\begin{aligned}\mathcal{H}_{r_1,r_3} = &\big((-2, n-\alpha), \dots, (-2, n-\alpha+r_1-2), (-2, n-\alpha+r_1-1),\\ &(-1, b_{r_1+r_2}-\alpha), \dots, (-1, b_{r_1+r_2}-1), (-1, b_{r_1+r_2}),\\ &(0, -(r_3-1)), \dots, (0, -1), (0, 0)\big),\end{aligned}$$

where $b_{r_1+r_2} \in \mathbb{Z}$. Let $D(\mathcal{H}_{r_1,r_3})$ be the derived category generated by $\mathcal{H}_{r_1,r_3}$. By Lemma 4.36, we get $\mathcal{O}(-1,y) \in D(\mathcal{H}_{r_1,r_3})$ for any $y \in \mathbb{Z}$. We consider the Koszul complex

$$0 \to \mathcal{O}(-E_2 - E_4) \to \mathcal{O}(-E_2) \oplus \mathcal{O}(-E_4) \to \mathcal{O} \to 0. \tag{4.5}$$

Sine $E_2 = (1,0)$ and $E_4 = (1,-n)$, we tensor the complex 4.5 with $\mathcal{O}(0,-\alpha)$ to get

$$0 \to \mathcal{O}(-2, n-\alpha) \to \mathcal{O}(-1,-\alpha) \oplus \mathcal{O}(-1, n-\alpha) \to \mathcal{O}(0,-\alpha) \to 0.$$

Thus $\mathcal{O}(0,-\alpha) \in D(\mathcal{H}_{r_1,r_3})$. Similarly, we get $\mathcal{O}(0,-\alpha+i) \in D(\mathcal{H}_{r_1,r_3})$ for $i = 1,\dots,r_1-1$ by tensoring the complex 4.5 with $\mathcal{O}(0,-\alpha+i)$. Since $r_1+r_3 = \alpha+1$, we have $-\alpha+(r_1-1) = -(r_3-1)-1$. So we get $\alpha+1$ line bundles $(0,-\alpha),(0,-\alpha+1),\dots,(0,-(r_3-1)-1),(0,-(r_3-1)),\dots,(0,-1),(0,0) \in D(\mathcal{H}_{r_1,r_3})$. Then by Lemma 4.36, we get $\mathcal{O}(0,y) \in D(\mathcal{H}_{r_1,r_3})$ for any $y \in \mathbb{Z}$. Then the final result is implied by the same arguments as in Proposition 4.38. □

The following example illustrate the lemma in a more concrete way.

Example 4.41. *Let us consider the exceptional collection*

$$\mathcal{H}_{1,3} = \big((-2,1),(-1,d-3),(-1,d-2),(-1,d-1),(-1,d),(0,-2),(0,-1),(0,0)\big)$$

on the toric DM stack in Example 4.14. By Lemma 4.36, we get $\mathcal{O}(-1,y) \in D(\mathcal{H}_{1,3})$ for any $y \in \mathbb{Z}$. Since $E_1 = (0,3), E_2 = (1,0), E_3 = (0,1), E_4 = (1,-4)$, the Koszul

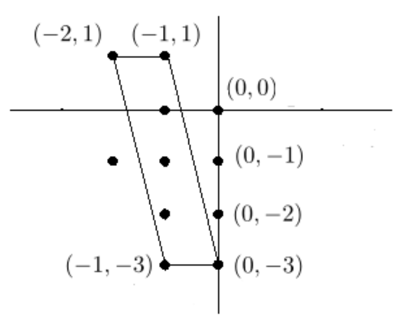

Figure 4.14:

complex 4.5 in Proposition 4.40 is

$$0 \to \mathcal{O}(-2,4) \to \mathcal{O}(-1,0) \oplus \mathcal{O}(-1,4) \to \mathcal{O} \to 0.$$

We tensor the complex by $\mathcal{O}(0,-3)$ *to get*

$$0 \to \mathcal{O}(-2,1) \to \mathcal{O}(-1,-3) \oplus \mathcal{O}(-1,1) \to \mathcal{O}(0,-3) \to 0.$$

See the parallelogram in Figure 4.14. Thus $\mathcal{O}(0,-3) \in \mathcal{D}(\mathcal{H}_{1,3})$. *Thus we have* $(0,-3)$, $(0,-2)$, $(0,-1)$, $(0,0) \in \mathcal{D}(\mathcal{H}_{1,3})$. *Then by Lemma 4.36, we get* $\mathcal{O}(0,y) \in D(\mathcal{H}_{r_1,r_3})$ *for any* $y \in \mathbb{Z}$. *Then the final result is implied by the same arguments as in Proposition 4.38.*

References

[1] V. Alexeev and D. Orlov. Derived categories of Burniat surfaces and exceptional collections. *Math. Ann.*, 357 (2013), no. 2, 743-759.

[2] C. Böhning, H-Ch. Graf von Bothmer, L. Katzarkov, and P. Sosna. Determinantal Barlow surfaces and phantom categories. *J. Eur. Math. Soc. (JEMS)*, 17(2015), no. 7, 1569-1592.

[3] C. Böhning, H-Ch. Graf von Bothmer, and P. Sosna. On the derived category of the classical Godeaux surface. *Adv. Math.*, 243(2013), 203-231.

[4] L. Borisov, L. Chen, and G. G. Smith. The orbifold Chow ring of toric Deligne-Mumforrd stacks. *J. Amer. Math. Soc.*, 18(2005), no. 1, 193-215.

[5] L. Borisov and R. Horja. On the K-theory of smooth toric DM stacks. *Snowbird lectures on string geometry*, 21-42, Contemp. Math., 401, Amer. Math. Soc., Providence, RI, 2006.

[6] L. Borisov and Z. Hua. On the conjecture of King for smooth toric Deligne-Mumford stacks. *Adv. Math.*, 221(2009), 277-301.

[7] L. Borisov and C. Wang. On H-trivial line bundles on toric DM stacks of dim $\geq$ 3. *preprint, 2019*, arXiv:1904.00799.

[8] T. Bridgeland. Derived categories of coherent sheaves. *In Proceedings of the international congress of mathematicians (ICM)*, Madrid, Spain, August 22-30, 2006. Volume II: Invited lectures, pages 563-582. Zürich: European Mathematical Society (EMS), 2006.

[9] H. Cartan and S. Eilenberg. Homological algebra. *Princeton Univ. Press, Princeton, NJ 1956*, Russian transl., Izdat. Inostr. Lit., Moscow 1960.

[10] D. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1995), no. 1, 17-50.

[11] V. I. Danilov. The geometry of toric varieties. *Russian Math. Surveys*, 33(1978), no. 2, 97-154.

[12] M. R. Douglas. D-branes, Categories and N=1 Supersymmetry. *J. Math. Phys.*, 42(2001), 2818-2843.

[13] A. I. Efimov. Maximal lengths of exceptional collections of line bundles. *J. London Math. Soc.*, (2)90(2014), 350-372.

[14] W. Fulton. Introduction to toric varieties. *Annals of Mathematics Studies 131*, Princeton University Press, Princeton, NJ, 1993.

[15] S. Galkin and E. Shinder. Exceptional collections of line bundles on the Beauville surface. *Adv. Math.*, 244(2013), 1033-1050.

[16] S. Gorchinskiy and D. Orlov. Geometric phantom categories. *Publ. Math. Inst. Hautes études Sci.*, 117(2013), 329-349.

[17] A. Grothendieck. Sur quelques points d'algébre homologique. *Tôhoku Math. J.*, (2) 9(1957), 119-221.

[18] L. Hille and M. Perling. A counterexample to King's conjecture. *Compos. Math*, 142(2006), 1507-1521.

[19] L. Hille and M. Perling. Exceptional sequences of invertible sheaves on rational surfaces. *Compos. Math*, 147(2011), 1230-1280.

[20] D. Huybrechts. Fourier-Mukai transforms in algebraic geometry. *Oxford Mathematical Monographs*, The Clarendon Press, Oxford University Press, 2006.

[21] A. Ishii and K. Ueda. Dimer models and exceptional collections. *preprint, 2009*, arXiv:0911.4529.

[22] Y. Kawamata. Derived categories of toric varieties. *Michigan Math. J.*, 54(2006), 517-535.

[23] A. D. King. Tilting bundles on some rational surfaces. *preprint, 1997.*

[24] M. Kontsevich. Homological algebra of mirror symmetry. *Proceedings of the International Congress of Mathematicians*, Vol. 1, 2 (Zürich, 1994), 120-139, Birkhäuser, Basel, 1995.

[25] M. Michalek. Family of counterexamples to King's conjecture. *C. R. Math. Acad. Sci. Pairs*, 349(2011), 67-69.

[26] S. Mukai. Duality between D(X) and D($\hat{\mathrm{X}}$) with its application to Picard sheaves. *Nagoya Math. J.*, 81 (1981), 153-175.

[27] S. Novaković. No phantoms in the derived category of curves over arbitrary fields, and derived characterizations of Brauer-Severi varieties. *preprint*, arXiv:1701.03020.

[28] N. Prabhu-Naik. Tilting bundles on toric Fano fourfolds. *J. Algebra*, 471(2017), 348-398.

[29] P. Sosna. Some remarks on phantom categories and motives. *preprint*, arXiv:1511.07711.

[30] J.-L. Verdier. Categories dérivées. *Séminaire de Géométrie Algebrique du Bois-Marie (SGA* $4\frac{1}{2}$ *)*, Springer-Verlag (Lecture Notes in Math., vol. 569), Berlin 1977, pp. 262-311.

[31] Š. Špenko and M. Van den Bergh with an appendix by J. P. Bell. On the noncommutative Bondal-Orlov conjecture for some toric varieties. *preprint*, arXiv:1804.02881.

[32] C. Wang. On H-trivial line bundles on toric DM stacks of dimension two. *preprint, 2018*, arXiv:1812.01758.